PEARSON mathematics

NEW SOUTH WALES

S.B. 10

STAGES 5.1, 5.2 AND 5.3

AUSTRALIAN CURRICULUM

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Breath
Water
I am Maths
Living in extremes
Sustaining the environment

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We would like to thank our authors for their extraordinary dedication and their contribution to the development of this project.

Pearson Australia

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Fibonacci poem

The poem on the inside front cover is called a Fib. The sequence of syllables in the lines follows a Fibonacci sequence: 1, 1, 2, 3, 5, 8.

Contents

Series features vi
Using Pearson Mathematics New South Wales viii
Stage 5 Curriculum Grid for Years 9 and 10 x

Chapter 1 Financial mathematics 2

Recall 1 2
Stage 5.1 **1.1** Interest 3
Technology Exploration: Spreadsheet 5
Stage 5.1 **1.2** Compound interest—the general formula 11
Problem solving: The oldest cake problem 15
Stage 5.2◊ **1.3** Compound interest—different compounding periods 16
Half-time 1 19
Stage 5.2◊ **1.4** Compound interest—further applications 20
Stage 5.2◊ **1.5** Comparing interest rates 24
Stage 5.2◊ **1.6** Depreciation 28
Problem solving: Weird and wonderful numbers 34
Chapter review 1 35
D.I.Y. Summary 35

Chapter 2 Algebra 38

Recall 2 38
Stage 5.2 **2.1** Expanding brackets 39
Stage 5.3§ **2.2** Expanding special products 43
Stage 5.2 **2.3** Factorising using common factors 47
Stage 5.2 **2.4** Factorising monic quadratic expressions 51
Stage 5.3§ **2.5** Factorising using special products 55
Problem solving: Playground swings 60
Half-time 2 61
Stage 5.3§ **2.6** Factorising by completing the square 62
Stage 5.3§ **2.7** Factorising non-monic quadratic trinomials 66
Stage 5.2 **2.8** Algebraic fractions 70
Stage 5.3§ **2.9** More complex algebraic fractions 74
Problem solving: How to square two-digit numbers quickly—without a calculator 79
Stage 5.3§ **2.10** Rearranging formulas 80
Problem solving: Gravity 84
Chapter review 2 85
D.I.Y. Summary 85

Chapter 3 Measurement 89

Recall 3 89
Stage 5.1 **3.1** Area 90
Stage 5.2 **3.2** Surface area of prisms and cylinders 97
Stage 5.2 **3.3** Volume of prisms and cylinders 101
Problem solving: Cola cost crisis 105
Half-time 3 106
Stage 5.3 **3.4** Surface area of tapered solids and spheres 107
Problem solving: Egyptian engineering 113
Stage 5.3 **3.5** Volume of tapered solids and spheres 114
Stage 5.3 **3.6** Applications of volume 120
Problem solving: A pinch of salt 124
Chapter review 3 125
D.I.Y. Summary 125
Mixed review A 129
Challenge A 131

Chapter 4 Linear relationships 132

Recall 4 132
Stage 5.2 **4.1** Linear equations 133
Stage 5.3§ **4.2** Solving more complex linear equations 139
Stage 5.2 **4.3** Gradient review 141
Stage 5.3§ **4.4** Calculating gradient 145
Stage 5.2 **4.5** Sketching linear graphs using gradient and y-intercept 150
Problem solving: Odd one out 155
Half-time 4 156
Stage 5.3§ **4.6** Sketching linear graphs using x- and y-intercepts 157
Stage 5.3§ **4.7** Parallel and perpendicular lines 159
Problem solving: Area of a triangle 161
Stage 5.2 **4.8** Linear inequalities 162
Problem solving: Win ratio 166
Stage 5.2 **4.9** Simultaneous equations 167
Chapter review 4 175
D.I.Y. Summary 175

Chapter 5 Trigonometry 178

Recall 5 178

Stage 5.2◊ **5.1** The trigonometric ratios 179

Technology Exploration: Calculator 180

Stage 5.2◊ **5.2** Finding lengths 185

Stage 5.2◊ **5.3** Finding angles 190

Technology Exploration: Calculator 190

Stage 5.2◊ **5.4** Angles of elevation and depression 194

Problem solving: Outback mail run 197

Half-time 5 198

Stage 5.2◊ **5.5** Bearings 199

Problem solving: Seeing the light 203

Stage 5.2◊ **5.6** Mixed two-dimensional problems 204

Chapter review 5 210

D.I.Y. Summary 210

Chapter 6 Advanced trigonometry 212

Recall 6 212

Stage 5.3§ **6.1** Solving three-dimensional problems 213

Problem solving: The maths challenge 219

Stage 5.3§ **6.2** The unit circle 220

Stage 5.3§ **6.3** Trigonometric graphs 226

Technology Exploration: GeoGebra 227

Stage 5.3§ **6.4** Solving trigonometric equations 232

Half-time 6 235

Stage 5.3§ **6.5** The sine and cosine rules 236

Problem solving: Trig chimps 242

Stage 5.3§ **6.6** Applications of the sine and the cosine rules 243

Stage 5.3§ **6.7** Area of a triangle using trigonometry 249

Problem solving: Triangles to the max 252

Chapter review 6 253

D.I.Y. Summary 253

Mixed review B 256

Challenge B 260

Chapter 7 Statistics 262

Recall 7 262

Stage 5.2◊ **7.1** Cumulative frequency curves 263

Problem solving: Product codes 267

Stage 5.2◊ **7.2** Box plots 268

Problem solving: Draw that plot 276

Stage 5.2◊ **7.3** Comparing data sets 277

Problem solving: What do you mean? 285

Stage 5.2 **7.4** Scatter plots 286

Technology Exploration: Spreadsheet 288

Half-time 7 293

Stage 5.2 **7.5** Time-related data 294

Stage 5.2◊ **7.6** Survey data: reliability and validity 299

Stage 5.3 **7.7** Standard deviation 304

Technology Exploration: Spreadsheet 306

Problem solving: Keeping score 309

Stage 5.3 **7.8** Lines of best fit 310

Technology Exploration: Spreadsheet 311

Problem solving: Making the data fit 315

Stage 5.3 **7.9** Survey data: planning and implementation 316

Chapter review 7 322

D.I.Y. Summary 322

Chapter 8 Geometry 328

Recall 8 328

Stage 5.2 **8.1** Congruent and similar triangles 329

Stage 5.2 **8.2** Proving congruence and similarity 336

Stage 5.3§ **8.3** Proofs using congruent triangles 340

Problem solving: How many triangles? 344

Stage 5.3§ **8.4** Proofs using similar triangles 345

Stage 5.3§ **8.5** Geometric properties of special quadrilaterals 349

Problem solving: Folding and cutting 356

Half-time 8 357

Stage 5.3# **8.6** Angles in circles 359

Technology Exploration: GeoGebra 361

Stage 5.3# **8.7** Chords of circles 371

Technology Exploration: GeoGebra 372

Stage 5.3# **8.8** Tangents and secants 381

Technology Exploration: GeoGebra 382

Chapter review 8 388

D.I.Y. Summary 388

Chapter 9 Non-linear relationships 393

Recall 9 393

Stage 5.2◊ 9.1 Non-linear graphs and their equations 394
Problem solving: How much is enough? 400
Stage 5.3§ 9.2 Quadratic transformations 401
Technology Exploration: GeoGebra 402
Problem solving: Finding curved areas 409
Stage 5.2 9.3 Solving quadratic equations using the Null Factor Law 410
Stage 5.3§ 9.4 Solving quadratic equations using special product rules 415
Stage 5.3§ 9.5 Solving quadratics by completing the square 418
Stage 5.3§ 9.6 Solution of non-monic quadratics 422
Problem solving: Projectiles 427
Half-time 9 428
Stage 5.3§ 9.7 Sketching parabolas 429
Stage 5.3§ 9.8 Rectangular hyperbolas and inverse proportion 435
Problem solving: The mass of Earth 443
Stage 5.3§ 9.9 Sketch graphs of circles, exponentials and rectangular hyperbolas 444
Technology Exploration: GeoGebra 445
Stage 5.3§ 9.10 Solving simultaneous equations where one equation is non-linear 453
Technology Exploration: GeoGebra 454
Technology Exploration: Spreadsheet 456
Stage 5.3 9.11 Curves and rates 461
Chapter review 9 466
D.I.Y. Summary 466
Mixed review C 471
Challenge C 473

Chapter 10 Polynomials 474

Recall 10 474
Stage 5.3§ 10.1 Equations of the form $y = ax^n$ and their graphs 475
Problem solving: Different paths 480
Stage 5.3§ 10.2 Transformations of $y = ax^n$ 481
Stage 5.3# 10.3 Polynomials 487
Half-time 10 495
Stage 5.3# 10.4 Remainder and factor theorems 496
Stage 5.3# 10.5 Sketching polynomials 501
Stage 5.3# 10.6 Functions and inverses 511
Chapter review 10 518
D.I.Y. Summary 518

Chapter 11 Probability 520

Recall 11 520
Stage 5.2 11.1 Probability review 521
Problem solving: A chance of freedom 529
Stage 5.2 11.2 Venn diagrams and two-way tables 530
Stage 5.2 11.3 Mutually exclusive events 536
Half-time 11 539
Stage 5.2 11.4 Probability tree diagrams 540
Problem solving: The origins of probability theory 546
Stage 5.2 11.5 Conditional statements 547
Stage 5.2 11.6 Dependent and independent events 551
Problem solving: Winning the lottery 555
Chapter review 11 556
D.I.Y. Summary 556

Chapter 12 Surds, indices and logarithms 560

Recall 12 560
Stage 5.3§ 12.1 Rational and irrational numbers 561
Problem solving: Surd shells 565
Stage 5.3§ 12.2 Multiplying and dividing surds 566
Stage 5.3§ 12.3 Adding and subtracting surds 571
Stage 5.3§ 12.4 Surds and the distributive law 575
Problem solving: Connecting squares 578
Stage 5.3§ 12.5 Rationalising the denominator 579
Half-time 12 583
Stage 5.2 12.6 Summary of index laws 584
Stage 5.3§ 12.7 Fractional indices 588
Technology Exploration: Calculator 588
Problem solving: Lagrange's four-square theorem 592
Stage 5.3# 12.8 Logarithms 593
Stage 5.3# 12.9 Laws of logarithms 596
Stage 5.3# 12.10 Logarithmic and exponential graphs and equations 600
Chapter review 12 605
D.I.Y. Summary 605
Mixed review D 608
Challenge D 610
Answers 611
Glossary and index 723

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Pearson Mathematics New South Wales provides a differentiated pathway for every student, enabling you to support the full range of student abilities.

Pearson Mathematics New South Wales differentiated pathway structure

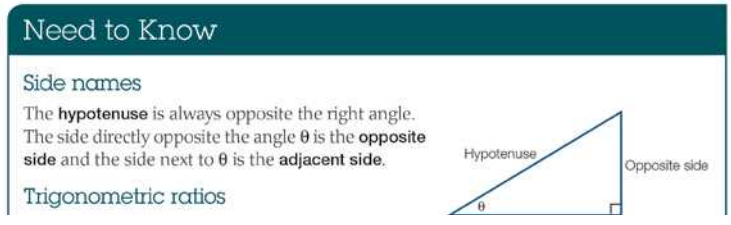
Recall	Each chapter has a Recall section that tests whether students have the prerequisite skills and understanding to begin a chapter. Every Recall question has a matching Recall Worksheet available free on ProductLink that teaches the relevant skill and allows students to practise it.	**Below Stage Level**
Bridging	Over 150 Bridging Worksheets are included with Pearson eBook 3.0 to support and scaffold students across Years 7 and 8 who struggle with secondary mathematics concepts, so that they are able to complete specific Student Book exercises.	
Foundation	A Foundation pathway through each exercise is shown in the Navigator feature to ensure students at this level are given sufficient skills practice as well as some exposure to the higher-order aspects of Working Mathematically.	**3 Navigator Pathways At Stage Level**
Standard	A Standard pathway through each exercise is shown in the Navigator feature to ensure students at this level are given a balance of skills practice and the higher-order aspects of Working Mathematically.	
Advanced	An Advanced pathway through each exercise is shown in the Navigator feature to ensure students at this level are given skills practice as well as an emphasis on the higher-order aspects of Working Mathematically.	
Challenge	Challenge pages feature high-level extension work for the most able students.	**Above Stage Level**

Pearson Mathematics New South Wales provides students with just the right amount of information in one no-frills book at each year level.

In Years 9 and 10, **Pearson Mathematics New South Wales** covers the full range of Stages 5.1, 5.2 and 5.3 (and Stage 4 revision where relevant), with content from all Stages explicitly labelled for maximum flexibility. To assist planning for Stage 6, the symbol ◊ indicates material for students intending to study Preliminary Mathematics General/ HSC Mathematics General 2; § is for Mathematics ('2 Unit'); and # is for Mathematics Extension 1.

Stage 4 | Stage 5.1 | Stage 5.2 | Stage 5.3
Stage 5.2◊ | Stage 5.3§ | Stage 5.3#

Need to Know sections provide concise, accurate theory that promotes understanding.


Need to Know

Side names

The **hypotenuse** is always opposite the right angle. The side directly opposite the angle θ is the **opposite side** and the side next to θ is the **adjacent side**.

Trigonometric ratios

Worked Examples clearly demonstrate the working that students will need to show to achieve full marks in a test.	**Worked Example 21** WE21 Factorise the following. (a) $ab + 5b + 3a + 15$ (b) $3x + 12 - xy - 4y$ **Working** (a) $ab + 5b + 3a + 15$ $= b(a + 5) + 3(a + 5)$ $= (a + 5)(b + 3)$ (b) $3x + 12 - xy - 4y$ $= 3(x + 4) - y(x + 4)$ $= (x + 4)(3 - y)$
Pearson Mathematics New South Wales is a perfect match to the NSW Syllabus for the Australian Curriculum, including all the higher-order aspects of Working Mathematically and the technology requirements. Syllabus references are provided at the start of every chapter, while a detailed Syllabus Correlation document (available free on ProductLink) shows where every element of syllabus content is covered.	**Syllabus references** **Number and Algebra: Non-Linear Relationships** ✓ Graph simple non-linear relationships, with and without the use of digital technologies, and solve simple related equations **(ACMNA296) (Stage 5.2◊)** ✓ Explore the connection between algebraic and graphical representations of relationships such as simple quadratics, circles and exponentials using digital technologies as appropriate **(ACMNA239) (Stage 5.2◊)** ✓ Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations **(ACMNA267) (Stage 5.3§)** **Number and Algebra** ✓ Solve simple quadrat strategies **(ACMNA241)** ✓ Solve a wide range of a variety of contexts ✓ Solve simultaneous e non-linear, using alg including the use of c **Number and Algebra** ✓ Solve problems involv the relationship betw corresponding to sim **(Stage 5.3)**
Questions in every exercise are categorised into 'Fluency and understanding' and 'Reasoning, communicating and open-ended'. High quality Problem solving tasks also appear regularly.	**Problem solving** **Odd one out** Four of these points lie on a single straight line. Which one is the odd one out? (2, 5) (-5, -7) (2, 4) (5, 13) (9, 21) Strategy options • Draw a diagram. • Look for a pattern. • Test all possible combinations.
Technology Explorations appear regularly throughout the text, providing students with a deeper understanding of concepts covered and making it easy to integrate ICT into the classroom.	**Technology Exploration: GeoGebra** **Non-linear graphs** Open GeoGebra. Use the Input bar at the bottom of the window to enter the equations to be graphed. Input: To see the graph clearly, select from the menu View > Layout… and then in the pop-up window go to Preferences - Layout > Basic. You can also adjust the preferences by

Pearson Mathematics New South Wales provides regular graded revision and reinforcement to help your students consolidate their learning.

Half-time—mid-chapter review	Half-time 1
Chapter review—end-of-chapter review	Chapter review 5
Mixed review—cumulative multi-chapter review	Mixed review A
Challenge—more challenging questions for students who need extension work	Challenge B

PEARSON mathematics

NEW SOUTH WALES

Stage 5 Curriculum Grid for Years 9 and 10

This maps the NSW Mathematics Syllabus (Stage 5) for the Australian Curriculum content to the *Pearson Mathematics New South Wales 9* and *10 Stages 5.1, 5.2 and 5.3* Student Books.

Detailed syllabus references are given at the start of each chapter. A full Stage 5 curriculum audit, including syllabus content dot points, is available on ProductLink.

Stage 5.1	Year 9	Year 10
Number and Algebra		
Financial Mathematics	Ch. 1	Ch. 1
Solve problems involving earning money	1.3, 1.4	
Solve problems involving simple interest **(ACMNA211)**	1.5, 1.6	1.1
Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies **(ACMNA229)**		1.1, 1.2
Indices	Ch. 2	
Extend and apply the index laws to variables, using positive-integer indices and the zero index **(ACMNA212)**	2.1, 2.2	
Simplify algebraic products and quotients using index laws **(ACMNA231)**	2.1, 2.2	
Apply index laws to numerical expressions with integer indices **(ACMNA209)**	2.2	
Linear Relationships	Ch. 4	
Find the midpoint and gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software **(ACMNA294)**	4.1, 4.6	
Find the distance between two points located on the Cartesian plane using a range of strategies, including graphing software **(ACMNA214)**	4.1	
Sketch linear graphs using the coordinates of two points **(ACMNA215)**	4.5, 4.7	
Solve problems involving parallel lines **(ACMNA238)**	4.6, 4.7	
Non-Linear Relationships	Ch. 8	
Graph simple non-linear relations, with and without the use of digital technologies **(ACMNA296)**	8.1	
Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technologies as appropriate **(ACMNA239)**	8.1	
Measurement and Geometry		
Area and Surface Area	Ch. 3	Ch. 3
Calculate the areas of composite shapes **(ACMMG216)**	3.2	3.1
Solve problems involving the surface areas of right prisms **(ACMMG218)**	3.3, 3.4	
Numbers of Any Magnitude	Ch. 2	
Investigate very small and very large time scales and intervals **(ACMMG219)**	2.4	
Express numbers in scientific notation **(ACMNA210)**	2.4	
Right-Angled Triangles (Trigonometry)	Ch. 6	
Use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles **(ACMMG223)**	6.1, 6.2, 6.3, 6.4	
Apply trigonometry to solve right-angled triangle problems **(ACMMG224)**	6.3, 6.4	
Solve right-angled triangle problems, including those involving angles of elevation and depression **(ACMMG245)**	6.5	
Properties of Geometrical Figures	Ch. 5	
Use the enlargement transformation to explain similarity **(ACMMG220)**	5.2	
Solve problems using ratio and scale factors in similar figures **(ACMMG221)**	5.2	
Statistics and Probability		
Single Variable Data Analysis	Ch. 7	
Identify everyday questions and issues involving at least one numerical and at least one categorical variable, and collect data directly from secondary sources **(ACMSP228)**	7.2	
Construct back-to-back stem-and-leaf plots and histograms and describe data, using terms including 'skewed', 'symmetric' and 'bi-modal' **(ACMSP282)**	7.3, 7.4	
Compare data displays using mean, median and range to describe and interpret numerical data sets in terms of location (centre) and spread **(ACMSP283)**	7.4	
Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data **(ACMSP253)**	7.2	

Probability	Ch. 7	Ch. 11
Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or' **(ACMSP226)**	7.5, 7.6	11.1, 11.2

Stage 5.2	**Year 9**	**Year 10**
Number and Algebra		
Financial Mathematics ◊		Ch. 1
Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies **(ACMNA229)**		1.3, 1.4, 1.5, 1.6
Ratios and Rates	Ch. 8	
Solve problems involving direct proportion; explore the relationship between graphs and equations corresponding to simple rate problems **(ACMNA208)**	8.8	
Algebraic Techniques	Ch. 2	Ch. 2
Apply the four operations to simple algebraic fractions with numerical denominators **(ACMNA232)**		2.8
Apply the four operations to algebraic fractions with pronumerals in the denominator		2.8
Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate **(ACMNA213)**	2.6	2.1
Factorise algebraic expressions by taking out a common algebraic factor **(ACMNA230)**	2.7	2.3
Expand binomial products and factorise monic quadratic expressions using a variety of strategies **(ACMNA233)**	2.6, 2.7, 2.8	2.1, 2.4
Indices	Ch. 2	Ch. 12
Apply index laws to algebraic expressions involving integer indices	2.3	12.6
Equations	Ch. 4, Ch. 8	Ch. 3, Ch. 4, Ch. 5, Ch. 9
Solve linear equations **(ACMNA215)**	4.3	4.1
Solve linear equations involving simple algebraic fractions **(ACMNA240)**	4.3	4.1
Solve simple quadratic equations using a range of strategies **(ACMNA241)**	8.3	9.3
Substitute values into formulas to determine an unknown **(ACMNA234)**	4.4	3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 5.2, 5.3, 5.4, 5.5, 5.6
Solve problems involving linear equations, including those derived from formulas **(ACMNA235)**	4.4	4.1
Solve linear inequalities and graph their solutions on a number line **(ACMNA236)**		4.8
Solve linear simultaneous equations, using algebraic and graphical techniques, including with the use of digital technologies **(ACMNA237)**		4.9
Linear Relationships	Ch. 4	Ch. 4
Interpret and graph linear relationships using the gradient–intercept form of the equation of a straight line	4.8	4.3, 4.5
Solve problems involving parallel and perpendicular lines **(ACMNA238)**	4.8	4.7
Non-Linear Relationships ◊	Ch. 8	Ch. 9
Graph simple non-linear relationships, with and without the use of digital technologies, and solve simple related equations **(ACMNA296)**	8.2, 8.5, 8.7	9.1
Explore the connection between algebraic and graphical representations of relationships such as simple quadratics, circles and exponentials using digital technologies as appropriate **(ACMNA239)**	8.2, 8.5, 8.7	9.1
Measurement and Geometry		
Area and Surface Area	Ch. 3	Ch. 3
Calculate the surface areas of cylinders and solve related problems **(ACMMG217)**	3.5	3.2
Solve problems involving surface area for a range of prisms, cylinders and composite solids **(ACMMG242)**	3.5	3.2
Volume	Ch. 3	Ch. 3
Solve problems involving the volumes of right prisms **(ACMMG218)**	3.6	3.3
Solve problems involving volume for a range of prisms, cylinders and composite solids **(ACMMG242)**	3.6	3.3
Right-Angled Triangles (Trigonometry) ◊	Ch. 6	Ch. 5
Apply trigonometry to solve right-angled triangle problems **(ACMMG224)**		5.1, 5.2, 5.3, 5.6
Solve right-angled triangle problems, including those involving direction and angles of elevation and depression **(ACMMG245)**	6.6	5.4, 5.5
Properties of Geometrical Figures	Ch. 5	Ch. 8
Formulate proofs involving congruent triangles and angle properties **(ACMMG243)**	5.5, 5.6	8.1, 8.2
Use the enlargement transformations to explain similarity and to develop the conditions for triangles to be similar **(ACMMG220)**	5.3	8.1, 8.2
Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes **(ACMMG244)**	5.4, 5.5, 5.6	8.1, 8.2

Statistics and Probability		
Single Variable Data Analysis ◊		Ch. 7
Determine quartiles and interquartile range **(ACMSP248)**		7.1
Construct and interpret box plots and use them to compare data sets **(ACMSP249)**		7.2
Compare shapes of box plots to corresponding histograms and dot plots **(ACMSP250)**		7.3
Investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians **(ACMSP227)**		7.6
Bivariate Data Analysis		Ch. 7
Investigate and describe bivariate numerical data where the independent variable is time **(ACMSP252)**		7.5
Use scatter plots to investigate and comment on relationships between two numerical variables **(ACMSP251)**		7.4
Probability	Ch. 7	Ch. 11
List all outcomes for two-step chance experiments, with and without replacement, using tree diagrams or arrays; assign probabilities to outcomes and determine probabilities for events **(ACMSP225)**	7.7	11.1, 11.2, 11.3, 11.4
Describe the results of two- and three-step chance experiments, with and without replacement, assign probabilities to outcomes, and determine probabilities of events; investigate the concept of independence **(ACMSP246)**		11.4, 11.6
Use the language of 'if ... then', 'given', 'of', 'knowing that' to investigate conditional statements and to identify common mistakes in interpreting such language **(ACMSP247)**		11.5

Stage 5.3	Year 9	Year 10
Number and Algebra		
Ratios and Rates		Ch. 9
Solve problems involving direct proportion; explore the relationship between graphs and equations corresponding to simple rate problems **(ACMNA208)**		9.11
Algebraic Techniques §	Ch. 2	Ch. 2
Add and subtract algebraic fractions with numerical denominators, including those with binomial numerators		2.9
Expand binomial products using a variety of strategies **(ACMNA233)**	2.9	2.2
Factorise monic and non-monic quadratic expressions **(ACMNA269)**	2.9	2.5, 2.6, 2.7, 2.9
Surds and Indices §		Ch. 12
Define rational and irrational numbers and perform operations with surds and fractional indices **(ACMNA264)**		12.1, 12.2, 12.3, 12.4, 12.5, 12.7
Equations §	Ch. 2	Ch. 2, Ch. 4, Ch. 9, Ch. 10
Solve complex linear equations involving algebraic fractions		4.2
Solve a wide range of quadratic equations derived from a variety of contexts **(ACMNA269)**		9.4, 9.5, 9.6
Solve simple cubic equations		10.5
Rearrange literal equations	2.5	2.10
Solve simultaneous equations, where one equation is non-linear, using algebraic and graphical techniques, including the use of digital technologies		9.10
Linear Relationships §	Ch. 4	Ch. 4
Find the midpoint and gradient of a line segment (interval) on the Cartesian plane **(ACMNA294)**	4.2	4.4
Find the distance between two points located on the Cartesian plane **(ACMNA214)**	4.2	
Sketch linear graphs using the coordinates of two points **(ACMNA215)**	4.9	4.6
Solve problems using various standard forms of the equation of a straight line		4.6
Solve problems involving parallel and perpendicular lines **(ACMNA238)**		4.4, 4.7
Non-Linear Relationships §	Ch. 8	Ch. 9, Ch. 10
Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations **(ACMNA267)**	8.4, 8.6	9.2, 9.7, 9.8, 9.9, 9.10
Describe, interpret and sketch cubics, other curves and their transformations		10.1, 10.2, 10.5

Polynomials #		Ch. 10
Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems **(ACMNA266)**		10.3, 10.4, 10.5
Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation **(ACMNA268)**		10.5
Logarithms #		Ch. 12
Use the definition of a logarithm to establish and apply the laws of logarithms **(ACMNA265)**		12.8, 12.9, 12.10
Solve simple exponential equations **(ACMNA270)**		12.10
Functions and Other Graphs #		Ch. 10
Describe, interpret and sketch functions		10.6

Measurement and Geometry		
Area and Surface Area		Ch. 3
Solve problems involving the surface areas of right pyramids, right cones, spheres and related composite solids **(ACMMG271)**		3.4
Volume		Ch. 3
Solve problems involving the volumes of right pyramids, right cones, spheres and related composite solids **(ACMMG271)**		3.5, 3.6
Trigonometry and Pythagoras' Theorem §		Ch. 6
Apply Pythagoras' theorem and trigonometry to solve three-dimensional problems in right-angled triangles **(ACMMG276)**		6.1
Use the unit circle to define trigonometric functions, and graph them, with and without the use of digital technologies **(ACMMG274)**		6.2, 6.3
Solve simple trigonometric equations **(ACMMG275)**		6.2, 6.4
Establish the sine, cosine and area rules for any triangle and solve related problems **(ACMMG273)**		6.5, 6.6, 6.7
Properties of Geometrical Figures §		Ch. 8
Formulate proofs involving congruent triangles and angle properties **(ACMMG243)**		8.3, 8.4
Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes **(ACMMG244)**		8.1, 8.3, 8.4, 8.5
Circle Geometry #		Ch. 8
Prove and apply angle and chord properties of circles **(ACMMG272)**		8.6, 8.7
Prove and apply tangent and secant properties of circles		8.8

Statistics and Probability		
Single Variable Data Analysis		Ch. 7
Calculate and interpret the mean and standard deviation of data and use these to compare data sets **(ACMSP278)**		7.7
Bivariate Data Analysis		Ch. 7
Use information technologies to investigate bivariate numerical data sets; where appropriate, students use a straight line to describe the relationship, allowing for variation **(ACMSP279)**		7.8
Investigate reports of studies in digital media and elsewhere for information on their planning and implementation **(ACMSP277)**		7.9

1 Financial mathematics

P Recall 1

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Convert the following percentages to decimals.

(a) 17% **(b)** 6% **(c)** 19.2%

(d) 4.8% **(e)** 2.01% **(f)** $\frac{1}{4}$ %

2 Calculate:

(a) 15% of \$20 000 **(b)** 6% of \$10 000

(c) 9.5% of \$350 000 **(d)** $7\frac{1}{2}$ % of \$45 000

3 **(a)** Increase 40 by:

(i) 100% **(ii)** 50% **(iii)** 25% **(iv)** 200%

(b) Decrease 60 by:

(i) 50% **(ii)** 30% **(iii)** 10% **(iv)** 75%

4 Evaluate:

(a) 7^2 **(b)** 2^3 **(c)** 10^6

(d) $(0.6)^2$ **(e)** $\left(\frac{1}{2}\right)^5$ **(f)** $(1.1)^4$

5 Convert the following to a decimal number of years. Assume a year has 52 weeks or 365 days.

(a) 6 months **(b)** 3 months **(c)** 8 years and 9 months

(d) 26 weeks **(e)** 13 weeks **(f)** 146 days

Syllabus references

Number and Algebra: Financial Mathematics

✓ Solve problems involving simple interest **(ACMNA211) (Stage 5.1)**

✓ Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies **(ACMNA229) (Stage 5.1)**

✓ Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies **(ACMNA229) (Stage 5.2◊)**

Interest

1.1

Stage 5.1

Need to Know

Borrowing and lending

There are many situations in which individuals, organisations or communities borrow money from, or lend money to, others.

For example:

- An individual may borrow money from a bank to buy a house, a car or to start a business.
- An investor who puts money in a bank account is lending money to the bank. The bank in turn uses the investor's money to lend to others.
- A government can borrow money from individuals by selling government bonds. This money can then be used to pay for some of the services they provide.

The original amount of money that is borrowed or loaned is called the **principal**.

The lender has provided a service for which they can expect to be paid. The borrower eventually repays the lender the principal and an extra agreed amount called the **interest**.

The interest is usually expressed as a percentage **per annum** (p.a.), which means 'for each year' in Latin.

Calculating simple interest

Simple interest (I) is the amount of interest accrued on the principal over a time period and depends on the principal (P), the interest rate per annum (R), and the time in years (N). It is calculated by finding the product of P, R and N. This gives the total interest that needs to be paid at the end of the time period. In the case of simple interest, the total **amount** is the sum of the principal and the interest.

Simple interest formula

$I = PRN$

where I = amount of simple interest
P = principal
R = rate of interest per annum (p.a.) as a decimal
N = time in years

Also:

$A = P + I$

where A = total amount repaid on the loan or collected from the investment
P = principal
I = simple interest

For example:

Calculate the simple interest and the amount on \$1000 at 6% p.a. for 4 years.

$I = PRN$

$= \$1000 \times 0.06 \times 4$

$= \$240$

$A = \$1240$

Compound interest

Compound interest is based on the idea of simple interest but it is designed to make the amount of interest grow larger each time period by increasing the principal. It differs from simple interest as the interest accrued after one year is added to the principal before the interest for the next year is calculated.

Compound interest achieves growth with two simple steps:

1. Calculating, at the end of the time period, the interest on the principal for that time period.
2. Adding this interest to the principal to make the new principal for the next time period.

For example:

Calculate the compound interest and the amount on \$1000 at 6% p.a. for 4 years.

Year	Principal	Interest	Amount
1	$P_1 = \$1000$	$I_1 = PRN$ $= \$1000 \times 0.06 \times 1$ $= \$60$	$A_1 = P_1 + I_1$ $= \$1000 + \$60$ $= \$1060$ $= P_2$
2	$P_2 = \$1060$	$I_2 = PRN$ $= \$1060 \times 0.06 \times 1$ $= \$63.60$	$A_2 = P_2 + I_2$ $= \$1060 + \$63.60$ $= \$1123.60$ $= P_3$
3	$P_3 = \$1123.60$	$I_3 = PRN$ $= \$1123.6 \times 0.06 \times 1$ $= \$67.416$	$A_3 = P_3 + I_3$ $= \$1123.60 + \$67.416$ $= \$1191.016$ $= P_4$
4	$P_4 = \$1191.016$	$I_4 = PRN$ $= \$1191.016 \times 0.06 \times 1$ $= \$71.46096$	$A_4 = P_4 + I_4$ $= \$1191.016 + \$71.46096$ $= \$1262.47696$

For practical reasons, the answer is finally rounded to \$1262.48. It is important to avoid rounding any numbers until the end of the calculation because rounding errors can occur if we round too early.

The value of this investment after 4 years of simple interest would be only \$1240.

Total compound interest = \$262.48

Total simple interest = \$240

In this case, compounding the investment earned \$22.48 more than simple interest would earn over the same time period.

Appreciation and inflation

Increase in cost or value is called **appreciation**. The rate of increase in prices within the economy is called the **inflation rate**. The simple interest formula can be used to calculate a **flat rate** of appreciation.

Technology Exploration: Spreadsheet

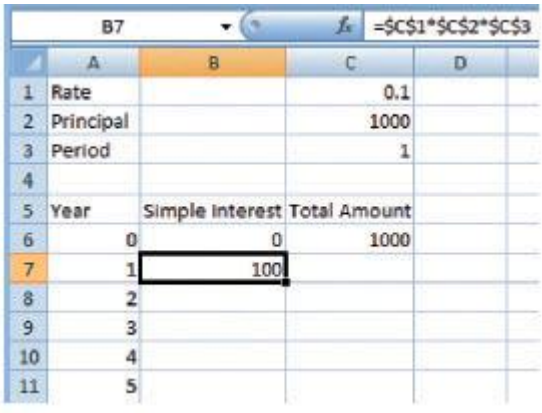
Comparing simple and compound interest

Equipment required: computer with a Microsoft Excel® spreadsheet (or similar)

Calculating simple interest

1 Open a new spreadsheet file to calculate the interest on a principal of $1000 invested at a rate of 10% p.a. for a year. Use the decimal equivalent of 10% and a time period of 1. Enter the headings 'Rate', 'Principal' and 'Period' in cells A1, A2 and A3 and 0.1, 1000 and 1 in cells C1, C2 and C3 as shown in the screen shot below. Enter the headings of 'Year', 'Simple Interest' and 'Total Amount' in cells A5, B5 and C5. It is important that you put the headings and values in the cells indicated as these cells are referenced for calculations later on.

2 Enter the year numbers under the heading of 'Year' in column A. Enter 0 in cell A6 then the formula =A6+1 in cell A7. To copy this formula down the column, you can 'Fill Down' by placing the cursor over the bottom right corner of the cell until the black cross appears, then clicking and dragging down. Alternatively, enter 1 in A7 and, highlighting A6 and A7, drag down the column. Under the headings of 'Simple Interest' and 'Total Amount', enter 0 and 1000 in cells B6 and C6 respectively. Widen column B by placing the cursor over the line between headings B and C until it changes to a cross with horizontal arrows. Double click on it. Repeat this process for column C.



3 To calculate the interest earned after one year, enter the formula =C1*C2*C3 into cell B7. We use a '$' symbol to 'fix' a particular cell, meaning that these cell addresses will still be used when the formula is copied down the column. Copy the formula down the column as described in Step **2**.

4 To add the interest to the total amount, enter the formula =B7+C6 into cell C7. Copy this formula down the column. Add the heading 'Amount of Interest' in cell D5 and 0 in D6. To calculate the total interest put the formula =D6+B7 into cell D7 and copy it down the column. Again, this column needs to be widened.

(a) What is the total amount of the investment after 8 years?

(b) How much of this total amount is interest?

Compounding interest

5 Extend the table with the headings 'Year' in E5, 'Compound Interest' in F5, and 'Total Amount' in G5. Manually enter or use a formula to enter the number of the year from 0 to 8, and an initial value of 0 for interest and 1000 for the total amount.

6 This time, we will add the interest earned after each year to the principal, and then use this amount as the new principal to calculate the interest for the following year. To do this, enter a new formula in the interest column, as in the table below.

F7 =G6*C3*C1

	A	B	C	D	E	F	G
1	Rate		0.1				
2	Principal		1000				
3	Period		1				
4							
5	Year	Simple Interest	Total Amount	Amount of Interest	Year	Compound Interest	Total Amount
6	0	0	1000	0	0	0	1000
7	1	100	1100	100	1	100	
8	2	100	1200	200	2		
9	3	100	1300	300	3		
10	4	100	1400	400	4		
11	5	100	1500	500	5		
12	6	100	1600	600	6		
13	7	100	1700	700	7		
14	8	100	1800	800	8		

7 To add the interest to the total amount, enter the formula =F7+G6 into cell G7. Highlight both cells F7 and G7, put the cursor in the bottom right corner of G7 and drag down to row 14.

(a) What is the total amount of the investment after 8 years?

(b) How much of this total amount is interest?

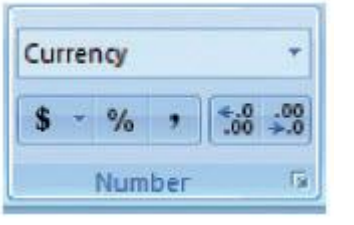
(c) How much extra interest is earned when it is calculated using a new amount each time, instead of the original principal?

(d) **(i)** Explain the difference in the interest formula used in step **3** to the one used in step **6**.

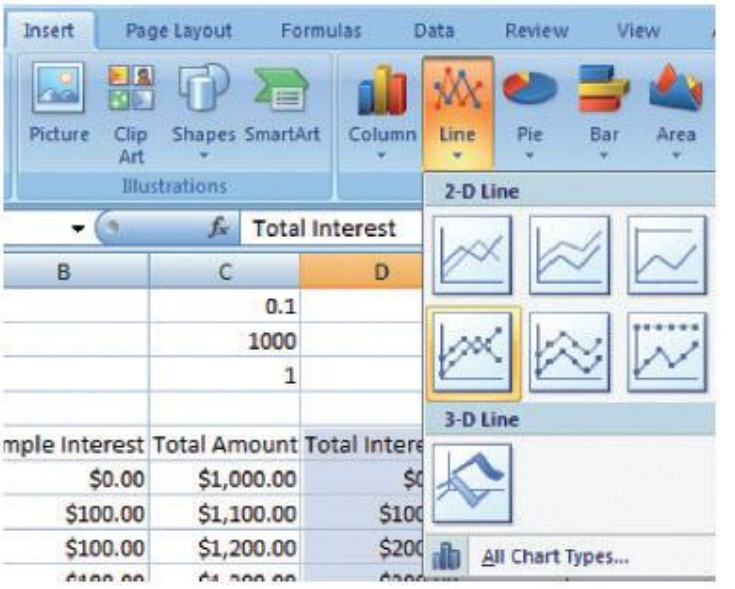
(ii) Why was it necessary to modify the formula?

8 Now put the heading 'Amount of Interest' in cell H5 and use a suitable formula to fill the values down the column.

9 Because the calculations involve money, format the entries in your table to show two decimal places. Highlight relevant cells and select Currency from the pop-up menu above the Number option under the Home tab.



10 The difference between the two types of interest can be clearly observed on a line graph. Highlight the 'Amount of Interest' columns (D and H). (Hold down the Ctrl key to highlight more than one column.) You need to highlight only rows 5 to 14. Go to the Insert tab and select the line graph as shown below.



11 Add axis labels by clicking on the Layout tab and using the Axis titles pop-up menu. Add a title by using the Chart title pop-up menu.

(a) Describe and then explain the shape of the simple interest graph.

(b) Describe and then explain the shape of the compound interest graph.

Extension

12 The amount of compound interest has been calculated by finding 10% of the principal each year and adding it on. It could have been calculated by multiplying the principal by 1.1 to find 110% of it for each year. By continuing to multiply each new amount obtained by 1.1, we have the total amount after each year. Create a third table with the headings of 'Year' and 'Total Amount'. Enter year numbers of 0 to 8 in the first column, and 1000 as the first value of the 'Total Amount' column. Under this, enter a formula that multiplies this initial principal of $1000 by 1.1. What do you notice about the values generated by the formula?

Worked Example 1

WE1

$20 000 is borrowed for 5 years at an interest rate of 9.5% p.a. If the interest is calculated as simple interest, find:

(a) the amount of interest charged over the period of the loan

(b) the total amount to be repaid at the end of the loan.

Working

(a) $P = 20\,000$, $R = 9.5\% = 0.095$, $N = 5$

$I = PRN$

$I = 20\,000 \times 0.095 \times 5$
$= \$9500$

(b) $P = 20\,000$, $I = 9500$

$A = P + I$

$A = 20\,000 + 9500$
$= \$29\,500$

Worked Example 2

WE 2

Find the information that is missing in each of the following simple interest situations.

(a) \$200 interest is paid on an investment of \$5000 after 6 months.

(b) \$350 interest is paid on an investment at 4% p.a. after 9 months.

(c) \$1250 interest is paid on an investment of \$3000 invested at 5% p.a.

Working

(a) $I = \$200$
$P = \$5000$
$R = ?$

$N = \frac{6}{12} = 0.5$

$I = PRN$
$200 = 5000 \times R \times 0.5$

$R = \frac{200}{5000 \times 0.5}$
$= 0.08$

The rate is 8% p.a.

(b) $I = \$350$
$P = ?$
$R = 0.04$

$N = \frac{9}{12} = 0.75$

$I = PRN$
$350 = P \times 0.04 \times 0.75$

$P = \frac{350}{0.04 \times 0.75}$
$= 11\,666.67$

The principal is \$11 666.67.

(c) $I = \$1250$
$P = \$3000$
$R = 0.05$
$N = ?$

$I = PRN$
$1250 = 3000 \times 0.05 \times N$

$N = \frac{1250}{3000 \times 0.05}$
$= 8.333$

The time is $8\frac{1}{3}$ years, which is 8 years and 4 months.

Worked Example 3

WE 3

\$2000 is invested at an interest rate of 7.5% p.a. Find the:

(a) simple interest earned after 3 years

(b) amount of the investment after 3 years earning simple interest

(c) amount of the investment after 3 years earning compound interest

(d) compound interest earned after 3 years

(e) difference between the amounts earned in parts **(b)** and **(c)**.

Working

(a) $I = PRN$
$= 2000 \times 0.075 \times 3$
$= \$450$

(b) $A = \$2450$

(c) $I_1 = P_1RN$
$= 2000 \times 0.075 \times 1$
$= 150$

$A_1 = P_1 + I_1$
$= 2000 + 150$
$= 2150$

$= P_2$

$I_2 = P_2RN$
$= 2150 \times 0.075 \times 1$
$= 161.25$

$A_2 = P_2 + I_2$
$= 2150 + 161.25$
$= 2311.25$

$= P_3$

$I_3 = P_3RN$
$= 2311.25 \times 0.075 \times 1$
$= 173.343\,75$

$A_3 = P_3 + I_3$
$= 2311.25 + 173.343\,75$
$= \$2484.59$

(d) $I = A_3 - P_1$
$= 2484.59 - 2000$
$= \$484.59$

(e) Difference in amounts earned
$= 2484.59 - 2450$
$= \$34.59$

Worked Example 4

WE4

Calculate the total amount owing on a loan of $320 000 after 2 years if the interest rate of 5% p.a. is:

(a) calculated as simple interest

(b) compounded annually.

Working

(a) $I = PRN$

$= 320\,000 \times 0.05 \times 2$

$= \$32\,000$

$A = \$352\,000$

(b) $I_1 = P_1RN$

$= 320\,000 \times 0.05 \times 1$

$= 16\,000$

$A_1 = P_1 + I_1$

$= 320\,000 + 16\,000$

$= 336\,000$

$P_2 = A_1$

$= 336\,000$

$I_2 = P_2RN$

$= 336\,000 \times 0.05 \times 1$

$= 16\,800$

$A_2 = P_2 + I_2$

$= 336\,000 + 16\,800$

$= \$352\,800$

1.1 Interest

Answers page 611

Stage 5.1

Navigator

Q1 (a)–(h), Q2, Q3, Q4, Q5, Q6, Q7, Q10	Q1 (d)–(k), Q2 (a), (c), (e), Q3, Q4, Q5, Q6, Q7, Q8, Q10	Q1 (g)–(n), Q2 (b), (d), (f), Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

If necessary, round the answers to the following questions to the nearest cent.

Fluency and understanding

WE1

1 For each of the following loans the interest is calculated as simple interest. Find:

(i) the amount of interest charged over the period of the loan

(ii) the total amount to be repaid at the end of the loan.

(a) $40 000 borrowed for 1 year at 8% p.a.

(b) $30 000 borrowed for 1 year at 9% p.a.

(c) $30 000 borrowed for 5 years at 7% p.a.

(d) $40 000 borrowed for 12 years at 5% p.a.

(e) $150 000 borrowed for 25 years at 7.5% p.a.

(f) $240 000 borrowed for 30 years at 6.5% p.a.

(g) $300 000 borrowed for 1.1 years at 6.75% p.a.

(h) $350 000 borrowed for 1.1 years at 7.25% p.a.

(i) \$13 000 borrowed for 5 years at $9\frac{1}{2}$ % p.a.

(j) \$15 800 borrowed for 4 years at $10\frac{3}{4}$ % p.a.

(k) \$45 000 borrowed for $12\frac{1}{2}$ years at 5% p.a.

(l) \$36 000 borrowed for $5\frac{1}{2}$ years at 7% p.a.

(m) \$10 000 borrowed for 2 years and 3 months at 15% p.a.

(n) \$16 000 borrowed for 3 years and 9 months at 17% p.a.

WE2

2 Find the information that is missing in each of the following simple interest situations.

(a) \$2400 interest is paid on an investment at 20% p.a. after 3 years.

(b) \$7200 interest is paid on an investment at 15% p.a. after 6 years.

(c) \$25 200 interest is paid on an investment of \$20 000 after 9 years.

(d) \$40 788 interest is paid on an investment of \$30 900 after 12 years.

(e) \$65 000 interest is paid on an investment of \$100 000 invested at 6.5% p.a.

(f) \$135 000 interest is paid on an investment of \$150 000 invested at 7.5% p.a.

WE3

3 \$5600 is invested at an interest rate of 4.5% p.a. Find the:

(a) simple interest earned after 3 years

(b) amount of the investment after 3 years earning simple interest

(c) amount of the investment after 3 years earning compound interest

(d) compound interest earned after 3 years

(e) difference between the amounts earned in parts **(b)** and part **(c)**.

WE4

4 Calculate the total amount owing on a loan of \$7000 after 2 years, assuming no payments have been made, if the interest rate of 16% p.a. is:

(a) calculated as simple interest (b) compounded annually.

Hint

Use $A = P + I$ to find the simple interest first. Then use $I = PRN$ to find R.

5 At what simple percentage rate will an initial investment of \$7500 grow to a total investment of \$8400 in 10 years?

6 If the simple interest on an investment over 18 years at 4% is \$93, what was the amount of the original investment?

Reasoning, communicating and open-ended

7 The time for an amount of money to grow to 120% of its initial value when invested at 10% p.a. simple interest would be:

A 1 year **B** 2 years **C** 4 years **D** 6 months

8 Jess bought a car with a loan of \$20 000 from her parents. They agreed to charge her simple interest for the time of the loan. Four years later, she repaid \$24 400. What interest rate did she pay?

9 When Bree was born, her grandparents invested \$2000 at 5.6% simple interest to mature on her 18th birthday. Two years later, they invested another \$2000 so that the amount of interest earned by this investment would be the same as the interest earned on the first investment. It would also mature on her 18th birthday. What was the rate of simple interest on the second investment?

10 Using simple interest rates between 3% p.a. and 12% p.a. and a time period of less than 10 years, find two interest rates that would give the same return on two investments of the same amount if one investment was for 2 years more than the other.

Compound interest—the general formula

1.2

Need to Know

Stage 5.1

Finding the compound interest formula

To calculate compound interest by using the simple interest formula on a year by year basis is time consuming and inefficient. If algebra is used, a more efficient way of calculating compound interest can be found.

$$\begin{aligned} A_2 &= A_1 + I_2 \\ &= P_1(1+R) + P_1(1+R)R \\ &= P_1(1+R)(1+R) \\ &= P_1(1+R)^2 \end{aligned}$$

Repeating this process gives

$$A_3 = P_1(1+R)^3$$

This gives a formula for calculating compound interest where the interest rate is written as r, a decimal, to indicate compound interest.

For example:

Consider the example used in Section 1.1 for calculating the compound interest and the amount on \$1000 at 6% p.a. for 4 years.

Year	Amount owed at the end of each year (in dollars)
1	$1000 \times 1.06 = 1000 \times (1.06)^1$ $= 1060$
2	$1000 \times (1.06)^1 \times 1.06 = 1000 \times (1.06)^2$ $= 1123.60$
3	$1000 \times (1.06)^2 \times 1.06 = 1000 \times (1.06)^3$ $= 1191.016$
4	$1000 \times (1.06)^3 \times 1.06 = 1000 \times (1.06)^4$ $= 1262.47696$

Note:

1. The total value of the investment at the end of the first year is 106% of \$1000. This is the same as multiplying the initial principal by 1.06.
2. Each year, the total value is 1.06 times the total value at the end of the previous year (or the beginning of the current year).
3. The index number is the same as the year number.

Comparing these amounts with those calculated in Section 1.1, we can see that they are identical after the same time period.

If A_n = the total amount owed after n years, then for the above problem we can write:

$$A_n = 1000(1 + 0.06)^n$$

Compound interest formula (compounded annually)

$A = P(1 + r)^n$

where A = amount of the loan or investment after n years
P = principal
r = rate of interest (p.a.) as a decimal
n = time in years

Also:

$I = A - P$

where I = compound interest owed or earned after a certain number of years
A = amount of the loan or investment after the same number of years
P = principal

Appreciation

The compound interest formula can be used to predict a future cost or value if the appreciation increases at the same rate every year (compounds annually).

Worked Example 5

WE 5

$12 500 is borrowed for 7 years at an interest rate of 15% p.a. If the interest is compounded annually, find:

(a) the total value of the loan after 7 years

(b) the amount of interest charged over the period of the loan.

Working

(a) $P = 12\,500$, $r = 15\% = 0.15$, $n = 7$

$A = P(1 + r)^n$
$= 12\,500(1 + 0.15)^7$
$= 12\,500 \times 1.15^7$
$= \$33\,250.25$
(rounded to the nearest cent)

(b) $A = 33\,250.25$, $P = 12\,500$

$I = A - P$
$= 33\,250.25 - 12\,500$
$= \$20\,750.25$

1.2 Compound interest—the general formula

Answers page 611

Stage 5.1

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q9, Q10, Q11	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11

Fluency and understanding

WE 5

1 For each of the following loans the interest is now compounded annually. Find:

(i) the total amount to be repaid at the end of the loan

(ii) the amount of interest charged over the period of the loan.

(a) \$40 000 borrowed for 1 year at 8% p.a.

(b) \$30 000 borrowed for 1 year at 9% p.a.

(c) \$30 000 borrowed for 5 years at 7% p.a.

(d) \$40 000 borrowed for 12 years at 5% p.a.

(e) \$150 000 borrowed for 25 years at 7.5% p.a.

(f) \$240 000 borrowed for 30 years at 6.5% p.a.

(g) \$300 000 borrowed for 1.1 years at 6.75% p.a.

(h) \$350 000 borrowed for 1.1 years at 7.25% p.a.

(i) \$13 000 borrowed for 5 years at $9\frac{1}{2}$ % p.a.

(j) \$15 800 borrowed for 4 years at $10\frac{3}{4}$ % p.a.

(k) \$45 000 borrowed for $12\frac{1}{2}$ years at 5% p.a.

(l) \$36 000 borrowed for $5\frac{1}{2}$ years at 7% p.a.

(m) \$10 000 borrowed for 2 years and 3 months at 15% p.a.

(n) \$16 000 borrowed for 3 years and 9 months at 17% p.a.

2 Lily invested \$20 000 for 5 years at an interest rate of 5.25% p.a. Jarrah invested \$45 000 for 7 years at an interest rate of 9.5% p.a.

(a) Calculate the interest earned on each investment assuming:

(i) annual compound interest **(ii)** simple interest.

(b) Compare the compounded interest with the simple interest.

3 \$4245 at 6.5% p.a. compound interest, compounded annually over 8 years, will amount to:

A $\$4245(1 + 0.065)$ **B** $\$4245(1 + 0.065) \times 8$

C $\$4245(1 + 0.065)^8$ **D** $\$4245(1 + 0.65)^8$

Hint

In some cases you may need to use 'guess and check' to solve the equation.

4 Complete the following table assuming an annual compounding period.

Amount (A)	Principal (P)	Interest rate p.a. (r)	Time in years (n)
\$1359.81		8%	13
	\$24 600	12.2%	5
\$1 048 209.60	\$1 000 200	4.8%	
\$151.71		3%	20
\$60.50	\$50		2

5 For each of the following appreciation questions:

(i) state whether the problem should be modelled by the simple interest formula or by the compound interest formula

(ii) solve the problem, rounding answers to the nearest whole number or dollar.

(a) Martina currently gets \$10 pocket money a week. She has been promised a \$4 increase each year. What is the promised rate of increase? How much more pocket money will she get in 3 years than she is getting now?

(b) An antique clock appreciates in value by 11% p.a. If it was valued 10 years ago at \$350, how much is it worth now?

(c) Yang paid \$20 000 for a new motorbike. If the inflation rate is 6.5% p.a., what is the estimated cost of a similar new motorbike in 4 years' time?

(d) Rhea has been awarded an annual salary increase rate of 7% p.a. Her current salary is \$42 000 p.a. Calculate how much she can expect to earn in 3 years' time if the same rate continues.

(e) Ethel was given a \$500 ring 30 years ago. Each year it appreciated at a flat rate of 5%. What is it worth now?

(f) In 1993, a house was purchased for \$214 500. It was sold in 2010 for \$652 000. What was the yearly average rate of appreciation?

6 Costs in a business are growing at an annually compounding rate of 8% p.a. Currently they run at \$780 per week but 7 years ago they were:

A $\dfrac{\$780}{1.08^7}$ B $\$780 \times 1.08^7$ C $\$780 \times 1.08^7 - 780$ D $\dfrac{\$780}{1.8^7}$

Reasoning, communicating and open-ended

Hint

Either choose a number for the amount Ahmed has to invest or let P be this amount.

7 Ahmed has some money to invest. He wants to at least quadruple his money.

(a) How long will it take if he invests at a simple interest rate of 3% p.a.?

(b) How long will it take if he invests at an annually compounded interest rate of 3%?

Give your answers in whole years.

Hint

Write two equations with P and r in them. Then use these two equations to write $\frac{51\,398.55}{48\,951}$ in terms of P and r.

8 A certain sum of money \$$P$ is deposited into a bank account at an annually compounding interest rate, r. After 2 years it has grown to \$48 951 and after 3 years it has grown to \$51 398.55.

What is the interest rate?

9 Predeep wanted to invest \$10 000 he inherited for 3 years. He was offered a simple interest rate over the time period of 6.3% p.a. or a compound interest rate of 5.9% p.a.

(a) Which investment will earn him more money in that time period? How much extra would it earn him over the return on the other investment?

(b) If he decided to invest the money for 4 years, would his choice remain the same? Explain your answer.

10 Find values for the principal of an investment and the annual interest rate, given that the investment compounds annually for 2 years, to give a final value within \$100 of \$15 000.

11 Calum's car was written off as a result of hail damage in a sudden and severe local storm. His interstate cousin Jessica also had to replace her car after flood damage. Both needed to borrow the same amount of money. Calum was able to get a flat rate loan at 10% p.a. for 8 years. Jessica will end up paying the same amount of interest as Calum but her loan has a simple interest rate between 10% and 20% p.a. and will need to be repaid over a shorter period of time. Find possible values for the interest rate and period of Jessica's loan.

Problem solving

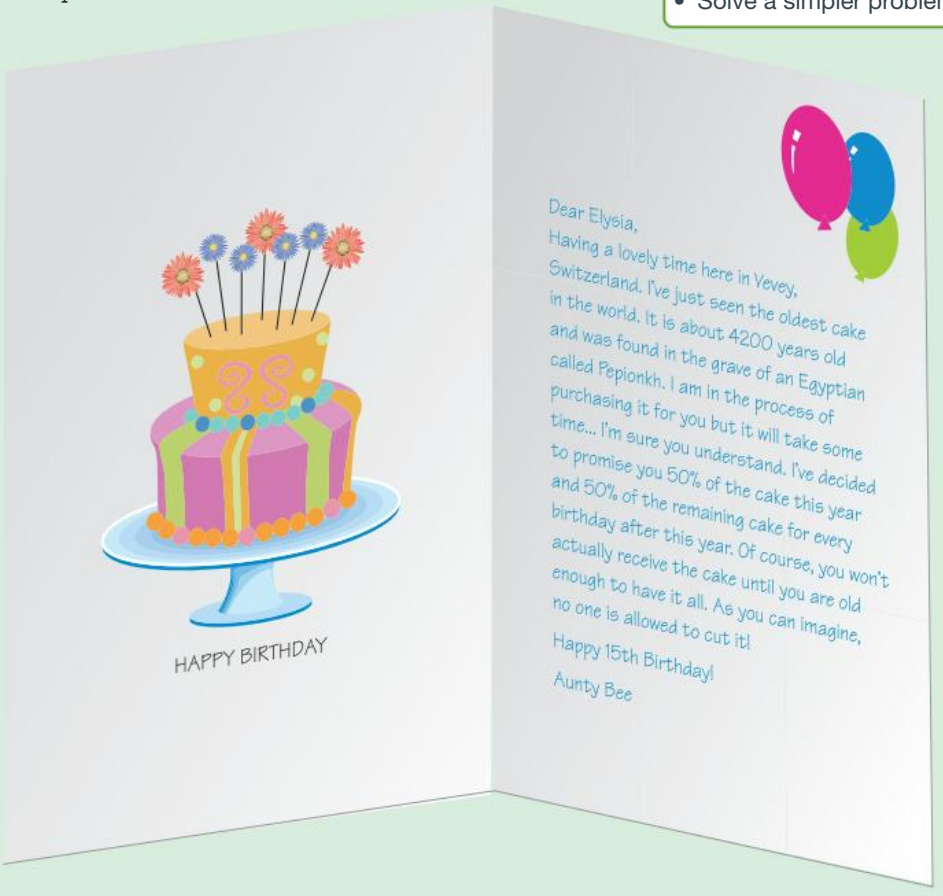
The oldest cake problem

1 What percentage of the cake has been promised to Elysia by her 21st birthday?

2 How old will Elysia be when she finally gets Aunty Bee's present?

Strategy options

- Draw a diagram.
- Solve a simpler problem.



1.3 Compound interest—different compounding periods

Stage 5.2◊

Need to Know

Different compounding periods

The annual compounding period is not always used in practical situations. For instance, the interest on a loan may compound daily, monthly, quarterly or every half year. A simple modification to the compound interest rule will allow us to deal with these different compounding periods.

The important thing is to ensure that the compounding period and the interest rate are expressed using the same time period. So, if a monthly compounding period is used, a monthly interest rate is needed.

Consider a loan at 12% p.a. If the compounding period is a:

- quarter (three months), the interest rate for each period = 3% per period and there will be 4 periods per year
- month, the interest rate for each period = 1% per period and there will be 12 periods per year
- day, the interest rate for each period = $\frac{12}{365}$% (≈ 0.0329%) per period and there will be 365 periods per year.

The quarterly rate is 3%, the monthly rate 1% and the daily rate $\frac{12}{365}$%. Rounding off the amount to an appropriate number of decimal places is done at the end of our calculations.

The compound interest formula

$A = P(1 + r)^n$

where A = amount owing or earned at the end of the complete process
P = principal (original amount borrowed or loaned)
r = interest rate for the period, expressed as a decimal
n = number of compounding periods

Worked Example 6

WE 6

Clive borrowed $12 000 to buy a lathe for his factory. He agreed to pay interest at a rate of 15% p.a. and to repay the loan in full in 3 years. Calculate the amount to be repaid if the interest is compounded:

(a) annually **(b)** half-yearly **(c)** quarterly.

Working

(a) $P = 12\,000$
$r = 0.15$
$n = 3 \times 1$
$= 3$

$A = P(1 + r)^n$

$A = 12\,000 \times (1.15)^3$
$= \$18\,250.50$

(b) $P = 12\,000$
$r = \frac{0.15}{2}$
$= 0.075$
$n = 3 \times 2$
$= 6$

$A = P(1 + r)^n$

$A = 12\,000 \times (1.075)^6$
$= \$18\,519.62$

(c) $P = 12\,000$
$r = \frac{0.15}{4}$
$= 0.0375$
$n = 3 \times 4$
$= 12$

$A = P(1 + r)^n$

$A = 12\,000 \times (1.0375)^{12}$
$= \$18\,665.45$

1.3 Compound interest—different compounding periods

Answers page 612

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4, Q5, Q9 (a), (b), Q10	Q1, Q3, Q4, Q5, Q6, Q8, Q9 (a), (b), Q10	Q1, Q4, Q5, Q6, Q7, Q8, Q9, Q11

If necessary, round answers to the nearest cent.

Fluency and understanding

1 Calculate the amount to be repaid after 2 years on a loan of $16 000 if the 14% interest p.a. is compounded: WE 6

(a) annually **(b)** half-yearly **(c)** quarterly.

2 Calculate how much an investor would repay on a debt of $8000 after $1\frac{1}{2}$ years if the 12% interest p.a. is compounded:

(a) half-yearly **(b)** quarterly, i.e. every 3 months.

3 If a loan of $5800 is made at 16% p.a. compounded every half year, over 6 years the debt would grow to:

A $\$5800(1 + 0.16)^6$ **B** $\$5800(1 + 0.08)^{12}$

C $\$5800(1 + 0.16)^{12}$ **D** $\$5800(1 + 0.8)^{12}$

4 If the loan in Question **3** was compounded quarterly, the interest accrued would be:

A $\$5800(1 + 0.04)^6$ **B** $\$5800(1 + 0.16)^6 - \$5800$

C $\$5800(1 + 0.04)^{24} - \$5800$ **D** $\$5800(1 + 0.4)^{24} - \$5800$

5 Calculate how much interest (to the nearest cent) is added over 10 years to an account paying 9% p.a. interest on an initial sum of $45 800 if the interest is compounded:

(a) half-yearly **(b)** quarterly **(c)** monthly.

6 How much more will an investor get on an investment of $32 000 over 4 years in an account offering 8.8% p.a. if the interest is compounded weekly rather than annually? (Assume 52 weeks in a year.)

7 Calculate the interest in Question **6** if it were to be compounded daily.

8 Martina won $5000 in the lottery and decided to invest it with a view to taking a trip to Paris sometime in the future. Martina was given two options when she invested the money. She could have an interest rate of 4.5% p.a. compounded quarterly or she could have 4.4% p.a. compounded daily. Which is the better deal, and by how much, if Martina invests the money for 2 years?

Reasoning, communicating and open-ended

9 **(a)** $20 000 is to be invested for 3 years. Calculate the interest earned on each of the following to find which would give the best return.

(i) 5% p.a. compounded quarterly

(ii) 4.5% p.a. compounded every 2 months

(iii) 4% p.a. compounded monthly

(b) Do you think the size of the investment would alter which option in part **(a)** would give the best return?

(c) Prove your answer to part **(b)** algebraically by redoing part **(a)** with $\$P$ in place of $20 000.

10 'The more frequent the compounding, the greater the interest.' Explain what this statement means. Use a principal of $10 000 invested at 8% p.a. for 5 years and three different compounding periods (such as weekly, quarterly or monthly) to help you demonstrate your answer. Express the percentage increase correct to one decimal place.

11 Tyson is having trouble understanding the difference between simple and compound interest. He can't see why compound interest is higher than simple interest after the first compounding period if the principal, interest rate and overall time period are the same. How would you explain the difference to Tyson? Use an example of $4000 invested for 3 years at 5% p.a. to help you explain your answer.

Half-time 1

Stage 5.1 Q1–5 **Stage 5.2◊ Q6, 7**

Round answers to the nearest cent, unless told otherwise.

1 A painting was purchased 11 years ago for $26 900. It has just been sold for $78 000. Calculate the flat rate of appreciation p.a. Round your answer to one decimal place. Ex.1.1

2 Find the simple interest earned on each of the following investments. Ex.1.1

(a) $6000 invested for 5 years at 6.3%

(b) $32 800 invested for 3 years at 5.8%

(c) $246 000 invested for 8 years at 8.2%.

3 Calculate the total amount owing on a loan of $6000 after 3 years if the 14% interest p.a. is compounded annually. Ex.1.2

4 Find the compound interest paid on each of the following loans. Ex.1.2

(a) $23 500 lent for 3 years at 7.2%

(b) $8000 lent for 4 years at 6.5%

(c) $452 000 lent for 20 years at 5.3%.

5 Catherine and Ken invested $20 000 for 6 years at an interest rate of 7.5% p.a. Round your answers to the nearest dollar. Ex.1.2

(a) Calculate the value of the interest earned on the investment if it was compounded annually.

(b) Calculate the value of the interest earned on the investment if simple interest was paid instead.

(c) Compare the compounded interest with the simple interest.

6 How much interest is added over 8 years to an account paying 8.25% p.a. interest on the initial sum of $55 700 if the interest is compounded: Ex.1.3

(a) half-yearly **(b)** monthly **(c)** daily?

7 If a loan of $7200 is made at 17% p.a. compounded quarterly over 5 years, the debt would grow to: Ex.1.3

A $7200(1+0.17)^5$ **B** $7200\left(1+\frac{0.17}{4}\right)^{20}$ **C** $7200\left(1+\frac{0.17}{4}\right)^{5}$ **D** $7200(1+0.17)^{20}$

1.4 Compound interest—further applications

Stage 5.2◊

Need to Know

Finding the principal

1. Write the compound interest formula.
2. Identify the values of the variables A, r and n. (Use the exact value for r.)
3. Substitute these values into the formula and simplify.
4. Transpose the equation to make P the subject.
5. Evaluate P.

Finding the rate

1. Write the compound interest formula.
2. Identify the values of A, r and n.
3. Substitute these values into the formula and simplify.
4. Transpose the equation to find r. *Remember*: Don't round until the end.

(The inverse operation to finding a power of 12 is the 12th root.)

5. Multiply by 100 to convert to a percentage and round the answer to the appropriate number of decimal places.

Finding the time period

1. Write down the formula.
2. Identify the values of A, P and r.
3. Substitute these values into the formula and simplify.
4. Transpose the equation as far as possible to make n the subject.
5. Use 'guess and check' to find the value of n.

Worked Example 7

WE 7

Aunt Eva leaves Heidi a legacy of some deposit stock that was invested for 10 years at 5.11% p.a. compounded daily. The value of the cheque received was \$21 669.68. Calculate the initial deposit. Give your answer rounded to the nearest dollar. (Ignore the occurrence of leap years in the 10-year period.)

Working

$A = P(1 + r)^n$

$A = 21\,669.68$

$r = \dfrac{0.0511}{365}$

$n = 10 \times 365$

$= 3650$

$$21\,669.68 = P\left(1 + \frac{0.0511}{365}\right)^{3650}$$

$$P = \frac{21\,669.68}{\left(1 + \frac{0.0511}{365}\right)^{3650}}$$

$P = \$13\,000.00595$

So, the principal was \$13 000.

Worked Example 8

WE 8

In the first year of his business Jack had takings of $228 000. They grew to $520 000 after 12 years, increasing at the same rate every year. At what annual percentage rate has his business been growing? Round your answer to two decimal places.

Working

$A = 520\,000$, $P = 228\,000$, $n = 12 \times 1 = 12$

$$A = P(1 + r)^n$$

$$520\,000 = 228\,000(1 + r)^{12}$$

$$\frac{520\,000}{228\,000} = (1 + r)^{12}$$

$$\sqrt[12]{\frac{520\,000}{228\,000}} = (1 + r)$$

$$r = \sqrt[12]{\frac{520\,000}{228\,000}} - 1$$

$$r = 0.071\,122\,251\,37$$

$$r = 7.11\% \text{ p.a.}$$

The business has been growing at a rate of 7.11% p.a.

Worked Example 9

WE 9

How many years will it take a $452 stamp to be worth $1000 if it increases in value by 7.5% p.a.? Assume that interest is compounding annually. Give your answer to the nearest whole number of years.

Working

$A = 1\,000$, $P = 452$, $r = 0.075$

$$A = P(1 + r)^n$$

$$1000 = 452(1 + 0.075)^n$$

$$1000 = 452(1.075)^n$$

$$\frac{1000}{452} = (1.075)^n$$

$$2.212\,389\,381 = (1.075)^n$$

If $n = 4$, $(1.075)^4 = 1.335\,469$
If $n = 10$, $(1.075)^{10} = 2.061\,031$
If $n = 12$, $(1.075)^{12} = 2.381\,779$
If $n = 11$, $(1.075)^{11} = 2.215\,608$

It will take 11 years for the stamp to increase its worth to $1000.

Answers page 612

1.4 Compound interest—further applications

Stage 5.2◊

Navigator

Q1, Q2, Q3, Q5, Q6, Q8	Q1 (b)–(d), Q2 (b)–(d), Q3 (a)–(b), Q4, Q5, Q6, Q8	Q1 (c)–(d), Q2 (c)–(d), Q3 (c), Q4, Q5, Q6, Q7, Q9

Fluency and understanding

WE7

1 In each of the following, use the compound interest formula to find the principal.

(a) Calculate how much would need to be invested at 8% p.a. compounded each half year to accumulate to \$9600 in 6 years. Round your answer to the nearest dollar.

(b) How much would Li have to deposit in order to receive \$10 000 in 7 years if she places her money in an account that pays 8.8% interest, compounded quarterly? Round your answer to the nearest dollar.

(c) How much would Sylvester have to deposit to receive \$8000 in 5 years' time if he places money in an account that pays 7.8% interest, compounded half yearly? Round your answer to the nearest cent.

(d) A deposit accumulates to \$4500 in 9 months at 12% p.a. compounded quarterly. The initial deposit, in dollars, was:

A $\frac{4500}{1.3^3}$ **B** $\frac{4500}{1.03^3}$ **C** $\frac{4500}{1.12^{3/4}}$ **D** 4500×1.04^3

WE8

2 In each of the following, use the compound interest formula to find the annual interest rate.

(a) Sales of \$26 900 grow to \$78 000 in 11 years. Calculate the percentage growth p.a. rounded to one decimal place. Assume the growth compounds annually.

(b) Costs of \$6200 increase to \$8000 in 8 years. Calculate the percentage increase p.a. rounded to two decimal places. Assume annual compounding.

(c) Calculate the rate of interest p.a. which would allow \$7900 to accumulate to \$9000 in 5 years if interest is compounded each half year. Round your answer to one decimal place.

(d) \$2800 accumulates to \$4500 in 3 years. The percentage rate of interest if it is compounded annually is:

A $\sqrt[3]{\frac{4500}{2800}} - 1$ **B** $\left(\sqrt[3]{\frac{4500}{2800}} - 1\right) \times 100$

C $\sqrt[3]{\frac{4500}{2800}} \times 100$ **D** $\frac{4500}{2800} \times 100$

WE9

3 In each of the following, use the compound interest formula and the method of guess and check (or a CAS) to find the number of years. In each case, assume that interest is compounding annually.

(a) How many years will it take for a \$2400 sapphire ring to be worth \$8000 if it increases in value by 10.6% p.a.? Round your answer to the nearest whole number of years.

(b) How many years will it take for a \$40 000 block of land to be worth \$90 000 if it increases in value by 8.5% p.a.? Round your answer to the nearest whole number.

(c) How many years will it take for a \$3400 porcelain dinner set to be worth \$5000 if it increases in value by 9.7% p.a.? Round your answer to the nearest tenth of a year.

4 \$5000 is invested for 4 years with the interest compounded quarterly. For the first 2 years the interest rate is 4% p.a. and for the other 2 years it is 4.8% p.a.

(a) How much will the investment be worth at the end of the 4 years?

(b) How much interest will be earned?

(c) Would the investor be better off to invest for the whole 4-year period at 4.4% p.a.? How much interest would this investment earn?

(d) How much interest would be earned if the first 2 years were at 4.8% p.a. and the rest at 4% p.a.?

(e) The conditions of investment change and interest is now compounded annually. What would the interest rate need to be to earn the same amount of interest as found in part **(b)**?

(f) The interest rate is now 5% p.a. compounded annually. After how many years will the investment be at least as great as the result shown in part **(a)**?

5 Pene's savings are in a bank account that attracts 3.5% p.a. If her savings are to grow from \$10 000 to over \$20 000 in 20 years, what is the smallest number of compounding periods per year required?

Reasoning, communicating and open-ended

6 A building society offered an interest rate of 7.25% p.a. on any investment over \$3000. Mary deposited \$3600 and received \$5944 after 7 years.

(a) How often was the interest compounded?

(b) What simple interest rate would have given the same return in the same time? Round your answer to one decimal place.

(c) Mary was a little disappointed because she'd calculated a return of \$5971. Her calculations were based on an incorrect assumption. What was that assumption?

7 If savings in an account are to be *at least* tripled over 20 years with an interest rate of 5.55% p.a., how often should the interest be compounded? Give your answer in months.

8 Jacob and Hafida are planning an overseas trip that will cost \$20 000. They have decided to pool their money and invest it at 6.5% p.a. compounding quarterly for 1 year. They did the following calculations:

How much do we need to invest?

$P = ?,\ A = 20\,000,\ n = 4,\ r = 0.065$

$$20\,000 = P\left(1 + \frac{0.065}{4}\right)^4$$

$$20\,000 = P(1.02)^4$$
$$20\,000 = P \times 1.08$$
$$P = \frac{20\,000}{1.08}$$
$$P = 18\,518.52$$

Checking:

$A = ?,\ P = 18\,518.52,\ n = 4,\ r = 0.065$

$$A = 18518.52\left(1 + \frac{0.065}{4}\right)^4$$

$$A = 18518.52(1 + 0.02)^4$$
$$A = 20\,045.04$$

When they collect their money a year later, they are almost \$250 below their target. Where did they go wrong?

9 Give different rates and compounding periods that will result in savings being increased by at least 50% over 4 years.

1.5 Comparing interest rates

Stage 5.2◊

Need to Know

Nominal and effective interest rates

The **nominal interest rate** (R) is the interest rate that would apply to a principal for one year, i.e. a rate per annum. If the compounding periods are less than one year the amount of interest is larger, which means the interest rate is in fact higher. To make comparisons easier, the **effective interest rate** (R_{ef}) is calculated. This is the rate that would give the same amount if simple interest was applied for 1 year and is found by comparing the amount of interest earned in the first year of an investment under various compounding periods.

For example:

A principal of \$10 000 and an annual interest rate of 6%.

If the compounding period is 1 year,

$A = 10\,000(1 + 0.06)^1$

$= 10\,600$

$I = A - P$

$= \$600$

If the compounding period is 6 months,

$A = 10\,000(1 + 0.03)^2$

$= 10\,609$

$I = A - P$

$= \$609$

$R_{ef} = \frac{I}{P} \times 100\%$

$= 6.09\%$

Case	Number of compounding periods	Interest in the first year	Percentage return
A	0 (simple interest)	\$600	$\frac{600}{10\,000} \times 100 = 6\%$ p.a.
B	1 (annual)	\$600	$\frac{600}{10\,000} \times 100 = 6\%$ p.a.
C	2 (6 monthly)	\$609	$\frac{609}{10\,000} \times 100 = 6.09\%$ p.a.
D	4 (quarterly)	\$613.64	$\frac{613.64}{10\,000} \times 100 = 6.14\%$ p.a.
E	12 (monthly)	\$616.78	$\frac{616.78}{10\,000} \times 100 = 6.17\%$ p.a.

This is equivalent to a simple interest rate of 6.09% p.a.

The table above summarises this information for a number of compounding periods, and includes the simple interest calculation as well. Remember that simple interest over 1 year is the same as compound interest, compounded annually, for 1 year. The effective interest rate for quarterly compounding periods is 6.14% p.a. and for monthly compounding periods is 6.17% p.a.

Effective interest rate formula

$$R_{ef} = \left(\frac{I_1}{P}\right) \times 100\%$$

where R_{ef} = effective interest rate or real interest rate (p.a.) as a percentage
I_1 = amount of interest after the first year
P = principal

The effective interest rate is useful for comparing situations with different nominal rates and different compounding periods.

$$R_{ef} = \left[\left(1 + \frac{R}{n}\right)^n - 1\right] \times 100\%$$

where R_{ef} = effective interest rate (p.a.) as a percentage
R = nominal interest rate as a decimal
n = number of compounding periods in 1 year

Worked Example 10

WE 10

Find the effective rate of interest (R_{ef}) if \$25 000 is invested for 1 year at the following rates. Where necessary, round your answers to three decimal places.

(a) a simple rate of 9.5% p.a.

(b) an annually compounding rate of 9.5% p.a.

(c) a monthly compounding rate of 9.5% p.a.

(d) a weekly compounding rate of 9.5% p.a.

Working

(a) $R_{ef} = 9.5\%$ p.a.

For simple interest, the effective rate is the same as the nominal rate.

(b) $R_{ef} = 9.5\%$ p.a.

For annual compounding, the effective rate is the same as the nominal rate.

(c) $A = P(1 + r)^n$

$$= 25\,000\left(1 + \frac{0.095}{12}\right)^{12}$$

$$= 27\,481.1896$$

$$I = A - P$$
$$= 27\,481.1896 - 25\,000$$
$$= 2481.1896$$

$$R_{ef} = \left(\frac{I_1}{P}\right) \times 100$$

$$= \frac{2481.1896}{25\,000} \times 100$$

$$= 9.9247$$

The effective interest rate is 9.925% p.a.

(d) $A = P(1 + r)^n$

$$= 25\,000\left(1 + \frac{0.095}{52}\right)^{52}$$

$$= 27\,489.0887$$

$$I = A - P$$
$$= 27\,489.0887 - 25\,000$$
$$= 2489.0887$$

$$R_{ef} = \left(\frac{I_1}{P}\right) \times 100$$

$$= \frac{2489.0887}{25\,000} \times 100$$

$$= 9.9563$$

The effective interest rate is 9.956% p.a.

Worked Example 11 WE11

A loan attracts an interest rate of 23% compounded daily. Find:

(a) the nominal rate, R, and the number of compounding periods, n

(b) the effective rate, R_{ef}, using the values found in **(a)**.

Working

(a) $R = 23\%$
$n = 365$

(b) $R = 23\% = 0.23$

$$R_{ef} = \left[\left(1 + \frac{R}{n}\right)^n - 1\right] \times 100$$

$$R_{ef} = \left[\left(1 + \frac{0.23}{365}\right)^{365} - 1\right] \times 100$$

$$= 0.2585 \times 100$$

$$= 25.85\% \text{ p.a.}$$

Answers page 612

1.5 Comparing interest rates

Stage 5.2◊

Navigator

Q1 (a)–(d), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q11	Q1 (a)–(c), (e), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11	Q1 (a), (b), (e), (f), Q2 (d)–(f), Q6, Q7, Q9, Q10, Q11, Q12

Fluency and understanding

WE10

1 For each of the following find the effective rate of interest (R_{ef}).

(a) \$40 000 invested at a simple interest rate of 7% p.a.

(b) \$10 000 invested at an annual compounding rate of 9.4% p.a.

(c) \$100 000 invested at a monthly compounding rate of 12.75% p.a.

(d) \$35 000 borrowed at a weekly compounding rate of 19.7% p.a.

(e) \$20 000 borrowed at a fortnightly compounding rate of 16.5% p.a.

(f) \$200 000 invested at a rate of $6\frac{1}{4}$ % p.a. compounded every 5 days.

WE11

2 For each of the following find:

(i) the nominal rate, R, and the number of compounding periods, n

(ii) the effective rate, R_{ef}, using the values found in **(i)**.

(a) 7.3% p.a. compounded 5 times a year

(b) 15.5% p.a. compounded monthly

(c) 21% p.a. compounded quarterly

(d) $16\frac{1}{4}$ % p.a. compounded every 4 months

(e) daily compounding rate of $20\frac{1}{3}$ % p.a.

(f) half-yearly compounded rate of 5.25% p.a.

3 The effective interest rate that corresponds to 5% p.a. compounded every 2 months is:

A $\left[\left(1+\frac{0.05}{2}\right)^2-1\right]\times 100$

B $\left[\left(1+\frac{0.05}{6}\right)^6-1\right]\times 100$

C $\left[\left(1+\frac{0.5}{6}\right)^6-1\right]\times 100$

D $\left[\left(1+\frac{5}{2}\right)^2-1\right]\times 100$

4 Two banks are currently offering the following rates on their investment savings accounts:

Q Bank: 15% p.a. compounded quarterly
D Bank: 14.9% p.a. compounded daily

(a) Find the effective rates being offered by the two banks.

(b) Which bank is offering the better deal?

5 Ned needs to buy a new laptop. He is planning to take out a personal loan with an interest rate of 20.5% p.a. compounded quarterly. The alternative is to use his credit card that has an interest rate of 20.1% compounded daily.

(a) What is the effective rate of this personal loan?

(b) What is the credit card's effective rate?

(c) Which of the two options should Ned choose and why?

6 A bed can be purchased on a 'get it now, pay in 12 months' plan.

The interest payment options are different depending on whether the purchase is made online or instore. The online purchase attracts a flat rate interest of 25% p.a. The interest on the instore purchase is 23% p.a. compounded daily.

(a) What is the real or effective interest rate charged on an instore purchase?

(b) Which is the cheaper option?

Reasoning, communicating and open-ended

7 The following represent investment options. In each case, choose the better option *without calculating the effective rate* and justify your choice.

(a) 12% p.a. compounded monthly
15% p.a. compounded daily

(b) 10.5% p.a. compounded weekly
10.5% p.a. compounded monthly

(c) 11% p.a. compounded daily
12% p.a. compounded daily

(d) 15% p.a. flat rate for 2 years
15% p.a. compounded annually for 2 years

8 The options in Question **7** now represent loans. Will your choice of option change? Justify your answer.

9 Inez and Sam need to choose between the following lucrative investments:

Option A yields 35.2% p.a. compounded daily.
Option B yields 42% p.a. compounded annually.
Option C yields 40% p.a. compounded quarterly.

Which option should they choose and why?

10 Simon has saved the 10% deposit for a new car. The car company can offer him an interest-free loan on the remaining balance for 6 months and then 15% p.a. compounded daily. Alternatively, he could borrow the money elsewhere for a flat rate of 9% p.a.

(a) If he pays out the loan plus interest in a year, which option will be cheaper?

(b) If he pays out the loan plus interest in two years, which option will be cheaper?

11 'Nominal interest rate' and 'simple interest rate' are the same thing. Explain with an example why this statement is untrue.

12 Two investments have different nominal interest rates and different compounding periods. However, the effective interest rate for both is exactly 46.41% p.a. What might be the nominal rates and respective compounding periods for these two investments?

1.6 Depreciation

Stage 5.2◊

Need to Know

Depreciation is the process by which items decrease in value.

Straight-line depreciation

Straight-line depreciation or flat-rate depreciation applies when items lose a constant amount of value each year. This amount of loss is measured in dollars and is known as the **depreciation value**. The rate of this depreciation can be written as a percentage or decimal. Depreciation over a number of years is known as **cumulative depreciation**.

Another important term is **initial cost** or **prime cost**, which refers to the original purchase price of the item. Finally, the value of the item after depreciation is known as the **written-down value** or **adjusted value** (previously called the 'book value'). With straight-line depreciation, the written-down value will eventually be zero.

For example:

A company buys a truck for \$75 000. Each year it depreciates at 20% of its prime cost.

The depreciation each year is 20% of \$75 000
$= 0.2 \times 75\,000$
$= \$15\,000$

The cumulative depreciation after 3 years will be 20% of \$75 000 × 3
$= 0.2 \times 75\,000 \times 3$
$= \$45\,000$

The written-down value after 3 years will be \$75 000 – \$45 000
$= \$30\,000$

After another 2 years the written-down value will reach zero.

When dealing with straight-line depreciation:

- cumulative depreciation = prime cost × rate of depreciation × time in years
- written-down value = prime cost – cumulative depreciation

These relationships resemble the formulas for simple interest. This connection can be written more formally using the formulas below.

Straight-line depreciation formula

$D_n = Prn$

where D_n = cumulative depreciation after n years, in dollars
P = prime cost or initial cost, in dollars
r = rate of depreciation (p.a.) as a decimal and
n = time in years

$$V_n = P - D_n$$

where V_n = written-down value or adjusted value after n years, in dollars

Reducing balance depreciation

Reducing balance depreciation applies when items depreciate by a smaller amount each year. Each year, the amount of depreciation is calculated as a percentage of the previous year's value. As the value of the item decreases so the corresponding depreciation also decreases. With reducing balance depreciation the written-down value will never reach zero.

For example:

Consider the company that purchased a truck for \$75 000 with a depreciation of 20% p.a., but this time calculate the depreciation on the written-down value each year, not on the prime cost.

Initial value = \$75 000

Depreciation year 1 = 20% of \$75 000
= 0.20 × \$75 000
= \$15 000

Written-down value year 1 = \$75 000 – \$15 000
= \$60 000

Depreciation year 2 = 0.20 × \$60 000
= \$12 000

Written-down value year 2 = \$60 000 – \$12 000
= \$48 000

Depreciation year 3 = 0.20 × \$48 000
= \$9600

Written-down value year 3 = \$48 000 – \$9600
= \$38 400

This can be set out neatly in a table for the first 5 years as shown.

Year number	Depreciation	Cumulative depreciation	Written-down value
0			75 000
1	15 000	15 000	60 000
2	12 000	27 000	48 000
3	9600	36 600	38 400
4	7680	44 280	30 720
5	6144	50 424	24 576

The table makes it easier to see that the written-down value is 80% of the previous written-down value. So, instead of calculating 0.20 × \$75 000 and subtracting that from \$75 000 to get \$60 000, 0.80 × \$75 000 can be calculated to obtain \$60 000.

The last column in the table opposite can be rewritten in the form shown in the table below that will make it easier to see the connection to compound interest.

So, after n years:

Written-down value
$= 75\,000(0.8)^n$
$= 75\,000(1 - 0.2)^n$

This is similar to the compound interest formula, except there is a subtraction sign instead of an addition sign in the brackets.

Year	Written-down value (\$)
0	75 000
1	$0.8 \times 75\,000$
2	$0.8 \times (0.8 \times 75\,000) = (0.8)^2 \times 75\,000$
3	$0.8 \times (0.8 \times (0.8 \times 75\,000)) = (0.8)^3 \times 75\,000$
4	$0.8 \times (0.8 \times (0.8 \times (0.8 \times 75\,000))) = (0.8)^4 \times 75\,000$
5	$0.8 \times (0.8 \times (0.8 \times (0.8 \times (0.8 \times 75\,000)))) = (0.8)^5 \times 75\,000$

Reducing balance depreciation formula

$V_n = P(1 - r)^n$

where V_n = written-down value after n years, in dollars
P = prime cost, in dollars
r = rate of depreciation (p.a.) as a decimal
n = time in years

The above formula is based on the assumption that the depreciation is an annual event, as is usually the case. If it was not an annual event, we could alter the formula in the same way we altered the compound interest formula to take account of this.

'Straight-line depreciation' can also be expressed as:

'depreciation at a percentage of its prime cost or initial value or purchase price every year'.

'Reducing balance depreciation' can also be expressed as:

'depreciation at a percentage of its reduced value every year'.

Unit cost depreciation

The **total possible depreciation** of an item is its purchase cost less any resale value or **scrap value** that it may still have at the end of its useful life. Scrap value is also known as **salvage value** or **residual value**.

The **expected life** of an item can be expressed as a unit of time (years/months/weeks/hours) or a unit of work like 'pages printed' for a printer or 'photos taken' for a camera.

The **unit cost depreciation** is a measure of how much the item depreciates in dollars per unit of its expected life.

Total possible depreciation = prime cost – scrap value (salvage value or residual value)

$$\text{Unit cost depreciation} = \frac{\text{total possible depreciation}}{\text{expected life}}$$

Depreciation (after a certain amount of use or work)
= Unit cost depreciation × number of units of work done

Worked Example 12

WE12

The initial cost of a washing machine 2 years ago was \$1950. It depreciates at 18.5% p.a. Assuming straight-line depreciation what is the:

(a) cumulative depreciation since the time of purchase

(b) current written-down value of the machine?

Working

(a) $P = 1950$, $r = 18.5\% = 0.185$, $n = 2$

$D_n = Prn$

$D_2 = 1950 \times 0.185 \times 2$
$= \$721.50$

(b) $P = 1950$, $D_2 = 721.50$

$V_n = P - D_n$

$V_2 = 1950 - 721.50$
$= \$1228.50$

Worked Example 13

WE13

The prime cost of a lawn mower 7 years ago was \$5600. If it depreciates at 9% p.a. assuming reducing balance depreciation, calculate the:

(a) current written-down value of the mower

(b) cumulative depreciation since the time of purchase.

Working

(a) $P = 5600,\ r = 9\% = 0.09,\ n = 7$

$V_n = P(1 - r)^n$

$V_7 = 5600(1 - 0.09)^7$

$= \$2893.86$

(b) $P = 5600,\ V_7 = 2893.86$

$D_n = P - V_n$

$D_7 = 5600 - 2893.86$

$= \$2706.14$

Worked Example 14

WE14

A laptop is expected to work for 6000 hours before it has a resale or scrap value of \$100. The purchase price is \$1300. Find:

(a) the total possible depreciation

(b) the unit cost depreciation

(c) the depreciation after 1000 hours of use

(d) the written-down value after 1000 hours of use.

Working

(a) Total possible depreciation

= prime cost − scrap value

Prime cost = \$1300

Scrap value = \$100

Total possible depreciation

= \$1300 − \$100

= \$1200

(b) Unit cost depreciation

$= \dfrac{\text{total possible depreciation}}{\text{expected life}}$

Total depreciation = \$1200

Expected life = 6000 hours

$\text{Unit cost depreciation} = \dfrac{1200}{6000}$

$= 0.2$

The laptop decreases in value by 20 cents for every hour of work.

(c) Depreciation after use

= unit cost depreciation × number of units of work done

Unit cost depreciation = \$0.20 per hour

Number of units of work done = 1000

Depreciation after use

$= 0.2 \times 1000$

= \$200

(d) $V = P - D$

$P = \$1300,\ D = \$200$

$V = \$1300 - \$200$

$= \$1100$

Answers page 612

1.6 Depreciation

Stage 5.2◊

Navigator

Q1 (a)–(f), Q2 (a)–(f), Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q19	Q1 (b)–(e), (g), Q2 (b)–(e), (g), Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q14, Q15, Q19	Q1 (b), (c), (f), (h), Q2 (b), (c), (f), (h), Q3, Q6, Q7, Q10, Q11, Q12, Q15, Q16, Q17, Q18, Q19, Q20

Fluency and understanding

WE12

1 For each of the following, assume straight-line depreciation. Find:

(i) the cumulative depreciation since the time of purchase

(ii) the current written-down value.

Round all answers to the nearest cent.

(a) The initial price 5 years ago was \$30 000. $r = 7\%$ p.a.

(b) The prime cost 12 years ago was \$47 500. $r = 5\%$ p.a.

(c) The purchase price 20 years ago was \$100 000. $r = 5\%$ p.a.

(d) The initial cost 23 years ago was \$790 000. $r = 4.25\%$ p.a.

(e) The prime cost 4 years ago was \$145 000. $r = 10.5\%$ p.a.

(f) The initial price $5\frac{1}{2}$ years ago was \$300 000. $r = 6.5\%$ p.a.

(g) The purchase price 6 months ago was \$47 500. $r = 20\frac{3}{4}\%$ p.a.

(h) The prime cost 18 months ago was \$152 700. $r = 33\frac{1}{4}\%$ p.a.

WE13

2 For each of the above, assume reducing balance depreciation. Find:

(i) the current written-down value

(ii) the cumulative depreciation since the item was purchased.

Round all answers to the nearest cent.

WE14

3 **(a)** A machine is expected to work for 3000 hours before it has a resale or scrap value of \$150. The purchase price is \$6150. Find:

(i) the total possible depreciation

(ii) the unit cost depreciation

(iii) the depreciation after 10 hours of use

(iv) the written-down value after 10 hours of use.

(b) A machine is expected to work for 2000 hours before it has no scrap value. The purchase price is \$1000. Find:

(i) the total possible depreciation

(ii) the unit cost depreciation

(iii) the depreciation after 1000 hours of use

(iv) the written-down value after 1000 hours of use.

4 What would be the total depreciation after 6 years on a \$780 workstation if it is depreciated at 11.25% p.a. of its prime cost?

5 A car costs \$24 000 new. The value of the car in 3 years, if it depreciates at 25% p.a. of its prime cost each year, will be:

A $\$\dfrac{24\,000 - 24\,000 \times 0.75}{3}$

B $\$24\,000 - 24\,000 \times 0.25 \times 3$

C $\$24\,000 \times 0.75^3$

D $\$24\,000 \times 0.25 \times 3$

6 Calculate the value of a computer (initial cost \$5600) after 4 years if it is depreciated at 15% p.a of its prime cost each year.

7 Calculate the depreciation in each year, cumulative depreciation and written-down or adjusted value for each of the first 3 years if a \$45 000 forklift is depreciated at 20% of its prime cost each year.

8 Which of the answers in Question **5** would be correct if the depreciation was calculated on the car's written-down value at the start of each year?

9 Calculate the value of a computer (cost \$5600) after 4 years if it is depreciated at 15% p.a. of its written-down value at the start of each year. Compare this answer with that obtained for Question **6**.

10 How much would the total depreciation be after 6 years on a \$780 workstation if it is depreciated at 15% p.a. of its written-down value at the start of each year?

11 Calculate the depreciation, cumulative depreciation and written-down value for each of the first 3 years if a \$45 000 forklift is depreciated at 20% p.a. of its written-down value at the start of each year. Compare this answer with that obtained in Question **7**.

12 A \$7500 machine has a scrap value of \$300, and is guaranteed for 11 000 cycles. In a year when it operates for 2630 cycles the amount of depreciation, in dollars, is:

A $\dfrac{7500 - 2630}{11\,000} \times 300$

B $\dfrac{11\,000 - 7500}{2630} \times 300$

C $\dfrac{7500 - 300}{11\,000} \times 2630$

D $\dfrac{7500}{11\,000} \times 2630$

13 **(a)** The amount of depreciation (in dollars) for a \$5000 machine with no scrap value, guaranteed for 8000 hours, in its first year when it works for 1350 hours, is:

A $\dfrac{8000}{1350} \times 5000$ **B** $\dfrac{8000}{5000} \times 1350$ **C** $\dfrac{5000}{6650} \times 1350$ **D** $\dfrac{5000}{8000} \times 1350$

(b) At the end of the first year the written-down value (in dollars) of the machine described in part **(a)** is:

A $5000 - \dfrac{5000}{8000} \times 1350$

B $6350 - \dfrac{8000}{5000} \times 1350$

C $8000 - \dfrac{5000}{8000} \times 1350$

D $8000 - \dfrac{1350}{8000} \times 5000$

14 **(a)** A \$10 000 soft-drink bottler with no scrap value is guaranteed for 120 000 bottles. Calculate the depreciation in its first year when it fills 47 000 bottles.

(b) Calculate the value of the soft-drink bottler after the first year.

Reasoning, communicating and open-ended

15 Calculate the value of an \$11 500 screen printer with a scrap value of \$200 guaranteed for 90 000 prints when it is used 12 400 times.

16 Calculate the value of a \$3500 photocopier with a scrap value of \$400 guaranteed for 600 000 copies after it produces 210 000 copies.

17 A camera costs $3500, has a scrap value of $900 and is guaranteed for 8000 photos. Calculate the value after the first and second year of its life if in the 2 years it takes 450 and 1800 photos respectively.

18 A $30 000 car depreciates at 20% p.a. of its written-down value at the start of each year every year for the first 3 years. An $80 000 car depreciates at 25% p.a. of its written-down value at the start of each year every year for the first 3 years. After that time, both cars depreciate at 10% p.a. of their written-down value at the start of each year for every subsequent year.

(a) What is the difference in their values:

(i) at purchase **(ii)** after 1 year **(iii)** after 2 years **(iv)** after 3 years?

(b) When will the difference in their values be under $3000?

19 An item was initially purchased for $10 000. It is now worth only $3000. Assuming that it depreciated at the same rate of its prime cost every year, what was the possible rate of decrease and corresponding elapsed time?

20 Repeat Question **19** assuming the depreciation followed a reducing balance model.

Problem solving

Weird and wonderful numbers

When you've been up and down the stairs several times, take time out and try to answer these questions:

1 What did you find at the bottom of the stairs? What number(s) sent you back to the top?

2 Why is there a special condition attached to step 1?

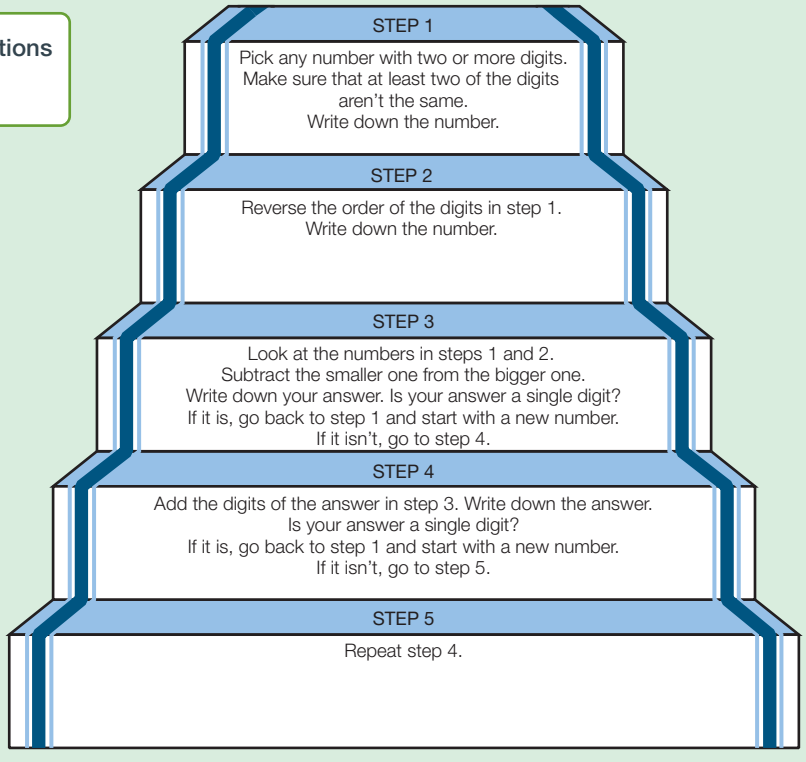
3 Will you ever find something else at the bottom of the stairs?

Strategy options

- Seek an exception.
- Solve a simpler problem.

Strategy options

- Act it out.



Chapter review 1

D.I.Y. Summary

Key Words

Stage 5.1	amount	interest
	appreciation	per annum
	compound interest	principal
	flat rate	simple interest
	inflation rate	
Stage 5.2◊	adjusted value	reducing balance depreciation
	cumulative depreciation	residual value
	depreciation	salvage value
	depreciation value	scrap value
	effective interest rate	straight-line depreciation
	expected life	total possible depreciation
	initial cost	unit cost depreciation
	nominal interest rate	written-down value
	prime cost	

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.1 Q1–4 **Stage 5.2◊ Q5–13**

1 The original amount of money that is borrowed or loaned is called the _____.

2 Interest rates are normally expressed with p.a. after them. This stands for _____ and it means for every year.

3 _____ is always higher than _____ after the first compounding period.

4 Increase in cost or value is called _____.

5 The stated interest rate is known as the _____. The adjusted rate that takes into account the effect of any compounding in the first year is called the _____.

6 The amount of depreciation on a certain item is \$10 each year for 7 years. This is an example of _____.

7 Depreciation that has a decreasing value from one year to the next is called _____.

8 If the depreciation is \$5 in the first year and \$4 in the second year, the _____ over the first two years is \$9.

9 The value of an item after depreciation is known as its _____ or _____.

10 The formula $V_n = P - D_n$ has three _____, V_n, P and D_n. In this formula, P stands for the _____ or _____.

11 Any value that an item still has at the end of its useful life is known as _____, _____ or _____.

12 The initial purchase price less any scrap value is the _____.

13 This value divided by the expected life gives the _____.

Stage 5.1 Q1, 2, 7 Stage 5.2◊ Q3–6, 8–14

Fluency and understanding

1 Find **(a)** the interest and **(b)** the amount owing on a loan of $36 520 for a period of two years if a rate of 7.5% simple interest is charged. Ex 1.1

2 **(a)** The amount an investment of $8200 amounts to after 2 years, if 9.5% p.a. interest is compounded annually, is closest to: Ex 1.2

A $1558 **B** $1632 **C** $7790 **D** $9832

(b) The amount an investment of $4000 amounts to after 2 years, if 3.5% p.a. interest is compounded annually, is closest to:

A $285 **B** $4192 **C** $4280 **D** $4285

3 Michael borrowed $14 000 to buy a new piano. He agreed to pay interest at a rate of 16% p.a. and to repay the loan in full after 4 years. Calculate the amount to be repaid if the interest is compounded: Ex 1.3

(a) half-yearly **(b)** quarterly **(c)** monthly.

4 Helga deposited some money into a savings account. The account guaranteed a return of 9.25% p.a. compounded daily. Calculate the initial deposit if the savings had grown to $47 638.38 after 5 years. Give your answer rounded to the nearest dollar. Ex 1.4

5 Find the effective interest rate for each of the following: Ex 1.5

(a) $38 000 invested at a simple rate of 8.7% p.a.

(b) $50 000 invested at an annually compounding rate of 9.5% p.a.

(c) the interest owed on a loan of $700 after 1 year was $28

(d) 17.4% p.a. compounded every 2 months.

6 The initial price 5 years ago was $60 000. The rate of depreciation is 7% p.a. Ex 1.6

(a) Assuming straight-line depreciation, find the current written-down value.

(b) Assuming reducing balance depreciation, find the current written-down value.

Round all answers to the nearest cent.

7 Simple interest is paid on an investment of $12 500 and yields $19 687.50 in 5 years. What is the rate of interest being paid?

8 Nina's salary has increased by 6% p.a. each year over the past 3 years to $70 500 p.a. Calculate her salary 3 years ago, to the nearest dollar.

9 Assume an inflation rate of 5.6% p.a. In 4 years, an ice-cream, currently $1.20, will cost:

A $\$1.20 \times (1.014)^4$ **B** $\$1.20 \times (1.056)^4$ **C** $\$1.20 \times (1.014)^{16}$ **D** $\$1.20 \times (1.56)^4$

10 Calculate the rate of interest p.a. (rounded to one decimal place) that would allow $6700 to accumulate to $10 000 in 5 years if interest is compounded each quarter.

11 A dining suite was purchased $3\frac{1}{2}$ years ago for $4600. It depreciated at a fixed percentage of its reduced value each year and has just sold for $2800. The rate of depreciation is closest to:

A 15.25% p.a. **B** 13.22% p.a. **C** 13% p.a. **D** 11.18% p.a.

12 Copy and complete the following.
If the prime cost of a photocopier is $10 800 and it has a scrap value of $500, then the total possible depreciation is $_____. If it has an expected life of 206 000 pages, then the unit cost depreciation is $_____ per page. After it copies 125 000, the depreciation is $_____ and the written-down value is $_____.

Reasoning and communicating

13 How many years will it take for an investment of $10 000 to increase by at least 50% if the interest rate of 10% p.a. is compounded half-yearly? Give your answer to the nearest tenth of a year.

14 The following represent loan options. In each case, choose the better option *without calculating the effective rate* and justify your choice.

(a) 11% p.a. compounded quarterly — 14% p.a. compounded monthly

(b) 10% p.a. compounded daily — 10% p.a. compounded monthly

(c) 17% p.a. compounded daily — 17.5% p.a. compounded daily

(d) 12% p.a. flat rate for 2 years — 12% p.a. compounded annually for 2 years

2 Algebra

Recall 2 P

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 How many terms are there in each expression?

(a) $x^2 + 6x + 9$ **(b)** $16x^2 - y^2$ **(c)** $xy + 2y + 4x + 8$

2 What is the coefficient of each of these terms?

(a) $4x$ **(b)** $-6ab$ **(c)** y **(d)** $-12x^2y$

3 Simplify:

(a) $3 \times a + 5 \times b$ **(b)** $8 \times 3x - 8 \times y$

4 Expand and simplify if possible.

(a) $3(x + 2)$ **(b)** $5(x - 3)$ **(c)** $3x + 2(x - 4)$

5 Expand and simplify if possible.

(a) $-2(x + 3)$ **(b)** $-8(x - 5)$ **(c)** $-(x + 2)$

6 Simplify:

(a) $(3x)^2$ **(b)** $(2y)^2$ **(c)** $(7k)^2$

7 Simplify:

(a) $(4x)^2 - 2x^2$ **(b)** $p^2 - (3p)^2$ **(c)** $(-y)^2 + y^2$

8 Rearrange to make x the subject.

(a) $y = 2x + 3$ **(b)** $2a = \dfrac{x - 4b}{c}$ **(c)** $s = \dfrac{5x - at^2}{3}$

9 Find two numbers that:

(a) multiply to give 21 and add to give 10 **(b)** multiply to give 24 and add to give 14

(c) multiply to give 20 and add to give -9 **(d)** multiply to give -18 and add to give 3

(e) multiply to give -20 and add to give -8 **(f)** multiply to give -40 and add to give -3.

10 Simplify the following.

(a) $\dfrac{2}{3} + \dfrac{1}{4}$ **(b)** $\dfrac{5}{6} - \dfrac{3}{8}$ **(c)** $\dfrac{a}{3} + \dfrac{4a}{3}$

11 Simplify the following.

(a) $\dfrac{3}{8} \times \dfrac{4}{9}$ **(b)** $\dfrac{x}{10} \times \dfrac{5}{x}$ **(c)** $\dfrac{5}{6} \div \dfrac{10}{9}$

Syllabus references

Number and Algebra: Algebraic Techniques

- ✓ Apply the four operations to simple algebraic fractions with numerical denominators **(ACMNA232) (Stage 5.2)**
- ✓ Apply the four operations to algebraic fractions with pronumerals in the denominator **(Stage 5.2)**
- ✓ Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate **(ACMNA213) (Stage 5.2)**
- ✓ Factorise algebraic expressions by taking out a common algebraic factor **(ACMNA230) (Stage 5.2)**
- ✓ Expand binomial products and factorise monic quadratic expressions using a variety of strategies **(ACMNA233) (Stage 5.2)**
- ✓ Add and subtract algebraic fractions with numerical denominators, including those with binomial numerators **(Stage 5.3§)**
- ✓ Expand binomial products using a variety of strategies **(ACMNA233) (Stage 5.3§)**
- ✓ Factorise monic and non-monic quadratic expressions **(ACMNA269) (Stage 5.3§)**

Number and Algebra: Equations

- ✓ Rearrange literal equations **(Stage 5.3§)**

Expanding brackets

2.1

Stage 5.2

Need to Know

Expanding brackets

The **distributive law** is used to expand brackets.

$a(b + c) = ab + ac$
$a(b - c) = ab - ac$

Expanding binomial products

A **binomial product** has two terms in each expression.

For example, $(x + 3)(x + 7)$ is a binomial product. A multiplication sign is not required between the factors to indicate a product.

To find the expansion of binomial products, consider the rectangle below.

If the length of a rectangle is $(x + 7)$ units and the breadth is $(x + 3)$ units, then an expression for the area is:

$$A = lb$$

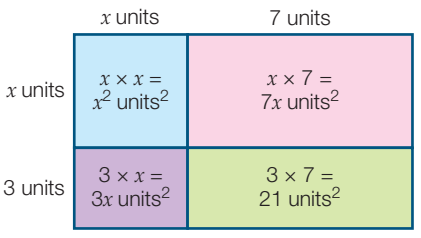
$$A = (x + 7)(x + 3) \text{ units}^2$$

Step 1 Separate the $x + 7$ into a length of x units and a length of 7 units, then separate $(x + 3)$ in a similar way.

Step 2 Add the areas of the four smaller rectangles and simplify like terms.

$A = x^2 + 7x + 3x + 21$

$A = x^2 + 10x + 21 \text{ units}^2$



Step 3 Equate the two expressions that represent the area of the rectangle.

$(x + 7)(x + 3) = x^2 + 10x + 21$

This is called the extended distributive law.

FOIL can be used to expand a binomial product by showing how to multiply the terms.

Multiply the:

First terms in each bracket

Outside terms in each bracket

Inside terms in each bracket

Last terms in each bracket

and then add them all together. Simplify like terms if appropriate.

To expand a binomial product, multiply each term in the second bracket by each term in the first bracket and add them together.

$$(a + b)(c + d) = \overset{F}{ac} + \overset{O}{ad} + \overset{I}{bc} + \overset{L}{bd}$$

Worked Example 1

WE1

Expand and simplify each of the following.

(a) $3(4x - 2)$

(b) $x(x + 4) - 2(x - 1)$

Working

(a) $3(4x - 2)$
$= 12x - 6$

(b) $x(x + 4) - 2(x - 1)$
$= x^2 + 4x - 2x + 2$
$= x^2 + 2x + 2$

Worked Example 2

WE2

Expand and simplify $(x + 5)(2x - 3)$.

Working

$(x + 5)(2x - 3)$
$= 2x^2 - 3x + 10x - 15$
$= 2x^2 + 7x - 15$

Answers page 613

2.1 Expanding brackets

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3, Q4, Q6, Q8	Q1 Column 2, Q2 Column 2, Q3, Q4, Q5, Q6, Q8	Q1 Column 3, Q2 Column 3, Q3, Q4, Q5, Q6, Q7, Q8

Fluency and understanding

WE1

1 Expand and simplify each of the following.

(a) $4(x + 3)$ **(b)** $8(x + 2)$ **(c)** $12(x + 11)$

(d) $2(6x - 1)$ **(e)** $7(2x - 5)$ **(f)** $x(5x - 4)$

(g) $-2(x + 1)$ **(h)** $-x(x - 7)$ **(i)** $-x(3x - 6)$

(j) $-3x(x + 6)$ **(k)** $-2x(x - 9)$ **(l)** $-4x(x + 2)$

(m) $x(x + 2) + 3(x + 2)$ **(n)** $x(2x + 1) + 2(2x + 1)$ **(o)** $x(5x + 2) + 3(5x + 2)$

(p) $2x(x - 3) - 5(x + 6)$ **(q)** $5x(2x + 5) - 4(3x - 8)$ **(r)** $4x(3x - 1) - 6(2x + 1)$

WE2

2 Expand and simplify each of the following.

(a) $(x+1)(x+3)$ **(b)** $(x+4)(x+2)$ **(c)** $(x+5)(x+4)$

(d) $(x+8)(x-3)$ **(e)** $(x+7)(x-9)$ **(f)** $(x-3)(x+7)$

(g) $(x-1)(x-7)$ **(h)** $(x-4)(x-5)$ **(i)** $(x-9)(x-3)$

(j) $(2x+3)(x-2)$ **(k)** $(4x-5)(x+3)$ **(l)** $(7x+1)(x-4)$

(m) $(3x-1)(2x-5)$ **(n)** $(5x-3)(7-x)$ **(o)** $(6x-5)(2-3x)$

3 Expand and simplify the following.

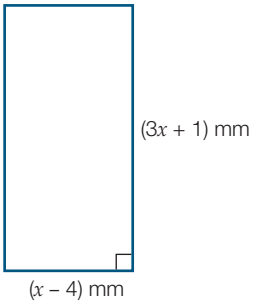
(a) $(4x-3y)(2x+5y)$ **(b)** $(5x+2y)(7x+6y)$ **(c)** $(3x-4y)(5x-8y)$

(d) $(3a+2b)(5a-4b)$ **(e)** $(4p-3r)(2p+7r)$ **(f)** $(9m-2n)(3m+n)$

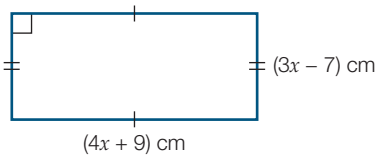
(g) $(5c+2d)(3c+5d)$ **(h)** $(2g-3h)(4g-5h)$ **(i)** $(6b-7f)(2b+3f)$

4 Find the area, in terms of x, of each of these figures.

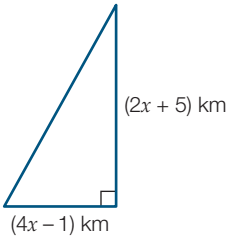
(a)



(b)



(c)



5 **(a)** Draw a rectangle of length $(x+a)$ units and breadth $(x+b)$ units.

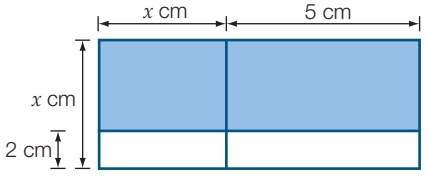
(b) Write an expression for the area of the rectangle in part **(a)** in factorised form.

(c) Expand and simplify your expression in part **(b)**.

(d) Draw a square of side length x units. Extend its length by a units and the breadth by b units to form four small rectangles.

(e) Add the areas of the four smaller rectangles and compare this answer to part **(c)**.

6 **(a)** Write down the dimensions of the shaded rectangle.



(b) Write the binomial product used to find the area of the shaded region, then expand and simplify.

(c) Find the area of the square of side x cm and the rectangle of length 5 cm and breadth x cm.

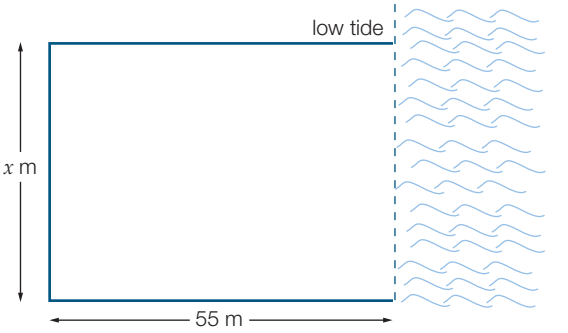
(d) Find the unshaded area, writing it as the sum of the areas of the two rectangles.

(e) Use your answers in part **(c)** and part **(d)** to find the area of the shaded region in simplest form.

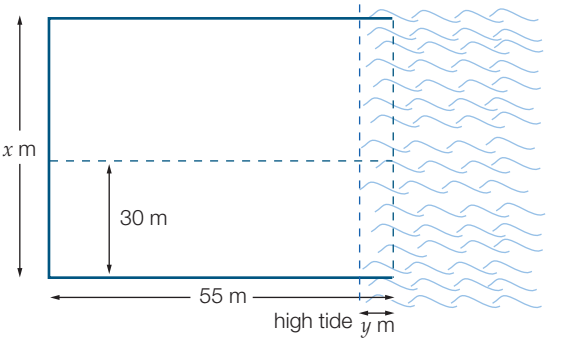
(f) Compare your answers in part **(b)** and part **(e)**.

Reasoning, communicating and open-ended

7 The Holmes family own a property at the beach. Their land is x m wide and extends down to the water's edge. At low tide, the length of the land is 55 m, as shown in the diagram at right.



As the tide comes in, the length is reduced by y m.



(a) Write an expression for the length of the land at any time of day.

(b) Write an expression for the area of the land at any time of day.

(c) What is the area of the land at low tide?

(d) The family have decided to subdivide the land by reducing their land breadth by 30 m. Write an expression for the area of the remaining land at any time of the day.

(e) Expand the expression.

(f) If the tide comes in 10 m at high tide, what is the area of land at high tide?

(g) What is the difference between the area of land at low tide before the subdivision and at high tide after the subdivision?

8 Choose values for a, b, c and d for $(ax + b)(cx + d)$, then expand and simplify the resulting product.

Expanding special products

2.2

Need to Know

Stage 5.3§

Expanding special binomial products

There are some expansions that are helpful to expand by rule.

The **Difference of Two Squares** rule:

$$(a+b)(a-b) = a^2 - ab + ba - b^2$$
$$= a^2 - b^2$$

The **Perfect Square** rules:

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2 + ab + ba + b^2$$
$$= a^2 + 2ab + b^2$$

$$(a-b)^2 = (a-b)(a-b)$$
$$= a^2 - ab - ba + b^2$$
$$= a^2 - 2ab + b^2$$

Rules for expanding special binomial products

Difference of Two Squares	$(a+b)(a-b) = a^2 - b^2$
Perfect Square	$(a+b)^2 = a^2 + 2ab + b^2$
Perfect Square	$(a-b)^2 = a^2 - 2ab + b^2$

Expanding three factors

If we have three or more factors, we first multiply any two factors together and simplify the result. We then repeat this process until the expression is fully expanded. Look for special product expansions to expand an expression more easily.

Worked Example 3

WE3

Expand and simplify:

(a) $(x-3)(x+3)$

(b) $(3x+1)^2$

Working

(a) $(x-3)(x+3)$
$= (x)^2 - (3)^2$
$= x^2 - 9$

(b) $(3x+1)^2$
$= (3x)^2 + 2 \times 3x \times 1 + (1)^2$
$= 9x^2 + 6x + 1$

Worked Example 4

WE4

Expand and simplify.

(a) $x(x-2)(x+3)$ **(b)** $(x+3)(x+1)(x-4)$ **(c)** $(x-1)(x-5)(x+1)$

Working

(a) $x(x-2)(x+3)$
$= x(x^2+3x-2x-6)$
$= x(x^2+x-6)$
$= x^3+x^2-6x$

(b) $(x+3)(x+1)(x-4)$
$= (x^2+x+3x+3)(x-4)$
$= (x^2+4x+3)(x-4)$
$= x^3-4x^2+4x^2-16x+3x-12$
$= x^3-13x-12$

(c) $(x-1)(x-5)(x+1)$
$= (x-1)(x+1)(x-5)$
$= (x^2-1)(x-5)$
$= x^3-5x^2-x+5$

Answers page 614

2.2 Expanding special products

Stage 5.3§

Navigator

Q1, Q2 Column 1, Q3, Q4, Q5, Q6, Q7, Q8, Q11, Q12	Q1, Q2 Column 2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11, Q12, Q13	Q1, Q2 Column 3, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Fluency and understanding

WE3

1 Expand each of the following, using the rules for special products.

(a) $(2-x)(2+x)$ **(b)** $(7-x)(7+x)$ **(c)** $(6+x)(6-x)$

(d) $(2x+1)(2x-1)$ **(e)** $(3x+4)(3x-4)$ **(f)** $(5-3x)(5+3x)$

(g) $(x+6)^2$ **(h)** $(x+11)^2$ **(i)** $(8+x)^2$

(j) $(x-3)^2$ **(k)** $(x-7)^2$ **(l)** $(x-12)^2$

(m) $(2x+5)^2$ **(n)** $(3x+2)^2$ **(o)** $(7x-2)^2$

WE4

2 Expand and simplify each of the following.

(a) $x(x+6)(x+4)$ (b) $x(x-1)(x+5)$ (c) $x(x+1)(x-4)$

(d) $-x(x-3)(x+2)$ (e) $-3x(x-2)(x-1)$ (f) $-4x(x-8)(x+3)$

(g) $(x+4)(x+2)(x+5)$ (h) $(x+2)(x-5)(x-1)$ (i) $(x-3)(x-6)(x-2)$

(j) $(x-3)(x+3)(x+5)$ (k) $(x+7)(x-2)(x+2)$ (l) $(2x+1)(x+4)(x-4)$

(m) $(x-3)(x+4)(x+4)$ (n) $(x+2)(x+2)(x-1)$ (o) $(x-6)(x+5)(x-6)$

3 The expression $(x+1)(x+3)(x-2)$ is *not* the same as:

A $(x+3)(x+1)(x-2)$ B $(x-3)(x+1)(x+2)$

C $(x+1)(x-2)(x+3)$ D $(x-2)(x+1)(x+3)$

4 Expand and simplify the following.

(a) $(2x+7y)(2x+7y)$ (b) $(7x+6y)(7x+6y)$ (c) $(5x+8y)(5x+8y)$

(d) $(4x-y)(4x-y)$ (e) $(6x-5y)(6x-5y)$ (f) $(9x-7y)(9x-7y)$

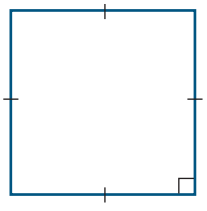
(g) $(5y-x)(5y+x)$ (h) $(3x-y)(3x+y)$ (i) $(2x+3y)(2x-3y)$

(j) $x(2x-1)(3x+5)$ (k) $x(3x+2)(5x-6)$ (l) $3x(4x+7)(2x+3)$

(m) $-x(3x-4)(5x+4)$ (n) $-2x(5x+3)(3x+4)$ (o) $-4x(2x-3)(3x-5)$

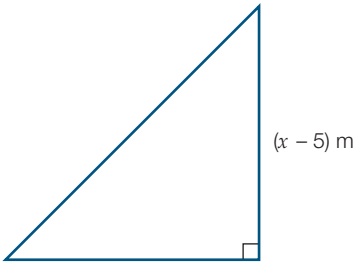
5 Find the area, in terms of x, of each of these figures.

(a)



$(2x+3)$ cm

(b)



6 (a) Expand $x(x+a)(x+b)$.

(b) Use your answer to part **(a)** to:

(i) write down the coefficient of x^2

(ii) write down the coefficient of x

(iii) show that $x(x+a)(x+b) = x^3 - 4x^2 - 12x$, when $a = -6$ and $b = 2$.

(c) For the expression $x(x+25)(x-4)$:

(i) write down the values of a and b

(ii) use the expansion found in part **(a)** to expand $x(x+25)(x-4)$.

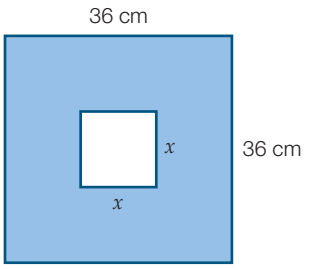
7 (a) If x is the second of three consecutive even numbers, write an expression for the product of the three numbers.

(b) Expand the expression in part **(a)**.

(c) Why is it simpler to make x the second and not the first number?

Reasoning, communicating and open-ended

8 A square blue tile of side length 36 cm has a small white square of side length x cm painted on it.



(a) Write the expression (in cm^2) for the area of the tile that is not white.

(b) Write this expression as a product of factors.

(c) Use this product to find the area that remains blue if the white square has a side of 6 cm.

9 **(a)** Expand and simplify $(x - a)^2 - a^2$.

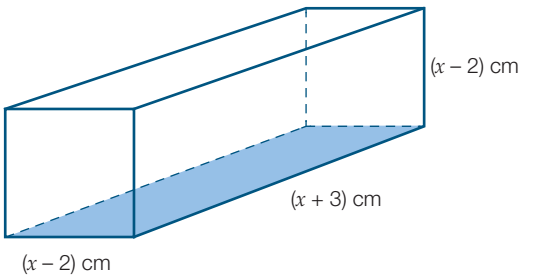
(b) Write down the expansion for $x(x - 2a)$.

(c) What can you conclude about $(x - a)^2 - a^2$ and $x(x - 2a)$ from your answers in part **(a)** and part **(b)**?

(d) A rectangular lawn has a length of x m and a breadth of $(x - 2a)$ m. A square lawn of length $(x - a)$ m has a square-based fountain of side length a m in the centre. If the price of turfing the lawn is based only on its area, is one lawn more expensive to lay with turf than the other? Justify your answer.

(e) If the square lawn has side length 40 m, the fountain has a side length of 3 m and the price of turf is \$12/m^2, what is the total cost of laying the lawn?

10 Consider this box.



(a) Write an expression (in cm^3) for the volume of the box in factorised form.

(b) Find the volume when x is 4 cm.

(c) Show that the volume of the box is 72 cm^3, when $x = 5$.

(d) Write the expression (in cm^3) for the volume in expanded form. Use this to calculate the volume when $x = 5$.

(e) State the dimensions (length, breadth and height) of the box when the volume is 72 cm^3.

(f) What is the smallest possible integer value of x so that the box exists?

(g) Find the volume of the box when x is the value found in part **(f)**.

11 Decide whether $(x^2 + a^2)^3$ can equal zero for some value(s) of x and a other than $x = 0$, $a = 0$.

12 Find two different sets of values for a, b, c and d so that the expanded, simplified form of $(ax + b)(cx + d)$ is a binomial expression.

Hint

A binomial expression contains two terms.

13 **(a)** Expand and simplify $\frac{1}{2}(x + y)^2 + \frac{1}{2}(x - y)^2$.

(b) Find two different sets of values for x and y so that the expression obtained in part **(a)** is the square of a whole number.

14 Find two different whole numbers representing x so that the value of the expression $(4 - 2x)(x + 3)(5x - 10)$ is:

(a) positive **(b)** negative **(c)** zero.

Factorising using common factors

2.3

Stage 5.2

Need to Know

Factorising expressions

Factorising is the opposite or reverse process to expanding.

Expressions can be converted from factorised form to expanded form and, in reverse, from expanded form to factorised form.

For example:

$5(x + 3) = 5 \times x + 5 \times 3$

$= 5x + 15$

factorised form $\rightarrow$ expanded form

In reverse, $5x + 15 = 5(x + 3)$

expanded form $\rightarrow$ factorised form

Factorise expressions by finding the **highest common factor (HCF)** of the terms and writing it outside a set of brackets that contains the remaining terms.

A common factor of -1 may also be taken out.

Factorising by grouping in pairs

Expressions with four terms may **factorise by grouping in pairs**. Take out a common factor from each pair to give two terms. Then take out a common factor from each term.

Worked Example 5

WE 5

Factorise:

(a) $15xy - 9y$ **(b)** $8x^2 + 12x$ **(c)** $-12a^2b - 9ab^2$

Working

(a) $15xy - 9y$
$= 3y(5x - 3)$

(b) $8x^2 + 12x$
$= 4x(2x + 3)$

(c) $-12a^2b - 9ab^2$
$= -3ab(4a + 3b)$

Worked Example 6

WE6

Factorise $k(m+3)+5(m+3)$.

Working

$k(m+3)+5(m+3)$
$=(m+3)(k+5)$

Worked Example 7

WE7

Factorise each of the following expressions by grouping in pairs.

(a) $2ax-8ay-cx+4cy$ **(b)** $uw+vx+xu+vw$

Working

(a) $2ax-8ay-cx+4cy$
$=2a(x-4y)-c(x-4y)$
$=(x-4y)(2a-c)$

(b) $uw+vx+xu+vw$
$=uw+xu+vw+vx$
$=u(w+x)+v(w+x)$
$=(w+x)(u+v)$

Answers page 615

2.3 Factorising using common factors

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q6, Q7, Q8, Q9, Q11, Q13	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13	Q1 Column 3, Q2 Column 2, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13

Fluency and understanding

WE5

1 Factorise each expression.

(a) $4x+8$ **(b)** $18x-15$ **(c)** $24x-9$
(d) $xy+7x$ **(e)** $12kx+21k$ **(f)** $32ab+20b$
(g) $7x^2y+7y^2$ **(h)** $9x^2+3xy$ **(i)** $48m^2+8m$
(j) $-6x-x^2$ **(k)** $-4x+x^2$ **(l)** $-24a^2+10a$
(m) $6ab-18a^2b$ **(n)** $40x^2y+16xy^2$ **(o)** $10cd^2-2cd$

WE6

2 Factorise each expression.

(a) $x(k+1)+4(k+1)$ **(b)** $a(x-4)+6(x-4)$
(c) $m(n-7)+2(n-7)$ **(d)** $x(p+3)-9(p+3)$
(e) $n(q-2)-7(q-2)$ **(f)** $x(u-4)-5(u-4)$
(g) $d(c+5)+(c+5)$ **(h)** $a(d+2)-(d+2)$
(i) $k(m+7)-(m+7)$ **(j)** $x(x-3)+5(x-3)$

WE7

3 Factorise each of the following expressions by grouping in pairs.

(a) $xy + 3y + 5x + 15$ **(b)** $ab + 7b + 2a + 14$ **(c)** $mn + kn + 4m + 4k$

(d) $xy - y + 3x - 3$ **(e)** $3a - 3b + ac - bc$ **(f)** $km - 4m + 2k - 8$

(g) $15ab - 10b - 12a + 8$ **(h)** $6ab - 4b - 15a + 10$ **(i)** $8mn - 6n - 12m + 9$

(j) $vx - vy - ax + ay$ **(k)** $pq - rq - np + nr$ **(l)** $gh - fh - dg + df$

(m) $m^2n + np + m^2t + pt$ **(n)** $a^2b + ab^2 - ac - bc$ **(o)** $d^2k - dk - dr + r$

(p) $27xy - 10 + 18y - 15x$ **(q)** $36a + 5c - 4ac - 45$ **(r)** $25vw + 5v - 15w - 3$

(s) $32a^3c + 15b - 12abc - 40a^2$ **(t)** $21d^2 + 20fh - 12dh - 35df$ **(u)** $m^2nr - 20p + 4mpr - 5mn$

4 Answer TRUE or FALSE to each of these statements.

(a) $8a + 4ab = 4a(2 + b)$ **(b)** $2x - 6xy = 2x(1 - y)$

(c) $mn + 3n - 2m - 6 = (m - 3)(n + 2)$ **(d)** $x^2 - 4x + x - 4 = (x - 4)(x + 1)$

(e) $3x^2 + 15x - 2x - 10 = (x - 2)(3x + 5)$ **(f)** $4x^2 + 20x - 3x - 15 = (x - 5)(4x + 3)$

5 When factorised, $16f^2g - 12gf$ becomes:

A $2fg(8f - 6)$ **B** $4f^2g(4g - 3f)$ **C** $16fg(f - 12)$ **D** $4fg(4f - 3)$

6 The total area of the two shapes shown is:

A $(a + 4)(b - c)$ **B** $(a + c)(b - 4)$ **C** $(a + b)(c - 4)$ **D** $(a - c)(b + 4)$

7 $10dk + 25d - 2k - 5$ factorises to:

A $(2k + 5)(5d - 1)$ **B** $5d(2k + 5)$ **C** $(2k - 5)(5d + 1)$ **D** $(10k + 1)(d - 5)$

8 Mandy has $24y$ cents. She buys x kg of potatoes that cost \$3.76 per kilogram. Express (in cents) in factorised form how much money Mandy has now.

9 Factorise the expression for the square of a given number subtracted from eight times that number.

10 Five small bulldog clips are added to a box containing p small bulldog clips. Five large bulldog clips are added to another box that contains p large bulldog clips. Each small clip costs a cents and each large clip costs b cents.

(a) Give the expression (in cents) for the combined cost of the clips.

(b) Factorise the expression in part **(a)**.

(c) If the cost of one large clip is twice the cost of one small clip write a factorised expression for the total cost of the clips in terms of a and p.

11 **(a)** Draw and label a rectangle that has a length of p m and a breadth of r m.

(b) The length is reduced by a m. Show this on your diagram and find the new area as the difference of two areas.

(c) The breadth of this shortened rectangle is now increased by b m. Show this on your diagram. Label the dimensions and find an expression for the added area.

(d) Show that the area of the newly formed rectangle is $(pr - ar + pb - ab)$ m^2.

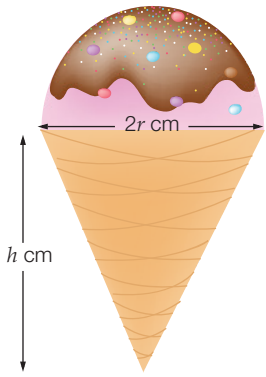
(e) Find the area as a product of the new length and breadth.

(f) Comment on your answers to part **(d)** and part **(e)**.

Reasoning, communicating and open-ended

12 The volume of a sphere of radius r is $\frac{4\pi r^3}{3}$ and the volume of a cone of base radius r and height h is $\frac{\pi r^2 h}{3}$.

Renata is holding an ice-cream cone, with a scoop of ice-cream on top. The ice-cream can be considered to be hemispherical (half a sphere).



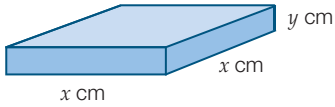
(a) Give an expression for the total volume of ice-cream (m^3), including the ice-cream in the cone and the scoop on top, if the ice-cream filled the cone.

(b) Factorise the expression in part **(a)**.

(c) If $r = 3$ cm, $h = 10$ cm and $\pi = 3.142$, what is the total volume? (Give your answer correct to two decimal places.)

(d) Assume that there is no ice-cream inside the cone, just the scoop on top, and Renata lets it melt inside the cone. If the height of the cone is $2r$, use your expression in part **(b)** to show that the total volume is twice the volume of the cone and therefore all the melted ice-cream will fill the cone exactly.

13 A paver has the shape and dimensions shown.



(a) Write an expression (in cm^2) for the total surface area.

(b) Factorise the expression in part **(a)**.

(c) Find two sets of values for x and y so that the volume (in cm^3) has numerically the same value as the surface area. The values do not have to be integers.

Factorising monic quadratic expressions

2.4

Need to Know

Stage 5.2

Quadratic expressions

A **quadratic expression** has the form of $ax^2 + bx + c$.

In a quadratic expression the highest power of the variable is 2. In a **monic quadratic** expression, the coefficient of the squared variable is 1.

For example: $x^2 + 5x - 4$, $6 - 3a^2$, h^2 and $4n^2 + 7n$ are all quadratic expressions. Of these, only $x^2 + 5x - 4$ and h^2 are monic quadratic expressions.

Quadratic expressions with four terms can sometimes be factorised by grouping in pairs. Look for a common factor in both pairs of terms and take out the factor from each pair.

Factorising quadratic trinomials

A **quadratic trinomial** consists of three terms.

For example: $x^2 + 5x - 4$ is a quadratic trinomial.

Quadratic trinomials can sometimes be factorised by splitting the middle term so that the expression can be factorised by grouping in pairs. In these expressions, the coefficients of the two middle terms multiply to give the constant term.

For example:

$$x^2 - 5x - 14 = x^2 - 7x + 2x - 14 \qquad -7 + 2 = -5 \qquad -7 \times 2 = -14$$
$$= x(x - 7) + 2(x - 7)$$
$$= (x - 7)(x + 2)$$

Sometimes a quadratic trinomial has a common factor that needs to be taken out before the monic trinomial can be factorised.

Monic trinomials can also be factorised using the cross method. Follow these steps.

Step 1 Draw a pair of crossed arrows and write the variable at the beginning of each arrow.

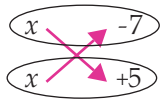
Step 2 Identify a factor pair for the constant term.

For the expression $x^2 - 2x - 35$, the constant term -35 has factor pairs of -35, 1; -7, 5; -5, 7 and -1, 35.

Step 3 Trial a factor pair by writing them at the end of each arrow.

For $x^2 - 2x - 35$, select a pair of factors -7 and 5. Follow the arrows to write the two terms as a sum or difference. If this does not give the middle term, try a new pair of factors.

Step 4 Circle the top and the bottom of the arrows as shown. Each circle contains a binomial factor.



$-7x + 5x = -2x$

Step 5 Write the factorisation.

$x^2 - 2x - 35 = (x - 7)(x + 5)$

Worked Example 8

WE 8

Factorise $x^2 - 4x + 5x - 20$ by grouping in pairs.

Working

$$\begin{aligned} & x^2 - 4x + 5x - 20 \\ &= x(x-4) + 5(x-4) \\ &= (x-4)(x+5) \end{aligned}$$

Worked Example 9

WE 9

Factorise these quadratic trinomials by splitting the middle term.

(a) $p^2 - 11p + 30$ **(b)** $3q^2 + 6q - 24$

Working

(a)

$$\begin{aligned} & p^2 - 11p + 30 \\ &= p^2 - 5p - 6p + 30 \\ &= p(p-5) - 6(p-5) \\ &= (p-5)(p-6) \end{aligned}$$

$\times \rightarrow +30$, $+ \rightarrow -11$ } -5, -6

(b)

$$\begin{aligned} & 3q^2 + 6q - 24 \\ &= 3(q^2 + 2q - 8) \\ &= 3(q^2 + 4q - 2q - 8) \\ &= 3[q(q+4) - 2(q+4)] \\ &= 3(q+4)(q-2) \end{aligned}$$

$\times \rightarrow -8$, $+ \rightarrow +2$ } -2, +4

Worked Example 10

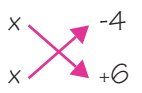
WE 10

Factorise the following quadratic expressions using the cross method.

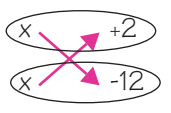
(a) $x^2 - 10x - 24$ **(b)** $x^2 - 9x + 14$

Working

(a) $x^2 - 10x - 24$



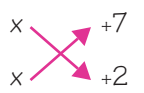
$6x - 4x = 2x$
(incorrect in magnitude and sign)



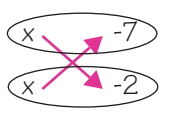
$-12x + 2x = -10x$
(correct in magnitude and sign)

$= (x+2)(x-12)$

(b) $x^2 - 9x + 14$



$2x + 7x = 9x$
(incorrect in sign)



$-2x - 7x = -9x$
(correct in magnitude and sign)

$= (x-2)(x-7)$

2.4 Factorising monic quadratic expressions

Answers page 615

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q9, Q10	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q9, Q10	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

1 Factorise each expression by grouping in pairs. WE 8

(a) $x^2 + 2x + 4x + 8$ **(b)** $x^2 - 3x + 6x - 18$ **(c)** $x^2 - x + 7x - 7$

(d) $x^2 + 5x - 2x - 10$ **(e)** $x^2 - 4x - 3x + 12$ **(f)** $x^2 + 3x - 4x - 12$

(g) $x^2 - 2x + 4x - 8$ **(h)** $x^2 + x - 3x - 3$ **(i)** $x^2 - 2x - 8x + 16$

2 Factorise these quadratic trinomials by splitting the middle term. WE 9

(a) $x^2 + 3x + 2$ **(b)** $y^2 + 9y + 14$ **(c)** $a^2 - 10a + 24$

(d) $k^2 + 11k + 24$ **(e)** $m^2 - 2m - 15$ **(f)** $x^2 - x - 30$

(g) $p^2 - 10p + 16$ **(h)** $d^2 - 5d + 4$ **(i)** $y^2 - 5y + 6$

(j) $m^2 - 4m - 12$ **(k)** $c^2 - 10c + 21$ **(l)** $g^2 + 2g - 35$

(m) $2y^2 + 14y + 24$ **(n)** $3m^2 + 18m + 24$ **(o)** $5x^2 - 5x - 100$

(p) $-4a^2 - 8a + 12$ **(q)** $-3k^2 + 12k + 63$ **(r)** $-5x^2 - 20x + 105$

3 Factorise these quadratic trinomials using the cross method. WE 10

(a) $z^2 - 8z + 15$ **(b)** $k^2 + 5k - 6$ **(c)** $z^2 + 10z + 21$

(d) $t^2 - t - 20$ **(e)** $r^2 + r - 30$ **(f)** $t^2 - 13t + 22$

(g) $z^2 - z - 42$ **(h)** $m^2 - 9m + 20$ **(i)** $y^2 + 7y - 60$

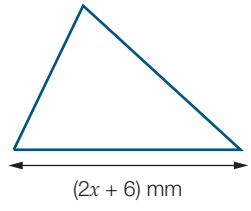
(j) $n^2 - 3n - 28$ **(k)** $p^2 + 9p - 36$ **(l)** $a^2 - 14a + 48$

4 If $A = 4x^2 + 6x + 29$ and $B = 1 + 9x + 5x^2$, then $A - B$ when factorised becomes:

A $(x - 4)(x + 7)$ **B** $(x + 4)(7 - x)$ **C** $(x + 7)(4 - x)$ **D** $(x - 7)(4 - x)$

5 A rectangle has a breadth of $(x + 3)$ cm. Find the length of the rectangle if its area is $(2x^2 + 10x + 12)$ cm^2.

6 A triangle has a base length of $(2x + 6)$ mm. Find the height of the triangle if its area is $(3x^2 - 3x - 36)$ mm^2.



Reasoning, communicating and open-ended

7 A rectangular table has an area of $(x^2 + 30x - 1800)$ cm^2.

(a) If the length of the table (cm) is one of the factors of the expression, find an expression for the breadth.

(b) If the length of the table is 210 cm, find the breadth.

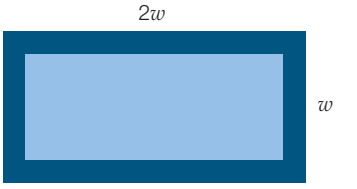
(c) If the breadth is 1 m, what is the length?

8 A rectangular block of land is twice as long as its breadth w m. A concrete path of uniform breadth is to be constructed around the inside border of the block and will occupy an area of $(6w - 4)$ m^2.

(a) What is the area left unconcreted?

(b) Factorise this area.

(c) If the factors represent the length and breadth of the unconcreted area, find the breadth of the path.



Hint

Rewrite your answer in **(b)** as the product of two algebraic factors by subtracting the same amount from the original length and breadth.

9 Write two monic quadratic trinomials that will factorise, and factorise them.

10 If $x^2 + 5x - 24$ represents a rectangular area:

(a) write down two possible sets of dimensions for the rectangle

(b) evaluate the area for each set of dimensions.

Factorising using special products

2.5

Stage 5.3§

Need to Know

Special products

Some quadratic expressions can be factorised easily if they are recognised as special products. These special products are the Difference of Two Squares and Perfect Squares.

The Difference of Two Squares rule

The Difference of Two Squares rule can be used to expand quadratic expressions.

$(a + b)(a - b) = a^2 - b^2$

By reversing this expansion:

$a^2 - b^2 = (a + b)(a - b)$

Using surds to factorise quadratics

A surd is an irrational number such as $\sqrt{7}$. It does not have an exact decimal equivalent.

Surds can be used to write an integer such as 7 as a square number $(\sqrt{7})^2 = 7$, so surds can be used to factorise quadratics using the Difference of Two Squares rule.

The Perfect Square rules

$(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$

By reversing these expansions we get:

$a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

We can use these rules to factorise many quadratic trinomials.

Special product rules

$a^2 - b^2 = (a + b)(a - b)$ (Difference of Two Squares)
$a^2 + 2ab + b^2 = (a + b)^2$ (Perfect Square)
$a^2 - 2ab + b^2 = (a - b)^2$ (Perfect Square)

Worked Example 11

WE11

Factorise these expressions using the Difference of Two Squares rule.

(a) $d^2 - 81$ **(b)** $180 - 5y^2$ **(c)** $4k^2 - 9m^2$

Working

(a) $d^2 - 81$
$= (d + 9)(d - 9)$

(b) $180 - 5y^2$
$= 5(36 - y^2)$
$= 5(6 + y)(6 - y)$

(c) $4k^2 - 9m^2$
$= (2k)^2 - (3m)^2$
$= (2k - 3m)(2k + 3m)$

Worked Example 12

WE12

Factorise $(x+3)^2 - 9$.

Working

$$
\begin{aligned}
&(x+3)^2 - 9\\
&= (x+3)^2 - 3^2\\
&= [(x+3) - 3][(x+3) + 3]\\
&= x(x+6)
\end{aligned}
$$

Worked Example 13

WE13

Factorise these expressions. (Your answers will include surds.)

(a) $x^2 - 3$

(b) $10 - 5x^2$

Working

(a)
$$
\begin{aligned}
&x^2 - 3\\
&= x^2 - (\sqrt{3})^2\\
&= (x - \sqrt{3})(x + \sqrt{3})
\end{aligned}
$$

(b)
$$
\begin{aligned}
&10 - 5x^2\\
&= 5(2 - x^2)\\
&= 5[(\sqrt{2})^2 - x^2]\\
&= 5(\sqrt{2} - x)(\sqrt{2} + x)
\end{aligned}
$$

Worked Example 14

WE14

Factorise these expressions.

(a) $(x+7)^2 - (x+2)^2$

(b) $(x-5)^2 - 7$

Working

(a)
$$
\begin{aligned}
&(x+7)^2 - (x+2)^2\\
&= [(x+7) - (x+2)][(x+7) + (x+2)]\\
&= (x+7-x-2)(x+7+x+2)\\
&= 5(2x+9)
\end{aligned}
$$

(b)
$$
\begin{aligned}
&(x-5)^2 - 7\\
&= (x-5)^2 - (\sqrt{7})^2\\
&= (x-5+\sqrt{7})(x-5-\sqrt{7})
\end{aligned}
$$

Worked Example 15

WE15

Factorise these expressions.

(a) $x^2 - y^2 + 5x - 5y$

(b) $25m^2 - n^2 - 5m - n$

Working

(a)
$$
\begin{aligned}
&x^2 - y^2 + 5x - 5y\\
&= (x^2 - y^2) + (5x - 5y)\\
&= (x-y)(x+y) + 5(x-y)\\
&= (x-y)(x+y+5)
\end{aligned}
$$

(b)
$$
\begin{aligned}
&25m^2 - n^2 - 5m - n\\
&= (5m+n)(5m-n) - 1(5m+n)\\
&= (5m+n)(5m-n-1)
\end{aligned}
$$

Worked Example 16

WE16

Factorise these expressions using a Perfect Square rule.

(a) $x^2 + 10x + 25$ **(b)** $40x^2 - 120x + 90$

Working

(a) $x^2 + 10x + 25$
$= x^2 + 2(x)(5) + 5^2$
$= (x + 5)^2$

(b) $40x^2 - 120x + 90$
$= 10(4x^2 - 12x + 9)$
$= 10[(2x)^2 - 2(2x)(3) + 3^2]$
$= 10(2x - 3)^2$

2.5 Factorising using special products

Answers page 616

Navigator

Stage 5.3§

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5 Column 1, Q6 Column 1, Q7, Q8, Q10 Column 1, Q11, Q13, Q15, Q18	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5 Column 2, Q6 Column 2, Q7, Q8, Q9, Q10 Column 2, Q11, Q12, Q13, Q15, Q18	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5 Column 2, Q6 Column 3, Q7, Q8, Q9, Q10 Column 3, Q11, Q12, Q14, Q15, Q16, Q17, Q19

Fluency and understanding

1 Factorise these expressions using the Difference of Two Squares rule. WE11

(a) $p^2 - 4$ **(b)** $d^2 - 100$ **(c)** $9 - k^2$
(d) $3p^2 - 75$ **(e)** $80 - 5m^2$ **(f)** $8k^2 - 32$
(g) $(2y)^2 - 49$ **(h)** $25 - (4x)^2$ **(i)** $(6m)^2 - (5n)^2$
(j) $9x^2 - 16$ **(k)** $4 - 25a^2$ **(l)** $81d^2 - 64$

2 Factorise these expressions. WE12

(a) $(x + 2)^2 - 9$ **(b)** $(a - 1)^2 - 25$ **(c)** $16 - (k + 5)^2$
(d) $36 - (p + 1)^2$ **(e)** $1 - (2y - 3)^2$ **(f)** $1 - (3x - 2)^2$
(g) $5(x + 3)^2 - 5y^2$ **(h)** $7(a + 5)^2 - 7b^2$ **(i)** $3(x - 4)^2 - 12$

3 Factorise these expressions. (Your answers will include surds.) WE13

(a) $x^2 - 2$ **(b)** $7 - p^2$ **(c)** $4k^2 - 5$
(d) $9p^2 - 2$ **(e)** $3a^2 - 33$ **(f)** $2b^2 - 6$

4 Factorise these expressions. WE14

(a) $(x + 2)^2 - (y + 1)^2$ **(b)** $(a + 2)^2 - (a + 3)^2$ **(c)** $(2k + 3)^2 - (m - 3)^2$
(d) $(x + 4)^2 - (x - 4)^2$ **(e)** $(2x + 1)^2 - (x + 3)^2$ **(f)** $(5x - 4)^2 - (x - 7)^2$
(g) $(x + 2)^2 - 3$ **(h)** $(y - 3)^2 - 6$ **(i)** $(x + 8)^2 - 5$
(j) $(x - 5)^2 - 13$ **(k)** $(x - 1)^2 - 19$ **(l)** $(2x + 1)^2 - 2$

WE15

5 Factorise these expressions.

(a) $x^2 - 9 + 4x + 12$ **(b)** $a^2 - 16 + 5a - 20$

(c) $x^2 - y^2 + 7x - 7y$ **(d)** $9a^2 - b^2 + 6a + 2b$

(e) $x^2 - 16y^2 + 2x - 8y$ **(f)** $4p^2 - q^2 - 6p - 3q$

WE16

6 Factorise each expression using a Perfect Square rule.

(a) $x^2 + 6x + 9$ **(b)** $x^2 + 4x + 4$ **(c)** $x^2 - 14x + 49$

(d) $x^2 - 8x + 16$ **(e)** $x^2 + 2x + 1$ **(f)** $x^2 + 12x + 36$

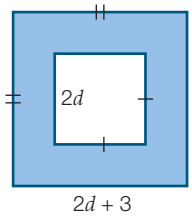
(g) $4x^2 + 20x + 25$ **(h)** $9x^2 + 24x + 16$ **(i)** $25x^2 - 10x + 1$

7 When factorised, $(2a - 1)^2 - (a + 1)^2$ becomes:

A $3(a - 1)(a + 1)$ **B** $3a(a - 2)$

C $(a - 1)(a + 1)$ **D** $(2a - 1)(a + 1)$

8 When factorised, the expression for the blue area shown becomes:



A $4d(d + 3)$ **B** $(4d + 3)(d + 1)$

C $(2d + 3)(d + 1)$ **D** $3(4d + 3)$

9 When factorised, $\left(\frac{a}{k}\right)^2 - \left(\frac{b}{2k}\right)^2$ is:

A $\frac{1}{4k^2}(2a - b)(2a + b)$ **B** $\frac{1}{2k^2}(a - b)(a + 2b)$

C $\frac{1}{4k^2}(a - 3b)(a + 3b)$ **D** $\frac{1}{k^2}(a - b)(a + b)$

10 Factorise these expressions.

(a) $8m^2 - 200n^2$ **(b)** $6x^2 - 24r^2$ **(c)** $5a^2 - 45c^2$

(d) $27a^2 - 48b^2$ **(e)** $72x^2 - 98y^2$ **(f)** $175t^2 - 28p^2$

(g) $54k^2 - 24m^2$ **(h)** $2x^2y^2 - 32z^2$ **(i)** $3p^2q^2 - 12n^2$

11 Factorise these expressions.

(a) $16c^2 - 25d^2 + 10d + 8c$ **(b)** $k^2 - 64 + km - 8m$

(c) $81 - 49a^2 - 27d + 21ad$ **(d)** $16 - 9y^2 - 16w + 12wy$

(e) $25m^2 - 9n^2 - 20m - 12n$ **(f)** $144p^2 - 49r^2 - 12pq + 7rq$

Reasoning, communicating and open-ended

12 The difference between the squares of two consecutive odd numbers is 56.

(a) If x and y are the two numbers and x is greater than y, write the equation describing the information.

(b) Factorise the left side of the equation in part **(a)**.

(c) Explain why $x = y + 2$.

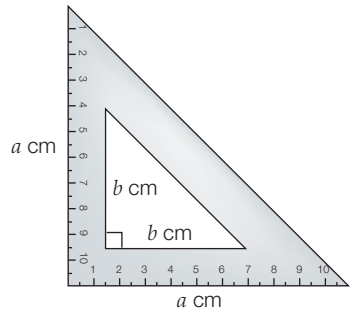
(d) Substitute $y + 2$ for x in the equation in part **(b)** and simplify.

(e) Solve the equation to obtain the value for y. Hence, state the value of x.

13 **(a)** Write the expression (in cm^2) for the shaded area of the set square shown in factorised form.

(b) What is the shaded area if $a = 3b$?

(c) What are the dimensions of the instrument if the shaded area is 100 cm^2 given $a = 3b$?



14 **(a)** Factorise $a^2 - b^2$.

(b) How many factors do you have?

(c) Find $\dfrac{a^2 - b^2}{a - b}$, $a \neq b$.

(d) Write $a^4 - b^4$ as the difference of two squares.

(e) Factorise $a^4 - b^4$ fully.

(f) How many factors did you have?

(g) Find $\dfrac{a^4 - b^4}{a - b}$, $a \neq b$.

(h) Write $a^8 - b^8$ as the difference of two squares.

(i) Factorise $a^8 - b^8$ fully using your answer from part **(e)**.

(j) Find $\dfrac{a^8 - b^8}{a - b}$, $a \neq b$.

(k) Use the pattern to write an expression for $\dfrac{a^{16} - b^{16}}{a - b}$, $a \neq b$.

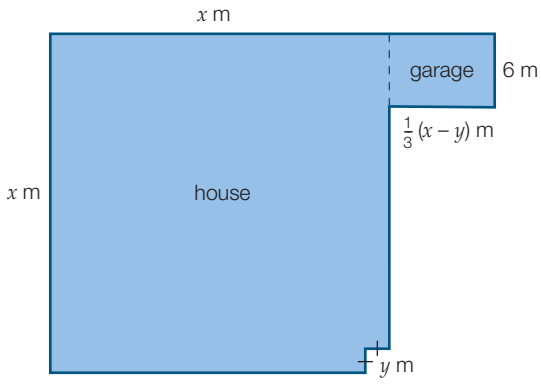
(l) Use the expression in part **(k)** to show that $2^{16} - 1 = 3 \times 5 \times 17 \times 257$.

15 The shape of a house with its garage is shown.

(a) Write the expression (in m^2) for the area of the house.

(b) Write the expression (in m^2) for the area of the garage.

(c) Write the expression (in m^2) for the total area in factorised form.



16 **(a)** If $a = m - n$, $b = 2\sqrt{mn}$ and $c = m + n$, show that $c^2 = a^2 + b^2$.

(b) Show that if $p = kr$ then the expression $t^2 = p^2 - r^2$ becomes $t^2 = r^2(k - 1)(k + 1)$.

(c) What values of k would make $t = 0$?

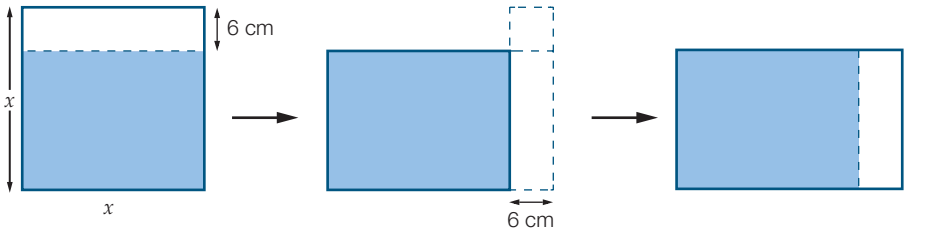
(d) What values of k are not possible? Justify your answer.

Hint

Can $k = 0$?

Can k be a fraction?

17 A square sheet of plywood is made into a rectangle by cutting 6 cm from one end and joining it to the other. The overhanging edge is then trimmed.



(a) If the length of the sides of the original sheet of plywood were x cm, find the area of the square sheet of plywood.

(b) What area has been trimmed off the rectangle?

(c) Use your answers in part **(a)** and part **(b)** to find the area of the rectangular sheet of plywood.

(d) Factorise your answer.

(e) Use the diagrams above to find the length and breadth of the rectangle obtained in terms of x.

(f) What is the relationship of the answer in part **(d)** to the length and breadth of the rectangle?

18 Write two trinomial expressions that have a difference that is a perfect square.

19 Write an expression with four terms so that two terms are the difference of the squares of two variables and the other two terms are a multiple of the sum of these variables. Factorise your expression.

Problem solving

Playground swings

The length (in metres) of the chains of a playground swing that takes t seconds for one swing is given by $l = \frac{5t^2}{M}$.

1 What is the difference (in metres) in the chain lengths of two swings that take t_1 seconds and t_2 seconds, respectively, to make one swing? (Assume t_1 is greater than t_2.) Express your answer in factorised form.

2 If $t_1 - t_2 = \frac{3M}{10}$ and $t_1 + t_2 = \frac{M}{5}$ then find the difference in the chain lengths.

Strategy options

- Look for a pattern.
- Have I seen a similar problem?

Half-time 2

Stage 5.2 Q1–5 **Stage 5.3§ Q6–11**

1 Expand and simplify. Ex.2.1

(a) $(x + 2)(x + 6)$ **(b)** $(x - 5)(x + 7)$

(c) $(3x - 2)(x + 5)$ **(d)** $(2x - 1)(x + 3)$

2 Factorise each of the following. Ex.2.4

(a) $x^2 + 2x - 15$ **(b)** $x^2 - 14x + 24$ **(c)** $x^2 - 3x - 40$

3 Factorise each of the following. Ex.2.4

(a) $x^2 + 5x + 7x + 35$ **(b)** $x^2 - 3x - 4x + 12$ **(c)** $x^2 + 7x - 3x - 21$

4 Factorise each of the following by taking out a common factor. Ex.2.3

(a) $7x + 14y$ **(b)** $18x^2 + 24xy$

(c) $3(x - 2) + y(x - 2)$ **(d)** $3(x + 5) + 2y(x + 5)$

5 The expression $acx^2 + adx + bcx + bd$ can be factorised to obtain: Ex.2.3

A $(bx + a)(cx + d)$ **B** $(ax + c)(bx + d)$

C $(dx + b)(ax + c)$ **D** $(ax + b)(cx + d)$

6 The expression $a^2 - b^2 + 3a + 3b$ can be factorised to obtain: Ex.2.5

A $(a - b)(a - b - 3)$ **B** $(a - b)(a + b - 3)$

C $(a + b)(a - b + 3)$ **D** $(a + b)(a + b - 3)$

7 The expression $a^2b^2 - c^2 + abd - cd$ can be factorised to obtain: Ex.2.5

A $(ad + b)(cd - b)$ **B** $(ab - c)(ab + c + d)$

C $(ac - d)(a + b + c + d)$ **D** $(cd - b)(ab - b)$

8 Factorise each of the following. Ex.2.5

(a) $x^2 + 14x + 49$ **(b)** $x^2 - 18x + 81$ **(c)** $9x^2 - 30x + 25$

9 Factorise each of the following. Ex.2.5

(a) $x^2 - 9$ **(b)** $9x^2 - 36y^2$

(c) $8x^2 - 50y^2$ **(d)** $27 - 48x^2$

10 Expand and simplify. Ex.2.2

(a) $(x - 3)(x + 3)$ **(b)** $(x + 8)(x - 8)$

(c) $(3x + y)(3x - y)$ **(d)** $(6x - 5y)(6x + 5y)$

11 Expand and simplify. Ex.2.2

(a) $x(x + 2)(x + 3)$ **(b)** $3(x - 2)(x + 2)$

(c) $(3x - 4)(7x + 19)(3x + 3)$ **(d)** $6x(2x - 6)(4x - 1)$

2.6 Factorising by completing the square

Stage 5.3§

Need to Know

Factorising four terms by grouping 'three and one'

Not all trinomials can be factorised using binomial products.

For example:

For the quadratic trinomial $x^2 + 6x - 2$, two rational numbers cannot be found that multiply to give -2 and add to 6.

If a group of three terms form a perfect square and a fourth term is subtracted, the Perfect Square rule and Difference of Two Squares rule can be used to factorise the expression.

Many trinomials can be converted to 'three and one' expressions so they can be factorised using the **completing the square** method of factorising.

Consider $x^2 + 6x - 2$ again.

- Add a third term to the first two terms to make a perfect square.

  For $x^2 + 6x - 2$, add 9 to obtain $(x^2 + 6x + 9) - 2$.
- Subtract the amount added to keep the expression equivalent.

  Subtract 9 to obtain $(x^2 + 6x + 9) - 2 - 9$.

A perfect square has been obtained by adding 9.

Subtract 9 to keep the whole expression equivalent to $x^2 + 6x - 2$.

$$x^2 + 6x - 2 = (x^2 + 6x + 9) - 2 - 9$$

$= (x^2 + 6x + 9) - 11$ This is now a 'three and one' expression involving a perfect square.

$= (x + 3)^2 - 11$ Use the Difference of Two Squares rule to factorise using surds.

$$= (x + 3)^2 - (\sqrt{11})^2$$

$$= (x + 3 - \sqrt{11})(x + 3 + \sqrt{11})$$

Completing the square

Factorise monic trinomials by completing the square.

To complete the square on $x^2 + bx + c$:

Step 1 Divide b, the coefficient of x, by 2 to obtain $\frac{b}{2}$.

Step 2 Add $\left(\frac{b}{2}\right)^2$ to make a perfect square.

Step 3 Subtract $\left(\frac{b}{2}\right)^2$ to keep the expression equivalent.

Step 4 Factorise the perfect square.

Step 5 Factorise the expression using the difference of the perfect square.

If the coefficient of x is not 1, look for a common factor that can be taken out of all terms.

Worked Example 17

WE17

Factorise $x^2 + 4x + 4 - y^2$ by grouping 'three and one'.

Working

$$
\begin{aligned}
& x^2 + 4x + 4 - y^2 \\
= & (x^2 + 4x + 4) - y^2 \\
= & (x + 2)^2 - y^2 \\
= & (x + 2 - y)(x + 2 + y)
\end{aligned}
$$

Worked Example 18

WE18

Factorise $x^2 - 8x + 9$ by completing the square.

Working

$$
\begin{aligned}
& x^2 - 8x + 9 \\
= & (x^2 - 8x + 16) - 16 + 9 \\
= & (x - 4)^2 - 7 \\
= & (x - 4)^2 - (\sqrt{7})^2 \\
= & (x - 4 - \sqrt{7})(x - 4 + \sqrt{7})
\end{aligned}
$$

Worked Example 19

WE19

Factorise $3x^2 + 12x - 30$ by completing the square.

Working

$$
\begin{aligned}
& 3x^2 + 12x - 30 \\
= & 3(x^2 + 4x - 10) \\
= & 3[(x^2 + 4x + 4) - 4 - 10] \\
= & 3[(x + 2)^2 - 14] \\
= & 3[(x + 2)^2 - (\sqrt{14})^2] \\
= & 3[(x + 2 - \sqrt{14})(x + 2 + \sqrt{14})] \\
= & 3(x + 2 - \sqrt{14})(x + 2 + \sqrt{14})
\end{aligned}
$$

Answers page 617

2.6 Factorising by completing the square

Stage 5.3§

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q9, Q10	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q9, Q10	Q1 Column 2, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

WE17

1 Factorise these expressions by grouping 'three and one'.

(a) $x^2 + 8x + 16 - y^2$ **(b)** $y^2 - 4y + 4 - k^2$

(c) $y^2 + 8y + 16 - m^2$ **(d)** $m^2 - 10m + 25 - 4n^2$

(e) $x^2 + 2x + 1 - 9y^2$ **(f)** $q^2 + 6q + 9 - 16p^2$

(g) $a^2 - 6ab + 9b^2 - 16$ **(h)** $x^2 - 10xy + 25y^2 - 36$

(i) $4a^2 - 12a + 9 - 16b^2$ **(j)** $9c^2 + 30c + 25 - 4d^2$

WE18

2 Factorise these expressions by completing the square.

(a) $x^2 + 6x + 2$ **(b)** $x^2 - 10x + 4$ **(c)** $x^2 - 8x - 3$

(d) $x^2 + 20x + 77$ **(e)** $x^2 - 12x + 21$ **(f)** $x^2 + 2x - 8$

(g) $x^2 - 14x - 2$ **(h)** $x^2 - 4x - 2$ **(i)** $x^2 + 16x + 47$

(j) $x^2 + 3x + 1$ **(k)** $x^2 + 7x + 9$ **(l)** $x^2 - x - 4$

(m) $x^2 - 5x + 5$ **(n)** $x^2 + 5x - 2$ **(o)** $x^2 + x - 5$

(p) $x^2 - 9x + 13$ **(q)** $x^2 - 3x - 2$ **(r)** $x^2 - 7x - 1$

WE19

3 Factorise each expression by completing the square.

(a) $2x^2 + 4x - 10$ **(b)** $4x^2 - 16x - 4$ **(c)** $10x^2 - 80x - 30$

(d) $3x^2 - 6x - 12$ **(e)** $5x^2 + 20x - 5$ **(f)** $7x^2 + 42x + 14$

4 The expression $2m^2 - 12m + 4$ when factorised is:

A $2(m - 6 - \sqrt{7})(m - 6 + \sqrt{7})$ **B** $2(m + 7 - \sqrt{3})(m + 7 + \sqrt{3})$

C $2(m - 6 - \sqrt{2})(m - 6 + \sqrt{2})$ **D** $2(m - 3 - \sqrt{7})(m - 3 + \sqrt{7})$

5 Factorising $x^2 + x - \frac{5}{16}$ gives:

A $(x - \frac{1}{4})(x + \frac{5}{4})$ **B** $(x - \frac{5}{16})(x + 1)$ **C** $(x - \frac{5}{4})(x + \frac{1}{16})$ **D** $(x + \frac{1}{4})(x - \frac{5}{4})$

6 A seagull is diving for a fish. Its height above sea level , in metres, at time t seconds is given by the expression $t^2 - 8t + 14$.

(a) How high is the seagull at $t = 0$?

(b) Factorise the expression by completing the square.

(c) How long is the time interval between $t = 4 - \sqrt{2}$ and $t = 4 + \sqrt{2}$, rounded to two decimal places.

(d) Explain what happens when $t = 4 \pm \sqrt{2}$.

(e) Explain what the seagull is doing between $t = 4 - \sqrt{2}$ and $t = 4 + \sqrt{2}$.

7 The profit made by a newspaper publisher in the first 8 months is given by $P = -5x^2 + 40x - 5$ where x is the time in months.

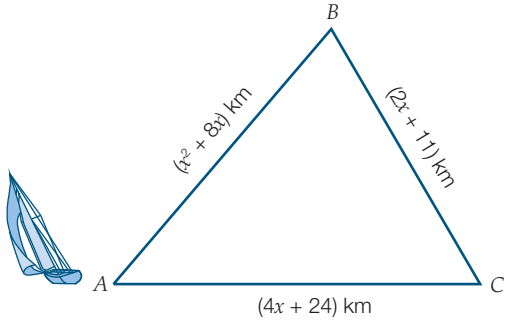
(a) Factorise the expression for P.

(b) Show that $P = 0$ when $x = 4 \pm \sqrt{15}$.

(c) Show what happened to the company in the first 8 months by substituting different values for x into the expression.

Reasoning, communicating and open-ended

8 A yacht sails a leg of a race by going in a straight course from point A to point B for $(x^2 + 8x)$ km. From point B it travels to point C for $(2x + 11)$ km, then back to point A for $(4x + 24)$ km, as shown in the diagram.



(a) Write the simplified expression (in km) for the total distance the yacht sailed.

(b) Factorise the expression in part **(a)**.

(c) If the factorised expression also represents the product of the length and breadth of a rectangle and if the length is 15 km, find the value of x. Give your answer as an exact distance.

(d) Find the length of the race, rounded to two decimal places, by substituting the exact value for x into the expression for the total distance found in **(a)**.

9 The number of earthworms in a compost heap after d days is given by $100 - 4d - d^2$.

(a) How many earthworms were there originally?

(b) Factorise the expression for the number of worms.

(c) Use your factorisation to show that when $d = 2(\sqrt{26} - 1)$, no worms remain.

(d) Evaluate $2(\sqrt{26} - 1)$ and round to two decimal places to find on which day the number of earthworms is reduced to zero.

(e) A second colony of earthworms is now in another compost heap. Their population is given by $64 - 2n - n^2$.

(i) Factorise the expression.

(ii) Use this factorisation to show that no earthworms remain when $n = \sqrt{65} - 1$.

(f) Use your answers to part **(c)** and part **(e) (ii)** to decide which colony stayed longer in their respective compost heaps.

10 **(a)** Write the expression $x^2 - 2ax + 2b$ in the form $(x - a)^2 + A$.

(b) Write an expression for A in terms of a and b.

(c) **(i)** What is the relationship between a and b if the expression is to be a perfect square?

(ii) Find a value for a and b that produces a perfect square.

2.7 Factorising non-monic quadratic trinomials

Stage 5.3§

Need to Know

Splitting the middle term

Factorise the **non-monic quadratic** trinomial $ax^2 + bx + c$ by splitting the middle term.

- Multiply the coefficient of the squared term a by the constant c.
- List factor pairs of ac and choose a factor pair that add to give b.
- Rewrite the expression using these factors as coefficients of the two middle terms to replace the middle term.
- Factorise using grouping in pairs.

Some strategies for splitting the middle term

For a positive constant term:

- factors will have the same sign as the middle term
- if the magnitude of the coefficient of the middle term is small, the factors will be close to each other
- if the magnitude of the coefficient of the middle term is large, the factors will be far apart.

For a negative constant term:

- factors will be opposite in sign
- if the magnitude of the coefficient of the middle term is small, choose factors that are close in magnitude but opposite in sign
- if the magnitude of the coefficient of the middle term is large, choose factors that are far apart in magnitude but opposite in sign.

The cross method

Factorise non-monic quadratic trinomials using the cross method.

- Choose a factor pair that multiply to give the squared term (ax^2).
- Choose a factor pair that multiply to give the constant term (c).
- Cross multiply the terms from one factor pair with the terms from the other pair.
- If the sum of the two multiplications gives the middle term (b) then we have our factorisation, if not, then try the other cross multiplication or use different factor pairs.

Completing the square with non-monic quadratic expressions

Not all quadratic trinomials factorise with rational binomial products and some do not factorise at all. 'Completing the square' can be used with non-monic quadratics to see if the expression can be factorised.

To complete the square, the coefficient of the quadratic must be 1. Take the coefficient of x out as a common factor before completing the square.

Worked Example 20

WE 20

Factorise the following quadratic expressions by splitting the middle term.

(a) $3x^2 - 11x - 4$ **(b)** $5x^2 - 43x + 24$ **(c)** $12x^2 + 11x - 36$

Working

(a) $3x^2 - 11x - 4$

$= 3x^2 - 12x + x - 4$

$= 3x(x - 4) + 1(x - 4)$

$= (x - 4)(3x + 1)$

× to give -12
+ to give -11
} -12, 1

(b) $5x^2 - 43x + 24$

$= 5x^2 - 40x - 3x + 24$

$= 5x(x - 8) - 3(x - 8)$

$= (x - 8)(5x - 3)$

× to give 120
+ to give -43
} -40, -3

(c) $12x^2 + 11x - 36$

$= 12x^2 + 27x - 16x - 36$

$= 3x(4x + 9) - 4(4x + 9)$

$= (4x + 9)(3x - 4)$

× to give -432
+ to give 11
} 27, -16

Worked Example 21

WE 21

Factorise the following quadratic expressions using the cross method.

(a) $3x^2 - 11x - 4$ **(b)** $5x^2 - 43x + 24$ **(c)** $12x^2 + 11x - 36$ **(d)** $15x^2 + 26x + 8$

Working

(a) $3x^2 - 11x - 4$

3x -4
x +1

$3x - 4x = -x$ (incorrect middle term)

3x +1
x -4

$-12x + x = -11x$ (correct middle term)

$3x^2 - 11x - 4$

$= (3x + 1)(x - 4)$

(b) $5x^2 - 43x + 24$

5x -8
x -3

$-15x - 8x = -23x$ (incorrect middle term)

5x -3
x -8

$-40x - 3x = -43x$ (correct middle term)

$5x^2 - 43x + 24$

$= (5x - 3)(x - 8)$

(c) $12x^2 + 11x - 36$

3x +4
4x -9

$-27x + 16x = -11x$ (incorrect middle term)

3x -4
4x +9

$27x - 16x = +11x$ (correct middle term)

$12x^2 + 11x - 36$

$= (4x + 9)(3x - 4)$

(d) $15x^2 + 26x + 8$

5x +4
3x +2

$10x + 12x = 22x$ (incorrect middle term)

5x +2
3x +4

$20x + 6x = 26x$ (correct middle term)

$15x^2 + 26x + 8$

$= (5x + 2)(3x + 4)$

Worked Example 22 WE22

Factorise the following expressions, if possible, by completing the square

(a) $2x^2 - 5x - 2$

(b) $2x^2 - 4x + 6$

Working

(a) $2x^2 - 5x - 2$

$= 2(x^2 - \frac{5}{2}x - 1)$

$= 2[(x^2 - \frac{5}{2}x + \frac{25}{16}) - \frac{25}{16} - 1]$

$= 2[(x - \frac{5}{4})^2 - \frac{41}{16}]$

$= 2(x - \frac{5}{4} + \frac{\sqrt{41}}{4})(x - \frac{5}{4} - \frac{\sqrt{41}}{4})$

(b) $2x^2 - 4x + 6$

$= 2(x^2 - 2x + 3)$

$= 2[(x^2 - 2x + 1) - 1 + 3]$

$= 2[(x - 1)^2 + 2]$

The expression cannot be factorised further.

Answers page 617

2.7 Factorising non-monic quadratic trinomials

Stage 5.3§

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q6, Q7	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7

Fluency and understanding

WE20

1 Factorise these expressions by splitting the middle term.

(a) $2x^2 + 5x + 2$ (b) $2y^2 + 9y + 7$ (c) $5b^2 + 8b + 3$

(d) $3k^2 - 11k + 6$ (e) $11a^2 - a - 12$ (f) $5x^2 - 11x - 12$

(g) $2m^2 + 3m - 9$ (h) $7x^2 - 23x + 18$ (i) $11c^2 - 25c + 6$

(j) $5y^2 + 19y + 12$ (k) $3p^2 + 8p - 16$ (l) $2x^2 + 3x - 20$

(m) $3x^2 + 8x + 5$ (n) $7y^2 - 18y + 8$ (o) $3t^2 + t - 4$

(p) $7p^2 + 22p + 16$ (q) $3q + 2q^2 - 5$ (r) $4x + 5x^2 - 12$

(s) $8 - 18r + 7r^2$ (t) $21 - 16c + 3c^2$ (u) $2 - 10t + 12t^2$

WE21

2 Factorise these expressions using the cross method.

(a) $4x^2 + 8x + 3$ (b) $10y^2 + 13y + 4$ (c) $9c^2 - 19c + 2$

(d) $6w^2 + 17w + 12$ (e) $15y^2 + 7y - 30$ (f) $8x^2 - 9x - 14$

(g) $4a^2 + 12a - 27$ (h) $24k^2 - 2k - 15$ (i) $20x^2 - 44x + 21$

(j) $12b^2 + 32b + 5$ (k) $6g^2 + 7g + 2$ (l) $8a^2 + 22a + 5$

(m) $4x^2 - 11x + 6$ (n) $10m^2 + 17m + 7$ (o) $12p^2 - 7p - 10$

(p) $9d^2 + 6d - 8$ (q) $6x^2 - x - 22$ (r) $18n^2 + 31n - 35$

(s) $24y^2 - 17y + 3$ (t) $15x^2 + 46x + 16$ (u) $12x^2 - 11x - 15$

WE 22

3 Factorise the following expressions, if possible, by completing the square.

(a) $2x^2 - 6x - 1$ (b) $3x^2 - 12x - 7$ (c) $2x^2 - 8x - 3$

(d) $3x^2 - 9x + 4$ (e) $5x^2 - 10x + 3$ (f) $2x^2 - 7x + 2$

(g) $2x^2 - 4x + 5$ (h) $2x^2 - 8x + 9$ (i) $3x^2 - 6x + 5$

4 The floor (in m^2) of a rectangular room of height x m is given by the expression $6x^2 - 7x - 3$.

(a) Factorise the expression.

(b) Assuming that the two factors represent the length and the breadth of the room and that the length is 11.5 m, find the height of the room.

(c) What is the breadth of the room?

Reasoning, communicating and open-ended

5 The number of bird nests in a park depends on the number of trees it has. In Byrd Park there are $12t^2 - 12t + 10$ bird nests, where t is $\frac{1}{100}$ of the number of trees. (The number of trees is estimated to the nearest multiple of 25.) Plume Park has $4t^2 + 6t + 1$ bird nests.

(a) Write an expression in terms of t for the difference in the number of bird nests in the two parks. (Assume that the number of bird nests in Byrd Park is greater than the number of bird nests in Plume Park for all values of t.)

(b) (i) Simplify the expression in part **(a)**.

(ii) Factorise the expression in part **(b) (i)**.

(c) Use your answer from part **(b) (ii)** to show that the parks will have equal numbers of birds when there are 75 or 150 trees.

6 Paul and Selina each have \$$x$. Later, each person has more money and the product of their money is $12x^2 - 8x - 15$.

(a) Factorise the expression.

(b) Find how much money each person had initially if Paul now has \$65 and has more money than Selina.

(c) How much money does Selina have now?

7 (a) Find a non-monic quadratic trinomial that will factorise to give binomial products. Factorise it.

(b) Find a non-monic quadratic trinomial that will factorise but not as binomial products. Factorise it.

(c) Find a non-monic quadratic trinomial that will not factorise. Explain why it cannot be factorised.

2.8 Algebraic fractions

Stage 5.2

Need to Know

Multiplication and division

Rules for multiplying and dividing numerical fractions also apply to **algebraic fractions** (expressions involving variables in fraction form).

To simplify a fraction, look for common factors in the numerator (top) and the denominator (bottom) of the fraction. Simplify by dividing the numerator and denominator by these common factors.

A common factor may be an algebraic term or expression.

Note that a fraction such as $\frac{x+4}{x-1}$ cannot be simplified further as there are no common factors.

The x in both the numerator and denominator cannot be cancelled because x is *not* a factor of $x + 4$ or $x - 1$.

Multiplying or dividing two or more fractions can also be simplified by cancelling common factors. The numerators and denominators may need to be factorised first to identify any common factors. As well as cancelling common factors within fractions, a factor that is common to the numerator of one fraction and the denominator of the other can also be cancelled.

When multiplying (or dividing) algebraic fractions, follow these steps.

Step 1 Factorise if possible.

Step 2 Change a division into a multiplication by inverting the second fraction.

Step 3 Identify common factors in the numerators and denominators of the fractions and divide the numerators and denominators by these common factors.

Step 4 Simplify by multiplying the remaining numerators and the remaining denominators.

Addition and subtraction

Use the rules for addition and subtraction of numerical fractions to deal with algebraic fractions.

When adding (or subtracting) algebraic fractions, follow these steps.

Step 1 Find the lowest common denominator (LCD). If there is a factor that is common to both denominators, only use it *once* in the common denominator.

Step 2 Rewrite each fraction as an equivalent fraction with this common denominator.

Step 3 Add (or subtract) the new numerators.

Worked Example 23

WE 23

Simplify **(a)** $\frac{4a}{6}$ **(b)** $\frac{15(x-3)}{10}$

Working

(a) $\frac{4a}{6}$

$= \frac{\cancel{4}^{2}a}{\cancel{6}_{3}}$

$= \frac{2a}{3}$

(b) $\frac{15(x-3)}{10}$

$= \frac{\cancel{15}^{3}(x-3)}{\cancel{10}_{2}}$

$= \frac{3(x-3)}{2}$

Worked Example 24

WE 24

Simplify **(a)** $\frac{12y^2}{9y^4}$ **(b)** $\frac{3m^2n^5}{15mn^2}$

Working

(a) $\frac{12y^2}{9y^4}$

$= \frac{\cancel{12}^{4}\cancel{y^2}}{\cancel{9}_{3}\cancel{y^4}_{y^2}}$

$= \frac{4}{3y^2}$

(b) $\frac{3m^2n^5}{15mn^2}$

$= \frac{\cancel{3}\cancel{m^2}^{m}\cancel{n^5}^{n^3}}{\cancel{15}_{5}\cancel{m}_{1}\cancel{n^2}_{1}}$

$= \frac{mn^3}{5}$

Worked Example 25

WE 25

Simplify **(a)** $\frac{8p}{9} \times \frac{3p}{28}$ **(b)** $\frac{2r}{7} \div \frac{8r}{35}$

Working

(a) $\frac{8p}{9} \times \frac{3p}{28}$

$= \frac{\cancel{8}^{2}p}{\cancel{9}_{3}} \times \frac{\cancel{3}^{1}p}{\cancel{28}_{7}}$

$= \frac{2p^2}{21}$

(b) $\frac{2r}{7} \div \frac{8r}{35}$

$= \frac{\cancel{2}^{1}\cancel{r}^{1}}{\cancel{7}_{1}} \times \frac{\cancel{35}^{5}}{\cancel{8}_{4}\cancel{r}_{1}}$

$= \frac{5}{4}$

Worked Example 26 WE26

Simplify $\frac{(x+2)(x+4)}{(x-1)(x+2)}$.

Working

$$\frac{(x+2)(x+4)}{(x-1)(x+2)}$$

$$= \frac{\overset{1}{\cancel{(x+2)}}(x+4)}{(x-1)\underset{1}{\cancel{(x+2)}}}$$

$$= \frac{x+4}{x-1}$$

Worked Example 27 WE27

Simplify:

(a) $\frac{x}{4}+\frac{x}{5}$

(b) $\frac{2f}{5}-\frac{3f}{7}$

Working

(a) $\frac{x}{4}+\frac{x}{5}$

$$= \frac{5x}{20}+\frac{4x}{20}$$

$$= \frac{5x+4x}{20}$$

$$= \frac{9x}{20}$$

(b) $\frac{2f}{5}-\frac{3f}{7}$

$$= \frac{14f}{35}-\frac{15f}{35}$$

$$= \frac{14f-15f}{35}$$

$$= -\frac{f}{35}$$

Answers page 618

2.8 Algebraic fractions

Stage 5.2

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

WE23

1 Simplify the following.

(a) $\frac{8x}{10}$ (b) $\frac{6k}{42}$ (c) $\frac{25p}{45}$

(d) $\frac{4(a-b)}{2}$ (e) $\frac{9(x+1)}{6}$ (f) $\frac{14(y-2)}{8}$

(g) $\frac{5y(a-4)}{10}$ (h) $\frac{8x(x+5)}{36}$ (i) $\frac{18x(x-4)}{24}$

2 Simplify. WE24

(a) $\frac{15a^3}{6a^2}$ (b) $\frac{8n^4}{20n^5}$ (c) $\frac{18r^5}{45r^3}$ (d) $\frac{7p^2q^3}{21p^3q^2}$ (e) $\frac{16x^4y^3}{24x^2y^8}$ (f) $\frac{63v^3w^4}{35v^6w^2}$

3 Simplify the following. WE25

(a) $\frac{9m}{10} \times \frac{15m}{16}$ (b) $\frac{7k}{12} \times \frac{9k}{14}$ (c) $\frac{5cd}{2} \times \frac{4}{3c}$

(d) $\frac{11b}{15} \div \frac{22b^4}{25}$ (e) $\frac{8pr}{5} \div \frac{14pr^2}{15}$ (f) $\frac{27d}{32} \div \frac{33}{12d^2}$

4 Simplify the following. WE26

(a) $\frac{(x+1)(x+6)}{(x-5)(x+1)}$ (b) $\frac{(x-3)(x+2)}{(x-3)(x-2)}$ (c) $\frac{(2x-3)(x+4)}{(x+4)(2x+1)}$

(d) $\frac{(3x+1)(x-2)}{(x-7)(3x+1)}$ (e) $\frac{(a+2)(b+5)}{(b+3)(a+2)}$ (f) $\frac{(x+7)(y-2)}{(y-2)(x-1)}$

5 Simplify the following. WE27

(a) $\frac{x}{2} + \frac{x}{3}$ (b) $\frac{x}{5} + \frac{x}{2}$ (c) $\frac{x}{6} + \frac{x}{4}$

(d) $\frac{2x}{3} - \frac{x}{6}$ (e) $\frac{4x}{5} - \frac{x}{15}$ (f) $\frac{2x}{3} - \frac{5x}{4}$

(g) $\frac{4x}{3} - \frac{2y}{4}$ (h) $\frac{5x}{7} - \frac{3y}{4}$ (i) $\frac{11x}{8} - \frac{8y}{9}$

6 $\frac{3x}{4} + \frac{5x}{9}$ simplifies to:

A $\frac{15x^2}{36}$ B $\frac{8x}{36}$ C $\frac{47x}{36}$ D $\frac{8x^2}{36}$

Reasoning, communicating and open-ended

7 At a party, a cake that contains x calories has been cut into eight equal pieces. Tim has 3 pieces. He also has an ice-cream that contains two-fifths of the calories of the cake. How many calories has Tim eaten?

8 A metal solid of mass m kg is cut into three pieces of equal mass. When one of the pieces is sanded and polished, it loses one-tenth of its mass. What is the mass of the polished piece?

9 Find two fractions with different denominators that add together to make $\frac{17n}{24}$.

10 Ollie simplified $\frac{4r}{32}$ as $8r$. His teacher told him he was incorrect. Explain to Ollie why he was wrong and show him the correct answer.

2.9 More complex algebraic fractions

Stage 5.3§

Need to Know

Addition and subtraction of algebraic fractions

All fractions are added and subtracted in the same way.

Step 1 Find the lowest common denominator (LCD). You may need to factorise first. If there is a factor that is common to both denominators, only use it once in the common denominator.

Step 2 Rewrite each fraction as an equivalent fraction with this common denominator placing each binomial numerator in brackets.

Step 3 Rewrite as one fraction with this common denominator.

Step 4 Expand each set of brackets before simplification. When subtracting, be careful to change signs when expanding the brackets.

Step 5 Simplify the numerator.

Multiplication and division of algebraic fractions involving trinomial expressions

Algebraic fractions involving trinomial expressions can often be simplified if the trinomials can be factorised. A factor of a denominator can be divided exactly into an identical factor in the numerator of the same fraction or of another fraction when fractions are being multiplied. Division must be converted into multiplication. Cancelling of factors does *not* apply to addition and subtraction.

Worked Example 28

WE 28

Simplify **(a)** $\frac{x-3}{5}+\frac{x+8}{3}$ **(b)** $\frac{x+7}{4}-\frac{x-3}{6}$

Working

(a)

$$\frac{x-3}{5}+\frac{x+8}{3}$$
$$=\frac{3(x-3)}{15}+\frac{5(x+8)}{15}$$
$$=\frac{3(x-3)+5(x+8)}{15}$$
$$=\frac{3x-9+5x+40}{15}$$
$$=\frac{8x+31}{15}$$

(b)

$$\frac{x+7}{4}-\frac{x-3}{6}$$
$$=\frac{3(x+7)}{12}-\frac{2(x-3)}{12}$$
$$=\frac{3(x+7)-2(x-3)}{12}$$
$$=\frac{3x+21-2x+6}{12}$$
$$=\frac{x+27}{12}$$

Worked Example 29

WE29

Simplify **(a)** $\dfrac{4y}{y^2-9}+\dfrac{12}{y-3}$ **(b)** $\dfrac{3}{x^2+2x+1}+\dfrac{3}{x^2-1}$

Working

(a) $\dfrac{4y}{y^2-9}+\dfrac{12}{y-3}$

$$=\frac{4y}{(y-3)(y+3)}+\frac{12}{y-3}$$

$$=\frac{4y}{(y-3)(y+3)}+\frac{12(y+3)}{(y-3)(y+3)}$$

$$=\frac{4y+12(y+3)}{(y-3)(y+3)}$$

$$=\frac{4y+12y+36}{(y-3)(y+3)}$$

$$=\frac{16y+36}{(y-3)(y+3)}$$

(b) $\dfrac{3}{x^2+2x+1}+\dfrac{3}{x^2-1}$

$$=\frac{3}{(x+1)^2}+\frac{3}{(x-1)(x+1)}$$

$$=\frac{3(x-1)}{(x+1)^2(x-1)}+\frac{3(x+1)}{(x-1)(x+1)^2}$$

$$=\frac{3(x-1)+3(x+1)}{(x+1)^2(x-1)}$$

$$=\frac{3x-3+3x+3}{(x+1)^2(x-1)}$$

$$=\frac{6x}{(x+1)^2(x-1)}$$

Worked Example 30

WE30

Simplify:

(a) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)}\times\dfrac{(x-2)(x+1)}{(x+5)(x+3)}$ **(b)** $\dfrac{(x+2)(x+5)}{(x-1)(x+1)}\div\dfrac{(x+5)(x+3)}{(x-2)(x+1)}$

Working

(a) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)}\times\dfrac{(x-2)(x+1)}{(x+5)(x+3)}$

$$=\frac{(x+2)\overset{1}{\cancel{(x+5)}}}{(x-1)\underset{1}{\cancel{(x+1)}}}\times\frac{(x-2)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+5)}}(x+3)}$$

$$=\frac{(x+2)(x-2)}{(x-1)(x+3)}$$

(b) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)}\div\dfrac{(x+5)(x+3)}{(x-2)(x+1)}$

$$=\frac{(x+2)(x+5)}{(x-1)(x+1)}\times\frac{(x-2)(x+1)}{(x+5)(x+3)}$$

$$=\frac{(x+2)\overset{1}{\cancel{(x+5)}}}{(x-1)\underset{1}{\cancel{(x+1)}}}\times\frac{(x-2)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+5)}}(x+3)}$$

$$=\frac{(x+2)(x-2)}{(x-1)(x+3)}$$

Worked Example 31

WE31

Simplify $\dfrac{4x^2-12x}{x^2-25}\times\dfrac{x^2+7x+10}{2x^2+4x}$.

Working

$$\frac{4x^2-12x}{x^2-25}\times\frac{x^2+7x+10}{2x^2+4x}$$

$$=\frac{4x(x-3)}{(x-5)(x+5)}\times\frac{(x+2)(x+5)}{2x(x+2)}$$

$$=\frac{\overset{2}{\cancel{4}}\overset{1}{\cancel{x}}(x-3)}{(x-5)\underset{1}{\cancel{(x+5)}}}\times\frac{\overset{1}{\cancel{(x+2)}}\overset{1}{\cancel{(x+5)}}}{\underset{1}{\cancel{2}}\underset{1}{\cancel{x}}\underset{1}{\cancel{(x+2)}}}$$

$$=\frac{2(x-3)}{x-5}$$

Worked Example 32

WE32

Simplify $\dfrac{x^2+x-12}{3x^2+12x}\div\dfrac{x^2-9}{12x}$.

Working

$$\frac{x^2+x-12}{3x^2+12x}\div\frac{x^2-9}{12x}$$

$$=\frac{x^2+x-12}{3x^2+12x}\times\frac{12x}{x^2-9}$$

$$=\frac{(x-3)(x+4)}{3x(x+4)}\times\frac{12x}{(x-3)(x+3)}$$

$$=\frac{\overset{1}{\cancel{(x-3)}}\overset{1}{\cancel{(x+4)}}}{\underset{1}{\cancel{3}}\underset{1}{\cancel{x}}\underset{1}{\cancel{(x+4)}}}\times\frac{\overset{4}{\cancel{12}}\overset{1}{\cancel{x}}}{\underset{1}{\cancel{(x-3)}}(x+3)}$$

$$=\frac{4}{x+3}$$

2.9 More complex algebraic fractions

Answers page 618

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q10, Q11, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q10, Q11, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13

Fluency and understanding

1 Simplify the following. WE 28

(a) $\frac{x+6}{2}+\frac{x+2}{3}$ (b) $\frac{x-3}{4}+\frac{x+4}{8}$ (c) $\frac{x-7}{3}+\frac{x+5}{9}$

(d) $\frac{3x+2}{6}-\frac{x+4}{4}$ (e) $\frac{3x-8}{4}-\frac{x-3}{10}$ (f) $\frac{4x-1}{9}-\frac{x-6}{12}$

(g) $\frac{3x+2y}{4}+\frac{5x-y}{3}$ (h) $\frac{7x-2y}{3}+\frac{x-3y}{5}$ (i) $\frac{8x-3y}{4}+\frac{5x+7y}{6}$

(j) $\frac{4x-3y}{5}-\frac{3x-2y}{6}$ (k) $\frac{5x-2y}{4}-\frac{x+y}{2}$ (l) $\frac{7x+3y}{2}-\frac{2x-y}{3}$

2 Simplify, leaving denominators in factorised form. WE 29

(a) $\frac{2m}{m-4}+\frac{1}{m^2-4m}$ (b) $\frac{5}{x^2+2x}+\frac{7}{3x+6}$ (c) $\frac{6}{x^2-25}+\frac{2}{x+5}$

(d) $\frac{3p}{p+1}-\frac{4}{p^2+p}$ (e) $\frac{3}{h^2-6h}-\frac{5}{h}$ (f) $\frac{3}{b^2+4b+4}-\frac{2}{b+2}$

(g) $\frac{5}{c^2-2c+1}+\frac{4}{c^2-1}$ (h) $\frac{4}{k^2+k}+\frac{5}{k^2-1}$ (i) $\frac{7}{4d^2-1}-\frac{5}{4d^2+2d}$

3 Simplify the following. WE 30

(a) $\frac{3}{x-2}\times\frac{x-2}{9}$ (b) $\frac{15}{5-2x}\times\frac{5+2x}{5}$

(c) $\frac{x-1}{x-4}\div\frac{x-1}{x+4}$ (d) $\frac{x+4}{x+3}\div\frac{x+4}{x-3}$

(e) $\frac{x(x+7)}{2(x-3)}\div\frac{4x}{x-3}$ (f) $\frac{7x(x-4)}{10x}\div\frac{2(x-4)}{5}$

(g) $\frac{(x+1)(x-3)}{(x-4)(x+7)}\times\frac{(x+7)(x+2)}{(x-1)(x+1)}$ (h) $\frac{(x-4)(x+4)}{(x+3)(x-4)}\times\frac{(x-2)(x+3)}{(x+1)(x-2)}$

(i) $\frac{(3x-2)(x+5)}{(2x+3)(x-6)}\div\frac{(x+2)(x+5)}{(2x+3)(x-3)}$ (j) $\frac{(5x-3)(2x+3)}{(x-4)(x+7)}\div\frac{(2x+3)(5x-4)}{(x-3)(x+7)}$

WE31

4 Simplify the following. Remember to factorise first.

(a) $\dfrac{x^2+11x+24}{x+3}$

(b) $\dfrac{x^2-12x+27}{x-3}$

(c) $\dfrac{x+2}{x^2-3x-10}$

(d) $\dfrac{x+5}{x^2-2x-35}$

(e) $\dfrac{8x^2-2}{8x+4}$

(f) $\dfrac{3x^2-12}{x+2}$

(g) $\dfrac{4x^2+12x+9}{4x+6}$

(h) $\dfrac{9x^2+30x+25}{6x+10}$

(i) $\dfrac{x^2+2x-3}{x^2-5x-24}$

(j) $\dfrac{x^2+3x-4}{x^2+4x-5}$

(k) $\dfrac{x^2-7x-8}{x^2-10x+16}$

(l) $\dfrac{x^2-7x+6}{x^2-2x-24}$

(m) $\dfrac{3x+6}{4x-24}\times\dfrac{8x-48}{6x+12}$

(n) $\dfrac{5x+10}{2x-8}\times\dfrac{3x-12}{9x+18}$

(o) $\dfrac{x^2-5x}{3x+3}\times\dfrac{x+1}{2x^2+6x}$

(p) $\dfrac{4x^2+8x}{x+5}\times\dfrac{2x-14}{x^2+2x}$

(q) $\dfrac{x^2-x-12}{x^2+9x+18}\times\dfrac{x^2+4x-12}{x^2-2x-8}$

(r) $\dfrac{x^2+7x+10}{x^2+x-20}\times\dfrac{x^2-7x+12}{x^2-5x-14}$

WE32

5 Simplify the following.

(a) $\dfrac{x^2-4}{x^2-9}\div\dfrac{4x+8}{12x-36}$

(b) $\dfrac{x^2-25}{x^2-49}\div\dfrac{3x-15}{5x+35}$

(c) $\dfrac{x^2-36}{x^2+6x-7}\div\dfrac{x^2-5x-6}{x^2-1}$

(d) $\dfrac{x^2-6x-16}{x^2-x-30}\div\dfrac{x^2-64}{x^2-5x-6}$

(e) $\dfrac{x^2+6x+9-y^2}{4x+6}\div\dfrac{2x^2+6x-2xy}{10x^2+15x}$

(f) $\dfrac{x^2-2x+1-4y^2}{12x^2-5x-2}\div\dfrac{3x-3+6y}{28x+7}$

6 If $4x+3y=5$, then $\dfrac{x+1}{3}+\dfrac{y+1}{4}$ simplifies to:

A $\dfrac{5}{12}$ B $\dfrac{5}{7}$ C 1 D $\dfrac{7}{12}$

Reasoning, communicating and open-ended

7 (a) Factorise and simplify as much as possible the expression

$$\frac{9b^2-9a^2}{6b^2+3bc+2ab+ac}\div\frac{3a+3b}{2b+c}.$$

(b) If $a=5$ and $b=7$, show that the expression in part **(a)** takes the value $\dfrac{3}{13}$.

8 (a) Show that $(a-b)(a^2+ab+b^2)=a^3-b^3$.

(b) Write a^6-b^6 as the difference of two cubes.

(c) Use the result from part **(b)** to show that $\dfrac{a^6-b^6}{a^2-b^2}=a^4+a^2b^2+b^4$.

9 A circular disc of diameter $2w$ m is cut from a square piece of timber. The side lengths of the timber are the same as the diameter of the disc.

(a) Write an expression for the area of timber wasted.

(b) What fraction of the area of the uncut timber is the area of the timber wasted? Give your answer in simplified form.

10 The density of an object of mass m kg and volume v m^3 is given by $d = \frac{m}{v}$ kg/m^3.

(a) What is the density of a sphere of mass m kg and radius r m if its volume is $\frac{4\pi r^3}{3}$?

(b) A golf ball has density d_1, radius r_1 and volume v_1. A basketball has density d_2, radius r_2 and volume v_2. Show that $\frac{d_2}{d_1} = \left(\frac{m_2}{m_1}\right)\left(\frac{r_1}{r_2}\right)^3$.

(c) If the basketball's mass is 20 times greater than the mass of the golf ball and its radius is 5 times greater than the radius of the golf ball, what is the value of $\frac{d_2}{d_1}$?

11 Write the sum of two terms with different denominators that can be added to give $\frac{x+2}{3}$.

12 Write two algebraic fractions with different denominators that when subtracted give $\frac{7x+6}{3}$.

13 Write two algebraic fractions with different denominators that when multiplied give $\frac{4}{x+2}$.

Problem solving

How to square two-digit numbers quickly—without a calculator

We start with the Difference of Two Squares rule:

$$a^2 - b^2 = (a - b)(a + b)$$

Adding b^2 to both sides we get:

$$a^2 - b^2 + b^2 = b^2 + (a - b)(a + b)$$

So $$a^2 = b^2 + (a - b)(a + b)$$

This expression can be used to find the square of a two-digit number quickly, without a calculator! In the expression:

a is the number to be squared

b can be any number (we choose a convenient value).

For example, to find 19^2, we have $a = 19$ and choose $b = 1$ (you'll see why in a moment).

Start with $a^2 = b^2 + (a - b)(a + b)$

Substitute $a = 19$ and $b = 1$.

$$\begin{aligned} 19^2 &= 1^2 + (19 - 1)(19 + 1) \\ &= 1^2 + (18)(20) \\ &= 1^2 + 360 \\ &= 361 \end{aligned}$$

Notice how the choice of 1 for b results in one of the brackets containing a multiple of 10, thus making multiplication by the contents of the other pair of brackets easier.

Use the method described above to calculate the following. Show the steps involved.

(a) 29^2

(b) 59^2

(c) 79^2

(d) 99^2

Strategy options

- Look for a pattern.
- Break problem into manageable parts.

2.10 Rearranging formulas

Stage 5.3§

Need to Know

Formulas

A formula is an equation showing the relationship between two or more variables or unknowns. The equation $T = 2\pi\sqrt{\frac{l}{g}}$ is an example of a formula used in physics. This formula can be rearranged to make l or g the subject ($l =$ or $g =$). To rearrange formulas like this, apply the same inverse operations to both sides, using the same methods as used to solve equations.

Equations involving roots

If equations involve square roots, square both sides.

If equations involve cube roots, cube both sides.

Equations involving powers

The inverse operation of squaring is taking the square root. The solution to $x^2 = 9$ is $x = 3$ or $x = -3$, because $(3)^2 = (-3)^2 = 9$. Therefore, when taking the square root, include ±.

In many cases, only the positive square root is an acceptable answer.

For example: time, area, volume and length can only have positive values.

Worked Example 33

WE33

(a) Rearrange $V = \frac{1}{3}\pi r^2 h$ to make r the subject of the formula.

(b) Find r when $V = 27\pi$ and $h = 4$ are subtituted into the rearranged formula.

Working

(a)

$$V = \frac{1}{3}\pi r^2 h$$
$$3V = \pi r^2 h$$
$$\frac{3V}{\pi} = r^2 h$$
$$r^2 = \frac{3V}{\pi h}$$
$$r = \pm\sqrt{\frac{3V}{\pi h}}$$

(b)

$$r = \pm\sqrt{\frac{3V}{\pi h}}$$
$$= \pm\sqrt{\frac{3 \times 27\cancel{\pi}}{\cancel{\pi} \times 4}}$$
$$= \pm\sqrt{\frac{81}{4}}$$
$$= \pm\frac{9}{2}$$
$$= \pm 4\frac{1}{2}$$

Worked Example 34

WE 34

(a) Rearrange $T = 2\pi\sqrt{\frac{l}{g}}$ to make **(i)** l and **(ii)** g the subject of the formula.

(b) **(i)** Find l when $T = 0.6\pi$ and $g = 10$ are substituted into the rearranged formula.

(ii) Find g when $T = \frac{4\pi}{7}$ and $l = 0.8$ are substituted into the rearranged formula.

Working

(a)

$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$$

$$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$$

$$l = g \times \left(\frac{T}{2\pi}\right)^2$$

(i) $$l = \frac{gT^2}{4\pi^2}$$

(ii) $$g = \frac{4\pi^2 l}{T^2}$$

(b) (i)

$$l = \frac{gT^2}{4\pi^2}$$

$$= \frac{10 \times (0.6\pi)^2}{4\pi^2}$$

$$= \frac{10 \times \overset{0.09}{\cancel{0.36}}\cancel{\pi^2}}{\cancel{4}\cancel{\pi^2}}$$

$$= 0.9$$

(ii)

$$g = \frac{4\pi^2 l}{T^2}$$

$$= \frac{4 \times \pi^2 \times 0.8}{\left(\frac{4\pi}{7}\right)^2}$$

$$= \frac{\cancel{4}\cancel{\pi^2} \times 49 \times \overset{0.2}{\cancel{0.8}}}{\cancel{16}\cancel{\pi^2}}$$

$$= 9.8$$

Answers page 619

2.10 Rearranging formulas

Stage 5.3§

Navigator

Q1 (a)–(c), Q2 (a)–(c), Q3, Q4, Q5, Q6, Q9	Q1 (d)–(f), Q2 (d)–(f), Q3, Q4, Q5, Q7, Q8, Q9	Q1 (g)–(i), Q2 (g)–(i), Q3, Q4, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

WE33

1 **(i)** Rearrange the following, to make the variable in each bracket the subject of the formula.

(ii) Find the value of this variable when the given values are substituted into the rearranged formula.

(a) $A = s^2$	$[s]$	$A = 27$
(b) $K = \frac{1}{2}mv^2$	$[v]$	$K = 20$ and $m = 8$
(c) $F = \frac{mv^2}{r}$	$[v]$	$F = 9$, $r = 7$ and $m = 6$
(d) $F = \frac{Gm_1m_2}{r^2}$	$[r]$	$F = 16$, $G = 18$, $m_1 = 6$ and $m_2 = 5$
(e) $g_1 = \frac{r_2^2}{r_1^2}g_2$	$[r_2]$	$g_1 = 9.8$, $g_2 = 10$ and $r_1 = 20$
(f) $P = \frac{V^2}{R}$	$[V]$	$P = 9$ and $R = 7$
(g) $P = I^2R$	$[I]$	$P = 10$ and $R = 4.5$
(h) $E = \frac{1}{2}cv^2$	$[v]$	$E = 96$ and $c = 6$
(i) $A = \frac{\pi D^2}{4}$	$[D]$	$A = 9\pi$

WE34

2 **(i)** Rearrange the following, to make the variable in each bracket the subject of the formula.

(ii) Find the value of this variable when the given values are substituted into the rearranged formula.

(a) $A = \sqrt{A_x + A_y}$	$[A_y]$	$A = 9$ and $A_x = 43$
(b) $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$[c]$	$b = 9$, $a = 4$ and $x = -1$
(c) $l = \frac{3r}{\sqrt{T}}$	$[T]$	$r = 8$ and $l = 3$
(d) $x = \sqrt{y - z}$	$[z]$	$y = 4$ and $x = 10$
(e) $x = \sqrt{\frac{y}{z}}$	$[z]$	$y = 7$ and $x = 5$
(f) $x = \sqrt{\frac{a + b}{c}}$	$[b]$	$c = 3$, $x = 2$ and $a = -4$
(g) $x = \sqrt{y - z}$	$[y]$	$x = 4$ and $z = 6$
(h) $x = \sqrt{\frac{a + b}{c}}$	$[c]$	$a = 4$, $b = -1$ and $x = -3$
(i) $x = \sqrt{\frac{y}{z}}$	$[y]$	$x = 4$ and $z = 6$

3 Which of the following expressions is equivalent to $S = \frac{4r^2}{\sqrt{t}}$?

A $t = \frac{16r^2}{S^2}$ B $r = \pm\sqrt{\frac{St}{4}}$ C $t = \frac{16r^4}{S^2}$ D $r = \frac{S\sqrt{t}}{4}$

4 The formula for the area of a circle is $A = \pi r^2$.

(a) Rearrange this equation to make r the subject.

(b) Explain why there is no ± sign required.

(c) Find the radius r when the area A is 40 cm^2.

5 The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(a) Rearrange this equation to make r the subject.

(b) Explain why there is no ± sign required.

(c) Find the radius r when the volume V is 125 cm^3.

6 The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. By first rearranging this equation to change the subject, find:

(a) the height of a cone with volume 312 m^3 and radius 14 m

(b) the radius of a cone with volume 160 cm^3 and height 8 cm.

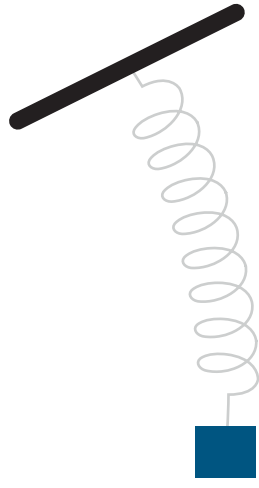
Reasoning, communicating and open-ended

7 The formula $v^2 = u^2 + 2as$ is regularly used in physics to find the final velocity v of an object that has initial velocity u, acceleration a and displacement s. (Displacement is the distance of the object from its starting point.)

(a) Calculate the initial velocity of a car that stops in 42 m with acceleration -20 m/s^2.

(b) What does a negative value for the acceleration mean in this situation?

8 The formula used to determine the frequency f of a spring, given the spring constant k and mass of the object m, is $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.



Frequency is measured in hertz (Hz) and is the number of cycles (up and down) per unit of time (that is, the frequency is the number of oscillations per second). A particular spring with an object of mass 3.5 kg has a frequency of 3 Hz.

(a) Calculate the spring constant for this spring.

(b) An object of mass 7.2 kg is attached to the same spring. Calculate the new frequency.

9 Elizabeth was changing the subject of the formula $g = 4h^2 - 9k$ to h and has completed the following working.

$$g = 4h^2 - 9k$$

$$g + 9k = 4h^2 \qquad \text{Line 1}$$

$$h^2 = \frac{g + 9k}{4} \qquad \text{Line 2}$$

$$h = \sqrt{\frac{g + 9k}{4}} \qquad \text{Line 3}$$

$$h = \frac{\sqrt{g + 9k}}{2} \qquad \text{Line 4}$$

(a) Identify the line with the mistake.

(b) Explain why this is mathematically inaccurate.

10 **(a)** Make up a formula of your own with at least three variables. One of the variables, that is not the subject of this formula, must be squared. Rearrange the formula to make this variable the subject.

(b) Make up a formula of your own with at least three variables. Take the square root of one of the variables that is not the subject of this formula. Rearrange the formula to make this variable the subject.

Problem solving

Gravity

The formula $v = \sqrt{2gx}$ is used to calculate the velocity, v, of an object that has already fallen x metres, where g is acceleration due to gravity. Gravity is different depending on the planet that you are on, as shown in the table.

Planet	Acceleration due to gravity (m/s^2)
Mercury	3.59
Venus	8.87
Earth	9.81
Mars	3.77
Jupiter	25.95
Saturn	11.08

1 On which planet would it take the longest amount of time for an object to fall 100 m?

2 According to this formula, does it take longer for a feather or a heavy boulder to fall?

3 How much higher do I have to drop a ball when on Mars, so that the velocity at which it hits the ground is the same as if it was dropped on Earth?

Strategy options

- Make a table.
- Work backwards.

Hint

The slowest velocity gives the longest time.

Chapter review 2

D.I.Y. Summary

Key Words

Stage 5.2	algebraic fractions	highest common factor (HCF)
	binomial product	monic quadratic
	factorise by grouping in pairs	quadratic expression
	factorising	quadratic trinomial
Stage 5.3§	completing the square	non-monic trinomial
	difference of two squares	perfect square

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.2 Q1–3 **Stage 5.3§ Q4–7**

1 $ax^2 + bx + c$ is called a _____. When the coefficient of x^2 is 1 the expression is a _____.

2 $(3a + b)(4 - 2a)$ is called a _____.

3 We _____ to write the expression $2ax + 2ay + 3x + 3y$ as a product of factors $(2a + 3)(x + y)$.

4 The first thing to do to factorise $50x^2 - 2y^2$ is to take out the _____. We can then use the _____ rule to factorise the expression.

5 $(3t - 2)^2$ is an example of a _____.

6 To factorise $x^2 - 4x - 6$ as $(x - 2 - \sqrt{10})(x - 2 + \sqrt{10})$ we use a method called _____.

7 $x^2 - 9$ is an example of the _____.

Stage 5.2 Q1, 4–6, 11, 12, 17, 18, 21 **Stage 5.3§ Q2, 3, 7–10, 13–16, 19, 20, 22–27**

Fluency and understanding

1 Expand and simplify the following. Ex.2.1

(a) $5(2x + 7)$ (b) $x(x - 3) + 2(x - 3)$

(c) $(x + 6)(x + 4)$ (d) $(4x - 1)(2x + 3)$

2 Expand and simplify the following. Ex.2.2

(a) $(x + 6)^2$ (b) $(2x - 3)^2$

(c) $(5 - x)(5 + x)$ (d) $(3x + 7)(3x - 7)$

3 Expand and simplify the following. Ex.2.2

(a) $4x(x - 7)(x + 2)$ (b) $-3x(x + 8)(x - 3)$

(c) $(x - 1)(x + 5)(x - 6)$ (d) $(2x + 3)(2 - x)(x + 4)$

4 Factorise these expressions. Ex.2.3

(a) $7x + 21$ (b) $12ab - 8a$ (c) $18x^2 + 24x$

5 Factorise these expressions. Ex.2.3

(a) $d(c - 4) + 6(c - 4)$ (b) $x(x + 2) - 3(x + 2)$

(c) $5a(a + 1) - 2(a + 1)$ (d) $2y(3y - 5) - 7(3y - 5)$

6 Factorise these expressions. **Ex. 2.4**

(a) $x^2 + 11x + 30$ **(b)** $x^2 - 10x - 24$

7 Factorise each of these quadratic trinomials. **Ex. 2.5**

(a) $x^2 + 10x + 25$ **(b)** $x^2 - 18x + 81$

(c) $9x^2 - 24x + 16$ **(d)** $4x^2 + 4x + 1$

8 Factorise these expressions. **Ex. 2.5**

(a) $x^2 - 9$ **(b)** $25 - y^2$ **(c)** $4x^2 - y^2$

(d) $49x^2 - 16$ **(e)** $36x^2 - 81y^2$ **(f)** $64 - 9x^2$

(g) $(2x - 5)^2 - 4$ **(h)** $(x - 5)^2 - (x + 8)^2$ **(i)** $(x + 2)^2 - (x - 3)^2$

(j) $x^2 - 3$ **(k)** $(x + 7)^2 - 5$ **(l)** $(x - 4)^2 - 7$

9 Factorise these expressions. **Ex. 2.6**

(a) $x^2 + 6x - 2$ **(b)** $x^2 - 10x + 23$

(c) $x^2 - 4x - 7$ **(d)** $x^2 + 8x + 13$

10 Factorise the following quadratic expressions. **Ex. 2.7**

(a) $12x^2 + x - 20$ **(b)** $12x^2 - 4x - 5$ **(c)** $6x^2 + 17x - 14$

(d) $3x^2 + 5x + 1$ **(e)** $4x^2 - 7x - 5$ **(f)** $2x^2 + 6x + 3$

11 Simplify the following. **Ex. 2.8**

(a) $\frac{18x}{45}$ **(b)** $\frac{2(a-3)}{6}$ **(c)** $\frac{10x(x-1)}{8x}$ **(d)** $\frac{12ab}{4b}$

12 Simplify the following. **Ex. 2.8**

(a) $\frac{2x}{3} + \frac{x}{2}$ **(b)** $\frac{6x}{7} - \frac{3y}{2}$ **(c)** $\frac{(x+4)(x-3)}{(x-3)(x+2)}$

13 Simplify the following. **Ex. 2.9**

(a) $\frac{x^2 + 3x + 2}{x + 1}$ **(b)** $\frac{(2x-1)(x+1)}{(x+7)(x-7)} \times \frac{(3x+7)(x+7)}{(x-1)(2x-1)}$

(c) $\frac{4x^2 - 4x + 1}{2x - 1}$ **(d)** $\frac{2x^2 - 8x}{x^2 + 3x} \div \frac{x^2 - 16}{x^2 + 8x + 15}$

14 Simplify the following. **Ex. 2.9**

(a) $\frac{x+1}{5} + \frac{x-2}{3}$ **(b)** $\frac{3x-2}{10} - \frac{x+5}{6}$ **(c)** $\frac{x-6}{5} - \frac{y-3}{10}$

15 Rearrange each of the following to make the pronumeral in brackets the subject. **Ex. 2.10**

(a) $A = lb$ $[b]$ **(b)** $A = 2\pi\sqrt{\frac{l}{g}}$ $[l]$

(c) $SA = 4\pi r^2$ $[r]$ **(d)** $a = x^2y - z$ $[x]$

16 If $9x^2 - 12x + 4$ is written as $(ax + b)^2$, state the values of a and b.

17 In January, John's weight was x kg. In April his weight was $(x + a)$ kg and in September it was $(x + b)$ kg.

(a) If the product of John's weight in April and September is $x^2 + x - 30$, determine the values of a and b.

(b) By how many kilograms had John's weight increased by April?

(c) What was John's overall weight gain/loss by the end of September?

(d) What happened to his weight from April to September?

18 The heights in centimetres of four people are ax, $2x$, ay and $2y$.

(a) Write down the expression for their total height.

(b) Factorise the expression in part **(a)**.

(c) If $x = 65$, $y = 75$ and $a = 3$, use your factorised expression to find the average height of the four people.

19 **(a)** Which of the following methods of factorising is the most appropriate for each of the expressions **(i)** to **(vi)** below?

A factorising using common factors

B factorising using the Difference of Two Squares rule

C factorising by grouping in pairs

D factorise by grouping (3 and 1)

E factorising quadratic trinomials (multiply to give … , add to give …)

F factorising by completing the square

(i) $x^2 + 5x - 24$

(ii) $25a^2 - b^2$

(iii) $x^2 + 6x + 9 - y^2$

(iv) $x^2 + 12x + 34$

(v) $16x^2 - 25x$

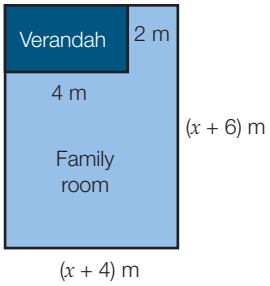
(vi) $4x - 4y + x^2 - y^2$

(b) Factorise each expression in part **(a)**.

20 Factorise the following quadratic expressions.

(a) $6x^2 + x - 35$ **(b)** $10x^2 - 17x + 3$ **(c)** $12x^2 + 19x + 4$

21 A house has had a rectangular extension added. The extension has a length of $(x + 6)$ m and a breadth of $(x + 4)$ m. An area 2 m wide and 4 m long is to be an untiled verandah. The remainder is to be a tiled family room. The cost of tiling a family room is \$18 per square metre.



(a) Write an expression for the area of the family room in factorised form.

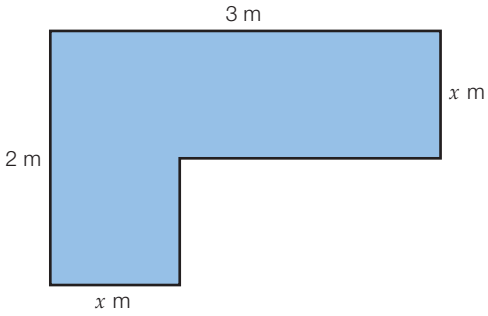
(b) Write an expression for the cost of tiling the family room.

(c) Find the cost of tiling the family room if x is 2 m.

22 Find, in expanded form, the square of the sum of $4x$ and $3y$.

Reasoning and communicating

23 **(a)** Write an expression (in m^2) for the area of the kitchen bench surface shown at right. Simplify the expression as far as possible.



(b) Factorise the expression for the area.

(c) Find the area of the kitchen bench surface when the breadth of the bench, x, is:

(i) 1 m

(ii) 0.5 m

(iii) 0.75 m

24 A biscuit box has a length 14 cm greater than its height and a breadth 9 cm greater than its height.

(a) Draw a diagram showing this information. Use h to represent height.

(b) Write an expression (in factorised form) to represent the volume of the biscuit box.

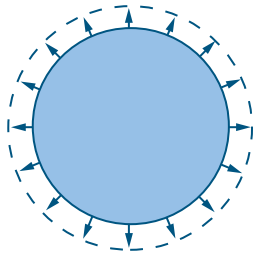
(c) Expand and simplify the expression for volume.

(d) Use your answer for part **(b)** to find the volume of the biscuit box if its height is 6 cm.

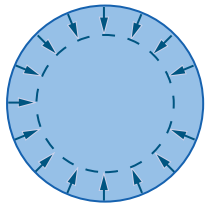
(e) Use your answer for part **(c)** to find the volume of the biscuit box if its height is 6 cm.

(f) Which form of the volume expression (factorised or expanded) did you find easier to work with? Explain your answer.

25 A circular disc of metal expands when it is heated. When it is heated to a particular temperature, its radius increases by 2 mm. When cooled to a particular temperature below room temperature, the disc contracts and its radius decreases by 2 mm. (Do not substitute for π in any parts of this question.)



Disc expands when heated



Disc contracts when cooled

(a) If the disc has a radius of x mm at room temperature (20°C), write down its area at room temperature.

(b) In terms of x, what is the radius of the heated disc?

(c) Hence, write an expression for the area of this larger disc.

(d) In terms of x, what is the radius of the cooled disc?

(e) Hence, write an expression for the area of this smaller disc.

(f) Write an expression for the difference between the area of the larger disc and that of the smaller disc.

(g) Is there a common factor that can be taken out of this expression for the difference in area? What is it?

(h) Factorise this expression for the difference in area.

(i) If the original disc (at 20°C) had a radius of 10 mm, find the difference in area between the larger and smaller discs.

(j) If the difference in area between the larger and smaller discs is 136π mm^2, find the radius of the disc at 20°C.

26 An exercise book is $(8x + 3)$ cm long and $(2y + 7)$ cm wide.

(a) Write an expression (in cm^2) for the area of the book.

(b) What is the expression for the area if $y = 4x - 5$?

(c) Expand and simplify the expression in part **(b)**.

(d) **(i)** In terms of x, what value of y will result in the expression for the area being a perfect square?

(ii) Expand and simplify the perfect square in **(d) (i)**.

27 The number of private cars observed during a period of x days is modelled by the expression $x^2 + p^2x$ where p is the number of passengers. The number of taxis observed with p passengers during y days is given by $p^2y - y^2$.

(a) Write the expression for the total number of cars and taxis.

(b) Factorise the expression in part **(a)**.

(c) Explain why x must be greater than y if the number of passengers is zero.

(d) **(i)** How many vehicles were there in total if $x = y$?

(ii) It is found that $2px$ of these vehicles were white. Write, in factorised form, the number of non-white vehicles.

Measurement 3

P Recall 3

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Use your calculator to evaluate the following.

(a) 25^2 **(b)** 3.1^3 **(c)** $\sqrt{11.9025}$

2 Given the formula $A = \frac{1}{2}(a + b)h$:

(a) find A if $h = 4$, $a = 3$ and $b = 5$

(b) find h if $A = 30$, $a = 7$ and $b = 5$

(c) find b if $A = 48$, $h = 6$ and $a = 9$.

3 Complete the following unit conversions.

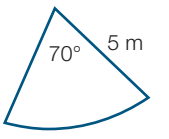
(a) 4.1 m = _____ cm **(b)** 4.25 km = _____ cm **(c)** 40 m^2 = _____ cm^2

(d) 578 cm^2 = _____ m^2 **(e)** 8.25 km^2 = _____ m^2 **(f)** 600 mL = _____ L

(g) 20 cm^3 = _____ mL **(h)** 30.8 m^3 = _____ L **(i)** 7250 cm^3 = _____ m^3

4 Find each of the following values in the sector shown, rounded to two decimal places.

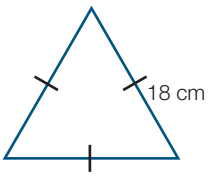
(a) the arc length **(b)** the perimeter **(c)** the area



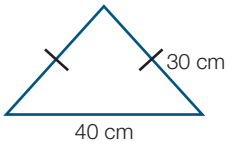
5 **(i)** Use Pythagoras' Theorem to find the perpendicular height of each of the following shapes, rounded to two decimal places.

(ii) Use the height to calculate the area of each shape.

(a)



(b)



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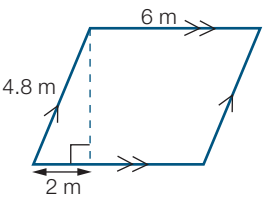
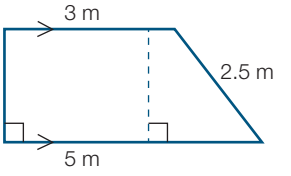


(d)



Syllabus references

Number and Algebra: Equations

✓ Substitute values into formulas to determine an unknown **(ACMNA234) (Stage 5.2)**

Measurement and Geometry: Area and Surface Area

✓ Calculate the areas of composite shapes **(ACMMG216) (Stage 5.1)**

✓ Calculate the surface areas of cylinders and solve related problems **(ACMMG217) (Stage 5.2)**

✓ Solve problems involving surface area for a range of prisms, cylinders and composite solids **(ACMMG242) (Stage 5.2)**

✓ Solve problems involving the surface areas of right pyramids, right cones, spheres and related composite solids **(ACMMG271) (Stage 5.3)**

Measurement and Geometry: Volume

✓ Solve problems involving the volumes of right prisms **(ACMMG218) (Stage 5.2)**

✓ Solve problems involving volume for a range of prisms, cylinders and composite solids **(ACMMG242) (Stage 5.2)**

✓ Solve problems involving the volumes of right pyramids, right cones, spheres and related composite solids **(ACMMG271) (Stage 5.3)**

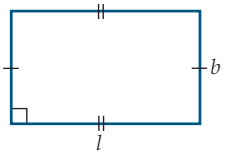
3.1 Area

Stage 5.1

Need to Know

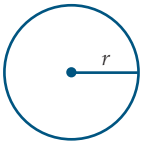
Area formulas for common shapes

Rectangle



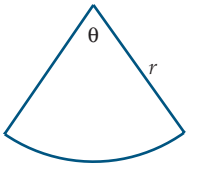
$A = lb$

Circle



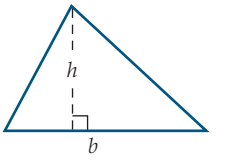
$A = \pi r^2$

Sector



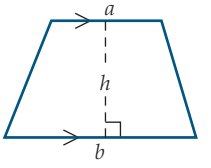
$A = \frac{\theta}{360} \times \pi r^2$

Triangle



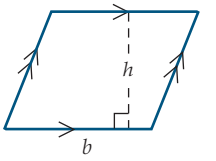
$A = \frac{1}{2}bh$

Trapezium



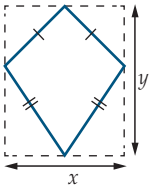
$A = \frac{1}{2}(a + b)h$

Parallelogram



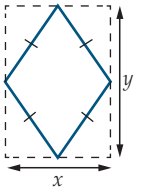
$A = bh$

Kite



$A = \frac{1}{2}xy$

Rhombus



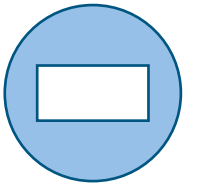
$A = \frac{1}{2}xy$

Areas of composite shapes

Finding the area of a **composite shape** involves adding or subtracting areas of simpler shapes.

For example:

Add the areas of the rectangle and the triangle to find the enclosed area



Subtract the area of the rectangle from the area of the circle to find the shaded area

Rounding and accuracy

A calculation should always have been simplified as much as possible before a calculator is used. (Simplifying involves cancelling common factors and/or performing simple whole number multiplications.) Rounding should only be done once, at the very end. This ensures that the most accurate final answer is obtained. Using rounded answers while working through a problem means errors are accumulated, or built up, each time, which results in an inaccurate final answer. To use a calculator efficiently for these calculations, the correct order of operations needs to be applied, making use of the bracket keys and the answer key to recall the previous answer.

Exact answers

Exact answers mean that answers are not approximated and rounded. Calculations involving π and surds such as $\sqrt{2}$ must be left in exact form, unless otherwise required.

Area of an annulus

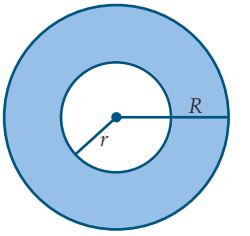
An **annulus** is a ring shape created by two concentric circles (circles that have the same centre).

To find the area, we subtract the area of the smaller circle from the area of the larger one.

$A = \pi R^2 - \pi r^2$

$A = \pi(R^2 - r^2)$ (factorised form)

$= \pi(R + r)(R - r)$ (factorising using the Difference of Two Squares rule—a useful form to work with)

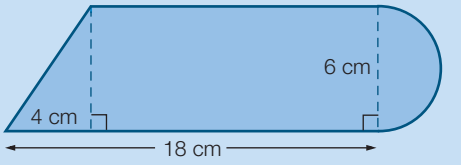


Use capital and lower case versions of a letter to indicate the same dimension in two figures that have the same shape. Here, R is used to represent the radius of the larger circle and r for the radius of the smaller circle.

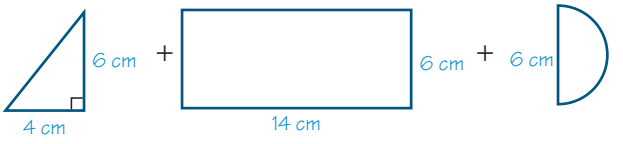
Worked Example 1

WE1

Find the area of this shape rounded to two decimal places.



Working



$l = 18 - 4$
$= 14$ cm

$r = 6 \div 2$
$= 3$ cm

$A = \frac{1}{2}bh + lb + \frac{\pi r^2}{2}$

$A = \frac{1}{\cancel{2}_1} \times \cancel{4}^2 \times 6 + 14 \times 6 + \frac{\pi \times 3^2}{2}$

$= 12 + 84 + \frac{9\pi}{2}$

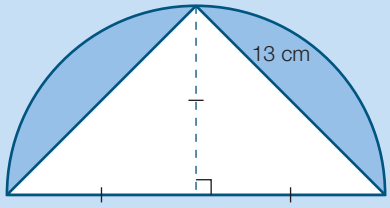
$= 96 + \frac{9\pi}{2}$

$= 110.14$ cm^2 (2 d.p.)

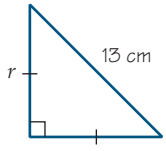
Worked Example 2

WE 2

Find the shaded area in the diagram, rounding your answer to two decimal places. (You may need to use Pythagoras' Theorem to calculate the lengths of some unknown sides.)



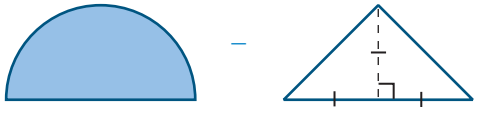
Working



$$r^2 + r^2 = 13^2$$
$$2r^2 = 13^2$$
$$r^2 = \frac{13^2}{2}$$
$$= 84.5 \text{ cm}$$

(note: $r = \sqrt{84.5} = 9.192388...$

Calculations below require the value of r^2 so use $r^2 = 84.5$)



$$A = \frac{1}{2}\pi r^2 - \frac{1}{2}bh$$

$b = 2h$ and $h = r$ so $b = 2r$

$$A = \frac{1}{2}\pi r^2 - \frac{1}{\cancel{2}_1} \times \cancel{2}^1 r \times r$$
$$= \frac{1}{2}\pi r^2 - r^2$$
$$= \frac{84.5\pi}{2} - 84.5$$
$$= 84.5\left(\frac{\pi}{2} - 1\right)$$
$$= 48.23 \text{ cm}^2 \text{ (2 d.p.)}$$

3.1 Area

Answers page 621

Stage 5.1

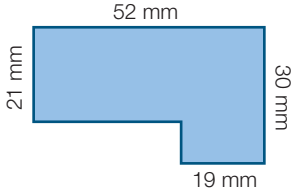
Navigator

Q1 (a)–(d), Q2 Columns 1 & 2, Q3, Q5 (a), (b), (g), Q8, Q9, Q11, Q14	Q1, Q2 Columns 2 & 3, Q3, Q4, Q5 Column 1, Q6, Q7, Q8, Q9, Q11, Q13, Q14	Q1 (c)–(f), Q2 Columns 2 & 3, Q4, Q5 Column 2, Q6, Q7, Q9, Q10, Q12, Q13, Q14, Q15

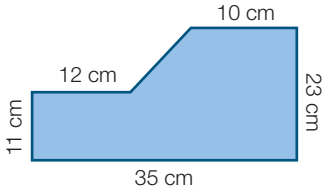
Fluency and understanding

1 Find the area of each of the following shapes. Round your answer to two decimal places where necessary. WE1

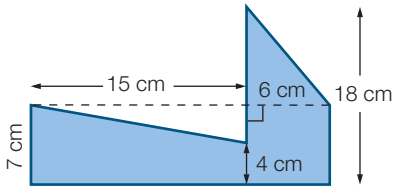
(a)



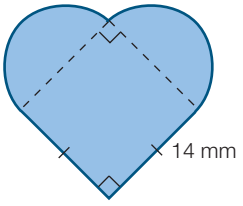
(b)



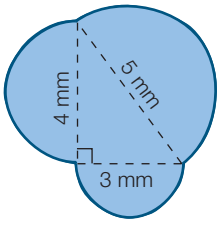
(c)



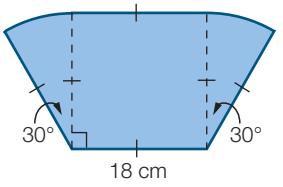
(d)



(e)

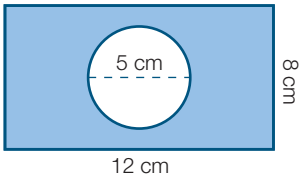


(f)

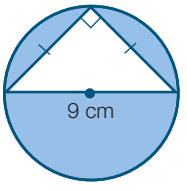


2 Find the shaded area in each of the following, rounding your answer to two decimal places. (You may need to use Pythagoras' Theorem to calculate the lengths of some unknown sides.) WE2

(a)



(b)



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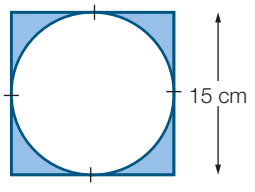

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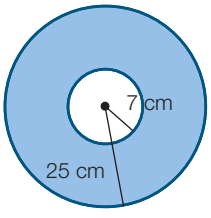

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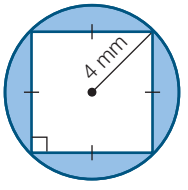

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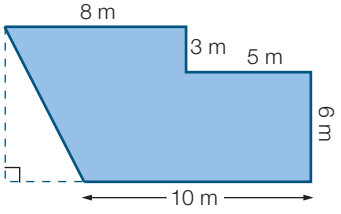
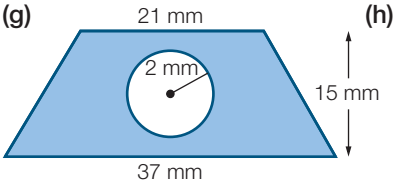
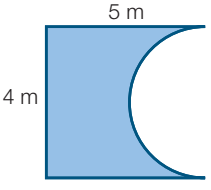


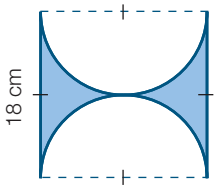
(g)



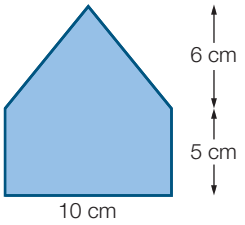
(h)



(i)



3 The area of this composite shape in cm^2 can be found by:



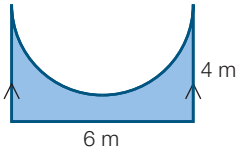
A $(10 \times 5) + (10 \times 6)$

B $(10 \times 11) + (10 \times 6)$

C $(10 \times 5) + (\frac{1}{2} \times 10 \times 6)$

D $(\frac{1}{2} \times 10 \times 5) + (\frac{1}{2} \times 10 \times 6)$

4 The area of this composite shape in m^2 can be found by:



A $(6 \times 4) + (\frac{1}{2} \times \pi \times 6^2)$

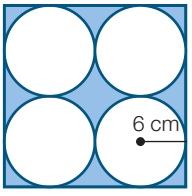
B $(6 \times 4) - (\frac{1}{2} \times \pi \times 6^2)$

C $(6 \times 4) - (\frac{1}{2} \times \pi \times 3^2)$

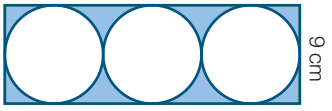
D $(6 \times 4) - (\frac{1}{2} \times \pi \times 32)$

5 Find the shaded area of each of the following, rounded to two decimal places.

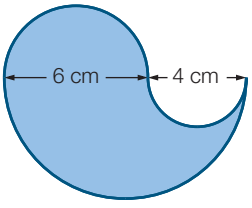
(a)



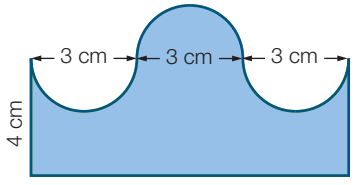
(b)



(c)



(d)



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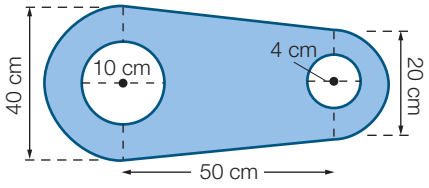
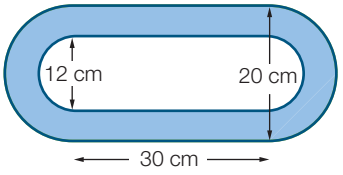
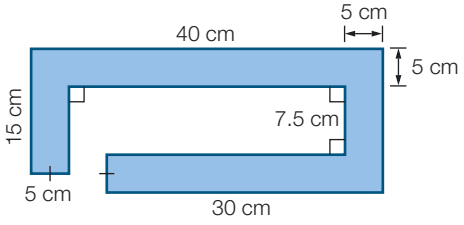


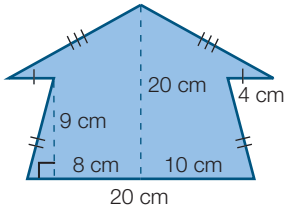
(f)



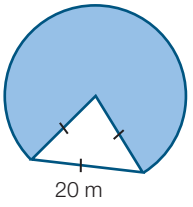
(g)



(h)



6 Find the area of both the sector and the triangle. (You will need to use Pythagoras' Theorem to calculate the height of the triangle.) Write both answers correct to the nearest m^2.



7 The Department of Roads has a machine that lays bitumen road at a rate of 20 metres in length every 15 minutes. The bitumen is 6 m wide and the machine works 7 hours per day. The department is building a road 12.5 km long.

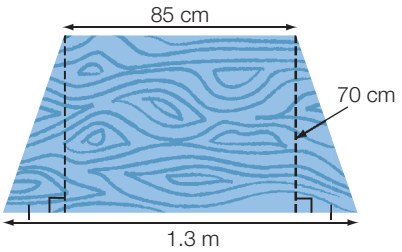
(a) How long will it take to complete the road? State your answer in both hours and days, correct to two decimal places.

(b) What will be the total area of bitumen laid by the machine?

Hint

Drawing a diagram can help you visualise the problem.

8 The tables in Joshua's classroom are trapezoidal in shape, as shown at right. The surface of each table needs to be laminated. If there are 13 tables in the classroom, what area (m^2) of laminate is required? Round your answer to two decimal places.

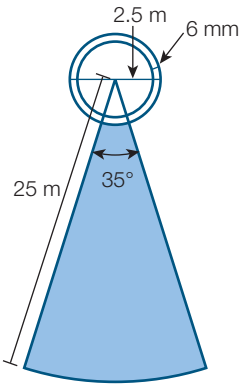


9 Elspeth has a circular swimming pool that is 7 m in diameter. She is installing a path 1 m wide all around the pool. What will be the area of this path, rounded to two decimal places?

10 The throwing area for the Olympic shot put event is shown in the diagram. The shot put is thrown from a circle that is 2.5 m in diameter to the outside of the rim. The rim is 6 mm wide (athletes cannot touch the rim when throwing). To be 'legal' a throw must land in the landing zone shown shaded in blue. Calculate, correct to two decimal places:

(a) the area of the rim of the circle

(b) the area of the landing zone.



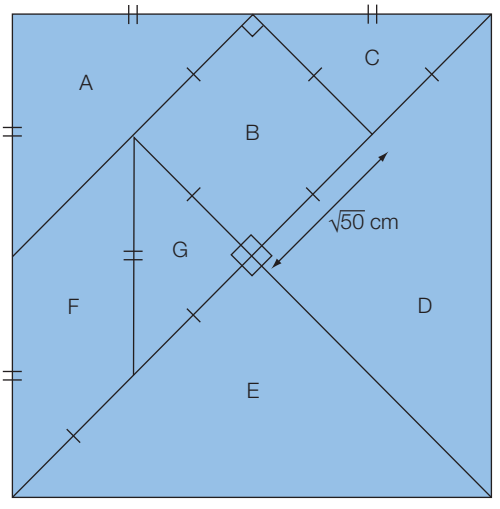
Reasoning, communicating and open-ended

11 Only two rectangles with integer dimensions (side lengths are whole numbers) have the same numerical value for their area and perimeter. Find both rectangles.

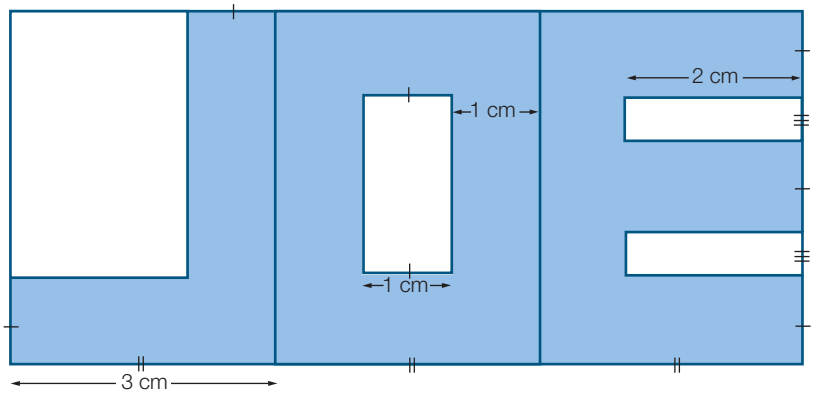
12 What is the radius of a circle with an area equal to that of a square 8 m wide? Round your answer to two decimal places.

13 Find the area of each piece of the tangram puzzle shown at right.

(Remember $\sqrt{50} \times \sqrt{50} = 50$.)



14 **(a)** To label his school folder, Joe cuts out the letters of his name from a 9 cm by 4 cm piece of coloured paper as shown below. What area of coloured paper do the letters cover?



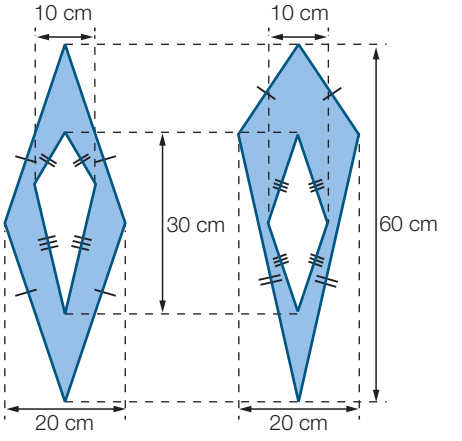
(b) Use the 9 cm by 4 cm rectangle in part **(a)** for another set of three letters. Calculate the area of paper covered by these letters. Is the area covered greater than, less than, or about the same as that covered in part **(a)**?

15 Refer to the diagram on the right.

(a) Without doing any calculations explain why the shaded area in both shapes is equal.

(b) Draw another pair of shapes with a similar relationship to the pair shown.

(c) Draw as many such pairs of shapes as you can.



Surface area of prisms and cylinders

3.2

Need to Know

Stage 5.2

Surface area of prisms

A **prism** is a solid with straight edges and a **uniform cross-section**. This means that the shape on one end (called the base) continues all the way through to the other end.

The **surface area** (SA) of a prism is the total area of all its external surfaces.

To calculate the surface area of a prism, it is often useful to draw a **net**—a 2D (flat) diagram that clearly shows each of the faces of the prism.

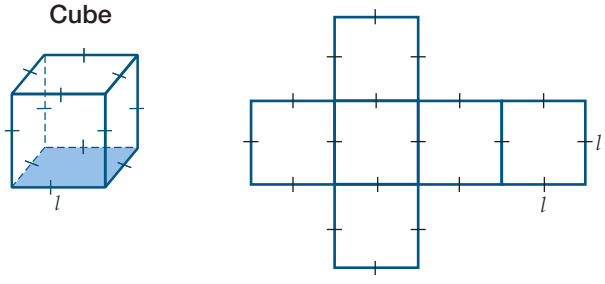
Surface area of cylinders

A **cylinder** also has a uniform cross-section, but it is not a prism as it does not have straight edges. The surface area of cylinders can also be found by considering their nets.

The surface area of a cylinder is the sum of the areas of the two circular faces at each end and the area of the curved surface. The area of each circular face is $A = \pi r^2$. When flattened, the curved surface is a rectangle with its length equal to the circumference of the circles on the ends: $C = 2\pi r$. Its width is equal to the height of the cylinder, h. The area of the curved surface is $A = 2\pi r \times h$.

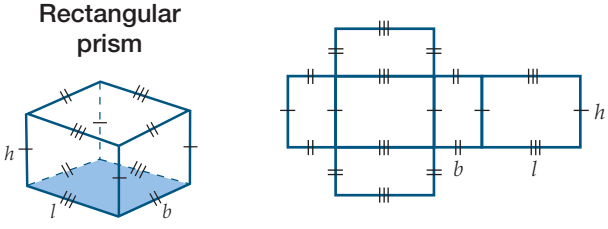
Formulas for the surface area of prisms and cylinders

Cube



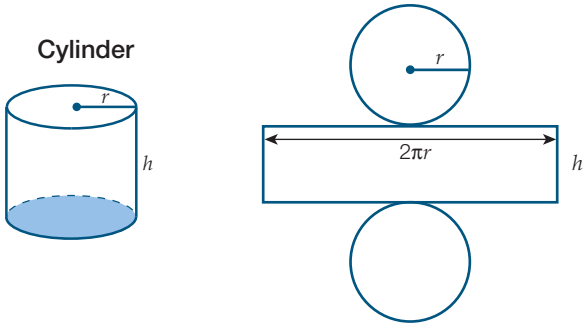
SA = area of 6 squares
$= 6l^2$

Rectangular prism



SA = area of 6 rectangles
$= 2lb + 2bh + 2lh$
$= 2(lb + bh + lh)$
(factorised form)

Cylinder

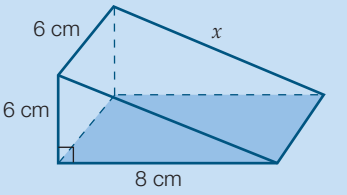


SA = area of 2 circles and area of curved surface
$= 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$
(factorised form)

Worked Example 3

WE3

Find the surface area of the triangular prism.



Working

$x^2 = 6^2 + 8^2$

$= 100$

$x = 10$ cm

$SA = A_1 + A_2 + 2 \times A_3 + A_4$

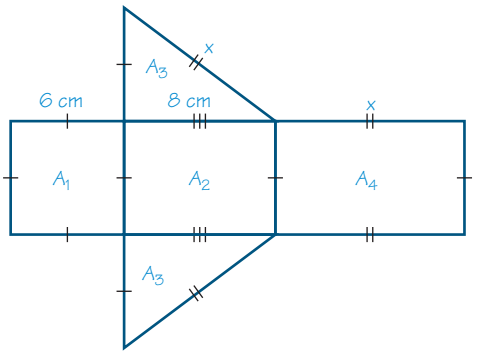
$= l^2 + lb + \cancel{2}^{1} \times \frac{bh}{\cancel{2}_{1}} + LB$

$= l^2 + lb + bh + LB$

$= 6^2 + 8 \times 6 + 8 \times 6 + 10 \times 6$

$= 36 + 48 + 48 + 60$

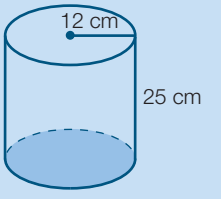
$= 192$ cm^2



Worked Example 4

WE4

Find the surface area of this cylinder, rounded to two decimal places.



Working

$r = 12$ cm

$h = 25$ cm

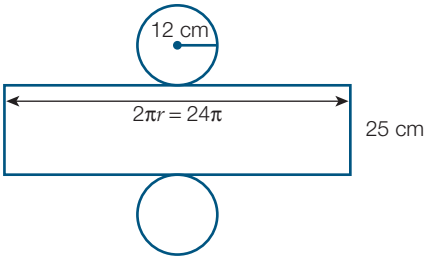
$SA = 2\pi r^2 + 2\pi rh$

$= 2 \times \pi \times 12^2 + 2 \times \pi \times 12 \times 25$

$= 288\pi + 600\pi$

$= 888\pi$

$= 2789.73$ cm^2 (2 d.p.)



3.2 Surface area of prisms and cylinders

Answers page 621

Navigator

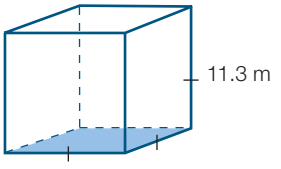
Stage 5.2

Q1, Q2, Q4, Q5, Q7	Q1 Column 2, Q2, Q3, Q4, Q5, Q6, Q7	Q1 Column 2, Q2, Q3, Q5, Q6, Q7, Q8

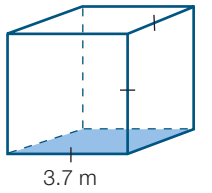
Fluency and understanding

1 Find the surface area of each solid, rounded to two decimal places where necessary. WE 3

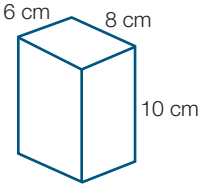
(a)



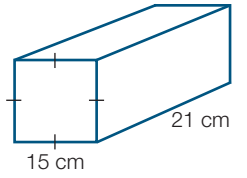
(b)



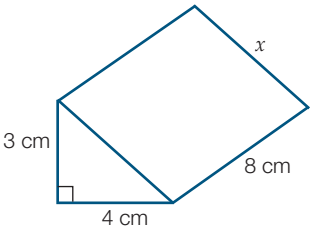
(c)



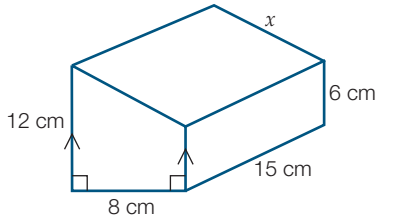
(d)



(e)

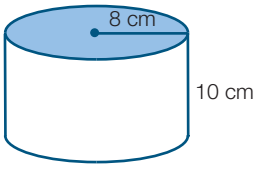


(f)

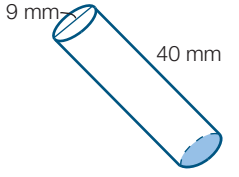


2 Find the surface area of each cylinder, rounded to two decimal places. WE 4

(a)

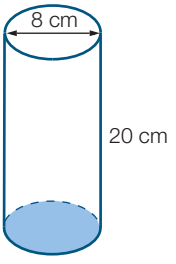


(b)



3 The calculation that gives the surface area of each of the following solids is:

(a)



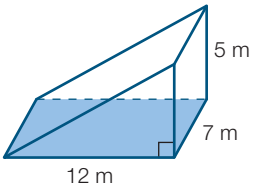
A $2 \times \pi \times 8(8 + 20)$

B $2 \times \pi \times 4(8 + 20)$

C $2 \times \pi \times 8(4 + 20)$

D $2 \times \pi \times 4(4 + 20)$

(b)



A $5 \times 12 + 7(5 + 12 + 17)$

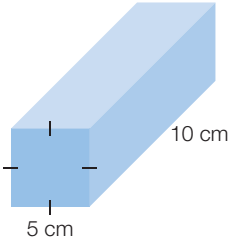
B $\frac{5 \times 12}{2} + 7(5 + 13)$

C $\frac{5 \times 12}{2} + 7(5 + 12 + 13)$

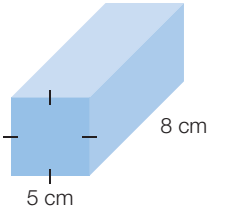
D $5 \times 12 + 7(5 + 12 + 13)$

4 Amanda has made a set of wooden building blocks as a present for her little brother. She built ten of each type of block and now she has to paint them. What is the total area that she has to paint?

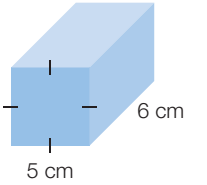
Block A



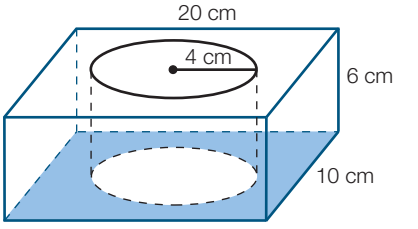
Block B



Block C

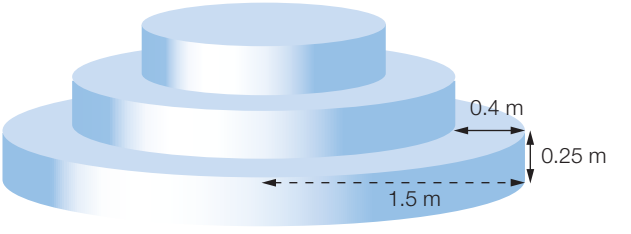


5 The diagram shows a plastic casing that has been specially made to surround a water pipe. Find the surface area of the casing, rounded to two decimal places.



Reasoning, communicating and open-ended

6 A circular podium has three steps as shown. The base of the podium has a radius of 1.5 m and each step has a width of 0.4 m. Each step is 0.25 m higher than the previous one. Calculate, correct to two decimal places, the amount of carpet needed to cover the visible surfaces of the podium.



7 **(a)** Write the formula for the surface area of a rectangular prism.

(b) By how many times does the surface area of the rectangular prism increase if all three of its dimensions are doubled? Demonstrate your answer with two suitable prisms, one of which has its length, breadth and height double that of the other one.

8 **(a)** Choose two variables to represent the radius and height of a cylinder. Find the surface area of this cylinder. Leave your answer in terms of π.

(b) Double the height of the cylinder and find the new surface area, leaving your answer in terms of π.

(c) Double the radius of the original cylinder and find the new surface area, leaving your answer in terms of π.

(d) Explain why your answers to part **(b)** and part **(c)** are not the same.

Volume of prisms and cylinders

3.3

Need to Know

Stage 5.2

Prisms and cylinders

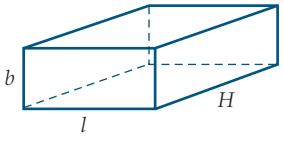
Prisms and cylinders have a uniform cross-section, so the volume of these solids is found by multiplying the area of the cross-sectional shape on the base (A) by the height of the solid (H), where the height is perpendicular to the base. The base is whichever face has the uniform cross-section, so it is not necessarily the face that the prism is sitting on.

The volume of solids with constant cross-section is:

$$V = AH$$

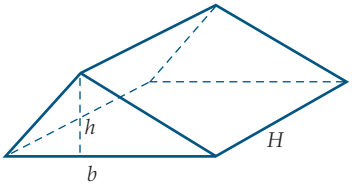
where A = the area of the cross-section
H = the perpendicular height of the solid.

Rectangular prism



$V = AH$
$A = lb$
$V = lbH$

Triangular prism

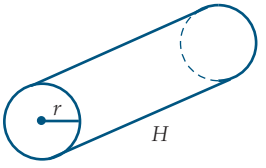


$V = AH$

$A = \frac{1}{2}bh$

$V = \frac{1}{2}bhH$

Cylinder



$V = AH$
$A = \pi r^2$
$V = \pi r^2 H$

In the formula for the triangular prism above we used the lower case h for the height of the triangular-shaped base, and the upper case H for the height of the prism.

Capacity

The **capacity** of a container refers to the volume of a substance (usually liquid or gas) that the container can hold.

1000 millilitres (mL) = 1 litre (L)

1000 litres = 1 kilolitre (kL)

1 cm^3 = 1 mL

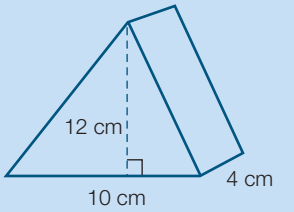
1000 cm^3 = 1 L

1 m^3 = 1000 L = 1 kL

Worked Example 5

WE 5

Find the volume of the triangular prism.



Working

$V = AH$

$= \frac{1}{2}bhH$

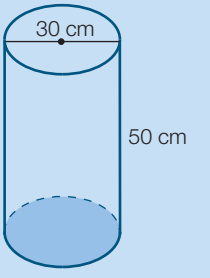
$= \frac{1}{2} \times 10 \times 12 \times 4$

$= 240\ cm^3$

Worked Example 6

WE 6

What is the capacity of this cylindrical tank? Answer in litres, rounded to two decimal places.



Working

$r = \frac{30}{2}$

$= 15\ cm$

$V = A \times H$

$= \pi r^2 H$

$= \pi \times 15^2 \times 50$

$= 11\,250\pi$

$= 35\,342.92\ cm^3\ (1000\ cm^3 = 1\ L)$

Capacity = 35.34 L (2 d.p.)

3.3 Volume of prisms and cylinders

Answers page 621

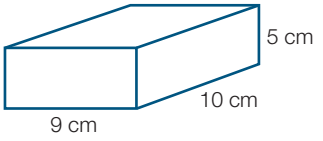
Navigator

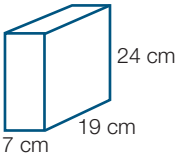
Stage 5.2

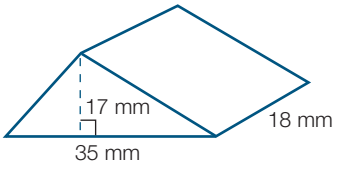
Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

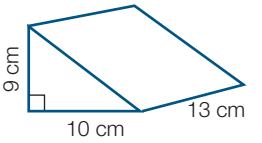
Fluency and understanding

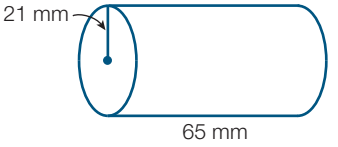
1 Find the volume of each solid with constant cross-section. Round to two decimal places, if necessary. WE 5

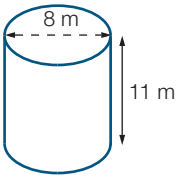
(a)


(b)


(c)


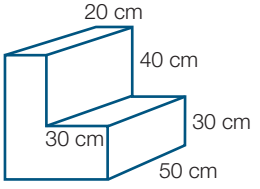
(d)


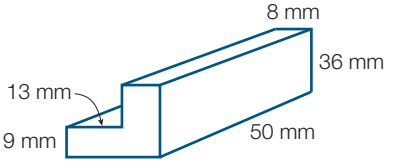
(e)


(f)


2 Find the capacity of the soft drink can in mL, rounded to two decimal places. Assume that the can is a regular cylinder. WE 6

3 Find the volume of each solid with constant cross-section. Round to two decimal places, if necessary.

(a)


(b)


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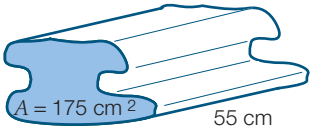

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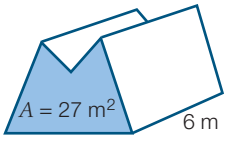
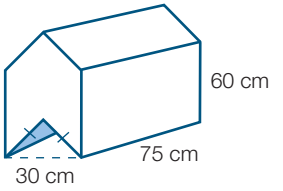
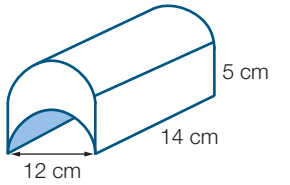


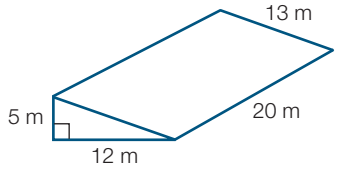
(e)



(f)



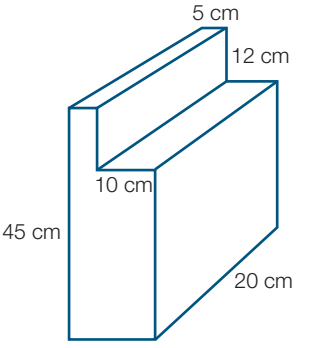
4 **(a)** The capacity of this prism in kL can be found by:



A $12 \times 5 \times 20$ **B** $\frac{1}{2} \times 5 \times 12 \times 20$

C $(\frac{1}{2} \times 5 \times 12) + (20 \times 12)$ **D** $\frac{1}{2} \times 12 \times 20 \times 5 \times 13$

(b) The capacity of this prism in mL can be found by:



A $5 \times 12 \times 20 + 10 \times (45 - 12) \times 20$

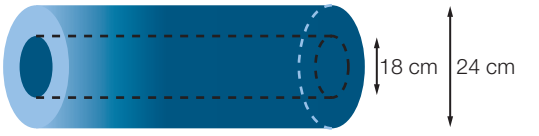
B $5 \times 12 \times 20 + 10 \times 45 \times 20$

C $5 \times 45 \times 20 + 10 \times 12 \times 20$

D $(5 + 10) \times 45 \times 20 - 10 \times 12 \times 20$

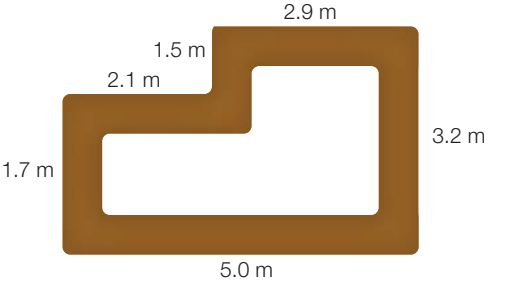
5 Ingrid needs 25 concrete columns for her new verandah. Each column will be cylindrical in shape with a base diameter of 28 cm and a height of 85 cm. What is the total volume of concrete required? Give your answer in cubic metres, correct to two decimal places. ($1\ m^3 = 1\,000\,000\ cm^3$)

6 A 3.5 m-long plastic pipe is to be used for drainage. If it has an outer diameter of 24 cm and an inner diameter of 18 cm, find the volume of plastic needed to construct it. Express your answer in m^3 correct to two decimal places.

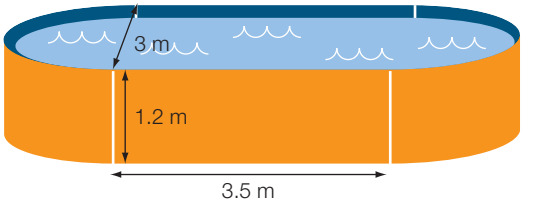


Reasoning, communicating and open-ended

7 Gael is building an extension onto her house. She has just finished digging a series of trenches. All the trenches are 85 cm deep and 55 cm wide. The plan for the trenches is shown. What was the volume of dirt removed? Round your answer to two decimal places.



8 Terry's swimming pool is rectangular with two semicircular ends. How many litres of water does it hold when full? State your answer to the nearest whole number.



9 A rectangular box has a volume of 200 cm^3.

(a) Find three possible whole number sets of dimensions that give the correct volume.

(b) Find three sets of dimensions with at least one non-whole number that give the correct volume.

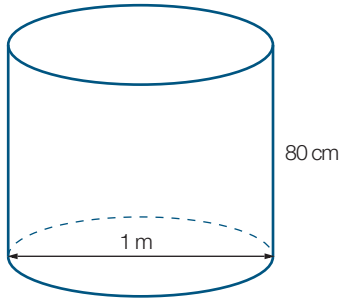
(c) Calculate the surface area of the box for each of the three sets of dimensions found in **(a)**.

10 Anita is attempting to find the volume of the cylinder in cm^3, correct to two decimal places.

(a) Identify the two mistakes in the working shown.

(b) Find the correct answer.

(c) Give Anita two tips to write in her notes so she can avoid similar mistakes in the future.



$$\begin{aligned} V &= AH \\ &= \pi r^2 H \\ &= \pi \times 1^2 \times 80 \\ &\approx 251.33 \text{ cm}^3 \end{aligned}$$

Problem solving

Cola cost crisis

Fizzy Cola comes in cylindrical cans of diameter 5 cm and height 15 cm. The cost of one can is \$2.35. The manufacturers have created a promotional, 'super-sized' can, which is the same height but twice as wide. The sales manager of Fizzy Cola is wondering at what price she should sell the promotional can. Her assistant thinks that the new can is 'twice as big' as the original can, so it must hold 'twice as much' cola and should cost 'twice the price'.

1 Suggest a price for the 'super-sized' can based on a comparision of its volume with the original can.

2 Is the assistant's reasoning correct? Why or why not?

Strategy options

- Break problem into manageable parts.
- Draw a diagram.
- Have I seen a similar problem?

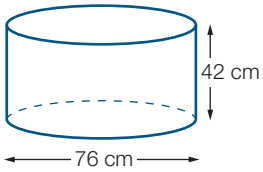
Half-time 3

Stage 5.1 Q3 **Stage 5.2 Q1–2, 4–6**

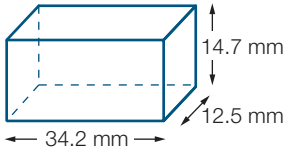
Ex.3.2

1 Find the total surface area of the following solids. Round your answers to the nearest whole number.

(a)

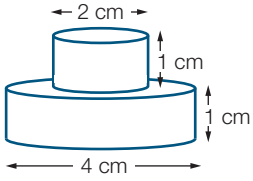


(b)



Ex.3.2, 3.3

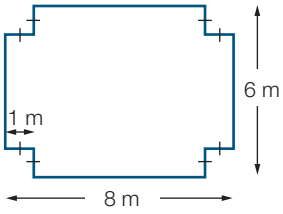
2 Find the exact surface area and exact volume of this composite shape. ('Exact' means writing your answer in terms of π.)



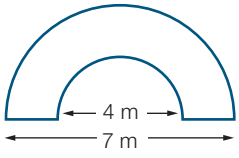
Ex.3.1

3 Find the area of the following shapes, rounded to two decimal places, if necessary.

(a)



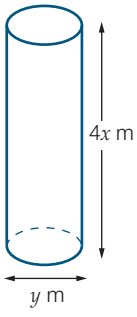
(b)



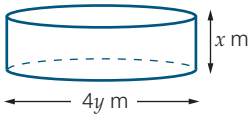
Ex.3.3

4 Which of the following solids, **A**, **B** or **C**, has the greatest volume? Justify your answer by evaluating the exact volume of each one.

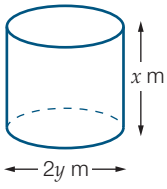
A



B



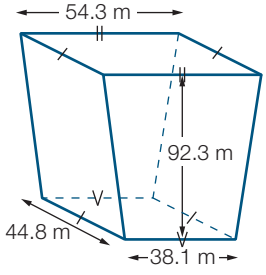
C



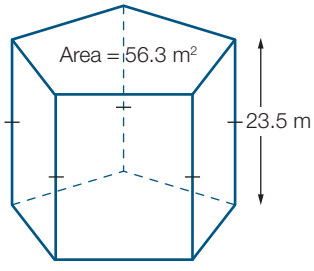
Ex.3.3

5 Find the volume of the following solids, rounded to two decimal places.

(a)



(b)



Ex.3.3

6 A cylindrical tank at a milk processing plant has a diameter of 2.6 m and a height of 3.4 m. How many 250 mL cartons could be completely filled from one tank full of milk?

Surface area of tapered solids and spheres

3.4

Stage 5.3

Need to Know

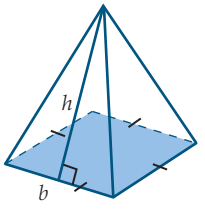
Tapered solids

Tapered solids do not have a uniform cross-section. The cross-section gets smaller as you move from the base of the solid to the top. The tapered solids that will be considered here are **pyramids** and **cones**.

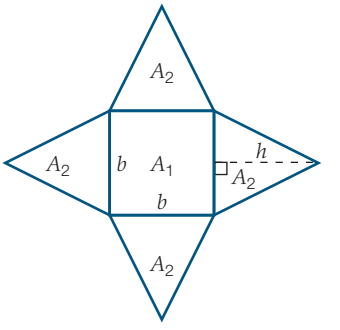
Pyramids

A pyramid is made up of a base and triangular sides that meet at a point, called the vertex. The number of triangles depends on the number of sides of the base shape. A pyramid with a square base has four triangles, whereas a pyramid with a pentagonal base has five triangles, and so on. The net of a square-based pyramid is shown here. The surface area is found by adding the individual areas of the base and sides.

For a square-based pyramid:

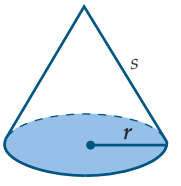


$$\begin{aligned} SA &= A_1 + 4 \times A_2 \\ &= b^2 + 4 \times \frac{bh}{2} \\ &= b^2 + 2bh \end{aligned}$$



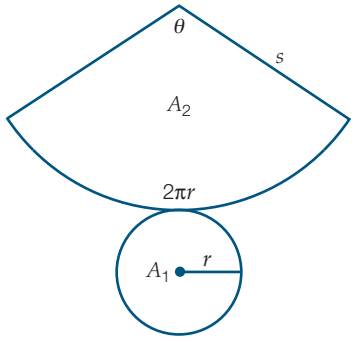
Cones

The surface area of a cone is the sum of the area of the circular base of radius, r, and the area of the sloping curved surface. The distance along this sloping surface from the base to the vertex is called the slant height, s.



When flattened, the curved surface is a sector of a circle that wraps around the base. This sector has a radius of s and an arc length of $2\pi r$ (i.e. the circumference of the circular base).

To find the area of the curved surface, first write the arc length in terms of r and s.



$$\text{Arc length} = \frac{\theta}{360} \times 2\pi \times s$$

$$2\pi r = \frac{\theta}{360} \times 2\pi \times s$$

$$r = \frac{\theta}{360} \times s$$

$$\frac{r}{s} = \frac{\theta}{360}$$

Use this expression in the formula for the area of the sector:

$$\text{Area of sector} = \frac{\theta}{360} \times \pi s^2 \quad (s = \text{radius of sector})$$

$$= \frac{r}{s} \times \pi \times s^2$$

$$A_2 = \pi rs$$

The area of the curved surface $= \pi rs$.

So SA of a cone = area of circle + area of curved surface

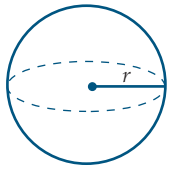
$$= A_1 + A_2$$

$$= \pi r^2 + \pi rs$$

$$= \pi r(r + s) \text{ (factorised form)}$$

Spheres

A **sphere** has only one single curved surface. The surface of a sphere is equivalent to four times the area of a circle of the same radius.

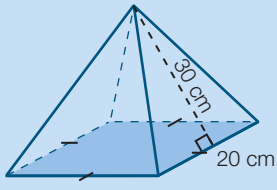


SA of a sphere $= 4\pi r^2$

Worked Example 7

WE7

Find the surface area of this square-based pyramid.



Working

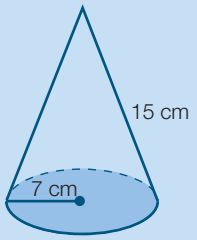
$b = 20$ cm $\quad h = 30$ cm

$$SA = b^2 + 2bh$$
$$= 20^2 + 2 \times 20 \times 30$$
$$= 400 + 1200$$
$$= 1600 \text{ cm}^2$$

Worked Example 8

WE 8

Find the surface area of this cone, rounded to two decimal places.



Working

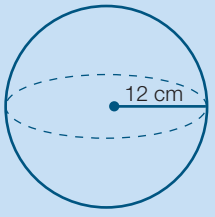
$r = 7$ cm $\quad s = 15$ cm

$$\begin{aligned} SA &= \pi r^2 + \pi rs \\ &= \pi \times 7^2 + \pi \times 7 \times 15 \\ &= 49\pi + 105\pi \\ &= 154\pi \\ &= 483.81 \text{ cm}^2 \text{ (2 d.p.)} \end{aligned}$$

Worked Example 9

WE 9

Find the surface area of this sphere, rounded to two decimal places.



Working

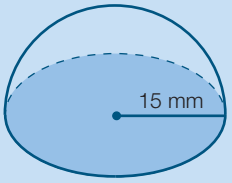
$r = 12$

$$\begin{aligned} SA &= 4\pi r^2 \\ &= 4 \times \pi \times 12^2 \\ &= 576\pi \\ &= 1809.56 \text{ cm}^2 \text{ (2 d.p.)} \end{aligned}$$

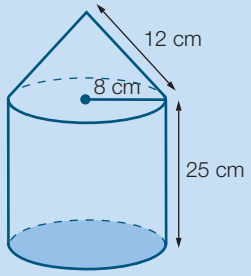
Worked Example 10 WE10

Find the surface area of each of the solids, correct to two decimal places.

(a)



(b)



Working

(a) $r = 15$

$SA = A_{hemisphere} + A_{circle}$

$= \frac{1}{\cancel{2}_1} \times \cancel{4}^2\pi r^2 + \pi r^2$

$= 2\pi r^2 + \pi r^2$

$= 3\pi r^2$

$= 3 \times \pi \times 15^2$

$= 675\pi$

$= 2120.58 \text{ mm}^2$ (2 d.p.)

(b) $r = 8 \quad s = 12 \quad h = 25$

$SA = A_{cone\ (curved)} + A_{cylinder\ (curved)} + A_{circle}$

$= \pi rs + 2\pi rh + \pi r^2$

$= \pi \times 8 \times 12 + 2 \times \pi \times 8 \times 25 + \pi \times 8^2$

$= 96\pi + 400\pi + 64\pi$

$= 560\pi$

$= 1759.29 \text{ cm}^2$ (2 d.p.)

Answers page 621

3.4 Surface area of tapered solids and spheres

Stage 5.3

Navigator

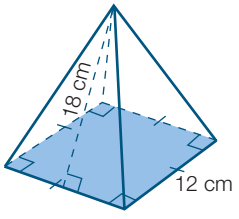
Q1, Q2, Q3, Q4 Column 1, Q6, Q7, Q9, Q11, Q12	Q1, Q2, Q3, Q4 Column 2, Q5, Q6, Q7, Q8, Q10, Q11, Q12	Q1 (b), Q2 (b), Q3 (b), Q4, Q5, Q7, Q8, Q9, Q10, Q12, Q13, Q14

Fluency and understanding

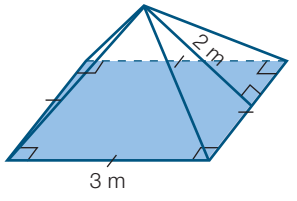
WE7

1 Find the surface area of each pyramid, rounded to the nearest whole number.

(a)



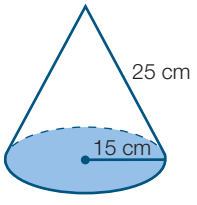
(b)



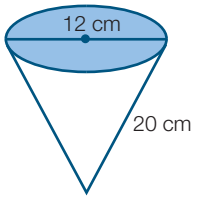
WE8

2 Find the surface area of each cone, rounded to two decimal places.

(a)

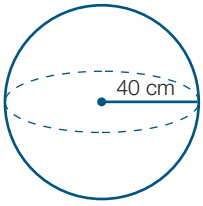


(b)

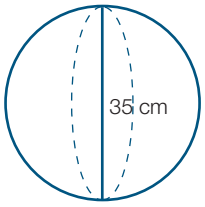


3 Find the surface area of each sphere, rounded to two decimal places. WE9

(a)

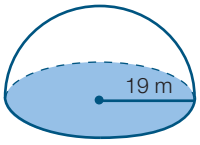


(b)

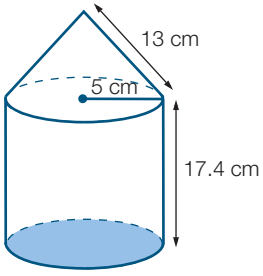


4 Find the surface area of each solid, rounded to two decimal places. WE10

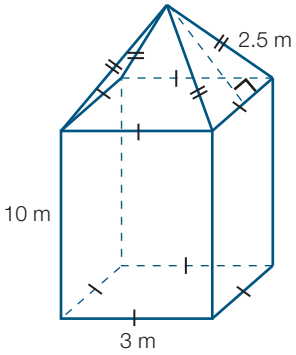
(a)



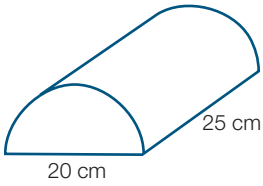
(b)



(c)



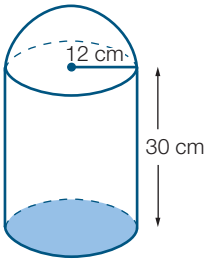
(d)



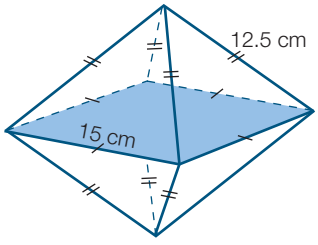
Hint

Use Pythagoras' Theorem to calculate the heights of the triangles.

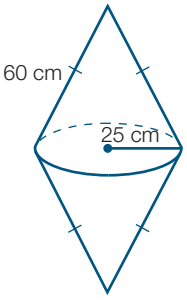
(e)



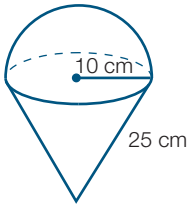
(f)



(g)

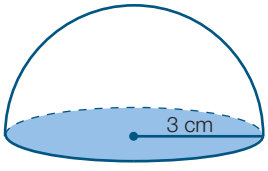


(h)



5 The calculation that gives the surface area of each of the following solids is:

(a)



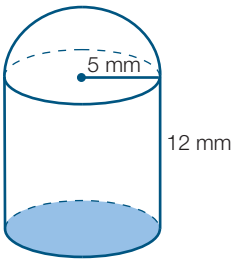
A $4 \times \pi \times 3^2$

B $\frac{1}{2} \times 4 \times \pi \times 3^2$

C $\frac{1}{2} \times 4 \times \pi \times 3^2 + \pi \times 3^2$

D $4 \times \pi \times 3^2 + \pi \times 3^2$

(b)



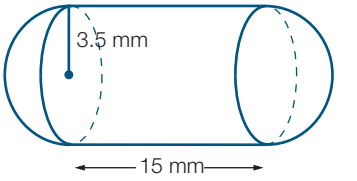
A $4 \times \pi \times 12^2 + 2 \times \pi \times 5 \times 12 + \pi \times 12^2$

B $2 \times \pi \times 5^2 + \pi \times 5 \times 12 + \pi \times 5^2$

C $4 \times \pi \times 5^2 + 2 \times \pi \times 5 \times 12 + 2 \times \pi \times 5^2$

D $\frac{1}{2} \times 4 \times \pi \times 5^2 + 2 \times \pi \times 5 \times 12 + \pi \times 5^2$

6 Find, correct to two decimal places, the surface area of a capsule that has the dimensions shown.



Hint

Determine the radius, then the slant height.

7 A conical party hat has a vertical height of 18 cm. The circumference of the circular base is 43 cm. Calculate the area of cardboard required to make the hat (assume no overlap). Round your answer to the nearest whole number.

Hint

Draw a labelled diagram.

8 A pyramid in Egypt has a square base that covers an area of 44 100 m^2 and a height perpendicular to the base of 130 m. Calculate the total surface area of the four triangular faces of the pyramid, to the nearest square metre.

Reasoning, communicating and open-ended

9 Marcus has a new tenpin bowling ball and he wants to sew together a protective bag for it. He has a rectangular piece of material 80 cm long and 40 cm wide. The bowling ball has a diameter of 30 cm.

(a) Show that the area of the material is sufficient to cover the surface area of the bowling ball.

(b) Even though Marcus has a sufficient area of material, explain why he should consider using a longer piece of material.

10 (a) Calculate the slant height of a cone of radius 25 cm and surface area 4398.23 cm^2, to the nearest centimetre.

(b) Calculate the radius of a cone with a surface area of 1809.56 cm^2 if the slant height is three times the radius.

11 Find two possible sets of dimensions (radius and slant height) for a cone with a surface area between 200 cm^2 and 300 cm^2.

12 Explain why a cone cannot have a radius and a slant height with the same length.

13 A cone has its height doubled, but its radius stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its total surface area.

(b) Evaluate its total surface area after its height is doubled.

(c) Choose another two values to represent another cone and evaluate its total surface area.

(d) Double the height of this cone and evaluate the new total surface area.

(e) Does doubling the height double the surface area?

14 A cone has its radius doubled, but its height stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its total surface area.

(b) Evaluate its total surface area after its radius is doubled.

(c) Choose another two values to represent another cone and evaluate its total surface area.

(d) Double the radius of this cone and evaluate the new total surface area.

(e) Does doubling the radius double the surface area?

Problem solving

Egyptian engineering

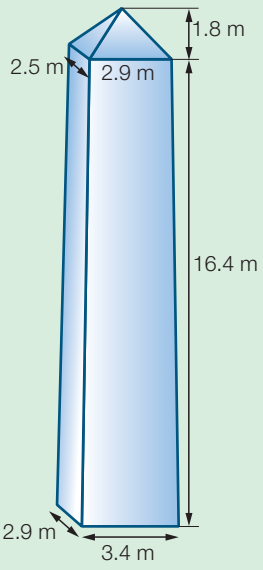
Egypt, four thousand years ago. Pharaoh Mentuhotep is constructing two massive stone obelisks on either side of the entrance to a temple. The obelisks are two tall columns that have four trapezoidal faces—they are narrower at the top than at the bottom. Each of the these faces will be covered in hieroglyphics.

At the top of the obelisk sits a small pyramid-shaped cap, called a 'pyramidion'. The four exposed faces of the pyramidion will be covered in gold, to reflect the sun.

Mentuhotep has given you, his chief engineer, a labelled drawing of what the obelisk should look like. Your job is to determine the area to be covered in:

(a) gold

(b) hieroglyphics.



Strategy options

- Break problem into manageable parts.
- Draw a diagram.
- Have I seen a similar problem?

3.5 Volume of tapered solids and spheres

Stage 5.3

Need to Know

Tapered solids

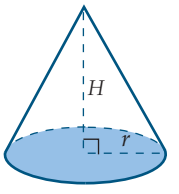
Cones and pyramids take up exactly one-third of the space of a cylinder or prism with corresponding base and height.

The volume of a tapered solid is:

$$V = \frac{1}{3}AH$$

where A = the area of the cross-section at the base
H = the perpendicular height of the solid.

Cone

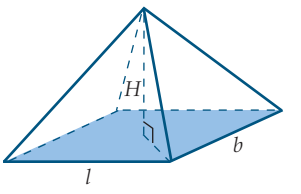


$A = \pi r^2$

$V = \frac{1}{3}AH$

$V = \frac{1}{3}\pi r^2 H$

Rectangular-based pyramid

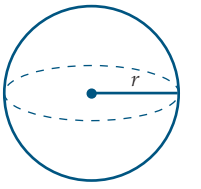


$A = lb$

$V = \frac{1}{3}AH$

$V = \frac{1}{3}lbH$

Sphere



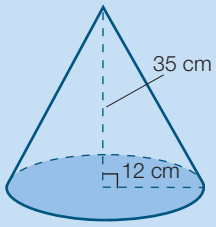
$V = \frac{4}{3}\pi r^3$

Worked Example 11

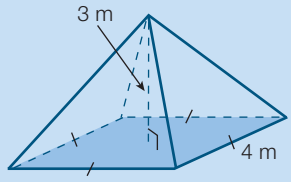
WE11

Find the volume of each of the tapered solids, rounded to two decimal places if necessary.

(a)



(b)



Working

(a) $r = 12 \quad H = 35$

$$V = \frac{1}{3}AH$$
$$= \frac{1}{3}\pi r^2 H$$
$$= \frac{1}{3} \times \pi \times 12^2 \times 35$$
$$= 1680\pi$$
$$= 5277.88 \text{ cm}^3 \text{ (2 d.p.)}$$

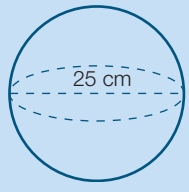
(b) $b = 4 \quad H = 3$

$$V = \frac{1}{3}AH$$
$$= \frac{1}{3}b^2 H$$
$$= \frac{1}{\cancel{3}} \times 4^2 \times \cancel{3}$$
$$= 16 \text{ m}^3$$

Worked Example 12

WE12

Find the volume of the sphere, rounded to two decimal places.

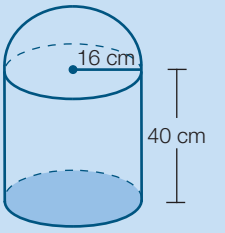


Working

$$r = \frac{25}{2}$$
$$= 12.5 \text{ cm}$$
$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times \pi \times 12.5^3$$
$$= 8181.23 \text{ cm}^3 \text{ (2 d.p.)}$$

Worked Example 13 WE13

Find the volume of this solid, rounded to two decimal places.



Working

$r = 16 \qquad H = 40$

$V = V_{cylinder} + V_{hemisphere}$

$= \pi r^2 H + \frac{1}{2} \times \frac{4}{3}\pi r^3$

$= \pi \times 16^2 \times 40 + \frac{1}{\cancel{2}_1} \times \frac{\cancel{4}^2}{3} \times \pi \times 16^3$

$= 10\,240\pi + \frac{8192}{3}\pi$

$= 40\,748.55 \text{ cm}^3$ (2 d.p.)

Answers page 622

3.5 Volume of tapered solids and spheres

Stage 5.3

Navigator

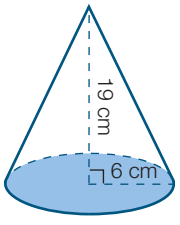
Q1, Q2, Q3, Q4, Q7, Q8, Q9, Q10, Q13	Q1, Q2, Q3, Q5, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14	Q1, Q2, Q3, Q5, Q6, Q8, Q10, Q11, Q12, Q14, Q15

Fluency and understanding

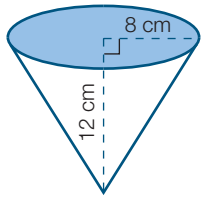
WE11

1 Find the volume of each of these tapered solids, rounded to two decimal places, if necessary.

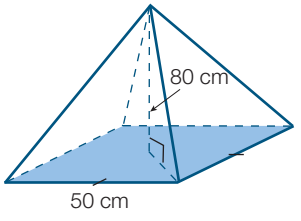
(a)



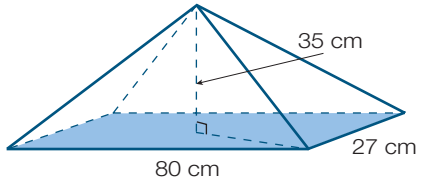
(b)



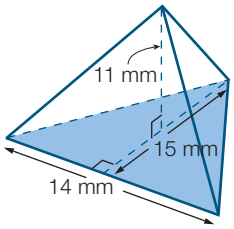
(c)



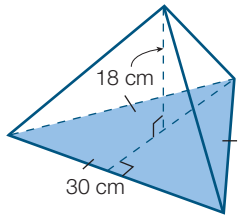
(d)



(e)

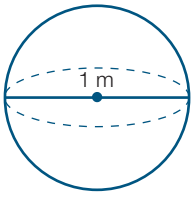


(f)

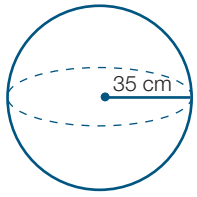


2 Find the volume of each sphere, rounded to two decimal places. WE12

(a)

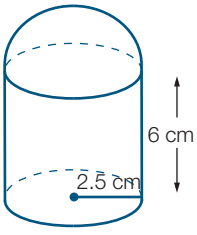


(b)

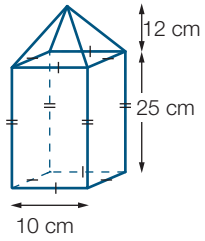


3 Find the volume of each of these solids, rounded to two decimal places. WE13

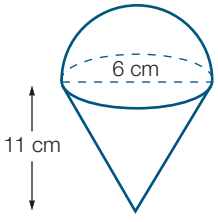
(a)



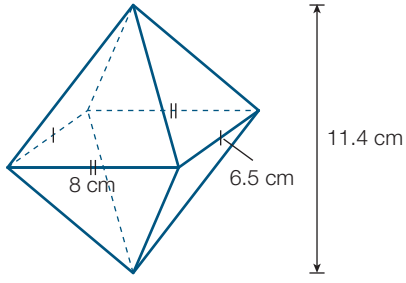
(b)



(c)



(d)



4 Earth has an approximate diameter of 12 800 km. If Earth is assumed to be spherical, what is its volume to the nearest 1000 km^3?

5 When the radius of a sphere is doubled:

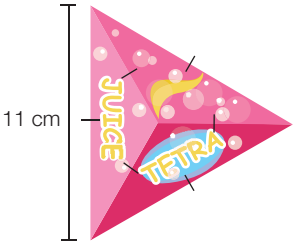
A the volume is doubled
B the volume is 4 times larger
C the volume is 6 times larger
D the volume is 8 times larger

6 The radius of a cone 25 cm high with a volume of 37 969.9 cm^3 is closest to:

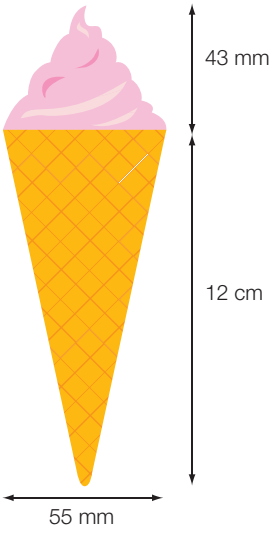
A 38 cm **B** 121 cm **C** 363 cm **D** 1450 cm

7 What is the difference between the volume of a cone of height 20 cm with a base diameter of 30 cm, and a pyramid of height 20 cm with a square base 30 cm wide? State your answer correct to two decimal places.

8 This 'tetra-pak' of frozen juice is a triangular-based pyramid. Each face is an equilateral triangle of side length 11 cm. If the height of the tetra-pak is 9 cm, find its capacity. State your answer correct to the nearest mL.



9 Anica buys an ice-cream cone at her local shop. The soft-serve ice-cream fills the cone and forms a conical (cone-shaped) peak as shown. Calculate the amount of ice-cream Anica receives to the nearest mL.



Reasoning, communicating and open-ended

10 Ball bearings are tiny metal balls used to reduce the friction between the moving parts of machines. How many ball bearings with a radius of 3 mm can be made from a solid cube of metal with a side length of 2 cm, assuming there is no metal wasted?

11 The table at right gives the diameters of the eight planets.

(a) How many times is Jupiter's diameter bigger than Mercury's (to the nearest whole number)?

(b) How many of Mercury would fit into Jupiter, by volume?

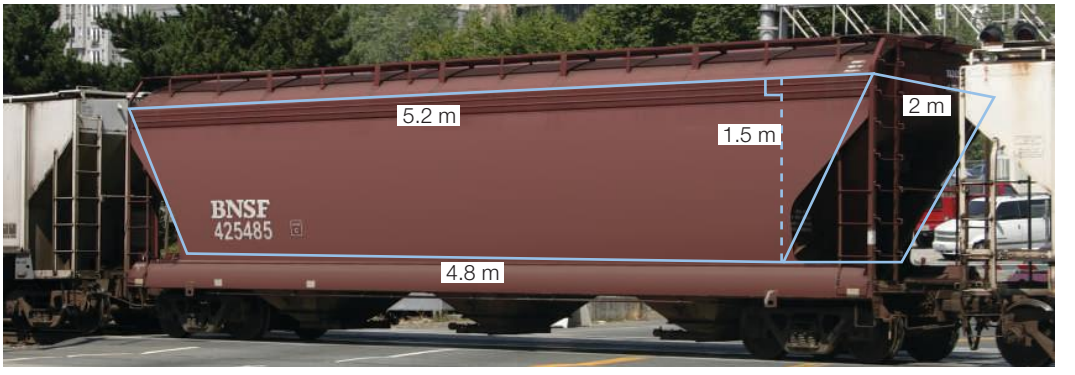
(c) Explain why your answers to part **(a)** and part **(b)** are so different.

(d) Calculate how many Earths would fit into each of the larger planets (by volume).

Planet	Diameter (km)
Mercury	4 900
Venus	12 200
Earth	12 800
Mars	6 800
Jupiter	138 000
Saturn	113 000
Uranus	47 000
Neptune	50 000

12 Six wheat silos are situated at a country railway siding, as shown below. The silos are all full and the grain will be moved to the city in railway trucks, as shown in the second diagram. How many trucks will be needed to move all the grain to town?

The silos are large cylinders with a cone on top. The trucks can be assumed to be trapezoidal prisms.



13 **(a)** Choose two values to represent the base length and the height of a square-based pyramid. Calculate the pyramid's volume.

(b) Double the length of the base and recalculate the volume.

(c) How many times greater is the volume of the second pyramid than the first?

(d) Repeat parts **(a)**–**(c)** with a new base and height. What can you conclude?

14 A cone has its height doubled, but its radius stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its volume.

(b) Evaluate its volume after its height is doubled.

(c) Choose another two values to represent another cone and evaluate its volume.

(d) Double the height of this cone and evaluate the new volume.

(e) Can you find a relationship between the volume of a cone after the height is doubled and its original volume? Show the relationship, if there is one, or explain why a relationship does not exist.

15 A cone has its radius doubled, but its height stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its volume.

(b) Evaluate its volume after its radius is doubled.

(c) Choose another two values to represent another cone and evaluate its volume.

(d) Double the radius of this cone and evaluate the new volume.

(e) Can you find a relationship between the volume of a cone after the radius is doubled and its original volume? Show the relationship, if there is one, or explain why a relationship does not exist.

3.6 Applications of volume

Stage 5.3

Need to Know

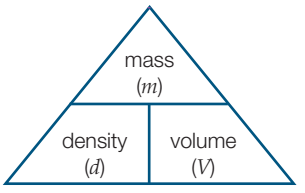
Density

The **density** of a substance is the amount of mass within a given unit of volume.

$$\text{Density} = \frac{\text{mass}}{\text{volume}} \text{ or } d = \frac{m}{V}$$

where d = density in g/cm^3 (g/mL) or kg/m^3 (kg/kL)
m = mass in grams (g) or kilograms (kg)
V = volume in centimetres cubed (cm^3 or mL) or metres cubed (m^3 or kL)

A triangle diagram is helpful in rearranging the density formula to find mass and volume.



- If two quantities are side by side, find their product.

  $m = d \times V$

- If two quantities are placed one over the other, find the quotient.

  $d = \frac{m}{V}$ $\qquad V = \frac{m}{d}$

Concentration

$$\textbf{Concentration} = \frac{\text{amount of solute (mass or volume)}}{\text{amount of solution (mass or volume)}}$$

Note that the concentration is the amount of solute compared to the final solution, not the amount of solvent added.

Worked Example 14

WE14

A rock has a mass of 80 g and volume of 25 cm^3. Find its density.

Working

$$d = \frac{m}{V}$$

$$= \frac{80 \text{ g}}{25 \text{ cm}^3}$$

$$= 3.2 \text{ g/cm}^3$$

Worked Example 15

WE15

Find the mass of 400 cm^3 of a block of metal with a density of 1.6 g/cm^3.

Working

$d = \frac{m}{V}$

$m = d \times V$

$m = 1.6 \times 400$

$= 640\ g$

Check: $\frac{640\ g}{400\ cm^3} = 1.6\ g/cm^3$

Worked Example 16

WE16

Find the volume of a 120 g piece of gold with a density of 19.3 g/cm^3 and round your answer to one decimal place.

Working

$d = \frac{m}{V}$

$V = \frac{m}{d}$

$V = \frac{120}{19.3}$

$= 6.22\ cm^3$ (2 d.p.)

Check: $\frac{120\ g}{6.22\ cm^3} = 19.3\ g/cm^3$ (1 d.p.)

Worked Example 17

WE17

A 3 L jug of cordial solution is made with 150 mL of cordial concentrate. Find the concentration of the cordial in mL/L.

Working

$$\text{Concentration} = \frac{\text{amount of solute}}{\text{amount of solution}}$$

$$= \frac{150\ mL}{3\ L}$$

$$= \frac{50\ mL}{1\ L}$$

The concentration of the cordial is 50 mL/L.

Worked Example 18 WE18

6 g of salt is dissolved in water to make 120 g of saline solution. Find the concentration of the solution in g/100 g.

Working

$$\text{Concentration} = \frac{\text{amount of solute}}{\text{amount of solution}}$$

$$= \frac{6\ g}{120\ g}$$

$$= \frac{6}{120} \times 100$$

Concentration is 5 g/100 g.

Answers page 622

3.6 Applications of volume

Stage 5.3

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 (a), Q7, Q8, Q11, Q13	Q1, Q2, Q3, Q5, Q6, Q7, Q9, Q12, Q13	Q1, Q2, Q3, Q6, Q7, Q9, Q10, Q11, Q12, Q13

Fluency and understanding

WE14

1 For each of the following, find the density.

(a) a metal sphere of volume 9 cm^3, and mass 63.9 g

(b) 20 cm^3 of liquid that has a mass of 128 g

(c) a solid of volume 24 cm^3 and mass 19.2 g

WE15

2 For each of the following, calculate the mass and state answers correct to two decimal places.

(a) A lump of metal with a volume of 800 cm^3 and a density of 7.8 g/cm^3.

(b) 450 mL of a liquid that has a density of 1.08 g/mL.

(c) 2 m^3 of a gas that has a density of 1.98 kg/m^3.

WE16

3 Calculate the volume of each of the following, rounded to two decimal places.

(a) A 150 g piece of lead with a density of 11.3 g/cm^3.

(b) 486 g of a liquid with a density of 0.7 g/mL.

(c) 13 kg of wood that has a density of 0.85 kg/m^3.

WE17

4 2.5 L of sugar solution contains 50 g of sugar. What is its concentration in g/L?

WE18

5 500 g of weedkiller solution contains 12 g of active ingredient. Find its concentration in g/100 g.

6 (a) A saline solution is made from 2.5 L of water and 28 g of salt. The concentration of salt in g/L is closest to:

A 28 B 14 C 12 D 11

(b) A solution of copper sulfate is made from 1350 L of water and 1.6 kg of copper sulfate. The concentration of copper sulfate in g/L is closest to:

A 0.8 B 1 C 800 D 1000

7 The Goulburn River carries sediment from erosion in its catchment. Barry takes a 5 L sample of water from the river. From this he extracts 38 g of sediment (1000 mg = 1 g). What is the concentration of sediment in the water, in mg/mL?

8 Marlon is making up 5 L of cordial for a party. The instructions on the cordial (concentrate) bottle say to add 1 cup of cordial to 4 cups of water.

(a) If Marlon uses a standard 250 mL cup, how many cups of each of the cordial concentrate and water are needed?

(b) What is the concentration of cordial concentrate in the final solution? Write your answer in mL/L.

9 (a) Crude oil is measured in units called 'barrels'. 1 barrel is equivalent to 159 litres. Calculate the mass of 1 barrel of crude oil if the density of the crude oil is 825 kg/m^3.

(b) Petrol has a density of 737 kg/m^3. A tanker truck used for transporting petrol has a capacity of 30 kL. Calculate the mass of petrol in a full tanker truck. Round your answer to the nearest tonne.

Hint

How many litres in 1 m^3?

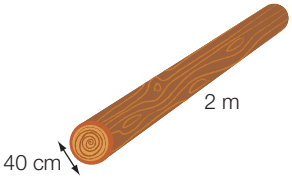
10 Jay's circular swimming pool has a diameter of 7 m and a depth of 1.6 m, but Jay only fills it to 20 cm below the rim. Calculate, correct to two decimal places, the chlorine concentration in g/L if he adds 2.1 kg of chlorine.

Reasoning, communicating and open-ended

11 The density of water is 1 g/cm^3. An object will *float* in water if its density is *less* than 1 g/cm^3.

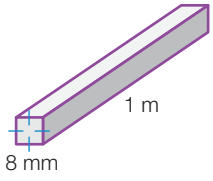
(a) By calculating the densities of the six objects below, determine which will float on water.

A wooden log



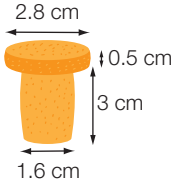
Mass: 100 kg

A steel rod



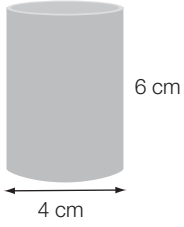
Mass: 0.5 kg

A cork



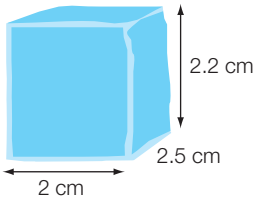
Mass: 2 g

An aluminium cylinder



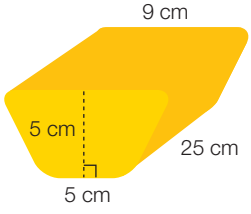
Mass: 200 g

An ice block



Mass: 10 g

A gold bar



Mass: 16 kg

(b) Find the mass of each object above if it was made of lead, which has a density of 11.35 g/cm^3. Write your answers correct to the nearest gram.

12 High phosphorus levels in water can lead to blue-green algal blooms. The town of Evanshill takes its water from Lake Farmer, which has phosphorus levels contributing to blue-green algal blooms. This means the water supply to the town must be shut down. Two main sources of phosphorus are:

- Johnstone's abattoir, which discharges 5 ML (1 ML = 1 000 000 L) per day with a phosphorus concentration of 10 mg/L
- the Arvon River, which has an average discharge of 200 ML per day at a phosphorus concentration of 0.2 mg/L.

(a) Which source is contributing more phosphorus per day?

(b) What is the actual concentration of phosphorus contributed to Lake Farmer each day in mg/L? Round your answer correct to two decimal places.

13 The Environment Protection Authority monitors the level of the common bacteria *E. coli* at swimming beaches. If the level on any day is above 1000 organisms/100 mL, the beach is closed. Tess collects a 200 mL water sample from a beach. An examination of the sample shows that in 80 mL there are 700 *E. coli* organisms.

(a) What is the concentration of *E. coli* in the sample in organisms/100 mL?

(b) Is the concentration in the sample high enough to close the beach?

(c) List three sample sizes and the number of *E. coli* that would be high enough to close the beach.

Problem solving

A pinch of salt

The salt concentration in the ocean is about 0.035 g/cm^3. How much salt could be extracted from a cubic kilometre of sea water?

Strategy options

- Break problem into manageable parts.
- Have I seen a similar problem?

Chapter review 3

D.I.Y. Summary

Key Words

Stage 5.1	annulus	composite shape
Stage 5.2	capacity	prism
	cylinder	surface area
	net	uniform cross-section
Stage 5.3	concentration	pyramid
	cone	sphere
	density	tapered solid

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.1 Q1 **Stage 5.2 Q2–5** **Stage 5.3 Q6–9**

1 To find the area of a _____, divide the shape into simple shapes, calculate the individual areas and then add them together.

2 Adding together the individual areas of each face of a solid will give the _____.

3 The volume of a _____ or _____ is found by multiplying the area of its uniform cross-section by its height.

4 Volume is measured in m^3 but _____ is measured in L.

5 A _____ is a 2D representation of the total surface area of a solid.

6 The volume of a _____ is calculated using the formula $\frac{4\pi r^3}{3}$, where r is the radius.

7 A pyramid is an example of a _____.

8 The formula for _____ is $\frac{\text{mass}}{\text{volume}}$.

9 The _____ of a dissolved substance in a solution can be expressed in many different units. These include mL/L, g/100 g, mg/L or g/L.

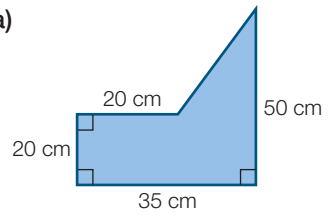
Stage 5.1 Q1, 2 **Stage 5.2 Q3, 5, 11–13, 15, 20–22** **Stage 5.3 Q4, 6–10, 14, 16–19, 23**

Fluency and understanding

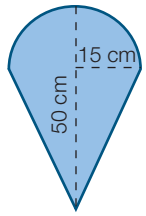
1 Find the shaded area of each of these shapes, rounded to two decimal places.

Ex.3.1

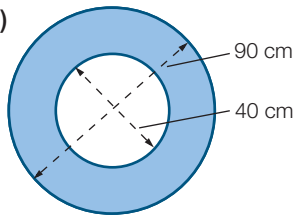
(a)



(b)



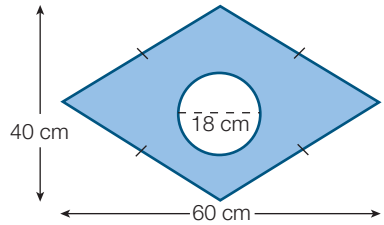
(c)



2 The shaded area correct to two decimal places is:

A 182.12 cm^2 **B** 945.53 cm^2

C 1143.45 cm^2 **D** 2145.53 cm^2

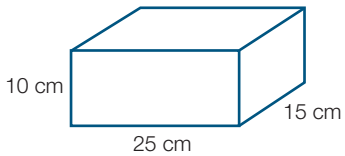


Ex.3.1

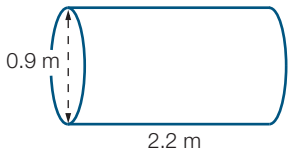
3 Find the surface area of each of these solids, rounded to two decimal places.

Ex.3.2

(a)



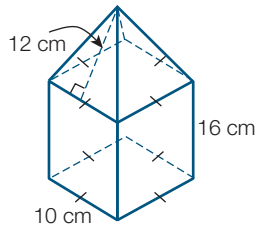
(b)



4 The surface area of this solid is equal to:

A 340 cm^2 **B** 840 cm^2

C 980 cm^2 **D** 1180 cm^2

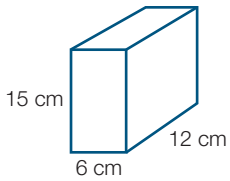


Ex.3.4

5 Find the volume of each of these solids, rounded to two decimal places.

Ex.3.3

(a)



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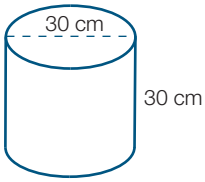
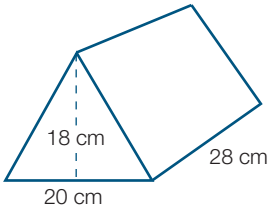


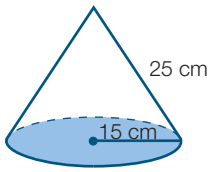
(c)



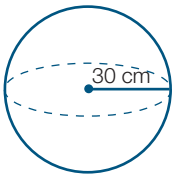
6 Find the surface area of each of these solids, rounded to two decimal places.

Ex.3.4

(a)



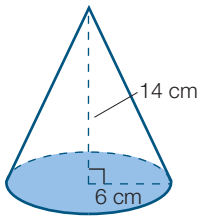
(b)



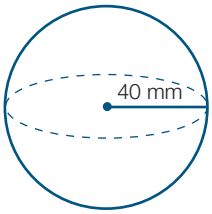
7 Find the volume of each of these solids, rounded to two decimal places.

Ex.3.5

(a)



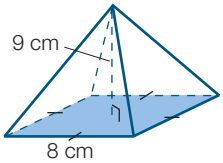
(b)



8 The volume of this solid is:

A 192 cm^3 **B** 208 cm^3

C 288 cm^3 **D** 576 cm^3



Ex.3.5

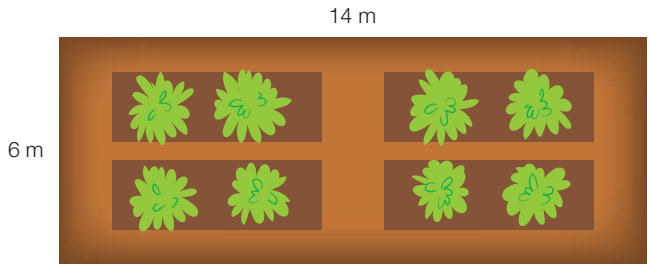
9 A rectangular block of wood with the dimensions of 1 m × 2 m × 2 m has a mass of 120 kg. What is its density?

Ex.3.6

10 A certain antiseptic should be mixed with warm water for best effects. If 30 mL should be mixed with 120 mL of water, express this as a concentration in mL/L.

Ex.3.6

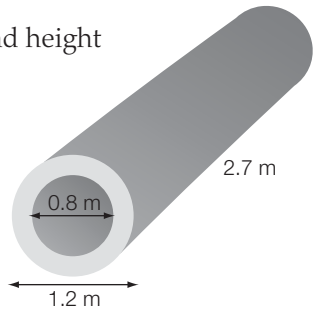
11 The diagram shows four garden beds, each 5.4 m long and 1.8 m wide, surrounded by paving bricks. Find the number of bricks needed to pave the area (to the nearest 100) if a brick measures 225 mm by 112 mm.



12 **(a)** Find the capacity of a cylindrical petrol tank of height 3.2 m and diameter 2.5 m. Express your answer to the nearest litre.

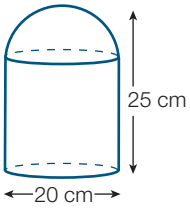
(b) The dimensions of a cylindrical rainwater tank are diameter 1760 mm and height 2100 mm. Calculate the capacity of the tank to the nearest litre.

13 Find the volume of material used to make this pipe in m^3, correct to two decimal places.

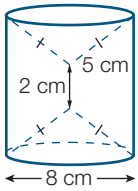


14 Find the surface area and volume of the following shapes rounded to the nearest whole number. Use Pythagoras' Theorem where necessary.

(a)



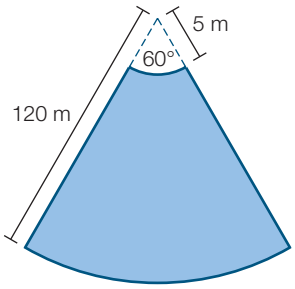
(b)



15 A snowball-throwing court has the measurements shown.

(a) Calculate the area in which the thrower has to land the snowball, to the nearest square metre.

(b) Find the perimeter of the shaded area, to the nearest metre.



16 **(a)** Find the volume of a 200 g block of polystyrene foam if its density is 0.1 g/cm^3.

(b) What mass of salt should be dissolved in 300 mL of pure water to give a concentration of 50 g/L?

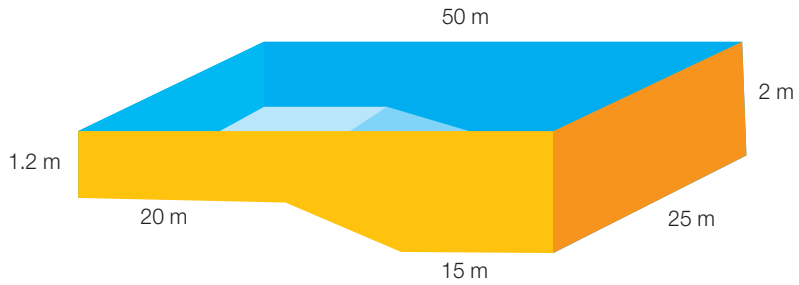
17 If the inside of this ice-cream cone is completely full of ice-cream, find the total volume of the ice-cream sitting in and on top of the cone, to the nearest cubic centimetre. (Assume the shape of the ice-cream on top is a hemisphere.)

18 Ayse is inflating balloons with helium gas for a birthday party. Each balloon is roughly spherical in shape and about 40 cm in diameter. If she has to inflate two dozen balloons, find the total capacity (in litres) of helium required, rounded to the nearest whole number.

19 The density of the material used to construct the pipe in Question **13** is 5.8 g/cm^3. Find the mass of the pipe.

Reasoning and communicating

20 The local council swimming pool is shaped as shown.



(a) Find the capacity of the pool in both litres and kilolitres.

(b) It is proposed to renovate the pool so that it is 2 m deep for its entire length. If its other dimensions remain the same, how much extra water will be required for the new pool?

21 A cylindrical container has radius 7 cm and height 10 cm. What must be the height of a rectangular container 7 cm long and 10 cm wide if the two containers are to have the same volume? Write your answer correct to two decimal places.

22 Engineers wish to investigate building a tunnel through a mountain rather than having to build a road over the top. The road will cost $45 000 for every 10 m length that is laid, whether it is over the top or through the tunnel.

(a) The tunnel would be a semicircle of radius 10 m and would be 400 m in length. Calculate the volume of earth to be removed, to the nearest cubic metre.

(b) Find the cost (in whole dollars) of tunnelling and removal of dirt if this cost is estimated to be $150/m^3.

(c) Calculate the surface area of the curved ceiling/wall, to the nearest square metre. (Assume that the entire floor will become road.)

(d) Find the cost (in whole dollars) of fortifying the ceiling wall if the cost is $100/m^2.

(e) Find the cost (in whole dollars) of laying the road through the tunnel.

(f) What is the total cost (in whole dollars) of construction of the road through the tunnel?

The road over the top of the mountain would be 1.4 km long.

(g) Find the cost of laying the road over the top of the mountain.

(h) Which is the cheaper option and by how much?

23 Mert owns a large pool initially containing 1000 L of fresh water. He adds 100 L of salt water, with a concentration of 52 g/L, to the pool, causing the water to become salty. Assuming the salt spreads evenly throughout the pool, what is the concentration of salt in the pool? Write your answer correct to two decimal places.

Mixed review A

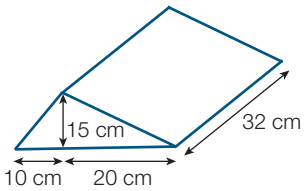
Stage 5.1 Q1, 17 **Stage 5.2** Q2–4, 6, 8–10, 14–16, 19 **Stage 5.3** Q5, 7, 11–13, 18, 20

Fluency and understanding

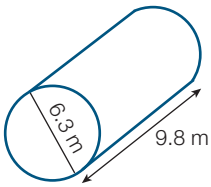
1 Find the interest earned, to the nearest cent, on an investment of \$2500 at 6.5% p.a. compounded monthly for 5 years. Ex.1.2

2 Find the surface area of each of the following solids. Where necessary, give your answer correct to two decimal places. Ex.3.2

(a)



(b)



3 Expand and simplify the following. Ex.2.1

(a) $(x - y)(3x - 2y)$ **(b)** $(3a - 6b)(2a - 5b)$ **(c)** $(5t + 4h)(3t - 9h)$

4 Find the effective rate of interest if \$10 000 is invested for 1 year at the following rates. Ex.1.4

(a) 7% simple interest **(b)** 7% compounded annually

(c) 7% compounded quarterly **(d)** 7% compounded monthly

5 Fully factorise the following expressions. Ex.2.5

(a) $x^2 - 4x + 4$ **(b)** $x^2 - 9$ **(c)** $x^2 + 14x + 49$

(d) $100x^2 - 25$ **(e)** $4xy^2 + 2xy - 8x$ **(f)** $4x^2 + 32x + 64$

6 Simplify the following. Ex.2.8

(a) $\frac{x}{5} - \frac{3x}{7}$ **(b)** $\frac{35a^4b^3}{56ab^5}$

7 Simplify the following. Ex.2.9

(a) $\frac{x^2 - 16}{3x - 12} \div \frac{x^2 - x - 20}{x^2 - 10x + 25}$ **(b)** $\frac{x + 3}{5} + \frac{x + 5}{6}$

8 Expand and simplify. Ex.2.1

(a) $6(x + 11) - x(x + 11)$ **(b)** $4(x + 6) - x(x + 6)$

(c) $2x(x + 4) - 3(x + 4)$ **(d)** $7y(y - 1) + 2(y - 1)$

9 Find, to the nearest cent, the total amount owing on a loan of \$4600 for 5 years if the 3.7% p.a. interest is compounded: Ex.1.3

(a) annually **(b)** quarterly **(c)** monthly.

Assume no repayments are made in the time period.

10 Factorise each expression. Ex.2.4

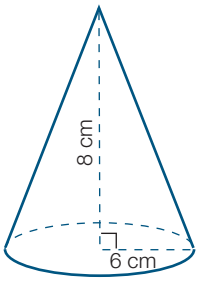
(a) $x^2 - 6x - 27$ **(b)** $x^2 - 7x + 10$ **(c)** $6x^2 - x - 2$

11 If the total mass of dissolved solids in a 30 mL sample of lake water is found to be 5 g, what is the concentration in g/L? State your answer correct to one decimal place. Ex.3.6

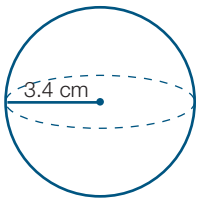
12 Find the volume of each of the following solids. Round your answers to two decimal places.

Ex. 3.5

(a)



(b)



13 Expand and simplify:

Ex. 2.2

(a) $(5x - 1)^2$ **(b)** $(4x - 3)^2$ **(c)** $(8 - 3x)^2$ **(d)** $(6 - 5x)^2$

14 Alana has \$5000 to invest for 3 years. Which is a better value arrangement?

Option A: 7.5% compounded quarterly Option B: 6.8% compounded monthly

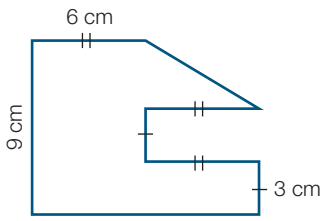
15 Calculate, to the nearest cent, the value after 4 years of a \$2300 laptop if it is depreciated in the following ways.

(a) 15% straight-line depreciation **(b)** 15% reducing balance depreciation

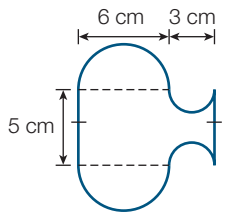
16 A cube has a volume of 512 cm^3. Calculate its surface area.

17 Find the area of each of the following shapes. Where necessary, give your answer correct to two decimal places.

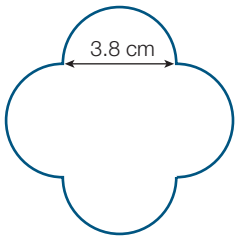
(a)



(b)



(c)

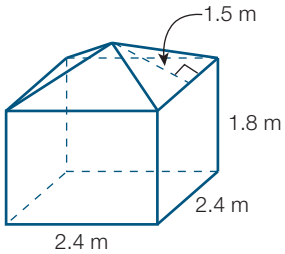


Reasoning and communicating

18 Scott wishes to erect a greenhouse. The support structure shown is made of metal. Using the plan shown, calculate correct to one decimal place:

(a) the amount of glass needed for the walls and roof

(b) the length of metal needed for the support structure.



19 A square piece of carpet of side length x is extended by 5 m in length and by 3 m in breadth.

(a) Write an algebraic expression for the area of the extended carpet, in factorised form.

(b) Expand your answer to part **(a)** and simplify by collecting like terms.

(c) If $x = 4$, find the area of:

(i) the original **(ii)** the extended piece of carpet.

(d) If the original piece of carpet cost \$128, suggest a comparable price for the extended piece of carpet.

20 How many mL of tea tree oil should be added to 180 mL of water to make a 10% solution?

Challenge A

1 Calculate $1000 - 1 + 3 - 5 + 7 \ldots$ to 102 terms (numbers in the sequence).

2 Damo has \$733 to spend in the airport shop before flying home. If he spent all his money on four different kinds of gifts for his 20 friends and relatives, priced at \$18, \$23, \$52 and \$69, how many of each gift did he buy?

3 A music store had a sale on CDs. For every two CDs that you bought at the normal price, you could purchase a third CD for \$1. If 12 CDs were bought for \$120, then the regular price for a CD was:

A \$15.00 **B** \$14.50 **C** \$14.00 **D** \$10.00

4 How many integer values of x satisfy $\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$?

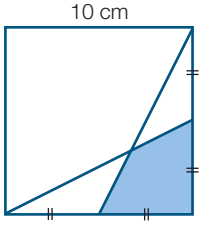
A 1 **B** 2 **C** 3 **D** 4

5 p, q and r are three different positive integers, and $(pq)^2 = pqr$.

Which of the following is a possible value for r?

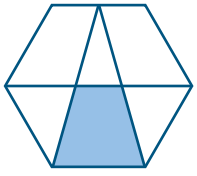
A 1 **B** 3 **C** 4 **D** 6

6 Find the area of the shaded region in the diagram.



7 A square fits tightly in an equilateral triangle with its base on the triangle's base. What percentage of the area of the triangle does it occupy?

8 Find the area of the shaded trapezium shown if the regular hexagon has a side length of 40 cm.

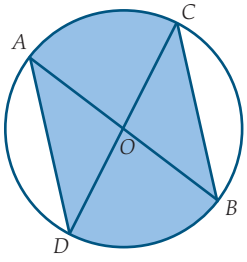


9 In the diagram, AB and CD are diameters of a circle of radius 4 cm.

If AB is perpendicular to CD, the area of the shaded region is:

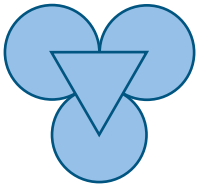
A $8\pi + 4$ **B** $8\pi + 8$

C $16 + 4\pi$ **D** $16 + 8\pi$



10 What is the side length of a square that just fits symmetrically within an isosceles triangle of sides 5 cm, 10 cm and 10 cm?

11 Find the volume of stone in a pillar, 6 m high, the cross-section of which is based on an equilateral triangle of 30 cm side length, having three circular sectors drawn with their centres as the vertices of the equilateral triangle. Give your answer in m^2 correct to three decimal places.



Hint

Remember that integers are positive or negative whole numbers or zero.

4 Linear relationships

P Recall 4

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 If $P = 2t - 7$, find P when:

(a) $t = 0$ **(b)** $t = 5$ **(c)** $t = -4$ **(d)** $t = 2$

2 Solve the following linear equations.

(a) $2a - 3 = 11$ **(b)** $\frac{b}{4} + 7 = 13$ **(c)** $3(2c - 5) = 45$ **(d)** $\frac{2d + 6}{5} = 4$

3 Plot these coordinate points on a Cartesian plane.

(a) $A(4, 3)$ **(b)** $B(0, 2)$ **(c)** $C(-1, 2)$ **(d)** $D(-2, -3)$

4 Copy and complete the table of values below for each of the following equations, then plot each graph for $-2 \le x \le 2$ on a Cartesian plane.

x	-2	-1	0	1	2
y					

(a) $y = x + 3$ **(b)** $y = x - 2$ **(c)** $y = 2 - x$ **(d)** $y = 3x$

5 Sketch the graphs of:

(a) $y = 3$ **(b)** $x = 2$

6 Write down **(i)** the gradient and **(ii)** the y-intercept of the graphs for each of the following equations.

(a) $y = 3x + 5$ **(b)** $y = 3 - 2x$ **(c)** $2x + 3y = 4$

7 Find the gradient of the line joining the points with coordinates:

(a) (1, 3) and (4, -2) **(b)** (-3, -6) and (2, -1)

8 **(a)** Sketch the graphs of $x = 1$ and $y = 3$ on the same set of axes.

(b) Find the point of intersection of the two graphs.

Syllabus references

Number and Algebra: Equations

- ✓ Solve linear equations **(ACMNA215) (Stage 5.2)**
- ✓ Solve linear equations involving simple algebraic fractions **(ACMNA240) (Stage 5.2)**
- ✓ Solve problems involving linear equations, including those derived from formulas **(ACMNA235) (Stage 5.2)**
- ✓ Solve linear inequalities and graph their solutions on a number line **(ACMNA236) (Stage 5.2)**
- ✓ Solve linear simultaneous equations, using algebraic and graphical techniques, including with the use of digital technologies **(ACMNA237) (Stage 5.2)**
- ✓ Solve complex linear equations involving algebraic fractions **(Stage 5.3§)**

Number and Algebra: Linear Relationships

- ✓ Interpret and graph linear relationships using the gradient-intercept form of the equation of a straight line **(Stage 5.2)**
- ✓ Solve problems involving parallel and perpendicular lines **(ACMNA238) (Stage 5.2)**
- ✓ Find the midpoint and gradient of a line segment (interval) on the Cartesian plane **(ACMNA294) (Stage 5.3§)**
- ✓ Sketch linear graphs using the coordinates of two points **(ACMNA215) (Stage 5.3§)**
- ✓ Solve problems using various standard forms of the equation of a straight line **(Stage 5.3§)**
- ✓ Solve problems involving parallel and perpendicular lines **(ACMNA238) (Stage 5.3§)**

Linear equations

4.1

Stage 5.2

Need to Know

Linear relationships

A **linear relationship** exists between two variables if the graph of the relationship is a **linear graph** (a straight line). In a linear relationship, a uniform change in one variable, the **independent variable**, produces a uniform change in the other, the **dependent variable**.

A linear relationship is described by a **linear equation**.

Solving linear equations

To solve linear equations of the type $5a - 3 = 2a + 6$ where the unknown appears on both sides of the equation, we must first add or subtract a variable term on both sides of the equation so that the variable appears on one side of the equation only (usually the LHS). If the equations involve brackets, we can expand the brackets using the distributive law $a(b + c) = ab + ac$.

Solving linear equations with fractions

To solve a linear equation containing fractions, we find the lowest common multiple (LCM) of the denominators, called the lowest common denominator (LCD). We then multiply both sides of the equation by the LCD. This simplifies the equation by eliminating the fractions.

Worked Example 1

WE1

Solve each of the following equations to find the value of the variable.

(a) $5x - 6 = 34$

(b) $\frac{3a}{2} + 4 = -1$

Working

(a) $5x - 6 = 34$

$5x = 40$

$x = 8$

Check: LHS $= 5x - 6$

$= 5 \times 8 - 6$

$= 34$

$=$ RHS

(b) $\frac{3a}{2} + 4 = -1$

$\frac{3a}{2} = -5$

$3a = -10$

$a = \frac{-10}{3}$

$= -3\frac{1}{3}$

Check: LHS $= \frac{3a}{2} + 4$

$= \frac{3 \times -\frac{10}{3}}{2} + 4$

$= -\frac{10}{2} + 4$

$= -5 + 4$

$= -1$

$=$ RHS

Worked Example 2

WE 2

Solve each of the following equations to find the value of the variable.

(a) $5a - 3 = 2a + 6$ **(b)** $2a - 5 = 6a + 1$ **(c)** $4(k + 1) = 3(6 - k)$

Working

(a)
$$\begin{aligned} 5a - 3 &= 2a + 6 \\ 3a - 3 &= 6 \\ 3a &= 9 \\ a &= 3 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= 5a - 3 \\ &= 5 \times 3 - 3 \\ &= 15 - 3 \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 2a + 6 \\ &= 2 \times 3 + 6 \\ &= 6 + 6 \\ &= 12 \end{aligned}$$

LHS = RHS

(b)
$$\begin{aligned} 2a - 5 &= 6a + 1 \\ 6a + 1 &= 2a - 5 \\ 4a + 1 &= -5 \\ 4a &= -6 \\ a &= -\frac{6}{4} \\ &= -\frac{3}{2} \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= 2a - 5 \\ &= 2 \times -\frac{3}{2} - 5 \\ &= -3 - 5 \\ &= -8 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 6a + 1 \\ &= 6 \times -\frac{3}{2} + 1 \\ &= -9 + 1 \\ &= -8 \end{aligned}$$

LHS = RHS

(c)
$$\begin{aligned} 4(k + 1) &= 3(6 - k) \\ 4k + 4 &= 18 - 3k \\ 7k + 4 &= 18 \\ 7k &= 14 \\ k &= 2 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= 4(k + 1) \\ &= 4(2 + 1) \\ &= 4 \times 3 \\ &= 12 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= 3(6 - k) \\ &= 3(6 - 2) \\ &= 3 \times 4 \\ &= 12 \end{aligned}$$

LHS = RHS

Worked Example 3

WE3

Solve each of the following equations to find the value of the variable.

(a) $\frac{2x}{5} - \frac{5x}{8} = 9$

(b) $\frac{2x+1}{3} = \frac{x-3}{5}$

Working

Method 1: Multiplying by the LCD

(a) LCD = 40

$$\frac{2x}{5} - \frac{5x}{8} = 9$$
$$\frac{40 \times 2x}{5} - \frac{40 \times 5x}{8} = 40 \times 9$$
$$16x - 25x = 40 \times 9$$
$$-9x = 40 \times 9$$
$$-x = 40$$
$$x = -40$$

Check:

$$\text{LHS} = \frac{2 \times -40}{5} - \frac{5 \times -40}{8}$$
$$= -16 + 25$$
$$= 9$$
$$= \text{RHS}$$

(b) LCD = 15

$$\frac{2x+1}{3} = \frac{x-3}{5}$$
$$\frac{15(2x+1)}{3} = \frac{15(x-3)}{5}$$
$$5(2x+1) = 3(x-3)$$
$$10x + 5 = 3x - 9$$
$$7x + 5 = -9$$
$$7x = -14$$
$$x = -2$$

Check:

$$\text{LHS} = \frac{2 \times -2 + 1}{3} = \frac{-3}{3} = -1$$

$$\text{RHS} = \frac{-2-3}{5} = \frac{-5}{5} = -1$$

LHS = RHS

Method 2: Writing all terms as fractions using the LCD

(a) LCD = 40

$$\frac{2x}{5} - \frac{5x}{8} = 9$$
$$\frac{8 \times 2x}{40} - \frac{5 \times 5x}{40} = 9$$
$$8 \times 2x - 5 \times 5x = 9 \times 40$$
$$16x - 25x = 9 \times 40$$
$$-9x = 9 \times 40$$
$$-x = 40$$
$$x = -40$$

Check:

$$\text{LHS} = \frac{2 \times -40}{5} - \frac{5 \times -40}{8}$$
$$= -16 + 25$$
$$= 9$$
$$= \text{RHS}$$

(b) LCD = 15

$$\frac{2x+1}{3} = \frac{x-3}{5}$$
$$\frac{5(2x+1)}{15} = \frac{3(x-3)}{15}$$
$$5(2x+1) = 3(x-3)$$
$$10x + 5 = 3x - 9$$
$$7x + 5 = -9$$
$$7x = -14$$
$$x = -2$$

Check:

$$\text{LHS} = \frac{2 \times -2 + 1}{3} = \frac{-3}{3} = -1$$

$$\text{RHS} = \frac{-2-3}{5} = \frac{-5}{5} = -1$$

LHS = RHS

Worked Example 4 WE4

Write an equation for the following and solve it to find the unknown numbers.

The sum of three consecutive numbers is 21.

Working

Let the first number be x.
Second number would be $x + 1$.
Third number would be $x + 2$.

$$x + (x + 1) + (x + 2) = 21$$
$$3x + 3 = 21$$
$$3x = 18$$
$$x = 6$$

Check: LHS $= 6 + (6 + 1) + (6 + 2)$
$= 6 + 7 + 8$
$= 21$
$=$ RHS

The three consecutive numbers are 6, 7 and 8.

(Alternatively, use $x - 1$, x and $x + 1$ as the three consecutive numbers.)

Answers page 625

4.1 Linear equations

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q15, Q18	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 (d)–(f), Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q15, Q16, Q18, Q19	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 (d)–(f), Q6, Q8, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

Fluency and understanding

WE1

1 Solve the following equations to find the value of the given variable.

(a) $4x + 3 = 15$ **(b)** $4x + 1 = 33$ **(c)** $3c + 7 = -2$

(d) $5a - 3 = 17$ **(e)** $7x - 2 = -44$ **(f)** $7d - 5 = -26$

(g) $6 - 5m = -4$ **(h)** $9 - 4n = 5$ **(i)** $-2 - 3x = 1$

(j) $19 = 6f + 7$ **(k)** $-9 = 4e - 1$ **(l)** $-3 = 5p - 4$

(m) $\frac{x}{8} - 5 = -5$ **(n)** $\frac{g}{7} + 2 = 5$ **(o)** $\frac{h}{4} - 7 = -3$

(p) $\frac{2k}{3} + 1 = 3$ **(q)** $\frac{5x}{3} - 2 = 8$ **(r)** $\frac{7y}{2} + 3 = 3$

(s) $3 - \frac{4x}{5} = 2$ **(t)** $6 - \frac{2x}{3} = 4$ **(u)** $4 - \frac{5x}{7} = -1$

WE2

2 Solve the following equations to find the value of the given variable.

(a) $3a + 7 = a + 9$ **(b)** $9d + 4 = 2d - 3$ **(c)** $5e + 2 = 7 - e$

(d) $7c - 1 = 4c + 5$ **(e)** $4b - 3 = 2b + 5$ **(f)** $4f - 5 = 8 - 2f$

(g) $g - 11 = 5 - g$ **(h)** $2h + 3 = 7 - 3h$ **(i)** $4k + 13 = 7 - 2k$

(j) $2 - 3x = 2x + 7$ **(k)** $3 - 4x = 2x - 3$ **(l)** $-6 - x = 2x + 1$

(m) $2(x + 3) = 6x$ **(n)** $3(2x + 1) = 18x$ **(o)** $5(7x + 2) = 65x$

(p) $3(m + 2) = m - 5$ **(q)** $2(3d - 5) = 2d + 7$ **(r)** $4(2d - 3) = 6d + 2$

(s) $5(a - 2) = 3(a + 2)$ **(t)** $7(a + 2) = 2(2a - 1)$ **(u)** $4(2g - 3) = 5(3g - 4)$

(v) $7(2t + 4) = 4(7 - 2t)$ **(w)** $5(2h + 3) = 4(3h + 1)$ **(x)** $2(4j + 3) = 3(1 - j)$

3 Solve the following equations to find the value of the given variable. WE3

(a) $\frac{a+7}{3} = \frac{a-3}{5}$ (b) $\frac{b-2}{2} = \frac{b+1}{3}$ (c) $\frac{2c-6}{3} = \frac{c+7}{4}$

(d) $\frac{3d+1}{2} = \frac{d-3}{4}$ (e) $\frac{2x+1}{3} = \frac{x-3}{5}$ (f) $\frac{5y-2}{3} = \frac{y+3}{4}$

(g) $\frac{3-2m}{3} = \frac{5-4m}{4}$ (h) $\frac{1-4k}{4} = \frac{3k-7}{2}$ (i) $\frac{2-3t}{4} = \frac{t-5}{3}$

(j) $\frac{x}{2} + \frac{3x}{7} = 13$ (k) $\frac{x}{3} - \frac{5x}{4} = -11$ (l) $\frac{6x}{7} - \frac{4x}{3} = -1$

(m) $\frac{a+7}{3} + 2 = -4$ (n) $\frac{p-3}{4} - 1 = 2$ (o) $\frac{m-8}{5} - 2 = -9$

4 Write an equation for each of the following and solve it to find the unknown numbers. WE4

(a) 6 is added to x to give a result of 13.

(b) 7 is subtracted from y to give a result of -2.

(c) Twice a certain number is 17.

(d) One-third of a certain number is 12.

(e) Two more than three times a number is 23.

(f) Five less than four times a number is -3.

5 (a) The equation $3a + 5 = -1$ has the solution:

A $a = \frac{4}{3}$ B $a = -\frac{1}{2}$ C $a = -2$ D $a = -18$

(b) The equation $\frac{d}{7} + 3 = -2$ has the solution:

A $d = 7$ B $d = -\frac{5}{7}$ C $d = -7$ D $d = -35$

(c) The equation $\frac{2x}{3} - 4 = 4$ has the solution:

A $x = -48$ B $x = 0$ C $x = 3$ D $x = 12$

6 (a) $3x + 8 = 2 - x$ has the solution:

A $x = -3$ B $x = -\frac{3}{2}$ C $x = \frac{2}{3}$ D $x = 3$

(b) $7(y - 3) = 5(15 - y)$ has the solution:

A $y = 9$ B $y = 8$ C $y = 7$ D $y = 0$

(c) $\frac{3m-2}{2} = \frac{m+11}{3}$ has the solution:

A $m = 0$ B $m = 2$ C $m = 4$ D $m = 7$

7 The sum of four consecutive numbers is 38. What numbers are they?

8 Find three consecutive even numbers whose sum is 126.

9 Sam divided his bag of jubes so that he and his friends Nikki and Minh could share them. He gave himself twice as many as he gave Nikki, and Minh received three times as many as Nikki. If the bag contained 42 jubes, how many did each person receive?

Hint

Add 2 to get the next even number.

10 The breadth of a rectangular block of land is $\frac{2}{7}$ of its length. If its perimeter is 252 m, find its length.

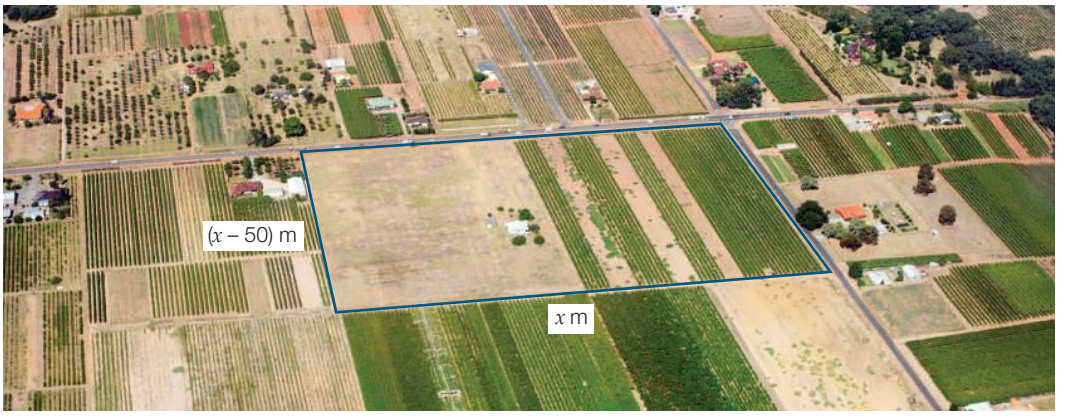
11 Suppose that p represents Penny's age.

(a) How old will she be in 7 years' time?

(b) Write an expression to represent twice her age in 7 years' time.

(c) What is Penny's age now if twice her age in 7 years' time is 48?

12 If the perimeter of the rectangular block of land shown in the photograph below is 300 m, find the length and breadth of the block.



13 To travel to a concert, Olivia and her friends decided to walk to the train station. The trip on the train covered 6 times the distance they had walked. They then rode in a taxi for the final 2 km to the concert venue. Altogether they travelled 12.5 km. How far did Olivia and her friends walk, and how far did they travel by train?

14 Ross is twenty years older than his niece Krystal. Krystal has worked out that in five years' time her uncle will be twice as old as she will be. Find Krystal's age.

15 If $\frac{3}{5}$ of a number is subtracted from $\frac{2}{3}$ of the same number, the result is 3. Find the number.

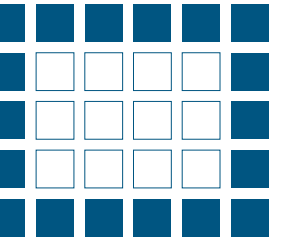
Reasoning, communicating and open-ended

16 Amy is twice as old as Lucy. Three years from now the sum of their ages will be 42. How old is Amy?

17 Two cars start at the same time from the same point and travel in opposite directions. One car is travelling 10 km/h faster than the other. In 3 hours they are 288 km apart. Find the speed of each car.

18 Write three different equations, each having a solution of -2, where at least three mathematical operations are used.

19 **(a)** Mrs Grey wants Tim the tiler to lay square tiles to make a bordered rectangle with white tiles on the interior and blue tiles on the border. For his first attempt, Tim comes up with the following design, which contains 18 blue and 12 white tiles. If the rectangle remains 3 white tiles wide, but the number of white tiles in the length increases to l, how many blue tiles will be required for the border?



(b) **(i)** Draw a different rectangular design that uses 12 white tiles.

(ii) How many blue tiles would now be needed?

(iii) If the number of white tiles in the width remains constant, but the number of white tiles in the length increases to t, how many blue tiles will be required for the border?

(c) Choose a different number of white tiles to make a rectangular pattern and find the number of blue tiles (n) needed to make a border.

Solving more complex linear equations

4.2

Stage 5.3§

Need to Know

Complex fractional equations

Complex fractional equations can be solved in the same manner as simple fractional equations. By using the LCD, the fractions can be simplified and solved in the same way as other linear equations. Remember to multiply every term by the LCD or write every term with the LCD as the denominator.

Worked Example 5

WE 5

Solve each of the following equations to find the value of the variable.

(a) $\frac{x}{3} - 2 = \frac{x-1}{4}$ **(b)** $\frac{2x-5}{3} - \frac{5x-3}{8} = -1$

Working

Method 1: Multiplying by the LCD

(a) LCD = 12

$$\frac{x}{3} - 2 = \frac{x-1}{4}$$
$$\frac{12x}{3} - 12 \times 2 = \frac{12(x-1)}{4}$$
$$4x - 24 = 3(x-1)$$
$$4x - 24 = 3x - 3$$
$$x - 24 = -3$$
$$x = 21$$

(b) LCD = 24

$$\frac{2x-5}{3} - \frac{5x-3}{8} = -1$$
$$\frac{24(2x-5)}{3} - \frac{24(5x-3)}{8} = 24 \times -1$$
$$8(2x-5) - 3(5x-3) = -24$$
$$16x - 40 - 15x + 9 = -24$$
$$x - 31 = -24$$
$$x = 31 - 24$$
$$x = 7$$

Method 2: Writing all terms as fractions using the LCD

(a) LCD = 12

$$\frac{x}{3} - 2 = \frac{x-1}{4}$$
$$\frac{4x}{12} - \frac{24}{12} = \frac{3(x-1)}{12}$$
$$4x - 24 = 3(x-1)$$
$$4x - 24 = 3x - 3$$
$$x = 21$$

Check:

$$\text{LHS} = \frac{21}{3} - 2 = 7 - 2 = 5$$

$$\text{RHS} = \frac{21-1}{4} = \frac{20}{4} = 5$$

LHS = RHS

(b) LCD = 24

$$\frac{2x-5}{3} - \frac{5x-3}{8} = -1$$
$$\frac{8(2x-5)}{24} - \frac{3(5x-3)}{24} = -1$$
$$8(2x-5) - 3(5x-3) = 24 \times -1$$
$$16x - 40 - 15x + 9 = -24$$
$$x - 31 = -24$$
$$x = 31 - 24$$
$$x = 7$$

Check: $\text{LHS} = \frac{2 \times 7 - 5}{3} - \frac{5 \times 7 - 3}{8}$

$$= \frac{14-5}{3} - \frac{35-3}{8}$$
$$= \frac{9}{3} - \frac{32}{8}$$
$$= 3 - 4$$
$$= -1$$
$$= \text{RHS}$$

Answers page 625

4.2 Solving more complex linear equations

Stage 5.3§

Navigator

Q1, Q2, Q3, Q5	Q1, Q2, Q3, Q5, Q6	Q1, Q2, Q3, Q4, Q5, Q6

Fluency and understanding

WE5

1 Solve the following equations to find the value of the given variable.

(a) $\frac{3p+1}{2} - \frac{2p}{5} = 6$ **(b)** $\frac{3n-1}{5} + \frac{n}{2} = 2$ **(c)** $\frac{3t+2}{4} - \frac{2t}{3} = -1$

(d) $\frac{n-4}{5} + \frac{4n-1}{7} = 6$ **(e)** $\frac{4m-3}{7} + \frac{2m+5}{2} = 3$ **(f)** $\frac{2k+3}{2} + \frac{5-k}{3} = 2$

(g) $\frac{2k-5}{3} - \frac{5k-3}{8} = -1$ **(h)** $\frac{4p+1}{9} - \frac{3p+1}{5} = 1$ **(i)** $\frac{2x+3}{4} - \frac{3x-5}{3} = 2$

2 The equation $\frac{2m+9}{3} - \frac{5m+8}{7} = 2$ has the solution:

A 3 **B** 9 **C** -9 **D** -3

3 Three times a number is added to five and the result is divided by four. This amount, when added to one-sixth of four more than five times that number, gives a total of negative six. What was the original number?

Reasoning, communicating and open-ended

4 Tom lives a certain number of kilometres from his school. Jane lives half of four kilometres less than three times Tom's distance from the school. Reena lives four kilometres closer to the school than Jane. She knows that the distance from her house to the school can be calculated if she adds ten to Tom's distance and divides the result by three. How far from the school do they each live?

5 Jet solved a linear equation. His working shown below contains an error. Where did he go wrong? Show Jet how to obtain the correct solution.

$$\frac{3n-1}{4} - \frac{2n+5}{3} = 2$$
$$3(3n-1) - 4(2n+5) = 12 \times 2$$
$$9n - 3 - 8n + 20 = 24$$
$$n + 17 = 24$$
$$n = 7$$

6 Make up your own linear equation containing two complex fractions and an integer, then solve it.

Gradient review

4.3

Stage 5.2

Need to Know

Gradient

Gradient is a measure of the steepness or slope of an incline.

Gradient can be positive ╱, negative ╲, zero — or undefined |.

Gradient is defined as the rise divided by the run and is usually represented by the letter m.

$$m = \frac{rise}{run}$$

The run is measured from left to right, while the rise is measured in the upward direction. So a downward direction will give a negative run and the gradient will be negative.

For $m > 0$, as m increases, the steepness of the incline increases.

For $m < 0$, as m decreases, the steepness of the incline increases.

Gradient and linear equations

A linear equation can be written as $y = mx + b$ where m is the gradient and b is the y-intercept.

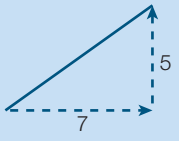
If a linear graph is drawn, the linear equation can be found by finding the gradient and y intercept, then substituting these values into the linear equation.

Worked Example 6

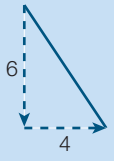
WE 6

Find the gradient of the following inclines.

(a)



(b)



Working

(a) $m = \frac{rise}{run}$

$= \frac{5}{7}$

(b) $m = \frac{rise}{run}$

$= \frac{-6}{4}$

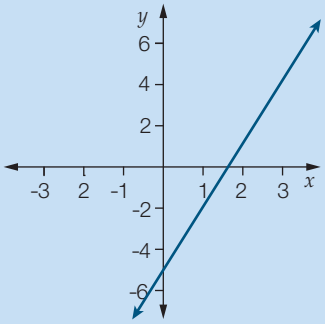
$= -\frac{3}{2}$

Worked Example 7

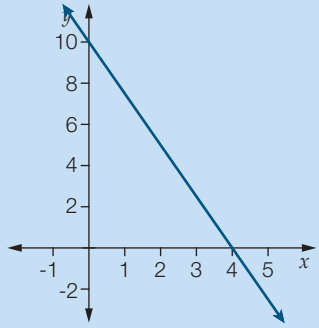
WE7

Find the linear equation for the following graphs by first finding the gradient and y intercept.

(a)

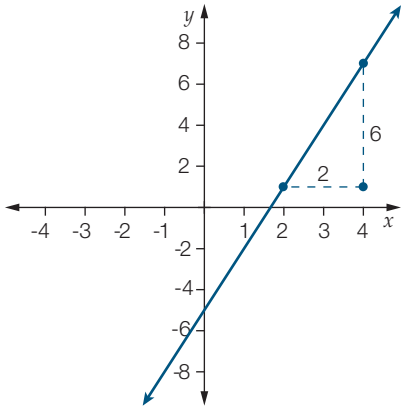


(b)



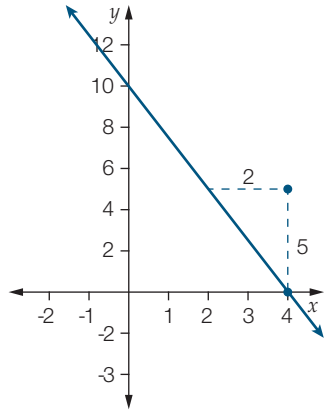
Working

(a)



$$m = \frac{rise}{run}$$
$$= \frac{6}{2}$$
$$= 3$$
$$b = -5$$
$$y = mx + b$$
$$y = 3x - 5$$

(b)



$$m = \frac{rise}{run}$$
$$= \frac{-5}{2}$$
$$b = 10$$
$$y = mx + b$$
$$y = -\frac{5}{2}x + 10$$

4.3 Gradient review

Answers page 626

Stage 5.2

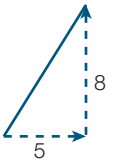
Navigator

Q1, Q2, Q3, Q4, Q6 (a), (b), Q7, Q8	Q1, Q2, Q3, Q4, Q5, Q6 (c), (d), Q7, Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

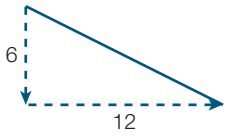
Fluency and understanding

1 Find the gradient of the following inclines. WE 6

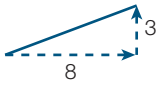
(a)



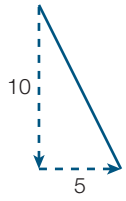
(b)



(c)

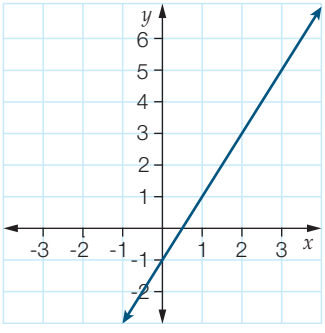


(d)

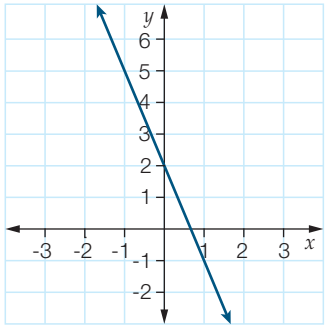


2 Find the linear equation for the following graphs by first finding the gradient and y intercept. WE 7

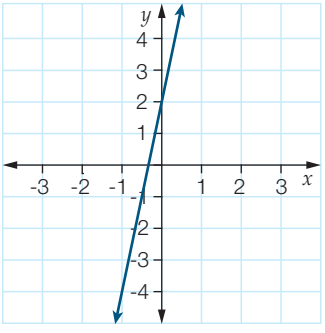
(a)



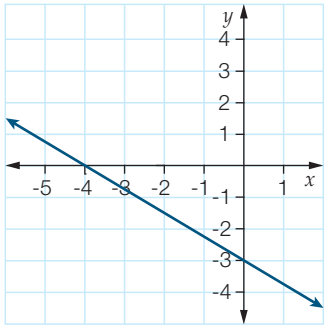
(b)



(c)



(d)

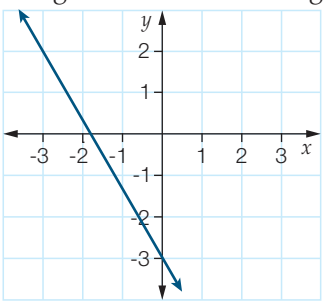


3 Find the gradient of the graph of each of the following equations.

(a) $y = 5x - 2$ **(b)** $y = -3x + 6$ **(c)** $y = 6x + 1$

(d) $y = 2x - 7$ **(e)** $y = 4x + 3$ **(f)** $y = -5x + 1$

4 The gradient of the linear graph shown is:



A $\frac{3}{5}$ **B** $-\frac{3}{5}$ **C** $\frac{5}{3}$ **D** $-\frac{5}{3}$

Reasoning, communicating and open-ended

5 For the equation $3x + 4y - 12 = 0$:

A $m = 0.75, b = -3$ **B** $m = -0.75, b = 3$ **C** $m = 3, b = 12$ **D** $m = 3, b = 4$

6 Find the gradient of the graph of each of the following equations.

(a) $4x - 3y = 2$ **(b)** $3x + 2y = 6$ **(c)** $5x + 4y = 1$

(d) $2x + 5y = 5$ **(e)** $6x + 2y = 8$ **(f)** $x + 5y = 10$

7 A ramp has a rise of 0.7 metres over a horizontal distance of 9 metres. Another ramp has a rise of 0.6 metres over a horizontal distance of 7.5 metres. Which ramp has the steeper gradient? Explain your decision.

8 The gradient of a linear graph is -2.5. If a point on the graph has the coordinates (2, 5), write the coordinates of two other points that could lie on the line.

Calculating gradient

4.4

Stage 5.3§

Need to Know

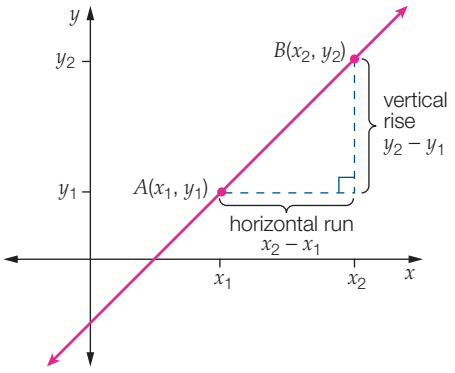
The gradient of a line connecting two points is the rate at which the line rises or falls.

The gradient is the same at any point along a straight line.

The steeper the line, the greater the magnitude of the gradient.

Calculating gradient

The gradient of a line can be found by considering any two points (x_1, y_1) and (x_2, y_2) on the line.



$$\text{Gradient of line } AB = \frac{\text{vertical rise}}{\text{horizontal run}}$$

$$= \frac{\text{difference in } y\text{-coordinates}}{\text{difference in } x\text{-coordinates}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) is one point and (x_2, y_2) is another point.

Usually (x_1, y_1) is the point to the left of (x_2, y_2) so that $x_2 - x_1$ is positive. If (x_2, y_2) is lower than (x_1, y_1), then $y_2 - y_1$ will be negative and the gradient will therefore be negative.

However, as $\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$ it makes no difference which point is labelled (x_1, y_1) or (x_2, y_2).

The gradient of a line is the gradient of any interval within that line. The gradient is constant at any point on the line. The gradient of a line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- The gradient is positive if the y-values increase as the x-values increase.
- The gradient is negative if the y-values decrease as the x-values increase.
- A horizontal line has zero gradient.
- A vertical line has an undefined gradient.

Standard forms of a linear equation

There are three standard forms of a linear equation:

1. Point-gradient form $\quad y - y_1 = m(x - x_1)$
2. Gradient-intercept form $\quad y = mx + b$
3. The general form $\quad ax + by + c = 0$

The general form of a linear equation

Any equation in the form of $y = mx + b$ can be transformed into the general form $ax + by + c = 0$.

For example:

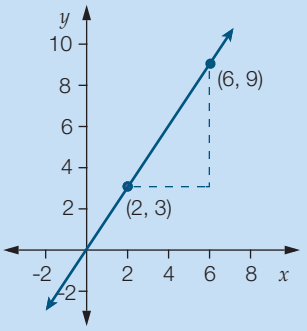
$y = 3x - 5$ is equivalent to $3x - y - 5 = 0$.

Worked Example 8

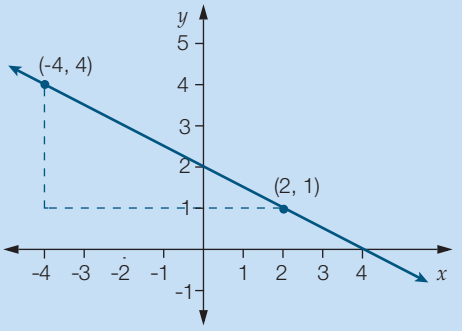
WE 8

Find the gradient of each of the following lines through the given points.

(a) (2, 3) and (6, 9)



(b) (-4, 4) and (2, 1)



Working

(a) Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (6, 9)$.

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{9 - 3}{6 - 2}$$
$$= \frac{6}{4}$$
$$= \frac{3}{2}$$

(b) Let $(x_1, y_1) = (-4, 4)$ and $(x_2, y_2) = (2, 1)$.

$$m = \frac{\text{rise}}{\text{run}}$$
$$= \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1 - 4}{2 - (-4)}$$
$$= \frac{-3}{6}$$
$$= -\frac{1}{2}$$

Worked Example 9

WE 9

Find the equation of a line with:

(a) a gradient of 2 that passes through the point (3, -5)

(b) a gradient of -4 that passes through the point (-4, 2).

Working

(a)
$$y - y_1 = m(x - x_1)$$
$$y - -5 = 2(x - 3)$$
$$y + 5 = 2x - 6$$
$$y = 2x - 6 - 5$$
$$y = 2x - 11$$

(b)
$$y - y_1 = m(x - x_1)$$
$$y - 2 = -4(x - -4)$$
$$y - 2 = -4(x + 4)$$
$$y - 2 = -4x - 16$$
$$y = -4x - 14$$

Worked Example 10 WE10

Find the equation of the line joining (2, -1) and (-3, 4).

Working

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{4 - -1}{-3 - 2}$$
$$= -1$$

$$y - y_1 = m(x - x_1)$$
$$y - -1 = -1(x - 2)$$
$$y + 1 = -x + 2$$
$$y = -x + 1$$

4.4 Calculating gradient

Answers page 626

Navigator

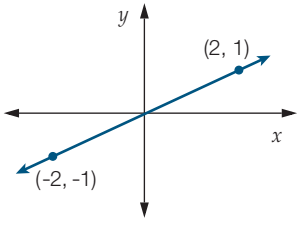
Stage 5.3§

Q1, Q2, Q3, Q4, Q5 Column 1, Q6, Q7, Q8, Q9, Q10, Q11, Q15	Q1, Q2, Q3, Q4, Q5 Column 2, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q15	Q1, Q2, Q3, Q4, Q5 Column 3, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16

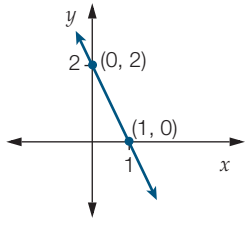
Fluency and understanding

1 Find the gradient of each of the following lines through the given points. WE8

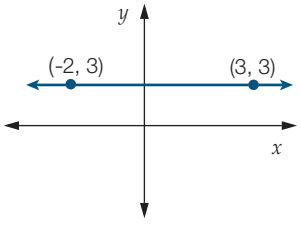
(a) (-2, -1) and (2, 1)



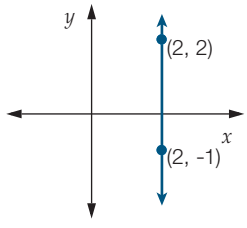
(b) (0, 2) and (1, 0)



(c) (-2, 3) and (3, 3)



(d) (2, -1) and (2, 2)



2 Find the equation of a line with: WE9

(a) a gradient of 6 that passes through the point (1, 3)

(b) a gradient of -2 that passes through the point (4, -1)

(c) a gradient of 3 that passes through the point (-4, -3)

(d) a gradient of -5 that passes through the point (3, -7).

3 Find the equation of the line joining: WE10

(a) (4, 7) and (5, 9) **(b)** (3, 11) and (6, 20) **(c)** (1, -2) and (3, 8)

(d) (3, 1) and (5, -5) **(e)** (5, 8) and (8, 2) **(f)** (-6, 5) and (-4, 3)

(g) (3, 2) and (6, 7) **(h)** (-6, 5) and (-4, 6) **(i)** (3, 8) and (6, -2)

4 Use the results from Question **1** to answer the following questions.

(a) If a line is sloping up to the right, does it have a positive or a negative gradient?

(b) If a line is sloping down to the right, does it have a positive or a negative gradient?

(c) What is the gradient of a horizontal line?

(d) What is the gradient of a vertical line?

5 Find the gradient of the line joining each of the following pairs of points.

(a) (2, 4) and (5, 5) **(b)** (0, 3) and (2, 7) **(c)** (4, 10) and (5, 6)

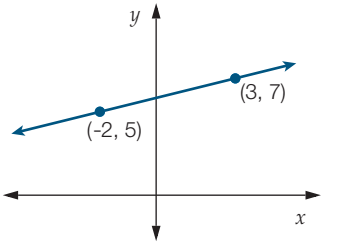
(d) (7, 5) and (9, 2) **(e)** (-2, 1) and (2, -4) **(f)** (-1, -3) and (0, -5)

(g) (0, 0) and (2, 6) **(h)** (2, 1) and (-2, -7) **(i)** (-2, 5) and (5, -2)

(j) (-4, 4) and (2, 0) **(k)** (-3, 2) and (2, 2) **(l)** (5, -4) and (-7, -4)

(m) (-8, 5) and (2, 1) **(n)** (-1, -1) and (6, 0) **(o)** (2, 5) and (2, 7)

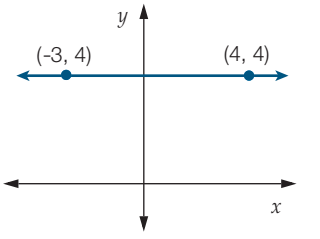
6 **(a)** The gradient of the line shown in the graph is:



A $\frac{5}{2}$ **B** 2

C $\frac{2}{5}$ **D** $-\frac{2}{5}$

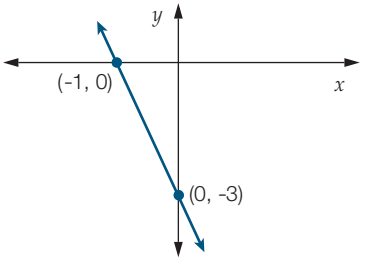
(b) The gradient of the line shown in the graph is:



A 8 **B** 0

C -7 **D** undefined

(c) The gradient of the line shown in the graph is:



A $\frac{1}{3}$ **B** $-\frac{1}{3}$

C 3 **D** -3

7 Write each of the equations given below in the general form of a linear equation $ax + by + c = 0$.

(a) $y - 3 = 5(x + 7)$ **(b)** $y + 5 = 6(x + 2)$ **(c)** $y + 3 = -4(x - 3)$

(d) $y = \frac{5}{3}x - 5$ **(e)** $y = \frac{1}{2}x + 8$ **(f)** $y = -\frac{10}{3}x + 18$

8 **(a)** Use the results from Question **5** to find a pair of points that lie on:

(i) a horizontal line

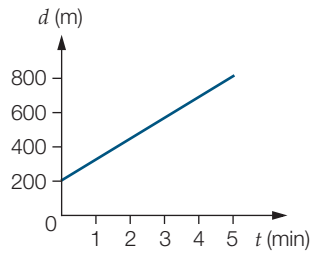
(ii) a vertical line

(iii) the line with the steepest negative gradient

(iv) the line with the smallest positive gradient.

(b) Use the results from Question **5** to find two pairs of points that would give two parallel non-horizontal lines.

9 Alex was 200 m from home and decided to walk to the local shop. This graph shows Alex's distance from home (d) at any particular time (t) on his 5-minute walk to the shop.



(a) How far was the shop from Alex's home?

(b) How far did Alex walk in 5 minutes?

(c) Find the gradient of this graph by first identifying two points that the line passes through.

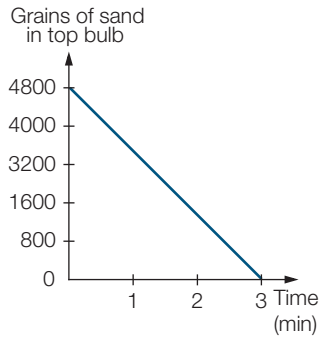
(d) Find Alex's speed on the walk.

(e) How far was Alex from home after 2 minutes of walking?

(f) Write down the relationship or rule to find the distance from home (d) after t minutes.

(g) What does the graph tell you about Alex's walking speed?

10 An egg-timer contains grains of sand which pour through the neck of the timer from the top bulb to the bottom in 3 minutes.



(a) How many grains of sand are in the egg-timer?

(b) How many grains of sand are left in the top bulb after 2 minutes?

(c) Find the gradient of this graph by identifying two points that the line passes through.

(d) Find the rate of change of sand grains with respect to time. In other words, what is the change in the number of sand grains left in the top bulb each minute?

(e) What does the negative represent in your answer to part **(c)**?

(f) Write down the relationship or rule to find the number of sand grains left in the top bulb (S) after t minutes.

(g) Use your rule to confirm your answer to part **(b)**.

(h) What are the restrictions on the values that t can take?

Reasoning, communicating and open-ended

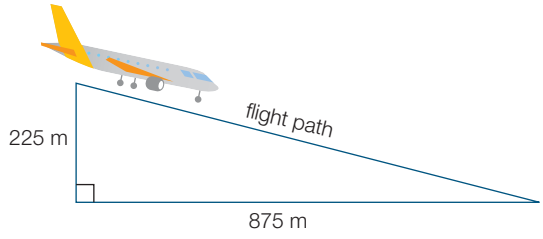
11 **(a)** Find the equation of the line that passes through (2, 4) and (4, 9):

(i) using the point (2, 4) as (x_1, y_1) **(ii)** using the point (4, 9) as (x_1, y_1)

(b) Did you get the same equation for parts **(i)** and **(ii)**? What conclusion can you make from your answers to parts **(i)** and **(ii)**?

12 The gradient of the line joining the points (2, 1) and (6, a) is 4. Find the value of a.

13 A passenger jet is landing on the flight path shown below. What is the gradient of the flight path?



Hint

Use the vertex of the right angle of the triangle as (0, 0).

14 Show that the points (-1, -5), (2, 4) and (5, 13) lie on the same straight line. This means that the points are collinear.

15 Identify three different sets of coordinates that when connected by a straight line produce each of the following gradients:

(a) -3 **(b)** -1 **(c)** $-\frac{1}{2}$ **(d)** 2 **(e)** 5

16 Write down the coordinates of two points such that a line joining them has the same gradient as the line joining (3, 5) and (7, -7).

4.5 Sketching linear graphs using gradient and y-intercept

Stage 5.2

Need to Know

Graphs of equations in gradient–intercept form

The equation of a linear graph can be written in the form:

$$y = mx + b$$

where m is the gradient and b is the y-coordinate of the y-intercept $(0, b)$.

To **sketch** a linear graph using the y-intercept and gradient, locate the point at which the graph crosses the vertical y-axis and then use the gradient to locate a second point. Once two points are known, a straight line can be drawn through them.

Linear graphs through the origin

The origin gives only one coordinate point (0, 0), so a second point is needed to be able to sketch the graph. A second point can be found by using the gradient. Taking the origin as the y-intercept, use the gradient to find a second point as for the gradient and y-intercept method.

Horizontal and vertical graphs

Other linear relationships between x and y can be of the form:

- $y = a$, where a is a constant, is the equation of a horizontal line parallel to the x-axis. The gradient of these lines is zero.
- $x = b$, where b is a constant, is the equation of a vertical line parallel to the y-axis. The gradient of these lines is undefined.

Worked Example 11

WE11

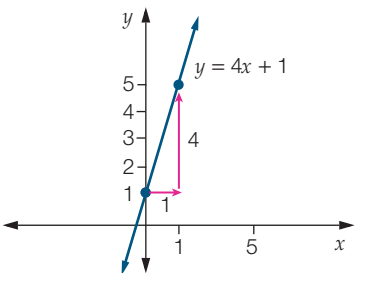
Find the y-intercept and the gradient, then sketch the graph of the equation $y = 4x + 1$.

Working

$y = mx + b$

$y = 4x + 1$
$b = 1$
y-intercept is $(0, 1)$.

$m = 4$
Gradient is 4, so the rise is 4 and the run is 1.



Worked Example 12

WE12

For the equation $3x + 2y = 4$, find the y-intercept and gradient and then sketch the graph.

Working

$y = mx + b$

$3x + 2y = 4$

$2y = -3x + 4$

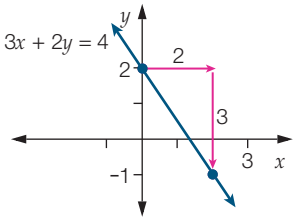
$y = \frac{-3x}{2} + 2$

$b = 2$

y-intercept is (0, 2).

$m = -\frac{3}{2}$

Gradient is $-\frac{3}{2}$, so the rise is -3 and the run is 2.



Worked Example 13

WE13

Sketch the graph of $y = 3x$, showing the y-intercept and one other point.

Working

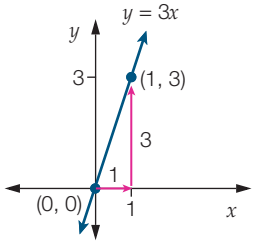
$y = mx + b$
$= 3x$

Gradient = 3,

so rise = 3 and run = 1.

$b = 0$

y-intercept = (0, 0)



Answers page 626

4.5 Sketching linear graphs using gradient and y-intercept

Stage 5.2

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 (a)–(h), Q7, Q8, Q9, Q15, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6 (c)–(j), Q7, Q8, Q9, Q11, Q12, Q14, Q15, Q16	Q1, Q2, Q3, Q4, Q5, Q6 (e)–(l), Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q16

Fluency and understanding

WE11

1 For each of the following equations, find the y-intercept and gradient and then sketch the graph.

(a) $y = 3x + 2$ **(b)** $y = 4x - 3$ **(c)** $y = -2x + 1$ **(d)** $y = -3x + 2$

WE12

2 Find the y-intercept and the gradient and then sketch a graph of each line.

(a) $x + 2y = -4$ **(b)** $x + 3y = 3$ **(c)** $3x + 4y = 8$

(d) $2x + 5y = -5$ **(e)** $2x - 3y = -1$ **(f)** $4x - 2y = 3$

WE13

3 Sketch graphs of the following, showing the intercept on the axes and one other point for each.

(a) $y = 4x$ **(b)** $y = x$ **(c)** $y = -2x$ **(d)** $y = -3x$

4 **(a)** The equation of a linear graph with a y-intercept of 2 and a gradient of -3 is:

A $y = 3x + 2$ **B** $x + 3y = 2$ **C** $x - 3y = 2$ **D** $3x + y = 2$

(b) The equation of a linear graph with a gradient of $\frac{1}{2}$ and a y-intercept of -3 is:

A $2x - y = 6$ **B** $x - 2y = 6$ **C** $x - 2y = -6$ **D** $2x - y = -6$

(c) The equation of a linear graph with a y-intercept of $\frac{2}{5}$ and a gradient of $-\frac{1}{4}$ is:

A $5x + 20y = 8$ **B** $20x + 5y = 8$ **C** $y = \frac{2x}{5} - \frac{1}{4}$ **D** $\frac{2x}{5} - y + \frac{1}{4} = 0$

5 When using the y-intercept and gradient to sketch a graph, first mark in the y-intercept and then use the gradient to locate a second coordinate point.

(a) A line has a gradient of $\frac{2}{3}$. From the y-intercept, you move:

A 2 units to the right and 3 units up **B** 2 units to the right and 3 units down

C 3 units to the right and 2 units up **D** 3 units to the right and 2 units down

(b) A line has a gradient of $-\frac{3}{4}$. From the y-intercept, you move:

A 4 units right and 3 units up **B** 3 units right and 4 units up

C 4 units right and 3 units down **D** 3 units left and 4 units down

(c) A line has a gradient of 4. From the y-intercept, you move:

A 4 units right and 4 units up **B** 1 unit right and 4 units up

C 4 units right and 4 units down **D** 1 unit right and 4 units down

6 Sketch graphs of the following, showing the intercepts on the axes.

(a) $y = 1$ (b) $y = 4$ (c) $y = -5$ (d) $y = -2$

(e) $y = 0$ (f) $y = \frac{1}{2}$ (g) $x = 2$ (h) $x = 5$

(i) $x = -3$ (j) $x = -4$ (k) $x = \frac{5}{2}$ (l) $x = 0$

7 (a) Two coordinate points that lie on the line $2x + y = 4$ are:

A (4, 0) and (0, 3) B (-4, 0) and (0, -2)

C (-2, 0) and (0, -4) D (2, 0) and (0, 4)

(b) Two coordinate points that lie on the line $y = 6x$ are:

A (0, 6) and (6, 0) B (0, 0) and (6, 6)

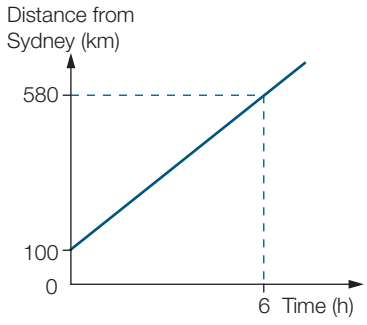
C (2, 12) and (0, 0) D (2, 0) and (0, 3)

(c) Two coordinate points that lie on the line $x = -3$ are:

A (0, 0) and (1, 3) B (-3, 1) and (1, -3)

C (-3, 0) and (0, 0) D (-3, -5) and (-3, 0)

8 (a) Consider the graph shown at right. Write down the intercept with the vertical axis.



(b) Find the gradient of this graph.

(c) Write an equation linking the distance from Sydney in kilometres (D) and the time in hours (T) in the form $y = mx + b$.

(d) What constant speed was maintained throughout the journey?

(e) If the journey started 250 km from Sydney and a constant speed of 100 km/h was maintained, what equation could be written to represent the relationship between D and T in the form $y = mx + b$.

(f) A particular trip from Sydney is represented by the equation $4D - 372T = 30$. What constant speed was maintained throughout the journey?

9 Darren has borrowed \$200 from his parents and has agreed to repay \$40 each week.

(a) After 3 weeks, how much does Darren still owe his parents?

(b) Write an equation linking the amount Darren owes his parents (A) with the number of weeks (n).

(c) Find the amount owed when $n = 0$.

(d) Find the number of weeks when $A = 0$.

(e) Use your answers from part **(c)** and part **(d)** to sketch the graph of this relationship. (Put A on the vertical axis, and n on the horizontal axis.)

Reasoning, communicating and open-ended

10 The annual yachting regatta of the Mathematical Club is to take place on Lake Tetrahedra. Starting and finishing at the marina, which can be considered to be coordinate point (0, 0), the yachts travel around six buoys in the order from 1 to 6, located at various positions on the lake. The first buoy is located at the point (2, 4). The second buoy is located at the point (-8, 8). The third buoy is located 10 units directly below the second buoy. The fourth buoy is located at the point (-3, 11). The fifth buoy is located 5 units to the right of the fourth buoy. The sixth buoy is located 4 units to the right of the first buoy.

(a) Plot each point on a Cartesian plane.

(b) Sketch the paths the yachts will take travelling to each buoy and back to the marina.

(c) Determine the equation of each path taken by the yachts between the:

(i) start of the race at the marina and the first buoy

(ii) first and second buoys

(iii) second and third buoys

(iv) third and fourth buoys

(v) fourth and fifth buoys

(vi) fifth and sixth buoys

(vii) sixth buoy and the finish of the race at the marina.

(d) The race referee's motor launch breaks down at the point (4, 7.5). Will this interfere with the paths taken by the yachts?

11 After her bath, Anne removes the plug to empty it. The water flows out of the bathtub at a rate of 0.75 L/s. There were originally 225 L of water in the bath.

(a) Write an equation relating the volume of water left in the bath (V) with the time in seconds (T).

(b) How long does it take to empty the bath?

12 Since the beginning of August, a local reservoir has been losing water at a constant rate. The Water Resources Department estimates that on 12 August the reservoir held 200 million kL of water, and on 21 August it held 164 million kL.

(a) Develop a linear equation relating V, the volume of water remaining (millions of kL), and n, the number of days since the beginning of August (when the reservoir was full).

(b) Draw a graph corresponding to the linear equation developed above to find the maximum capacity of the reservoir.

(c) Why should the graph only be drawn in the first quadrant?

13 A cubic block of ice has an edge of 10 cm. It melts so that its volume decreases at a constant rate (k) and the block remains a cube.

(a) What is the volume of the cube before it starts to melt?

(b) Write an equation relating (V) the volume of the cube after it has been melting for t minutes at a constant rate (k).

(c) What is the volume of the cube if it has an edge of 5 cm after 50 minutes?

(d) Find the value of k.

(e) What does the sign of k tell you about what is happening to the ice cube?

(f) Substitute the value of k found in part **(d)** to rewrite the equation found in part **(b)**.

(g) Draw a graph corresponding to the linear equation developed above to find how long it would take for the block of ice to completely melt.

Hint

The change in volume = kt.

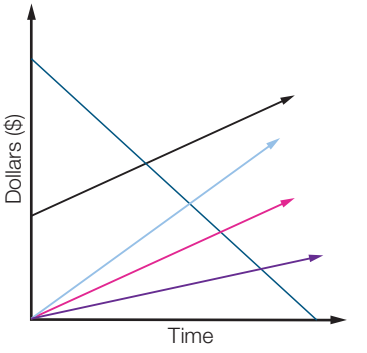
14 At the start of the year, Ringo had $600 in the bank. He has been saving nothing and spending $20 a week. In the same time, Sophia, his sister, had $100 in the bank. She has been saving $30 a week and spending nothing.

(a) Develop two linear equations relating the amounts of money that Ringo (R) and Sophia (S) each have in their bank accounts, to t, the number of weeks since the start of the year.

(b) Draw a graph corresponding to the linear equations to identify how many weeks it will take for Sophia to have twice the amount of money in the bank as her brother.

15 Write three different linear equations that each have the same gradient, but with a different y-intercept. What property of the three lines indicates that they have the same gradient? Explain.

16 Linear equations are used every day to model real-life situations. The graph shows five lines that model the costs of five different mobile phone plans. Match each graph with a description below and provide an explanation behind each choice.



Note:

- The specific values of gradients and y-intercepts are not needed for this activity.
- The connection fee (also referred to as flagfall) is the fixed cost when a call is made.
- The call rate is the number of cents per minute that is charged based on the duration of the call.

(1) This graph is for the mobile phone plan that has no connection costs and the cost of the call is dependent on the length of the call.

(2) This graph models the cost of the phone plan with no connection fee and the rate at which the call is charged is half the rate of phone plan graph (1).

(3) This graph models the amount of credit remaining on a pre-paid monthly mobile phone plan.

(4) This graph models the cost of a phone plan with a connection fee for every call made and the call rate is equivalent to (1).

(5) This graph models the cost of the phone plan with no connection fee as in (1) but the rate at which the call is charged is one and a half times greater.

17 Sketch three different graphs that have gradients between -1 and -2, and y-intercepts between 1 and 4.

Problem solving

Odd one out

Four of these points lie on a single straight line. Which one is the odd one out?

(2, 5) (-5, -7) (2, 4) (5, 13) (9, 21)

Strategy options

- Draw a diagram.
- Look for a pattern.
- Test all possible combinations.

Half-time 4

Stage 5.2 Q1–6 **Stage 5.3§ Q7–9**

Ex.4.5

1 For each of the following equations, find the y-intercept and gradient and then sketch the graph.

(a) $y = 3x + 2$ **(b)** $y = 5 - x$

(c) $y = 2x - 3$ **(d)** $y = 6 - 3x$

Ex.4.1

2 The Celsius temperature scale is used in Australia and most of the rest of the world, whereas the Fahrenheit temperature scale is used in the USA. In the Celsius scale, the freezing point of water is 0° and the boiling point is 100°. In the Fahrenheit scale, the freezing point of water is 32° and the boiling point is 212°. Show that the formula to convert a Fahrenheit temperature to a Celsius temperature can be expressed as $C = \frac{5}{9}(F - 32)$.

Ex.4.5

3 Sketch the graph of the following, showing the intercept with the axes and one other point.

(a) $y = -3x$ **(b)** $y = 4x$ **(c)** $x + 2y = 0$ **(d)** $3y = 5x$

Ex.4.1

4 Solve each of the following equations.

(a) $6x + 2 = 14$ **(b)** $3(x + 5) = 21$

(c) $\frac{2x - 3}{7} = 5$ **(d)** $3x + 4 = 16 - x$

(e) $\frac{5x}{6} - \frac{7x}{9} = 1$ **(f)** $\frac{2x + 4}{5} = \frac{3x - 3}{3}$

Ex.4.5

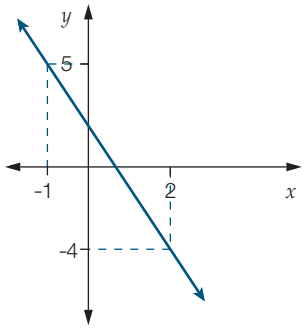
5 Sketch the graph of each of the following.

(a) $y = 4$ **(b)** $x = -2$ **(c)** $y = -3$ **(d)** $x = 8$

Ex.4.3

6 **(a)** Find the gradient of the linear graph shown.

(b) Find its equation.



Ex.4.2

7 Solve the following equations to find the value of the variable.

(a) $\frac{3x + 2}{2} - \frac{x + 10}{7} = 5$ **(b)** $\frac{3p - 5}{4} + \frac{6 - p}{3} = 2$

(c) $\frac{4m + 7}{3} - \frac{3m + 5}{8} = -5$ **(d)** $\frac{4k - 1}{3} - \frac{3 - 9k}{2} = 39$

Ex.4.4

8 Find the equation of the line with a:

(a) gradient of 4 passing through the point (-1, 5)

(b) gradient of -1 passing through the point (3, -7).

Ex.4.4

9 Calculate the gradient of the line passing through each pair of points, using $m = \frac{y_2 - y_1}{x_2 - x_1}$.

(a) (1, 3) and (5, 5) **(b)** (2, 4) and (6, -4)

(c) (-5, -3) and (5, 2) **(d)** (-4, 8) and (-2, -8)

Sketching linear graphs using x- and y-intercepts

4.6

Stage 5.3§

Need to Know

Linear equations are not always written in gradient-intercept form. The **general form** of a linear equation is an equation written in the form of $ax + by + c = 0$ where the gradient $= -\frac{a}{b}$ and the y-intercept $= -\frac{c}{b}$.

Graphs of equations in the general form

Linear graphs that are in the form of $ax + by + c = b$ can be sketched using x- and y-intercepts. Note that for a sketch graph, only the x- and y-intercepts need to be shown on the axes; no other values for x and y need be shown.

- To find the x-intercept, substitute $y = 0$ in the equation.
- To find the y-intercept, substitute $x = 0$ in the equation.

Worked Example 14

WE14

Find the x- and y-intercepts for $2x + 3y = 12$ and sketch the graph.

Working

When $y = 0$, $2x + 3 \times 0 = 12$

$2x = 12$

$x = 6$

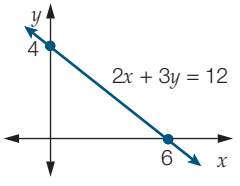
x-intercept is $(6, 0)$.

When $x = 0$, $2 \times 0 + 3y = 12$

$3y = 12$

$y = 4$

y-intercept is $(0, 4)$.



Answers page 629

4.6 Sketching linear graphs using x- and y-intercepts

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q7	Q1, Q2, Q3, Q4, Q5, Q7	Q1, Q2, Q3, Q4, Q5, Q6, Q7

Fluency and understanding

WE14

1 Find the x- and y-intercepts and then sketch a graph of each line.

(a) $2x + 3y - 6 = 0$ **(b)** $4x + y - 4 = 0$ **(c)** $x - 2y - 4 = 0$

(d) $x - 3y - 9 = 0$ **(e)** $4x + y - 2 = 0$ **(f)** $3x + y - 1 = 0$

2 The x- and y-intercepts of the graph of the equation $4x - 5y - 40 = 0$ are:

A (10, 0) and (0, 8) **B** (10, 0) and (0, -8)

C (-10, 0) and (0, 8) **D** (-10, 0) and (0, -8)

3 Sketch the graph of each of the following by finding x- and y-intercepts.

(a) $5x - 2y + 10 = 0$ **(b)** $6x + 5y + 30 = 0$ **(c)** $3x - 8y - 12 = 0$

Reasoning, communicating and open-ended

4 A rainwater tank holds 10 000 litres of water when full. It takes eight hours to empty.

(a) Write an equation in the general form to represent this situation. Let x represent the time and y represent capacity.

(b) Sketch a graph of the equation you obtained in part **(a)**.

(c) Use your graph to find the volume of water in the tank after it has been emptying for 5 hours.

5 A swimming pool of capacity 120 000 litres has a slow leak. It is estimated that unless the leak is found and repaired, the pool will be empty after 60 days.

(a) Write an equation in the general form to represent this situation. Let x represent the time and y represent capacity.

(b) Sketch a graph of the equation you obtained in part **(a)**.

(c) Find the rate at which the water is leaking in litres per minute.

(d) If the leak continued for 6 days, how much water would be lost?

6 A bank loan can be represented by the equation $16000x + y - 320\,000 = 0$ where x represents the length of time of the loan in years and y represents the amount of the loan to be repaid.

(a) Sketch the graph that represents this situation.

(b) How much money was borrowed initially?

(c) How much will still need to be repaid after 4 years?

(d) How long will it take to reduce the loan to \$100 000?

7 **(a)** Write a linear equation in general form that has a negative gradient and a y-intercept of (0, 4).

(b) Write a linear equation in general form that has a positive gradient and an x-intercept of (-3, 0).

Parallel and perpendicular lines

4.7

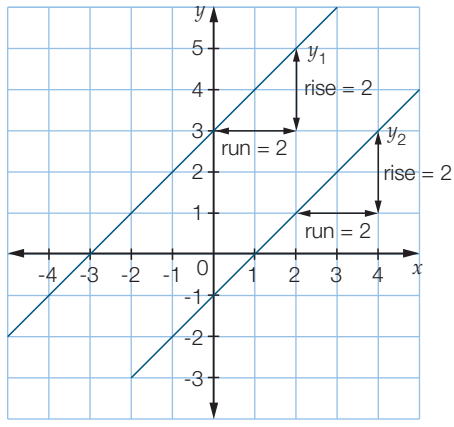
Need to Know

Stage 5.3§

Parallel lines

Two lines that are parallel never intersect as they have the same gradient.

$$m_1 = m_2$$



Perpendicular lines

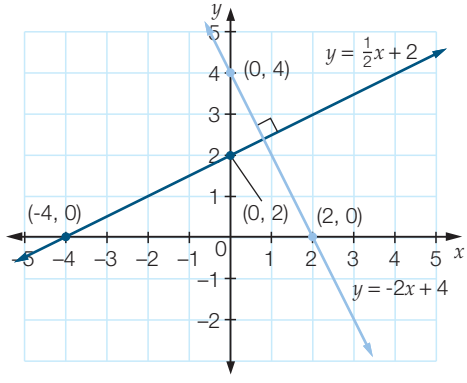
Rotating points on a line by 90° produces a line **perpendicular** to the original.

When lines are perpendicular, the product of their gradients is -1.

$$m_1 m_2 = -1$$

Therefore, the gradients m_1 and m_2 of perpendicular lines are the negative reciprocal of each other.

$$m_1 = -\frac{1}{m_2} \text{ and } m_2 = -\frac{1}{m_1}$$



Worked Example 15

WE15

Show that the line with equation $2y = 4x - 9$ is parallel to the line with equation $3y - 6x + 5 = 0$.

Working

$2y = 4x - 9$

$y = 2x - \frac{9}{2}$

$m_1 = 2$

$3y - 6x + 5 = 0$

$3y = 6x - 5$

$y = 2x - \frac{5}{3}$

$m_2 = 2$

$m_1 = m_2$

Therefore, the lines are parallel.

Worked Example 16 WE16

Show that the line with equation $y = 3x - 10$ is perpendicular to the line with equation $6y + 2x + 12 = 0$.

Working

$y = 3x - 10$

$m_1 = 3$

$6y + 2x + 12 = 0$

$6y = -2x - 12$

$y = -\frac{1}{3}x - 2$

$m_2 = -\frac{1}{3}$

$m_1m_2 = 3 \times -\frac{1}{3}$

$= -1$

Therefore, the lines are perpendicular.

Answers page 630

4.7 Parallel and perpendicular lines

Stage 5.3§

Navigator

Q1, Q2, Q3 Column 1, Q4 Column 1, Q5, Q6, Q7, Q8, Q9, Q10, Q17	Q1, Q2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q15, Q17	Q1, Q2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q8, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Fluency and understanding

WE15

1 (a) Show that the line with equation $5x - 2y = 7$ is parallel to the line with equation $2y = 5x + 11$.

(b) Show that the line with equation $10y = 5x + 3$ is parallel to the line with equation $3x - 6y = 2$.

WE16

2 (a) Show that the line with equation $4x + 3y = 7$ is perpendicular to the line with equation $8y = 6x - 2$.

(b) Show that the line with equation $2x - 5y - 1 = 0$ is perpendicular to the line with equation $4y + 10x = 3$.

3 Find the gradient of each of the following pairs of lines and determine if they are parallel.

(a) $y = 3x - 10$ and $y = 3x + 10$

(b) $y = 2x + 5$ and $y = -2x + 6$

(c) $y = \frac{1}{2}x - 4$ and $y = \frac{1}{2}x + 5$

(d) $y + 4x = 4$ and $y = 4x - 8$

(e) $y - 5x = 9$ and $y = 9x + 5$

(f) $y - 2x = 9$ and $y - 2x - 4 = 0$

(g) $2y + 6x - 4 = 0$ and $y + 3x - 8 = 0$

(h) $4y + 12x - 4 = 0$ and $3y + 12x - 9 = 0$

(i) $-2y + 8x - 4 = 0$ and $4y - 16x - 8 = 0$

(j) $5x + 3y = 15$ and $3x - 15y = 10$

4 Find the gradient of each of the following pairs of lines and determine if they are perpendicular.

(a) $y = 2x - 10$ and $2y + x + 12 = 0$

(b) $y - 4x = 10$ and $4y + 2x + 12 = 0$

(c) $y = \frac{1}{3}x$ and $6y + 2x + 12 = 0$

(d) $6y - 3x = 24$ and $2y + 4x + 12 = 0$

(e) $-4y + 3x = 16$ and $8y + 12x + 12 = 0$

(f) $5y - 4x = 25$ and $8y + 5x + 12 = 0$

5 Which of the following lines is parallel to $y = 4x + 12$?

A $y = -\frac{1}{4}x + 12$ B $4y = 4x + 12$ C $2y - 8x + 6 = 0$ D $2y + 8x + 12 = 0$

6 Which of the following lines is perpendicular to $y = 3x - 9$?

A $y = 3x + 9$ B $6y = 9x + 12$ C $y = -3x - 9$ D $9y + 3x = 27$

7 Show that the line joining (-2, 2) and (4, 8) is parallel to $y = x + 10$.

8 Show that the line joining (-2, 5) and (4, -7) is parallel to $2x + y - 10 = 0$.

9 Show that the line joining (-4, 1) to (0, 4) is perpendicular to $y = -\frac{4}{3}x - 2$.

10 Show that the line joining (4, 5) to (9, 8) is perpendicular to $y = -\frac{5}{3}x + 10$.

Reasoning, communicating and open-ended

11 (a) Which two of the following lines **(i)–(v)** are parallel?

(b) Which two of the following lines **(i)–(v)** are perpendicular?

(i) $y = 3x + 3$ (ii) $3x + y = 3$

(iii) $3x - y = 3$ (iv) $3x + 3y = 2$

(v) $3y = x + 3$

12 Find the equation of the line through (2, -3) that is perpendicular to $y = -\frac{1}{3}x + 11$.

13 Find the equations of the lines through (0, 5) that are parallel and perpendicular to $2x + y - 10 = 0$.

14 Show that the points $A(-1, -1)$, $B(1, -4)$ and $C(2, 1)$ form a right-angled triangle ABC.

15 Show that the points $A(-3, 1)$, $B(-1, 3)$, $C(3, 1)$ and $D(1, -1)$ form the vertices of a parallelogram.

16 Show that the points $A(-4, 3)$, $B(-2, -2)$, $C(3, 0)$ and $D(1, 5)$ form the vertices of a square.

17 Identify three different straight-line equations that are:

(a) parallel to $3x - 6y + 9 = 0$

(b) perpendicular to $4x - 12y - 18 = 0$.

Problem solving

Area of a triangle

Find the area (in square units) of the triangle formed by the three lines whose equations are:

1 $y - x - 6 = 0$

2 $x - 2y = 3$

3 $x = -y + 6$

Strategy options

- Draw a diagram.
- Break problem into manageable parts.

4.8 Linear inequalities

Stage 5.2

Need to Know

Inequality symbols

$<$ means less than

$>$ means greater than

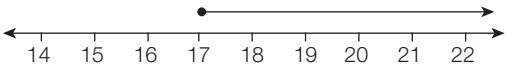
$\leq$ means less than or equal to

$\geq$ means greater than or equal to

Linear inequalities

An equation is a statement of equality, such as $x = 17$. A **linear inequality** is a statement such as $x \geq 17$, which means that x can be any value greater than or equal to 17.

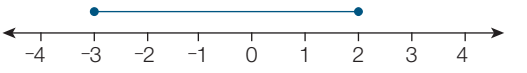
On a number line, $x \geq 17$ can be represented as:



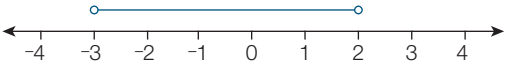
- If the inequality includes the end point, a closed circle • is used; these are inequalities that contain $\geq$ or $\leq$.
- If the inequality does not include the end point, an open circle ∘ is used; these are inequalities that contain $>$ or $<$.

For example:

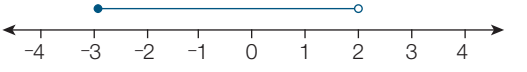
$-3 \leq x \leq 2$



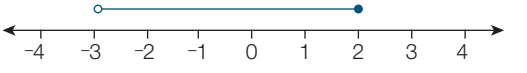
$-3 < x < 2$



$-3 \leq x < 2$



$-3 < x \leq 2$



Solving inequalities

We solve linear inequalities using the same inverse operations in the opposite order as we use to solve linear equations.

When we multiply or divide both sides of the inequality by a negative number, the inequality sign is reversed.

Worked Example 17

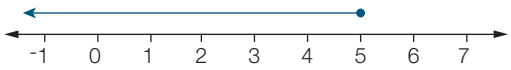
WE17

Represent the values of x on a number line for each of the following.

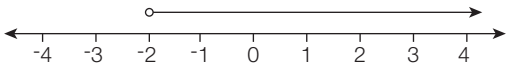
(a) $x \le 5$ **(b)** $x > -2$ **(c)** $3 < x \le 7$

Working

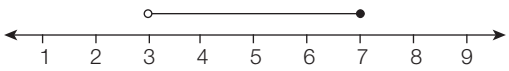
(a) $x \le 5$



(b) $x > -2$



(c) $3 < x \le 7$



Worked Example 18

WE18

Solve for x in each of the following inequalities.

(a) $x + 3 \ge 12$ **(b)** $4x - 3 < -7$

Working

(a)

$$x + 3 \ge 12$$
$$x + 3 - 3 \ge 12 - 3$$
$$x \ge 9$$

(b)

$$4x - 3 < -7$$
$$4x - 3 + 3 < -7 + 3$$
$$4x < -4$$
$$\frac{4x}{4} < \frac{-4}{4}$$
$$x < -1$$

Worked Example 19

WE19

Solve for x in each of the following.

(a) $-4x \le 20$ **(b)** $9 - 2x \ge 5$

Working

(a) $-4x \le 20$

$$\frac{-4x}{-4} \ge \frac{20}{-4}$$

(dividing by a negative number reverses the inequality symbol)

$$x \ge -5$$

(b) $9 - 2x - 9 \ge 5 - 9$

$$\frac{-2x}{-2} \le \frac{-4}{-2}$$

(dividing by a negative number reverses the inequality symbol)

$$x \le 2$$

Answers page 631

4.8 Linear inequalities

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q7, Q8, Q9, Q10 Column 1, Q11 Column 1, Q13, Q15	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8, Q9, Q10 Column 2, Q11 Column 2, Q13, Q14, Q16	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q8, Q9, Q10 Column 3, Q11 Column 3, Q12, Q13, Q14, Q15, Q16

Fluency and understanding

WE17

1 Represent the values of x on a number line for each of the following.

(a) $x \geq 5$ **(b)** $x \leq 8$ **(c)** $x \geq -2$

(d) $x < -4$ **(e)** $x > -5$ **(f)** $x > 7$

(g) $x \leq 0$ **(h)** $x \geq 2$ **(i)** $x \leq -5$

(j) $x > 1$ **(k)** $x > 0$ **(l)** $x < -2$

(m) $-2 \leq x \leq 4$ **(n)** $1 \leq x \leq 6$ **(o)** $0 < x < 5$

(p) $-3 < x < -1$ **(q)** $-4 \leq x < -2$ **(r)** $-4 < x \leq 0$

WE18

2 Solve for x in each of the following.

(a) $x + 6 > -4$ **(b)** $x - 3 < -8$ **(c)** $x + 12 \leq 2$

(d) $\frac{x}{5} \leq 7$ **(e)** $\frac{x}{9} \geq -1$ **(f)** $\frac{x}{7} > -4$

(g) $3x < -16$ **(h)** $5x > -27$ **(i)** $9x \leq -81$

(j) $7x - 3 \geq 53$ **(k)** $4x + 11 \leq 3$ **(l)** $3x + 24 > 0$

(m) $\frac{4x - 1}{5} > 3$ **(n)** $\frac{2x + 5}{3} < 9$ **(o)** $\frac{3x + 2}{7} \leq 2$

(p) $\frac{3x}{4} + 1 \leq -5$ **(q)** $\frac{7x}{11} - 2 > 5$ **(r)** $\frac{4x}{5} - 10 > -2$

WE19

3 Solve for y in each of the following.

(a) $-3y \geq 15$ **(b)** $-8y \leq 48$ **(c)** $-5y \leq 25$

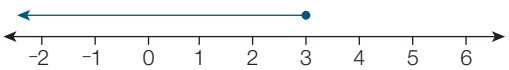
(d) $-y < -12$ **(e)** $-2y > -42$ **(f)** $-3y > -36$

(g) $1 - 2y \leq 17$ **(h)** $8 - 5y \geq -2$ **(i)** $5 - 3y \leq 17$

(j) $5(2 - 3y) \geq 11$ **(k)** $4(3 - 7y) \leq -2$ **(l)** $3(5 - 2y) \geq -4$

(m) $\frac{8 - 3y}{5} > -1$ **(n)** $\frac{14 - 6y}{3} < -3$ **(o)** $\frac{12 - 5y}{4} > -2$

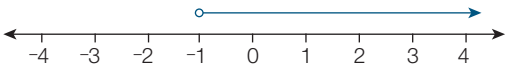
4 (a)



This number line represents:

A $x < 3$ **B** $x \leq 3$ **C** $x \geq 3$ **D** $x > 3$

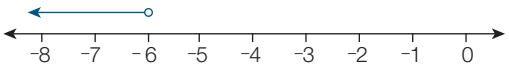
(b)



This number line represents:

A $x < -1$ **B** $x \leq -1$ **C** $x \geq -1$ **D** $x > -1$

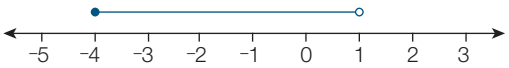
(c)



This number line represents:

A $x < -6$ **B** $x \leq -6$ **C** $x \geq -6$ **D** $x > -6$

(d)



This number line represents:

A $-4 < x \leq 1$ **B** $-4 \geq x > 1$ **C** $-4 \leq x \leq 1$ **D** $-4 \leq x < 1$

5 Which alternative is *not* a possible solution to the inequality given?

(a) $2x + 3 \geq 7$

A $x = 1$ **B** $x = 2.1$ **C** $x = 3$ **D** $x = 5.2$

(b) $5x - 2 < -9$

A $x = -5$ **B** $x = -3$ **C** $x = -2$ **D** $x = 0$

(c) $11 - 4x < -1$

A $x = 10$ **B** $x = 7$ **C** $x = 4$ **D** $x = 2$

(d) $\frac{6x+1}{2} > 5x - 2$

A $x = -2$ **B** $x = -1$ **C** $x = 1$ **D** $x = 2$

6 The reserve price at an auction of a house is \$270 000. The house is sold at auction.

(a) Represent the selling price (P) on a number line.

(b) Write down the linear inequality for the selling price (P).

(c) Write down the inequality for the selling price (P) given that the house sold for less than \$320 000.

Hint

The reserve price is the smallest amount that the house could be sold for.

7 The speed limit on a particular freeway is 100 km/h. Write down an inequality that represents all possible legal speeds on that freeway.

8 A sales representative needs to sell at least five vacuum cleaners a week to receive a bonus commission. Write down an inequality for the number of vacuum cleaners (n) the sales representative needs to sell if he wants to earn a bonus.

9 For the wool and other materials to make seven hand-knitted jumpers, Lucy and Lindsay spent \$385. Write down an inequality to show the selling price (j) for each jumper if they wish to make a profit.

10 Find the solutions for each of the following.

(a) $5x - 3 \geq 3 - 4x$ (b) $2x + 7 \leq 4 - 9x$ (c) $6x + 2 < 3x + 2$

(d) $2(x + 10) > 10 - 3x$ (e) $15x - 4 \leq 4(3x + 2)$ (f) $5(2x - 3) \geq 8x + 9$

(g) $\frac{2x+1}{7} > \frac{x-3}{4}$ (h) $\frac{3x-4}{2} < \frac{4x+7}{5}$ (i) $\frac{2-3x}{2} \leq \frac{5x-16}{3}$

Hint

When multiplying or dividing by a negative, the inequality sign reverses.

11 Find the solutions for each of the following.

(a) $y + 6 \leq 2y + 7$ **(b)** $20 - 3y > 5 + 2y$ **(c)** $7y - 4 \geq 3 + 10y$

(d) $13 - y < 5y + 11$ **(e)** $4 - \frac{y}{3} \leq -6$ **(f)** $7 - \frac{y}{6} \geq 1$

(g) $-3 - \frac{5y}{12} < 2$ **(h)** $\frac{-7y}{3} + 2 > 0$ **(i)** $\frac{3y + 2}{11} > \frac{2y - 5}{3}$

Reasoning, communicating and open-ended

12 A shopkeeper can fit a total of only 120 2-litre cartons of milk (both full cream and low fat) into the refrigerator. He knows that he needs at least 50 2-litre cartons of full cream milk and that no more than 40 2-litre cartons of low fat milk will be needed.

(a) Let f represent the number of cartons of full cream milk and l represent the number of cartons of low fat milk. Write down the two inequalities to show the number of cartons of each type of milk required by the shopkeeper and show them on a number line.

(b) Write down an inequality to show the number of cartons of milk in the refrigerator.

13 Tony has spent \$428 to buy 20 calculators wholesale.

(a) Write an inequality to represent the price (p) he could sell each calculator for and make a profit.

(b) Solve the inequality to find the minimum price that he could sell the calculators for and still make a profit.

14 At the local milk bar, Sue spends \$3.45 on milk and bread. With the change from her \$5 note, she buys some liquorice straps at 40c each.

(a) Write an inequality to represent the number of liquorice straps (n) she could buy.

(b) Solve the inequality to find the maximum number of liquorice straps she could buy.

15 Draw a number line showing numbers that could be greater than or equal to a negative number, but less than a positive number.

16 **(a)** Make up an inequality, with the variable on both sides of the inequality, that will produce solutions that are all negative.

(b) Make up an inequality, with the variable on both sides of the inequality, that will produce solutions that are all positive.

Hint

Let the number of wins at the start of the weekend be w. Given the win ratio was 0.5 or $\frac{1}{2}$, what is the relationship between the number of wins, w, and the total number of games, n?

Problem solving

Win ratio

A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 0.5 or $\frac{1}{2}$. Over the course of the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her overall win ratio is greater than 0.503. What is the greatest number of matches she could have won before the weekend began?

Strategy options

- Break problem into manageable parts.
- Test all possible combinations.

Simultaneous equations

4.9

Stage 5.2

Need to Know

Solutions of simultaneous equations

Any two linear equations containing two variables can graph as parallel lines, **coincident** lines or intersecting lines.

Parallel lines have the same gradient and never intersect, so equations that graph as parallel lines have no simultaneous solution.

Coincident lines are lines that have the same gradient but also contain all of the same points, so equations that graph as coincident lines have an infinite number of solutions.

Equations that give non-parallel and non-coincident graphs are called **unique equations** and have one solution.

Solving simultaneous equations

Simultaneous equations can be solved graphically, by elimination or by substitution.

Solving graphically

Graphs can be used to solve simultaneous equations because graphs of unique equations have only one point of intersection. This point satisfies both equations and its coordinates give the solution.

Solving by elimination

One of the variables is eliminated, then a value is found for the remaining variable. This value is then substituted back into one of the original equations to find the value for the second variable. These two values will give the solution to the equations. It doesn't matter which variable is eliminated when solving simultaneous equations, as there is only one solution.

Strategies for eliminating a variable

Multiply one equation by an amount or both equations by different amounts to obtain a coefficient for one of the variables in each equation that is equal in magnitude but opposite in sign. The equations can now be added to eliminate one of the variables. If the coefficients are equal in magnitude and have the same sign, the equations can be subtracted.

- If only one equation needs to be multiplied, eliminate that variable.
- If the signs in front of one of the variables are opposite, eliminate that variable.
- If the signs in front of each of the variables are the same, eliminate the variable where the coefficients are smallest in magnitude. Multiplying one equation by a negative number and then adding the equations is often easier than multiplying by a positive number and then subtracting.

Solving by substitution

This method is commonly used when one or both equations are written with one of the variables as the subject of the equation.

For example:

$y = 3x + 5$ in gradient-intercept form or $x = 4 - 3y$

One variable is replaced in the second equation, so a value can be found for the remaining variable. This value can then be used to find the solution.

Solving problems using simultaneous equations

Step 1 Identify the unknowns. Define each unknown with a different variable.

Step 2 Express the information given as equations involving the unknowns. (If there are two unknowns, i.e. two pieces of information required, there must be two unique equations.)

Step 3 Solve the equations simultaneously using elimination, substitution or graphical means.

Step 4 Check your solution by substitution.

Step 5 Express the final answer to the problem in words.

Worked Example 20

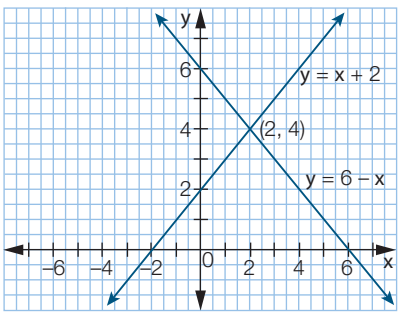
WE20

(a) Sketch the pair of linear equations $y = x + 2$ and $y = 6 - x$ and find the point of intersection.

(b) Hence, find the solution to $y = x + 2$ and $y = 6 - x$.

Working

(a)



Point of intersection is (2, 4)

(b) $x = 2, y = 4$

Check: $y = x + 2$ $\quad$ $y = 6 - x$

$4 = 2 + 2$ $\quad$ $4 = 6 - 2$

$4 = 4$ ✔ $\quad$ $4 = 4$ ✔

Worked Example 21

WE 21

Solve the following simultaneous equations algebraically using the elimination method.

(a) $3x + 5y = -3$
$x - 5y = 19$

(b) $4x + 5y = -6$
$4x - 3y = 10$

Working

(a)

$3x + 5y = -3$ (1)
$x - 5y = 19$ (2)

$4x = 16$ (1) + (2)
$x = 4$

$3x + 5y = -3$ (1)
$3 \times 4 + 5y = -3$
$12 + 5y = -3$
$5y = -15$
$y = -3$

$x = 4$, $y = -3$ or $(x, y) = (4, -3)$

Check: Use $x - 5y = 19$

$\text{LHS} = x - 5y$
$= 4 - 5 \times -3$
$= 4 + 15$
$= 19$
$= \text{RHS}$

(b)

$4x + 5y = -6$ (1)
$4x - 3y = 10$ (2)

$8y = -16$ (1) – (2)
$y = -2$

$4x + 5y = -6$ (1)
$4x + 5 \times -2 = -6$
$4x - 10 = -6$
$4x = 4$
$x = 1$

$x = 1$, $y = -2$ or $(x, y) = (1, -2)$

Check: Use $4x - 3y = 10$

$\text{LHS} = 4x - 3y$
$= 4 \times 1 - 3 \times -2$
$= 4 + 6$
$= 10$
$= \text{RHS}$

Worked Example 22

WE22

Solve the following simultaneous equations algebraically using the elimination method.

(a) $3x + 2y = 5$
$4x - 3y = -16$

(b) $2x + 5y = 9$
$x + 3y - 5 = 0$

Working

(a)

$3x + 2y = 5$ (1)
$4x - 3y = -16$ (2)
$9x + 6y = 15$ (3) (1) × 3
$8x - 6y = -32$ (4) (2) × 2
$17x = -17$ (3) + (4)
$x = -1$

$3x + 2y = 5$ (1)
$3 \times -1 + 2y = 5$
$-3 + 2y = 5$
$2y = 8$
$y = 4$

$x = -1, y = 4$ or $(x, y) = (-1, 4)$

Check: Use $4x - 3y = -16$

LHS $= 4x - 3y$
$= 4 \times -1 - 3 \times 4$
$= -4 - 12$
$= -16$
$=$ RHS

(b)

$2x + 5y = 9$ (1)
$x + 3y - 5 = 0$ (2)
$x + 3y = 5$ (3)
$2x + 6y = 10$ (4) (3) × 2
$2x + 5y = 9$ (1)
$y = 1$ (4) − (1)

$x + 3y - 5 = 0$ (2)
$x + 3 \times 1 - 5 = 0$
$x - 2 = 0$
$x = 2$

$x = 2, y = 1$ or $(x, y) = (2, 1)$

Check: Use $2x + 5y = 9$

LHS $= 2x + 5y$
$= 2 \times 2 + 5 \times 1$
$= 4 + 5$
$= 9$
$=$ RHS

Worked Example 23

WE 23

Solve the following simultaneous equations algebraically using the substitution method.

(a) $y = 2x$
$x + y = 18$

(b) $2x + 3y = -4$
$x = y + 3$

Working

(a)

$y = 2x$ (1)
$x + y = 18$ (2)

Substituting $2x$ for y in (2):

$x + (2x) = 18$ (3)
$3x = 18$
$x = 6$

$y = 2x$ (1)
$= 2 \times 6$
$= 12$

$x = 6, y = 12$ or $(x, y) = (6, 12)$

Check: Use $x + y = 18$

$\text{LHS} = x + y$
$= 6 + 12$
$= 18$
$= \text{RHS}$

(b)

$2x + 3y = -4$ (1)
$x = y + 3$ (2)

Substituting $y + 3$ for x in (1):

$2(y + 3) + 3y = -4$ (3)
$2y + 6 + 3y = -4$
$5y + 6 = -4$
$5y = -10$
$y = -2$

$x = y + 3$ (2)
$= -2 + 3$
$= 1$

$x = 1, y = -2$ or $(x, y) = (1, -2)$

Check: Use $2x + 3y = -4$

$\text{LHS} = 2x + 3y$
$= 2 \times 1 + 3 \times -2$
$= 2 - 6$
$= -4$
$= \text{RHS}$

4.9 Simultaneous equations

Answers page 632

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q19	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q19	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5, Q6, Q8, Q9, Q10, Q13, Q14, Q15, Q16, Q17, Q18, Q20, Q21

Equipment required: graph paper for Question **1**

Fluency and understanding

WE 20

1 **(i)** Sketch each pair of linear equations on graph paper and find the point of intersection.

(ii) Hence, find the solution for each pair of simultaneous equations.

(a) $y = x + 3$
$y = 5 - x$

(b) $y = x - 7$
$y = 3 - x$

(c) $x - y = 6$
$2x - y = 8$

(d) $3x - y = 6$
$x - 2y = 2$

(e) $y = 2x$
$y = 3x - 4$

(f) $y = 6 - 2x$
$y = 4x$

(g) $x = -2$
$y = 3x$

(h) $y = -x$
$y = 3$

(i) $y = x + 2$
$y = x + 4$

WE21

2 Solve the following pairs of simultaneous equations using the elimination method.

(a) $x + y = 5$
$x - y = 3$

(b) $x + y = 1$
$x - y = 9$

(c) $2x + 3y = 9$
$5x - 3y = 12$

(d) $3x + 2y = 13$
$9x - 2y = 47$

(e) $3x + 2y = 9$
$-3x + y = 0$

(f) $7x + y = 8$
$-x + y = -8$

(g) $-2x + y = 7$
$2x + 3y = 5$

(h) $2x - 3y = -2$
$5x - 3y = 4$

(i) $-3x + y = -3$
$-3x - 2y = -12$

WE22

3 Solve the following pairs of simultaneous equations using the elimination method.

(a) $3x + 2y = 1$
$-2x + y = -3$

(b) $5x + 2y = -7$
$x + 7y = 25$

(c) $2x + 6y = 4$
$4x + 3y = -1$

(d) $2x + 5y = -1$
$4x + 3y = 5$

(e) $3x + 2y - 1 = 0$
$-2x + y + 3 = 0$

(f) $4x + 3y + 9 = 0$
$2x - 5y + 11 = 0$

(g) $3x - 2y = 7$
$2x - 3y = 3$

(h) $5x - 2y = -23$
$2x + 7y = 22$

(i) $4x + 2y - 6 = 0$
$-3x - 3y + 6 = 0$

(j) $\frac{x}{2} + y = 0$
$x - 3y = 10$

(k) $x + \frac{y}{4} = 8$
$2x - y = -2$

(l) $5x - 6y = 21$
$\frac{x}{3} + \frac{y}{2} = 5$

(m) $7x - y = -25$
$\frac{2x}{5} - \frac{y}{2} = 3$

(n) $\frac{2x}{3} + y + 10 = 0$
$4x - \frac{y}{4} + 10 = 0$

(o) $\frac{5x}{6} + \frac{3y}{5} + 2 = 0$
$7x + 3y + 27 = 0$

WE23

4 Solve the following pairs of simultaneous equations using the substitution method.

(a) $y = 2x$
$x + 5y = 22$

(b) $x + 3y = -30$
$y = 3x$

(c) $x = -2y$
$y = 3x - 14$

(d) $2x + y = 7$
$x = 6 - y$

(e) $x = y + 1$
$3x - y = 7$

(f) $y = 7 - 3x$
$4x - 3y = 18$

(g) $y = 1 - 2x$
$5x + 2y = 0$

(h) $x = 2 - 7y$
$2y = x - 11$

(i) $y = 6 - 3x$
$x = 2y - 19$

5 **(a)** For simultaneous equations $y = 2x - 5$ and $y = 3 - 2x$, the solution is:

A (0, -5) **B** (0, 3) **C** (2, -1) **D** (2, 1)

(b) For simultaneous equations $x + 2y = 8$ and $3x - y = 3$, the solution is:

A (6, 1) **B** (3, 6) **C** (4, 2) **D** (2, 3)

6 **(a)** Adding which pair of equations below will eliminate y?

A $x + y = 4$
$2x + y = 3$

B $3x + 2y = 6$
$-3x + y = 0$

C $x - y = 7$
$3x - y = 7$

D $2x - 3y = 4$
$x + 3y = 2$

(b) Adding which pair of equations in part **(a)** will eliminate x?

7 **(a)** $5x - 6y = 4$ (1)
$-3x + 8y = 2$ (2)

To eliminate x in the pair of equations above, we need to:

A multiply (1) by 3 and add to 5 times (2)

B multiply (2) by 3 and add to (1)

C multiply (1) by -3 and add to 5 times (2)

D multiply (1) by -3 and add to 5 times (2)

(b) $9x + 7y = 3$ (1)
$5x + 3y = -1$ (2)

To eliminate y in the pair of equations above, we need to obtain a new equation formed by:

A $3 \times (1) + 7 \times (2)$ **B** $5 \times (1) + 9 \times (2)$ **C** $3 \times (1) - 7 \times (2)$ **D** $5 \times (1) - 9 \times (2)$

8 **(a)** Simultaneous equations $y = x + 3$ and $2x - y = -11$ have the solution:

A $x = 3, y = 6$ **B** $x = -5, y = 1$ **C** $x = -5, y = -8$ **D** $x = -8, y = -5$

(b) Simultaneous equations $x = 2y$ and $4x - 7y - 3 = 0$ have the solution:

A (-3, -6) **B** (2, 4) **C** (3, 6) **D** (6, 3)

9 The fruit below cost $14.05.

The fruit below cost $8.50.

How much did each pineapple and each kiwifruit cost? Assume that all pineapples are the same price and all kiwifruit are the same price.

10 Three salad rolls and four pies cost $20.56 altogether. A salad roll costs $1.16 more than a pie. How much does each cost?

11 The sum of x and y is 12. When y is subtracted from $3x$ the result is 16. Find x and y.

12 The sum of two numbers is 38 and their difference is 24. What are the numbers?

Reasoning, communicating and open-ended

13 Two books have a total of 500 pages. If the number of pages in one book is 12 pages less than 3 times the number of pages in the other, how many pages are there in each book?

14 A farmer uses 140 m of fencing wire to enclose a rectangular paddock. If the length of the paddock is 10 m longer than the breadth, find the dimensions of the paddock.

15 After buying a book, Libby received 18 coins totalling $1.65 in change. If she was given only 5- and 10-cent coins, how many of each did she receive?

16 Five less than twice Michelle's present age is equal to Yesmin's present age. Two years ago, twice Yesmin's age was the same as 10 years more than triple Michelle's age. Find the present ages of the two girls.

17 Wes and Tony live in two cities 360 km apart. They leave their houses at the same time by car and travel towards each other at 80 km/h and 100 km/h, respectively.

(a) Write an equation for each journey representing the distance from Wes's house (D) at any time (t).

(b) How much time will pass before they meet on the highway?

18 It takes a barge 3 hours to travel 27 km upstream against the current. On its return journey, with the current, the barge can travel the same distance downstream and an additional 3 km in only 2 hours. Find the speed of the boat and the current.

19 The pirate Captain Greybeard left careful instructions for his crew to locate his buried treasure. Using the old oak tree as the origin, north–south as the y-axis, east–west as the x-axis and each unit on the number plane as 20 paces, the solution of four different pairs of simultaneous equations would lead to the burial site.

The first point, A, is at the intersection of the two lines $x = -3$ and $y = -7$. The second point, B, is at the intersection of $2x - y = 7$ and $2x - 3y = -3$. The third point, C, is at the intersection of $2y + x = 0$ and $2y = x - 8$. The final point, D, is at the intersection of $y = 7x$ and $x + y = 8$.

(a) Find the coordinates of each of the points A, B, C and D.

(b) Plot the points A, B, C and D on the same number plane and join AB and CD.

(c) Find the coordinates of the intersection of the two lines, which will give the location of the buried treasure.

(d) Determine the equations of the lines AB and CD and solve algebraically to show that the coordinates found in **(c)** are the solution of these equations.

20 Make up three different pairs of simultaneous equations, each having the same solution, where $x = 3$ and $y = 4$.

21 Emmanuel has three numbers. When he adds them together in pairs he gets three different totals: 11, 17 and 22. What are the three numbers Emmanuel had to start with? Can you describe a method that would enable you to work out the three numbers given any three totals?

Chapter review 4

D.I.Y. Summary

Key Words

Stage 5.2	coincident	linear equation	simultaneous equations
	dependent variable	linear graph	sketch
	gradient	linear inequality	unique equations
	independent variable	linear relationships	
Stage 5.3§	general form		

Stage 5.2 Q1–7

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1. A _____ indicates how two quantities that change at a constant rate are related to one another.
2. A _____ can be solved by performing inverse operations on both sides of the equation.
3. The measure of a line's steepness is called the _____.
4. _____ show that the relationship between two variables is a straight line.
5. For a _____, only two points need to be shown.
6. Two linear equations that contain the same variables and are not parallel or coincident can be solved using _____.
7. $3x + 4y = 7$ and $8y = 14 - 6x$ are examples of _____.

Stage 5.2 Q1, 3, 6, 10–14, 16, 19, 20, 22, 23 **Stage 5.3§ Q2, 4, 5, 7–9, 15, 17, 18, 21**

Fluency and understanding

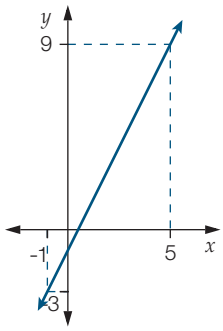
1. Solve the following equations. **Ex.4.1**

   (a) $2c - 7 = 9$ (b) $\frac{2x-3}{4} + 6 = -1$ (c) $4v - 5 = v + 4$

2. Solve the following equations. **Ex.4.2**

   (a) $\frac{3a+1}{2} - \frac{2a+3}{5} = 1$ (b) $\frac{8k-1}{7} - \frac{4k-3}{5} = -4$

3. Find the equation of the graph shown. **Ex.4.3**



4 The gradient of the line joining the points (2, 9) and (5, -6) is: Ex.4.4

A 1 **B** 5 **C** $-\frac{1}{5}$ **D** -5

5 Find the gradient and y-intercept for each of the following. Ex.4.4

(a) $y = 5x + 2$ **(b)** $y = -2x$ **(c)** $2x - 5y = 10$ **(d)** $4x + y = 8$

6 Graph the following lines by finding the y-intercept and gradient. Ex.4.5

(a) $y = 2x - 5$ **(b)** $y = -2x$ **(c)** $x + 5y = 10$ **(d)** $y = 4x$

7 Sketch graphs of the following, by finding the x- and y-intercepts on the axes. Ex.4.6

(a) $4x - 5y - 20 = 0$ **(b)** $3x + 7y - 42 = 0$ **(c)** $3 - 5x - y = 0$ **(d)** $-2x + 6 - y = 0$

8 Show whether or not each pair of lines is parallel, using their gradients. Ex.4.7

(a) $x + 2y = 7$ and $x + 2y = 3$ **(b)** $x - y + 6 = 0$ and $3x - 2y + 4 = 0$

(c) $4x + y = 6$ and $4x + y + 6 = 0$ **(d)** $y = 4x - 3$ and $8x - 2y = 0$

9 Show whether or not each pair of lines is perpendicular, using their gradients. Ex.4.7

(a) $2x + 2y = 4$ and $2x + y = 8$ **(b)** $x - y + 9 = 0$ and $2x + 2y - 3 = 0$

(c) $2x - 4y = 9$ and $4x + 2y + 6 = 7$ **(d)** $3x + 2y = 8$ and $3y = 2x + 4$

10 Find the solution set for the following linear inequalities. Ex.4.8

(a) $9x \geq 21$ **(b)** $\frac{3x}{5} + 2 > -5$ **(c)** $7 - 6x < 2$

11 Solve the following simultaneous equations using the elimination method. Ex.4.9

$5x - 2y = 8$ $3x - 4y = -12$

12 Solve the following simultaneous equations using the substitution method. Ex.4.9

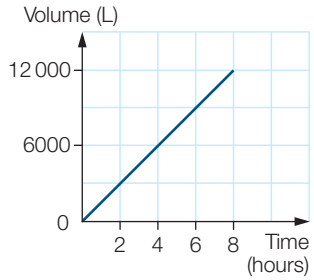
$6x - 5y = -21$ $y = 2x + 5$

13 In an enclosure at the zoo, there are twice as many wombats as emus. Altogether, they have a total of 90 legs. How many wombats are there?

14 To earn extra money, Martin works part-time in a butcher's shop. In a particular week during the school holidays he worked twice as many hours on Tuesday as he did on Monday but only $\frac{3}{4}$ the hours on Wednesday compared to Monday. For these 3 days he received \$127.50. If he is paid \$8.50 per hour, how many hours did Martin work on Monday?

15 A swimming pool has been filled with water.

The graph shows the volume of water (in litres) in the pool after a number of hours as it filled.



(a) How long did it take for the pool to contain 12 000 L of water?

(b) How many litres of water flowed into the pool each hour?

(c) What is the gradient of the graph?

(d) Write down the relationship for the volume (V) in terms of the time (T).

16 Manny, a motorcycle courier, notes his distance from the centre of Sydney on a delivery trip:

Time (min)	0	10	20	30	40	50	60
Distance from Sydney (km)	20	26	32	38	44	50	56

(a) Find an equation describing Manny's distance from Sydney (d km) in terms of the time spent travelling (t minutes).

(b) Draw a graph of Manny's distance from Sydney as a function of the time spent travelling.

(c) What does the gradient of the graph represent?

(d) What does the intercept with the vertical d-axis represent?

(e) Find Manny's distance from Sydney after he has been travelling for 14 minutes.

(f) How long does it take for Manny to be 48 km from Sydney?

17 What is the equation of the line that passes through (6, 3) and is perpendicular to $y - x - 1 = 0$?

18 What is the equation of the line that passes through (8, -2) and is parallel to $y - 5x = -1$?

19 Find the solution set for the following linear inequalities.

(a) $\frac{3x-4}{2} < \frac{2x+1}{3}$ **(b)** $3(2x - 4) \geq 4(3x + 1)$

20 Twice one number added to three times another number equals -3. If the first number is tripled then subtracted from 8 and the result is divided by 2, the second number is obtained. Find the two numbers.

21 Find the values of a and b in the following equations given that the lines are perpendicular and intersect on the y-axis.

$ax + 3y = 1$ $by = -x - 1$

Reasoning and communicating

22 Houses for sale at auction often have an EPR (Expected Price Range) in the advertisement. This means that it is expected that the house will be sold for a price somewhere between the values given.

(a) If the EPR for a house is advertised as \$570 000–\$620 000, write this as an inequality for the selling price of the house (p).

(b) If the deposit must be 10% of the price paid, write an inequality to show how much deposit is needed if the price falls within the EPR.

(c) If the house sold for more than \$50 000 above the highest value in the EPR, write an inequality to show how much the house sold for.

(d) If the house sold for less than \$50 000 above the highest value in the EPR, write an inequality to show how much the house sold for.

23 Three times the distance (d) between two towns subtracted from 823 km would take 4 hours travelling at an average speed (s). However, five times the same distance subtracted from 1071 km would only take 3 hours travelling at the same average speed. How far is the distance (d) between the two towns and what is the average speed (s)?

5 Trigonometry

P Recall 5

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Solve these equations for x. Express your answer as a decimal, correct to the number of decimal places (d.p.) given in brackets next to the equation.

(a) $3x = 24.5$ (2 d.p.)

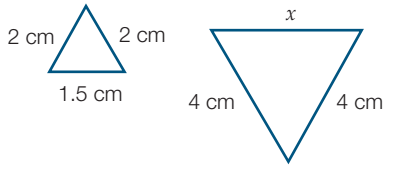
(b) $\frac{x}{20} = 0.7825$ (1 d.p.)

(c) $\frac{12}{x} = 2.307\,02$ (2 d.p.)

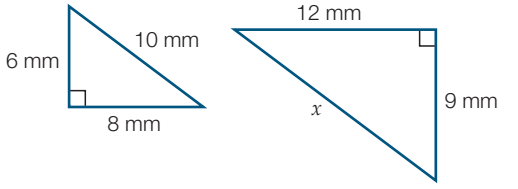
(d) $0.345 = \frac{16}{x}$ (3 d.p.)

2 Determine the length of the missing side x in each pair of similar triangles.

(a)

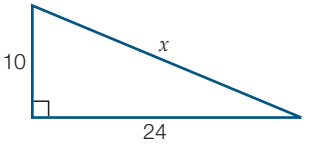


(b)

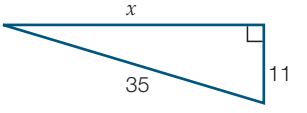


3 Use Pythagoras' Theorem to calculate the value of x in each triangle. Round your answer to two decimal places where necessary.

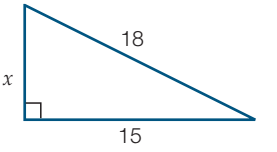
(a)



(b)

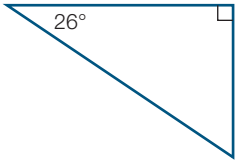


(c)

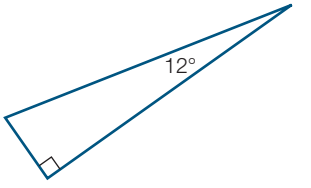


4 Copy each right-angled triangle shown below and label the hypotenuse, the opposite side and the adjacent side for the angle given.

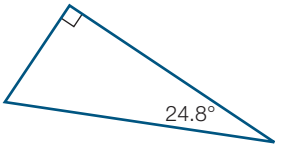
(a)



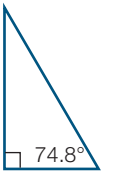
(b)



(c)



(d)



Syllabus references

Number and Algebra: Equations

✓ Substitute values into formulas to determine an unknown **(ACMNA234) (Stage 5.2)**

Measurement and Geometry: Right-Angled Triangles (Trigonometry)

✓ Apply trigonometry to solve right-angled triangle problems **(ACMMG224) (Stage 5.2◊)**

✓ Solve right-angled triangle problems, including those involving direction and angles of elevation and depression **(ACMMG245) (Stage 5.2◊)**

The trigonometric ratios

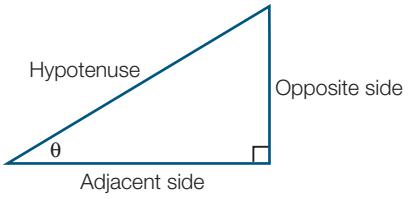
5.1

Stage 5.2◊

Need to Know

Side names

The **hypotenuse** is always opposite the right angle. The side directly opposite the angle θ is the **opposite side** and the side next to θ is the **adjacent side**.



Trigonometric ratios

Pairs of side lengths in right-angled triangles can be used to calculate three **trigonometric ratios**. The ratios are written in fraction forms.

The ratio $\frac{\text{Opposite}}{\text{Hypotenuse}}$ is referred to as the **sine** of θ and abbreviated to sin θ.

The ratio $\frac{\text{Adjacent}}{\text{Hypotenuse}}$ is referred to as the **cosine** of θ and abbreviated to cos θ.

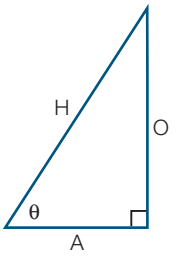
The ratio $\frac{\text{Opposite}}{\text{Adjacent}}$ is referred to as the **tangent** of θ and abbreviated to tan θ.

In a right-angled triangle:

$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$ or $\frac{\text{O}}{\text{H}}$

$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ or $\frac{\text{A}}{\text{H}}$

$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ or $\frac{\text{O}}{\text{A}}$



The phrase 'SOH-CAH-TOA' can help you remember which two sides belong to each ratio.

Angles in degrees, minutes and seconds

The units used to measure angles are degrees (°). There are 360° in one revolution. One degree can be broken up into smaller units called **minutes** (′) and **seconds** (″). There are 60 seconds in a minute and 60 minutes in a degree.

That is, 60′ = 1° and 60″ = 1′ so there are 3600″ in 1°.

An angle of 24°30′ is equal to 24.5°, because $\frac{30'}{60'} = 0.5$.

An angle of 57°45′18″ is equal to 57.755°, because $\frac{45'}{60'} = 0.75$ and $\frac{18'}{3600'} = 0.005$.

Make sure that your calculator is in 'degree' mode for the calculations in this chapter.

Technology Exploration: Calculator

Degrees, minutes and seconds on the calculator

Most scientific calculators have a key marked [° ′ ″] or [D°M′S], which converts angles in degrees, minutes and seconds to decimal degrees (a degree measurement with a decimal part).

To change 24°30′ into decimal degrees using a scientific calculator, press [2] [4] [° ′ ″] [3] [0] [° ′ ″] [=].

The screen will show 24.5.

Pressing [° ′ ″] will convert between the form of degrees, minutes and seconds and the form of decimal degrees.

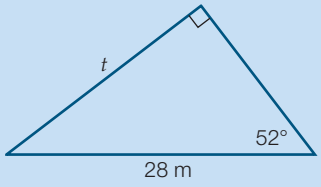
If your calculator works differently, consult its instructions.

Worked Example 1

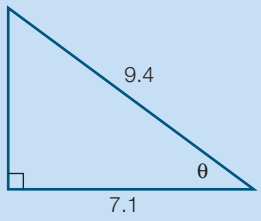
WE1

Label the Hypotenuse, Opposite and Adjacent sides on these right-angled triangles and hence decide which trigonometric ratio we could use to find the unknown.

(a)

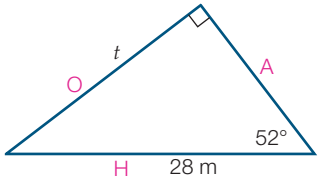


(b)



Working

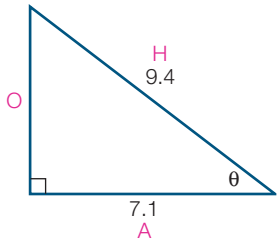
(a)



$$\sin\theta = \frac{O}{H}$$

$$\sin 52° = \frac{t}{28}$$

(b)



$$\cos\theta = \frac{A}{H}$$

$$\cos\theta = \frac{7.1}{9.4}$$

Worked Example 2

WE 2

Use a calculator to find each of the following values, rounded to four decimal places.

(a) sin 29° **(b)** cos 58° **(c)** tan 60.5°

Working

(a) sin 29° = 0.4848 (4 d.p.)

(Press the **sin** key, followed by the value of the angle, then '='.)

(b) cos 58° = 0.5299 (4 d.p.)

(Press the **cos** key, followed by the value of the angle, then '='.)

(c) tan 60.5° = 1.7675 (4 d.p.)

(Press the **tan** key, followed by the value of the angle, then '='.)

Worked Example 3

WE 3

Convert the following angles to decimal degrees, rounded to two decimal places.

(a) 56°24′ **(b)** 23°16′45″

Working

(a) 56°24′ = 56.40°

(Press the following keys:

5 **6** **° ′ ″** **2** **4** **° ′ ″** **=** **° ′ ″**)

(b) 23°16′45″ = 23.28° (2 d.p.)

(Press the following keys:

2 **3** **° ′ ″** **1** **6** **° ′ ″** **4** **5** **° ′ ″** **=** **° ′ ″**)

Worked Example 4

WE 4

Use a calculator to find each of the following values, rounded to four decimal places.

(a) sin 46°36′ **(b)** cos 6°12′34″

Working

(a) sin 46°36′ = 0.7266 (4 d.p.)

(Press the following keys:

sin **4** **6** **° ′ ″** **3** **6** **° ′ ″** **=**)

(b) cos 6°12′34″ = 0.9941 (4 d.p.)

(Press the following keys:

cos **6** **° ′ ″** **1** **2** **° ′ ″** **3** **4** **° ′ ″** **=**)

Answers page 634

5.1 The trigonometric ratios

Stage 5.2◊

Navigator

Q1, Q2 Columns 1 & 2, Q3, Q4, Q5, Q6, Q10, Q11, Q14, Q15, Q16	Q1, Q2 Column 2, Q3, Q4, Q6, Q7, Q8, Q10, Q11, Q13, Q14, Q15, Q16, Q17	Q1, Q2 Column 3, Q3 (d)–(f), Q4 (d)–(f), Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17

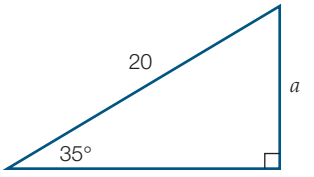
Equipment required: ruler and protractor for Questions **6**, **14–16**

Fluency and understanding

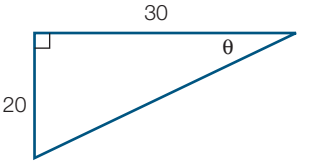
WE1

1 Label the hypotenuse (H), opposite (O) and adjacent (A) sides on these right-angled triangles and hence decide which trigonometric ratio should be used to find the unknown in each case.

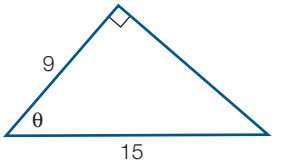
(a)



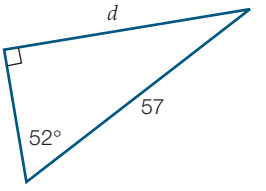
(b)



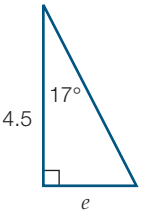
(c)



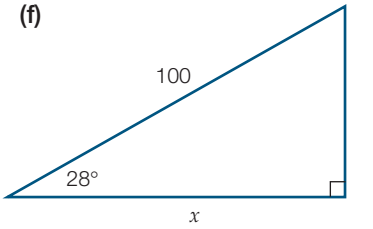
(d)



(e)



(f)



WE2

2 Use your calculator to find the following values, rounded to four decimal places.

(a) sin 23° **(b)** cos 17° **(c)** tan 39°

(d) cos 5° **(e)** sin 41° **(f)** cos 49°

(g) tan 19.7° **(h)** cos 23.5° **(i)** tan 43.65°

WE3

3 Convert these angles to decimal degrees. Round your answers to two decimal places if necessary.

(a) 15°30′ **(b)** 35°16′ **(c)** 23°10′

(d) 76°47′50″ **(e)** 44°59′20″ **(f)** 8°45′9″

WE4

4 Calculate the following, rounding values to four decimal places.

(a) sin 25°47′ **(b)** tan 88°30′ **(c)** cos 64°17′

(d) tan 36°30′46″ **(e)** sin 20°25′39″ **(f)** cos 81°44′16″

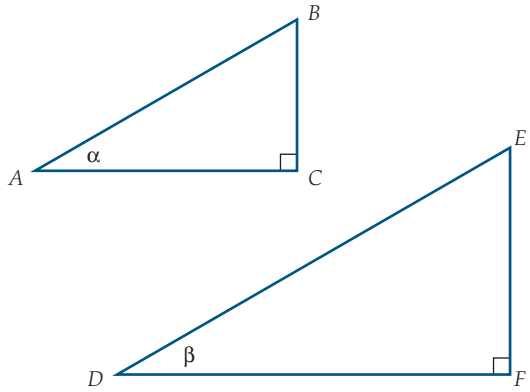
5 **(a)** Using a calculator, and evaluating the result to four decimal places, the value of sin 37°34′ is:

A 0.6064 **B** 0.6065 **C** 0.6097 **D** 0.7951

(b) Using a calculator, and evaluating the result to four decimal places, the value of cos 17.25° is:

A 0.2965 **B** 0.3105 **C** 0.9542 **D** 0.9550

6 Measure the side lengths of triangles *ABC* and *DEF* to the nearest millimetre.



(a) Calculate the ratios of the following pairs of sides. Round your answers to one decimal place.

(i) $\frac{DF}{AC}$ **(ii)** $\frac{EF}{BC}$ **(iii)** $\frac{DE}{AB}$

(b) Calculate the value for sin α.

(c) Calculate the value for sin β.

(d) Are these sine ratios about the same? Why?

(e) Measure the sizes of angles α and β with a protractor.

(f) Would the cosine ratios be the same for each of the triangles?

(g) Would the tangent ratios be the same for each of the triangles?

(h) Give a reason for your answers to **(f)** and **(g)**.

7 **(a)** Calculate the value of:

(i) sin 35° **(ii)** cos 35° **(iii)** tan 35° **(iv)** $\frac{\sin 35^\circ}{\cos 35^\circ}$

(b) Comment on your results.

8 If sin θ = 0.755 (3 d.p.) and cos θ = 0.656 (3 d.p.), use your result from Question **7** to calculate the value of tan θ, to three decimal places.

9 The cosine of an angle is always less than 1 because:

A the Adjacent side is always shorter than the Opposite side

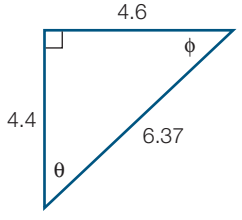
B the Adjacent side is always shorter than the Hypotenuse

C the Adjacent side is always longer than the Hypotenuse

D the Adjacent side is always longer than the Opposite side.

Reasoning, communicating and open-ended

10 **(a)** Calculate the following values using the triangle shown, rounded to three decimal places:



(i) sin θ **(ii)** cos ϕ

(b) Explain your results.

11 A right-angled triangle has an angle of 54° as one of its acute angles.

(a) Using three different pronumerals to represent the side lengths, draw a diagram of the triangle, including the measurement of the other angle.

(b) Write down values of sin 54° and cos 54° using the pronumerals that represent the side lengths.

(c) Find the values of sin 54° and cos 54° using a calculator, and round to four decimal places.

(d) Use your answers to **(c)** to show that the value of $(\sin 54^\circ)^2 + (\cos 54^\circ)^2$ equals 1. (*Note*: This is usually written as $\sin^2 54^\circ + \cos^2 54^\circ$.)

(e) Use your answers to **(b)** and Pythagoras' Theorem to show that $(\sin 54^\circ)^2 + (\cos 54^\circ)^2 = 1$.

12 The value of the tangent ratio for each of the two base angles in a right-angled isosceles triangle is 1. Explain why.

13 Only one of the trigonometric ratios can have a value greater than 1. Name it, and explain why this is not possible for the other two ratios.

14 **(a)** Draw a right-angled triangle with sides of any length.

- **(i)** Carefully measure all the angles of the triangle.
- **(ii)** Carefully measure all the sides of the triangle.
- **(iii)** Find the sine, cosine and tangent ratios for the two acute angles, rounded to two decimal places.
- **(iv)** Write down as many relationships between the angles and the ratios as you can find.

(b) Draw a second triangle with the same angles, but with different side lengths (i.e. a similar triangle). Repeat steps **(ii)** and **(iii)** from part **(a)** for this triangle.

(c) Compare the values of sin, cos and tan that you calculated for the two triangles. How similar are they? Did you expect them to be the same?

(d) Compare the values of the three ratios you calculated for each angle to the values produced by your calculator. Are they the same? Suggest a reason for your answer.

15 A right-angled triangle has a tangent ratio of $\frac{7}{10}$ for one angle.

(a) Use the tangent ratio to draw the triangle. Measure the length of the hypotenuse.

(b) Calculate the other ratios, rounded to four decimal places.

Hint

Which values of O and A give a ratio of $\frac{7}{10}$?

16 Angle θ in a right-angled triangle has a sine ratio of 0.4. Draw three different triangles that could contain angle θ. Use Pythagoras' Theorem to help you find any unknown side lengths.

17 **(a)** Find two angles θ and ϕ, so that $\sin\theta = \cos\phi$.

(b) What is the relationship between θ and ϕ if $\sin\theta = \cos\phi$ for all values of θ and ϕ?

Finding lengths

5.2

Stage 5.2◊

Need to Know

Using the ratios

Because the value of each of the trigonometric ratios is unique for a particular acute angle θ, one of the three trigonometric ratios can be used to find the length of an unknown side of a right-angled triangle if the value of θ and the length of one other side are known.

Finding an unknown length in a right-angled triangle

Step 1 Draw the right-angled triangle, showing all known information and the unknown value. A side and an acute angle must be known.

Step 2 Label the sides O, A and H, and write down the known information.

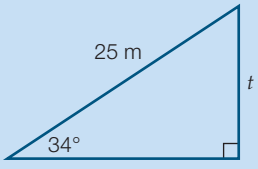
Step 3 Identify which trigonometric ratio to use. (SOH-CAH-TOA is helpful here.)

Step 4 Substitute the known values, form an equation and solve for the unknown using a calculator.

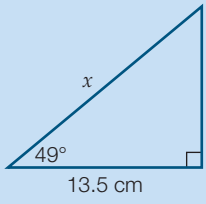
Worked Example 5

WE 5

(a) Find the value of t, rounded to two decimal places.

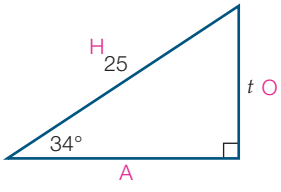


(b) Find the value of x, rounded to two decimal places.



Working

(a)



$\theta = 34°$

$O = t$

$H = 25$

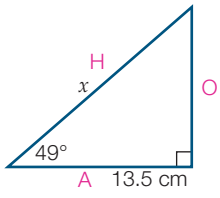
$\sin\theta = \frac{O}{H}$

$\sin 34° = \frac{t}{25}$

$t = 25 \times \sin 34°$

$t = 13.98$ m (2 d.p.)

(b)



$\theta = 49°$

$A = 13.5$

$H = x$

$\cos\theta = \frac{A}{H}$

$\cos 49° = \frac{13.5}{x}$

$x = \frac{13.5}{\cos 49°}$

$x = 20.58$ cm (2 d.p.)

Worked Example 6

WE 6

A 3.2 m ladder rests against a wall so that its angle with the wall is 65°. How high up the wall does the ladder reach? State your answer correct to the nearest centimetre.

Working

$\theta = 65°$

$H = 3.2$ m

$A = d$

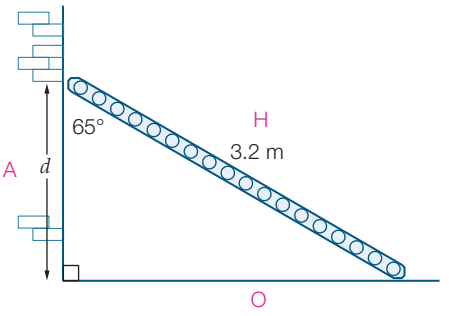
$\cos\theta = \frac{A}{H}$

$\cos 65° = \frac{d}{3.2}$

$d = 3.2 \times \cos 65°$

$d = 1.35$ (2 d.p.)

The ladder rests 1.35 m up the wall.



Worked Example 7

WE 7

A rectangular wooden frame has a diagonal piece of wood joining the opposite corners for support. The vertical height of the frame is 2.5 m. The angle between the diagonal and vertical part of the frame is 57°15′. How long is the diagonal piece of the frame?

Working

$\theta = 57°15'$

$A = 2.5$ m

$H = x$

$\cos\theta = \frac{A}{H}$

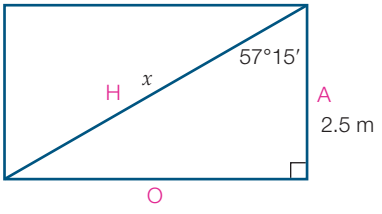
$\cos 57°15' = \frac{2.5}{x}$

$x \cos 57°15' = 2.5$

$x = \frac{2.5}{\cos 57°15'}$

$x = 4.62$ (2 d.p.)

The diagonal of the frame is 4.62 m long.



5.2 Finding lengths

Answers page 636

Stage 5.2◊

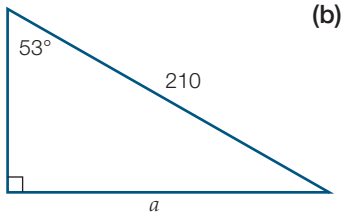
Navigator

Q1, Q2, Q3, Q5, Q7, Q9, Q11, Q12, Q13, Q14	Q1 Columns 2 & 3, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q11, Q12, Q14	Q1 Column 3, Q2, Q3, Q5, Q6, Q8, Q9, Q10, Q12, Q13, Q14

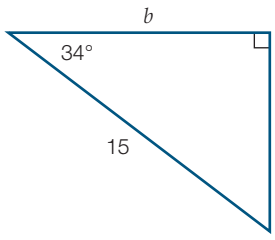
Fluency and understanding

1 Find the value of the unknown length in each of the following. Round your answers to two decimal places. WE 5

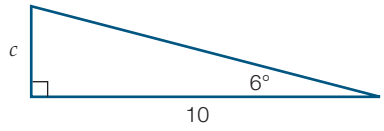
(a)



(b)



(c)



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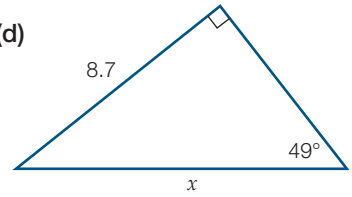
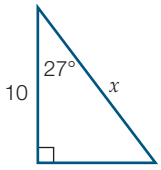
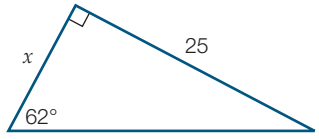


(e)



(f)



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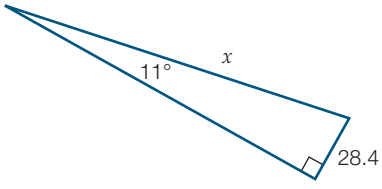

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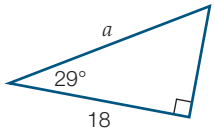
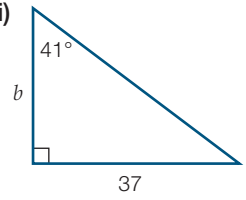


(i)



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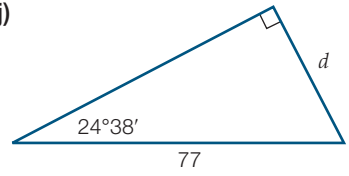

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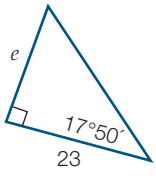
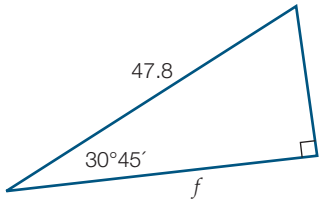
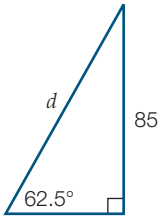


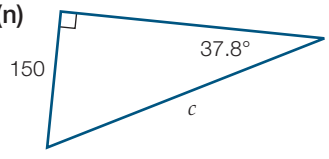
(l)



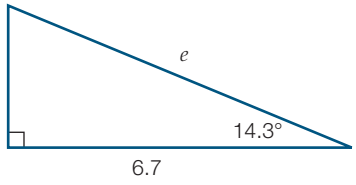
(m)



(n)



(o)



2 A 3 m rope is stretched from the top of a vertical tent pole and pegged into the ground so that it makes an angle of 50° with the ground. How far away from the pole is the rope pegged? Give your answer to the nearest centimetre. WE 6

3 A rectangular gate has diagonal metal supports joining the opposite corners. The supports make an angle of 25°18′ with the horizontal sections of the frame. If the gate is 1.6 m high, how long is one of the diagonal supports? WE 7

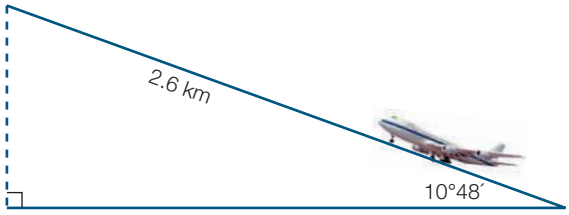
4 **(a)** A plane takes off and rises steadily at an angle of 10°48′ to the horizontal. The altitude of the plane after it has travelled along a flight path of 2.6 km (to the nearest metre) is:

A 472 m

B 487 m

C 2554 m

D 13 875 m



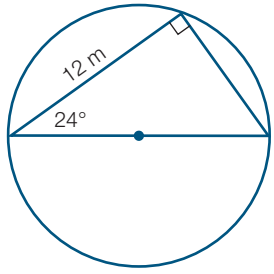
(b) The radius of the circle at right is closest to:

A 6.57 m

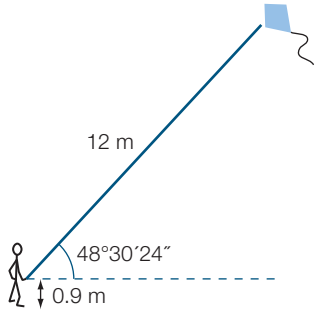
B 13.14 m

C 14.75 m

D 29.50 m



5 Jessica is flying a kite attached to a string 12 m long. She is holding the string at a height of 0.9 m from the ground. How high is the kite if it makes an angle of 48°30′24″ with the horizontal? Round your answer to two decimal places.

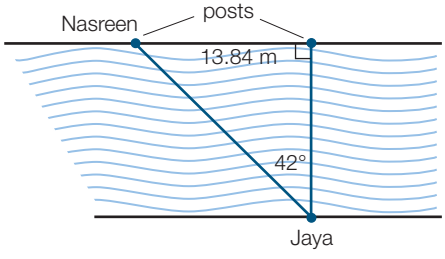


6 To measure a tall building, Charlie stands 150 m from the base of the building and holds an inclinometer (a device for measuring angles) to his eye, as shown in the diagram. He looks along the top of the inclinometer and tilts it up until he can see the top of the building. The inclinometer measures an angle of 58°. This is the angle between Charlie's line of sight to the top of the building and the horizontal. Charlie's eye-level is 1.63 m from the ground.

(a) Draw a diagram that shows all the known information.

(b) Calculate the height of the building, correct to one decimal place.

7 Jaya wants to calculate the width of a river. She finds that lines joining her point of observation to the two posts on the opposite bank are 42° apart. Her sister Nasreen, who is on the opposite bank, measures the distance between the two posts to be 13.84 m. How wide is the river?



8 A mine shaft 350 m long is to be drilled at an angle of 21°24′ to the vertical. How far directly below the ground will the end of the shaft be, rounded to two decimal places?

9 A road sign is in the shape of an equilateral triangle. The line from one vertex to the midpoint of the opposite side is 34.6 cm long. How long is each side of the road sign?

Reasoning, communicating and open-ended

10 Members of a stunt team wish to make a daring climb along a rope, one end of which has been attached to the top of a building 23 m high and the other end joined to the top of a nearby building 38 m high. The rope is inclined at an angle of 49°30′ to the horizontal. Answer the following correct to two decimal places.

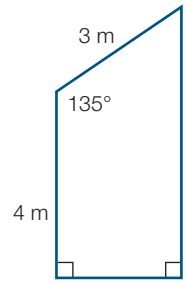
(a) How far along the rope will each member of the stunt team need to climb to reach the other building?

(b) One of the stunt team wants to drop a ball so that a man standing midway between the two buildings can catch it. From how far along the rope does she need to drop it, and how far will it fall if the man catches it 1.8 m above the ground?

11 James is trying to decide which room to use for a party. He has two rooms in mind but needs to use the larger one. One room is a rectangular shape with length 4 m and breadth 3 m. The other room is in the shape of a trapezium as shown in the diagram.

(a) Calculate the area of each of the rooms.

(b) Which room is larger?



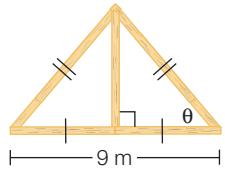
Hint

Formula for the area of a trapezium: $A = \frac{1}{2}(a + b)h$

12 The 'pitch angle' of the roof of a house is the angle that the roof makes with the horizontal. It is usually between 10° and 30°. If the length of the base of this roof frame is 9 m, find two possible values for the:

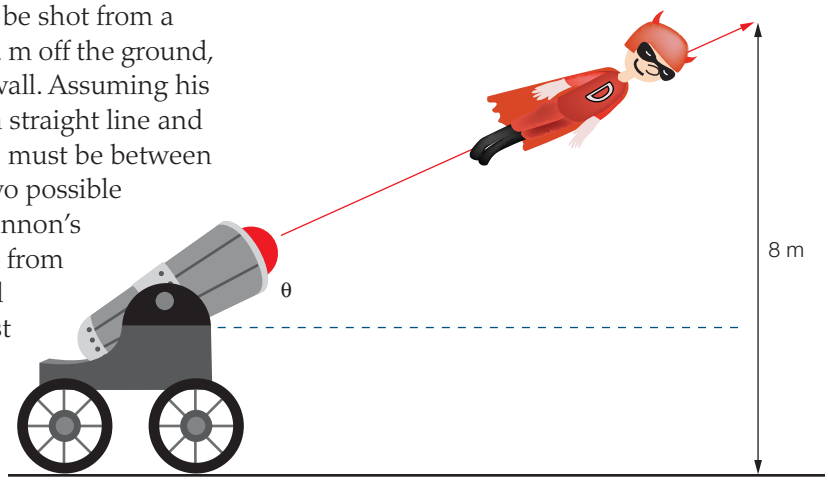
(a) height of the roof at its tallest point

(b) total length of wood needed to build the frame shown.



13 A right-angled triangle ABC has the right angle at B. If the side opposite to angle A has a length of 10 cm, write down at least three possible lengths of the hypotenuse given that angle A is between 35° and 40°.

14 Daredevil Dan will be shot from a cannon that sits 1.2 m off the ground, to fly over an 8 m wall. Assuming his flight path will be a straight line and the trajectory angle must be between 30° and 50°, find two possible distances for the cannon's horizontal distance from the base of the wall if this distance must be between 8 m and 10 m.



5.3 Finding angles

Stage 5.2◊

Need to Know

Because the value of each of the trigonometric ratios is unique for a particular acute angle θ, one of the three trigonometric ratios can also be used to find an acute angle, θ, of a right-angled triangle if any two sides are known.

To find an unknown angle in a right-angled triangle:

1. Draw the right-angled triangle, showing all known information and the unknown angle. Two sides must be known.
2. Label the sides O, A and H, and write down the known information.
3. Identify which trigonometric ratio to use. (SOH-CAH-TOA is helpful here.)
4. Substitute the known values, form an equation and solve for the unknown using a calculator.

Technology Exploration: Calculator

Finding an inverse trigonometric ratio

To find θ we need to 'undo' the sine ratio to find the angle that corresponds to the sine ratio with that particular value. Calculators usually have an INV, 2nd F or SHIFT key. This key, used before the sin key, should apply the *inverse* sine ratio, which we write as $\sin^{-1}$.

For example: If $\theta = \sin^{-1}(0.72)$, θ may be found on a calculator by pressing SHIFT sin . 7 2 = .

The screen should show 46.054 480 44. So θ, to the nearest degree, is 46°. The angle could also be converted to degrees, minutes, seconds.

Make sure your calculator is in degree mode. (A 'D' or 'DEG' will be shown on the screen.)

Worked Example 8

WE 8

If cos a = 0.6434, find:

(a) a in decimal degrees rounded to two decimal places

(b) a in degrees, minutes and seconds.

Working

(a) a = 49.95° (2 d.p.)

(Press the following keys:
SHIFT cos . 6
4 3 4 =)

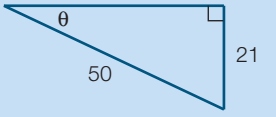
(b) a = 49°57′15″ (nearest second)

(Press °′″)

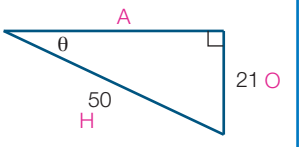
Worked Example 9

WE9

Find the value of θ in degrees, minutes and seconds.



Working



$\sin\theta = \frac{O}{H}$

$\sin\theta = \frac{21}{50}$

$\theta = \sin^{-1}\left(\frac{21}{50}\right)$

$\theta = 24°50'4.51''$

$\theta = 24°50'5''$ (nearest second)

Worked Example 10

WE10

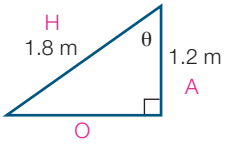
The top of a 1.8 m ladder that is leaning against a wall, reaches a height of 1.2 m. Draw a diagram that shows this situation, then use it to calculate the angle that the top of the ladder makes with the wall, correct to the nearest degree.

Working

$\theta = ?$

$A = 1.2$ m

$H = 1.8$ m



$\cos\theta = \frac{A}{H}$

$\theta = \cos^{-1}\left(\frac{1.2}{1.8}\right)$

$\theta = 48.189\,685\,11$

$\theta = 48°$ (nearest degree)

The ladder makes an angle of approximately 48° with the wall.

Answers page 636

5.3 Finding angles

Stage 5.2◊

Navigator

Q1, Q2, Q3, Q4, Q5, Q7, Q10, Q11, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q10, Q11, Q13	Q2 (a), (c), (e), Q3, Q5, Q6, Q8, Q9, Q11, Q12, Q13

Fluency and understanding

WE8

1 Find the value of each pronumeral **(i)** in decimal degrees rounded to two decimal places, **(ii)** in degrees, minutes and seconds.

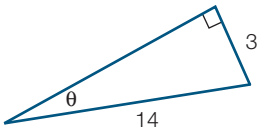
(a) $\cos a = 0.1538$ **(b)** $\sin b = 0.6719$ **(c)** $\tan c = 2.3085$

(d) $\cos k = 0.556$ **(e)** $\tan m = 4.687$ **(f)** $\sin p = 0.9715$

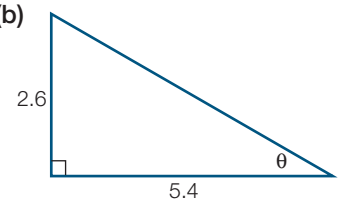
WE9

2 Find the value of θ in each triangle in degrees, minutes and seconds.

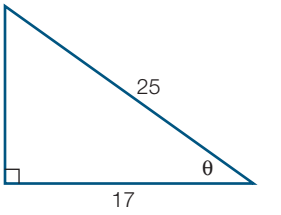
(a)



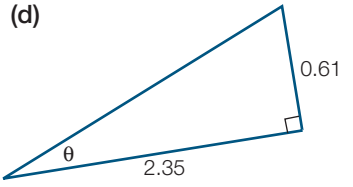
(b)



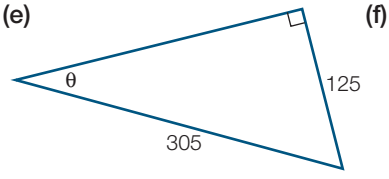
(c)



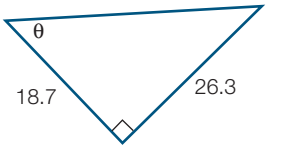
(d)



(e)



(f)



WE10

3 A tree that is 8.4 m tall makes a shadow of 12 m.

(a) Draw a diagram to show this situation.

(b) To the nearest degree, calculate the angle that the sun makes with the ground to form the shadow.

4 **(a)** The value of the pronumeral a in the equation $\tan a = 0.6395$, to the nearest degree is:

A 32° **B** 33° **C** 40° **D** 50°

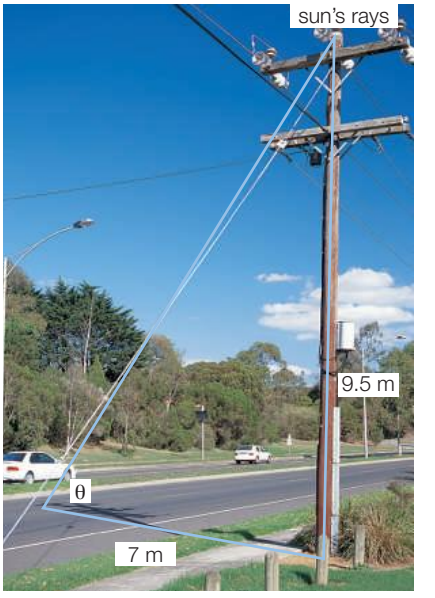
(b) The value of θ in degrees, minutes and seconds in the equation $\sin \theta = 0.8938$ is:

A 26°38′43″ **B** 41°47′25″ **C** 63°21′17″ **D** 63°35′47″

5 On a slope, a railway rises 35 cm for every 50 m of track. Find the angle at which the track is inclined to the horizontal, to two decimal places.

6 Scott is dragging a heavy box along the floor holding the end of a 1.5 m piece of rope. The rope is attached to the top of the box and makes an angle θ with the horizontal. Scott is exactly 1.0 m in front of the box. To the nearest degree, what angle does the rope make with the horizontal?

7 Find the angle of inclination, θ, of the sun's rays when a 9.5 m high telephone pole casts a shadow 7 m long, as shown in the photograph at right. Write your answer in degrees, minutes and seconds.



8 A 2.75 m long children's slide has a ladder that reaches 1.43 m vertically above the ground. To the nearest degree, the angle the slide makes with the horizontal is:

A 27° **B** 31°

C 42° **D** 58°

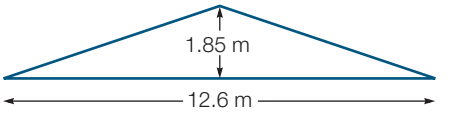
Reasoning, communicating and open-ended

9 A furniture removal company uses a ramp to wheel furniture into its vans. The back of a van is 82 cm above the ground and the ramp is 1.5 m long.

(a) To the nearest degree, what is the angle of the ramp with the ground?

(b) How much longer does the ramp need to be so that the angle is 30°?

10 A symmetrical roof frame of height 1.85 m has a span of 12.6 m as shown in the diagram at right.



(a) Find the angle at the apex of the roof, to the nearest degree. (The apex is the highest point on the roof.)

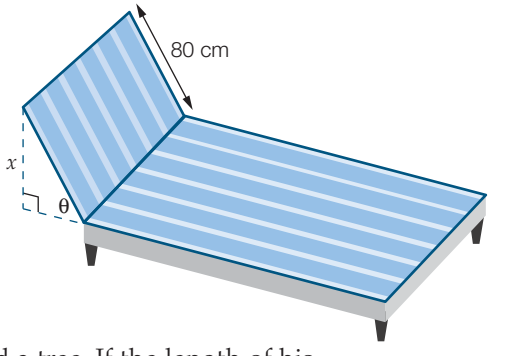
(b) Find the total length of material needed to make the frame.

11 A 10 m ladder is to be used to reach different window sills in a building.

(a) If the heights of the window sills range from 2 m to 6 m above the ground, calculate the range of angles that the ladder makes with the ground.

(b) Is it practical to use the ladder to reach all the window sills?

12 A sun lounge can be adjusted from a flat, lying position through to an upright, sitting position. The adjustable back is 80 cm long. Choose five different vertical heights that the back could be raised to. For each height (x) calculate the angle the back makes with the horizontal (θ).



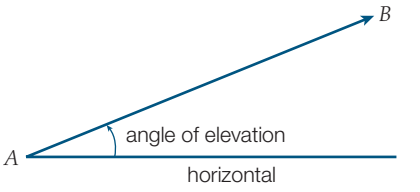
13 Bilal is 1.85 m tall and is standing near a lamp post and a tree. If the length of his shadow is 94 cm, use your own reasonable values for the heights of the lamp post and the tree to determine the length of their shadows at the same time.

5.4 Angles of elevation and depression

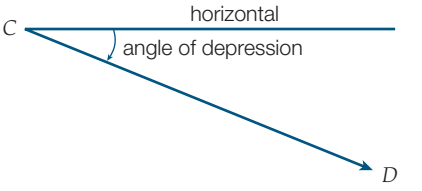
Stage 5.2◊

Need to Know

The **angle of elevation** from A to B is the angle that the line AB makes with the horizontal.



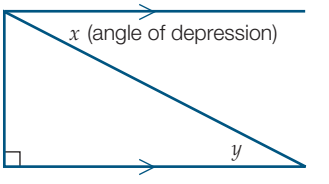
The **angle of depression** from C to D is the angle the line CD makes with the horizontal.



The angle of depression may not always be inside the triangle that we draw. However, we can use alternate angles to find an angle within the triangle.

x and y are alternate angles on parallel lines

$\therefore x = y$



Worked Example 11

WE 11

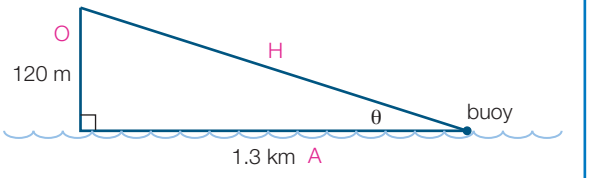
A marker buoy used to guide ships is 1.3 km away from the base of a 120 m cliff. To the nearest degree, what is the angle of elevation from the buoy to the top of the cliff? Answer correct to the nearest degree.

Working

$\theta = ?$

$O = 120$ m

$A = 1.3$ km



$$\tan\theta = \frac{O}{A}$$

$$\tan\theta = \frac{120}{1300}$$

$$\theta = \tan^{-1}\left(\frac{120}{1300}\right)$$

$\theta = 5.273\,89\ldots$

$\theta = 5°$ (nearest degree)

The angle of elevation from the buoy to the top of the cliff is approximately 5°.

Worked Example 12

WE 12

A ranger at the top of a lookout tower spots a small fire at an angle of depression of 12.5°. If the tower is 76 m high, calculate the distance from the base of the tower to the fire. Round your answer to the nearest metre.

Working

$\theta = 12.5°$

$O = 76$ m

$A = x$

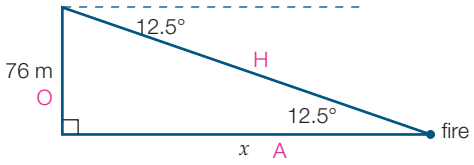
$\tan\theta = \frac{O}{A}$

$\tan 12.5° = \frac{76}{x}$

$x \tan 12.5° = 76$

$x = \frac{76}{\tan 12.5°}$

$x = 343$ m (nearest metre)



The fire is approximately 343 m from the base of the tower.

5.4 Angles of elevation and depression

Answers page 636

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4, Q6, Q8, Q12, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14

Fluency and understanding

1 Hayden is standing 1.2 km away from the base of a 305 m tall building. To the nearest degree, what is the angle of elevation from the point where Hayden is standing to the top of the building? WE 11

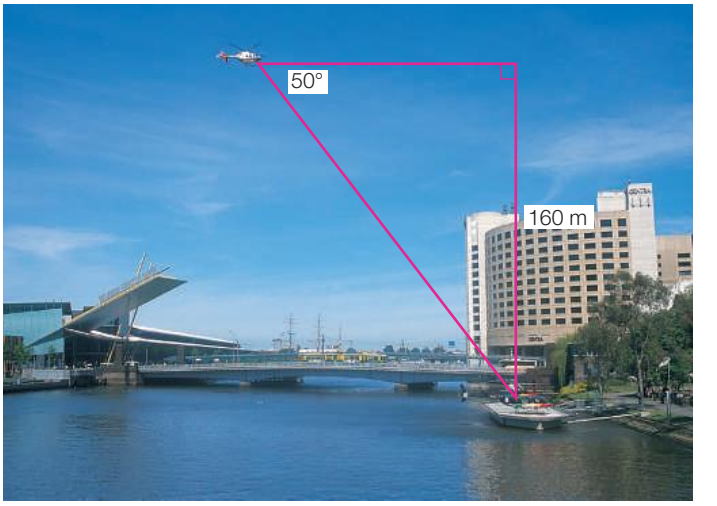
2 A person in a hot-air balloon 150 m above the ground spots a parked car at an angle of depression of 16.5°. Calculate the distance of the car to the point that is directly underneath the balloon to the nearest metre. WE 12

3 During the day, Kristy notices that a post that is 3 m tall casts a shadow of 2.3 m. What is the angle of elevation of the sun at that time? Answer to the nearest degree.

4 The angle of depression from the top of a pine tree to a point on the ground 50 m away is 38°. Find the height of the tree to the nearest metre.

5 A traffic signal 3.95 m high casts a shadow across the road. If the angle of elevation of the sun is 24.8°, find the length of the shadow, correct to two decimal places.

6 A helicopter is flying at a height of 160 m above a river. The pilot can see another helicopter on a pontoon along the river located at an angle of depression of 50°. Find, to the nearest metre:



(a) the distance from the first helicopter to the second along the line of sight

(b) how far the first helicopter must fly at the same altitude to be directly over the second.

7 (a) A railway rises 400 cm for every 2 km of track. From the lower end of the track, the angle of elevation (to two decimal places) is:

A 0.11° B 11.54° C 25° D 78.69°

(b) From a hot-air balloon, the angle of depression to a marked landing site is 6°50′30″. If the observation deck in the balloon is 190 m above the ground, the horizontal distance the balloon needs to travel to be above the landing site is closest to:

A 191.23 m B 1.58 km C 1.67 km D 1.68 km

8 Rachel uses an inclinometer (a device for measuring angles) to measure the angle of elevation from her eye level to the top of a billboard 25 m away. She obtains a value of 42°. She also measures the distance from the ground to her eye level and finds it to be 153 cm. Find the height of the billboard to the nearest metre.

9 A television antenna is mounted on the top of a nearby hill. From a position 4.5 km away, the angle of elevation to the bottom of the antenna is 17°30′, whereas the angle of elevation to the top of the antenna is 17°50′. Find the height of the antenna in metres, correct to two decimal places.

10 A yacht is 1.8 km away from the base of a 120 m cliff. The deck of the yacht is 3 m above sea level. To the nearest degree, what is the angle of elevation from the deck of the yacht to the top of a 4.2 m flagpole on the edge of the cliff?

Reasoning, communicating and open-ended

11 Peter views a stationary beetle on the ground, 2.6 m away, at an angle of depression of 32°. Peter's eyes are 15 cm below the top of his head. He is standing upright.

(a) Find Peter's height to the nearest centimetre.

(b) The beetle moves directly ahead to a new position. If the distance from Peter to the beetle along his line of sight is 4.7 m, find the new angle of depression, correct to one decimal place.

12 Sonya is standing 20 m from a hot-air balloon that is rising vertically at a speed of 1.5 m/s. When the balloon's position is at eye-level, she follows its path for 7 seconds.

(a) Calculate the angle of elevation of the balloon from Sonya's line of sight after 7 seconds.

(b) If Sonya is 175 cm tall, what is the height of the balloon above the ground after 7 seconds?

13 A flagpole 24 m high is to be supported by a cable that is 18 m long when pulled taut. It must be fastened at least halfway up the pole at an angle of elevation between 45° and 60°.

(a) Find three possible angles of elevation for the cable.

(b) For each angle of elevation, find the distance from the flagpole that the cable must be fastened.

14 From the top of a 20 m lookout, Ranger Sam spots a hiker 200 m due east of his position, heading due east away from him at a constant speed of 50 m/min. Ranger Sam notes the angle of depression of the hiker from his position every minute for a period of one hour. Find two values for the angles of depression that Ranger Sam recorded.

Problem solving

Outback mail run

In outback Australia, remote houses and communities receive their mail by plane.

The pilot flying the mail plane is cruising at an altitude of 4500 m when she looks out of the left window and sees one house at an angle of depression of 17°. Out of the right window, she spots another house at an angle of depression of 13°.

How far apart are the two houses, to the nearest kilometre?

Strategy options

- Draw a diagram.
- Have I seen a similar problem?
- Break problem into manageable parts.

Half-time 5

Stage 5.2◊ Q1–6

Ex.5.3

1 A ladder of length 3 m reaches 1.85 m up a vertical wall. The ladder is moved closer to the wall and now reaches 2.10 m up the wall. What is the change in the angle the ladder makes with the ground?

Ex.5.3

2 An ant travels 76 cm in a straight line, turns 90°, follows a straight path for 80 cm and then travels directly to its starting point. The smallest interior angle of the triangle formed is:

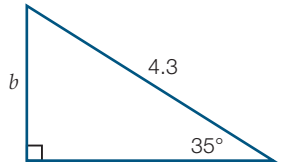
A 40.2° **B** 43.5° **C** 44.6° **D** 45.5°

Ex.5.1

3 Which calculation can be used to find the length of side b?

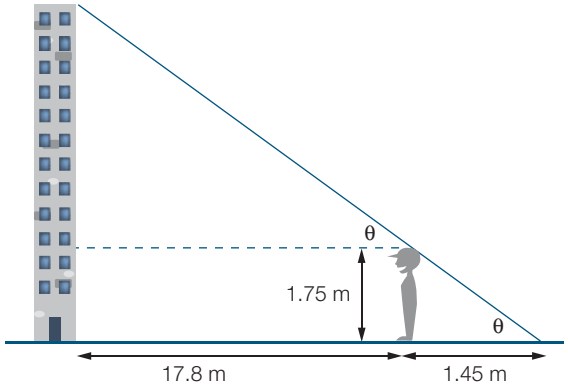
A $b = \dfrac{4.3}{\cos 35°}$ **B** $b = 4.3 \times \sin 35°$

C $b = \dfrac{\tan 35°}{4.3}$ **D** $b = 4.3 \times \cos 35°$



Ex.5.3

4 Find the angle θ in the small triangle and use the result to calculate the height of the building to the nearest centimetre.

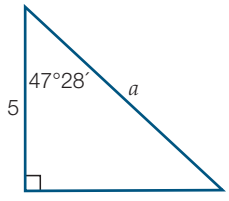


Ex.5.2

5 The value of a in the triangle on the right, rounded to two decimal places, is:

A 2.41 **B** 3.39

C 3.68 **D** 7.40



Ex.5.4

6 From the top of a lookout, a ranger spots a small scrub fire at an angle of depression of 6°14′. How far is the fire from the base of the lookout if the eye-level of the ranger is 35 m above the ground? Give your answer correct to the nearest metre.

Bearings

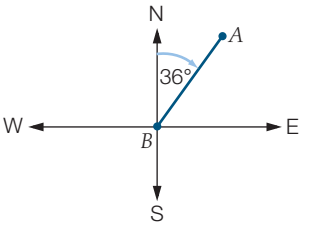
5.5

Stage 5.2◊

Need to Know

True bearings

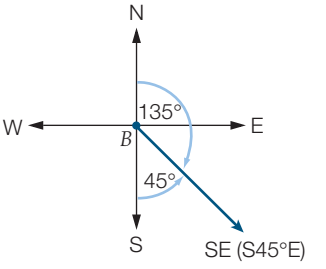
True bearings give the angle in degrees in a clockwise direction from north. The true bearing of A from B is 036°T. True bearings are always written with three digits, so the bearing of 36° is written as 036°T.



Compass bearings

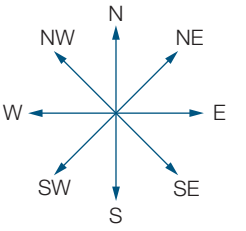
Compass bearings are directions given in terms of turning a number of degrees east or west of north or south (whichever direction is closer to the position). The compass bearing of A from B is N36°E.

The direction of southeast (SE) is shown here. As a compass bearing, it is written as S45°E. As a true bearing, it is written as 135°T. Compass bearings are sometimes called conventional bearings.



The directions exactly in between the four compass points of north, east, south and west have the following bearings.

northeast (NE) is 45°T or N45°E
southeast (SE) is 135°T or S45°E
southwest (SW) is 225°T or S45°W
northwest (NW) is 315°T or N45°W

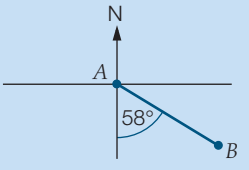


Worked Example 13

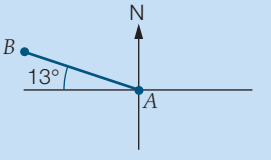
WE13

Find **(i)** the compass bearing and **(ii)** the true bearing of point B from point A, for each of the following.

(a)



(b)



Working

(a) **(i)** S58°E
(It is closer to south, so write 'S' and then 58° to the east 'E')

(ii) 180 – 58 = 122
122°T
(Find the difference between the compass bearing and 180°.)

(b) **(i)** 90 – 13 = 77
N77°W
(It is closer to north, so write 'N' and then 77° to the west 'W')

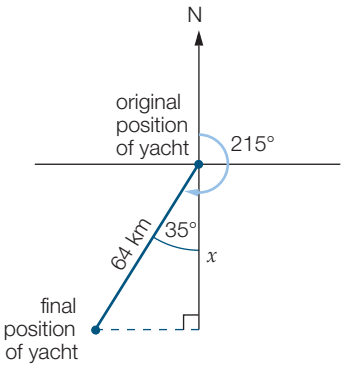
(ii) 270 + 13 = 283
283°T
(Add 13° to 270°.)

Worked Example 14

WE14

A yacht sails 64 km on a true bearing of 215°T. How far south is the yacht from its original position? Round the answer to two decimal places.

Working



$\theta = 35°$

$H = 64$ km

$A = x$

$\cos\theta = \frac{A}{H}$

$\cos 35° = \frac{x}{64}$

$x = 64 \times \cos 35°$

$x = 52.43$ km (2 d.p.)

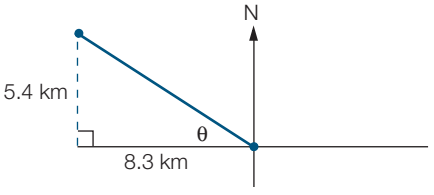
The yacht is 52.43 km south from its original position.

Worked Example 15

WE15

On a hiking trip, Sarah walks 8.3 km west and then 5.4 km north. Calculate her true bearing and compass bearing from her original position, to the nearest degree.

Working



$O = 5.4$ km

$A = 8.3$ km

$\theta = ?$

$\tan\theta = \frac{O}{A}$

$\tan\theta = \frac{5.4}{8.3}$

$\theta = \tan^{-1}\left(\frac{5.4}{8.3}\right)$

$\theta = 33°$ (nearest degree)

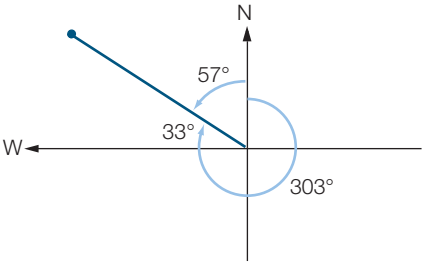
True bearing $= 270° + 33°$

$= 303°$T

Compass bearing $= 90° - 33°$

$= 57°$

$=$ N57°W



True bearing = 303°T

Compass bearing = N 57° W

5.5 Bearings

Answers page 637

Stage 5.2◊

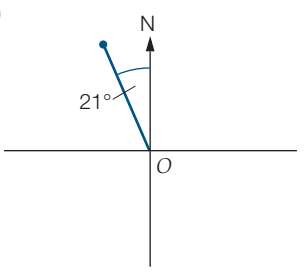
Navigator

Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q11, Q14	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9, Q11, Q13, Q14, Q15	Q1, Q2, Q3, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14, Q15

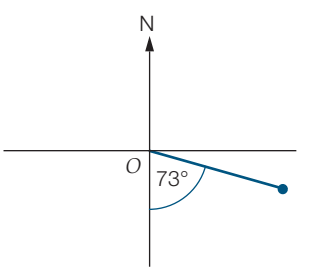
Fluency and understanding

1 Find **(i)** the compass bearing and **(ii)** the true bearing for each of the following points from O. WE13

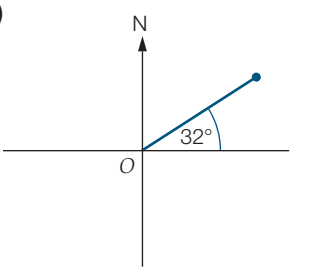
(a)



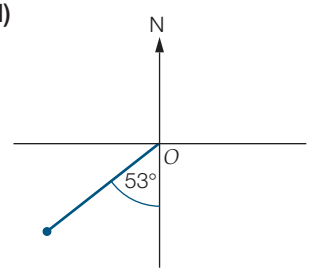
(b)



(c)



(d)



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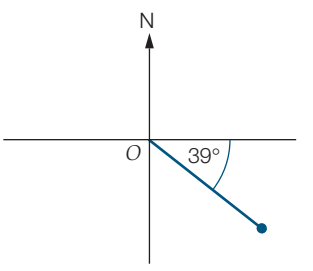
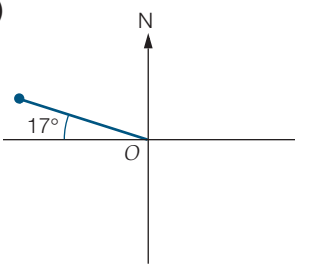


(f)

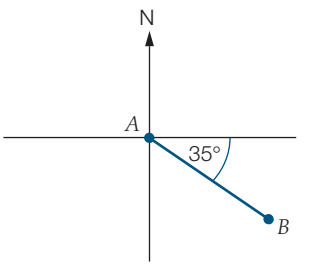


2 A ship sails for 150 km on a bearing of N40°E. How far north of its starting point is it, to the nearest kilometre? WE14

3 **(a)** Pedro sails his yacht 2.5 km east and then 4.7 km south. To the nearest degree, state Pedro's true bearing and compass bearing from his original position. WE15

(b) A taxi driver drives 630 m south and then 410 m east. To the nearest degree, what is the bearing of the taxi from its starting position? Give your answer as a compass bearing and a true bearing.

4 **(a)** The compass bearing of point B from point A, as shown in the diagram, is:



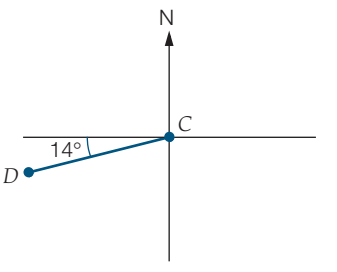
A N35°E

B S55°E

C N35°W

D S35°E

(b) The true bearing of point D from point C, as shown in the diagram, is:



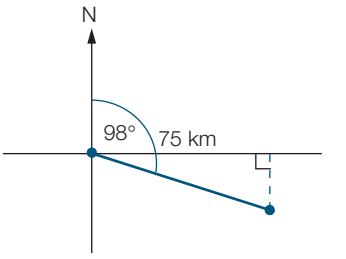
A 076°T

B 194°T

C 256°T

D 284°T

5 A helicopter travelling on a true bearing of 098°T is now 75 km east of its original position. How far has the helicopter travelled, to two decimal places?



6 A plane flies on a true bearing of 280°T at 600 km/h after taking off at 1300 hours. To the nearest kilometre, how far north of its starting position is it at 1530 hours?

7 Jack drives north at 80 km/h for 2 hours. He then turns and drives east at the same speed for 3 hours.

(a) To the nearest degree, what is Jack's true bearing from his original position?

(b) If Jack could drive from his current position straight back to his starting position:

(i) how far would he travel, to the nearest kilometre

(ii) what true bearing should he follow?

8 A hiker walks 5 km on a true bearing of 025°T from camp *A* to camp *B*.

Round your answers to two decimal places.

(a) How far east has she walked from *A* to *B*?

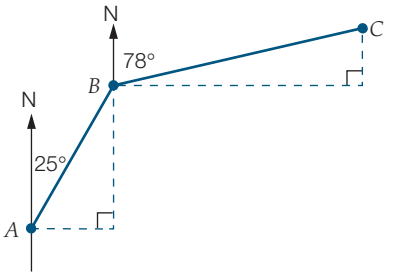
She then continues for another 6 km on a true bearing of 078°T from camp *B* to camp *C*.

(b) **(i)** How far east of camp *B* is she now?

(ii) How far east of camp *A* is she now?

(c) **(i)** How far north of camp B is she now?

(ii) How far north of camp A is she now?



Warning

Round your final answers, but do not use rounded answers in further calculations. Use the original unrounded answers in further calculations.

9 Sasha is running an orienteering course. He runs 750 m to the first checkpoint on a bearing of S53°W. At the checkpoint he is instructed to run another 600 m on a bearing of N68°W to the next checkpoint.

(a) To the nearest metre, how far west is Sasha from his starting position?

(b) Is Sasha's current position north or south of his starting position? How far north or south?

10 A toy robot walks 16 m on a bearing of S62°W, and then turns and walks another 23 m NW. How far west is the robot from its starting point? Round your answer to two decimal places.

Reasoning, communicating and open-ended

11 A crew row their boat from point *A* to point *B*, a distance of 200 m on a bearing of S55°E. They then row due north a further 440 m to point *C*, as shown in the diagram.

(a) Find, to the nearest metre:

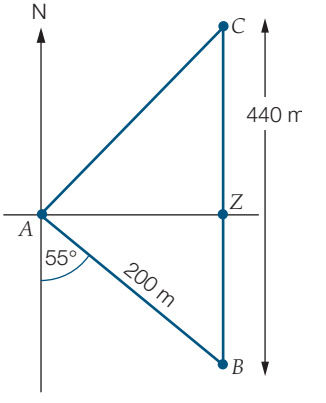
(i) the distance from *B* to *Z*

(ii) the distance from *A* to *Z*.

(b) Find the true bearing of *C* from *A*, to the nearest degree.

Finally, the crew row back from *C* to *A*.

(c) What is the total distance rowed by the crew, to the nearest metre?



Hint

Pythagoras' Theorem is helpful for finding missing sides in a right-angled triangle.

12 **(a)** From his home, Minh jogs 2.7 km NE and then 7.4 km SE. To the nearest degree, find his true bearing from his starting position.

(b) If Minh then jogs 6.9 km west, how far south is he from his home? Round your answer to two decimal places.

13 An air force jet takes off from an airfield and flys on a bearing of N37°E. The pilot is asked by ground staff to alter her course, 600 km into the trip, by turning 120° in a clockwise direction.

(a) State the new compass bearing of the jet after it has travelled 600 km.

(b) The jet travels for a further 800 km on its new bearing. How far east has it travelled from its starting position, to the nearest kilometre?

(c) How far north or south of its position at take-off, is the current position of the jet, to the nearest kilometre?

(d) What compass bearing should the jet now follow to return to the airstrip?

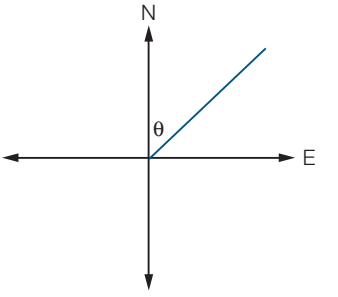
(e) How far will it travel on its return, if it follows the bearing you determined in **(d)**? Answer to the nearest kilometre.

14 Choose a true bearing θ less than 90° and state its true bearing and compass bearing after it is:

(a) reflected about the east-west line (the x-axis of the Cartesian plane)

(b) reflected about the north-south line (the y-axis of the Cartesian plane)

(c) rotated 180°.



15 Design a sailing course around five buoys labelled A, B, C, D and E. On each leg of the course, write down the true bearing to be taken by the sailors competing in the race. Remember that the course finishes at the starting point.

Problem solving

Seeing the light

1 A ship leaves its dock, which is 3 km due west of a lighthouse, and travels NE at a constant speed of 15 km/h.

(a) To the nearest minute, when is the ship due north of the lighthouse?

(b) At this time, how far is the ship from the lighthouse?

(c) How far has the ship travelled?

2 Rework these answers if the ship's bearing is N20°E.

Strategy options

- Draw a diagram.
- Break problem into manageable parts.

5.6 Mixed two-dimensional problems

Stage 5.2◊

Need to Know

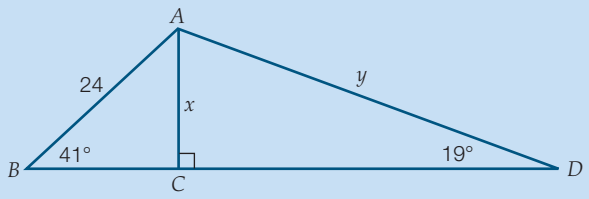
Situations arise when more than one trigonometric ratio is required to find an unknown side or angle. In such cases, it is important to determine which ratio is used first.

Also note that it is important to use unrounded values when using answers in further calculations. This avoids the accumulation of errors due to rounding. Always use the unrounded value if it is required in further calculations. Use your calculator's memory key to store values, if necessary, and then round off once only for your final answer.

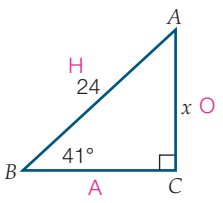
Worked Example 16

WE16

Find the values of x and y, rounded to two decimal places.



Working



$\theta = 41°$

$H = 24$

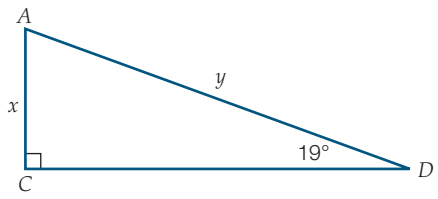
$O = x$

$\sin\theta = \frac{O}{H}$

$\sin 41° = \frac{x}{24}$

$x = 24 \times \sin 41°$

$x = 15.745\,416\,7$



$\theta = 19°$

$O = 15.745\,416\,7$

$H = y$

$\sin\theta = \frac{O}{H}$

$\sin 19° = \frac{x}{y}$

$y \times \sin 19° = x$

$y = \frac{x}{\sin 19°}$

$y = \frac{15.745\,416\,7}{\sin 19°}$

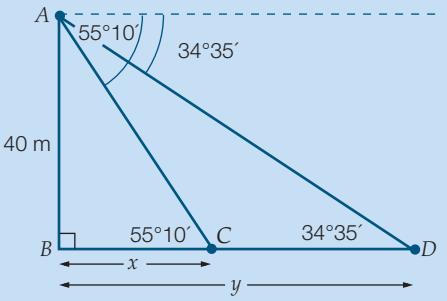
$y = 48.362\,889\,55$

$x = 15.75$ units $\quad y = 48.36$ units (2 d.p.)

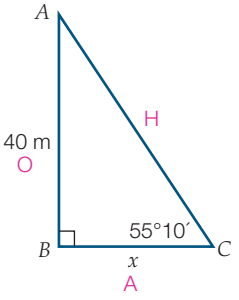
Worked Example 17

WE17

Thea sights two swimmers (both on the same bearing from her) at angles of depression of 34°35′ and 55°10′ from her vantage point on the cliff. If her eye-level is 40 m above the water level, find the distance between the two swimmers, rounded to two decimal places.



Working



$\theta = 55°10'$

$O = 40$ m

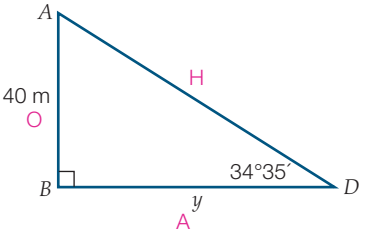
$A = x$

$\tan\theta = \frac{O}{A}$

$\tan 55°10' = \frac{40}{x}$

$x = \frac{40}{\tan 55°10'}$

$x = 27.835\,250\,33$



$\theta = 34°35'$

$O = 40$ m

$A = y$

$\tan\theta = \frac{O}{A}$

$\tan 34°35' = \frac{40}{y}$

$y = \frac{40}{\tan 34°35'}$

$y = 58.019\,401\,28$

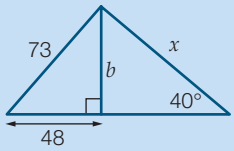
$y - x = 30.184\,150\,95$

$CD = 30.18$ m (2 d.p.)

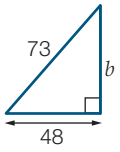
The distance between the swimmers is approximately 30.18 m.

Worked Example 18 WE18

Calculate the value of x, rounded to two decimal places.



Working



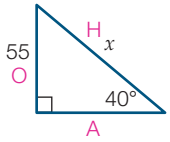
$c^2 = a^2 + b^2$, where $c = 73$, $a = 48$

$73^2 = 48^2 + b^2$

$b^2 = 73^2 - 48^2$

$b^2 = 3025$

$b = 55$



$\theta = 40°$

$H = x$

$O = 55$

$\sin\theta = \frac{O}{H}$

$\sin 40° = \frac{55}{x}$

$x = \frac{55}{\sin 40°}$

$x = 85.56$ units (2 d.p.)

Answers page 637

5.6 Mixed two-dimensional problems

Stage 5.2◊

Navigator

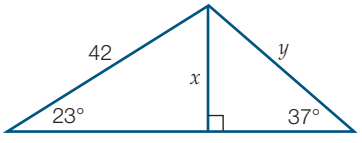
Q1, Q2, Q3, Q4 (a), (b), Q5, Q6, Q7, Q8, Q10, Q12, Q15, Q19	Q1, Q2, Q3, Q4 (c), (d), Q6, Q7, Q9, Q10, Q12, Q13, Q14, Q15, Q16, Q19	Q1 (b), (d), (f), Q2, Q3, Q4, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q15, Q16, Q17, Q18, Q19

Fluency and understanding

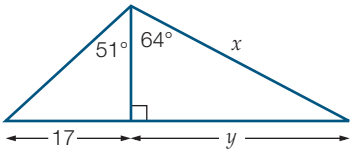
WE16

1 Find the values of x and y, rounded to two decimal places.

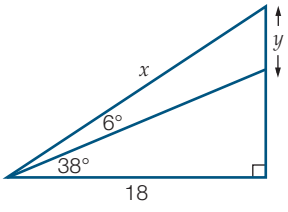
(a)



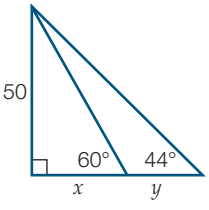
(b)



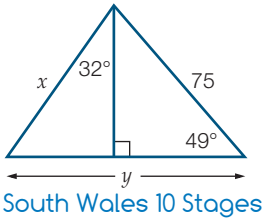
(c)



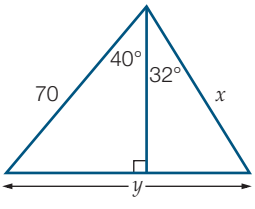
(d)



(e)



(f)



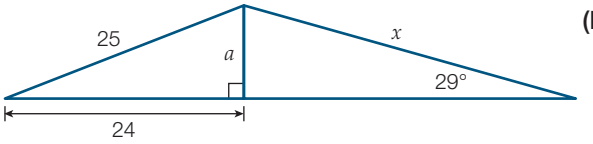
Hint

Use unrounded answers in further calculations to avoid rounding errors.

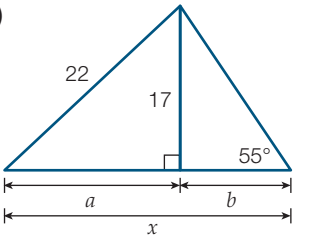
2 From the top of a lighthouse 75 m above sea level, two whales on the same bearing can be seen in the ocean at angles of depression of 28°40′ and 46°25′. Find the distance between the two whales, rounded to two decimal places. WE17

3 Calculate the value of the x in the following, rounded to two decimal places. WE18

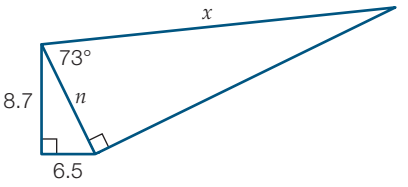
(a)



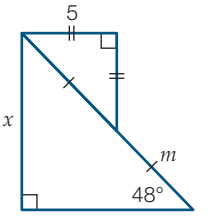
(b)



(c)

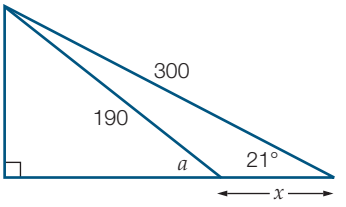


(d)

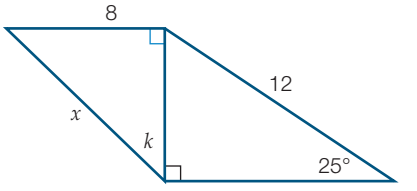


4 Find the values of the pronumerals. Round your answers to one decimal place.

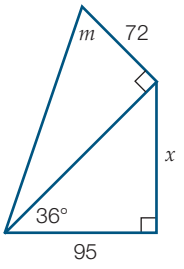
(a)



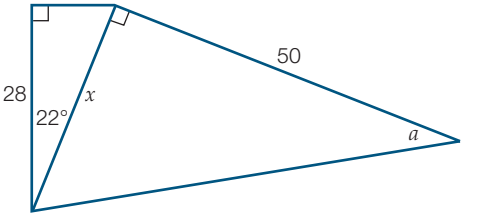
(b)



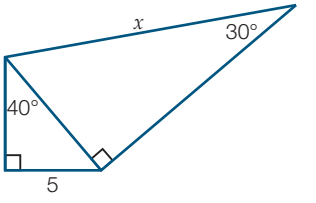
(c)



(d)



5 The length of x in the diagram is closest to:



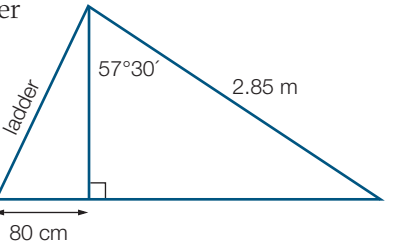
A 15.56

B 17.96

C 31.11

D 42.04

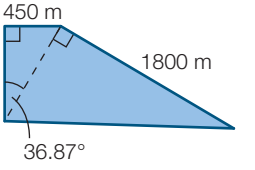
6 A slide in a playground has a vertical support and a ladder as shown in the diagram. The slide is 2.85 m long and makes an angle of 57°30′ with the vertical support. If the base of the ladder is 80 cm away from the support, the length of the ladder, to two decimal places, is:



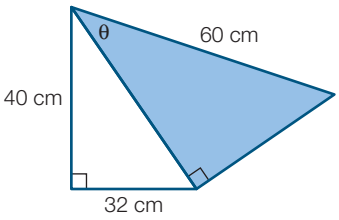
A 0.37 m B 0.42 m

C 1.53 m D 1.73 m

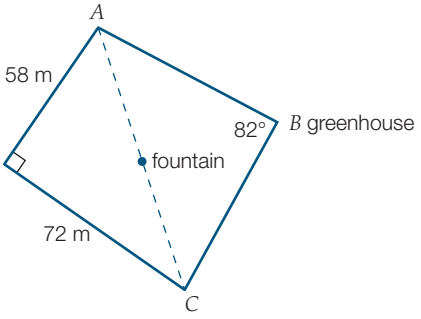
7 A farm whose perimeter shape is shown requires wire fencing costing $5 per metre. Calculate the total cost of fencing, to the nearest dollar.



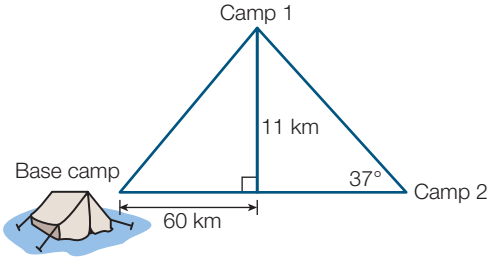
8 A blue tile and a white tile are to be joined as part of a mosaic design. What angle θ is required? Express your answer in degrees and minutes.



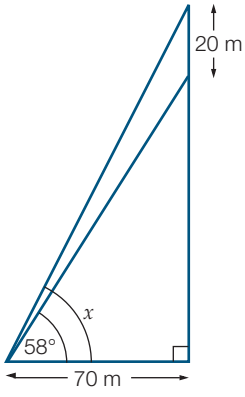
9 A surveyor is hired to peg out the garden shown. Shape *ABC* will be an isosceles triangle and the fountain will be positioned midway between *A* and *C*. To the nearest metre, what will be the length of the direct path from the fountain to the greenhouse?



10 Miguel left his base camp and trekked to Camp 1, then to Camp 2 and finally back to base camp. To the nearest kilometre, how far did he walk in total?



11 When Liz is 70 m away from a building she notices a stunt actor being filmed for a movie at an angle of elevation of 58° from her horizontal line of sight. He is climbing the side of the building. What will be the angle of elevation after the stunt actor has climbed a further 20 m? Round your answer to the nearest degree.

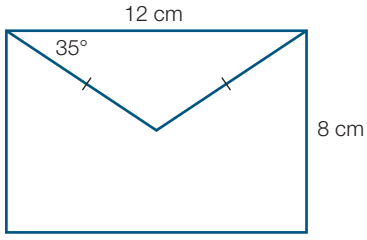


12 A 4.5 m ladder leaning against a wall makes an angle of 64°5′ with the ground. If the ladder is moved 83 cm down the wall, what angle does it now make with the ground, in degrees, minutes and seconds?

13 A pilot is landing a plane. She views two lights positioned in a line on the runway at angles of depression of 18° and 22°. If the plane is 300 m above the runway, find the distance between the two lights. State your answer correct to two decimal places.

Reasoning, communicating and open-ended

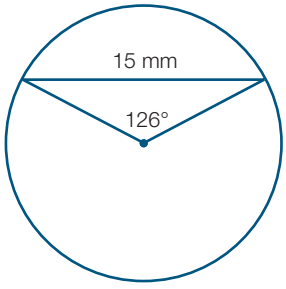
14 An envelope has a length of 12 cm and a breadth of 8 cm. The back flap is an isosceles triangle with an edge that makes an angle of 35° with the top of the envelope. Find the distance between the bottom of the flap and the bottom of the envelope. Round your answer to two decimal places.



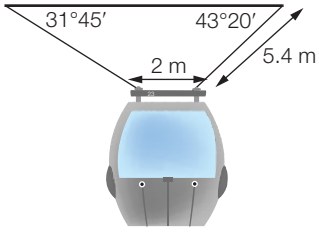
Hint

Draw a vertical line from the bottom of the flap to the top of the envelope.

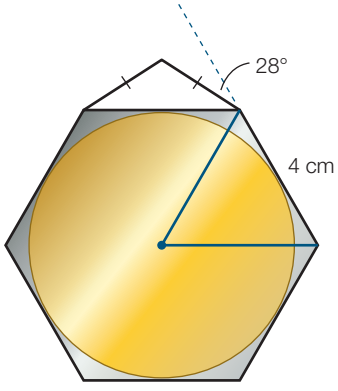
15 A chord 15 mm long forms an angle of 126° at the centre of a circle. Find the length of the diameter of the circle, correct to two decimal places.



16 A cable car is supported by two wires attached to a horizontal cable. One of the wires is 5.4 m long and makes an angle of 43°20′ with the horizontal cable. The other wire makes an angle of 31°45′ with the cable. If the cable car is 2 m long, how far apart along the cable are the wires, correct to two decimal places?



17 A jeweller has designed a medal in the shape of a circle mounted on a regular hexagon. The circular inset is made from gold. The medallion will hang by a chain passing through the triangular shape shown at the top of the medal.



Hint

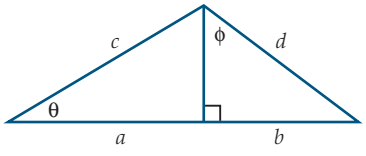
The sum of the interior angles of an n-sided polygon is $(n - 2) \times 180°$.

(a) Use the triangle marked on the medal to find the radius of the circular inset.

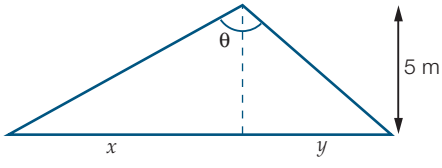
(b) Use your answer to **(a)** to find the area of gold required to make the circular inset. Round your answer to the nearest square centimetre.

(c) What length of metal is required for the two sides of the triangular hanging top? Round your answer to two decimal places.

18 Determine values for a, b, c, d, θ and ϕ so that $a > b$ and $c > d$.



19 The cross-section of a roof (not necessarily symmetrical in shape) of height 5 m is shown. Determine values for x, y and θ so that the base has a length between 20 m and 25 m, and θ is greater than 100°.



Chapter review 5

D.I.Y. Summary

Key Words

Stage 5.2◊	adjacent side	cosine	opposite side	tangent
	angle of depression	hypotenuse	second	trigonometric ratio
	angle of elevation	minute	sine	true bearing
	compass bearing			

Stage 5.2◊ Q1–8

Copy and complete the following using words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 The _____ ratio is obtained by dividing the adjacent side by the hypotenuse.

2 The _____ ratio is obtained by dividing the opposite side by the hypotenuse.

3 The _____ ratio is obtained by dividing the opposite side by the adjacent side.

4 When a direction is given with a measurement east or west from north or south, this is a _____.

5 A _____ always has three digits and shows the number of degrees from north in a clockwise direction.

6 One degree is divided into sixty equal parts. Each part is called a _____.

7 There are 3600 _____s in one degree.

8 The angle from the horizontal as you look up is called an _____. The angle from the horizontal as you look down is called an _____.

Stage 5.2◊ Q1–18

Fluency and understanding

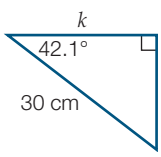
1 Use a calculator to find the following values, rounded to four decimal places. Ex.5.1

(a) sin 35° (b) tan 51.3° (c) cos 6.045°

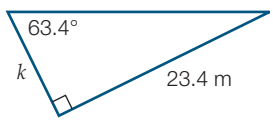
(d) tan 29°45′ (e) sin 16°24′ (f) cos 73°6′40″

2 Find the value of k in each of these triangles, rounded to two decimal places. Ex.5.2

(a)



(b)



3 For each of the following trigonometric ratios, find the value of θ in: Ex.5.3

(i) decimal degrees, rounded to two decimal places

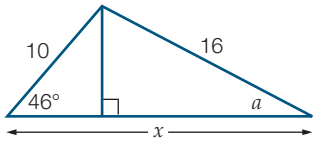
(ii) degrees, minutes and seconds.

(a) $\cos\theta = 0.74$ (b) $\tan\theta = 2.645$ (c) $\sin\theta = 0.4609$

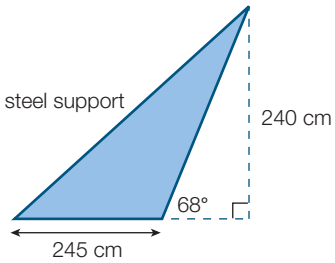
4 A mast of height 5 m is on top of a building whose base is 100 m away from a drain on the road. The angle of elevation of the top of the mast from the drain is 34°. Calculate the height of the building, rounded to two decimal places. Ex.5.4

5 A plane flies due north for 70 km, then due west for a further 145 km. What is the compass bearing of the plane from its starting point, to the nearest degree? Ex 5.5

6 Find the values of the pronumerals in this diagram. Round your answers correct to one decimal place. Ex 5.6



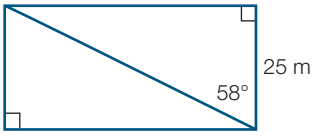
7 The sculpture, 'A study in triangles', is supported by a rigid steel rod. What is the length of the steel support, rounded to the nearest centimetre? Ex 5.6



8 A conveyor belt in an underground mine is 650 m long and moves to the surface at an angle of 14°48′ from the horizontal. The depth of the mine is: Ex 5.6

A 2544.57 m **B** 672.30 m **C** 628.44 m **D** 166.04 m

9 The length of the diagonal in this rectangle is: Ex 5.6



A 13.25 m **B** 29.48 m

C 40.01 m **D** 47.18 m

10 A balloon on the end of a string attached to the ground is being blown by the wind. If the balloon is 12.75 m above the ground and the string makes an angle of 48° to the horizontal, find the length of the string, rounded to two decimal places.

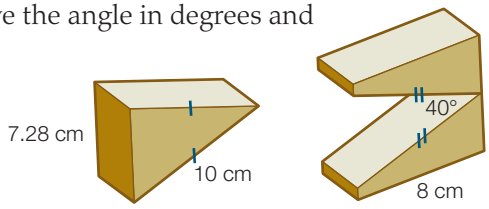
11 A ramp is 9.4 m long and joins two levels, one 3.1 m higher than the other. What angle does the ramp make with the lower level, rounded to one decimal place?

12 A lifeguard on watch at a beach spots a shark at an angle of depression of 9°. If the lifeguard views the shark from a height (to his eye-level) of 16 m above sea-level, find the horizontal distance between the lifeguard and the shark to the nearest metre.

13 A speedboat travels 26 km on a true bearing of 065°T. Find how far the boat is to the east of its starting point. Round your answer to two decimal places.

14 An aeroplane leaves the airport and flies 35 km in the direction N70°E. It then flies due east for a further 38 km. What is the aeroplane's distance and compass bearing from the airport? Round the distance to the nearest kilometre and give the angle in degrees and minutes.

15 Decide whether the wedge shown will fit exactly into the cavity in the block of wood. Give reasons for your answer.

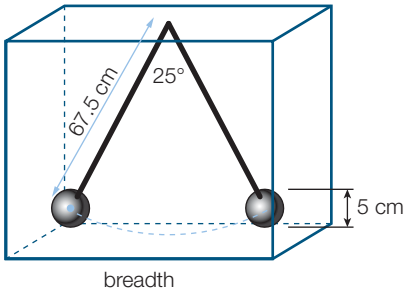


Reasoning and communicating

16 Amos leaves home and cycles for a distance of 35 km on a bearing of N30°E. He changes direction and cycles in a direction of S60°E before heading home on a bearing of S49°W. To the nearest kilometre, find the total distance Amos travelled.

17 For an equilateral triangle, explain two different ways the height can be found using trigonometric ratios.

18 A case is to be constructed to hold a pendulum. The shaft of the pendulum is connected to the centre of the bob and is 67.5 cm in length. The diameter of the bob is 5 cm. If the pendulum swings through an angle of 25°, what must the minimum breadth of the case be? Round your answer to two decimal places.



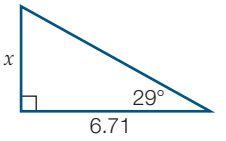
6 Advanced trigonometry

P Recall 6

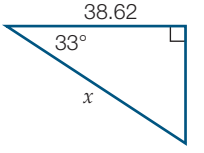
Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Calculate the value of x and round to two decimal places.

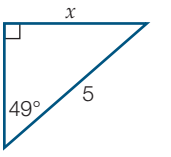
(a)



(b)

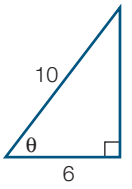


(c)



2 Determine the size of the angle θ, in degrees and minutes.

(a)



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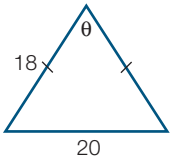

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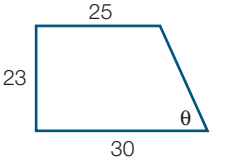
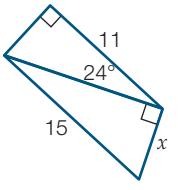
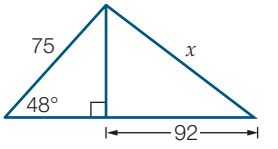


3 Find the length of x. Round your answer to two decimal places.

(a)



(b)



4 Jake is flying a kite on the end of a 20 m rope. If the rope has an angle of elevation from Jake's line of sight of 69°, how high is the kite above the ground, given the distance from the ground to Jake's eye-level is 1.65 m? Round your answer to two decimal places.

5 A hiker leaves her campsite and walks 10 km on a bearing of 143°T, then walks 6 km on a bearing of 039°T. Calculate the following and round your answers to two decimal places.

(a) How far is she from her campsite?

(b) What bearing must she take to walk directly back to the camp?

Syllabus references

Measurement and Geometry: Trigonometry and Pythagoras' Theorem

- ✓ Apply Pythagoras' theorem and trigonometry to solve three-dimensional problems in right-angled triangles **(ACMMG276) (Stage 5.3§)**
- ✓ Use the unit circle to define trigonometric functions, and graph them, with and without the use of digital technologies **(ACMMG274) (Stage 5.3§)**
- ✓ Solve simple trigonometric equations **(ACMMG275) (Stage 5.3§)**
- ✓ Establish the sine, cosine and area rules for any triangle and solve related problems **(ACMMG273) (Stage 5.3§)**

Solving three-dimensional problems

6.1

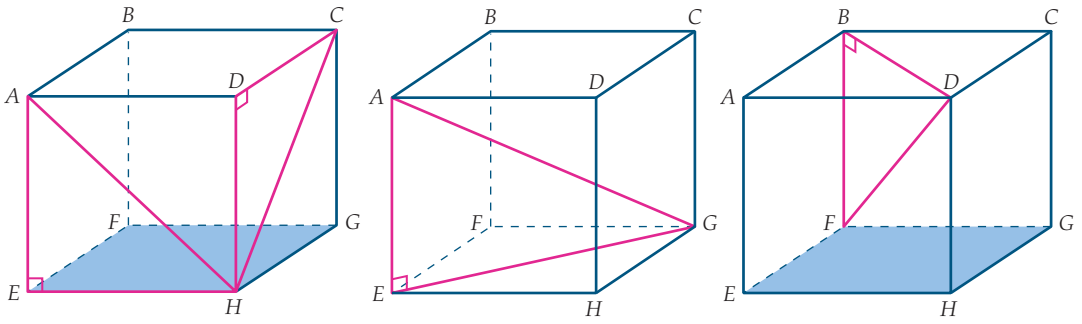
Need to Know

Stage 5.3§

Three-dimensional problems

A three-dimensional problem involving right-angled triangles may require the use of any combination of the sine, cosine and tangent ratios, and Pythagoras' Theorem.

A diagram drawn to appear three-dimensional (3D) has right-angled triangles drawn as two-dimensional (2D) figures. Note that right angles may not appear as exactly 90° on your page in a 3D representation.



Right-angled triangles such as ΔAEH and ΔCDH can be found quite easily. Inside the cube, right-angled triangles such as ΔAEG and ΔDBF can be located where the hypotenuse of the triangle is the diagonal connecting opposite vertices of the cube.

It helps to redraw the right-angled triangle to be solved as a separate two-dimensional diagram. More than one right-angled triangle may need to be solved in the three-dimensional diagram to obtain the required unknown.

To solve a three-dimensional problem involving right-angled triangles:

Step 1 Draw a diagram that represents the problem.

Step 2 Identify right-angled triangles that contain the unknown you want or that will help you find it.

Step 3 Redraw these right-angled triangles.

Step 4 Solve these triangles using trigonometric ratios and/or Pythagoras' Theorem.

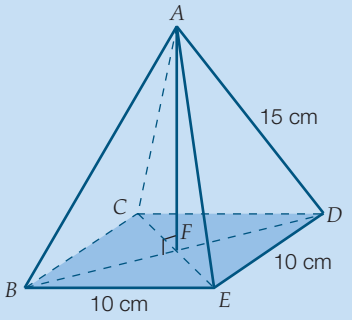
Worked Example 1

WE1

A right pyramid *ABCDE* stands on a square base of edge length 10 cm. The sloping edges are each of length 15 cm. *AF* is the perpendicular height.

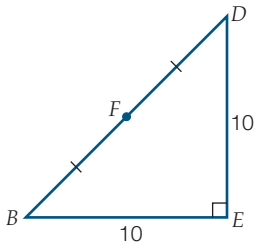
(a) Find the angle that a sloping edge makes with the base of the pyramid, rounded to the nearest degree.

(b) Hence, find the height of the pyramid, rounded to two decimal places.



Working

(a)



(We will use ΔBDE to find BD as DF is half of BD.)

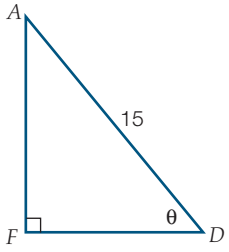
In ΔBDE:

$(BD)^2 = 10^2 + 10^2$

$(BD)^2 = 200$

$BD = \sqrt{200}$

$DF = \dfrac{\sqrt{200}}{2}$



(We will use ΔADF to find θ.)

In ΔADF:

$AD = 15,\ DF = \dfrac{\sqrt{200}}{2},\ \theta = ?$

$\cos\theta = \dfrac{A}{H}$

$\cos\theta = \dfrac{DF}{15}$

$\theta = \cos^{-1}\left(\dfrac{DF}{15}\right)$

$\theta = \cos^{-1}\left(\dfrac{\sqrt{200}}{30}\right)$

$\theta = 61.874\,474\,3$

$\theta = 62°$ (nearest degree)

A sloping edge makes an angle of about 62° with the base of the pyramid.

(b) $\sin\theta = \dfrac{O}{H}$

$\sin\theta = \dfrac{AF}{15}$

$AF = 15\sin\theta$

$AF = 15\sin 61.874\,494\,3°$

$AF = 13.228\,756\,56$

$AF = 13.23$ (2 d.p.)

The height of the pyramid is about 13.23 cm.

6.1 Solving three-dimensional problems

Answers page 637

Navigator

Stage 5.3§

Q1, Q2, Q3, Q5, Q9, Q12, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q9, Q10, Q11, Q12, Q15, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

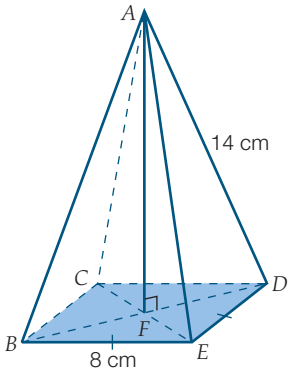
Fluency and understanding

WE1

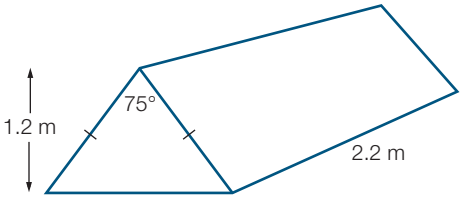
1 A right pyramid *ABCDE* stands on a square base of edge length 8 cm. The sloping edges are each 14 cm long. *AF* is the perpendicular height.

(a) Find the angle that a sloping edge makes with the base of the pyramid, rounded to the nearest degree.

(b) Hence, find the height of the pyramid, rounded to two decimal places.



2 A tent has the dimensions shown in the diagram. Find the area of the groundsheet needed to fit inside the tent. Round your answer to two decimal places.



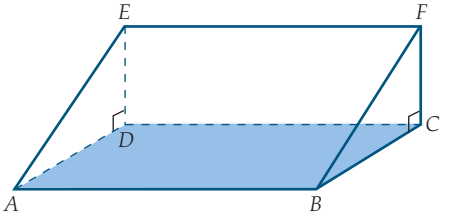
3 **(a)** To find the angle that plane *ABFE* makes with plane *ABCD*, we could calculate:

A $\angle FCB$ **B** $\angle BFC$

C $\angle DAB$ **D** $\angle DAE$

(b) To find the length *AF*, we could use the triangle:

A *ABC* **B** *AEF* **C** *BCF* **D** *ABD*



For Questions **4** to **11**, unless stated otherwise, round angles to one decimal place and round other answers to two decimal places.

4 *PQRST* is a square-based pyramid with a base of side length 14 mm and a height of 6 mm. The point *X* on the base is directly below *T* at the apex, and point *Y* is midway along *PQ*.

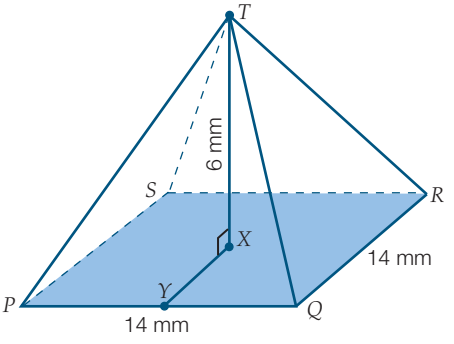
(a) Find the angle *TXY*.

(b) What angle does the sloping face *PQT* make with the base?

(c) Find the length of *TY*.

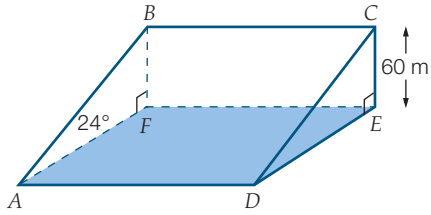
(d) Find the length of a sloping edge of the pyramid.

(e) Find the angle that a sloping edge makes with the base of the pyramid.



5 A new road sign warning of merging traffic consists of a circle inside an equilateral triangle. What is the radius of the circle?

6 A hill *ABCD* rises at an angle of 24° to the horizontal. The hill is 60 m high.



(a) If you walked straight up the hill along *AB*, what distance would you travel?

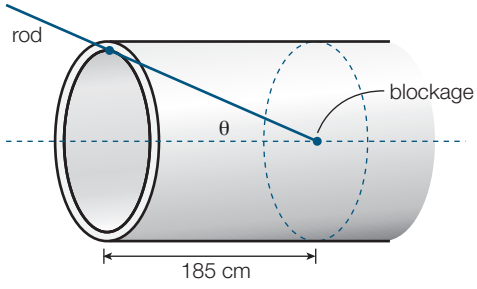
(b) What horizontal distance would you cover walking along *AB*?

(c) Your friend walks along a track from *A* to *C*, taking 30 minutes to reach the top. Horizontally, the distance she covers is 510 m. How fast does she walk in m/min?

(d) Find the angle of the track *AC* to the horizontal.

(e) Gradient represents the steepness of a slope and can be calculated by using the formula $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{vertical distance}}{\text{horizontal distance}}$. Calculate the gradients of *AB* and *AC*, and hence determine which track is steeper.

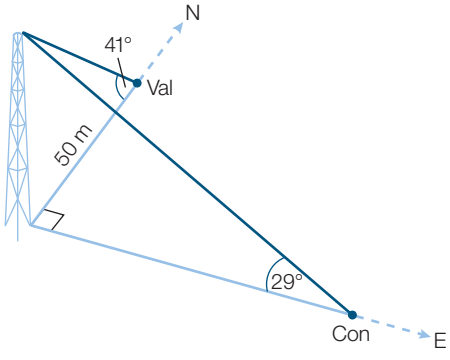
7 A plumber inserts a metal rod into a cylindrical pipe whose inner radius is 9 cm and discovers that there is a blockage in the centre of the inner circle of the pipe at the point shown.



(a) What is the maximum length of the rod that can be inserted inside the pipe to reach the blockage?

(b) What is the maximum angle θ possible? Give your answer in degrees, minutes and seconds.

8 Val is 50 m due north of a communications tower. The top of the tower is at an angle of elevation of 41° from the ground at Val's observation point. From Con's observation point, due east of the tower, the angle of elevation of the top of the tower is 29°.



Find:

(a) the height of the tower

(b) the distance from Con to the foot of the tower

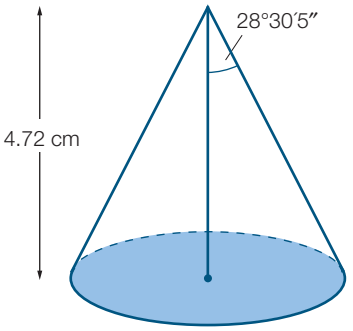
(c) the distance between Val and Con

(d) Con's true bearing from Val's observation point.

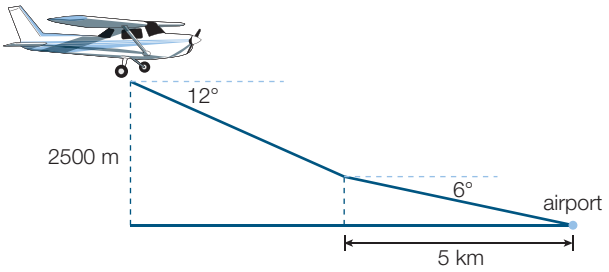
9 The height of a vertical cone is 4.72 cm, and its semi-vertical angle is 28°30′5″. Find:

(a) the slant height of the cone

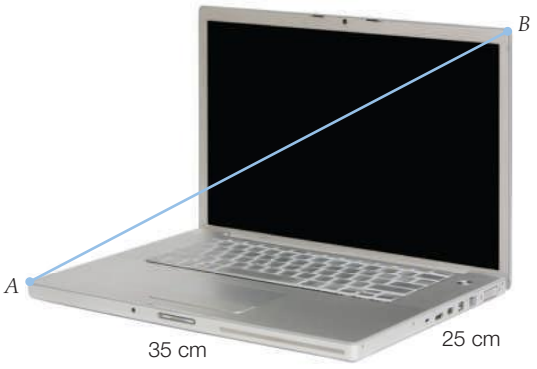
(b) the diameter of the cone.



10 A light plane at a height of 2500 m begins to descend at an angle of 12° to the horizontal. When the plane is 5 km from the airport, the descent angle is reduced to 6°. What horizontal distance from the airport is the plane when it is 2500 m above the ground? Round your answer to the nearest metre.



11 A laptop sends information via infrared from a transmitter at point *A* to a receiver at point *B*. When the screen is at right angles to the keyboard, the distance from the receiver to the transmitter must be no more than 50 cm. Is this true for the laptop shown?



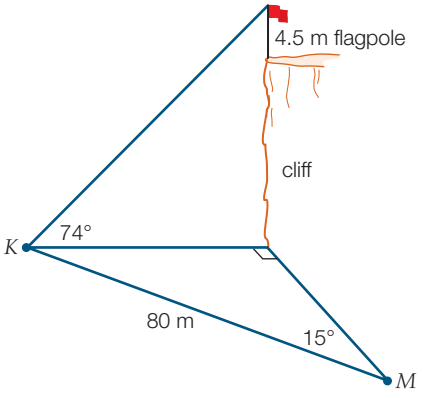
Reasoning, communicating and open-ended

12 A passenger on a cruise ship is positioned 42 m above the water level and is admiring a tropical island in the distance. One end of the island is at a true bearing of 028° and has an angle of depression of 11°, while the other end is at a true bearing of 118° and has an angle of depression of 19°. How long is the island to the nearest metre?

13 A ship's navigator needs to find the height of a cliff. She takes measurements from two different positions in the water, as shown in the diagram. There is a flagpole on top of the cliff.

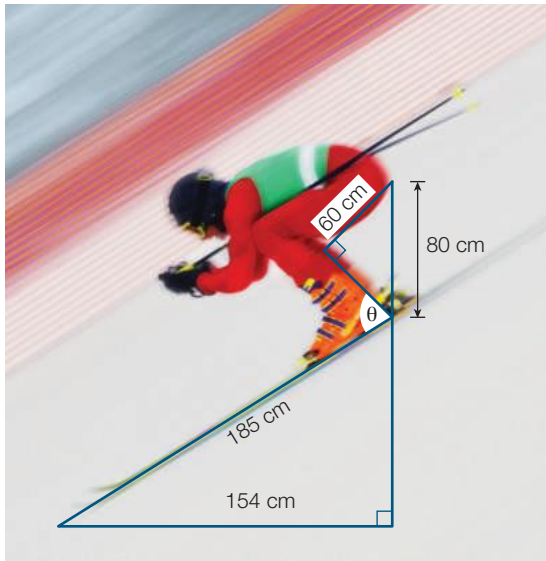
(a) If the flagpole is known to be 4.5 m tall, find the height of the cliff.

(b) What is the distance from the top of the flagpole to the navigator at position *K*?

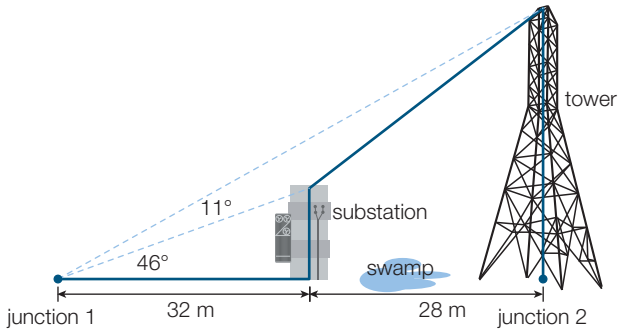


14 Liam wants to construct a rectangular desktop with a diagonal length of 70 cm. When the desktop is tilted at 15° to the horizontal, it is raised 8 cm. Find the area of the desktop, to the nearest square centimetre.

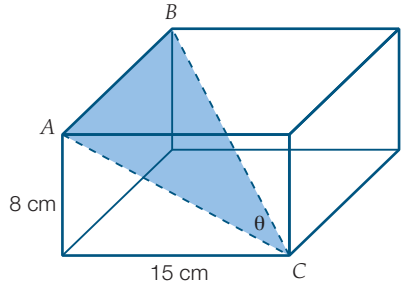
15 To minimise a skier's air resistance, the angle θ shown must be less than 77°. Decide if this is true for the skier shown.



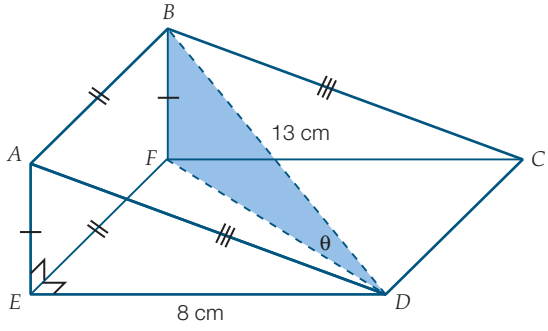
16 Electric cable (shown in blue) will be laid underground from junction 1 to the base of the substation. It will continue vertically up the wall of the substation. From the top of the substation it will continue over the swamp to the top of the tower and down to the ground to junction 2. What is the total length of cable required?
Round your answer to the nearest metre.



17 A jewellery box in the shape of a rectangular prism has the dimensions shown. Using three different values for θ, obtain the area of triangle *ABC* and calculate the ratio $\frac{\text{area of triangle } ABC}{\text{volume of box}}$.
What can you conclude about your results?



18 A wedge of cheese in the shape of a triangular prism has the dimensions shown. By choosing two values for the angle θ, calculate the following, correct to two decimal places.



(a) Find the height, *BF*, of the wedge.

(b) Find the lengths *DF*, *EF* and *BC*.

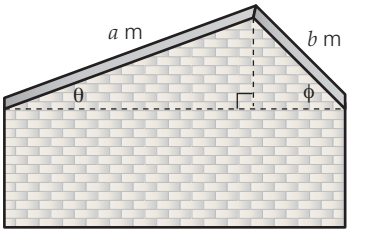
(c) Calculate the area of rectangle *ABCD*.

(d) What effect did changing the value of θ have on the area of *ABCD*?

(e) Calculate the total area of plastic needed to wrap the cheese (no overlap).

(f) For which value of θ was the least plastic used? Compare this to your answer in **(d)**.

19 The cross-section of a roof is shown. a and b are the lengths of the roofing, and θ and ϕ are the angles each roofline makes with the horizontal.



(a) Show that $\frac{a}{b} = \frac{\sin\phi}{\sin\theta}$.

(b) If $a = 2b$ and angles θ and ϕ must be greater than 20° and less than 60°, state one pair of possible values for the angles θ and ϕ. State your answer to the nearest degree.

Problem solving

The maths challenge

You are in one of two 'tribes' that are competing on a desert island. Both tribes are led to the edge of a deep, fast flowing river, where the weekly challenge is issued. Each tribe is given a large protractor, a 10 m long measuring tape, and some instructions, written on a piece of bark. You crowd around the bark with your fellow tribe members. It reads:

Calculate the height of the tree directly across the river from where you are standing. Use only the equipment you have been given, plus any stick, stone, leaf or vine from the jungle around you. The first team to accurately determine the height of the tree wins double food rations for the week.

P.S. Two triangles could be useful.

'That's impossible!' moan half the tribe members. 'We don't even know the width of the river!'

Describe the method and calculations your tribe could use to calculate the height of the tree and win the challenge.

Strategy options

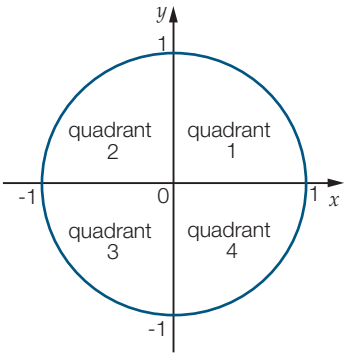
- Make a model.
- Act it out.
- Break problem into manageable parts.

6.2 The unit circle

Stage 5.3§

Need to Know

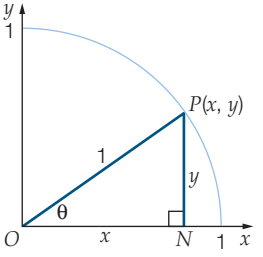
A **unit circle** has a radius of 1 and a centre at the origin (0, 0) of the Cartesian plane. The unit circle is divided into four **quadrants** as shown on the diagram.



Angles in the first quadrant

The right-angled triangle *NOP* has been drawn in quadrant 1 of the unit circle with its radius as the hypotenuse. From Δ*NOP*, we can define two trigonometric ratios:

- sine θ, abbreviated to sin θ
- cosine θ, abbreviated to cos θ.



$$\sin\theta = \frac{O}{H} = \frac{NP}{OP} = \frac{y}{1} = y$$

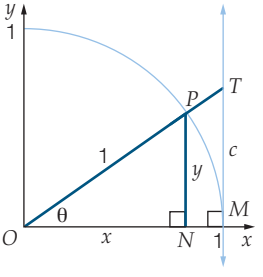
$$\cos\theta = \frac{A}{H} = \frac{ON}{OP} = \frac{x}{1} = x$$

In the right-angled triangle *MOT*, *TM* is a tangent to the circle at *M* and parallel to the *y*-axis at the point *T*. The length of *TM* is marked here as *c*.

From Δ*MOT*, we can define the trigonometric ratio:

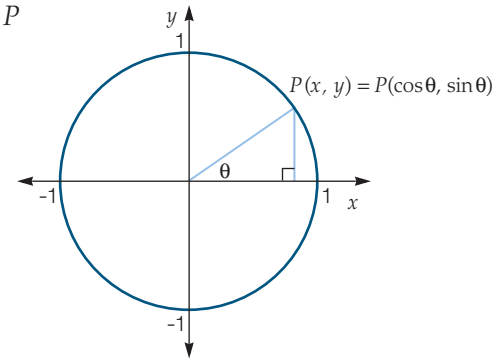
- tangent θ, abbreviated to tan θ.

$$\tan\theta = \frac{O}{A} = \frac{TM}{OM} = \frac{c}{1} = c = \frac{y}{x} \quad (\Delta ONP \,|||\, \Delta OMT)$$



Note that $\tan\theta = \frac{\text{rise}}{\text{run}}$ for the line *OT*, so this tangent is also the gradient *m* of the line *OT*.

The angle θ determines the position of a point P on the circumference of the unit circle. $P(x, y)$ can now be written as $P(\cos\theta, \sin\theta)$, where:



- $\cos\theta = x$ (the x-coordinate of P)
- $\sin\theta = y$ (the y-coordinate of P)
- $\tan\theta = \dfrac{y}{x}$

  $= \dfrac{\sin\theta}{\cos\theta}$

In quadrant 1, all trigonometric ratios are positive.

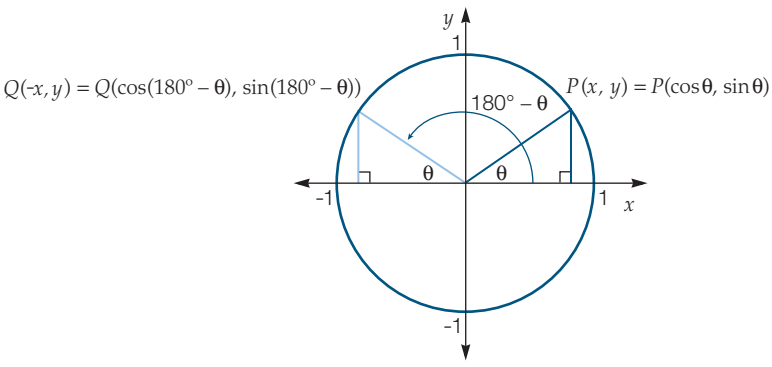
As θ increases from 0° to 90°:

- $\sin\theta$ increases from 0 to 1
- $\cos\theta$ decreases from 1 to 0
- $\tan\theta$ increases from 0 to an undefined positive ratio.

Equivalent angles

The unit circle can also be used to find trigonometric ratios for angles in the other three quadrants using **equivalent angles**. By drawing a congruent right-angled triangle in each quadrant we can determine trigonometric ratios for angles that are not acute.

Angles in the second quadrant



$\sin(180° - \theta) = \dfrac{O}{H}$

$= \dfrac{y}{1}$

$= y$

$= \sin\theta$

$\cos(180° - \theta) = \dfrac{A}{H}$

$= \dfrac{-x}{1}$

$= -x$

$= -\cos\theta$

$\tan(180° - \theta) = \dfrac{O}{A}$

$= \dfrac{y}{-x}$

$= -\dfrac{y}{x}$

$= -\tan\theta$

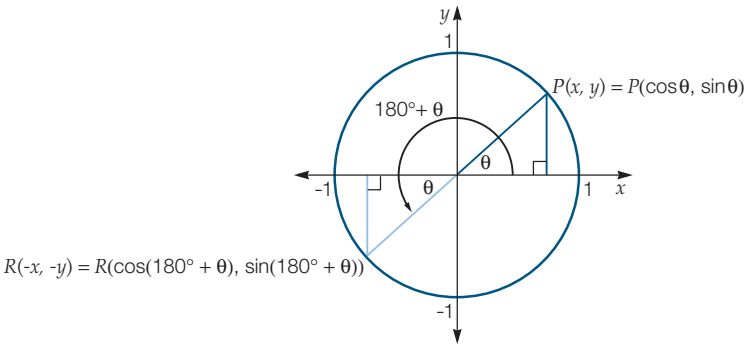
In quadrant 2, only the sin ratio is positive.

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\cos(180° - \theta) = -\cos\theta$ $\quad$ $\tan(180° - \theta) = -\tan\theta$

As θ increases from 90° to 180°:

- $\sin\theta$ decreases from 1 to 0
- $\cos\theta$ decreases from 0 to -1
- $\tan\theta$ decreases from an undefined positive ratio to 0.

Angles in the third quadrant



$$\sin(180° + \theta) = \frac{O}{H} = \frac{-y}{1} = -y = -\sin\theta$$

$$\cos(180° + \theta) = \frac{A}{H} = \frac{-x}{1} = -x = -\cos\theta$$

$$\tan(180° + \theta) = \frac{O}{A} = \frac{-y}{-x} = \frac{y}{x} = \tan\theta$$

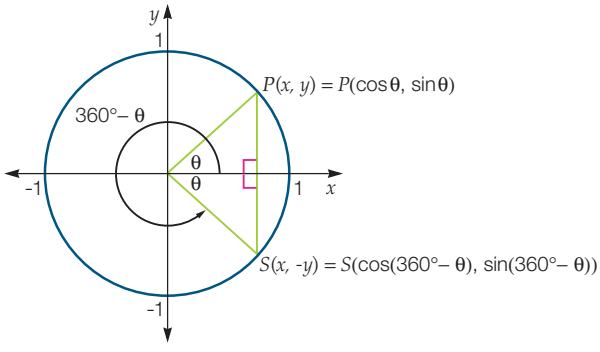
Only the tan ratio is positive in quadrant 3.

$\tan(180° + \theta) = \tan\theta$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$

As θ increases from 180° to 270°:

- $\sin\theta$ decreases from 0 to -1
- $\cos\theta$ increases from -1 to 0
- $\tan\theta$ decreases from 0 to an undefined negative ratio.

Angles in the fourth quadrant



$$\sin(360° - \theta) = \frac{O}{H} = \frac{-y}{1} = -y = -\sin\theta$$

$$\cos(360° - \theta) = \frac{A}{H} = \frac{x}{1} = x = \cos\theta$$

$$\tan(360° - \theta) = \frac{O}{A} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta$$

In quadrant 4, only the cos ratio is positive.

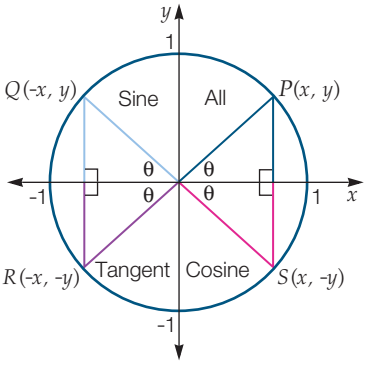
$\cos(360° - \theta) = \cos\theta$ $\sin(360° - \theta) = -\sin\theta$ $\tan(360° - \theta) = -\tan\theta$

As θ increases from 270° to 360°:

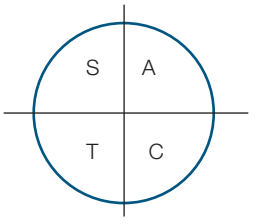
- $\sin\theta$ increases from -1 to 0
- $\cos\theta$ increases from 0 to 1
- $\tan\theta$ increases from an undefined negative ratio to 0.

Summary of trigonometric ratios in the unit circle

	Angle			
	0°	90°	180°	270°
sin θ	0	1	0	-1
cos θ	1	0	-1	0
tan θ	0	undefined	0	undefined



- In quadrant 1, all the ratios are positive. (ALL, A)
- In quadrant 2, only the sine ratio is positive. (SINE, S)
- In quadrant 3, only the tangent ratio is positive. (TANGENT, T)
- In quadrant 4, only the cosine ratio is positive. (COSINE, C)



This can be remembered by using the CAST diagram shown. You can make up your own mnemonic (a word or phrase that you can use to help you remember).

Angles of any magnitude

Only angles from 0° to 360° have been used until now. Note that cos θ = 1 for θ = 0° and θ = 360° as both angles represent the same position on the unit circle. In fact, if θ is any given angle, every multiple of 360° added to θ or subtracted from θ represents the same point on the unit circle.

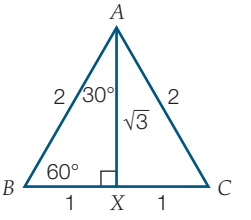
For example:

As θ values of 25°, 25° + 360°, 25° + 720°, 25° – 360° and 25° – 720° all occupy the same position on the unit circle, they have the same trigonometric ratio values.

sin 385° = sin 25° cos 745° = cos 25° tan (-335°) = tan 25°

Deriving exact trigonometric ratios for 30° and 60°

Consider an equilateral triangle of side length 2 units. All angles are 60°. If a perpendicular line is drawn from one vertex to the opposite side, two congruent triangles are formed with the hypotenuse 2 units long and one side of 1 unit in length. Using Pythagoras' Theorem, the third side is $\sqrt{3}$ units long.



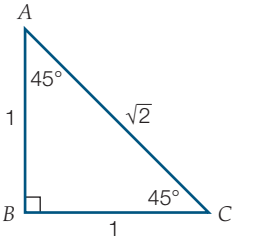
The exact trigonometric ratios for 30° and 60° can now be found.

$\sin 30° = \frac{1}{2}$ $\cos 30° = \frac{\sqrt{3}}{2}$ $\tan 30° = \frac{1}{\sqrt{3}}$

$\sin 60° = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{1}{2}$ $\tan 60° = \frac{\sqrt{3}}{1}$

Deriving exact trigonometric ratios for 45°

Consider an isosceles right-angled triangle of side length 1 unit. Using Pythagoras' Theorem, the hypotenuse is of length $\sqrt{2}$.



$\sin 45° = \frac{1}{\sqrt{2}}$ $\cos 45° = \frac{1}{\sqrt{2}}$ $\tan 45° = 1$

Exact values for related values in other quadrants can now also be found.

For example: sin 135°, cos 225°, tan 315°

Worked Example 2

WE2

State the quadrant in which each of the following angles is found.

(a) 35° **(b)** 265° **(c)** 135° **(d)** 342°

Working

(a) quadrant 1
(35° is between 0° and 90°)

(b) quadrant 3
(265° is between 180° and 270°)

(c) quadrant 2
(135° is between 90° and 180°)

(d) quadrant 4
(342° is between 270° and 360°)

Worked Example 3

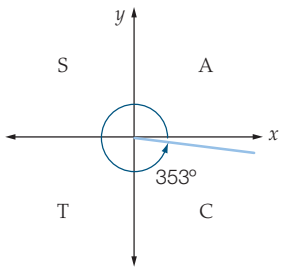
WE3

For each of the following relationships, find the equivalent trigonometric ratio in quadrant 1, taking care to include the appropriate sign.

(a) tan 353° **(b)** sin 124° **(c)** cos 234°

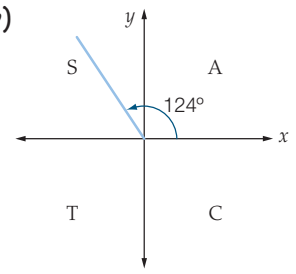
Working

(a)



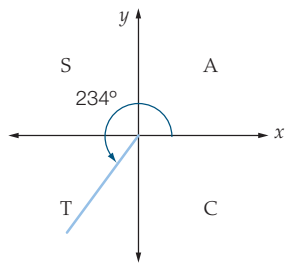
353° = (360 – 7)°
tan 353° = -tan 7°
(tan is negative in quadrant 4)

(b)



124° = (180 – 56)°
sin 124° = sin 56°
(sin is positive in quadrant 2)

(c)



234° = (180 + 54)°
cos 234° = -cos 54°
(cos is negative in quadrant 3)

Worked Example 4

WE4

Find the exact value for each of the following trigonometric ratios.

(a) sin (225)° **(b)** cos (330)° **(c)** tan (120)°

Working

(a) sin (225)°
= sin (180 + 45)°
= -sin 45°
$= -\frac{1}{\sqrt{2}}$

(b) cos (330)°
= cos (360 – 30)°
= cos 30°
$= \frac{\sqrt{3}}{2}$

(c) tan (120)°
= tan (180 – 60)°
= -tan 60°
$= -\sqrt{3}$

6.2 The unit circle

Answers page 638

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

WE 2

1 State the quadrant in which each of the following angles is found.

(a) 355° **(b)** 222° **(c)** 111° **(d)** 102°

(e) 191° **(f)** 23° **(g)** 263° **(h)** 97°

(i) 286° **(j)** 342° **(k)** 156° **(l)** 260°

WE 3

2 For each of the following, find the equivalent trigonometric ratio in quadrant 1, taking care to include the appropriate sign.

(a) sin 135° **(b)** cos 224° **(c)** tan 111° **(d)** sin 264°

(e) cos 321° **(f)** tan 123° **(g)** sin 263° **(h)** cos 97°

(i) tan 186° **(j)** sin 342° **(k)** cos 166° **(l)** tan 207°

WE 4

3 Find the exact value for each of the following trigonometric ratios.

(a) cos 225° **(b)** tan 330° **(c)** sin 120°

(d) tan 225° **(e)** sin 330° **(f)** cos 120°

(g) sin 135° **(h)** cos 210° **(i)** tan 240°

(j) cos 135° **(k)** tan 210° **(l)** sin 240°

4 Which of the following groups of angles are all found in quadrant 3?

A 213°, 182° and 156° **B** 187°, 245° and 194°

C 98°, 124° and 156° **D** 342°, 276° and 199°

5 The equivalent trigonometric ratio to sin 25° is:

A sin 205° **B** sin 345° **C** -sin 155° **D** sin 155°

6 The exact value of sin 300° is:

A 0.866 **B** $\frac{\sqrt{3}}{2}$ **C** $-\frac{\sqrt{3}}{2}$ **D** -0.866

7 Jay knows the values of sine, cosine and tangent when the angles are 30°, 45° and 60°. For which other angles greater than 0° but less than 360° can she find trigonometric ratios using what she already knows?

Reasoning, communicating and open-ended

8 By using a right-angled triangle and labelling one of the acute angles as θ, show that:

(a) $\sin\theta = \cos(90 - \theta)$ **(b)** $\cos\theta = \sin(90 - \theta)$

9 Write down two angles that are 180° apart. For each of your chosen angles, write down the sign for the ratios: sin, cos and tan. Use a calculator to check that you are correct.

10 **(a)** Explain why $\cos(-\theta) = \cos\theta$ for any angle θ.

(b) Explain why $\sin(-\theta) = -\sin\theta$ for any angle θ.

(c) Using your answers to **(a)**, **(b)** and the relationship $\tan(\theta) = \frac{\sin\theta}{\cos\theta}$, explain why $\tan(-\theta) = -\tan\theta$.

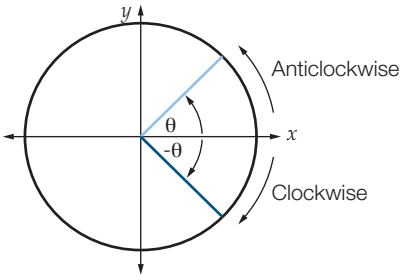
6.3 Trigonometric graphs

Stage 5.3§

Need to Know

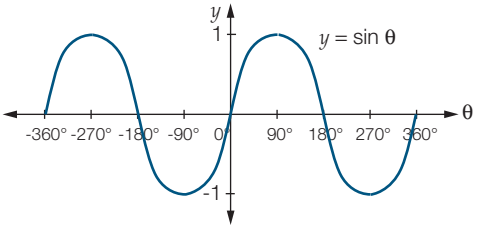
Trigonometric graphs

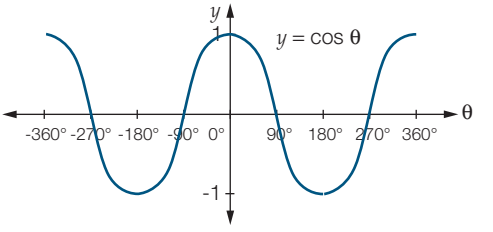
The unit circle can be used to sketch trigonometric graphs. This involves using symmetry properties and using coordinate values of the unit circle.



Note that, by convention, positive angles are measured anticlockwise from the positive direction of the x-axis, whereas negative angles are measured clockwise from the positive direction of the x-axis.

The shape of trigonometric graphs keeps repeating in cycles infinitely in both directions, so they are called cyclic graphs.



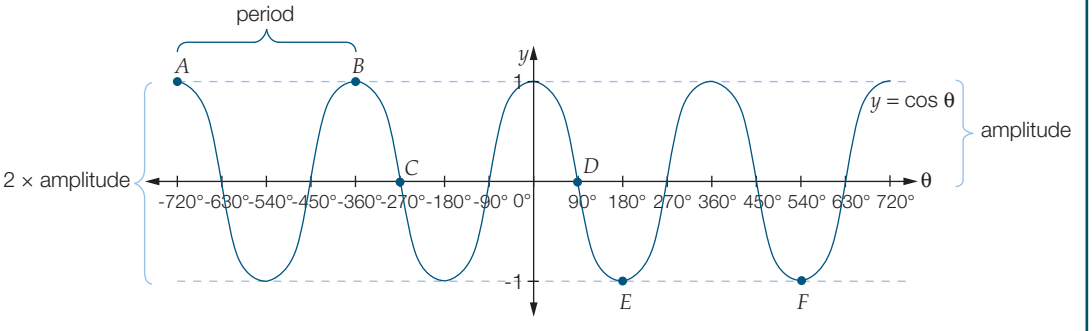


Period

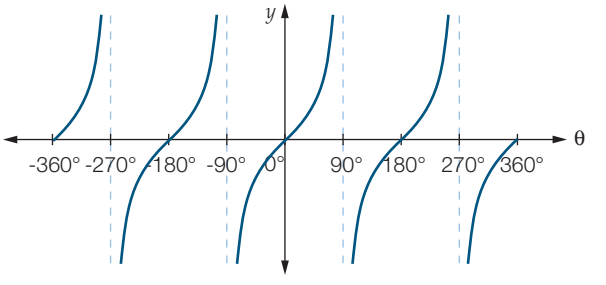
The **period** of a trigonometric graph is the horizontal distance it takes to complete one full cycle of the graph.

Amplitude

The **amplitude** of a trigonometric graph is half the distance between the minimum y-value and the maximum y-value.



The graph of tan θ is quite different to those of sin θ and cos θ. This is because it is not bounded by a maximum value of 1 and a minimum value of -1 and it has points where it is undefined. This means that the graph is not continuous at these points (there are breaks in the graph).



A tangent graph approaches **asymptotes**, where the graph is undefined. Asymptotes are lines that the graph approaches but never touches or crosses. The equations of the asymptotes are $\theta = -270°$, $\theta = -90°$, $\theta = 90°$ and $\theta = 270°$. Asymptotes occur every 180° and the graph repeats between these asymptotes, so the period of $y = \tan\theta$ is 180°.

	$y = \cos\theta$	$y = \sin\theta$	$y = \tan\theta$
period	360°	360°	180°
amplitude	1	1	undefined
asymptotes	none	none	$(90 + 180n)°$, where $n = 0, 1, 2, \ldots$

Technology Exploration: GeoGebra

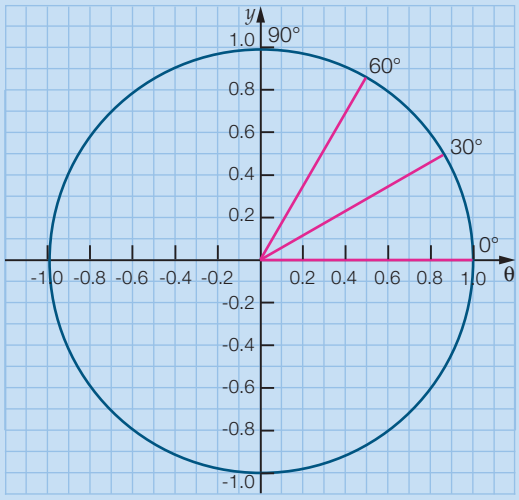
Trigonometric graphs

1. Open GeoGebra and select View > Layout > Preferences > Basic. In Dimension, enter -10 as x-min and 370 as x-max, -3 as y-min and 3 as y-max. (Alternatively, right-click on Graph View, select Graphics… and input the x and y values.)
2. In the Input bar, enter y=sin(x°). Click on α in the Input bar to find the degree sign.
3. Now enter y=2sin(x°) and y=3sin(x°). Describe the relationship between the amplitude of the graph and the coefficient of sin (x°).
4. What do you notice about the period for these three graphs?
5. Click on the green circle in Algebra View for the equations y=2sin(x°) and y=3sin(x°). This will remove these graphs from Graph View but retain them for future use.
6. Now enter y=sin(2x°) and y=sin(3x°). Describe the relationship between the period of the graph and the coefficient of x.
7. What do you notice about the amplitude for these three graphs?
8. Open a new GeoGebra window and set the dimensions as previously. In the Input bar enter y=cos(x°).
9. By altering the coefficient of cos (x°) as sine was in step 3, investigate the relationship between the amplitude and period of the graph and the coefficient of cos (x°).
10. Remove the graphs drawn in step 9 and, by altering the coefficient of x as in step 6, investigate the relationship between the amplitude and period of the graph and the coefficient of x.
11. What is the relationship between y = sin (x°) and y = cos (x°)?
12. Open a new GeoGebra window and set the Dimensions with -10 as x-min and 370 as x-max, -20 as y-min and 20 as y-max.
13. In the Input bar, enter y=tan(x°). What is the period of this graph? Can an amplitude be specified for y = tan (x°)? Explain.
14. Describe what has happened to the graph at x = 90° and x = 270°? Explain why this has occurred.
15. In the Input bar, enter y=2tan(x°) and y=3tan(x°). Describe the effect on the period of the graph by altering the coefficient of tan (x°).
16. Remove the graphs drawn in step 15 and, in the Input bar, enter y=tan(2x°) and y=tan(3x°). Describe the effect on the period of the graph by altering the coefficient of x.

Worked Example 5

WE 5

Use the unit circle to sketch the graph of the equation $y = \cos\theta$ for values of θ ranging from 0° to 360°.



Working

θ	0°	30°	60°	90°
$\cos\theta$	1	0.87	0.5	0

(Find the x-coordinates of the points on the unit circle for the angles 0°, 30°, 60° and 90° in the unit circle.)

θ	120°	150°	180°
$\cos\theta$	-0.5	-0.87	-1

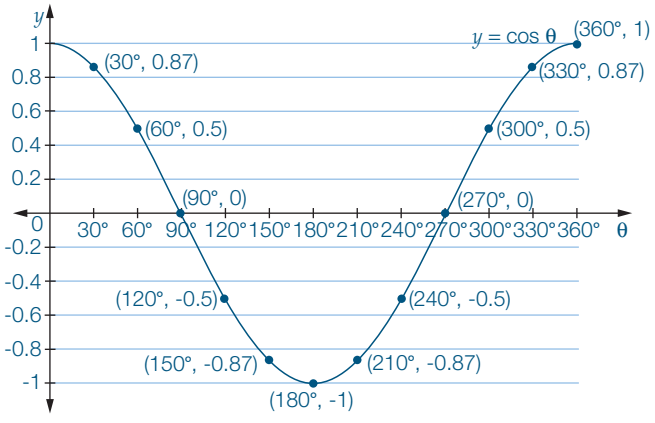
(Reflect the angles 0°, 30°, 60° and 90° in the y-axis of the unit circle to find angles in the second quadrant. Find the x-coordinates of the points on the unit circle for these angles.)

θ	210°	240°	270°
$\cos\theta$	-0.87	-0.5	0

(Reflect the angles 120°, 150° and 180° in the x-axis of the unit circle to find angles in the third quadrant. Find the x-coordinates of the points on the unit circle for these angles.)

θ	300°	330°	360°
$\cos\theta$	0.5	0.87	1

(Reflect the angles 210°, 240° and 270° in the y-axis of the unit circle to find angles in the fourth quadrant. Find the x-coordinates of the points on the unit circle for these angles.)



6.3 Trigonometric graphs

Answers page 638

Stage 5.3§

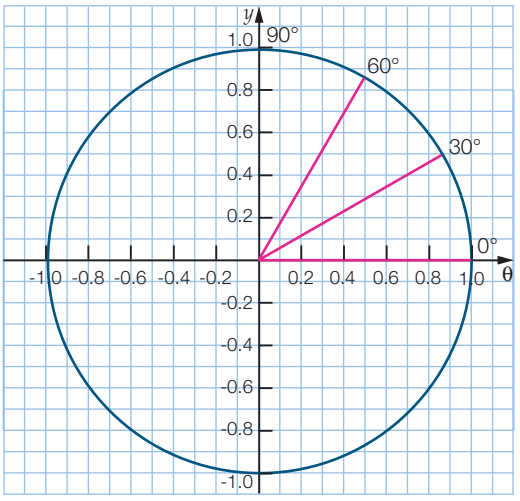
Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q9	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9

Fluency and understanding

WE 5

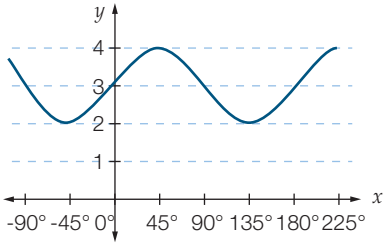
1 Use the unit circle shown to sketch the graph of the equation $y = \sin\theta$ for values of θ ranging from 0° to 360°.



2 The period and the amplitude of the trigonometric relationship shown is:

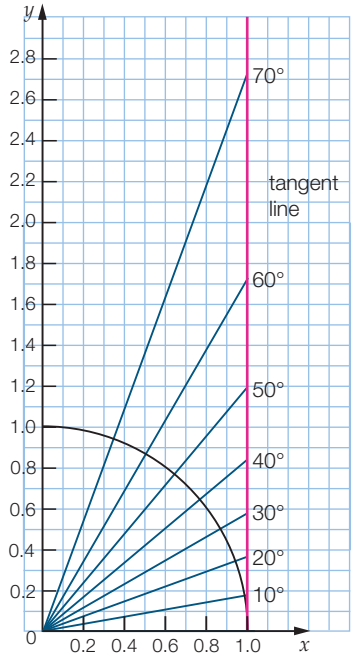
A 90°, 1 **B** 90°, 2

C 180°, 1 **D** 180°, 2



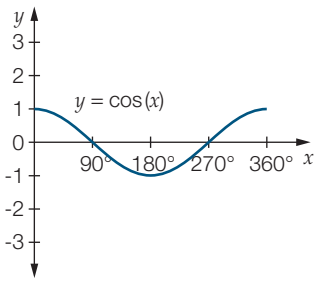
3 **(a)** Use the unit circle with the tangent line shown in the diagram on the right to sketch the graph of the equation $y = \tan x$ for values of x ranging from 0° to 70° in intervals of 10°.

(b) What do you think is happening to the value of $\tan x$ as the value of x gets closer to 90°?

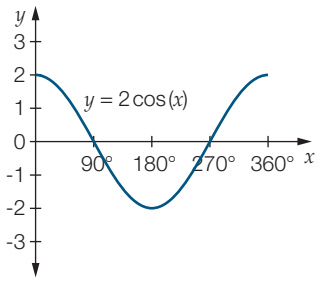


4 Consider the three cosine graphs shown below.

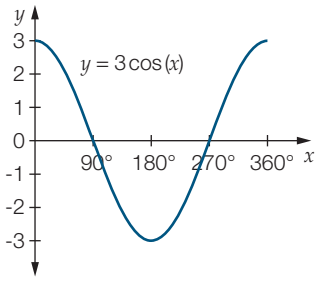
(i)



(ii)



(iii)



(a) State the period for each graph.

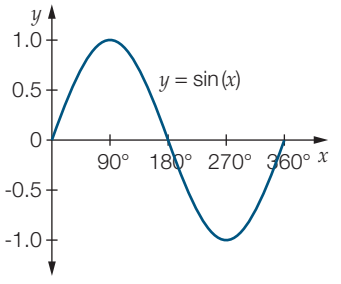
(b) State the amplitude for each graph.

(c) What conclusion can you draw about the effect of the a in $y = a \times \cos x$?

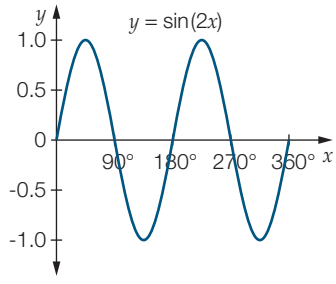
(d) Sketch what you think $y = 4 \cos x$ for $0° < x < 360°$ would look like. Check your answer using digital technology.

5 Consider the three sine graphs shown below.

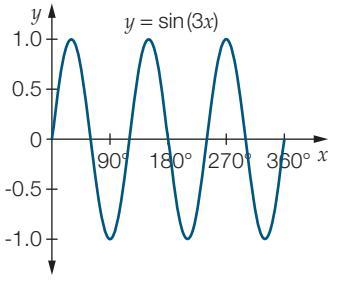
(i)



(ii)



(iii)



(a) State the period for each graph.

(b) State the amplitude for each graph.

(c) What conclusion can you draw about the effect of the n in $y = \sin(nx)$?

(d) Sketch what you think $y = \sin 6x$ for $0° < x < 360°$ would look like. Check your answer using digital technology.

Reasoning, communicating and open-ended

6 **(a)** On the same set of axes plot the graphs of: **(i)** $y = \sin(x)°$ and **(ii)** $y = \cos(90 - x)°$ using the table of values given below.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin(x)°$									

x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\cos(90 - x)°$									

(b) What did you find?

(c) What conclusion can you draw?

7 **(a)** On the same set of axes plot the graphs of: **(i)** $y = \cos(x)°$ and **(ii)** $y = \sin(90 - x)°$ using the table of values given below.

x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\cos(x)°$									

x	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\sin(90 - x)°$									

(b) What did you find?

(c) What conclusion can you draw?

8 At a harbour, a siren sounds every time the tide is at 5 m. Since tides are based on sine curves, and the siren first sounds when the angle for the sine ratio is 24°, calculate the angle when the siren will sound next.

9 **(a)** Sketch a sine graph with an amplitude of 2.

(b) Sketch a sine graph with a period of 180°.

(c) Sketch a cosine graph with an amplitude of 3.

(d) Sketch a cosine graph with a period of 120°.

6.4 Solving trigonometric equations

Stage 5.3§

Need to Know

Using inverse trigonometric ratios to solve equations

Solutions to the trigonometric equation $\sin x = 0.5$ can be found by using the inverse sin function on a calculator. By finding $\sin^{-1}(0.5)$, we will get the solution $x = 30°$, which is an angle in quadrant 1. However, we know that $\sin(180° - \theta) = \sin\theta$, so $x = 150°$, a quadrant 2 angle, is also a solution.

In fact, there is an infinite number of solutions, since adding multiples of 360° to each solution produces more solutions. So $x = 30° + 360° = 390°$ and $x = 150° + 360° = 510°$ are another two solutions.

Coterminal angles

Coterminal angles are angles that are in the same position on the unit circle, so their trigonometric ratios are the same. Coterminal angles can be used to give an infinite number of solutions to some problems, as adding or subtracting multiples of 360° to an angle returns the angle to the same position on the unit circle.

Worked Example 6

WE 6

Find all solutions for x that satisfy each of the following equations.

(a) $2\cos x - 1 = 0$ for $0° \le x \le 360°$

(b) $3 - \tan x = 2$ for $0° \le x \le 180°$

Working

(a) $2\cos x - 1 = 0$

$2\cos x = 1$

$\cos x = 0.5$

$x = \cos^{-1}(0.5)$

$x = 60°$

Cosine is positive in quadrants 1 and 4.

The equivalent angle of 60° in quadrant 4 is $360° - 60° = 300°$.

$2\cos x - 1 = 0$ has solutions $x = 60°$, and $x = 300°$ for $0° \le x \le 360°$.

(b) $3 - \tan x = 2$

$-\tan x = -1$

$\tan x = 1$

$x = \tan^{-1}(1)$

$x = 45°$

Tangent is positive in quadrants 1 and 3.

The equivalent angle of 45° in quadrant 3 is $180° + 45° = 225°$.

225° is not within the range $0° \le x \le 180°$ so it is not a solution.

$3 - \tan x = 2$ has the solution $x = 45°$ for $0° \le x \le 180°$.

Worked Example 7

WE 7

Determine all solutions to the equation $\sin x = 0.5$.

Working

Sine is positive in quadrants 1 and 2.

In quadrant 1, $\sin\theta$ is positive.

$\sin x = 0.5$

$x = \sin^{-1} 0.5$

$x = 30°$

In quadrant 2, the equivalent angle to 30° is $180° - 30° = 150°$.

The solutions in the range 0° to 360° are $x = 30°$, $x = 150°$.

$x = (30 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$

and $x = (150 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$

or $x = [(2n + 1)180 - 30]°$, $n = 0, 1, 2, 3, \ldots$

Worked Example 8

WE 8

Determine all solutions to the equation $2\tan x + 1 = 0$. Write your answers correct to two decimal places.

Working

$2\tan x + 1 = 0$

$2\tan x = -1$

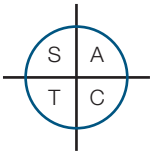
$\tan x = -\frac{1}{2}$

Tangent is negative in quadrants 2 and 4.

In quadrant 1, $\tan\theta$ is positive.

$x = \tan^{-1}\left(\frac{1}{2}\right)$

$x = 26.565°$



In quadrant 2, the equivalent angle is $(180 - 26.565)° = 153.435°$
$= 153.44°$ (2 d.p.)

In quadrant 4, the equivalent angle is $(360 - 26.565)° = 333.435°$
$= 333.44°$ (2 d.p.)

The solutions for $0° \le x \le 360°$ are $x = 153.44°$, $x = 333.44°$

$x = (153.44 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$

and $x = (333.44 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$

or $x = (153.44 + 180n)°$, where $n = 0, 1, 2, 3, \ldots$

6.4 Solving trigonometric equations

Answers page 640

Navigator

Stage 5.3§

Q1, Q2, Q3, Q4, Q7, Q8, Q11	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q12	Q1, Q2, Q3, Q5, Q6, Q9, Q10, Q12

Fluency and understanding

WE 6

1 Find all solutions for x that satisfy each of the following:

(a) $2\cos x + 1 = 0$ for x for $0° \le x \le 360°$

(b) $4 - 2\sin x = 3$ for x for $0° \le x \le 180°$

WE7

2 Determine all solutions to the equation $\cos x = \frac{-1}{2}$.

WE8

3 Determine all solutions to the equation $4\sin x - 1 = 0$. Write all answers correct to two decimal places.

4 The solutions to $4 - 5\sin x = 0$ for x for $0° \le x \le 180°$ are closest to:

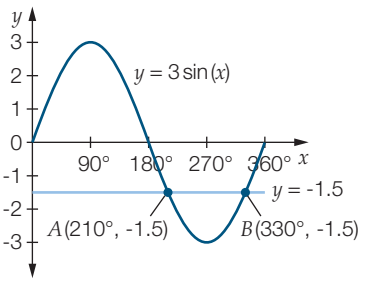
A 127°, 127° **B** 53°, 127° **C** 37°, 143° **D** 53°, 307°

5 Use digital technology to find the solutions for $0° \le x \le 360°$ for each of the following equations. Where necessary, round your answers to two decimal places.

(a) $4\sin x - 3 = 0$ **(b)** $5\tan x + 2 = 0$ **(c)** $3 - 5\cos x = 0$ **(d)** $\frac{3\sin x}{2} - 1 = 0$

6 The depth D metres below sea level of a submarine after T minutes is given by the trigonometric equation $D = 4\cos T - 4$ where $T \ge 0$. When will be the next time the submarine surfaces ($D = 0$) if it was at the surface at 3:10 p.m. initially?

7 The graph shown at right could be used to solve which of the following trigonometric equations?



A $3\sin x - 1.5 = 0$

B $1.5 - 3\sin x = 0$

C $3\sin x + 1.5 = 0$

D $3\sin x = 1.5$

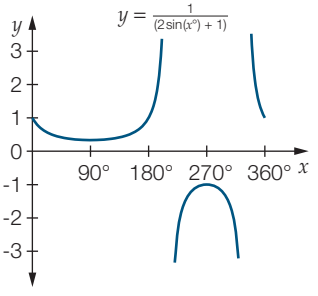
8 Considering the graph shown in Question **7** the solutions shown are in:

A quadrants 1 and 2 **B** quadrants 2 and 3

C quadrants 3 and 4 **D** quadrants 1 and 4.

Reasoning, communicating and open-ended

9 A function is defined as $y = \frac{1}{2\sin x + 1}$. Its graph is shown at right. The graph appears to be undefined for two values of x in the domain [0°, 360°].



(a) What would cause this graph to be undefined?

(b) Find the values of x for which the graph is undefined in the domain [0°, 360°].

10 Find all solutions to the equation $\tan x = \sin x$ for $0° \le x \le 360°$.

11 For each of the sine, cosine and tangent ratios, write at least one equation similar to those in Question **5** and solve it for $0° \le x \le 360°$ using digital technology.

12 **(a)** Write any trigonometric equation that has solutions in:

(i) quadrants 1 and 3

(ii) quadrants 3 and 4.

(b) How do you know which trigonometric ratio to use in each of parts **(i)** and **(ii)**?

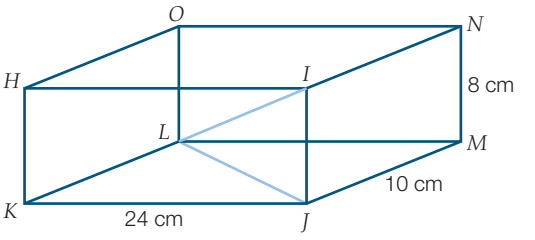
Hint

Simplify the equation using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ to find an equivalent equation and then factorise it using $\sin(x)$ as a common factor.

Half-time 6

Stage 5.3§ Q1–6

1 A rectangular prism is shown. **Ex.6.1**



(a) Calculate $\angle ILJ$, by first finding the length of LJ.

(b) Calculate the length of IL.

Round both answers to two decimal places.

2 Find all values for x, correct to one decimal place, that satisfy the following equations and conditions: **Ex.6.3**

(a) $3\sin x - 2 = 0,\ 0° \le x \le 180°$

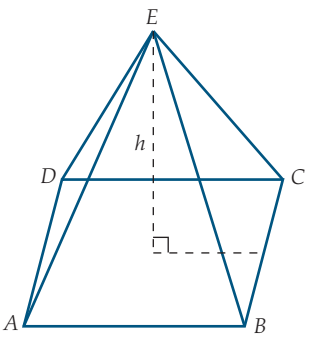
(b) $5\tan x + 1 = 3,\ 0° \le x \le 360°$

3 Find the equivalent first quadrant ratio for each of the following, taking care to include the appropriate sign. **Ex.6.2**

(a) $\sin 135°$ **(b)** $\cos 98°$ **(c)** $\tan 200°$

(d) $\sin 247°$ **(e)** $\cos 300°$ **(f)** $\tan 339°$

4 A right pyramid $ABCDE$ has a square base of side length 9 cm. The sloping edges are 15 cm in length. **Ex.6.1**



(a) Find the angle that the sloping edge makes with the base of the pyramid, to the nearest degree.

(b) Using your answer to **(a)**, or any other method, find the height of the pyramid, h. State your answer correct to one decimal place.

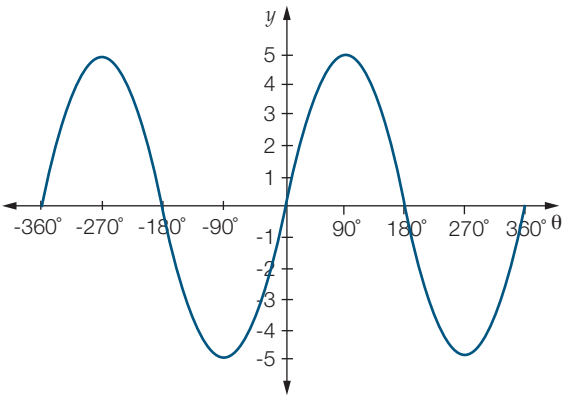
(c) The side length of the square base of the pyramid is increased by 3 cm, but the length of the sloping edges remains the same.

(i) Will the angle you found in **(a)** increase or decrease?

(ii) By how many degrees will it increase or decrease?

5 The period and amplitude of the trigonometric graph shown is: **Ex.6.3**

A 360°, 10 **B** 180°, 5 **C** 360°, 5 **D** 180°, 10



6 Determine all solutions to the following equations. State your solutions correct to two decimal places, where necessary. **Ex.6.4**

(a) $\cos x = 0.5$ **(b)** $3\sin x - 1 = 0$

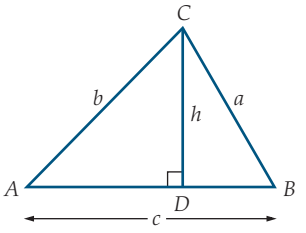
6.5 The sine and cosine rules

Stage 5.3§

Need to Know

Sine rule

Consider a triangle with vertices at A, B and C and side lengths a, b, c, where a is opposite angle A, b is opposite angle B and c is opposite angle C. D is a point on AB such that CD is perpendicular to AB forming triangle CAD and triangle DBC, each of height h.



$\sin A = \dfrac{h}{b}$ (from ΔCAD)

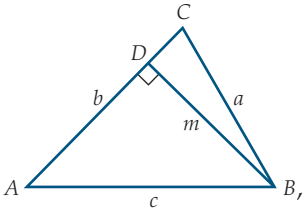
$\therefore h = b \sin A$ (1)

$\sin B = \dfrac{h}{a}$ (from ΔDBC)

$\therefore h = a \sin B$ (2)

$b \sin A = a \sin B$ equating (1) and (2)

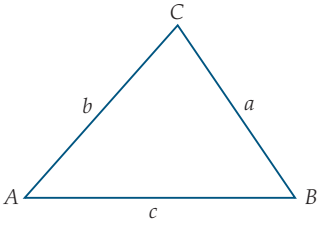
$\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ (3) rearranging this expression

Similarly, by using 

, it can be shown that $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$.

Therefore,

$\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ and is known as the **sine rule**.



This form is used to find an unknown angle when two sides and one angle opposite one of these sides are known.

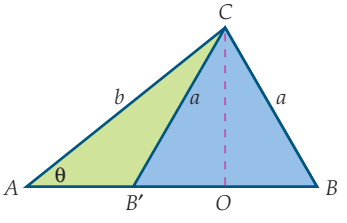
By taking the reciprocal, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

This form is used to find an unknown side when two angles and one side opposite one of these angles are known.

Only two equated expressions can be used at any one time.

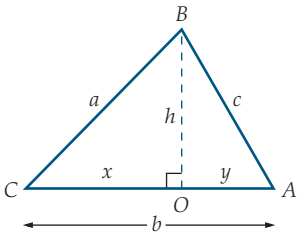
The ambiguous case of the sine rule

Consider ΔCAB, where two sides a, b and a non-included angle θ are known. By reflecting side BC about the vertical line OC, triangle CAB' also satisfies the information and $\angle AB'C = 180° - \angle ABC$.



Cosine rule

If side lengths a and b and the included angle C are known, the **cosine rule** can be used.



To derive the formula, draw the perpendicular height, h, from B to O on side b.

$b = x + y$

$y = b - x$

Squaring both sides,

$y^2 = b^2 - 2bx + x^2$ (1)

Using Pythagoras' Theorem in ΔAOB,

$c^2 = h^2 + y^2$ (2)

In ΔCOB, $h^2 = a^2 - x^2$ (3)

Substituting for h^2 using equation (3) in equation (2)

$c^2 = a^2 - x^2 + y^2$ (4)

Substituting for y^2 using equation (1) in equation (4)

$c^2 = a^2 - x^2 + b^2 - 2bx + x^2$

$c^2 = a^2 + b^2 - 2bx$ (5)

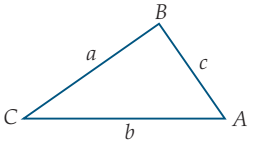
In ΔCOB, $x = a \cos C$ (6)

Substituting for x using equation (6) in equation (5)

$c^2 = a^2 + b^2 - 2ab \cos C$

This simplifies to Pythagoras' Theorem when $C = 90°$.

Depending on the angle to be used, the cosine rule can be written in three ways.



$c^2 = a^2 + b^2 - 2ab \cos C$

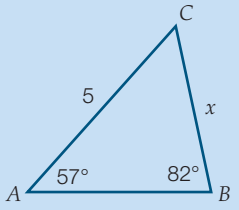
$b^2 = a^2 + c^2 - 2ac \cos B$

$a^2 = b^2 + c^2 - 2bc \cos A$

Worked Example 9

WE9

Calculate the value of x, correct to two decimal places.



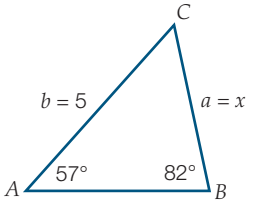
Working

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{x}{\sin 57°} = \frac{5}{\sin 82°}$$

$$x = \frac{5 \times \sin 57°}{\sin 82°}$$

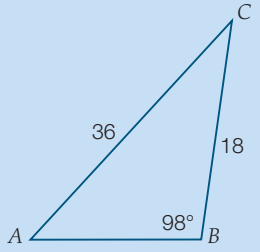
$$x = 4.23 \text{ units (2 d.p.)}$$



Worked Example 10

WE10

Calculate the value of $\angle CAB$ in degrees, rounded to two decimal places.



Working

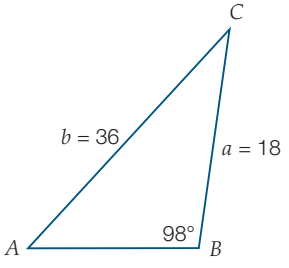
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{18} = \frac{\sin 98°}{36}$$

$$\sin A = \frac{18 \times \sin 98°}{36}$$

$$A = \sin^{-1}\left(\frac{18 \times \sin 98°}{36}\right)$$

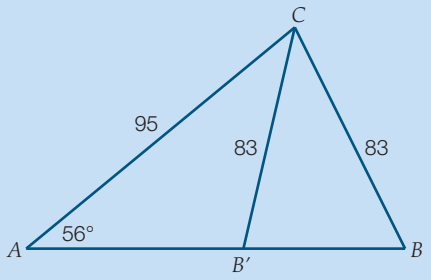
$\angle CAB = 29.68°$ (2 d.p.)



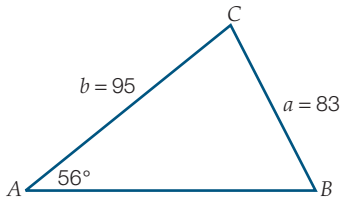
Worked Example 11

WE11

Calculate the values of $\angle ABC$ and $\angle AB'C$, correct to two decimal places.



Working

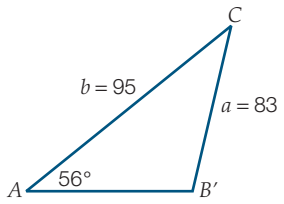


$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin 56°}{83} = \frac{\sin B}{95}$$

$$B = \sin^{-1}\left(\frac{95 \times \sin 56°}{83}\right)$$

$$\angle ABC = 71.60°$$

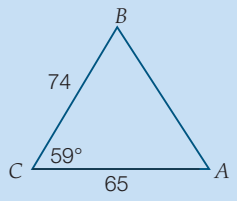


$$\angle AB'C = 180° - 71.60°$$
$$= 108.40°$$

Worked Example 12

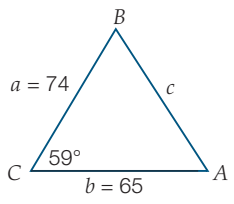
WE12

Calculate the length of AB, correct to two decimal places.



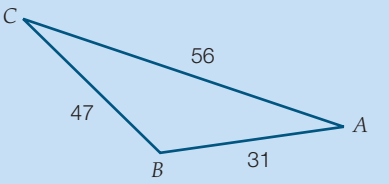
Working

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$c^2 = 74^2 + 65^2 - 2 \times 74 \times 65 \times \cos 59°$$
$$c^2 = 9701 - 9620\cos 59°$$
$$c = 68.89 \text{ units (2 d.p.)}$$



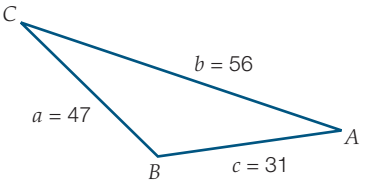
Worked Example 13 WE13

Calculate the value of $\angle ACB$ in degrees, correct to two decimal places.



Working

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$31^2 = 47^2 + 56^2 - 2 \times 47 \times 56 \times \cos C$$
$$961 = 5345 - 5264 \times \cos C$$
$$-4384 = -5264 \times \cos C$$
$$\cos C = \frac{-4384}{-5264}$$
$$\angle ACB = \cos^{-1}\left(\frac{-4384}{-5264}\right)$$
$$\angle ACB = 33.61°$$



Answers page 640

6.5 The sine and cosine rules

Stage 5.3§

Navigator

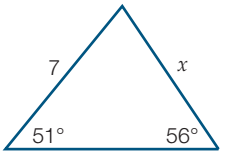
Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q12, Q13, Q18	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9, Q11, Q12, Q13, Q15, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q8, Q10, Q12, Q14, Q15, Q16, Q17, Q18

Fluency and understanding

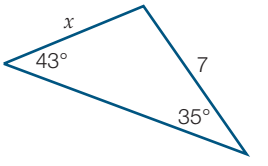
WE9

1 Calculate the value of x, correct to two decimal places.

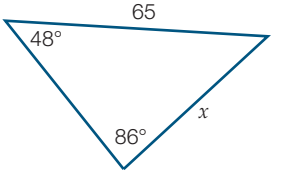
(a)



(b)



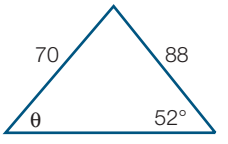
(c)



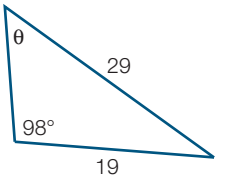
WE10

2 Calculate the value of θ in degrees, correct to two decimal places.

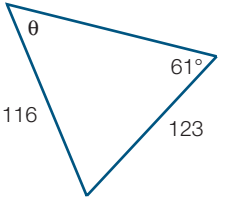
(a)



(b)



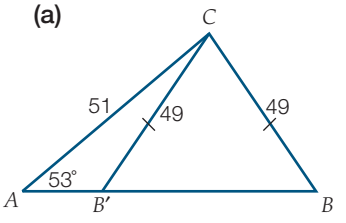
(c)



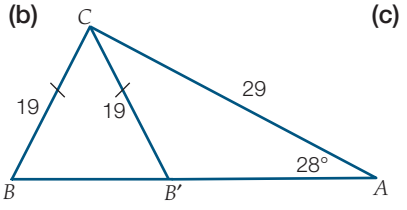
WE11

3 Calculate the values of $\angle ABC$ and $\angle AB'C$, correct to two decimal places.

(a)



(b)



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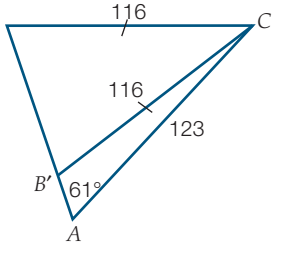
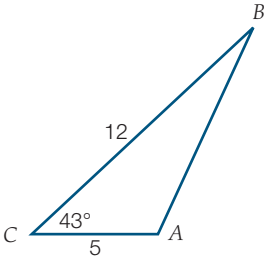
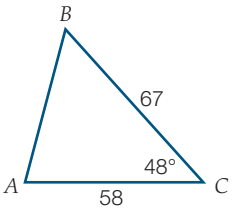


4 Calculate the length AB, correct to two decimal places. WE12

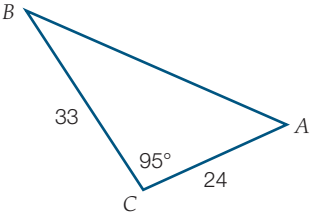
(a)



(b)

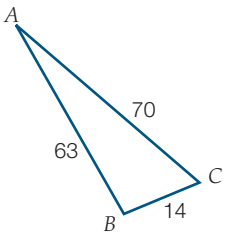


(c)

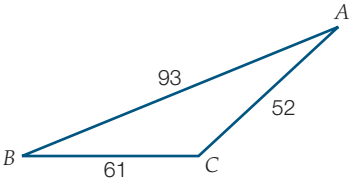


5 Calculate the value of angle $\angle ACB$, correct to two decimal places. WE13

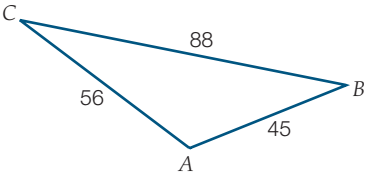
(a)



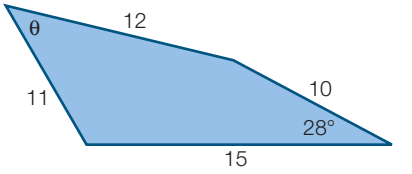
(b)



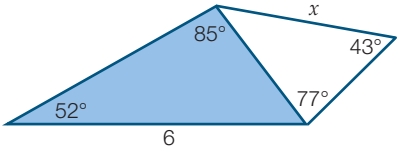
(c)



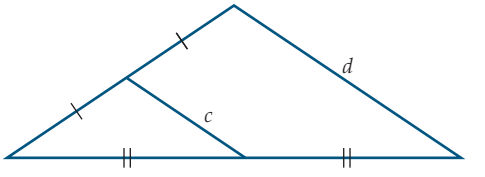
6 Calculate the size of angle θ, correct to the nearest degree.



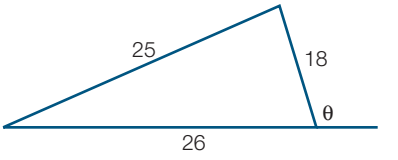
7 Calculate the value of x and round to two decimal places.



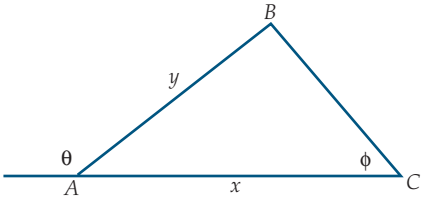
8 Use the cosine rule to show that $d = 2c$.



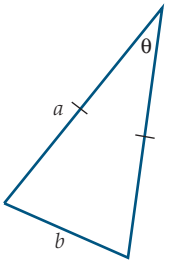
9 Calculate the value of θ, giving your answer as a decimal correct to two decimal places.



10 Show that $x = \dfrac{y\sin(\theta - \phi)}{\sin\phi}$.



11 For the triangle shown, which is the correct statement?



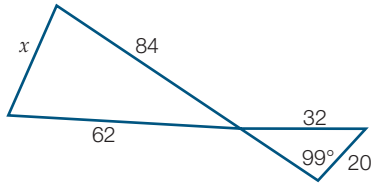
A $a^2 = b^2(2 - \cos\theta)$

B $b^2 = 2a^2(1 + \cos\theta)$

C $a^2 = b^2(2 + \cos\theta)$

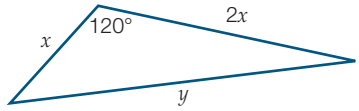
D $b^2 = 2a^2(1 - \cos\theta)$

12 Calculate the value of x and round to two decimal places.

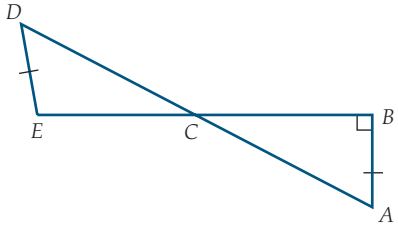


13 The value of y is:

A $7x^2$ **B** $\sqrt{7x}$ **C** $\sqrt{7x^2}$ **D** $49x$



14 Show that $\sin D = \dfrac{EC}{AC}$.



Reasoning, communicating and open-ended

15 Show that the sine rule reduces to $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$ when applied to a right-angled triangle.

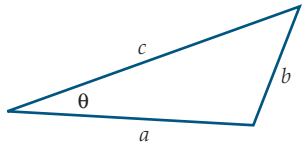
16 Apply the cosine rule to an equilateral triangle to show that each interior angle is 60°.

17 **(a)** Using the cosine formula $c^2 = a^2 + b^2 - 2ab\cos C$, show that if $a = b$ and $c = ka$, then $\cos C = \dfrac{2 - k^2}{2}$.

(b) State two values for k and use them to evaluate $\cos\theta$, rounded to two decimal places.

(c) What are the minimum and maximum possible values for k?

18 Determine the angle θ in the following triangle if sides b and c are positive integer multiples of a.



Problem solving

Trig chimps

Equipment required: 1 brain, calculator

For all calculations round your answers to two decimal places.

A certain group of monkeys are renowned for their longevity.

The ages of three chimps, Syn, Coss and Ten satisfy the following conditions.

- The sum of their ages is 158.
- Coss' age is a multiple of 13 and is an odd number.
- Expressed in degrees, Ten's age is in quadrant 2.
- Expressed in degrees, Syn's age and Coss' age are in quadrant 1.
- The tangent of Ten's age is 15.08 less than the cosine of Coss' age.
- The two digits after the decimal point in the cosine of Coss' age are consecutive digits.

What are the ages of the three chimps?

Strategy options

- Guess and check.
- Test all possible combinations.

Applications of the sine and the cosine rules

6.6

Stage 5.3§

Need to Know

Many problems that require lengths or angles to be found can be solved using the sine or cosine rule.

Strategies for solving problems

Step 1 Draw a diagram to represent the situation.

Step 2 Identify known information.

Step 3 Identify what needs to be found.

Step 4 Decide whether the sine or cosine rule is appropriate to use.

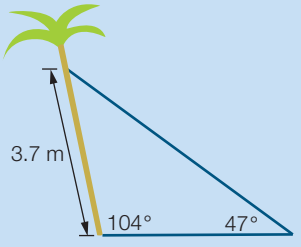
Step 5 Substitute values into the chosen rule.

Step 6 Solve the equation to find the unknown length or angle.

Worked Example 14

WE14

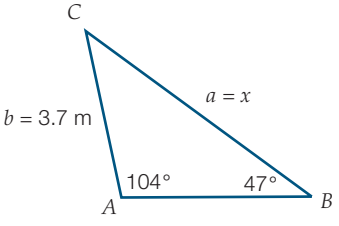
A tree growing at an angle of 104° to the horizontal is in danger of falling. It has been supported by a rope tied to the tree 3.7 m along the tree trunk and pegged to the ground at an angle of 47° as shown in the diagram. Find the length of rope used, correct to one decimal place.



Working

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$a = x$, $\angle A = 104$,
$b = 3.7$, $\angle B = 47$



$$\frac{x}{\sin 104°} = \frac{3.7}{\sin 47°}$$

$$x = \frac{3.7 \times \sin 104°}{\sin 47°}$$

$$x = 4.9$$

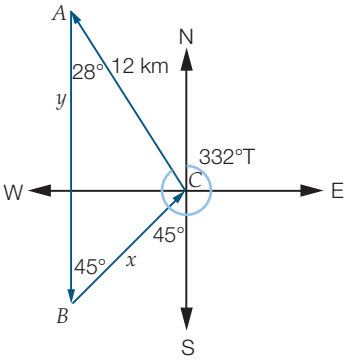
The length of rope is 4.9 m.

Worked Example 15

WE15

A helicopter leaves base and flies for 12 km on a bearing of 332° T. It then heads due south for a certain distance before flying back to base on a bearing of 45° T. What is the total flight distance? Round your answer to two decimal places.

Working



$360 - 332 = 28°$

$\angle A = 28°$

$\angle B = 45°$

$\angle C = 180 - (28 + 45) = 107°$

$a = x$
$b = 12$
$c = y$

(Here, $\angle A$ is obtained using alternate angles and $\angle C$ by $180° - (\angle A + \angle B)$.)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{x}{\sin 28°} = \frac{12}{\sin 45°}$$

$$x = \frac{12 \times \sin 28°}{\sin 45°}$$

$x = 7.967\,196\,615$

$$\frac{y}{\sin 107°} = \frac{12}{\sin 45°}$$

$$y = \frac{12 \times \sin 107°}{\sin 45°}$$

$y = 16.229\,029\,87$

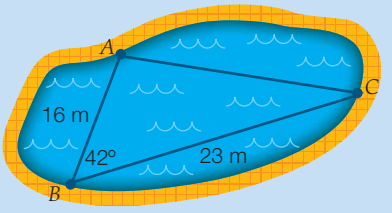
Perimeter $= 12 + 16.229\,029\,87 + 7.967\,196\,615$
$= 36.196\,226\,49$

The total flight is 36.20 km.

Worked Example 16

WE16

A walkway is to be constructed across a pond from A to C. There is an existing walkway of length 16 m from A to B. A rope measuring 23 m is attached to point B and point C. The angle between the rope and the existing walkway is 42°. Calculate the length of the new walkway, correct to two decimal places.



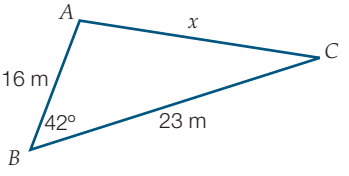
Working

$b = x$, $a = 23$, $c = 16$, $B = 42°$
$b^2 = a^2 + c^2 - 2ac \cos B$

$x^2 = 23^2 + 16^2 - 2 \times 23 \times 16 \times \cos 42°$

$x^2 = 238.045\,408\,4$
$x = 15.428\,720\,25$

The required length for the walkway is 15.43 m.



6.6 Applications of the sine and the cosine rules

Answers page 640

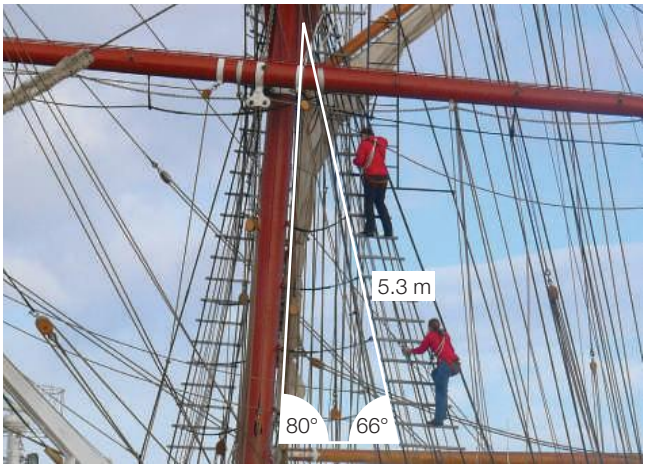
Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q17	Q1, Q2, Q3, Q6, Q7, Q11, Q12, Q14, Q15, Q16, Q17, Q18	Q1, Q2, Q3, Q8, Q9, Q12, Q13, Q15, Q16, Q17, Q18

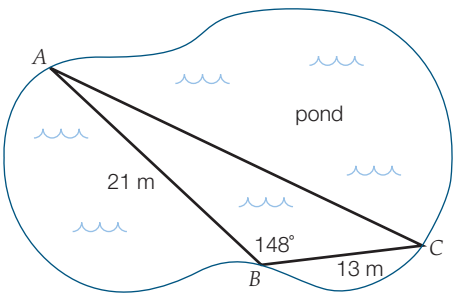
Fluency and understanding

1 Part of a ship's rigging shown forms a triangle with base angles of 80° and 66° and rigging of length 5.3 m. Calculate the length of the section of the mast in the triangle shown, correct to two decimal places. WE14

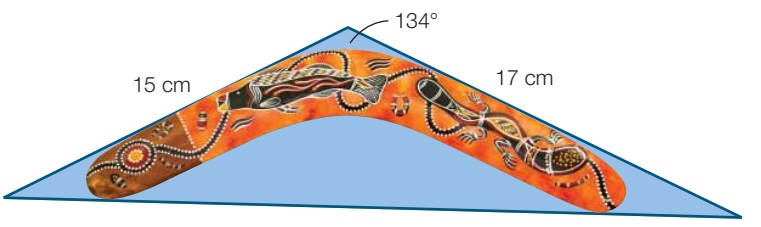


2 Postman Pete leaves the post office and makes deliveries for 2.5 km on Aiden Rd, which is on a bearing of 072°T from his starting point. Pete then heads due north on Sari Boulevard for a certain distance before returning to the post office on a bearing of 210°T. Find the total distance of Postman Pete's route, rounded to two decimal places. WE15

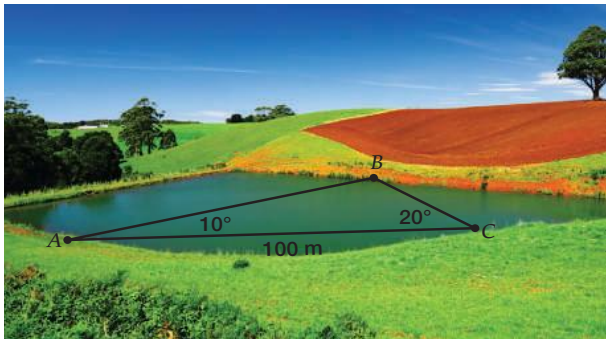
3 A walkway is to be constructed across a pond from *A* to *C*. There is an existing walkway of length 21 m from *A* to *B*. A rope measuring 13 m is attached to point *B* and point *C*. The angle between the rope and the existing walkway is 148°. Calculate the length of the new walkway, correct to two decimal places. WE16



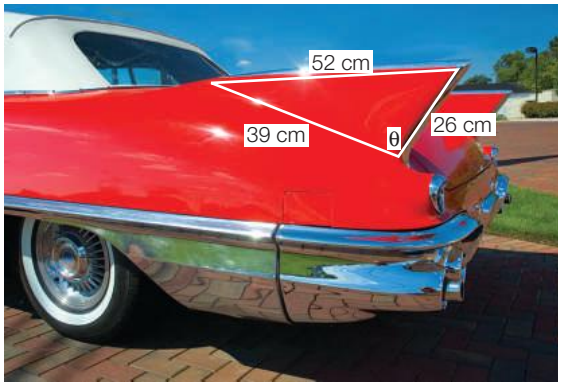
4 A boomerang is mounted on a triangular frame as shown. If the two shorter sides of the frame are 15 cm and 17 cm respectively and the included angle is 134°, calculate the perimeter of the frame, rounded to two decimal places.



5 A pipe is laid across a section of a water dam from A to B. A 100 m rope is attached from point A to point C and the angles between the rope and AB, and the rope and CB are measured. If $\angle A$ is 10° and $\angle C$ is 20°, what length of pipe is required? State the answer to the nearest metre.

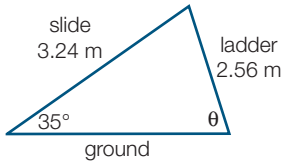


6 The fins of this classic car are triangular with the dimensions of 26 cm, 39 cm and 52 cm as shown. For aerodynamic stability, it is recommended that the angle θ be less than 105°. Is this true for this vehicle?

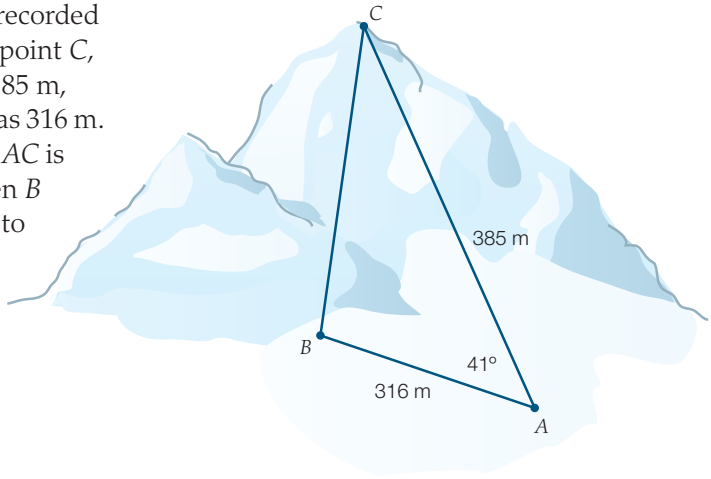


7 Rangers Tim and Zoe are 10 km apart, with Tim due west of Zoe. They notice a fire that is N50°E of Tim and N35°W of Zoe. Decide which ranger is closer to the fire by finding the distance each ranger is from the fire.

8 A children's slide is of length 3.24 m and makes an angle of 35° with the ground. If the ladder is 2.56 m, what angle θ does it make with the ground? Round your answer to two decimal places.



9 A ski lift is to be constructed at point B on a mountain. A surveyor recorded the distance from point A to point C, the top of the mountain, as 385 m, and the distance from A to B as 316 m. If the angle between AB and AC is 41°, find the distance between B and the top of the mountain to the nearest metre.



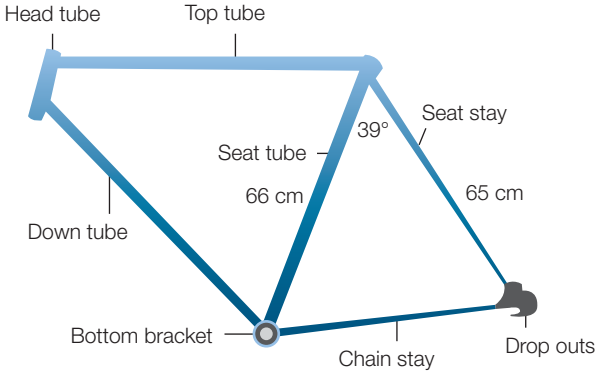
10 To the nearest centimetre, the length of the chain stay in the bicycle frame is:

A 44 cm

B 47 cm

C 51 cm

D 62 cm



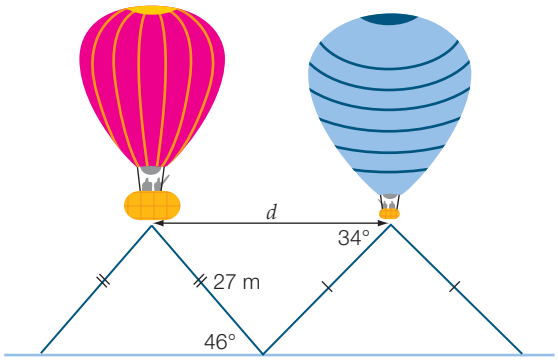
Reasoning, communicating and open-ended

11 Ken and Sofia set out in different directions from their campsite. After Ken walks 2150 m on a bearing of 316° T and Sofia walks for 2000 m, the distance between them is 2696 m. What are the two possible bearings that Sofia could have used? Give your answer correct to the nearest degree.

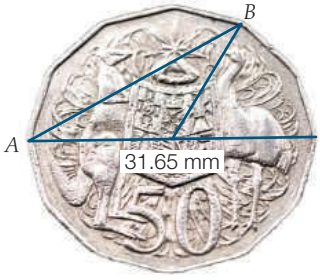
12 Two hot air balloons are stationary at the same height above the ground. They are secured by ropes as shown.

(a) Calculate the horizontal distance between the base of each balloon, correct to two decimal places.

(b) What is the total length of rope required to secure the balloons?



13 The longest diagonal of a 50 cent coin is 31.65 mm. Calculate the length *AB*, correct to two decimal places.



Hint

A 50c coin has the shape of a regular dodecagon (a 12-sided polygon).

14 The dial on a sundial casts a shadow as shown. The dial is 24 cm long, the angle between the dial and the base is 38° and there is 15 cm between the top of the dial and the end of the shadow. Find the length of the shadow, correct to two decimal places.

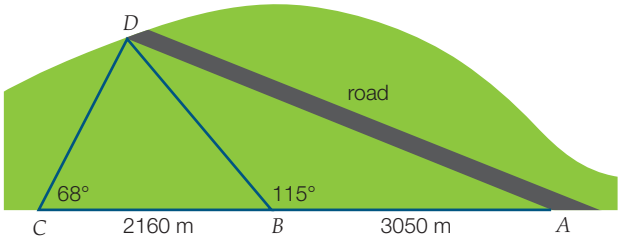
Hint

Be sure to find all possible solutions.

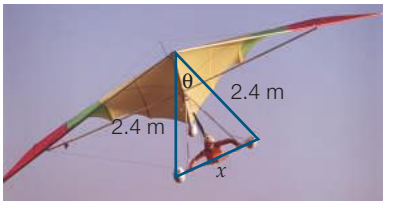
15 A pendant is in the shape of a scalene triangle and its two smaller interior angles are 25° and 28°. If the longest side is of length 4 cm, the lengths of the two other sides (in cm), correct to two decimal places are:

A 2.26, 2.35 **B** 2.12, 2.35 **C** 2.12, 2.18 **D** 2.11, 2.35

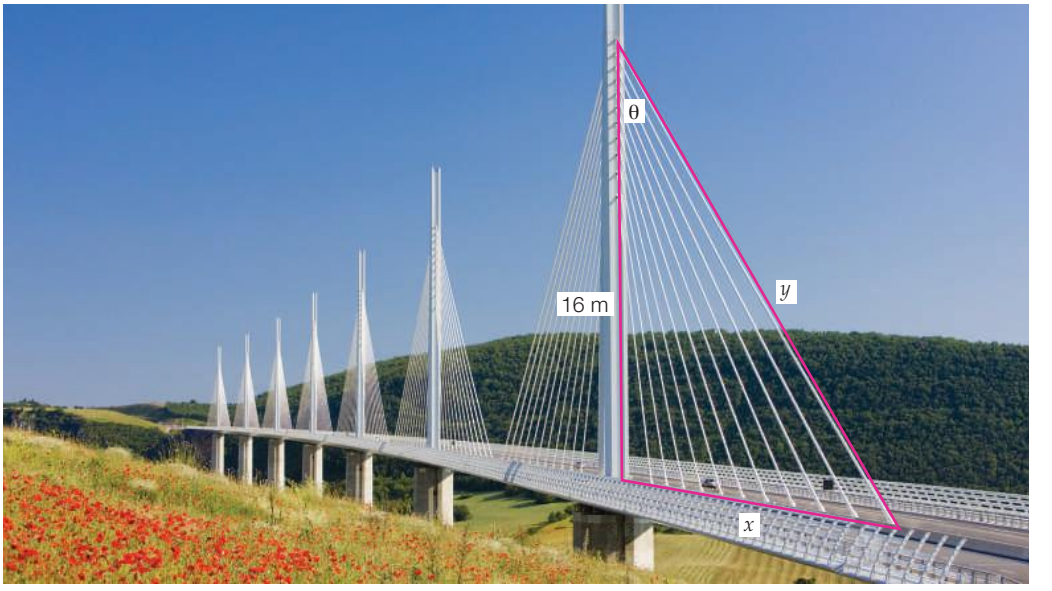
16 A new highway is being built across a paddock, from point *A* to point *D* as shown on the diagram. This highway currently runs in a straight line from *A* to *C* through *B*, 3050 m from *A*. The angle between *AB* and *BD* is 115°. The distance from *B* to *C* is a further 2160 m and the angle between *BC* and *CD* is 68°. Find the length of the new highway *AD*, correct to the nearest metre.



17 For maximum strength and stability, the angle θ between the support wires in the hang glider must be no more than 40° and the distance x between the wheels must be between 80 cm and 1.2 m. Determine two sets of allowable values for θ and x.



18 The length y of one of the bridge's support cables is less than 40 m and the angle θ is in the range $15° < \theta < 75°$. By choosing reasonable values for θ, find two possible values for the length x, and then find the value of y for each x-value.



Area of a triangle using trigonometry

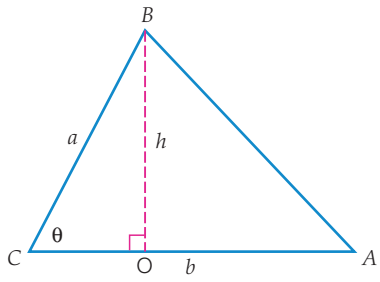
6.7

Need to Know

Stage 5.3§

Area of a triangle

The area of any triangle can be found using the formula $A = \frac{1}{2}bh$, where b is the base and h is the perpendicular height.



If h is not known but two sides and the angle included between them are known, a new formula can be used.

In ΔBOC, $h = a \sin \theta$ so we can substitute $a \sin \theta$ for h in the area formula.

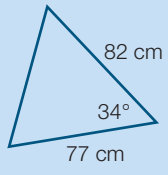
$$A = \frac{1}{2} \times b \times h$$
$$= \frac{1}{2} \times b \times a \sin \theta$$
$$= \frac{1}{2} ab \sin \theta$$

$A = \frac{1}{2} ab \sin \theta$, where θ is the angle included between sides a and b.

Worked Example 17

WE 17

Calculate the area of the triangle, correct to two decimal places.



Working

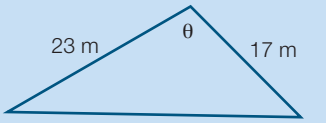
$a = 77$ cm, $b = 82$ cm, $\theta = 34$

$$\text{Area} = \frac{1}{2} ab \sin \theta$$
$$= \frac{1}{2} \times 77 \times 82 \times \sin 34°$$
$$= 1765.371\,996$$

The area is 1765.37 cm^2. (2 d.p.)

Worked Example 18 WE18

Find the size of the angle required if the area of the triangle shown is 195 m^2. Round the answer to the nearest degree.



Working

$area = 195\ m^2, \quad a = 23\ m, \quad b = 17\ m$

$$\text{Area} = \frac{1}{2}ab\sin\theta$$

$$195 = \frac{1}{2} \times 23 \times 17 \times \sin\theta$$

$$\sin\theta = \frac{195 \times 2}{23 \times 17}$$
$$= 0.9974$$
$$\theta = \sin^{-1}(0.9974)$$
$$= 85.901\,340\,36$$

The angle is 86°.

Answers page 640

6.7 Area of a triangle using trigonometry

Stage 5.3§

Navigator

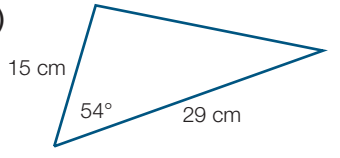
Q1, Q2, Q3, Q4, Q5, Q6, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13

Fluency and understanding

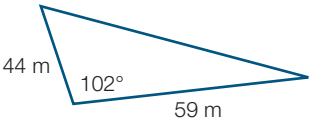
WE17

1 Calculate the area of each of the following triangles, correct to two decimal places.

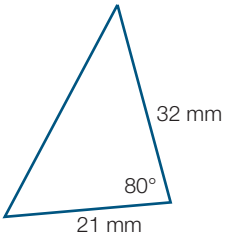
(a)



(b)

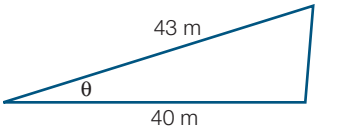


(c)



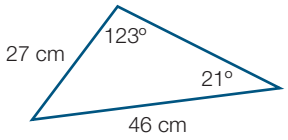
WE18

2 Find the size of the angle required if the area of the triangle shown is 529 m^2. Write the answer correct to the nearest degree. Assume θ is an acute angle.

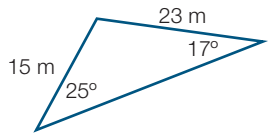


3 Calculate the area of the following triangles, correct to two decimal places.

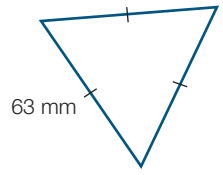
(a)



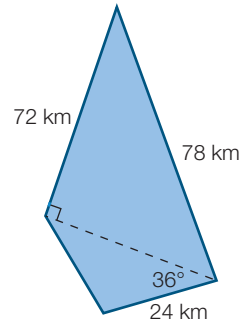
(b)



(c)

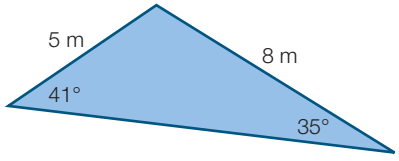


4 Calculate the area of the shape, correct to the nearest square kilometre.



5 The ends of the cells of a bee's honeycomb are regular hexagons. If each edge length is 1.5 mm, what is the area of the end of one cell? Round your answer to two decimal places.

6 The area of the triangle in square metres is closest to:



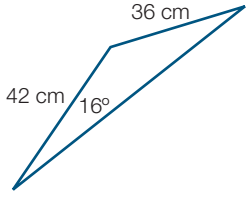
A 11.47 **B** 13.12

C 19.41 **D** 20.00

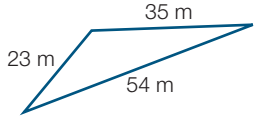
Reasoning, communicating and open-ended

7 Calculate the area of the following triangles, correct to two decimal places.

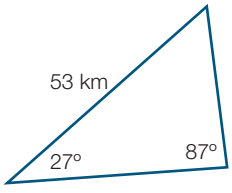
(a)



(b)



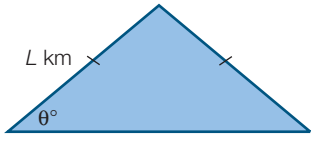
(c)



8 A plot of land is in the shape of an isosceles triangle.

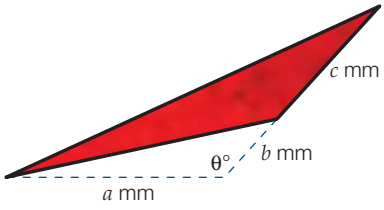
(a) Show that the area of land is $L^2 \sin\theta \cos\theta$ km^2.

(b) What is the area when $L = 5.6$ km and $\theta = 45°$?



9 **(a)** Show that the area of the section of red stained glass, shown in the diagram on the right, is $\frac{1}{2}ac\sin\theta$ mm^2.

(b) For what value of θ will the area be a maximum? What is this area?



Hint

How many equilateral triangles fit together to make a hexagon?

Hint

Are there any ambiguous cases?

Hint

Divide this triangle in half and use trigonometry to find the base and the height of the small triangles.

Hint

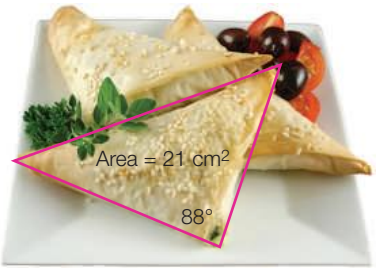
Use the sine or cosine rule to find the unknown sides or angles.

10 The sail shown, consisting of two layers of cotton fabric, is to be replaced at a cost of \$135 per square metre. If the length of the sail along the mast is 3.8 m and the two angles at the base of the sail are 85° and 57° as shown in the diagram, calculate the total replacement cost of the sail. Answer to the nearest dollar.

11 **(a)** In the formula for the area of a triangle, $\frac{1}{2}ab\sin\theta$, what is the range of values possible for the angle θ?

(b) Explain why $\sin\theta$ cannot take on values less than zero.

12 The shape of the cheese savoury can be approximated by an isosceles triangle. The angle between the two equal sides is 88° and the area covered by the savoury is 21 cm^2. If the perimeter is to be covered with a strip of cheese flavouring, what length will be used? Round your answer to the nearest centimetre.



13 Find values for the lengths of two equal sides of an isosceles triangle and the size of the angle between them, so that the area is a square number. The triangle must not be right-angled.

Problem solving

Triangles to the max

Servane has a square piece of cardboard, which is 30 cm × 30 cm. She wants to cut out as many small triangles as possible that have one side of 12 cm, another of 15 cm and the angle between them is 30°. Using area formulas she calculates that she will get 20 triangles. Her friend, Sophie, disagrees and says that 16 triangles are the most she can cut. Is either of them correct? Explain your answer. Draw a diagram to show the maximum number of triangles you could cut out from a piece of cardboard with the same dimensions.

Strategy options

- Make a model.
- Act it out.

Chapter review 6

D.I.Y. Summary

Key Words

Stage 5.3§	amplitude	coterminal angles	quadrant
	asymptote	equivalent angle	sine rule
	cosine rule	period	unit circle

Stage 5.3§ Q1–9

Copy and complete the following using words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 One half the difference between the minimum value and the maximum value of the graph of a trigonometric ratio is the _____.

2 The _____ relates the angle between two sides of a triangle and the side length opposite the angle.

3 Angles in quadrant 1 that have the same values for trigonometric ratios in the other quadrants are _____.

4 The trigonometric ratios are derived from the _____.

5 The smallest interval after which the graph of a trigonometric relationship repeats is the _____.

6 The _____ can be used when two angles in a triangle and a side length opposite one of the angles is known.

7 Each of the four sections of the unit circle is a _____.

8 Each _____ on the graph $y = \tan\theta$ is a vertical line for which $\tan\theta$ is not defined.

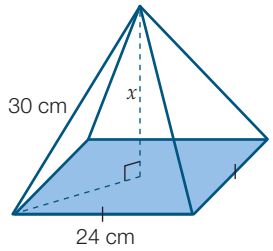
9 _____ are in the same position on the unit circle so their trigonometric ratios are exactly the same.

Stage 5.3§ Q1–26

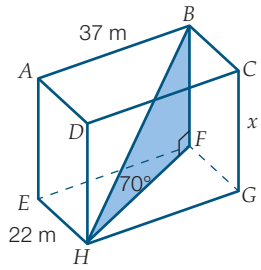
Fluency and understanding

1 Calculate the length of x in each shape. Round your answers to two decimal places. Ex 6.1

(a)



(b)



2 To the nearest centimetre, what is the length of the longest possible timber pole that can fit in a shed in the shape of a rectangular-based prism of length 24 m, breadth 10 m and height 4 m? Ex 6.1

3 In which quadrant are each the following angles? Ex 6.2

(a) 37° (b) 274° (c) 93° (d) 182°

4 Find the equivalent ratio to each of the following in quadrant 1 using the correct sign. Ex 6.2

(a) cos 173° (b) sin 254° (c) tan 312°

5 The equivalent trigonometric ratio in quadrant 1 of cos 253° is:

A cos 17° **B** -cos 73° **C** -cos 107° **D** cos 37°

Ex. 6.2

6 Use a unit circle to sketch the graph of the equation $y = \cos x$ for $0° \le x \le 360°$.

Ex. 6.3

7 Find all solutions to $\tan x + 1 = 0$ for $0° \le x \le 360°$.

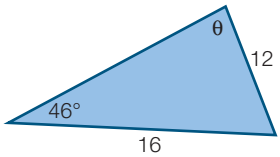
Ex. 6.4

8 Determine all solutions to the equation $4 \cos x - 1 = 0$. Round your answers to two decimal places.

Ex. 6.4

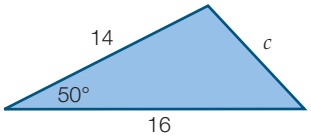
9 In degrees, the angle θ shown is closest to:

A 70.42 **B** 72.10 **C** 73.56 **D** 75.78



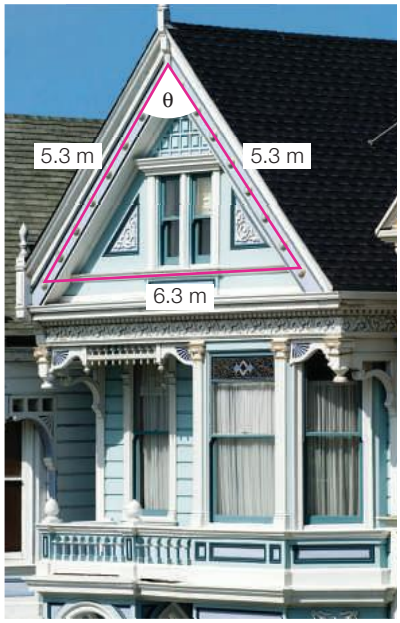
Ex. 6.5

10 Calculate the value of c, rounded to two decimal places.



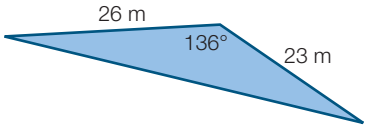
Ex. 6.5

11 A roof consists of rooflines of 5.3 m as shown in the diagram. The rooflines determine the pitch, which is the angle formed where the rooflines meet. Calculate the pitch θ of the roof, correct to the nearest degree.



Ex. 6.6

12 Calculate the area of the shape, rounded to two decimal places.

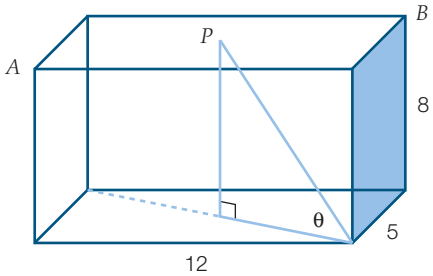


Ex. 6.7

13 P is the midpoint of AB. The angle θ is closest to:

A 22.6° **B** 33.7°

C 42.2° **D** 50.9°



14 State the period and the amplitude, where appropriate, of each relationship.

(a) $y = 7 \sin 2x$

(b) $y = -6 \cos 5x$

(c) $y = 3 \tan 4x$

15 The period of $\tan nx$ is 15°. The value of n must be:

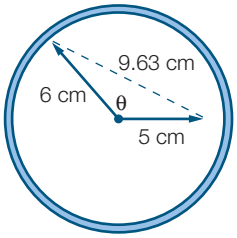
A 10 **B** 12 **C** 15 **D** 24

16 Sketch the graph of $y = 2 \cos x$ for $-180° \le x \le 180°$ using the graph of the basic function $y = \cos x$ as a guide.

17 **(a)** In which quadrant is 400°?

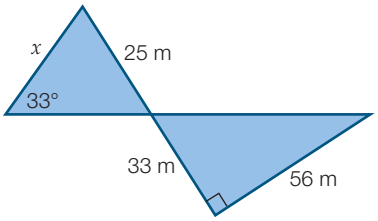
(b) What is the first angle measured clockwise from the positive x-axis that will be coterminal with 400°?

18 The lengths of the hour and minute hand of a clock face are 5 cm and 6 cm respectively as shown on the right. To the nearest degree, what is the size of the obtuse angle formed between the hands when the tips of the hands are 9.63 cm apart.



19 Calculate the area of an isosceles triangle where the angle between the two equal sides each of length 10 cm is 67°. State your answer correct to two decimal places.

20 Calculate the value of x in the diagram at right, rounded to two decimal places.



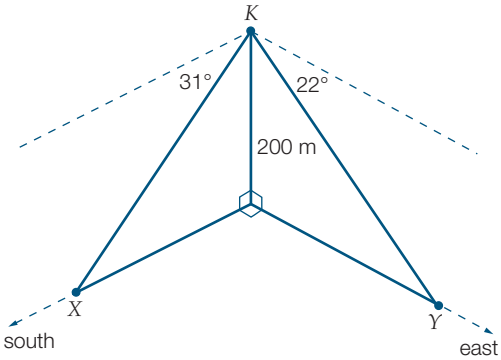
Reasoning and communicating

21 The longest drinking straw that can fit into a cylindrical soft-drink can is 15.7 cm. If the can has a radius of 36 mm, find:

(a) the angle, correct to one decimal place, that the straw makes with the base of the can

(b) the height of the can, rounded to two decimal places.

22 From a point K on top of a 200 m high building, the angle of depression to a point X on the ground due south of K is 31° and the angle of depression to a point Y on the ground due east of K is 22°.



(a) Find, correct to two decimal places, the distance between:

(i) the building and point X

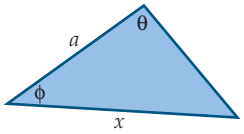
(ii) the building and point Y.

(b) Hence, find the distance between X and Y rounded to two decimal places. (Use Pythagoras' Theorem.)

(c) What is the true bearing of Y from X, to the nearest degree?

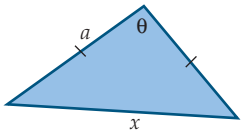
23 Which angles in the unit circle do not have equivalent angles? Use angles from 0° to 360°.

24 **(a)** **(i)** For given values of a, θ and ϕ explain how the sine rule can be used to find the value of x in the triangle at right.



(ii) Find the value of x when $\phi = 41°$, $\theta = 93°$ and $a = 10$. Write your answer correct to two decimal places.

(b) **(i)** For given values for a and θ, explain how the sine rule can be used to find the value of x in the triangle at right.

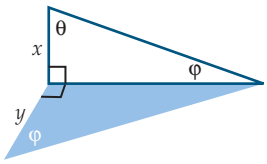


(ii) Find the value of x when $\theta = 93°$ and $a = 10$. Write your answer correct to two decimal places.

25 A fishing boat leaves port on a bearing of 136°T. Following this bearing, it drops x m of netting. The vessel then travels north-east, dropping y m of netting along the way. Finally, it heads directly back to its starting point, dropping netting as it moves. The expression that gives the area in square metres enclosed by the netting is:

A $\frac{1}{2}xy\sin 44°$ **B** $\frac{1}{2}xy\sin 45°$ **C** $\frac{1}{2}xy\sin 46°$ **D** $\frac{1}{2}xy\sin 89°$

26 The shadow of a triangular sculpture is shown. If $y = 1.5x$, show that $2\tan\theta = 3\tan\varphi$.



Mixed review B

Stage 5.1 Q7 **Stage 5.2** Q1–5, 11, 13, 14, 17–22, 27, 29, 33, 35, 36 **Stage 5.3** Q6, 8–10, 12, 15, 16, 23–26, 28, 30–32, 34

Fluency and understanding

1 Find the interest earned on an investment of \$6000 at 4.2% p.a. for 3 years, where the interest is compounded: Ex.1.3

(a) annually **(b)** half-yearly **(c)** quarterly.

2 A painting has increased in value, solely due to the effects of inflation, from \$2500 to \$3100 over 5 years. Find the average annual inflation rate. State your answer correct to one decimal place. Ex.1.4

3 Expand and simplify each of the following. Ex.2.1

(a) $(3x-4)(x+2)$ **(b)** $(2x+5)(3x-1)$ **(c)** $(4x-3)(2x-5)$

4 Factorise the following expression fully. Ex.2.2

(a) $(3x-7)(2x+6)-5(3x-7)$ **(b)** $2x(3x-2)(4x+5)+2x(3x-2)$

5 Factorise the following quadratic trinomials. Ex.2.3

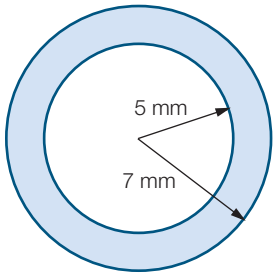
(a) x^2-3x-4 **(b)** $x^2-14x+45$ **(c)** $x^2-7x+12$

6 Simplify the following. Ex.2.9

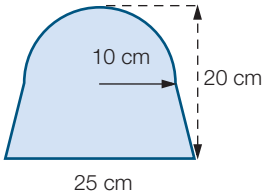
(a) $\frac{1}{3x}-\frac{2}{x}$ **(b)** $\frac{3x}{x+1}-\frac{2}{x-1}$ **(c)** $\frac{2x+1}{x}+\frac{3x-3}{3x}$

7 Find the shaded area of the following shapes, correct to two decimal places. Ex.3.1

(a)



(b)



8 Find the total surface area, correct to two decimal places, of a shape consisting of a cube of side length 7 cm with a pyramid of base 7 cm and perpendicular height 4 cm sitting on top of it. Ex.3.4

9 Find the volume of a square pyramid with base length 12 cm and perpendicular height 15 cm. Ex.3.5

10 A metal sphere has a volume of 12 cm^3. If its mass is 76.8 g, what is its density? Ex.3.6

11 Solve each of the following equations. Ex.4.1

(a) $\frac{2x+3}{7}=3$ **(b)** $\frac{4(3x-2)}{5}=4$ **(c)** $2(4x+5)=26$

(d) $7(2x-1)=21$ **(e)** $6x+2=7x$ **(f)** $8x-9=7x-4$

(g) $4(x+5)=2x-4$ **(h)** $\frac{2(2x-4)}{3}=4$

12 Calculate the gradient of the line that passes through each pair of coordinates. Ex.4.2

(a) (-3, 0) and (3, 9) **(b)** (-4, -1) and (-2, 5)

(c) (0, 6) and (4, -2) **(d)** (-3, 2) and (-1, 6)

13 Identify which of the following points lie on each given line. Ex.4.3

(a) (-2, -1) or (2, 8) $y = 3x + 5$

(b) (1, -1), (-2, -3) or (-3, 5) $y = -2x + 1$

(c) (0, -2), (4, 1) or (-8, 0) $y = \frac{1}{4}x + 2$

14 Graph each of the following lines using the gradient and y-intercept method. Ex.4.3

(a) $y = -2x + 1$ **(b)** $y = 3x + 3$ **(c)** $y = \frac{1}{4}x - 4$ **(d)** $y + 6x = 9$

15 Determine whether each pair of lines is perpendicular. Write 'Yes' or 'No' and justify your answer with appropriate calculations. Ex.4.7

(a) $2x + 4y = 8$ and $6x = 3y - 3$ **(b)** $2x - y = 3$ and $x + 2y = 8$

(c) $5x + 2y = 8$ and $5y = 2x + 10$ **(d)** $3y = 4x - 2$ and $4y = 3x - 4$

16 Determine whether each pair of lines is parallel. Write 'Yes' or 'No' and justify your answer with appropriate calculations. Ex.4.7

(a) $6x - 3y = 9$ and $3x = 3y - 3$ **(b)** $x - y = 8$ and $x + y - 2 = 0$

(c) $2x + 5y = 7$ and $5y = 2x - 8$ **(d)** $10y = 3x + 2$ and $6x - 20y + 4 = 0$

17 Solve the following inequalities. Ex.4.8

(a) $2x - 1 \leq 4x + 2$ **(b)** $\frac{4x - 1}{3} > \frac{3x + 5}{2}$

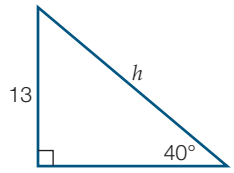
18 Solve the following simultaneous equations. Ex.4.9

(a) $3x - 5y = 7$
$x + 5y = -11$

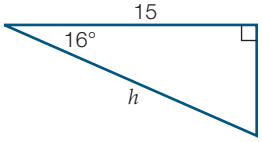
(b) $2x - 3y = 17$
$y = 3x - 22$

19 Find the values of the pronumerals, correct to two decimal places. Ex.5.2

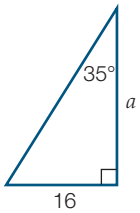
(a)



(b)



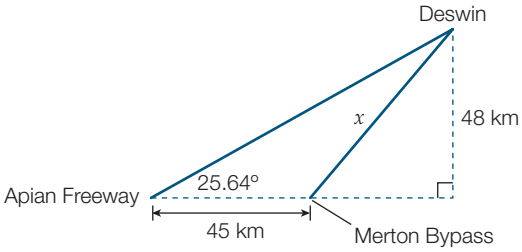
(c)



20 A man standing 8 m from a flagpole looks up to the top of it. Through what angle of elevation does he look (to two decimal places) if the top is 16 m higher than his eye-level? Ex.5.4

21 A scout walks 3 km on a true bearing of 050°T and then 5 km on a true bearing of 077°T. What is the scout's distance from the starting position (to two decimal places)? Ex.5.5

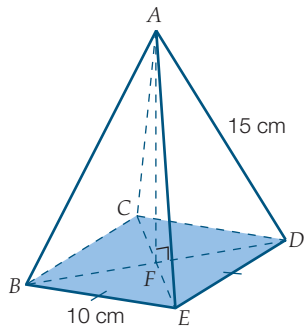
22 A proposed section of highway that was to connect the Apian Freeway to the town of Deswin will now connect the Merton Bypass to Deswin as shown. What will be the length, x km, of the new section of highway? Write your answer to the nearest kilometre. Ex.5.6



23 A right pyramid *ABCDE* stands on a square base of edge length 10 cm. The sloping edges are each 15 cm long. *AF* is the perpendicular height.

(a) Find, correct to one decimal place, the angle that a sloping edge makes with the base of the pyramid.

(b) Hence, find the height of the pyramid, correct to one decimal place.

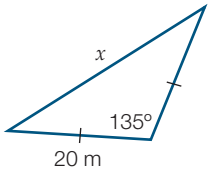


Ex. 6.1

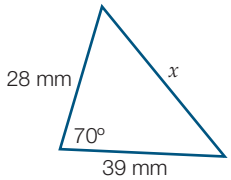
24 Find the value of x in the following triangles, correct to two decimal places.

Ex. 6.6

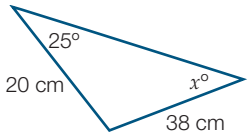
(a)



(b)



(c)



25 Find the area of each of the triangles in Question **24**, correct to two decimal places.

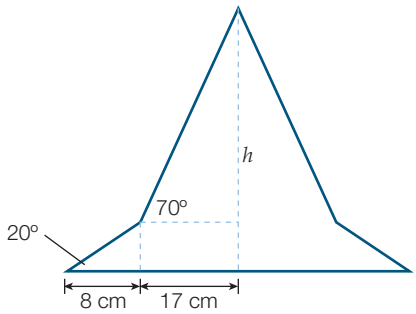
Ex. 6.7

26 The height of a particular young pine tree is found to increase in a linear manner each month in the first few years after planting. What was the initial height of the tree, if after 2 months the height is 34 cm, and after 5 months the height is 52 cm?

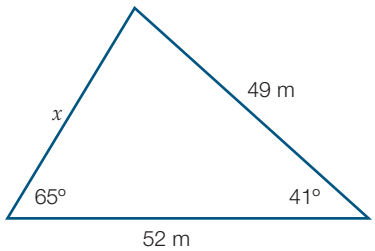
27 The sides of a rectangle differ by 4 m. If all sides are increased by 2 m, the area increases by 52 m^2. What are the dimensions of the original rectangle?

28 Determine whether the following points are collinear: $A(1, -1)$, $B(3, 3)$, $C(6, 9)$.

29 A witch's hat, used on roads to divert traffic, rises from the outside of the brim at an angle of 20° for a horizontal distance of 8 cm then rises to an apex at an angle of 70° for a further horizontal run of 17 cm. Find h, the height of the witch's hat, correct to two decimal places.



30 Which of the following equations cannot be used to find the value of x from the diagram?



A $\dfrac{x}{\sin 41°} = \dfrac{49}{\sin 65°}$

B $\dfrac{x}{\sin 41°} = \dfrac{52}{\sin 74°}$

C $x^2 = 52^2 + 49^2 - 2 \times 52 \times 49 \cos 41°$

D $\dfrac{x}{\sin 41°} = \dfrac{52}{\sin 65°}$

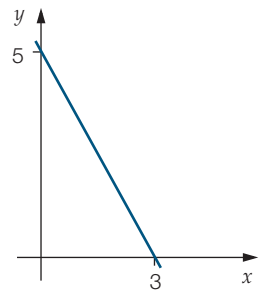
Reasoning and communicating

31 Prove that a right-angled triangle is produced by connecting the points $A(-1, 3)$, $B(5, -1)$ and $C(0, -2)$.

32 Prove that a rectangle is produced by connecting the points $A(-6, 0)$, $B(3, -6)$, $C(5, -3)$ and $D(-4, 3)$.

33 A quadrilateral has the angles $x°$, $(2x + 3)°$, $(2x - 3)°$ and $(x + 6)°$. Find the size of each angle.

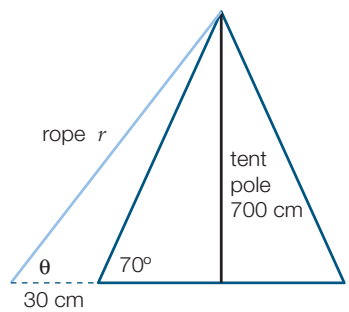
34 Find the equation of the line that is perpendicular to the line shown on the diagram, such that the perpendicular line passes through the point (-2, 3).



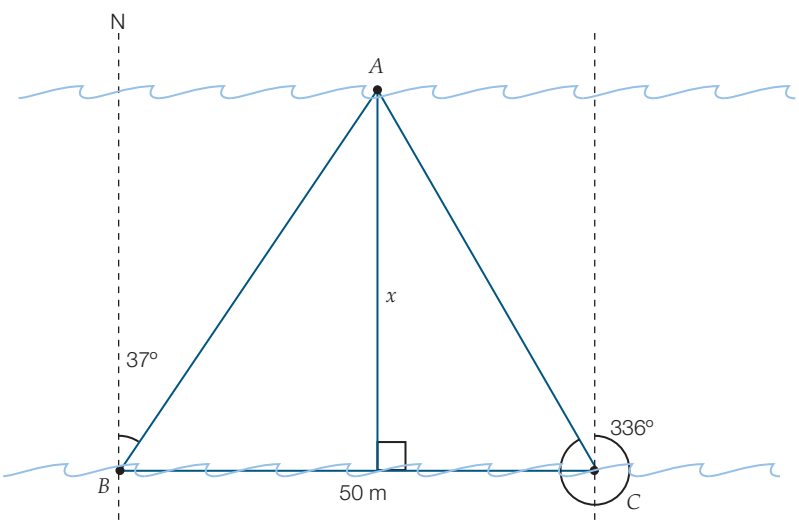
35 The sides of a tent of height 700 cm make an angle of 70° with the ground. A rope used to hold the tent pole upright is secured 30 cm from the base of the tent.

(a) Find the angle it makes with the ground, correct to two decimal places.

(b) Find the length of the rope, to the nearest centimetre.

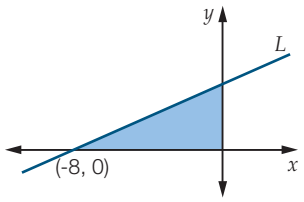


36 A point A on one side of a river is at a bearing of 037°T from point B on the other side of the river. From point C, 50 m due east of point B, the bearing of point A is 336°T. How wide is the river, in metres, correct to two decimal places?



Challenge B

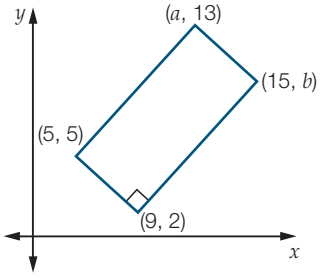
1 The line L crosses the x-axis at (-8, 0). The area of the shaded region is 16. What is the slope of the line L?



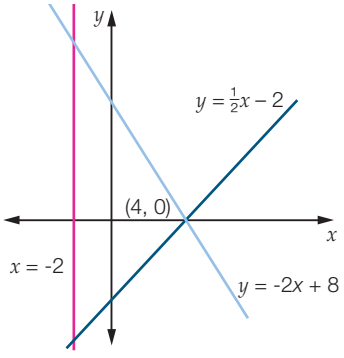
A $-\frac{1}{2}$ **B** $\frac{1}{2}$

C 2 **D** 4

2 In the rectangle shown, what is the value of $a - b$?



3 The lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ meet at (4, 0), as shown. The area of the triangle formed by these two lines and the line $x = -2$ is:



A 15 **B** 27

C 36 **D** 45

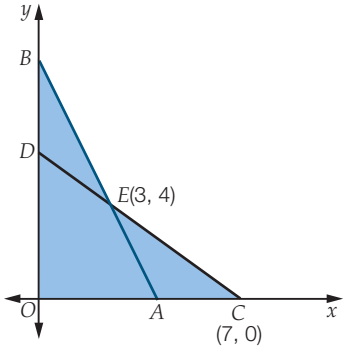
4 ΔABC has vertices $A(0, 0)$, $B(9, 0)$ and $C(0, 6)$. The points P and Q lie on the side AB such that $AP = PQ = QB$. Similarly, the points R and S lie on the side AC so that $AR = RS = SC$. The vertex C is joined to each of the points P and Q. The vertex B is joined to R and S.

(a) Find the equation of the line through the points R and B.

(b) Find the equation of PC.

(c) The line segments PC and RB intersect at X, and the line segments QC and SB intersect at Y. Show, by calculation, that the points A, X and Y lie on the same straight line.

5 A line with slope -3 intersects the positive x-axis at A and the positive y-axis at B. A second line intersects the x-axis at $C(7, 0)$ and the y-axis at D. The lines intersect at $E(3, 4)$.



(a) Find the slope of the line through C and E.

(b) Find the equation of the line through C and E, and the coordinates of the point D.

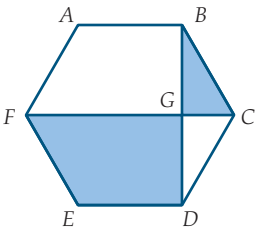
(c) Find the equation of the line through A and B, and the coordinates of the point B.

(d) Determine the area of the shaded region.

6 The point Y lies on the line $y = 5x + 3$. The coordinates of the point Z are (3, -2). If X is the midpoint of YZ, then X must lie on the line:

A $y = \frac{5x}{2} - \frac{7}{2}$ **B** $y = 5x + 1$ **C** $y = \frac{5x}{2} + \frac{1}{2}$ **D** $y = 5x - 7$

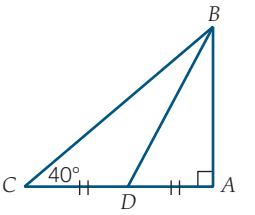
7 In the regular hexagon $ABCDEF$, the lines BD and CF intersect at G. The ratio of the area of quadrilateral $DEFG$ to ΔBCG is:



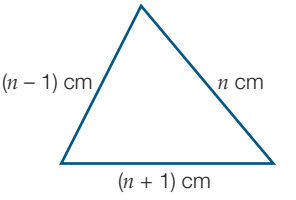
A $3\sqrt{3}:1$ **B** $5:1$

C $4:1$ **D** $2\sqrt{3}:1$

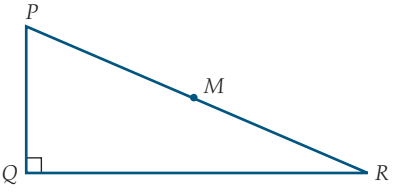
8 A flagpole AB is 20 metres tall. From a point C on flat ground, the angle of elevation to the top of the flagpole is 40°. If $CD = DA$, calculate the size of $\angle ADB$ rounded to the nearest degree.



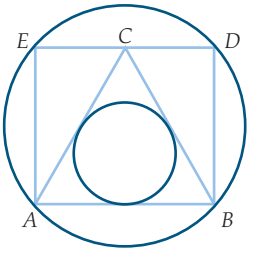
9 Find the area of this triangle if n is a whole number and the area of the triangle is a whole number of square centimetres. All the sides are between 10 cm and 20 cm. You may wish to use Heron's formula for the area of a triangle: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ and a, b and c are the sides of the triangle.



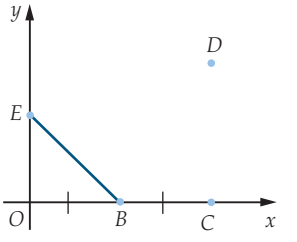
10 Triangle PQR is right-angled at Q and has side lengths $PQ = 14$ and $QR = 48$. M is the midpoint of PR. Draw the interval MS, where S is on PQ, such that $MS \| RQ$. Find the value of the cosine of $\angle MQP$, correct to two decimal places.



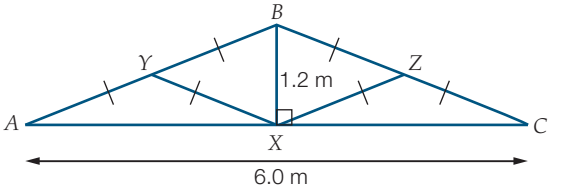
11 ΔABC is an equilateral triangle. The radius of the circle just touching the sides of ABC is 1 unit. A larger circle is drawn through the vertices of the rectangle $ABDE$. Calculate the diameter of the larger circle.



12 In the diagram, E has coordinates (0, 2), and B lies on the positive x-axis so that $BE = 7$. The point C lies on the positive x-axis so that $BC = OB$. If D lies in the first quadrant such that $\angle CBD = 30°$ and $\angle BCD = 90°$, what is the length of ED correct to two decimal places?



13 The diagram shows a roof truss. Triangle ABC is isosceles with $AB = BC$. The midpoints of AB and BC are Y and Z respectively. $AC = 6.0$ m and $BX = 1.2$ m. Calculate the length of AB correct to one decimal place. Find the total length of timber required to make the truss.



Calculate the size of $\angle BAX$ and hence, find the size of $\angle BYX$. Round both angles to the nearest degree.

7 Statistics

P Recall 7

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Find the mean, median (middle number), mode (most common number) and range (difference between largest and smallest numbers) for each of these sets of data.

(a) The number of siblings for your five best friends:
2 4 5 1 4

(b) The shoe size of your nine young cousins:
3.5 4.5 5.0 5.5 5.0 4.5 5.5 5.0 3.5

2 Classify each of the following as being nominal, discrete or continuous data.

(a) Number of goals scored by a hockey team each week.

(b) Height of the members of your maths class.

(c) Size of blouses on a rack.

3 The heights (in centimetres) of 20 students are as follows:

179.7 167.3 168.6 164.7 182.8 160.5 179.2 163.0 177.7 171.3
169.8 173.2 164.7 172.1 165.6 171.9 164.8 166.7 169.1 166.7

Round the heights to the nearest centimetre and:

(a) using intervals of 5 cm, draw a stem plot

(b) find the median.

4 The distance jumped by each competitor in the long jump was recorded and is shown in the frequency table.

(a) Find an estimate for the value of the mean.

(b) In which class interval does the median fall?

Class interval (m)	Frequency
5.75–<6.00	4
6.00–<6.25	6
6.25–<6.50	9
6.50–<6.75	13
6.75–<7.00	5
7.00–<7.25	3

Syllabus references

Statistics and Probability: Single Variable Data Analysis

✓ Determine quartiles and interquartile range **(ACMSP248) (Stage 5.2◊)**
✓ Construct and interpret box plots and use them to compare data sets **(ACMSP249) (Stage 5.2◊)**
✓ Compare shapes of box plots to corresponding histograms and dot plots **(ACMSP250) (Stage 5.2◊)**
✓ Investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians **(ACMSP227) (Stage 5.2◊)**
✓ Calculate and interpret the mean and standard deviation of data and use these to compare data sets **(ACMSP278) (Stage 5.3)**

Statistics and Probability: Bivariate Data Analysis

✓ Investigate and describe bivariate numerical data where the independent variable is time **(ACMSP252) (Stage 5.2)**
✓ Use scatter plots to investigate and comment on relationships between two numerical variables **(ACMSP251) (Stage 5.2)**
✓ Use information technologies to investigate bivariate numerical data sets; where appropriate, students use a straight line to describe the relationship, allowing for variation **(ACMSP279) (Stage 5.3)**
✓ Investigate reports of studies in digital media and elsewhere for information on their planning and implementation **(ACMSP277) (Stage 5.3)**

Cumulative frequency curves

7.1

Need to Know

Stage 5.2◊

Finding the median of grouped data

Finding statistics from ungrouped data is time consuming and tedious when there is a large amount of data involved. It is often convenient to group data for statistical analysis but, as individual data values are lost, only estimates for statistics can now be found.

When large amounts of data are grouped, finding the median is no longer possible using the methods used for ungrouped data. A **cumulative frequency curve**, sometimes called an **ogive**, is used to obtain an estimate for the median and other associated statistics. The size of the error introduced by estimating the median is small and insignificant.

Cumulative frequency curve

A cumulative frequency curve shows the number of results less than a given value in grouped data.

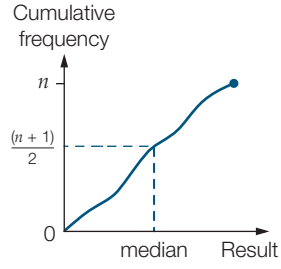
The median of n results is the $\left(\frac{n+1}{2}\right)$th result.

To find the median of data grouped in class intervals, follow these steps.

Step 1 Construct a cumulative frequency table.

Step 2 Plot the cumulative frequency curve.

Step 3 Find the median by reading the $\left(\frac{n+1}{2}\right)$th result from the curve.



Worked Example 1

WE1

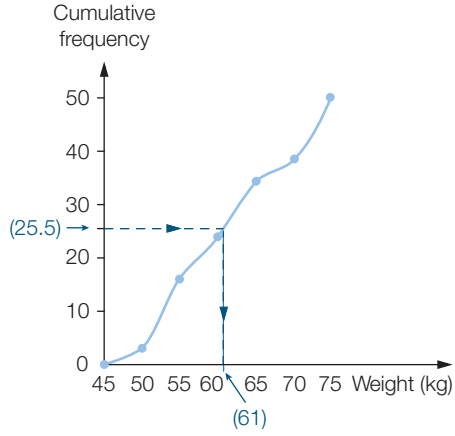
The following frequency table lists the weights of 50 packing crates, measured in kilograms. Find the median by constructing a cumulative frequency curve.

Class interval (kg)	Frequency
45–<50	3
50–<55	13
55–<60	8
60–<65	10
65–<70	4
70–<75	12

Working

Class interval (kg)	Cumulative frequency
<45	0
<50	3
<55	16
<60	24
<65	34
<70	38
<75	50

$\left(\frac{50+1}{2}\right) = 25.5$th result



The median is about 61 kg.

Worked Example 2

WE2

The following table shows the data on the number of components produced during a shift. Find the median by constructing a cumulative frequency curve.

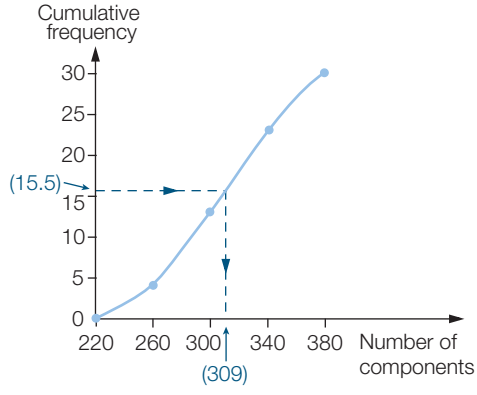
Class interval	Frequency
220–259	4
260–299	9
300–339	10
340–379	7

Working

Number of components	Cumulative frequency
<220	0
<260	4
<300	13
<340	23
<380	30

$\left(\frac{30+1}{2}\right) = 15.5$th result

The median is about 309 components per shift.



7.1 Cumulative frequency curves

Answers page 643

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4, Q5, Q7	Q1, Q2, Q4, Q5, Q6, Q7	Q1 Column 2, Q2, Q4, Q5, Q6, Q7

Fluency and understanding

1 Find the median for each of the following grouped data sets by constructing a cumulative frequency curve. WE1

(a)

Class interval	Frequency
0–<10	4
10–<20	5
20–<30	8
30–<40	11
40–<50	7

(b)

Class interval	Frequency
8–<10	1
10–<12	4
12–<14	6
14–<16	9
16–<18	7

(c)

Class interval	Frequency
60–<70	9
70–<80	12
80–<90	19
90–<100	17
100–<110	13

(d)

Class interval	Frequency
250–<300	37
300–<350	58
350–<400	26
400–<450	16
450–<500	9
500–<550	4

2 (a) The frequency table on the right shows the number of students present each day at a particular school during first term. Find the median number of students present by constructing a cumulative frequency curve. WE2

Class interval	Frequency
450–459	2
460–469	7
470–479	8
480–489	15
490–499	10
500–509	8

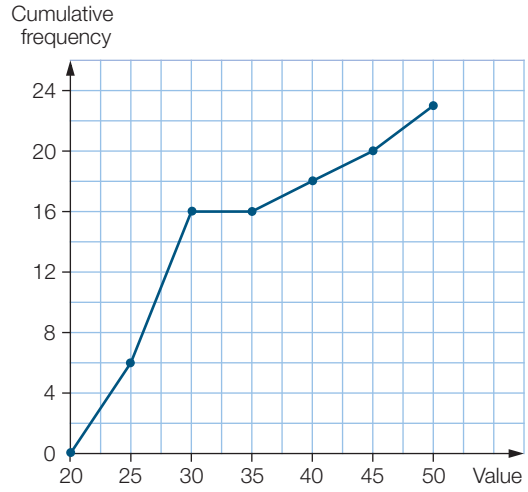
(b) The frequency table on the right shows the number of points scored by each team in one round of a rugby competition. Find the median score by constructing a cumulative frequency curve.

Class interval	Frequency
0–<10	2
10–<20	3
20–<30	3
30–<40	4
40–<50	1
50–<60	1

3 The frequency table on the right lists the lengths of fish caught, measured in centimetres. Find the median length by constructing a cumulative frequency curve.

Class interval (cm)	Frequency
30–<35	7
35–<40	10
40–<45	5
45–<50	12
50–<55	13
55–<60	4

4 The following questions relate to the cumulative frequency graph shown on the right.



(a) The number of values in the data set is:

A 23 **B** 24

C 26 **D** 50

(b) The median is closest to:

A 12 **B** 27

C 28 **D** 29

(c) The class interval that has a frequency of zero is:

A 20–<25 **B** 30–<35

C 30–<36 **D** 31–<36

5 The number of people boarding each of 25 successive trains at a station was as follows:

6	16	21	29	39	46	42	29	21	29	35	21	17
10	8	18	17	26	7	9	12	19	26	20	21	

(a) Find the median of this raw data.

(b) Group the data into the intervals 1–<10, 11–<20 etc. to draw a cumulative frequency curve and find the median.

(c) On how many trains was the number of people boarding less than 18? Compare the answer obtained from the raw data with the one obtained using a cumulative frequency curve.

Reasoning, communicating and open-ended

6 There is another way to estimate the median for grouped data. It uses the likely location of the median within the class interval in which the median lies. If we know that the median is the third of eight results in the class interval 20–<25, we say that the median is approximated by the value $20 + \frac{3}{8} \times 5 \approx 21.9$. We would be unlikely to go beyond even one decimal place in a calculation such as this.

(a) Use this method to find an approximation for the median for Question **1(b)**.

(b) Compare this with the value obtained using the cumulative frequency curve.

(c) Use this method to find an approximation for the median for Question **2(a)**.

(d) Compare this with the value obtained using the cumulative frequency curve.

(e) What do you think about this alternative method? List some positives and some negatives of the method.

7 Use the following information to construct a discrete data set with at least 19 results and then calculate the mean.

(a) median = 39, odd number of results

(b) median = 72.5, even number of results

Problem solving

Product codes

Most products that you buy have a barcode attached to them. This barcode represents the European Article Number (EAN) and is a 13-digit unique code for that product. As part of the checking system to ensure the correct numbers are scanned by a barcode reader, the 13th digit is actually a check digit.

The following are all legal product codes:

9316090060604	9337090060609
9326090060603	9338090060606
9336090060602	9339090060603

1 In the first column of barcodes, apart from the check digit, which is the only digit that changes? What do you notice about the sum of the two digits that are changing?

2 Predict the final digit in the code 934609006060*.

3 In the second column of barcodes, apart from the check digit, which is the only digit that is changing? Do these digits follow the same pattern as those in the first column of barcodes?

Not every column acts in the same way. In fact, the even columns all act in the same way and the odd columns all act in the same way. We will place the very first EAN-13 into a table to show what happens.

×1	×3	×1	×3	×1	×3	×1	×3	×1	×3	×1	×3	×1
9	3	1	6	0	9	0	0	6	0	6	0	4

4 Calculate the sum of these multiplied values.

5 Do the same for each of the EAN-13 codes given in the table to the left. What conclusion can you reach?

6 Now calculate the final digit in each of the following EAN-13 codes:

(a) 933919006060*

(b) 933929006060*

(c) 933928006060*

(d) 933927006060*

Strategy options

- Work backwards.
- Test all possible combinations.

7.2 Box plots

Stage 5.2◊

Need to Know

Percentiles and quartiles

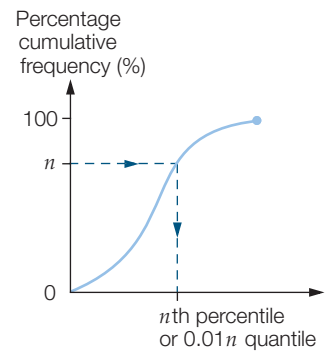
The median is not the only statistic that can be found from a cumulative frequency curve. By converting grouped data to a proportion (percentage) and drawing a **percentage cumulative frequency curve**, the value below which a percentage of the results occur can be found.

For any percentage cumulative frequency curve, n% of results lie below the nth **percentile**. A percentile value, when written as a decimal, is called the **quantile**.

A cumulative frequency is the number of results below a given value.

$$\text{Percentage cumulative frequency} = \frac{\text{cumulative frequency}}{\text{total number of results}} \times 100\%$$

Percentile and quantile values can be read from a percentage cumulative frequency graph. The nth percentile (or $0.01n$ quantile) is the value on the horizontal axis corresponding to n% on the vertical axis, where $0 \le n \le 100$.



For example:

A percentage cumulative frequency curve can be used to find the value below which 70% of the results lie. This value is called the 70th percentile or the 0.7 quantile.

Measures of spread

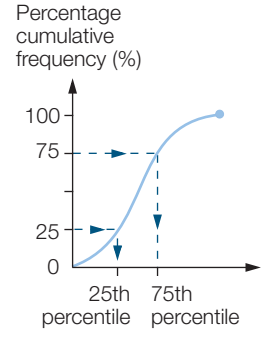
The mean and median are measures of the location (or central tendency) of a set of data. These measures by themselves do not give much information about the data. Statistics showing the spread of the data help to interpret the data better.

There are three commonly used measures of spread (or variation): **range, interquartile range** (IQR) and **standard deviation**. Standard deviation is related to the mean and is covered in a later section.

Range and interquartile range

The range of ungrouped data can be found by subtracting the smallest data value from the largest data value. The range of grouped data can be found by subtracting the smallest value in the class intervals from the largest value in the class intervals.

A data set can be divided into four sections where each section, called a **quartile**, contains 25% of the data. The 75th percentile and 25th percentile are called the upper and lower quartile, respectively. The percentage cumulative frequency curve can be used to find the upper and lower quartile.



The interquartile range (IQR) is the difference between the upper quartile (Q_U) and the lower quartile (Q_L).

$$\begin{aligned} \text{IQR} &= Q_U - Q_L \\ &= \text{75th percentile} - \text{25th percentile} \end{aligned}$$

The upper quarter of results occurs above the upper quartile, whereas the lower quarter of results occurs below the lower quartile. The median may be found on the percentage cumulative frequency curve by evaluating the 50th percentile.

If you have the raw data set, then the interquartile range can be found by breaking the data into two equal subsets. The lower quartile is the median of the lower half of the data and the upper quartile is the median of the upper half of the data.

The quartiles can also be written as Q_1 and Q_3.

In summary:

lower quartile = $Q_L = Q_1$ = 25th percentile = 0.25 quantile

median = Q_2 = 50th percentile = 0.50 quantile

upper quartile = $Q_U = Q_3$ = 75th percentile = 0.75 quantile

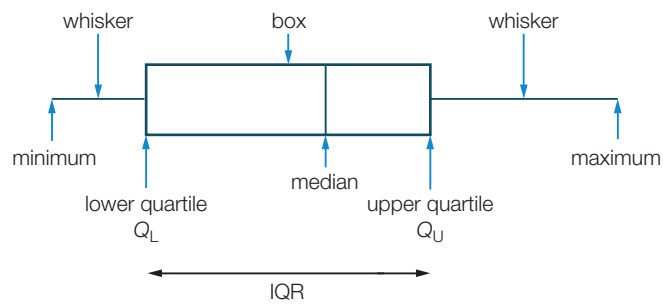
When these statistics have been found, they can be used to draw **box plots**.

Box plots

A box plot, also known as a box-and-whisker plot, is a graphical summary of five key pieces of data known as the **five-number summary**:

- minimum value
- lower quartile Q_L (25th percentile)
- median (50th percentile)
- upper quartile Q_U (75th percentile)
- maximum value

The central part of the diagram, between the two quartiles, is the box, and the single lines extending to the minimum and maximum values are the whiskers.



Outliers

When one or more of the surveyed results lie outside a certain range, they may be considered as **outliers.**

In any particular data set, an outlier may be smaller than most of the other data values, it might be larger than most of the other data values or there might be outliers at both ends of the data range.

It can be difficult to decide if a value is an outlier. Unless there is a good reason to believe that the value is an outlier, the value should be included in your calculations. Outliers often occur as a result of errors in the collection or recording of data values.

In general, the median is not affected by outliers, whereas the mean can change considerably depending on how far away the outlier falls from the rest of the data set. It is important to acknowledge that a data set may contain outliers and to comment on their possible origin.

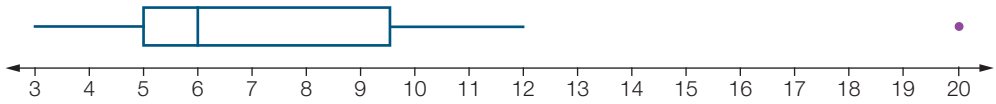
Drawing a graph, such as a box plot, helps to decide whether or not a value is an outlier. A mathematical formula classifies a value as an outlier if it is more than 1.5 × IQR above Q_U or more than 1.5 × IQR below Q_L.

Showing outliers

When drawing a box plot that includes an outlier, or some outliers, highlight these values by drawing them as dots instead of stretching the whisker all the way out to the outlier value.

For example:

This is a box plot for the data set: 3, 5, 5, 6, 6, 7, 7, 12, 20. Outliers are <-1.75 and >16.25, therefore 20 is an outlier.



The right-hand whisker ends at 12, which is the highest value that is not an outlier. The outlier, 20, is shown on the box plot as a dot.

Worked Example 3

WE3

The following table shows the data in class intervals from a survey of the weights of packing crates. Use the data to:

(a) draw a cumulative frequency table and then a percentage cumulative frequency table

(b) draw a percentage cumulative frequency curve

(c) evaluate the 80th percentile and the 0.3 quantile.

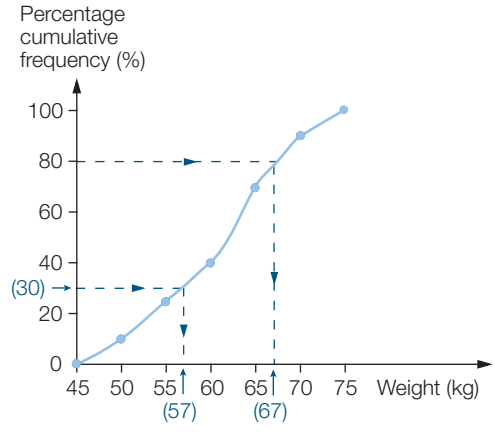
Weight (kg)	Frequency
45–<50	2
50–<55	3
55–<60	3
60–<65	6
65–<70	4
70–<75	2

Working

(a)

Weight (kg)	Cumulative frequency	Percentage cumulative frequency
<45	0	$\frac{0}{20} \times 100 = 0$
<50	2	$\frac{2}{20} \times 100 = 10$
<55	5	$\frac{5}{20} \times 100 = 25$
<60	8	$\frac{8}{20} \times 100 = 40$
<65	14	$\frac{14}{20} \times 100 = 70$
<70	18	$\frac{18}{20} \times 100 = 90$
<75	20	$\frac{20}{20} \times 100 = 100$

(b)



(c) The 80th percentile is 67 kg.

The 0.3 quantile (30%) is 57 kg.

Worked Example 4

WE4

The weights of 20 packing crates (kg) are shown in the frequency table.

(a) Calculate the range and interquartile range.

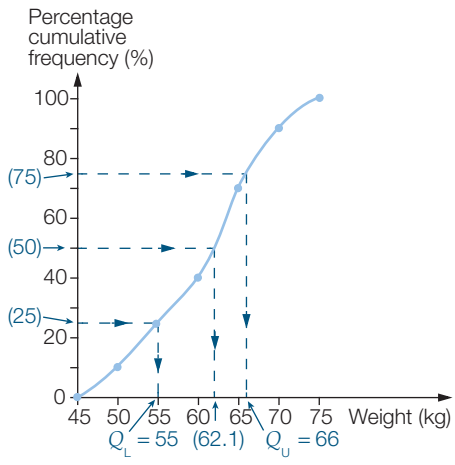
(b) Draw a box plot to represent the data.

Weight (kg)	Frequency
45–<50	2
50–<55	3
55–<60	3
60–<65	6
65–<70	4
70–<75	2

Working

(a) Range = largest result − smallest result
= 75 − 45
= 30

Weight (kg)	Cumulative frequency	Percentage cumulative frequency
<45	0	$\frac{0}{20} \times 100 = 0$
<50	2	$\frac{2}{20} \times 100 = 10$
<55	5	$\frac{5}{20} \times 100 = 25$
<60	8	$\frac{8}{20} \times 100 = 40$
<65	14	$\frac{14}{20} \times 100 = 70$
<70	18	$\frac{18}{20} \times 100 = 90$
<75	20	$\frac{20}{20} \times 100 = 100$

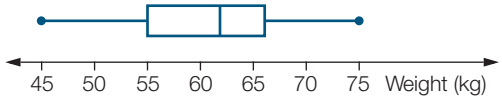


$Q_U = 66$
$Q_L = 55$

IQR = 66 − 55
= 11

The range of weights is 30 kg and the interquartile range is 11 kg.

(b) Five-number summary: minimum value = 45, Q_L = 55, median = 62.1, Q_U = 66, maximum value = 75



Worked Example 5

WE 5

Consider the following results from a survey of the number of cans of baked beans produced on an assembly line per hour:

244 246 247 248 242 24 248 248 247 247 246 244 242 242 246

Identify the outlier and suggest a possible reason for it.

Working

The result 24 is more than 200 less than the next smallest result; it is clearly an outlier.

Machine breakdown, operator error, operator injury or result incorrectly recorded could all be reasons for this outlier.

Worked Example 6

WE 6

Determine if there are any outliers in this list of the weights of 20 students (kg):

60.4	53.9	61.1	62.3	50.4	57.3	45.7	59.4	68.6	65.5
60.8	67.9	57.6	62.9	65.8	70.0	73.7	49.6	54.8	62.7

Working

Ordered data list:

45.7	49.6	50.4	53.9	54.8
57.3	57.6	59.4	60.4	60.8
61.1	62.3	62.7	62.9	65.5
65.8	67.9	68.6	70.0	73.7

The data list seems relatively evenly spread out; no outlier is obvious.

$$Q_L = \frac{54.8 + 57.3}{2}$$
$$= 56.05$$

$$Q_U = \frac{65.5 + 65.8}{2}$$
$$= 65.65$$

$$\text{IQR} = Q_U - Q_L$$
$$= 65.65 - 56.05$$
$$= 9.6$$

$$1.5 \times \text{IQR} = 1.5 \times 9.6$$
$$= 14.4$$

$$Q_L - 1.5 \times \text{IQR} = 56.05 - 14.4$$
$$= 41.65$$

$$Q_U + 1.5 \times \text{IQR} = 65.65 + 14.4$$
$$= 80.05$$

None of the values lie outside the designated range, confirming the original view that there are no outliers.

7.2 Box plots

Answers page 644

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q11, Q13, Q14, Q15	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q11, Q12, Q13, Q15

Fluency and understanding

1 Draw cumulative frequency and percentage cumulative frequency curves to represent the following data, and find the 60th percentile for each data set. WE3

(a)

Class interval	Frequency
15–<16	4
16–<17	5
17–<18	7
18–<19	3
19–<20	4

(b)

Class interval	Frequency
150–<155	1
155–<160	2
160–<165	3
165–<170	4
170–<175	3

2 For each given data set, calculate **(i)** the range and **(ii)** the interquartile range, then **(iii)** draw a box plot. WE4

(a)

Class interval	Frequency
0–<10	4
10–<20	5
20–<30	8
30–<40	11
40–<50	7

(b)

Class interval	Frequency
60–69	9
70–79	12
80–89	19
90–99	17
100–109	13

3 The number of points scored by a basketballer in a season of games is: WE5

32	30	16	18	24	16	28	26	19	18	8	26
34	36	41	18	16	14	17	21	26	36	54	8

Without grouping the data, identify any outliers and suggest possible reasons for them.

4 The following is a list of the heights, measured to the nearest centimetre, of the players at the local football club. WE6

155	167	189	156	178	179	190	200	206	196	187	156
166	178	168	178	168	170	167	159	167	160	163	181
191	157	156	187	157	159	156	161	153	173	174	168
177	164	175	185	186	164	171	170	180	141	167	159

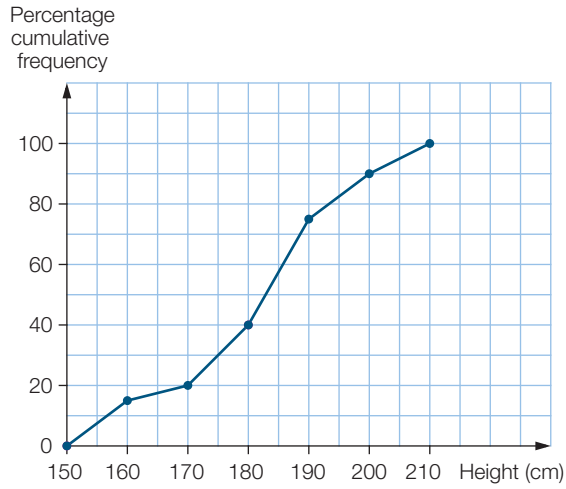
(a) Find the range of the heights.

(b) Calculate the interquartile range by finding the upper and lower quartiles.

(c) Calculate the values that are the cut-off values at either end of the data set after which the results would be considered outliers.

(d) Are there any outliers in the data set?

5 Each question below refers to the percentage cumulative frequency graph shown at right, which represents the heights of players in a football club.



(a) The 30th percentile is closest to:

A 170 **B** 172.5

C 175 **D** 184

(b) The 0.8 quantile is closest to:

A 170 **B** 188

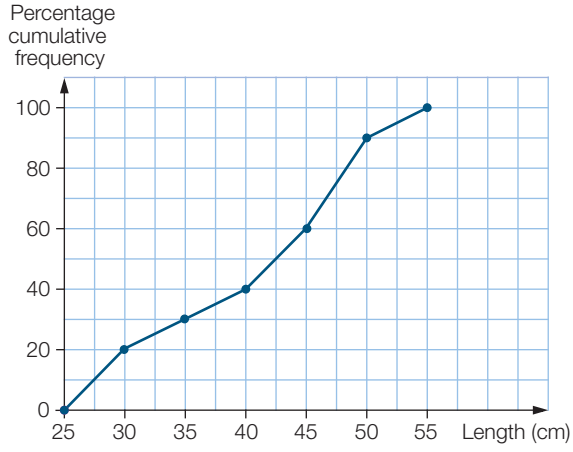
C 191 **D** 194

(c) The interquartile range is closest to:

A 16 **B** 18

C 22 **D** 50

6 For the percentage cumulative frequency graph shown at right, outliers would be:



A results less than 32 cm

B results greater than 63 cm

C results less than 16.5 cm and greater than 63 cm

D none

7 The distances run (km) by athletes training for a marathon are summarised in this frequency table.

Class interval	Frequency
10–<15	4
15–<20	6
20–<25	13
25–<30	16
30–<35	11
35–<40	9

(a) Draw a cumulative frequency curve for this data.

(b) How many of the distances run were less than 17 km?

(c) Draw a percentage cumulative frequency curve.

(d) Find the 20th percentile and the 0.8 quantile.

8 The data below shows the distances travelled (km) by car in a week for 30 employees.

895	245	697	1098	573	1569	1296	1880	972	843
1108	1037	986	1265	1398	1107	1275	1117	1023	1062
1197	1047	1284	1402	438	569	974	1328	1194	1073

(a) Calculate the range of the data.

(b) Calculate the interquartile range by finding the upper and lower quartiles.

(c) Draw a box plot and identify any outliers.

9 A survey of the number of components produced on an assembly line per shift gave the following results:

223	246	288	295	374	278	286	251	299	370	345
313	368	312	309	333	357	328	347	281	299	304
305	346	321	287	315	305	274	228	963		

(a) Is it possible to have an outlier that is very high as well as one that is very low?

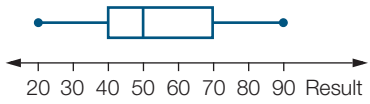
(b) Suggest possible explanations for the 963 components produced in the last shift.

10 For each of the following box plots:

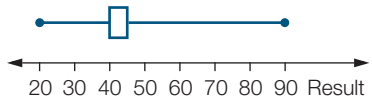
(i) indicate the values below which, and above which, outliers would lie.

(ii) decide whether there are any outliers in the data set displayed.

(a)



(b)



Reasoning, communicating and open-ended

11 The following data represents the weight, measured in kilograms, of players at a sporting club:

80	84	100	85	84	91	80	80	86	85	106	92
104	83	79	81	85	81	87	105	88	95	82	100
114	86	85	92	88	88	77	101	89	99	86	76
86	83	91	80	82	74	107	79	91	82	87	

(a) Calculate the five-number summary for the data.

(b) What percentage of the values lie below the lower quartile?

(c) What percentage of the values lie above the upper quartile?

(d) What does the theory tell us these two values should be?

(e) Describe how well this data matches theory and offer an explanation for any difference.

(f) Calculate the interquartile range for the data and use this to decide if there are any outliers in the data set.

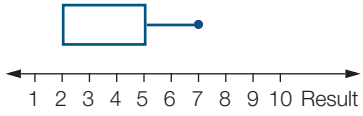
(g) Draw a box plot for the data.

12 For each of the following box plots:

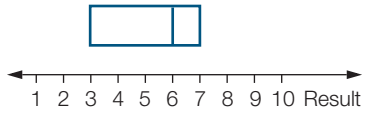
(i) construct a data set of at least 20 values that matches it

(ii) explain the reasoning you used to do this, including any assumptions made and alternative actions that might have been taken.

(a)



(b)



13 Consider this table, which shows the annual earnings for all the employees of a company.

Annual income	Frequency
\$35 000–<\$40 000	17
\$40 000–<\$45 000	24
\$45 000–<\$50 000	18
\$50 000–<\$55 000	16
\$55 000–<\$60 000	1
\$60 000–<\$65 000	1
\$65 000–<\$70 000	2
\$70 000–<\$75 000	1
\$75 000–<\$80 000	1
\$80 000–<\$85 000	1

(a) Calculate estimates for the mean and median annual incomes.

(b) Calculate estimates for the range and interquartile range.

(c) Are there any outliers in the data set? If so, list them.

(d) Remove any outlier(s) from the results and recalculate the mean, median, range and interquartile range.

(e) Comment on your results.

14 **(a)** Draw a cumulative frequency graph with an interquartile range of 35.

(b) Draw a percentage cumulative frequency graph with an interquartile range of 16.

15 **(a)** Construct a data set with at least 20 values that has an outlier at the upper end of the data set. The outlier is to be less than $Q_U + 2 \times \text{IQR}$.

(b) Construct a data set with at least 20 values that has an outlier at the lower end of the data set. The outlier is to be greater than $Q_L - 2 \times \text{IQR}$.

Problem solving

Draw that plot

A data set has the following characteristics:

- 50th percentile is 56
- 75th percentile is 80
- IQR is 50
- Range is 80.

Draw a box plot to represent this information. Which of the five key values are you certain about? Give the maximum and minimum values possible for the values about which you are unsure.

Strategy options

- Work backwards.
- Draw a diagram.

Comparing data sets

7.3

Stage 5.2◊

Need to Know

To compare sets of data, two measures of centre or location, the mean and the median, and two measures of the spread, the range and the interquartile range, can be used.

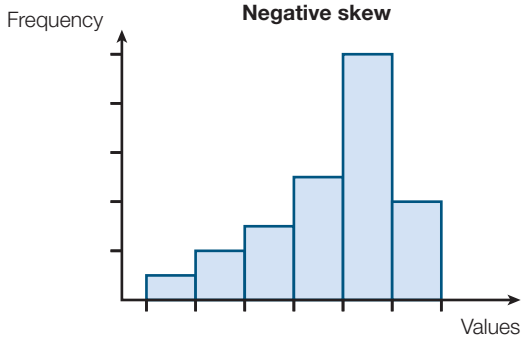
Using box plots and dot plots to compare data

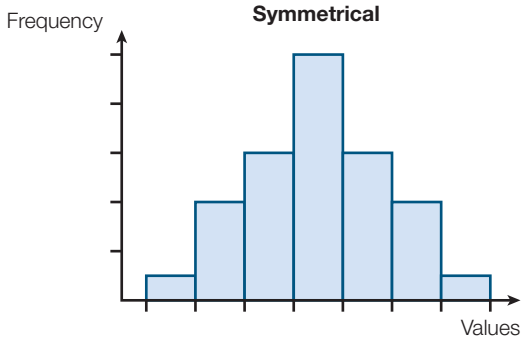
Box plots are useful when comparing data sets. Using a **parallel box plot**, which is two or more box plots drawn on the same set of axes, the median, range and IQR of the data sets can be compared. A parallel box plot can also be used to decide which data set is more consistent. It is important that the scale used covers all values in both sets.

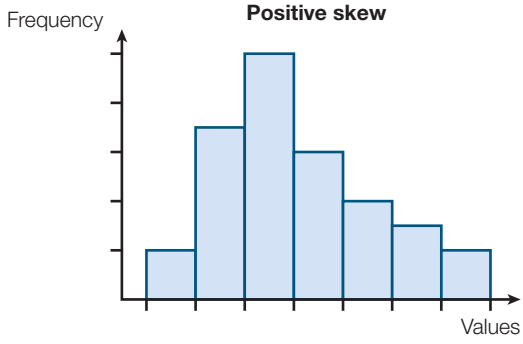
In the same way, a **parallel dot plot** can also be useful. When drawing a parallel dot plot it is common to put the scale in twice; it effectively serves as the dividing line between the two dot plots.

Histograms

Data can also be represented by histograms, which can be skewed (positively or negatively) or symmetrical. The following histograms show examples.







To help remember which is which, negative skew has the long tail on the left (or negative) side of the peak (mode) and positive skew has the long tail on the right (or positive) side of the peak (mode).

In a negatively-skewed histogram or dot plot, there are more values at the right-hand end of the range as opposed to the left-hand end. This is shown in a box plot by the median being pushed towards the right-hand side of the box. The opposite is true for a positively skewed histogram or dot plot.

A dot plot provides more information than a histogram as every data element is retained.

Worked Example 7

WE 7

For each of the following sets of data:

Set A: 1 1 1 2 3 3 3 4 4 4 5 5 5 5 6
6 6 6 7 7 7 8 8 8 8 8 9 9 9

Set B: 1 1 1 2 3 3 3 4 4 4 5 5 5 5 6
6 6 6 7 7 7 8 8 8 8 8 9 9 19

(a) find the mean

(b) find the median

(c) find the range

(d) calculate the interquartile range

(e) use your findings to compare the two sets of data.

Working

(a) Set A mean $= \frac{158}{29}$
$= 5.48$

Set B mean $= \frac{168}{29}$
$= 5.79$

(b) Set A median $= 6$

Set B median $= 6$

(c) Set A range $= 9 - 1$
$= 8$

Set B range $= 19 - 1$
$= 18$

(d) Set A $Q_L = 3.5$

Set A $Q_U = 8$

Set A IQR $= 8 - 3.5$
$= 4.5$

Set B $Q_L = 3.5$

Set B $Q_U = 8$

Set B IQR $= 8 - 3.5$
$= 4.5$

(e) The two sets have the same median, but the mean of Set B is slightly higher than Set A. The two sets have the same interquartile range, but the range of Set B is greater than Set A. This is because Set B has the score of 19, which affects the mean and the range, but not the median (middle score) nor the interquartile range (spread of the middle half of the data).

Worked Example 8

WE 8

The heights (cm) of the students in two Year 8 classes were recorded.

8A:	148	149	150	150	153	154	154	154	155	155	156	156	156
	157	157	158	158	158	158	158	160	161	163	165	168	
8B:	150	151	152	152	152	153	153	154	155	155	155	156	156
	156	156	157	157	158	160	160	160	161	161	162	163	

(a) Find the median height for each class.

(b) Find the range for each class.

(c) Find the interquartile range for each class.

(d) Draw a parallel box plot for both classes.

(e) Compare the statistics for the two classes.

Working

(a) 8A median = 156

8B median = 156

(b) 8A range = 168 − 148
= 20

8B range = 163 − 150
= 13

(c) 8A Q_L = 154

8A Q_U = 158

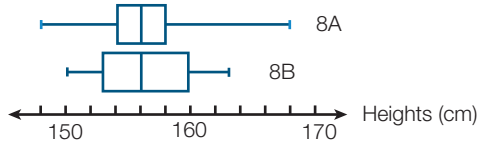
8A IQR = 158 − 154
= 4

8B Q_L = 153

8B Q_U = 160

8B IQR = 160 − 153
= 7

(d)



(e) The medians for the two classes are the same but the range for 8A is much larger than for 8B. However, the IQR for 8A is much smaller than the IQR for 8B.

Answers page 646

7.3 Comparing data sets

Stage 5.2◊

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13

Fluency and understanding

WE7

1 Calculate the following values for each pair of data sets below. Where necessary, round your answers to two decimal places.

(i) mean **(ii)** median **(iii)** range **(iv)** interquartile range

(a) Set A: 22 22 24 25 26 27 28 28 29 30
31 32 35 35 36 37 38 39 40

Set B: 12 22 24 25 26 27 28 28 29 30
31 32 35 35 36 37 38 39 40

(b) Set A: 44 44 46 47 47 48 48 48 51
52 52 52 54 55 58 58 58 60

Set B: 24 44 46 47 47 48 48 48 51
52 52 52 54 55 58 58 58 80

(c) Set A: 11 12 12 12 13 13 13 13 14 14 15
16 17 18 18 18 21 22 24 24 24 26

Set B: 11 12 12 12 13 13 13 13 13 13 13
16 17 18 18 18 21 22 24 24 24 26

(d) Make a few comments about what you have found by completing these calculations.

WE8

2 The heights (cm) of the students in two Year 8 classes were recorded.

8C: 145 146 147 147 149 150 150 151 152 153 154 155 155
156 157 158 158 159 160 162 163 163 164 166 169

8D: 147 148 149 150 150 150 152 153 154 154 155 155 155
156 156 156 157 157 158 160 160 160 161 161 162

(a) Find the median height for each class.

(b) Find the range for each class.

(c) Find the interquartile range for each class.

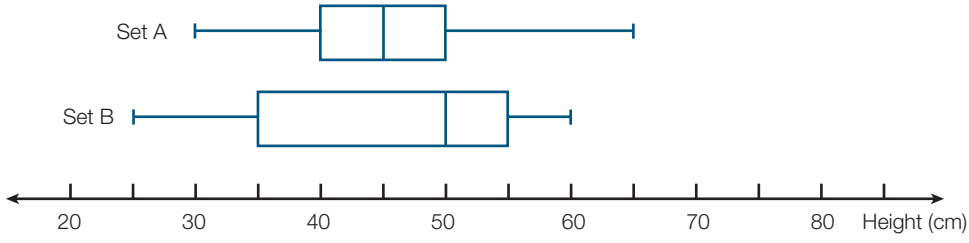
(d) Draw a parallel box plot for both classes.

(e) Compare the statistics for each class.

3 **(a)** For each data set in the following parallel box plot, find the:

(i) median **(ii)** range **(iii)** IQR.

(b) Compare the statistics for the two data sets.



4 The following data list shows the weight (kg) of each player on the 2011 playing list for the Brisbane Broncos in the NRL.

104	106	83	95	110	105	94	110	101	93	104	100
109	108	99	93	90	86	95	106	106	85	105	91
105	89	94	100	98	106	110	111	88	100	96	

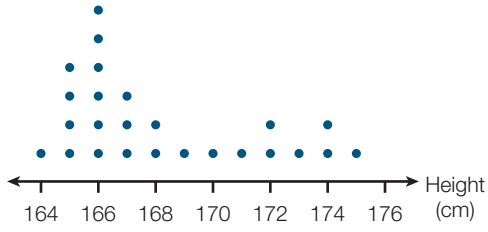
(a) Draw a histogram of the heights. Use a class interval of 5, with the first interval being 80–<85.

(b) Calculate the five-number summary for the data and use it to construct a box plot.

(c) Draw a dot plot of the data.

(d) Make some comments about what the three displays tell you about the data set.

5 **(a)** The students in a class had their heights (cm) recorded. The following dot plot shows the results.

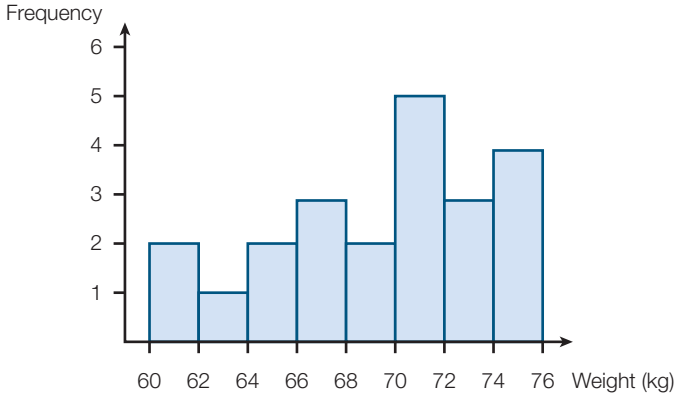


The interquartile range for this data set is closest to:

A 5 **B** 5.5

C 7.5 **D** 11

(b) The histogram below shows the weights, in kg, of the students in another class.



The median of this data set is likely to be closest to:

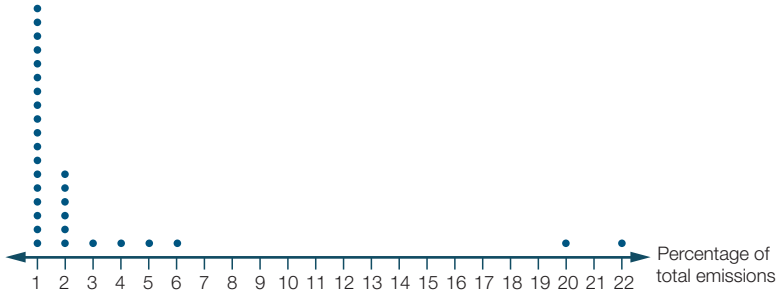
A 66 **B** 70 **C** 74 **D** 75

6 Three classes sat the same mathematics test, marked out of 100. The results for the three classes are summarised in the following table.

Class	Mean	Median	Range	IQR
A	64.32	68	72	27.5
B	64.12	63	31	19
C	60.92	63	55	38.5

Which class do you think performed best on the test?

7 The following dot plot shows the percentage of the world's total emissions of carbon dioxide for 30 countries, which contribute at least 1% of the total emissions when rounded to the nearest whole number.

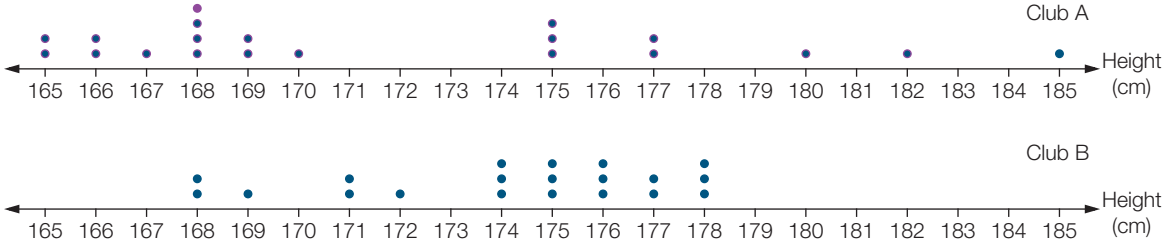


(a) Calculate the **(i)** mean and **(ii)** median for the data.

(b) Calculate the **(i)** range and **(ii)** interquartile range for the data.

(c) Which pair of values better describe the data set: mean and range; or median and interquartile range? Give a reason to support your answer.

8 The following parallel dot plot shows the heights of players at two netball clubs who play in the same competition. The heights have been rounded to the nearest centimetre.



(a) Which club appears to have the taller players?

(b) Calculate the mean and median of the given data for the two clubs.

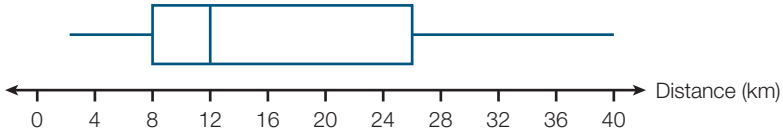
(c) Calculate the range and IQR for each of the two clubs.

(d) Draw a parallel box plot for the data.

(e) Compare the shape of the parallel dot plot with that of the parallel box plot.

(f) Do you still agree with your answer to part **(a)**? Use statistics to justify your answer.

9 A school collected information about the distance (km) that students travel to school. The results were published only in box plot form, as shown below.



(a) Write a sentence or two describing what information the box plot gives you about the travel distances for the students at this school.

(b) What would you estimate to be the mean distance travelled by the students? Describe how you made your estimation.

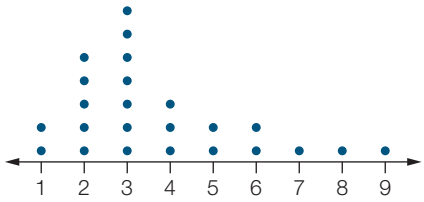
(c) **(i)** Draw a histogram that might represent the data collected. Use a bin size of 4 with a first class interval of 0–<4, and so on. Information was collected from 160 students.

(ii) Show that your histogram supports the values shown in the box plot.

(iii) Use your histogram to estimate the mean distance. How does this value compare to your estimate in **(b)**?

Reasoning, communicating and open-ended

10 **(a)** Look at the following dot plot.



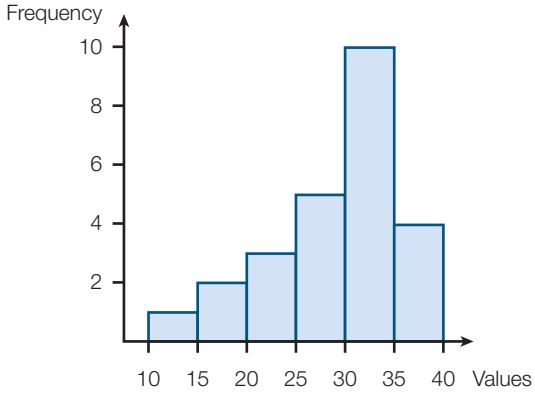
(i) Comment on the skewness of the dot plot.

(ii) Calculate the five-number summary for the data.

(iii) Draw a box plot of the data.

(iv) How does positive skew 'look' in a box plot?

(b) Look at the following histogram.



(i) Comment on the skewness of the dot plot.

(ii) Calculate the five-number summary for the data. Use the midpoint of each interval to represent all the values for that interval.

(iii) Draw a box plot of the data.

(iv) How does negative skew 'look' in a box plot?

11 The following table gives the height (cm) and weight (kg) of each of the players on the 2010 playing lists of the AFL teams Adelaide Crows and Essendon.

Adelaide

Height:	200	184	201	185	201	185	184	178	179	178
	194	192	182	187	195	188	180	198	185	192
	181	185	191	181	195	191	196	195	182	188
	176	191	196	184	193	183	192	178	180	194
	193	191	182	190	189	179	181			
Weight:	99	85	100	85	99	84	82	81	80	80
	93	97	75	82	90	86	84	102	84	87
	82	83	99	76	87	74	98	92	79	89
	79	82	98	86	89	75	88	82	77	92
	86	87	80	87	84	81	80			

Essendon

Height:	191	182	197	190	182	184	178	188	186	184
	182	188	187	182	186	191	185	193	201	185
	197	190	194	196	201	191	173	197	198	173
	180	184	189	182	192	187	190	180	201	187
	178	180	185	186	205	182				
Weight:	100	83	94	92	81	83	77	81	83	81
	77	88	74	80	86	94	74	89	108	85
	89	90	89	100	105	91	72	95	93	68
	86	86	87	85	81	90	86	84	105	84
	80	76	78	95	89	87				

(a) Find the mean heights and weights for each club.

(b) Find the median heights and weights for each club.

(c) Find the range and IQR for the heights and weights for each club.

(d) Draw a parallel box plot for the heights of players in each club.

(e) Draw a parallel dot plot for the weights of players in each club.

(f) Which team is taller overall? Use your statistics to justify your decision.

(g) Which team is heavier overall? Use your statistics to justify your decision.

12 The following table shows the percentage of the Australian population, and the percentage of the Aboriginal and Torres Strait Islander (ATSI) population, in particular age ranges.

	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44
Australia	6.7	6.3	6.5	6.9	7.6	7.6	6.9	7.2	6.9
ATSI	12.3	11.2	11.3	11.2	9.5	7.7	6.3	6.2	5.9

	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–85	85+
Australia	7.0	6.5	5.9	5.4	4.0	3.1	2.3	1.7	1.2
ATSI	5.1	4.2	3.3	2.3	1.5	1.0	0.6	0.3	0.2

Source: ABS Census of Population and Housing: Understanding the increase in Aboriginal and Torres Strait Islander counts between 2006 and 2011 (cat. no. 2077.0).

(a) Draw cumulative frequency graphs for the Australian population and for the ATSI population.

(b) Use the cumulative frequency curves to estimate the values for the five-number summary for each population. For the purposes of this work you should use 90 as the maximum value.

(c) Draw a parallel box plot showing the two populations.

(d) Write a couple of sentences comparing the two populations.

13 Collect two sets of data from students in your class or year level. Use the data to calculate some statistics. Compare the data sets using the statistics you have found.

Problem solving

What do you mean?

Task 1

It was reported that most Australians have more than the average number of legs.

How can this be true?

Task 2

Each week, for 4 weeks in a row, a market research company interviewed people asking questions about the performance of the Prime Minister. The results are as follows:

Week 1:	47 approved	42 disapproved	11 were unsure
Week 2:	68 approved	57 disapproved	7 were unsure
Week 3:	112 approved	106 disapproved	8 were unsure
Week 4:	77 approved	68 disapproved	7 were unsure

What percentage approval would you say the Prime Minister enjoys?
Give some explanation for why you think your answer is justified.

Strategy options

- Act it out.
- Make a table.

7.4 Scatter plots

Stage 5.2

Need to Know

Identifying relationships

Data collected on two variables is called **bivariate data**. It is often useful to see if a relationship exists between the two variables. A relationship may exist but one may not cause the other. They might both be related to another variable.

For example:

The variables 'head circumference' and 'ability to recall times tables' may be strongly related when data is collected on primary school students of varying age. However, a student's ability to recall times tables is not caused by their head circumference. Both variables are most likely related to age: as a student gets older their head gets bigger and they get better at their times tables.

Scatter plots

A **scatter plot** can be used to display data and see if any connection exists between the variables. A scatter plot is a series of points drawn on a pair of axes. If a scatter plot indicates at least a strong relationship (positive or negative) then you can make a rough prediction about other values based on those plotted.

For example:

If data is collected on the height and weight of a group of people, height can be plotted on the horizontal axis and weight on the vertical axis.

The scatter plot can show if there is a trend in the placement of the points.

It would be expected that a relationship existed between the two variables *height* and *weight* for the students who have been measured. However a lot of data would need to be collected before it could be concluded that taller students weigh more.

Describing relationships

If there is a relationship between bivariate data, a scatter plot will show a linear pattern.

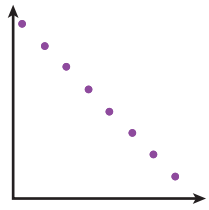
The following diagrams give some examples of possible relationships and the strength of those relationships.

They show whether the trend is linear or non-linear and, if linear, whether the strength of the relationship between the variables is strong, moderate or weak.

Perfect negative linear relationship

Description: As one variable increases so the other decreases. The points form a perfect linear relationship.

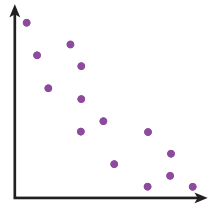
Example: Length and breadth of a rectangle for a fixed perimeter



Strong negative linear relationship

Description: As one variable increases so the other generally decreases. There may be a few points that don't fit a linear pattern exactly.

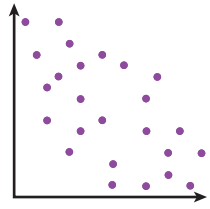
Example: Age of a second-hand car and its value (excluding vintage cars)



Weak negative linear relationship

Description: There is some trend evident of one variable increasing as the other decreases, although there would certainly be a number of points that don't fit a linear pattern.

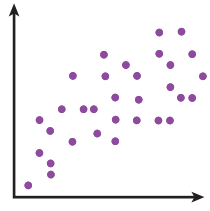
Example: Average number of hours spent keeping fit/week and the number of annual visits to a doctor



Weak positive linear relationship

Description: There is some trend evident of either both variables increasing or both variables decreasing, although there would certainly be a number of points that don't fit a linear pattern.

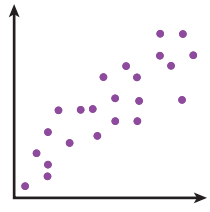
Example: Money spent on a holiday and money spent buying a car



Strong positive linear relationship

Description: As one variable increases so the other generally increases. There may be a few points that don't fit a linear pattern exactly.

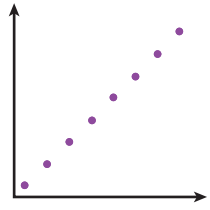
Example: Smoking and lung cancer



Perfect positive linear relationship

Description: As one variable increases so does the other. The points fit a perfect linear relationship.

Example: Volume and height of water in a tank



No linear relationship

Description: There is no linear pattern evident in the placement of the points.

Example: People who like cycling and people who like playing a musical instrument

Non-linear trends

An apparent trend will not always be linear. A non-linear relationship may be a more appropriate choice but this topic will not be covered at this level.

Worked Example 9

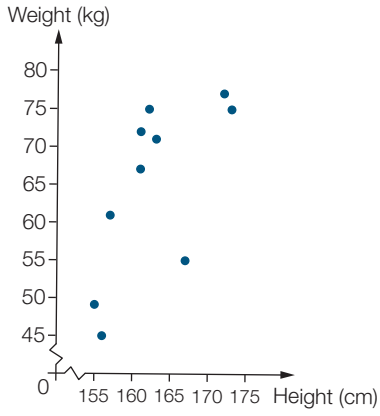
WE 9

The following table represents the heights (cm) and weights (kg) of 10 students.

Height (cm)	156	167	157	155	161	173	163	162	161	172
Weight (kg)	45	55	61	49	67	75	71	75	72	77

Draw a scatter plot of the data with height on the horizontal axis.

Working



Technology Exploration: Spreadsheet

Drawing a scatter plot

A scatter plot is one of the graph types that can be easily drawn using a spreadsheet program such as Microsoft Excel®.

When dealing with bivariate data remember that the data comes in pairs. If you are entering the data and miss a value, it must be entered in the correct position within the list so that it matches its pair in the other list.

The following data lists the height (cm) and weight (kg) of the playing members of an elite sporting club.

Height (cm)

184	184	196	185	184	183	175	195	182	182	194	194
188	183	182	203	202	185	183	190	177	198	195	189
190	185	183	188	183	191	192	193	189	192	190	199
184	192	179	187	187	185	186	196	197	195		

Weight (kg)

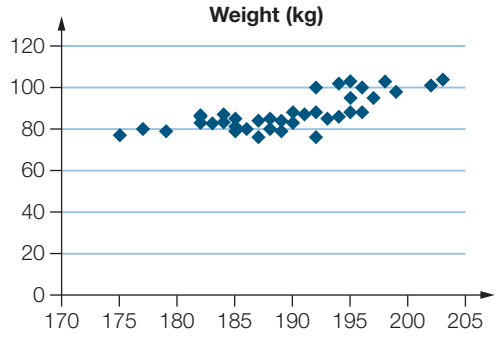
84	83	100	81	83	83	77	103	87	86	102	86
85	83	83	104	101	85	83	88	80	103	95	79
88	79	82	80	82	87	88	85	84	100	83	98
87	76	79	76	84	85	80	88	95	88		

(a) Open a new Excel sheet.

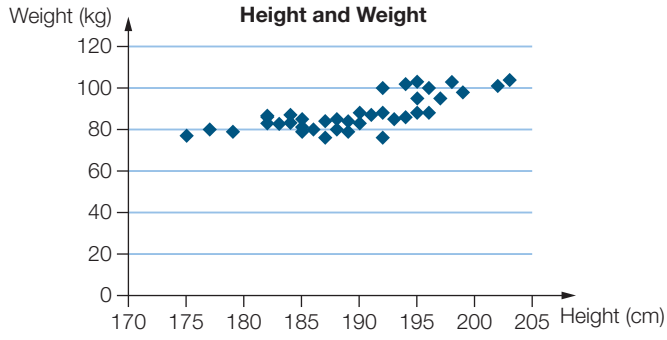
(b) In cell A1 put the heading 'Height (cm)' and in A2 'Weight (kg)'.

(c) Enter the values in the appropriate columns taking care to retain the order of the values.

(d) From the menu, select Insert > Scatter and choose the first option. A graph similar to the one shown should appear.



(e) You can then make some minor modifications to the graph to make it more acceptable as a mathematical graph.



From the graph we can see that there is a strong positive relationship existing between the two variables.

If we only have a few data pairs it is probably best to draw the scatter plot by hand. However, with 48 pairs of values as we have here, the use of technology is a more sensible option.

Answers page 649

7.4 Scatter plots

Stage 5.2

Navigator

Q1, Q2, Q3, Q4, Q6, Q7, Q9	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9

Fluency and understanding

WE9

1 The following table represents the heights (cm) and weights (kg) of 10 football players. Draw a scatter plot of the data. Put height on the horizontal axis. Make a comment about the type of relationship, if any, that is displayed by the data.

Height (cm)	203	186	184	189	186	197	188	181	182	185
Weight (kg)	103	84	84	86	89	97	87	82	80	86

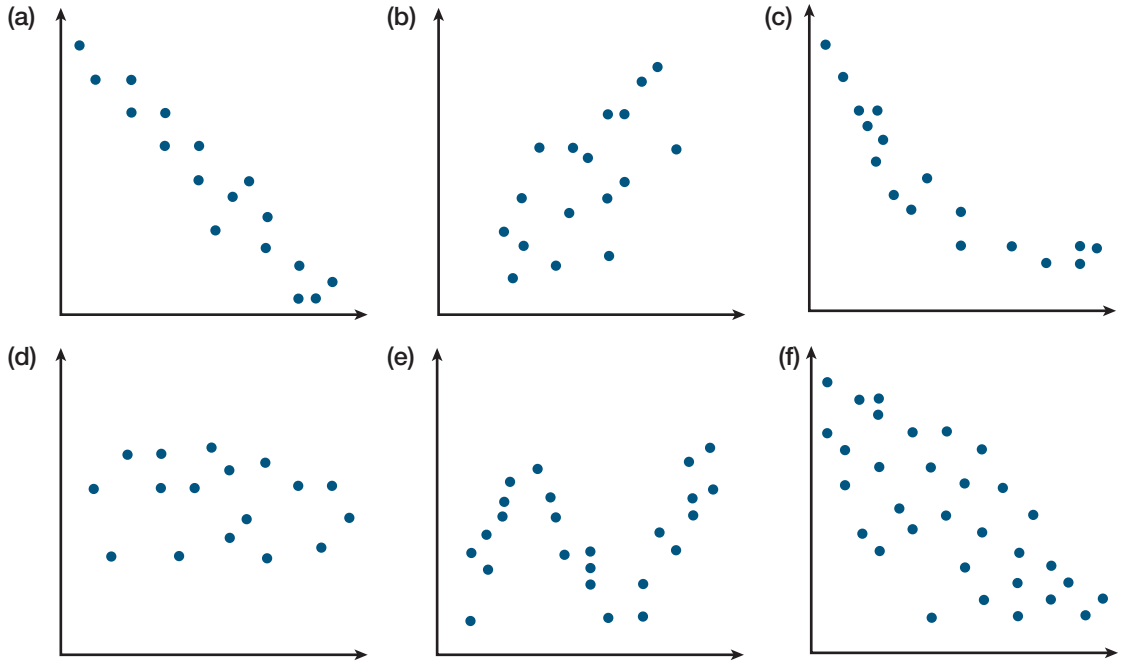
2 The data below show the marks obtained by 10 students taking physics and mathematics tests.

Student	A	B	C	D	E	F	G	H	I	J
Physics mark (out of 50)	42	38	44	41	46	31	25	30	33	36
Mathematics mark (out of 70)	66	60	65	59	65	50	38	58	60	60

(a) Draw a scatter plot of the data with the physics mark on the horizontal axis.

(b) Comment on any relationship evident between the marks.

3 For each of the following scatter plots state whether the trend is linear or non-linear. If linear, also state whether the trend is positive or negative and give an indication of strength.



4 The following table shows the breadth and length, both measured in centimetres, of 88 bivalves of the same species collected on a beach.

B **(cm)**	1.3	1.4	1.4	1.5	1.9	2.1	2.1	2.1	2.2	2.2	2.3	2.4	2.4	2.5
L **(cm)**	1.7	1.9	1.8	1.9	2.3	3.5	2.8	2.7	3.0	2.7	3.0	3.2	2.9	3.5

B **(cm)**	2.5	2.6	2.6	2.6	2.7	2.8	2.8	2.9	2.9	2.9	2.9	3.0	3.0	3.0
L **(cm)**	3.3	3.5	3.4	3.4	3.2	3.8	3.5	3.8	3.9	3.8	3.7	3.8	4.0	3.7

B **(cm)**	3.1	3.4	3.4	3.5	3.6	3.6	3.6	3.6	3.7	3.9	3.9	4.1	4.1	4.1
L **(cm)**	4.1	4.4	4.3	4.5	4.6	4.8	4.9	4.5	4.9	5.4	5.2	5.3	5.4	5.2

B **(cm)**	4.2	4.2	4.2	4.2	4.3	4.7	4.8	4.8	4.8	4.9	5.0	5.0	5.1	5.2
L **(cm)**	5.4	5.5	5.6	5.2	5.6	6.1	6.6	6.1	6.0	6.1	6.6	6.2	6.6	6.4

B **(cm)**	5.2	5.4	5.6	5.6	5.8	5.8	5.8	5.9	6.0	6.0	6.0	6.2	6.2	6.3
L **(cm)**	6.8	6.3	7.0	7.9	7.4	7.4	6.9	7.9	7.7	7.8	7.3	8.5	7.5	7.6

B **(cm)**	6.5	6.6	6.6	6.7	6.7	6.7	6.7	6.9	6.9	7.0	7.1	7.1	7.1	7.2
L **(cm)**	9.3	8.1	8.0	8.2	9.2	8.0	7.9	8.6	8.1	9.5	9.1	9.1	9.4	8.4

B **(cm)**	7.3	7.3	7.4	7.5
L **(cm)**	9.3	8.9	8.9	8.6

(a) Use a spreadsheet or any other form of technology to draw a scatter plot.

(b) Comment on any relationship evident between the measurements.

5 CensusAtSchool is a program run by the Australian Bureau of Statistics. The following data is a random sample of 30 values related to Year 10 students in NSW schools in 2012. The two variables listed are height (cm) and right foot length (cm).

Height (cm)

162	176	158	164	164	173	188	171	165	147	158	155
140	162	160	165	194	160	154	170	165	182	164	172
170	167	161	175	160	172						

Right foot length (cm)

23	30	24	23	22	20	28	26	26	22	24	24
20	22	23	23	30	12	21	23	17	26	15	25
27	27	25	26	22	25						

(a) What sort of relationship, if any, would you expect to exist between these two variables?

(b) Plot the data, putting height on the horizontal axis.

(c) Does the scatter plot support the opinion you expressed in **(a)**.

(d) Did any of the data values concern you? If so, which ones were they, and what did you do about them?

6 The data below represent the arm span (cm) and height (cm) of ten Year 10 students.

Arm span (cm)	162	153	140	166	144	168	191	180	163	143
Height (cm)	162	176	158	164	164	173	188	171	165	147

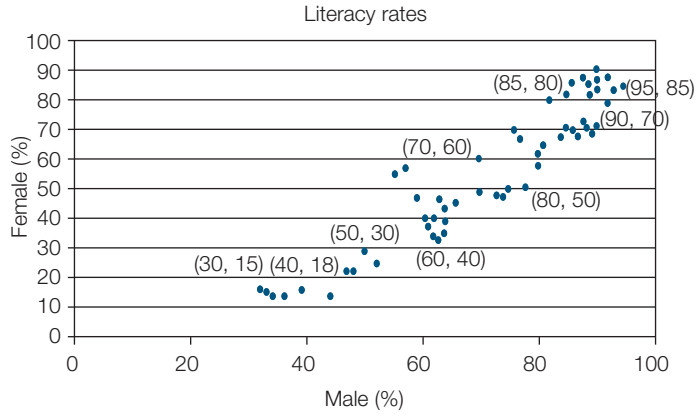
(a) Draw a scatter plot with arm span on the horizontal axis. Use the plot to describe any relationship between the two variables.

(b) Draw a scatter plot with height on the horizontal axis. Use the plot to describe any relationship between the two variables.

(c) Did it make any difference where the variables were plotted on the scatter plot?

Reasoning, communicating and open-ended

7 A search of the internet was conducted on literacy rates for countries with literacy rates regarded as low or medium. The following graph was produced as part of the report.

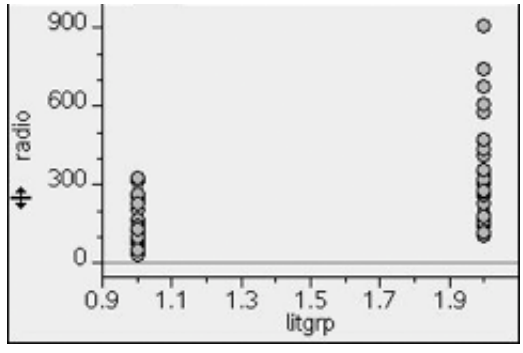


(a) Is the data primary or secondary?

(b) Describe any relationship you can see between male and female literacy rates. As part of this answer state whether you think, in general, male or female literacy rates are higher.

(c) Could your conclusions in **(b)** be applied to countries regarded as having high literacy rates?

8 The internet search of literacy rates also produced data regarding the number of radios per 1000 people in the population. From countries with a literacy rate listed as low (litgrp of 1) or medium (litgrp of 2) the following graph was produced.



(a) What information can you gather from this graph?

(b) The student doing the investigation had meant to construct a box plot. Given the following statistics, construct a parallel box plot for the data.

	Min	Q_L	Median	Q_U	Maximum
Low literacy	34	79	130	226	329
Medium literacy	107	178	306	434	907

(c) Use your parallel box plot to help you make a few statements about the two data sets.

(d) There is one data pair that looks like an outlier. Which pair is this?

9 Identify two variables, for which you can collect data, that you think will show either a strong positive or a strong negative relationship. Collect at least 10 pairs of data values and plot them on a scatter plot. Write a brief comment as to whether or not the collected data supported your view about the relationship between them.

Half-time 7

Stage 5.2◊ Q1–3 **Stage 5.2 Q4**

1 Complete the following tasks for the data shown in the table.

(a) Find the range.

(b) Find the interquartile range.

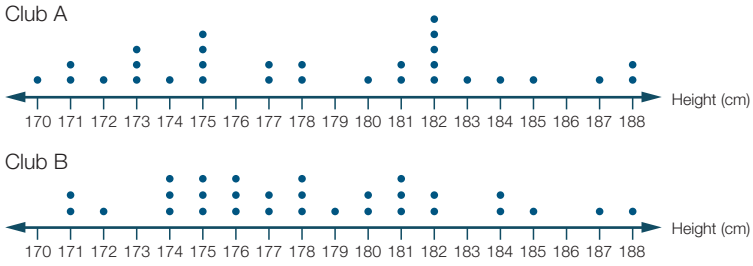
(c) Draw a box plot.

Class interval	Frequency
10–<15	9
15–<20	15
20–<25	12
25–<30	17
30–<35	13
35–<40	11

Ex.7.2

2 The following parallel dot plot shows the heights of players at two hockey clubs who play in the same competition. The heights have been rounded to the nearest centimetre.

Ex.7.3



(a) Which club appears to have the taller players?

(b) Calculate the mean, median, range and IQR for each club.

(c) Draw a parallel box plot for the data.

(d) Reassess your answer for part **(a)**. Justify your assessment using the statistics you have calculated.

(e) Which statistic is the most helpful in making your assessment?

3 Construct a cumulative frequency curve and use it to find an estimate for the median of the following data set.

Class interval	Frequency
0–<20	12
20–<40	19
40–<60	21
60–<80	32
80–<100	19

Ex.7.1

4 The following table gives the head circumferences, in cm, and the lengths, in cm, of 12 babies at birth. Draw a scatter plot of the data. Put head circumference on the horizontal axis. Make a comment about the type of relationship, if any, that is displayed by the data.

Ex.7.4

Head circumference (cm)	38	35	39	37	34	32
Length (cm)	51	50	54	53.5	50.5	47.5

Head circumference (cm)	33	35	35	37	39.5	36
Length (cm)	50	53	49	51	51	50.5

7.5 Time-related data

Stage 5.2

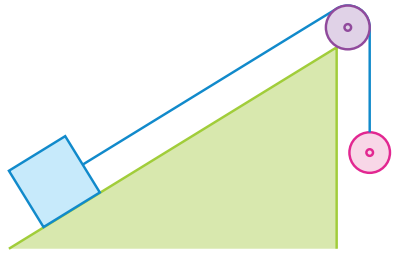
Need to Know

Independent and dependent variables

In bivariate data, one of the variables often causes the other variable to change. The variable that causes the other to change is called the **independent variable**, and the one that changes is called the **dependent variable**.

For example:

If we were conducting an experiment to see the effect that changing the mass on the end of the pulley had on the distance a block moved up an inclined plane, the mass would be the independent variable and the distance moved up the plane would be the dependent variable. It is the mass that causes the block to move up the plane, it is not the distance moved up the plane that causes the mass to change.



Plotting time-related variables

In statistics, if time is one of the variables it is almost always the independent variable. Scatter plots are used as a way of examining time-related data. When we have large data sets, it may be appropriate to use technology such as a spreadsheet program like Microsoft Excel® to draw the scatter plot.

Worked Example 10

WE 10

The following table shows the temperatures, in °C, for a city over a 24-hour period from 8 a.m. one morning to 7 a.m. the next morning.

Time	8 a.m.	9 a.m.	10 a.m.	11 a.m.	noon	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.
Temperature (°C)	18.4	21	21.5	22.3	23.3	24.7	24.9	25.3	24.2	23.2	22	20.4

Time	8 p.m.	9 p.m.	10 p.m.	11 p.m.	mid-night	1 a.m.	2 a.m.	3 a.m.	4 a.m.	5 a.m.	6 a.m.	7 a.m.
Temperature (°C)	18.9	18.2	18.2	17.9	17.1	16.5	16.36	15.9	15.6	15.6	15.4	16.5

(a) Draw a scatter plot to show these temperatures.

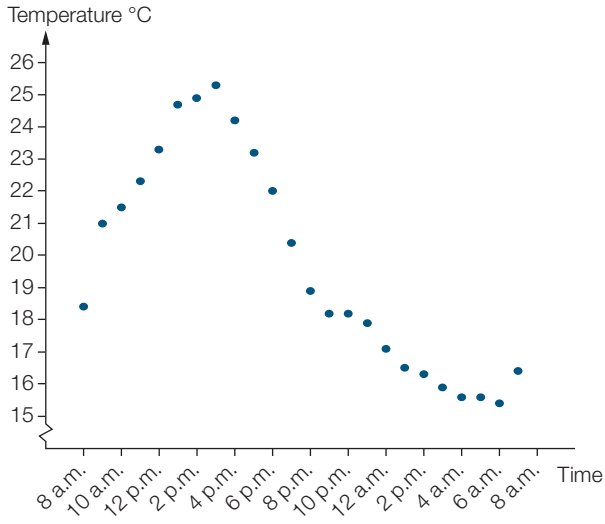
(b) What is the maximum temperature shown on the graph?

(c) Is this necessarily the maximum daily temperature? If not, what might that maximum temperature have been?

(d) Do the temperatures follow a similar pattern to what you would expect for where you live?

Working

(a)



(b) The maximum temperature recorded is 25.3°C.

(c) If this is not the absolute maximum it is likely to have been in the preceding hour, as the temperature falls away more quickly after 3 p.m. than it was rising before.

(d) It doesn't really matter where you live, it is hotter during the day than at night.

7.5 Time-related data

Answers page 651

Navigator

Stage 5.2

Q1, Q2, Q3, Q5, Q9, Q10	Q1, Q2, Q3, Q4, Q6, Q7, Q9	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9

Equipment required: a spreadsheet program would be useful for Questions **3–5**

Fluency and understanding

WE 10

1 The table below gives the wind speeds, in km/h, for Quebec in Canada over a 24-hour period.

Time	mid-night	1 a.m.	2 a.m.	3 a.m.	4 a.m.	5 a.m.	6 a.m.	7 a.m.	8 a.m.	9 a.m.	10 a.m.	11 a.m.
Wind speed (km/h)	9	9	13	7	6	6	4	4	6	4	7	6

Time	noon	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.	8 p.m.	9 p.m.	10 p.m.	11 p.m.
Wind speed (km/h)	6	7	9	7	11	13	11	13	11	9	9	11

(a) Draw a scatter plot to show the wind speed.

(b) What seems unusual about the 2 a.m. recording?

(c) Describe what is happening to the wind speed between 4 p.m. and 8 p.m.

2 For the following pairs of variables, state which one you think is the independent variable. Give a brief reason for your answer.

(a) Test score and number of hours spent studying.

(b) Height, measured in cm, and age, measured in months, for children less than 3 years of age.

(c) Age at death and average number of cigarettes smoked per day.

(d) Amount of rubbish collected, measured in kg, and the number of people in a household.

3 The following data set indicates the number of bicycles that crossed northbound on the Sydney Harbour bridge for 29 consecutive Fridays.

332 372 367 370 361 398 387 363 447 331 530 454 460 428 283
515 444 498 481 500 396 440 402 383 339 325 402 517 416

Draw a scatter plot to represent the data. Use technology or draw the graph by hand. Comment on the graph produced.

4 The following table gives the number of 10-year-old Australians in June of each year from 1971 to 2009.

1971	260 439	1981	282 850	1991	252 907	2001	275 465
1972	261 046	1982	280 163	1992	254 099	2002	275 707
1973	258 136	1983	266 496	1993	258 857	2003	275 806
1974	256 359	1984	257 190	1994	258 625	2004	276 100
1975	244 424	1985	248 865	1995	259 391	2005	278 212
1976	242 216	1986	242 895	1996	261 345	2006	273 908
1977	244 927	1987	239 390	1997	259 506	2007	274 506
1978	248 919	1988	241 462	1998	261 417	2008	274 009
1979	261 286	1989	242 343	1999	264 986	2009	276 519
1980	262 714	1990	243 448	2000	271 193		

(a) Use a suitable form of technology to create a scatter plot displaying the data.

(b) Comment on the trend seen in the number of 10-year-old Australians over this time period.

5 The following table gives the number of road deaths for the years 2001–2012 for New South Wales, Victoria and Queensland. The data was taken from the Australian Road Deaths Database provided by the Department of Infrastructure and Transport.

State	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
NSW	524	561	539	510	508	496	435	374	453	405	364	374
Vic	444	397	330	343	346	337	332	303	290	288	287	282
Qld	324	322	310	311	330	335	360	328	331	249	269	280

Source: www.bitre.gov.au/statistics/safety/fatal_road_crash_database.aspx, accessed 16 April 2013

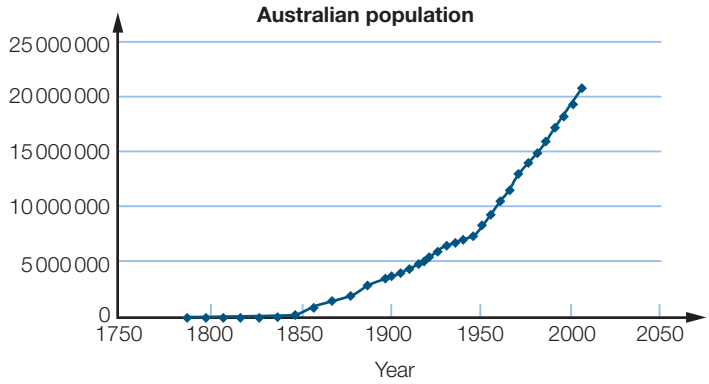
(a) Show the data for all three states on the same scatter plot. If possible, use some form of technology to assist you in the construction of this scatter plot.

(b) Describe any trends you see in the data.

(c) What other information would you like to have to make your analysis more accurate?

Reasoning, communicating and open-ended

6 The graph on the right shows the estimated Australian population from 1788 until 2006.



(a) Describe the trend you see in the graph.

The figures prior to 1961 do not include the Indigenous population. Estimates of the size of the Indigenous population prior to European settlement range from 300 000 to 1 000 000.

(b) How do you think the inclusion of the Indigenous population prior to 1961 would affect the graph?

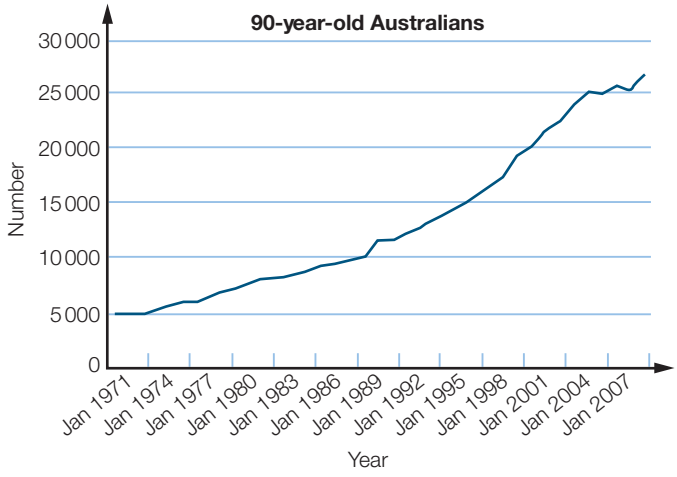
(c) Why does the graph seem to go along the horizontal axis for the first 50 years or so?

(d) How do you think the early figures were collected?

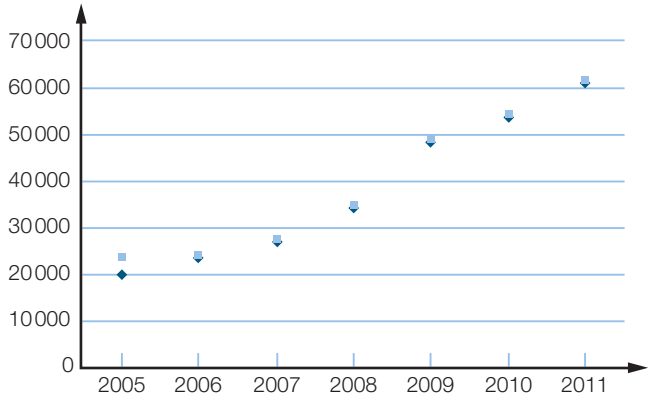
(e) Which figures do you think are more reliable, the early ones or the later ones?

(f) How do we now estimate the Australian population?

7 The graph on the right shows the number of 90-year-old Australians from 1971 through to 2007. Comment on the trend you see in the graph and make some attempt to explain why this is the case. For what other ages would you expect to see this type of trend?



8 Sonika produced the graph below for the information given in the table. Suggest how she could improve her graph.



Mercedes-Benz E320 used car prices (August 2014)

Year	Elegance	Avantgarde
2005	19 700	23 600
2006	23 400	24 100
2007	26 700	27 600
2008	34 000	35 000
2009	48 200	49 000
2010	53 500	54 400
2011	60 800	61 800

9 The table below gives some information about extinct and endangered species in Australia. These figures relate to the period after European settlement of Australia.

	Extinct	Critically endangered	Endangered	Vulnerable	Total
Total plant species and subspecies	49	81	523	665	1318
Mammals	27	4	33	55	119
Birds	23	6	41	61	131
Reptiles	0	2	14	37	53
Frogs	4	2	14	12	32
Fish	1	3	16	25	45
Invertebrates	0	16	12	6	34
Total animal species and subspecies	55	33	130	196	414

Source: www.climatechange.gov.au/publications/biodiversity/~/media/publications/biodiversity/biodiversity-vulnerability-assessment-lowres.ashx

Of the 27 mammal species that are extinct, 10 of these have occurred since 1900. Some other information about Australia's biodiversity is as follows:

- 82% of our mammals, 45% of our land birds, 89% of our reptiles and 93% of our frogs occur nowhere else
- estimates of probable global losses of species are as high as 25% over the next 30 years
- 90% of native vegetation in the eastern temperate zone has been removed or replaced by introduced crops
- 50% of rainforests have been cleared
- the proportion of Australia covered by woodland or forest has been reduced by more than one-third.

Your task is to produce some graphs to indicate how you think Australia's biodiversity has changed since European settlement. You may wish to research some more information.

10 Write a story about Scott's car journey in his 4WD that lasted at least 8 hours and during which time he travelled at least 550 km. Draw a line graph to represent the story you write.

Survey data: reliability and validity

7.6

Stage 5.2◊

Need to Know

The **reliability of data** refers to the ability to get the same general results if the survey is conducted again under the same conditions.

Validity refers to the conclusions we draw from the data collected. Just because our survey results indicate that the further you travel to school the more your parents earn, does not mean that either variable causes the other to occur.

Results can be reliable but not valid, but valid results must be reliable.

Bias in a survey can arise from the sampling method chosen, the types of question asked or who funded the research.

Worked Example 11

WE 11

Essential Media Communications is a company that uses free media, paid media and social media platforms as part of their promotional strategies. They also use sample surveys to establish the opinions of the Australian population with regard to the issues they are promoting. The company has produced the following table to indicate the accuracy of survey findings based on sample size and the strength of response to the particular question.

Sample size	Variation where the amount is near the percentage of				
	10% or 90%	20% or 80%	30% or 70%	40% or 60%	50%
100	6%	8%	9%	9%	10%
200	4%	5%	6%	7%	7%
300	3%	5%	5%	6%	6%
400	3%	4%	4%	5%	5%
500	3%	4%	4%	4%	4%
600	2%	3%	4%	4%	4%
700	2%	3%	3%	4%	4%
800	2%	3%	3%	3%	3%
900	2%	3%	3%	3%	3%
1000	2%	3%	3%	3%	3%

Assuming the values in the table are accurate, between which two values would you expect the population proportion to lie if:

(a) a sample of 300 was taken and the result was **(i)** 22% **(ii)** 49%

(b) a sample of 900 was taken and the result was **(i)** 32% **(ii)** 93%?

Working

(a) **(i)** Find the accuracy level from the table. For a sample size of 300 and a result near 20%, the accuracy level is 5%. Add and subtract this percentage from the result to find the range. Expect the population result to be between 17% and 27%.

(ii) Find the accuracy level from the table. For a sample size of 300 and a result near 50%, the accuracy level is 6%. Add and subtract this percentage from the result to find the range. Expect the population result to be between 43% and 55%.

(b) **(i)** Find the accuracy level from the table. For a sample size of 900 and a result near 30%, the accuracy level is 3%. Add and subtract this percentage from the result to find the range. Expect the population result to be between 29% and 35%.

(ii) Find the accuracy level from the table. For a sample size of 900 and a result near 90%, the accuracy level is 2%. Add and subtract this percentage from the result to find the range. Expect the population result to be between 91% and 95%.

Answers page 652

7.6 Survey data: reliability and validity

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4	Q1, Q2, Q3, Q4, Q5	Q1, Q2, Q3, Q4, Q5, Q6

Fluency and understanding

WE11

1 Essential Media Communications is a company that uses free media, paid media and social media platforms as part of their promotional strategies. They also use sample surveys to establish the opinions of the Australian population with regard to the issues they are promoting. The company has produced the following table to indicate the accuracy of survey findings based on sample size and the strength of response to the particular question.

Sample size	Variation where the amount is near the percentage of				
	10% or 90%	20% or 80%	30% or 70%	40% or 60%	50%
100	6%	8%	9%	9%	10%
200	4%	5%	6%	7%	7%
300	3%	5%	5%	6%	6%
400	3%	4%	4%	5%	5%
500	3%	4%	4%	4%	4%
600	2%	3%	4%	4%	4%
700	2%	3%	3%	4%	4%
800	2%	3%	3%	3%	3%
900	2%	3%	3%	3%	3%
1000	2%	3%	3%	3%	3%

Assuming the values in the table are accurate, between which two values would you expect the population proportion to lie if:

(a) a sample of 500 was taken and the result was **(i)** 12% **(ii)** 52%

(b) a sample of 1000 was taken and the result was **(i)** 42% **(ii)** 88%.

2 Some newspapers conduct a daily poll on a question of current interest. To vote you need to SMS a particular number for Yes or a different number for No.

(a) Do you think this type of survey is reliable?

(b) One such question was 'Do you like the new Qantas uniform as modelled by Miranda Kerr?' Do you think the question is clear? What doubts, if any, do you have about its wording?

(c) The next day it was reported that 57% of results were Yes. This represented 42 of the 75 votes received. What do you think about the number of votes received and the validity of the conclusion?

(d) Another day the question was 'Do you think Australia is a racist country?' It attracted 498 votes, 137 Yes and 361 No. Who would be interested in the results of this poll?

(e) Why do you think newspapers run polls such as these?

3 There are a number of companies using the internet to run their surveys. These companies have a large database of people who are invited to take part in their surveys. Payment of some sort is often involved. This may be in the form of entries into a monthly draw for, say, $5000 or credit towards gift cards. This credit may be $2–$5 depending on the length of the survey.

(a) Do you think this survey method would exclude particular groups of people?

(b) Do you think paying people to take part in surveys is a good thing or a bad thing? Write a sentence or two explaining your thoughts.

(c) Often there are screening questions before the survey proper commences. Here information such as location, gender, age, household income and employment status is obtained. Why do you think this sort of information is used as a screen?

(d) Another survey method is the telephone survey in which a company rings a list of numbers and tries to get people to respond to their questions. Is this a better method than the internet-based type? Write a sentence or two explaining your thoughts.

Reasoning, communicating and open-ended

4 The following is the headline and part of a story about a survey related to body image conducted with preschool children.

Discrimination starts young as fat phobia begins

KIDS as young as three are fat-phobic, a new study has found, and it means larger children are being shunned.

A La Trobe University study has found three-to-five-year-olds don't want to invite their larger-sized peers to birthday parties.

And they think fatter children won't have many friends and are meaner, sadder and lazier than skinnier kids.

The study, of 118 Victorian preschoolers, is one of the first to reveal anti-fat attitudes in children this young.

Across all genders and ages, the children studied chose larger figures as having negative rather than positive characteristics.

Source: *Herald Sun*, 17 November 2012, by Susie O'Brien

The article does not give any specific percentages for any of the statements made.

(a) What do you think of the sample size? Look back at the table in Question **1** to help you frame your answer.

(b) If 53% of the respondents had recorded anti-fat attitudes, within what range would you expect the population percentage to fall?

(c) Do you think the article gives us cause for concern? What do you think should be done to follow up on these findings?

5 Here is an article related to a survey of flight attendants.

What hosties want: what flight attendants hate about you

Don't snap your fingers at me! ... a survey of 700 cabin crew from around the world has revealed flight attendants' biggest gripes about passengers.

Whatever you do when you fly, don't snap your fingers at the airline stewards.

Passengers who click their fingers to get the staff's attention is the biggest gripe for a quarter of cabin crew (26 per cent), according to a new survey.

The Skyscanner survey of more than 700 international cabin crew members in 85 countries found the second most annoying passenger habit is the scramble to leave the plane before the seat belt signs are switched off (13 per cent).

Coming in third are passengers trying to cram too much hand baggage into the overhead lockers (11 per cent).

Other irritations include talking through the safety demonstration (nine per cent); asking for more blankets and pillows (eight per cent); and stuffing rubbish in the seat pocket (seven per cent).

The results also showed that, according to cabin crew, the perfect passenger is male, travels alone for pleasure rather than business and is aged in his 30s.

Economy passengers are also favoured over those who travel in premium class.

Skyscanner's Sam Poullain said the results were unexpected.

'Contrary to stereotypes, it's interesting to see that it's the men who get the vote for the perfect passenger,' he said in a media release, 'while cabin crew prefer to work in economy rather than premium class.'

Flight attendants' biggest gripes about passengers:

1. Clicking fingers to get your attention 26%
2. Leaving seat at the end before the light goes off 13%
3. Stuffing too many bags into overhead locker 11%
4. Complaining there's no space for their bag in the locker 10%
5. Talking through the safety demo 9%
6. Asking for more blankets/pillows 8%
7. Stuffing rubbish in the seat pocket 7%
8. Asking for a different meal 6%
9. Ringing the attendant bell to complain about temperature 6%
10. Asking for a specific brand of drink 4%.

Source: AAP, 15 November 2012, http://www.smh.com.au/travel/what-hosties-want-what-flight-attendants-hate-about-you-20121115-29e1s.html

(a) How many flight attendants were surveyed?

(b) Why is the word 'hosties' used in the headline whereas elsewhere the term used is 'flight attendant'?

(c) What other information would you like to have access to before acting on the results reported in this article?

(d) Why do you think Skyscanner's Sam Poullain thought the results were unexpected?

(e) What do the reported percentages add up to? What does this suggest about the structure of the survey instrument?

6 The following media release was issued by the Australian Council of Trade Unions (ACTU).

Employers using 'cash-in-hand' work to rip off workers and taxpayers, new research finds

One in four young workers are not receiving their legal entitlements to penalty rates, sick leave, superannuation and other conditions because employers are using a black market cash economy to avoid their obligations, according to new research.

A survey of more than 1000 workers commissioned by the Victorian Trades Hall Council and the ACTU reveals the shocking abuse of 'cash-in-hand' in the modern Australian economy.

The survey is to be launched today at the Young Workers Conference in Melbourne.

ACTU President Ged Kearney said employers paid workers cash-in-hand to avoid their legal obligations and to dodge taxes.

'The illegal use of cash-in-hand payments to workers is one of the dirty secrets of the Australian economy, and is being used by employers who don't mind ripping off the taxpayer and denying entitlements to their workers,' Ms Kearney said.

'It is appalling that at the same time we have employers running a major campaign to abolish penalty rates, we find that between 50 to 60% of all young workers in black market jobs are denied their basic legal entitlements including penalty rates, superannuation, annual leave, sick leave and basic notice before they get the sack.'

Ms Kearney said cash-in-hand work was another example of the rise of insecure work in Australia.

'We are seeing a rise in casual, contract and other forms of work that mean workers do not have sick leave and annual leave, and which shift risk from employers to workers.'

'Workers are increasingly doing jobs that have irregular and inconvenient hours, have limited entitlements and have no job security. Cash-in-hand work is just part of this trend, which makes it tougher for workers.'

The national survey, conducted by Essential Research between 12 and 23 September, found that 13% of workers had worked 'cash-in-hand' in the last three years, and 5% were working cash-in-hand in one of their current jobs.

A clear majority of cash-in-hand workers were not receiving penalty rates, annual leave or sick leave, and a worrying 48% not being paid superannuation.

Younger workers were more likely to work cash-in-hand than older workers, with 24% of people aged 18-29 saying they had worked cash-in-hand.

'There is a real concern that young workers are being told they must work cash-in-hand by employers if they want to get the job,' Ms Kearney said.

'The majority of businesses do the right thing and pay the taxes they are required to. They should be outraged by this behaviour, because it means they are forced to compete with companies that have an unfair advantage. Cash-in-hand work, like sham contracting, hurts workers and businesses that do the right thing and pay their tax and entitlements.'

Source: Media Release, ACTU, 27 September 2012

(a) What is the general thrust of the information provided in this media release?

(b) How would you describe the language used in the headline?

The following information comes from the methodology report released by the polling company.

Of the 2014 that were polled, 1097 qualified for the VTHC questions by being either employed full time (more than 35 hours per week) or employed part time (less than 35 hours per week). Respondents that were self-employed, unemployed, retired, students, home duties or on welfare payments were filtered out of the sample.

All results in this report are therefore based on a filtered sample of 1097 respondents. For two of the questions, the results are based on a sub-sample of 142 respondents. Caution should be applied to the statistical reliability of these results, noting a higher margin of error of around 7%.

In the results section the following comment can be found:

Those respondents that worked, or had worked, in a cash-in-hand job ($n = 142$) were asked the length of time they were employed in that job.

(c) On how many responders does the media release indicate the figures are based?

(d) Based on the last comment, above, how many people are indicated as being in the sub-category of being paid cash in hand?

(e) Why do you think there is some confusion as to the numbers involved in the media release?

7.7 Standard deviation

Stage 5.3

Need to Know

Traditionally, the most commonly used measure of spread was the **standard deviation**. Compared to the IQR, it has the advantage of using all the data but has the disadvantage of being time consuming to calculate without the use of technology. To find the standard deviation, the **variance** needs to be found first.

Variance

Consider the following data set:

5, 6, 7, 10, 10, 12, 14, 16

The mean of the data is 10. How far each of these values is away from the mean is called the score's deviation from the mean. The deviation values are:

-5, -4, -3, 0, 0, 2, 4, 6

Adding these together gives 0. This will always be the case for any set of data values. To overcome this problem, the deviations are squared before they are added together, making all the values positive.

The squared deviation values are:

25, 16, 9, 0, 0, 4, 16, 36 which sum to 106.

This sum is then divided by the number of scores to find the average deviation from the mean. This is called the variance.

$$\text{Variance} = \frac{\text{sum of (deviations from the mean)}^2}{\text{number of values}}$$

$$\text{In this case, Variance} = \frac{106}{8}$$
$$= 13.25$$

Standard deviation

The square root of the variance gives the standard deviation. The symbol σ is commonly used to represent the standard deviation.

For example:

For a variance of 13.25, the standard deviation is approximately 3.64.

Most scientific calculators can calculate the standard deviation. This calculation can also be completed using a spreadsheet program such as Microsoft Excel®. When using technology, you will be given the choice of finding σ_n, the standard deviation of a population or σ_{n-1}, the standard deviation of a sample. At this level, the σ_n value is always used.

$$\text{Standard deviation} = \sqrt{\text{variance}}$$

Interpreting the standard deviation

For two data sets with the same mean, the smaller the value of the standard deviation the more closely packed is the data set. The data is more consistently located with respect to the mean value.

Worked Example 12

WE 12

Find the standard deviation, rounded to two decimal places, for the following data:

2, 3, 3, 3, 5, 7, 8, 11, 12

Working

$$\text{Mean} = \frac{2+3+3+3+5+7+8+11+12}{9}$$

$$= 6$$

Deviations: -4, -3, -3, -3, -1, 1, 2, 5, 6

Square of deviations: 16, 9, 9, 9, 1, 1, 4, 25, 36

$$\text{Variance} = \frac{(16+9+9+9+1+1+4+25+36)}{9}$$

$$= \frac{110}{9}$$

$$\text{Standard deviation} = \sqrt{\frac{110}{9}}$$

$$= 3.50 \text{ (to 2 d.p.)}$$

Worked Example 13

WE 13

The table below shows the test results for 15 students in English and mathematics. Both tests were scored out of 50. Using a scientific calculator or a Microsoft Excel® spreadsheet, find the mean and standard deviation for each data set and use these values to compare them. Round the values to two decimal places, where necessary.

English	42	44	31	20	25	34	45	50	44	12	13	23	44	47	42
Maths	38	34	42	39	40	44	43	27	34	27	40	32	31	22	23

Working

English: $\text{mean} = \frac{516}{15} = 34.4$

$$\text{variance} = \frac{2303.6}{15}$$

$$\text{std dev} = \sqrt{\frac{2303.6}{15}}$$

$$= 12.39$$

Maths: $\text{mean} = \frac{516}{15} = 34.4$

$$\text{variance} = \frac{731.6}{15}$$

$$\text{std dev} = \sqrt{\frac{731.6}{15}} = 6.98$$

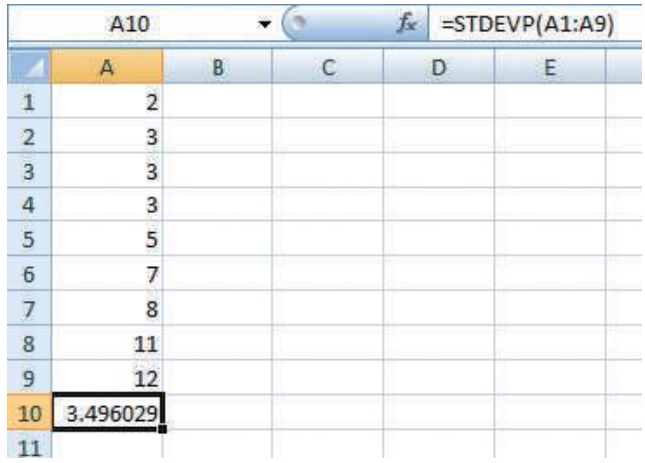
The two tests had the same mean score but the standard deviation was much less for the mathematics test. This means the scores for the mathematics test were more consistent. However, we can see that in the English test both the best and worst results, overall, were recorded.

Technology Exploration: Spreadsheet

Standard deviation

Given the following data, use a spreadsheet program such as Microsoft Excel® to calculate standard deviation.

Enter the data on a new sheet and then use the function STDEVP (or if not in Excel, use a similar function that will find standard deviation) with the data. You need to include the cell range where the data is to be found, as shown.



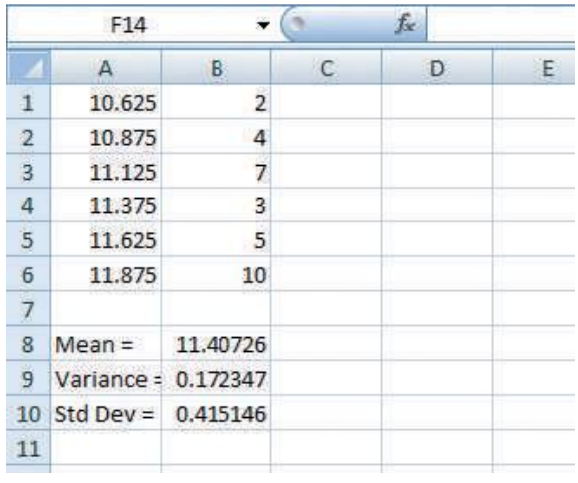
If you have a large data set that is set out in a frequency table, you can still use a spreadsheet to find an estimate for the standard deviation. You need to use the class centre to represent the data values. The following process basically mimics the by-hand calculation method.

For example, consider the data shown that represents the time taken by students to complete a 100 m sprint at a school sports carnival.

Time (s)	Frequency
10.50–<10.75	2
10.75–<11.00	4
11.00–<11.25	7
11.25–<11.50	3
11.50–<11.75	5
11.75–<12.00	10

1. Open a new sheet and put the midpoint values in column A and the frequencies in column B.
2. Highlight the values in column A and select Formulas > Define Name. Enter 'Values' as the name.
3. Highlight the values in column B and select Formulas > Define Name. Enter 'Frequencies' as the name.
4. In a convenient cell write 'Mean =' and then in the cell beside it enter the formula =SUMPRODUCT(Values,Frequencies)/SUM(Frequencies). Name this cell 'Mean'.
5. In the next cell down write 'Variance =' and then in the cell beside it enter the formula =SUMPRODUCT((Values–Mean)^2,Frequencies)/SUM(Frequencies). Name this cell 'Variance'.
6. In the next cell down write 'Std Dev =' and then in the cell beside it enter the formula =SQRT(Variance).

Your completed sheet should look something like the screenshot at right.



7.7 Standard deviation

Answers page 652

Stage 5.3

Navigator

Q1, Q2, Q3, Q6, Q8, Q11, Q12	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9, Q11	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9, Q11

Equipment required: access to a spreadsheet program

Fluency and understanding

1 Find the standard deviation, rounded to two decimal places, for the following data: WE12

11 12 15 15 17 19 21 24 26 29

2 The table below shows the test results for 15 students in English and mathematics. Both tests were scored out of 50. Find the mean and standard deviation for each data set and use these values to compare them. Express the values rounded to two decimal places, where necessary. WE13

English	40	40	34	23	24	39	46	47	41	19	23	23	34	37	45
Maths	33	44	32	49	36	38	41	29	37	29	41	35	30	22	24

3 Find the mean and standard deviation, rounded to two decimal places where required, for the grouped data shown in the table on the right. The data represents the distances, in km, travelled to school by a group of students.

Distance (km)	Frequency
0–<2	22
2–<4	28
4–<6	15
6–<8	7
8–<10	4
10–<12	2

Hint

Find the class centre to calculate the mean and standard deviation of a frequency table with class intervals, e.g. class centre of 6–<8 is 7.

4 The following table shows the heights, in metres, of the ten highest mountains in each of Australia, New Zealand, Brazil and Sri Lanka.

Australia	2228	2185	2178	2166	2132	2048	2047	2045	2050	1979
New Zealand	3754	3500	3440	3279	3201	3183	3176	3163	3160	3157
Brazil	2973	2890	2882	2861	2849	2798	2791	2770	2734	2033
Sri Lanka	2524	2243	2240	2100	2100	2076	2036	2016	2010	1896

(a) Calculate the range for each country.

(b) Calculate the standard deviation for each country, rounded to two decimal places.

(c) Write a sentence or two about what you have found.

5 (a) Find the standard deviation, rounded to two decimal places, for each of the following data sets:

(i) 13 14 14 15 15 15 16 16 19 20 22 25 26 28

(ii) 13 14 14 15 15 15 16 16 19 20 22 25 26 48

(iii) 3 14 14 15 15 15 16 16 19 20 22 25 26 28

(iv) 3 14 14 15 15 15 16 16 19 20 22 25 26 48

(b) Write a sentence or two about what this tells you.

6 You have recorded a data set as:

23 45 46 47 55 57 58 59 62 65

and calculated the mean and standard deviation. When checking your work you discover that the '23' should have been '43'. Changing this will:

A increase both the mean and standard deviation

B increase the mean and decrease the standard deviation

C decrease the mean and increase the standard deviation

D decrease both the mean and the standard deviation.

7 (a) Find the standard deviation, rounded to two decimal places, for each of the following data sets:

(i) 1 1 2 4 4 5 5 6 6 7 7 9 9 9

(ii) 3 3 6 12 12 15 15 18 18 21 21 27 27 27

(iii) 1 1 2 4 4 5 5 6 6 7 7 9 9 9

(iv) 3 3 4 6 6 7 7 8 8 9 9 11 11 11

(b) Write a sentence or two about what this tells you.

8 (a) Calculate the mean and standard deviation for the following data set:

2 2 4 5 5 6 6 7 9 9

(b) A new value is added to the data set. It is 5.5. What effect does this have on the standard deviation?

(c) A different new value is added. It is 2. What effect does this have on the standard deviation?

(d) A different new value is added. It is 9. What effect does this have on the standard deviation?

9 You have recorded a data set as:

5 5 6 7 9 11 13 15 19

and calculated the mean and standard deviation. When checking your work you discover you left out a value of 6. Adding this value to the data set will:

A increase both the mean and standard deviation

B increase the mean and decrease the standard deviation

C decrease the mean and increase the standard deviation

D decrease both the mean and the standard deviation.

Reasoning, communicating and open-ended

10 The screenshot shows some information taken from a music playlist. You will need to be careful how you deal with this data. Give your answers correct to the nearest second.

(a) Find the mean and standard deviation for songs listed as *Rock* (including *Rock*, *Classic Rock*, *RockNRoll*).

(b) Find the mean and standard deviation for songs listed as *Pop*.

(c) Find the mean and standard deviation for songs listed as neither *Rock* nor *Pop*.

(d) Draw a parallel box plot showing the three categories *Rock*, *Pop* and *Other*. Use decimal time values here.

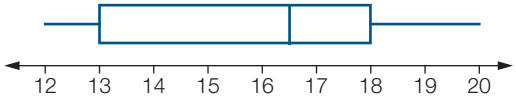
(e) Write a couple of sentences comparing the three categories. Is there any evidence to support the view that *Rock* songs are of a different length to *Pop* songs, for instance?

11 **(a)** Construct a data set with at least 20 values that has a mean of between 20 and 22, a range of at least 15 and a standard deviation of between 2 and 4.

(b) Write a couple of sentences about how you went about this task.

(c) What effect did the condition about the range have on this task?

12 Construct a data set, of at least 20 values, that would fit the box plot on the right and then find the mean and standard deviation of your data set.



Genre	Time	Genre	Time
Rock	2:07	Pop	2:25
Blues	3:25	Soundtrack	2:12
RnB	2:26	Rock	4:15
Blues	5:27	R&B	3:59
Rock	3:35	Blues	2:33
Country & F...	2:16	Blues	15:30
Rock	5:28	Rock	2:47
Rock	4:02	Rock	5:30
Blues	3:31	Rock	3:32
Blues	2:51	Rock	12:28
Rock	3:26	R&B	4:36
Rock	2:34	Rock	3:30
Pop	3:46	Pop	2:15
Classic Rock	4:01	Blues	3:17
Grunge	3:50	Country & F...	2:48
Pop	3:13	RockNRoll	3:05
Pop	2:25	Pop	3:00
Pop	2:30	Rock	3:51
Blues	2:09	Rock	3:43
Rock	3:31	R&B	3:56
Rock	5:51	Soundtrack	5:56
RockNRoll	1:59	Rock	2:52
Rock	3:32	R&B	4:10
Pop	3:49	Pop	3:52
Blues	4:20	Pop	2:40
Blues	2:46	Rock	3:34
Rock	3:20	Blues	6:00
Rock	3:32	R&B	4:47
Rock	3:32	Rock	5:55
Blues	4:07	Classical	3:57
Rock	8:33	Rock	3:42
Rock	8:53	Pop	4:14

Hint

Change the time in seconds to decimal parts of a minute.

Problem solving

Keeping score

From her attempts to shoot 40 points in the season's first basketball game, Angela scored only 10 points. This gave her a shooting average of 25%. In the second game she had the chance to score 30 points but only scored enough to raise her shooting average over the two games to 50%.

(a) How many points did she score in the second game?

(b) What was her shooting average for the second game?

(c) What is the greatest shooting average Angela could have had after the second game, assuming her maximum in the second game was still 30 points?

Strategy options

- Guess and check.
- Work backwards.

7.8 Lines of best fit

Stage 5.3

Need to Know

If scatter plots of bivariate data shows a linear relationship, it can be used to make predictions about other data pairs by drawing a **line of best fit** and finding its equation.

A line of best fit can be drawn approximately by eye. The line should be positioned so that there is the same number of points above and below the line.

A spreadsheet program such as Microsoft Excel® can be used to draw an exact line of best fit. This line of best fit is called a *least squares line of best fit* and it minimises the vertical distance between each point and the line of best fit. These values are squared so that distances from points above the line do not cancel out the distances from the points below the line. A least squares line of best fit is also known as a **regression line**. A spreadsheet can also be used to find the equation of the regression line.

A word of warning: A line of best fit should be used only if the data displays at least a medium to strong linear relationship. Always look at the scatter plot first before finding the equation of the linear regression line using technology.

Making predictions

If the equation of the line of best fit can be found, it can be used to make predictions. When making predictions, using a value from within the known range is called **interpolation**. Using a value from outside this range is called **extrapolation**. In general, values obtained using interpolation can be used with greater confidence than those obtained using extrapolation because the same trend may not continue outside the known range.

For example:

If an equation for the line of best fit connecting a student's mathematics mark with their science result was found to be

Mathematics mark = 0.627 × Science mark + 19.132

then, if a student scored 22 for science, their mathematics mark could be predicted by substituting the known value into the equation.

Mathematics mark = 0.627 × 22 + 19.132

= 32.926

For a student who scored 22 for science, the predicted mathematics mark is 33.

However, this equation should not just be rearranged to predict the science mark for another student if their mathematics mark was known. The horizontal axis would need to be used for the mathematics results and a new equation found for the new line of best fit. Technology makes this switch of variables quite simple.

If a student scored 29 for mathematics their science mark could be predicted by substituting the known value into the new equation, for example:

Science mark = 1.298 × Mathematics mark – 21.564

Technology Exploration: Spreadsheet

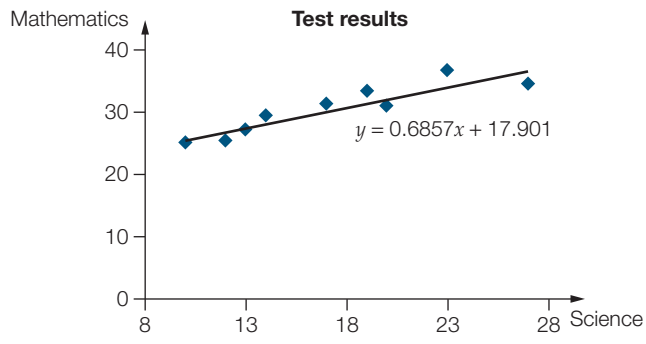
Lines of best fit

Given the following data, use a spreadsheet program such as Microsoft Excel® to find a line of best fit.

Pupil	A	B	C	D	E	F	G	H	I	J
Science mark (out of 30)	20	23	10	27	14	13	12	20	17	19
Mathematics mark (out of 40)	31	36	23	34	29	27	25	30	31	33

1. Enter the data pairs (science mark and mathematics mark) using two columns.
2. Draw a scatter plot by selecting the columns, then from the menu selecting Insert > Scatter and choosing the first option.
3. Right-click one of the points of your scatter plot and select 'Add Trendline'.
4. Ensure 'Linear' and 'Display Equation on Chart' are selected and then Close.

You should get a graph like the one shown below.



For a student who scored 25 for science, the predicted mathematics mark is 35:

Mathematics mark $= 0.6857 \times 25 + 17.901$

$= 35.0435$

This value is within the range of known data values for that variable. For science scores the range was 10–27 and for mathematics scores it was 25–36.

Answers page 654

7.8 Lines of best fit

Stage 5.3

Navigator

Q1, Q2, Q3, Q4, Q8, Q9	Q1, Q2, Q3, Q5, Q7, Q8	Q1, Q2, Q3, Q5, Q6, Q7, Q8

Equipment required: access to a spreadsheet program

Fluency and understanding

1 Find the equation of the line of best fit, using Microsoft Excel® or some other form of technology, for each of the following sets of data. Round your answers to three decimal places, where necessary. In each case, the required rule is of the form $y = \ldots$.

(a)

x	0	1	2	3	4	5	6
y	4	16	29	42	54	65	76

(b)

x	0	2	5	7	10	12	14
y	-7	8	60	92	150	170	200

(c)

x	1	3	7	11	13	16	20
y	45	28	18	0	-5	-15	-45

(d)

x	3	6	8	10	13	15	18
y	26	12	1	-12	-24	-30	-48

2 It is believed that the height of a male at 30 months is usually about half his full adult height. (This kind of approximate rule is called a 'rule of thumb'.) The table below gives the height (cm) for 15 boys measured at 30 months of age and again at 19 years of age (taken to be their full adult height).

30-month height (cm)	89.0	89.9	90.3	90.8	90.9	91.0	91.1	91.2
19-year height (cm)	178.0	177.1	179.6	181.8	184.0	180.5	182.0	183.1

30-month height (cm)	91.2	91.9	92.9	93.3	94.7	95.4	96.1
19-year height (cm)	180.1	185.1	182.0	186.3	187.4	187.9	189.4

(a) Draw a scatter plot of the data, putting the 30-month height on the horizontal axis.

(b) Draw, by eye, a line of best fit over your scatter plot.

(c) Do you think that the 30-month height is a good predictor of full adult height?

(d) The rule of thumb says that a 94 cm 30-month-old boy will grow to a full height of 188 cm. What height does your line of best fit predict? Give your answer correct to the nearest cm.

3 **(a)** Using the line of best fit equation $d = 23.2t - 4.6$, the interpolated value for d when $t = 40$ is closest to:

A 59 **B** 88 **C** 923 **D** 928

(b) **(i)** For the table of values, use technology to find the equation of the line of best fit. Give your answer in the form $q = \ldots$.

p	0	3	5	12	15
q	2	7	9	15	21

(ii) Assuming the relationship continues to be linear, the extrapolated value for q when $p = 25$ is closest to:

A 32 **B** 30 **C** 28 **D** 25

4 The following table gives the national running records for various distances for men for 10 different countries.

100 m (s)	10.39	10.31	10.17	12.18	10.35	10.11	10.98	10.60	10.61	10.46
200 m (s)	20.81	20.06	20.22	23.20	20.65	20.38	21.82	21.42	20.96	20.66
5000 m (min)	14.04	13.28	13.55	16.70	13.42	13.34	14.16	13.77	13.32	13.10

(a) Draw a scatter plot with the 100 m on the horizontal axis and the 200 m on the vertical axis.

(b) Does there appear to be a connection between the times for the 100 m and the 200 m?

(c) Draw a scatter plot with the 100 m on the horizontal axis and the 5000 m on the vertical axis.

(d) Does there appear to be a connection between the times for the 100 m and the 5000 m?

(e) Can you explain what you have found?

5 An engineer conducted an experiment modelling the force needed to set an object in motion. Her experiment consisted of different-sized blocks of wood of mass (m kilograms) being set in motion on a horizontal surface by exerting a force (F newtons). The size of the force needed to move each block of wood was recorded and the following results were obtained.

Mass m (kg)	1	2	3	5	7.5	10	12.5	15	20
Force F (N)	5	9	15	24	38	50	60	70	100

(a) Use technology to find the equation of the linear line of best fit, if this is appropriate for this data. Round your answer to two decimal places.

(b) The ratio $\frac{F}{m}$ is proportional to the *coefficient of friction* between the blocks of wood and the horizontal surface. What is the value of this ratio?

(c) Find the force, rounded to two decimal places, needed to move a block of wood of mass:

(i) 4 kg **(ii)** 11 kg **(iii)** 18.5 kg.

(d) Assuming the relationship between the force and the mass continues to hold for heavier blocks of wood, find the force needed to move a block of wood of mass:

(i) 50 kg **(ii)** 200 kg **(iii)** 1 tonne.

6 A chemist conducted an experiment in which the solubility (S units) of sodium chloride in water was tested for different temperatures (T°C). The results of the experiment are recorded in the following table.

Temperature T (°C)	0	10	20	30	40	50	60	70	80	90	100
Solubility S (units)	35.0	35.4	36.1	36.5	37.1	37.7	38.2	38.6	38.9	39.4	40.0

(a) Use technology to find the equation of the line of best fit when temperature is on the horizontal axis. Round your answer to two decimal places.

(b) Find the solubility of sodium chloride in water at:

(i) 25°C **(ii)** 78°C **(iii)** 120°C

(c) Use technology to find the equation of the line of best fit when solubility is on the horizontal axis.

(d) Find the temperature at which sodium chloride has the following solubilities in water:

(i) 36.0 units **(ii)** 38.5 units **(iii)** 45.0 units

(e) What is the potential problem with the answers to **(b) (iii)** and **(d) (iii)**?

Reasoning, communicating and open-ended

7 A complex spring system is tested under compression and the results obtained are shown in the table below.

Compression Δx (mm)	0	20	40	60	80	100	120	140
Applied force F (N)	0	1.8	3.9	6.1	8.2	9.9	11.9	14.1

Compression Δx (mm)	160	180	200	220	240	260	280	300
Applied force F (N)	16.1	17.9	20.0	26.7	32.9	38.4	44.6	50.2

(a) Use technology to draw a scatter plot of the data, with compression on the horizontal axis. You should be able to see that the data splits into two groups. At which compression value does this occur?

(b) Find an equation for each of the two groups of data. Round your answers to three decimal places.

(c) Find the applied force required, rounded to one decimal place, to produce a compression of:

(i) 75 mm **(ii)** 150 mm **(iii)** 225 mm

(d) Breaking the data into the same two groups, find the equations of the lines of best fit if applied force was on the horizontal axis.

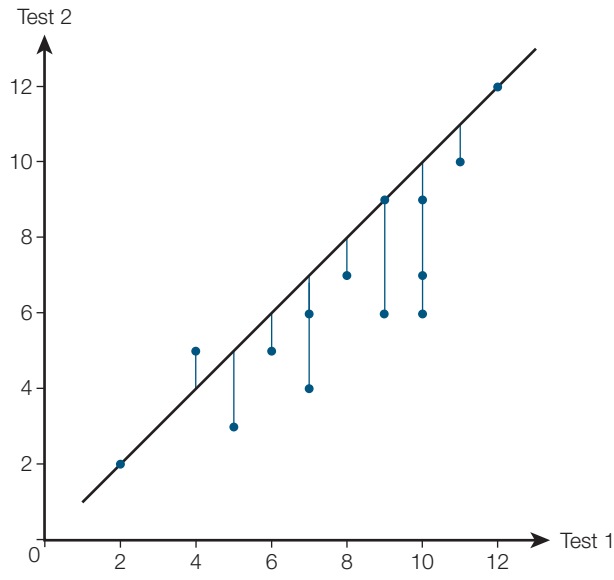
(e) Find the compression, to the nearest whole number, if the spring system has an applied force of:

(i) 5 N **(ii)** 30 N **(iii)** 45 N

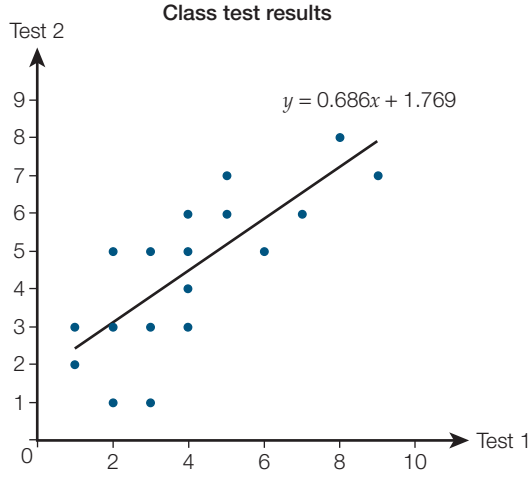
(f) The spring system has a maximum compression of 500 mm with an applied force of 90.0 N. Have we found an appropriate equation for this part of the graph? Explain why, listing any assumptions you have made. (Determine a new equation if necessary.)

8 (a) What mistake has been made when drawing this line of best fit?

(b) Redraw the graph in your exercise book (assuming that the coordinate pairs are all whole number pairs) and draw the line of best fit correctly.



9 A scatter plot has been drawn with its line of best fit drawn over it. The equation of the line of best fit has been given. Make up three questions you could ask using this information. The data refers to results on two mathematics tests.



Problem solving

Making the data fit

Equipment required: a spreadsheet program such as Microsoft Excel® would be useful

Imagine that you have drawn a scatter plot and found that the equation of the line of best fit is $y = 3x - 4$, with the values rounded to the nearest whole number.

Find a data set of at least 10 number pairs that would produce such a line of best fit.

Resist the temptation to make the data points all fit the line perfectly!

Strategy options

- Look for a pattern.
- Work backwards.

7.9 Survey data: planning and implementation

Stage 5.3

Need to Know

When conducting a survey, care needs to be taken with the way the sample is compiled. The most common types of sampling are listed below.

Random sampling: Every person in the population has an equal chance of being selected.

Stratified random sampling: Randomly select from the whole population but in proportion to strata in the whole population. For example, if your school has 40% boys, then 40% of people surveyed on a general issue should be boys.

Convenience sampling: This means selecting in a cheap or easy manner. For example, you might survey your friends and family.

Worked Example 14

WE14

A school has 900 students distributed across the year levels as shown below.

Year	Number
Year 7	120
Year 8	120
Year 9	120
Year 10	140
Year 11	220
Year 12	180
Total	900

A sample of 30 students is to be selected.

(a) How could you select a random sample of the school population?

(b) How could you select a stratified random sample of the school population?

Working

(a) Number the students from 1 to 900. Then use a random number generator to randomly select 30 values. (If a number is generated twice just choose another number to replace it.)

(b) The strata used will be the six year levels. Calculate the proportion of students at each year level. Year 7 is used as a sample calculation: $\frac{120}{900} \times 30 = 4$.

So, you need to select 4 students from each of Years 7–9; 5 students from Year 10; 7 students from Year 11; 6 students from Year 12. (Some rounding of numbers is necessary which may lead to a number that is not exactly 30.)

You could then use a similar process to that described in **(a)** to select these particular students.

7.9 Survey data: planning and implementation

Answers page 655

Stage 5.3

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10

Fluency and understanding

WE 14

1 A school has 1200 students, distributed across the year levels as shown below.

Year	Number
Year 7	180
Year 8	180
Year 9	180
Year 10	220
Year 11	240
Year 12	200
Total	1200

A sample of 60 students is to be selected.

(a) How could you select a random sample of the school population?

(b) How many students should be chosen from each year level if you are using a stratified random sample of the school population?

2 Newspapers and other media outlets conduct polls related to political parties at least every month. The polling details for one such poll include: 'The poll is conducted by telephone nationwide. The maximum margin of error to apply to this sample is approximately 2.6%.'

(a) A particular poll indicates support for Labor at 29%, the Coalition at 49% and the Greens at 12%.

(i) What is the worst figure for Labor that this result might indicate, given the margin of error?

(ii) What is the best result for the Coalition that this result might indicate?

(b) Another poll with the same polling details indicates support for Labor at 45%, the Coalition at 46% and the Greens at 6%. The report in the newspaper had the headline 'Coalition in the lead'. Do you think this headline is justified? Write a sentence or two outlining your thoughts.

3 A principal asks the Student Representative Council of her school for its opinion of a proposal for a change in school hours. What kind of sampling is this?

A random sampling **B** convenience sampling

C stratified random sampling **D** not enough information to say

4 In a sample of 50 seedlings, 9 had white flowers, 12 had mauve flowers, 18 had crimson flowers, 2 had pink flowers, 8 had magenta flowers, and 1 had purple flowers. If this sample was representative of the general population, what mean number would be needed if a garden designer required:

(a) 32 magenta **(b)** 10 pink **(c)** 120 mauve?

5 Yvette has a large packet of jellybeans. She has made a random selection of 20 jellybeans and found 2 black, 6 pink, 3 green, 4 blue and 5 orange. If this sample truly reflects the mix of colours, how many of each colour would you expect if you knew the bag contained 130 jellybeans?

6 The Sensis Social Media Report (2011) contains the following information under the heading 'Methodology'.

The research consisted of a telephone survey among 803 randomly selected respondents across Australia. Quotas were applied to provide for adequate representations in each age group and both genders. Quotas were also applied on location to ensure coverage across metropolitan and regional areas in each state and territory. The interview took approximately 10 minutes to administer and fieldwork was conducted in late February and early March 2011. The data was weighted by age and gender within location so that the results would be reflective of the actual Australian population. Sample sizes by age and gender are shown below.

	Total	Male	Female
14 to 19 years of age	100	47	53
20 to 29 years of age	140	73	67
30 to 39 years of age	134	67	67
40 to 49 years of age	135	68	67
50 to 64 years of age	160	80	80
65 plus years of age	134	67	67
Total	803	402	401

Source: http://about.sensis.com.au/ignitionsuite/uploads/docs/sensis%20social%20media%20report.pdf, p. 7

(a) Based on the figures in the table, are there more males or females in the 14 to 19 years of age range?

(b) Express this as a decimal showing the number of males per 100 females.

According to the Australian Bureau of Statistics Population Pyramid for 2012, the number of males per 100 females in the age range 14 to 19 years of age is as shown in the following table.

Age	14	15	16	17	18	19
Number of males per 100 females	105.66	105.6	105.4	105.53	105.37	104.77

Source: http://www.abs.gov.au/websitedbs/d3310114.nsf/home/Population%20Pyramid%20-%20Australia

(c) What comment would you make about the claim made about this being reflective of the Australian population?

(d) The survey was about access to, and use of, social media. Can you think of a group of people who would have been excluded from the survey? Would the views and actions of these people have any bearing on the overall survey results?

Reasoning, communicating and open-ended

7 The following information is included under the heading 'Methodology' as Appendix B to the ME Bank *Household Financial Comfort Index Report* for December 2012.

> ME Bank commissioned DBM Consultants to develop the *Household Financial Comfort Index* with Economics & Beyond and Baker Group. The research includes an online survey of approximately 1500 Australians aged 18 years and older who do not work in the market research or public relations industries. Waves of research have been conducted in October 2011, June 2012 and this third wave of research was conducted in mid-December 2012 with 1534 respondents). For analysis, the population sample was weighted according to ABS statistics on household composition, age, state, and employment status to ensure that the results reflected Australian households.
>
> *Source*: http://www.mebank.com.au/media/1233682/household_financial_comfort_report_dec_2012.pdf, p. 25

In the foreword to the report the following information is provided:

> The *Household Financial Comfort Index* specifically measures: overall household financial comfort by asking respondents to estimate their financial comfort as well as their expectations and confidence across 11 measures.
>
> *Source*: http://www.mebank.com.au/media/1233682/household_financial_comfort_report_dec_2012.pdf, p. 5

(a) Why do you think the survey is limited to individuals aged 18 years and over?

(b) Why is it mentioned that the 'the population sample was weighted according to ABS statistics on household composition, age, state and employment status'?

(c) Why do you think the third wave of research included 1534 respondents when the aim was approximately 1500? Why would it not be exactly 1500?

One of the key findings of the report was:

> The slight decrease in financial comfort from June 2012 was driven by decreases in how comfortable households with children ('Couples with young children', 'Couples with older children' and 'Single parents') were feeling about their financial situation. This downshift in comfort for households with children may reflect a return to 'normal levels' after a temporary positive effect of increased government payments that came through in the June quarter 2012.
>
> *Source*: http://www.mebank.com.au/media/1233682/household_financial_comfort_report_dec_2012.pdf, p. 4

(d) What does this tell you about the way household groups are broken up in the survey?

(e) What does this tell you about the effect of Government policy in this area?

8 The following is an extract from *The Citizen*. It is related to the polls conducted by reputable polling companies in the lead-up to a Federal election.

> All analysts point to the method of selecting people and contacting them as key determinants of reliable polling. Representative polling is based on the views of a random sample. But if the sample does not represent all groups in the community, the results will be skewed.
>
> And yet, a reality of modern polling and whether a voter's views are captured by the pollsters will almost certainly depend on whether they have a landline, whether theirs is a silent number and whether they are at home in the evenings.
>
> Polls using only landline phone numbers exclude mobile phone users – more likely to be young people and professionals. And nearly all polls rely on methods that exclude responses from non-English speaking voters.
>
> *Source*: See more at http://www.thecitizen.org.au/media/opinion-polls-often-matter-opinion-say-pundits#sthash.DGpEOnid.dpuf

(a) Only one of the major polling organisations calls mobile telephone numbers. Why do you think this is the case?

(b) What effect would having a silent telephone number have on your chance of being surveyed?

(c) How would you go about trying to determine the percentage of the population that has a landline telephone connection?

(d) Only one of the major polling organisations uses face-to-face surveying techniques. Why do you think this is the case?

(e) How is it that you think the methods used exclude non-English speakers?

9 The Australian Bureau of Statistics (ABS) has embarked upon an ambitious project related to the health and well-being of Australians. The National Health Survey (NHS) is one component of the Australian Health Survey (AHS).

The following information comes from the ABS website.

- The 2011–13 Australian Health Survey (AHS) is the largest and most comprehensive health survey ever conducted in Australia. This survey, conducted throughout Australia, was designed to collect a range of information from Australians about health related issues, including health status, risk factors, actions, and socioeconomic circumstances.
- The 2011–12 NHS collected information about:
  - the health status of the population, including long-term health conditions experienced;
  - health-related aspects of people's lifestyles, such as smoking, Body Mass Index, diet, exercise and alcohol consumption;
  - use of health services such as consultations with health practitioners and actions people have recently taken for their health; and
  - demographic and socioeconomic characteristics.
- The NHS was conducted from a sample of approximately 15 600 private dwellings across Australia. Urban and rural areas in all states and territories were included, while Very Remote areas of Australia and discrete Aboriginal and Torres Strait Islander communities (and the remainder of the Collection Districts in which these communities were located) were excluded. These exclusions are unlikely to affect national estimates, and will only have a minor effect on aggregate estimates produced for individual states and territories, excepting the Northern Territory where the population living in Very Remote areas accounts for around 23% of persons.
- Non-private dwellings such as hotels, motels, hospitals, nursing homes and short-stay caravan parks were excluded from the survey. This may affect estimates of the number of people with some long-term health conditions (for example, conditions which may require periods of hospitalisation).
- Within each selected dwelling, one adult (aged 18 years and over) and one child were randomly selected for inclusion in the survey. Sub-sampling within households enabled more information to be collected from each respondent than would have been possible had all usual residents of selected dwellings been included in the survey.

APPROACHED SAMPLE, FINAL SAMPLE AND RESPONSE RATES

	New South Wales	Victoria	Queensland	South Australia	Western Australia	Tasmania	Northern Territory	Australian Capital Territory	Australia
Households approached (after sample loss)	3269	3051	2907	2285	2385	1635	1307	1516	18 355
Households in sample	2736	2516	2457	1962	2144	1469	975	1306	15 565
Response rate (%)	83.7	82.5	84.5	85.9	89.9	89.8	74.6	86.1	84.8
Persons in sample	3602	3287	3244	2508	2847	1909	1304	1725	20 426

- Sampling error in this survey is measured by Relative Standard Errors (RSEs). In this publication, estimates with an RSE of 25% to 50% are preceded by an asterisk (e.g. *3.4) to indicate the estimate should be used with caution. Estimates with an RSE over 50% are indicated by a double asterisk (e.g. **0.6) and are considered too unreliable for most purposes.

(a) How many households are represented in the survey?

(b) How many people are represented in the survey?

(c) Why are these two numbers different? Write a sentence or two explaining your thoughts.

(d) Who are likely to be the organisations or authorities most interested in the results from this survey?

(e) Why is the information included in the last arrow point provided?

(f) What do you think this point means: 'Sub-sampling within households enabled more information to be collected from each respondent than would have been possible had all usual residents of selected dwellings been included in the survey'?

(g) Re-read the arrow point about the exclusion of non-private dwellings. Give an example for each type of dwelling listed that explains why that type of dwelling may have been excluded.

(h) The number of households approached from each state and territory varies quite a bit. Why do you think this is the case?

(i) What do you think is meant by this point: 'demographic and socioeconomic characteristics'?

The initial findings from the NHS included the following:

Of the National Health Priority Areas, the top long-term health conditions experienced in Australia in 2011–12 were:

- arthritis – 3.3 million people (14.8%)
- mental and behavioural conditions – 3.0 million people (13.6%)
- asthma – 2.3 million people (10.2%)
- heart disease – 1.0 million people (4.7%)

(j) What do you think are the implications of these findings? Who will act on them and what are they likely to do?

10 It can be difficult sometimes to understand how census data is used. The Australian Bureau of Statistics has a possible answer to this!

RUN THAT TOWN!

The Australian Bureau of Statistics (ABS) has launched an interactive and innovative game that promotes the release and application of Census data. Run That Town is the first game of its kind to be released by a national statistical agency using real Census data.

Run That Town puts the player in charge of any Australian neighbourhood and challenges them to make planning decisions based on 2011 Census data to boost popularity, grow the town and keep the locals happy.

Sue Taylor, Director in the 2016 Census, said by using core demographic data, the game demonstrates, in a simple and visual way, the many ways Census data can be used by everyday Australians in a light-hearted and entertaining way.

'Run That Town offers a unique insight into how Census data can be used. The game is an innovative and fun way to communicate with the public, while embracing digital media.'

'Run That Town is a strategy game with a twist that is intended for a wide audience offering both one-time and ongoing users a unique insight into the value of Census data.' Ms Taylor said.

Run That Town can be downloaded for free from the App Store and played on all current Apple devices.

'I encourage people to download the game, take control of any Australian suburb and shape it the way you want by using real Census data and see how your decisions affect your town's demographics.'

Source: For more information visit http://runthattown.abs.gov.au/
http://www.abs.gov.au/ausstats/abs@.nsf/Lookup/1376.0main+features4002013

As part of a small group investigate this game, or develop your own ideas for a statistics-based game.

Chapter review 7

D.I.Y. Summary

Key Words

Stage 5.2	bivariate data	independent variable
	dependent variable	scatter plot
Stage 5.2◊	bias	parallel dot plot
	box plot	percentage cumulative frequency curve
	cumulative frequency curve	percentile
	five-number summary	quantile
	interquartile range	quartile
	ogive	range
	outlier	reliability of data
	parallel box plot	validity
Stage 5.3	convenience sampling	regression line
	extrapolation	standard deviation
	interpolation	stratified random sampling
	line of best fit	variance
	random sampling	

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.2◊ Q1–6 **Stage 5.3 Q7–8**

1 The difference between the upper quartile and the lower quartile is called the _____.

2 Another name for a cumulative frequency curve is _____.

3 A _____ is a graph that shows the minimum, lower quartile, median, upper quartile and maximum values for a data set.

4 The value below which 80% of values lie on a cumulative frequency curve can also be referred to as the 0.8 _____.

5 The difference between the highest observed value and the lowest is called the _____.

6 If a value is more than $1.5 \times \text{IQR}$ above Q_U or more than $1.5 \times \text{IQR}$ below Q_L then it is an _____.

7 When we use a regression line to predict a value within the known range we are using _____. If the value is outside the known range we are using _____.

8 The _____ is the square root of the _____.

Stage 5.2◊ Q1–4, 7, 11–13, 15 Stage 5.2 Q5–6, 14 Stage 5.3 Q8–10, 16–18

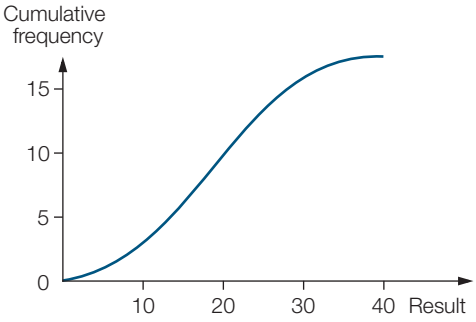
Fluency and understanding

1 For each of the following cumulative frequency curves: Ex.7.1, 7.2

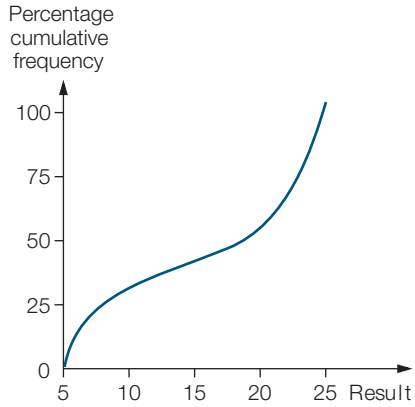
(i) read off the median **(ii)** read off the interquartile range

(iii) draw a box plot.

(a)



(b)



2 For the set of data shown at right: Ex.7.2

(a) find the median by drawing a percentage cumulative frequency curve

(b) read off the interquartile range

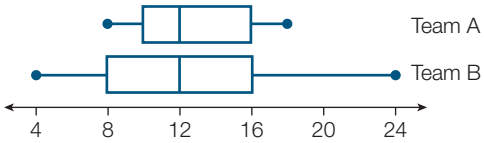
(c) draw a box plot.

Class interval	Frequency
120–<125	5
125–<130	17
130–<135	32
135–<140	54
140–<145	24
145–<150	13

3 A frog in training for the big jump-off achieves the following jump lengths (in centimetres). On the basis of these jumps, how long would a jump need to be to be considered an outlier? Ex.7.2

19 25 17 22 20 19 16 15 17 21

4 You have been working in a group on an analysis task. One member of the group has produced the following parallel box plot for the data that has been collected. Your task is to write the first draft of the analysis. This should include a comment about which team you think has been more successful. Ex.7.3



5 The Changi climate station in Singapore has the following long-term climate values. Ex.7.4

	Jan	Feb	Mar	Apr	May	Jun
Rainfall (mm)	243.3	159.4	186.6	179.5	175.2	161.3
Temperature (°C)	30.1	31.2	31.6	31.7	31.6	31.3

	Jul	Aug	Sep	Oct	Nov	Dec
Rainfall (mm)	158.5	175.4	168.7	193.3	256.4	287.9
Temperature (°C)	30.9	30.9	30.9	31.1	30.6	30.0

(a) Draw a scatter plot of the data putting rainfall on the horizontal axis.

(b) Make a comment about the type of relationship, if any, displayed by the data.

6 For the following pairs of variables, state which one you think is the independent variable. Give a brief reason for your answer. **Ex. 7.5**

(a) Sprinter's race time and number of hours spent training

(b) Amount of fertiliser applied per hectare on a farm, measured in kilograms, and weight yield per hectare of a crop, measured in tonnes

(c) Cost of producing eggs on a farm and the average amount of space allocated for each hen

(d) Amount of water used by a household, measured in kilolitres, and the number of people in a household.

7 Some TV stations conduct a daily poll on a question of current interest. To vote you need to telephone a particular number for Yes or a different number for No. **Ex. 7.6**

(a) Do you think this type of survey is reliable? Give reasons for your answer.

(b) One such question was 'Do you think more money should be spent on roads than public transport?' Do you think the question is clear? What doubts, if any, do you have about its wording?

(c) The next day it was reported that 56% of results were Yes. This represented 84 of the 150 votes received. What do you think about the number of votes received and the validity of the conclusion?

(d) Another day the question was 'Do you think the government should subsidise manufacturing in Australia?' It attracted 1025 votes, 642 Yes and 383 No. Who would be interested in the results of this poll?

(e) Why do you think TV stations run polls such as this?

8 Find, rounded to two decimal places, the standard deviation for the following data sets. **Ex. 7.7**

(a) 3 4 4 5 6 7 7 7 8 9 10 11 11 12 12 12 14 15 16

(b)

Number of flaws	Frequency
0	2
1	19
2	16
3	15
4	7

(c)

Height (m)	Frequency
0–<5	6
5–<10	11
10–<15	15
15–<20	8
20–<25	2

9 Find the equation of the line of best fit, using a spreadsheet or some other form of technology, for each of the following sets of data. Round your answers to three decimal places, where necessary. In each case, the required rule is of the form $y = \ldots$. **Ex. 7.8**

(a)

x	0	1	2	3	4	5	6
y	3	7	13	20	26	30	35

(b)

x	0	2	5	7	10	12	14
y	18	15	11	5	0	-6	-11

10 In a sample of 50 cars in a car park, 13 were Toyotas, 7 Hyundais, 11 Fords, 10 Holdens, 4 BMWs, 2 Mercedes-Benzes and 3 Volvos. If this sample was representative of the general population, what mean number would be in the car park if there were: **Ex. 7.9**

(a) 24 Fords **(b)** 5 Volvos **(c)** 32 Toyotas?

11 Consider the following data:

72.8	64.8	55.4	44.3	61.8	52.9	62.0	73.6	34.5	72.3
65.9	57.7	64.4	55.3	45.2	71.2	69.6	54.2	59.9	51.1

(a) Group the data into intervals of 10 (i.e. 30–<40, 40–<50 etc.) and draw a cumulative frequency curve.

(b) Calculate the five-number summary for the data.

(c) Find the median from the cumulative frequency curve and compare it with the median calculated from the raw data.

12 The rainfall (in millimetres) recorded on each day of a month in a particular location was:

12	15	17	16	25	2	44	3	2	0
4	6	8	10	0	11	3	6	3	8
6	15	15	17	0	22	17	21	30	33

(a) Calculate the range for the rainfall figures.

(b) Calculate the five-number summary for the data.

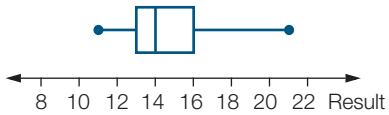
(c) Draw a box plot to represent the data.

13 For each of the following box plots:

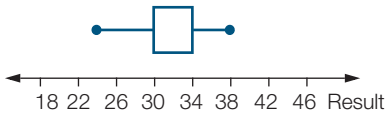
(i) indicate what values would cause results to be considered as outliers

(ii) decide whether there are any outliers in the data set displayed.

(a)



(b)



Reasoning and communicating

14 Some of the data available from the Bureau of Meteorology website for Perth (Metro) is shown below.

Temperature (°C)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Annual
Mean maximum	30.9	31.3	29.5	25.6	22.4	19.3	18.3	18.8	20.0	22.9	26.2	28.8	24
Mean minimum	17.9	18.0	16.5	13.5	10.5	8.4	7.8	8.1	9.5	11.2	14.1	16.2	12

(a) Find the mean difference in temperature (°C) for each month.

(b) Draw a scatter plot showing mean maximum temperature, mean minimum temperature and mean difference in temperature.

(c) Describe the temperature pattern for Perth (Metro) using the graph as an aid.

(d) Could you use this graph to describe the temperature pattern for Kalgoorlie, in inland Western Australia? Give a reason for your answer.

(e) The daily maximum and minimum temperatures in Perth for June 2010 are given.

Minimum:	10.9	6.8	5.6	4.8	9.9	5.7	6.5	6.6	8.7	7.3	9.6
	6.9	4.3	2.2	2.3	6.1	12.0	9.7	10.8	13.1	6.4	9.4
	8.2	10.8	11.1	6.4	3.1	6.3	8.4	5.2			
Maximum:	16.2	17.9	17.6	17.0	16.9	14.3	15.5	15.9	15.1	16.3	16.1
	16.0	13.4	13.8	14.0	16.9	16.3	16.5	17.0	17.3	16.4	17.3
	18.2	18.2	15.2	14.2	14.5	14.6	11.9	14.0			

How did June 2010 compare to the overall data?

15 At his head office, the owner of a company holds a record of the salaries paid to his employees. Annual salaries 3 years ago and today are given in the following table.

Salary range (× $10 000)	Number of employees (3 years ago)	Number of employees (currently)
1–<3	12	9
3–<5	15	10
5–<7	9	9
7–<9	5	8
9–<11	3	7
11–<13	0	5

(a) Represent the above data by drawing a graph that emphasises the difference between salaries paid 3 years ago and those paid today.

(b) Contrast the means, medians and interquartile ranges for the two sets of data.

(c) Draw a box plot for each set of data.

(d) Describe, in your own words, the change in salaries over the past 3 years.

(e) How much more is the company paying in salaries today than it was 3 years ago?

(f) By what percentage has the average salary risen?

16 The number of colony forming units per 100 mL of water of two different bacteria taken from a particular lake at weekly periods is recorded in the following table. The table also shows the maximum temperature (°C) for the day of recording.

Fecal coliform (CFU/100 mL)	350	73	180	150	200	130	2200	6300
***E. coli* (CFU/100 mL)**	350	90	120	330	200	120	1600	5800
Temp. (°C)	15.3	14.6	16.8	16.8	18	24	19.6	21.7

Fecal coliform (CFU/100 mL)	360	1200	560	1700	19	430	320	87
***E. coli* (CFU/100 mL)**	300	980	630	1800	16	270	270	160
Temp. (°C)	20.1	19.4	22.1	24.1	22	23	19	18

(a) In how many different ways can the data be grouped?

Use technology to:

(b) draw a scatter plot with fecal coliform on the x-axis and *E. coli* bacteria on the y-axis

(c) find the equation of the line of best fit, assuming the data is linear.

17 The grouped frequency table below shows the age distribution of the employees of a local company. The age given is the nearest whole number of years on 1 January of the current year. So, if a worker was 22 years and 5 months old they would be recorded as 22, whereas a worker 22 years and 7 months old would be recorded as 23.

Age at 1 Jan (years)	16–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55
Frequency	25	30	32	28	33	42	55	10

(a) Estimate the mean age, to one decimal place, giving a reason for your estimate.

(b) Calculate the mean and standard deviation for the ages of the employees, giving your answers rounded to two decimal places, where appropriate.

(c) Draw a cumulative frequency curve and use it to find the five-number summary.

(d) Draw a box plot of the data.

(e) Use your cumulative frequency graph to estimate the number of employees who are:

(i) under 32 **(ii)** over 48.

18 The following article was published on the *Sydney Morning Herald* website in November 2012.

Smaller banks outrank big four in consumer survey

None of the big four rank in the top 10 when it comes to Australia's preferred bank, new research shows.

Commonwealth Bank, Westpac, National Australia Bank and ANZ are all outranked by their smaller rivals, according to data compiled by consumer feedback website ProductReview.com.au.

ING Direct is number one, according to more than 3500 reviews on the website.

It is followed by Heritage Bank, a customer-owned business, and ME Bank, which was established by super funds and unions.

Other top-10 banks include Bendigo Bank, Bank of Cyprus and Macquarie Bank.

All four of the big banks received an average rating of 2.6 out of 5.0 on the ProductReview website, leaving them well down the list.

'Typically, people are trending away from the big providers as they look for better deals and personalised service,' website spokesman Samuel Williamson said in statement on Friday.

But, the latest industry statistics show the majors remain a dominant force in the banking industry, holding many times the funds of those at the top of ProductReview's latest rankings.

Commonwealth is the largest holder of deposits with more than $344 billion on its books, and Westpac has issued the greatest volume of loans at $405 billion, data from the Australian Prudential Regulation Authority shows.

ING holds just $29 billion in deposits and $41 billion in loans.

Heritage and ME Bank are even smaller, with just $5 billion and $8 billion in both deposits and loans, respectively.

Source: AAP, 16 November 2012. Read more: http://www.smh.com.au/business/smaller-banks-outrank-big-four-in-consumer-survey-20121116-29fut.html

(a) The story suggests the results are based on more than 3500 reviews on the website. Do you think this counts as an unbiased survey? Explain your thoughts in a sentence or two.

(b) What does the size of a bank have to do with the results of this research?

(c) Do you think you would be more likely to log into a website, such as the one described here, to make positive or negative comments about a company?

(d) Are the comments about Heritage Bank being a customer-owned business and ME Bank being established by superannuation funds and unions of any relevance to the results recorded?

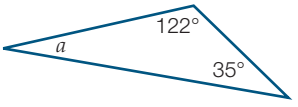
8 Geometry

P Recall 8

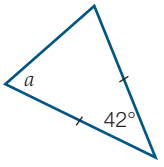
Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Find the value of the variable in each of the following.

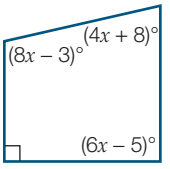
(a)



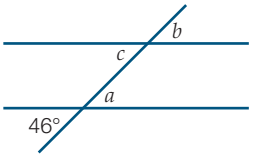
(b)



(c)

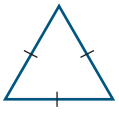


2 Find the value of the variables in this diagram.

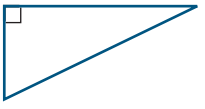


3 State whether the following triangles are equilateral, scalene, isosceles or right-angled.

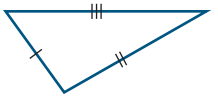
(a)



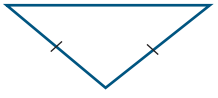
(b)



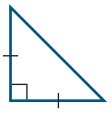
(c)



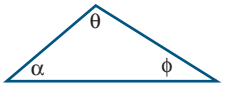
(d)



(e)



(f)

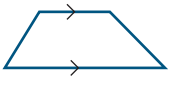


4 Match the name to its shape.

(a) rectangle **(b)** scalene triangle **(c)** isosceles triangle

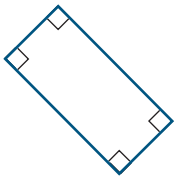
(d) parallelogram **(e)** trapezium

A

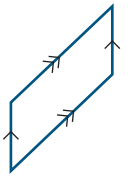


B

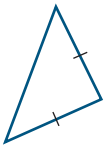
C



D



E



Syllabus references

Measurement and Geometry: Properties of Geometrical Figures

- ✓ Formulate proofs involving congruent triangles and angle properties **(ACMMG243) (Stage 5.2)**
- ✓ Use the enlargement transformations to explain similarity and to develop the conditions for triangles to be similar **(ACMMG220) (Stage 5.2)**
- ✓ Formulate proofs involving congruent triangles and angle properties **(ACMMG243) (Stage 5.3§)**
- ✓ Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes **(ACMMG244) (Stages 5.2, 5.3§)**

Measurement and Geometry: Circle Geometry

- ✓ Prove and apply angle and chord properties of circles **(ACMMG272) (Stage 5.3#)**
- ✓ Prove and apply tangent and secant properties of circles **(Stage 5.3#)**

Congruent and similar triangles

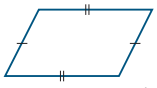
8.1

Stage 5.2

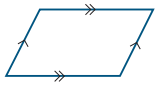
Need to Know

Geometric notation

- Sides with the same markings are equal in length.

  For example: 

- Sides with the same number of arrows are parallel.

  For example: 

- The symbol || means 'is parallel to'.

  For example: $AB \parallel CD$

- The symbol ⊥ means 'is perpendicular to'.

  For example: $AB \perp CD$

- The symbol ≡ means is 'congruent to'.

  For example: $\Delta ABC \equiv \Delta DEF$

- The symbol ∴ means 'therefore'.

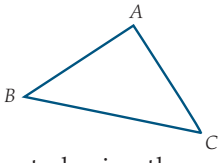
  For example:

  $$a = 60^\circ$$
  $$a = b$$
  $$\therefore\ b = 60^\circ$$

- The symbol ||| means 'is similar to'.

  For example: $\Delta ABC \parallel\!| \ \Delta DEF$

- The triangle with vertices A, B and C is written as ΔABC.

  For example: 

- Angles are denoted using the symbol ∠.

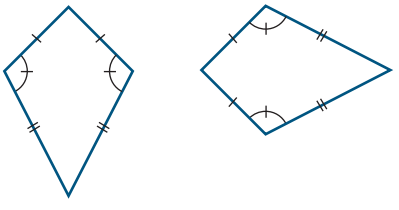
  For example: the angle at vertex A is written as $\angle BAC$. They can also be given letter names such as α, β, ϕ.

- Intervals are named according to their end points.

  For example: the interval between A and B is written as AB.

Congruency

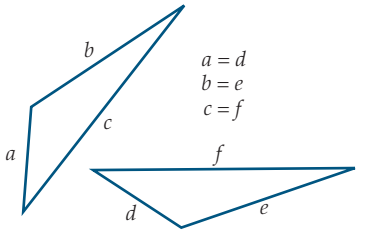
Congruent figures are identical in size and shape, but their orientation may be different.



Tests for congruent triangles

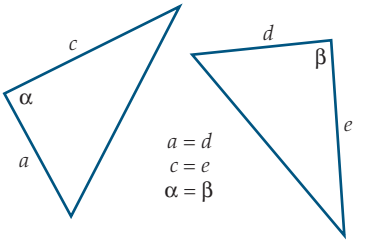
1 Side Side Side (SSS)

The three pairs of sides are equal in length.



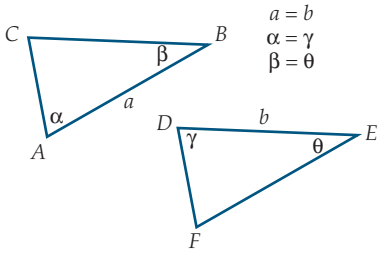
2 Side Angle Side (SAS)

Two pairs of sides are equal in length and the **included angles** are equal.



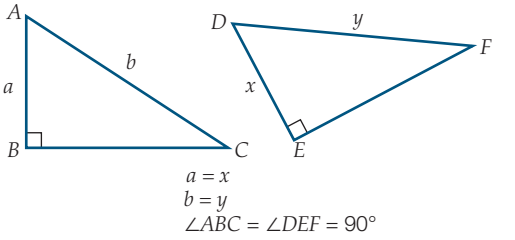
3 Angle Angle Side (AAS)

Two pairs of angles are equal and one pair of matching sides is equal in length.



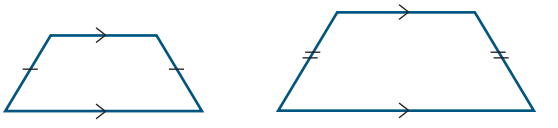
4 Right Angle Hypotenuse Side (RHS)

Both triangles contain a right angle, the hypotenuse in both triangles are equal in length and another pair of sides is equal in length.



Similarity

Similar shapes are those where all side lengths have undergone the same dilation.



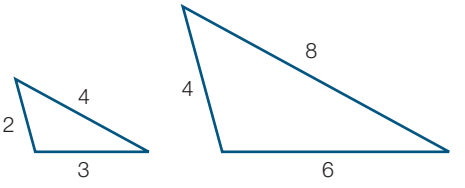
When a figure is dilated:

- matching angles are the same
- matching side lengths are in the same ratio.

Tests for similar triangles

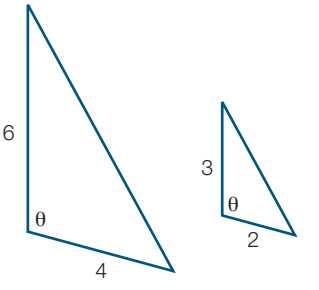
1 Side Side Side (SSS)

Three sides of one triangle are in the same proportion to the three sides of another triangle.



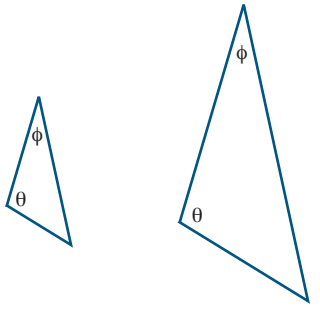
2 Side Angle Side (SAS)

Two sides of one triangle are in the same proportion to two sides of another triangle, and the included angles are equal.



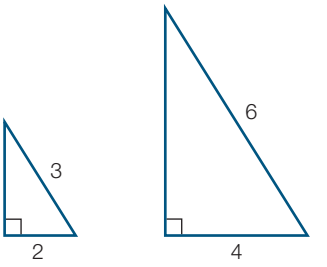
3 Angle Angle Angle (AAA)

Two angles of one triangle are equal to two angles in another triangle. All three matching angles are therefore equal.



4 Right Angle Hypotenuse Side (RHS)

The hypotenuse and one side of a right-angled triangle are in the same proportion to the hypotenuse and one side of another right-angled triangle.



Note: These tests are similar to the four tests for congruence, the difference being that where congruence requires that matching sides be the same length, similarity requires that matching sides be in the same proportion.

To find the length of an unknown side in a similar triangle:

Step 1 Find two pairs of matching sides that include the unknown side.

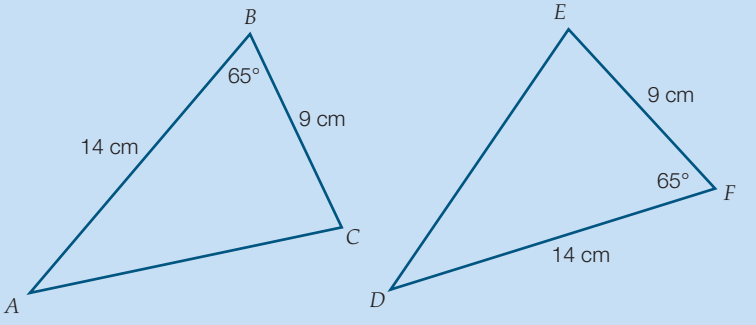
Step 2 Equate the two ratios.

Step 3 Solve the equation to find the exact value of the unknown, unless told otherwise.

Worked Example 1

WE1

Show that ΔABC and ΔDFE are congruent, stating the congruency test used.



Working

$AB = DF$ (both 14 cm)
$\angle ABC = \angle DFE$ (both 65°)
$BC = FE$ (both 9 cm)

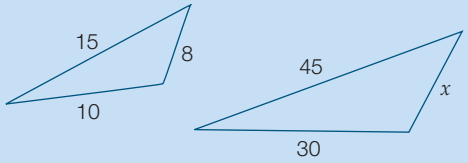
The triangles are congruent. (SAS)

Worked Example 2

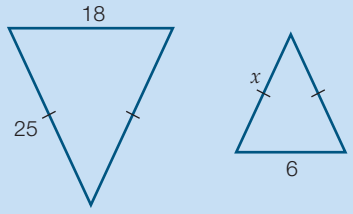
WE2

Given that these pairs of triangles are similar, find x.

(a)



(b)



Working

(a) $x:8 = 45:15$

$$\frac{x}{8} = \frac{45}{15}$$

$$x = \frac{45}{15} \times 8$$

$$x = 24 \text{ units}$$

(b) $x:25 = 6:18$

$$\frac{x}{25} = \frac{6}{18}$$

$$x = \frac{6}{18} \times 25$$

$$x = 8\frac{1}{3} \text{ units}$$

Answers page 658

8.1 Congruent and similar triangles

Stage 5.2

Navigator

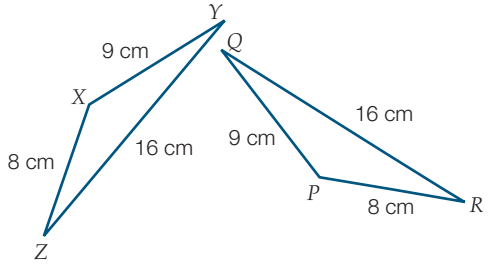
Q1 Column 1, Q2 Column 1, Q3, Q4, Q5, Q6, Q7, Q8 Column 1, Q10, Q12, Q14, Q15	Q1 Column 2, Q2 Column 2, Q3, Q4, Q5, Q6, Q7, Q8 Column 2, Q9, Q10, Q12, Q14, Q15	Q1 Column 2, Q2 Column 2, Q3, Q4, Q5, Q6, Q7, Q8 Column 2, Q9, Q10, Q11, Q12, Q13, Q15

Fluency and understanding

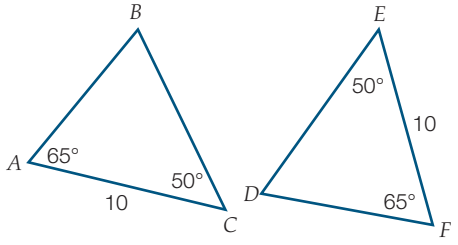
WE1

1 Show that the following pairs of triangles are congruent, stating the congruency test used.

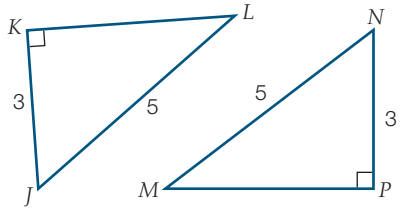
(a)



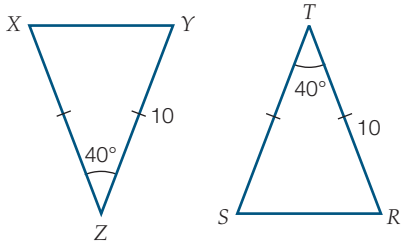
(b)



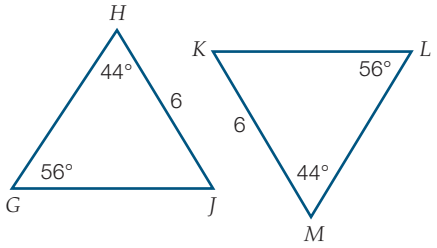
(c)



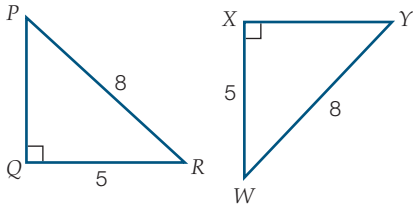
(d)



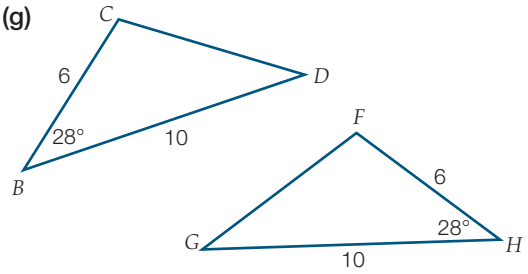
(e)



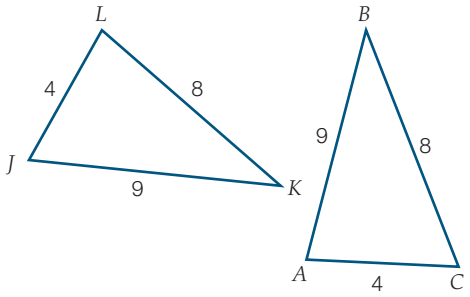
(f)



(g)



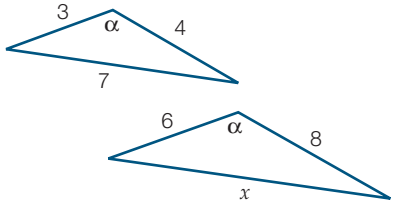
(h)



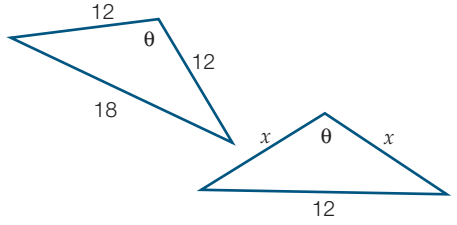
2 Given that these pairs of triangles are similar, find x.

WE2

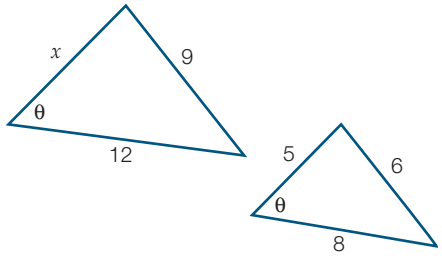
(a)



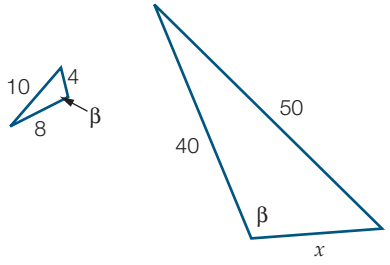
(b)



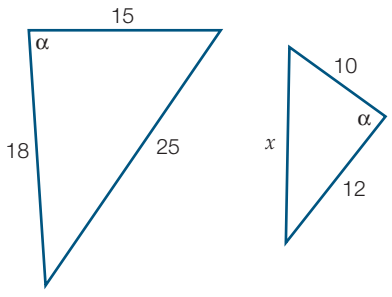
(c)



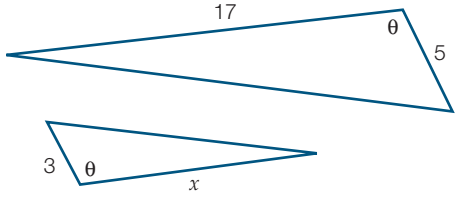
(d)



(e)

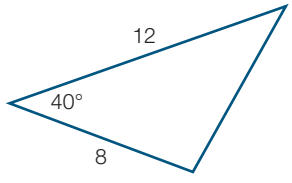


(f)

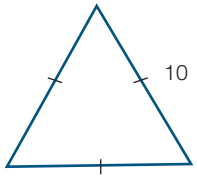


3 Match the pairs of congruent triangles given below.

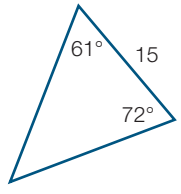
(a)



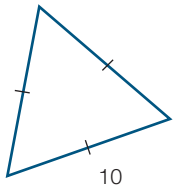
(b)



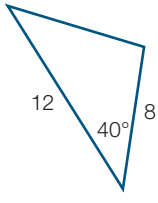
(c)



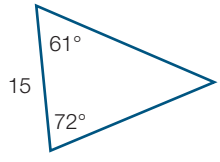
(d)



(e)



(f)

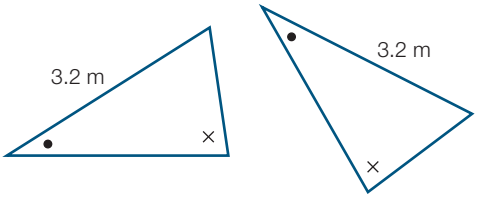


4 Which one of the following conditions does *not* ensure congruence for two triangles?

A SSS **B** SAS **C** AAS **D** AAA

5 Which condition verifies that these two triangles are congruent?

A SSS **B** SAS
C AAS **D** RHS



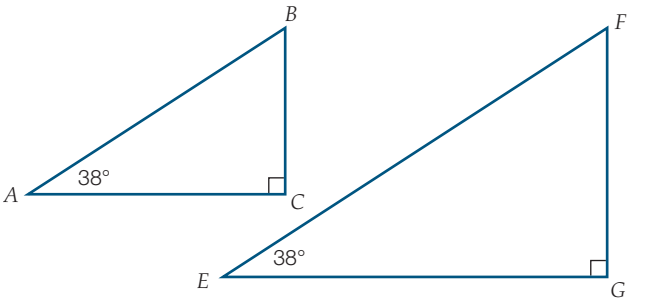
6 Given $\Delta ABC \parallel\!| \Delta EFG$:

(a) the matching side to AB is:

A FG **B** EF
C BC **D** AC

(b) the matching side to FG is:

A BC **B** AC
C EF **D** EG



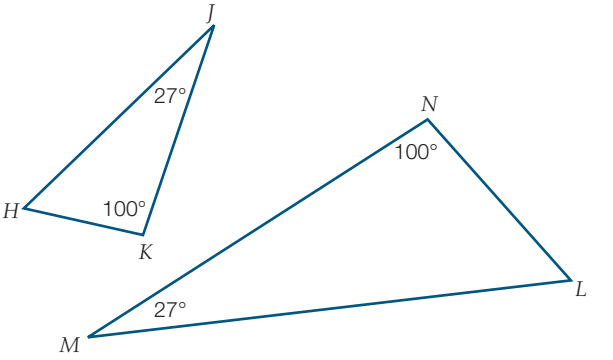
7 Given $\Delta HJK \parallel\!| \Delta LMN$:

(a) the matching side to LM is:

A HJ **B** MN
C JK **D** LN

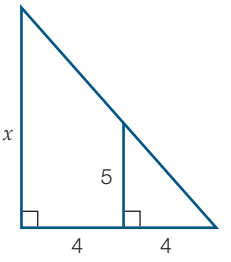
(b) the matching side to LN is:

A JK **B** LM
C HK **D** NL

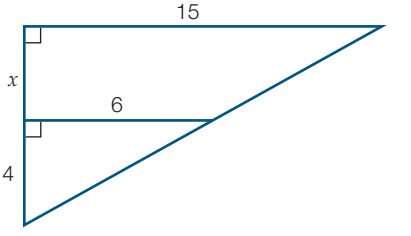


8 In the following diagrams, first determine the similar triangles and then find the value of x.

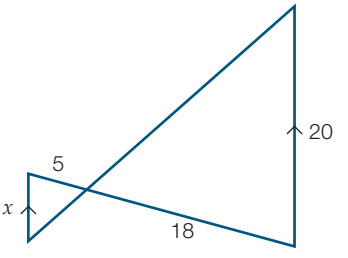
(a)



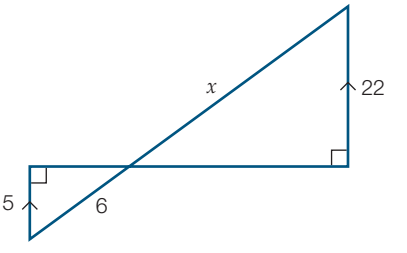
(b)



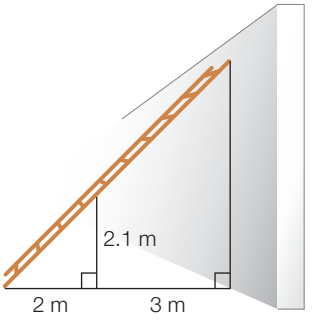
(c)



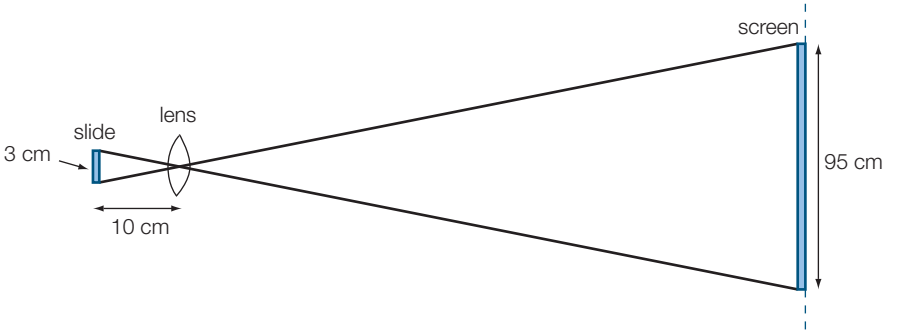
(d)



9 Dirk is climbing up a long ladder leaning against a wall. His friend Sean is holding up the ladder to prevent it wobbling. Sean is 3 m from the wall and 2 m from the base of the ladder. With his arms stretched above his head he holds the ladder at a point 2.1 m above the ground. How far up the wall does the ladder reach?



10 A light source projects the image of a photographic slide onto a screen, as shown in the diagram. The slide is 3 cm high and positioned 10 cm behind a lens. Calculate the distance between the lens and the screen if the image on the screen is 95 cm high. Give your answer correct to two decimal places.



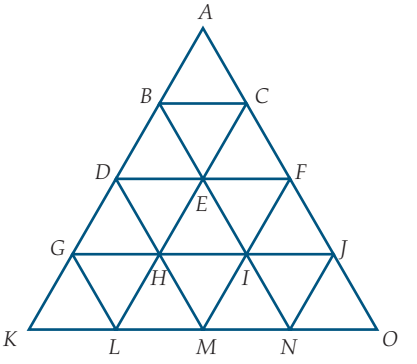
11 Ziva plays tennis at night under floodlights. When she stands 4.5 m from the base of a floodlight, she casts a shadow 1.4 m long. If Ziva is 1.7 m tall, how high is the floodlight?

Reasoning, communicating and open-ended

12 The small triangles in the following diagram are all equilateral.

(a) Find six triangles congruent to ΔDFM.

(b) Find two pairs of congruent triangles larger than ΔDFM.



13 The Akela (leader) of the local scout group was explaining to the scouts how to use a stick to estimate the height of a tree.

Step 1: Have someone hold the stick vertically on the ground.

Step 2: Lie on the ground in a position so that when looking at the tree through the stick the top of the tree and the top of the stick are in line.

Step 3: Unfortunately, Akela has forgotten Step 3.

(a) Draw a diagram of the situation and using your knowledge of similar triangles complete the instructions Akela needs to give.

(b) List any practical problems you can see with this method.

14 Explain why two triangles with three equal angles may not necessarily be congruent. Show this by drawing diagrams.

15 $\Delta ABC \parallel\!| \Delta DEF$. The dilation factor is $\frac{3}{2}$. Draw a diagram to show both triangles and write possible values for the sides of both triangles if ΔABC is the smaller triangle and BC is equal to 4 cm.

8.2 Proving congruence and similarity

Stage 5.2

Need to Know

Proving congruence

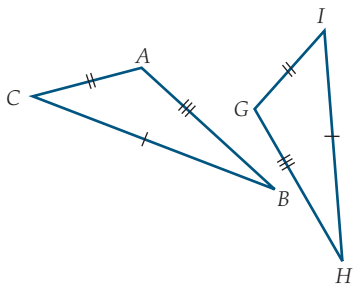
It is often necessary to **prove** that two triangles are congruent by using one of the four tests for congruence: SSS, AAS, SAS and RHS. If triangles are congruent, matching sides and angles will be equal.

When showing that two shapes are congruent, write the matching vertices in the same order.

For example, in the congruent triangles shown, the matching vertices are A and G, B and H, and C and I.

This congruency can be shown by writing $\Delta ABC \equiv \Delta GHI$.

This allows easy identification of matching sides and angles because they are in the same position in the statement.



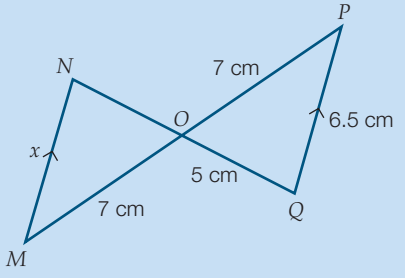
Proving similarity

If two triangles can be proved to be similar using the tests for similarity, lengths of unknown sides and the size of unknown angles can be found.

Worked Example 3

WE 3

Prove that the triangles ΔMNO and ΔPQO are congruent, and hence find the value of the given variable.



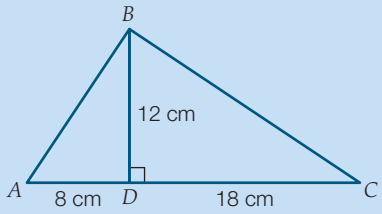
Working

$\angle MNO = \angle PQO$	(alternate angles, $MN \parallel PQ$)
$\angle MON = \angle POQ$	(vertically opposite angles)
$MO = PO$	(given)
$\Delta MNO \equiv \Delta PQO$	(AAS)
$MN = PQ$	(matching sides in congruent triangles)
$x = 6.5$ cm	

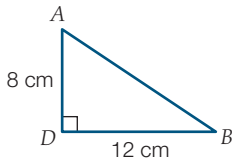
Worked Example 4

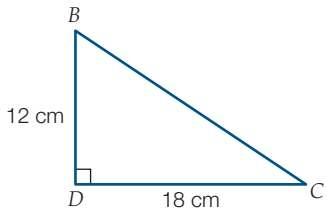
WE4

Prove that the triangles ΔABD and ΔBCD are similar.



Working





$$\frac{BD}{CD} = \frac{12}{18} = \frac{2}{3}$$

$$\frac{AD}{BD} = \frac{8}{12} = \frac{2}{3}$$

BD and CD are in the ratio $2:3$.
AD and BD are in the ratio $2:3$.

$\angle ADB = \angle BDC = 90°$

$\Delta ABD \parallel\!| \Delta BCD$ because two sides of one triangle are in the same ratio as two matching sides of the other triangle and the included angles are equal (SAS).

Answers page 659

8.2 Proving congruence and similarity

Stage 5.2

Navigator

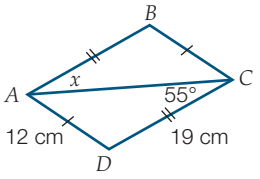
Q1, Q2, Q3, Q4, Q5 (a), Q6 (a), Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7 (a), Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

Fluency and understanding

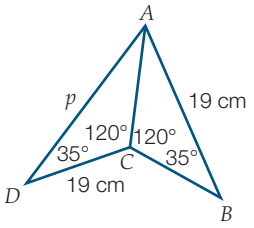
WE3

1 Prove that the following pairs of triangles are congruent and hence find the value of the given variable.

(a)



(b)



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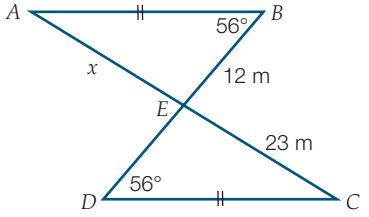

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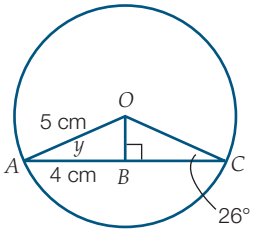

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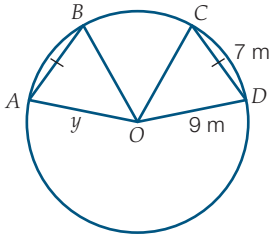

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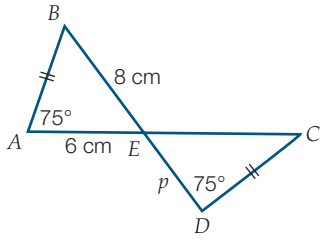
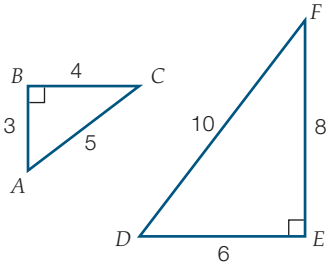


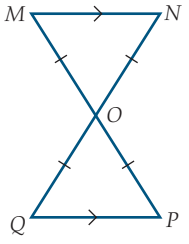
WE4

2 Prove that the following pairs of triangles are similar.

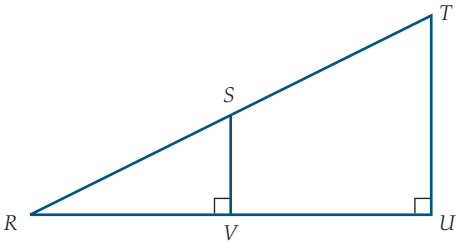
(a)



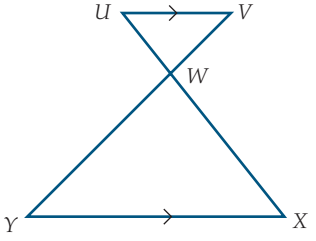
(b)



(c)



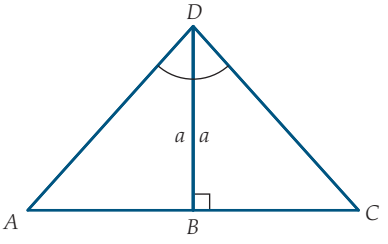
(d)



3 DB is the perpendicular bisector of AC and bisects $\angle ADC$. Which test for congruence can be used to prove that $\Delta ABD \equiv \Delta CBD$?

A SSS **B** SAS

C AAS **D** RHS



4 Which of the following is not enough to determine similarity?

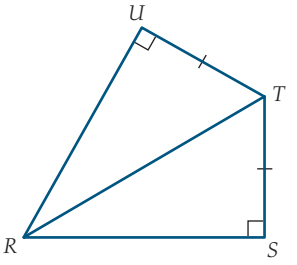
A Two angles of one triangle are equal to two angles of the other triangle.

B Two sides of one triangle are proportional to two sides of another triangle.

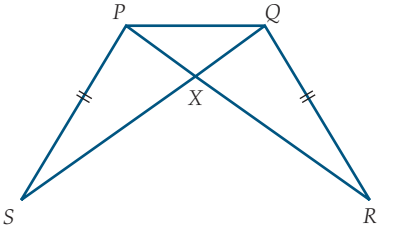
C The hypotenuse and second side of a right-angled triangle are proportional to the hypotenuse and second side of another right-angled triangle.

D Two sides of one triangle are proportional to two sides of another triangle and the included angles are equal.

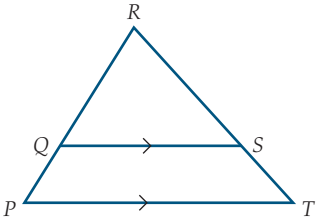
5 **(a)** Prove that $\Delta RUT \equiv \Delta RST$



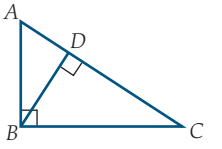
(b) $SP = RQ$ and $\angle SPQ = \angle RQP$.
Prove that $\Delta SPQ \equiv \Delta RQP$.



6 **(a)** Prove that $\Delta RQS \parallel\!| \Delta RPT$.



(b) Prove that $\Delta ABC \parallel\!| \Delta BDC \parallel\!| \Delta ADB$.



Reasoning, communicating and open-ended

7 **(a)** Explain why or why not a pair of shapes is similar if they are any two:

(i) rectangles **(ii)** squares

(iii) isosceles triangles **(iv)** equilateral triangles

(v) isosceles right-angled triangles **(vi)** circles.

(b) For each of the similar shapes in part **(a)**, state the conditions necessary for the pairs of shapes to be congruent.

8 Draw four pairs of congruent triangles and label them with enough information to show congruence. Each pair should represent a different test. Write the name of the congruence test that applies to each pair.

8.3 Proofs using congruent triangles

Stage 5.3§

Need to Know

Definitions and properties

A **definition** is the minimum amount of information needed to identify a particular figure.

A proven geometric relationship in geometry is known as a **theorem**.

A **property** is a theorem about a particular figure that has been proved from the definition.

Using one of the four tests for congruence: SSS, AAS, SAS and RHS, congruent triangles can be used to prove properties for isosceles and equilateral triangles and special quadrilaterals.

Definitions of triangles

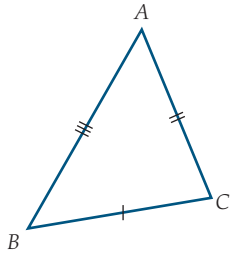
1. A scalene triangle is a triangle with no two sides equal in length.
2. An isosceles triangle is a triangle with only two sides equal in length.
3. An equilateral triangle is a triangle with all sides equal in length.

Properties of triangles

Scalene triangle

A scalene triangle has no particular property as all angles are of different magnitude.

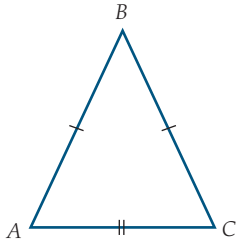
$\angle BAC \neq \angle ABC \neq \angle BCA$



Isosceles triangle

An isosceles triangle has two angles of equal magnitude.

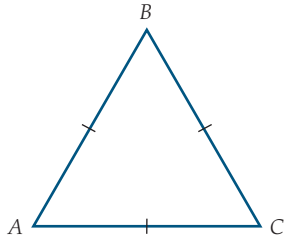
$\angle BAC = \angle BCA \neq \angle ABC$



Equilateral triangle

An equilateral triangle has all angles of equal magnitude of 60°.

$\angle BAC = \angle ABC = \angle BCA = 60°$



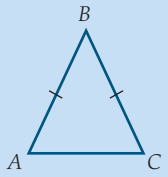
Worked Example 5

WE 5

In ΔABC, $BA = BC$.

(a) Prove that $\angle BAC = \angle BCA$.

(b) What property of isosceles triangles have you just proved?

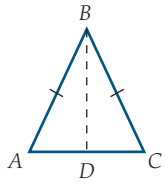


Working

(a) Line BD meets the midpoint of AC.

Consider ΔABD and ΔCBD.

$BA = BC$	(given)
$BD = BD$	(common side)
$AD = CD$	(by construction)
$\therefore \Delta ABD \equiv \Delta CBD$	(SSS)
$\angle BAD = \angle BCD$	(matching angles in congruent triangles)
$\therefore \angle BAC = \angle BCA$	



(b) The result proved that angles opposite equal sides are equal in isosceles triangles.

Hint

Construct a line from B to the midpoint of AC.

8.3 Proofs using congruent triangles

Answers page 660

Navigator

Stage 5.3§

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q13, Q14, Q15, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q13, Q14, Q15, Q16	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q15, Q16

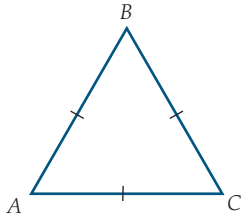
Fluency and understanding

WE 5

1 In ΔABC, $AB = BC = CA$

(a) Prove that $\angle ABC = \angle BCA = \angle CAB$.

(b) What property of an equilateral triangle have you just proved?



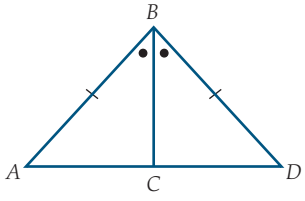
Hint

Construct a line from B to the midpoint of AC and a line from C to the midpoint of AB.

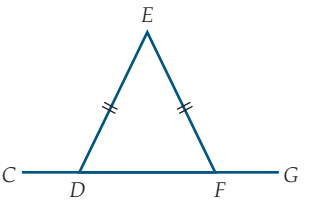
2 Use the triangles shown to prove the following properties of isosceles triangles.

(a) The bisector of the angle between the two equal sides of an isosceles triangle bisects the third side.

(b) The bisector of the angle between the two equal sides of an isosceles triangle is perpendicular to the third side.

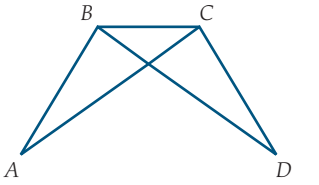


3 ΔDEF is isosceles. Prove that $\angle CDE = \angle GFE$.



4 The diagram shows two triangles, ΔABC and ΔDCB, with $AB = DC$ and $\angle ABC = \angle DCB$.

Which of the following tests can be used to prove that $\Delta ABC \equiv \Delta DCB$?

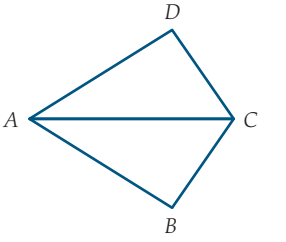


A SAS **B** AAS **C** SSS

D There is insufficient information to determine whether the triangles are congruent.

5 In the diagram opposite, AC bisects $\angle BAD$ and $\angle BCD$.

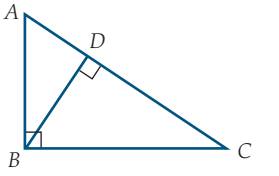
Which of the following tests can be used to prove that $\Delta ABC \equiv \Delta ADC$?



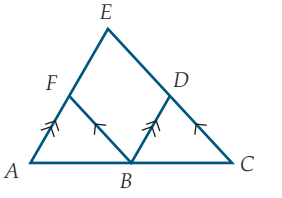
A SAS **B** AAS **C** SSS

D There is insufficient information to determine whether the triangles are congruent.

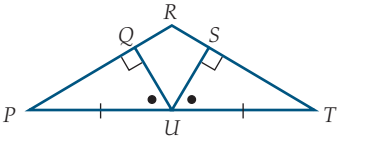
6 For the triangle shown, show that $AC^2 = DC^2 + 2BD^2 + AD^2$.



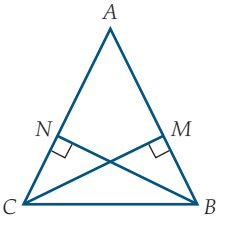
7 B is the midpoint of AC. Show that $\Delta ABF \equiv \Delta BCD$.



8 Prove that ΔPRT is isosceles.

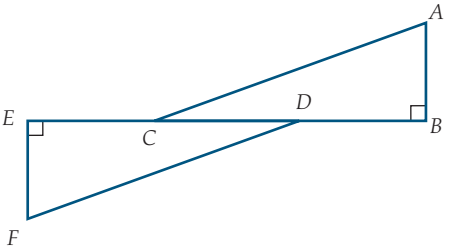


9 In ΔABC, $AB = AC$. Prove that $AM = AN$.

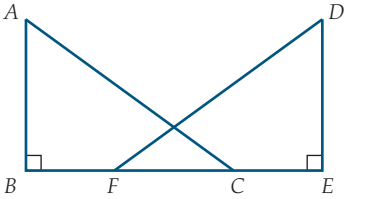


Reasoning, communicating and open-ended

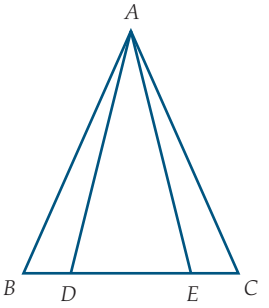
10 If $\angle BAC = \angle DFE$, $\angle ABC = \angle DEF$ and $BD = EC$, prove that $\Delta ABC \equiv \Delta FED$.



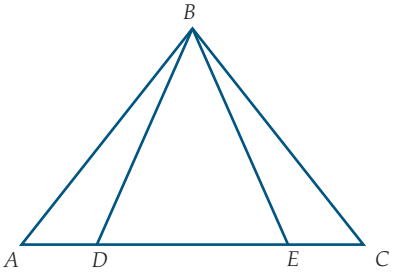
11 If $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BF = EC$, prove that $\Delta ABC \equiv \Delta DEF$.



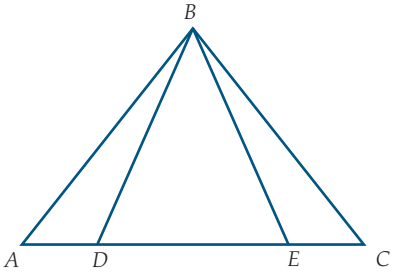
12 **(a)** In the diagram, $\angle ABD = \angle ACE$. Also, $AB = AC$ and $BD = CE$. Prove that $\Delta ABE \equiv \Delta ACD$.



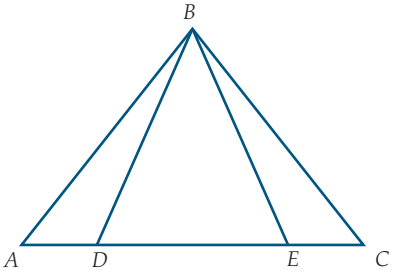
(b) Given $\angle BAD = \angle BCE$ and $\angle ABD = \angle CBE$, prove that $\Delta ABE \equiv \Delta CBD$.



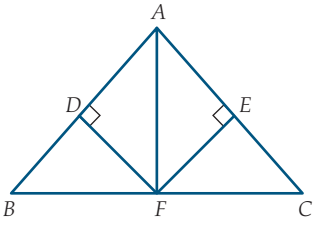
(c) Given $AE = CD$ and $\angle BAD = \angle BCE$, prove that $\Delta ABD \equiv \Delta CBE$.



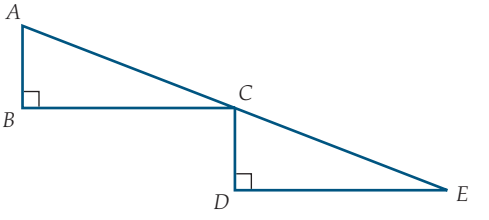
(d) Given $AB = CB$ and $AE = CD$, prove that $\Delta ABD \equiv \Delta CBE$.



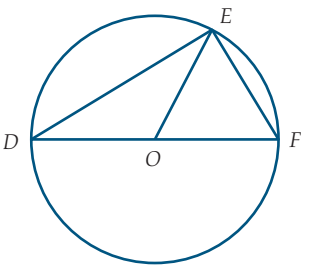
13 In the diagram, $FD = FE$, $FD \perp AB$ and $FE \perp AC$. Prove that $\Delta ADF \equiv \Delta AEF$.



14 ΔABC and ΔCDE are right-angled triangles and $AB = CD$. If C is the midpoint of AE, prove that $\Delta ABC \equiv \Delta CDE$.



15 When you draw a line from each end of a diameter to a point on the circumference of a circle, the angle formed is called an angle in a semicircle. Prove, using isosceles triangles, that the angle in a semicircle is a right angle.



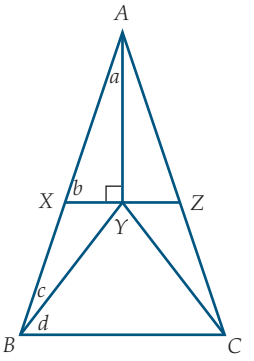
16 ABC is an isosceles triangle with $AB = AC$. X is a point on AB and Z is a point on AC such that XZ is parallel to BC. Y is the midpoint of XZ and $AY = YC$. Let $\angle BAY = a$.

(a) Prove $AX = AZ$.

(b) Prove $\Delta AXY \equiv \Delta AZY$.

(c) Prove $\Delta BXY \equiv \Delta CZY$.

(d) Find b, c and d in terms of a.

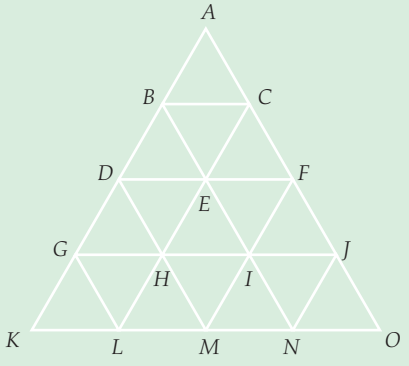


17 Draw a diagram of your own that contains two congruent triangles. Prove that they are congruent using one of the congruency tests.

Problem solving

How many triangles?

How many triangles can you find in this diagram?



Strategy options

- Solve a simpler problem.
- Have I seen a similar problem?
- Draw a diagram.
- Look for a pattern.

Proofs using similar triangles

8.4

Need to Know

Stage 5.3§

The tests for similar and congruent triangles can be used to prove general results known as **theorems** about triangles and quadrilaterals. Once these theorems have been proved, they are known as properties of triangles and quadrilaterals, and may be used to prove other relationships in these and other geometric figures, and to solve geometric problems.

Procedure for proving theorems

Theorems are stated in general terms. For instance, this means that a theorem proved for one isosceles triangle applies to all isosceles triangles.

To prove a theorem:

1. Use a general example that contains some given information.
2. Draw a diagram to display the given information.
3. Make a statement of the relationship to be proved.
4. Use congruence or similarity in the proof.
5. Generalise the proof as a property of the figure.

This generalisation becomes the theorem, which can then be used for other geometric proofs.

Similarity is used to prove the three theorems stated below.

The midpoint theorem

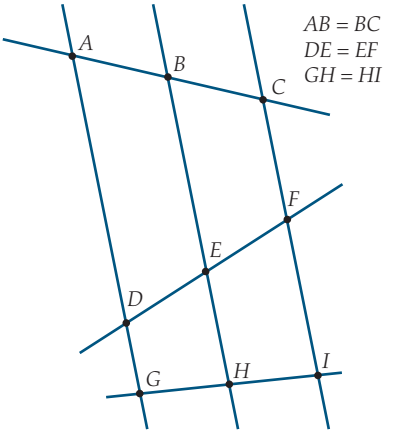
The interval joining the midpoints of two sides of a triangle is parallel to the third side and is half its length.

The converse to the midpoint theorem (the inverse theorem)

A line drawn through the midpoint of one side of a triangle and parallel to a second side **bisects** the third side.

The intercept theorem

When a set of parallel lines makes equal intercepts on one transversal, it makes equal intercepts on any other transversal.



Worked Example 6

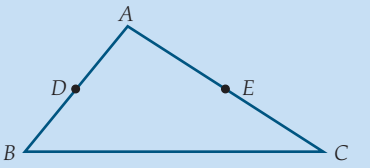
WE 6

In ΔABC, D is the midpoint of AB and E is the midpoint of AC.

Draw DE and prove that:

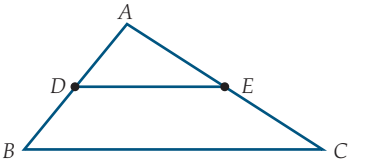
(a) DE is parallel to BC.

(b) $DE = \frac{1}{2}BC$.



Working

(a) Consider ΔADE and ΔABC.



$\angle DAE = \angle BAC$ (common angles)

$AD = DB$ (D is the midpoint of AB)

$\therefore AB = 2AD$

$AE = EC$ (E is the midpoint of AC)

$\therefore AC = 2AE$

$\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$

$\therefore \Delta ADE \,|||\, \Delta ABC$ (two sides are proportional and their included angles are equal)

$\angle ADE = \angle ABC$ (matching angles of similar triangles)

$\angle AED = \angle ACB$ (matching angles of similar triangles)

$\angle ADE$ and $\angle ABC$ are a pair of equal matching angles formed when the transversal AB cuts the lines DE and BC.

$\therefore DE \parallel BC$ (Corresponding angles are equal so the lines are parallel.)

$\therefore$ The line joining the midpoints of two sides of a triangle (DE) is parallel to the third side (BC).

(b) As the triangles are similar, the matching sides are proportional.

$\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \frac{1}{2}$

$\therefore \frac{DE}{BC} = \frac{1}{2}$

$\therefore DE = \frac{1}{2}BC$

$\therefore$ The line joining the midpoints of two sides of a triangle (DE) is half the length of the third side (BC).

This proves the midpoint theorem.

8.4 Proofs using similar triangles

Answers page 662

Navigator

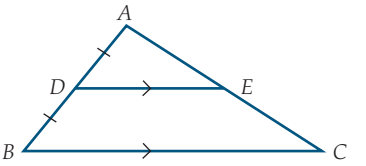
Stage 5.3§

Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12

Fluency and understanding

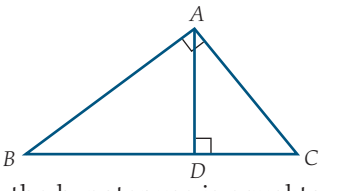
WE 6

1 The line through the midpoint of a side of a triangle parallel to another side bisects the third side. In triangle ΔABC, D is the midpoint of AB and DE is parallel to BC, meeting AC at E. Prove that:



(a) $\Delta ABC \,|||\, \Delta ADE$ **(b)** $AE = EC$.

2 In a right-angled triangle, if a perpendicular line is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to each another. Triangle ΔABC is right-angled at A and $AD \perp BC$. Prove that triangles ΔADB, ΔDCA and ΔCAB are similar.



3 In a right-angled triangle, the square on the perpendicular to the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse. Using the diagram for Question **2**, prove that $AD^2 = BD \times DC$.

4 In a right-angled triangle, the square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side. Using the diagram for Question **2**, prove that $BA^2 = BD \times BC$ and $CA^2 = CD \times CB$.

5 Use the result from Question **4** to prove Pythagoras' Theorem: $BC^2 = BA^2 + CA^2$

6 Which of the following is not sufficient to prove that two triangles are similar?

A The two triangles have two angles the same.

B The three sides of one triangle are proportional to the three sides of the other triangle.

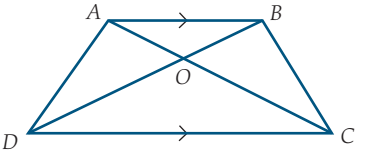
C Two sides of one triangle are proportional to two sides of another triangle.

D Two sides of one triangle are proportional to two sides of another triangle and the included angles are equal.

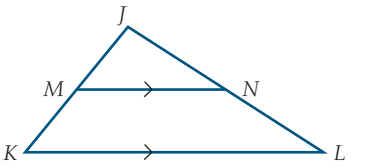
7 In the trapezium $ABCD$, AB is parallel to DC and the diagonals intersect at O.

(a) Show that $OA:OC = OB:OD$.

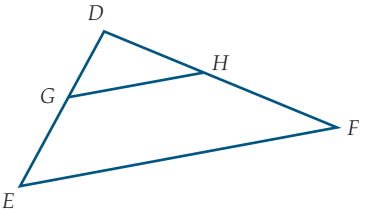
(b) If $DC = 2AB$, show that O is a point of trisection of both diagonals. (That is, show that $AO:OC = BO:OD = 1:2$ and hence that $AO:AC = BO:BD = 1:3$.)



8 A straight line drawn parallel to one side of a triangle cuts the other two sides proportionally. In triangle ΔJKL, M is a point on JK and MN is parallel to KL, meeting JL at N. Prove that $\frac{JM}{MK} = \frac{JN}{NL}$.

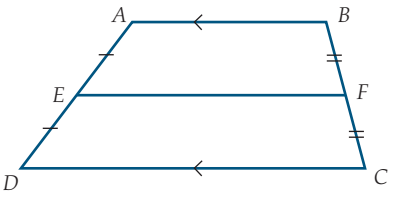


9 Conversely, if a line cuts two sides of a triangle proportionally, it is parallel to the third side. In ΔDEF, $\frac{DG}{GE} = \frac{DH}{HF}$. Prove that $GH \parallel EF$.



Reasoning, communicating and open-ended

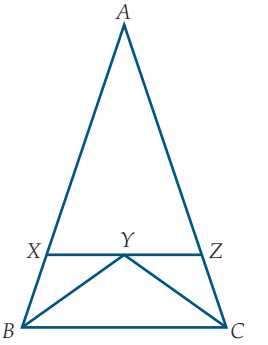
10 The straight line that joins the midpoints of the slanted sides of a trapezium is parallel to the parallel sides. $ABCD$ is a trapezium in which $AB \parallel DC$. E is the midpoint of AD and F is the midpoint of BC. Prove that $EF \parallel AB$.



Hint

Extend DA and CB and let them intersect at G.

11 ΔABC is an isosceles triangle such that $AB = AC$. X is a point on AB and Z is a point on AC such that XZ is parallel to BC. Y is the midpoint of XZ and $BX = XY$. Prove that $\Delta BXY \parallel\!| \Delta BYC$.



Hint

Prove $\Delta BXY \equiv \Delta CZY$ and let them intersect at G.

12 Draw a figure different from any given in this exercise that contains at least two similar triangles. Prove that your triangles are similar.

Geometric properties of special quadrilaterals

8.5

Stage 5.3§

Need to Know

Perpendicular lines

- Two lines are perpendicular if they intersect at right angles.
- A **perpendicular bisector** bisects another line (cuts it in half) at right angles.

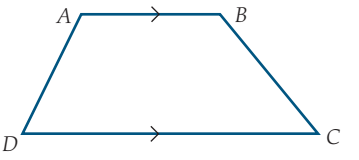
Definitions of special quadrilaterals

1. A trapezium is a quadrilateral with one pair of opposite sides parallel.
2. A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
3. A kite is a quadrilateral with two pairs of adjacent sides equal in length.
4. A rectangle is a quadrilateral in which all angles are right angles.
5. A rhombus is a quadrilateral with all sides equal in length.
6. A square is a rectangle with two adjacent sides equal in length or a rhombus with all angles equal to 90°.

Properties of special quadrilaterals

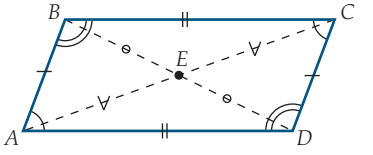
Trapezium

- A trapezium has no particular properties.



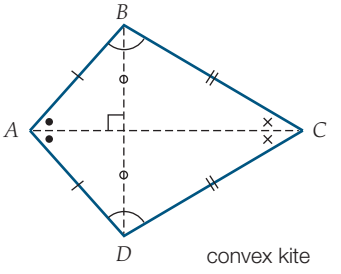
Parallelogram

- Both pairs of opposite angles are equal in magnitude: $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$.
- Both pairs of opposite sides are equal: $AB = DC$ and $AD = BC$.
- Diagonals bisect each other: $AE = EC$ and $BE = ED$.

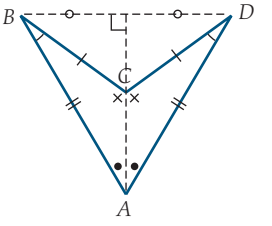


Kite

- One pair of opposite angles are equal in magnitude: $\angle ABC = \angle ADC$.
- A kite has an axis of symmetry that bisects the non-equal vertex angles through which it passes.
- The axis of symmetry of the kite bisects the other diagonal at right angles.



convex kite

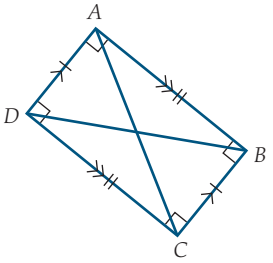


concave kite

Rectangle

All of the properties of a parallelogram, plus:

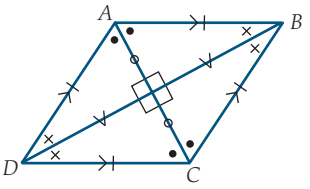
- Diagonals are equal in length: $AC = BD$.
- The lines joining the midpoints of opposite sides are lines of symmetry.



Rhombus

All of the properties of a parallelogram, plus:

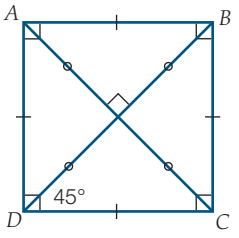
- Diagonals bisect each other at right angles: $AC \perp BD$.
- Diagonals bisect the vertex angles: $\angle DAC = \angle BAC = \angle DCA = \angle BCA$, $\angle ADB = \angle CDB = \angle ABD = \angle CBD$.
- Both diagonals are lines of symmetry.



Square

All of the properties of a rectangle and a rhombus, plus:

- Diagonals make an angle of 45° with each side.
- Each diagonal has length equal to $\sqrt{2}$ times the side length.

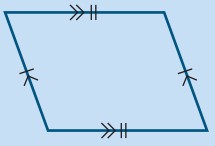


Worked Example 7

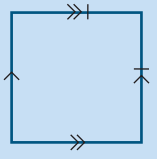
WE7

For each of the following shapes, write down the information shown, then use the definitions for quadrilaterals to determine the type of shape.

(a)



(b)



Working

(a) Both pairs of opposite sides are equal and parallel.

The shape is a parallelogram because both pairs of opposite sides are parallel.

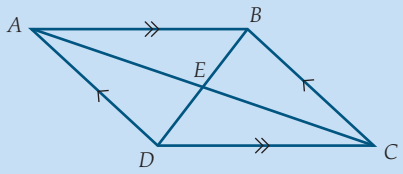
(b) Opposite sides parallel means that the shape is a parallelogram.

Two adjacent sides equal in length means that the parallelogram is a rhombus.

Worked Example 8

WE 8

ABCD is a parallelogram.



(a) By drawing the diagonal *AC*, prove that $\Delta ADC \equiv \Delta CBA$.

(b) Use these triangles to prove that $\angle ADC = \angle ABC$ and $\angle DAB = \angle BCD$.

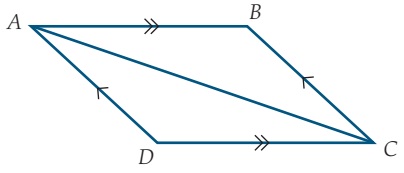
(c) What special property of a parallelogram does this prove?

(d) By now drawing the diagonal *BD*, prove that $\Delta CDE \equiv \Delta ABE$.

(e) Use these triangles to prove that $AE = CE$ and $DE = BE$.

(f) What special property of a parallelogram does this prove?

Working



(a) Consider ΔADC and ΔCBA.

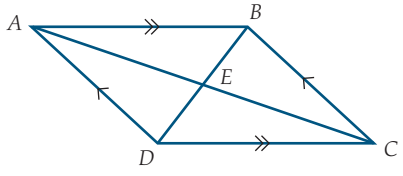
$\angle DCA = \angle BAC$ (alternate angles, $AB \parallel DC$)
$\angle DAC = \angle BCA$ (alternate angles, $AD \parallel BC$)
AC is a common side

$\Delta ADC \equiv \Delta CBA$ (AAS)

(b) $\angle ADC = \angle ABC$ (matching angles in congruent triangle)

$\angle DAB = \angle DAC + \angle BAC$
$\angle BCD = \angle BCA + \angle DCA$
But $\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$ (from part **(a)**)

(c) The property proved is that the opposite angles of a parallelogram are equal.



(d) Consider ΔABE and ΔCDE.

$AB = CD$ ($\Delta ADC \equiv \Delta CBA$)
$\angle DCE = \angle BAE$ (alternate angles, $AB \parallel DC$)
$\angle CDE = \angle ABE$ (alternate angles, $AB \parallel DC$)
$\angle ABE = \angle CDE$ (AAS)

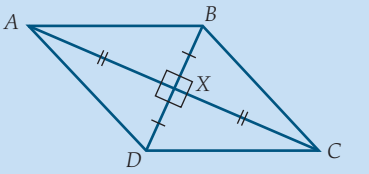
(e) $AE = CE$ and $DE = BE$ (matching sides in congruent triangles)

(f) The property proved is that the diagonals of a parallelogram bisect each other.

Worked Example 9

WE9

The diagonals of a quadrilateral bisect each other at right angles. Prove that this quadrilateral is a rhombus.



Working

Consider ΔAXD and ΔAXB.

$\angle AXD = \angle AXB = 90°$	(given)
AX is a common side	
$DX = BX$	(given)
$\Delta AXD \equiv \Delta AXB$	(SAS)
$AD = AB$	(matching sides in congruent triangles)

Consider ΔAXD and ΔCXB.

$\angle AXD = \angle CXB = 90°$	(given)
$AX = CX$	(given)
$DX = BX$	(given)
$\Delta AXD \equiv \Delta CXB$	(SAS)
$\angle DAX = \angle BCX$	(matching angles in congruent triangles)
$AD \parallel BC$	(alternate angles $\angle DAX$ and $\angle BCX$ are equal)

Similarly, $AB \parallel CD$.

$AB \parallel CD$ and $AD \parallel BC$	(opposite sides parallel)
$AD = AB$	(two adjacent sides equal)

So the shape is a rhombus as two adjacent sides are equal in length and the opposite sides are parallel.

Answers page 664

8.5 Geometric properties of special quadrilaterals

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11, Q12, Q13, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q19	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

Fluency and understanding

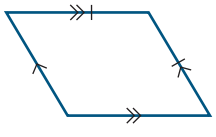
WE7

1 For each of the following shapes:

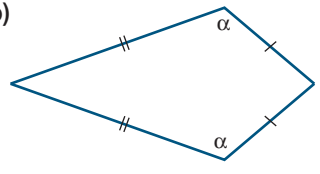
(i) write down the information shown

(ii) use the definitions for quadrilaterals to determine the type of shape.

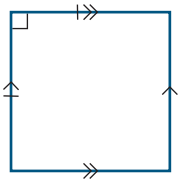
(a)



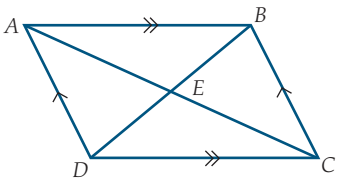
(b)



(c)



2 *ABCD* is a parallelogram. Use the diagram shown to prove the following properties of parallelograms. WE 8



(a) By drawing the diagonal *BD*, prove that $\Delta DAB \equiv \Delta BCD$.

(b) Use these triangles to prove that $\angle DAB = \angle DCB$ and $\angle ADC = \angle ABC$.

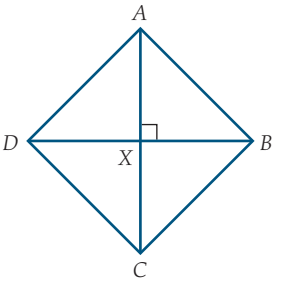
(c) What special property of a parallelogram does this prove?

(d) By now drawing the diagonal *AC*, prove that $\Delta DAE \equiv \Delta BCE$.

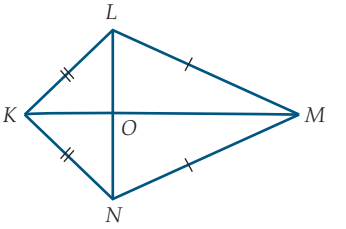
(e) Use these triangles to prove that $AE = CE$ and $DE = BE$.

(f) What special property of a parallelogram does this prove?

3 The diagonals of a quadrilateral are equal in length and bisect each other at right angles. Prove that this quadrilateral is a square. WE 9



4 Use the diagram shown to prove the following properties of a kite.



(a) A kite has one pair of opposite equal angles.

(b) A kite has diagonals that cross at right angles.

5 When proving that a quadrilateral is a trapezium but not a parallelogram, it is necessary to show:

A only one pair of parallel sides

B one pair of parallel sides and one pair of non-parallel sides

C one pair of parallel sides and one pair of equal sides

D two pairs of parallel sides.

6 A quadrilateral must be a parallelogram if one pair of opposite sides is:

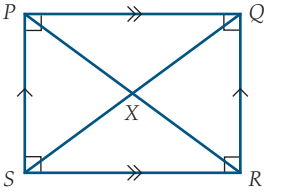
A only equal in length

B only parallel

C equal in length and parallel

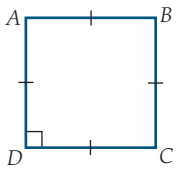
D parallel and the other pair of opposite sides is equal in length.

7 (a) For the diagram shown, prove $PQ = RS$.



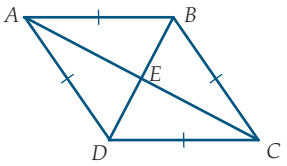
(b) What special property of a rectangle have you just proved?

8 Prove the following using the diagram shown:



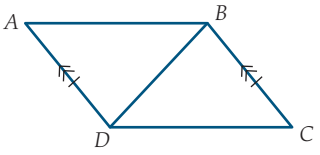
(a) If a quadrilateral has all sides equal and one angle is a right angle, then it is a square.

(b) If all sides of a quadrilateral are equal, then the quadrilateral is a rhombus.

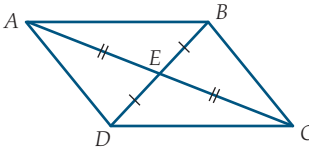


9 Prove the following using the diagrams.

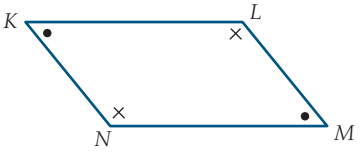
(a) If a pair of opposite sides in a quadrilateral is parallel and equal, then the quadrilateral is a parallelogram.



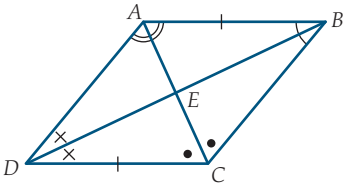
(b) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



(c) If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.



(d) If both diagonals of a quadrilateral bisect the vertex angles through which they pass, then the quadrilateral is a rhombus.



10 To prove that a parallelogram must be a rectangle it is necessary to prove that its diagonals:

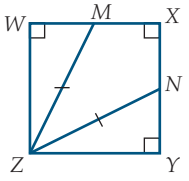
A are equal in length

B bisect each other

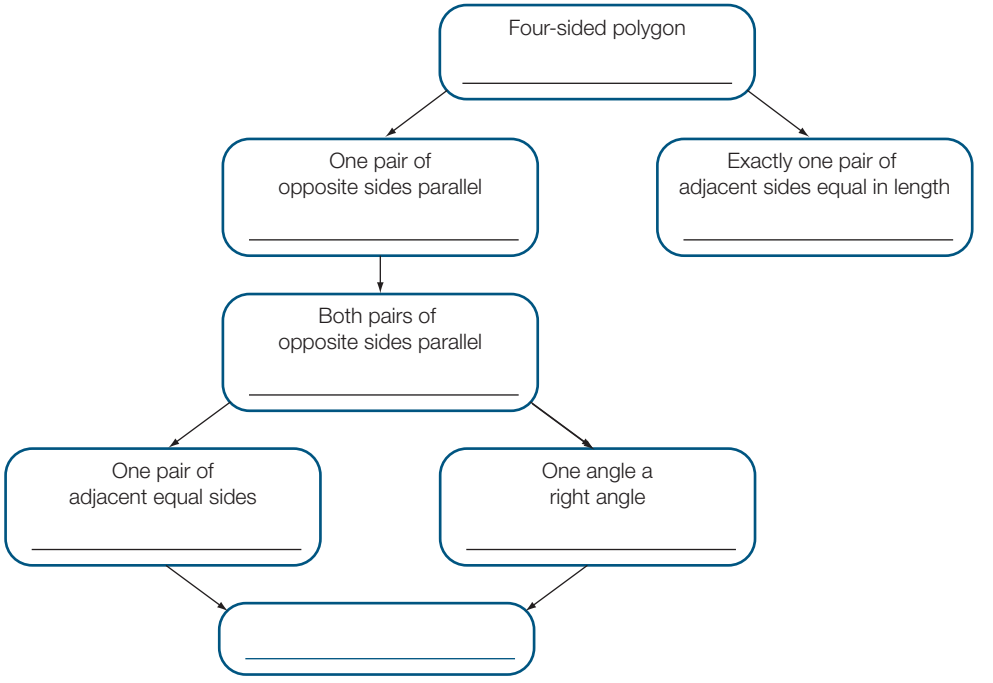
C bisect the angles of the parallelogram

D are perpendicular to each other.

11 $WXYZ$ is a square. Show that $\Delta WMZ \equiv \Delta YNZ$.

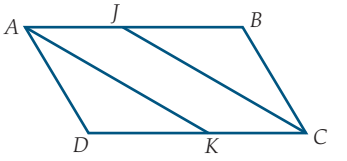


12 Copy and complete the following flowchart to show the relationships between different quadrilaterals.



Reasoning, communicating and open-ended

13 $ABCD$ is a parallelogram and $AJ = CK$.
Show that $\Delta AKD \equiv \Delta CJB$.



14 Explain why a square is a rhombus, but a rhombus is not necessarily a square.

15 Explain why a square is a rectangle, but a rectangle is not necessarily a square.

16 Adam and Alvin were discussing how many different quadrilateral shapes they could use to make biscuits. Choose three different quadrilateral shapes that they could use. The shapes need to have one of the following properties in common:

- at least one pair of parallel sides
- at least one pair of equal sides
- at least one pair of equal angles.

Draw the shapes and name them, and then identify the property you have used.

17 **(a)** The statement 'All squares have equal sides' is true. The converse statement is 'All shapes with four equal sides are squares'. Is this true? Explain why or why not.

(b) Write two or more true statements about special quadrilaterals and their converse statements. Examine whether or not the converse statements are true.

18 **(a)** Draw any three rhombuses and mark in the diagonals.

(b) Measure the angles at the centre where the diagonals intersect.

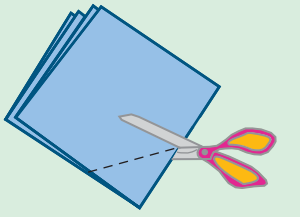
(c) What property of rhombuses does this seem to show?

(d) Is this sufficient proof of this property?

19 Gary, Matt and George are arguing about the geometric figure that could have the property that the diagonals bisect at right angles. Matt says there is only one such figure, Gary says there are two and George says there are three or more. Who is right and what shape could the figure(s) be?

Problem solving

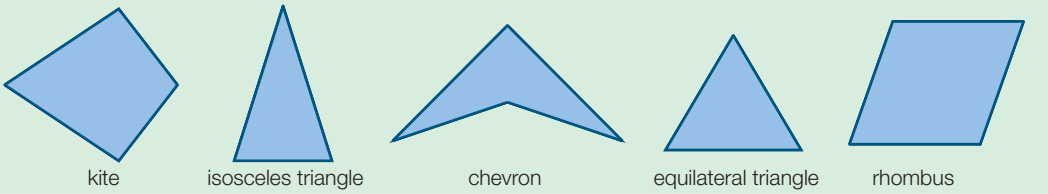
Folding and cutting



Equipment required: some loose sheets of paper and scissors

1 You are going to fold a piece of paper twice and then cut across the dotted corner. Can you predict the shape of the resulting hole?

The shape of the resulting hole will depend on the angle at which you cut and also the angle of the second fold. See if you can produce the following shaped holes.

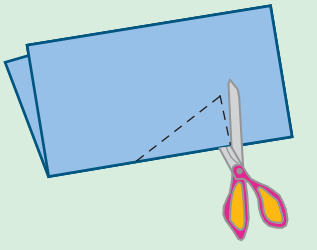


kite | isosceles triangle | chevron | equilateral triangle | rhombus

Can you find the maximum number of sides of the hole such a single cut produces for a piece of paper folded twice?

2 Now try folding the sheet of paper just once, but using two cuts to produce a hole when unfolded. What shape will your hole be?

Can you produce an equilateral triangle, an isosceles triangle and a scalene triangle?



Strategy options

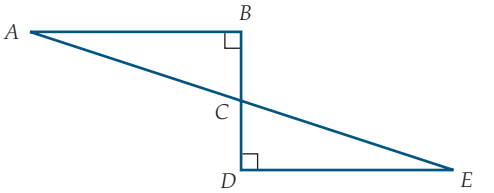
- Act it out.
- Make a model.

Half-time 8

Stage 5.2 Q1–7 **Stage 5.3§ Q8–13**

1 **(a)** Given that $BC = DC$, prove that $\Delta ABC \equiv \Delta EDC$.

(b) Hence, prove that $AB = ED$.

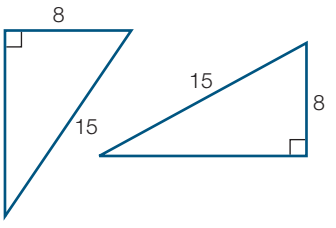


Ex. 8.2

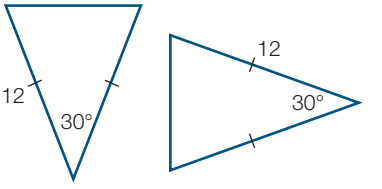
2 Give reasons why each of the following pairs of triangles are congruent.

Ex. 8.1

(a)



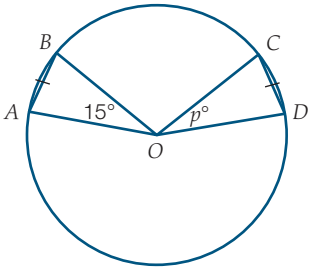
(b)



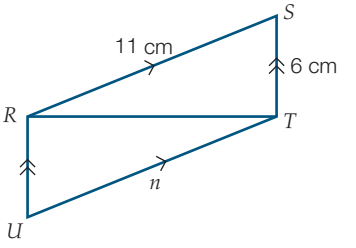
3 For each of the following, prove that the two triangles are congruent and hence find the value of the pronumeral.

Ex. 8.2

(a)



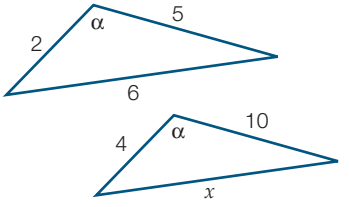
(b)



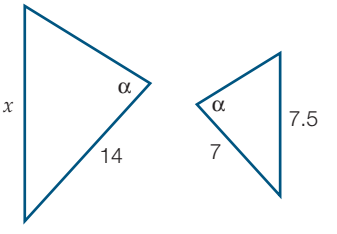
4 Given that these pairs of triangles are similar, find x.

Ex. 8.2

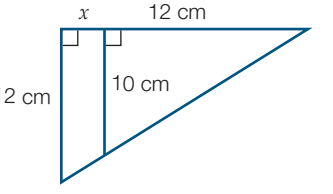
(a)



(b)

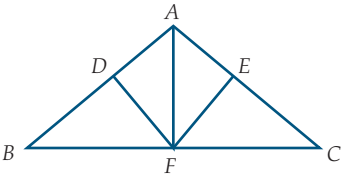


5 Find the value of x.



Ex. 8.1

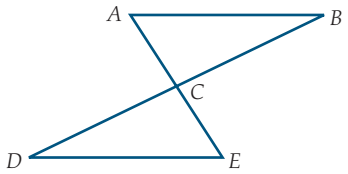
6 In the diagram, $FD = FE$. Also, $FD \perp AB$ and $FE \perp AC$. Prove that $\Delta ADF \equiv \Delta AEF$.



Ex. 8.2

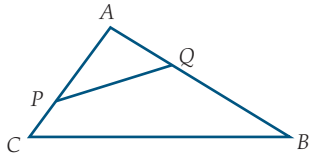
Ex. 8.2

7 Given $AB \parallel DE$ and that C is the midpoint of BD, prove that $\Delta ABC \equiv \Delta DEC$.



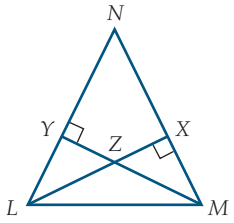
Ex. 8.4

8 Given that $\angle ABC = \angle APQ$, prove that $\Delta ABC \parallel\!| \Delta APQ$.



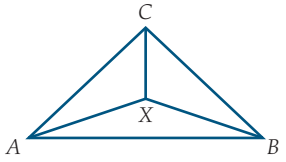
Ex. 8.4

9 In ΔLMN, LX is perpendicular to MN and MY is perpendicular to LN, as shown in the diagram. The point of intersection of LX and YM is Z. Prove that $\Delta LYZ \parallel\!| \Delta MXZ$, giving reasons.



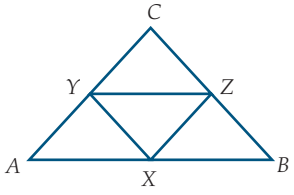
Ex. 8.3

10 Given that $AC = BC$ and $AX = BX$, prove that $\Delta ACX \equiv \Delta BCX$.



Ex. 8.3

11 In the diagram, ΔABC is isosceles with $AC = BC$. X, Y and Z are the midpoints of AB, AC and BC respectively. Prove that $XY = XZ$.

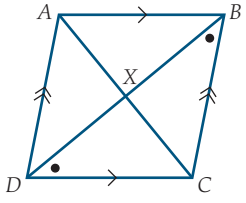


Ex. 8.5

12 A parallelogram $ABCD$ is drawn with diagonals AC and BD intersecting at point X. If $\angle DBC = \angle BDC$:

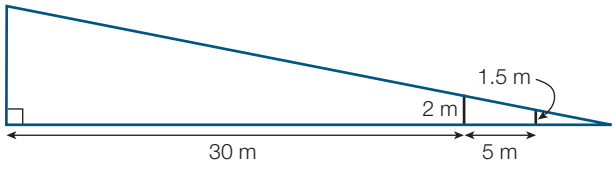
(a) prove $ABCD$ is a rhombus

(b) prove that the diagonals of a rhombus intersect at right angles.



Ex. 8.4

13 A student 1.5 m tall, standing 5 m in front of a 2 m brick wall, can just see the top of a tree growing 30 m behind the wall. How tall is the tree?



Angles in circles

8.6

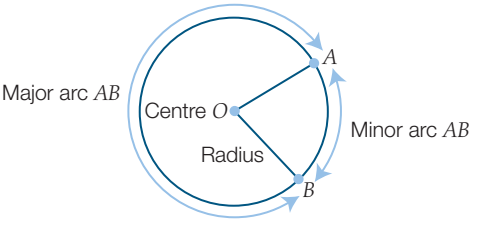
Stage 5.3#

Need to Know

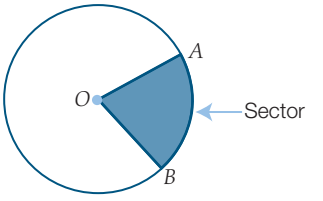
Terms related to circles

A circle is a plane curve consisting of all the points at a given distance (called the radius) from a fixed point in the plane (called the centre).

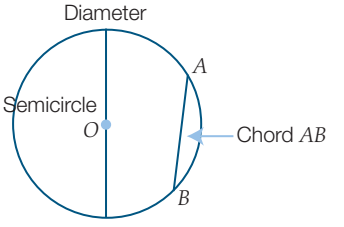
- An **arc** of a circle is a portion of the circle bounded by two points on the circle. The shorter arc is the **minor arc** and the longer arc is the **major arc**. An arc that goes completely around the circle is called the circumference.



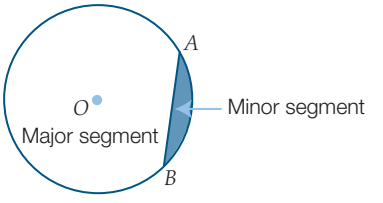
- A **sector of a circle** is the plane figure bounded by two radii of the circle and one of the arcs that they form.



- A **chord** of a circle is an interval joining any two points on the circle. A chord that passes through the centre of a circle is called a **diameter**. The length of the diameter is twice the radius. A diameter divides the circle into two semicircles.

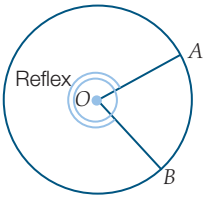


- A **segment** of a circle is the plane figure bounded by a chord and an arc. A **minor segment** is bounded by a chord and a minor arc. A **major segment** is bounded by a chord and a major arc.



- The word **subtend** means 'to be opposite to' or 'measure off'. The side of a triangle subtends the opposite angle, and an arc of a circle subtends an angle at the centre of a circle. The angle at the centre of a circle is **subtended** by the arc or the chord of the circle.

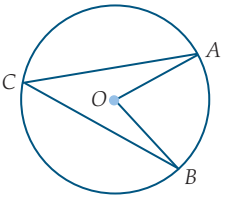
The major arc AB subtends the reflex angle AOB at the centre, O, of the circle.



The minor arc AB subtends the acute or obtuse angle AOB at the centre, O, of the circle.

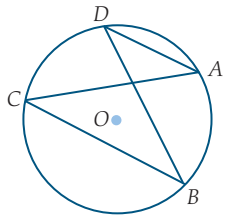
- Angles may be subtended at the centre of the circle or at the circumference. They stand on the same arc or they are subtended by the same chord if the arms of the angle start from and end at the same points on the circumference of the circle.

  $\angle ACB$ is called the **angle at the circumference** standing on the arc AB. $\angle AOB$ is the **angle at the centre** standing on the arc AB.



- Angles at the circumference of a circle drawn from the same two points on the circumference are said to be standing on the same arc. If they are on the same side of the chord they are called **angles in the same segment**.

  $\angle BCA$ and $\angle BDA$ are angles at the circumference standing on the minor arc AB. They are in the same segment because they are on the same side of the chord AB.



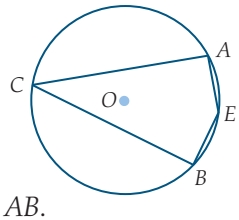
- A **cyclic quadrilateral** is a quadrilateral whose vertices lie on the circumference of a circle. $AEBC$ is a cyclic quadrilateral.

  $\angle CBD$ and $\angle CAD$ are angles at the circumference standing on the minor arc CD. They are in the same segment because they are on the same side of the chord CD.

  $\angle BCA$ is an angle at the circumference standing on the minor arc AB.

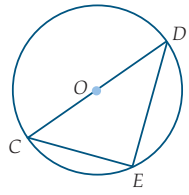
  $\angle BEA$ is an angle at the circumference standing on the major arc AB.

  They are in opposite segments as they are on opposite sides of the chord AB.



- The **angle in a semicircle** is formed when the ends of a diameter are joined to a point on the circumference.

  $\angle DEC$ is an angle in a semicircle.

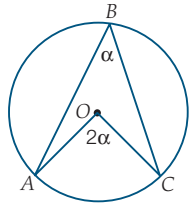


Five circle theorems

Angles at the centre and circumference

The angle at the centre is twice the angle at the circumference standing on the same arc.

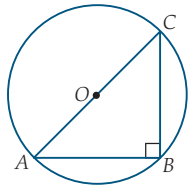
$\angle AOC = 2 \times \angle ABC$



Angles in a semicircle

The angle in a semicircle is a right angle.

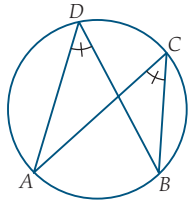
$\angle ABC = 90°$



Angles at the circumference on the same arc

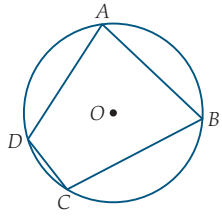
Angles in the same segment subtended by the same arc at the circumference are equal.

$\angle ADB = \angle ACB$



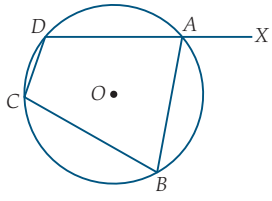
Opposite angles of a cyclic quadrilateral are supplementary.

$\angle ADC + \angle ABC = 180°$



An exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

$\angle XAB = \angle BCD$



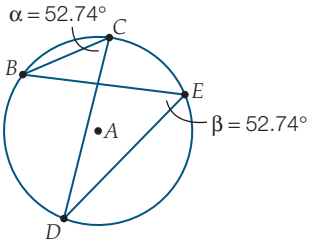
Technology Exploration: GeoGebra

Angles in circles

Open the GeoGebra program. Right-click in the Graphics View and un-select Axes.

1 (a) Use the Circle with Centre through Point tool to draw a circle with centre A.

(b) Use the Segment between Two Points tool to draw BC and CD where B, C and D are points on the circumference, hence forming $\angle BCD$, an angle at the circumference. Use the Segment between Two Points tool to draw another angle $\angle BED$ at the circumference on the same arc. The angles will be displayed as α and β in the Algebra View.



(c) Use the Angle tool to measure the size of both $\angle BCD$ and $\angle BED$. They will be equal, i.e. $\alpha = \beta$.

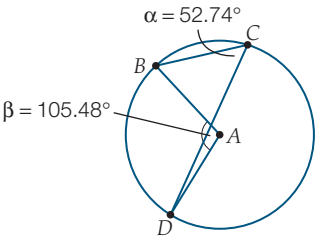
(d) Use the Move tool to move points C and E around the circumference. The angles will remain equal.

(e) Use the Move tool to move point B or D around the circumference. The angle size will change but the two angles will remain equal to each other.

2 (a) Open a new window.

(b) Use the Circle with Centre through Point tool to draw a circle with centre A.

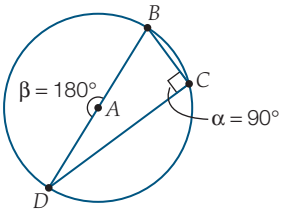
(c) Use the Segment between Two Points tool to draw BC and CD where B, C and D are points on the circumference, hence forming $\angle BCD$, an angle at the circumference. Use the Segment between Two Points tool to draw angle $\angle BAD$, the angle at the centre on the same arc. The angles will be displayed as α and β in the Algebra View.



(d) Use the Angle tool to measure the size of both $\angle BCD$ and $\angle BAD$.

(e) Use the Input bar to enter $\gamma = 2*\alpha$. Enter the Greek letters using the α symbol on the right-hand side of the Input bar. You will see that $\gamma = \beta$, so $2\ \angle BCD = \angle BAD$.

(f) Move point B so that $\angle BAD$ is a straight line through the centre of the circle. $\angle BCD$ will equal 90°.

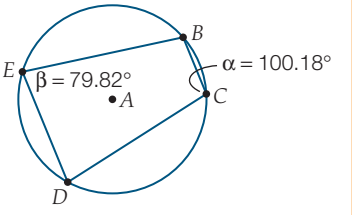


3 Open a new window.

(a) Use the Circle with Centre through Point tool to draw a circle with centre A.

(b) Use the Segment between Two Points tool to draw quadrilateral $BCDE$ where B, C, D and E are points on the circumference.

(c) Use the Angle tool to measure the size of both interior angles, $\angle BCD$ and $\angle DEB$. These will be displayed as α and β in the Algebra View.



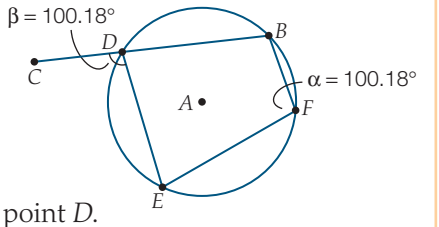
(d) In the Input bar, enter sum[{α, β}]. In the Algebra View, this sum will be displayed as γ and will be equal to 180°.

(e) Use the Move tool to move points B, C, D and E to change the size of $\angle BCD$ and $\angle DEB$. The sum of these angles will remain 180°.

4 Open a new window.

(a) Use the Circle with Centre through Point tool to draw a circle with centre A.

(b) Use the Segment between Two Points tool to draw a line BC, where B is a point on the circle and C is a point outside the circle such that BC intersects with the circle at point D.



(c) Use the Segment between Two Points tool to draw a cyclic quadrilateral $BDEF$. (Points B, D, E and F all lie on the circumference of the circle.)

(d) Use the Angle tool to measure the size of angles, $\angle EFB$ and $\angle EDC$. These will be displayed as α and β in the Algebra View. They will be equal.

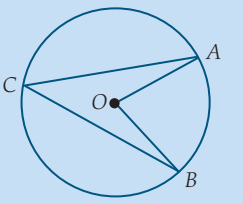
(e) Use the Move tool to move points B, D, E and F to change the location and/or size of $\angle EFB$ and $\angle EDC$. They will remain equal to each other.

Worked Example 10

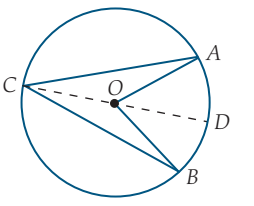
WE10

A, B and C are three points on the circumference of a circle with centre O.

Prove that $\angle AOB = 2\angle ACB$.



Working



$OC = OA$	(radii of a circle)
$\therefore \Delta OCA$ is isosceles	
$\therefore \angle OCA = \angle OAC$	(base angles of isosceles triangles are equal)
$OC = OB$	(radii of a circle)
$\therefore \Delta OCB$ is isosceles	
$\therefore \angle OCB = \angle OBC$	(base angles of isosceles trianges are equal)
$\angle AOD = \angle ACO + \angle CAO$	(exterior angle of ΔAOC equals sum of interior opposite angles)
$\therefore \angle AOD = 2\angle ACO$	($\angle OCA = \angle OAC$)
$\angle BOD = \angle BCO + \angle CBO$	(exterior angle of ΔBOC equals sum of interior opposite angles)
$\therefore \angle BOD = 2\angle BCO$	($\angle OCB = \angle OBC$)

Now $\angle ACB = \angle ACO + \angle BCO$

and $\angle AOB = \angle AOD + \angle BOD$

$\therefore \angle AOB = 2\angle ACO + 2\angle BCO$ (from above)

$\angle AOB = 2(\angle ACO + \angle BCO)$

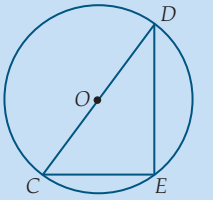
$\therefore \angle AOB = 2\angle ACB$

Hence, the angle at the centre is twice the angle at the circumference standing on the same arc.

Worked Example 11

WE11

CD is a diameter of a circle with centre O. E is a point on the circumference. Prove that $\angle CED$ is a right angle.



Working

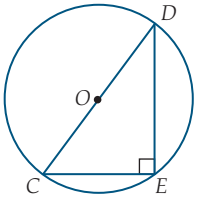
$\angle COD = 2\angle CED$ (angle at the centre is twice the angle at the circumference standing on the same arc CD)

But $\angle COD = 180^\circ$ (COD is a straight line)

$\therefore \angle CED = 90^\circ$

$\angle CED$ is a right angle.

Hence, the angle in a semicircle is a right angle.

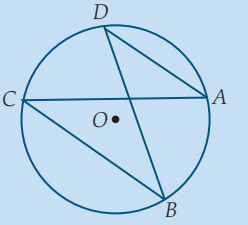


Worked Example 12

WE12

A, B, C and D are four points on the circumference of a circle with centre O.

Prove that $\angle ACB = \angle ADB$.



Working

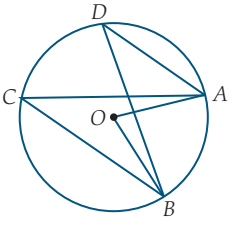
$\angle AOB = 2\angle ADB$ (angle at the centre is twice the angle at the circumference standing on the same arc)

Similarly:

$\angle AOB = 2\angle ACB$

$\angle ACB = \angle ADB$ (both half of $\angle AOB$)

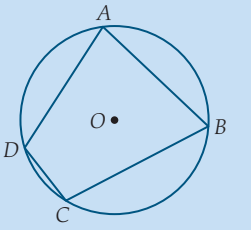
Hence, angles in the same segment subtended by the same arc at the circumference are equal.



Worked Example 13

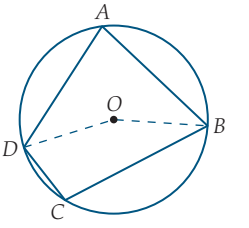
WE13

$ABCD$ is a cyclic quadrilateral in a circle of centre O. By joining OB and OD, show that $\angle BAD + \angle BCD = 180°$ and $\angle ADC + \angle ABC = 180°$.



Working

Join OB and OD where O is the centre of the circle.



Obtuse $\angle BOD = 2\angle BAD$ (angle at the centre = 2 × angle at circumference)

Similarly,

Reflex $\angle BOD = 2\angle BCD$ (angle at the centre = 2 × angle at circumference)

Obtuse $\angle BOD$ + Reflex $\angle BOD = 360°$

$2\angle BAD + 2\angle BCD = 360°$

$\angle BAD + \angle BCD = 180°$

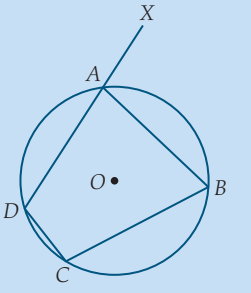
$\angle ADC + \angle ABC = 180°$ (sum of angles in a quadrilateral = 360°)

Hence, opposite angles of a cyclic quadrilateral are supplementary.

Worked Example 14

WE14

$ABCD$ is a cyclic quadrilateral in circle of centre O. Side DA is extended to point X to form an external angle $\angle BAX$. Show that $\angle BCD = \angle BAX$.



Working

$\angle BAD + \angle BCD = 180°$ (opposite angles of a cyclic quadrilateral are supplementary)

and $\angle BAD + \angle BAX = 180°$ (adjacent angles on a straight line)

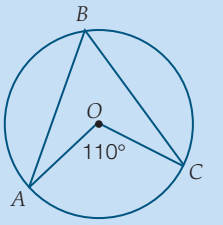
$\angle BCD = \angle BAX$

Hence, an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Worked Example 15

WE15

Find the size of $\angle ABC$, giving reasons for your answer.



Working

$\angle AOC = 2\angle ABC$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\angle AOC = 110°$ (given)

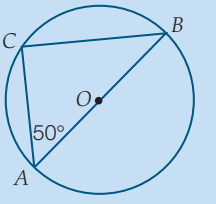
$\therefore 2\angle ABC = 110°$

$\therefore \angle ABC = 55°$

Worked Example 16

WE16

AB is a diameter of the circle with centre O. $\angle CAB = 50°$. Find the size of $\angle ABC$, giving reasons for your answer.



Working

$\angle ACB = 90°$ (angle in a semicircle is 90°)

$\angle BAC = 50°$ (given)

$\angle ABC + \angle BAC + \angle ACB = 180°$ (angle sum of a triangle is 180°)

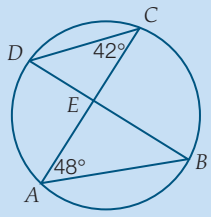
$\angle ABC + 50° + 90° = 180°$

$\therefore \angle ABC = 40°$

Worked Example 17

WE17

A, B, C and D are points on the circumference of the circle, AC and DB intersect at E. If $\angle CAB = 48°$ and $\angle ACD = 42°$, find the size of $\angle AEB$, giving reasons for your answer.



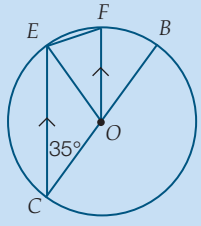
Working

$\angle ACD = 42°$	(given)
$\angle ACD = \angle ABD$	(angles at the circumference on the same arc)
$\therefore \angle ABD = \angle EBA = 42°$	
$\angle EAB = 48°$	(given)
$\angle AEB + \angle EBA + \angle EAB = 180°$	(angle sum of a triangle is 180°)
$\angle AEB + 42° + 48° = 180°$	
$\angle AEB + 90° = 180°$	
$\therefore \angle AEB = 90°$	

Worked Example 18

WE18

BC is a diameter of the circle with centre O. E and F are points on the circumference of the circle such that $OF \parallel CE$. If $\angle OCE = 35°$, find the size of $\angle OFE$, giving reasons for your answer.



Working

$\angle FOB = \angle ECB$	(corresponding angles, $FO \parallel EC$)
$\therefore \angle FOB = 35°$	
$\angle EOB = 2\angle ECB$	(angle at the centre is twice the angle at the circumference standing on the same arc)
$\angle EOB = 2 \times 35°$	
$= 70°$	
$\angle EOF + \angle FOB = \angle EOB$	
$\angle EOF + 35° = 70°$	
$\therefore \angle EOF = 35°$	
ΔOEF is isosceles	($OE = OF$, radii of the circle)
$\angle OEF = \angle OFE$	(base angles of an isosceles triangle)
But $\angle OEF + \angle OFE + 35° = 180°$	(angle sum of a triangle)
$\therefore 2\angle OFE = 145°$	
$\angle OFE = 72.5°$	

8.6 Angles in circles

Answers page 667

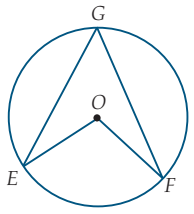
Stage 5.3#

Navigator

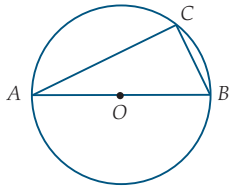
Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12 (a)–(d), Q13, Q17, Q18, Q23, Q25, Q27	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q14, Q16, Q18, Q19, Q20, Q21, Q22, Q25, Q26, Q27	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22, Q24, Q25, Q26, Q27

Fluency and understanding

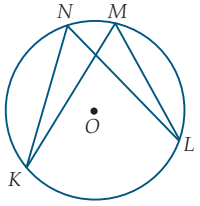
1 E, F and G are three points on the circumference of a circle with centre O. Prove that $\angle EOF = 2\angle EGF$. WE10



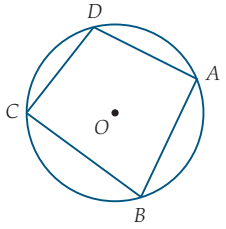
2 AB is a diameter of a circle with centre O. C is a point on the circumference. Prove that $\angle ACB = 90°$. WE11



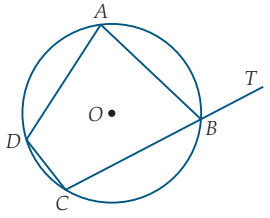
3 K, L, M and N are four points on the circumference of a circle with centre O. Prove that $\angle KML = \angle KNL$. WE12



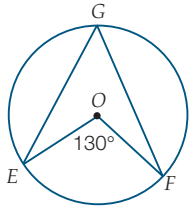
4 $ABCD$ is a cyclic quadrilateral in a circle with centre O. By joining CO and OA, show that $\angle CDA + \angle CBA = 180°$. WE13



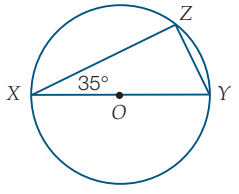
5 $ABCD$ is a cyclic quadrilateral in a circle of centre O. Side CB is extended to point T to form an external angle $\angle TBA$. Show that $\angle ADC = \angle ABT$. WE14



6 $\angle EOF = 130°$. Find the size of $\angle EGF$, giving reasons for your answer. WE15

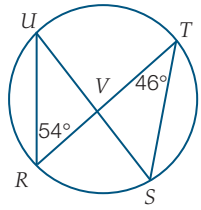


7 XY is a diameter of the circle with centre O. $\angle ZXY = 35°$. Find the size of $\angle ZYX$, giving reasons for your answer. WE16



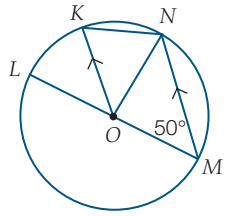
WE17

8 R, S, T and U are points on the circumference of a circle. RT and SU intersect at V. If $\angle URT = 54°$ and $\angle RTS = 46°$, find the size of $\angle TVS$, giving reasons for your answer.



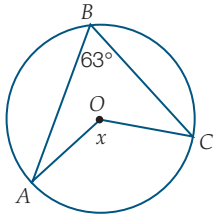
WE18

9 LM is a diameter of the circle with centre O. K and N are points on the circumference of the circle such that $OK \parallel MN$. If $\angle OMN = 50°$, find the size of $\angle OKN$, giving reasons for your answer.

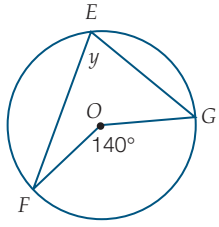


10 Find the value of the variables in each figure. Give reasons for your answer.

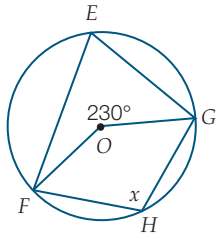
(a)



(b)

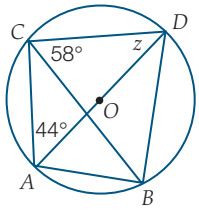


(c)

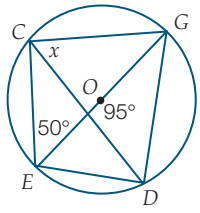


11 Find the value of the variables in each figure. Give reasons for your answer.

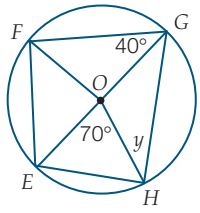
(a)



(b)

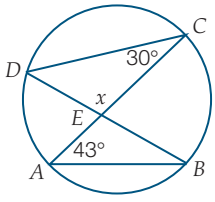


(c)

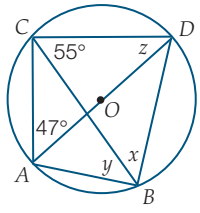


12 Find the value of the variables in each figure. Give reasons for your answer.

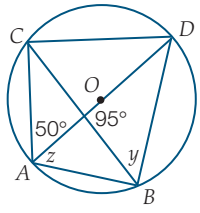
(a)



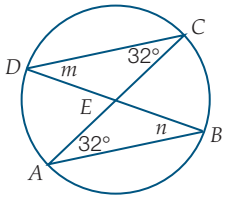
(b)



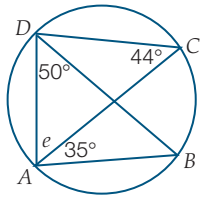
(c)



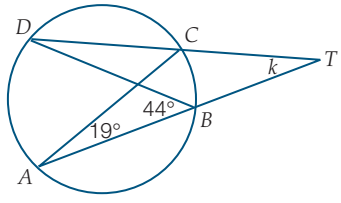
(d)



(e)

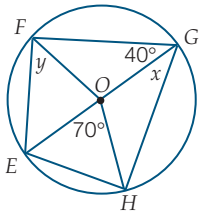


(f)

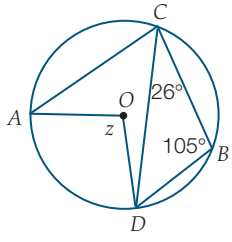


13 Find the value of the variables in each figure. Give reasons for your answer.

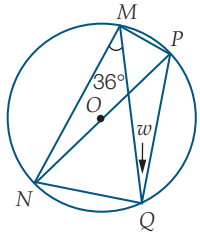
(a) EG is the diameter of the circle



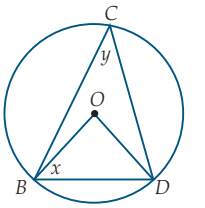
(b) $AC \parallel DB$



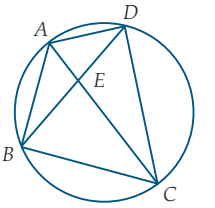
(c) $NM = MQ$, NP is the diameter



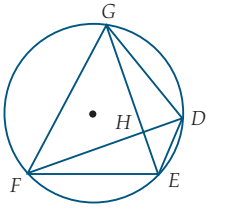
14 O is the centre of the circle. Prove that $x + y = 90°$.



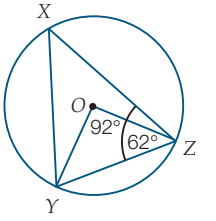
15 AC is the diameter of the circle. If $\angle BDC = 52°$ and $BD = DC$, calculate the size of $\angle ACD$, giving reasons for your answer.



16 If $\angle FGE = 48°$, $\angle GED = 42°$ and $\angle DFE = 20°$, calculate the size of each angle in the quadrilateral $GDEF$, giving reasons for your answers. What can you say about the diagonals of this quadrilateral?

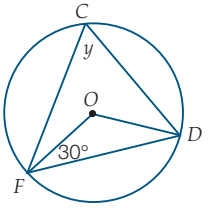


17 If $\angle YOZ = 92°$ and $\angle XZY = 62°$, calculate the size of $\angle OZX$ giving reasons for your answer.

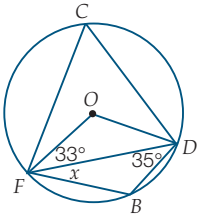


18 Find the value of the variable in each figure. Give reasons for your answer.

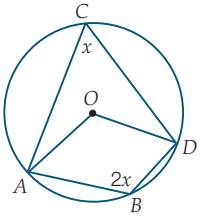
(a)



(b)

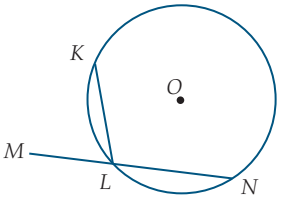


(c)



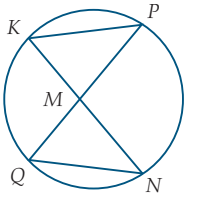
Reasoning, communicating and open-ended

19 $KL = LN$ in the circle with centre O. Prove that $\angle KLM = \angle LON$.

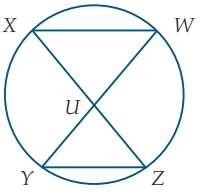


20 **(a)** Prove that $\Delta KPM \,|||\, \Delta NQM$.

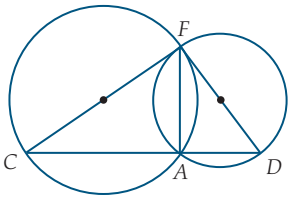
(b) If $KM = MQ$ prove that $MP = MN$.



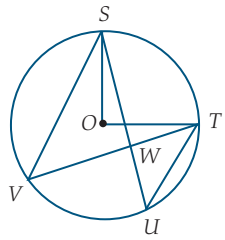
21 XZ and WY intersect at U. If $XU = UW$ prove that $XW \parallel YZ$.



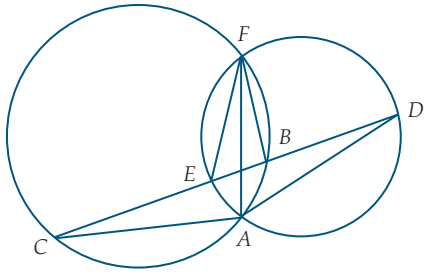
22 CF is a diameter of the circle CFA. FD is a diameter of the circle FDA. Prove that C, A and D are collinear.



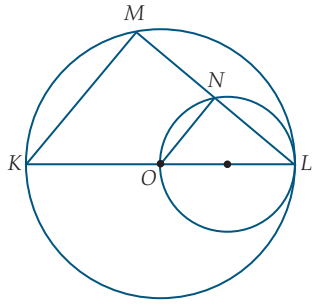
23 In the circle with centre O, $VS \parallel UT$ and $SO \perp OT$. Calculate the size of $\angle TWU$. Give reasons for your answer.



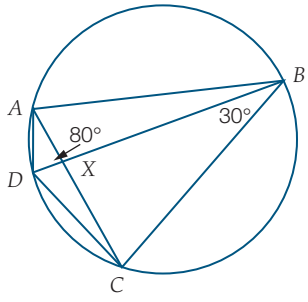
24 If $CA = AD$, prove that AF bisects $\angle EFB$.



25 O is the centre of the circle KML. KL is a diameter. OL is a diameter of the circle ONL. Prove that $KM \parallel ON$.



26 $ABCD$ is a cyclic quadrilateral whose diagonals intersect at X. $\angle DBC = 30°$ and $\angle AXD = 80°$.



(a) Find the size of:

(i) $\angle DAC$ **(ii)** $\angle ADB$ **(iii)** $\angle ACB$.

(b) If $\angle BAC < 80°$, choose a size for $\angle BAC$ and then find the size of:

(i) $\angle BDC$ **(ii)** $\angle DCA$ **(iii)** $\angle DBA$.

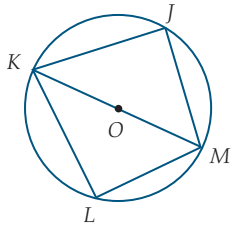
(c) If $\angle BAC = 50°$, write something about:

(i) $\angle ADC$ **(ii)** $\angle ABC$.

(d) If $\angle BAC = 10°$, write something about:

(i) $\angle ADB$ **(ii)** $\angle BCD$.

27 KM is a diameter of the circle with centre O. J and L are points on the circumference. Claire claims that $JKLM$ is a square; James claims that $JKLM$ is a kite. Who is correct, or is neither correct? Explain your decision.



Chords of circles

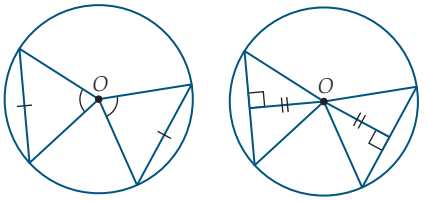
8.7

Stage 5.3#

Need to Know

Chords of equal length

Chords of equal length in a circle subtend equal angles at the centre and are **equidistant** from the centre.

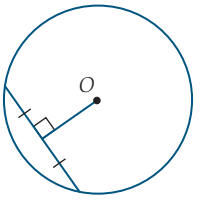


Perpendicular to a chord from the centre

The perpendicular from the centre of a circle to a chord bisects the chord.

Midpoint of a chord to the centre

The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

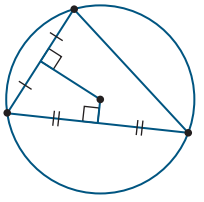


Perpendicular bisector of a chord

The perpendicular bisector of a chord of a circle passes through the centre of the circle.

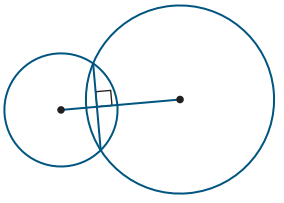
Perpendicular bisectors of a triangle

Given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle through three points.



Line joining centres of two circles and the common chord

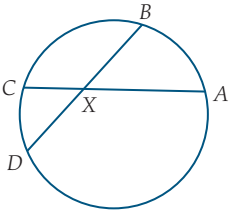
When two circles intersect, the line joining their centres bisects their common chord at right angles.



Two intersecting chords of a circle

The products of the intercepts of two intersecting chords of a circle are equal.

$AX \times CX = DX \times BX$



Technology Exploration: GeoGebra

Chords of circles

Open the GeoGebra program. Right-click in the Graphics View and un-select Axes.

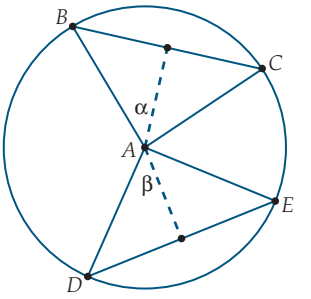
(a) Use the Circle with Centre through Point tool to draw a circle with centre A.

(b) Use the Segment between Two Points tool to draw a chord BC where B and C are points on the circle. Use the Segment between Two Points tool to join AB and AC.

(c) Repeat this process to draw another triangle ADE where D and E are also points on the circle.

(d) Use the Distance or Length tool to measure BC and DE. Use the Move tool to make $BC = DE$.

(e) Use the Angle tool to measure the size of both $\angle BAC$ and $\angle DAE$. They will be equal, i.e. $\alpha = \beta$.



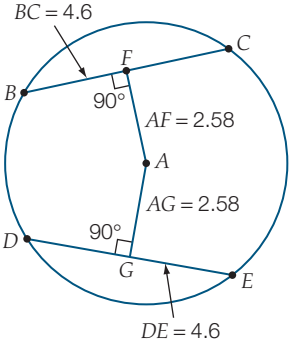
(f) Use the Perpendicular Line tool to draw a perpendicular line to BC through point A. Use the Segment between Two Points tool to draw a line AF from the centre of the circle to point F on BC. Use the Move tool to move point F so that AF is concurrent with the perpendicular line. Right-click on the perpendicular line and delete it.

(g) Repeat this process to draw a line AG from centre A, perpendicular to chord DE such that G is a point on DE.

(h) Use the Distance or Length tool to measure AF and AG. These lengths will be equal.

(i) Use the Distance or Length tool to measure BF and CF. These lengths will also be equal.

(j) Use the Angle tool to measure $\angle AFB$ and $\angle AGD$. They should both be 90°.



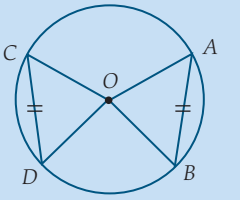
Worked Example 19

WE 19

AB and CD are equal chords of the circle with centre O.

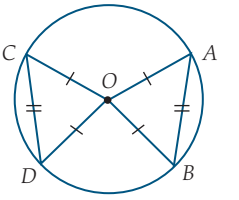
(a) Prove that $\angle AOB = \angle COD$.

(b) OE and OF are the perpendiculars drawn from O to AB and CD, respectively. Prove that $OE = OF$.



Working

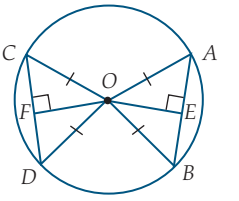
(a) Consider ΔOAB and ΔOCD.



$OA = OC$	(radii of circle)
$AB = CD$	(given, equal chords)
$BO = DO$	(radii of circle)
$\Delta OAB \equiv \Delta OCD$	(SSS)
$\angle AOB = \angle COD$	(matching angles of congruent triangles)

(b) $\angle BAO = \angle DCO$ (matching angles in congruent triangles)

Consider ΔEAO and ΔFCO.



$OA = OC$	(radii of circle)
$\angle EAO = \angle FCO$	(matching angles in congruent triangles OAB and OCD)
$\angle AEO = \angle CFO$	(both right angles, $OE \perp AB$ and $FO \perp CD$)
$\Delta EAO \equiv \Delta FCO$	(AAS)
$OE = OF$	(matching sides of congruent triangles)

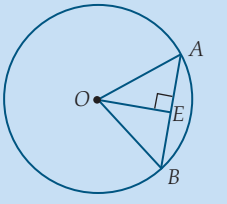
Hence, chords of equal length in a circle subtend equal angles at the centre and are equidistant from the centre.

Worked Example 20

WE 20

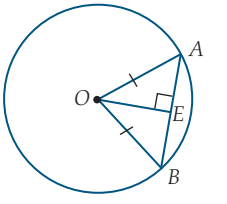
AB is a chord of a circle with centre O. $OE \perp AB$.

Prove that $AE = EB$.



Working

Consider ΔEAO and ΔEBO.



$\angle OEA = \angle OEB = 90°$	($OE \perp AB$)
$OA = OB$	(radii of circle)
OE is a common side.	
$\Delta EAO \equiv \Delta EBO$	(RHS)
$AE = EB$	(matching sides in congruent triangles)

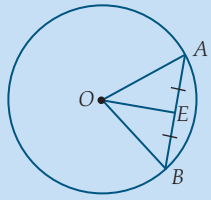
Hence, the perpendicular from the centre of a circle to a chord bisects the chord.

Worked Example 21

WE21

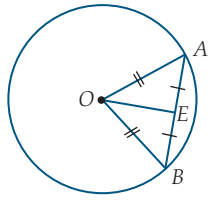
AB is a chord of a circle with centre O. E is the midpoint of AB.

Prove that $OE \perp AB$.



Working

Consider ΔEAO and ΔEBO.



$AE = EB$	(E is midpoint of AB)
$OA = OB$	(radii of circle)
OE is a common side.	
$\Delta EAO \equiv \Delta EBO$	(SSS)
$\angle OEA = \angle OEB$	(matching angles of congruent triangles)
But $\angle OEA = 180° - \angle OEB$	(straight angle)

$\therefore \angle OEA = \angle OEB = 90°$

$\therefore OE \perp AB$

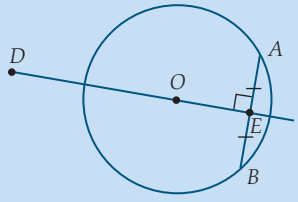
Hence, the line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Worked Example 22

WE22

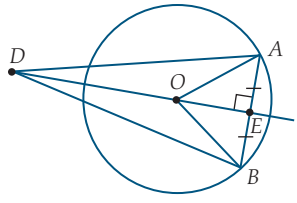
E is the midpoint of the chord AB in a circle with centre O. DE is the perpendicular bisector of AB. Prove that O lies on DE.

When constructed, DE appears to pass through O. To prove this is true we need to use a different sort of proof.



Working

Consider ΔDAE and ΔDBE.



$DA = DB$	(matching sides in congruent triangles)
$OA = OB$	(radii of the circle centre O on which A and B lie)

O is equidistant from A and B (it must lie on the perpendicular bisector of AB)

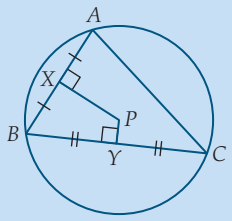
$\therefore O$ lies on DE

Hence, the perpendicular bisector of a chord of a circle passes through the centre of the circle.

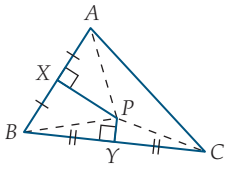
Worked Example 23

WE23

A, B and C are three non-collinear points forming ΔABC. P is a point inside the triangle such that PX and PY are the perpendicular bisectors of AB and BC respectively. Show that P is the centre of a circle through points A, B and C.



Working



Join PA, PB and PC.

In ΔAPX and ΔBPX

$AX = BX$ (given)

$\angle PXA = \angle PXB$ (both equal 90°)

PX is common

$\Delta APX \equiv \Delta BPX$ (SAS)

$AP = BP$

Similarly, $\Delta CPY \equiv \Delta BPY$ (SAS)

$CP = BP$

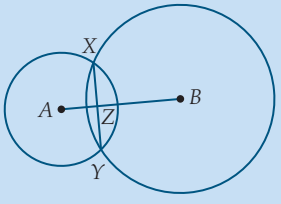
$\therefore$ P is the centre of a circle through A, B and C ($AP = BP = CP =$ radii).

Hence, given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle through three points.

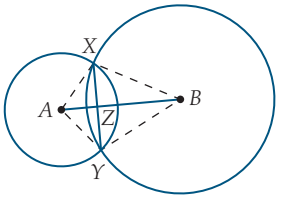
Worked Example 24

WE24

Two circles with centres A and B respectively intersect at points X and Z. Show that the line, AB, joining the centres of the two circles, is the perpendicular bisector of XZ.



Working



Join AX, AZ, BX and BZ.

$AX = AY$ (radii of circle with centre A)

$BX = BY$ (radii of circle with centre B)

$\therefore$ Quadrilateral $AXBY$ is a kite.

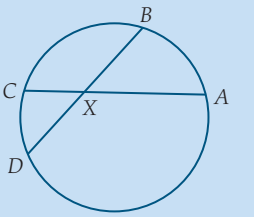
$AB \perp XY$ and $XZ = YZ$ (diagonal joining the non-equal angles of a kite bisect the other diagonal at right angles)

Hence, when two circles intersect, the line joining their centres bisects their common chord at right angles.

Worked Example 25

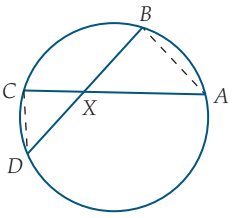
WE25

AC and *BD* are two chords of a circle intersecting at point *X*. Show that the products of the intercepts of *AC* and *BD* are equal; i.e. $AX \times XC = BX \times XD$.



Working

Join *AB* and *CD*.



$\angle ABD = \angle ACD$ and $\angle BAC = \angle BDC$ (angles subtended by the same arc at the circumference are equal)

$\Delta ABX \,|||\, \Delta DCX$ (AAA)

$$\frac{AX}{BX} = \frac{DX}{CX}$$

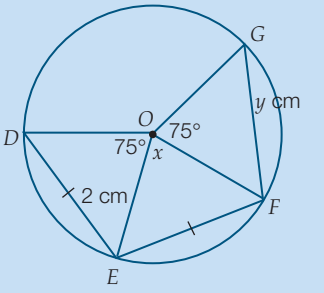
$\therefore AX \times CX = BX \times DX$

The products of the intercepts of two intersecting chords of a circle are equal.

Worked Example 26

WE26

Calculate the value of the pronumerals. Give reasons for your answers.



Working

$DE = EF$ (equal chords marked on diagram)

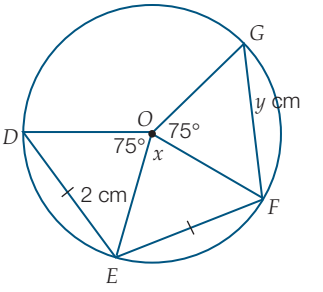
$\therefore \angle DOE = \angle EOF$ (equal chords subtend equal angles at the centre of a circle)

$\therefore x = 75°$

$\angle DOE = \angle FOG = 75°$

$\therefore DE = GF$ (equal angles at the centre of a circle are subtended by equal chords)

$\therefore y = 2$ cm

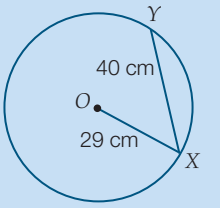


Worked Example 27

WE 27

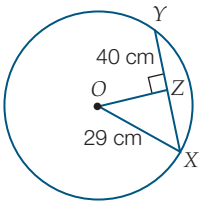
XY is a chord of a circle with centre O.

If $XY = 40$ cm and $OX = 29$ cm, calculate the perpendicular distance of XY from O. Give reasons for your answer.



Working

Draw OZ, the perpendicular bisector of XY.



$XZ = ZY$ (perpendicular from the centre of a circle to a chord bisects the chord)

$\therefore XZ = 20$ cm (half of XY)

$OX^2 = XZ^2 + OZ^2$ (Pythagoras' Theorem)

Hence, $OZ^2 = OX^2 - XZ^2$

$\therefore OZ^2 = 29^2 - 20^2$

$= 841 - 400$

$= 441$

$\therefore OZ = 21$ cm

XY is 21 cm from O.

8.7 Chords of circles

Answers page 669

Navigator

Stage 5.3#

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10 (a), (c), Q11 (a), (b), Q13 (a), Q14, Q17, Q19, Q20	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10 (b), (c), Q11, Q12 (a), (b), Q13 (a), (b), Q14, Q15, Q17, Q19, Q20	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21

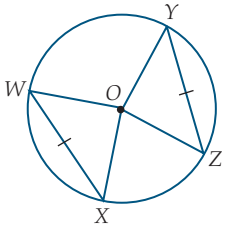
Fluency and understanding

WE 19

1 WX and YZ are equal chords of a circle with centre O.

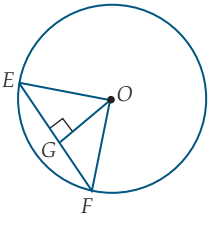
(a) Prove that $\angle WOX = \angle YOZ$.

(b) OP and OQ are the perpendiculars drawn from O to WX and YZ, respectively. Prove that $OP = OQ$.



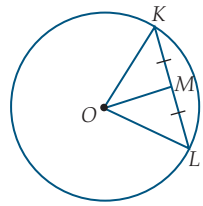
WE 20

2 EF is a chord of a circle with centre O. $OG \perp EF$. Prove that $EG = GF$.



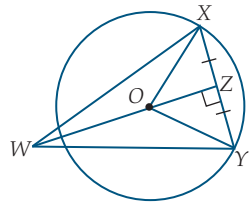
WE21

3 KL is a chord of a circle with centre O. M is the midpoint of KL. Prove that $OM \perp KL$.



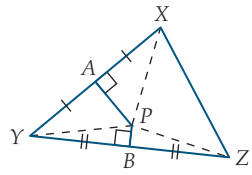
WE22

4 Z is the midpoint of the chord XY in a circle with centre O. WZ is the perpendicular bisector of XY. Prove that O lies on WZ.



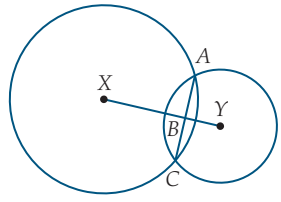
WE23

5 X, Y and Z are three non-collinear points forming ΔXYZ. P is a point inside the triangle such that PA and PB are the perpendicular bisectors of XY and YZ respectively. Show that P is the centre of a circle through points X, Y and Z.



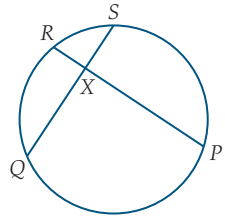
WE24

6 Two circles with centres X and Y respectively intersect at points A and C. Show that the line, XY, joining the centres of the two circles, is the perpendicular bisector of AC.



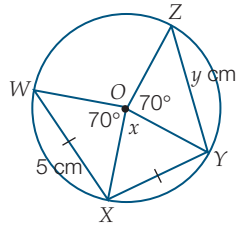
WE25

7 PR and QS are two chords of a circle intersecting at point X. Show that the products of the intercepts of PR and QS are equal, i.e. $RX \times PX = SX \times QX$.



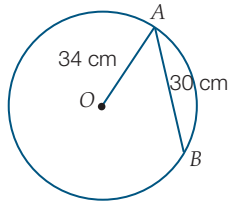
WE26

8 Calculate the value of the pronumerals. Give reasons for your answers.



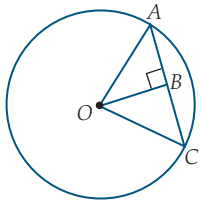
WE27

9 AB is a chord of a circle with centre O. If $AB = 30$ cm and $OA = 34$ cm, calculate the perpendicular distance of AB from O. Give reasons for your answer.

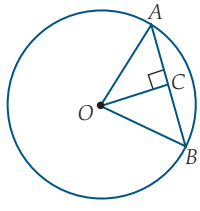


10 Calculate the length of AB in each diagram, giving reasons for your answer.

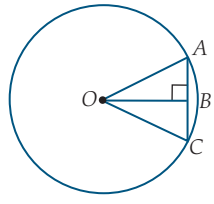
(a) $AC = 10$ cm, $OB \perp AC$



(b) $AC = 10$ cm, $OC \perp AB$

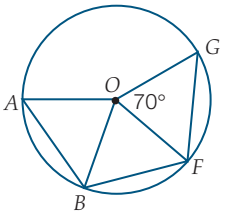


(c) $OA = 17$ cm, $OB = 8$ cm

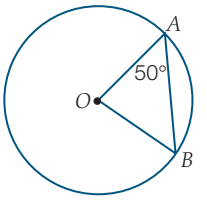


11 Calculate the size of $\angle AOB$ in each figure, giving reasons for your answer.

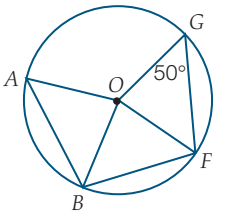
(a) $AB = GF$, $\angle GOF = 70°$



(b) $\angle OAB = 50°$

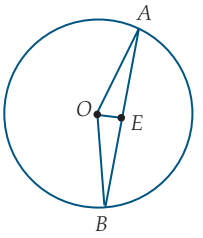


(c) $AB = GF$, $\angle OGF = 50°$

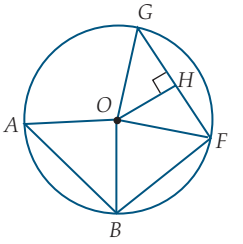


12 Calculate the length of OA in each figure, giving reasons for your answer.

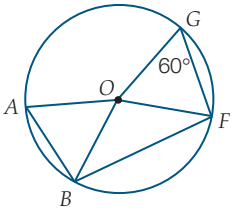
(a) $AB = 48$ cm, $OE = 7$ cm, E is the midpoint of AB.



(b) $GH = HF = 21$ cm, $OH = 20$ cm, $AB = GF$

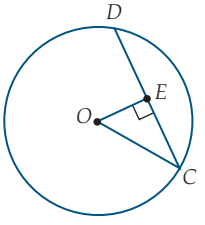


(c) $\angle OGF = 60°$, $AB = GF = 10$ cm

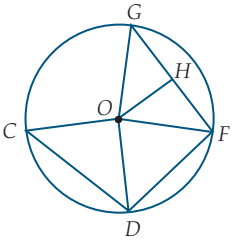


13 Calculate the length of CD in each diagram, giving reasons for your answer.

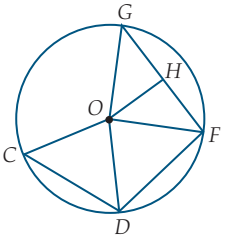
(a) $OE \perp CD$, $OC = 53$ cm, $OE = 28$ cm



(b) $\angle FHO = 90°$, $\angle FOD = \angle DOC$, $GF = FD$, $OF = 73$ cm, $OH = 48$ cm

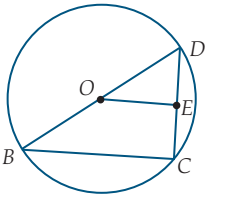


(c) $\angle GOF = \angle COD$, $GH = HF$, $OG = 85$ cm $OH = 77$ cm

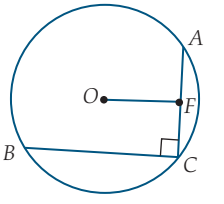


Reasoning, communicating and open-ended

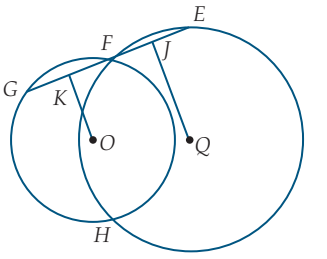
14 BD is the diameter of a circle with centre O. C is a point on the circumference of the circle. E is the midpoint of CD. Prove that $OE \parallel BC$, and hence that $\angle BCD = 90°$.



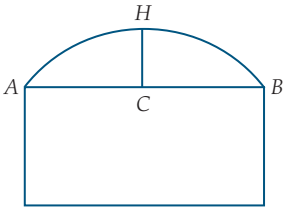
15 O is the centre of the circle ACB. $AF = FC$ and $\angle ACB = 90°$. Mark D, the midpoint of BC, and use it to prove that $BC = 2OF$.



16 The circles *GFH*, with centre *O*, and *FEH*, with centre *Q*, intersect at *F* and *H*. *GFE* is a straight line. *K* is the midpoint of *GF* and *J* is the midpoint of *FE*. Prove that $\angle KOQ + \angle OQJ = 180°$.



17 The entrance to a shopping centre is a rectangular doorway with a circular arch above it as shown in the diagram. The doorway *AB* is 4 m wide and the radius of the arch is 2.5 m. Calculate the height of the arch, *CH*. Justify your calculations.



18 *KLM* is an isosceles triangle inscribed in a circle of radius 25 mm. If $KL = KM$ and $LM = 48$ mm, calculate the length of *KL*. Give reasons to justify your calculation. (A triangle inscribed in a circle has its vertices on the circumference of the circle.)

19 The doorways to a shopping centre are being designed. The shape is to be similar to that in the diagram for Question **17**. The rectangular doorways are to be between 4 and 6 m wide and at least 2.5 m high. Each doorway is to have a circular arch on top of the rectangular door opening such that the total height of the door and the arch is between 3 and 4.5 m. Create two designs, giving the breadth and height of the rectangular doorway, the radius of the arch and the height of the arch above the centre of the doorway.

20 Construct the perpendicular bisectors of the sides *KL* and *LM* of a scalene triangle *KLM*. Explain why their point of intersection, *X*, is the centre of the circle through *L*, *K* and *M*.

21 Circles with centres *E* and *F* intersect at *G* and *H*. Draw this information on a diagram. Use your diagram to explain why **(i)** *EF* bisects *GH* and **(ii)** $CE \perp GH$.

Tangents and secants 8.8

Stage 5.3#

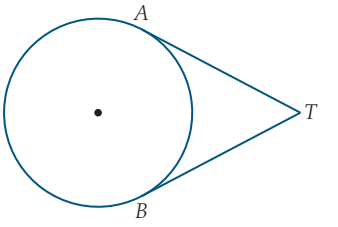
Need to Know

A **tangent** to a circle is a line drawn from an external point that intersects with a circle at one point only.

Tangents from an external point

Two tangents drawn to a circle from an external point are equal.

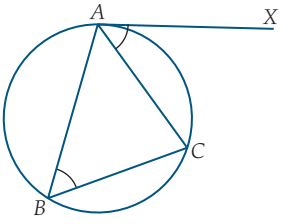
$TA = TB$



Angle between tangent and chord of a circle and the angle in the alternate segment

The angle formed between a tangent and a chord drawn to the point of contact is equal to the angle in the alternate segment.

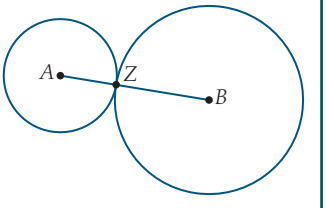
$\angle XAC = \angle ABC$



Two touching circles

When two circles touch, their centres and the point of contact are collinear.

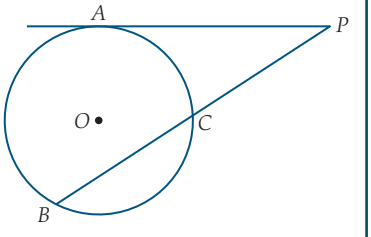
A, Z and B lie on the same straight line.



A tangent and a secant to a circle from an external point

The square of the tangent to a circle from an external point equals the product of the intercepts of any **secants** from the point.

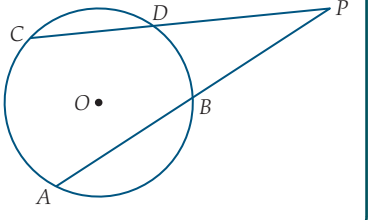
$PA^2 = PB \times PC$.



Two intersecting secants to a circle from an external point

The products of the intercepts of two intersecting secants to a circle from an external point are equal.

$PA \times PB = PC \times PD$.

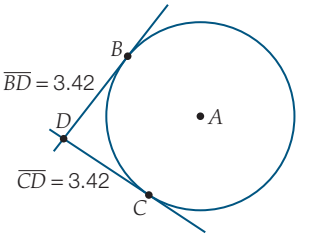


Technology Exploration: GeoGebra

Tangents and secants

Open the GeoGebra program. Right-click in the Graphics View and un-select Axes.

1 Use the Circle with Centre through Point tool to draw a circle with centre *A*.



(a) Use the Tangents tool to draw a tangent to the circle at point *B*.

(b) Use the New Point tool to place a point *C* on the circle.

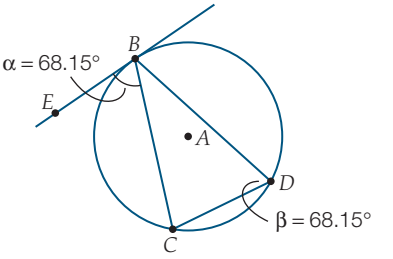
(c) Use the Tangents tool to draw a tangent to the circle at point *C*.

(d) Use the Intersect Two Objects tool to find point *D*, the intersection of the two tangents.

(e) Use the Distance or Length tool to measure *BD* and *CD*. The lengths *BD* and *CD* will be equal.

(f) Use the Move tool to move points *B*, *C* and *D*. The lengths of the tangents will change but they will remain equal to each other.

2 Open a new window.



(a) Use the Circle with Centre through Point tool to draw a circle with centre *A*.

(b) Use the Tangents tool to draw a tangent to the circle at point *B*.

(c) Use the Segment between Two Points tool to draw chords *BC*, *CD* and *DB*, where *C* and *D* are points on the circumference.

(d) Use the New Point tool to place a point *E* on the tangent.

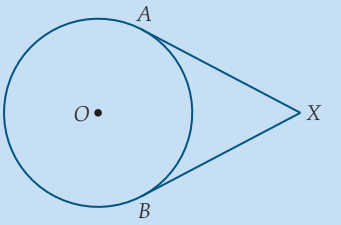
(e) Use the Angle tool to measure the size of the angles $\angle EBC$ and $\angle BDC$. These will be displayed as α and β in the Algebra View and will be equal.

(f) Use the Move tool to move points *B*, *C* and *D*. The angle size will change but the angles will remain equal to each other.

Worked Example 28

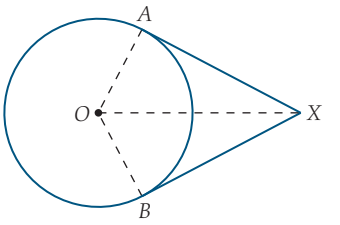
WE28

XA and XB are two tangents drawn from an external point X to a circle with centre O intersecting at A and B respectively. Show that $AX = BX$.



Working

Join OA, OB and OX where O is the centre of the circle.



$\angle OBX = \angle OAX = 90°$ (angle between tangent and chord = 90°)

OX is common

$OA = OB$ (both radii of circle)

$\Delta OAX \equiv \Delta OBX$ (RHS congruency)

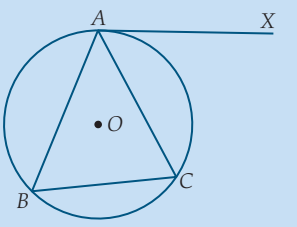
$\therefore\ AX = BX$

Hence, two tangents drawn to a circle from an external point are equal.

Worked Example 29

WE29

A tangent XA is drawn to a circle of centre O from an external point X, intersecting the circle at point A. A chord XAC subtends $\angle ABC$ at point B on the circumference. Show that $\angle XAC = \angle ABC$.



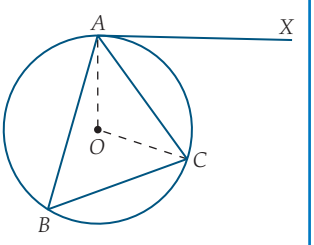
Working

Join OA, OC where O is the centre of the circle.

$\angle AOC = 2\angle ABC$ (angle at centre = 2 × angle at circumference)

$\angle AOC + \angle OAC + \angle OCA = 180°$ (angles in a triangle = 180°)

$\angle OAC = \angle OCA$ (base angles of isosceles triangle are equal)



$\angle AOC + 2\angle OAC = 180°$

$\angle AOC = 2\angle ABC$

$2\angle ABC + 2\angle OAC = 180°$

$\angle ABC + \angle OAC = 90°$

$\angle CAX + \angle OAC = 90°$ (angle between radius and tangent = 90°)

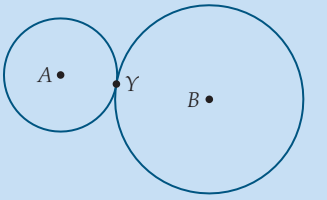
$\angle CAX = \angle ABC$

Hence, the angle formed between a tangent and a chord drawn to the point of contact is equal to the angle in the alternate segment.

Worked Example 30

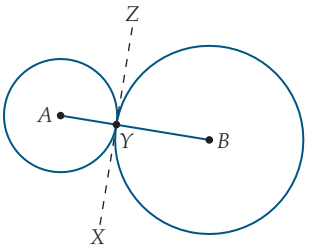
WE30

Two circles with centres A and B respectively touch at point Y. Show that points A, B and Y are collinear.



Working

Draw XYZ, a tangent to both circles at point of contact Y, and join AY and BY.



$\angle AYZ = 90°$ (angle between radius and tangent = 90°)

$\angle BYZ = 90°$ (angle between radius and tangent = 90°)

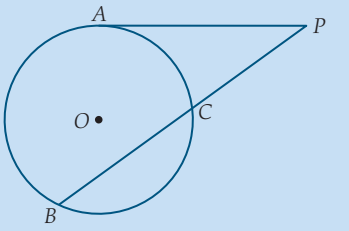
$\angle AYZ + \angle BYZ = 180°$ (angles on a straight line are supplementary)

AYB is a straight line.

Worked Example 31

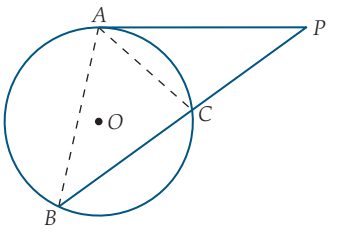
WE31

A tangent PA is drawn to a circle with centre O from an external point P. A secant PB is drawn from point P intersecting with the circle at points C and B. Show that the square of the tangent PA to the circle equals the product of the intercepts PC and PB of the secant PB.



Working

Join AC and AB.



$\angle PAC = \angle ABP$ (angle between tangent and chord = angle in alternate segment)

$\angle APB$ is common to ΔAPC and ΔBPA

$\Delta APC \,|||\, \Delta BPA$ (AAA)

$$\therefore \frac{PA}{PC} = \frac{PB}{PA}$$

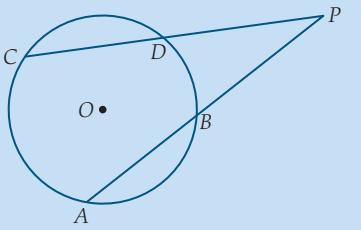
$$PA^2 = PB \times PC$$

Hence, the square of the tangent to a circle from an external point equals the product of the intercepts of any secants from the point.

Worked Example 32

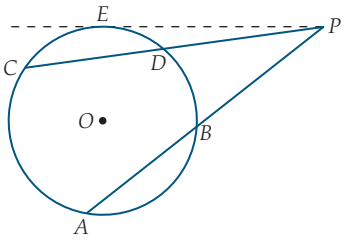
WE 32

PA is a secant drawn to a circle of centre O from an external point P and intersecting with the circle at points B and A. Another secant PC is drawn to the circle from point P intersecting at points C and D. Show that $PA \times PB = PC \times PD$.



Working

Draw tangent PE.



$PE^2 = PD \times PC$ (the square on the tangent is equal to the product of the intercepts of a secant)

and $PE^2 = PB \times PA$ (the square on the tangent is equal to the product of the intercepts of a secant)

$\therefore PD \times PC = PB \times PA$ (both equal to PE^2)

The product of the intercepts of two secants to a circle from an external point are equal.

8.8 Tangents and secants

Answers page 672

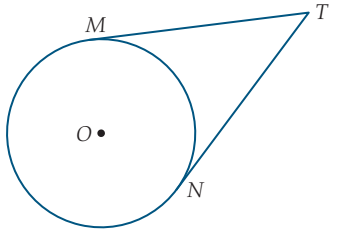
Navigator

Stage 5.3#

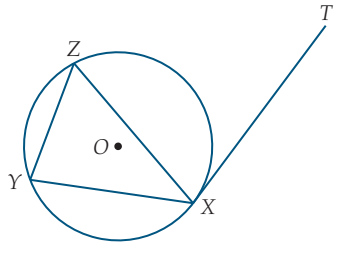
Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q16	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q16	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16

Fluency and understanding

1 TM and TN are two tangents drawn from an external point T to a circle with centre O, intersecting at M and N respectively. Show that $MT = NT$. WE 28

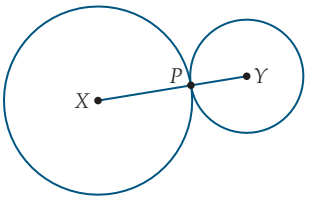


2 A tangent TX is drawn to a circle of centre O from an external point T, intersecting the circle at point X. A chord XZ subtends $\angle XYZ$ at point Y on the circumference. Show that $\angle TXZ = \angle XYZ$. WE 29



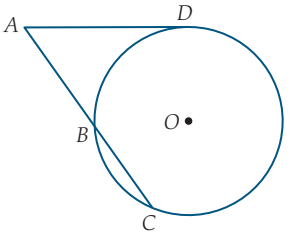
WE30

3 Two circles with centres X and Y respectively touch at point P. Show that points A, B and P are collinear.



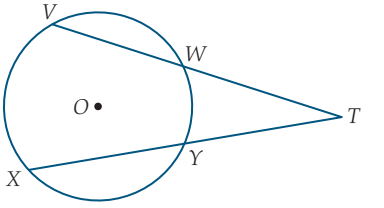
WE31

4 A tangent AD is drawn to a circle with centre O from an external point A. A secant AC is drawn from point A, intersecting with the circle at points B and C. Show that the square of the tangent AD to the circle equals the product of the intercepts AB and AC of the secant AC.

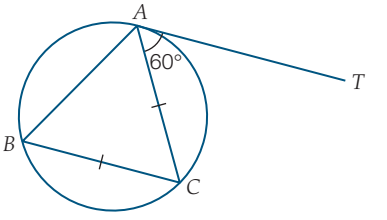


WE32

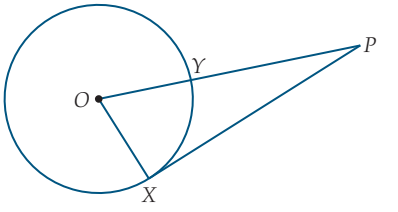
5 TX is a secant drawn to a circle of centre O from an external point T and intersecting with the circle at points X and Y. Another secant TV is drawn to the circle from point T, intersecting at points V and W. Show that $TX \times TY = TV \times TW$.



6 TA is a tangent to point A on the circumference of a circle from an external point T. C is another point on the circumference. If $\angle TAC = 60°$ and $AC = BC$, show that ΔABC is equilateral.



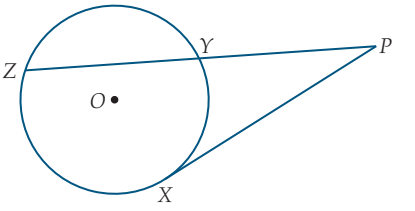
7 A circle with centre O has a radius of 5 mm. A tangent PX of length 12 mm is drawn from an external point P to a point X on the circumference of the circle. The straight line joining P and O intersects with the circumference at Y. The length of PY is



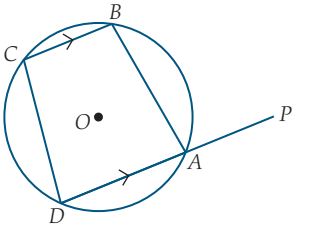
A 7 mm **B** 8 mm

C 9 mm **D** 10 mm

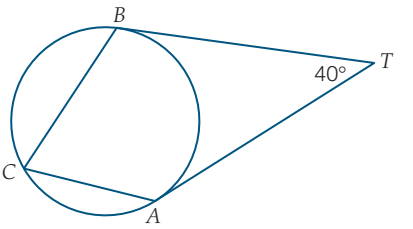
8 PX is a tangent from an external point P to a circle of centre O touching the circle at X such that $PX = 6$ m. PZ is a secant from P intersecting with the circle at points Y and Z as shown. If $PY = 4$ m, find the length of YZ.



9 The side DA of a cyclic quadrilateral $ABCD$ is extended to an external point P. If $\angle PAB = 106°$ and $AD \parallel BC$, find $\angle ADC$.

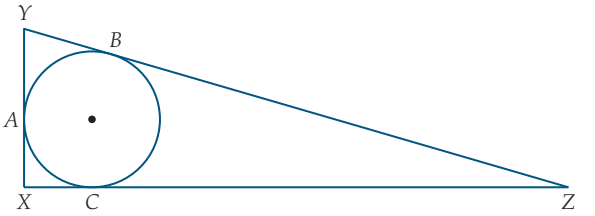


10 TA and TB are tangents to points A and B from an external point T. C is another point on the circumference. Find $\angle ACB$.

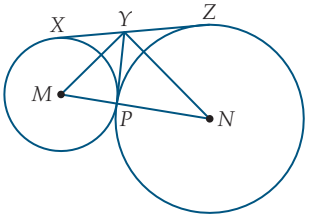


Reasoning, communicating and open-ended

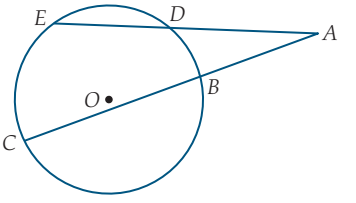
11 XY and XZ are tangents to a circle with centre O such that XY is perpendicular to XZ. If $AX = 3$ cm and $BZ = 21$ cm, find the length of XY.



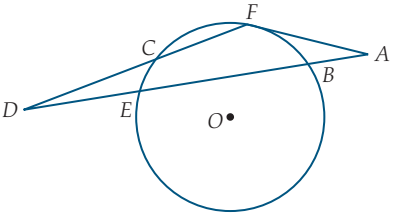
12 Two circles with centres M and N respectively touch at point P. XZ is a tangent to both circles, touching circle with centre M at X and circle with centre N at Z. PY is also a tangent to both circles, intersecting XZ at Y. Show that $\angle MYN = 90°$.



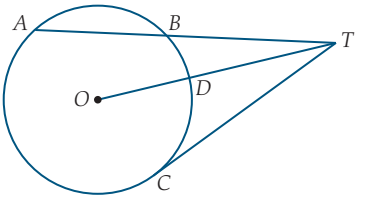
13 AC and AE are secants to a circle with centre O from an external point A. The secant AC intersects with the circle at points B and C such that $AB = 5$ m and $BC = 7$ m. The secant AE intersects with the circle at D and E such that $AD = 6$ m. Find the length of DE.



14 AF is a tangent to a circle with centre O from an external point A, touching the circle at F. AD is a straight line that intersects with the circle at points B and E. Points D and F are joined, intersecting with the circle at C. Given that $AF = 4$ cm, $AB = 2$ cm, $BD = 10$ cm and $CD = 5$ cm, show that $CF = \frac{1}{2}BE$.



15 TC is a tangent from an external point T to a circle of centre O and radius 5 cm, touching the circle at point C. TO is a line that intersects with the circle at D such that $TD = 8$ cm. TA is a secant from point T, intersecting with the circle at points A and B. If $AB = 7$ cm, find the length of TB.



16 Find a possible length for a secant to a circle from an external point if the length of the tangent from that point equals 20 cm. What is the length of the chord section of this secant?

Chapter review 8

D.I.Y. Summary

Key Words

Stage 5.2	congruent	prove	
	included angle	similar	
Stage 5.3§	bisect	perpendicular	property
	definition	perpendicular bisector	theorem
Stage 5.3#	angle at the centre	cyclic quadrilateral	secant
	angle at the circumference	equidistant	sector of a circle
	angle in a semicircle	major arc	segment
	angles in the same segment	major segment	subtend
	arc	minor arc	subtended
	chord	minor segment	tangent

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.2 Q1–3 Stage 5.3§ Q4–7 Stage 5.3# Q8–11

1 Triangles are _____ if they are identical in all aspects; i.e. they have exactly the same shape and size.

2 When applying SAS to test for congruence it is important that the angle is the _____.

3 Triangles are _____ if they have exactly the same shape, although their sizes may be different.

4 The minimum amount of information needed to identify a particular figure is called a _____.

5 A _____ is true about a particular figure but is not enough to define that figure.

6 A line at right angles to another line that also cuts that line in half is called a _____.

7 If we divide a line or angle into two equal parts, we _____ the line or angle.

8 If two radii are drawn to different points on the circumference of a circle they divide the circle into a _____ and a _____.

9 A line joining one point on the circumference of the circle to another is called a _____.

10 An arc will _____ an angle at the circumference.

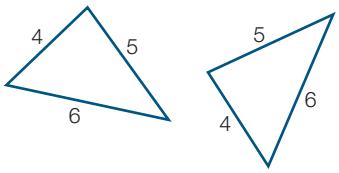
11 A chord, that is not a diameter, drawn on a circle divides the circle into a _____ and a _____.

Stage 5.2 Q1–5, 14 Stage 5.3§ Q6–9, 15–18, 20–23 Stage 5.3# Q10–13, 19, 24–26

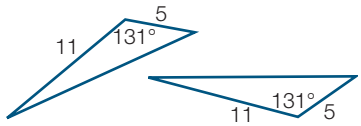
Fluency and understanding

1 Give reasons why the following pairs of triangles are congruent. Ex 8.1

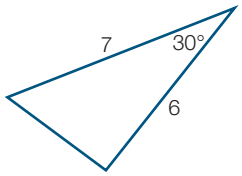
(a)



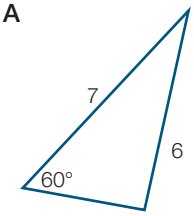
(b)



2 The triangle congruent with the triangle shown is: Ex 8.1



A



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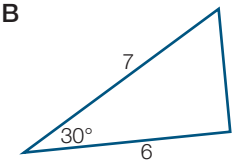
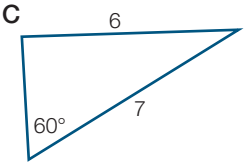
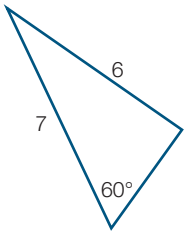


C

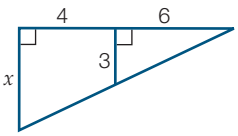


D

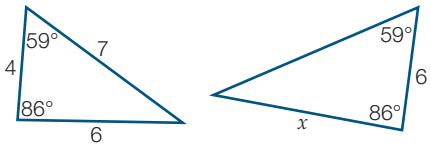


3 Find the value of x in each of the following pairs of triangles. Ex 8.1

(a)

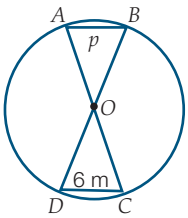


(b)



4 Prove that the following pairs of triangles are congruent and hence find the value of the pronumeral. Ex 8.2

(a)



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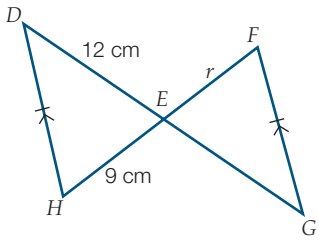
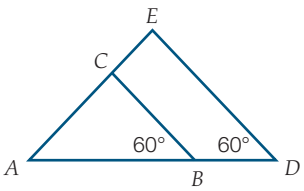
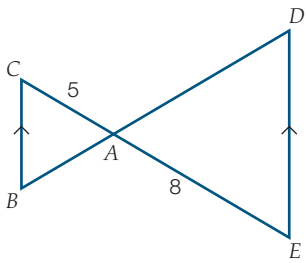


5 Prove that the following pairs of triangles, ΔABC and ΔADE, are similar. Ex 8.2

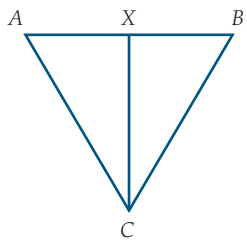
(a)



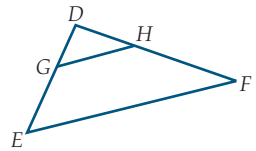
(b)



6 Given that CX bisects $\angle ACB$, and $CX \perp AB$, prove that $AX = XB$. Ex 8.3

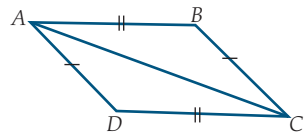


7 In ΔDEF, G is a point on DE and H is a point on DF such that $GH \parallel EF$. Prove that $\dfrac{DG}{GE} = \dfrac{DH}{HF}$.



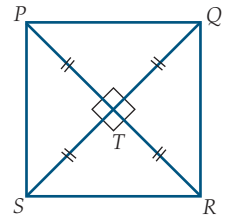
Ex. 8.4

8 If both pairs of opposite sides of a quadrilateral are equal, prove that the quadrilateral is a parallelogram.



Ex. 8.5

9 If the diagonals of a quadrilateral are equal and bisect each other at right angles, prove that the quadrilateral is a square.

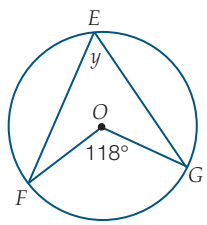


Ex. 8.5

10 Find the value of the pronumerals in each figure. Give reasons for your answer.

Ex. 8.6

(a)



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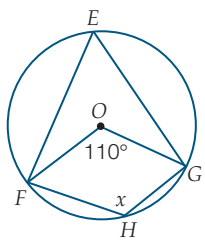
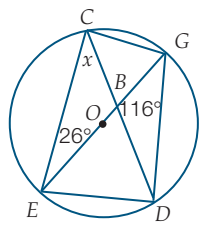


(c)



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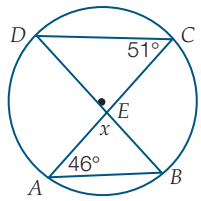
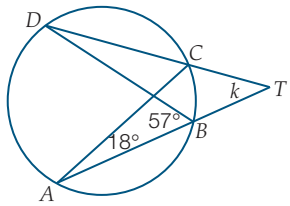


(e)



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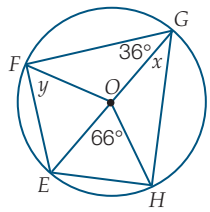
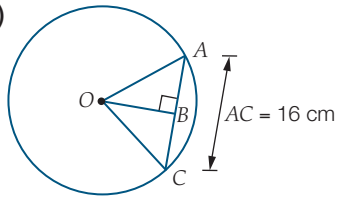


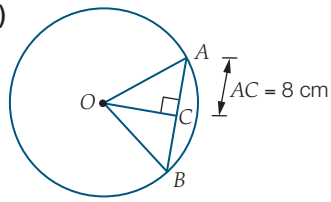
11 Calculate the length of AB in each diagram, giving reasons for your answer.

Ex. 8.7

(a)



(b)



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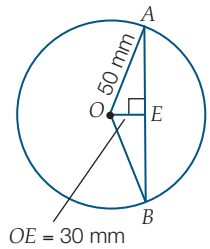
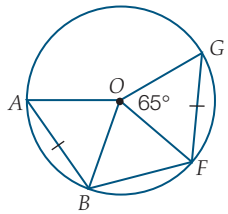


12 Calculate the size of $\angle AOB$ in each figure, giving reasons for your answer.

Ex. 8.7

(a)



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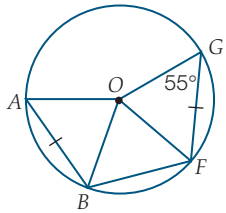

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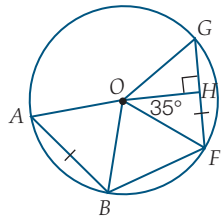
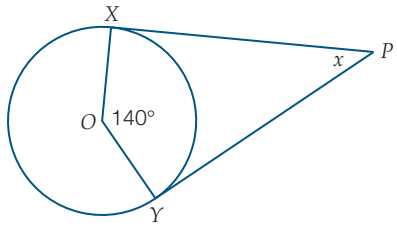


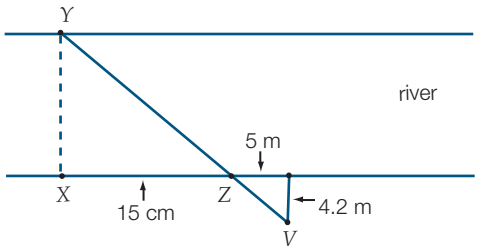
13 Tangents PX and PY are drawn to a circle with centre O, intersecting at points X and Y.

(a) Prove $PXOY$ is a cyclic quadrilateral.

(b) Find the size of angle x given $\angle XOY = 140°$.



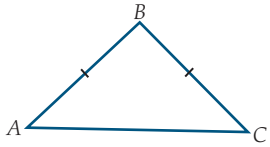
14 Angela is on a walk through the forest when she comes to a river. She wants to know the distance across the river. She draws triangles as shown in the diagram to help her. Find the distance across the river.



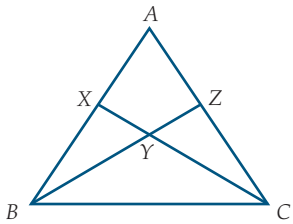
15 In ΔABC, $BA = BC$.

(a) Prove that $\angle BAC = \angle BCA$, by drawing the line BD perpendicular to AC, meeting it at D.

(b) What property of isosceles triangles have you just proved?



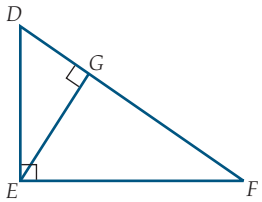
16 $AB = AC$ and $AX = AZ$. If $\angle YBC = \angle YCB$, prove that $XY = ZY$.



17 ΔDEF is right-angled at E. $EG \perp DF$.

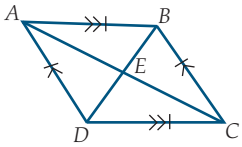
(a) Prove that triangles DEF, DEG and EGF are similar.

(b) Prove that $EG^2 = DG \times GF$.



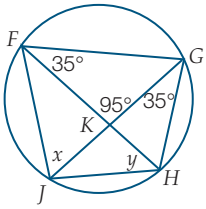
18 **(a)** For the diagram shown, prove $AC \perp BD$.

(b) What special property of a rhombus have you just proved?

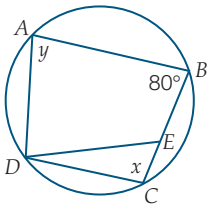


19 Find the value of the variables in each of the following, giving reasons for your answers.

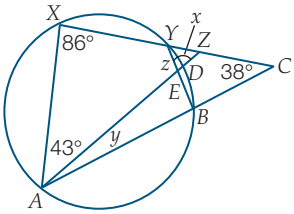
(a) $GH = HJ$



(b) $AB \parallel DC$

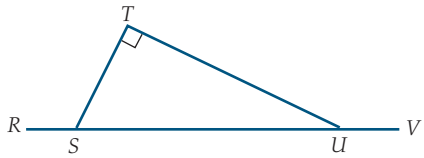


(c) $\angle YDZ = x$

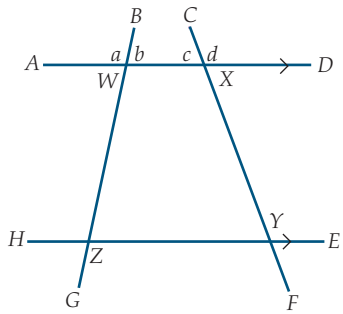


Reasoning and communicating

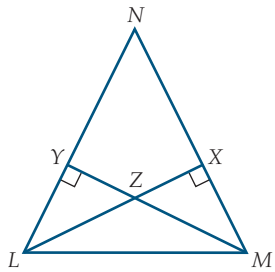
20 Prove that $\angle TSR + \angle TUV = 270°$.



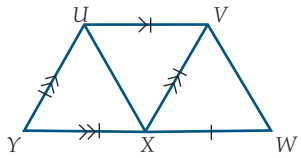
21 Use the diagram to prove that the sum of the interior angles of a trapezium is 360°.



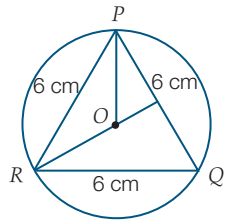
22 In triangle LMN, LX is perpendicular to NM and MY is perpendicular to LN, as shown in the diagram. The point of intersection of LX and MY is Z. Prove that $\Delta LYZ \,|||\, \Delta MXZ$, giving reasons.



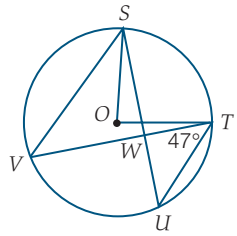
23 $UVXY$ is a rhombus and $YX = XW$. Show that $\Delta UXY \equiv \Delta VWX$.



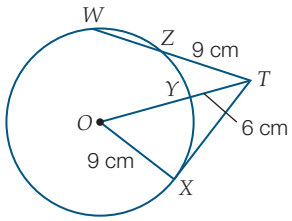
24 PQR is an equilateral triangle inscribed in a circle with centre O. (P, Q, R lie on the circle.) If $PQ = 6$ cm, calculate OP, the radius of the circle, giving reasons for your answer.



25 In the circle with centre O, $SU \perp VT$ and $\angle VTU = 47°$. Find the size of $\angle SOT$, giving reasons for your answer.



26 TX is a tangent to circle with centre O and radius 9 cm. The line TO joining T to the centre O intersects with the circle at Y such that $TY = 6$ cm. TW is a secant that intersects with the circle at Z such that $TZ = 9$ cm. Find the length WZ. Give reasons for your answer.



Non-linear relationships

9

P Recall 9

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 (a) If $p = 2m^2 - 3m + 5$, find the value of p when:

(i) $m = 1$ (ii) $m = 0$ (iii) $m = -1$

(b) Find the value of $x(x - 3)$ when:

(i) $x = 5$ (ii) $x = -3$ (iii) $x = 0$

2 Factorise each of these quadratic expressions.

(a) $x^2 - 7x$ (b) $2x^2 + 10x$ (c) $x^2 - 4$

3 Expand and simplify each of the following.

(a) $3x(x + 4)$ (b) $(x + 2)(x + 5)$ (c) $(x - 7)(x + 3)$

(d) $(x + 8)^2$ (e) $(x - 3)(x - 1)$ (f) $(x - 4)(x + 4)$

4 Factorise each of the following expressions.

(a) $x^2 - 25$ (b) $9x^2 - 16$ (c) $x^2 - 5$

(d) $x^2 - 8x + 16$ (e) $x^2 + 14x + 49$ (f) $3x^2 - 66x + 363$

5 Factorise each of the following.

(a) $x^2 + 7x + 10$ (b) $x^2 - 5x + 6$ (c) $x^2 - 13x - 14$

(d) $x^2 + 5x - 24$ (e) $x^2 + 5x - 36$ (f) $2x^2 + 14x + 12$

(g) $-x^2 - 3x - 2$ (h) $-x^2 + 7x - 10$ (i) $-2x^2 + 2x + 24$

6 Write each of the following as a number without an index.

(a) 3^3 (b) 4^{-1} (c) $5^{-2} + 3$

7 Solve the following using the Null Factor Law.

(a) $(x - 1)(x + 3) = 0$ (b) $(x - 1)^2 = 0$ (c) $x(x - 5) = 0$

8 Solve the following using special product rules to factorise first.

(a) $x^2 - 6x + 9 = 0$ (b) $x^2 + 8x + 16 = 0$ (c) $121x^2 - 36 = 0$

9 Factorise each of the following by completing the square.

(a) $x^2 + 4x + 2$ (b) $x^2 - 6x + 6$ (c) $x^2 - 3x - 5$

Syllabus references

Number and Algebra: Non-Linear Relationships

✓ Graph simple non-linear relationships, with and without the use of digital technologies, and solve simple related equations **(ACMNA296) (Stage 5.2◊)**

✓ Explore the connection between algebraic and graphical representations of relationships such as simple quadratics, circles and exponentials using digital technologies as appropriate **(ACMNA239) (Stage 5.2◊)**

✓ Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations **(ACMNA267) (Stage 5.3§)**

Number and Algebra: Equations

✓ Solve simple quadratic equations using a range of strategies **(ACMNA241) (Stage 5.2)**

✓ Solve a wide range of quadratic equations derived from a variety of contexts **(ACMNA 269) (Stage 5.3§)**

✓ Solve simultaneous equations, where one equation is non-linear, using algebraic and graphical techniques, including the use of digital technologies **(Stage 5.3§)**

Number and Algebra: Ratios and Rates

✓ Solve problems involving direct proportion; explore the relationship between graphs and equations corresponding to simple rate problems **(ACMNA208) (Stage 5.3)**

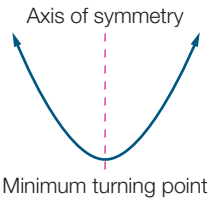
9.1 Non-linear graphs and their equations

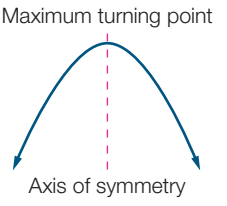
Stage 5.2◊

Need to Know

Parabolas

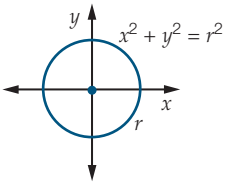
- A **parabola** is a graph of the quadratic relationship $y = ax^2 + bx + c$.
- Every parabola has a **vertex** or **turning point**.
- A parabola can have a minimum turning point or a maximum turning point.
- Every parabola has an **axis of symmetry**.





Circles

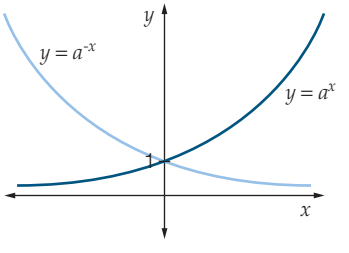
- The equation of a circle with its centre at the origin (0, 0) and with radius r is given by $x^2 + y^2 = r^2$.
- The graphs of circle equations written in the form of $(x - h)^2 + (y - k)^2 = r^2$ have the centre at (h, k) and radius r.
- The four points plotted on the circumference are $(h + r, k)$, $(h, k + r)$, $(h - r, k)$ and $(h, k - r)$.



Exponentials

An **exponential** relationship occurs when a quantity grows or decays at a rate proportional to its current value. It can be recognised when the independent variable is the index or exponent of a constant base. All exponential graphs have a limiting value, a value they approach but never reach. This limiting value gives us a horizontal **asymptote**. An asymptote is a line that a curve approaches but never cuts or crosses as the independent variable approaches particular values.

The graphs of exponential equations in the form of $y = a^x$ and $y = a^{-x}$ pass through the point (0, 1) because $a^0 = 1$, for all values of a.



Worked Example 1

WE1

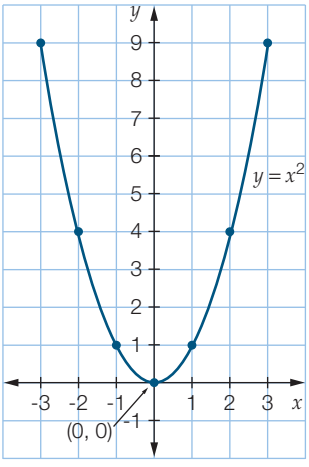
Plot the following parabolas for values of x between -3 and 3 inclusive:

(a) $y = x^2$ **(b)** $y = 2x^2 - 1$ **(c)** $y = 5 - x^2$

Working

(a) $y = x^2$

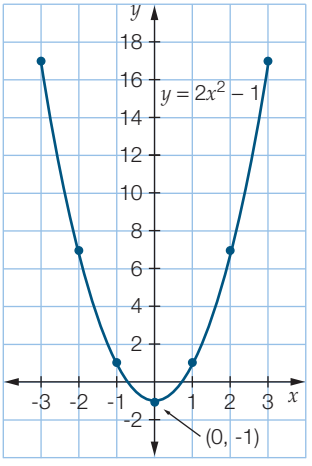
x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



The equation of the axis of symmetry is $x = 0$.

(b) $y = 2x^2 - 1$

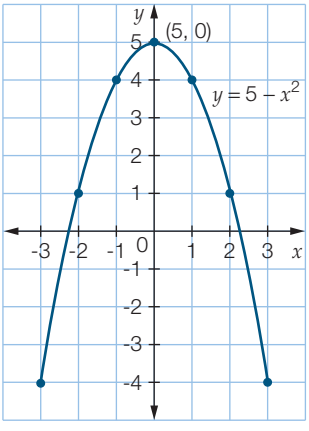
x	-3	-2	-1	0	1	2	3
y	17	7	1	-1	1	7	17



The equation of the axis of symmetry is $x = 0$.

(c) $y = 5 - x^2$

x	-3	-2	-1	0	1	2	3
y	-4	1	4	5	4	1	-4



The equation of the axis of symmetry is $x = 0$.

Worked Example 2

WE2

Plot the graphs of the following equations.

(a) $x^2 + y^2 = 25$

(b) $(x - 1)^2 + (y + 2)^2 = 36$

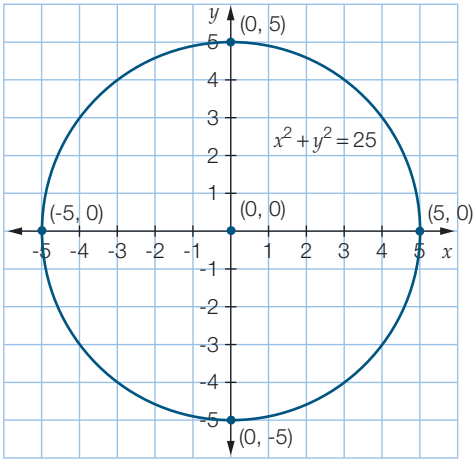
Working

(a) $x^2 + y^2 = 25$

$r = 5,\ h = 0,\ k = 0$

so $(h - r) = -5,\ (h + r) = 5$

x	-5	0	5
y	0	±5	0

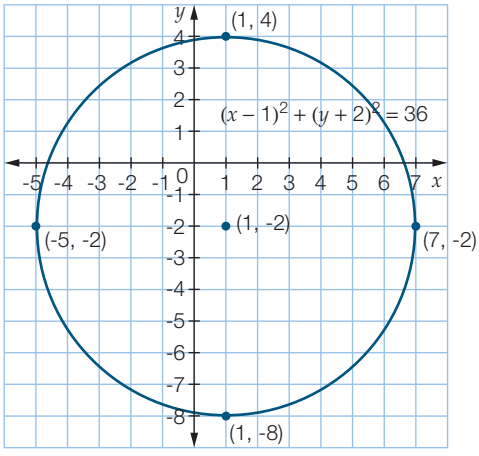


(b) $(x - 1)^2 + (y + 2)^2 = 36$

$r = 6,\ h = 1,\ k = -2$

so $(h - r) = -5,\ (h + r) = 7$

x	-5	1	7
y	-2	-8, 4	-2



Worked Example 3

WE 3

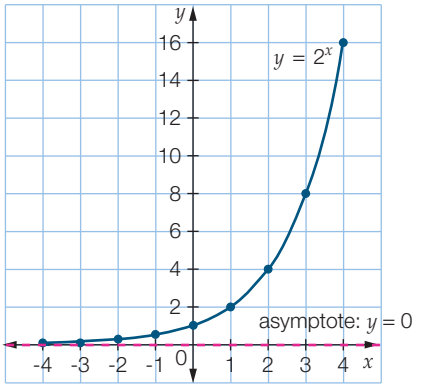
Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = 2^x$ **(b)** $y = 2^{-x} + 3$

Working

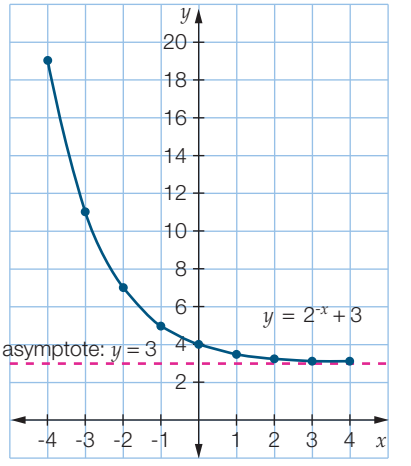
(a) $y = 2^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



(b) $y = 2^{-x} + 3$

x	-4	-3	-2	-1	0	1	2	3	4
y	19	11	7	5	4	$3\frac{1}{2}$	$3\frac{1}{4}$	$3\frac{1}{8}$	$3\frac{1}{16}$



9.1 Non-linear graphs and their equations

Answers page 676

Navigator

Stage 5.2◊

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9, Q10, Q11, Q13, Q15, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q7, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q18

Equipment required: graph paper for Questions **7** and **8**, calculator for Questions **11** and **14**

Fluency and understanding

1 Plot the graphs of each of the following using integer values of x between -3 and 3 inclusive. WE 1

(a) $y = x^2 + 1$ **(b)** $y = x^2 - 3$ **(c)** $y = x^2 + 5$

(d) $y = 2x^2$ **(e)** $y = -2x^2$ **(f)** $y = -2x^2 + 1$

2 Plot the graphs of each of the following equations. WE 2

(a) $(x + 3)^2 + (y + 2)^2 = 16$ **(b)** $(x - 3)^2 + (y - 2)^2 = 9$

(c) $(x - 1)^2 + (y + 1)^2 = 25$ **(d)** $x^2 + (y + 4)^2 = 36$

(e) $(x + 4)^2 + y^2 = 8$ **(f)** $(x - 4)^2 + (y + 2)^2 = 12$

WE3

3 Plot the graphs of the following equations for $-3 \le x \le 3$.

(a) $y = 2^x + 2$ **(b)** $y = 2^x - 1$ **(c)** $y = 2^x - 3$

(d) $y = 3^x - 1$ **(e)** $y = 3^x - 2$ **(f)** $y = 3^x + 4$

4 **(a)** Complete a table of values and then plot the graph for each of the following using integer values for x between -4 and 4 inclusive.

(i) $y = (x - 1)(x + 3)$ **(ii)** $y = x(x + 3)$ **(iii)** $y = (4 + x)(2 - x)$

(b) Write down the coordinates of the turning point for each graph and state whether it is a maximum or minimum.

5 The graph of $y = 2^{x-2} + 1$ would be:

A a circle **B** a rectangular hyperbola

C an exponential **D** a parabola

6 A path around an area on a coordinate map is described by the equation $(x + 1)^2 + (y - 3)^2 = 16$.

(a) Make a simple sketch of the path.

(b) Determine whether the following points are within or outside the circle.

(i) (1, 2) **(ii)** (3, 6) **(iii)** (1.5, 2.5)

Hint

Will the temperature ever reach 25°C exactly?

7 A cup of coffee cools exponentially in a room where the temperature is 25°C. The temperature of the coffee was taken every 30 minutes and the results recorded in the table below:

t (min)	0	30	60	90	120	150
T (°C)	100	54	36.2	29.3	26.7	25.6

(a) Use graph paper to plot a graph of time, t, against temperature, T, (show t on the horizontal axis). Join the points with a smooth curve. Use your graph to answer parts **(b)** to **(e)**.

(b) What is the initial temperature of the coffee?

(c) Use the graph to find the temperature of the coffee after:

(i) 10 min **(ii)** 20 min **(iii)** 40 min **(iv)** 50 min

Give your answer to the nearest degree.

(d) During which 10-minute period in the first hour did the coffee cool the fastest? Justify your answer.

(e) When will the temperature be 27.5°C?

(f) Extend the curve. Predict the temperature after 3 hours.

8 The volume of a child's circular swimming pool, with a depth of $\frac{3}{\pi}$ (≈ 0.96 m), depends on the radius of the pool. The volume of the pool is given by $V = \pi r^2 h$, where V is the volume of the pool in cubic metres, r is the radius (in metres) and h is the height (in metres).

(a) Show that $V = 3r^2$.

(b) Explain why this relationship is quadratic.

(c) Complete the table of values for $0 \le r \le 5$.

r	0	1	2	3	4	5
r^2						
V						

(d) Plot a graph of the relationship $V = 3r^2$ on graph paper. (Represent r on the horizontal axis.)

(e) Find the volume of the pool when the radius is:

(i) 2.5 m **(ii)** 3.8 m

(f) Find the radius of the pool if its volume is:

(i) 30 m^3 **(ii)** 65 m^3

(g) What is the difference in volume between a pool of radius 1.5 m and one of radius 3.5 m?

9 **(a)** Write down the equation of a circle with:

(i) centre (3, 1) and radius 7

(ii) centre (-2, -4) and radius 3

(iii) centre (2, -5) and radius 2.

(b) Plot the graph for each of the equations in part **(a)**.

10 A point that does not lie on the graph of $(x - 3)^2 + (y + 2)^2 = 25$ would be:

A (7, 1) **B** (-1, -5) **C** (7, -5) **D** (-7, 1)

11 **(a)** For the relationship $y = 2^x$, copy and complete this table of values for x and y.

x	-2	-1	0	1	2	3	4
y							

(b) Use this table of values to plot a graph of $y = 2^x$. Join the points with a smooth curve.

(c) Use your graph to determine approximate values for:

(i) $2^{1.5}$ **(ii)** $2^{0.8}$ **(iii)** $2^{-0.5}$

(iv) $2^{-1.2}$ **(v)** $\sqrt{2}$ **(vi)** $(\sqrt{2})^3$

(d) Use a calculator to check the answers you obtained in part **(c)**.

(e) Use your graph to find the value of the exponent, correct to one decimal place, when y is:

(i) 5 **(ii)** 12 **(iii)** 15

Reasoning, communicating and open-ended

12 A tablecloth has a length 0.4 m longer than its breadth.

(a) Write an equation for the area of the tablecloth in terms of its length if the length is represented by the variable l.

(b) Plot a graph of the area against the length.

(c) Use this graph to find the dimensions of the tablecloth if the area of the tablecloth is 3.2 m^2.

13 A town has an initial population of 4000, but the establishment of a new industry in the town is expected to increase the population by 6% per year for future years.

(a) What will be the predicted population 1 year after the industry is established?

(b) Complete a table of values to show the predicted population for the first 5 years after the industry is established. Round the population to the nearest whole number. Use P to represent the population after n years.

(c) Write an equation to model the predicted population over future years.

(d) Sketch a graph of this relationship using your equation.

(e) Use your graph to predict when the population will have doubled in size.

(f) Use your equation to confirm your prediction in **(e)** is correct.

(g) Explain why the graph is a first quadrant graph only.

14 The water coming out of a fountain in a pond forms a parabolic shape. At any point p from the place where the water comes out, the height of the water, h, can be found using the equation: $h = -\frac{1}{16}(p - 40)^2 + 100$ (h and p are in centimetres).

(a) Plot the graph of the relationship between h and p, with p on the horizontal axis and the values of p between 0 and 80.

(b) How high does the water go up in the air?

(c) How far from the water outlet does the water from the fountain reach?

(d) Assuming that the fountain is in the centre of a pond and reaches to the edge of the pond, what area would be covered by water?

15 The centre of a circular water fountain is 3 m east and 2 m south of the corner of a building. The water fountain has a radius of 1 m. Could a tree be planted 2.4 m east and 2.7 m south of the corner of the building. Justify your answer by using an equation to plot a graph that represents this situation.

16 **(a)** Write an equation for a circle that does not have its centre at the origin and plot its graph.

(b) Write an equation for a parabola that has a maximum turning point and plot the graph.

(c) Write an equation for an exponential relationship that has an asymptote at $y = -1$.

17 When Sian threw a ball up in the air and then caught it, the path of the ball traced a parabolic shape.

(a) Sketch what you think might have happened on a set of axes.

(b) Describe the main features of the graph and explain why it has a parabolic shape.

18 Find the equations of at least two circles that pass through the point (1, 2) and have different centres.

Problem solving

How much is enough?

Bob needs to build a picket fence from his house to the garage. The curve formed by the top of the pickets follows the path of a parabola with equation:

$$y = \frac{3}{2.56}(x^2 - 3.2x + 2.56) + 1.2$$

where y is the height above the ground and x is the distance from the wall of the house. The pickets are 50 mm wide and have a gap of 50 mm between them. The length of the fence is 3.2 m. Each picket is shaped to a point at the top. What length of full width wood is used for the pickets? (A half-width picket is used at each end.)

Strategy options

- Draw a diagram.
- Make a model.
- Break problem into manageable parts.

Quadratic transformations

9.2

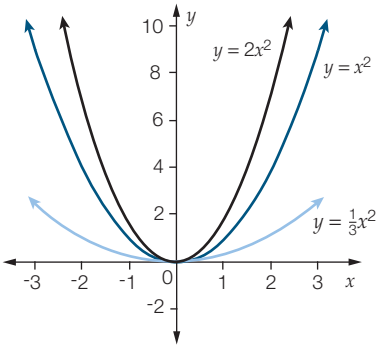
Stage 5.3§

Need to Know

Simple quadratic graphs can be sketched instead of plotting them when performing the **transformations** of **dilation**, **reflection** and **translation** of quadratics.

Dilations

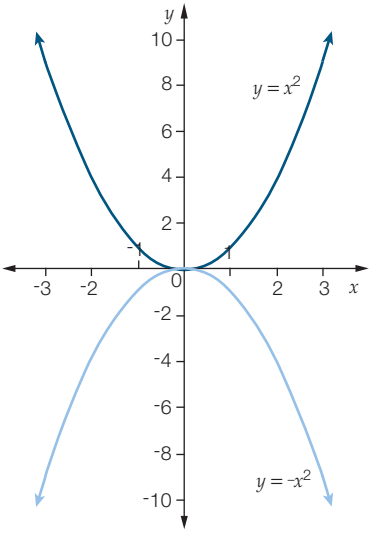
Graphing $y = x^2$, $y = \frac{1}{3}x^2$ and $y = 2x^2$ on the same set of axes shows that the coefficient of x^2 determines the shape of the graph, but the turning point is still the same. This effect is called dilation. The coefficient of x^2 is called the **dilation factor**.



- A factor greater than 1 will make the graph narrower.
- A factor less than 1 but greater than 0 will make the graph wider.

Reflections

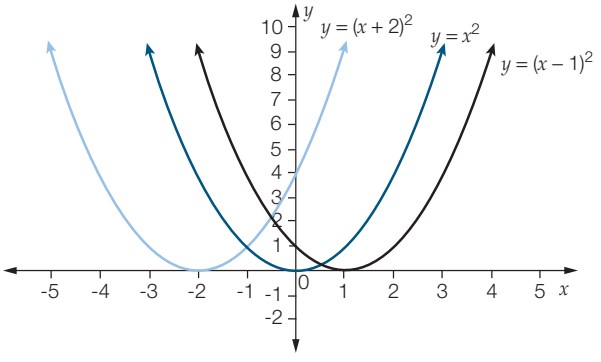
The graph of $y = -x^2$ is the same shape as the graph of $y = x^2$ except that it is upside down. We say it is *inverted* as it is a reflection in the x-axis. All graphs that have a negative coefficient for x^2 will have this same basic shape. The turning point of these graphs is a maximum.



Horizontal translations

Graphing $y = x^2$ with $y = (x - 1)^2$ and $y = (x + 2)^2$ shows that when a number is added or subtracted to x in the brackets, the basic graph $y = x^2$ is moved left or right. This movement is called a horizontal translation.

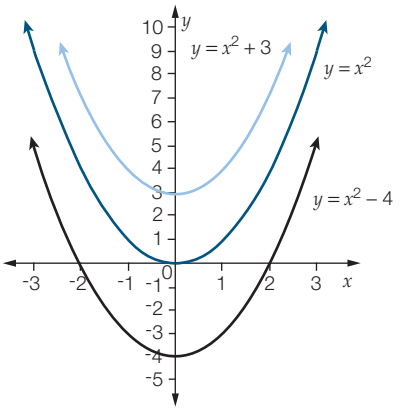
If $y = (x - h)^2$, h indicates the size and direction of the horizontal translation.



- When h is positive, the graph is moved right h units.
- When h is negative, the graph is moved left h units.

Vertical translations

Graphing $y = x^2$, $y = x^2 + 3$ and $y = x^2 - 4$ shows that the basic graph $y = x^2$ will move up when adding a number and down when subtracting a number. This movement is called a vertical translation.



If $y = x^2 + k$, the constant term k indicates the size and direction of the vertical translation.

- When k is positive, the graph is moved up k units.
- When k is negative, the graph is moved down k units.

Combined transformations

The basic graph of $y = x^2$ can be used to obtain other parabolas in the form of $y = a(x - h)^2 + k$ by:

- dilating it by a factor of a, making it narrower ($a > 1$) or wider ($0 < a < 1$)
- reflecting it in the x-axis (inverting it) ($a < 0$)
- translating it h units horizontally, k units vertically or in both directions.

Technology Exploration: GeoGebra

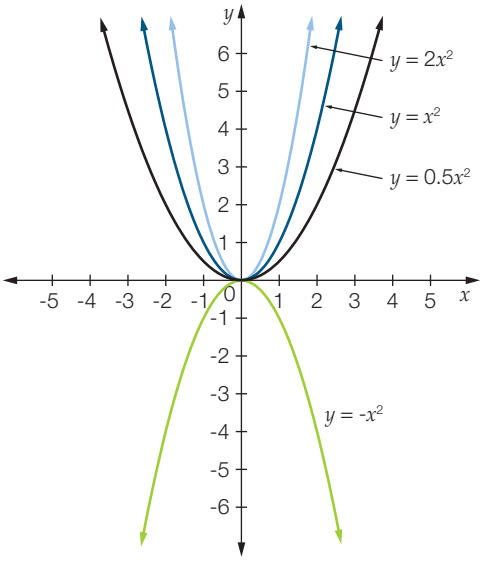
Exploring quadratic transformations

1 Open GeoGebra and right-click on the Graph View. Set the x-axis values from -6 to 6 and the y-axis values from -7 to 7. Label the x and y axes and set the Distance to 1.

2 In the Input bar enter **(a)** $y = x^2$. (Use ∧ to show the index.) The graph will appear on the screen.

On the same set of axes now draw **(b)** $y = 2x^2$, **(c)** $y = 0.5x^2$ and **(d)** $y = -x^2$.

Label each graph using the Insert Text icon.



3 What do you notice as the coefficient of x^2 changes?

4 Open a new window. In the Input bar enter:

(a) $y = x^2$ **(b)** $y = (x - 2)^2$ **(c)** $y = (x - 5)^2$

(d) $y = (x + 1)^2$ **(e)** $y = (x + 3)^2$

Label each graph using the Insert Text icon.

5 What do you notice as the number added to x before squaring changes?

6 Open a new window. In the Input bar enter:

(a) $y = x^2$ **(b)** $y = x^2 - 1$ **(c)** $y = x^2 - 3$

(d) $y = x^2 + 4$ **(e)** $y = x^2 + 6$

Label each graph using the Insert Text icon.

7 What do you notice when a constant is added to x^2?

8 Open a new window. In the Input bar enter:

(a) $y = x^2$ **(b)** $y = (x - 3)^2 - 4$ **(c)** $y = (x - 2)^2 + 1$

(d) $y = (x + 1)^2 - 3$ **(e)** $y = (x + 3)^2 + 5$

Label each graph using the Insert Text icon.

9 Describe the transformation for each of the graphs in question **8 (b)**, **(c)**, **(d)** and **(e)** from the graph of $y = x^2$.

10 For equations in the form of $y = a(x - h)^2 + k$:

(i) describe the effect changing a has on the graph

(ii) describe the effect changing h has on the graph

(iii) describe the effect changing k has on the graph.

11 Write down the turning points for all the graphs drawn in question **8**. Describe the relationship of the turning point to the values h and k in the equation $y = a(x - h)^2 + k$.

Worked Example 4

WE4

For each of the following equations:

(a) $y = 4x^2$ **(b)** $y = \frac{1}{4}x^2$ **(c)** $y = -3x^2$

(i) state the dilation factor

(ii) determine whether the graph is narrower or wider than the parabola $y = x^2$

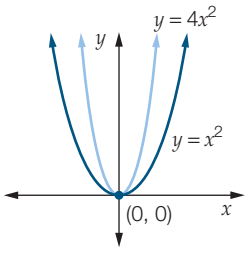
(iii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Working

(a) **(i)** The dilation factor is 4.

(ii) The graph is narrower than $y = x^2$.

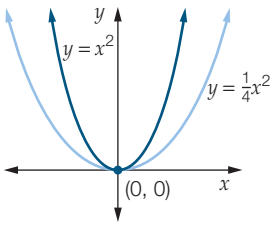
(iii)



(b) **(i)** The dilation factor is $\frac{1}{4}$.

(ii) The graph is wider than $y = x^2$.

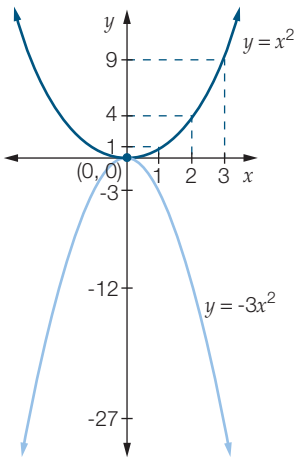
(iii)



(c) **(i)** The dilation factor is -3.

(ii) The graph is narrower than $y = x^2$.

(iii)



Worked Example 5

WE 5

For each of the following equations:

(a) $y = (x + 1.3)^2$ **(b)** $y = (x - 2)^2$

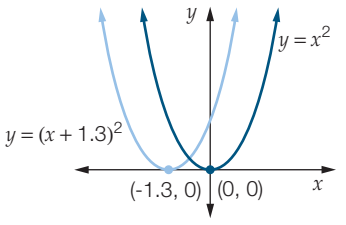
(i) state the translation of the graph of $y = x^2$

(ii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Working

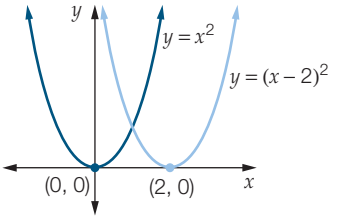
(a) **(i)** $y = x^2$ is translated 1.3 units to the left.

(ii)



(b) **(i)** $y = x^2$ is translated 2 units to the right.

(ii)



Worked Example 6

WE 6

For each of the following equations:

(a) $y = x^2 - 3$ **(b)** $y = x^2 + 2.5$

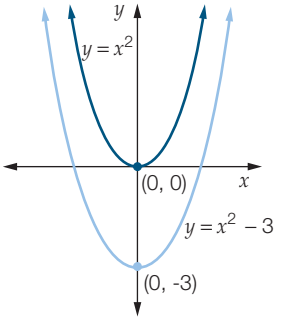
(i) state the translation of the graph of $y = x^2$

(ii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Working

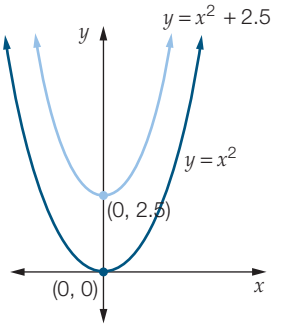
(a) **(i)** $y = x^2$ is translated 3 units down.

(ii)



(b) **(i)** $y = x^2$ is translated 2.5 units up.

(ii)



Worked Example 7

WE 7

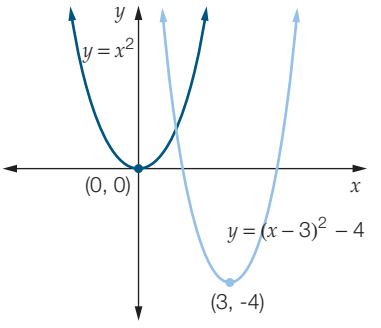
For each of the following equations, sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

(a) $y = (x - 3)^2 - 4$

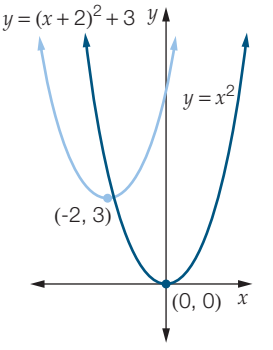
(b) $y = (x + 2)^2 + 3$

Working

(a) $y = x^2$ is translated 3 units right and 4 units down



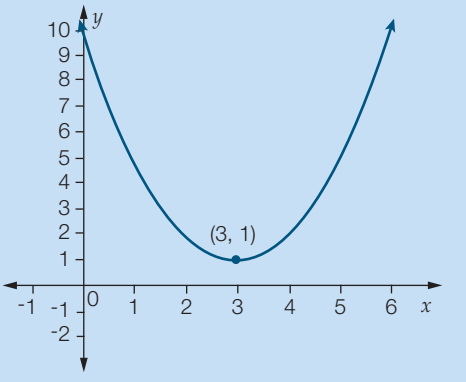
(b) $y = x^2$ is translated 2 units left and 3 units up



Worked Example 8

WE 8

Using the graph shown, find the equation of the curve, assuming there is no dilation.



Working

$y = a(x - h)^2 + k$

$a = 1$ (no dilation)

The turning point is at (3, 1) so $h = 3$ and $k = 1$.

$y = (x - 3)^2 + 1$

9.2 Quadratic transformations

Answers page 680

Navigator

Stage 5.3§

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17

Fluency and understanding

1 For each of the following equations: WE 4

(a) $y = 2x^2$ **(b)** $y = \frac{x^2}{2}$

(i) state the dilation factor

(ii) determine whether the graph is narrower or wider than the parabola $y = x^2$

(iii) sketch the graph on the same set of axes as the graph of $y = x^2$.

2 For each of the following equations: WE 5

(a) $y = (x - 1.3)^2$ **(b)** $y = (x + 2)^2$

(i) sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation

(ii) state the translation of the graph of $y = x^2$.

3 For each of the following equations: WE 6

(a) $y = x^2 + 6$ **(b)** $y = x^2 - 1.2$

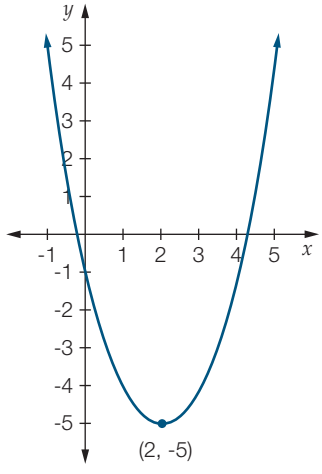
(i) sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation

(ii) state the translation of the graph of $y = x^2$.

4 For each of the following equations, sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation on the same set of axes. WE 7

(a) $y = (x + 3)^2 - 4$ **(b)** $y = (x - 2)^2 - 3$

5 Using the graph shown on the right, find the equation of the curve, assuming there is no dilation. WE 8



6 To sketch the graph of $y = (x + 5)^2 - 2$, the graph of $y = x^2$ will need to be translated:

A 5 units right and 2 units down **B** 5 units left and 2 units down

C 5 units left and 2 units up **D** 5 units right and 2 units up

7 On one set of axes, sketch and label the graphs of:

(a) $y = x^2$ **(b)** $y = x^2 - 5$ **(c)** $y = (x + 4)^2$ **(d)** $y = (x - 1)^2 + 5$

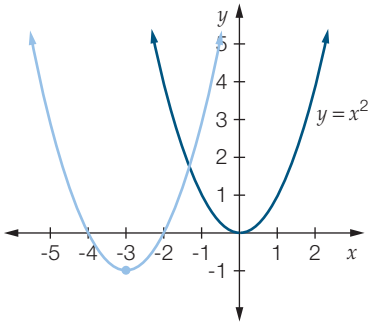
8 On one set of axes, sketch and label the graphs of:

(a) $y = x^2$ **(b)** $y = 3x^2$ **(c)** $y = \frac{x^2}{3}$ **(d)** $y = -x^2$

9 The equation of the parabola shown on the right is:

A $y = (x - 3)^2 + 1$ **B** $y = (x + 3)^2 + 1$

C $y = (x + 1)^2 + 3$ **D** $y = (x + 3)^2 - 1$



10 **(a)** Sketch and label the graph of $y = (x + 2)^2$.

(b) Use the graph from part **(a)** to sketch and label the graph of $y = -(x + 2)^2$ on the same set of axes.

11 **(a)** Sketch the graph of $y = x^2 + 4$.

(b) On the same set of axes, sketch and label the graph formed by dilating the graph in part **(a)** by a factor of 2.

(c) Write the equation for this graph.

Hint

It will have the same turning point as the graph in part **(a)**.

12 **(a)** Sketch the graph of $y = (x - 3)^2$.

(b) On the same set of axes, sketch and label the graph formed by dilating the graph in part **(a)** by a factor of 2.

(c) Write the equation for this graph.

Reasoning, communicating and open-ended

13 A quadratic graph has a turning point at (-3, 2) and a dilation factor of 1.

(a) Find the equation of this graph. **(b)** Sketch the graph of this equation.

14 A rope tied at both ends to a horizontal bar hangs in the shape of a parabola $y = x^2$. A photo of the rope shows that the bottom of the rope is 2 m left of a person in the photo and 0.5 m above ground level. If the person is standing on a spot marked (0, 0), what would be the equation of the rope?

15 An arch is built in the shape of a parabola to span a roadway. It is modelled by the equation $y = -(x - h)^2 + k$ where the origin (0, 0) is the base of the left-hand side of the arch. If the road is 6 m wide and the highest point of the arch is to be 9 m above ground level (assume the ground is flat), find the coordinates of lights if they are to be placed on the arch so they are horizontally 1 m apart and start and end at ground level.

16 **(a)** Write down an equation of a parabola that is a vertical translation of $y = x^2$ and sketch a graph of both $y = x^2$ and your equation on the same set of axes.

(b) Write down an equation of a parabola that is a horizontal translation of $y = x^2$ and sketch a graph of both $y = x^2$ and your equation on the same set of axes.

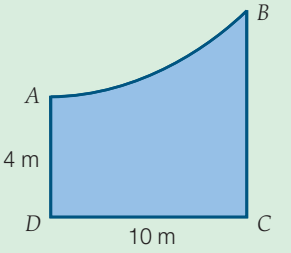
17 Freya and Syd are asked to sketch the graph of $y = (x + 6)^2 - 5$. Syd's graph was not correct. Freya's graph is correct. Syd thinks that the + sign in front of the 6 means that the graph should be moved 6 units to the right and the negative sign in front of the 5 means that the graph should be moved 5 units down. Where did Syd go wrong? What explanation would Freya give to him so that he doesn't make the same mistake again?

Problem solving

Finding curved areas

Erica wants to paint a wall of her architect-designed house. It has a shape that is described by the equation:

$$y = \frac{x^2}{50}$$



Some wall dimensions are shown on the diagram.

1 How high is the highest point on the wall?

2 Estimate the area by considering the shape to be a trapezium ($ABCD$).

Erica uses her CAS and finds that the actual area of the wall is 46.6 m^2.

3 How close can you get to this answer? Explain the method you used.

Strategy options

- Look for a pattern.
- Break problem into manageable parts.

9.3 Solving quadratic equations using the Null Factor Law

Stage 5.2

Need to Know

Null Factor Law

The x-intercepts of a quadratic graph can be found by solving quadratic equations. When sketching graphs and solving problems that involve quadratic relationships, solutions to quadratic equations can be obtained by factorising the equations and using the **Null Factor Law**. This law is based on the fact that if the product of two or more expressions is equal to zero, then at least one of the factors must equal zero.

If $ab = 0$, then $a = 0$, $b = 0$ or $a = b = 0$.

Solving quadratic equations algebraically

1. Equate one side (usually RHS) to zero.
2. Factorise the expression by looking for:
   - a common factor
   - a monic trinomial where the middle term can be split so that the coefficients of the two middle terms multiply to give the constant term.
3. Use the Null Factor Law to find solutions.

You may need to take out the coefficient of x^2 as a common factor and divide both sides by the common factor before factorising.

Solving algebraically gives exact solutions. Not all quadratics cross or touch the x-axis, so not all quadratic equations will have solutions.

Worked Example 9

WE 9

Solve the following equations for x using the Null Factor Law.

(a) $x(x - 2) = 0$

(b) $(x + 1)(x - 6) = 0$

Working

(a) $x(x - 2) = 0$
$x = 0$ or $x - 2 = 0$
$x = 0$ or $x = 2$

(b) $(x + 1)(x - 6) = 0$
$x + 1 = 0$ or $x - 6 = 0$
$x = -1$ or $x = 6$

Worked Example 10

WE 10

Solve each equation for x:

(a) $4x^2 = 0$

(b) $(x + 1)^2 = 0$

Working

(a)
$$4x^2 = 0$$
$$x^2 = 0$$
$$x = 0$$

(b)
$$(x + 1)^2 = 0$$
$$x + 1 = 0$$
$$x = -1$$

Worked Example 11

WE 11

Solve each equation for x:

(a) $3x^2 - 6x = 0$

(b) $11x^2 = x$

Working

(a)
$$3x^2 - 6x = 0$$
$$3x(x - 2) = 0$$
$$3x = 0 \text{ or } x - 2 = 0$$
$$x = 0 \text{ or } x = 2$$

(b)
$$11x^2 = x$$
$$11x^2 - x = 0$$
$$x(11x - 1) = 0$$
$$x = 0 \text{ or } 11x - 1 = 0$$
$$x = 0 \text{ or } 11x = 1$$
$$x = 0 \text{ or } x = \frac{1}{11}$$

Worked Example 12

WE 12

Solve the following quadratic equations for x.

(a) $x^2 - 6x + 8 = 0$

(b) $2x^2 + 8x = 24$

(c) $-x^2 - 4x - 3 = 0$

Working

(a)
$$x^2 - 6x + 8 = 0$$
$$x^2 - 2x - 4x + 8 = 0$$
(splitting the middle term)
$$x(x - 2) - 4(x - 2) = 0$$
$$(x - 2)(x - 4) = 0$$
$$x - 2 = 0 \text{ or } x - 4 = 0$$
$$x = 2 \text{ or } x = 4$$

(b)
$$2x^2 + 8x = 24$$
$$2x^2 + 8x - 24 = 0$$
$$2(x^2 + 4x - 12) = 0$$
$$x^2 + 4x - 12 = 0$$
$$x^2 + 6x - 2x - 12 = 0$$
(splitting the middle term)
$$x(x + 6) - 2(x + 6) = 0$$
$$(x + 6)(x - 2) = 0$$
$$x + 6 = 0 \text{ or } x - 2 = 0$$
$$x = -6 \text{ or } x = 2$$

(c)
$$-x^2 - 4x - 3 = 0$$
$$-1(x^2 + 4x + 3) = 0$$
$$x^2 + 4x + 3 = 0$$
$$x^2 + x + 3x + 3 = 0$$
(splitting the middle term)
$$x(x + 1) + 3(x + 1) = 0$$
$$(x + 1)(x + 3) = 0$$
$$x + 1 = 0 \text{ or } x + 3 = 0$$
$$x = -1 \text{ or } x = -3$$

Answers page 681

9.3 Solving quadratic equations using the Null Factor Law

Stage 5.2

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5 Column 1, Q6 Column 1, Q7 Column 1, Q8, Q9 Column 1, Q10, Q11, Q12, Q13, Q14, Q15, Q18	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5 Column 2, Q6 Column 2, Q7 Column 2, Q8, Q9 Column 2, Q10, Q11, Q12, Q13, Q15, Q16, Q18	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5 Column 3, Q6 Column 3, Q7 Column 3, Q8, Q9 Column 3, Q10, Q11, Q12, Q13, Q15, Q16, Q17

Fluency and understanding

WE9

1 Solve the following equations for x using the Null Factor Law.

(a) $x(x-11)=0$ **(b)** $x(x+6)=0$ **(c)** $x(x+4)=0$

(d) $2x(x-1)=0$ **(e)** $3x(x-1)=0$ **(f)** $-4x(x-6)=0$

(g) $(x-3)(x+7)=0$ **(h)** $(x-4)(x+1)=0$ **(i)** $(x-2)(x-6)=0$

(j) $(4-x)(x-5)=0$ **(k)** $(7-x)(x+2)=0$ **(l)** $(3-x)(x+10)=0$

WE10

2 Solve the following quadratic equations for x.

(a) $-6x^2=0$ **(b)** $3x^2=0$ **(c)** $2x^2=0$

(d) $(x+7)^2=0$ **(e)** $(x+1)^2=0$ **(f)** $(x-9)^2=0$

WE11

3 Solve the following equations for x.

(a) $x^2-4x=0$ **(b)** $x^2+x=0$ **(c)** $x^2-6x=0$

(d) $3x^2=15x$ **(e)** $4x^2=2x$ **(f)** $4x^2=-12x$

WE12

4 Solve the following quadratic equations for x.

(a) $x^2+3x+2=0$ **(b)** $x^2+9x+18=0$ **(c)** $x^2-11x+28=0$

(d) $x^2-10x+9=0$ **(e)** $x^2-x-30=0$ **(f)** $x^2+2x-8=0$

(g) $2x^2-24x+64=0$ **(h)** $2x^2+4x-30=0$ **(i)** $3x^2-18x-48=0$

(j) $5x^2+20x-60=0$ **(k)** $4x^2-4x-8=0$ **(l)** $6x^2-12x-18=0$

5 Rewrite each of the following equations so that the right-hand side equals zero, then solve each equation.

(a) $x^2+7x=-12$ **(b)** $x^2+8x=-12$ **(c)** $x^2-2x=63$

(d) $x^2+5x=24$ **(e)** $-x^2-5x=6$ **(f)** $-x^2+2x=-15$

(g) $x^2+5x-20=30$ **(h)** $x^2-6x-13=14$ **(i)** $x^2+66=17x$

(j) $x^2+5=6x$ **(k)** $-x^2+16=6x$ **(l)** $-x^2+21=4x$

6 **(a)** The solution/s to $x(x-2)=0$ is/are:

A $x=2$ **B** $x=-2$ **C** $x=0, x=-2$ **D** $x=0, x=2$

(b) The solutions to $(x+3)(x-4)=0$ are:

A $x=-3, x=4$ **B** $x=3, x=-4$ **C** $x=-3, x=-4$ **D** $x=3, x=4$

7 Solve the following equations.

(a) $x^2 + 3x = 10$ (b) $x^2 - 2x = 3$ (c) $x^2 - 3x - 18 = 10$

(d) $x^2 + 8x + 12 = 5$ (e) $(x + 4)(x + 3) = 6$ (f) $(x - 5)(x + 6) = 12$

8 (a) The solution/s to $x^2 - 9x + 18 = 0$ is/are:

A $x = 3, x = 6$ B $x = -9$ C $x = 3, x = -6$ D $x = -3, x = -6$

(b) The solutions to $x^2 + 6x + 5 = 0$ are:

A $x = 1, x = 5$ B $x = -1, x = 5$ C $x = 1, x = -5$ D $x = -1, x = -5$

(c) The solutions to $3x^2 - 12x - 63 = 0$ are:

A $x = 3, x = 7$ B $x = -3, x = 7$ C $x = 3$ D $x = 3, x = -7$

(d) The solutions to $-x^2 + x + 12 = 0$ are:

A $x = 3, x = 4$ B $x = -3, x = 4$ C $x = 3, x = -4$ D $x = -3, x = -4$

(e) The solutions to $x^2 + 3x = 4$ are:

A $x = 1, x = 3$ B $x = 1, x = -4$ C $x = 1, x = 4$ D $x = -1, x = 4$

9 The height, h centimetres, of a ball above the ground when it is being bounced is given by $h = t(50 - t)$, where t is the time in seconds. Find the two times (in seconds) when the ball is on the ground, that is, when its height (h) is 0 cm.

10 A graph has the equation $y = (x + 1)(x - 4)$. Find the x-intercepts of this graph.

11 Determine the x-intercepts of the following quadratic relationships.

(a) $y = (x - 8)(x + 5)$ (b) $y = 6x^2 + 24x$ (c) $y = x^2 - 11x - 12$

Reasoning, communicating and open-ended

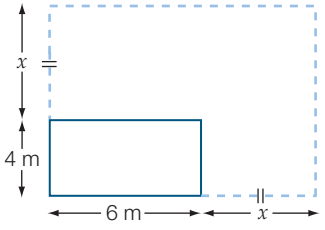
12 The temperature, T degrees Celsius, is measured during the day and found to follow the quadratic relationship $T = -t^2 + 10t + 5$, where t represents the time in hours after 8 a.m.

(a) Find the temperature, T, when t is equal to 0.

(b) By substituting for T, find the values of t when the temperature is 21°C.

(c) At what times is the temperature 21°C?

13 At the zoo, a wombat enclosure is to be enlarged to accommodate recent arrivals.



(a) Write an expression for the area of the new enclosure if the same length is to be added to two sides, as in the diagram.

(b) The new enclosure is to have double the area of the existing one. Find the dimensions of the new wombat enclosure.

14 Gerald hit his golf ball down the fairway. The path the ball followed was parabolic in shape with the relationship:

$$h = 2d - \frac{1}{30}d^2$$

where h is the height above the ground of the golf ball when it is a horizontal distance d from him. Both h and d are in metres. How far from Gerald did the golf ball land?

15 To determine the number of diagonals (D) in a polygon containing n sides, the following quadratic equation can be used:

$$D = \frac{n}{2}(n - 3)$$

(a) Show that a triangle contains no diagonals.

(b) Show that the given quadratic equation is true for each of these polygons—quadrilateral, pentagon, hexagon—by determining the number of diagonals algebraically and then diagrammatically.

(c) Calculate the number of diagonals in:

(i) an octagon **(ii)** a decagon **(iii)** a dodecagon.

(d) How many sides has a polygon that has:

(i) 170 diagonals **(ii)** 299 diagonals?

(e) A similar equation can be used to determine the number of consecutive integers, beginning at 1, required to sum to a particular value.

$$S = \frac{n}{2}(n + 1)$$

Use the equation to find the number of integers, beginning at 1, required to sum to:

(i) 105 **(ii)** 903

16 Choose pairs of values for x. For each pair, write equations for which these are the solution.

17 Decide on two values of x that will be the x-intercepts of a quadratic graph. Write down at least two possible equations for the graph.

18 Write three different quadratic equations that have two solutions lying between $x = -3$ and $x = 3$. At least one of the equations should produce a solution containing a surd.

Solving quadratic equations using special product rules

9.4

Stage 5.3§

Need to Know

Some quadratic equations can be solved easily if the quadratic expression can be recognised as a special product. The quadratic expression may be a perfect square or the difference of two squares.

Perfect Square rules: $a^2 + 2ab + b^2 = (a + b)^2$

$a^2 - 2ab + b^2 = (a - b)^2$

Difference of Two Squares rule: $a^2 - b^2 = (a + b)(a - b)$

Solving quadratic equations using special products

1. Equate one side of the equation to zero.
2. Factorise the expression by looking for:
   - a common factor
   - a perfect square
   - difference of two squares.
3. Use the Null Factor Law to find solutions.

Worked Example 13

WE13

Solve the following equations for x.

(a) $x^2 - 49 = 0$ **(b)** $4x^2 = 25$ **(c)** $5x^2 = 3$

Working

(a)

$$x^2 - 49 = 0$$
$$(x+7)(x-7) = 0$$

(Using the Difference of Two Squares rule)

$x + 7 = 0$ or $x - 7 = 0$

$x = -7$ or $x = 7$

$x = \pm 7$

(b)

$$4x^2 = 25$$
$$4x^2 - 25 = 0$$
$$(2x)^2 - (5)^2 = 0$$
$$(2x+5)(2x-5) = 0$$

(Using the Difference of Two Squares rule)

$2x + 5 = 0$ or $2x - 5 = 0$

$x = -\frac{5}{2}$ or $x = \frac{5}{2}$

$x = \pm\frac{5}{2}$

(c)

$$5x^2 = 3$$
$$5x^2 - 3 = 0$$
$$x^2 - \frac{3}{5} = 0$$
$$\left(x + \sqrt{\frac{3}{5}}\right)\left(x - \sqrt{\frac{3}{5}}\right) = 0$$

(Using the Difference of Two Squares rule)

$x + \sqrt{\frac{3}{5}} = 0$ or

$x - \sqrt{\frac{3}{5}} = 0$

$x = -\sqrt{\frac{3}{5}}$ or $x = \sqrt{\frac{3}{5}}$

$x = \pm\sqrt{\frac{3}{5}}$

Worked Example 14

WE14

Solve the following equations for x.

(a) $x^2 + 10x + 25 = 0$ **(b)** $2x^2 - 36x = -162$

Working

(a) $x^2 + 10x + 25 = 0$

$(x+5)(x+5) = 0$

(Using the Perfect Square rule)

$$(x+5)^2 = 0$$
$$x + 5 = 0$$
$$x = -5$$

(b)

$$2x^2 - 36x = -162$$
$$2x^2 - 36x + 162 = 0$$
$$2(x^2 - 18x + 81) = 0$$
$$2(x-9)^2 = 0$$

(Using the Perfect Square rule)

$$(x-9)^2 = 0$$
$$x - 9 = 0$$
$$x = 9$$

9.4 Solving quadratic equations using special product rules

Answers page 682

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

WE13

1 Solve the following quadratic equations for x.

(a) $x^2 - 25 = 0$ (b) $x^2 - 81 = 0$ (c) $x^2 - 4 = 0$

(d) $16x^2 - 25 = 0$ (e) $9x^2 - 49 = 0$ (f) $100x^2 - 81 = 0$

(g) $3x^2 - 75 = 0$ (h) $4x^2 - 4 = 0$ (i) $7x^2 - 28 = 0$

(j) $49 - x^2 = 0$ (k) $81 - x^2 = 0$ (l) $121 - x^2 = 0$

(m) $x^2 - 13 = 0$ (n) $x^2 - 24 = 0$ (o) $x^2 - 32 = 0$

WE14

2 Solve the following quadratic equations for x.

(a) $x^2 - 14x + 49 = 0$ (b) $x^2 + 8x + 16 = 0$ (c) $x^2 + 16x + 64 = 0$

(d) $x^2 - 4x + 4 = 0$ (e) $x^2 + 2x + 1 = 0$ (f) $x^2 - 6x + 9 = 0$

(g) $2x^2 - 20x + 50 = 0$ (h) $5x^2 - 10x + 5 = 0$ (i) $6x^2 + 24x + 24 = 0$

(j) $3x^2 - 18x = -27$ (k) $5x^2 - 50x = -125$ (l) $24x - 3x^2 = 48$

3 (a) The solution/s to $x^2 - 12x + 36 = 0$ is/are:

A $x = 0, x = -6$ B $x = -6$ C $x = 6, x = 0$ D $x = 6$

(b) The solution/s to $x^2 - 64 = 0$ is/are:

A $x = 8, x = -8$ B $x = 8$ C $x = 0, x = 8$ D $x = 0, x = -8$

4 Solve the following equations.

(a) $3x^2 - 28 = 20$ (b) $2x^2 - 15 = 3$ (c) $(x - 6)^2 = 36$

5 The solution/s to $x^2 - 4x = -4$ is/are:

A $x = 2, x = -2$ B $x = 4, x = -1$ C $x = 2$ D $x = 4, x = 1$

6 Determine the x-intercepts of the following quadratic equations.

(a) $y = x^2 - 144$ (b) $y = x^2 - 12x + 36$

Reasoning, communicating and open-ended

7 Solve the following quadratic equations.

(a) $x^4 - 10x^2 + 9 = 0$ (b) $x^4 - 29x^2 + 100 = 0$

8 The dimensions of a square room are altered so that the length is increased by 80 cm and the breadth is decreased by the same amount. The area is now 9.6 m^2. What are the new dimensions of the room?

9 A block of land has a house with square dimensions built on it leaving 105 m^2 for lawn and garden. Given that the length of the block of land was 14 m and the breadth was 4 m wider than the dimensions of the house, find the area of the block of land.

10 Choose a single value of x. Write down a quadratic equation for which this x value is the only solution.

9.5 Solving quadratics by completing the square

Stage 5.3§

Need to Know

If a quadratic equation cannot be solved to give rational solutions, it may have irrational solutions that involve surds. These quadratic equations can be solved by completing the square to produce the difference of two squares. Not all quadratic equations have real solutions.

Solving by completing the square

1. Equate one side of the equation to zero.
2. Divide each term by the coefficient of x^2, if it is not 1, to produce a monic quadratic equation.
3. Complete the square on the equation. (Divide the coefficient of x by 2, square this value, add it on to the quadratic expression and subtract the same amount to keep the equation equivalent. Factorise the perfect square formed and then use the difference of two squares to complete the factorisation.)
4. Use the Null Factor Law to solve the quadratic.

Worked Example 15

WE15

For each of the following expressions:

(a) $x^2 - 6x + 3$ **(b)** $x^2 + 5x - 2$

(i) Complete the square and factorise.

(ii) Equate the expression to zero and use your answer for part **(i)** to solve the quadratic equation formed.

Working

(a) **(i)** $x^2 - 6x + 3$

$= [x^2 - 6x + \left(\frac{6}{2}\right)^2] - \left(\frac{6}{2}\right)^2 + 3$

(completing the square)

$= (x^2 - 6x + 9) - 9 + 3$

$= (x - 3)^2 - 6$

$= (x - 3)^2 - (\sqrt{6})^2$

$= (x - 3 + \sqrt{6})(x - 3 - \sqrt{6})$

(Difference of Two Squares rule)

(ii) $x^2 - 6x + 3 = 0$

$(x - 3 + \sqrt{6})(x - 3 - \sqrt{6}) = 0$

$(x - 3 + \sqrt{6}) = 0$

or $(x - 3 - \sqrt{6}) = 0$

$x = 3 - \sqrt{6}$ or $x = 3 + \sqrt{6}$

$x = 3 \pm \sqrt{6}$

(b) **(i)** $x^2 + 5x - 2$

$= [x^2 + 5x + \left(\frac{5}{2}\right)^2] - \left(\frac{5}{2}\right)^2 - 2$

(completing the square)

$= (x + \frac{5}{2})^2 - \frac{25}{4} - 2$

$= (x + \frac{5}{2})^2 - \frac{33}{4}$

$= (x + \frac{5}{2})^2 - \left(\sqrt{\frac{33}{4}}\right)^2$

$= (x + \frac{5}{2} - \sqrt{\frac{33}{4}})(x + \frac{5}{2} + \sqrt{\frac{33}{4}})$

(Difference of Two Squares rule)

(ii) $x^2 + 5x - 2 = 0$

$(x + \frac{5}{2} - \sqrt{\frac{33}{4}})(x + \frac{5}{2} + \sqrt{\frac{33}{4}}) = 0$

$(x + \frac{5}{2} - \sqrt{\frac{33}{4}}) = 0$

or $(x + \frac{5}{2} + \sqrt{\frac{33}{4}}) = 0$

$x = -\frac{5}{2} + \sqrt{\frac{33}{4}}$ or $x = -\frac{5}{2} - \sqrt{\frac{33}{4}}$

$x = \frac{-5}{2} \pm \sqrt{\frac{33}{4}}$ or $x = \frac{-5 \pm \sqrt{33}}{2}$

Worked Example 16 WE16

Solve $x^2 + 6x - 2 = 0$ using the method of completing the square. Leave your answer in exact surd form.

Working

$$x^2 + 6x - 2 = 0$$

$$[x^2 + 6x + \left(\frac{6}{2}\right)^2] - \left(\frac{6}{2}\right)^2 - 2 = 0 \qquad \text{(completing the square)}$$

$$(x^2 + 6x + 9) - 9 - 2 = 0$$

$$(x + 3)^2 - 11 = 0$$

$$(x + 3)^2 - (\sqrt{11})^2 = 0$$

$$(x + 3 - \sqrt{11})(x + 3 + \sqrt{11}) = 0 \qquad \text{(Difference of Two Squares rule)}$$

$x + 3 - \sqrt{11} = 0$ or $x + 3 + \sqrt{11} = 0$

$x = -3 + \sqrt{11}$ or $x = -3 - \sqrt{11}$

$x = -3 \pm \sqrt{11}$

Answers page 682

9.5 Solving quadratics by completing the square

Stage 5.3§

Navigator

Q1 Column 1, Q2 Column 1, Q3, Q4 Column 1, Q5, Q11	Q1 Column 2, Q2 Column 2, Q3, Q4 Column 2, Q5, Q6, Q7, Q8, Q10, Q11	Q1 Column 3, Q2 Column 3, Q3, Q4 Column 3, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12

Fluency and understanding

WE15

1 For each of the following expressions:

(a) $x^2 + 2x - 5$ **(b)** $x^2 + 4x - 7$ **(c)** $x^2 + 8x + 5$

(d) $x^2 + 3x - 1$ **(e)** $x^2 + 5x + 3$ **(f)** $x^2 + 3x - 5$

(i) Complete the square and factorise.

(ii) Equate the expression to zero and use your answer for part **(i)** to solve the quadratic equation formed.

WE16

2 Solve these quadratic equations using the method of completing the square. Leave your answers in exact surd form.

(a) $x^2 - 8x + 6 = 0$ **(b)** $x^2 + 6x + 4 = 0$ **(c)** $x^2 - 3x + 1 = 0$

(d) $x^2 - 5x - 2 = 0$ **(e)** $x^2 + 4x - 3 = 0$ **(f)** $x^2 - 20x - 12 = 0$

(g) $2x^2 - 8x - 10 = 0$ **(h)** $3x^2 + 12x + 3 = 0$ **(i)** $4x^2 + 16x - 20 = 0$

3 (a) The solution to $2x^2 + 4x - 4 = 0$ is:

A $x = 1 \pm \sqrt{3}$ B $x = -2$ or 4 C $x = -1 \pm \sqrt{3}$ D $x = -4$ or 2

(b) The solution to $x^2 - 4x - 3 = 0$ is:

A $x = 4$ or -1 B $x = 2 \pm \sqrt{7}$ C $x = -2 \pm \sqrt{7}$ D $x = -4$ or 1

(c) The solution to $x^2 - 8x + 14 = 0$ is:

A $x = -4$ or $-\sqrt{2}$ B $x = 4\sqrt{2}$ or $-4\sqrt{2}$

C $x = -4 - \sqrt{2}$ or $-4 + \sqrt{2}$ D $x = 4 + \sqrt{2}$ or $4 - \sqrt{2}$

4 Solve the following quadratic equations.

(a) $x^2 - 4x = 2$ (b) $x^2 = 6x - 2$ (c) $3 + 4x = x^2$

(d) $x^2 = 5 + 3x$ (e) $x^2 - 5x = -2$ (f) $x^2 = 5 - 7x$

5 (a) Explain why the quadratic equation $x^2 + 4x + 7$ cannot be factorised.

(b) Explain why the quadratic equation $x^2 + 4x + 7 = 0$ has no solution.

(c) As $x^2 + 4x + 7 = 0$ has no solution, what does this tell you about the graph of $y = x^2 + 4x + 7$?

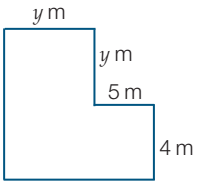
Reasoning, communicating and open-ended

6 An envelope has an area of 80 cm^2. If the length is 4 cm longer than the breadth, find the dimensions of the envelope:

(a) exactly

(b) rounded to one decimal place.

7 An area of lawn as shown in the diagram is to be mown. Find the value y represents if the total area mown is 44 m^2.



8 If the dimensions of a rectangular park are $(x + 4)$ m and $(x - 5)$ m:

(a) find the exact value of x when the area of the park is 180 m^2

(b) find the value of x rounded to two decimal places.

9 A photograph of length l cm and breadth $(l - 2)$ cm is surrounded by a 2 cm rectangular frame. Altogether, the photo and the frame occupy an area of 215 cm^2.

(a) Write the dimensions of the photo and frame in terms of l.

(b) Write an equation in terms of l for the area of the photo and frame.

(c) Find the exact dimensions of the photo.

(d) Find the dimensions of the photo and frame correct to one decimal place.

10 Write a quadratic equation that gives solutions containing surds. Solve your equation.

11 Write a quadratic equation that has no solution. Demonstrate how you know that it has no solutions.

12 By using the process of completing the square, the solution $x = -a \pm \sqrt{3}$ was obtained. By choosing a value for a between 0 and 5, write a possible quadratic equation that could have this solution.

9.6 Solution of non-monic quadratics

Stage 5.3§

Need to Know

Solving non-monic quadratic equations: $ax^2 + bx + c = 0$

1 Factorise non-monic quadratic expressions by:

- splitting the middle term
- using the cross method
- completing the square.

2 Use the Null Factor Law to solve for x.

The quadratic formula

Non-monic equations can also be solved using the quadratic formula. This method is useful when it is hard to find rational factors for the quadratic expression or when the solutions are irrational.

The solution to $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Testing for solutions

If the quadratic expression does not factorise, there will be no solutions for the quadratic equation.

Using the quadratic formula, if $b^2 - 4ac$ is negative, there will be no solutions.

If the quadratic expression is a perfect square, there will be only one solution for the quadratic equation.

Using the quadratic formula, if $b^2 - 4ac$ equals zero, there will be only one solution.

Worked Example 17

WE17

Solve the quadratic equation $3x^2 - 11x - 4 = 0$. Factorise the expression first by splitting the middle term.

Working

$a = 3,\ b = -11,\ c = -4$

$3x^2 - 11x - 4 = 0$

$ac = 3 \times (-4) = -12$

$\left.\begin{array}{l} \times \text{ to give } -12 \\ + \text{ to give } -11 \end{array}\right\} -12, 1$

$3x^2 - 12x + x - 4 = 0$

$3x(x - 4) + 1(x - 4) = 0$

$(x - 4)(3x + 1) = 0$

$x - 4 = 0$ or $3x + 1 = 0$

$x = 4$ or $x = -\dfrac{1}{3}$

Worked Example 18

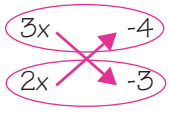
WE18

Solve $6x^2 - 17x + 12 = 0$. Factorise the expression first using the cross method.

Working

$a = 6,\ b = -17,\ c = 12$

$6x^2 - 17x + 12 = 0$



$-9x - 8x = -17x$

$(3x - 4)(2x - 3) = 0$

$3x - 4 = 0 \text{ or } 2x - 3 = 0$

$3x = 4 \text{ or } 2x = 3$

$x = \frac{4}{3} \text{ or } x = \frac{3}{2}$

Worked Example 19

WE19

Solve $2x^2 - 7x + 2 = 0$. Factorise the expression first, by completing the square on x.

Working

$$2x^2 - 7x + 2 = 0$$

$$2\left(x^2 - \frac{7}{2}x + 1\right) = 0$$

$$x^2 - \frac{7}{2}x + 1 = 0$$

$$\left[x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2\right] + 1 - \left(\frac{49}{16}\right) = 0$$

$$\left(x - \frac{7}{4}\right)^2 - \frac{33}{16} = 0$$

$$\left(x - \frac{7}{4} + \frac{\sqrt{33}}{4}\right)\left(x - \frac{7}{4} - \frac{\sqrt{33}}{4}\right) = 0$$

$$x - \frac{7}{4} + \frac{\sqrt{33}}{4} = 0 \text{ or } x - \frac{7}{4} - \frac{\sqrt{33}}{4} = 0$$

$$x = \frac{7}{4} - \frac{\sqrt{33}}{4} \text{ or } x = \frac{7}{4} + \frac{\sqrt{33}}{4}$$

$$x = \frac{7 \pm \sqrt{33}}{4}$$

Worked Example 20 WE20

Use the quadratic formula to solve $3x^2 - 4x - 5 = 0$. Give your answers in exact form and rounded to two decimal places.

Working

$a = 3,\ b = -4,\ c = -5$

$3x^2 - 4x - 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-5)}}{2 \times 3}$$

$$= \frac{4 \pm \sqrt{16 + 60}}{6}$$

$$= \frac{4 \pm \sqrt{76}}{6}$$

$x = 2.12$ or $x = -0.79$

Answers page 683

9.6 Solution of non-monic quadratics

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q13, Q14, Q16, Q17, Q20	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19 (a), (b), Q20	Q1, Q2, Q3 (c), (d), Q4 (b), (c), Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22

Equipment required: calculator for Question **4**

Fluency and understanding

WE17

1 Solve each quadratic equation. Factorise the expression first by splitting the middle term:

(a) $3x^2 - 4x - 4 = 0$ **(b)** $3x^2 + 8x + 4 = 0$

(c) $5x^2 - 11x + 6 = 0$ **(d)** $4x^2 + 3x - 10 = 0$

WE18

2 Solve each equation. Factorise the expression first using the cross method.

(a) $5x^2 - 17x + 12 = 0$ **(b)** $3x^2 + 8x + 5 = 0$

(c) $6x^2 - x - 2 = 0$ **(d)** $8x^2 + 6x - 9 = 0$

WE19

3 Solve each equation. Factorise the expression first by completing the square on x.

(a) $2x^2 - 5x + 2 = 0$ **(b)** $3x^2 + 10x + 1 = 0$

(c) $5x^2 - 2x - 6 = 0$ **(d)** $6x^2 + 9x - 2 = 0$

4 Use the quadratic formula to solve each equation. Give your answers in **(i)** exact form and **(ii)** rounded to two decimal places. WE20

(a) $3x^2 - 5x - 4 = 0$ **(b)** $5x^2 + 10x + 1 = 0$

(c) $2x^2 - 11x + 6 = 0$ **(d)** $x^2 + 5x - 3 = 0$

5 For each of these quadratic equations:

(a) $x^2 + 5x + 2 = 0$ **(b)** $2x^2 - 3x + 1 = 0$

(c) $-3x^2 - x + 3 = 0$ **(d)** $5x^2 + 4x = 0$

(i) list the values that a, b and c represent if the equation is of the form $ax^2 + bx + c = 0$

(ii) find the solution to each equation using the quadratic formula.

6 The solutions to the equation $10x^2 - 23x + 12 = 0$ are:

A $x = 3$ or $x = 4$ **B** $x = \frac{2}{3}$ or $x = \frac{5}{4}$

C $x = 1.5$ or $x = 0.8$ **D** $x = -1.5$ or $x = -0.8$

7 Solve each quadratic equation using an appropriate method.

(a) $3x^2 - 5x + 2 = 0$ **(b)** $5x^2 + 10x + 2 = 0$

(c) $4x^2 - 9 = 0$ **(d)** $x^2 + 7x - 9 = 0$

(e) $6x^2 - 7x - 10 = 0$ **(f)** $4x^2 + 4x + 1 = 0$

(g) $7x^2 - 6 = 0$ **(h)** $8x^2 + 6x - 4 = 0$

(i) $36x^2 - 60x + 25 = 0$ **(j)** $3x^2 - 11x + 7 = 0$

8 Which of the following equations has the solutions $x = 1.5$ or $x = 2.5$?

A $x^2 - 16x + 15 = 0$ **B** $4x^2 - 16x + 15 = 0$

C $x^2 - 8x + 15 = 0$ **D** $4x^2 + 16x + 15 = 0$

9 The solutions to $x^2 + px + q = 0$ are:

A $x = \dfrac{-p \pm \sqrt{q}}{2}$ **B** $x = \dfrac{p \pm \sqrt{p^2 + 4q}}{2}$

C $x = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$ **D** $x = \dfrac{-2p \pm \sqrt{p^2 - 4q}}{2}$

10 Rick's driving speed was measured over a 10-minute period and the following relationship was found to exist: $s = -4t^2 + 31t$, where s is the speed in kilometres per hour after t minutes. When was Rick travelling at 60 km/h?

11 The temperature inside a tent was measured over a period of time and the following quadratic relationship was found to exist: $T = -2h^2 + 11h + 21$, where T is the temperature in degrees Celsius after h hours.

(a) When was the temperature 0°C?

(b) When was the temperature 26°C?

12 A tourist, high above the ground enjoying the sights from a hot-air balloon, unfortunately drops a camera and watches it fall to the ground. The height of the camera above the ground, h (in metres), t seconds after it has been dropped can be represented by the relationship $h = -5t^2 + 192$.

(a) At what height above the ground is the camera dropped?

(b) How long does it take for the camera to fall to the ground? Round your answer to one decimal place.

Reasoning, communicating and open-ended

13 Use the quadratic formula to show that none of the following equations has a solution. Write the part of the quadratic formula that indicates that no solutions are possible.

(a) $2x^2 - 3x + 2 = 0$

(b) $5x^2 + 5x + 2 = 0$

(c) $4x^2 - 6x + 9 = 0$

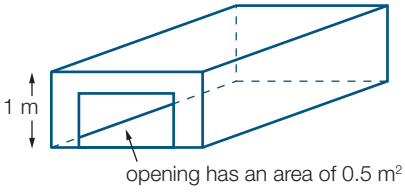
(d) $2x^2 + 7x + 9 = 0$

14 A policeman on a motorbike is pursuing a car along a highway. After a short time, the driver of the car notices the policeman and starts to slow down, finally stopping on the side of the road. The speed of the car during this time can be represented by the quadratic function $S = -3t^2 + 17t + 70$, where S is the speed of the car in kilometres per hour, t minutes after the policeman started following.

(a) How long was the car under surveillance until it stopped?

(b) During this time, did the driver break the speed limit of 100 km/h?

15 Yeti's owners are going to build a dog kennel for him. It will be 1 m high and twice as long as it is wide, and it will have an opening at one end. When they are finished, they are going to paint the outside of it, including the base of the kennel, to keep it waterproof.



(a) Write an algebraic expression for the area of the kennel that is to be painted.

(b) If they buy a 4 L tin of paint, which will cover 31.4 m^2, and plan to give the kennel two coats of paint, find the dimensions of the largest kennel that could be constructed. Round your answer to one decimal place.

16 Sally is a metalworker. She has several rectangular pieces of sheetmetal with dimensions of 30 cm by 25 cm and she wants to make three simple trays by cutting out the same size square from each corner of the material and then folding up the edges to complete the tray. When completed, each tray will have a different base area: 400 cm^2, 350 cm^2 and 300 cm^2.

(a) Draw a diagram of the net for each rectangular tray with the corners cut out. Draw broken lines to indicate where the trays will be folded.

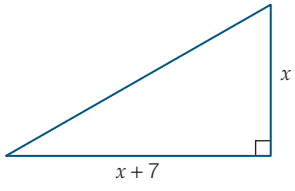
(b) Write an algebraic expression for the length of the base of each tray.

(c) Write an algebraic expression for the breadth of the base of each tray.

(d) Calculate the length of the square corner cut out in each tray (correct to one decimal place).

(e) What are the dimensions of each tray?

17 A right-angled triangle with shorter sides x cm and $(x + 7)$ cm has an area of 60 cm^2. Find the perimeter of the triangle.



18 Solve the following. The sum of the first n terms of the sequence 2, 5, 8, 11 … is given by the formula $S = \frac{n}{2}[4 + 3(n - 1)]$, where n is the number of terms in the sequence. Use the formula to find the number of terms needed to make a sum of 805.

19 Solve the following quadratic equations by first multiplying by the lowest common denominator. Note that the denominator can never be zero.

(a) $2x - 1 = \frac{21}{x}$ **(b)** $x = \frac{9x - 14}{x}$ **(c)** $x = \frac{x - 9}{x - 5}$

(d) $\frac{3}{x - 1} = \frac{x}{x + 1}$ **(e)** $\frac{5}{x + 2} + \frac{3}{x} = 1$ **(f)** $\frac{x - 1}{x} + 5 = \frac{3}{x^2}$

20 Make up three non-monic quadratic equations that have as their solutions $x = 2.5$ or $x = -3$.

21 By using the process of completing the square, the solution $x = \frac{-a \pm \sqrt{5}}{2}$ was obtained.

By choosing a value of a between -3 and 3, make up at least three possible quadratic equations that would have this solution.

22 By choosing values for a, b and c in the general quadratic equation $ax^2 + bx + c = 0$, make up an equation that has:

(a) no solution **(b)** one solution **(c)** two solutions.

Write the solutions to your equations where they exist.

Problem solving

Projectiles

When an object is projected vertically upward with an initial velocity v m/s, its height h m above the ground after t seconds is approximately given by the equation $h = vt - 5t^2$.

(a) A rocket is fired vertically upward at an initial velocity of 25 m/s. At what times is the height of the rocket above ground going to be zero?

(b) The equation $h = vt - 5t^2$ can also be expressed as $5t^2 - vt + h = 0$.

(i) When the height of the rocket is 30 m and $v = 25$, show that the expression becomes $t^2 - 5t + 6 = 0$.

(ii) By factorising $t^2 - 5t + 6$, find at what times the rocket is at a height of 30 m.

(c) The rocket is now projected with an initial velocity of 35 m/s. How long does it take for the rocket to be at a height of 30 m on its way down?

(d) The rocket is again projected, but with an initial velocity of 10 m/s.

(i) Show that a perfect square is obtained when the left side of the equation $5t^2 - vt + h = 0$ is factorised using $h = 5$.

(ii) For how long is the rocket in flight?

(iii) What can you conclude about the maximum height the rocket can reach?

Strategy options

- Draw a diagram.
- Look for a pattern.

Half-time 9

Stage 5.2 Q1, 4 **Stage 5.2◊ Q2, 3, 5** **Stage 5.3§ Q6–11**

Ex.9.3

1 Solve each of the following.

(a) $x^2 + 4x = -3$ **(b)** $x^2 - 5x = 6$ **(c)** $x^2 + 8 = 9x$

(d) $x^2 - 4x = 5$ **(e)** $2x^2 + 6x = -4$ **(f)** $3x^2 + 9x = 30$

Ex.9.1

2 Find the coordinates of the centre and the radius of the circle with equation $(x + 1)^2 + (y - 2)^2 = 4$.

Ex.9.1

3 Plot the graph of:

(a) $y = -2^{-x}$ for values of x from -3 to 3 inclusive

(b) $(x - 2)^2 + (y - 3)^2 = 4$, showing the four points $(h + r, k)$, $(h, k + r)$, $(h - r, k)$, $(h, k - r)$ on the circumference of the circle with centre (h, k) and radius r.

Ex.9.3

4 Solve these quadratic equations.

(a) $x^2 + 6x + 5 = 0$ **(b)** $x^2 - 9x = 0$ **(c)** $3x^2 + 9x = 0$

(d) $x^2 + x - 6 = 0$ **(e)** $x^2 - 4x + 3 = 0$ **(f)** $x^2 - 6x + 8 = 0$

Ex.9.1

5 **(a)** Plot the graph of $y = 2^{-x} - 1$ for values of x from -3 to 3 inclusive.

(b) What is the equation of the asymptote for the graph?

(c) What are the coordinates of the y-intercept?

Ex.9.2

6 Sketch each parabola, stating the changes that should be made to the graph $y = x^2$.

(a) $y = 2(x + 3)^2 - 4$ **(b)** $y = \frac{1}{2}(x - 1)^2 + 3$ **(c)** $y = (x - 2)^2 - 4$

Ex.9.2

7 The graph $y = x^2$ is moved 3 units right and 2 units up. What is the equation of the new graph?

Ex.9.2

8 Consider the equation $y = -x^2 + 1$.

(a) Set up a table of values using integer values of x from -3 to 3 inclusive.

(b) Plot the points on a number plane and join them with a smooth curve.

(c) Find the coordinates of the turning point from your graph.

(d) Find the equation of the axis of symmetry.

Ex.9.4

9 Solve each of the following.

(a) $x^2 - 64 = 0$ **(b)** $(x + 1)^2 = 0$ **(c)** $(x - 2)^2 = 4$

Ex.9.5

10 Solve the following quadratic equations by completing the square. Leave your answer in exact form.

(a) $x^2 + 2x - 6 = 0$ **(b)** $x^2 - 7x + 3 = 0$ **(c)** $2x^2 + 8x - 4 = 0$

Ex.9.6

11 Solve the following quadratic equations. Leave your answers in exact form.

(a) $2x^2 - 5x - 1 = 0$ **(b)** $3x^2 + 9x + 2 = 0$ **(c)** $-3x^2 + x + 3 = 0$

Sketching parabolas

9.7

Stage 5.3§

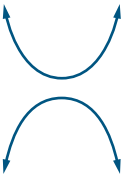
Need to Know

Key features of a parabola

To sketch a quadratic graph, identify the following key features:

1. The shape of the graph

- A quadratic equation with a positive coefficient of x^2 will produce a graph that is a positive parabola.
- A quadratic equation with a negative coefficient of x^2 will produce a graph that is a negative parabola.



2. The *y*-intercept

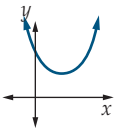
This is where the graph crosses the y-axis and so $x = 0$.

3. The *x*-intercept(s)

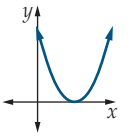
This is where the graph crosses the x-axis and so $y = 0$.

x-intercepts are found by equating the quadratic to zero and solving.

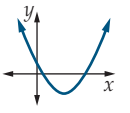
- If there is no solution, there are no x-intercepts.



- If there is one solution, the parabola touches the x-axis. There is only one x-intercept.



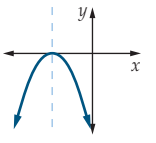
- If there are two solutions, there are two distinct x-intercepts. These may involve surds.



When a graph has two x-intercepts, the x-coordinate of the turning point is halfway between the two x-intercepts.

4. Axis of symmetry

The axis of symmetry is a vertical line that passes through the turning point, dividing the graph into mirror image parts.



5. The turning point

- The turning point can be found by completing the square to transform the equation into **turning point form**.

The graph of $y = (x - h)^2 + k$ has a turning point at (h, k).

Worked Example 21

WE21

Sketch the graph of the quadratic relationship $y = x^2 + 2x - 8$ by finding:

(a) the shape of the graph

(b) the y-intercept

(c) the x-intercepts

(d) the turning point

(e) the axis of symmetry.

Working

(a) This is a shaped parabola as the coefficient of x^2 is positive.

(b) When $x = 0$, $y = 0^2 + 2 \times 0 - 8$

$y = 0 + 0 - 8$

$y = -8$

The y-intercept is $(0, -8)$.

(c) When $y = 0$, $x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x + 4 = 0$ or $x - 2 = 0$

$x = -4$ or $x = 2$

The x-intercepts are $(-4, 0)$ and $(2, 0)$.

(d) For the turning point, $x = \dfrac{-4 + 2}{2}$

$= -1$

When $x = -1$

$y = (-1)^2 + 2 \times -1 - 8$

$y = 1 - 2 - 8$

$y = -9$

The turning point is at $(-1, -9)$.

(Alternatively)

$y = x^2 + 2x - 8$

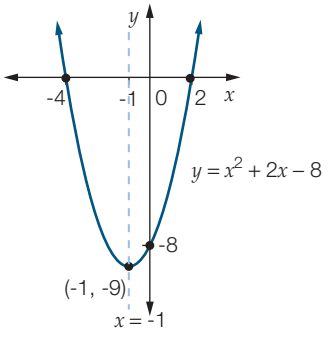
(completing the square)

$y = x^2 + 2x + 1 - 8 - 1$

$y = (x + 1)^2 - 9$

The turning point is at $(-1, -9)$.

(e)



Worked Example 22

WE 22

Sketch the parabola $y = x^2 + 2x - 15$, showing the x- and y-intercepts, turning point and the axis of symmetry. Find the coordinates of the turning point by transforming the equation into turning point form.

Working

$y = x^2 + 2x - 15$

This is a ∪ shaped parabola as the coefficient of x^2 is positive.

When $x = 0$, $y = 0 + 0 - 15$
$= -15$

The y-intercept is $(0, -15)$.

When $y = 0$, $x^2 + 2x - 15 = 0$

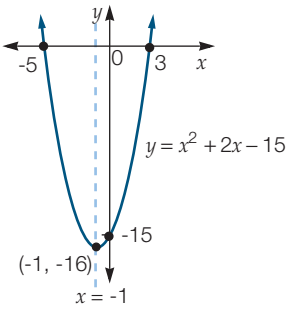
$(x + 5)(x - 3) = 0$

$x + 5 = 0$ or $x - 3 = 0$
$x = -5$ or $x = 3$

The x-intercepts are $(-5, 0)$ and $(3, 0)$.

$y = x^2 + 2x - 15$
$y = (x^2 + 2x + 1) - 1 - 15$
$y = (x + 1)^2 - 16$

The turning point is at $(-1, -16)$.



Worked Example 23

WE 23

Sketch the graph of the equation $y = 4 - x^2$, showing all the key features.

Working

$y = 4 - x^2$

This is a ∩ shaped parabola as the coefficient of x^2 is negative.

When $x = 0$, $y = 4 - 0^2$
$y = 4$

The y-intercept is $(0, 4)$.

When $y = 0$, $4 - x^2 = 0$
$(2 + x)(2 - x) = 0$
$2 + x = 0$ or $2 - x = 0$
$x = \pm 2$

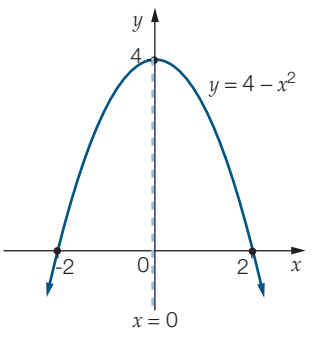
The x-intercepts are $(-2, 0)$ and $(2, 0)$.

For the turning point:

$x = \dfrac{-2 + 2}{2}$
$= 0$

When $x = 0$, $y = 4 - 0$
$y = 4$

The turning point is at $(0, 4)$.



Worked Example 24 WE24

Sketch the graph of the equation $y = -(x-2)^2 + 5$, showing all the key features.

Working

$y = -(x-2)^2 + 5$

The graph of $y = x^2$ needs to be inverted and translated 2 units right and 5 units up.

$\therefore\ h = 2$ and $k = 5$

The turning point is at (2, 5).

When $x = 0$, $y = -(0-2)^2 + 5$
$= -(4) + 5$
$= 1$

The y-intercept is (0, 1).

When $y = 0$, $(x-2)^2 - 5 = 0$

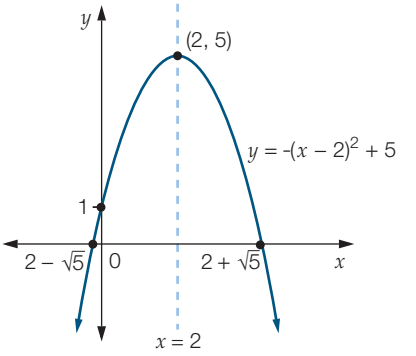
$(x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) = 0$

$x = 2 - \sqrt{5}$ or $x = 2 + \sqrt{5}$

$x = 2 \pm \sqrt{5}$

The x-intercepts are $(2 - \sqrt{5}, 0)$ and $(2 + \sqrt{5}, 0)$.

$x \approx 4.2$ and -0.2



Answers page 685

9.7 Sketching parabolas

Stage 5.3§

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5 Column 1, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q16, Q17	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5 Column 2, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q16, Q17	Q1, Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5 Column 3, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q18

Fluency and understanding

WE21

1 Sketch the graph of each of the following quadratic relationships by finding **(i)** the shape of the graph, **(ii)** the y-intercept, **(iii)** the x-intercepts, **(iv)** the turning point and **(v)** the axis of symmetry.

(a) $y = x^2 - 8x + 15$ **(b)** $y = x^2 - 6x + 8$ **(c)** $y = x^2 - 10x + 21$

(d) $y = -x^2 + 4x + 12$ **(e)** $y = -x^2 + 2x + 3$ **(f)** $y = -x^2 + 2x + 8$

(g) $y = x^2 - 8x + 12$ **(h)** $y = x^2 + 2x - 3$ **(i)** $y = x^2 + 4x - 5$

WE22

2 Sketch the following parabolas showing the x- and y-intercepts, the turning point and the axis of symmetry. Find the coordinates of the turning point by transforming the equations into turning point form.

(a) $y = x^2 + 2x - 8$ **(b)** $y = x^2 + 8x + 15$ **(c)** $y = x^2 + 3x + 2$

(d) $y = x^2 - 4$ **(e)** $y = 9 - x^2$ **(f)** $y = x^2 - 7$

(g) $y = x^2 - 2x$ **(h)** $y = x^2 + 3x$ **(i)** $y = x^2 + 2x$

3 Sketch the graphs of the following equations, showing all the key features. WE23

(a) $y = x^2 - 25$ (b) $y = x^2 - 9$ (c) $y = x^2 - 16$

(d) $y = 81 - x^2$ (e) $y = 4 - x^2$ (f) $y = 1 - x^2$

(g) $y = x^2 - 4x$ (h) $y = x^2 + 6x$ (i) $y = 3x^2 + 6x$

4 Sketch the graphs of the following equations, showing all the key features. WE24

(a) $y = (x + 2)^2 - 9$ (b) $y = (x + 5)^2 - 1$ (c) $y = -(x + 1)^2 + 5$

(d) $y = (x - 3)^2 + 2$ (e) $y = (x - 2)^2 + 4$ (f) $y = (x - 4)^2 + 2$

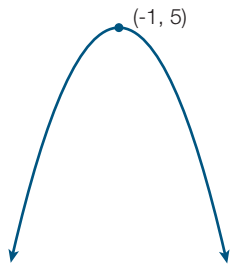
5 Sketch the graphs of the following equations, showing all the key features.

(a) $y = x^2 + 10x + 16$ (b) $y = x^2 + 8x + 7$ (c) $y = -x^2 - 6x - 5$

(d) $y = -x^2 + 4x - 3$ (e) $y = x^2 + 6x + 9$ (f) $y = x^2 + 10x + 25$

(g) $y = -x^2 + 3x$ (h) $y = -x^2 - 5x$ (i) $y = -x^2 + 7x - 12$

6 When expressed graphically, which of the following equations would have a shape like that of the parabola shown on the right?



A $y = x^2 + 2x - 5$ B $y = -x^2 - 2x + 4$

C $y = 1 - x^2$ D $y = 6 - 2x - x^2$

7 (a) The coordinates of the y-intercept of $y = x^2 + 4x + 3$ are:

A (0, 8) B (-1, 0) C (3, 0) D (0, 3)

(b) The coordinates of the x-intercepts of $y = x^2 - 9$ are:

A (0, -3), (0, 3) B (-3, 0), (3, 0) C (-9, 0), (9, 0) D (0, -9), (0, 9)

8 (a) The x-intercepts of the parabola $y = x^2 + 5x - 6$ are:

A -2 and 3 B -3 and 2 C -6 and 1 D -1 and 6

(b) If the x-intercepts of a parabola are -1 and 5, the turning point could be:

A (3, 7) B (2, 4) C (1, -6) D (-2, 2)

9 A ball is thrown into the air and follows a path given by the relationship $h = -t^2 + 10t$, where h is the height of the ball in metres after t seconds of flight. Sketch the graph of the relationship. Use the sketch graph to find:

(a) the total time the ball was in flight

(b) the maximum height the ball reached.

10 (a) The equation of parabola A is:

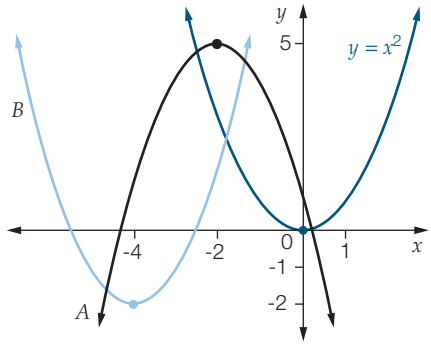
A $y = (x - 2)^2 + 5$ B $y = -(x + 2)^2 + 5$

C $y = -(x - 2)^2 + 5$ D $y = -(x + 2)^2 - 5$

(b) The equation of parabola B is:

A $y = (x - 4)^2 - 2$ B $y = (x + 4)^2 - 2$

C $y = (x - 4)^2 + 2$ D $y = (x + 4)^2 + 2$



11 **(a)** In turning point form, the quadratic equation $y = x^2 - 8x + 18$ can be written as:

A $y = (x - 4)^2 + 34$ **B** $y = (x + 4)^2 + 2$

C $y = (x - 4)^2 - 2$ **D** $y = (x - 4)^2 + 2$

(b) Hence, write down the coordinates of the turning point for the parabola $y = x^2 - 8x + 18$.

12 **(a)** Sketch the graph of the equation $y = x^2 - 6x + 5$, showing all its key features.

(b) Use the graph to find the solution to the equation $x^2 - 6x - 7 = 0$.

Hint

What can you add to -7 to equal 5?

13 Over a period of 8 hours, the temperature of a room follows the relationship $T = h^2 - 6h + 15$, where T is the temperature in degrees Celsius, h hours after commencing the experiment.

(a) Express the equation in turning point form and hence, sketch the graph of this quadratic relationship.

(b) What is the initial temperature?

(c) After 2 hours, is the temperature of the room increasing or decreasing?

(d) After 6 hours, is the temperature of the room increasing or decreasing?

(e) Find the minimum temperature.

(f) When did the minimum temperature occur?

(g) Would it be realistic to use the equation over a 24-hour period? Explain why or why not.

Reasoning, communicating and open-ended

14 Determine the equation of a graph that has x-intercepts of -3 and 5 and a coefficient of x^2 equal to 1.

15 A rectangular area of land is to be fenced off using 44 m of continuous fencing.

(a) Write an equation for the area enclosed by the fencing. (Let x = length of one side of the rectangle.)

(b) Sketch a graph of this quadratic relationship showing all key features.

(c) What is the smallest value x can be?

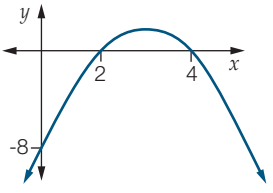
(d) What is the largest value x can be?

(e) What is the maximum area that can be enclosed by this fencing?

(f) What should the dimensions of the rectangle be for the enclosed area to be as big as possible?

16 Explain with examples how you could determine if x-intercepts will exist for a quadratic graph.

17 Archie was graphing $y = x^2 - 2x - 8$. He drew the graph shown below.



(a) Explain why he is wrong and how he could have avoided these mistakes.

(b) Draw the correct graph.

18 Write down the equations of three parabolas that have $x = 5$ as an axis of symmetry. For each parabola, find the turning point.

Rectangular hyperbolas and inverse proportion

9.8

Need to Know

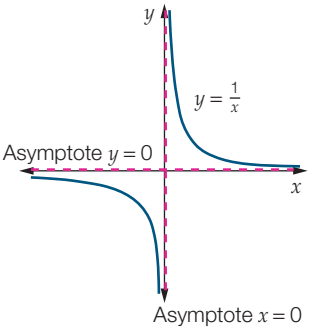
Stage 5.3§

Rectangular hyperbolas

The simplest **rectangular hyperbola** is represented by the equation $y = \frac{1}{x}$.

This hyperbola takes the shape opposite, with both horizontal and vertical asymptotes.

A table of values for the equation $y = \frac{1}{x}$ is shown below.



x	-100	-10	-5	-1	-0.1	-0.01	0	0.01	0.1	1	5	10	100
y	-0.01	-0.1	-0.2	-1	-10	-100	cannot be evaluated	100	10	1	0.2	0.1	0.001

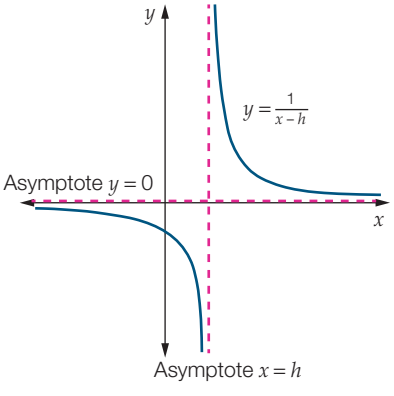
Plotting points, there are two separate parts to the graph. When $x = 0$, the y-value cannot be evaluated. When x is just greater than 0, y is extremely large. When x is just a little smaller than 0, y is extremely small (negatively large). The graph is discontinuous at $x = 0$ and there is an asymptote at $x = 0$, the y-axis. There is also no solution to the equation $0 = \frac{1}{x}$.

When x is extremely large, y becomes extremely small but never reaches 0. When x is extremely small (i.e. negatively large), y becomes extremely small but negative. Once again, it can never reach 0.

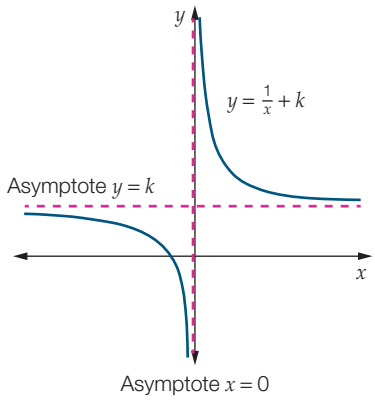
The line $y = 0$, or the x-axis, is also an asymptote. A graph of this relationship is called a rectangular hyperbola.

Translations of rectangular hyperbolas

$y = \frac{1}{x - h}$ is the graph of $y = \frac{1}{x}$ translated h units to the right.

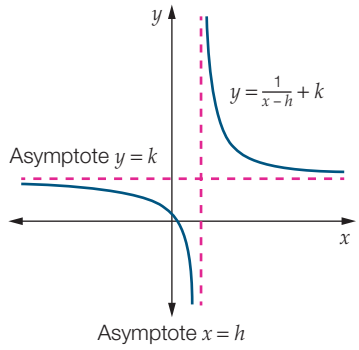


$y = \frac{1}{x} + k$ is the graph of $y = \frac{1}{x}$ translated k units upwards.



$y = \frac{1}{x-h} + k$ is the graph of $y = \frac{1}{x}$ translated h units to the right and k units upwards.

The general equation of a rectangular hyperbola is $y = \frac{a}{x-h} + k$, where a is the dilation factor, h is the translation to the right and k is the translation upwards.



Inverse proportion

It is possible that an increase in one quantity will result in the decrease of the other quantity, or vice versa.

For example:

When a gas is confined to a certain volume, it exerts a pressure on the sides of the container. If the volume of the container increases, the pressure exerted by the gas decreases. If the volume of the container decreases, the pressure exerted by the gas increases. The product of the pressure and volume stays constant:

$PV = K$, where P is the pressure, V is the volume and K is a constant. This can be written as $P = \frac{K}{V}$.

This type of proportion is called **inverse proportion**, and the graph of P plotted against V will be a 1st-quadrant hyperbola.

In inverse proportionality, we say that one variable varies inversely as another variable is raised to a power. The simplest case is the one in which this power is 1. If y is inversely proportional to x, $y \propto \frac{1}{x}$ where '$\propto$' means 'is proportional to', or, as an equation:

$y = \frac{k}{x}$ or $xy = k$, where k is the constant of proportionality.

This means that when x is doubled, y is halved. If x is tripled, y is divided by 3.

In general, two variables, x and y, are in inverse proportion when:

$y \propto \frac{1}{x^n}$, where n is any positive number; or, as an equation, when

$y = \frac{1}{x^n}$ or $x^n y = k$, where k is the constant of proportionality.

As x increases, y decreases. The graph of an inversely proportional relationship is a rectangular hyperbola.

Worked Example 25

WE 25

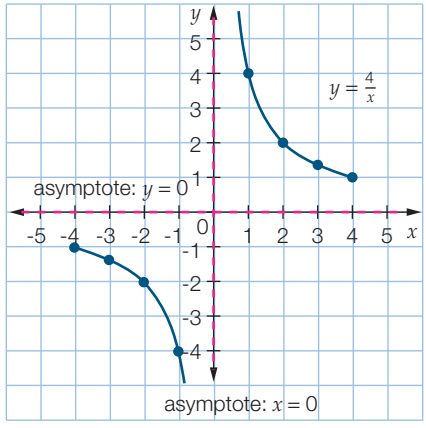
Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = \frac{4}{x}$ (b) $y = \frac{6}{x-1}$ (c) $y = \frac{8}{x} - 3$

Working

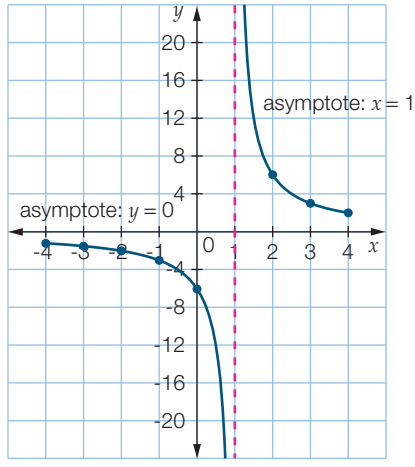
(a) $y = \frac{4}{x}$

x	-4	-3	-2	-1	0	1	2	3	4
y	-1	$-\frac{4}{3}$	-2	-4	undefined	4	2	$\frac{4}{3}$	1



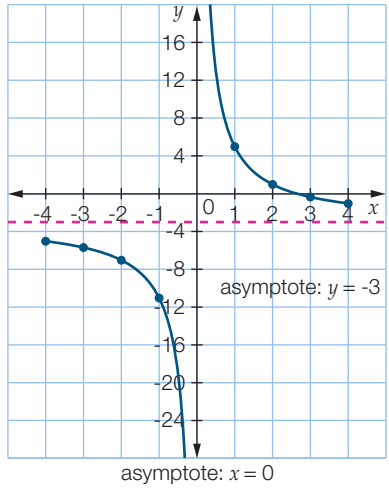
(b) $y = \frac{6}{x-1}$

x	-4	-3	-2	-1	0	1	2	3	4
y	$-\frac{6}{5}$	$-\frac{3}{2}$	-2	-3	-6	undefined	6	3	2



(c) $y = \frac{8}{x} - 3$

x	-4	-3	-2	-1	0	1	2	3	4
y	-5	$-5\frac{2}{3}$	-7	-11	undefined	5	1	$-\frac{1}{3}$	-1



Worked Example 26

WE26

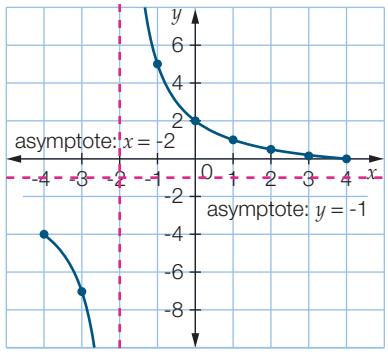
Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = \dfrac{6}{x+2} - 1$ **(b)** $y = \dfrac{1}{x-1} + 3$

Working

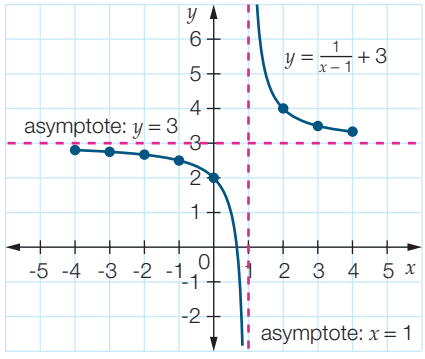
(a) $y = \dfrac{6}{x+2} - 1$

x	-4	-3	-2	-1	0	1	2	3	4
y	-4	-7	undefined	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	0



(b) $y = \dfrac{1}{x-1} + 3$

x	-4	-3	-2	-1	0	1	2	3	4
y	$2\frac{4}{5}$	$2\frac{3}{4}$	$2\frac{2}{3}$	$2\frac{1}{2}$	2	undefined	4	$3\frac{1}{2}$	$3\frac{1}{3}$

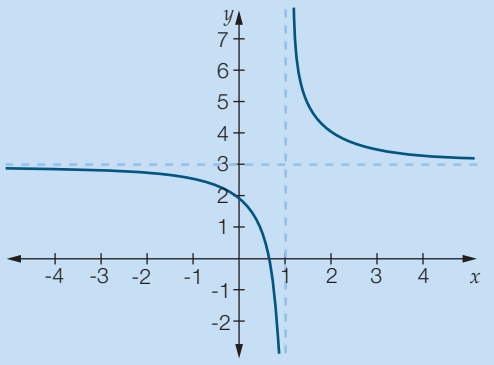


Worked Example 27

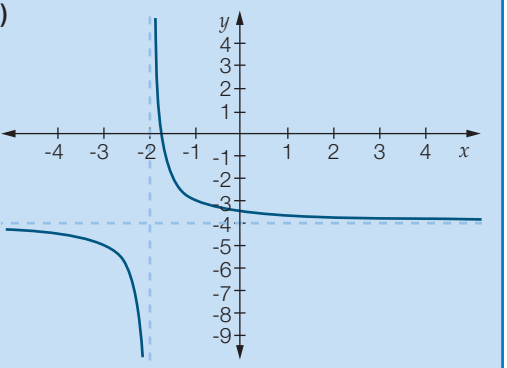
WE 27

Determine the equation of each of the following hyperbolas from their graphs. Use $a = 1$ in the general equation $y = \frac{a}{x-h} + k$.

(a)



(b)



Working

(a) Asymptotes: $x = 1$, $y = 3$

$a = 1$

$h = 1$ and $k = 3$

Equation: $y = \frac{1}{x-1} + 3$

(b) Asymptotes: $x = -2$, $y = -4$

$a = 1$

$h = -2$ and $k = -4$

Equation: $y = \frac{1}{x - -2} + -4$

$y = \frac{1}{x+2} - 4$

Worked Example 28

WE 28

Which of the following could be examples of inverse proportion?

(a)

a	1	3	6	8
b	2	6	12	16

(b)

c	1	2	3	5
d	10	5	3	1

(c)

e	1	2	3	4
f	4	2	1	0.5

Working

(a) Not inverse proportion. (b is increasing as a is increasing.)

(b) Could be inverse proportion. (d is decreasing as c is increasing.)

(c) Could be inverse proportion. (f is decreasing as e is increasing.)

Worked Example 29 WE29

Find the equation of the relationship between the variables in each of the following tables of values.

(a)

x	5	10	15	20	30
y	60	30	20	15	10

(b)

F	1	2	5	10
d	200	50	8	2

Working

(a) $y \propto \frac{1}{x^n}$

$5 \times 60 = 300$
$10 \times 30 = 300$
$15 \times 20 = 300$
$20 \times 15 = 300$
$30 \times 10 = 300$

$y \propto \frac{1}{x}$ and $k = 300$

$y = \frac{k}{x}$ or $xy = k$

$y = \frac{300}{x}$ or $xy = 300$

(b) $d \propto \frac{1}{F^n}$

$1 \times 200 = 200$
$2 \times 50 = 100$
$5 \times 8 = 40$
$10 \times 2 = 20$

F^2	1	4	25	100
d	200	50	8	2

$1 \times 200 = 200$
$4 \times 50 = 200$
$25 \times 8 = 200$
$100 \times 2 = 200$

$d \propto \frac{1}{F^2}$ and $k = 200$

$d = \frac{k}{F^2}$ or $dF^2 = k$

$d = \frac{200}{F^2}$ or $dF^2 = 200$

Answers page 688

9.8 Rectangular hyperbolas and inverse proportion

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18

Fluency and understanding

WE25

1 Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = \frac{2}{x}$ (b) $y = \frac{3}{x}$ (c) $y = -\frac{4}{x}$

(d) $y = \frac{2}{x+2}$ (e) $y = \frac{6}{x-3}$ (f) $y = \frac{4}{x-2}$

(g) $y = \frac{4}{x} - 1$ (h) $y = \frac{3}{x} - 2$ (i) $y = \frac{8}{x} + 3$

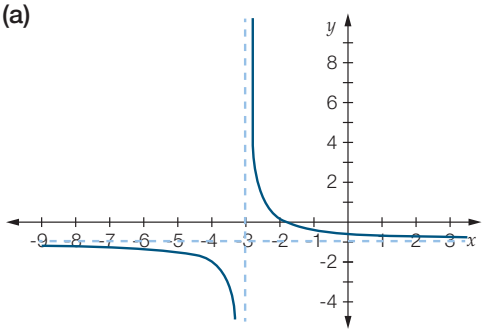
2 Plot the graphs of the following equations for $-4 \le x \le 4$. WE26

(a) $y = \frac{4}{x-1} + 1$ (b) $y = \frac{6}{x+2} - 1$ (c) $y = -\frac{4}{x+1} + 2$

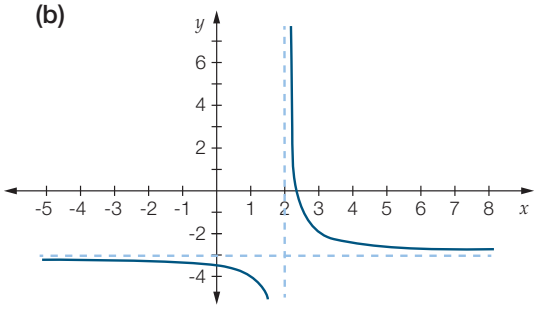
(d) $y = \frac{3}{x-2} - 4$ (e) $y = \frac{4}{x-3} + 3$ (f) $y = \frac{4}{x-1} - 3$

3 Determine the equation of each of the following hyperbolas from their graphs. Use $a = 1$ in the general equation $y = \frac{a}{x-h} + k$. WE27

(a)



(b)



4 Which of the following could be examples of inverse proportion? WE28

(a)

v	1	2	4	5
w	7	3	2	1

(b)

p	1	2	3	4
q	3	4	5	0

(c)

x	1	2	3	4
y	24	12	8	6

(d)

s	1	2	3
t	9	2.25	1

5 Find the equation of the relationship between the variables in each of the following tables of values. WE29

(a)

a	1	2	4	5
b	40	20	10	8

(b)

c	3	4	5	6
d	20	15	12	10

(c)

e	3	6	9	12
f	72	36	24	18

(d)

g	4	8	16	24
h	36	18	9	6

(e)

k	1	2	4	10
l	240	60	15	2.4

(f)

m	1	2	4	6
n	144	36	9	4

6 Which of the following equations will *not* graph as a hyperbola?

A $y = \frac{2}{x+5} - 6$ B $y = 3 + \frac{2}{x}$ C $x(y-3) = 1$ D $y = \frac{2}{x^2} - 1$

7 The graph of the equation $y = \frac{1}{x-1} + 2$ has asymptotes at:

A $x = 2, y = 1$ B $x = 1, y = 2$ C $x = -1, y = 2$ D $x = 1, y = -2$

8 Express each of these relationships involving inverse proportion as an equation, using k as the constant.

(a) $p \propto \frac{1}{q}$ (b) $y \propto \frac{1}{x^2}$ (c) $a \propto \frac{1}{\sqrt{b}}$

(d) t is inversely proportional to w.

(e) V is inversely proportional to h^2.

(f) f is inversely proportional to the cube of g.

(g) L is inversely proportional to the square root of m.

(h) a is inversely proportional to the square root of d.

9 **(a)** The cost per passenger on a bus tour, c, is inversely proportional to the number of passengers on the tour, n. This may be written as:

A $c \propto n$ **B** $c \propto n^2$ **C** $c \propto \frac{1}{n}$ **D** $c \propto \frac{1}{n^2}$

(b) The thickness, t, of a pizza is inversely proportional to the square of its radius, r. This may be written as:

A $t \propto r$ **B** $t \propto r^2$ **C** $t \propto \frac{1}{r}$ **D** $t \propto \frac{1}{r^2}$

(c) The radius, r, of a cylinder is inversely proportional to the square root of its height, h. This may be written as:

A $r \propto h$ **B** $r \propto \frac{1}{h}$ **C** $r \propto \sqrt{h}$ **D** $r \propto \frac{1}{\sqrt{h}}$

10 State the value of the constant of proportionality in each of the following.

(a) $y = \frac{7}{x}$ **(b)** $a = \frac{6.3}{b^2}$ **(c)** $d = \frac{12}{g}$

(d) $p = \frac{0.97}{q^2}$ **(e)** $w = 5 \times \frac{1}{\sqrt{n}}$ **(f)** $B = 3 \times \frac{1}{\sqrt{c}}$

(g) $M = \frac{1}{2p^3}$ **(h)** $L = \frac{1}{5a^3}$ **(i)** $g = \frac{1}{0.4h}$

(j) $w = \frac{1}{1.6x}$ **(k)** $V = \frac{1}{0.08c^2}$ **(l)** $n = \frac{1}{1.25q^3}$

11 y is inversely proportional to the cube of x. If $y = 16$ where $x = 2$, find:

(a) y where $x = 5$ **(b)** x where $y = 2$.

12 Average speed is inversely proportional to the time taken to travel a given distance. At an average speed of 100 km/h, a certain journey takes 2 hours to complete. However, due to a roadworks' speed limit the average speed was reduced to 40 km/h. How long does it now take to travel this distance?

13 The frequency of waves of electromagnetic radiation is inversely proportional to their wavelength. If gamma rays have a frequency of 3×10^{22} Hz with a wavelength of 1×10^{-14} m, what is the wavelength of ultraviolet waves with a frequency of 1.5×10^{16} Hz?

14 The gravitational force of attraction between two bodies is inversely proportional to the square of their distance apart. The force between two objects 40 km apart is 85 units. What is the distance between the two objects when the force is 340 units?

Reasoning, communicating and open-ended

15 The following results were obtained during an experiment measuring light intensity, I, from a light source on an object a distance, d metres, away from the light.

Distance, *d*	1	2	3	4
Intensity, *I*	720	180	80	45

(a) Plot the graph of light intensity (I) against distance (d).

(b) Plot the graph of light intensity (I) against the reciprocal of d, i.e. $\frac{1}{d}$.

(c) Is I inversely proportional to d? Justify your answer.

(d) Copy and complete the following table.

d	1	2	3	4	6	8	12
d^2	1						
I	720	180	80	45			

(e) Calculate the value d^2I for each pair of values. Is this value constant?

(f) Write an equation showing the relationship between I and d.

(g) Plot the graph of I against $\frac{1}{d^2}$. Comment on the result.

(h) What is the effect on the value for I when d is doubled?

(i) What is the effect on the value for I when d is tripled?

(j) What is the effect of halving the value of d?

16 Write down an equation for a rectangular hyperbola that has asymptotes at $x = -2$ and $y = 6$.

17 Substitute your own values for h and k into the equation of a hyperbola in the form of $y = \frac{1}{x-h} + k$, and draw a graph of this relationship.

18 Write three pairs of coordinate points for three different equations of the type $y = \frac{k}{x^n}$. One pair of coordinate points must be common to all three equations.

Problem solving

The mass of Earth

In 1611, Johannes Kepler (1571–1630) proposed a law relating the radius (R) of orbit and time (T) for one orbit of a planet or satellite orbiting a very massive central body (e.g. the Sun).

Kepler's law of planetary motion states that for planets or satellites orbiting the same central body, $\frac{R^3}{T^2}$ = a constant.

The constant can be written as: $\frac{GM}{4\pi^2}$, where $G = 6.67 \times 10^{-11}$ (in units of m^3 per kg per s^2) and M is the mass of the central body (in kg). Use Kepler's law to calculate the mass of the Earth.

- The Earth orbits the Sun every year and its radius of orbit is 228 gigametres.
- The Moon orbits the Earth every 27.3 days and its radius of orbit is 380 000 km.

Strategy options

- Break the problem into manageable parts.
- Guess and check.

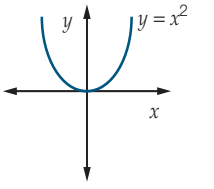
9.9 Sketch graphs of circles, exponentials and rectangular hyperbolas

Stage 5.3§

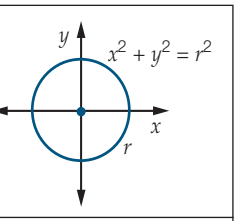
Need to Know

The graphs of circles, exponentials and hyperbolas are transformed by translations and dilations in the same way as quadratics. The graphs can be sketched easily if the basic shape and key features are recognised. In reverse, if a graph is sketched, its equation can be found by recognising the basic shape and then writing the basic equation with the noted translations and/or dilations.

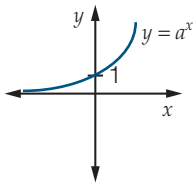
Quadratics

Basic equation $y = x^2$	Basic shape is a parabola. 
Translated equation $y = (x - h)^2 + k$	Basic shape is moved h units to right and k units up.
Dilated equation $y = ax^2$	If $a < -1$ or $a > 1$, then the parabola is narrower than $y = x^2$. If $-1 < a < 1$, then the parabola is wider than $y = x^2$. If the equation has a negative coefficient, $a < 0$, then the basic shape of the parabola is reflected in the x-axis.

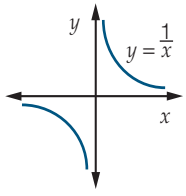
Transforming circles

Basic equation $x^2 + y^2 = r^2$	Basic shape is a circle. The centre is (0, 0) and radius is r. 
Translated equation $(x - h)^2 + (y - k)^2 = r^2$	Basic shape is moved h units to right and k units up. The centre is (h, k) and the radius is r.
Dilated equation $ax^2 + ay^2 = r^2$	The radius of the circle is changed. Note that the dilation must be the same for x and y. The centre is (0, 0) and the radius is $\frac{r}{\sqrt{a}}$.

Transforming exponential graphs

Basic equation $y = a^x$, $a > 1$	Basic shape is an exponential graph. The graph is continually increasing with an asymptote at $y = 0$ and a y-intercept of (0, 1). If $0 < a < 1$, the graph is reflected in the y-axis. Note that $\left(\frac{1}{a}\right) = a^{-1}$ so $\left(\frac{1}{a}\right)^x = a^{-x}$ and the graph of $y = a^{-x}$ is the same as $y = \left(\frac{1}{a}\right)^x$. 
Translated equation $y = a^{(x-h)} + k$	Basic shape is moved h units to the right and k units up. The asymptote is $y = k$ and the y-intercept is $(0, a^{-h} + k)$. The point (0, 1) has moved to the point $(h, k + 1)$.

Transforming rectangular hyperbolas

Basic equation $y = \frac{1}{x}$	The basic shape is a hyperbola. It has two asymptotes, the x- and y-axes. There are no x-and y-intercepts. 
Translated equation $y = \frac{1}{x-h} + k$	Basic shape is moved h units to the right and k units up. There are asymptotes at $x = h$ and $y = k$ and graph is discontinuous for $x = h$.

Technology Exploration: GeoGebra

Exploring transformations of circles, exponentials and rectangular hyperbolas

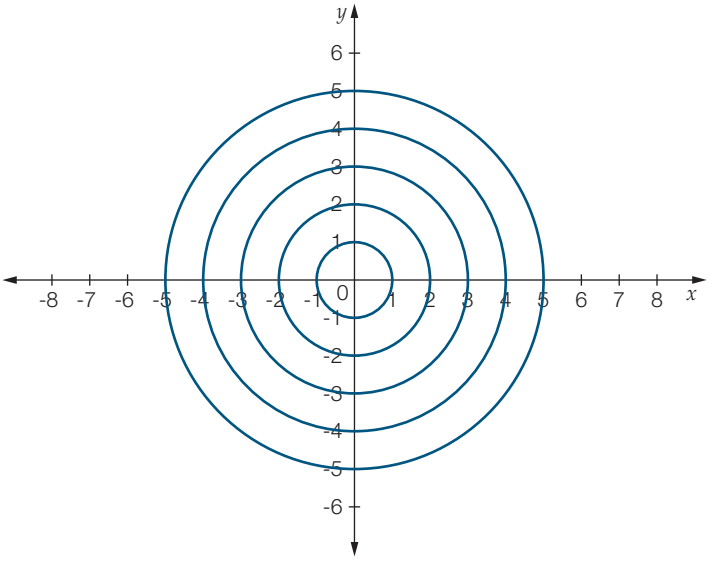
Circle graphs

1 Open GeoGebra and right-click on the Graph View. Set the x-axis values from -9 to 9 and the y-axis values from -5.35 to 5.35. Adjust the x-axis : y-axis ratio to 1:1 if necessary. Label the x- and y-axes and set the Distance to 1.

2 In the Input bar enter **(a)** $x^2 + y^2 = 4$. (Use the ^ to show the index.) Press Enter and the graph will appear on the screen.

On the same set of axes, draw **(b)** $x^2 + y^2 = 1$, **(c)** $x^2 + y^2 = 9$, **(d)** $x^2 + y^2 = 16$ and **(e)** $x^2 + y^2 = 25$.

Label each graph using the Insert Text tool.



3 What do you notice as the constant changes? What is the relationship of the radius of the circle to the constant in the equation?

4 Open a new window. In the Input bar enter:

(a) $x^2 + y^2 = 4$ **(b)** $(x - 2)^2 + y^2 = 4$ **(c)** $(x - 3)^2 + y^2 = 4$

(d) $(x - 4)^2 + y^2 = 4$ **(e)** $(x + 1)^2 + y^2 = 4$

Label each graph using the Insert Text tool.

5 What do you notice as the number added to x before squaring changes?

6 Open a new window. In the Input bar enter:

(a) $x^2 + y^2 = 4$ **(b)** $x^2 + (y - 2)^2 = 4$ **(c)** $x^2 + (y - 4)^2 = 4$

(d) $x^2 + (y + 1)^2 = 4$ **(e)** $x^2 + (y + 3)^2 = 4$

Alter the Dimensions to see the full graph, ensuring that the ratio of the x-axis : y-axis remains **1:1**.

Label each graph using the Insert Text tool.

7 What do you notice as the number added to y before squaring changes?

8 Open a new window. In the Input bar enter:

(a) $x^2 + y^2 = 4$ **(b)** $(x - 2)^2 + (y - 4)^2 = 4$ **(c)** $(x - 3)^2 + (y + 2)^2 = 4$

(d) $(x + 4)^2 + (y - 1)^2 = 4$ **(e)** $(x + 1)^2 + (y + 3)^2 = 4$

Label each graph using the Insert Text tool.

9 Describe the transformation for each of the graphs **(b)**, **(c)**, **(d)** and **(e)** from the graph of $y = x^2$.

10 For equations in the form of $(x - h)^2 + (y - k)^2 = r^2$:

(a) describe the effect changing r has on the graph

(b) describe the effect changing h has on the graph

(c) describe the effect changing k has on the graph.

11 Write down the centre of each circle for the graphs drawn in question **8**. Describe the relationship of the centre coordinates to the values h and k in the equation $(x - h)^2 + (y - k)^2 = r^2$.

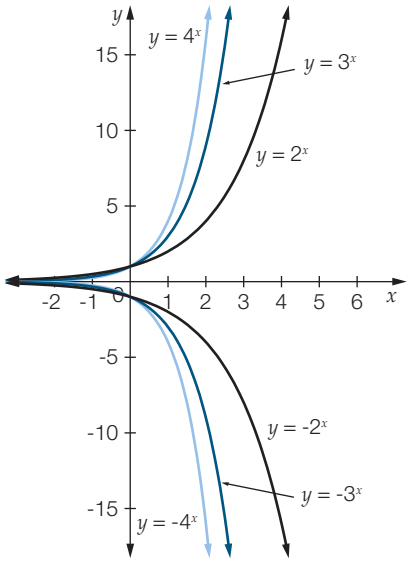
Exponential graphs

1 Open GeoGebra and right-click on the Graph View. Set the x-axis values from -3 to 7 and the y-axis values from -20 to 30. Label the x- and y-axes and set the Distance to 1.

2 In the Input bar enter **(a)** $y = 2^x$. (Use the ^ to show the index.) Press Enter and the graph will appear on the screen.

On the same set of axes, draw **(b)** $y = 3^x$, **(c)** $y = 4^x$, **(d)** $y = -2^x$ and **(e)** $y = -4^x$.

Label each graph using the Insert Text tool.



3 What do you notice as the base changes? What was the effect of the negative sign in front of the base?

4 Open a new window. In the Input bar enter:

(a) $y = 2^x$ **(b)** $y = 2^{x-1}$ **(c)** $y = 2^{x-3}$

(d) $y = 2^{x+2}$ **(e)** $y = 2^{x+4}$

Label each graph using the Insert Text tool.

5 What do you notice as the number added to the index x changes?

6 Open a new window. In the Input bar enter:

(a) $y = 2^x$ **(b)** $y = 2^{x-4}$ **(c)** $y = 2^{x-1}$

(d) $y = 2^{x+2}$ **(e)** $y = 2^{x+3}$

Label each graph using the Insert Text tool.

7 What do you notice as the number added to 2^x changes?

8 Open a new window. In the Input bar enter:

(a) $y = 2^x$ **(b)** $y = 2^{x-1} - 4$ **(c)** $y = 2^{x-3} - 1$

(d) $y = 2^{x+2} + 2$ **(e)** $y = 2^{x+4} + 3$

Label each graph using the Insert Text tool.

9 Describe the transformation for each of the graphs **(b)**, **(c)**, **(d)** and **(e)** from the graph of $y = 2^x$.

10 For equations in the form of $y = a^{x-h} + k$:

(i) describe the effect changing a has on the graph

(ii) describe the effect changing h has on the graph

(iii) describe the effect changing k has on the graph.

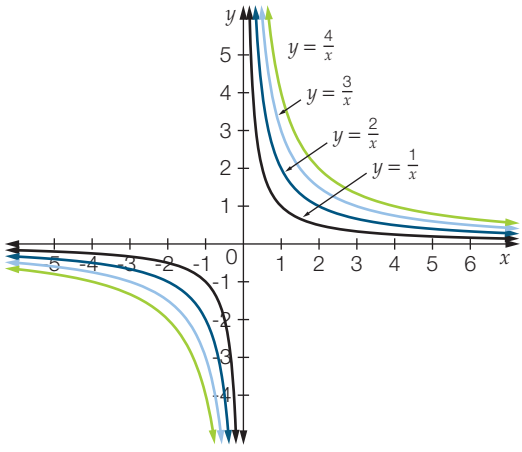
11 Using the graphs drawn in question **8**, describe the relationship of the coordinates of the point (0, 1) to the values h and k in the equation $y = a^{x-h} + k$.

Rectangular hyperbolas

1 Open GeoGebra and right-click on the Graph View. Set the x-axis values from -9 to 9 and the y-axis values from -5.35 to 5.35. Adjust the x-axis : y-axis ratio to 1:1 if necessary. Label the x- and y-axes and set the Distance to 1.

2 In the Input bar enter **(a)** $y = \frac{1}{x}$. Press Enter and the graph will appear on the screen.

On the same set of axes, draw **(b)** $y = \frac{2}{x}$, **(c)** $y = \frac{3}{x}$ and **(d)** $y = \frac{4}{x}$.



3 What do you notice as the constant changes?

4 Open a new window. In the Input bar enter:

(a) $y = \frac{1}{x}$ **(b)** $y = \frac{1}{x-1}$ **(c)** $y = \frac{1}{x-4}$

(d) $y = \frac{1}{x+2}$ **(e)** $y = \frac{1}{x+3}$

5 What do you notice as the denominator changes?

6 Open a new window. In the Input bar enter:

(a) $y = \frac{1}{x}$ **(b)** $y = \frac{1}{x} + 1$ **(c)** $y = \frac{1}{x} + 3$

(d) $y = \frac{1}{x} - 2$ **(e)** $y = \frac{1}{x} - 4$

7 What do you notice as a constant is added to $\frac{1}{x}$?

8 Open a new window. In the Input bar enter:

(a) $y = \frac{1}{x}$ **(b)** $y = \frac{1}{x-1} + 1$ **(c)** $y = \frac{1}{x-4} + 2$

(d) $y = \frac{1}{x+2} - 3$ **(e)** $y = \frac{1}{x+3} - 4$

9 What do you notice as the denominator changes and a constant is added?

10 For equations in the form of $y = \frac{a}{x-h} + k - 4$:

(i) describe the effect changing a has on the graph

(ii) describe the effect changing h has on the graph

(iii) describe the effect changing k has on the graph.

Worked Example 30

WE30

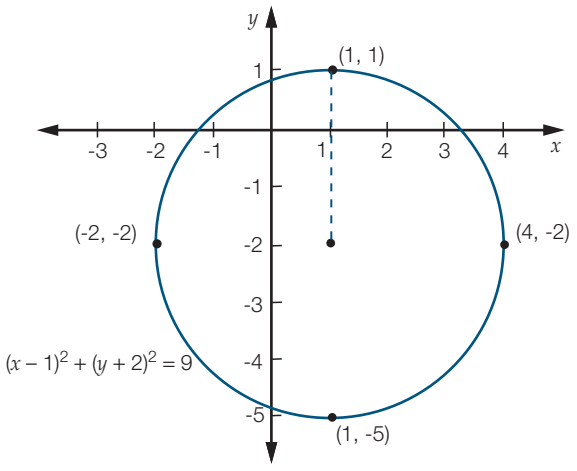
Sketch the graph of $(x-1)^2 + (y+2)^2 = 9$, clearly showing the centre of the circle and the radius.

Working

Coordinates of the centre are (1, -2).

$r = \sqrt{9}$
$= 3$

$\therefore$ radius is 3 units



Worked Example 31

WE31

Sketch the graph of $y = 3^{x-1} + 2$, clearly showing the asymptote and other key features.

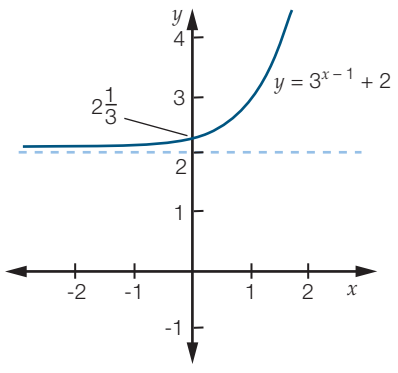
Working

$y = 3^{x-1} + 2$ is the graph $y = 3^x$ translated 1 unit right and 2 units up.

The asymptote is $y = 2$.

When $x = 0$, $y = 3^{-1} + 2$
$= 2\frac{1}{3}$

The y-intercept is $(0, 2\frac{1}{3})$.



Worked Example 32 WE32

Sketch the graph of $y = \frac{1}{x-4} + 2$, clearly showing the key features (asymptotes, and x- and y-intercepts).

Working

$y = \frac{1}{x-4} + 2$

The asymptotes are $x = 4$ and $y = 2$.

When $x = 0$, $y = -\frac{1}{4} + 2$

$= 1\frac{3}{4}$

The y-intercept is $(0, 1\frac{3}{4})$.

When $y = 0$, $0 = \frac{1}{x-4} + 2$

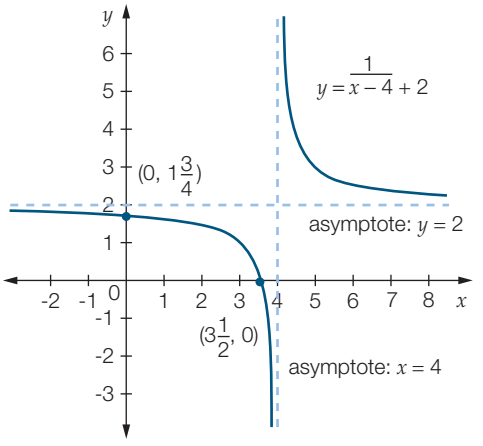
$-2 = \frac{1}{x-4}$

$-2x + 8 = 1$

$-2x = -7$

$x = 3\frac{1}{2}$

The x-intercept is $(3\frac{1}{2}, 0)$.



Answers page 691

9.9 Sketch graphs of circles, exponentials and rectangular hyperbolas

Stage 5.3§

Navigator

Q1, Q3, Q4, Q5, Q7, Q11	Q1, Q3, Q4, Q7, Q9, Q11	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9, Q10, Q11

Fluency and understanding

WE30

1 Sketch the graph of each of the following, clearly showing the centre of the circle and the radius.

(a) $(x-2)^2 + (y+1)^2 = 16$

(b) $(x+3)^2 + (y-4)^2 = 25$

(c) $(x-2)^2 + (y-2)^2 = 4$

(d) $(x+4)^2 + (y+1)^2 = 16$

2 Sketch the graph of the following, clearly showing the asymptote and other key features for each graph. WE31

(a) $y = 2^{x-2} + 1$ (b) $y = 2^{x+1} + 3$

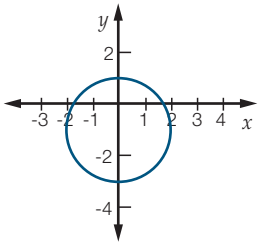
(c) $y = 2^{x} - 3$ (d) $y = 3^{x-4} - 1$

3 Sketch the graph of the following, clearly showing the key features (asymptotes, and x- and y-intercepts) for each graph. WE32

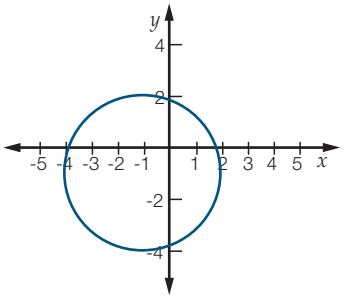
(a) $y = \dfrac{1}{x+3} - 2$ (b) $y = \dfrac{1}{x-5} + 1$ (c) $y = \dfrac{1}{x+2} + 3$

4 What is the equation of each of the following graphs?

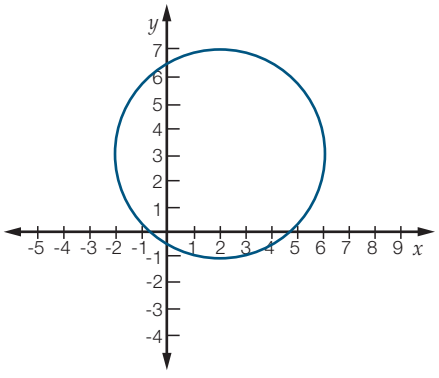
(a)



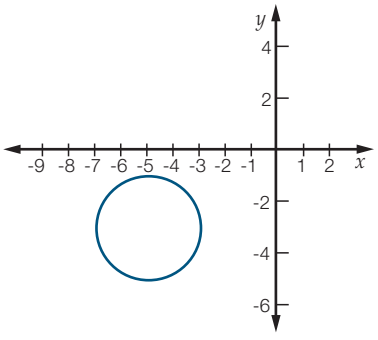
(b)



(c)



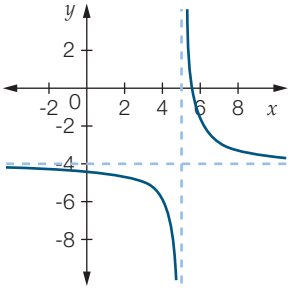
(d)



5 (a) Write the equation formed if the graph of $y = \dfrac{1}{x}$ is translated 3 units left and 4 units up.

(b) Graph this new equation showing any relevant asymptotes.

6 The equation for the graph at right would be:



A $y = \dfrac{1}{x+4} + 5$ B $y = \dfrac{1}{x-4} - 5$

C $y = \dfrac{1}{x-5} - 4$ D $y = \dfrac{1}{x+5} - 4$

7 (a) Write the new equation formed if a circle with equation $(x-3)^2 + (y+4)^2 = 25$ is translated 2 units left and 3 units down.

(b) Write the new equation formed if a circle with equation $(x-3)^2 + (y+4)^2 = 25$ has its radius increased by 3 units.

Reasoning, communicating and open-ended

8 **(a)** Show that $\frac{1}{x-1} + 3 = \frac{3x-2}{x-1}$.

(b) Hence, sketch the graph of $y = \frac{3x-2}{x-1}$.

9 A circular rose garden is to be planted in a rectangular lawn in front of a house. The lawn is 15 m from the front boundary to the house and 25 m along the front boundary. A diagram of the lawn and the rose garden is to be drawn before the garden is prepared for planting. The left-hand side of the front boundary is used as the origin.

(a) The landscape gardener who is creating this garden initially places the centre of the garden in the centre of the lawn. Write down the coordinates of the centre of the circle.

(b) If the rose garden is to have a radius of 3 m, write the equation that would represent the border of the garden.

(c) Sketch a graph of this garden.

(d) If the garden is moved 0.5 m closer to the front boundary and 6.5 m further to the left:

(i) write the new equation **(ii)** sketch the new graph.

(e) If the radius is now reduced by 1 m after the move:

(i) write the new equation **(ii)** sketch the new graph.

10 **(a)** Write the equation of a rectangular hyperbola that has been translated from the origin.

(b) Identify the asymptotes for a graph of your equation.

(c) Sketch the graph.

11 A circle whose centre is not at the origin passes through the point (1, 1). What could be the equation of this circle?

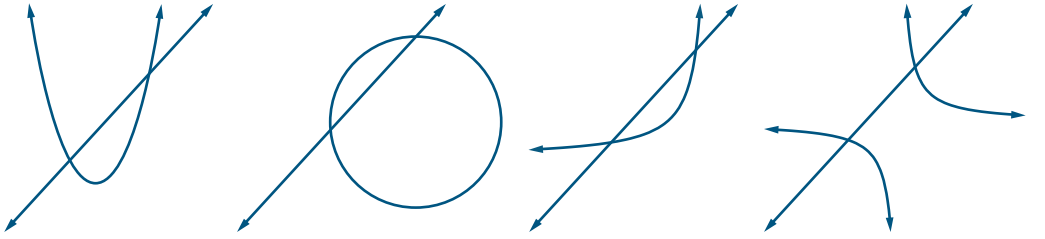
Solving simultaneous equations where one equation is non-linear

9.10

Stage 5.3§

Need to Know

A straight line can intersect with any of the non-linear curves that have been explored, a parabola, a circle, an exponential or a hyperbola.



Finding points of intersection

The points of intersection of a linear and a non-linear graph can be found by solving simultaneous equations using algebraic or graphical methods. A spreadsheet may also be used.

If the line is a tangent to a non-linear graph, the simultaneous equations will have only one solution.

If a linear and a non-linear graph do not intersect, the simultaneous equations will have no real solution.

Solving simultaneously

Given a linear and a non-linear equation both with y as the subject, the sides of each equation containing the x variable can be equated. A new equation will be formed which will be able to be solved if the graphs of the equations intersect. Solutions may be rational or irrational, and the quadratic formula (or completing the square) may need to be used. If the solutions are irrational, a calculator can be used to find approximate solutions. The solution(s) for the x value can then be substituted into either of the original equations (usually the linear) to find the y value(s). These x and y values are the coordinates of the point(s) of intersection.

If y is not the subject of the non-linear equation, as is the case with a circle equation, the y value may be replaced using substitution.

The intersection of a linear equation and an exponential equation can only be solved graphically or numerically using a spreadsheet.

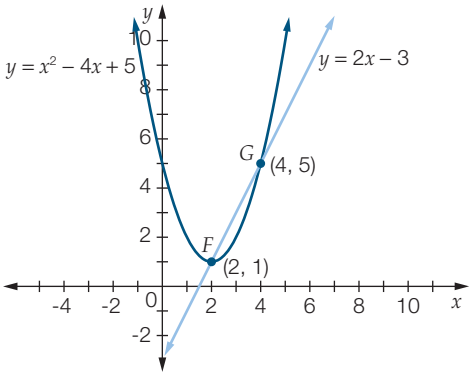
Technology Exploration: GeoGebra

Intersection of a line with non-linear curves

1 Open GeoGebra and right-click on the Graph View. Set the x-axis values from -6 to 12 and the y-axis values from -3 to 11. Label the x- and y-axes and set the Distance to 1.

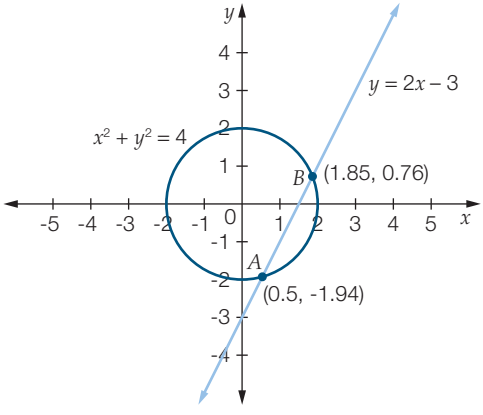
2 In the Input bar enter **(a)** $y = x^2 - 4x + 5$ and $y = 2x - 3$. (Use the ^ to show the index.) Press Enter and the graph will appear on the screen. Use the New Point tool to place a point on each point of intersection.

Identify the points of intersection as (2, 1) and (4, 5).



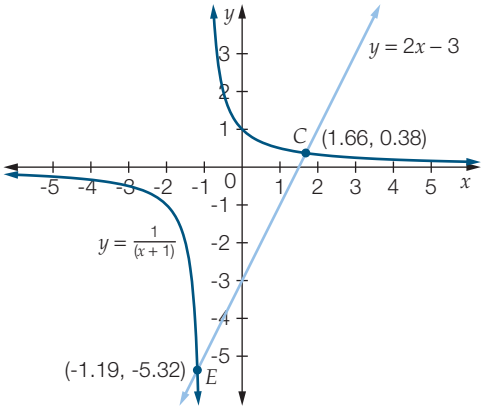
3 In Algebra View, deselect the equation $y = x^2 - 4x + 5$ and the points of intersection. In the Input bar enter **(b)** $x^2 + y^2 = 4$. Adjust the x-axis : y-axis ratio to 1 : 1. Use the New Point tool to place a point on each point of intersection.

Identify the points of intersection as (0.5, -1.94) and (1.85, 0.76).



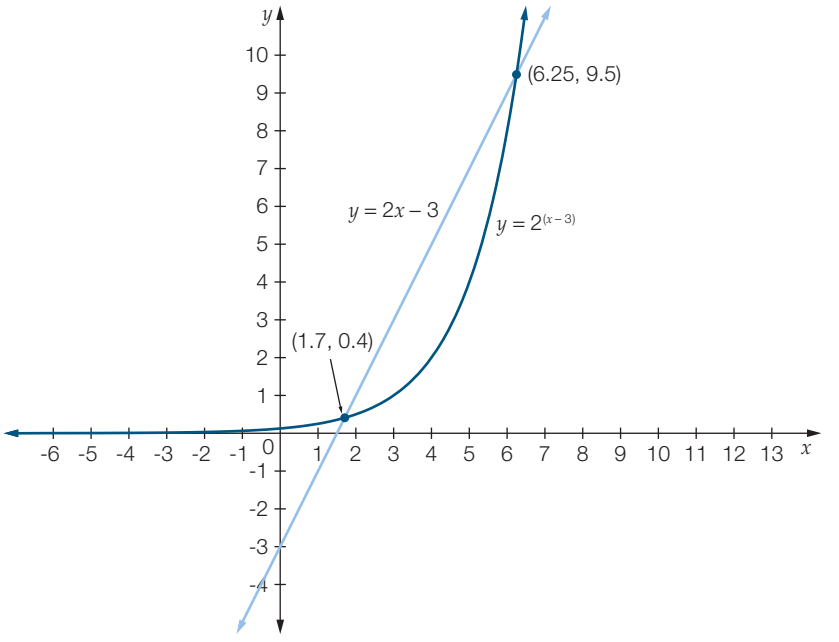
4 In Algebra View, deselect the equation $x^2 + y^2 = 4$ and the points of intersection. In the Input bar enter **(c)** $y = \dfrac{1}{x+1}$. Adjust the window as required. Use the New Point tool to place a point on each point of intersection.

Identify the points of intersection as (1.66, 0.38) and (-1.19, -5.32).



5 In Algebra View, deselect the equation $y = \dfrac{1}{x+1}$ and the points of intersection. In the Input bar enter **(d)** $y = 2^x$. Adjust the window as required. Use the New Point tool to place a point on each point of intersection.

Identify the points of intersection as (1.7, 0.4) and (6.25, 9.5).



Technology Exploration: Spreadsheet

Points of intersection of graphs

1 Open a spreadsheet program such as Microsoft Excel®. Insert labels in cells A1, B1 and C1 of x, y=2x-3 and y=2^(x-3) respectively. Insert 0 and 1 into cells A2 and A3 and fill down to A12.

2 In cell B2, insert =2*A2-3 and in cell C2, insert =2^(x-3). Highlight B2 and C2 and fill down to row 12.

3 In cell D2, insert =C2-B2. Fill down to D12.

4 The values of each relationship are closest for x values between 1 and 2 and again between 6 and 7. Insert a new row between rows 3 and 4 and insert 1.5 in the new A4 and fill down the formulas in columns B, C and D. Insert another row between A4 and A5 and try values between 1.5 and 2 until the value in D5 is close to 0. When $x = 1.7$, the y values for each relationship are approximately equal to 0.4.

5 Repeat this procedure for x values between 6 and 7. When $x = 6.25$, the y values for each relationship are approximately equal to 9.5. Your spreadsheet should look like the image shown.

	A	B	C	D
1	x	y=2x-3	y=2^(x-3)	
2	0	-3	0.125	3.125
3	1	-1	0.25	1.25
4	1.5	0	0.353553	0.353553
5	1.7	0.4	0.406126	0.006126
6	2	1	0.5	-0.5
7	3	3	1	-2
8	4	5	2	-3
9	5	7	4	-3
10	6	9	8	-1
11	6.25	9.5	9.513657	0.013657
12	6.5	10	11.31371	1.313708
13	7	11	16	5
14	8	13	32	19
15	9	15	64	49
16	10	17	128	111

The points of intersection are approximately (1.7, 0. 4) and (6.25, 9.5).

Worked Example 33

WE33

Find the point(s) of intersection of the line given by the equation $y = 2x - 3$ and the parabola given by the quadratic equation $y = x^2 - 4x + 5$ using algebraic techniques. Give your answers correct to 2 decimal places.

Working

Given $y = 2x - 3$ and $y = x^2 - 4x + 5$

$$x^2 - 4x + 5 = 2x - 3$$

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$x = 2 \text{ or } 4$$

$$y = 2x - 3$$

$$= 2 \times 2 - 3;\ x = 2$$

$$= 1$$

and $$y = 2x - 3$$

$$= 2 \times 4 - 3;\ x = 4$$

$$= 5$$

Points of intersection (2, 1) and (4, 5).

Worked Example 34

WE34

Find the point(s) of intersection of the line given by the equation $y = 2x - 3$ and the circle $x^2 + y^2 = 4$. Give your answers correct to 2 decimal places.

Working

Given $y = 2x - 3$ and $x^2 + y^2 = 4$

$$x^2 + (2x - 3)^2 = 4$$

$$x^2 + 4x^2 - 12x + 9 = 4$$

$$5x^2 - 12x + 5 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{12 \pm \sqrt{144 - 100}}{10}$$

$$= \frac{12 \pm \sqrt{44}}{10}$$

$$= \frac{6 \pm \sqrt{11}}{5}$$

$$\approx \frac{6 \pm 3.32}{5}$$

$$\approx 1.86 \text{ and } 0.54$$

$$y \approx 0.72 \text{ and } -1.92$$

Points of intersection (1.86, 0.72) and (0.54, -1.92).

Worked Example 35

WE35

Find the point(s) of intersection of the line given by the equation $y = 2x - 3$ and the rectangular hyperbola $y = \frac{1}{x+1}$. Give your answers correct to 2 decimal places.

Working

Given $y = 2x - 3$ and $y = \frac{1}{x+1}$

$$2x - 3 = \frac{1}{x+1}$$

$$(2x-3)(x+1) = 1$$

$$2x^2 - x - 3 = 1$$

$$2x^2 - x - 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1+32}}{4}$$

$$= \frac{1 \pm \sqrt{33}}{4}$$

$$\approx \frac{1 \pm 5.74}{4}$$

$$\approx 1.69 \text{ and } -1.19$$

$$y \approx 0.38 \text{ and } -5.37$$

Points of intersection (1.69, 0.38) and (-1.19, -5.37).

Worked Example 36

WE36

Show that the line given by the equation $y = 3x - 10$ is a tangent to the curve given by the quadratic equation $y = x^2 - 5x + 6$.

Working

Given $y = 3x - 10$ and $y = x^2 - 5x + 6$

$$x^2 - 5x + 6 = 3x - 10$$

$$x^2 - 8x + 16 = 0$$

$$(x-4)^2 = 0$$

$$x = 4$$

$$y = 3 \times 4 - 10$$

$$= 2$$

As there is only one point of intersection, at (4, 2), $y = 3x - 10$ is a tangent to the curve given by $y = x^2 - 5x + 6$.

Worked Example 37

WE 37

Show that the line given by the equation $y = 2x - 7$ does not intersect with the circle given by the equation $x^2 + y^2 = 9$.

Working

Given $y = 2x - 7$ and $x^2 + y^2 = 9$

Substituting for y,

$$x^2 + (2x - 7)^2 = 9$$

$$x^2 + 4x^2 - 28x + 49 - 9 = 0$$

$$5x^2 - 28x + 40 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{28 \pm \sqrt{28^2 - 800}}{4}$$

$$= \frac{28 \pm \sqrt{-16}}{4}$$

This has no solution as the square root of a negative number cannot be found.

Therefore, there is no point of intersection.

9.10 Solving simultaneous equations where one equation is non-linear

Answers page 693

Navigator

Stage 5.3§

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Equipment required: computer for Question **6**

Approximate any irrational values in this exercise correct to 2 decimal places.

Fluency and understanding

WE 33

1 **(a)** Find the point(s) of intersection of the line given by the equation $y = 3x + 2$ and the parabola given by the quadratic equation $y = x^2 + 6x - 2$ using algebraic techniques.

(b) Find the point(s) of intersection of the line given by the equation $y = 2x + 5$ and the parabola given by the quadratic equation $y = x^2 + 7x + 11$ using algebraic techniques.

(c) Find the point(s) of intersection of the line given by the equation $y = 5x - 4$ and the parabola given by the quadratic equation $y = x^2 - 2x + 6$ using algebraic techniques.

WE34

2 (a) Find the point(s) of intersection of the line given by the equation $y = x - 3$ and the circle given by the equation $x^2 + y^2 = 9$.

(b) Find the point(s) of intersection of the line given by the equation $y = 3x + 5$ and the circle given by the equation $x^2 + y^2 = 5$.

(c) Find the point(s) of intersection of the line given by the equation $y = 2x - 1$ and the circle given by the equation $x^2 + y^2 = 5$.

WE35

3 (a) Find the point(s) of intersection of the line given by the equation $y = 4x + 1$ and the rectangular hyperbola given by the equation $y = \dfrac{3}{2x+1}$.

(b) Find the point(s) of intersection of the line given by the equation $y = 2x - 7$ and the rectangular hyperbola given by the equation $y = \dfrac{-5}{x+2}$.

(c) Find the point(s) of intersection of the line given by the equation $y = x - 1$ and the rectangular hyperbola given by the equation $y = \dfrac{1}{x+1}$.

WE36

4 (a) Show that the line given by the equation $y = 2x - 3$ is a tangent to the curve given by the quadratic equation $y = x^2 - 4x + 6$.

(b) Show that the line given by the equation $y = 4x - 7$ is a tangent to the curve given by the quadratic equation $y = x^2 + 14x + 18$.

(c) Show that the line given by the equation $y = x - 7$ is a tangent to the curve given by the quadratic equation $y = x^2 + 5x - 3$.

WE37

5 (a) Show that the line given by the equation $y = x + 3$ does not intersect with the circle given by the equation $x^2 + y^2 = 4$.

(b) Show that the line given by the equation $y = 3x - 5$ does not intersect with the circle given by the equation $x^2 + y^2 = 2$.

(c) Show that the line given by the equation $y = 2x - 5$ does not intersect with the circle given by the equation $x^2 + y^2 = 3$.

6 Find the point of intersection of the exponential curve with the equation given by $y = 2^{x-1}$ and the line with equation $y = 2x$ using digital technology.

Reasoning, communicating and open-ended

7 The line with equation $y = 5 - x$ and the rectangular hyperbola $y = \dfrac{8}{3x-4}$:

A intersect at the points (4, 1) and $\left(2\frac{1}{3}, 2\frac{2}{3}\right)$

B touch at the point (4, 1)

C touch at the point $\left(4, 2\frac{1}{3}\right)$

D do not intersect

8 Solve the equations $y = x + 3$ and $(x + 2)^2 + y^2 = 1$ simultaneously and decide whether the graphs of these equations would touch, intersect or not intersect at all. Justify your decision.

9 A line intersects with the graph of a parabola with equation $y = x^2 - 5x + 6$. The x-coordinates of the points of intersection are 1 and 6 respectively. Find the equation of the line.

10 Write the equation of a circle with radius 2 and the equation of a line with gradient 3. Solve these equations simultaneously to find if they intersect, touch or not intersect at all.

Curves and rates

9.11

Stage 5.3

Need to Know

Rates

A rate describes how one variable, the dependent variable, changes as another variable, the independent variable, changes. A rate can be constant or variable. Rates can be positive where both variables increase or negative where one variable decreases as the other variable increases.

Constant rates

Constant rates occur when the rate of change is always the same.

For example:

If the price of petrol is $1.53/litre every litre costs the same amount, so this is a constant rate.

Constant rates can be shown graphically by a straight-line graph where the gradient of the graph shows the constant rate of change. If the gradient is positive, the rate is positive. If the straight line graph is a first quadrant graph going through the origin, it shows that the variables are in direct proportion. The equation can be written as $y = kx$, $x \geq 0$ where k, the constant of proportionality, is the gradient of the graph. If the gradient is negative, the rate is negative. The steeper the line the larger the rate will be.

Variable rates

Not all rates are constant. Any graph that is not linear, such as those that graph variables in inverse proportion, shows variables that are changing at a variable rate.

For example:

A car travelling over a distance does not maintain a constant speed. The speed changes throughout the journey. The rate increases rapidly as the vehicle starts off and varies even when the vehicle is trying to maintain a constant speed. The rate changes as the vehicle changes speed limit zones and when it encounters traffic or road obstructions. The rate decreases (increases at a negative rate) also as the vehicle comes to a stop. Speeding up and slowing down indicate a variable rate of change.

Parabolas, exponentials, circles and rectangular hyperbolas all show variable rates of change. The gradient of these graphs changes as the value of the independent variable varies.

Constant rates can alter over time. The rate then becomes a variable rate.

For example:

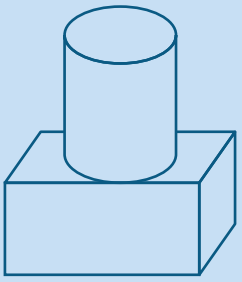
The conversion rate for A$ to US$ may be A$1 = US$0.93 one day but be A$1 = US$0.95 the next day. While each of these rates is a constant rate, the conversion rate is a variable rate over an extended time period. A graph would be a line graph consisting of several linear parts.

Worked Example 38

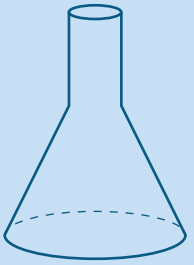
WE38

These containers are being filled with water at a constant rate. Draw a graph to show the relationship between the height of the water, h, and the time, t, taken to fill the container.

(a)



(b)



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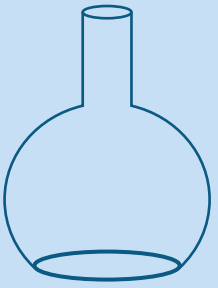
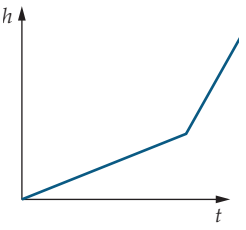
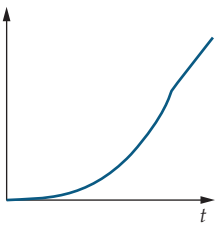


Working

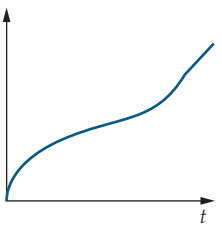
(a)



(b)



(c)



Worked Example 39

WE39

State whether the following rates are constant or variable:

(a) The cost of potatoes per kilogram at the supermarket on any day

(b) The cost of potatoes per kilogram at the supermarket throughout the year

(c) The hourly rate of pay of a retail worker

(d) The speed of a tennis ball being thrown straight up into the air

(e) The pulse rate of a person during exercise.

Working

(a) Constant

(b) Variable

(c) Constant

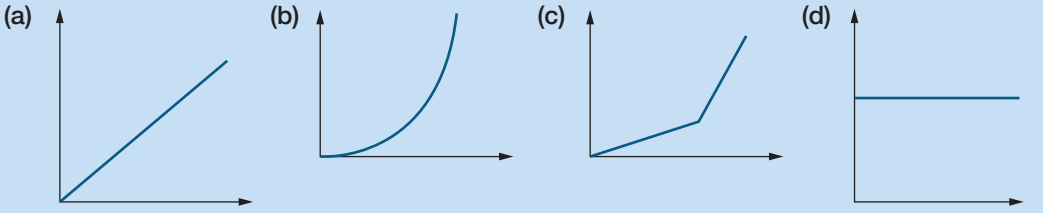
(d) Variable

(e) Variable

Worked Example 40

WE 40

State whether the following graphs show a constant or variable rate of change.



Working

(a) Constant

(b) Variable

(c) Variable

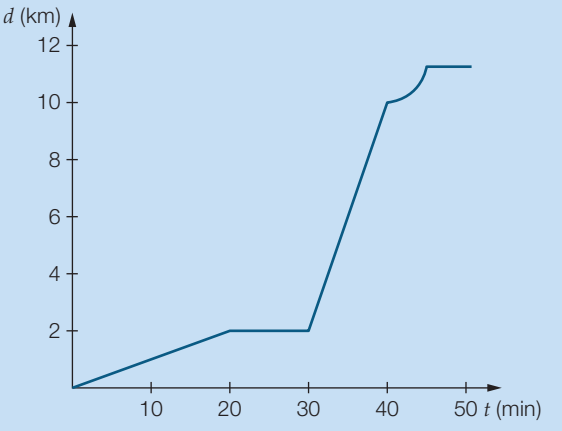
(d) Constant

Worked Example 41

WE 41

Jo's journey to school is shown by the following graph. During her trip she walks, runs and travels by bus, and she goes straight to her locker when she gets to school.

Describe her trip stating the time each part of her journey took her.



Working

Jo walks 2 kilometres to the bus stop at a constant rate for 20 minutes, waits for a bus for 10 minutes, travels 8 kilometres on the bus for another 10 minutes at a constant rate, starts walking the last kilometre but begins to jog at a variable rate getting faster and faster. She runs for 5 minutes until she reaches the lockers where she stays for another 5 minutes.

Answers page 693

9.11 Curves and rates

Stage 5.3

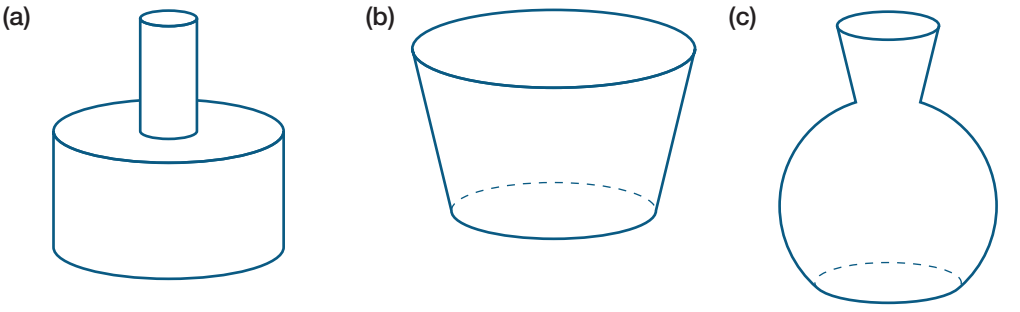
Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12

Fluency and understanding

WE38

1 These containers are being filled with water at a constant rate. Draw a graph to show the relationship between the height of the water, h, and the time, t, taken to fill the container.



WE39

2 State whether the following rates are constant or variable:

(a) The cost of oranges per kilogram at the supermarket on any day

(b) The cost of oranges per kilogram at the supermarket throughout the year

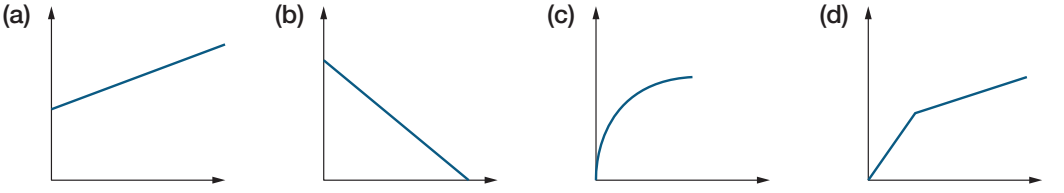
(c) The rate at which a car uses petrol

(d) The speed of a stone dropped from the top of a cliff

(e) The rent charged per week for an apartment during a lease period.

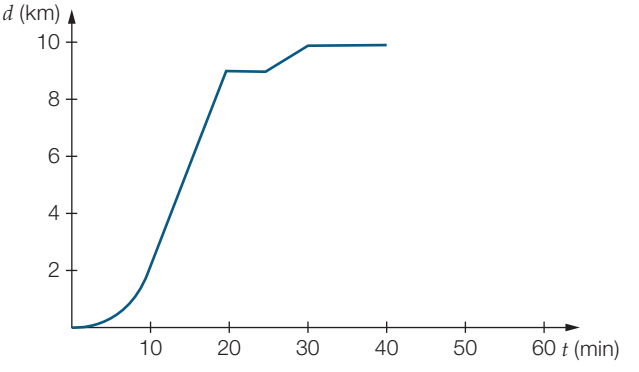
WE40

3 State whether the following graphs show a constant or variable rate of change.



WE41

4 Pete's journey to school is shown by the following graph. During his trip he walks, runs and travels by bus and he goes straight to his locker when he gets to school. Describe his trip stating the time each part of his journey took him.



5 A chef is paid \$704 for 32 hours' work. What is his rate of pay?

6 A train takes 25 minutes to travel a 50-kilometre stretch at a constant speed. What is the speed in kilometres per hour?

7 Which of the following is not a constant rate?

A An hourly rate of pay

B The number of hours of sunlight per day

C The cost of mushroom compost per cubic metre on a particular day at a garden centre

D The number of cans of soup produced by a robotic assembly line every hour in a certain factory

Reasoning, communicating and open-ended

8 A phone company charges $80 per month for a mobile phone on a 24-month plan with a $600 cap.

Is this an example of a constant rate? Explain your answer.

9 An electricity company has a supply charge of 99c per day and a consumption charge of 27c per kWh.

A rival company has a supply charge of 95c per day and a consumption charge of 30c per kWh.

Compare the bills from each company for a household that has used 1500 KWh over a 90-day period and decide which company would be the cheaper electricity provider.

10 Sketch a graph to show the following relationships:

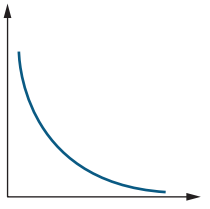
(a) decreasing at an increasing rate

(b) increasing at an increasing rate

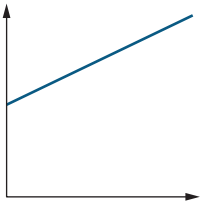
(c) decreasing at a constant rate.

11 Describe the rate of change of the following graphs and state whether the relationship is increasing or decreasing

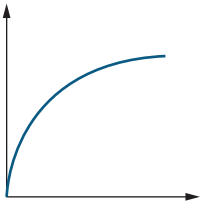
(a)



(b)



(c)



12 While travelling on a straight stretch of a country road, Tim notes that his car has travelled 6 kilometres in 10 minutes. He states that the car has travelled at a constant rate of 36 km/h. Emma disagrees. Decide who is correct and explain your decision.

Chapter review 9

D.I.Y. Summary

Key Words

Stage 5.2	Null Factor Law		
Stage 5.2◊	asymptote	exponential	turning point
	axis of symmetry	parabola	vertex
Stage 5.3§	dilation	inverse proportion	transformation
	dilation factor	rectangular hyperbola	translation
	hyperbola	reflection	turning point form

Stage 5.2◊ Q2, 3, 5, 6 **Stage 5.3§ Q1, 4**

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 The basic graph of $y = x^2$ can be used to obtain other parabolas by using a combination of any of these _____s: _____s, _____s or _____s.

2 A _____ is the type of graph produced by a quadratic relationship.

3 To graph a quadratic it is important to show the x- and y-intercepts and the _____.

4 To change a quadratic that is in the form $y = ax^2 + bx + c$ ($a = 1$) into _____, it is necessary to complete the square.

5 An _____ is a line on a Cartesian plane that a curve approaches but does not cut or touch for certain values of the independent variable.

6 An _____ equation has a horizontal asymptote and may be increasing or decreasing at a rapid rate.

Stage 5.2◊ Q1, 2, 16, 18–20 **Stage 5.2 Q4–6, 17** **Stage 5.3§ Q3, 7–14, 21–26** **Stage 5.3 Q15**

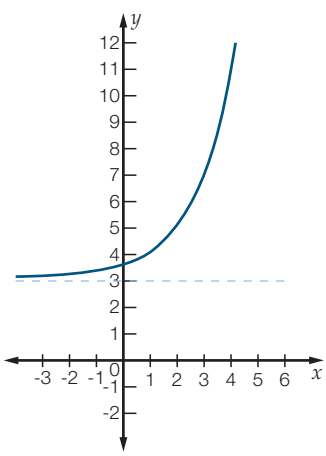
Fluency and understanding

1 For each graph below, identify the type of graph and give the:

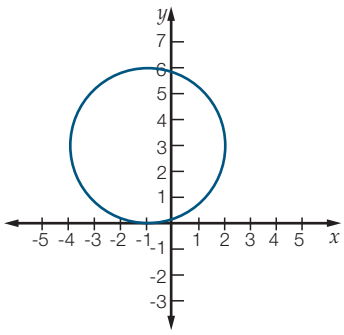
Ex. 9.1

- equation of the asymptote if it is an exponential
- coordinates of the centre and the radius if it is a circle.

(a)



(b)



2 Plot the graph of $y = x^2 - 2x - 8$ for $-5 \le x \le 5$. **Ex.9.1**

(a) Write down the coordinates of the x- and y-intercepts.

(b) Write down the coordinates of the turning point. Does the parabola have a maximum or a minimum turning point?

(c) Write down the equation of the axis of symmetry.

(d) From the graph, find the value of y when $x = 3$.

(e) What are the values of x when $y = 7$?

(f) Hence, state the values of x when $x^2 - 2x - 8 = 7$.

(g) Write down the values of x when $x^2 - 2x - 8 = 0$.

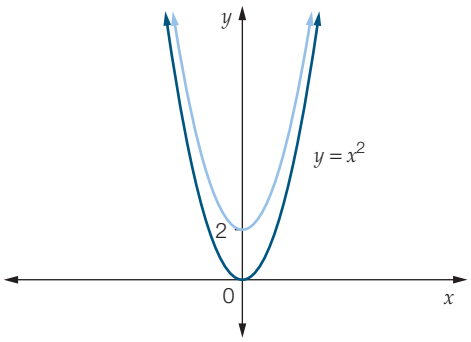
3 (a) The equation of the parabola shown could be: **Ex.9.2**

A $y = x^2 + 2$

B $y = x^2 - 2$

C $y = (x - 2)^2$

D $y = (x + 2)^2$



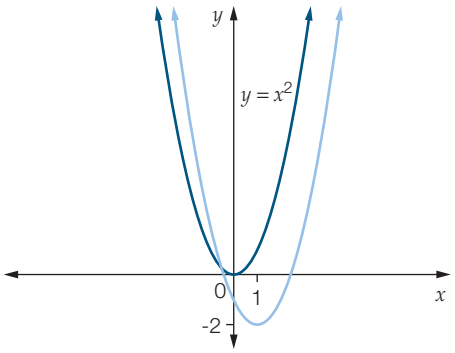
(b) The equation of the parabola shown could be:

A $y = (x - 1)^2 - 2$

B $y = (x - 1)^2 + 2$

C $y = (x + 1)^2 + 2$

D $y = (x + 1)^2 - 2$



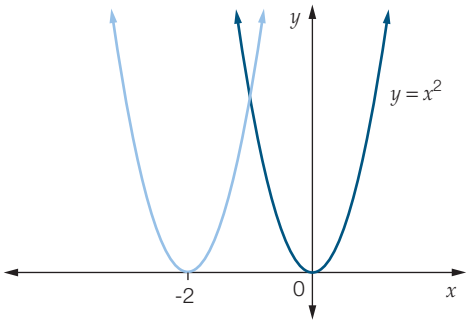
(c) The equation of the parabola shown could be:

A $y = (x - 2)^2$

B $y = (x + 2)^2$

C $y = x^2 - 2$

D $y = x^2 + 2$



4 Solve these quadratic equations using the Null Factor Law. **Ex.9.3**

(a) $x^2 - 6x = 0$ (b) $x^2 - 5x - 14 = 0$ (c) $x^2 = 8x$

(d) $2x(x + 4) = 0$ (e) $(x - 7)(x + 1) = 0$ (f) $x^2 + 15x - 16 = 0$

5 Solve these quadratic equations. **Ex.9.3**

(a) $x^2 + 12x + 35 = 0$ (b) $x^2 + x - 6 = 0$ (c) $-x^2 - 8x + 9 = 0$

(d) $-x^2 + 9x - 8 = 0$ (e) $3x^2 + 9x - 12 = 0$ (f) $2x^2 - 8x + 6 = 0$

6 Solve these quadratic equations. **Ex.9.3**

(a) $x^2 - 4x = 5$ (b) $x^2 - 6 = 5x$ (c) $x^2 + 12 = -7x$

(d) $3x^2 = 27x$ (e) $(x - 3)(x + 7) = 11$ (f) $(x - 3)(x + 2) = 6$

7 Solve these quadratic equations using the Null Factor Law. Ex 9.4

(a) $4x^2 - 81 = 0$ **(b)** $x^2 + 6x + 9 = 0$ **(c)** $x^2 + 36 = 12x$

8 Solve each of the following by completing the square. Ex 9.5

(a) $x^2 - 4x + 2 = 0$ **(b)** $x^2 + 8x + 9 = 0$ **(c)** $-x^2 - 2x + 4 = 0$

9 Solve each quadratic equation using an appropriate method. Ex 9.6

(a) $2x^2 - 18 = 0$ **(b)** $2x^2 - 9x = 5$ **(c)** $(2x - 5)^2 = 3$

(d) $2x^2 + 5x + 1 = 0$ **(e)** $3x^2 - 6x + 2 = 0$ **(f)** $13x^2 + 7x - 3 = 0$

10 For each of the following: Ex 9.7

(i) state the changes that have been made to the graph of $y = x^2$

(ii) state the coordinates of the turning point

(iii) find the x- and y-intercepts

(iv) sketch a graph of each equation showing the turning point and intercepts.

(a) $y = x^2 - 4$ **(b)** $y = -2(x + 1)^2 - 4$ **(c)** $y = \frac{1}{3}(x - 7)^2$

11 For each of the following equations: Ex 9.7

(i) find the y-intercept **(ii)** find the x-intercepts

(iii) find the coordinates of the turning point **(iv)** sketch the graph.

(a) $y = x^2 - 4x - 12$ **(b)** $y = -(x + 1)^2 + 4$

(c) $y = 2(x - 3)^2 - 6$ **(d)** $y = -x^2 + 3x + 4$

12 Sketch the graph of each of the following. Ex 9.8

(a) $y = \dfrac{1}{x - 4} + 1$ **(b)** $y = \dfrac{1}{x + 2} - 1$

13 Sketch the graph of each of the following. Ex 9.9

(a) $(x + 2)^2 + (y - 3)^2 = 16$ **(b)** $x^2 + (y - 3)^2 = 4$

(c) $y = 2^{(x - 2)} + 1$ **(d)** $y = 2^{x + 2} - 4$

14 Find the points of intersection of the line with equation $y = 3x + 7$ and the parabola of the equation $y = x^2 + 2x + 5$. Ex 9.10

15 State whether the following are constant or variable rates: Ex 9.11

(a) The cost of bananas/kg at a supermarket on any day

(b) The conversion rate of A$ to US$ over a month

(c) The weekly rate for a car hire for a two-week period.

16 A softball thrown by an outfielder to the pitcher followed a path where the height h, in metres, of the softball after t seconds, is described by the relationship:

$h = -t^2 + 6t + 1$

(a) Complete a table of values for this relationship.

(b) Plot the graph of this relationship on graph paper.

(c) What was the height of the softball after 2 seconds?

(d) Use your graph to find when the ball was first at a height of 7 m above the ground?

(e) Use your graph to find during which time interval the ball was above a height of 9 m?

(f) What was the maximum height of the softball during its flight?

(g) How long was the ball in flight if it was caught at a height of 1.6 m above the ground on its downward path?

17 A train ticket is 5 cm longer than it is wide.

(a) Draw a diagram of the ticket, labelling the breadth x.

(b) Write down the length of the ticket in terms of x.

(c) Write an expression for the area of the train ticket in terms of x.

(d) If each ticket requires 24 cm^2 of paper, write an equation that represents this area.

(e) Solve this equation to find x.

(f) Hence, state the dimensions of the train ticket.

(g) Did you use both of the values for x that you found using algebra to find the dimensions of the ticket? Explain your answer.

Reasoning and communicating

18 A fountain sends a jet of water on a parabolic path, which can be described by the equation $h = -2x^2 + 0.8x$, where h is the height of the water, in metres, at a horizontal distance x, in metres, from the water outlet.

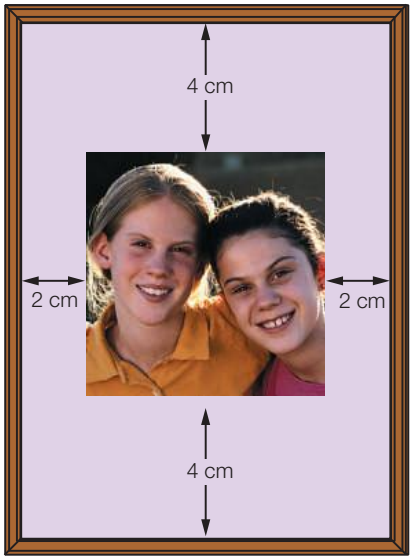
(a) Where is the jet of water again at the height that it left the outlet?

(b) If it hits the base of a pond at a horizontal distance of 0.8 m from the outlet, how far below the outlet does the water land?

(c) What is the maximum height above the base of the pond that the jet of water reaches?

(d) Sketch the graph of the path of the water.

19 A square photograph is to be attached to a coloured mounting and then framed.



(a) Find the length of a side of the photo if the frame encloses a total area of 140 cm^2.

(b) Hence find the dimensions of the photo frame.

(c) Calculate the area of the photograph.

(d) Calculate the area of coloured mounting that can be seen.

(e) What area of coloured mounting would be seen if the same photograph was to have a 2 cm coloured mounting all around?

20 The sum of two numbers is 20.

(a) The two numbers are x and y. Write an equation to represent the sum.

(b) Rearrange the equation so that x is the subject (i.e. in the form $y =$).

(c) Write an expression for the product of the two numbers using x.

(d) Graph the expression and use it to find the maximum product and the numbers that obtain it.

21 A tennis ball is struck in the centre of a tennis court 23.78 m long, 1 m behind the base line. It follows a parabolic path parallel to the sidelines and its height is determined by the equation $h = -0.0045x^2 + 0.135x$, where x is the distance from a fence, which is 6.11 m from the base line.

(a) At what height was the tennis ball struck?

(b) Show that the ball clears the net, which is 0.913 m high in the centre.

(c) Determine whether the ball lands inside or outside the tennis court.

22 The graph of $y = x^2$ is translated so that it passes through two points (3, 8) and (9, 8).

(a) Write down the x-coordinate of the turning point of the graph.

(b) Find the equation of the curve.

(c) Sketch this graph.

23 A pattern is drawn with four circles with equations:

A $(x-3)^2 + (y-2)^2 = 16$

B $(x+1)^2 + (y-3)^2 = 4$

C $(x+5)^2 + (y+4)^2 = 9$

D $(x+2)^2 + (y+1)^2 = 16$

(a) Which two circles are the same size?

(b) By sketching the graphs, find which circle intersects with the other three circles.

24 Solve $\dfrac{3(x+1)}{5x} - \dfrac{1}{2x} = 4x$.

25 Substitute $m = x^2$ into $x^4 - x^2 - 12 = 0$ to solve the equation for x.

26 The sum of a sequence is given by $S = \dfrac{n}{2}[2a + (n-1)d]$, where n stands for the number of terms in the sequence. If $a = 5$ and $d = 4$, find the number of terms in the sequence needed to make a sum of 860.

Mixed review C

Stage 5.1 Q3 **Stage 5.2** Q1, 2, 4, 5, 7, 9, 10, 17 **Stage 5.3** Q6, 8, 11–16, 18, 19

Fluency and understanding

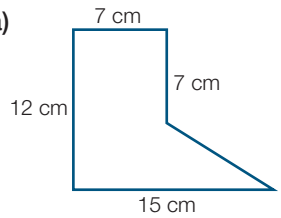
1 An investment of \$5000 is made at 3.5% p.a. for 6 years. Find the difference in interest earned if the interest is compounded monthly compared to annually. Ex.1.3

2 Factorise the following quadratic trinomials. Ex.2.4

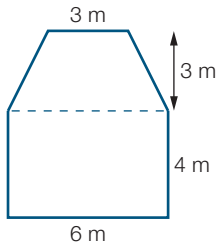
(a) $a^2 - 5a - 36$ **(b)** $a^2 - 8a + 12$ **(c)** $4a^2 + 20a + 25$

3 Find the area of each of the following shapes. Ex.3.1

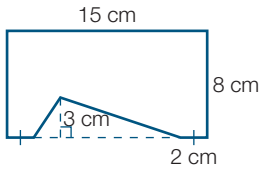
(a)



(b)



(c)

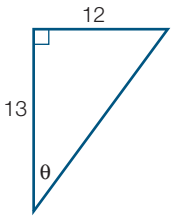


4 Solve the following equations. Ex.4.1

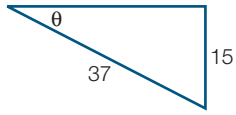
(a) $\frac{3a+4}{7} = -2$ **(b)** $\frac{4a+3}{2} = \frac{2a}{5}$ **(c)** $\frac{3a-2}{3} = \frac{4a-5}{2}$

5 Find the value of the pronumeral in each, correct to one decimal place. Ex.5.3

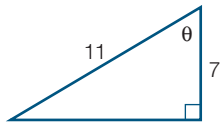
(a)



(b)



(c)



6 Find the length of the third side of a scalene triangle with two sides of length 6 cm and 7 cm and an included angle of 43°. Round your answer to two decimal places. Ex.6.4

7 Find the range and the interquartile range for each of the following grouped data frequency tables. Give your answers correct to two decimal places. Ex.7.2

(a)

Class interval	Frequency
20–<25	8
25–<30	6
30–<35	3
35–<40	2
40–<45	5
45–<50	5

(b)

Class interval	Frequency
110–<115	12
115–<120	15
120–<125	17
125–<130	19
130–<135	13
135–<140	11

8 Prove that the diagonals of a kite intersect at right angles. Ex.8.5

9 Consider the equation $y = 2x^2 + x$.

Ex. 9.1

(a) Set up a table of values for values of x between -3 to 3 inclusive.

(b) Plot the points on a Cartesian plane.

(c) Find the turning point of the graph.

(d) Determine the x- and y-intercepts for the graph.

10 What are the asymptotes of the graph of $y = \frac{1}{x+1} - 3$?

Ex. 9.1

11 Solve each of the following equations.

Ex. 9.4

(a) $2x^2 - 50 = 0$ (b) $x^2 - 100 = 0$ (c) $x^2 - 14x + 49 = 0$

12 Sketch the graph of each of the following quadratics, showing all important features.

(a) $y = x^2 + 6x - 4$ (b) $y = x^2 - 4x + 4$ (c) $y = -2x^2 + 3x$

13 Solve each of the following for x.

(a) $x^2 + 13x + 42 = 0$ (b) $4x^2 + 2x = 0$ (c) $x^2 - 5x + 6 = 0$

14 Write down the turning point of each of the following graphs.

(a) $y = (x - 2)^2 + 3$ (b) $y = (x + 4)^2 - 1$ (c) $y = (x - 7)^2 - 6$

15 Write the equation formed when $x^2 + y^2 = 25$ is translated 4 units right and 2 units up.

16 A point that does not lie on the graph of $(x - 3)^2 + (y - 2)^2 = 20$ would be:

A (7, 0) B (5, 6) C (6, 5) D (-1, 4)

17 The graph of the equation $y = 2^{x-1} + 1$ has an asymptote at:

A $x = 1$ B $x = -1$ C $y = 1$ D $y = -1$

Reasoning and communicating

18 A mobile phone screen is 2.5 cm longer than it is wide.

(a) Write an equation for the area of the screen in terms of its breadth, in cm^2.

(b) Plot a graph of the area against the breadth.

(c) Use this graph to find the dimensions of the screen if the area of the screen is 37.5 cm^2.

19 A graph has x-intercepts at $x = 4$ and $x = 6$.

(a) Determine the x-coordinate of the turning point.

(b) If the x^2 coefficient is 1, determine the equation of the graph.

Challenge C

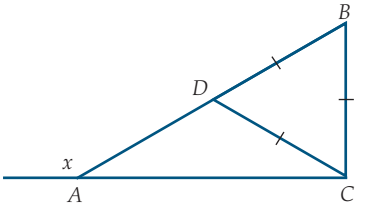
1 The mean of 19 consecutive integers is 99. The largest of these integers is:

A 118 **B** 117 **C** 108 **D** 107

2 Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is:

A 34 **B** 35 **C** 36 **D** 37

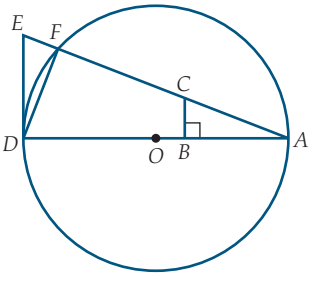
3 Find the value of x if the large triangle has been broken into two triangles of equal area. The large triangle is not marked as being right angled.



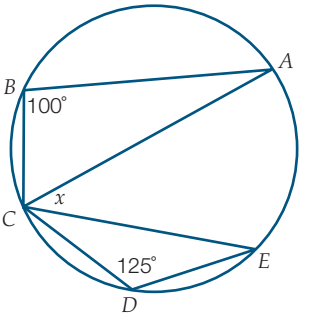
4 In the diagram, $ABC = 90°$, $CB \parallel ED$, $AB = DF$, $AD = 24$, $AE = 25$ and O is the centre of the circle. The perimeter of $CBDF$ is:

A 40 units **B** 42 units

C 43 units **D** 44 units



5 Find x in the diagram on the right.



6 **(a)** Determine the values of x for which $(x + 3)(x - 6) = -14$.

(b) Determine the values of x for which $(x^2 - 3x)^2 = 4 - 3(3x - x^2)$.

7 **(a)** What is the sum of the two numbers that satisfy the equation $x^2 - 6x - 7 = 0$?

(b) What is the product of the two numbers that satisfy the equation $5x^2 - 20 = 0$?

(c) Find the average of the numbers that satisfy the equation $x^3 - 6x^2 + 5x = 0$.

8 The equation of a parabola is written in the form $y = (x - 2)(x - 8) + 7$.

(a) If $(c, 7)$ lies on the parabola, find all possible values of c.

(b) Find the coordinates of the vertex of the parabola.

9 The circle $x^2 + y^2 = 9$ and the parabola $y = x^2 - 3$ intersect at three points.

(a) Find the y-coordinates of the points of intersection.

(b) Hence, find the exact values of the coordinates of the points of intersection.

10 Polynomials

P Recall 10

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Plot the points in each of the following tables of values and join them with a straight line or a smooth curve to obtain the graph of each relationship.

(a)

x	-3	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4	6

(b)

x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11

2 State whether the relationship is linear, quadratic or neither.

(a) $y = x^3 + 2x^2 + 3$ **(b)** $b = 2c + 3$ **(c)** $z = 4 - 3w^2$

3 **(a)** Complete this table of values for $y = x^2 + 2x - 8$.

x	-5	-4	-3	-2	-1	0	1	2	3
y									

(b) On graph paper, plot the graph of the parabola $y = x^2 + 2x - 8$.

(c) From the graph, write the:

(i) y-intercept

(ii) x-intercepts

(iii) coordinates of the turning point.

(d) Decide whether the graph has a maximum or a minimum turning point.

(e) What is the equation of the axis of symmetry?

4 Factorise each quadratic expression:

(a) $x^2 - 5x - 6$ **(b)** $4x^2 - 81$ **(c)** $3x^2 - 3x - 36$

(d) $x^2 - 14x + 49$ **(e)** $-x^2 + 8x + 20$ **(f)** $x^2 - 6x + 1$

5 Solve these equations using the Null Factor Law.

(a) $(x + 2)(x - 1) = 0$ **(b)** $(x + 9)(x - 9) = 0$ **(c)** $3x(2x - 5) = 0$

Syllabus references

Number and Algebra: Equations

✓ Solve simple cubic equations **(Stage 5.3§)**

Number and Algebra: Non-Linear Relationships

✓ Describe, interpret and sketch cubics, other curves and their transformations **(Stage 5.3§)**

Number and Algebra: Polynomials

✓ Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems **(ACMNA266) (Stage 5.3#)**

✓ Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation **(ACMNA268) (Stage 5.3#)**

Number and Algebra: Functions and Other Graphs

✓ Describe, interpret and sketch functions **(Stage 5.3#)**

Equations of the form $y = ax^n$ and their graphs

10.1

Need to Know

Stage 5.3§

Equations of the graph for $y = ax^n$

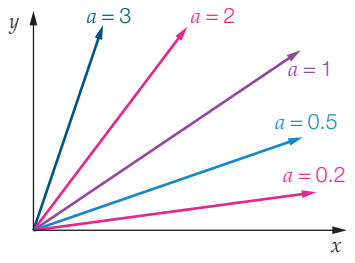
The linear equation $y = 2x$, the quadratic equation $y = x^2$, the **cubic** equation $y = x^3$ and the **quartic** equation $y = 5x^4$ belong to a class of equations described by $y = ax^n$, where a is a constant and n is an integer.

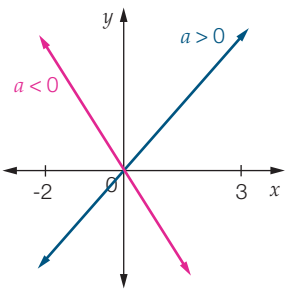
The properties of graphs of $y = ax^n$ will be investigated for different non-zero values of a and for $n = 1, 2, 3$ and 4.

Linear equations: $y = ax$

When $n = 1$, $y = ax^n$ defines a linear equation $y = ax$, a straight line that passes through the origin (0, 0) and has gradient a.

- If $a > 0$, the gradient is positive.
- If $a < 0$, the gradient is negative.



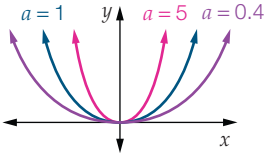


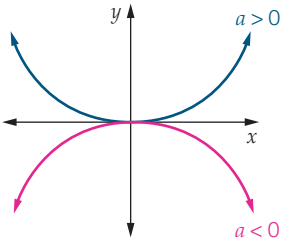
The coefficient of x is the dilation factor. The larger the size of a (ignoring the sign), the steeper the line. Every x-value of the graph of $y = x$ is multiplied by a to find the y-values of $y = ax$.

Quadratic equations: $y = ax^2$

For $n = 2$, the quadratic equation $y = ax^2$ graphs as a parabola that passes through the origin and is symmetrical about the y-axis. The axis of symmetry is the line $x = 0$. The coefficient of x^2, a, is the dilation factor.

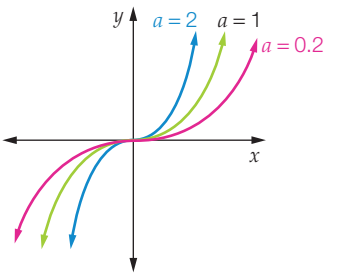
- If $a > 0$, the graph has a minimum turning point at (0, 0).
- If $a > 1$, the graph becomes narrower than the graph of $y = x^2$.
- If $0 < a < 1$, then the graph becomes wider than the graph of $y = x^2$.
- If $a < 0$, the graph is inverted. It is a reflection in the x-axis of the graph for $a > 0$ and it now has a maximum turning point at (0, 0).



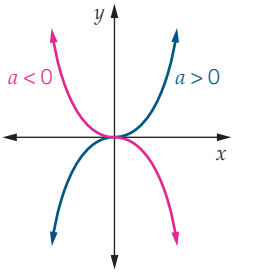


Cubic equations: $y = ax^3$

For $n = 3$, $y = ax^3$ is a cubic graph that passes through the origin. It does not have a turning point (maximum or minimum) where the graph changes direction, but it does have a special point at the origin called a **point of inflection**. It is not symmetrical about the x-axis or the y-axis, but the graph on one side of the origin is reflected in both the x- and y-axes to produce the other half of the graph.

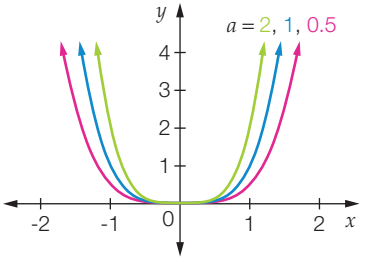


- If $a > 1$, the graph becomes steeper (narrower) than the graph of $y = x^3$.
- If $0 < a < 1$, the graph becomes flatter (wider) than the graph of $y = x^3$.
- If $a < 0$, the graph is a reflection in the x-axis of the graph for $a > 0$.

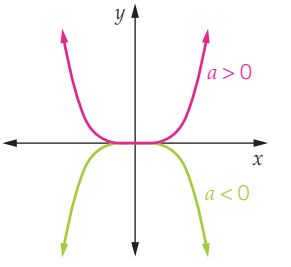


Quartic equations: $y = ax^4$

When $n = 4$, the equation $y = ax^4$ graphs as a quartic that passes through the origin and is symmetrical about the y-axis. A quartic graph is steeper than a quadratic graph with the same dilation factor for $x > 1$ and $x < -1$. It is squarer than a quadratic graph with the same dilation factor for $-1 < x < 1$.



- If $a > 0$, the graph has a minimum turning point at (0, 0).
- If $a > 1$, the graph becomes narrower than the graph of $y = x^4$.
- If $0 < a < 1$, the graph becomes wider than the graph of $y = x^4$.
- If $a < 0$, the graph is a reflection in the x-axis of the graph for $a > 0$ and it now has a maximum turning point at (0, 0).



Worked Example 1

WE1

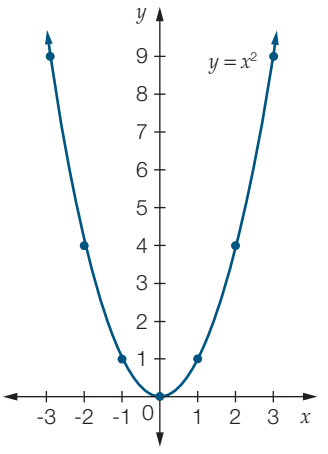
Construct a table of values for $x = -3$ to 3 and then plot the graph of the following equations:

(a) $y = x^2$ **(b)** $y = x^3$ **(c)** $y = x^4$

Working

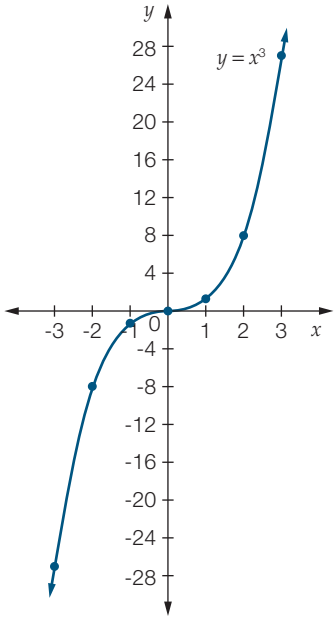
(a)

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



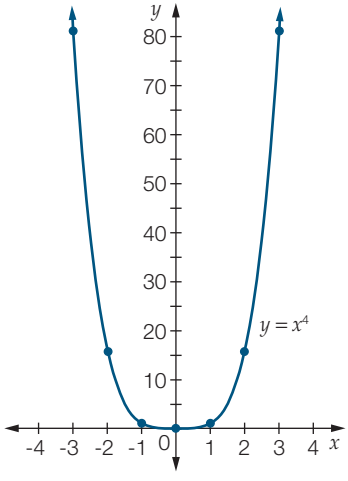
(b)

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	2	8	27



(c)

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81



Answers page 697

10.1 Equations of the form $y = ax^n$ and their graphs

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Fluency and understanding

WE1

1 Construct a table of values for $x = -3$ to 3 and then plot the following equations.

(a) $y = 2x^2$ **(b)** $y = 2x^3$ **(c)** $y = 2x^4$

2 Match each equation to its graph.

U $y = -\frac{2}{3}x$

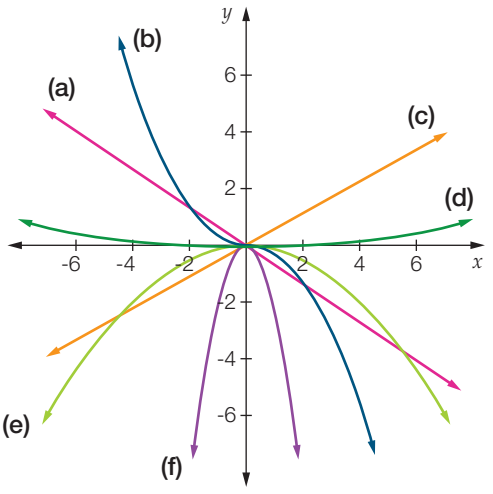
V $y = -6x^2$

W $y = \frac{x}{2}$

X $y = -\frac{1}{10}x^2$

Y $y = 0.01x^2$

Z $y = \left(-\frac{1}{2}x\right)^3$



3 **(a)** Plot the graph of each linear equation on the same number plane. Use x-values from -3 to 3.

(i) $y = x$ **(ii)** $y = 2x$ **(iii)** $y = 4x$

(b) What can you conclude about the steepness of the line in relation to the coefficient of x?

(c) Does your answer to part **(b)** apply to the equations $y = -x$, $y = -2x$ and $y = -4x$?

4 **(a)** Plot the graph of each quadratic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = 2x^2$ **(ii)** $y = 3x^2$ **(iii)** $y = 4x^2$

(b) Describe what happens to the shape of the parabola as the coefficient of x^2 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -2x^2$, $y = -3x^2$ and $y = -4x^2$?

Warning

Is -4 > -2?

5 **(a)** Plot the graph of each cubic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = x^3$ **(ii)** $y = 2x^3$ **(iii)** $y = 3x^3$

(b) Describe what happens to the shape of the cubic graph as the coefficient of x^3 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -x^3$, $y = -2x^3$ and $y = -3x^3$?

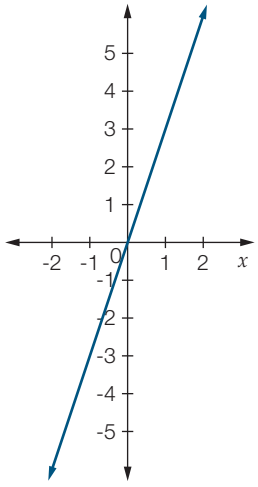
6 **(a)** Plot the graph of each quartic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = 0.5x^4$ **(ii)** $y = x^4$ **(iii)** $y = 3x^4$

(b) Describe what happens to the shape of the quartic graph as the coefficient of x^4 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -x^4$, $y = -3x^4$, $y = -0.5x^4$?

7 The straight line shown is reflected in the x-axis and then reflected in the y-axis. The equation describing the new graph is:



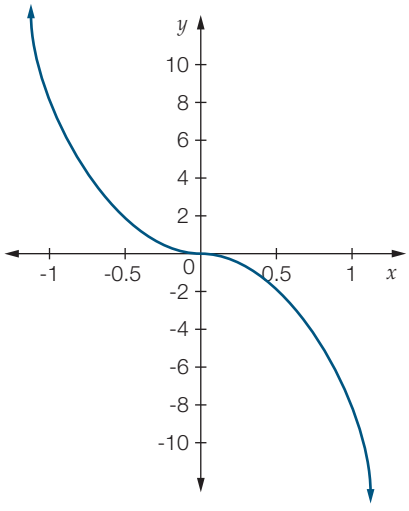
A $y = \frac{1}{3}x$

B $y = 3x$

C $y = -\frac{1}{3}x$

D $y = -3x$

8 A possible equation to describe the graph shown is:



A $y = -8x^2$

B $y = x^3$

C $y = -x^2$

D $y = -8x^3$

Reasoning, communicating and open-ended

9 The shape of a skate ramp is described by the equation $y = \frac{1}{3}x^2$, where x and y are the horizontal and vertical distances respectively, in metres, from the bottom of the ramp.

(a) Draw the graph of the equation using values of x from 0 to 3.

(b) One end of a taut length of rope is attached to the bottom of the ramp and the other is attached to a point on the ramp with x-coordinate 1.5. What is the equation of the line describing the position of the rope?

10 Tim and Matthew set off from their campsite in different directions. Tim's distance, d km, from the campsite at t minutes is given by the equation $d = 0.1t$, and Matthew's distance from the campsite is given by $d = -0.1t$.

(a) Represent each equation on the same number plane using values of t from 0 to 10.

(b) How far apart are the boys after 10 minutes?

(c) What is the equation of the line describing the path Tim could have taken had he walked at a constant speed and travelled 2.4 km in 20 minutes?

Hint

What does a negative distance represent?

11 The shape of a flat weathervane is defined by the two equations $y = 4x$ and $y = 2x^2$ for x-values from 0 to 2, where x and y represent length in metres and the origin is taken to be the base of the weathervane.

(a) Sketch the two graphs on the same number plane using x-values from 0 to 2.

(b) If the area under the curve $y = 2x^2$ from $x = 0$ to $x = 2$ and bounded by the x-axis is $5\frac{1}{3}$ square metres, what is the area covered by one face of the weathervane?

(c) The wind changes direction so that the weathervane swings around 180° about the origin. What are the equations describing the shape of the weathervane now? State the values of x that apply.

12 **(a)** On the same set of axes draw the curves $y = x^3$ and $y = \frac{1}{2}x^4$ for $x = -2$ to 2.

(b) Shade in the region between these two curves between $x = 0$ and $x = 2$.

(c) Reflect this region in the y-axis and draw it on your graph.

(d) What is the equation of the top boundary of this new region?

13 **(a)** Find two consecutive whole number values for c so that the gradient of the straight line described by the equation $y = -\left(\frac{c-9}{1-c}\right)x$ is positive.

(b) What can be concluded about the gradient when $c = 1$? What does this tell you about the line?

14 Consider the graphs of $y = ax^2$ and $y = 2a - ax^2$. Draw the graphs on the same number plane for $a = 1, 2, 3, 4$. What have you discovered about the points of intersection of the two graphs? Repeat for negative values of a and comment on your answer.

Problem solving

Different paths

Mark wants to build a path from his front door to the gate. He draws up a grid and determines that his front door is at grid reference (0, 0) and the gate is at (5, 12). He decides that as he has been learning about graphs of the type $y = ax^n$ he will see how many different graphs he can get that fit these two points.

1 Find the equations of the curves that he would need for $n = 1, 2, 3$ and 4.

2 For each value of n, determine at least one other grid reference that he should use when making the path.

Strategy options

- Draw a diagram.
- Test all possible combinations.
- Look for a pattern.

Transformations of $y = ax^n$

10.2

Need to Know

Stage 5.3§

Horizontal translations: graphs of equations of the form $y = a(x - h)^n$

The graph of $y = a(x - h)^n$ is a translation of the graph of $y = ax^n$ by h units horizontally.

If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.

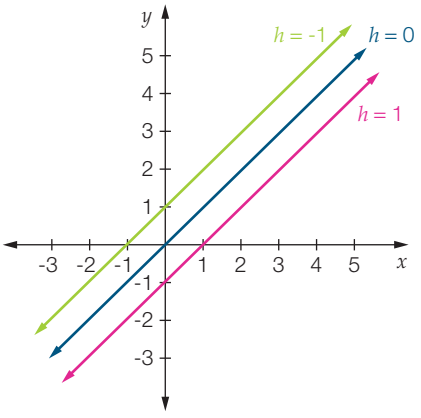
Horizontal translations of linear graphs

Substituting $a = 1$, $n = 1$ and $h = -1, 0, 1$ into $y = a(x - h)^n$ gives $y = x - 1$, $y = x$ and $y = x + 1$.

The graphs of these equations are straight lines with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively.

The value of h represents the x-coordinate of the x-intercept and the graph of $y = ax$ is translated h units in the positive direction of the x-axis.

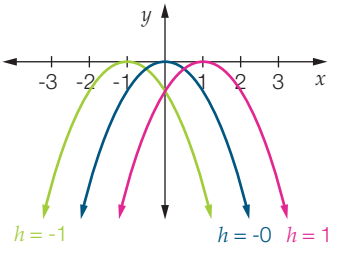
If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.



Horizontal translations of quadratic graphs

Substituting $a = -3$, $n = 2$ and $h = -1, 0, 1$ into $y = a(x - h)^n$ gives $y = -3(x + 1)^2$, $y = -3x^2$ and $y = -3(x - 1)^2$.

The graphs of the equations are parabolas, symmetrical about the line $x = h$ and the y-value of the y-intercept is $a(-h)^2$, found by substituting $x = 0$ in $y = a(x - h)^2$. The value of h represents the x-coordinate of the x-intercept and the graph of $y = ax^2$ is translated h units in the direction of the x-axis.



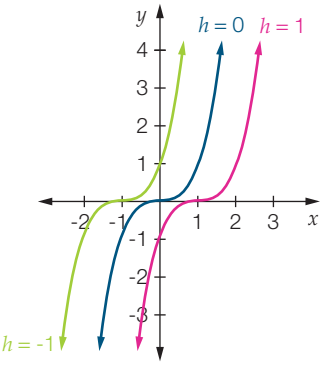
If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.

Horizontal translations of cubic graphs

Substituting $a = 1$, $n = 3$ and $h = -1, 0, 1$ into $y = a(x - h)^n$ gives $y = (x + 1)^3$, $y = x^3$ and $y = (x - 1)^3$.

The y-value of the y-intercept is $a(-h)^3$, found by substituting $x = 0$ in $y = a(x - h)^3$. This type of cubic has only one x-value for the x-intercept. The value of h represents the x-intercept and the graph of $y = ax^3$ is translated h units in the direction of the x-axis.

If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.



Vertical translations: graphs of equations of the form $y = ax^n + k$

The graph of $y = ax^n + k$ is a translation of the graph of $y = ax^n$ by k units vertically.

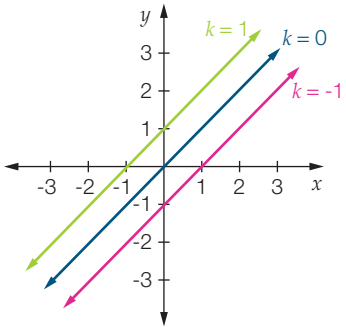
If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

Vertical translations of linear graphs

Substituting $a = 1$, $n = 1$ and $k = -1, 0, 1$ into $y = ax^n + k$ gives $y = x - 1$, $y = x$ and $y = x + 1$.

The graphs are straight lines with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively. The value of k represents the y-coordinate of the y-intercept and has the effect of translating the graph of $y = ax$ a distance of k units in the positive direction of the y-axis.

If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

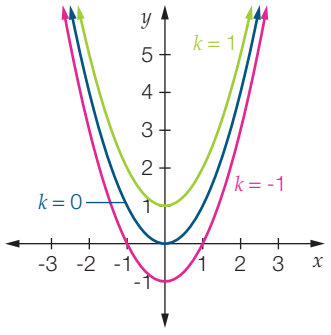


Vertical translations of quadratic graphs

Substituting $a = 1$, $n = 2$ and $k = -1, 0, 1$ into $y = ax^n + k$ gives $y = x^2 - 1$, $y = x^2$, and $y = x^2 + 1$.

The graphs are parabolas, symmetrical about the line $x = h$. The graph may have zero, one or two x-intercepts. The value of k represents the y-coordinate of the y-intercept and is found by substituting $x = 0$ in $y = ax^n + k$. The graph of $y = ax^2$ is translated k units in the positive direction of the y-axis.

If $k > 0$ the translation is up, and if $k < 0$ the translation is down.



Vertical translations of cubic graphs

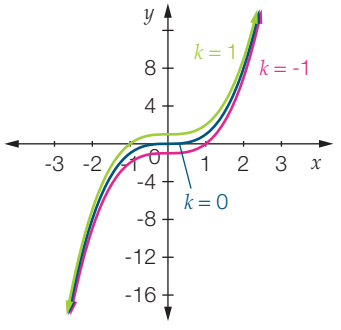
Substituting $a = 1$, $n = 3$ and $k = -1, 0, 1$ into $y = ax^n + k$ gives $y = x^3 - 1$, $y = x^3$ and $y = x^3 + 1$.

The graphs are cubics with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively.

The value of k represents the y-coordinate of the y-intercept and is found by substituting $x = 0$ in $y = ax^3 + k$, and the graph of $y = ax^3$ is translated k units in the positive direction of the y-axis.

If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

This type of cubic has only one x-intercept.



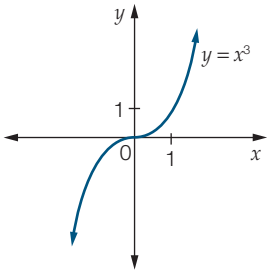
Worked Example 2

WE2

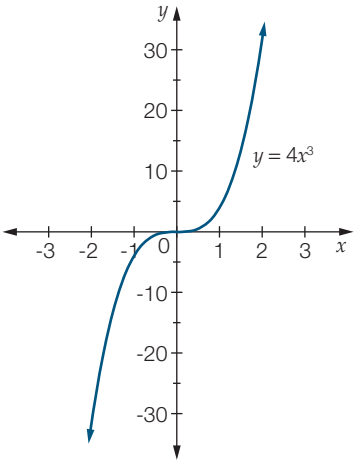
Sketch the graph of $y = -4(x - 1)^3$ by transforming the basic graph of $y = x^3$.

Working

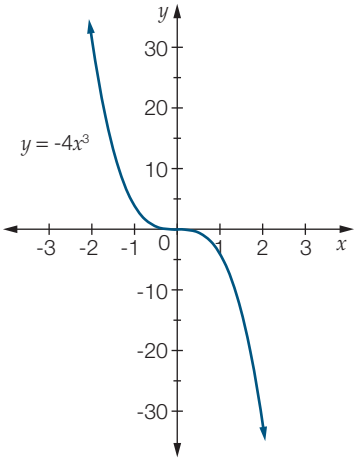
Step 1: Sketch the graph of $y = x^3$.



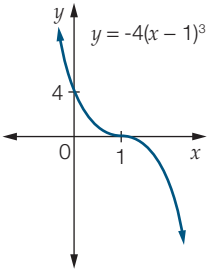
Step 2: Dilate the graph by a factor of 4.



Step 3: Reflect the graph in the x-axis.



Step 4: Translate the graph 1 unit to the right.



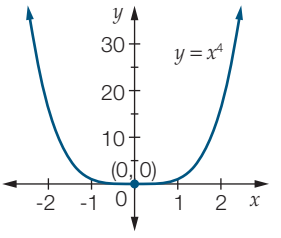
Worked Example 3

WE3

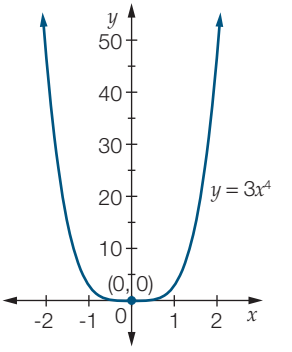
Sketch the graph of $y = -3x^4 + 2$ by transforming the basic graph of $y = x^4$.

Working

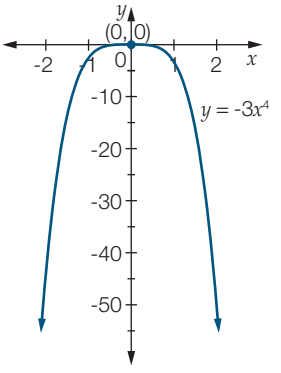
Step 1: Sketch the graph of $y = x^4$.



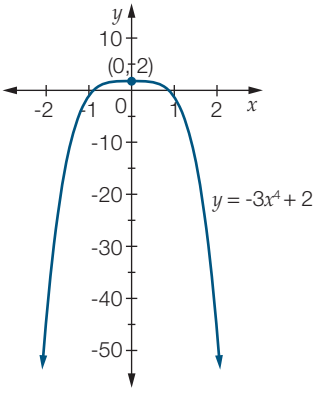
Step 2: Dilate the graph by a factor of 3.



Step 3: Reflect the graph in the x-axis.



Step 4: Translate the graph 2 units up.



Answers page 699

10.2 Transformations of $y = ax^n$

Stage 5.3§

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Fluency and understanding

WE2

1 Sketch the graph of $y = -3(x + 2)^3$ by transforming the graph of $y = x^3$.

WE3

2 Sketch the graph of $y = -x^2 + 1$ by transforming the graph of $y = x^2$.

3 Sketch the graph of $y = 4x - 2$ by transforming the graph of $y = x$.

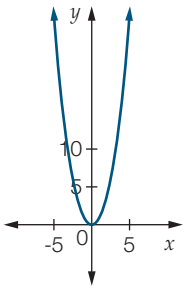
4 Sketch the graph of $y = 3(x - 1)^4$ by transforming the graph of $y = x^4$.

5 Sketch the graph of $y = \frac{1}{2}(x - 3)^2$ by transforming the graph of $y = x^2$.

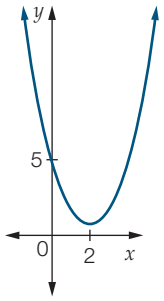
6 Sketch the graph of $y = -\frac{1}{10}(x+2)^4$ by transforming the graph of $y = x^4$.

7 The graph of the equation $y = x^2$ is transformed to the graph of the equation $y = -2(x+2)^2 - 1$. The graph of the transformed equation is:

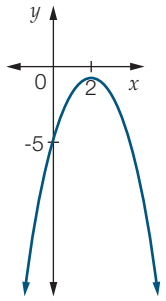
A



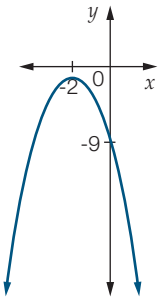
B



C



D



8 To obtain the graph of $y = -3(x+2)^3 + 1$ from $y = x^3$:

A Dilate by a factor of 3 in the y-direction, reflect in the x-axis, translate 2 units right and translate one unit up.

B Dilate by a factor of 3 in the y-direction, reflect in the x-axis and translate 2 units left.

C Dilate by a factor of 3 in the y-direction, reflect in the x-axis, translate 2 units left and translate one unit up.

D Dilate by a factor of 3 in the y-direction, translate 1 unit left and translate 2 units down.

9 The y-intercept of the graph of the equation $y = (16x - 3)^3$ is:

A (0, 54) **B** (0, -24) **C** (0, -27) **D** (0, 48)

Reasoning, communicating and open-ended

10 A quadratic in the form $y = a(x-h)^2 + k$ has a turning point at (1, 2) and passes through the point (2, -1).

(a) State the values of h and k.

(b) Will the value of a be positive or negative? Give a reason for your answer.

(c) Find the value of a.

11 The shape and position of a snake's body at time t seconds is described by the equation $y = \frac{(-1)^t x^3}{10} + 100t$, where x and y are in centimetres and $-10 \le x \le 10$.

(a) What is the equation describing the snake's initial shape? (Initial means $t = 0$.)

(b) Sketch the graph of the equation in part **(a)** using the basic graph $y = x^3$.

(c) On the same axes, sketch the snake's shape and position:

(i) 1 second later **(ii)** 2 seconds later.

(d) Compare the position of the snake after 1 second with its initial position. State the y-value about which the snake's new position is a reflection of its initial position.

12 The cross-section of the entrance to a tunnel has the shape of the parabola $y = 5 - 0.2(x - 5)^2$, where x and y are given in metres and $0 \le x \le 10$. A second tunnel with the same basic shape is 5 m higher and has its centre 15 m to the right of the centre of the first tunnel.

(a) What is the equation describing the second tunnel's entrance?

(b) Sketch the graphs of the two equations on the same number plane.

(c) Use Pythagoras' theorem to calculate the distance, to one decimal place, between the tops of the entrances to the tunnels.

13 The shape of the front leg of an antique chair is described by the equation $y = (x - 2)^3 + 8$, where x and y are given in centimetres. The second front leg is a reflection of the first leg in the line $x = 12$.

(a) Sketch the graph of the equation describing the shape of the first leg for $0 \le x \le 6$.

(b) State the equation describing the shape of the second leg.

(c) Sketch the graph of the second leg on the same number plane as your answer to part **(a)**.

(d) What is the minimum distance between the two legs?

14 Starting with the equation $y = x^3$, write two cubic equations so that each equation has a reflection in the x-axis, a dilation factor between 1 and 10, a horizontal translation to the left and a vertical translation downwards.

Polynomials

10.3

Stage 5.3#

Need to Know

Defining a polynomial

A **polynomial** is:

- an expression that contains only one variable
- each term in the expression contains the variable raised to a different positive integer power or the zero index.

Terms of a polynomial

- The terms are arranged in descending powers of the variable.
- Each term has a numerical coefficient.
- The first term, called the **leading term**, has an index of n.

Coefficients of a polynomial

- The **leading coefficient**, a_n, is the coefficient of the leading term.
- The leading coefficient of a monic polynomial is equal to 1, $a_n = 1$.
- The power of the last term is zero. As $x^0 = 1$, the last term is a constant equal to a_0 (the coefficient of x^0).
- The coefficient of every term, except for the leading coefficient, a_n, may be zero. If the coefficient of a term is zero, its value is zero, so the term is not written.
- Where x^n is the highest power of x, there are $(n + 1)$ terms. However, as some coefficients may be zero, fewer than $(n + 1)$ terms may appear in the expression.

Writing polynomials

- $P(x)$ represents a polynomial in x. (Take care not to confuse polynomial notation with probability notation.) Other capital letters such as $Q(x)$ or $R(x)$ can be used to represent polynomials.
- The **degree** of the polynomial $P(x)$ is the highest power of x that appears in the expression. In a polynomial of $(n + 1)$ terms, the polynomial is of degree n.

A polynomial can be written as $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0x^0$, where $a_n \neq 0$, n is a positive integer and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are the coefficients of the powers of x.

For example:

The expression $5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$ is a polynomial in x as all the powers are positive integers.

That is, $P(x) = 5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$. The degree of the polynomial is 6. As there is no x^3 term, the coefficient of x^3 is 0.

$5x^6$	$-8x^5$	$+7x^4$	$+0x^3$	$+2x^2$	$-3x$	$+9$
$a_n = a_6$	$a_{n-1} = a_5$	$a_{n-2} = a_4$	$a_{n-3} = a_3$	$a_{n-4} = a_2$	$a_{n-5} = a_1$	$a_{n-6} = a_0$
$= 5$	$= -8$	$= 7$	$= 0$	$= 2$	$= -3$	$= 9$

Substitution into a polynomial

When $x = a$ is substituted into $P(x)$, $P(x)$ is written as $P(a)$.

That is, $P(a)$ is the value of $P(x)$ when $x = a$.

For example:

For $P(x) = 5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$,

$$\begin{aligned} P(2) &= 5(2)^6 - 8(2)^5 + 7(2)^4 + 2(2)^2 - 3(2) + 9 \\ &= 187 \end{aligned}$$

Addition and subtraction of polynomials

Polynomials can be added and subtracted by combining like terms (terms of the same degree).

Multiplication of polynomials

The product will be a polynomial whose degree is the sum of the degrees of the two polynomials being multiplied.

- Write the polynomial product in brackets.
- Expand the brackets by multiplying each term in the second bracket by each term in the first bracket.
- Simplify by adding like terms.

Polynomial division

Polynomial division is carried out the same way as numerical division.

For example: $25 \div 4 = 6\frac{1}{4}$ is equivalent to $25 = 6 \times 4 + 1$

where 25 is the **dividend**, 4 is the **divisor**, 6 is the **quotient** and 1 is the **remainder**.

When a polynomial of degree n, $P(x)$, is divided by a linear expression $D(x)$, the result is a polynomial $Q(x)$ of degree $(n - 1)$ with a remainder $R(x)$.

$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ is equivalent to $P(x) = D(x) \times Q(x) + R(x)$, where $P(x)$ is the dividend, $D(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder. If $D(x)$ is linear, $R(x)$ is an integer. To find $Q(x)$ and $R(x)$, a method similar to numerical long division of whole numbers is used.

Numerical long division

If the divisor is greater than 12 and has no common factors with the dividend, long division is required. The process is the same as short division except that all the steps are written down instead of doing them mentally.

Polynomial long division

1. Write the dividend in polynomial form including any terms that have a zero coefficient.
2. Divide the first term of the dividend by the leading term of the divisor and place the result above the second term in the dividend aligning the powers of x.
3. Multiply each term of the divisor by the first term in the quotient. Write the result underneath.
4. Insert brackets and subtract the like terms.
5. Repeat the process until the degree of the remainder is less than the degree of the divisor.
6. State the result. When dividing by a linear expression, the remainder will be a constant.

Worked Example 4 WE4

State the degree, leading coefficient and constant term of each polynomial.

(a) $5x^4 + 3x^3 - x^2 + x - 2$

(b) $12 + 6x - x^3$

Working

(a) $5x^4 + 3x^3 - x^2 + x - 2$

The leading term is $5x^4$.

The degree is 4.

The leading coefficient is 5.

The constant term is -2.

(b) $-x^3 + 6x + 12$

The leading term is $-x^3$.

The degree is 3.

The leading coefficient -1.

The constant term is 12.

Worked Example 5 WE5

$P(x) = 2x^3 - x^2 + 4x - 7$ and $Q(x) = x^3 + 3x^2 - x + 1$

(a) Evaluate $P(2)$ and $Q(-2)$.

(b) Write the expression for $P(x) + 2Q(x)$.

Working

(a) $P(x) = 2x^3 - x^2 + 4x - 7$

$P(2) = 2 \times 2^3 - 2^2 + 4 \times 2 - 7$

$P(2) = 16 - 4 + 8 - 7$

$= 13$

$Q(x) = x^3 + 3x^2 - x + 1$

$Q(-2) = (-2)^3 + 3 \times (-2)^2 - (-2) + 1$

$Q(-2) = -8 + 12 + 2 + 1$

$= 7$

(b) $P(x) + 2Q(x)$

$= 2x^3 - x^2 + 4x - 7 + 2(x^3 + 3x^2 - x + 1)$

$= 2x^3 - x^2 + 4x - 7 + 2x^3 + 6x^2 - 2x + 2$

$= 4x^3 + 5x^2 + 2x - 5$

Worked Example 6 WE6

If $P(x) = 3x - 2$ and $Q(x) = 4x^2 + x - 5$, write $P(x).Q(x)$ as a polynomial.

Working

$P(x).Q(x) = (3x - 2)(4x^2 + x - 5)$

$= 3x(4x^2 + x - 5) - 2(4x^2 + x - 5)$

$= 12x^3 + 3x^2 - 15x - 8x^2 - 2x + 10$

$= 12x^3 - 5x^2 - 17x + 10$

Worked Example 7

WE 7

If $P(x) = x^2 - 2x - 3$ and $Q(x) = 4x^3 + x^2 - 6$, write $P(x).Q(x)$ as a polynomial.

Working

$$\begin{aligned} P(x).Q(x) &= (x^2 - 2x - 3)(4x^3 + x^2 - 6) \\ &= x^2(4x^3 + x^2 - 6) - 2x(4x^3 + x^2 - 6) - 3(4x^3 + x^2 - 6) \\ &= 4x^5 + x^4 - 6x^2 - 8x^4 - 2x^3 + 12x - 12x^3 - 3x^2 + 18 \\ &= 4x^5 - 7x^4 - 14x^3 - 9x^2 + 12x + 18 \end{aligned}$$

Worked Example 8

WE 8

Evaluate each expression using long division. Do not use a calculator.

(a) 3689 ÷ 17 **(b)** 16 319 ÷ 23

Working

(a)

$$\begin{array}{r} 217 \\ 17\overline{)3689} \\ -\underline{34} \\ 28 \\ -\underline{17} \\ 119 \\ -\underline{119} \\ 0 \end{array}$$

$3689 \div 17 = 217$

(b)

$$\begin{array}{r} 709 \\ 23\overline{)16319} \\ -\underline{161} \\ 219 \\ -\underline{207} \\ 12 \end{array}$$

$16\,319 \div 23 = 709\frac{12}{23}$

Worked Example 9

WE 9

Show the step-by-step long division process to find the quotient and remainder for $\dfrac{x^2 + 6x + 10}{x + 2}$.

Working	Explanations
$x + 2 \overline{)\, x^2 + 6x + 10}$	
$\begin{array}{r} x \\ x + 2 \overline{)\, x^2 + 6x + 10} \end{array}$	Divide the first term of the dividend by the leading term of the divisor to find the first term of the quotient.
$\begin{array}{r} x \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ x^2 + 2x \end{array}$	Multiply each term of the divisor by the first term of a quotient and write underneath the dividend, aligning like terms.
$\begin{array}{r} x \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ -\,(x^2 + 2x) \\ \hline 0 + 4x \end{array}$	Bracket and subtract like terms.
$\begin{array}{r} x \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ x^2 + 2x \quad \downarrow \\ \hline 0 + 4x + 10 \end{array}$	Add the next term of the dividend to the previous remainder.
$\begin{array}{r} x + 4 \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ x^2 + 2x \quad \downarrow \\ \hline 4x + 10 \end{array}$	Divide the first term of the new dividend by the first term of the divisor to find the next term of the quotient.
$\begin{array}{r} x + 4 \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ x^2 + 2x \\ \hline 4x + 10 \\ 4x + 8 \end{array}$	Multiply each term of the divisor by the second term in the quotient and write underneath the new dividend, aligning like terms.
$\begin{array}{r} x + 4 \quad \text{quotient} \\ x + 2 \overline{)\, x^2 + 6x + 10} \\ x^2 + 2x \\ \hline 4x + 10 \\ -\,(4x + 8) \\ \hline 0 + 2 \quad \text{remainder} \end{array}$	Bracket and subtract like terms.
$\dfrac{x^2 + 6x + 10}{x + 2} = x + 4 + \dfrac{2}{x + 2}$	Write the remainder as a fraction of the divisor.

Worked Example 10 WE10

Use long division to find the quotient and remainder for $\frac{8x^3 + 4x - 1}{2x + 3}$.

Working

$$
\begin{array}{r}
4x^2 - 6x + 11 \\
2x+3\overline{)\,8x^3 + 0x^2 + 4x - 1} \\
\underline{8x^3 - 12x^2} \\
0 - 12x^2 + 4x \\
\underline{-(-12x^2 - 18x)} \\
0 + 22x - 1 \\
\underline{-(22x + 33)} \\
0 - 34
\end{array}
$$

$$\frac{8x^3 + 4x - 1}{2x + 3} = 4x^2 - 6x + 11 - \frac{34}{2x + 3}$$

Answers page 700

10.3 Polynomials

Stage 5.3#

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 Column 1, Q7, Q8, Q9 Column 1, Q10, Q11, Q12, Q13, Q14, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9 Column 2, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q19	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9 Column 2, Q10, Q11, Q12, Q14, Q15, Q16, Q17, Q19, Q20

Fluency and understanding

WE4

1 State the degree, leading coefficient and constant term of each polynomial.

(a) $2x^3 + 5x^2 - x + 2$ **(b)** $5 - 4x^4 - 3x^3 + 7x^2$

(c) x **(d)** $-6 - x - x^2$

WE5

2 If $P(x) = 3x^3 - 2x^2 - x + 2$ and $Q(x) = -x^3 - 4x^2 - 5x + 6$, evaluate:

(a) $P(3)$ **(b)** $Q(4)$ **(c)** $P(-2)$ **(d)** $Q(-1)$

WE6

3 Write each expression as a polynomial if $P(x) = 5x + 1$, $Q(x) = x^2 + 2x - 3$ and $R(x) = 2x^3 - x^2 - x + 6$.

(a) $P(x).\ Q(x)$ **(b)** $P(x).\ R(x)$

WE7

4 If $P(x) = x^2 - 3x + 4$ and $Q(x) = 2x^3 + 3x^2 - 5$, write $P(x).\ Q(x)$ as a polynomial.

WE8

5 Evaluate each expression by performing long division. Do not use a calculator.

(a) $3715 \div 25$ **(b)** $4061 \div 12$ **(c)** $5924 \div 17$

(d) $2018 \div 19$ **(e)** $14\,797 \div 29$ **(f)** $10\,946 \div 123$

WE9

6 Show the step-by-step long division process to find the quotient and remainder for:

(a) $\frac{x^2 + 6x + 10}{x + 3}$ **(b)** $\frac{x^2 + 3x - 1}{x - 2}$

WE10

7 Use long division to find the quotient and remainder for:

(a) $\dfrac{x^3+6x+5}{x+2}$ (b) $\dfrac{x^4-3x^2+1}{x-3}$

8 Write $-3A(x)+B(x)+C(x)$ as a polynomial if $A(x)=-x+4$, $B(x)=4x^2-2x+1$ and $C(x)=-x^2+3x-8$.

9 Find the quotient and remainder for $\dfrac{P(x)}{Q(x)}$ if:

(a) $P(x)=x^2+7x+13$, $Q(x)=x+3$ (b) $P(x)=4x^2+4$, $Q(x)=2x-1$

(c) $P(x)=2x^2-3x+1$, $Q(x)=x+1$ (d) $P(x)=3x^3+2x-4$, $Q(x)=x-2$

10 If $P(x)=-5x^3-8x^2-3x+2$ and $Q(x)=x^3+4x^2-x+1$ and $R(x)=2P(x)-2Q(x)$, the leading coefficient of $R(x)$ is:

A -12 B -5 C 1 D 8

11 Use long division to show that the remainder is zero when $4x^2+x-3$ is divided by $4x-3$.

12 The remainder when x^3+x^2+x+1 is divided by $x-1$ is:

A -5 B -3 C 2 D 4

13 Calculate $\dfrac{A(x)}{B(x)}$ when $A(x)=x^4-6x^3+5$ and $B(x)=x-5$.

14 $P(x)=3x^3-2x^2-x+2$ and $Q(x)=-x^3-4x^2-5x+6$.

(a) If $R(x)=5P(x)-Q(x)$, write the polynomial $R(x)$ and evaluate $R(2)$.

(b) If $R(x)=2P(x)+3Q(x)$, write the polynomial $R(x)$ and evaluate $R(-3)$.

(c) If $6P(x)=2R(x)-10Q(x)$, write the polynomial $R(x)$ and evaluate $R(-1)$.

Reasoning, communicating and open-ended

15 (a) Calculate $\dfrac{A(x)}{B(x)}$ when $A(x)=2x^5-3x^4-6x^2-3x+18$ and $B(x)=2x-3$.

(b) What is the value of $\dfrac{A(x)}{B(x)}$ for $x=2$?

(c) Evaluate $A(2)$ and $B(2)$ and find $\dfrac{A(2)}{B(2)}$. Compare your answer to your answer in **(b)**.

16 Alana has $P(x)$ dollars, Bruno has $Q(x)$ dollars and Cien has $\dfrac{P(x)}{Q(x)}$ dollars, where $P(x)=12x^4+16x^3+3x^2+28x+32$ and $Q(x)=3x+4$.

(a) Write the polynomial that describes the amount of money Cien has.

(b) (i) Dylan has x dollars. What is the ratio of Cien's amount of money to Dylan's?

(ii) Use your answer to part **(i)** to explain why x cannot be zero.

17 A sheet of pastry is $(2x^2-9x)$ cm long and $(4x+3)$ cm wide.

(a) Write an expression for the perimeter of the sheet of pastry.

(b) Write an expression for the area of the sheet of pastry.

(c) (i) The sheet of pastry is to be divided into small rectangles of side lengths x cm and $(x-1)$ cm. Write an expression for the largest number of small rectangles that could be cut out of the pastry if any unused pastry is rerolled to the same thickness to maximise the number of squares that could be formed.

(ii) If all the pastry is used (there is no remainder), what is the smallest possible value of x and how many rectangles can be formed?

18 Write a non-monic polynomial of degree 5 that contains four terms and has a negative constant term.

19 Find a polynomial $P(x)$ of degree 3 so that $\frac{P(x)}{x-3}$ leaves no remainder.

20 $P_n(x)$ is a polynomial of degree n. It has 1 for all its coefficients so that, for example, $P_2(x) = x^2 + x + 1$ and $P_3(x) = x^3 + x^2 + x + 1$.

(a) Write an expression for $\frac{P_n(x)}{x+1}$ for $n = 1, 2, 3, 4, 5$.

(b) By considering your results in part **(a)**, show that if n is odd, $\frac{P_n(x)}{x+1}$ is a polynomial of degree $n - 1$ consisting of even powers of x.

(c) By considering your results in part **(a)**, show that if n is even, $\frac{P_n(x)}{x+1}$ is a polynomial of degree $n - 1$ consisting of odd powers of x plus a remainder of $\frac{1}{x+1}$.

Half-time 10

Stage 5.3§ Q1, 3, 5 **Stage 5.3# Q2, 4, 6**

1 Graph each equation by constructing a table of values for $x = -3$ to 3. Ex.10.1

(a) $y = 4x^2$ **(b)** $y = \frac{1}{4}x^3$

2 **(a)** Write the expression $4x^2 - 5 - 4x + 3x^4$ in polynomial form. Ex.10.3

(b) What is the coefficient of x^3 in the polynomial given in part **(a)**?

(c) Divide the polynomial given in part **(a)** by $x - 1$.

3 **(a)** Sketch the graph of $y = 2x^2 + 3$ by starting with $y = x^2$. Ex.10.2

(b) Sketch the graph of $y = -3(x - 2)^3 + 1$ by starting with $y = x^3$.

(c) Sketch the graph of $y = 2(x + 1)^4$ by starting with the graph of $y = x^4$.

4 **(a)** Given $P(x) = x^3 - 2x^2 - 5x + 6$, find $P(3)$. Ex.10.3

(b) Use long division to find the quotient when the cubic expression in part **(a)** is divided by $x - 3$.

5 The vertical cross-section of a bottle is described by the equations $y = (x - 4)^3 + 1$ and $y = -(x - 14)^3 + 1$. Ex.10.2

(a) Graph both equations on the same axes using x-values from 1 to 17.

(b) State the equation of the axis of symmetry of the bottle.

6 $P(x) = 2x^3 + 3x^2 - 5x - 2$ and $Q(x) = 4x^2 - 3x + 1$ Ex.10.3

(a) Evaluate $P(x) + 2Q(x)$.

(b) Find the product of $P(x)$ and $Q(x)$.

10.4 Remainder and factor theorems

Stage 5.3#

Need to Know

Remainder theorem

The **remainder theorem** can be used to find the remainder when a polynomial $P(x)$ is divided by a linear factor $(x - a)$ or $(ax - b)$.

For example:

$$\begin{aligned} P(x) &= x^2 + x + 2 \\ &= x^2 + x - 6 + 8 \\ &= (x + 3)(x - 2) + 8 \end{aligned}$$

So $\dfrac{P(x)}{(x-2)} = (x+3) + \dfrac{8}{(x-2)}$ and $\dfrac{P(x)}{(x+3)} = (x-2) + \dfrac{8}{(x+3)}$.

When $P(x)$ is divided by $(x - 2)$ or $(x + 3)$ the remainder is 8.

Substituting $x = 2$ into $P(x)$,

$$\begin{aligned} P(2) &= 4 + 2 + 2 \\ &= 8 \end{aligned}$$

Substituting $x = -3$ into $P(x)$,

$$\begin{aligned} P(-3) &= 9 - 3 + 2 \\ &= 8 \end{aligned}$$

$P(2)$ and $P(-3)$ both give the remainder of 8.

For the polynomial $P(x)$:

- If the polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.
- If the polynomial $P(x)$ is divided by $(ax - b)$, then the remainder is $P\left(\dfrac{b}{a}\right)$.

The factor theorem

The **factor theorem** can be used to find factors of a polynomial $P(x)$ in the form of $(x - a)$ or $(ax - b)$.

For example:

$$\begin{aligned} P(x) &= x^3 + x^2 - 6x \\ &= x(x - 2)(x + 3) \end{aligned}$$

So x, $(x - 2)$ and $(x + 3)$ are factors of $P(x)$.

However, using the remainder theorem, $P(0) = 0$, $P(2) = 0$ and $P(-3) = 0$.

So x, $(x - 2)$ and $(x + 3)$ are factors of $P(x)$.

If the remainder is 0, then the expression that you are dividing by is a factor of the polynomial.

For the polynomial $P(x)$:

- If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.
- If $P\left(\dfrac{b}{a}\right) = 0$, for the polynomial $P(x)$, then $(ax - b)$ is a factor of $P(x)$.

Worked Example 11

WE 11

If $P(x) = x^3 + 3x^2 - 4x + 5$ and $Q(x) = x - 2$:

(a) find $\dfrac{P(x)}{Q(x)}$ **(b)** evaluate $P(2)$ **(c)** comment on the result.

Working

(a)

$$
\begin{array}{r}
x^2 + 5x + 6 \\
x - 2 \overline{\big) \, x^3 + 3x^2 - 4x + 5} \\
-(x^3 - 2x^2) \\
\hline
5x^2 - 4x \\
-(5x^2 - 10x) \\
\hline
6x + 5 \\
-(6x - 12) \\
\hline
17
\end{array}
$$

(b) $P(2) = (2)^3 + 3 \times 2^2 - 4 \times 2 + 5$

$= 8 + 12 - 8 + 5$

$= 17$

(c) The remainder when $P(x)$ is divided by $x - 2$ is the same as the value of $P(2)$.

Worked Example 12

WE 12

Use the remainder theorem to find the remainder for:

(a) $\dfrac{2x^2 - x + 1}{x - 2}$ **(b)** $\dfrac{-4x^3 + x^2 - x + 5}{2x + 3}$

Working

(a) $P(x) = 2x^2 - x + 1$

$x - 2 = x - a$

hence $a = 2$

$P(2) = 8 - 2 + 1$

$= 7$

The remainder is 7.

(b) $P(x) = -4x^3 + x^2 - x + 5$

$2x + 3 = ax - b$

$\therefore \dfrac{b}{a} = \dfrac{-3}{2}$

$P(\dfrac{-3}{2}) = \dfrac{27}{2} + \dfrac{9}{4} + \dfrac{3}{2} + 5$

$= 22\dfrac{1}{4}$

The remainder is $22\dfrac{1}{4}$.

Worked Example 13

WE13

Use the factor theorem to fully factorise $x^3 + 6x^2 + 11x + 6$.

Working

$P(x) = x^3 + 6x^2 + 11x + 6$

Try $x = 1$

$P(1) = 1 + 6 + 11 + 6$
$= 24$

$\therefore$ $(x - 1)$ is not a factor.

Try $x = -1$

$P(-1) = -1 + 6 - 11 + 6$
$= 0$

Hence, $(x + 1)$ is a factor.

$$
\begin{array}{r}
x^2 + 5x + 6 \\
x + 1 \overline{\big) x^3 + 6x^2 + 11x + 6} \\
-(x^3 + x^2) \\
\hline
5x^2 + 11x \\
-(5x^2 + 5x) \\
\hline
6x + 6 \\
-(6x + 6) \\
\hline
0
\end{array}
$$

$P(x) = (x + 1)(x^2 + 5x + 6)$
$= (x + 1)(x + 2)(x + 3)$

Worked Example 14

WE14

Use the factor theorem to fully factorise $x^4 + x^3 - 7x^2 - x + 6$.

Working

$P(x) = x^4 + x^3 - 7x^2 - x + 6$

Try $x = 1$

$P(1) = 1 + 1 - 7 - 1 + 6$
$= 0$

Hence, $(x - 1)$ is a factor.

$$
\begin{array}{r}
x^3 + 2x^2 - 5x - 6 \\
x - 1 \overline{\big) x^4 + x^3 - 7x^2 - x + 6} \\
-(x^4 - x^3) \\
\hline
2x^3 - 7x^2 \\
-(2x^3 - 2x^2) \\
\hline
-5x^2 - x \\
-(-5x^2 + 5x) \\
\hline
-6x + 6 \\
-(-6x + 6) \\
\hline
0
\end{array}
$$

$P(x) = (x - 1)(x^3 + 2x^2 - 5x - 6)$

$P(x) = (x - 1)M(x)$

Try $x = -1$ ($x = 1$ has been used)

$M(x) = x^3 + 2x^2 - 5x - 6$

$M(-1) = -1 + 2 + 5 - 6$
$= 0$

Hence, $(x + 1)$ is a factor.

$M(x) = (x + 1)(x^2 + x - 6)$
$= (x + 1)(x - 2))(x + 3)$

$P(x) = (x - 1)(x + 1)(x - 2)(x + 3)$

10.4 Remainder and factor theorems

Answers page 701

Navigator

Stage 5.3#

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14, Q15

Fluency and understanding

1 If $P(x) = x^3 + 4x^2 - 5x + 3$ and $Q(x) = x - 2$, find $\frac{P(x)}{Q(x)}$, evaluate $P(2)$ and comment on the result. WE11

2 Use the remainder theorem to find the remainder for: WE12

(a) $\frac{x^2 - 2x + 1}{x - 2}$ **(b)** $\frac{-3x^3 + 2x^2 - x + 2}{2x + 6}$

3 Use the factor theorem to fully factorise $x^3 - 6x^2 + 11x - 6$. WE13

4 Use the factor theorem to fully factorise $x^4 - x^3 - 7x^2 + x + 6$. WE14

5 Use the remainder theorem to find the remainder if:

(a) $4x^2 + 3x - 1$ is divided by $x - 3$ **(b)** $5x^3 - 2x^2 + 6x - 10$ is divided by $x + 1$

(c) $x^4 + x^3 - 3x^2 - x - 1$ is divided by $2x + 1$ **(d)** $x^5 - 1$ is divided by $2x - 1$

(e) $(6x - 8)(4x + 5)$ is divided by $x - 5$ **(f)** $49x^2 - 144$ is divided by $7x + 2$.

6 Use the factor theorem to factorise the following polynomials.

(a) $x^3 + x^2 - 10x + 8$ **(b)** $x^3 - 5x^2 + 3x + 9$

(c) $x^3 - 6x^2 + 3x + 10$ **(d)** $x^3 + 3x^2 - 16x + 12$

(e) $x^3 + 6x^2 + 12x + 8$ **(f)** $x^3 - 3x^2 - 9x - 5$

Hint

Find the factors of the constant term and use them to test for factors of the polynomial.

7 Calculate the remainder for $\frac{P(x)}{Q(x)}$ given:

(a) $P(x) = 4x^3 + 2x^2 - x + 1$, $Q(x) = 5x + 15$

(b) $P(x) = -x^4 - 3x^2 + 2x + 1$, $Q(x) = -x - 1$

(c) $P(x) = -x^3 - 9x^2 + 10x - 12$, $Q(x) = -2x$

(d) $P(x) = x^5 - x^4 - x^3 + 2x^2 - x - 1$, $Q(x) = 4x + 8$.

8 Use the factor theorem to factorise the following polynomials.

(a) $x^4 + 6x^3 + 7x^2 - 6x - 8$ **(b)** $x^4 - 7x^3 + 14x^2 - 8x$

(c) $x^4 - x^3 - 7x^2 + 13x - 6$ **(d)** $x^4 + 2x^3 - 12x^2 + 14x - 5$

(e) $x^4 - 4x^3 + 3x^2 + 4x - 4$ **(f)** $x^4 + 2x^3 - 13x^2 - 14x + 24$

9 When $2x^3 - 4x^2 + x - 1$ is divided by $x + \frac{1}{2}$, the remainder is:

A $1\frac{1}{2}$ **B** $-2\frac{3}{4}$ **C** $1\frac{3}{4}$ **D** $-3\frac{1}{2}$

Reasoning, communicating and open-ended

10 If $\dfrac{x^3 - 4x^2 - 4x + 16}{x - 4} = ax^2 + bx + c$ the values of a, b and c are:

A 1, -1, -4 **B** -3, 4, 0 **C** 4, 3, -1 **D** 1, 0, -4

11 If $x + 2$ is a factor of $6x^3 + 19x^2 + 11x + a$, find the value of a.

12 Use the factor theorem to fully factorise each polynomial. Then solve the polynomial equation.

(a) $x^3 - 6x^2 + 11x - 6 = 0$

(b) $x^3 + x^2 - 4x - 4 = 0$

(c) $x^3 + 9x^2 + 26x + 24 = 0$

(d) $x^4 + x^3 - 7x^2 - x + 6 = 0$

(e) $x^4 - 5x^2 + 4 = 0$

(f) $x^4 - x^3 - 6x^2 + 6x = 0$

13 A toy top is moving along the floor as it spins. Its distance, d cm, from a fixed point at time t seconds, is given by the equation $d = t^4 - 6t^3 + 13t^2 - 12t + 4$.

(a) Factorise the polynomial.

(b) How far is the top from the fixed point initially?

(c) At what times is the top at the fixed point?

(d) By inspection of your answer to part **(a)**, or by sketching the graph, describe the motion of the top after 2 seconds.

14 Give two monic polynomials of degree 3 that leave a remainder of -2 when they are divided by $x - 3$.

15 A polynomial equation of degree 4 has solutions $x = 0$, $x = -1$, $x = 2$, $x = 3$. Write down two possible equations.

Sketching polynomials

10.5

Need to Know

Stage 5.3#

Key features of a polynomial

Before a polynomial can be sketched, its key features need to be identified.

Step 1 Identify the shape of $y = a(x - h)^2 + k$ using transformations of $y = ax^n$.

Step 2 Find the y-intercept. All polynomial graphs have a y-intercept. Find the y-value of the y-intercept by substituting $x = 0$ into the polynomial.

Step 3 Find the x-intercepts. When a polynomial $P(x)$ is written as the product of its factors, the solution of $P(x) = 0$ can be found by equating each factor equal to zero and solving for x. The solutions are the x-values of the x-intercepts of the graph of $y = P(x)$.

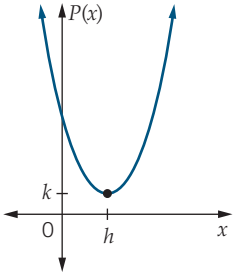
- If the degree of $P(x)$ is 1, the linear polynomial $P(x) = mx + b$ has one solution for $P(x) = 0$. There is always exactly one x-intercept.
- If the degree of $P(x)$ is 2, the quadratic polynomial $P(x) = ax^2 + bx + c$ has at most two solutions for $P(x) = 0$. There may be zero, one or two x-intercepts. The x-values of the x-intercepts may not be rational.
- If the degree of $P(x)$ is 3, the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ has at most three solutions for $P(x) = 0$. There will always be at least one x-intercept. The x-values of the x-intercepts may not be rational.
- The polynomial $P(x)$ of degree n has at most n solutions for $P(x) = 0$. If n is odd, there will always be at least one x-intercept. If n is even, there may be no x-intercepts.
- Identify whether the polynomial cuts or touches the x-axis. A repeated factor with an even power indicates that the graph touches the x-axis. A repeated factor with an odd power indicates a point of inflection on the x-axis.

Step 4 Find the turning point of graphs of quadratic equations. Finding the turning points (maximum or minimum) or points of inflection for polynomials of degree >2 are optional, but they may be found using a computer program such as GeoGebra if desired. A point of inflection (h, k) is a special point where the reflection in both the lines $x = h$ and $y = k$ of one half of the graph will give the other half.

Quadratics

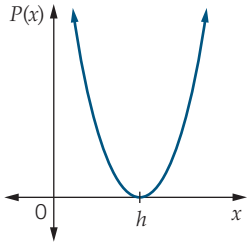
The graphs of a quadratic fit one of the following three types for $a > 0$. If $a < 0$, the graphs will be reflected in the x-axis.

$P(x) = a(x - h)^2 + k, k > 0$



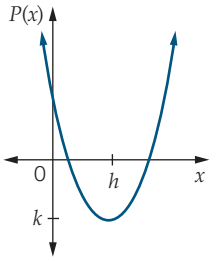
- no solution for $P(x)$
- no x-intercept

$P(x) = a(x - h)^2, k = 0$



- 1 solution for $P(x)$
- 1 x-intercept

$P(x) = a(x - h)^2 + k, k < 0$

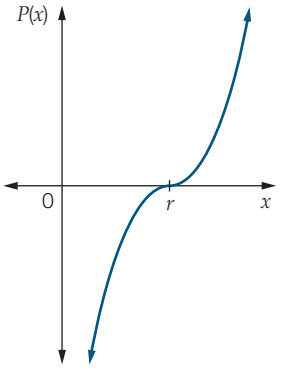


- 2 solutions for $P(x)$
- 2 x-intercepts

Cubics

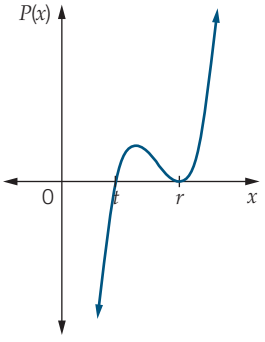
A cubic polynomial expressed in the form $P(x) = a(x - r)(x - s)(x - t)$ is easy to sketch as the x-intercepts are given by $x = r$, $x = s$ and $x = t$. Depending on the values of r, s and t there will be one, two or three solutions to the equation $P(x) = 0$.

$P(x) = a(x - r)^3$



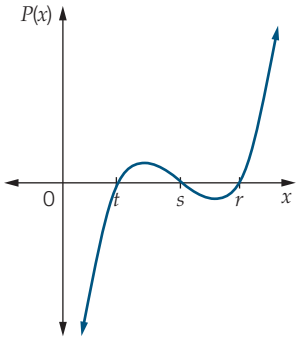
- r, s and t all have the same value
- 1 solution for $P(x)$
- cuts the x-axis once at $(r, 0)$

$P(x) = a(x - r)^2(x - t)$



- r and s have the same value
- 2 solutions for $P(x)$
- cuts the x-axis once at $(t, 0)$ and touches the x-axis once at $(r, 0)$

$P(x) = a(x - r)(x - s)(x - t)$

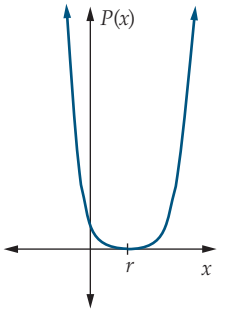


- r, s and t all have distinct values
- 3 solutions for $P(x)$
- cuts the x-axis 3 times at $(t, 0)$, $(s, 0)$, $(r, 0)$

Quartics

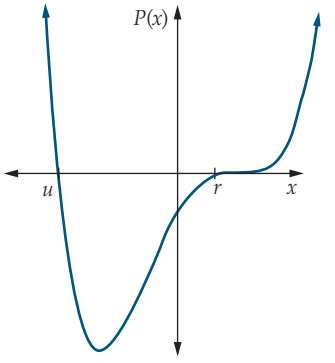
Some quartic polynomials of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ can be written in the form $P(x) = a(x - r)(x - s)(x - t)(x - u)$. In this case, there can be up to four solutions for $P(x) = 0$. Five possibilities are shown for $a > 0$. If $a < 0$ then the graphs are reflected in the x-axis and both ends would be pointing downwards.

$P(x) = (x - r)(x - r)(x - r)(x - r)$
$= (x - r)^4$



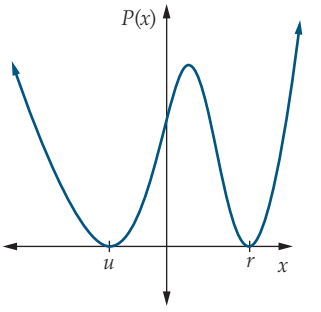
- r, s, t and u all have the same value
- 1 solution for $P(x)$
- touches the x-axis at $(r, 0)$

$P(x) = (x - r)(x - r)(x - r)(x - u)$
$= (x - r)^3(x - u)$



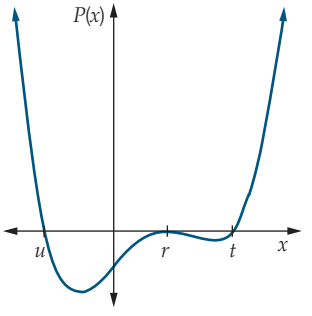
- r, s and t all have the same value
- 2 solutions for $P(x)$
- cuts the x-axis at $(u, 0)$ and touches the x-axis at $(r, 0)$

$P(x) = (x - r)(x - r)(x - u)(x - u)$
$= (x - r)^2(x - u)^2$



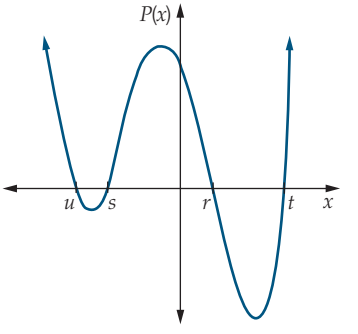
- r and s have the same value, t and u have the same value
- 2 solutions for $P(x)$
- touches the x-axis at $(r, 0)$ and $(u, 0)$

$P(x) = (x - r)(x - r)(x - t)(x - u)$
$= (x - r)^2(x - t)(x - u)$



- r and s have the same value
- 3 solutions for $P(x)$
- touches x-axis at $(r, 0)$ and cuts it at $(t, 0)$ and $(u, 0)$

$y = (x - r)(x - s)(x - t)(x - u)$



- r, s, t and u all have different values
- 4 solutions for $P(x)$
- cuts the x-axis at $(r, 0)$, $(s, 0)$, $(t, 0)$ and $(u, 0)$

If a factor occurs only once, it corresponds to a point where the curve cuts the x-axis.

If a factor is repeated an even number of times, it corresponds to a point where the curve touches the x-axis, creating a turning point, either a maximum or a minimum.

If a factor is repeated an odd number of times, it corresponds to a point of inflection, where the graph cuts the x-axis and flattens out slightly.

A quartic can have one, two or three turning points. If it has two turning points, it must also have a point of inflection as the ends of the graph have to point in the same direction.

Like parabolas, some quartics also have no solutions. These quartics cannot be factorised.

Worked Example 15

WE15

Sketch the graph of each quadratic polynomial, showing the x- and y-intercepts and the coordinates of the turning point.

(a) $P(x) = 2x^2 + 2x + 1$ **(b)** $P(x) = 3x^2 + 6x + 3$ **(c)** $P(x) = -6x^2 + x + 2$

Working

(a) $P(x) = 2x^2 + 2x + 1$

$x = 0$

$P(0) = 2(0)^2 + 2(0) + 1$
$= 1$

The y-intercept is $(0, 1)$.

There are no obvious factors so use the quadratic formula.

$a = 2,\ b = 2,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 8}}{2 \times 2}$$

$$= \frac{-2 \pm \sqrt{-4}}{4}$$

This gives no solution so the graph has no x-intercepts.

As a is positive, the graph has a minimum turning point.

$$x = -\frac{b}{2a}$$

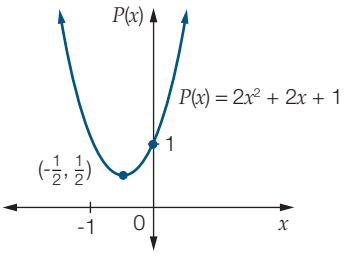
$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

when $x = -\frac{1}{2}$, $y = \frac{2}{4} - \frac{2}{2} + 1$

$$= \frac{1}{2}$$

The turning point is at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.



(b) $P(x) = 3x^2 + 6x + 3 = 0$

$x = 0$

$P(0) = 3(0)^2 + 6(0) + 3$
$= 3$

The y-intercept is $(0, 3)$.

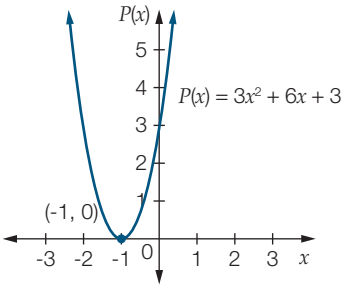
$P(x) = 0$

$3x^2 + 6x + 3 = 0$
$3(x^2 + 2x + 1) = 0$
$3(x + 1)^2 = 0$
$x = -1$

The x-intercept is $(-1, 0)$.

As a is positive, the graph has a minimum turning point.

The turning point is at $(-1, 0)$.



(c) $P(x) = -6x^2 + x + 2$

$x = 0$

$P(0) = -6(0)^2 + 0 + 2$
$= 2$

The y-intercept is (0, 2).

$P(x) = 0$
$-6x^2 + x + 2 = 0$
$-(6x^2 - x - 2) = 0$
$-(3x - 2)(2x + 1) = 0$
$x = \frac{2}{3}$ or $x = -\frac{1}{2}$

The x-intercepts are $\left(\frac{2}{3}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.

As a is negative, the graph has a maximum turning point.

$x = -\frac{b}{2a}$

$= -\frac{1}{2 \times -6}$

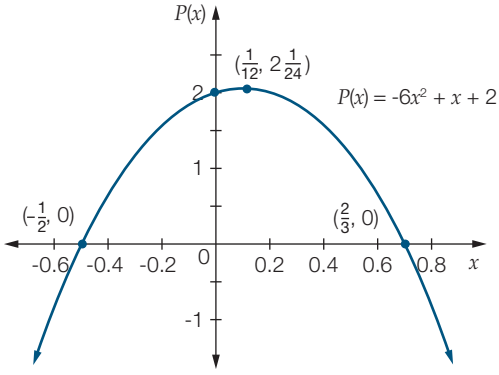
$= \frac{1}{12}$

$y = -6\left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right) + 2$

$= -6 \times \frac{1}{144} + \frac{1}{12} + 2$

$= 2\frac{1}{24}$

The turning point is at $\left(\frac{1}{12}, 2\frac{1}{24}\right)$.



Worked Example 16

WE16

Sketch the graph of each cubic polynomial, showing all x- and y-intercepts. Optional: A computer program such as GeoGebra may be used to find the coordinates of the turning points or points of inflection.

(a) $P(x) = -4(x - 1)^3$ **(b)** $P(x) = (x + 2)(x - 1)^2$ **(c)** $P(x) = x^3 - 3x^2 - x + 3$

Working

(a) $P(x) = -4(x - 1)^3$

$x = 0$

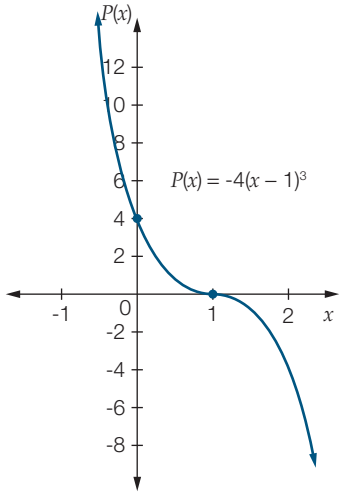
$P(0) = -4 \times (-1)^3$
$= 4$

The y-intercept is (0, 4).

$P(x) = 0$
$-4(x - 1)^3 = 0$
$x - 1 = 0$

$x = 1$ is the only solution.
The curve cuts the x-axis at (1, 0).

Optional: (1, 0) is a point of inflection.



(b) $P(x) = (x + 2)(x - 1)^2$

$x = 0$, $P(0) = 2 \times (-1)^2$
$= 2$

The y-intercept is (0, 2).

$P(x) = 0$

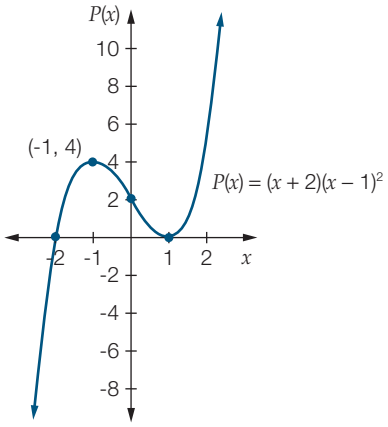
$(x + 2) = 0$ or $(x - 1)^2 = 0$

$x = -2, x = 1$

The curve cuts the x-axis at (-2, 0) and touches it at (1, 0).

Optional: (1, 0) is a minimum turning point.

(-1, 4) is a maximum turning point.



(c) $P(x) = x^3 - 3x^2 - x + 3$

$x = 0$, $P(0) = 3$

The y-intercept is (0, 3).

$x^3 - 3x^2 - x + 3 = 0$

$P(-1) = -1 - 3 + 1 + 3$
$= 0$

$(x + 1)$ is a factor.

$P(3) = 27 - 27 - 3 + 3$
$= 0$

$(x - 3)$ is a factor.

$P(1) = 1 - 3 - 1 + 3$
$= 0$

$(x - 1)$ is a factor.

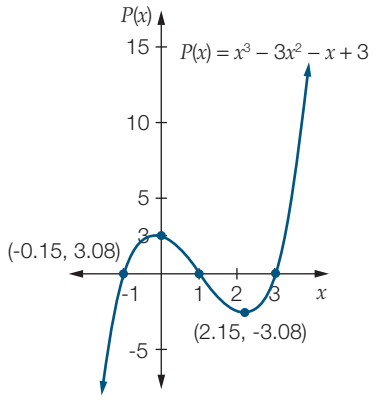
$(x + 1)(x - 1)(x - 3) = 0$

$x = -1, x = 1, x = 3$

The curve cuts the x-axis at (-1, 0), (1, 0) and (3, 0).

Optional: (-0.15, 3.08) is a maximum turning point.

(2.15, -3.08) is a minimum turning point.



Worked Example 17

WE17

Sketch the graph of $P(x) = x^4 - 9x^2 + 4x + 12$, showing all x- and y-intercepts. A computer program such as GeoGebra may be used to find the turning points or points of inflection.

Working

$P(x) = x^4 - 9x^2 + 4x + 12$

$P(0) = (0)^4 - 9(0)^2 + 4(0) + 12$
$= 12$

The y-intercept is $(0, 12)$.

Factorise using the factor theorem.
$P(-1) = 1 - 9 - 4 + 12$
$= 0$

$(x + 1)$ is a factor.

$$
\begin{array}{r}
x^3 - x^2 - 8x - 12 \\
x + 1 \overline{\big) x^4 + 0x^3 - 9x^2 + 4x + 12} \\
\underline{x^4 + x^3} \\
-x^3 - 9x^2 \\
\underline{-x^3 - x^2} \\
-8x^2 + 4x \\
\underline{-8x^2 - 8x} \\
12x + 12 \\
\underline{12x + 12} \\
0
\end{array}
$$

$P(x) = (x + 1)(x^3 - x^2 - 8x + 12)$

$M(x) = x^3 - x^2 - 8x + 12$

$M(2) = 8 - 4 - 16 + 12$
$= 0$

$(x - 2)$ is a factor.

$$
\begin{array}{r}
x^2 + x - 6 \\
x - 2 \overline{\big) x^3 - x^2 - 8x + 12} \\
\underline{x^3 - 2x^2} \\
x^2 - 8x \\
\underline{x^2 - 2x} \\
-6x + 12 \\
\underline{-6x + 12} \\
0
\end{array}
$$

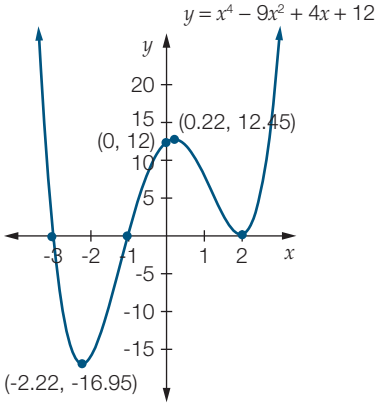
$P(x) = (x + 1)(x - 2)(x^2 + x - 6)$

$P(x) = (x + 1)(x - 2)(x - 2)(x + 3)$

$P(x) = (x + 1)(x + 3)(x - 2)^2$

For $P(x) = 0$, $x = -1$, $x = -3$, $x = 2$ (repeated solution)

The curve cuts the x-axis at $(-3, 0)$ and $(-1, 0)$, and touches it at $(2, 0)$ (repeated factor).



The curve cuts the x-axis twice, touches it once and has three turning points.

Optional: $(-2.22, -16.95)$ and $(2, 0)$ are minimum turning points.

$(0.22, 12.45)$ is a maximum turning point.

Answers page 702

10.5 Sketching polynomials

Stage 5.3#

Navigator

Q1, Q2, Q3, Q4 Column 1, Q5 (a)–(d), Q6 (a)–(d), Q7, Q8, Q9, Q11, Q12, Q16	Q1, Q2, Q3, Q4 Column 2, Q5 (b)–(e), Q6 (b)–(e), Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q15, Q16	Q1, Q2, Q3, Q4 Column 3, Q5 (c)–(f), Q6 (c)–(f), Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Optional: For polynomials of degree > 2, a computer package such as GeoGebra may be used to find turning points or points of inflection.

Fluency and understanding

WE15

1 Sketch the graph of each quadratic polynomial, showing all x- and y-intercepts and the coordinates of the turning point.

(a) $P(x) = 2x^2 + 6x + 5$ **(b)** $P(x) = 3x^2 + 12x + 12$ **(c)** $P(x) = -5x^2 - 6x + 8$

WE16

2 Sketch the graph of each cubic polynomial, showing all x- and y-intercepts.

(a) $P(x) = -2(x + 1)^3$ **(b)** $P(x) = (x - 2)(x + 1)^2$ **(c)** $P(x) = x^3 + 3x^2 - x - 3$

WE17

3 Sketch the graph of $P(x) = x^4 - 9x^2 - 4x + 12$, showing all x- and y-intercepts.

4 Sketch the graph of each polynomial.

(a) $P(x) = x^2 - 5x - 14$ **(b)** $P(x) = x^2 - 6x + 8$ **(c)** $P(x) = x^2 - 7x - 12$

(d) $P(x) = x^2 - 12x + 36$ **(e)** $P(x) = x^2 - 15x + 56$ **(f)** $P(x) = x^2 - x - 90$

5 Sketch the graph of each polynomial.

(a) $P(x) = x^3 + 7x^2 + 7x - 15$ **(b)** $P(x) = -x^3 + 9x^2 - 27x + 27$

(c) $P(x) = x^3 - 12x + 16$ **(d)** $P(x) = x^3 + x^2 - 12x$

(e) $P(x) = x^3 - 2x^2 - 11x + 12$ **(f)** $P(x) = x^3 + 5x^2 - x - 5$

6 Sketch the graph of each polynomial.

(a) $P(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$ **(b)** $P(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

(c) $P(x) = x^4 - 2x^3 + x^2$ **(d)** $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$

(e) $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ **(f)** $P(x) = x^4 + 3x^3 - 5x^2 - 3x + 4$

7 The equation $x^3 - 7x^2 + 11x - 5 = 0$ has:

A no solutions **B** only one solution at $x = 1$

C three solutions **D** two solutions

8 The zeroes of the polynomial $P(x) = x(x - 8)(x - 2)(x - 8)$ are:

A 0 and 8 **B** 0 and 2 **C** 2 and 8 **D** 0, 2 and 8

9 $P(x) = 2x^3 + 5x^2 + x - 20$ and $Q(x) = 2x^3 + 4x^2 - x + 15$. Find the solutions to $P(x) = Q(x)$.

10 $P(x) = 5x^4 + 4x^3 - 3x^2 - 5x + 5$ and $Q(x) = 3x^4 + 2x^3 - x^2 - 2x + 2$. Find the solutions to $2P(x) = 3Q(x)$.

Reasoning, communicating and open-ended

11 A flying fish, initially under water, suddenly leaps above the surface before re-entering the water. Its motion is given by $P(t) = -t^3 + 3t^2 - 2t$, where $P(t)$ is the vertical distance of the fish from sea level at time t seconds, with $0 \le t \le 2$.

(a) Factorise the polynomial and sketch the graph for values of t from 0 to 2.

(b) For how much time (after $t = 0$) was the fish swimming until it broke the sea surface? Approximately what was its greatest depth?

(c) For how much time was the fish above the surface and approximately what was its maximum height?

12 A plumber inserts a long flexible rod in a length of water pipe to test for blockages. The pipe is described by the equation $y = x^3 - 6x^2 + 9x$, where x is the horizontal distance and y is the vertical distance from the rod's insertion point (0, 0) and the units are in metres.

(a) Factorise the expression and sketch the graph for values of x from 0 to 4.

(b) There are blockages located at the turning points of the graph. If the first blockage is at the point (1, 4), find the coordinates of the second blockage.

(c) To avoid the first blockage, the plumber decides to install a new system of pipes starting at (0, 0) and ending at the second blockage. The pipes take the shape of a parabola. Give one equation for this parabola.

(d) While installing the new pipes, the plumber is informed that there is an underground power cable located at (2.5, 1.2), which must be avoided. Can the plumber still use the parabola pipe layout?

13 A wave pattern on a cathode ray oscilloscope is described by the polynomial $P(x) = x^4 + 5x^3 - x^2 - 5x$. A negative pulse is the section of the wave that is below the x-axis and between the x-intercepts. Similarly, a positive pulse is the section of the wave that is above the x-axis and between the x-intercepts.

(a) Factorise the polynomial and sketch its graph.

(b) Describe the location of each type of pulse.

(c) The wave pattern is a model for part of the cross-section of an island that consists of two lagoons and a small rocky outcrop with mountains on either side. The x-axis represents sea level, and each unit on the axis is 100 metres. Each unit on the y-axis is in metres.

(i) How wide and approximately how deep is the larger lagoon?

(ii) A research observation tower planned for the rocky outcrop requires a minimum length of land of 110 m. Can it be built?

14 A dam is located at the bottom of a hill. If $y = 0$ is taken as ground level, the cross-section (west to east) of both shapes can be described by the polynomial $x^3 - 5x^2 - 2x + 24$ over $-2 \le x \le 5$, where each unit of x represents 100 m and each unit of y represents 1 m.

(a) Factorise the polynomial.

(b) Sketch the graph of the polynomial for $-2 \le x \le 5$.

(c) What is the horizontal distance between the western base of the hill and the eastern bank of the dam?

15 A section of a roller-coaster track is described by the equation $y = 0.001(x - r)(x - s)^2$, where x is the horizontal distance in metres from Daredevil Corner and y is the height of the roller-coaster in metres. The graph has intercepts at $x = 0$ and $x = 60$ and the y-intercept is zero. The turning points occur at $x = 60$ and $x = 20$.

(a) Use this information to write the equation.

(b) Sketch the graph of the equation for $0 \le x \le 80$.

(c) The top of Heartstop Hill is at a horizontal distance of 80 m from Daredevil Corner. The length of the track from Daredevil Corner to Heartstop Hill is six times the height of the roller-coaster at Heartstop Hill. The roller-coaster on average moves on the track at 32 km/h. How many minutes does it take to travel from Daredevil Corner to Heartstop Hill?

16 Find two monic quartic polynomials that have three zeroes, and sketch their graphs.

17 A quartic polynomial has factors of $(x + 1)$ and $(x - 3)$. Using at least these factors, write down two quartic polynomials that:

(a) have exactly two zeroes

(b) have three zeroes

(c) have four zeroes

(d) are non-monic with only one turning point.

Functions and inverses

10.6

Stage 5.3#

Need to Know

Sets and relations

A **set** is a collection of objects. The members of a set are called **elements**. The elements of sets are written inside curly brackets (braces), { }.

For example:

$X = \{1, 2, 3\}$ is a set containing the numbers 1, 2 and 3. 1, 2 and 3 are the members or elements of the set.

A **relation** is a set of ordered pairs, (x, y), which may be connected by a rule.

For example:

$A = \{(1, 3), (4, -2), (5, 13)\}$ and $B = \{(1, 3), (2, 6), (3, 9)\}$ are both relations.

The set A has no rule connecting the ordered pairs but the set B is a relation connected by the rule $y = 3x$.

Domain, range and interval notation

All the x-values given by the relation form the **domain**. All the y-values form the **range**.

For example:

The domain of $A = \{1, 4, 5\}$ and the domain of $B = \{1, 2, 3\}$.

The range of $A = \{3, -2, 13\}$ and the range of $B = \{3, 6, 9\}$.

When the domain and/or range correspond to an interval, **interval notation** is used. Do not confuse interval notation with an ordered pair.

$(a, b) = \{x: a < x < b\}$	$[a, \infty) = \{x: x \geq a\}$
$[a, b] = \{x: a \leq x \leq b\}$	$(a, \infty) = \{x: x > a\}$
$(a, b] = \{x: a < x \leq b\}$	$(-\infty, b] = \{x: x \leq b\}$
$[a, b) = \{x: a \leq x < b\}$	$(-\infty, b) = \{x: x < b\}$

Implied (maximal) domain

When a rule for a relation is given without a domain, the **implied (maximal) domain** is the largest domain possible.

For example:

Given $y = 3x + 2$, the implied domain is $(-\infty, \infty)$, while the implied domain for $x^2 + y^2 = 9$ is $[-3, 3]$.

Types of correspondence

A relation may have **many-to-many, one-to-many, many-to-one or one-to-one correspondence**.

In many-to-many correspondence, many values for x correspond to a y-value and many values of y correspond to an x-value.

In one-to-many correspondence, one value of x corresponds to many values of y.

In many-to-one correspondence, many values of x correspond to one value of y.

In one-to-one correspondence, one value of x corresponds to one value of y.

For example:

$C = \{(1, 2), (1, 3), (2, 3)\}$ is a many-to-many relation

$D = \{(3, -1), (3, 1), (3, 2)\}$ is a one-to-many relation

$E = \{(1, 2), (3, 2), (5, 2)\}$ is a many-to-one relation

$F = \{(1, 2), (2, 3), (3, 4)\}$ is a one-to-one relation

Functions

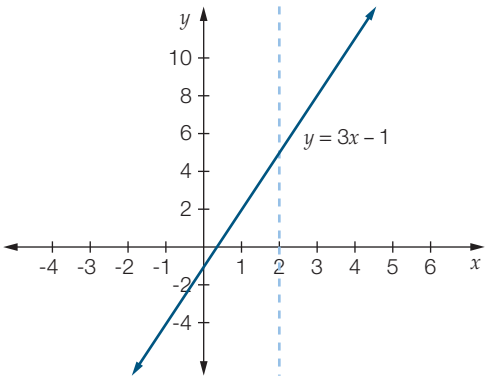
A function is a relation where there is many-to-one or one-to-one correspondence. A function is denoted by $f(x)$. Other letters can also be used to indicate functions, such as $g(x)$ and $h(x)$.

For example:

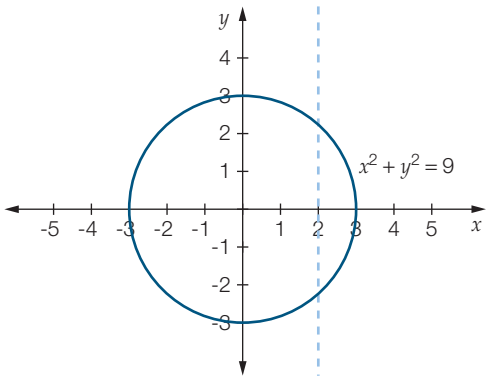
The sets E and F given above are functions.

The graph of a function can be identified by using a **vertical line test**. If a vertical line is drawn anywhere on the graph and only intersects with the graph of the relation once, it is a function.

For example:



$y = 3x - 1$ is a function and can be written as $f(x) = 3x - 1$



$x^2 + y^2 = 9$ is not a function.

Substitution into a function

$f(a)$ means that a is substituted for x in the function $f(x)$.

Inverse functions

In an inverse function, the x and y elements are interchanged.

For example:

For $f(x) = 3x - 1$, by interchanging x and y, $y = 3x - 1$ becomes $x = 3y - 1$.

Rearranging to make y the subject gives $y = \frac{x}{3} + \frac{1}{3}$, the inverse function denoted by $f^{-1}(x)$.

The domain and range of the function are also interchanged for the inverse:

dom $f^{-1}(x)$ = ran $f(x)$ and ran $f^{-1}(x)$ = dom $f(x)$.

A function has an inverse function if and only if it is one-to-one.

An inverse function can be found graphically by reflecting the graph of the function in the line $y = x$.

If the function and its inverse intersect, the point of intersection also lies on the line $y = x$.

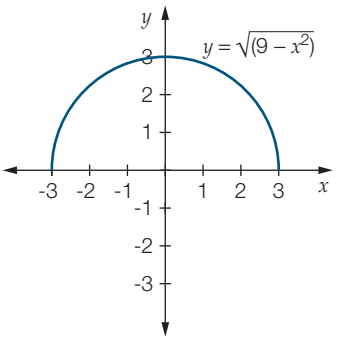
Restricting the domain

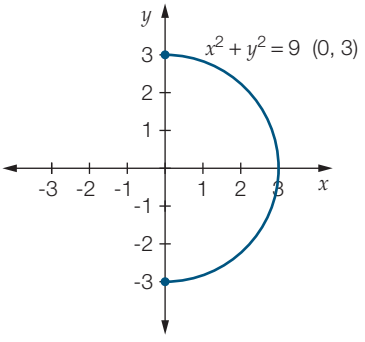
The domain of a function can be restricted so that an inverse function can exist.

For example:

$x^2 + y^2 = 9$ is not a function but $y = \sqrt{9 - x^2}$ is a function where dom $f(x)$ = [-3, 3] and ran $f(x)$ = [0, 3].

Interchanging the range and domain for an inverse shows that the inverse is a relation not a function.





If the domain is restricted so that dom $f(x)$ = [0, 3], then an inverse function,

$f^{-1}(x) = \sqrt{9 - x^2}$ would exist.

Worked Example 18

WE18

Write down the domain and range for the following relations.

(a) $S = \{(1, 5), (3, 9), (7, 16)\}$

(b) $T = \{(-8, 2), (0, 0), (4, -1)\}$

Working

(a) $S = \{(1, 5), (3, 9), (7, 16)\}$

Dom $S = \{1, 3, 7\}$

Ran $S = \{5, 9, 16\}$

(b) $T = \{(-8, 2), (0, 0), (4, -1)\}$

Dom $T = \{-8, 0, 4\}$

Ran $T = \{-1, 0, 2\}$

Worked Example 19

WE19

Write the following using interval notation.

(a) $\{x: x > 4\}$ **(b)** $\{x: -1 \le x < 8\}$ **(c)** $\{x: x \le -5\}$

Working

(a) $\{x: x > 4\} = (4, \infty)$

(b) $\{x: -1 \le x < 8\} = [-1, 8)$

(c) $\{x: x \le -5\} = [-\infty, 5)$

Worked Example 20

WE20

State the implied domain and range for the following relations. Identify the shape of the graph of the relation first.

(a) $y = x^2 + 2$ **(b)** $(x + 1)^2 + (y - 1)^2 = 4$ **(c)** $y = 2^x$

Working

(a) $y = x^2 + 2$

This graphs as a parabola.

The implied domain is $(-\infty, \infty)$.

The range is $[2, \infty)$.

(b) $(x + 1)^2 + (y - 1)^2 = 4$

This graphs as a circle.

The implied domain is $[-3, 1]$.

The range is $[-1, 3]$.

(c) $y = 2^x$

This graphs as an exponential.

The implied domain is $(-\infty, \infty)$.

The range is $(0, \infty)$.

Worked Example 21

WE21

State the type of correspondence in the following relations.

(a) $y = 4 - 3^x$ **(b)** $y = 5x^2 - 11$ **(c)** $x^2 + (y - 1)^2 = 25$

Working

(a) $y = 4 - 3^x$

This graphs as an exponential.

It has 1-to-1 correspondence.

(b) $y = 5x^2 - 11$

This graphs as a parabola.

It has many-to-1 correspondence.

(c) $x^2 + (y - 1)^2 = 25$

This graphs as a circle.

It has many-to-many correspondence.

Worked Example 22

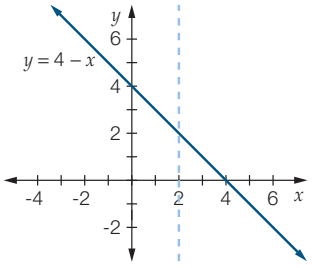
WE 22

Sketch a graph of the following relations and identify which of them are functions by using the vertical line test.

(a) $y = 4 - x$ **(b)** $y = (x - 4)^2 - 1$ **(c)** $(x + 2)^2 + (y + 1)^2 = 1$

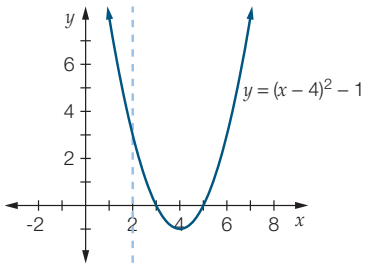
Working

(a) $y = 4 - x$



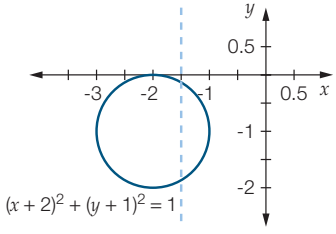
$y = 4 - x$ is a function.

(b) $y = (x - 4)^2 - 1$



$y = (x - 4)^2 - 1$ is a function.

(c) $(x + 2)^2 + (y + 1)^2 = 1$



$(x + 2)^2 + (y + 1)^2 = 1$ is not a function.

Worked Example 23

WE 23

Find the inverse of the function $f(x) = 4 - 2x$ and sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. State the domain and range for $f(x)$ and $f^{-1}(x)$.

Working

$f(x) = 4 - 2x$

$y = 4 - 2x$

To find the inverse $f^{-1}(x)$, interchange x and y.

$x = 4 - 2y$

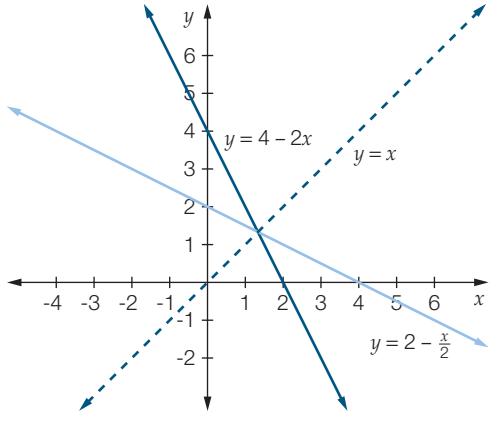
Rearranging to make y the subject

$2y = 4 - x$

$y = 2 - \frac{x}{2}$

dom $f(x)$ = ran $f^{-1}(x) = (-\infty, \infty)$

ran $f(x)$ = dom $f^{-1}(x) = (-\infty, \infty)$



Answers page 705

10.6 Functions and inverses

Stage 5.3#

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q14, Q15, Q17, Q18, Q19	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14, Q15, Q17, Q18, Q19	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

Fluency and understanding

WE18

1 Write down the domain and range for the following relations.

(a) $P = \{(3, 4), (2, -1), (5, 11)\}$ **(b)** $Q = \{(-6, 2)(-4, 4), (-2, 6)\}$ **(c)** $R = \{(0, 4)(3, -2), (6, -6)\}$

WE19

2 Write the following using interval notation.

(a) $\{x: x < 6\}$ **(b)** $\{x: -5 < x \le 7\}$ **(c)** $\{x: x \ge 3\}$

(d) $\{x: 1 < x < 6\}$ **(e)** $\{x: x \le 9\}$ **(f)** $\{x: -2 \le x \le 4\}$

WE20

3 State the implied domain and range for the following relations. Identify the shape of the graph of the relation first.

(a) $(x - 5)^2 + (y + 3)^2 = 25$ **(b)** $y = (x + 1)^2 + 4$ **(c)** $y = 3x + 4$

WE21

4 State the type of correspondence in the following relations.

(a) $y = \dfrac{1}{x - 4}$ **(b)** $y = x^3 - 2$ **(c)** $y = x^2 + 2$

(d) $y = 2x + 5$ **(e)** $(x + 4)^2 + (y - 1)^2 = 9$ **(f)** $x = 2$

WE22

5 Sketch a graph of the following relations and identify which of them are functions by using the vertical line test.

(a) $y = (x + 5)^3$ **(b)** $(x - 4)^2 + y^2 = 16$ **(c)** $y = (x + 3)^3 + 1$

WE23

6 Find the inverse of the function $f(x) = 2x + 5$ and sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. State the domain and range for $f(x)$ and $f^{-1}(x)$.

7 Given $f(x) = 4x - 7$, find:

(a) $f(3)$ **(b)** $f(0)$ **(c)** $f(-1)$ **(d)** $f(10)$ **(e)** $f(6)$ **(f)** $f(-5)$

8 Sketch the graphs of the following for the given domain then state whether the equation represents a function or a relation.

(a) $y = 3x - 5, x > 5$ **(b)** $y = x^2 - 5, -1 < x < 3$ **(c)** $y = -\sqrt{36 - x^2}, 0 \le x \le 6$

(d) $y^2 = x, [0, \infty)$ **(e)** $y = 4, [-3, 5]$ **(f)** $y = x^3 + 2, (-2, 2]$

9 State the largest possible domain and range for the following functions.

(a) $y = \sqrt{x + 1}$ **(b)** $y = x^2 - 2$ **(c)** $y = -\sqrt{4 - x^2}$

Reasoning, communicating and open-ended

10 Given $f(x) = x^2 + 4$, find:

(a) $f(x + 1)$ **(b)** $f(2x)$ **(c)** $f(4 - x)$ **(d)** $f\left(\frac{x}{2}\right)$

11 Given $f(x) = 1 - \frac{1}{x}$, find:

(a) $f(x + 1)$ **(b)** $f(2x)$ **(c)** $f(4 - x)$ **(d)** $f\left(\frac{1}{x}\right)$

12 From the relation $x^2 + y^2 = 4$, $[0, 2]$, form two functions and state the range of each.

13 From the function $f(x) = (x - 2)^2$, by restricting the domain, form two one-to-one functions that have the same rule.

14 Find the inverse function for each of the following, stating the domain and range of the inverse.

(a) $f(x) = (x + 3)^2 + 2$, $[0, \infty)$ **(b)** $f(x) = \sqrt{25 - x^2}$, $[0, 5]$

15 Find the point of intersection of $f(x) = 3x - 2$ and its inverse $f^{-1}(x)$.

16 Find the inverse of $f(x) = (x - 3)^2 + 2$, $(-\infty, 3]$. Will the function and its inverse ever intersect? Explain.

17 **(a)** Write down a quadratic equation and restrict its domain so that it is a one-to-one function.

(b) Find the inverse function, stating its range and domain.

18 In a test, Sol has written that $(x - 3)^2 + (y - 1)^2 = 4$, $[3, 5]$ is a function. Is he correct or incorrect? Give reasons for your decision.

19 Rohani thinks that all linear equations are one-to-one functions. Chee says they are all functions but not necessarily one-to-one. Dominic argues that some linear equations are not functions. Who is correct? Give examples to justify your answer.

Chapter review 10

D.I.Y. Summary

Key Words

Stage 5.3§	cubic	point of inflection	quartic
Stage 5.3#	degree	leading coefficient	range
	dividend	leading term	relation
	divisor	many-to-many correspondence	remainder
	domain	many-to-one correspondence	remainder theorem
	element	one-to-many correspondence	set
	factor theorem	one-to-one correspondence	vertical line test
	implied (maximal) domain	polynomial	
	interval notation	quotient	

Stage 5.3# Q1–4

1 The highest power of a polynomial gives the _____ of the polynomial.

2 The first term of a polynomial written with powers in descending order is called the _____.

3 A linear factor is divided into a _____ to find a _____ and a _____.

4 A polynomial can be factorised by using the _____.

Stage 5.3§ Q1, 2, 9, 12–14 **Stage 5.3# Q3–8, 10, 11**

Fluency and understanding

1 Graph each equation by constructing a table of values for $x = -3$ to 3. **Ex. 10.1**

(a) $y = 3x^2$ (b) $y = \frac{1}{2}x^3$ (c) $y = -x^4$

2 (a) Sketch the graph of $y = 2x^2 - 3$ starting with $y = x^2$. **Ex. 10.2**

(b) Sketch the graph of $y = -2(x + 2)^3 + 1$ starting with $y = x^3$.

(c) Sketch the graph of $y = -3(x + 2)(x - 4)(x + 1)$.

3 (a) If $P(x) = 4x - 3$ and $Q(x) = 3x^2 - x + 2$, evaluate: **Ex. 10.3**

(i) $P(4)$ (ii) $Q(-2)$ (iii) $P(-1).Q(-1)$

(b) Write an expression for:

(i) $2P(x) + 3Q(x)$ (ii) $P(x) \times Q(x)$ (iii) $Q(x) - P(x)$

4 Find the quotient and remainder for $\frac{4x^3 + 6x^2 + 26x + 13}{4x + 2}$. **Ex. 10.3**

5 Use the remainder theorem to find the remainder produced by each division.

Ex.10.4

(a) $\dfrac{10x^3 + 7x^2 - 4x - 3}{x + 1}$ (b) $\dfrac{x^4 - x^3 + x^2 + x - 1}{2x - 1}$

6 Use the factor theorem to factorise:

Ex.10.4

(a) $x^3 + 6x^2 - 9x - 14$ (b) $x^4 - 5x^2 + 4$

7 Sketch the graph of each polynomial.

Ex.10.5

(a) $P(x) = x^3 - 8x^2 - x + 8$ (b) $P(x) = x^4 + 2x^3 - 4x^2 - 2x + 3$

8 Write the following using interval notation.

Ex.10.6

(a) $\{x: x \le 6\}$ (b) $\{x: 0 \le x \le 10\}$ (c) $\{x: x > -4\}$

9 An equation $y = x^4$ is dilated by a factor of $\frac{1}{2}$ in the y-direction, reflected in the x-axis and translated 2 units left and 3 units up.

(a) Write the new equation.

(b) Sketch the graph of the equation. Factorise the quartic to find the x-intercepts, correct to two decimal places.

10 Find the remainder using long division when $P(x) = x^3 - 3x^2 - 6x + 1$ is divided by $x + 2$. Confirm your answer by finding the remainder using the remainder theorem.

11 (a) Find the inverse of the function $f(x) = \dfrac{x}{2} - 1$ and sketch the graph of $f(x)$ and $f^{-1}(x)$ on the same set of axes.

(b) State the range and domain of $f(x)$ and $f^{-1}(x)$.

Reasoning and communicating

12 A bowl has a shape that can be described by the parabola $y = \dfrac{1}{10}x^2$ for $-10 \le x \le 10$.

The position of a length of uncooked spaghetti that just touches the rim of the bowl is described by the line $y = x$. The coordinate (0, 0) is the middle of the bottom of the bowl and all units are in centimetres.

(a) Graph both equations on the same axes.

(b) What is the vertical distance, above the centre of the bowl, of the end of the spaghetti?

13 The vertical cross-section of a bottle is described by the equations $y = (x - 5)^3 + 2$ and $y = -(x - 15)^3 + 2$.

(a) Graph both equations on the same axes using x-values from 2 to 18.

(b) State the equation of the axis of symmetry of the bottle.

14 A capsule less dense than water is released under water and moves upwards subject to ocean currents. Its height, h metres, from sea level at time, t seconds, is given by $h = t^3 - 9t^2 + 26t - 24$.

(a) Factorise the equation and sketch the graph for $0 \le t \le 4$.

(b) Describe the motion of the capsule in this time period.

11 Probability

P Recall 11

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Express in simplest form.

(a) $\frac{3}{15}$ **(b)** $\frac{16}{52}$

2 Evaluate the following.

(a) $\frac{1}{8} + \frac{3}{4}$ **(b)** $\frac{2}{5} + \frac{1}{3}$ **(c)** $1 - \frac{3}{5}$

(d) $\frac{2}{5} \times \frac{3}{4}$ **(e)** 0.8×0.5 **(f)** 0.05×0.2

3 **(a)** Two students are to be chosen from a class of girls and boys. Complete a list of the gender pairings possible, in the order first student, second student.

(b) What are the two possible outcomes when a coin is tossed?

4 **(a)** A container has 25 numbered tokens: 6 are red, 10 are blue and the rest are yellow. A token is chosen at random from the container.

(i) How many different tokens can be chosen?

(ii) How many possible outcomes are there?

(iii) How many of these outcomes result in a red token being chosen?

(iv) How many of these outcomes result in a yellow token being chosen?

(v) How many of these outcomes result in a token that is not blue being chosen?

(b) A fair six-sided *die* (singular of *dice*) is rolled.

(i) List all the possible outcomes.

(ii) How many of these outcomes are an even number?

Syllabus references

Statistics and Probability: Probability

✓ Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or' **(ACMSP226) (Stage 5.1)**

✓ List all outcomes for two-step chance experiments, with and without replacement, using tree diagrams or arrays; assign probabilities to outcomes and determine probabilities for events **(ACMSP225) (Stage 5.2)**

✓ Describe the results of two- and three-step chance experiments, with and without replacement, assign probabilities to outcomes, and determine probabilities of events; investigate the concept of independence **(ACMSP246) (Stage 5.2)**

✓ Use the language of 'if … then', 'given', 'of', 'knowing that' to investigate conditional statements and to identify common mistakes in interpreting such language **(ACMSP247) (Stage 5.2)**

Probability review

11.1

Stage 5.2

Need to Know

Probability is the study of the chance, or likelihood, of a particular **event** occurring.

The probability of an event lies between 0 and 1 with 0 being impossible and 1 being certain.

$$0 \le P(\text{event}) \le 1$$

Estimating probability

An estimate of the probability of an outcome can sometimes be found by examining past results. The long-term pattern of results can be used to indicate a similar event occurring in the future. This process is known as estimating probability and assumes that conditions will remain the same for future events. The greater the number of trials, the more accurate this estimate will be.

An estimate, based on **long-run proportion**, of $P(\text{success}) = \dfrac{\text{number of past successes}}{\text{total number of past trials}}$.

For example:

Greg is playing tennis and has served 15 aces in the last 80 points. An estimate of the probability of Greg serving an ace is:

$$P(\text{Greg serving an ace}) = \frac{15}{80} = \frac{3}{16}$$

Assuming this will always be the proportion of aces Greg will serve, then $\frac{3}{16}$ can be used as the long-run proportion. However, it is important to understand that this is an estimate.

Theoretical probability

Probabilities can be deduced in situations where all the possible outcomes can be listed, if the outcomes can be assumed to be equally likely.

For example:

If a coin is flipped, it would be reasonable to assume that heads and tails are equally likely, but if two different footballers were lined up to take a shot at goal from the same position it would not be reasonable to assume that they had the same probability of scoring a goal.

The full list of possible outcomes in a probability experiment is called the **sample space**.

$$P(\text{success}) = \frac{\text{number of equally likely successful outcomes}}{\text{total number of equally likely possible outcomes}}$$

Complementary events

A **complementary event** is the opposite of the event under consideration. The sum of the probability of an event and its complementary event is 1.

$$P(A) + P(\text{not } A) = 1$$

Multiple events

Consider flipping a coin once, twice or three times.

Flipping a coin once has two possible outcomes, a head H or a tail T.

Flipping a coin twice has four possible outcomes, HH, HT, TH, TT.

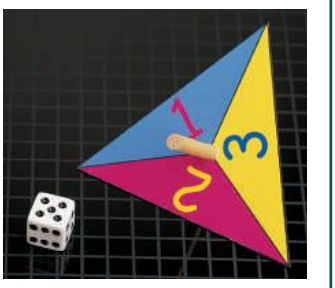
Flipping a coin three times has eight possible outcomes, HHH, HHT, HTH, HTT, TTH, THT, THH, TTT.

Multiple events can comprise two events with different numbers of equally likely outcomes.

For example:

Flipping a coin (2 outcomes: H or T) and rolling a die (6 outcomes 1, 2, 3, 4, 5, 6). There are 12 possible equally likely outcomes, H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6.

Two-way tables and tree diagrams

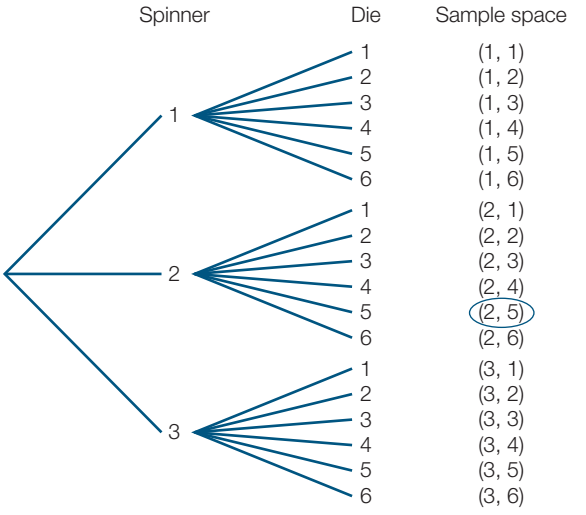


Two-way tables and tree diagrams are used to list the sample space.

Consider spinning a spinner with three possible outcomes and rolling a die.

All possible outcomes can be listed using a two-way table or using a tree diagram. From this, we can find the probability of obtaining a 2 with the spinner and rolling a 5 on the die.

Possible outcomes		Die					
		1	2	3	4	5	6
Spinner	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)



The probability of obtaining a 2 on the spinner and a 5 on the die is $\frac{1}{18}$.

Worked Example 1

WE1

Over the past 6 months Jake has been late to school on 12 occasions in 100 school days.

(a) What is the estimated probability he will be late next Wednesday?

(b) What is the estimated P(Jake being on time next Friday)?

Working

(a) Jake has been late 12 out of 100 days.

$$P(\text{late}) = \frac{12}{100}$$
$$= \frac{3}{25}$$

(b) $P(\text{late}) + P(\text{on time}) = 1$

$$P(\text{on time}) = 1 - P(\text{late})$$
$$= 1 - \frac{3}{25}$$
$$= \frac{22}{25}$$

Worked Example 2

WE2

Robin loves to play football with his mates after school but he also loves to play his guitar. He can't decide what to do, so he flips a coin. A head means he will play football and a tail means he will play his guitar. Find the probability that he:

(a) plays football 3 afternoons in a row

(b) plays football at least twice in 3 afternoons

(c) plays his guitar 2 afternoons in a row

(d) doesn't play football 3 afternoons in a row.

Working

HHH, HHT, HTH, HTT, THH, TTH, THT, TTT are all equally likely as long as the coin is fair.

(a) P(football 3 afternoons in a row)

$= P(\text{HHH})$

$= \dfrac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}}$

$= \dfrac{1}{8}$

(b) P(football at least twice in 3 afternoons)

$= P(\text{HHT or HTH or THH or HHH})$

$= \dfrac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}}$

$= \dfrac{4}{8}$

$= \dfrac{1}{2}$

(c) P(plays guitar 2 afternoons in a row)

$= P(\text{TTH or HTT or TTT})$

$= \dfrac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}}$

$= \dfrac{3}{8}$

(d) P(doesn't play football 3 afternoons in a row)

$= 1 - P$(plays football 3 afternoons in a row)

$= 1 - \dfrac{1}{8}$

$= \dfrac{7}{8}$

Worked Example 3

WE 3

Jemma has a dilemma. She has six favourite outfits: a black, a red, a blue, a green, a purple and a multi-coloured outfit. She has two favourite pairs of shoes; one has a pattern in the leather and one is plain. She can never decide what to wear so she rolls a die to choose her outfit and flips a coin to choose her shoes. On the die: 1 represents the black outfit, 2 the red, 3 the blue, 4 the green, 5 the purple and 6 the multi-coloured outfit. On the coin: heads represents the shoes with pattern, tails represents the plain pair. Find the probability that she will wear:

(a) the black outfit with the plain shoes

(b) the green, purple or red outfit

(c) the multi-coloured outfit and the patterned shoes

(d) the red or green outfit with plain shoes

(e) the red outfit

(f) the plain pair of shoes.

Working

A two-way table can be used to find the sample space.

		Die					
		1	2	3	4	5	6
Coin	H	H1	H2	H3	H4	H5	H6
	T	T1	T2	T3	T4	T5	T6

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

There are 12 equally likely outcomes.

(a) P(black outfit, plain shoes)
$= P(\text{T1})$
$= \frac{1}{12}$

(b) P(green or purple or red outfit)
$= P(\text{H2, H4, H5, T2, T4, T5})$
$= \frac{6}{12}$
$= \frac{1}{2}$

(c) P(multi-coloured outfit and shoes with pattern)
$= P(\text{H6})$
$= \frac{1}{12}$

(d) P(red or green outfit with plain shoes)
$= P(\text{T2, T4})$
$= \frac{2}{12}$
$= \frac{1}{6}$

(e) P(red outfit)
$= P(\text{T2, H2})$
$= \frac{2}{12}$
$= \frac{1}{6}$

(f) P(plain shoes)
$= P(\text{T1, T2, T3, T4, T5, T6})$
$= \frac{6}{12}$
$= \frac{1}{2}$

Worked Example 4

WE4

Libby and Chris wish to start a family. They plan to have three children. The chance of any particular child being a boy or a girl is considered to be equally likely.

(a) Represent all the possible outcomes using a tree diagram.

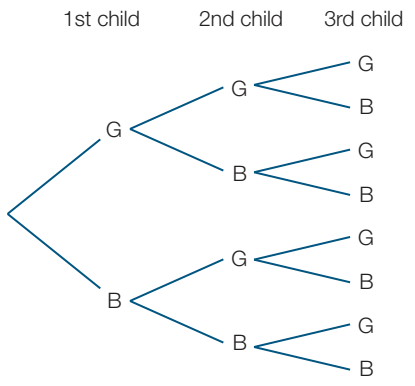
(b) List the sample space.

(c) Find the probability that they have:

- **(i)** a girl, a boy, then another girl
- **(ii)** 2 girls and a boy (in any order)
- **(iii)** 3 boys
- **(iv)** 3 children of the same gender
- **(v)** at least 2 girls

Working

(a)



(b) (G, G, G), (G, G, B), (G, B, G), (G, B, B), (B, G, G), (B, G, B), (B, B, G), (B, B, B)

There are 8 equally likely outcomes. These outcomes are equally likely because each child is considered equally likely to be a boy or a girl.

(c) **(i)** $P(G, B, G)$

$= \frac{1}{8}$

(ii) $P(2 \text{ girls and } 1 \text{ boy})$

$= \frac{3}{8}$

(iii) $P(3 \text{ boys})$

$= \frac{1}{8}$

(iv) $P(3 \text{ children of the same gender})$

$= \frac{2}{8}$

$= \frac{1}{4}$

(v) $P(\text{at least } 2 \text{ girls})$

$= \frac{4}{8}$

$= \frac{1}{2}$

Answers page 709

11.1 Probability review

Stage 5.2

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q16, Q18, Q20, Q22	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q15, Q17, Q18, Q21, Q23	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q15, Q17, Q18, Q19, Q21, Q22, Q23

Fluency and understanding

WE1

1 Ben's car has been hard to start on 8 days over the last 4 weeks.

(a) Estimate P(Ben's car will be hard to start tomorrow).

(b) Estimate P(Ben's car will be easy to start in 3 days' time).

(c) What assumptions have you made to calculate these answers?

WE2

2 Fiona is bushwalking through a national park. She finds that the path keeps dividing and she has to decide whether to take the left path or the path to the right. If she takes 3 left turns, she will arrive at a beach. If she takes 3 right turns, she will reach a waterfall. If she turns left twice and right once, not necessarily in that order, she will get to a lookout. If she turns left once and right twice, not necessarily in that order, she will reach a picnic area. She can't decide where to go so she flips a coin. Heads will mean she turns right and tails will mean that she turns left. Find the probability that she:

(a) gets to the beach

(b) arrives at the picnic area or waterfall

(c) reaches the lookout

(d) doesn't arrive at the beach.

WE3

3 Mario likes to eat at a local restaurant but he has difficulty choosing his meal as he likes 6 main courses equally and there are 2 desserts he loves. He decides to roll a die to choose his main course and flip a coin to decide his dessert. He assigns the meals this way: chicken parmigiana as 1, chilli prawns as 2, spaghetti bolognese as 3, grilled barramundi as 4, pepper steak as 5 and spicy calamari as 6. He assigns his desserts this way: sticky date pudding as H and tiramisu as T. Find the probability that he chooses:

(a) chicken parmigiana and tiramisu

(b) a seafood dish—chilli prawns, grilled barramundi or spicy calamari

(c) pepper steak and a sticky date pudding

(d) grilled barramundi or spaghetti bolognese as a main course with a tiramisu dessert

(e) chilli prawns

(f) sticky date pudding.

WE4

4 A coin is tossed three times.

(a) Represent all the possible outcomes using a tree diagram.

(b) List the sample space.

(c) Find the probability of tossing:

(i) a head, a tail then a head

(ii) a head, a head then a tail

(iii) 3 tails

(iv) all heads or all tails

(v) at least two heads.

5 Nikola and Brad are in a class containing 14 girls and 10 boys. To select a class representative, a person's name is chosen at random from the class roll. Find the probability that the class representative is:

(a) a girl

(b) not a girl

(c) Nikola

(d) Brad

(e) Nikola or Brad

(f) neither Nikola nor Brad.

6 Pedro has two drawers containing loose socks. There are a black, a white and a green sock in each. He is half asleep and grabs one sock, at random, from each drawer.

(a) Represent all the possible outcomes using a tree diagram.

(b) Why is it reasonable to regard these as equally likely outcomes?

(c) Find the probability of Pedro obtaining:

(i) a pair of black socks

(ii) one green and one white sock

(iii) a matching pair of socks

(iv) two socks that do not match.

7 Jacob and Ted are 2 members of a 12-member cricket team. The captain is chosen randomly each week. Find the probability that the captain is:

(a) Ted **(b)** not Jacob **(c)** Jacob or Ted **(d)** neither Jacob nor Ted.

8 The local supermarket is flooded and the labels are washed off 12 cans of peaches, 5 cans of apples and 3 cans of beans. The supermarket now has them all for sale for 50 cents a can, but you will not know what is inside each can until it is opened.

(a) If you buy only 1 can the probability of it being apples is:

A $\frac{1}{5}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{5}{8}$

(b) If you buy 1 can the probability of it containing fruit (not beans) is:

A $\frac{3}{20}$ **B** $\frac{5}{20}$ **C** $\frac{12}{20}$ **D** $\frac{17}{20}$

(c) If you want to be sure of purchasing at least 1 can of peaches, the minimum number of cans you must purchase is:

A 1 **B** 3 **C** 9 **D** 12

9 A spinner evenly divided and showing the numbers 1, 2, 3, 4 is spun once, and an eight-sided die showing 1 to 8 is rolled once.

(a) The probability that the total of the two numbers is 7 is:

A $\frac{3}{11}$ **B** $\frac{3}{32}$ **C** $\frac{1}{8}$ **D** $\frac{5}{32}$

(b) The probability that the total of the two numbers is 11 is:

A $\frac{1}{32}$ **B** $\frac{1}{16}$ **C** $\frac{1}{8}$ **D** $\frac{3}{32}$

(c) The probability that the total of the two numbers is a multiple of 3 is:

A $\frac{5}{16}$ **B** $\frac{11}{32}$ **C** $\frac{3}{8}$ **D** $\frac{15}{32}$

10 Suresh and Julian regularly play tennis against one another. Over the past 2 years, Suresh has won 16 times and Julian has won 20 times. What is the estimated probability that Suresh will win their next match?

11 Of the 125 students in Year 10, 15 ride to school every day and 30 ride sometimes. A student is chosen at random from the Year 10 students.

(a) What is the probability that the student always rides to school?

A $\frac{3}{25}$ **B** $\frac{9}{25}$ **C** $\frac{17}{25}$ **D** $\frac{22}{25}$

(b) What is the probability that the student never rides to school?

A $\frac{3}{25}$ **B** $\frac{9}{25}$ **C** $\frac{16}{25}$ **D** $\frac{22}{25}$

12 If the probability of it raining tomorrow is $\frac{2}{9}$, what is the probability of it not raining?

13 In a game of chance, 45 balls numbered from 1 to 45 are placed in a barrel. If one ball is drawn at random, find the probability that its number is:

(a) 2 **(b)** less than 10 **(c)** even

(d) odd **(e)** greater than 50 **(f)** between 10 and 20 inclusive.

14 In the game of Monopoly, players advance clockwise around the 40 squares of the board according to the total of the numbers shown on the roll of two dice.

(a) Use a two-way table to represent all the possible totals obtained from the roll of two dice.

(b) How many possible outcomes are there?

(c) Why is it reasonable to regard these as equally likely outcomes?

(d) If your token is on Whitechapel Road, what is the probability that your next throw will result in your landing on Euston Road?

(e) Suppose you do land on Euston Road. Find the probability of advancing to Bow Street or further on your next throw.

(f) One way of getting out of jail in the game of Monopoly is to roll a 'double' with the two dice (i.e. to roll the dice and obtain the same result on both). Supposing you are in jail, find the probability of getting out of jail with your next roll of the dice.

15 Skye has the phone numbers of 40 friends saved on her phone. If we were trying to estimate the probability of Skye messaging one of these friends, would it be reasonable to assume they were equally likely outcomes? Explain your thoughts in a sentence.

16 Val and Peter have a bag of sweets each. Val's bag of sweets contains 3 licorice blocks, 2 jelly babies and 1 chocolate caramel. Peter has 2 licorice blocks and 2 jelly babies in his bag. Both Val and Peter offer a sweet to Narelle. If Narelle randomly selects a sweet from each bag, find the probability that she obtains:

(a) 2 licorice blocks **(b)** 2 jelly babies **(c)** no licorice blocks

(d) no jelly babies **(e)** 1 licorice block **(f)** at least 1 jelly baby.

17 In a round-robin competition in tenpin bowling, Shelley plays 4 rounds. She estimates that she has an equal chance of winning or losing each round.

(a) Draw a tree diagram showing all the possible outcomes. (W = a win; L = a loss)

(b) List the sample space. How many possible outcomes are there?

(c) Find the probability that Shelley:

(i) wins all 4 rounds **(ii)** wins only 2 of the 4 rounds

(iii) wins at least 3 rounds **(iv)** wins the second round

(v) doesn't win the third round

(vi) doesn't win the first or the fourth round.

18 There are 12 houses in the court where Zoran lives. Of the 20 cars owned between the people who live in the court, 7 are Holdens, 5 are Fords and the rest are other makes. The owner of one of the Holdens moves out and a new person, who owns a car, moves into the court. Is it reasonable to say that the probability this new person owns a Holden is $\frac{7}{20}$? Explain your thoughts in a sentence.

19 Two fair six-sided dice are rolled. One is numbered 1, 1, 2, 2, 3, 3 and the other is numbered 2, 4, 6, 8, 10, 12.

(a) Draw up a two-way table to show the sample space.

(b) Draw up a two-way table to show the sum of the two numbers shown.

(c) Find the following probabilities.

(i) P(sum is even)

(ii) P(sum is multiple of 5)

(iii) P(sum is greater than 10)

(iv) P(sum is not less than 11)

(d) Why is it harder to answer the questions in part **(c)** for this two-way table than for a diagram with consecutive numbers on each side?

(e) Draw up another two-way table that shows the product of the two numbers.

(f) Draw up a table that shows the probability for each possible product.

Reasoning, communicating and open-ended

20 Explain the following observations.

(a) In the long run, the observed probability of Aiden wearing a tie to work is $\frac{19}{20}$, but on three consecutive days he is seen not wearing a tie.

(b) Lesley has kept track of the lottery numbers and has noticed that 16 has not occurred for 12 months. Thinking the probability of it being drawn soon must be high, she includes 16 in her numbers for the next 2 months, but it still isn't drawn.

21 You are given a 12-sided die, numbered 1 to 12. Write an event that would have the following probabilities.

(a) $P(A) = \frac{1}{6}$

(b) $P(B) = \frac{1}{3}$

(c) $P(C) = \frac{5}{12}$

22 The long-term probability of Maria's team beating Abbey's team at netball is $\frac{4}{5}$. List three possible sets of long-term results that would produce this figure.

23 Sharanpreet calculates the probability of a family having at least one boy in a family of three children is $\frac{3}{8}$. What is the correct answer and what mistake has Sharanpreet made?

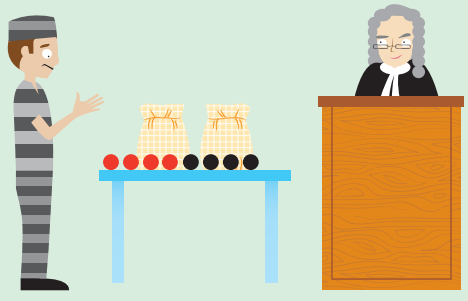
Hint

Assume that there is an equal probability of having a boy as having a girl.

Problem solving

A chance of freedom

A prisoner is brought before a judge who says: "I have decided to give you a chance of freedom. Here are four red balls and four black balls. Divide the eight balls into two groups, any way you wish, and then place each group in a bag. I will hide the two bags behind my back and you will choose a bag from either my right or my left hand. You can then choose a ball at random from the bag. If it is red you can go free. If it is black your sentence will double."



(a) How can the prisoner divide the balls to give him the best chance of freedom?

(b) What division would give him the worst chance?

Strategy options

- Act it out.
- Test all possible combinations.
- Solve a simpler problem.

11.2 Venn diagrams and two-way tables

Stage 5.2

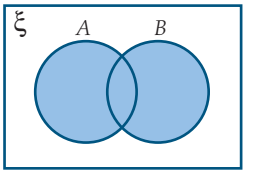
Need to Know

Venn diagrams

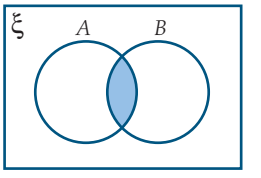
A **Venn diagram** consists of a rectangle with one or more circles inside it. The rectangle represents the **universal set**, which is every element under consideration for the particular situation being examined. On the Venn diagram it is represented by ξ. The circles represent particular events and are represented by capital letters.

There are three situations for which Venn diagrams are really useful. These are when dealing with the words 'and', 'or' and 'not'.

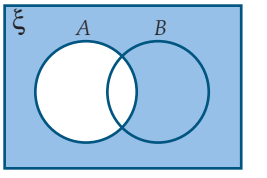
These can be shown on Venn diagrams as follows.



A or *B*



A and *B*



not *A*

Two-way tables

We can use a two-way table to show similar information.

	B	**not *B***
A	*A* and *B*	*A* and not *B*
not *A*	*B* and not *A*	not *A* and not *B*

Together, the sections of the table make up the universal set. In addition to this, no section has any overlap with any other section.

Worked Example 5

WE 5

A single card is drawn from a set of 25 cards numbered from 1 to 25.

Let A = 'odd numbers' and B = 'multiples of 3'.

(a) Represent this on a Venn diagram.

(b) Use the Venn diagram to find:

(i) $P(A)$

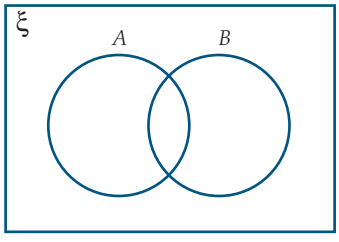
(ii) $P(A \text{ and } B)$

(iii) $P(\text{not } B)$

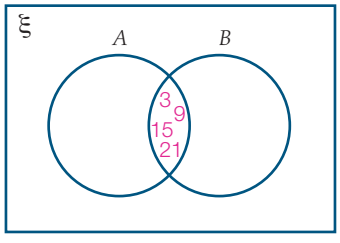
(c) Draw a two-way table to display the probability of possible events.

Working

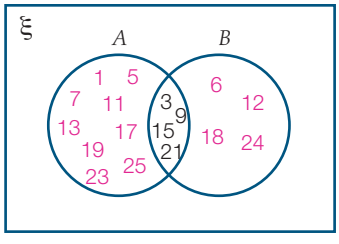
(a) Step 1: Draw a Venn diagram with sets A and B.



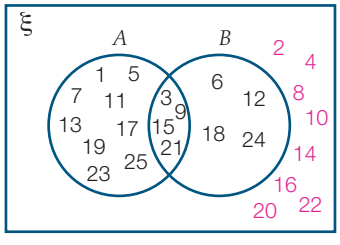
Step 2 : Write in the elements common to both sets.



Step 3: Write the remaining elements in either set.



Step 4: Write the remaining elements in the universal set.



(b) **(i)** $P(A) = \dfrac{13}{25}$

(ii) $P(A \text{ and } B) = \dfrac{4}{25}$

(iii) $P(\text{not } B) = 1 - P(B)$

$= 1 - \dfrac{8}{25}$

$= \dfrac{17}{25}$

(c) Step 1: Draw a two-way table with the known information.

	B	not B	
A	$\frac{4}{25}$		$\frac{13}{25}$
not A			
		$\frac{17}{25}$	

Step 2: Calculate $P(A \text{ and not } B)$.

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A			
		$\frac{17}{25}$	

Step 3: Calculate $P(\text{not } A)$.

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A			$\frac{12}{25}$
		$\frac{17}{25}$	

Step 4: Calculate $P(B)$ and $P(\text{not } A \text{ and not } B)$.

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A		$\frac{8}{25}$	$\frac{12}{25}$
	$\frac{8}{25}$	$\frac{17}{25}$	

Step 5: Calculate $P(B \text{ and not } A)$.

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{12}{25}$
	$\frac{8}{25}$	$\frac{17}{25}$	

Answers page 710

11.2 Venn diagrams and two-way tables

Stage 5.2

Navigator

Q1, Q2, Q3, Q5, Q8, Q9, Q11, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q12, Q13	Q1, Q2, Q3, Q5, Q6, Q7, Q9, Q10, Q12, Q13

Fluency and understanding

WE5

1 In a non-leap year, there are 28 days in February. A date in February is to be chosen to celebrate the 100th birthday of a city.

Let A = 'even numbered dates' and B = 'dates that are multiples of 5'.

(a) Represent this information on a Venn diagram.

(b) Use the Venn diagram to find:

(i) $P(A)$ **(ii)** $P(A \text{ and } B)$ **(iii)** $P(\text{not } B)$

(c) Draw a two-way table to display the probability of possible events.

2 Use the two-way table shown at right to find:

(a) $P(A \text{ and } B)$ **(b)** $P(\text{not } A \text{ and not } B)$

(c) $P(B \text{ and not } A)$ **(d)** $P(A \text{ and not } B)$

(e) $P(A)$ **(f)** $P(B)$

	B	not B
A	$\frac{3}{16}$	$\frac{5}{16}$
not A	$\frac{1}{16}$	$\frac{7}{16}$

3 In a class of 20, 8 students learn the piano, 6 take singing lessons and 10 students neither learn the piano nor have singing lessons.

(a) Represent this information in a two-way table.

(b) Use the two-way table to find the number of people who learn piano and take singing lessons.

(c) Use the two-way table to find the probability of:

(i) taking singing lessons but not piano **(ii)** taking piano lessons but not singing.

4 In a group of 30 people, there are 15 who like cats, 20 who like dogs and 6 who dislike both cats and dogs. One person is chosen from the group at random.

(a) Represent this information on a Venn diagram.

(b) Use the Venn diagram to find the probability that the person chosen at random is:

(i) a cat lover only

(ii) a dog lover only

(iii) a person who likes both.

5 Complete these two-way tables.

(a) $P(A) = \frac{7}{16}$

	B	**not *B***
A		$\frac{5}{16}$
not *A*	$\frac{1}{16}$	

(b) $P(D) = 0.7$

	C	**not *C***
D	0.2	
not *D*		0.1

6 Given the following Venn diagram, identify the numbers represented by each of the following.

(a) A or B

(b) A or C

(c) B or C

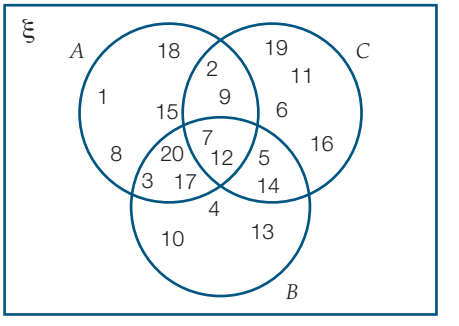
(d) A and B

(e) A and C

(f) B and C

(g) A and B and C

(h) not A and not B



7 A coin is tossed, a six-sided die is rolled and a ball is selected from a bag containing 4 red and 3 green balls. M is the set of outcomes that contains 'coin shows heads', N is the set of outcomes that contains 'a 2 is not rolled' and P is the set of outcomes that contains 'a red ball is selected'.

(a) Draw a Venn diagram to show the sample space.

(b) Write down the elements of not M, not N and not P.

(c) Is the probability of H1R the same as H1G? Explain your answer.

8 A poll of 2000 students revealed that 554 respondents only watch TV and do not browse the internet while they watch. 1200 respondents only use the internet and the TV is not on while they're browsing. 10 students claim to neither watch TV nor use the internet.

(a) Draw a two-way table to represent this situation.

(b) What is the probability of a student in this survey watching TV and browsing the internet at the same time?

Reasoning, communicating and open-ended

9 A survey of 1200 university students revealed that $\frac{3}{8}$ use Twitter, $\frac{3}{4}$ use Facebook, and $\frac{1}{4}$ use both.

(a) Represent the information in a two-way table.

(b) If a student is chosen at random, find the probability that the student:

(i) uses either

(ii) uses either but not on both.

10 Ms Lauder is creating the timetable for Year 12 students. The request forms have been entered online:

- 163 want to study mathematics, 95 physics, 124 chemistry
- 61 requested maths and physics
- 67 requested maths and chemistry
- 49 requested physics and chemistry
- 27 want to study all three subjects
- 17 want to study none of these subjects.

(a) Draw a Venn diagram to illustrate the situation.

(b) How many students requested only mathematics?

(c) How many students requested only physics?

(d) How many students requested only chemistry?

(e) What was the total number of requests?

(f) What is the probability of a Year 12 student at Ms Lauder's school selecting mathematics as one of their subjects? Will the answer be the same at every school?

11 In a scout troop there are 15 boys and 12 girls. 10 of the boys and 8 of the girls say they are interested in tennis.

(a) Show this information on a Venn diagram.

(b) A student is selected at random. Use the diagram to find the probability that the student selected is:

(i) a girl who is interested in tennis

(ii) a boy who is not interested in tennis

(iii) someone who is interested in tennis

(iv) someone who is not interested in tennis.

12 Human blood can contain up to three different antigens: A, B and Rh. The following table outlines the antigens present in each blood type. (The actual situation is slightly more complicated for the O group.) The values given in the table refer to Australia; countries with different ethnic make-up would have different values.

Blood group	A	B	Rh	% in Australia
A+	Yes	No	Yes	31
A–	Yes	No	No	7
B+	No	Yes	Yes	8
B–	No	Yes	No	2
AB+	Yes	Yes	Yes	2
AB–	Yes	Yes	No	1
O+	No	No	Yes	40
O–	No	No	No	9

The antigens present in the blood of patients at a clinic were recorded for 1 week:

- 39 patients had the A antigen
- 2 had the A and B antigens
- 12 had the B antigen
- 9 had the B and Rh antigens
- 67 had the Rh antigen
- 30 had the A and Rh antigens
- 2 had all three antigens
- 5 had no antigens.

(a) Draw a Venn diagram to show this situation.

(b) How many patients were there?

(c) What was the P(exactly one antigen)?

(d) What was the P(exactly two antigens)?

(e) Use the data to calculate the probability of each of the eight blood groups.

(f) Compare the answers from part **(e)** with those expected from long-term data as shown in the table.

13 From a class of 25 students, 14 like watching the news on TV and 16 said they use the internet to find out the news. Mert says there must be something wrong with the statistics because there are 30 different opinions from only 25 students. Sasha says Mert hasn't taken into account students who do both. Neither of them is correct. What is the correct interpretation of the statistics and what mistakes have Mert and Sasha made?

14 What is wrong with the following question?

In a survey of 200 people with a driving licence, 18% have never received a speeding fine, 44 people have received a parking ticket but have never received a speeding fine, and 78% have received both a speeding fine and a parking ticket. How many people have received a speeding fine but have never received a parking ticket?

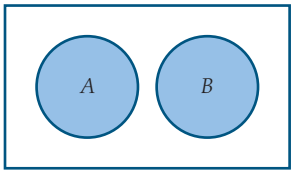
11.3 Mutually exclusive events

Stage 5.2

Need to Know

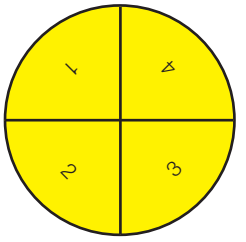
Mutually exclusive events

If two events cannot occur at the same time they are called **mutually exclusive** events. In a Venn diagram, mutually exclusive sets do not intersect. They are called **disjoint sets**.



For example:

The spinner at right is spun. Since the spinner is divided into four equal pieces, each number has an equal chance of occurring.



Consider two events.

A = obtaining an even number

= obtaining a 2 or a 4

B = obtaining a 3

These events are said to be mutually exclusive as it is impossible for both of them to occur at the same time.

$$P(A \text{ and } B) = 0$$

Addition law for mutually exclusive events

If A and B are mutually exclusive events (i.e. they cannot occur at the same time), then:

$$P(A \text{ or } B) = P(A) + P(B)$$

Worked Example 6

WE 6

A six-sided die is rolled. What is the probability that the number obtained is:

(a) a 2 or a 5?

(b) neither odd nor a 4?

Working

(a) $P(2 \text{ or } 5)$

$= P(2) + P(5)$

$= \frac{1}{6} + \frac{1}{6}$

$= \frac{2}{6}$

$= \frac{1}{3}$

(b) $P(\text{neither odd nor a } 4)$

$= P[\text{not}(\text{odd or } 4)]$

$= 1 - P(\text{odd or } 4)$

$= 1 - [P(\text{odd}) + P(4)]$

$= 1 - \left(\frac{1}{2} + \frac{1}{6}\right)$

$= 1 - \frac{2}{3}$

$= \frac{1}{3}$

11.3 Mutually exclusive events

Answers page 711

Navigator

Stage 5.2

Q1, Q2, Q3, Q4, Q5, Q6 Column 1, Q7, Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q10	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9, Q10

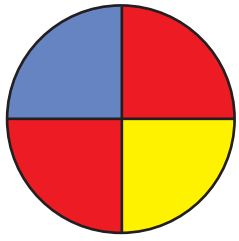
Fluency and understanding

WE 6

1 A fair eight-sided die numbered 1 to 8 is rolled. What is the probability that the number obtained is:

(a) a 3 or a 6 **(b)** neither 5 nor even?

2 A spinner is coloured as shown.



(a) What is the probability of spinning blue?

(b) What is the probability of spinning red?

(c) What is the probability of spinning blue or red?

(d) What is the probability of spinning blue or yellow?

3 Are any of the following pairs of events mutually exclusive?

(a) Obtaining a head on the toss of a coin and a tail on the toss of a coin.

(b) Choosing an odd-numbered horse in a race and horse number 7 in another race.

(c) Winning first prize in a raffle with a blue ticket and first prize in a raffle with a yellow ticket.

4 **(a)** A basket contains 3 red, 2 white and 5 black scarves. The probability that a scarf drawn at random will be red or black is:

A $\frac{1}{5}$ **B** $\frac{2}{3}$ **C** $\frac{7}{10}$ **D** $\frac{4}{5}$

(b) A jar contains 6 red lollies, 3 green lollies and 2 yellow lollies. The probability that a lolly drawn at random will be green or yellow is:

A $\frac{5}{11}$ **B** $\frac{6}{11}$ **C** $\frac{3}{11}$ **D** $\frac{5}{6}$

(c) A drawer contains 6 blue socks, 4 black socks and 2 brown socks. The probability that a sock drawn at random will *not* be black is:

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{2}{3}$

5 These probabilities apply to different modes of transport students use to travel to and from a particular school. The modes are all mutually exclusive.

$P(\text{car}) = \frac{7}{20}$ $P(\text{train}) = \frac{9}{20}$

$P(\text{bus}) = \frac{1}{20}$ $P(\text{walking}) = \frac{3}{20}$

A student is asked how he travels to school. Find the probability that:

(a) he travels by car or train

(b) he walks or catches a bus

(c) he does not walk nor catches a bus

(d) he does not travel by car nor by bus

(e) he travels by car, train or bus.

6 Twenty tokens numbered in order from 1 to 20 are placed in a bag and one is randomly selected. Find the probability that the number on the token is:

(a) 19 or 20

(b) a multiple of 3

(c) a multiple of 3 or a multiple of 7

(d) even or the number 15

(e) less than 10 or greater than 18

(f) neither 1 nor 2

(g) neither odd nor a multiple of 4

(h) 5, 6 or 7.

7 A standard pack of playing cards contains 52 cards, half red and half black. The red cards are divided into two suits: Hearts and Diamonds. The black cards are divided into two suits: Clubs and Spades. Each suit is made up of the cards Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen and King.

A card is drawn at random from the pack. Find the probability that the card is:

(a) red

(b) a Heart

(c) an Ace

(d) the King of Hearts

(e) red or the Ace of Spades

(f) a Diamond or the 4 of Hearts

(g) the 5 of Clubs or the Queen of Spades

(h) a King or an Ace

(i) any King or any Ace or the 10 of Hearts

(j) the Jack of Diamonds, any 2, or the 7 of Clubs

(k) neither a 6 nor a 9 of any suit

(l) neither a Spade nor red.

Reasoning, communicating and open-ended

8 A normal six-sided die is rolled. The following events have been defined:

A = rolling an even number | B = rolling a 2 or a 6

C = rolling a 1 or a 3 | D = rolling a multiple of 3

(a) List all possible pairs of events (order does not matter).

(b) Are any of these pairs mutually exclusive?

(c) Find the probability of each event.

(d) Find the probability that either event occurs for each of the pairs listed in part **(a)**.

(e) Find the probability that both of the events occur for each of the pairs listed in part **(a)**.

(f) Find the sum of the probabilities of each event in all the pairs listed in part **(a)**.

(g) Compare your answers to parts **(d)**, **(e)** and **(f)**. What do you find? Write an equation to summarise your findings.

9 A class conducted a traffic survey. The types of vehicles they recorded were:

sedan	station wagon	4WD	hatchback
truck	van	bus	2 door

(a) What difficulties can you see arising from the data collection?

(b) How could these difficulties have been avoided?

10 Consider a fair 20-sided die numbered 1 to 20. Write down at least eight pairs of mutually exclusive events related to this die.

Half-time 11

Stage 5.2 Q1–5

1 **(a)** Complete the following two-way table which relates to the number of surveyed students who own either, or both, an iPhone or an iPad. Ex.11.2

	iPhone	not iPhone	
iPad	60		75
not iPad			
	210		400

(b) Now use the completed two-way table to state the probability of a randomly chosen student owning:

(i) an iPhone but not an iPad **(ii)** an iPad but not an iPhone.

2 Ten balls numbered from 1 to 10 are placed in a bag and one is drawn at random. Find the probability that the number on the ball is: Ex.11.1

(a) 1 or 2

(b) an even number

(c) a 6 or an odd number

(d) neither an odd number nor a multiple of 4.

3 The probability of it raining on a given day of the weekend is 0.3. What is the probability that it rains on both Saturday and Sunday? Ex.11.1

4 In a chess tournament, each contestant must play three others. Suzanne estimates that she has an equal chance of winning or losing each game. Ex.11.1

(a) Draw a tree diagram and list the sample space for the possible outcomes of Suzanne's three games.

(b) How many possible outcomes are there?

(c) Find the probability of Suzanne:

(i) winning all three games **(ii)** winning only one game

(iii) winning the first game **(iv)** not winning the third game

(v) winning at least two games **(vi)** not winning either of the first two games.

5 A fair eight-sided die is numbered 1 to 8. The die is rolled once. The following events are defined: Ex.11.3

A = 'rolling an even number'

B = 'rolling a prime number'

C = 'rolling a multiple of 5'

(a) Given that order does not matter, list the pairs of events that are possible.

(b) Find the probability of each individual event.

(c) State which, if any, of the pairs of events are mutually exclusive.

11.4 Probability tree diagrams

Stage 5.2

Need to Know

Tree diagrams

Consider the case of flipping a fair coin three times. There are eight equally likely outcomes:

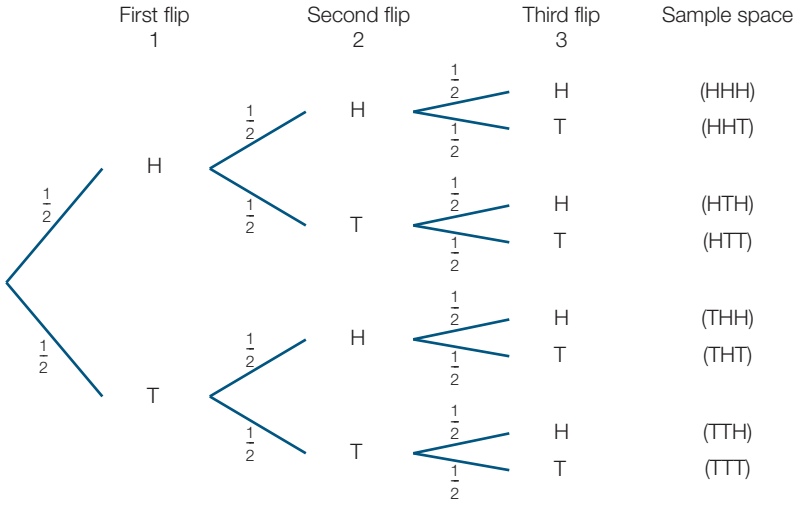
HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

So $P(\text{HHH}) = P(\text{HHT}) = P(\text{HTH}) = P(\text{HTT}) = P(\text{THH}) = P(\text{THT}) = P(\text{TTH}) = P(\text{TTT}) = \frac{1}{8}$

and

$P(\text{HHH}) + P(\text{HHT}) + P(\text{HTH}) + P(\text{HTT}) + P(\text{THH}) + P(\text{THT}) + P(\text{TTH}) + P(\text{TTT}) = 1.$

These outcomes can be represented with a probability tree diagram.



Calculating probabilities using a probability tree diagram

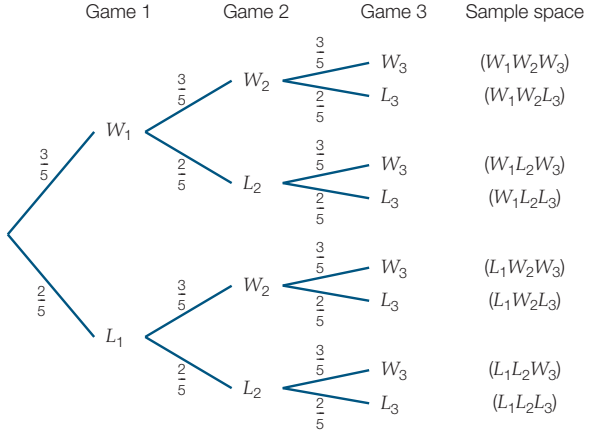
Probabilities can be calculated by multiplying the probabilities along the branches.

A **probability tree diagram** can also be used for events that do not have equally likely outcomes.

Consider a person playing a game three times with $P(\text{winning}) = \frac{3}{5}$ and $P(\text{losing}) = \frac{2}{5}$.

There are only two outcomes as $P(\text{winning}) + P(\text{losing}) = 1$.

A probability tree diagram can represent this information.



Now consider the probability of winning two out of three games. There are three possible outcomes $W_1W_2L_3$, $W_1L_2W_3$ and $L_1W_2W_3$ and the outcomes are not equally likely.

$$P(W_1W_2L_3) = \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$$

$$= \frac{18}{125}$$

Likewise, $P(W_1L_2W_3) = P(L_1W_2W_3) = \frac{18}{125}$.

As these events are mutually exclusive, these probabilities can be added.

$$P(\text{winning 2 out of 3 games}) = P(W_1W_2L_3) + P(W_1L_2W_3) + P(L_1W_2W_3)$$
$$= \frac{18}{125} + \frac{18}{125} + \frac{18}{125}$$
$$= \frac{54}{125} \text{ or } 0.432$$

Worked Example 7

WE7

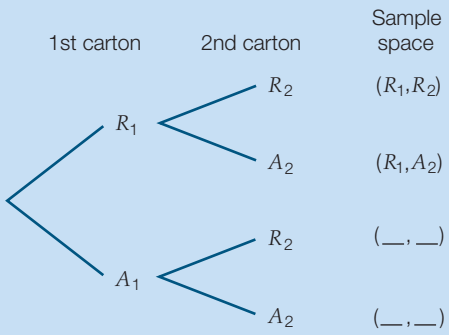
At supermarkets, cartons of eggs are inspected for breakages by the checkout assistant before payment is made. The probability that a carton of eggs will be rejected is found to be 0.1. Two cartons of eggs are brought to a checkout.

(a) Copy and complete the tree diagram on the right. Evaluate the probability of each outcome.

Let R = carton of eggs is rejected and A = carton of eggs is not rejected (is accepted).

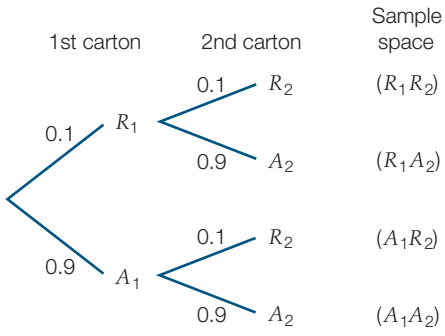
(b) Find the probability that:

- **(i)** both cartons will be rejected
- **(ii)** only one carton will be rejected
- **(iii)** neither carton will be rejected
- **(iv)** at least one carton will be rejected.



Working

(a) $P(R) = 0.1$, so $P(A) = 0.9$



$P(R_1R_2) = 0.1 \times 0.1$
$= 0.01$

$P(R_1A_2) = 0.1 \times 0.9$
$= 0.09$

$P(A_1R_2) = 0.9 \times 0.1$
$= 0.09$

$P(A_1A_2) = 0.9 \times 0.9$
$= 0.81$

(b) (i) $P(\text{both cartons will be rejected})$
$= P(R_1R_2)$
$= 0.01$

(ii) $P(\text{only one carton will be rejected})$
$= P(R_1A_2 \text{ or } A_1R_2)$
$= P(R_1A_2) + P(A_1R_2)$
$= 0.09 + 0.09$
$= 0.18$

(iii) $P(\text{neither carton will be rejected})$
$= P(A_1A_2)$
$= 0.81$

(iv) $P(\text{at least one carton will be rejected})$
$= P(\text{one carton rejected or both cartons rejected})$
$= P(\text{one carton rejected}) + P(\text{both cartons rejected})$
$= 0.18 + 0.01$
$= 0.19$

Worked Example 8

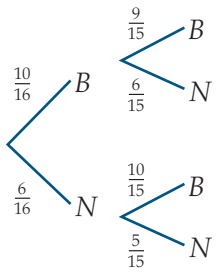
WE 8

The lucky dip at the school fete contains 16 different gifts, 10 for the 4–7 age group and 6 for the 8–11 age group. Unfortunately, the 16 gifts have been mixed together. As you cannot tell what is inside without opening the gift, the lucky dips are offered as a special two for the price of one. A 5-year-old takes his two gifts without replacement. Let B represent a gift is chosen in the 4–7 age group and N represent a gift in the 4–7 age group is not chosen.

(a) Find P(both gifts are in the 4–7 age group).

(b) Find P(the 5-year-old chooses at least one suitable gift).

Working



BB, BN, NB, NN

(a) $P(BB) = \frac{10}{16} \times \frac{9}{15} = \frac{3}{8}$

(b) $P(NN) = \frac{6}{16} \times \frac{5}{15} = \frac{1}{8}$

P(the 5-year-old chooses at least one suitable gift)

$= 1 - \frac{1}{8}$

$= \frac{7}{8}$

Answers page 712

11.4 Probability tree diagrams

Stage 5.2

Navigator

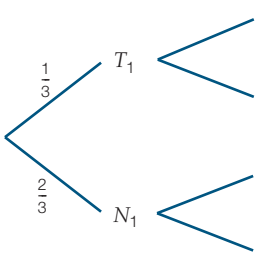
Q1, Q2, Q3, Q4, Q6, Q7, Q9, Q10, Q12, Q14, Q17	Q1, Q2, Q3, Q4, Q7, Q8, Q10, Q13, Q15, Q16	Q1, Q2, Q4, Q5, Q7, Q9, Q11, Q12, Q16, Q17

Fluency and understanding

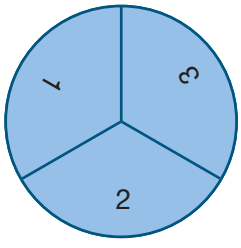
WE 7

1 A spinner numbered 1 to 3 is spun twice. Let T represent 'the number 2 is spun' and N represent 'the number 2 is not spun'.

(a) Copy and complete the following tree diagram. Evaluate the probability of each outcome.



Sample space	Probability of each outcome
(T_1T_2)	$P(T_1T_2) = \frac{1}{3} \times \frac{1}{3}$ $= \frac{1}{9}$
(T_1N_2)	$P(T_1N_2) = \frac{1}{3} \times \frac{2}{3}$ $=$
( )	$P(\quad) =$ $=$
( )	$P(\quad) =$ $=$



(b) Find the probability of:

(i) spinning 2 twice

(ii) spinning 2 once only

(iii) *not* spinning a 2

(iv) spinning 2 once at least.

2 Two balls are drawn without replacement from a bag containing 4 green marbles and 7 blue marbles. Draw a tree diagram to represent the situation and find: WE 8

(a) P(both marbles being blue) (b) P(getting at least one blue marble).

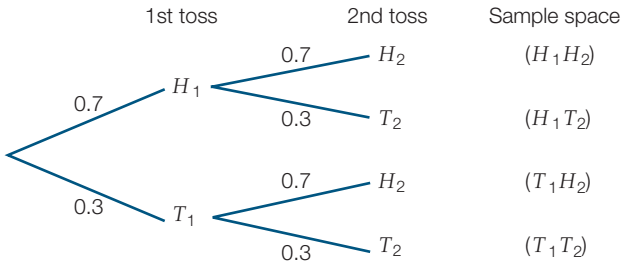
3 The spinner described in Question **1** is now spun three times.

(a) Draw a probability tree diagram for this situation.

(b) What is the probability of spinning a 2 three times?

(c) What is the probability of *not* obtaining a 2 on any of the three spins?

4 A biased coin is tossed twice. Let H represent 'obtaining a head with the coin' and T represent 'obtaining a tail with the coin'. The following probability tree diagram can be drawn.



(a) The probability of obtaining two heads is:

A $0.7 + 0.7$ **B** 0.7×0.7 **C** 0.25 **D** 0.7×0.3

(b) The probability of obtaining two tails is:

A 0.6 **B** 0.3 **C** 0.21 **D** 0.09

(c) The probability of obtaining a tail first, and then a head is:

A 0.25 **B** 0.21 **C** 0.49 **D** 1.0

(d) The probability of obtaining only one tail is:

A 0.21 **B** 0.3 **C** 0.42 **D** 0.63

(e) The probability of obtaining at least one tail is:

A 0.75 **B** 0.51 **C** 0.42 **D** 0.39

5 Two cards are drawn, without replacement, from a set of 16 cards numbered from 1 to 16.

Let M = the event a multiple of four is drawn and N = a multiple of four is not drawn.

(a) Draw a tree diagram and assign a probability to each branch.

(b) List the sample space.

(c) Find P(both cards are a multiple of 4).

(d) Find P(only one card is a multiple of 4).

(e) Find P(neither card is a multiple of 4).

6 For afternoon tea, patrons of a cafe can select from tea, coffee or hot chocolate, and either scones or carrot cake. The probability of a person selecting tea is 0.45, whereas the probability of a person selecting coffee is 0.35. The probability of a person selecting scones is 0.6.

(a) What is the probability of a person selecting hot chocolate?

(b) What is the probability of a person selecting carrot cake?

(c) Draw a probability tree diagram to represent this situation.

(d) Find the probability of a person:

(i) selecting tea and scones

(ii) selecting hot chocolate and carrot cake

(iii) selecting tea and scones or hot chocolate and carrot cake

(iv) *not* selecting coffee and scones.

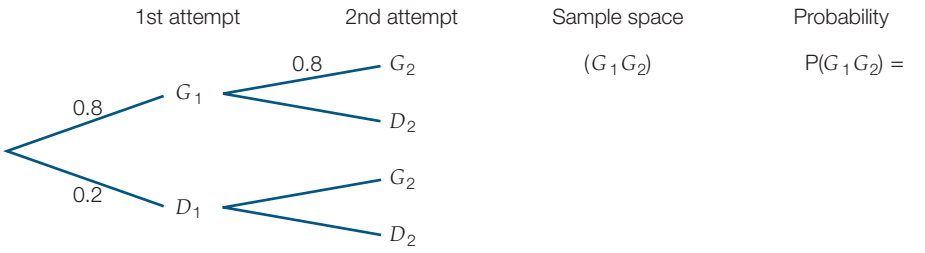
7 For each of the following:

(i) copy and complete the probability tree diagram

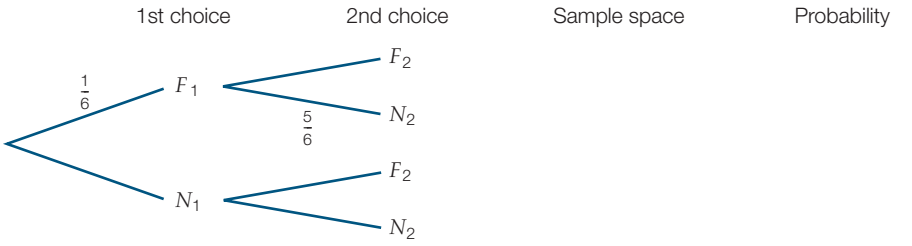
(ii) write down the probability of each possible outcome

(iii) find the sum of the probabilities of all outcomes.

(a) Let G = scores a goal and D = does not score a goal.



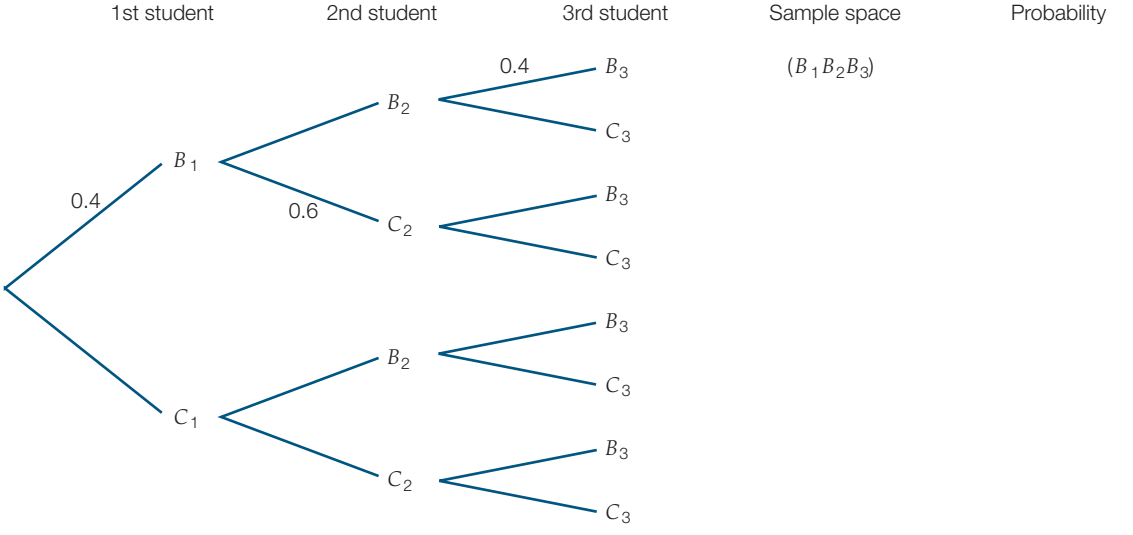
(b) Let F = a 4 is chosen from the numbers 1 to 6 and N = a 4 is not chosen.



(c) H = Henry is chosen from a group of boys.

A = Alia is chosen from a group of girls.

(d) Let B = student travels to school by bus and C = student does not travel by bus.



8 In a game of darts, each player has a turn to throw three darts one at a time. Seline estimates her chances of scoring a bullseye with any one throw of a dart to be 0.2. Find the probability that:

(a) Seline scores three bullseyes in her turn

(b) she scores two bullseyes only

(c) she scores at least one bullseye.

9 Two fair six-sided dice are rolled. Let T represent the event 'a 3 is rolled'.

(a) What would not T represent?

(b) Draw a probability tree diagram to illustrate this situation.

(c) Find the probability of:

(i) rolling two 3s **(ii)** not rolling any 3s **(iii)** rolling one 3 only

(iv) rolling at least one 3 **(v)** rolling no more than one 3.

10 A card is selected from a standard pack of 52 playing cards and then replaced. Another card is selected. Find the probability that:

(a) both cards are Aces

(b) the first card is a Club and the second card is red

(c) both cards are of the same colour

(d) one card is red and the other is black.

11 Wesley thinks his lucky number is 5. At the races, he decides to bet on horse number 5 in the first three events. In race number 1, there is a field of 12 horses. In race number 2, there is a field of 9 horses. In race number 3, there is a field of 10 horses.

(a) If each horse in an event has an equal chance of winning, draw a probability tree diagram to represent Wesley's chances of winning or losing on each race.

(b) Find the probability that:

(i) Wesley wins all three racing events

(ii) only one of his chosen horses wins

(iii) at least one of his chosen horses wins.

Reasoning, communicating and open-ended

12 A mathematics quick quiz consists of five multiple-choice questions, each with five choices for the solution. Draw up a table to show the probability of obtaining each of the totals of correct answers from 0 to 5 if you randomly guess the answers.

13 A fair six-sided die is rolled until a 6 is obtained.

(a) Copy and complete the following table, which shows the probability that this will take the indicated number of rolls. Round your answers to three decimal places.

Number of rolls	1	2	3	4	5	6
Probability						

(b) Add the probabilities you found in part **(a)**. What sum do you get?

(c) Explain what this answer means.

14 A party game for 29 people starts by assigning everyone a number from 1 to 29. In the first round, 7 numbers are drawn at random and those people eliminated. In the second round 6 people are eliminated, the third round eliminates 5 people, then 4, 3, 2 and finally 1, to leave the winner.

(a) Find P(being eliminated in round 1).

(b) Assuming you are not eliminated in the first round, find P(being eliminated in round 2).

(c) Assuming you are not eliminated in the previous rounds, find the probability of being eliminated in each round.

(d) Add the results for the probability of being eliminated in each round. Explain the result.

15 A set of volleyball is won when one side reaches 25 points. If we knew nothing about the ability of the teams, we would assign the chance of winning as 0.5 each. If, however, one side was leading 15–0, we may rethink the original assignment of 0.5 each. Estimate the new probability for each team to win the set. Justify your answer.

16 Draw up a probability tree diagram that finishes with at least 9 branches where one of the final branches has a probability of $\frac{1}{9}$ and another has a probability of $\frac{1}{8}$. Explain what your tree diagram represents.

17 The estimated probability of a 25-year-old male playing cricket is 0.2 and of a 25-year-old male playing golf is 0.18. Armapreet calculates that the probability of a man playing cricket or golf must be 0.38. Is she right? If not, what mistake has she made?

Problem solving

The origins of probability theory

The origins of modern probability theory involve a problem known as the 'problem of the points' or 'division of the stakes'. It involves a two-person game where each has made an equal contribution to the prize pot and the agreement is that the first person to get to a certain number of points will be the winner and collect the entire prize pot. The problem arises when the game is interrupted, and cannot be concluded. How should the prize pot be divided?

Let's say the winner is the first to 10 points.

1 How would you divide the pot if the score was 7 to 5?

2 One of the early solutions suggested that the pot should be divided in proportion to the number of games won. Using this solution, the pot above would be divided $\frac{7}{12} : \frac{5}{12}$. What do you think of this solution? What would happen if the score was 1 to 0? Pascal and Fermat thought the crucial issue was the possible ways the game could have continued rather than what had already happened.

3 Think of a game where the score is 8 to 7. Write all the ways the game could continue and use this to decide how to divide the pot.

4 What if the score was 2 to 5? What is the problem with this sort of score?

5 What do you think would be the easiest scores to deal with if the game was interrupted?

Strategy options

- Act it out.
- Draw a diagram.

Extension

Pascal and Fermat are considered to have solved this problem. Conduct some research to discover how they resolved the issue.

Conditional statements

11.5

Need to Know

Stage 5.2

Identifying conditional statements

When we are dealing with a **conditional statement** we are given information that has restricted the universal set. Words such as 'given', 'knowing that' and 'if … then' are indications of conditional statements.

Consider rolling two fair six-sided dice, one green and one red. All outcomes are listed in the array shown. There are 36 equally likely outcomes. They are equally likely as each die is assumed to be fair.

		Green die					
		1	2	3	4	5	6
Red die	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

So, $P(1, 1) = \frac{1}{36}$.

If the number on the green die is known to be 1, then $P(1, 1)$ would be $\frac{1}{6}$ because only the first column of results is possible from our array. The universal set has been restricted in this case, so effectively only the outcomes that have a 1 on the green die are considered.

$P(1, 1, \text{ given the green die is } 1) = \frac{1}{6}$.

The word 'given' implies a conditional statement. The information after the word 'given' is used to restrict the universal set. The number of outcomes in the restricted set becomes the denominator for the probability fraction.

The order of the statements is important.

For example:

		Green die	
		3	6
Red die	1	(3, 1)	(6, 1)
	2	(3, 2)	(6, 2)
	3	(3, 3)	(6, 3)
	4	(3, 4)	(6, 4)
	5	(3, 5)	(6, 5)
	6	(3, 6)	(6, 6)

		Green die					
		1	2	3	4	5	6
Red die	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Using the arrays above,

$P(\text{red die is even given green die is a multiple of 3}) = \frac{1}{2}$, whereas

$P(\text{green die is a multiple of three given the red die is even}) = \frac{1}{3}$.

Venn diagrams and tree diagrams can also be used to find conditional probability.

Worked Example 9

WE 9

A 16-sided die is rolled, with faces numbered 1–16. The following events are defined.

A = 'numbers less than 7' and B = 'multiples of 4'

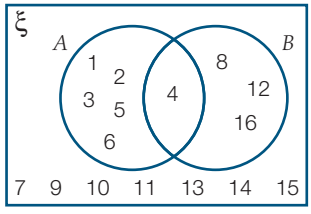
(a) Draw a Venn diagram to show this information.

(b) Calculate P(a number less than 7 given it is a multiple of 4).

(c) Calculate P(a multiple of 4 given it is a number less than 7).

Working

(a)



(b) The denominator is the number of elements in B (a multiple of 4): 4.

The numerator is the number of elements in B and in A (a number less than 7): 1.

P(a number less than 7 given it is a multiple of 4) $= \frac{1}{4}$.

(c) The denominator is the number of elements in A (a number less than 7): 6.

The numerator is the number of elements in A and in B (a multiple of 4): 1.

P(a multiple of 4 given it is a number less than 7) $= \frac{1}{6}$.

Worked Example 10

WE 10

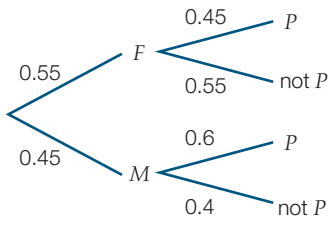
In a particular electorate, 55% of the voting population is female. It is also known that 60% of male voters support the Progressive Party, whereas only 45% of female voters support this party. Find the probability that a randomly selected voter from that electorate is male if you know that they support the Progressive Party. Round your answer to four decimal places.

Working

F = 'female'

M = 'male'

P = 'Progressive Party supporter'



$P(FP) = 0.55 \times 0.45$
$= 0.2475$

$P(MP) = 0.45 \times 0.6$
$= 0.27$

P(support Progressive Party)

$= P(FP) + P(MP)$

$= 0.2475 + 0.27$

$= 0.5175$

P(male and support Progressive Party)

$= P(MP)$

$= 0.27$

P(male given they support Progressive Party)

$= P(M \text{ given } P)$

$= \dfrac{0.27}{0.5175}$

$= 0.5217$

11.5 Conditional statements

Answers page 714

Stage 5.2

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q9 (a)–(c), Q11	Q1, Q2, Q4, Q5, Q7, Q9 (a)–(c), Q11	Q1, Q2, Q4, Q5, Q7, Q8, Q9, Q10, Q11

Fluency and understanding

1 **(a)** A 16-sided die is rolled, with faces numbered 1–16. The following events are defined. WE 9

A = 'numbers less than 9' and B = 'multiples of 3'

(i) Draw a Venn diagram to show this information.

(ii) Calculate P(number less than 9 given it is a multiple of 3).

(iii) Calculate P(multiple of 3 given it is a number less than 9).

(b) A 16-sided die, with faces numbered 1–16, is rolled and the following events are defined.

A = 'numbers greater than 5' and B = 'prime numbers'

(i) Draw a Venn diagram to show this information.

(ii) Calculate P(number greater than 5 given it is a prime number).

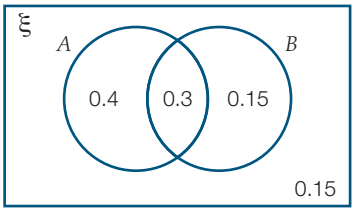
(iii) Calculate P(prime number given it is a number greater than 5).

2 In a particular electorate, 65% of the voting population are pensioners. It is also known that 55% of non-pensioners support the Progressive Party, whereas only 40% of pensioners support this party. Find the probability that a randomly selected voter from that electorate is a pensioner if you know that they support the Progressive Party. Round your answer to four decimal places. WE 10

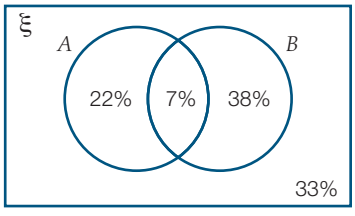
3 Use the information in the Venn diagram to find:

(i) $P(A \text{ given } B)$ **(ii)** $P(B \text{ given } A)$

(a)



(b)



4 A fair (unbiased) coin is flipped three times.

(a) The probability that the third flip results in a head, given the previous two flips were both heads, is:

A $\frac{1}{8}$ **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** 1

(b) The probability that the third flip results in a head, given at least two of the three flips are heads, is:

A $\frac{3}{8}$ **B** $\frac{1}{2}$ **C** $\frac{2}{3}$ **D** $\frac{3}{4}$

5 You have a reduced pack of playing cards consisting only of the 26 red cards. A card is selected at random from the pack. Find the probability that the card is:

(a) an Ace given it is a Heart **(b)** a Heart given it is a picture card (J, Q, K)

(c) the five of Diamonds given it is a Diamond **(d)** the five of Diamonds given it is a five.

6 A new disease is known to infect 2.5% of the population. A test for the disease has been developed, but it is not completely accurate. It correctly identifies the disease in infected patients 95% of the time, but also (incorrectly) identifies the disease in 7% of the uninfected population.

(a) Draw a tree diagram to show this information.

(b) Find P(have disease given test positive), rounding your answer to four decimal places.

(c) Find P(don't have disease given test negative), rounding your answer to four decimal places.

7 A sportswear company obtains its shoes from two suppliers, A and B. It is known that 5% of the shoes from supplier A are defective, whereas the figure is 9% for supplier B. Artemis obtains 65% of its shoes from supplier A. Where necessary, round your answers to four decimal places.

(a) Draw a tree diagram to show this information.

(b) Find $P(A$ given the shoe is defective).

(c) Find $P(B$ given the shoe is not defective).

8 A university student is studying both Economics and Asian Studies. The probability that she will pass Economics is 0.95 and that she will pass Asian Studies is 0.8. There is a 0.75 chance she will pass both. Find:

(a) P(passes Economics given passes Asian Studies)

(b) P(passes Asian Studies given passes Economics)

(c) P(passes Economics given passes only one of the subjects)

Reasoning, communicating and open-ended

9 Abbud plays a lot of table-tennis. He has a 70% chance of winning any particular game. Find the probability that Abbud wins a match decided by the:

(a) best of three games

(b) best of three games, given that he won the first game

(c) best of three games, given that it lasts three games

(d) best of five games

(e) best of five games, given that he won the first two games.

10 A new cola is produced which the maker claims has the same taste as the market-leading cola. A test is set up whereby people are offered a drink and they say whether it is the market leader or the new cola. The new cola is correctly identified 40% of the time and the market leader is correctly identified 75% of the time. If a person has an equal chance of being given either cola and they say the cup contains the market leader, what is the probability that this is true?

11 A tired teacher wrote this question:

Given $P(A) = 0.6$ and $P(B) = 0.4$, find $P(A$ given $B)$.

(a) What is wrong with the question?

(b) Rewrite the question so it is not wrong, and then find the answer.

Dependent and independent events

11.6

Stage 5.2

Need to Know

Independent events

Two events A and B are **independent events** if the result of one event has no effect on the result of the other event.

When A and B are independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

For example:

Consider the situation where a fair coin is tossed and a spinner that is equally divided into three sections labelled 1, 2 and 3 is spun. The outcomes are equally likely as the coin is assumed to be fair and the spinner is equally divided.

$P(\text{H}) = \frac{1}{2}$ $\quad P(1) = \frac{1}{3}$ $\quad P(\text{H}, 1) = \frac{1}{6}$

$$\begin{aligned} P(\text{H}) \times P(1) &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \\ &= P(\text{H}, 1) \end{aligned}$$

$P(\text{H}, 1)$ is the product of $P(\text{H})$ and $P(1)$ because the events are independent.

Sampling with replacement is a common example of an independent event.

Dependent events

Dependent events are influenced by the outcome of another event; sampling without replacement is an example of a dependent event.

For example:

A bag contains 5 red and 5 green marbles. Two marbles are selected without replacement. Find the probability that both are red.

$$P(\text{Red 1st draw}) = \frac{5}{10}$$

$$P(\text{Red 2nd draw}) = \frac{4}{9}$$

$$\begin{aligned} P(\text{Red, Red}) &= \frac{5}{10} \times \frac{4}{9} \\ &= \frac{2}{9} \end{aligned}$$

Worked Example 11

WE 11

Two representatives, one girl and one boy, are chosen at random from a group of 6 girls and 4 boys. If Linda and Adam are in this group, what is the probability that both Linda and Adam will be chosen?

Working

Let L = Linda chosen
and A = Adam chosen

L and A are independent events.

$P(L \text{ and } A) = P(L) \times P(A)$

$P(L) = \frac{1}{6}$

$P(A) = \frac{1}{4}$

$P(L \text{ and } A) = \frac{1}{6} \times \frac{1}{4}$

$= \frac{1}{24}$

The probability that both Linda and Adam will be chosen is $\frac{1}{24}$ (i.e. 1 chance in 24).

Worked Example 12

WE 12

Two cards are drawn out of a 52-card deck of playing cards without replacement.

(a) Find the P(drawing 2 Kings).

(b) Find the P(both cards are from the same suit).

Working

(a) $P(\text{King on 1st draw}) = \frac{4}{52}$

$= \frac{1}{13}$

$P(\text{King on 2nd draw}) = \frac{3}{51}$

$= \frac{1}{17}$

$P(\text{both cards being a King}) = \frac{1}{13} \times \frac{1}{17}$

$= \frac{1}{221}$

(b) $P(\text{Club, Club}) = \frac{13}{52} \times \frac{12}{51}$

$= \frac{1}{17}$

Similarly,
$P(\text{Spade, Spade})$
$= P(\text{Heart, Heart})$
$= P(\text{Diamond, Diamond})$
$= \frac{1}{17}$

$P(\text{both cards are from the same suit})$

$= \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17}$

$= \frac{4}{17}$

11.6 Dependent and independent events

Answers page 714

Navigator

Stage 5.2

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q15, Q18, Q19	Q1, Q2, Q3, Q5, Q7, Q8, Q10, Q12, Q13, Q14, Q15, Q17, Q18	Q1, Q2, Q4, Q6, Q7, Q9, Q10, Q11, Q13, Q14, Q16, Q17, Q19

Fluency and understanding

1 Katrina finds that on average her train is late 2 mornings out of 5, whereas Frank's bus is late 1 morning in every 4. Find the probability that on any given day both Katrina and Frank are late. WE11

2 A box containing 40 apples, 10 of each of 4 different varieties, has 5 rotten ones. If 2 apples are taken out without replacement, calculate: WE12

(a) P(2 rotten apples) **(b)** P(2 apples of the same variety)

3 Phillip has 2 pairs of blue and 1 pair of black jeans in his wardrobe. In a drawer, he has 1 red, 3 black and 5 white T-shirts. If he randomly chooses a pair of jeans and a T-shirt to wear, find the probability that he chooses:

(a) a white T-shirt and blue jeans

(b) a black T-shirt and black jeans

(c) black jeans and a T-shirt that is not black

(d) blue jeans and a T-shirt that is not red.

4 In a group of 20 people, 2 have red hair, 6 are blonde and the rest have dark hair. Of the group, 15 people are right-handed. If a person is chosen at random from the group, find the probability that the person:

(a) has red hair and is right-handed

(b) is blonde and is left-handed

(c) is right-handed and does not have red hair.

5 Label each of these pairs of events as either independent (I) or dependent (D).

(a) Tossing a coin and rolling a die.

(b) Drawing two cards from a pack without replacing the first before drawing the second.

(c) Drawing two cards from a pack replacing the first before drawing the second.

(d) Raining in two cities 100 km apart on the same day.

(e) Hours of study and test result.

(f) The number of hours of practice a learner driver does before getting their license and the chance of having an accident in the first month of having their license.

6 An urn contains 3 red beads, 4 blue beads and 8 green beads. If two beads are drawn without replacement, find:

(a) P(RR) **(b)** P(BB) **(c)** P(GG) **(d)** P(RB)

(e) P(BG) **(f)** P(GB) **(g)** P(same colour) **(h)** P(different colours)

7 **(a)** If a 20-cent coin is tossed in the air twice, the probability of obtaining two heads is:

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$

(b) If a six-sided die is rolled twice, the probability of obtaining a double 6 is:

A $\frac{1}{3}$ **B** $\frac{1}{6}$ **C** $\frac{1}{12}$ **D** $\frac{1}{36}$

(c) If a 20-cent coin is tossed and a six-sided die is rolled, the probability of getting a head and a multiple of 3 is:

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$

8 Joshua and Tania are competing in different sections of a chess tournament. Joshua estimates that his chance of winning against his opponent is 3 in 5, whereas Tania estimates that her chance of winning against her opponent is 5 in 7. Find the probability that:

(a) both Joshua and Tania win their chess game

(b) they both lose their chess game

(c) Joshua loses his game but Tania wins hers.

9 A drawer contains 6 black socks, 8 red socks and 4 white socks. If two socks are drawn without replacement, find:

(a) P(2 black socks)

(b) P(2 red socks)

(c) P(2 white socks)

(d) P(a pair of socks of matching colour)

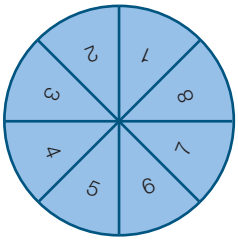
(e) P(two odd socks)

(f) P(one black sock)

(g) P(at least one black sock)

(h) P(no white socks)

10 A spinner numbered 1 to 8 is spun twice. Find the probability that:



(a) a 5 is obtained on the first spin

(b) a 5, then a 7, is spun

(c) a 5 is obtained on the first spin and a number less than 4 is obtained on the second

(d) an even number is obtained on the first spin and an odd number is obtained on the second.

11 If the probability that person A will be alive in 20 years is 0.7 and the probability that person B will be alive in 20 years is 0.5, find:

(a) the probability that they will both be alive in 20 years

(b) the probability that only one is alive in 20 years.

12 Natalie and her team play in a weekly squash competition. The probability that her team wins in any given week is 0.7. The outcome is not affected by any previous wins or losses. Find the probability that:

(a) Natalie's team does not win in any given week

(b) her team wins 2 weeks running

(c) her team loses 3 weeks running

(d) her team loses 4 weeks running.

13 A bag contains 6 white marbles, 5 black marbles and 4 green marbles. In each draw, a marble is drawn from the bag and not replaced. In three draws, find:

(a) P(WBG in that order)

(b) P(WBG in any order).

Reasoning, communicating and open-ended

14 Two normal unbiased six-sided dice are rolled. These events are defined:

A = rolling a 6 with the first die B = rolling a 5 with the second die

C = rolling a total of 8 D = rolling a total greater than 10

(a) Write the sample space for this experiment.

(b) Find the probability for each event.

(c) How many pairs of events are possible when order is not important? List them.

(d) How can you check which pairs are independent?

(e) Which, if any, of these pairs of events are independent?

15 You know that two events, A and B, are independent and that $P(A) = 0.3$ and $P(A \text{ and } B) = 0.15$.

(a) Find $P(B)$. **(b)** Now find $P(A \text{ or } B)$. Explain how you obtained this value.

16 A secondary school has 900 students, of whom 475 have a part-time job and 870 study mathematics. There are 470 mathematics students who have part-time jobs. If a student is picked at random, show that the events 'picking a student with a part-time job' and 'picking a maths student' are not independent.

17 For two events, A and B, we are given $P(A) = 0.35$ and $P(B) = 0.45$. Find $P(A \text{ and } B)$ if:

(a) A and B are mutually exclusive

(b) A and B are independent.

18 You have an unbiased ten-sided die, numbered 1 to 10. Write at least five questions related to independent events that deal with this die. Include the answers for your questions.

19 **(a)** Give two examples of pairs of dependent events.

(b) Give two examples of pairs of independent events.

Problem solving

Winning the lottery

In a particular raffle you purchase a ticket that has a unique six-digit number from 000001 to 999999 printed on it. The winning number is drawn at random and the ticket with that number printed on it wins the first prize. In this case, the prize is \$100 000. Any ticket that has the last five of the winning digits in order in the last five positions in their number wins second prize (\$20 000). Any ticket with the last four digits in order in the last four positions wins third prize (\$10 000), and any ticket with the last three digits in order in the last three positions wins fourth prize (\$2000). Each ticket can claim only one prize.

1 What is the probability that you win:

(a) \$100 000? **(b)** \$20 000?

(c) \$10 000? **(d)** \$2000?

2 What is the total payout on the lottery, assuming every prize is claimed?

3 How much do the tickets have to sell for so that the organisers make a profit?

Strategy options

- Break problem into manageable parts.
- Have I seen a similar problem?

Chapter review 11

D.I.Y. Summary

Key Words

Stage 5.2	addition law for mutually exclusive events	mutually exclusive
	complementary events	probability
	conditional statement	probability tree diagram
	dependent events	sample space
	disjoint sets	tree diagram
	events	two-way table
	independent events	universal set
	long-run proportion	Venn diagram

Stage 5.2 Q1–8

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 Every element under consideration in a particular question is called the _____.

2 If two events, A and B, are _____, then $P(A) \times P(B) = P(A \text{ and } B)$.

3 The events 'having brown hair' and 'not having brown hair' are an example of _____.

4 When we put the probabilities on each of the branches of a tree diagram we then have a _____.

5 Events that cannot occur at the same time are said to be _____.

6 A _____ or a _____ can be used to help find the sample space.

7 Spinning a spinner that has been equally divided into four pieces labelled 1, 2, 3 and 4, then calculating $P(1) + P(2) + P(3) + P(4) = 1$ is an example of the _____.

8 Words and phrases such as 'given', 'knowing that' and 'if … then' are indications we are dealing with a _____.

Stage 5.2 Q1–18

Fluency and understanding

1 A spinner numbered 1 to 5 is spun. Find: Ex. 11.1

(a) P(spinning a 2)
(b) P(not spinning a 2)
(c) P(spinning a number greater than 2)
(d) P(spinning an even number)
(e) P(spinning an odd number)
(f) P(spinning a number less than 5).

2 A box contains 15 blocks. Seven are red, five are white and the rest are blue. A block is chosen at random from the box. Find the probability that the block chosen is: Ex. 11.1

(a) blue
(b) not red
(c) red or blue.

3 A fair six-sided die is rolled. What is the probability of obtaining an even number or a 5? Ex. 11.1

4 A spinner numbered 1 to 4 is spun twice. **Ex.11.1**

(a) Represent all the possible outcomes using a two-way table.

(b) Find:

(i) P(spinning a pair of 3s) **(ii)** P(spinning only one 3)

(iii) P(not spinning any 3s) **(iv)** P(the sum of the two numbers obtained is 5).

5 A 50-cent coin is tossed twice. Find the probability of obtaining two tails. **Ex.11.1**

6 This two-way table shows the number of elements in each of three sets. **Ex.11.2**

	A and *B*	*A* and not *B*	not *A* and *B*	not *A* and not *B*	
C	1	4	3	2	**10**
not C	2	1	2	3	**8**
	3	**5**	**5**	**5**	**18**

Write down the number of elements in:

(a) the universal set **(b)** A

(c) A or B **(d)** B and C

(e) not B **(f)** not A and not B and not C.

7 A cutlery rack contains 6 knives, 5 forks and 2 spoons. **Ex.11.3**

(a) The probability that a piece of cutlery drawn at random will be a knife or a spoon is:

A $\frac{2}{13}$ **B** $\frac{2}{5}$ **C** $\frac{6}{13}$ **D** $\frac{8}{13}$

(b) The probability that a piece of cutlery drawn at random will not be a knife is:

A $\frac{6}{13}$ **B** $\frac{7}{13}$ **C** $\frac{6}{7}$ **D** $\frac{7}{6}$

8 Two cards are drawn, without replacement, from a set of 20 cards numbered 1 to 20. **Ex.11.4**

Let R = 'a prime number is drawn' and N = 'a prime number is not drawn'

(a) Draw a tree diagram to show this information.

(b) Find P(both cards show a prime number).

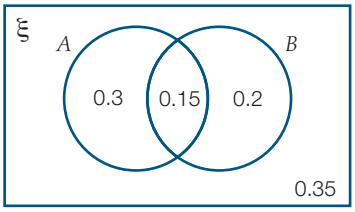
(c) Find P(no cards show a prime number).

(d) Find P(at least one card shows a prime number).

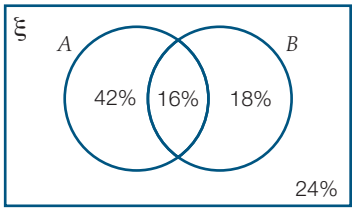
9 Use the information in each Venn diagram to find the following probabilities. Where necessary, give your answers correct to four decimal places. **Ex.11.5**

(i) $P(A \text{ given } B)$ **(ii)** $P(B \text{ given } A)$

(a)



(b)



10 John and Petula are in the same class at school and both have trouble getting to school on time. John is late 2 days out of 9, whereas Petula is late 3 days out of 11. What is the probability that they will both be late on any particular day? **Ex.11.6**

11 A fair seven-sided spinner, numbered 1 to 7, is spun. Find the probability of:

(a) obtaining a 5

(b) obtaining an even number

(c) obtaining a number less than 4

(d) not obtaining a 6

(e) obtaining a multiple of 3

(f) not obtaining a prime number.

12 In clay pigeon shooting, a clay block in the shape of a pigeon is used as a target for shooting. Jackie estimates her chance of hitting the moving clay target with one shot to be 0.75.

(a) What is the probability of her not hitting the target with one shot?

(b) Draw a probability tree diagram to represent the possible outcomes of shooting at three clay targets.

(c) Find the probability that Jackie hits:

(i) all three clay targets **(ii)** only one clay target **(iii)** at least one clay target.

13 In a survey of 60 people, it was found that 40 people liked lemonade, whereas 30 people liked cola. Two people said they disliked both.

(a) Present this information in a two-way table.

(b) If one person is chosen at random, find the probability that this person:

(i) likes lemonade

(ii) likes lemonade only

(iii) likes cola

(iv) likes cola only

(v) likes both lemonade and cola

(vi) does not like lemonade

(vii) likes lemonade or cola

(viii) likes neither cola nor lemonade.

14 These probabilities apply to eye colour for students at a particular school. The colours are all mutually exclusive.

$P(\text{green}) = \frac{3}{16}$ $P(\text{blue}) = \frac{1}{4}$ $P(\text{hazel}) = \frac{3}{8}$ $P(\text{brown}) = \frac{3}{16}$

A student is chosen at random. What is the probability that he or she has:

(a) blue or green eyes

(b) neither blue nor brown eyes

(c) blue, hazel or brown eyes

(d) neither hazel nor blue eyes?

15 Nerida and Ngaio play each other at badminton every week. The probability that Nerida wins any particular game is 0.65. Find, rounded to three decimal places where necessary, the probability that:

(a) Ngaio wins in any particular week

(b) Nerida wins for 3 weeks in a row

(c) Ngaio wins for 4 weeks in a row

(d) over a 6-week period the winning sequence is Nerida, Ngaio, Nerida, Ngaio, Nerida, Ngaio.

16 You toss three coins: a 10-cent, a 20-cent and a 50-cent. You note the results in that order.

(a) Find the probability of obtaining:

(i) three heads

(ii) two heads

(iii) one head

(iv) no heads.

(b) What do the answers in part **(a)** add up to? Is this an expected result? Explain your reasoning.

(c) Without doing any further calculations, explain how you can find the probability of each number of tails.

(d) Find the probability of getting:

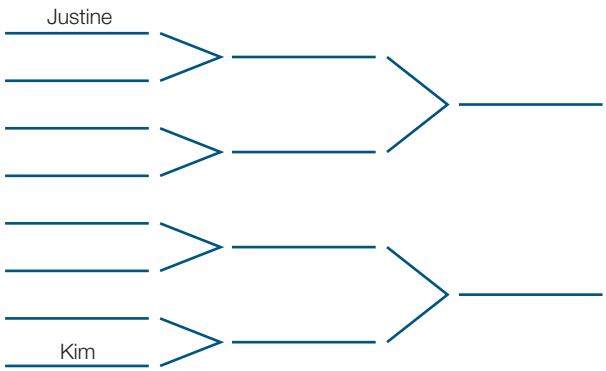
(i) HHT **(ii)** HTH **(iii)** THH.

(e) Write a statement about what you have found in part **(d)**.

Reasoning and communicating

17 Justine and Kim are entered for a tennis tournament that begins with eight players. The draw profile looks like the following (because they are so good, Justine and Kim are never scheduled to play until the final).

Justine and Kim are very confident players and they think they have a probability of $\frac{2}{3}$ of beating any other player, except for each other. If they play each other, Kim has a $\frac{3}{5}$ chance of winning. Find the following probabilities.



(a) The two girls play each other in the final.

(b) Kim wins the tournament (not necessarily against Justine).

(c) Justine wins the tournament (not necessarily against Kim).

18 A weather forecast predicting the chance of rain in the next 24 hours in some capital cities is recorded below.

City	Brisbane	Sydney	Canberra	Melbourne	Hobart	Adelaide	Perth	Darwin
Chance of rain	0.15	0.4	0.3	0.1	0.2	0.4	0.4	0.1

Assuming the events to be independent, find:

(a) P(rain in Melbourne and Brisbane)

(b) P(rain in Sydney and Perth)

(c) P(rain in Adelaide, Hobart and Perth)

(d) P(rain in Sydney and Canberra).

Long-term statistics are kept and found that the probability of rain in both Sydney and Canberra on the same day is actually 0.2.

(e) Explain the difference between your answer to **(d)** and the long-term probability.

12 Surds, indices and logarithms

P Recall 12

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet.

1 Write in mathematical symbols:

(a) the square root of 3

(b) the cube root of 5

(c) the number that when squared equals 7

(d) the number that when cubed equals 6

(e) the number that when raised to the fourth power equals 10.

2 Evaluate each of the following. Do not use a calculator.

(a) $\sqrt{4}$ **(b)** $\sqrt{36}$ **(c)** $\sqrt{9}$ **(d)** $\sqrt{1}$ **(e)** $\sqrt[3]{8}$

(f) $\sqrt[3]{64}$ **(g)** $\sqrt[3]{-27}$ **(h)** $\sqrt[3]{1}$ **(i)** $\sqrt[4]{16}$ **(j)** $\sqrt[5]{-1}$

3 Use a calculator to evaluate each of the following, correct to two decimal places where necessary.

(a) $\sqrt{11}$ **(b)** $-\sqrt{7}$ **(c)** $\sqrt[3]{15}$ **(d)** 34^2 **(e)** 3^9

4 Expand each of the following.

(a) $3(x + 4)$ **(b)** $-4(2y - 3)$

5 Simplify each of the following.

(a) $m^3 \times m^2$ **(b)** $3k^2p^4 \times 4kp^5$ **(c)** $\dfrac{x^3 \times x^6}{x^2}$

(d) $3x^0$ **(e)** $(m^3n)^4$ **(f)** $(x^{-3}y^2)^{-2}$

6 Expand and simplify each of the following.

(a) $(x - y)(x + y)$ **(b)** $(c - 9)(c + 9)$ **(c)** $(2p + 3)(2p - 3)$

Syllabus references

Number and Algebra: Indices

✓ Apply index laws to algebraic expressions involving integer indices **(Stage 5.2)**

Number and Algebra: Surds and Indices

✓ Define rational and irrational numbers and perform operations with surds and fractional indices **(ACMNA264) (Stage 5.3§)**

Number and Algebra: Logarithms

✓ Use the definition of a logarithm to establish and apply the laws of logarithms **(ACMNA265) (Stage 5.3#)**

✓ Solve simple exponential equations **(ACMNA270) (Stage 5.3#)**

Rational and irrational numbers

12.1

Stage 5.3§

Need to Know

Real numbers

Real numbers are numbers that can be shown on a continuous number line. They consist of all **rational numbers** combined with all **irrational numbers**.

Rational numbers

A rational number can be written as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Rational numbers include whole numbers, fractions and repeating decimals.

Irrational numbers

An irrational number cannot be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Irrational numbers include all surds and other special numbers, such as π. When represented in decimal form, an irrational number has an infinite number of non-repeating decimal places.

Surds

Surds are numbers that can only be expressed in exact form using the radical or root sign $\sqrt[n]{x}$. When $n = 2$, $\sqrt[n]{x}$ is written simply as $\sqrt{x}$. Note that $\sqrt[n]{x}$ represents a positive number. If a negative root is required, it must be written as $-\sqrt[n]{x}$.

Writing a number using a root sign does not, of itself, make the number a surd.

For example:

The square root of a perfect square, such as 36, will not be a surd as $\sqrt{36} = 6$.

The square root of a whole number that is not a perfect square is a surd. So $\sqrt{5}$ is a surd because 5 is not a perfect square. $\sqrt{5}$ cannot be written in fraction form, and if entered on a calculator gives 2.236 067..., a decimal with an infinite number of non-repeating decimal places.

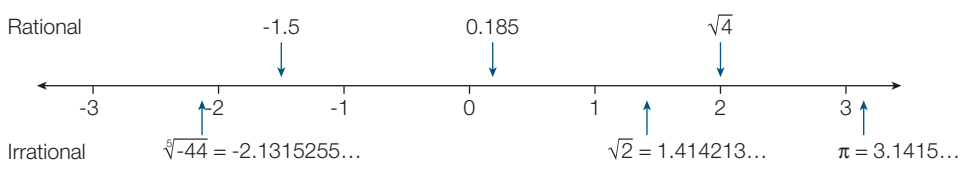
No number can be squared to make a negative number, so for $\sqrt{x}$ to exist, $x \geq 0$.

Surds are not just square roots of non-perfect squares. Cube roots or roots to any degree n are surds if the value inside the radical sign cannot be written as x^n.

For example:

$\sqrt[3]{8}$ is not a surd because 8 can be expressed as 2^3, but $\sqrt[3]{9}$ is a surd because 9 is not a perfect cube.

A calculator may be needed to identify large perfect squares, perfect cubes and higher order powers.



Rational numbers

- Integers

  $8 = \frac{8}{1}$, $\sqrt{4} = 2 = \frac{2}{1}$, $-15 = -\frac{15}{1}$

- Terminating decimals

  $0.8 = \frac{4}{5}$, $1.23 = \frac{123}{100}$, $0.0006 = \frac{6}{10\,000}$

- Recurring decimals

  $0.3\ldots = \frac{1}{3}$, $0.636\,363\ldots = \frac{7}{11}$

Irrational numbers

- Surds

  $-\sqrt{3}$, $\sqrt[3]{7}$, $4\sqrt{19}$

- Other infinite non-recurring decimals

  $\pi = 3.141\,592\ldots$

Recurring decimals

All recurring decimals can be written as fractions by multiplying the decimal by an appropriate power of 10 and subtracting the original from this new amount.

For example:

$0.\dot{8}$ can be multiplied by 10 to make $8.\dot{8}$. When the original is subtracted, $8.\dot{8} - 0.\dot{8} = 8$. This will be 9 times the original amount, so $0.\dot{8} = \frac{8}{9}$.

Worked Example 1

WE1

For each of the following numbers, state whether it is rational or irrational.

(a) $\sqrt{12}$

(b) $\sqrt{361}$

Working

(a) 12 is not a perfect square, so $\sqrt{12}$ is irrational.

(b) $361 = 19^2$, so $\sqrt{361}$ is rational.

Worked Example 2

WE2

Evaluate the following. Do not use a calculator.

(a) $-7\sqrt{81}$

(b) $3\sqrt{100} - 2\sqrt{16} + \sqrt{9}$

Working

(a) $-7\sqrt{81}$
$= -7 \times 9$
$= -63$

(b) $3\sqrt{100} - 2\sqrt{16} + \sqrt{9}$
$= 3 \times 10 - 2 \times 4 + 3$
$= 30 - 8 + 3$
$= 25$

Worked Example 3

WE 3

Use a calculator to evaluate each of the following, correct to three decimal places.

(a) $\sqrt{3} + \sqrt{5}$ **(b)** $-7 \times \sqrt[3]{9}$

Working

(a) $\sqrt{3} + \sqrt{5} \approx 3.968\,118\,785$
$= 3.968$ (3 d.p.)

(b) $-7 \times \sqrt[3]{9} \approx -14.560\,586\,76$
$= -14.561$ (3 d.p.)

Worked Example 4

WE 4

Convert the following recurring decimals to fractions in the form of $\frac{a}{b}$.

(a) $0.\dot{5}$ **(b)** $0.\dot{2}\dot{1}$ **(c)** $0.3\dot{2}$

Working

(a)
$x = 0.\dot{5}$
$10x = 5.\dot{5}$
$9x = 5$
$x = \frac{5}{9}$

(b)
$x = 0.\dot{2}\dot{1}$
$100x = 21.\dot{2}\dot{1}$
$99x = 21$
$x = \frac{21}{99}$
$= \frac{7}{33}$

(c)
$x = 0.3\dot{2}$
$10x = 3.\dot{2}$
$9x = 2.9$
$x = \frac{2.9}{9}$
$= \frac{29}{90}$

12.1 Rational and irrational numbers

Answers page 716

Navigator

Stage 5.3§

Q1 (a)–(f), Q2 (a)–(c), (g)–(i), Q3, Q4, Q5, Q6, Q7, Q8 (a), (d), (g), Q9 (a), (c), Q10, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9 (b), (c), Q10, Q11 (b), (c), Q14	Q1 (g)–(l), Q2 (d)–(f), (j)–(l), Q3, Q5, Q6, Q7, Q8, Q9 (c), (d), Q11, Q12, Q14

Equipment required: calculator for Questions **1**, **3**, **5** and **8**

Fluency and understanding

WE 1

1 For each of the following numbers, state whether it is rational or irrational.

(a) $\sqrt{5}$ **(b)** $\sqrt{36}$ **(c)** $-\frac{2}{7}$ **(d)** $\sqrt{10}$

(e) 4.6 **(f)** $\sqrt[3]{4}$ **(g)** $-\sqrt[3]{2}$ **(h)** $\sqrt[4]{1}$

(i) 2π **(j)** $\sqrt{20}$ **(k)** $-\sqrt{3}$ **(l)** 8.621

WE2

2 Evaluate the following. Do not use a calculator.

(a) $\sqrt{64}$ (b) $-\sqrt{25}$ (c) $-\sqrt{36}$

(d) $3\sqrt{9}$ (e) $7\sqrt{36}$ (f) $-4\sqrt{49}$

(g) $\sqrt{25+144}$ (h) $\sqrt{9}+\sqrt{16}$ (i) $7\sqrt{49}+4\sqrt{25}$

(j) $3\sqrt{16}+2\sqrt{81}$ (k) $5\sqrt{36}-7\sqrt{4}$ (l) $9\sqrt{100}-8\sqrt{16}+5\sqrt{9}$

WE3

3 Use a calculator to find the approximate value of each of the following surds, correct to three decimal places where necessary.

(a) $\sqrt{3}+\sqrt{2}$ (b) $\sqrt{8}-\sqrt{5}$ (c) $\sqrt{7}-\sqrt{13}$

(d) $\sqrt{35}$ (e) $-\sqrt{3}$ (f) $\sqrt[3]{17}$

(g) $\sqrt[3]{10}$ (h) $\sqrt{6}$ (i) $-\sqrt[3]{72}$

WE4

4 Convert the following recurring decimals to fractions in the form of $\frac{a}{b}$.

(a) $0.\dot{7}$ (b) $0.\dot{5}\dot{4}$ (c) $0.6\dot{5}$

(d) $0.0\dot{2}$ (e) $0.0\dot{7}\dot{2}9$ (f) $0.006\dot{3}4\dot{2}$

(g) $0.00\dot{6}$ (h) $0.\dot{8}14\dot{5}$ (i) $5.71\dot{6}$

5 (a) Which one of the following is a surd?

A $\sqrt{9}$ B $-\sqrt{1}$ C $\sqrt{8}$ D $\sqrt[3]{8}$

(b) Which one of the following is *not* a surd?

A $\sqrt[3]{7}$ B $\sqrt{11}$ C $\sqrt[3]{16}$ D $\sqrt{25}$

(c) Which one of the following is a rational number?

A $\sqrt{3}$ B $\sqrt[3]{2}$ C $\sqrt[3]{4}$ D $\sqrt{4}$

(d) Which one of the following is *not* a rational number?

A $\frac{1}{3}$ B 1.45 C 2.717 171 … D 3.721 452 …

6 (a) $4\sqrt{3}$ is the same as:

A $4+\sqrt{3}$ B $4-\sqrt{3}$ C $4\times\sqrt{3}$ D $\sqrt[4]{3}$

(b) $8\sqrt{16}$ is the same as:

A 8×4 B $8\div4$ C $8+4$ D 8×16

7 Do not use a calculator for this question.

(a) $\sqrt{7}$ is between $\sqrt{4}$ and $\sqrt{9}$, so $\sqrt{7}$ is between:

A 1 and 2 B 2 and 3 C 3 and 4 D 4 and 5

(b) $\sqrt{31}$ is between:

A 2 and 3 B 3 and 4 C 4 and 5 D 5 and 6

(c) $\sqrt[3]{11}$ is between:

A 1 and 2 B 2 and 3 C 3 and 4 D 4 and 5

8 Use a calculator to find the approximate value of each of the following, correct to three decimal places.

(a) $\sqrt{3+2}$ (b) $\sqrt{8-5}$ (c) $2\sqrt{6}+\sqrt{10}$

(d) $3\sqrt{2}+\sqrt{8}$ (e) $4\sqrt{5}-3\sqrt{6}$ (f) $6\sqrt{3}-6\sqrt{7}$

(g) $\sqrt[3]{-14}$ (h) $-\sqrt{43}$ (i) $2\sqrt{3}$

(j) $6\sqrt{5}$ (k) $-3\sqrt{11}$ (l) $5\times6\sqrt{5}$

Reasoning, communicating and open-ended

9 Find the exact side length of a square with area:

(a) $25\ m^2$ **(b)** $36\ cm^2$ **(c)** $46\ mm^2$ **(d)** $17\ m^2$

10 The area of a square vegetable garden is $60\ m^2$.

(a) What is the exact length of each side of the vegetable garden?

(b) If a fence of chicken wire is to be constructed around the vegetable garden, what length (to the nearest metre) of chicken wire is needed?

(c) Why do you think the measurement was to the nearest metre?

11 Find the exact side length of a cube with volume:

(a) $8\ m^3$ **(b)** $27\ m^3$ **(c)** $35\ cm^3$ **(d)** $15\ cm^3$

12 Two identical dice have a combined volume of $14\ cm^3$.

(a) Find the exact side length of each die.

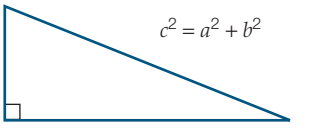
(b) What is the side length of each die to the nearest mm?

13 **(a)** Write three irrational numbers greater than zero.

(b) Write three irrational numbers less than zero.

(c) Draw a number line and mark the approximate places of these numbers.

14 Pythagoras' Theorem is used to determine the side lengths of right-angled triangles. Explain how to cut a piece of string exactly $\sqrt{2}$ m long using a pencil, a metre ruler and a piece of string.

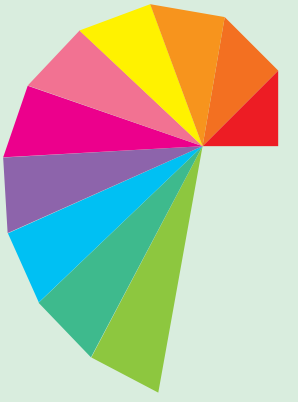


Problem solving

Surd shells

Equipment required: graph paper

A spiral can be created using straight lines to form a spiral made with ever increasing right-angled triangles. A very special spiral is created when starting with an isosceles right-angled triangle with equal sides of length 1 cm. Each subsequent triangle has a height of 1 cm and a base that is the length of the hypotenuse of the previous triangle.



(a) What is the perimeter of the tenth right-angled triangle?

(b) Calculate the exact perimeter of the entire spiral.

(c) Does the entire spiral have a rational or an irrational perimeter?

(d) Draw the spiral on graph paper. Measure the length of the perimeter of the entire spiral to check your answer to **(b)**.

(e) The first triangle has a rational area of $\frac{1}{2}\ cm^2$.

(i) Which is the next triangle to have a rational area?

(ii) Find this area and compare it to the area of the first triangle.

(iii) Find the next triangle to have a rational area.

(iv) Find this area and compare it to the area of the first triangle.

Strategy options

- Look for a pattern.
- Make a table.

12.2 Multiplying and dividing surds

Stage 5.3§

Need to Know

Multiplying surds

In general, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$, where a and b are positive numbers.

Two surds may be multiplied to produce a rational number.

For example:

$$\begin{aligned}\sqrt{5} \times \sqrt{20} &= \sqrt{5 \times 20}\\ &= \sqrt{100}\\ &= 10\end{aligned}$$

Surds should always be written in their simplest form. Surds can often be simplified by expressing the number under the root sign as a product of factors, one of which is a perfect square. Use the largest perfect square that is a factor.

For example:

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= \sqrt{4} \times \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

Squaring a square root

When a square root is multiplied by itself, the result is a rational number.

For example:

$\sqrt{5} \times \sqrt{5} = 5$

In general, $(\sqrt{a})^2 = \sqrt{a} \times \sqrt{a} = a$.

$\sqrt{12}$ is an **entire surd** because the entire number is under the square root sign. $2\sqrt{3}$ is considered to be simpler than $\sqrt{12}$ as it contains a smaller surd.

Sometimes it is necessary to convert a surd into an entire surd.

For example:

$$\begin{aligned}2\sqrt{6} &= \sqrt{4} \times \sqrt{6}\\ &= \sqrt{24}\end{aligned}$$

Dividing surds

In general, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$, where a and b are positive numbers and $b \neq 0$.

Multiplying and dividing surds

1. Group rational numbers together and surds together.
2. Multiply or divide rational numbers and multiply or divide surds.
3. Simplify surds if possible.

Worked Example 5

WE 5

Simplify:

(a) $\sqrt{2} \times \sqrt{12}$

(b) $4\sqrt{3} \times 5\sqrt{7}$

Working

(a) $\sqrt{2} \times \sqrt{12}$

$= \sqrt{2 \times 12}$

$= \sqrt{24}$

$= \sqrt{4 \times 6}$

$= \sqrt{4} \times \sqrt{6}$

$= 2\sqrt{6}$

(b) $4\sqrt{3} \times 5\sqrt{7}$

$= 4 \times 5 \times \sqrt{3} \times \sqrt{7}$

$= 20\sqrt{21}$

Worked Example 6

WE 6

Simplify:

(a) $\sqrt{21} \div \sqrt{7}$

(b) $\dfrac{12\sqrt{40}}{3\sqrt{5}}$

Working

(a) $\sqrt{21} \div \sqrt{7}$

$= \dfrac{\sqrt{21}}{\sqrt{7}}$

$= \sqrt{\dfrac{21}{7}}$

$= \sqrt{3}$

(b) $\dfrac{12\sqrt{40}}{3\sqrt{5}}$

$= \dfrac{12}{3} \times \sqrt{\dfrac{40}{5}}$

$= 4\sqrt{8}$

$= 4 \times \sqrt{4 \times 2}$

$= 4 \times \sqrt{4} \times \sqrt{2}$

$= 4 \times 2 \times \sqrt{2}$

$= 8\sqrt{2}$

Answers page 716

12.2 Multiplying and dividing surds

Stage 5.3§

Navigator

Q1 Column 1, Q2 Column 1, Q3 (a)–(f), Q4 (a)–(f), Q5 (a)–(c), Q6 Column 1, Q7 (a)–(f), Q8 (a), (b), Q10, Q12, Q13	Q1 Column 2, Q2 Column 2, Q3 (a)–(h), Q4 (d)–(l), Q5 (b), (c), (d), Q6 Column 2, Q7 every 2nd, Q8 (a), (c), Q10, Q13	Q1 Column 3, Q2 Column 3, Q3 (g)–(l), Q4 (g)–(l), Q5 (c), (d), (e), Q6 Column 3, Q7 (g)–(l), Q8 (c), (d), Q9, Q11, Q13

Fluency and understanding

WE5

1 Simplify each of the following.

(a) $\sqrt{3} \times \sqrt{2}$ (b) $\sqrt{11} \times \sqrt{5}$ (c) $\sqrt{7} \times \sqrt{10}$

(d) $\sqrt{7} \times \sqrt{7}$ (e) $\sqrt{4} \times \sqrt{4}$ (f) $\sqrt{16} \times \sqrt{4}$

(g) $\sqrt{36} \times \sqrt{2}$ (h) $\sqrt{25} \times \sqrt{7}$ (i) $\sqrt{2} \times \sqrt{9}$

(j) $2\sqrt{3} \times 5\sqrt{2}$ (k) $\sqrt{11} \times 4\sqrt{3}$ (l) $5 \times 8\sqrt{2}$

(m) $7\sqrt{5} \times \sqrt{5}$ (n) $\sqrt{6} \times 4\sqrt{6}$ (o) $5\sqrt{2} \times 4\sqrt{2}$

(p) $3\sqrt{7} \times 2\sqrt{7}$ (q) $5\sqrt{3} \times 3\sqrt{2}$ (r) $8\sqrt{6} \times 3\sqrt{12}$

WE6

2 Simplify each of the following.

(a) $\sqrt{\frac{27}{3}}$ (b) $\sqrt{\frac{32}{2}}$ (c) $\frac{\sqrt{125}}{\sqrt{5}}$

(d) $\frac{\sqrt{60}}{\sqrt{15}}$ (e) $\sqrt{72} \div \sqrt{2}$ (f) $\sqrt{45} \div \sqrt{5}$

(g) $\sqrt{7} \div \sqrt{7}$ (h) $\sqrt{\frac{64}{16}}$ (i) $\sqrt{25} \div \sqrt{9}$

(j) $\sqrt{\frac{100}{49}}$ (k) $\frac{\sqrt{4}}{\sqrt{81}}$ (l) $\sqrt{\frac{36}{121}}$

(m) $\sqrt{4} \div \sqrt{64}$ (n) $\sqrt{3} \div \sqrt{27}$ (o) $\frac{2\sqrt{36}}{3}$

(p) $\frac{5\sqrt{49}}{7}$ (q) $\frac{5\sqrt{9}}{3\sqrt{25}}$ (r) $\frac{6\sqrt{9}}{9\sqrt{36}}$

(s) $\frac{18\sqrt{75}}{3\sqrt{3}}$ (t) $5\sqrt{20} \div \sqrt{5}$ (u) $18\sqrt{48} \div 3\sqrt{8}$

(v) $\frac{5\sqrt{30}}{25\sqrt{6}}$ (w) $6\sqrt{14} \div 15\sqrt{2}$ (x) $12\sqrt{30} \div 18\sqrt{10}$

3 Simplify each of the following.

(a) $\sqrt{27}$ (b) $\sqrt{108}$ (c) $\sqrt{162}$ (d) $\sqrt{96}$

(e) $\sqrt{99}$ (f) $\sqrt{500}$ (g) $\sqrt{486}$ (h) $\sqrt{405}$

(i) $5\sqrt{8}$ (j) $-4\sqrt{24}$ (k) $-5\sqrt{63}$ (l) $4\sqrt{243}$

4 Express each of the following as an entire surd.

(a) $2\sqrt{3}$ (b) $3\sqrt{7}$ (c) $6\sqrt{2}$ (d) $3\sqrt{3}$

(e) $-10\sqrt{6}$ (f) $-2\sqrt{10}$ (g) $-8\sqrt{3}$ (h) $5\sqrt{7}$

(i) $8\sqrt{2}$ (j) $7\sqrt{5}$ (k) $-6\sqrt{3}$ (l) $-5\sqrt{11}$

Hint

Leave the negative sign outside the square root sign.

5 (a) $\sqrt{32}$ written in simplest form is:

A $2\sqrt{4}$ B $2\sqrt{8}$ C $4\sqrt{2}$ D $4\sqrt{8}$

(b) $\sqrt{3} \times \sqrt{7}$ simplifies to:

A $\sqrt{10}$ B $\sqrt{21}$ C $\sqrt{37}$ D $3\sqrt{7}$

(c) $\dfrac{\sqrt{49}}{\sqrt{4}}$ simplifies to:

A $\dfrac{2}{7}$ B $\dfrac{7}{4}$ C $\dfrac{7}{2}$ D 14

(d) $3\sqrt{5} \times 2\sqrt{6}$ simplifies to:

A $5\sqrt{11}$ B $6\sqrt{11}$ C $5\sqrt{30}$ D $6\sqrt{30}$

(e) $\dfrac{15\sqrt{6}}{3\sqrt{2}}$ simplifies to:

A $\sqrt{15}$ B $5\sqrt{3}$ C $5\sqrt{4}$ D $12\sqrt{3}$

6 Simplify each of the following.

(a) $3\sqrt{6} \times 5\sqrt{3}$ (b) $4\sqrt{2} \times 2\sqrt{14}$ (c) $-2\sqrt{6} \times 7\sqrt{10}$

(d) $-4\sqrt{8} \times 2\sqrt{10}$ (e) $-5\sqrt{3} \times \sqrt{15}$ (f) $10\sqrt{2} \times 3\sqrt{8}$

(g) $2\sqrt{12} \times 4\sqrt{3}$ (h) $2\sqrt{18} \times 3\sqrt{12}$ (i) $4\sqrt{27} \times 7\sqrt{28}$

(j) $-2\sqrt{20} \times -5\sqrt{75}$ (k) $-3\sqrt{72} \times -4\sqrt{63}$ (l) $\sqrt{54} \times \sqrt{162}$

7 Simplify each of the following.

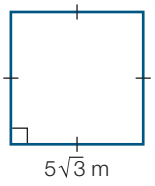
(a) $\sqrt{\dfrac{24}{3}}$ (b) $\sqrt{\dfrac{54}{2}}$ (c) $3\sqrt{\dfrac{90}{2}}$ (d) $\dfrac{-6\sqrt{120}}{\sqrt{5}}$

(e) $\dfrac{\sqrt{27}}{\sqrt{4}}$ (f) $\dfrac{\sqrt{32}}{\sqrt{25}}$ (g) $\dfrac{9\sqrt{8}}{\sqrt{9}}$ (h) $\dfrac{15\sqrt{28}}{2\sqrt{25}}$

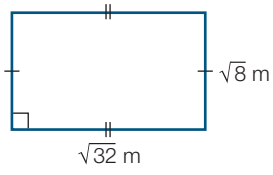
(i) $\sqrt{\dfrac{125}{36}}$ (j) $-3\sqrt{\dfrac{44}{81}}$ (k) $-2\sqrt{\dfrac{27}{100}}$ (l) $\dfrac{4}{5}\sqrt{\dfrac{147}{64}}$

8 Calculate the area of the following shapes, writing the answer as a surd in its simplest form.

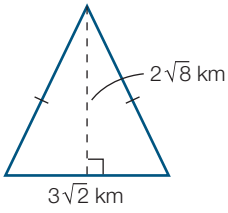
(a)



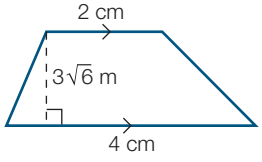
(b)



(c)



(d)



Hint

Area formulas:

Square: $A = l^2$

Rectangle: $A = lb$

Triangle: $A = \dfrac{bh}{2}$

Trapezium: $A = \dfrac{(a+b)h}{2}$

9 Given $\sqrt{a} \times \sqrt{b} = 3\sqrt{c}$, what are possible values of b and c if $a = 2$?

Reasoning, communicating and open-ended

Hint

Exact means write in surd form if necessary.

10 **(a)** Chris calculated the side of a triangle and found it was $\sqrt{41}$ mm. His friends' calculations gave lengths of $3\sqrt{5}$ mm and $2\sqrt{10}$ mm. If the actual length was 6 mm, which measurement was closest to this?

(b) If the area of a circular swimming pool is 45π m^2, find its *exact* circumference.

11 Adrianna has $\sqrt{50}$ m of tape that is to go across the two diagonals of rectangular windows with side lengths 40 cm and 35 cm.

(a) What is the length of the diagonal?

(b) How many windows can she tape? (She may have some tape left over.)

12 Find at least three surd calculations that simplify to $2\sqrt{6}$.

13 **(a)** Andrew multiplied two surds and got the answer 5. What could the two surds have been?

(b) Fiona divided two surds and got the answer 4. What could the two surds have been?

Adding and subtracting surds

12.3

Stage 5.3§

Need to Know

Like surds

Consider $3\sqrt{2} + 5\sqrt{2}$.

Notice that the surd $\sqrt{2}$ is contained in each term. This expression can be factorised by taking out the common factor of $\sqrt{2}$.

$$\begin{aligned} & 3\sqrt{2} + 5\sqrt{2} \\ = \; & 3 \times \sqrt{2} + 5 \times \sqrt{2} \\ = \; & \sqrt{2}(3+5) && \text{(take out common factor)} \\ = \; & \sqrt{2} \times 8 && \text{(simplify)} \\ = \; & 8\sqrt{2} \end{aligned}$$

So, $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

Numbers with the same surd are called *like surds*. To see whether there are like surds that can be added or subtracted, it may be necessary to express each term in simplest form. Only terms that contain like surds can be added or subtracted.

Terms that do not contain the same surd cannot be simplified. (The expression can be evaluated approximately using a calculator.)

Worked Example 7

WE 7

Simplify:

(a) $10\sqrt{3} - 4\sqrt{3}$

(b) $3\sqrt{5} + 2\sqrt{6} + \sqrt{5}$

Working

(a) $10\sqrt{3} - 4\sqrt{3}$

$= 6\sqrt{3}$

(b) $3\sqrt{5} + 2\sqrt{6} + \sqrt{5}$

$= 4\sqrt{5} + 2\sqrt{6}$

Worked Example 8 WE8

Simplify:

(a) $\sqrt{18}+\sqrt{8}$

(b) $2\sqrt{12}-\sqrt{20}-\sqrt{27}+2\sqrt{45}$

Working

(a) $\sqrt{18}+\sqrt{8}$

$=\sqrt{9\times 2}+\sqrt{4\times 2}$

$=3\sqrt{2}+2\sqrt{2}$

$=5\sqrt{2}$

(b) $2\sqrt{(12)}-\sqrt{20}-\sqrt{27}+2\sqrt{45}$

$=2\sqrt{4\times 3}-\sqrt{4\times 5}-\sqrt{9\times 3}+2\sqrt{9\times 5}$

$=2\times 2\sqrt{3}-2\sqrt{5}-3\sqrt{3}+2\times 3\sqrt{5}$

$=4\sqrt{3}-2\sqrt{5}-3\sqrt{3}+6\sqrt{5}$

$=4\sqrt{3}-3\sqrt{3}-2\sqrt{5}+6\sqrt{5}$

$=\sqrt{3}+4\sqrt{5}$

Answers page 717

12.3 Adding and subtracting surds

Stage 5.3§

Navigator

Q1 (a)–(c), (g)–(i), Q2 (a)–(c), (g)–(i), Q3, Q5 (a), (b), Q7, Q9, Q10, Q11, Q12	Q1 Column 1, Q2 Column 1, Q4, Q5 (b), (c), Q6 (a), Q7, Q9, Q11, Q12	Q1 Column 2, Q2 Column 2, Q4, Q5 (c)–(d), Q6, Q8, Q9, Q11, Q12

Fluency and understanding

WE7

1 Simplify each of the following.

(a) $3\sqrt{2}+7\sqrt{2}$

(b) $8\sqrt{5}-5\sqrt{5}$

(c) $11\sqrt{3}-6\sqrt{3}$

(d) $\sqrt{7}+7\sqrt{7}$

(e) $5\sqrt{3}-9\sqrt{3}$

(f) $12\sqrt{11}-35\sqrt{11}$

(g) $5\sqrt{2}-3\sqrt{2}+9\sqrt{2}$

(h) $8\sqrt{3}-9\sqrt{3}+\sqrt{3}$

(i) $7\sqrt{5}-13\sqrt{5}+4\sqrt{5}-\sqrt{5}$

(j) $10\sqrt{6}+3\sqrt{3}-5\sqrt{6}-8\sqrt{3}$

(k) $5\sqrt{2}-7\sqrt{5}+2\sqrt{5}+2\sqrt{2}$

(l) $7\sqrt{3}+6\sqrt{11}-8\sqrt{3}-10\sqrt{11}$

WE8

2 Simplify each of the following.

(a) $\sqrt{2}+\sqrt{18}$

(b) $\sqrt{28}-\sqrt{7}$

(c) $\sqrt{125}-2\sqrt{5}$

(d) $7\sqrt{18}+3\sqrt{50}$

(e) $5\sqrt{63}-7\sqrt{28}$

(f) $4\sqrt{90}-4\sqrt{160}$

(g) $\sqrt{40}+\sqrt{90}+\sqrt{160}$

(h) $2\sqrt{32}+7\sqrt{2}+4\sqrt{18}$

(i) $\sqrt{150}+5\sqrt{54}-6\sqrt{24}$

(j) $9\sqrt{48}-\sqrt{27}+2\sqrt{192}$

(k) $6\sqrt{80}-3\sqrt{180}-4\sqrt{405}$

(l) $\sqrt{8}+\sqrt{162}+\sqrt{18}+\sqrt{50}$

(m) $\sqrt{28}-\sqrt{112}+\sqrt{567}-\sqrt{175}$

(n) $2\sqrt{300}-6\sqrt{27}-3\sqrt{108}+4\sqrt{48}$

3 **(a)** $7\sqrt{3} - 2\sqrt{3}$ simplifies to:

A 5 **B** $5\sqrt{3}$ **C** $5\sqrt{6}$ **D** $9\sqrt{3}$

(b) $5\sqrt{5} - 3\sqrt{5}$ simplifies to:

A $2\sqrt{2}$ **B** $2\sqrt{5}$ **C** $2\sqrt{7}$ **D** $8\sqrt{12}$

(c) $5\sqrt{3} + 7\sqrt{3} + 3\sqrt{2}$ simplifies to:

A $15\sqrt{3}$ **B** $12\sqrt{6} + 3\sqrt{2}$ **C** $15\sqrt{8}$ **D** $12\sqrt{3} + 3\sqrt{2}$

4 **(a)** $\sqrt{3} + \sqrt{12}$ simplifies to:

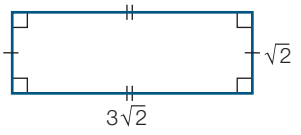
A $\sqrt{15}$ **B** $2\sqrt{3}$ **C** $3\sqrt{3}$ **D** $5\sqrt{3}$

(b) $5\sqrt{2} - 2\sqrt{8}$ simplifies to:

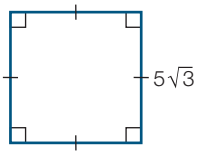
A $3\sqrt{-6}$ **B** $-3\sqrt{2}$ **C** 1 **D** $\sqrt{2}$

5 Calculate the perimeter of the following shapes, writing your answer as a surd in its simplest form.

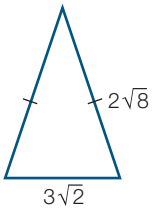
(a)



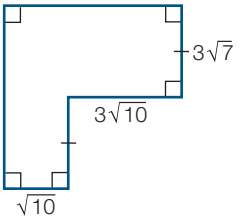
(b)



(c)

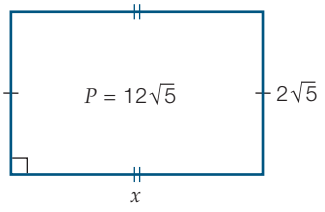


(d)

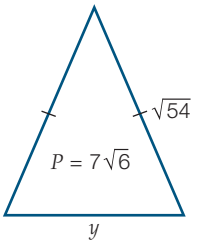


6 Find the value of the variable in each of the following shapes.

(a)



(b)



7 Find the exact area of a circle if its radius is $\sqrt{5}$ cm.

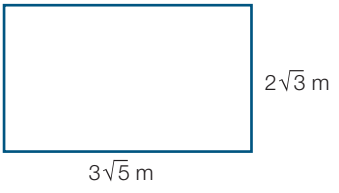
8 Find the exact area of a circular coin with diameter $\sqrt{3}$ cm.

Reasoning, communicating and open-ended

9 **(a)** Write down the perimeter of the rectangle shown.

(b) What is the exact area of this rectangle?

(c) If this rectangle represents the available floor space of a bedroom, can the floor be covered by a piece of carpet with area 22 m^2?



10 Make up at least three questions, using addition and/or subtraction, where $3\sqrt{5}$ is the answer.

11 A rectangle has an area of $\sqrt{96}$ cm^2. What might its perimeter be?

12 Ruby's answers to a surds quiz are shown.

(i) Find the perimeter of a triangle with side lengths $2\sqrt{3},\ 2\sqrt{3},\ 3\sqrt{3}$.

$$\begin{aligned} \text{Perimeter} &= 2\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} \\ &= 7\sqrt{9} \\ &= 7 \times 3 \\ \therefore P &= 21 \end{aligned}$$

(ii) The perimeter of a rectangle is $8\sqrt{6}$. If the length of one pair of sides is $\sqrt{6}$, find the length of the other pair.

$$\begin{aligned} \text{Length of other pair of sides} &= \frac{8\sqrt{6} - 2\sqrt{6}}{2} \\ &= \frac{6}{2} \\ \therefore l &= 3 \end{aligned}$$

(a) How would you explain to Ruby the mistake she made in each question?

(b) Write the correct answer for each question.

(c) Draw a diagram that could accompany each question.

Surds and the distributive law

12.4

Need to Know

Stage 5.3§

Consider the rectangle shown.

The side lengths AB and BC include surd terms. The two terms are not like, so they cannot be simplified to one term. To find the area of this rectangle, the length and breadth have to be multiplied:

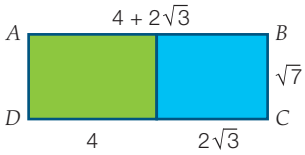
$A = lb$

$= (4 + 2\sqrt{3}) \times \sqrt{7}$ or $\sqrt{7}(4 + 2\sqrt{3})$

$= 4 \times \sqrt{7} + 2\sqrt{3}) \times \sqrt{7}$

$= 4\sqrt{7} + 2\sqrt{3 \times 7}$

$= 4\sqrt{7} + 2\sqrt{21}$ units^2



Using the distributive law

To simplify surd expressions involving brackets, the **distributive law** is used.

$a(b + c) = ab + ac$ or $(a + b)(c + d) = ac + ad + bc + bd$

Worked Example 9

WE 9

Expand and simplify:

(a) $2\sqrt{3}(5 - \sqrt{3})$

(b) $(2\sqrt{2} + 1)(\sqrt{6} - 4\sqrt{3})$

Working

(a) $2\sqrt{3}(5 - \sqrt{3})$

$= 10\sqrt{3} - 2\sqrt{9}$

$= 10\sqrt{3} - 2 \times 3$

$= 10\sqrt{3} - 6$

(b) $(2\sqrt{2} + 1)(\sqrt{6} - 4\sqrt{3})$

$= 2\sqrt{12} - 8\sqrt{6} + \sqrt{6} - 4\sqrt{3}$

$= 2 \times 2\sqrt{3} - 8\sqrt{6} + \sqrt{6} - 4\sqrt{3}$

$= 4\sqrt{3} - 8\sqrt{6} + \sqrt{6} - 4\sqrt{3}$

$= -7\sqrt{6}$

Answers page 717

12.4 Surds and the distributive law

Stage 5.3§

Navigator

Q1 Column 1, Q2, Q3, Q4, Q5, Q6, Q8, Q10	Q1 Column 2, Q2, Q3, Q5, Q6, Q8, Q9, Q10	Q1 Column 2, Q2, Q3, Q4, Q6, Q7, Q9, Q11

Equipment required: calculator for Questions **5**, **6**, **7**, **8** and **9**

Fluency and understanding

WE9

1 Expand and simplify each of the following.

(a) $3(\sqrt{7}+2)$

(b) $-4(\sqrt{11}+\sqrt{3})$

(c) $9(2\sqrt{5}-\sqrt{2})$

(d) $-\sqrt{7}(5\sqrt{2}-3\sqrt{5})$

(e) $4\sqrt{2}(2\sqrt{5}+6\sqrt{3})$

(f) $3\sqrt{5}(\sqrt{2}-7\sqrt{3}+3\sqrt{7})$

(g) $\sqrt{3}(\sqrt{3}+2)$

(h) $5\sqrt{7}(3\sqrt{2}-2\sqrt{7})$

(i) $-3\sqrt{5}(2-8\sqrt{3}+\sqrt{5})$

(j) $3\sqrt{2}(\sqrt{14}-\sqrt{8})$

(k) $4\sqrt{3}(\sqrt{3}-\sqrt{6}+5)$

(l) $-4\sqrt{5}(1-2\sqrt{5}+\sqrt{15})$

(m) $(\sqrt{5}+4)(\sqrt{2}+3)$

(n) $(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{7})$

(o) $(\sqrt{2}+\sqrt{3})^2$

(p) $(4\sqrt{5}-2\sqrt{2})^2$

(q) $(2\sqrt{7}+\sqrt{3})(2\sqrt{7}-\sqrt{3})$

(r) $(3\sqrt{5}-7\sqrt{2})(3\sqrt{5}+7\sqrt{2})$

2 **(a)** $5(2\sqrt{5}-\sqrt{2})$ is the same as:

A $10\sqrt{5}-\sqrt{10}$ **B** $10\sqrt{5}-5\sqrt{2}$ **C** $10\sqrt{5}-\sqrt{2}$ **D** $10\sqrt{3}$

(b) $4\sqrt{6}(\sqrt{2}-\sqrt{3})$ is the same as:

A $4\sqrt{8}-4\sqrt{18}$ **B** $8\sqrt{6}-12\sqrt{2}$ **C** $4\sqrt{8}-12\sqrt{2}$ **D** $8\sqrt{3}-12\sqrt{2}$

(c) $(2-\sqrt{5})(\sqrt{5}-1)$ expands to:

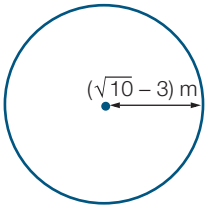
A $3\sqrt{5}+7$ **B** $3\sqrt{5}$ **C** $3\sqrt{5}-7$ **D** $3\sqrt{5}-3$

(d) $(\sqrt{7}+\sqrt{2})^2$ expands and simplifies to:

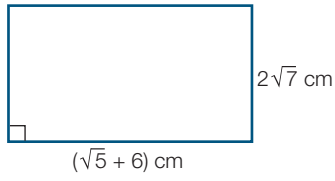
A $9-2\sqrt{14}$ **B** 9 **C** $9+2\sqrt{14}$ **D** $11\sqrt{14}$

3 Find the exact area of each shape.

(a)



(b)



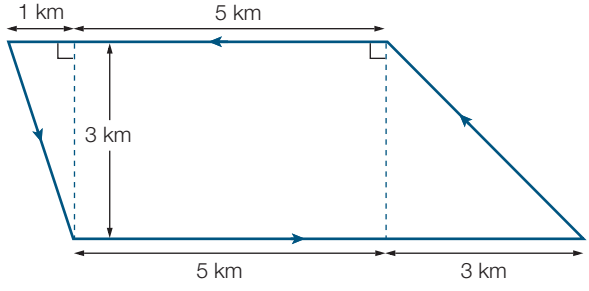
4 Find the exact area of a CD if it has a radius of $(4+\sqrt{5})$ cm.

5 On a walk around the city, Luke followed the path shown in the diagram, finishing back where he started.

(a) Find the area enclosed by his path.

(b) Find the exact distance he walked.

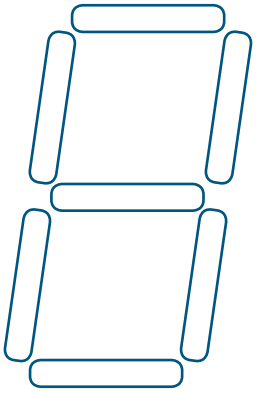
(c) State the distance he walked, to the nearest metre.



Hint

What shape can you see? Pythagoras' Theorem will help.

6 On a scoreboard, tube lighting is set up so that any digit from 0 to 9 can be displayed as shown in the diagram at right.



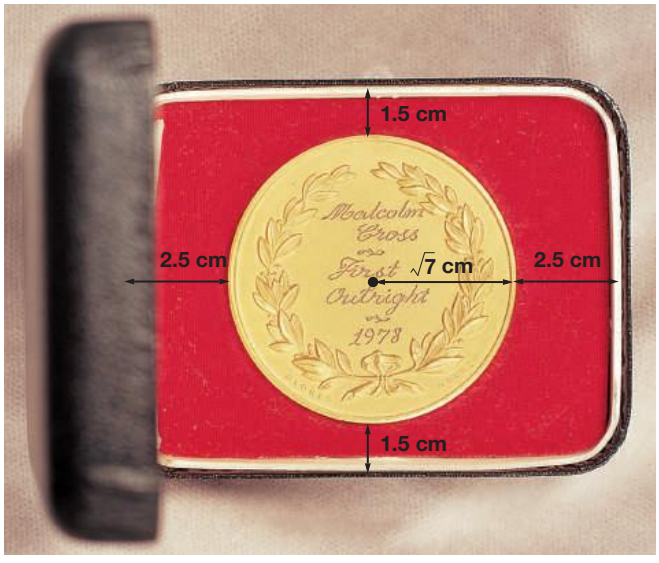
(a) If each individual section of tubing is $(3\sqrt{2}-2)$ m long, what exact length of tubing would be lit to produce the digit:

(i) 0 **(ii)** 4

(iii) 5 **(iv)** 8?

(b) Use your calculator to express each length found correct to two decimal places.

7 A sports medal is to be placed into a presentation box as shown. Give exact answers for parts **(a)** to **(d)**.



(a) What area of the base of the box does the medal take up?

(b) What are the dimensions (length and breadth) of the base of the box?

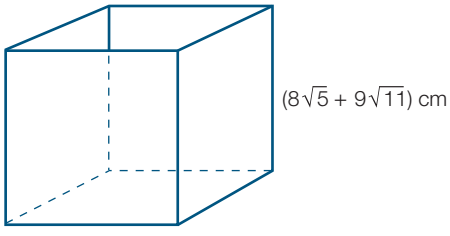
(c) What is the area of the base of the box?

(d) To stop the medal moving freely in the box, a thick layer of velvet will surround it. What area of velvet will be seen around the medal?

(e) Express your answer to part **(d)** correct to two decimal places.

Reasoning, communicating and open-ended

8 Zev has a toy box in the shape of a cube (open at the top). He wants to cover its surface (both inside and out) with adhesive-backed plastic and needs to know how much he should buy.



(a) What is the length of a side of the toy box, correct to one decimal place?

(b) Use your answer to part **(a)** to find the area to be covered with plastic. Assume that the box has walls of negligible thickness.

(c) Find the area to be covered with plastic using the value $8\sqrt{5}+9\sqrt{11}$ cm and keeping your answer as a surd expression.

(d) Use a calculator to express your answer to part **(c)** correct to one decimal place.

(e) Compare your answers to parts **(b)** and **(d)**. Which answer should Zev use when purchasing the plastic?

9 Square biscuits are to be cut out of a length of biscuit dough. Each biscuit has a line of chocolate icing running along the 6 cm diagonal of the square.

(a) Use Pythagoras' Theorem to find the exact side length of each biscuit.

(b) What exact length of biscuit dough would be needed to cut out a row of 50 biscuits? (Keep your answer in surd form.)

(c) Use a calculator to express your answer to part **(b)** correct to one decimal place.

(d) Use a calculator to express the side length of each biscuit correct to one decimal place.

(e) Use your answer to part **(d)** to calculate the length of biscuit dough needed to cut out a row of 50 biscuits.

(f) Compare your answers to parts **(c)** and **(e)**. If the biscuits are cut out by a machine that can be calibrated to $\frac{1}{100}$th of a centimetre, which answer should be used?

10 The area of a triangle is $\sqrt{48}$ cm^2. Draw at least two diagrams with different dimensions that could be the triangle.

11 The perimeter of a rectangle is given by $2(m\sqrt{n} + p)$.

(a) What might be the dimensions of the rectangle if the perimeter is greater than 20 m? Draw a diagram, showing lengths $m\sqrt{n}$ and p.

(b) Write an equation for the rectangle's area, using side lengths $m\sqrt{n}$ and p.

(c) State values of m, n and p so that the area of the rectangle is less than 10 m^2.

Problem solving

Connecting squares

The diagram below shows increasingly larger squares connecting to the smaller square along one edge.

(a) If the side lengths of the smallest squares are 1 cm, what is the area of the entire rectangle?

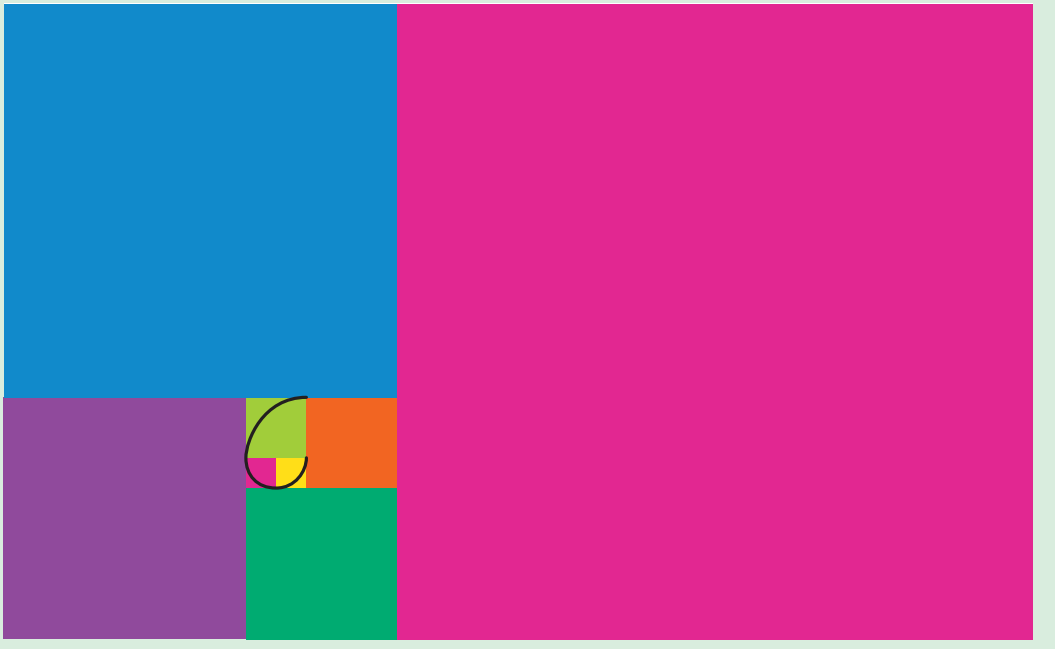
(b) This diagram is a spiral pattern that started with the yellow square. Complete the spiral pattern that has been started. If the pattern was continued, what would be the dimensions and area of the next square?

(c) Write down the side lengths of each square in ascending order. Have you seen this number pattern before? If so, what is its name?

(d) Explain the pattern and state the side lengths of the next five squares.

Strategy options

- Look for a pattern.
- Break problem into manageable parts.



Rationalising the denominator

12.5

Need to Know

Stage 5.3§

Rationalising the denominator of a fraction changes the denominator to a rational number but does not change the value of the fraction.

Rationalising a single surd term denominator

When the denominator is a single surd term, the numerator and denominator are multiplied by the smallest surd that will give a rational denominator.

Consider the number $\frac{1}{\sqrt{3}}$. The denominator $\sqrt{3}$ can be changed to a rational number if we multiply it by itself. The numerator must also be multiplied by the same number to keep the fraction equivalent.

For example:

$$\begin{aligned}\frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times 1\\ &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

So $\frac{1}{\sqrt{3}}$ can be written as $\frac{\sqrt{3}}{3}$ which has a rational denominator.

Using conjugate surds to rationalise denominators

When $\sqrt{a} + \sqrt{b}$ is multiplied by $\sqrt{a} - \sqrt{b}$, the result is a rational number.

Using the Difference of Two Squares rule:

$$\begin{aligned}(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) &= (\sqrt{a})^2 - (\sqrt{b})^2\\ &= a - b\end{aligned}$$

$\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are **conjugate surds**.

If the denominator has two terms where either or both contain surds, conjugate surds are used because their product does not contain any surd terms.

The denominator of these fractional surds is rationalised by multiplying the numerator and denominator by the conjugate of the denominator.

Worked Example 10

WE10

Rationalise the denominator of:

(a) $\dfrac{3\sqrt{5}}{\sqrt{6}}$

(b) $\dfrac{2\sqrt{3}}{\sqrt{20}}$

Working

(a) $\dfrac{3\sqrt{5}}{\sqrt{6}}$

$= \dfrac{3\sqrt{5}}{\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$

$= \dfrac{3\sqrt{30}}{6}$

$= \dfrac{\sqrt{30}}{2}$

(b) $\dfrac{2\sqrt{3}}{\sqrt{20}}$

$= \dfrac{2\sqrt{3}}{2\sqrt{5}}$

$= \dfrac{\cancel{2}\sqrt{3}}{\cancel{2}\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}}$

$= \dfrac{\sqrt{15}}{5}$

Worked Example 11

WE11

Rationalise the denominator of each of the following.

(a) $\dfrac{3}{3-\sqrt{2}}$

(b) $\dfrac{\sqrt{2}}{\sqrt{11}+3}$

Working

(a) $\dfrac{3}{3-\sqrt{2}}$

$= \dfrac{3}{(3-\sqrt{2})} \times \dfrac{(3+\sqrt{2})}{(3+\sqrt{2})}$

$= \dfrac{3(3+\sqrt{2})}{9-2}$

$= \dfrac{9+3\sqrt{2}}{7}$

(b) $\dfrac{\sqrt{2}}{\sqrt{11}+3}$

$= \dfrac{\sqrt{2}}{(\sqrt{11}+3)} \times \dfrac{(\sqrt{11}-3)}{(\sqrt{11}-3)}$

$= \dfrac{\sqrt{2}(\sqrt{11}-3)}{11-9}$

$= \dfrac{\sqrt{22}-3\sqrt{2}}{2}$

12.5 Rationalising the denominator

Answers page 718

Stage 5.3§

Navigator

Q1 (a)–(d), Q2, Q3, Q4, Q5, Q6, Q9, Q12	Q1 (c)–(f), Q2, Q3, Q4, Q5, Q6, Q7, Q11, Q12	Q1 (e)–(h), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11

Fluency and understanding

1 Express each of the following in simplest form with a rational denominator. WE 10

(a) $\frac{1}{\sqrt{7}}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{-12}{\sqrt{6}}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$

(e) $\frac{5\sqrt{11}}{\sqrt{5}}$ (f) $3\sqrt{\frac{5}{6}}$ (g) $-\frac{3}{4\sqrt{7}}$ (h) $\frac{6\sqrt{5}}{7\sqrt{3}}$

2 Express each of the following in simplest form with a rational denominator. WE 11

(a) $\frac{1}{3-\sqrt{7}}$ (b) $\frac{4}{2+\sqrt{3}}$ (c) $\frac{\sqrt{3}}{\sqrt{5}-\sqrt{2}}$ (d) $\frac{2}{2\sqrt{7}+5}$

(e) $\frac{6+\sqrt{3}}{2\sqrt{3}-3}$ (f) $\frac{\sqrt{5}+1}{7-3\sqrt{5}}$ (g) $\frac{\sqrt{6}-1}{3\sqrt{2}+\sqrt{7}}$ (h) $\frac{\sqrt{2}-3}{2\sqrt{3}+\sqrt{6}}$

3 (a) $\frac{5}{\sqrt{2}}$ expressed with a rational denominator is:

A $\frac{\sqrt{10}}{2}$ B $\frac{2\sqrt{5}}{5}$ C $\frac{5\sqrt{2}}{\sqrt{2}}$ D $\frac{5\sqrt{2}}{2}$

(b) $\frac{4}{3\sqrt{3}}$ expressed with a rational denominator is:

A $\frac{4\sqrt{3}}{9}$ B $\frac{\sqrt{12}}{9}$ C $\frac{4\sqrt{3}}{3}$ D $\frac{4\sqrt{3}}{27}$

(c) $\frac{2\sqrt{2}}{3\sqrt{7}}$ expressed with a rational denominator is:

A $\frac{2\sqrt{6}}{21}$ B $\frac{2\sqrt{14}}{21}$ C $\frac{14\sqrt{2}}{3}$ D $\frac{2\sqrt{14}}{3}$

4 (a) The conjugate of $2\sqrt{3}-5$ is:

A $2\sqrt{3}-5$ B $2\sqrt{3}+5$ C $\sqrt{3}+5$ D $5\sqrt{3}+2$

(b) To rationalise the denominator of $\frac{4+\sqrt{7}}{2-3\sqrt{3}}$ it is necessary to multiply by:

A $\frac{4-\sqrt{7}}{4-\sqrt{7}}$ B $\frac{1}{2+3\sqrt{3}}$ C $2+3\sqrt{3}$ D $\frac{2+3\sqrt{3}}{2+3\sqrt{3}}$

5 Simplify each of the following. (Rationalise each denominator first.)

(a) $\frac{1}{\sqrt{3}+1}+\frac{1}{3-\sqrt{5}}$ (b) $\frac{1}{\sqrt{7}+1}+\frac{1}{3-\sqrt{6}}$

(c) $\frac{1}{\sqrt{6}+\sqrt{3}}-\frac{1}{4-\sqrt{11}}$ (d) $\frac{1}{\sqrt{5}+\sqrt{3}}-\frac{1}{\sqrt{7}-2}$

Reasoning, communicating and open-ended

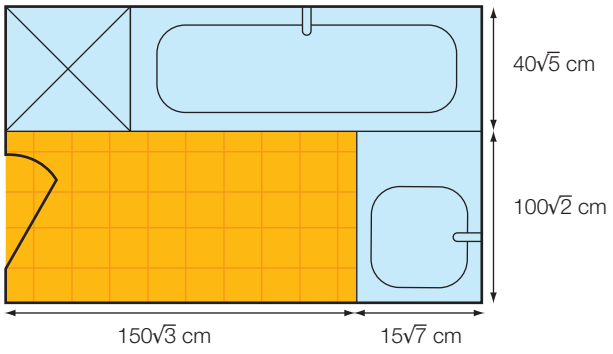
6 A piece of wood with length $5\sqrt{6}$ m is cut to make shelves for a bookcase with length $2\sqrt{2}$ m.

(a) Show $\dfrac{5\sqrt{6}}{2\sqrt{2}}$ as a fraction with a rational denominator.

(b) How many shelves can be cut from the wood?

(c) What is the exact length of wood left over?

7 The floor plan of a bathroom to be tiled is shown.



(a) Calculate the area to be tiled (shaded orange).

(b) The square tiles chosen have side lengths of $10\sqrt{3}$ cm. How many tiles are required for the bathroom floor? State the answer as a simplified surd with a rational denominator.

(c) How many tiles would you order?

(d) Find the exact area of the entire bathroom in square centimetres.

8 If $\dfrac{3+2\sqrt{3}}{4-\sqrt{2}}a = \dfrac{4+\sqrt{2}}{5-6\sqrt{3}}$, find the value of a, expressing your answer with a rational denominator.

9 Find at least three fractions that are equivalent to $\dfrac{4\sqrt{2}}{\sqrt{3}}$, one of which must have a rational denominator.

10 For each of the following, find two fractions that the stated number is between, with a rational denominator formed from the original fraction.

(a) $\dfrac{1}{\sqrt{3}}$ **(b)** $\dfrac{3}{\sqrt{7}}$ **(c)** $\dfrac{2\sqrt{3}}{\sqrt{5}}$ **(d)** $\dfrac{2}{3\sqrt{2}}$

11 Find at least two expressions that $2+\sqrt{3}$ can be multiplied by to give a rational answer.

12 Sura is trying to rationalise the denominator of the fraction $\dfrac{\sqrt{2}-3}{2\sqrt{3}+\sqrt{6}}$. She is struggling to understand why the denominator can be rationalised by multiplying both the numerator and denominator by the denominator's conjugate $2\sqrt{3}-\sqrt{6}$. How could you explain this to Sura?

$$\frac{\sqrt{2}-3}{(2\sqrt{3}+\sqrt{6})} \times \frac{(2\sqrt{3}-\sqrt{6})}{(2\sqrt{3}-\sqrt{6})}$$

Half-time 12

Stage 5.3§ Q1–8

1 Simplify each of the following. Ex.12.2

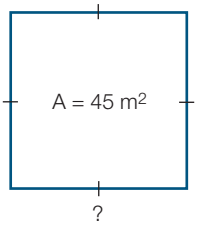
(a) $2\sqrt{3} \times \sqrt{3}$ **(b)** $4\sqrt{11} \times 2\sqrt{11}$ **(c)** $\sqrt{\dfrac{81}{16}}$

2 What is the area of a rectangular patio with dimensions of $(5 + \sqrt{3})$ m and $\sqrt{6}$ m? Ex.12.4

3 Express each of the following in simplest form with a rational denominator. Ex.12.5

(a) $\dfrac{\sqrt{5}}{\sqrt{6}}$ **(b)** $\dfrac{5}{2\sqrt{3}}$ **(c)** $\dfrac{3}{4 - \sqrt{11}}$

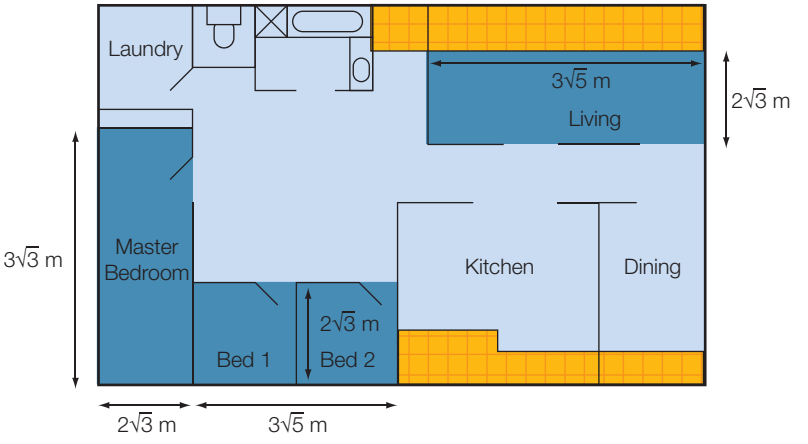
4 What is the exact side length of a square with an area of 45 m^2? Ex.12.1



5 For each of the numbers below, state whether it is rational (R) or irrational (I). Ex.12.1

(a) 3.656565… **(b)** 0.924 **(c)** $\sqrt[3]{9}$ **(d)** $-\sqrt{25}$

6 Calculate the exact total area to be carpeted in the bedrooms and the living room in the house below: Ex.12.4



7 Simplify each of the following. Ex.12.2

(a) $\sqrt{72}$ **(b)** $3\sqrt{20} \times 4\sqrt{45}$ **(c)** $\dfrac{3\sqrt{8}}{\sqrt{200}}$

8 Evaluate the following without the use of a calculator. Ex.12.3

(a) $-5\sqrt{9}$ **(b)** $\sqrt{81} + \sqrt{64}$ **(c)** $4\sqrt{36} - \sqrt{16} + 2\sqrt{49}$

12.6 Summary of index laws

Stage 5.2

Need to Know

$a^m \times a^n = a^{m+n}$	When multiplying numbers that are in index form and have the same base, keep the base and add the indices.
$\frac{a^m}{a^n} = a^{m-n}$	When dividing numbers that are in index form and have the same base, keep the base and subtract the indices.
$a^0 = 1$	Any number to the power of zero equals one, $a \neq 0$.
$(a^m)^n = a^{mn}$	When raising a number in index form to a power, keep the base and multiply the indices.
$(ab)^m = a^m b^m$	When a product in brackets is raised to a power m, each factor in the brackets is raised to the power m.
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	When a quotient in brackets is raised to a power m, both the numerator and the denominator are raised to the power m.
$a^{-m} = \frac{1}{a^m}$	A number raised to a negative power is the same as the reciprocal of the number raised to the positive of that power.

Simplifying indices

When performing calculations using indices, remember to follow these steps:

1. Cancel any numerical common factors outside brackets.
2. Expand brackets by multiplying indices.
3. Simplify terms with the same base using the index laws. With fractions:
   - multiply each numerator together and each denominator together by adding indices
   - divide terms with the same base by subtracting indices.
4. Apply any other index laws if necessary.
5. Express your answer with positive indices.

Worked Example 12

WE12

Simplify: $(a^4)^3b^7 \times (a^3b^{-4})^2$

Working

$$(a^4)^3b^7 \times (a^3b^{-4})^2$$
$$= a^{12}b^7 \times a^6b^{-8}$$
$$= a^{18}b^{-1}$$
$$= \frac{a^{18}}{b}$$

Worked Example 13

WE13

Simplify: $\left(\frac{m^2n}{m^5n^3}\right)^4 \div \frac{n^{-5}}{m^{12}}$

Working

$$\left(\frac{m^2n}{m^5n^3}\right)^4 \div \frac{n^{-5}}{m^{12}}$$
$$= \left(\frac{m^2n^1}{m^5n^3}\right)^4 \times \frac{m^{12}}{n^{-5}}$$
$$= \frac{m^8n^4}{m^{20}n^{12}} \times \frac{m^{12}}{n^{-5}}$$
$$= \frac{m^{20}n^4}{m^{20}n^7}$$
$$= m^0n^{-3}$$
$$= n^{-3}$$
$$= \frac{1}{n^3}$$

12.6 Summary of index laws

Answers page 718

Stage 5.2

Navigator

Q1 (a)–(e), (j), (l), Q2 (a), (c), (e), (j), Q3, Q4 (a), (b), (f), (h), Q5 (a), (c), (g), (i), Q6, Q7 (a), (d), (h), Q8, Q9, Q11	Q1 (c), (d), (f), (g), (m), (n), Q2 (b), (d), (f), (j), (k), Q3, Q4 (c), (e), (h), (i), Q5 (b), (e), (f), (j), (k), Q7 (b), (d), (f), (h), Q8, Q10, Q11	Q1 (d), (h), (i), (k), (n), (o), Q2 (g)–(l), Q3, Q4 (d), (g), (i)–(l), Q5 (d), (h)–(l), Q7 (c), (e), (g)–(i), Q8, Q10, Q11

Fluency and understanding

WE12

1 Simplify each of the following.

(a) $4x^7 \times 3x^{-2}$

(b) $a^3b^{-5} \times a^2b^2$

(c) $\frac{p^4q^6}{p^3q^9}$

(d) $\frac{(x^5)^{-2} \times x^8}{x^{-4}}$

(e) $(a^4)^3b^5 \times (a^2b)^4$

(f) $2(m^3)^2n^{-3} \times 3(mn)^4$

(g) $\frac{(p^2q^3)^{-1} \times p^5q^{-2}}{p^3q^2}$

(h) $\frac{(c^{-3}d^{-4})^2 \times c^2d^6}{c^5d^3}$

(i) $\frac{4(a^2b)^3 \times 3a^{-5}b^{-2}}{6ab}$

(j) $\frac{a^5b^{-2} \times a^2b^3}{a^4b^7 \times ab^{-4}}$

(k) $\frac{m^4n^{-3} \times m^5n^2}{mn^6 \times m^3n^{-2}}$

(l) $\frac{x^3y^{-6}}{x^{-4}y^7} \times \frac{x^{-2}y^4}{x^5y^{-9}}$

(m) $\frac{p^4q^{-2}}{p^2q^{-4}} \times \frac{pq^5}{p^3q^6}$

(n) $\frac{3k^2h^4}{4k^{-3}h^2} \times \frac{2k^{-1}h}{5kh^4}$

(o) $\frac{4a^4b}{9a^{-5}b^3} \times \frac{3a^{-2}b^5}{2ab^{-1}}$

WE13

2 Simplify each of the following.

(a) $\frac{k^9p^6}{k^4p^2} \div \frac{k^3p^2}{k^5p^4}$ (b) $\frac{x^6y^4}{x^3y^2} \div \frac{xy^5}{x^3y^8}$ (c) $\frac{6m^8n^6}{2m^4n^3} \div \frac{5m^3n^2}{2m^5n^4}$

(d) $\frac{2p^5q^3}{p^2q^{-4}} \div \frac{p^7}{q^4}$ (e) $\frac{7x^3y^5}{xy^{-3}} \div \frac{x^6}{y^2}$ (f) $\frac{m^{-4}n^{-2}}{m^5n^3} \div \frac{m^2n}{m^5n^4}$

(g) $\frac{k^{-5}p^2}{k^3p} \div \frac{k^5p^2}{k^6p^{-3}}$ (h) $\frac{2a^5b}{a^{-4}b^{-3}} \div \frac{3a^2b^6}{a^{-2}b}$ (i) $\frac{5f^4g^6}{f^{-3}g^{-1}} \div \frac{2f^2g^4}{f^{-4}g}$

(j) $\left(\frac{x^5}{y^2}\right)^2 \div \left(\frac{x^2}{y^2}\right)^4$ (k) $\left(\frac{4m^3n^4}{3mn}\right)^2 \div \frac{(2m^2n^3)^3}{3m^5n^3}$ (l) $\left(\frac{5p^2q^4}{2pq^3}\right)^2 \div \frac{(5p^3q^2)^3}{4p^7q^6}$

3 (a) $\frac{k^3 \times k^4}{k^9}$ is the same as:

A k^2 B $\frac{1}{k^2}$ C k^3 D $\frac{1}{k^3}$

(b) $(2x^3y^{-2})^3$ is the same as:

A $\frac{8x^9}{y^6}$ B $2x^9y^6$ C $8x^3y^6$ D $\frac{8}{x^9y^6}$

(c) $4(3a)^0$ is the same as:

A 0 B 1 C 4 D $12a$

4 Simplify each of the following.

(a) $\frac{(x^4y^{-2})^3 \times (xy^3)^2}{x^5y^{-4}}$ (b) $\frac{(a^3b^{-4})^5 \times (a^2b)^3}{a^9b^{-6}}$

(c) $\frac{(2k^{-3}p^2)^4 \times (k^2p^4)^{-2}}{(2kp)^2}$ (d) $\frac{(3m^4n^2)^4 \times (m^{-3}n^2)^{-2}}{(3mn)^2}$

(e) $\frac{(x^5y^{-3})^3}{(x^{-2}y)^2} \times \frac{(x^{-2}y^{-3})^2}{(x^4y^2)^{-3}}$ (f) $\left(\frac{m^3}{n^2}\right)^4 \times \left(\frac{m^{-2}}{n^4}\right)^3$

(g) $\left(\frac{p^2}{q^4}\right)^3 \times \left(\frac{p^{-3}}{q^5}\right)^2$ (h) $\left(\frac{f^{-3}}{g^2}\right)^{-4} \div \left(\frac{f^2}{g}\right)^3$

(i) $\left(\frac{a^{-2}b^2}{ab}\right)^3 \times \left(\frac{a^4b^{-3}}{a^{-1}b^2}\right)^2$ (j) $\left(\frac{m^{-3}n}{m^2n^2}\right)^2 \times \left(\frac{m^5n^{-2}}{m^{-3}n}\right)^3$

(k) $\frac{4x^3(y^{-2})^4}{(xy)^2} \times \left(\frac{x^{-4}y}{x^2y^{-3}}\right)^{-2}$ (l) $\left(\frac{3c^4d^3}{2cd}\right)^2 \div \frac{3cd}{(2c^{-2}d^3)^4}$

5 Simplify each of these, expressing your answers with positive indices.

(a) $x^{-4} \times x^6$ (b) $a^3b^{-2} \times b^{-3}$ (c) $p^{-4}q^3 \times p^5q^{-2}$

(d) $d^7f^{-2} \times d^{-3}f^4$ (e) $x^5y^2 \times y^{-4}x^{-5}$ (f) $a^3b \times b^{-1}a^{-6}$

(g) $(m^3)^{-2}n^3 \times m^4(n^{-1})^2$ (h) $(p^4)^{-3}q^6 \times p^5(q^{-2})^2$ (i) $(a^{-2}b^3)^2 \times (a^4b^{-1})^{-3}$

(j) $(x^4y^{-3})^2 \times (x^{-5}y)^{-1}$ (k) $4(p^{-3})^5q^6 \times 3(pq)^{-2}$ (l) $(2x^2y)^{-3} \times (3x^{-4}y^5)^2$

6 (a) Using base 5 index notation, write the numbers 125, 25, 5 and 1.

(b) Using base 5 index notation, write the next five numbers in the sequence.

(c) Write down the whole sequence without using index notation.

Reasoning, communicating and open-ended

7 Simplify each of these, expressing your answers with positive indices.

(a) $\dfrac{m^7n^{-3}}{m^{-2}n^{-1}}$ **(b)** $\dfrac{a^{-5}b^3}{a^2b^{-4}} \times \dfrac{a^3b^2}{a^{-3}b^6}$ **(c)** $\dfrac{x^7y^{-5}}{x^{-4}y^3} \times \dfrac{x^{-4}y^3}{x^2y^{-1}}$

(d) $\dfrac{x^{-4}y^7}{x^{-2}y^{-5}} \div \dfrac{x^3y^{-6}}{x^{-1}y}$ **(e)** $\dfrac{(3a^{-4}b^{-2})^2}{(a^{-2}b)^3}$ **(f)** $\dfrac{(2d^{-3}f^{-1})^3}{(d^{-2}f^2)^2}$

(g) $\dfrac{5(a^{-4})^3b^{-6}}{(a^2b^{-1})^4} \times \dfrac{2a^7(b^{-3})^2}{a^{-4}b^{-9}}$ **(h)** $\left(\dfrac{m^5n^{-2}}{m^{-3}n}\right)^{-1}$ **(i)** $\left(\dfrac{x^3}{y^2}\right)^{-4} \div \dfrac{x^{-10}}{(y^{-4})^2}$

8 **(a)** If I divide a number by its square, what is the result?

(b) I divide a room into squares that have a side length of a units. If the room has a breadth of $16a^2$ units and a length of $12a$ units, how many squares would I require to carpet the room?

9 **(a)** Write two expressions involving powers and multiplication that, when simplified, equal $36m^4n^8$.

(b) Write two expressions involving powers and division that can be simplified to $\dfrac{4m^4}{5n^6}$.

10 For the expression a^{-b}, write three different sets of values for a and b, so that its value is less than $\frac{1}{4}$. Mark each value on a number line.

11 Raphael and Murray were arguing over a question involving indices. Murray believes that $a^{-2} \times a^{-4}$ simplifies to $\dfrac{1}{a^8}$. Raphael disagrees.

(a) How did Murray calculate his answer? Is he correct?

(b) Why does Raphael disagree? What might Raphael's answer be, assuming it is correct?

(c) How would you explain the correct answer to the students?

12.7 Fractional indices

Stage 5.3§

Need to Know

The index laws can also be used when the index or power is a fraction.

Consider $\left(3^{\frac{1}{2}}\right)^2$. Using the index law $(a^m)^n = a^{mn}$, keep the base 3 and multiply the powers.

$$\left(3^{\frac{1}{2}}\right)^2 = 3^{\frac{1}{2}\times 2}$$
$$= 3^1$$
$$= 3$$

However, $(\sqrt{3})^2 = 3$

So $\left(3^{\frac{1}{2}}\right)^2 = (\sqrt{3})^2$

$3^{\frac{1}{2}} = \sqrt{3}$

So $a^{\frac{1}{2}} = \sqrt{a}$, $a > 0$

Similarly, $a^{\frac{1}{3}} = \sqrt[3]{a}$, $a > 0$

In general, $a^{\frac{1}{n}} = \sqrt[n]{a}$, $a > 0$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, $a > 0$.

A number raised to a fractional power $\frac{1}{n}$ is the nth root of the number.

Numbers raised to a fractional power obey all the index laws.

Technology Exploration: Calculator

Fractional indices on a calculator

Consider calculating $625^{\frac{1}{4}}$. Recall that $625^{\frac{1}{4}}$ is the same as $\sqrt[4]{625}$.

On your calculator look for $\sqrt[x]{\ }$ or $\sqrt[x]{y}$ or $\sqrt[y]{x}$ or $x^{1/y}$.

You may need to press the second function 2nd or Shift or inverse INV key before this key. The root, which is the small number in the radical sign ($\sqrt{\ }$), is usually entered first.

For $625^{\frac{1}{4}}$, press: 4 2nd $\sqrt[x]{\ }$ 6 2 5 = . The screen should show 5.

(Your calculator may require a different sequence, such as 6 2 5 INV $x^{1/y}$ 4 = . Learn how your calculator works.)

Worked Example 14

WE14

Simplify:

(a) $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$

(b) $\sqrt{x^6}$

(c) $\dfrac{(m^3)^{\frac{1}{3}} \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$

Working

(a) $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$

$= a^{\frac{1}{2}+\frac{1}{4}}$

$= a^{\frac{(2+1)}{4}}$

$= a^{\frac{3}{4}}$

(b) $\sqrt{x^6}$

$= (x^6)^{\frac{1}{2}}$

$= x^{6 \times \frac{1}{2}}$

$= x^3$

(c) $\dfrac{(m^3)^{\frac{1}{3}} \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$

$= \dfrac{m^1 \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$

$= \dfrac{m^{1-\frac{1}{3}}}{m^{\frac{1}{3}}}$

$= \dfrac{m^{\frac{2}{3}}}{m^{\frac{1}{3}}}$

$= m^{\frac{2}{3}-\frac{1}{3}}$

$= m^{\frac{1}{3}}$

Worked Example 15

WE15

Use your calculator to evaluate the following fractional powers.

(a) $7776^{\frac{1}{5}}$

(b) $289^{\frac{1}{2}}$

(c) $1331^{\frac{1}{3}}$

Working

(a) $7776^{\frac{1}{5}}$

$= \sqrt[5]{7776}$

$= 6$

(b) $289^{\frac{1}{2}}$

$= \sqrt{289}$

$= 17$

(c) $1331^{\frac{1}{3}}$

$= \sqrt[3]{1331}$

$= 11$

Answers page 719

12.7 Fractional indices

Stage 5.3§

Navigator

Q1 Column 1, Q2 Columns 1–2, Q3, Q4, Q5 (a)–(d), Q7 (a)–(d), Q9, Q10, Q11, Q12	Q1 Column 2, Q2 Columns 2–3, Q3, Q4, Q5 Column 1, Q6, Q7 (c)–(f), Q9, Q10, Q11, Q12	Q1 Column 3, Q2 Columns 3–4, Q3, Q5 Column 2, Q6, Q7 (e)–(h), Q8, Q9, Q10, Q11, Q13

Equipment required: calculator for Questions **2**, **3**, **4**, **7**, **10**, **11** and **12**

Fluency and understanding

WE14

1 Simplify each of the following.

(a) $x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ **(b)** $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$ **(c)** $k^{\frac{1}{3}} \times k^{\frac{1}{5}}$

(d) $\dfrac{m^{\frac{1}{2}}}{m^{\frac{1}{4}}}$ **(e)** $\dfrac{p^{\frac{1}{2}}}{p^{\frac{1}{3}}}$ **(f)** $\left(y^{\frac{1}{2}}\right)^{\frac{1}{2}}$

(g) $(25p^4)^{\frac{1}{2}}$ **(h)** $(49x^6)^{\frac{1}{2}}$ **(i)** $(8ab^3)^{\frac{1}{3}}$

(j) $\sqrt{x^4y}$ **(k)** $\sqrt{kn^8}$ **(l)** $\left(\dfrac{p^4}{q^{10}}\right)^{\frac{1}{2}}$

WE15

2 Evaluate each of the following. Express your answer correct to two decimal places where appropriate.

(a) $196^{\frac{1}{2}}$ **(b)** $529^{\frac{1}{2}}$ **(c)** $6859^{\frac{1}{3}}$ **(d)** $512^{\frac{1}{3}}$

(e) $1296^{\frac{1}{4}}$ **(f)** $4096^{\frac{1}{4}}$ **(g)** $3125^{\frac{1}{5}}$ **(h)** $1024^{\frac{1}{5}}$

(i) $500^{\frac{1}{3}}$ **(j)** $3000^{\frac{1}{3}}$ **(k)** $98^{\frac{1}{6}}$ **(l)** $145^{\frac{1}{6}}$

3 Using your calculator to assist you, place these numbers in ascending order.

$2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 4^{-1}, 6^{-\frac{1}{5}}$

4 **(a)** 3^5 is closest to:

A 1.71 **B** 15 **C** 125 **D** 243

(b) 4^{-3} is closest to:

A 0.012 **B** 0.016 **C** 64 **D** 81

(c) $20^{\frac{1}{4}}$ is closest to:

A 0.000 006 **B** 2.1 **C** 5 **D** 160 000

(d) $75^{\frac{1}{3}}$ is closest to:

A -25 **B** 0.24 **C** 4.22 **D** 421 875

5 Simplify each of the following, expressing your answers with positive indices.

(a) $\dfrac{\left(a^{\frac{1}{3}}\right)^{6} \times a}{a^{2}}$

(b) $\dfrac{\left(d^{\frac{1}{2}}\right)^{8} \times d^{2}}{d^{3}}$

(c) $(x^{4}y^{6})^{\frac{1}{2}} \times (x^{-3}y^{-12})^{\frac{1}{3}}$

(d) $(m^{6}n^{3})^{\frac{1}{3}} \times (m^{-8}n^{-10})^{\frac{1}{2}}$

(e) $\dfrac{(pq^{3})^{\frac{1}{2}} \times p^{\frac{1}{2}}q^{\frac{1}{2}}}{pq^{4}}$

(f) $\dfrac{(a^{5}b)^{\frac{1}{2}} \times a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{3}b^{5}}$

(g) $\left(\dfrac{m^{2}n}{n^{-\frac{1}{2}}}\right)^{-\frac{1}{3}} \times \dfrac{m\left(n^{\frac{1}{3}}\right)^{2}}{2n^{\frac{1}{3}}}$

(h) $\left(\dfrac{x^{3}y^{2}}{y^{\frac{1}{2}}}\right)^{-\frac{1}{3}} \times \dfrac{(x^{2})^{\frac{1}{3}}y}{5y^{-\frac{2}{3}}}$

6 **(a)** Find the expression that is *not* the same as $a^{\frac{3}{2}}b^{\frac{1}{2}}$.

A $\sqrt{a^{3}b}$ **B** $(\sqrt{ab})^{3}$ **C** $(a^{3}b)^{\frac{1}{2}}$ **D** $(\sqrt{a})^{3}\sqrt{b}$

(b) Simplifying $\left(\dfrac{a^{\frac{1}{2}}b^{3}}{b^{2}}\right)^{2} \div \dfrac{b^{\frac{3}{2}}a}{b^{4}}$ gives:

A $b^{\frac{9}{2}}$ **B** $b^{\frac{13}{2}}$ **C** $\dfrac{b^{\frac{7}{2}}}{a^{\frac{1}{2}}}$ **D** $\dfrac{b^{7}}{a^{\frac{1}{2}}}$

7 Evaluate each of the following, correct to three decimal places.

(a) $8^{-\frac{1}{3}}$ **(b)** $12^{-\frac{1}{3}}$ **(c)** $37^{-\frac{1}{2}}$ **(d)** $26^{-\frac{1}{2}}$

(e) $54^{-\frac{1}{4}}$ **(f)** $72^{-\frac{1}{4}}$ **(g)** $100^{-\frac{1}{5}}$ **(h)** $167^{-\frac{1}{5}}$

8 By simplifying first, find the value of x in each of the following equations.

(a) $\dfrac{72x^{2}}{\left(x \times x^{\frac{3}{2}}\right)^{2}} = 24$

(b) $\dfrac{\left(x^{\frac{1}{2}}\right)^{5}}{x^{-2}} \times \dfrac{5x^{-1}}{x^{2}} = 40$

(c) $\dfrac{7x^{-3}}{x^{3}}\left(\dfrac{x^{\frac{3}{2}}}{x}\right)^{6} = 56$

9 Simplify the following.

(a) $\sqrt{x^{6}}$ **(b)** $\sqrt[3]{a^{9}}$ **(c)** $\sqrt[4]{16m^{8}n^{4}}$

Reasoning, communicating and open-ended

10 Find the value of x in each of the following equations.

(a) $x^{-2} = \frac{1}{4}$ **(b)** $3^x = 27$ **(c)** $3^x = \frac{1}{9}$ **(d)** $2 \times 5^x = 50$

11 **(a)** Using your calculator, find $16\,807^{\frac{1}{5}}$.

(b) Hence, find the value of x and y in the expression $x^y = 16\,807$.

(c) Using your calculator, find $64^{\frac{4}{3}}$.

(d) Hence, find the value of d and e in the expression $d^e = 64$.

12 Using your calculator, find at least three ways of writing 343 as a power. When the base number is not exact, use three decimal places. For example, $343 = (18.520)^2$.

13 Write the expression $n^{\frac{3}{4}} m^{\frac{3}{5}}$ in at least three different ways.

Problem solving

Lagrange's four-square theorem

Diophantus was a Greek mathematician who lived in the 3rd century. He wrote a famous book called *Arithmetica*. In this book he discussed the following conjecture (a statement that has not been proven).

Every whole number is either a square itself or the sum of two, three or four squares.

This conjecture was subsequently proven in 1770 by Joseph-Louis Lagrange.

(a) Write a square number in each box so that each column and row adds to the total at the end.

(b) Can any of the totals in the grid be written using fewer square numbers?

(c) Starting at 1, write each whole number as a sum of four or fewer square numbers. How far can you get in 3 minutes?

			1^2	39
		1^2		27
	1^2			34
1^2				62
36	30	66	54	

Strategy options

- Make a table.
- Break problem into manageable parts.

Logarithms

12.8

Stage 5.3#

Need to Know

Defining logarithms

Logarithm is another word for power or index.

For example:

As $10^2 = 100$, the logarithm of 100 to base 10 is 2. It is written as $\log_{10} 100 = 2$.

logarithm

$$100 = 10^2$$

numerical value base

In this case, 10 is the base and 2 is the logarithm, but any number can be used.

For example, $2^3 = 8$, so the logarithm of 8 to base 2 is 3. This is written $\log_2 8 = 3$.

While any number can be used as the base number, the most frequently used rational values are 2 and 10. The **log** button on your calculator will give you the base 10 logarithm for any number keyed in.

If $a^y = x$, the logarithm of x to base a is y. It is written $\log_a x = y$.

Worked Example 16

WE16

(a) Write $2^6 = 64$ in its equivalent logarithmic form.

(b) Write $2^{-4} = \frac{1}{16}$ in its equivalent logarithmic form.

Working

(a) 6 is the logarithm
2 is the base
64 is the numerical value

$6 = \log_2 64$

(b) -4 is the logarithm
2 is the base
$\frac{1}{16}$ is the numerical value

$-4 = \log_2\left(\frac{1}{16}\right)$

Worked Example 17 WE17

Write $\log_{10}(10\,000) = 4$ in its equivalent index form.

Working

4 is the logarithm
10 is the base
10 000 is the numerical value

$10^4 = 10\,000$

Answers page 719

12.8 Logarithms

Stage 5.3#

Navigator

Q1 Column 1, Q2 Column 1, Q3, Q4 Column 1, Q5 Column 1, Q6, Q9, Q11	Q1 Column 2, Q2 Column 2, Q3, Q4 Column 2, Q5 Column 2, Q6, Q7, Q9, Q11	Q1 Column 3, Q2 Column 3, Q3, Q4 Column 3, Q5 Column 3, Q6, Q8, Q9, Q10

Equipment required: calculator for Questions **4–5**

Fluency and understanding

WE16

1 Write each of the following in its equivalent logarithmic form.

(a) $2^5 = 32$ **(b)** $2^8 = 256$ **(c)** $2^{12} = 4096$

(d) $3^3 = 27$ **(e)** $3^5 = 243$ **(f)** $3^8 = 6561$

(g) $10^2 = 100$ **(h)** $10^6 = 1\,000\,000$ **(i)** $10^8 = 100\,000\,000$

(j) $2^{-2} = \frac{1}{4}$ **(k)** $2^{-3} = \frac{1}{8}$ **(l)** $2^{-4} = \frac{1}{16}$

WE17

2 Write each of the following in its equivalent index form.

(a) $\log_{10}100 = 2$ **(b)** $\log_{10}1000 = 3$ **(c)** $\log_{10}100\,000 = 5$

(d) $\log_2 16 = 4$ **(e)** $\log_2 128 = 7$ **(f)** $\log_2 512 = 9$

(g) $\log_{10}\left(\frac{1}{1000}\right) = -3$ **(h)** $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **(i)** $\log_{10}\left(\frac{1}{1\,000\,000}\right) = -6$

(j) $\log_2\left(\frac{1}{8}\right) = -3$ **(k)** $\log_2\left(\frac{1}{32}\right) = -5$ **(l)** $\log_2\left(\frac{1}{128}\right) = -7$

3 **(a)** If $\log_{10}x = -2$ then x is equal to:

A -200 **B** $-\frac{1}{100}$ **C** $\frac{1}{100}$ **D** 200

(b) Writing $2^x = 2048$ in logarithmic form gives:

A $\log_2 x = 2048$ **B** $\log_x 2 = 2048$ **C** $\log_x 2048 = 2$ **D** $\log_2 2048 = x$

Hint

Write the number on the RHS as an index number, using the same base as the LHS.

4 **(a)** Write each of the following equations in index form so that both sides of the equation have the same base.

(b) Solve each equation.

(i) $3^x = 81$ **(ii)** $3^x = 6561$ **(iii)** $3^x = 19\,683$

(iv) $5^x = \frac{1}{25}$ **(v)** $5^x = \frac{1}{125}$ **(vi)** $5^x = \frac{1}{15\,625}$

(vii) $6^{-x} = 216$ **(viii)** $6^{-x} = 7776$ **(ix)** $6^{-x} = \frac{1}{1296}$

5 **(a)** Write each of the following equations in index form.

(b) Solve each equation.

(i) $\log_2 x = 4$ **(ii)** $\log_2 x = 6$ **(iii)** $\log_2 x = 7$

(iv) $\log_3\left(\frac{1}{81}\right) = x$ **(v)** $\log_3\left(\frac{1}{243}\right) = x$ **(vi)** $\log_3\left(\frac{1}{6561}\right) = x$

(vii) $\log_x 625 = 4$ **(viii)** $\log_x 343 = 3$ **(ix)** $\log_x\left(\frac{1}{3125}\right) = -5$

Reasoning, communicating and open-ended

6 **(a)** Write the following in index form to find the value of x.

(i) $\log_{10} x = 1$ **(ii)** $\log_5 x = 1$ **(iii)** $\log_7 x = 1$

(b) Write a general rule based on the above, using a as your base.

7 **(a)** Write the following in index form to find the value of x.

(i) $\log_{10} x = 0$ **(ii)** $\log_5 x = 0$ **(iii)** $\log_7 x = 0$

(b) Write a general rule to explain what you have just found out.

8 In the equation $\sqrt{a} = a^{\frac{1}{2}}$, why does a have to be greater than or equal to 0?

9 **(a)** Check that both of the following logarithmic equations are correct:

$\log_2 64 = 6$ and $\log_4 64 = 3$

(b) Now, given $\log_2(4096) = 12$, find the value of y in $\log_4(4096) = y$.

(c) Explain how you got your answer for **(b)**.

10 Given $\log_3(531\,441) = 12$, write two other related log statements that are true.

11 Lee wrote:

$-(6)^2 = -36$
because $-\log_6(-36) = 2$

Explain why this is incorrect and how the question should be tackled.

12.9 Laws of logarithms

Stage 5.3#

Need to Know

Laws for simplifying logarithms can be deduced as were the laws for simplifying indices.

Consider $b^m = x$ and $b^n = y$.

$$\begin{aligned} xy &= b^m \times b^n \\ &= b^{m+n} \end{aligned}$$

$\log_b(xy) = m + n$

Similarly, it can be shown that $\log_b\left(\frac{x}{y}\right) = m - n$.

Now consider $\log_b x = m$.

$$\begin{aligned} x &= b^m \\ x^n &= (b^m)^n \\ &= b^{mn} \end{aligned}$$

$$\begin{aligned} \log_b x^n &= mn \\ &= nm \\ &= n\log_b x \end{aligned}$$

If $\log_b b = y$, then $b = b^y$.

This means $y = 1$, so $\log_b b = 1$.

Also $\log_b b^n = n\log_b b$
$= n$

and $\log_b 1 = \log_b b^0$
$= 0 \log_b b$
$= 0$

Logarithmic laws

Logarithmic expressions can be simplified using the following logarithmic laws:

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b x^n = n \log_b x$
4. $\log_b b = 1$
5. $\log_b x^b = n$
6. $\log_b 1 = 0$

Worked Example 18

WE 18

Simplify $\log_2 8 + \log_2 16$, expressing your answer in logarithmic form.

Working

$\log_b x + \log_b y = \log_b xy$

$\log_2 8 + \log_2 16 = \log_2(8 \times 16)$
$= \log_2 128$

Worked Example 19

WE 19

Simplify $\log_2 12 - \log_2 4$, expressing your answer in logarithmic form.

Working

$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$

$\log_2 12 - \log_2 4 = \log_2\left(\frac{12}{4}\right)$
$= \log_2 3$

Worked Example 20

WE 20

Simplify $\log_2 5^2$, expressing your answer in logarithmic form without a power.

Working

$\log_b x^n = n \log_b x$

$\log_2 5^2 = 2 \log_2 5$

Worked Example 21

WE 21

Simplify the following, writing your answer as a basic numeral if possible.

(a) $\log_2 16$

(b) $\log_{10} 1000 - \log_{10} 10^2$

Working

(a) $\log_2 16$
$= \log_2 2^4$
$= 4 \log_2 2$
$= 4 \times 1$
$= 4$

(b) $\log_{10} 1000 - \log_{10} 10^2$
$= \log_{10} 10^3 - \log_{10} 10^2$
$= 3 \log_{10} 10 - 2 \log_{10} 10$
$= 3 - 2$
$= 1$

Answers page 720

12.9 Laws of logarithms

Stage 5.3#

Navigator

Q1, Q2, Q3, Q4, Q6 (a)–(b), Q7 (a)–(c), Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6 (a)–(d), Q7, Q8, Q10, Q11	Q1, Q2, Q3, Q4, Q5, Q6 (c)–(f), Q7, Q9, Q10, Q11

Fluency and understanding

WE18

1 Simplify the following, expressing your answer in logarithmic form.

(a) $\log_2 4 + \log_2 8$ **(b)** $\log_2 16 + \log_2 4$

(c) $\log_2 8 + \log_2 32$ **(d)** $\log_{10} 5 + \log_{10} 6$

(e) $\log_{10} 7 + \log_{10} 3$ **(f)** $\log_{10} 4 + \log_{10} 9$

WE19

2 Simplify the following, expressing your answer in logarithmic form.

(a) $\log_2 16 - \log_2 8$ **(b)** $\log_2 20 - \log_2 4$

(c) $\log_2 25 - \log_2 5$ **(d)** $\log_{10} 70 - \log_{10} 5$

(e) $\log_{10} 35 - \log_{10} 7$ **(f)** $\log_{10} 36 - \log_{10} 6$

WE20

3 Simplify the following, expressing your answer in logarithmic form without a power.

(a) $\log_2 5^4$ **(b)** $\log_2 6^3$ **(c)** $\log_2 7^4$

(d) $\log_{10} 7^3$ **(e)** $\log_{10} 8^4$ **(f)** $\log_{10} 11^5$

WE21

4 Simplify the following, expressing your answer as a basic numeral.

(a) $\log_2 32$ **(b)** $\log_2 128$ **(c)** $\log_2 256$

(d) $\log_{10} 10\,000 - \log_{10} 100$ **(e)** $\log_3 81 - \log_3 27$ **(f)** $\log_4 64 - \log_4 16$

5 **(a)** $\dfrac{\log_2 32}{\log_2 4}$ is equal to:

A $\dfrac{5}{2}$ **B** 3 **C** $\log_2(8)$ **D** $\log_2(28)$

(b) $\log_a(mn)$ can be written as:

A $\log_a m \times \log_a n$ **B** $\log_a m^n$

C $\log_a m + \log_a n$ **D** $\log_a m - \log_a n$

6 Simplify:

(a) $\log_{10} 4 + \log_{10} 250$ **(b)** $\log_2 64 + \log_2 32$

(c) $\log_2 4 + \log_2 8 - \log_2 16$ **(d)** $\log_3 27 + \log_3 81 - \log_3 243$

(e) $2\log_2 2 - \log_2\left(\frac{1}{8}\right) + \log_2 8$ **(f)** $\log_3 3 + 2\log_3\left(\frac{1}{9}\right) - 3\log_3 9$

7 Simplify:

(a) $\dfrac{\log_5 x^2}{\log_5 x}$ **(b)** $\dfrac{\log_3 x^5}{\log_3 x^6}$ **(c)** $\dfrac{\log_7 x^9}{\log_7 x^3}$

(d) $\dfrac{\log_2 (x+4)^2}{\log_2 (x+4)}$ **(e)** $\dfrac{3\log_5 (2x-1)}{\log_5 (2x-1)^2}$ **(f)** $\dfrac{\log_7 (3x+4)^4}{3\log_7 (3x+4)}$

Reasoning, communicating and open-ended

8 Simplify:

(a) $\dfrac{\log_x 2}{\log_x 16}$ **(b)** $\dfrac{\log_x 25}{\log_x 5}$ **(c)** $\dfrac{\log_x 49}{\log_x 7^3}$

9 **(a)** If $\log_a y = \log_a 4 + 3\log_a x$, express y in terms of x.

(b) If $\log_a y = \log_a 11 + 5\log_a x$, express y in terms of x.

10 If $5\log_2 y = \log_2(x-1) + 2$, show that $y^5 = 4(x-1)$.

11 One of the questions on a test was:

Simplify $\dfrac{\log_2 x}{\log_2 \sqrt{x}}$

Danielle wrote: $\dfrac{\log_2 x}{\log_2 \sqrt{x}} = \dfrac{\log_2 x}{\log_2 x^{\frac{1}{2}}} = \dfrac{1}{2} \times \dfrac{\log_2 x}{\log_2 x} = \dfrac{1}{2}$

Is Danielle's answer correct? If not, what is the correct answer and how would you explain her mistake to Danielle?

12.10 Logarithmic and exponential graphs and equations

Stage 5.3#

Need to Know

Exponential and logarithmic equations

An exponential equation is written in the form of $y = a^x$. This can also be written as $x = \log_a y$. If x and y are interchanged, the result gives $y = \log_a x$. This means that $y = \log_a x$ is the inverse of $y = a^x$.

Exponential and logarithmic graphs

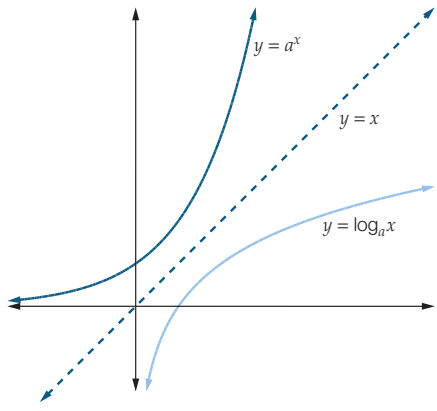
A logarithmic function is the inverse of an exponential function. The graph of an exponential function is reflected in the line $y = x$ to produce the graph of the inverse logarithmic function.

The point (0, 1) lies on all exponential graphs in the form of $y = a^x$.

The point (1, 0) lies on all logarithmic graphs in the form of $y = \log_a x$.

The x-axis is the asymptote of exponential graphs in the form of $y = a^x$.

The y-axis is the asymptote of logarithmic graphs in the form of $y = \log_a x$.



Simple logarithmic equations

Logarithmic equations can be solved by using the log laws.

For example:

$x = \log_2 8$
$x = \log_2 2^3$
$x = 3\log_2 2$
$x = 3$

Alternatively, a logarithmic equation can be transformed into an exponential equation that can be simplified.

For example:

$x = \log_2 8$
$2^x = 8$
$2^x = 2^3$
$x = 3$

A calculator can be used to solve other logarithmic equations but they need to be expressed using base 10. Logarithms expressed in other bases need to be transformed using the log law:

$$\log_a b = \frac{\log_{10} b}{\log_{10} a}$$

Simple exponential equations

Simple exponential equations can be solved by converting all numbers in the equation to exponential form with the same base.

For example:

$2^x = 64$
$2^x = 2^6$
$x = 6$

An exponential equation can be transformed into a logarithmic equation and the log laws and a calculator used to solve it.

For example:

$2^x = 5$
$x = \log_2 5$
$x = \frac{\log_{10} 5}{\log_{10} 2}$

Worked Example 22

WE 22

On the same set of axes, plot the graphs of $y = 2^x$ and $y = \log_2 x$, using a table of values for $-3 \le x \le 3$ for each graph. Using a dotted line, draw the graph of $y = x$.

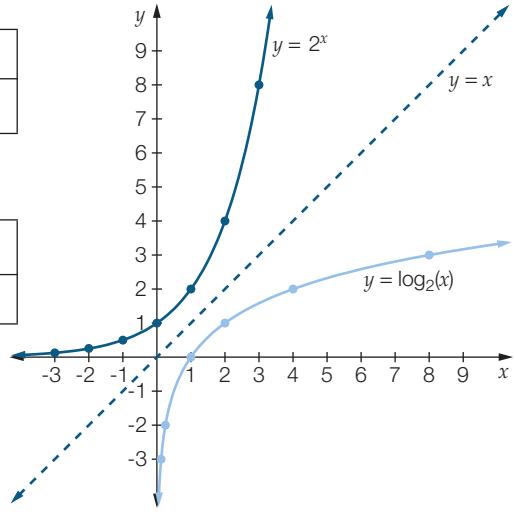
Working

(a) $y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

(b) $y = \log_2 x$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y	-3	-2	-1	0	1	2	3



Worked Example 23

WE 23

Solve the following simple logarithmic equations.

(a) $\log_{32}2 = x$ **(b)** $\log_x 27 = -3$ **(c)** $\log_5 x = 3$

Working

(a) $\log_{32}2 = x$

$2 = 32^x$

$2 = 2^{5x}$

$1 = 5x$

$x = \frac{1}{5}$

(b) $\log_x 27 = -3$

$27 = x^{-3}$

$3^3 = x^{-3}$

$3^{-3} = x^3$

$3^{-1} = x$

$x = \frac{1}{3}$

(c) $\log_5 x = 3$

$x = 5^3$

$x = 125$

Worked Example 24

WE 24

Solve the following simple exponential equations.

(a) $3^x = \frac{1}{81}$ **(b)** $4^x = 8$ **(c)** $9^{x-1} = \sqrt{3}$

Working

(a) $3^x = \frac{1}{81}$

$3^x = \frac{1}{3^4}$

$3^x = 3^{-4}$

$x = -4$

(b) $4^x = 8$

$2^{2x} = 2^3$

$2x = 3$

$x = 1.5$

(c) $9^{x-1} = \sqrt{3}$

$3^{3(x-1)} = 3^{\frac{1}{2}}$

$3(x-1) = \frac{1}{2}$

$x - 1 = \frac{1}{6}$

$x = 1\frac{1}{6}$

12.10 Logarithmic and exponential graphs and equations

Answers page 720

Navigator

Stage 5.3#

Q1, Q2 Columns 1 & 2, Q3 Columns 1 & 2, Q4, Q5, Q6, Q7 (a), (b), Q8 (a)–(f), Q9, Q10	Q1, Q2 Columns 1 & 2, Q3 Columns 1 & 2, Q4, Q5, Q6, Q7, Q8, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency and understanding

1 On the same set of axes, plot the graphs of $y = 3^x$ and $y = \log_3 x$, using a table of values for $-2 \le x \le 2$ for each graph. Using a dotted line, draw the graph of $y = x$. WE22

2 Solve the following simple logarithmic equations. WE23

(a) $\log_{25} 5 = x$ **(b)** $\log_{27} 9 = x$ **(c)** $\log_8 \frac{1}{2} = x$

(d) $\log_x 36 = 2$ **(e)** $\log_x 64 = -3$ **(f)** $\log_x 125 = -3$

(g) $\log_6 x = 3$ **(h)** $\log_2 x = -3$ **(i)** $\log_7 x = \frac{1}{2}$

3 Solve the following simple exponential equations. WE24

(a) $7^x = \frac{1}{49}$ **(b)** $3^x = \frac{1}{81}$ **(c)** $6^x = \frac{1}{216}$

(d) $9^x = 243$ **(e)** $8^x = 32$ **(f)** $25^x = \frac{1}{5}$

(g) $36^{2x+1} = 6$ **(h)** $49^{3x+1} = \frac{1}{7}$ **(i)** $64^{x+1} = 2\sqrt{2}$

4 The solution to the logarithmic equation $\log_x \frac{1}{4} = \frac{1}{2}$ is:

A -16 **B** -4 **C** $\frac{1}{4}$ **D** $\frac{1}{16}$

5 The solution to the exponential equation $16^{2x-1} = \frac{1}{8}$ is:

A 8 **B** -8 **C** $\frac{1}{8}$ **D** $-\frac{1}{8}$

Reasoning, communicating and open-ended

6 Solutions with a pH less than 7 are said to be acidic and solutions with a pH greater than 7 are basic or alkaline. Pure water has a pH very close to 7.

pH is defined as the decimal logarithm of the reciprocal of the hydrogen ion activity, a_{H^+}, in a solution.

$$\text{pH} = -\log_{10}(a_{H^+}) = \log_{10}\left(\frac{1}{a_{H^+}}\right)$$

Find the hydrogen ion activity, a_{H^+}, for pure water.

7 The Richter scale was developed to compare the strength of earthquakes. Ground vibrations caused by earthquakes are recorded as a wave-like trace, a seismogram, on instruments called seismographs. These seismograms plot ground motion, indicating the strength of the earthquake against time.

In simple terms, $M = \log_{10}S$, where M is the magnitude of the earthquake on the Richter scale and S is the strength of the earthquake. This equation is used to compare the strength of earthquakes.

(a) What is the strength of an earthquake that measures:

(i) 4 on the Richter scale **(ii)** 5 on the Richter scale?

(b) How much stronger is an earthquake that measures 5 on the Richter scale compared to one that measures 4?

(c) In 2008, an earthquake that measured 7.9 on the Richter scale hit the Sichuan region in China. The following day another earthquake hit China measuring 6.1. How much stronger was the Sichuan earthquake than the one the next day?

(d) An earthquake in San Francisco in 1906 measured 7.8 on the Richter scale and was 8 times stronger than the 1989 San Francisco earthquake. What was the magnitude of the 1989 earthquake?

8 The threshold of human hearing is defined as 0 decibels with an intensity, I, of 1×10^{-12} W/m^2 (watts per m^2). The level of intensity or loudness, L, measured in decibels (dB), is given by the formula $L = 10\log_{10}\left(\frac{I}{1 \times 10^{-12}}\right)$ dB.

(a) A vacuum cleaner has a sound intensity level of 1×10^{-5} W/m^2. How loud is this in decibels?

(b) A jet aircraft has a sound intensity of 100 W/m^2. How loud is this in decibels?

(c) How much louder is the jet than the vacuum cleaner?

(d) The loudness of a large orchestra is measured at 98 dB. Find the intensity of sound emitted by this orchestra.

(e) The intensity of sound in the front row seats of a rock concert is measured at 0.1 W/m^2. Find the loudness of sound emitted at this concert.

(f) Which is louder, the rock concert or the orchestra?

(g) Which is louder, thunder at a sound level of 110 dB or a pneumatic drill at a sound intensity of 1 W/m^2?

(h) If the threshold of discomfort is 120 dB and the threshold of pain is 130 dB, do any of the above noises:

(i) cause discomfort but not pain **(ii)** cause pain?

Justify your answers.

9 Write an exponential equation equal to a surd with a base of 2 and an index as a variable. Solve this equation.

10 Write a logarithmic equation using a base of 3 and equal to a negative number. Solve this equation.

Chapter review 12

D.I.Y. Summary

Key Words

Stage 5.3§	conjugate surds	rational numbers
	distributive law	rationalising the denominator
	entire surd	real numbers
	irrational numbers	surds
Stage 5.3#	logarithm	

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

Stage 5.3§ Q1–7 **Stage 5.3# Q8**

1 The real number system contains _____ and _____.

2 $\sqrt{2}$ and π are examples of _____.

3 _____ are numbers that contain a radical or root sign and cannot be written as an exact number.

4 To simplify surd expressions involving brackets, we can use the _____.

5 When _____ of a fraction it is necessary to multiply both the numerator and the denominator by a surd or _____ in order to make the denominator rational.

6 The _____ $\sqrt{75}$ can be simplified to $3\sqrt{5}$.

7 **(a)** Three examples of surds are _____.

(b) The conjugate of $\sqrt{3}+1$ is _____.

8 When we write statements such as $3^4 = 81$ is equivalent to $\log_3(81) = 4$, we refer to 4 as the _____.

Stage 5.3§ Q1–8, 11, 12, 17, 18, 21, 22 **Stage 5.2 Q9, 10** **Stage 5.3# Q13–16, 19, 20, 23**

Fluency and understanding

1 Use a calculator to evaluate the following, correct to three decimal places. Ex.12.1

(a) $\sqrt{17}$ **(b)** $\sqrt[3]{25}$ **(c)** $5 \times 4\sqrt{3}$ **(d)** $3\sqrt{7} - 4\sqrt{2}$

2 Simplify each of the following. Ex.12.2

(a) $4\sqrt{5} \times 3\sqrt{6}$ **(b)** $5\sqrt{7} \times 2\sqrt{7}$ **(c)** $\sqrt{\frac{64}{25}}$ **(d)** $\frac{18\sqrt{6}}{2\sqrt{3}}$

(e) $\sqrt{40}$ **(f)** $2\sqrt{125}$ **(g)** $3\sqrt{2} \times 5\sqrt{10}$ **(h)** $4\sqrt{\frac{75}{16}}$

3 Express each of the following as an entire surd. Ex.12.2

(a) $6\sqrt{2}$ **(b)** $-3\sqrt{7}$

4 Simplify each of the following. **Ex. 12.3**

(a) $6\sqrt{2} - 8\sqrt{2} + 3\sqrt{2}$ **(b)** $5\sqrt{7} - 2\sqrt{3} - \sqrt{7} + \sqrt{3}$

(c) $4\sqrt{12} + 7\sqrt{27}$ **(d)** $6\sqrt{50} - 2\sqrt{24} + 4\sqrt{32} + \sqrt{54}$

5 Expand and simplify each of the following. **Ex. 12.4**

(a) $\sqrt{5}(4\sqrt{3} - 2\sqrt{5} + \sqrt{2})$ **(b)** $-6\sqrt{2}(3\sqrt{7} - \sqrt{2} + 1)$

(c) $(2\sqrt{5} + 9)(\sqrt{5} - 2)$ **(d)** $(3\sqrt{7} + 4\sqrt{2})^2$

6 **(a)** $2\sqrt{6}(\sqrt{10} + 3\sqrt{3})$ can be simplified to: **Ex. 12.4**

A $2\sqrt{60} + 5\sqrt{18}$ **B** $8\sqrt{15} + 18\sqrt{2}$ **C** $4\sqrt{15} + 18\sqrt{2}$ **D** $30\sqrt{2} + 5\sqrt{18}$

(b) $8 \times \sqrt{\frac{81}{64}}$ can be simplified to:

A $\sqrt{9}$ **B** $24 \times \sqrt{\frac{1}{8}}$ **C** $8 \times \sqrt{\frac{9}{8}}$ **D** 9

7 Express each of the following in simplest form with a rational denominator. **Ex. 12.5**

(a) $\frac{5}{\sqrt{2}}$ **(b)** $\frac{4\sqrt{3}}{\sqrt{6}}$ **(c)** $\frac{7}{2\sqrt{5}}$

8 Express each of the following in simplest form with a rational denominator. **Ex. 12.5**

(a) $\frac{7}{3 - \sqrt{5}}$ **(b)** $\frac{\sqrt{2} - 2}{2\sqrt{3} + \sqrt{7}}$

9 Simplify each of the following. **Ex. 12.6**

(a) $(3ab^5)^2$ **(b)** $\frac{(cd^2)^8}{(c^3d)^2}$ **(c)** $\left(\frac{t^2}{u^3}\right)^4 \times \left(\frac{u^2}{t}\right)^7$

(d) $\frac{8x^5y^6}{6x^3y^2} \times \frac{3x^4y^7}{2x^5y}$ **(e)** $\frac{(m^4n)^3 \times (m^2n^5)^2}{(mn)^4}$ **(f)** $\left(\frac{a^4b^2}{ab^3}\right)^2 \div \left(\frac{a^3b}{a^5b^2}\right)^3$

10 Simplify, expressing your answers with positive indices. **Ex. 12.6**

(a) $3a^{-4} \times 2a^{-2}$ **(b)** $(4xy^2)^{-2} \times (3x^2y)^3$ **(c)** $\frac{(m^{-3}n^2)^4}{2(mn^3)^{-2}} \times \left(\frac{m^2n^{-1}}{m^{-3}n^3}\right)^2$

11 Simplify each of the following. **Ex. 12.7**

(a) $(64m^3n)^{\frac{1}{3}}$ **(b)** $\sqrt{p^6q}$

(c) $\frac{(a^6b^3)^{\frac{1}{2}} \times a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^3b^2}$ **(d)** $(a^2b^{-3})^{\frac{1}{2}} \times (a^{-4}b)^{\frac{5}{2}}$

(e) $\left(\frac{m^{\frac{1}{3}}n^{\frac{2}{3}}}{mn^{\frac{1}{2}}}\right)^2 \times \frac{2m^3n^5}{\left(m^{\frac{1}{2}}n\right)^4}$ **(f)** $\left(\frac{3^2m^{-2}n^2}{mn}\right)^3 \div \left(\frac{m^{\frac{1}{3}}n^{\frac{2}{3}}}{n^2m^{-3}}\right)^3$

12 Express each of the following in index form. **Ex. 12.7**

(a) $(\sqrt{k})^5$ **(b)** $\sqrt[4]{x^3}$ **(c)** $\frac{1}{\sqrt[3]{m^2}}$

13 Write each of the following in its equivalent logarithmic form. **Ex. 12.8**

(a) $4^7 = 16\,384$ **(b)** $2^{10} = 1024$ **(c)** $10^{-3} = \frac{1}{1000}$

14 Write each of the following in its equivalent index form.

Ex.12.8

(a) $\log_2(16\,384) = 14$ **(b)** $\log_3 243 = 5$ **(c)** $\log_{10}\left(\frac{1}{100\,000}\right) = -5$

15 Simplify the following, expressing your answer as a basic numeral where possible.

Ex.12.9

(a) $\log_2 8 + \log_2 64$ **(b)** $\log_2 16 - \log_2 64$

(c) $\log_3 5 + \log_3 10$ **(d)** $\log_5 110 - \log_5 11$

16 Solve for x.

Ex.12.10

(a) $5^x = \frac{1}{15625}$ **(b)** $6^{-x} = 216$

(c) $\log_x 625 = 4$ **(d)** $\log_3\left(\frac{1}{243}\right) = x$

17 A DVD has an area of 232π cm^2.

(a) What is its exact radius? (Leave your answer in surd form.)

(b) State the radius of the disc, correct to two decimal places.

18 A gift box is to have its four sides covered by adhesive paper. It has the dimensions of length $8\sqrt{10}$ cm, breadth 18 cm and height $(5\sqrt{6} + 3\sqrt{7})$ cm.

(a) What are the approximate length and height of the box, correct to one decimal place?

(b) Using your answers to part **(a)**, calculate the area to be covered by adhesive paper, correct to one decimal place.

(c) Using the surd expression, find the area to be covered by the paper. Express your answer as a surd.

(d) Use a calculator to express your answer to part **(c)** correct to one decimal place.

(e) Compare your answers to parts **(b)** and **(d)**.

19 Solve for x:

(a) $4^x = 1024$ **(b)** $3^x = 2187$ **(c)** $2^x = \frac{1}{64}$

(d) $\log_2(x) = -4$ **(e)** $\log_3(6561) = x$ **(f)** $\log_5\left(\frac{1}{625}\right) = x$

20 Simplify each of the following, expressing your answer as a basic numeral where possible.

(a) $\log_{10} 2 + \log_{10} 500$ **(b)** $\log_3 81 - \log_3 27 + \log_3 9$

(c) $\frac{\log_5 x^4}{2\log_5 x}$ **(d)** $\frac{\log_8 (5x-7)^3}{4\log_8 (5x-7)}$

Reasoning and communicating

21 **(a)** Find the exact side length of a square with an area of 50 cm^2.

(b) Find the exact side length of an equilateral triangle with an area of 40 cm^2.

(c) Find the exact circumference of a circular swimming pool with an area of 75π m^2.

22 If $\left(\frac{4-3\sqrt{5}}{5+2\sqrt{2}}\right)a = \frac{5-2\sqrt{2}}{4+\sqrt{5}}$, find the value of a, expressing your answer with a rational denominator.

23 **(a)** If $\log_a y = \log_a 7 + 3\log_a x$, express y in terms of x.

(b) If $\log_a y = \log_a 5 - 2\log_a x$, express y in terms of x.

Mixed review D

Stage 5.1 Q2 **Stage 5.2** Q7, 8, 15–19 **Stage 5.2◊** Q1, 9 **Stage 5.3** Q11 **Stage 5.3§** Q3–6, 10, 12–14, 20, 21

Fluency and understanding

1 An investment of \$5000 is made at 3.5% p.a. for 6 years. Find the difference in interest earned, to the nearest cent, if interest is compounded monthly compared to annually. Ex.1.3

2 Find the area and perimeter, correct to one decimal place, of a semicircle with a diameter of 9.6 cm. Ex.3.1

3 Write the following in simplest form. Ex.12.2

(a) $\sqrt{3} \times \sqrt{8}$ **(b)** $\sqrt{3} \times 3\sqrt{12}$ **(c)** $\sqrt{2} \times \sqrt{18}$

(d) $2\sqrt{5} \times 4\sqrt{10}$ **(e)** $2\sqrt{6} \times 3\sqrt{10}$ **(f)** $3\sqrt{6} \times 7\sqrt{12}$

4 Simplify the following. Ex.12.2

(a) $\sqrt{80}$ **(b)** $3\sqrt{240}$ **(c)** $7\sqrt{525}$

5 Solve the following equations. Ex.4.2

(a) $\frac{3a+4}{7} = -2$ **(b)** $\frac{4a+3}{2} = \frac{2a}{5}$ **(c)** $\frac{3a-2}{3} = \frac{4a-5}{2}$

6 Express each of the following in simplest form with a rational denominator. Ex.12.5

(a) $\frac{4\sqrt{3}}{\sqrt{5}}$ **(b)** $\frac{2}{5-\sqrt{3}}$ **(c)** $\frac{\sqrt{2}}{\sqrt{3}+\sqrt{5}}$

7 Factorise each of the following expressions. Ex.2.3

(a) $k(k-3)-(k-3)$ **(b)** $3t(t+2)-5(t+2)$

8 Simplify the expressions below: Ex.12.6

(a) $\frac{4a^4b}{9a^{-5}b^3} \times \frac{3a^{-2}b^5}{2ab^{-1}}$ **(b)** $\frac{(a^4b^2)^2}{(a^2b^3)^2} \times \frac{(a^{-3}b)^3}{(a^5b^2)^{-3}}$ **(c)** $\frac{k^{-8}}{n^4} \div \frac{k^9}{n}$

9 Find the value of each pronumeral, correct to two decimal places. Ex.5.3

(a) $\cos a = 0.1314$ **(b)** $\tan b = 2.621$ **(c)** $\sin c = 0.5231$

10 Using $m = \frac{y_2 - y_1}{x_2 - x_1}$, find the gradient of the line joining: Ex.4.4

(a) points (1, 7) and (4, 14) **(b)** points (3, 6) and (7, -2).

11 Find **(a)** the mean and **(b)** standard deviation for the following data set: Ex.7.7
18, 20, 32, 28, 17, 22, 24, 29, 34, 19.
Give your answers correct to two decimal places.

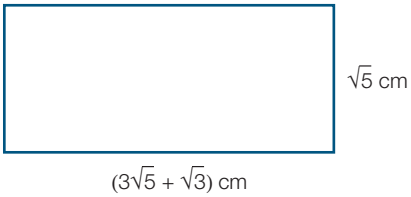
12 Sketch the graphs of each of the following, clearly showing the turning point. Ex.9.5

(a) $y = (x+3)(x-2)$ **(b)** $y = x^2 + 2x + 5$ **(c)** $y = 2x^2 - 4x$

13 For the rectangle shown:

(a) state its perimeter in simplest form

(b) state its area in simplest form.



Ex 12.2, 12.3

14 Write a definition for each of the following quadrilaterals.

(a) rectangle **(b)** rhombus **(c)** kite

Ex 8.5

15 Two dice, both numbered 1 to 6, are rolled at the same time. Find the probability that the sum of the two faces showing will be:

(a) 4 **(b)** 6 **(c)** greater than 9.

Ex 11.2

16 Without using a calculator, determine which is the larger of the following pairs, placing <, > or = between the numbers below.

(a) 3^{-5} ____ 5^{-3} **(b)** $9^{\frac{1}{2}}$ ____ 9^{-1} **(c)** $8^{-\frac{1}{3}}$ ____ $\dfrac{2^{-3}}{2^{-2}}$

Ex 12.6

17 On a Venn diagram, show the region described by not A and B and not C.

Ex 11.2

18 Expand and simplify each of the following.

(a) $2x(x + 4) - 4(3x - 5)$

(b) $(3x - 5)(2x + 7)$

Ex 2.1

19 Leonard and Faye want to develop a vegetable garden in their backyard. They have 20 m of chicken wire and enough posts to erect a fence around the four sides of the rectangular garden.

(a) If the length of the rectangular garden is x m, find an expression (in terms of x) for the breadth of the garden.

(b) Show that the area of the garden, A m^2, is given by the quadratic expression $A = -x^2 + 10x$.

(c) Sketch the graph of the area of the vegetable garden, A m^2, against the length of the garden, x m. (Show length on the horizontal axis.)

(d) What should be the dimensions of the garden if the area is to be as large as possible?

(e) What is the area of the largest vegetable garden that Leonard and Faye can make?

Reasoning and communicating

20 A square with side lengths $\sqrt{2}$ has been drawn using four squares with side lengths of 1 cm, as shown. What is the perimeter of a square that is made by joining the diagonals of four squares with side lengths of 2 cm?

21 **(a)** Draw an isosceles triangle with base length 10 cm and height 10 cm.

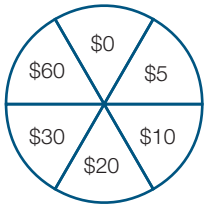
(b) Show that the length of the equal sides in the triangle in part **(a)** is an irrational number.

Challenge D

1 In a dice game, a player rolls two dice. The player's score is the larger of the two numbers on the dice. For example, if the player rolls 3 and 5 then the score is 5, or if the player rolls 4 and 4 then the score is 4. What is the probability that the score is 3 or less?

A $\frac{5}{36}$ **B** $\frac{1}{6}$ **C** $\frac{7}{36}$ **D** $\frac{1}{4}$

2 A prize wheel is divided into 6 equal sectors. Each time the wheel is spun, it stops on one of the sectors, resulting in a prize of the amount shown.



(a) On the first spin, \$5 is won. What is the probability of winning \$5 on the second spin?

(b) The wheel is spun a total of 50 times and the results recorded in the table.

(i) Copy and complete the table by writing in the two missing pieces of information.

(ii) How much prize money was won?

Prize	Frequency	Relative frequency
\$0	8	0.16
\$5		0.14
\$10	9	
\$20	10	0.20
\$30	8	0.16
\$60	8	0.16
Total	**50**	**1**

(c) To play this game you have to pay \$20 per throw.

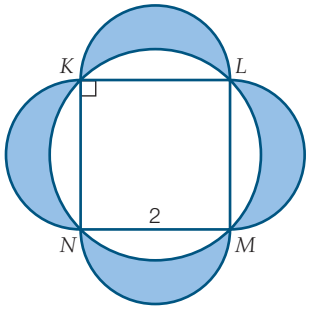
(i) If each outcome was equally likely and you spun the wheel 48 times, would you expect to make a profit?

(ii) If you spun the wheel 50 times and the results were as shown in the table in part **(b)**, would you make a profit?

3 A pyramid has a square base with an area of 1440 cm^2. Each of the pyramid's triangular faces is identical and has an area of 840 cm^2. The height of the pyramid is:

A $20\sqrt{3}$ cm **B** 40 cm **C** $30\sqrt{2}$ cm **D** $4\sqrt{145}$ cm

4 The square *KLMN*, with side length 2 units, is inscribed in a circle as shown. Using each side of the square as a diameter, semicircular arcs are drawn. The total area of the shaded region outside the circle and inside the semicircles is:



A 4 units^2 **B** π units^2

C 2π units^2 **D** $(2\pi - 4)$ units^2

5 Find the value of x in each of the following.

(a) $x^{-1} = 3^{-1} + 4^{-1}$ **(b)** $x + 27^{\frac{1}{3}} = 125^{\frac{1}{3}}$ **(c)** $2^x = 2(16^{12}) + 2(8^{16})$

(d) $2^{2x-4} = 8$ **(e)** $16^x = 2^{x+5} - 2^{x+4}$

Answers

Worked solutions and answers to *all* activities appear with the teacher resources on the eBook.

Chapter 1

Recall 1 (p. 2)

1 (a) 0.17 (b) 0.06 (c) 0.192
(d) 0.048 (e) 0.0201 (f) 0.0025

2 (a) \$3000 (b) \$600 (c) \$33 250 (d) \$3375

3 (a) (i) 80 (ii) 60 (iii) 50 (iv) 120
(b) (i) 30 (ii) 42 (iii) 54 (iv) 15

4 (a) 49 (b) 8 (c) 1 000 000
(d) 0.36 (e) $\frac{1}{32}$ (f) 1.4641

5 (a) 0.5 (b) 0.25 (c) 8.75
(d) 0.5 (e) 0.25 (f) 0.4

Exercise 1.1 (p. 9)

1 (a) (i) \$3200 (ii) \$43 200
(b) (i) \$2700 (ii) \$32 700
(c) (i) \$10 500 (ii) \$40 500
(d) (i) \$24 000 (ii) \$64 000
(e) (i) \$281 250 (ii) \$431 250
(f) (i) \$468 000 (ii) \$708 000
(g) (i) \$22 275 (ii) \$322 275
(h) (i) \$27 912.50 (ii) \$377 912.50
(i) (i) \$6175 (ii) \$19 175
(j) (i) \$6794 (ii) \$22 594
(k) (i) \$28 125 (ii) \$73 125
(l) (i) \$13 860 (ii) \$49 860
(m) (i) \$3375 (ii) \$13 375
(n) (i) \$10 200 (ii) \$26 200

2 (a) $P = \$4000$ (b) $P = \$8000$ (c) $I = 14\%$ p.a.
(d) $I = 11\%$ p.a. (e) $T = 10$ years (f) $T = 12$ years

3 (a) \$756 (b) \$6356 (c) \$6390.53
(d) \$790.53 (e) \$34.53

4 (a) \$9240 (b) \$9419.20

5 1.2% p.a. 6 \$129.17

7 B 8 5.5% 9 6.3%

10 Students' own answers. E.g. if the times were 5 and 7 years, the rates would be 7% and 5% respectively.

Exercise 1.2 (p. 13)

1 (a) (i) \$43 200 (ii) \$3200
(b) (i) \$32 700 (ii) \$2700
(c) (i) \$42 076.55 (ii) \$12 076.55
(d) (i) \$71 834.25 (ii) \$31 834.25
(e) (i) \$914 750.94 (ii) \$764 750.94
(f) (i) \$1 587 447.88 (ii) \$1 347 447.88
(g) (i) \$322 348.70 (ii) \$22 348.70
(h) (i) \$378 011.55 (ii) \$28 011.55
(i) (i) \$20 465.10 (ii) \$7465.10
(j) (i) \$23 770.16 (ii) \$7970.16
(k) (i) \$82 809.23 (ii) \$37 809.23
(l) (i) \$52 229.19 (ii) \$16 229.19
(m) (i) \$13 695.26 (ii) \$3695.26
(n) (i) \$28 828.16 (ii) \$12 828.16

2 Lily (a) (i) \$5830.96 (ii) \$5250
(b) Compound interest is \$580.96 higher.
Jarrah (a) (i) \$39 939.82 (ii) \$29 925
(b) Compound interest is \$10 014.82 higher.

3 C

4

Amount (A)	Principal (P)	Interest rate p.a. (r)	Time in years (n)
\$1359.81	\$500	8%	13
\$43 742.08	\$24 600	12.2%	5
\$1 048 209.60	\$1 000 200	4.8%	1
\$151.71	\$84	3%	20
\$60.50	\$50	10%	2

5 (a) (i) simple interest (ii) 40% p.a., \$12
(b) (i) compound interest (ii) \$994
(c) (i) compound interest (ii) \$25 729
(d) (i) compound interest (ii) \$51 452
(e) (i) simple interest (ii) \$1250
(f) (i) simple interest (ii) 12% p.a.

6 A

7 (a) 100 years (b) 47 years

8 5% p.a.

9 (a) The simple interest by \$13.52.
(b) No, the compound rate would be better, earning him \$57.20 more than the S.I. investment.

10 Students' own answers. E.g. \$12 000 invested at 11.8% p.a. for 2 years would give a final value of \$14 999 (clearly within the \$100 buffer!).

11 Students' own answers. E.g. $R \times n = 0.8$, e.g. 16% p.a. for 5 years, 15% p.a. for 5 years and 4 months, 12% p.a. for 6 years and 8 months.

Exercise 1.3 (p. 17)

1 (a) \$20 793.60 (b) \$20 972.74 (c) \$21 068.94

2 (a) \$9528.13 (b) \$9552.42

3 B 4 C

5 (a) \$64 656.50 (b) \$65 731.65 (c) \$66 472.15

6 \$647.54 7 \$13 499.14

8 \$5468.12 and \$5459.91; Martina is better off with the 4.5% p.a. compounded quarterly. The difference is \$8.21.

9 (a) (i) \$3215.09 (best return) (ii) \$2879.21

(iii) \$2545.44

(b) No

(c) (i) $I \approx 16\%$ of P (ii) $I \approx 14\%$ of P (iii) $I \approx 13\%$ of P

10 The statement means that if compound interest is calculated more frequently (e.g. every month instead of every quarter), it will result in a greater amount of interest. Quarterly compounding gives interest of \$4859.47. (48.6%) Monthly compounding gives interest of \$4898.46. (49.0%) Weekly compounding gives interest of \$4913.66. (49.1%)

11 Using the suggested example, simple interest is \$600 and interest compounded annually is \$630.50. The principal gets bigger after each time period.

Half-time 1 (p. 19)

1 17.27% p.a.

2 (a) \$1890 (b) \$5707.20 (c) \$161 376

3 \$8889.26

4 (a) \$5450.24 (b) \$2291.73 (c) \$817 713.85

5 (a) \$10 866 (b) \$9000

(c) Compound interest is \$1866 more than simple interest.

6 (a) \$50 649.49 (b) \$51 824.82 (c) \$52 059.90

7 B

Exercise 1.4 (p. 22)

1 (a) \$5996 (b) \$5437 (c) \$5456.76 (d) B

2 (a) 10.2% p.a. (b) 3.24% p.a. (c) 2.6% p.a. (d) B

3 (a) 12 years (b) 10 years (c) 4.2 years

4 (a) \$5956.42 (b) \$956.42

(c) only marginally better, I = \$956.46

(d) \$956.42 (e) 4.47% p.a.

(f) At least 4 years, as it would be worth \$5788.13 at the end of the third year and \$6077.53 at the end of the fourth year.

5 2

6 (a) every 4 months (b) 9.3% p.a.

(c) monthly compounding

7 At least every 4 months.

8 Rounding error: $\frac{0.065}{4} = 0.01625$, P = \$18 751.14

9 11% p.a. with annual compounding, 10.5% p.a. compounding half-yearly, 10.2% p.a. compounding daily

Exercise 1.5 (p. 26)

1 (a) 7% p.a. (b) 9.4% p.a. (c) 13.52% p.a.

(d) 21.73% p.a. (e) 17.88% p.a. (f) 6.45% p.a.

2 (a) (i) $R = 7.3\%$ p.a., $n = 5$ (ii) 7.52% p.a.

(b) (i) $R = 15.5\%$ p.a., $n = 12$ (ii) 16.65% p.a.

(c) (i) $R = 21\%$ p.a., $n = 4$ (ii) 22.71% p.a.

(d) (i) $R = 16\frac{1}{4}\%$ p.a., $n = 3$ (ii) 17.15% p.a.

(e) (i) $R = 20\frac{1}{3}\%$ p.a., $n = 365$ (ii) 22.54% p.a.

(f) (i) $R = 5.25\%$ p.a., $n = 2$ (ii) 5.32% p.a.

3 B

4 (a) Q Bank: 15.87% p.a., D Bank: 16.06% p.a.

(b) D Bank

5 (a) 22.13% p.a. (b) 22.26% p.a.

(c) Personal loan because the effective rate is lower.

6 (a) 25.85% p.a. (b) online

7 (a) 15% p.a. compounded daily—higher rate and more frequent compounding

(b) 10.5% p.a. compounded weekly—rate is the same so more frequent compounding is better

(c) 12% p.a. compounded daily—compounding frequency is the same so higher rate is better

(d) 15% p.a. compounded annually for 2 years—compounded beats flat rate over 2 years

8 Yes. In each case choose the other option because for loans we want the lowest effective rate.

9 Option C has the highest effective rate but all the options look too good to be true.

10 (a) car company loan (b) flat rate

11 1-year loan of \$10 000 at 6% p.a. compounded daily. Nominal rate is 6% p.a. The interest is compounded, not simple. However, the same amount of interest will be paid if \$10 000 is borrowed for a year at a simple interest rate of 6.18% p.a. Either way nominal isn't the same as simple.

12 42% p.a. compounded half-yearly, 40% p.a. compounded quarterly

Exercise 1.6 (p. 32)

1 (a) (i) \$10 500 (ii) \$19 500

(b) (i) \$28 500 (ii) \$19 000

(c) (i) \$100 000 (ii) No value

(d) (i) \$772 225 (ii) \$17 775

(e) (i) \$60 900 (ii) \$84 100

(f) (i) \$107 250 (ii) \$192 750

(g) (i) \$4928.13 (ii) \$42 571.87

(h) (i) \$76 159.13 (ii) \$76 540.87

2 (a) (i) \$20 870.65 (ii) \$9 129.35

(b) (i) \$25 667.10 (ii) \$21 832.90

(c) (i) \$35 848.59 (ii) \$64 151.41

(d) (i) \$290 950.49 (ii) \$499 049.51

(e) (i) \$93 037.95 (ii) \$51 962.05

(f) (i) \$207 293.23 (ii) \$92 706.77

(g) (i) \$42 285.67 (ii) \$5214.33

(h) (i) \$83 275.25 (ii) \$69 424.75

3 (a) (i) \$6000 (ii) \$2/hour

(iii) \$20 (iv) \$6130

(b) (i) \$1000 (ii) 50c/hour

(iii) \$500 (iv) \$500

4 \$526.50 5 B 6 \$2240

7 Dep 1st year = \$9000, C dep 1st year = \$9000

Written-down value 1st year = \$36 000

Dep 2nd year = \$9000, C dep 2nd year = \$18 000

Written-down value 2nd year = \$27 000

Dep 3rd year = \$9000, C dep 3rd year = \$27 000

Written-down value 3rd year = \$18 000

8 C

9 \$2923.24. Reducing balance depreciation is less so written-down value after 4 years is higher than under straight-line depreciation.

10 \$485.82

11 Written-down value 1st year = \$36 000, C dep 1st year = \$9000 Dep 1st year = \$9000

Written-down value 2nd year = \$28 800, C dep 2nd year = \$16 200 Dep 2nd year = \$7200

Written-down value 3rd year = \$23 040, C dep 3rd year = \$21 960 Dep 3rd year = \$5760

1st year is the same as it was for linear. After that, the depreciation slows down and depreciation for each year is less than the year before. After the 1st year, written-down value is also better than it was for linear.

12 C

13 (a) D (b) A

14 (a) \$3916.67 (b) \$6083.33

15 \$9943.11 16 \$2415

17 Written-down value after the 1st year is \$3353.75

Written-down value after the 2nd year is \$2768.75

18 (a) (i) \$50 000 (ii) \$36 000 (iii) \$25 800 (iv) \$18 390

(b) after 21 years

19 5% p.a. and 14 years, 10% p.a. and 7 years, 20% p.a. and 3.5 years, 70% p.a. and 1 year. Must have $r \times n = 0.7$

20 Any values that satisfy $0.3 = (1 - r)^n$; e.g. 70% p.a. and 1 year, 45.23% p.a. and 2 years, 21.4% p.a. and 5 years

Chapter review 1 (p. 35)

1 (a) \$5478 (b) \$41 998

2 (a) D (b) D

3 (a) \$25 913.02 (b) \$26 221.74 (c) \$26 438.68

4 \$30 000

5 (a) 8.7% p.a. (b) 9.5% p.a.

(c) 4% p.a. (d) 18.7% p.a.

6 (a) \$39 000 (b) \$41 741.30

7 11.5% p.a. 8 \$59 193 9 B

10 8.1% p.a. 11 B

12 If the prime cost of a photocopier is \$10 800 and it has a scrap value of \$500, then the total possible depreciation is \$10 300. If it has an expected life of 206 000 pages, then the unit cost depreciation is \$0.05 per page. After it copies 125 000 pages, the depreciation is \$6250 and the written-down value is \$4550.

13 4.2 years

14 (a) 11% p.a. compounded quarterly—lower nominal rate and less frequent compounding.

(b) 10% p.a. compounded monthly—same nominal rate but less frequent compounding.

(c) 17% p.a. compounded daily—same compounding period but lower rate.

(d) 12% p.a. flat rate for 2 years—after 1 year, flat rate is cheaper than compounded.

Chapter 2

Recall 2 (p. 38)

1 (a) 3 (b) 2 (c) 4

2 (a) 4 (b) -6 (c) 1 (d) -12

3 (a) $3a + 5b$ (b) $24x - 8y$

4 (a) $3x + 6$ (b) $5x - 15$ (c) $5x - 8$

5 (a) $-2x - 6$ (b) $-8x + 40$ (c) $-x - 2$

6 (a) $9x^2$ (b) $4y^2$ (c) $49k^2$

7 (a) $14x^2$ (b) $-8p^2$ (c) $2y^2$

8 (a) $x = \frac{y-3}{2}$ (b) $x = 2ac + 4b$ (c) $x = \frac{3s + at^2}{5}$

9 (a) 3, 7 (b) 2, 12 (c) -5, -4

(d) -3, 6 (e) -10, 2 (f) -8, 5

10 (a) $\frac{11}{12}$ (b) $\frac{11}{24}$ (c) $\frac{5a}{3}$

11 (a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$

Exercise 2.1 (p. 40)

1 (a) $4x + 12$ (b) $8x + 16$ (c) $12x + 132$

(d) $12x - 2$ (e) $14x - 35$ (f) $5x^2 - 4x$

(g) $-2x - 2$ (h) $-x^2 + 7x$ (i) $-3x^2 + 6x$

(j) $-3x^2 - 18x$ (k) $-2x^2 + 18x$ (l) $-4x^2 - 8x$

(m) $x^2 + 5x + 6$ (n) $2x^2 + 5x + 2$ (o) $5x^2 + 17x + 6$

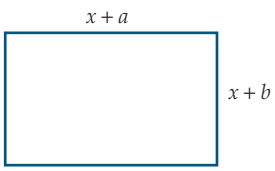
(p) $2x^2 - 11x - 30$ (q) $10x^2 + 13x + 32$ (r) $12x^2 - 16x - 6$

2 (a) $x^2 + 4x + 3$ (b) $x^2 + 6x + 8$ (c) $x^2 + 9x + 20$
(d) $x^2 + 5x - 24$ (e) $x^2 - 2x - 63$ (f) $x^2 + 4x - 21$
(g) $x^2 - 8x + 7$ (h) $x^2 - 9x + 20$ (i) $x^2 - 12x + 27$
(j) $2x^2 - x - 6$ (k) $4x^2 + 7x - 15$ (l) $7x^2 - 27x - 4$
(m) $6x^2 - 17x + 5$ (n) $-5x^2 + 38x - 21$ (o) $-18x^2 + 27x - 10$

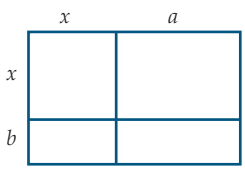
3 (a) $8x^2 + 14xy + 15y^2$ (b) $35x^2 + 44xy + 12y^2$
(c) $15x^2 - 44xy + 32y^2$ (d) $15a^2 - 2ab - 8b^2$
(e) $8p^2 + 22pr - 21r^2$ (f) $27m^2 + 3mn - 2n^2$
(g) $15c^2 + 31cd + 10d^2$ (h) $8g^2 - 22gh + 15h^2$
(i) $12b^2 + 4bf - 21f^2$

4 (a) $3x^2 - 11x - 4$ mm^2 (b) $12x^2 - x - 63$ cm^2
(c) $4x^2 + 9x - 2.5$ km^2

5 (a)



(b) $A = (x + a)(x + b)$ (c) $A = x^2 + ax + bx + ab$
(d)



(e) $x^2 + ax + bx + ab$. This is the same as the area found in part (c).

6 (a) $x + 5$ and $x - 2$
(b) $A = (x + 5)(x - 2) = x^2 + 3x - 10$
(c) x^2 and $5x$
(d) $2x + 10$
(e) $A = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$
(f) The answers are the same.

7 (a) $55 - y$ m (b) $x(55 - y)$ m^2
(c) $55x$ m^2 (d) $(x - 30)(55 - y)$ m^2
(e) $55x - xy - 1650 + 30y$ (f) $45x - 1350$ m^2
(g) $10x + 1350$

8 Students' own answers, e.g. $(3x - 4)(2x + 5) = 6x^2 + 7x - 20$.

Exercise 2.2 (p. 44)

1 (a) $4 - x^2$ (b) $49 - x^2$ (c) $36 - x^2$
(d) $4x^2 - 1$ (e) $9x^2 - 16$ (f) $25 - 9x^2$
(g) $x^2 + 12x + 36$ (h) $x^2 + 22x + 121$ (i) $64 + 16x + x^2$
(j) $x^2 - 6x + 9$ (k) $x^2 - 14x + 49$ (l) $x^2 - 24x + 144$
(m) $4x^2 + 20x + 25$ (n) $9x^2 + 12x + 4$ (o) $49x^2 - 28x + 4$

2 (a) $x^3 + 10x^2 + 24x$ (b) $x^3 + 4x^2 - 5x$
(c) $x^3 - 3x^2 - 4x$ (d) $-x^3 + x^2 + 6x$
(e) $-3x^3 + 9x^2 - 6x$ (f) $-4x^3 + 20x^2 + 96x$
(g) $x^3 + 11x^2 + 38x + 40$ (h) $x^3 - 4x^2 - 7x + 10$
(i) $x^3 - 11x^2 + 36x - 36$ (j) $x^3 + 5x^2 - 9x - 45$
(k) $x^3 + 7x^2 - 4x - 28$ (l) $2x^3 + x^2 - 32x - 16$
(m) $x^3 + 5x^2 - 8x - 48$ (n) $x^3 + 3x^2 - 4$
(o) $x^3 - 7x^2 - 24x + 180$

3 B

4 (a) $4x^2 + 28xy + 49y^2$ (b) $49x^2 + 84xy + 36y^2$
(c) $25x^2 + 80xy + 64y^2$ (d) $16x^2 - 8xy + y^2$
(e) $36x^2 - 60xy + 25x^2$ (f) $81x^2 - 126xy + 49y^2$
(g) $25y^2 - x^2$ (h) $9x^2 - y^2$
(i) $4x^2 - 9y^2$ (j) $6x^3 + 7x^2 - 5x$
(k) $15x^3 - 8x^2 - 12x$ (l) $24x^3 + 78x^2 + 63x$
(m) $-15x^3 + 8x^2 + 16x$ (n) $-30x^3 - 58x^2 - 24x$
(o) $-24x^3 + 76x^2 - 60x$

5 (a) $(2x + 3)(2x + 3) = 4x^2 + 12x + 9$
(b) $\frac{1}{2}(x + 5)(x - 5) = \frac{1}{2}(x^2 - 25)$ or $\frac{1}{2}x^2 - \frac{25}{2}$

6 (a) $x^3 + (a + b)x^2 + abx$
(b) (i) $a + b$ (ii) ab
(iii) $a + b = -4$ and $ab = -12$. Using the expansion from (a) this shows that $x(x - 6)(x + 2) = x^3 - 4x^2 - 12x$.
(c) (i) $a = 25$, $b = -4$ (ii) $x^3 + 21x^2 - 100x$

7 (a) $(x - 2)(x)(x + 2)$
(b) $x^3 - 4x$
(c) It is easier to have x as the second number as the other two expressions, $(x - 2)$ and $(x + 2)$, form a difference of two squares which is simpler to multiply.

8 (a) $1296 - x^2$ cm^2 (b) $(36 - x)(36 + x)$ (c) 1260 cm^2

9 (a) $x^2 - 2ax$ (b) $x^2 - 2ax$
(c) They are equivalent.
(d) Rectangular lawn area $= x(x - 2a)$, square lawn area $= (x - a)^2 - a^2$. They will cost equal amounts to turf as we have shown that these expressions are equivalent.
(e) \$19 092

10 (a) $(x - 2)^2(x + 3)$ (b) 28 cm^3
(c) When $x = 5$, dimensions are 3 cm, 8 cm and 3 cm, so the volume $= 3 \times 8 \times 3 = 72$ cm^3.
(d) $x^3 - x^2 - 8x + 12$; 72 cm^3
(e) height and breadth 3 cm, length 8 cm
(f) $x = 3$ (g) 6 cm^3

11 If $(x^2 + a^2)^3 = 0$ then $(x^2 + a^2) = 0$ and $x^2 = -a^2$, which can only be true if x and a are both equal to zero. No other values of x and a are possible as you cannot have a square number equal to a negative number.

12 Examples: $a = 2, b = 1, c = 0, d = 1$; $a = 1, b = 0, c = 1, d = 1$

13 (a) $x^2 + y^2$
(b) Examples: $x = 3, y = 4$; $x = 5, y = 12$

14 (a) -4, -5 (b) 4, 5 (c) 2, -3

Exercise 2.3 (p. 48)

1 (a) $4(x+2)$ (b) $3(6x-5)$ (c) $3(8x-3)$
(d) $x(y+7)$ (e) $3k(4x+7)$ (f) $4b(8a+5)$
(g) $7y(x^2+y)$ (h) $3x(3x+y)$ (i) $8m(6m+1)$
(j) $-x(6+x)$ (k) $-x(4-x)$ (l) $-2a(12a-5)$
(m) $6ab(1-3a)$ (n) $8xy(5x+2y)$ (o) $2cd(5d-1)$

2 (a) $(k+1)(x+4)$ (b) $(x-4)(a+6)$
(c) $(n-7)(m+2)$ (d) $(p+3)(x-9)$
(e) $(q-2)(n-7)$ (f) $(u-4)(x-5)$
(g) $(c+5)(d+1)$ (h) $(d+2)(a-1)$
(i) $(m+7)(k-1)$ (j) $(x-3)(x+5)$

3 (a) $(y+5)(x+3)$ (b) $(b+2)(a+7)$
(c) $(n+4)(m+k)$ (d) $(y+3)(x-1)$
(e) $(3+c)(a-b)$ (f) $(m+2)(k-4)$
(g) $(5b-4)(3a-2)$ (h) $(2b-5)(3a-2)$
(i) $(2n-3)(4m-3)$ (j) $(v-a)(x-y)$
(k) $(q-n)(p-r)$ (l) $(h-d)(g-f)$
(m) $(n+t)(m^2+p)$ (n) $(ab-c)(a+b)$
(o) $(dk-r)(d-1)$ (p) $(9y-5)(3x+2)$
(q) $(4a-5)(9-c)$ (r) $(5v-3)(5w+1)$
(s) $(8a^2-3b)(4ac-5)$ (t) $(3d-5f)(7d-4h)$
(u) $(mr-5)(4p+mn)$

4 (a) True (b) False (c) False
(d) True (e) False (f) False

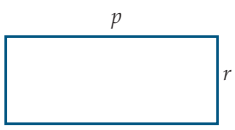
5 D 6 B 7 A

8 $8(3y-47x)$ cents

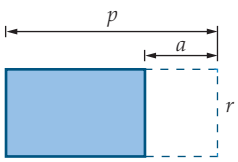
9 $8x-x^2=x(8-x)$

10 (a) $a(p+5)+b(p+5)$ (b) $(a+b)(p+5)$
(c) $3a(p+5)$

11 (a)

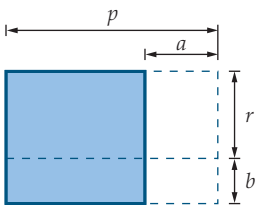


(b)



Area $= pr - ar$

(c)



Area added $= pb - ab$

(d) $A = lb$
$= (p-a)(r+b)$
$= pr + pb - ar - ab$

(e) Area $= (p-a)(r+b)$

(f) part (e) is the factorised form of part (d)

12 (a) $\frac{2\pi r^3}{3}+\frac{\pi r^2 h}{3}$ (b) $\frac{\pi r^2}{3}(2r+h)$ (c) 150.82 cm^3

(d) Cone volume is $\frac{\pi r^2(2r)}{3}=\frac{2\pi r^3}{3}$, which equals the half-sphere volume of the ice-cream.

13 (a) $2x^2+4xy\text{ cm}^2$ (b) $2x(x+2y)$
(c) $x=12, y=3; x=8, y=4$

Exercise 2.4 (p. 53)

1 (a) $(x+2)(x+4)$ (b) $(x-3)(x+6)$
(c) $(x-1)(x+7)$ (d) $(x+5)(x-2)$
(e) $(x-4)(x-3)$ (f) $(x-4)(x+3)$
(g) $(x+4)(x-2)$ (h) $(x+1)(x-3)$
(i) $(x-8)(x-2)$

2 (a) $(x+1)(x+2)$ (b) $(y+2)(y+7)$
(c) $(a-4)(a-6)$ (d) $(k+3)(k+8)$
(e) $(m-5)(m+3)$ (f) $(x-6)(x+5)$
(g) $(p-2)(p-8)$ (h) $(d-1)(d-4)$
(i) $(y-3)(y-2)$ (j) $(m-6)(m+2)$
(k) $(c-7)(c-3)$ (l) $(g+7)(g-5)$
(m) $2(y+3)(y+4)$ (n) $3(m+2)(m+4)$
(o) $5(x+4)(x-5)$ (p) $-4(a+3)(a-1)$
(q) $-3(k+3)(k-7)$ (r) $-5(x+7)(x-3)$

3 (a) $(z-5)(z-3)$ (b) $(k+6)(k-1)$
(c) $(z+3)(z+7)$ (d) $(t+4)(t-5)$
(e) $(r+6)(r-5)$ (f) $(t-11)(t-2)$
(g) $(z+6)(z-7)$ (h) $(m-5)(m-4)$
(i) $(y+12)(y-5)$ (j) $(n-7)(n+4)$
(k) $(p+12)(p-3)$ (l) $(a-6)(a-8)$

4 C

5 $(2x+4)$ cm

6 $(3x-12)$ mm

7 (a) $x-30$ (b) 120 cm (c) 190 cm

8 (a) $2w^2-(6w-4)=2w^2-6w+4$
(b) $(2w-2)(w-2)=2(w-1)(w-2)$
(c) 1 m

9 $x^2-x-2=(x+1)(x-2)$ and $x^2+x-42=(x+7)(x-6)$

10 (a) $x+8$ and $x-3$ are the factors so possible dimensions could be 12 units by 1 unit or 14 units by 3 units.
(b) Area $=12\times1=12$ units^2 or Area $=14\times3=42$ units^2.

Exercise 2.5 (p. 57)

1 (a) $(p-2)(p+2)$ (b) $(d-10)(d+10)$
(c) $(3-k)(3+k)$ (d) $3(p-5)(p+5)$
(e) $5(4-m)(4+m)$ (f) $8(k-2)(k+2)$
(g) $(2y-7)(2y+7)$ (h) $(5-4x)(5+4x)$
(i) $(6m-5n)(6m+5n)$ (j) $(3x-4)(3x+4)$
(k) $(2-5a)(2+5a)$ (l) $(9d-8)(9d+8)$

2 (a) $(x-1)(x+5)$ (b) $(a-6)(a+4)$
(c) $(-k-1)(k+9)$ or $-(k+1)(k+9)$
(d) $(5-p)(7+p)$ (e) $4(2-y)(y-1)$
(f) $3(1-x)(3x-1)$ (g) $5(x+3-y)(x+3+y)$
(h) $7(a+5-b)(a+5+b)$ (i) $3(x-6)(x-2)$

3 (a) $(x-\sqrt{2})(x+\sqrt{2})$ (b) $(\sqrt{7}-p)(\sqrt{7}+p)$
(c) $(2k-\sqrt{5})(2k+\sqrt{5})$ (d) $(3p-\sqrt{2})(3p+\sqrt{2})$
(e) $3(a-\sqrt{11})(a+\sqrt{11})$ (f) $2(b-\sqrt{3})(b+\sqrt{3})$

4 (a) $(x-y+1)(x+y+3)$ (b) $-(2a+5)$
(c) $(2k-m+6)(2k+m)$ (d) $16x$
(e) $(x-2)(3x+4)$ (f) $(4x+3)(6x-11)$
(g) $(x+2-\sqrt{3})(x+2+\sqrt{3})$ (h) $(y-3-\sqrt{6})(y-3+\sqrt{6})$
(i) $(x+8-\sqrt{5})(x+8+\sqrt{5})$
(j) $(x-5-\sqrt{13})(x-5+\sqrt{13})$
(k) $(x-1-\sqrt{19})(x-1+\sqrt{19})$
(l) $(2x+1-\sqrt{2})(2x+1+\sqrt{2})$

5 (a) $(x+3)(x+1)$ (b) $(a-4)(a+9)$
(c) $(x-y)(x+y+7)$ (d) $(3a+b)(3a-b+2)$
(e) $(x-4y)(x+4y+2)$ (f) $(2p+q)(2p-q-3)$

6 (a) $(x+3)^2$ (b) $(x+2)^2$ (c) $(x-7)^2$
(d) $(x-4)^2$ (e) $(x+1)^2$ (f) $(x+6)^2$
(g) $(2x+5)^2$ (h) $(3x+4)^2$ (i) $(5x-1)^2$

7 B 8 D 9 A

10 (a) $8(m+5n)(m-5n)$ (b) $6(x-2r)(x+2r)$
(c) $5(a-3c)(a+3c)$ (d) $3(3a-4b)(3a+4b)$
(e) $2(6x-7y)(6x+7y)$ (f) $7(5t-2p)(5t+2p)$
(g) $6(3k-2m)(3k+2m)$ (h) $2(xy-4z)(xy+4z)$
(i) $3(pq-2n)(pq+2n)$

11 (a) $(4c+5d)(4c-5d+2)$ (b) $(k-8)(k+8+m)$
(c) $(9-7a)(9+7a-3d)$ (d) $(4-3y)(4+3y-4w)$
(e) $(5m+3n)(5m-3n-4)$ (f) $(12p-7r)(12p-q+7r)$

12 (a) $x^2-y^2=56$ (b) $(x-y)(x+y)=56$
(c) The difference between consecutive odd numbers is 2.
(d) $4(y+1)=56$ (e) $x=15$, $y=13$

13 (a) $\frac{1}{2}(a-b)(a+b)$ cm^2 (b) $4b^2$ cm^2
(c) $a=15$ cm, $b=5$ cm

14 (a) $(a-b)(a+b)$ (b) two
(c) $a+b$ (d) $(a^2-b^2)(a^2+b^2)$
(e) $(a-b)(a+b)(a^2+b^2)$ (f) three
(g) $(a+b)(a^2+b^2)$ (h) $(a^4-b^4)(a^4+b^4)$
(i) $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$
(j) $(a+b)(a^2+b^2)(a^4+b^4)$
(k) $(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$
(l) $(2^{16}-1) = (2^{16}-1^{16}) \div (2-1)$
$= (2+1)(2^2+1^2)(2^4+1^4)(2^8+1^8)$
$= 3\times5\times17\times257$

15 (a) x^2-y^2 m^2 (b) $2(x-y)$ m^2
(c) $(x-y)(x+y+2)$ m^2

16 (a) $a^2+b^2=(m-n)^2+(2\sqrt{mn})^2$
$=m^2-2mn+n^2+4mn$
$=m^2+2mn+n^2$
$=(m+n)^2$
$=c^2$
(b) $t^2=(kr)^2-r^2$
$=(kr-r)(kr+r)$
$=r(k-1)r(k+1)$
$=r^2(k-1)(k+1)$
(c) $k=1$ or -1
(d) k cannot be equal to zero as this creates the impossible equation $a^2=-b^2$, and k cannot be a fraction between -1 and 1 as this creates a similar situation as when k is equal to zero, where a^2 is equal to a negative fraction of b^2.

17 (a) x^2 cm^2 (b) 36 cm^2
(c) x^2-36 cm^2 (d) $(x-6)(x+6)$
(e) Length is $x+6$ cm and breadth is $x-6$ cm.
(f) The factors in part **(d)** correspond to the length and breadth of the rectangle.

18 $2x^2+8x-40$ and x^2-8x-9

19 $a^2-b^2+3a+3b=(a+b)(a-b+3)$

Half-time 2 (p. 61)

1 (a) $x^2+8x+12$ (b) $x^2+2x-35$
(c) $3x^2+13x-10$ (d) $2x^2+5x-3$

2 (a) $(x+5)(x-3)$ (b) $(x-2)(x-12)$
(c) $(x+5)(x-8)$

3 (a) $(x+5)(x+7)$ (b) $(x-3)(x-4)$
(c) $(x+7)(x-3)$

4 (a) $7(x+2y)$ (b) $6x(3x+4y)$
(c) $(x-2)(3+y)$ (d) $(x+5)(3+2y)$

5 D 6 C 7 B

8 (a) $(x+7)^2$ (b) $(x-9)^2$ (c) $(3x-5)^2$

9 (a) $(x-3)(x+3)$ (b) $9(x-2y)(x+2y)$
(c) $2(2x-5y)(2x+5y)$ (d) $3(3-4x)(3+4x)$

10 (a) $x^2 - 9$ (b) $x^2 - 64$

(c) $9x^2 - y^2$ (d) $36x^2 - 25y^2$

11 (a) $x^3 + 5x^2 + 6x$ (b) $3x^2 - 12$

(c) $63x^3 + 150x^2 - 141x - 228$ (d) $48x^3 - 156x^2 + 36x$

Exercise 2.6 (p. 64)

1 (a) $(x + 4 - y)(x + 4 + y)$ (b) $(y - 2 - k)(y - 2 + k)$

(c) $(y + 4 - m)(y + 4 + m)$ (d) $(m - 5 - 2n)(m - 5 + 2n)$

(e) $(x + 1 - 3y)(x + 1 + 3y)$ (f) $(q + 3 - 4p)(q + 3 + 4p)$

(g) $(a - 3b - 4)(a - 3b + 4)$ (h) $(x - 5y - 6)(x - 5y + 6)$

(i) $(2a - 3 - 4b)(2a - 3 + 4b)$ (j) $(3c + 5 - 2d)(3c + 5 + 2d)$

2 (a) $(x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$

(b) $(x - 5 - \sqrt{21})(x - 5 + \sqrt{21})$

(c) $(x - 4 - \sqrt{19})(x - 4 + \sqrt{19})$

(d) $(x + 10 - \sqrt{23})(x + 10 + \sqrt{23})$

(e) $(x - 6 - \sqrt{15})(x - 6 + \sqrt{15})$

(f) $(x - 2)(x + 4)$

(g) $(x - 7 - \sqrt{51})(x - 7 + \sqrt{51})$

(h) $(x - 2 - \sqrt{6})(x - 2 + \sqrt{6})$

(i) $(x + 8 - \sqrt{17})(x + 8 + \sqrt{17})$

(j) $\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$

(k) $\left(x + \frac{7}{2} - \frac{\sqrt{13}}{2}\right)\left(x + \frac{7}{2} + \frac{\sqrt{13}}{2}\right)$

(l) $\left(x - \frac{1}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{1}{2} + \frac{\sqrt{17}}{2}\right)$

(m) $\left(x - \frac{5}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{5}}{2}\right)$

(n) $\left(x + \frac{5}{2} - \frac{\sqrt{33}}{2}\right)\left(x + \frac{5}{2} + \frac{\sqrt{33}}{2}\right)$

(o) $\left(x + \frac{1}{2} - \frac{\sqrt{21}}{2}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{2}\right)$

(p) $\left(x - \frac{9}{2} - \frac{\sqrt{29}}{2}\right)\left(x - \frac{9}{2} + \frac{\sqrt{29}}{2}\right)$

(q) $\left(x - \frac{3}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{3}{2} + \frac{\sqrt{17}}{2}\right)$

(r) $\left(x - \frac{7}{2} - \frac{\sqrt{53}}{2}\right)\left(x - \frac{7}{2} + \frac{\sqrt{53}}{2}\right)$

3 (a) $2(x + 1 - \sqrt{6})(x + 1 + \sqrt{6})$

(b) $4(x - 2 - \sqrt{5})(x - 2 + \sqrt{5})$

(c) $10(x - 4 - \sqrt{19})(x - 4 + \sqrt{19})$

(d) $3(x - 1 - \sqrt{5})(x - 1 + \sqrt{5})$

(e) $5(x + 2 - \sqrt{5})(x + 2 + \sqrt{5})$

(f) $7(x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$

4 D 5 A

6 (a) 14 m (b) $(t - 4 - \sqrt{2})(t - 4 + \sqrt{2})$

(c) 2.83 seconds (d) The seagull is at sea level.

(e) The seagull is below sea level, catching a fish.

7 (a) $-5(x - 4)^2 + 75 = -5(x - 4 - \sqrt{15})(x - 4 + \sqrt{15})$

(b) $P = -5(4 \pm \sqrt{15} - 4)^2 + 75$
$= -5(\pm\sqrt{15})^2 + 75$
$= -5 \times 15 + 75 = 0.$

(c) When $x = 0$, $P = -5$, so the company initially operated at a loss. When $x = 4$, $P = 75$, so the company had made a healthy profit after 4 months. When $x = 8$, $P = -5$, so the company made a loss again after 8 months.

8 (a) $x^2 + 14x + 35$ (b) $(x + 7 - \sqrt{14})(x + 7 + \sqrt{14})$

(c) $8 - \sqrt{14}$ km (d) 112.75 km

9 (a) 100 worms

(b) $-(d + 2 + 2\sqrt{26})(d + 2 - 2\sqrt{26})$

(c) When $d = 2(\sqrt{26} - 1)$, $(d + 2 - 2\sqrt{26}) = 0$, so no earthworms remain.

(d) 8.20, so the number of earthworms reduced to zero during the 9th day.

(e) (i) $-(n + 1 + \sqrt{65})(n + 1 - \sqrt{65})$

(ii) When $n = \sqrt{65} - 1$, $(n + 1 - \sqrt{65}) = 0$, so no earthworms remain.

(f) $\sqrt{65} - 1 = 7.06$, so the first colony will stay longer in their compost heap.

10 (a) $(x - a)^2 + 2b - a^2$ (b) $A = 2b - a^2$

(c) (i) $2b - a^2 = 0$ (ii) $a = 2, b = 2$

Exercise 2.7 (p. 68)

1 (a) $(2x + 1)(x + 2)$ (b) $(2y + 7)(y + 1)$

(c) $(5b + 3)(b + 1)$ (d) $(3k - 2)(k - 3)$

(e) $(11a - 12)(a + 1)$ (f) $(5x + 4)(x - 3)$

(g) $(2m - 3)(m + 3)$ (h) $(7x - 9)(x - 2)$

(i) $(11c - 3)(c - 2)$ (j) $(5y + 4)(y + 3)$

(k) $(3p - 4)(p + 4)$ (l) $(2x - 5)(x + 4)$

(m) $(x + 1)(3x + 5)$ (n) $(y - 2)(7y - 4)$

(o) $(3t + 4)(t - 1)$ (p) $(p + 2)(7p + 8)$

(q) $(q - 1)(2q + 5)$ (r) $(x + 2)(5x - 6)$

(s) $(r - 2)(7r - 4)$ (t) $(c - 3)(3c - 7)$

(u) $2(2t - 1)(3t - 1)$

2 (a) $(2x + 3)(2x + 1)$ (b) $(5y + 4)(2y + 1)$

(c) $(9c - 1)(c - 2)$ (d) $(3w + 4)(2w + 3)$

(e) $(5y - 6)(3y + 5)$ (f) $(8x + 7)(x - 2)$

(g) $(2a + 9)(2a - 3)$ (h) $(6k - 5)(4k + 3)$

(i) $(10x - 7)(2x - 3)$ (j) $(6b + 1)(2b + 5)$

(k) $(3g + 2)(2g + 1)$ (l) $(4a + 1)(2a + 5)$

(m) $(4x - 3)(x - 2)$ (n) $(10m + 7)(m + 1)$

(o) $(4p - 5)(3p + 2)$ (p) $(3d - 2)(3d + 4)$

(q) $(6x+11)(x-2)$ (r) $(9n-7)(2n+5)$

(s) $(8y-3)(3y-1)$ (t) $(5x+2)(3x+8)$

(u) $(4x+3)(3x-5)$

3 (a) $2\left(x-\frac{3}{2}-\frac{\sqrt{11}}{2}\right)\left(x-\frac{3}{2}+\frac{\sqrt{11}}{2}\right)$

(b) $3\left(x-2-\sqrt{\frac{19}{3}}\right)\left(x-2+\sqrt{\frac{19}{3}}\right)$

(c) $2\left(x-2-\sqrt{\frac{11}{2}}\right)\left(x-2+\sqrt{\frac{11}{2}}\right)$

(d) $3\left(x-\frac{3}{2}-\sqrt{\frac{11}{12}}\right)\left(x-\frac{3}{2}+\sqrt{\frac{11}{12}}\right)$

(e) $5\left(x-1-\sqrt{\frac{2}{5}}\right)\left(x-1+\sqrt{\frac{2}{5}}\right)$

(f) $2\left(x-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(x-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)$

(g) not possible

(h) not possible

(i) not possible

4 (a) $(3x+1)(2x-3)$ (b) 3.5 m (c) 4 m

5 (a) $12t^2-12t+10-(4t^2+6t+1)$

(b) (i) $8t^2-18t+9$ (ii) $(4t-3)(2t-3)$

(c) The difference in the number of birds must be zero. $0=(2t-3)(4t-3)$. For this to be true either $2t-3=0$ ($t=1.5$) or $4t-3=0$ ($t=0.75$). Hence, when there are 75 or 150 trees there are equal numbers of birds.

6 (a) $(6x+5)(2x-3)$

(b) They each had \$10 initially.

(c) \$17

7 (a) $5x^2+10x=5x(x+2)$

(b) $60x^2+80x-15=5(6x-1)(2x+3)$

(c) $2x^2-4x+20$. This will not factorise completely as once you have factored out the 2 and completed the square, the last term is positive—not the difference of two squares.

Exercise 2.8 (p. 72)

1 (a) $\frac{4x}{5}$ (b) $\frac{k}{7}$ (c) $\frac{5p}{9}$

(d) $2(a-b)$ (e) $\frac{3(x+1)}{2}$ (f) $\frac{7(y-2)}{4}$

(g) $\frac{y(a-4)}{2}$ (h) $\frac{2x(x+5)}{9}$ (i) $\frac{3x(x-4)}{4}$

2 (a) $\frac{5a}{2}$ (b) $\frac{2}{5n}$ (c) $\frac{2r^2}{5}$

(d) $\frac{q}{3p}$ (e) $\frac{2x^2}{3y^5}$ (f) $\frac{9w^2}{5v^2}$

3 (a) $\frac{27m^2}{32}$ (b) $\frac{3k^2}{8}$ (c) $\frac{10d}{3}$

(d) $\frac{5}{6b^3}$ (e) $\frac{12}{7r}$ (f) $\frac{27d^3}{88}$

4 (a) $\frac{x+6}{x-5}$ (b) $\frac{x+2}{x-2}$ (c) $\frac{2x-3}{2x+1}$

(d) $\frac{x-2}{x-7}$ (e) $\frac{b+5}{b+3}$ (f) $\frac{x+7}{x-1}$

5 (a) $\frac{5x}{6}$ (b) $\frac{7x}{10}$ (c) $\frac{5x}{12}$

(d) $\frac{x}{2}$ (e) $\frac{11x}{15}$ (f) $\frac{-7x}{12}$

(g) $\frac{8x-3y}{6}$ (h) $\frac{20x-21y}{28}$ (i) $\frac{99x-64y}{72}$

6 C

7 $\frac{31x}{40}$ 8 $\frac{3m}{10}$

9 Students' own answers, e.g. $\frac{3n}{4}+\frac{2n}{3}$.

10 Ollie divided the numerator and denominator by 4, but wrote the 8 in the numerator instead of the denominator. His answer should have been $\frac{r}{8}$.

Exercise 2.9 (p. 77)

1 (a) $\frac{5x+22}{6}$ (b) $\frac{3x-2}{8}$ (c) $\frac{4x-16}{9}$

(d) $\frac{3x-8}{12}$ (e) $\frac{13x-34}{20}$ (f) $\frac{13x+14}{36}$

(g) $\frac{29x+2y}{12}$ (h) $\frac{38x-19y}{15}$ (i) $\frac{34x+5y}{12}$

(j) $\frac{9x-8y}{30}$ (k) $\frac{3x-4y}{4}$ (l) $\frac{17x+11y}{6}$

2 (a) $\frac{2m^2+1}{m^2-4m}$ (b) $\frac{15+7x}{3x^2+6x}$ (c) $\frac{2x-4}{x^2-25}$

(d) $\frac{3p^2-4}{p^2+p}$ (e) $\frac{33-5h}{h^2-6h}$ (f) $\frac{-2b-1}{b^2+4b+4}$

(g) $\frac{9c+1}{(c+1)(c-1)^2}$ (h) $\frac{9k-4}{k(k+1)(k-1)}$

(i) $\frac{4d+5}{2d(2d+1)(2d-1)}$

3 (a) $\frac{1}{3}$ (b) $\frac{3(5+2x)}{5-2x}$ (c) $\frac{x+4}{x-4}$

(d) $\frac{x-3}{x+3}$ (e) $\frac{x+7}{8}$ (f) $\frac{7}{4}$

(g) $\frac{(x-3)(x+2)}{(x-4)(x-1)}$ (h) $\frac{x+4}{x+1}$

(i) $\frac{(3x-2)(x-3)}{(x-6)(x+2)}$ (j) $\frac{(5x-3)(x-3)}{(x-4)(5x-4)}$

4 (a) $x+8$ (b) $x-9$ (c) $\frac{1}{x-5}$

(d) $\frac{1}{x-7}$ (e) $\frac{2x-1}{2}$ (f) $3(x-2)$

(g) $\frac{2x+3}{2}$ (h) $\frac{3x+5}{2}$ (i) $\frac{x-1}{x-8}$

(j) $\frac{x+4}{x+5}$ (k) $\frac{x+1}{x-2}$ (l) $\frac{x-1}{x+4}$

(m) 1 (n) $\frac{5}{6}$ (o) $\frac{x-5}{6(x+3)}$

(p) $\frac{8(x-7)}{x+5}$ (q) $\frac{x-2}{x+2}$ (r) $\frac{x-3}{x-7}$

5 (a) $\frac{3(x-2)}{x+3}$ (b) $\frac{5(x+5)}{3(x-7)}$ (c) $\frac{x+6}{x+7}$

(d) $\frac{(x+2)(x+1)}{(x+5)(x+8)}$ (e) $\frac{5(x+y+3)}{4}$ (f) $\frac{7(x-1-2y)}{3(3x-2)}$

6 C

7 (a) $\frac{3(b-a)}{3b+a}$ (b) $\frac{3(7-5)}{3\times 7+5}=\frac{3}{13}$

8 (a) $(a-b)(a^2+ab+b^2)=a^3+a^2b+ab^2-a^2b-ab^2-b^3=a^3-b^3$

(b) $(a^2-b^2)(a^4+a^2b^2+b^4)$

(c) $\frac{a^6-b^6}{a^2-b^2}=\frac{(a^2-b^2)(a^4-a^2b^2+b^4)}{a^2-b^2}=a^4+a^2b^2+b^4$

9 (a) $4w^2-\pi w^2$ (b) $\frac{4-\pi}{4}$

10 (a) $\frac{3m}{4\pi r^3}$ kg/m^3

(b) $\frac{d_2}{d_1}=\frac{3m_2}{4\pi r_2^3}\div\frac{3m_1}{4\pi r_1^3}$

$=\frac{3m_2}{4\pi r_2^3}\times\frac{4\pi r_1^3}{3m_1}$

$=\frac{m_2}{m_1}\times\frac{r_1^3}{r_2^3}$

$=\left(\frac{m_2}{m_1}\right)\left(\frac{r_1}{r_2}\right)^3$

(c) $\frac{4}{25}$

11 Students' own answers, e.g. $\frac{x+1}{2}+\frac{1-x}{6}$.

12 Students' own answers, e.g. $\frac{3x+2}{4}-\frac{2x+1}{5}$.

13 Students' own answers, e.g. $\frac{x+1}{x+2}\times\frac{4}{x+1}$.

Exercise 2.10 (p. 82)

1 (a) (i) $s=\pm\sqrt{A}$ (ii) $s=\pm 3\sqrt{3}$

(b) (i) $v=\pm\sqrt{\frac{2K}{m}}$ (ii) $v=\pm\sqrt{5}$

(c) (i) $v=\pm\sqrt{\frac{Fr}{m}}$ (ii) $v=\pm\sqrt{10.5}$

(d) (i) $r=\pm\sqrt{\frac{Gm_1m_2}{F}}$ (ii) $r=\pm\sqrt{33.75}$

(e) (i) $r_2=\pm\sqrt{\frac{g_1r_1^2}{g_2}}$ (ii) $r_2=\pm 14\sqrt{2}$

(f) (i) $V=\pm\sqrt{PR}$ (ii) $V=\pm 3\sqrt{7}$

(g) (i) $I=\pm\sqrt{\frac{P}{R}}$ (ii) $I=\pm\sqrt{\frac{20}{9}}$

(h) (i) $v=\pm\sqrt{\frac{2E}{c}}$ (ii) $v=\pm 4\sqrt{2}$

(i) (i) $D=\pm\sqrt{\frac{4A}{\pi}}$ (ii) $D=\pm 6$

2 (a) (i) $A_y=A^2-A_x$ (ii) $A_y=38$

(b) (i) $c=\frac{b^2-(2ax+b)^2}{4a}$ (ii) $c=5$

(c) (i) $T=\frac{9r^2}{l^2}$ (ii) $T=64$

(d) (i) $z=y-x^2$ (ii) $z=-96$

(e) (i) $z=\frac{y}{x^2}$ (ii) $z=0.28$

(f) (i) $b=cx^2-a$ (ii) $b=16$

(g) (i) $y=x^2+z$ (ii) $y=22$

(h) (i) $c=\frac{a+b}{x^2}$ (ii) $c=\frac{1}{3}$

(i) (i) $y=zx^2$ (ii) $y=96$

3 C

4 (a) $r=\sqrt{\frac{A}{\pi}}$

(b) The radius is a distance and must be positive.

(c) 3.57 cm (2 d.p.)

5 (a) $r=\sqrt[3]{\frac{3V}{4\pi}}$

(b) Because it is a cube root and not a square root.

(c) 3.10 cm (2 d.p.)

6 (a) $h=\frac{3V}{\pi r^2}=1.52$ m (2 d.p.)

(b) $r=\sqrt{\frac{3V}{\pi h}}=4.37$ cm (2 d.p.)

7 (a) 41m/s (b) The car is slowing down.

8 (a) $k=1243.57$ (b) 2.1 Hz

9 (a) Line 3 does not have ± symbol.

(b) One possible explanation is that if $x^2=9$, then x can be 3 or -3, since $(3)^2=9$ and $(-3)^2=9$.

10 Sample answers:

(a) $a=2b+c^2$ $c=\pm\sqrt{a-2b}$

(b) $d=2e+\sqrt{f}$ $f=d^2-4de+4e^2$

Chapter review 2 (p. 85)

1 (a) $10x + 35$ (b) $x^2 - x - 6$
(c) $x^2 + 10x + 24$ (d) $8x^2 + 10x - 3$

2 (a) $x^2 + 12x + 36$ (b) $4x^2 - 12x + 9$
(c) $25 - x^2$ (d) $9x^2 - 49$

3 (a) $4x^2 - 20x - 56$ (b) $-3x^3 - 15x^2 + 72x$
(c) $x^3 - 2x^2 - 29x + 30$ (d) $-2x^3 - 7x^2 + 10x + 24$

4 (a) $7(x + 3)$ (b) $4a(3b - 2)$ (c) $6x(3x + 4)$

5 (a) $(c - 4)(d + 6)$ (b) $(x + 2)(x - 3)$
(c) $(a + 1)(5a - 2)$ (d) $(3y - 5)(2y - 7)$

6 (a) $(x + 5)(x + 6)$ (b) $(x + 2)(x - 12)$

7 (a) $(x + 5)^2$ (b) $(x - 9)^2$ (c) $(3x - 4)^2$ (d) $(2x + 1)^2$

8 (a) $(x - 3)(x + 3)$ (b) $(5 - y)(5 + y)$
(c) $(2x - y)(2x + y)$ (d) $(7x - 4)(7x + 4)$
(e) $(6x - 9y)(6x + 9y) = 9(2x - 3y)(2x + 3y)$
(f) $(8 - 3x)(8 + 3x)$ (g) $(2x - 7)(2x - 3)$
(h) $-13(2x + 3)$ (i) $5(2x - 1)$
(j) $(x - \sqrt{3})(x + \sqrt{3})$ (k) $(x + 7 - \sqrt{5})(x + 7 + \sqrt{5})$
(l) $(x - 4 - \sqrt{7})(x - 4 + \sqrt{7})$

9 (a) $(x + 3 - \sqrt{11})(x + 3 + \sqrt{11})$
(b) $(x - 5 - \sqrt{2})(x - 5 + \sqrt{2})$
(c) $(x - 2 - \sqrt{11})(x - 2 + \sqrt{11})$
(d) $(x + 4 - \sqrt{3})(x + 4 + \sqrt{3})$

10 (a) $(3x + 4)(4x - 5)$
(b) $(2x + 1)(6x - 5)$
(c) $(2x + 7)(3x - 2)$
(d) $3\left(x + \frac{5}{6} - \frac{\sqrt{13}}{6}\right)\left(x + \frac{5}{6} + \frac{\sqrt{13}}{6}\right)$
(e) $4\left(x - \frac{7}{8} - \frac{\sqrt{129}}{8}\right)\left(x + \frac{7}{8} + \frac{\sqrt{129}}{8}\right)$
(f) $2\left(x + \frac{3}{2} - \frac{\sqrt{3}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{3}}{2}\right)$

11 (a) $\frac{2x}{5}$ (b) $\frac{a - 3}{3}$ (c) $\frac{5(x - 1)}{4}$ (d) $3a$

12 (a) $\frac{7x}{6}$ (b) $\frac{12x + 21y}{14}$ (c) $\frac{x + 4}{x + 2}$

13 (a) $x + 2$ (b) $\frac{(x + 1)(3x + 7)}{(x - 7)(x - 1)}$
(c) $2x - 1$ (d) $\frac{2(x + 5)}{x + 4}$

14 (a) $\frac{8x - 7}{15}$ (b) $\frac{4x - 31}{30}$ (c) $\frac{2x - y - 9}{10}$

15 (a) $b = \frac{A}{l}$ (b) $l = \frac{A^2g}{4\pi^2}$
(c) $r = \pm\sqrt{\frac{SA}{4\pi}}$ (d) $x = \pm\sqrt{\frac{2(a + z)}{y}}$

16 $a = 3, b = -2$

17 (a) -5, 6 (b) 6 kg gain (c) 5 kg loss
(d) From April to September John lost 11 kg in weight.

18 (a) $ax + 2x + ay + 2y$ (b) $(a + 2)(x + y)$
(c) 175 cm

19 (a) (i) E (ii) B (iii) D
(iv) F (v) A (vi) C
(b) (i) $(x + 8)(x - 3)$ (ii) $(5a - b)(5a + b)$
(iii) $(x + 3 - y)(x + 3 + y)$ (iv) $(x + 6 - \sqrt{2})(x + 6 + \sqrt{2})$
(v) $x(16x - 25)$ (vi) $(x - y)(4 + x + y)$

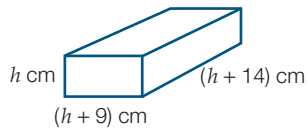
20 (a) $(3x - 7)(2x + 5)$ (b) $(2x - 3)(5x - 1)$ (c) $(3x + 4)(4x + 1)$

21 (a) $(x + 2)(x + 8)$ (b) $18(x + 2)(x + 8)$ (c) \$720

22 $16x^2 + 24xy + 9y^2$

23 (a) $5x - x^2$ m^2 (b) $x(5 - x)$
(c) (i) 4 m^2 (ii) 2.25 m^2 (iii) 3.1875 m^2

24 (a)



(b) $h(h + 9)(h + 14)$
(c) $h^3 + 23h^2 + 126h$ (d) 1800 cm^3
(e) 1800 cm^3 (f) Students' own answers.

25 (a) πx^2 (b) $x + 2$ (c) $\pi(x + 2)^2$
(d) $x - 2$ (e) $\pi(x - 2)^2$
(f) $\pi(x + 2)^2 - \pi(x - 2)^2$ (g) π
(h) $8\pi x$ (i) 80π (j) 17 mm

26 (a) $(8x + 3)(2y + 7)$ cm^2 (b) $(8x + 3)(8x - 3)$ cm^2
(c) $64x^2 - 9$ cm^2
(d) (i) $y = 4x - 2$ (ii) $64x^2 + 48x + 9$

27 (a) $x^2 + p^2x + p^2y - y^2$ (b) $(x + y)(x - y + p^2)$
(c) $(x + y)(x - y + p^2) = (x + y)(x - y)$; $x - y$ must be positive, so x must be greater than y.
(d) (i) $2p^2x$ or $2p^2y$ vehicles
(ii) $2px(p - 1)$ non-white vehicles

Chapter 3

Recall 3 (p. 89)

1 (a) 625 (b) 29.791 (c) 3.45

2 (a) $A = 16$ (b) $h = 5$ (c) $b = 7$

3 (a) 410 cm (b) 425 000 cm (c) 400 000 cm^2
(d) 0.0578 m^2 (e) 8 250 000 m^2 (f) 0.6 L
(g) 20 mL (h) 30 800 L (i) 0.007 25 m^3

4 (a) 6.11 m (b) 16.11 m (c) 15.27 m^2

5 (a) (i) 15.59 cm (ii) 140.3 cm^2
(b) (i) 22.36 cm (ii) 447.21 cm^2
(c) (i) 4.36 m (ii) 26.18 m^2
(d) (i) 1.5 m (ii) 6 m^2

Exercise 3.1 (p. 93)

1 (a) 1263 mm^2 (b) 583 cm^2 (c) 157.5 cm^2
(d) 349.94 mm^2 (e) 25.63 mm^2 (f) 493.65 cm^2

2 (a) 76.37 cm^2 (b) 43.37 cm^2 (c) 48.29 cm^2
(d) 1809.56 cm^2 (e) 18.27 mm^2 (f) 88.5 m^2
(g) 422.43 mm^2 (h) 13.72 m^2 (i) 69.53 m^2

3 C 4 C

5 (a) 123.61 cm^2 (b) 52.15 cm^2 (c) 47.12 cm^2
(d) 32.47 cm^2 (e) 2194.29 cm^2 (f) 441.06 cm^2
(g) 437.5 cm^2 (h) 294 cm^2

6 triangle: 173 m^2; sector: 1047 m^2

7 (a) 156.25 hours = 22.32 days (b) 75 000 m^2

8 9.78 m^2 9 25.13 m^2

10 (a) 0.05 m^2 (b) 190.42 m^2

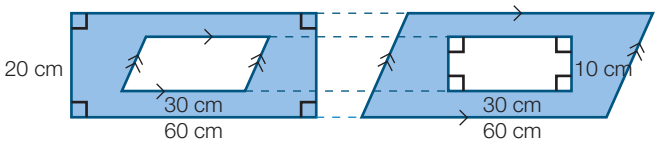
11 4×4; 3×6 12 4.51 m

13 A, B, F: 50 cm^2; C, G: 25 cm^2; D, E: 100 cm^2

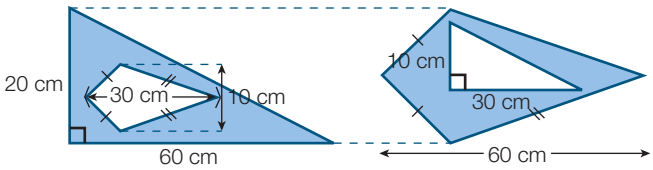
14 (a) 26 cm^2 (b) various answers

15 (a) The formulas for the overall area and the subtracted area are the same. The substituted values are the same hence the calculations are identical.

(b) Possible pair of shapes:



(c) Another possible pair of shapes:



Exercise 3.2 (p. 99)

1 (a) 766.14 m^2 (b) 82.14 m^2 (c) 376 cm^2
(d) 1710 cm^2 (e) 108 cm^2 (f) 684 cm^2

2 (a) 904.78 cm^2 (b) 1258.21 mm^2

3 (a) D (b) D

4 6300 cm^2 5 810.27 cm^2 6 12.25 m^2

7 (a) $2(lb + bh + lh)$

(b) Prism 1: $l = 10$ cm, $b = 5$ cm, $h = 2$ cm, SA = 160 cm^2

Prism 2: $l = 20$ cm, $b = 10$ cm, $h = 4$ cm, SA = 640 cm^2

The surface area quadruples when the dimensions are doubled.

8 (a) $r = 2$, $h = 5$, surface area = 28π

(b) $r = 2$, $h = 10$, surface area = 48π

(c) $r = 4$, $h = 5$, surface area = 72π

(d) Doubling the height increases the area of the curved face but not the circular ends. Doubling the radius increases the area of both the curved face and the circular ends. Also, the effect of doubling the radius is magnified, as the radius is then squared in the formula.

Exercise 3.3 (p. 103)

1 (a) 450 cm^3 (b) 3192 cm^3 (c) 5355 mm^3
(d) 585 cm^3 (e) 90 053.75 mm^3 (f) 552.92 m^3

2 374.36 mL

3 (a) 115 000 cm^3 (b) 20 250 mm^3 (c) 9625 cm^3
(d) 162 m^3 (e) 135 000 cm^3 (f) 840 cm^3

4 (a) B (b) D

5 1.31 m^3 6 0.07 m^3 7 6.64 m^3 8 21 082 L

9 (a) $4 \times 5 \times 10$ cm, $2 \times 10 \times 10$ cm, $1 \times 10 \times 20$ cm

(b) $8 \times 2.5 \times 10$ cm, $4 \times 4 \times 12.5$ cm, $0.5 \times 40 \times 10$ cm

(c) $4 \times 5 \times 10$: $SA = 220$ cm^2

$2 \times 10 \times 10$: $SA = 280$ cm^2

$1 \times 10 \times 20$: $SA = 460$ cm^2

10 (a) Anita has used the diameter value for the radius and did not convert all values to the same unit of measurement.

(b) 628 318.53 cm^3

(c) Point 1: Identify radius or diameter before using circle formulas; Point 2: Check for matching units before substituting values into formulas.

Half-time 3 (p. 106)

1 (a) 19 101 cm^2 (b) 2228 mm^2

2 Surface area = 14π cm^2, volume = 5π cm^3

3 (a) 44 m^2 (b) 12.96 m^2

4 Volume A = $xy^2\pi$ m^3, Volume B = $4xy^2\pi$ m^3, Volume C = $xy^2\pi$ m^3

B has the greatest volume

5 (a) 191 038.85 m^3 (b) 1323.05 m^3

6 72 206 cartons

Exercise 3.4 (p. 110)

1 (a) 576 cm^2 (b) 21 m^2

2 (a) 1884.96 cm^2 (b) 490.09 cm^2

3 (a) 20 106.19 cm^2 (b) 3848.45 cm^2

4 (a) 3402.34 m^2 (b) 829.38 cm^2 (c) 141 m^2
(d) 1599.56 cm^2 (e) 3619.11 cm^2 (f) 600 cm^2
(g) 9424.78 cm^2 (h) 1413.72 cm^2

5 (a) C (b) D

6 483.81 cm^2 7 414 cm^2 8 70 185 m^2

9 (a) Surface area of bowling ball = 2827.43 cm^2 (2 d.p.); area of material = 3200 cm^2. The area of material is larger than the surface area of the bowling ball.

(b) The material is not long enough to wrap around the ball, so he will need to cut and sew his existing material to wrap it. $C = \pi d = \pi \times 30 = 94.25$ cm (2 d.p.)

10 (a) 31 cm (b) 12 cm

11 $r = 5$, $s = 10$ gives a surface area of 75π cm^2 (235.62 cm^2)
$r = 6$, $s = 8$ gives a surface area of 84π cm^2 (263.89 cm^2)

12 The slant height is the hypotenuse of a right-angled triangle. The radius is one of the shorter sides.

13 (a) $h = 4$ m, $r = 3$ m, total surface area = 24π m^2
(b) total surface area = $3\pi(3 + \sqrt{73})$ m^2
(c) $h = 12$ m, $r = 5$m, total surface area = 90π m^2
(d) total surface area = $5\pi(5 + \sqrt{601})$ m^2
(e) No

14 (a) $h = 15$ m, $r = 8$m, total surface area = 200π m^2
(b) total surface area = $16\pi(16 + \sqrt{481})$ m^2
(c) $h = 24$ m, $r = 7$ m, total surface area = 224π m^2
(d) total surface area = $14\pi(14 + \sqrt{821})$ m^2
(e) No

Exercise 3.5 (p. 116)

1 (a) 716.28 cm^3 (b) 804.25 cm^3 (c) 66 666.67 cm^3
(d) 25 200 cm^3 (e) 385 mm^3 (f) 2338.27 cm^3

2 (a) 0.52 m^3 (b) 179 594.38 cm^3

3 (a) 150.53 cm^3 (b) 2900 cm^3
(c) 160.22 cm^3 (d) 197.60 cm^3

4 1 098 066 219 000 km^3

5 D 6 A

7 Pyramid is 1287.61 cm^3 larger.

8 157 mL 9 129 mL 10 70

11 (a) 28 (b) 22 338
(c) Calculations for volume involves cubing the radius.
(d) 1253 Earths in Jupiter, 688 Earths in Saturn, 50 Earths in Uranus, 60 Earths in Neptune

12 340 trucks

13 (a) $b = 5$ cm, $h = 9$ cm; volume = 75 cm^3
(b) Volume = 300 cm^3
(c) 4 times greater ($300 \div 75 = 4$)
(d) $b = 6$ cm, $h = 10$ cm; Volume of first pyramid = 120 cm^3
Volume of second pyramid = 480 cm^3
Conclusion: When the base of any square-based pyramid is doubled, the volume is quadrupled.

14 (a) $h = 4$ m, $r = 3$m, volume = 12π m^3
(b) volume = 24π m^3
(c) $h = 12$ m, $r = 5$m, volume = 100π m^3
(d) volume = 200π m^3
(e) When the height of a cone is doubled, the volume of the cone is also doubled.

15 (a) $h = 15$ m, $r = 8$ m, volume = 320π m^3
(b) volume = 1280π m^3
(c) $h = 24$ m, $r = 7$ m, volume = 392π m^3
(d) volume = 1568π m^3
(e) When the radius of a cone is doubled, the volume of the cone is multiplied by 4.

Exercise 3.6 (p. 122)

1 (a) 7.1 g/cm^3 (b) 6.4 g/cm^3 (c) 0.8 g/cm^3

2 (a) 6240 g (b) 486 g (c) 3.96 kg

3 (a) 13.27 cm^3 (b) 694.29 mL (c) 15.29 m^3

4 20 g/L 5 2.4 g/100 g

6 (a) D (b) B

7 7.6 mg/mL

8 (a) 4 cups, 16 cups (b) 200 mL/L

9 (a) 131.175 kg (b) 22 t

10 0.04 g/L

11 (a) wooden log: 0.40 g/cm^3; steel rod: 7.81 g/cm^3; cork: 0.22 g/cm^3; aluminium cylinder: 2.65 g/cm^3; ice block: 0.91 g/cm^3; gold bar: 18.29 g/cm^3
The wooden log, cork and ice block will all float on water.
(b) lead log: 2 852 562 g; lead rod: 726 g; lead cork: 103 g; lead cylinder: 856 g; lead block: 125 g; lead bar: 9931 g

12 (a) Johnstone's abattoir (50 kg/day compared with Arvon River 40 kg/day)
(b) 0.44 mg/L

13 (a) 875 organisms/100 mL (b) no
(c) 600 organisms/50 mL; 2000 organisms/100 mL; 800 organisms/60 mL

Chapter review 3 (p. 125)

1 (a) 925 cm^2 (b) 878.43 cm^2 (c) 5105.09 cm^2

2 B

3 (a) 1550 cm^2 (b) 7.49 m^2

4 C

5 (a) 1080 cm^3 (b) 21 205.75 cm^3 (c) 5040 cm^3

6 (a) 1884.96 cm^2 (b) 11 309.73 cm^2

7 (a) 527.79 cm^3 (b) 268 082.57 mm^3

8 A 9 30 kg/m^3 10 250 mL/L

11 1800 bricks

12 (a) 15 708 L (b) 5109 L

13 1.70 m^3

14 (a) 1885 cm^2, 6807 cm^3 (b) 327 cm^2, 302 cm^3

15 (a) 7527 m^2 (b) 361 m

16 (a) 2000 cm^3 (b) 15 g

17 335 cm^3 18 804 L 19 9839 kg

20 (a) 1950 kL (b) 550 kL

21 21.99 cm

22 (a) 62 832 m^3 (b) \$9 424 778
(c) 12 566 m^2 (d) \$1 256 637
(e) \$1 800 000 (f) \$12 481 415
(g) \$6 300 000 (h) the road over the top by \$6 181 415

23 4.73 g/L

Mixed review A (p. 129)

1 \$957.04

2 (a) 2786.89 cm^2 (b) 256.31 m^2

3 (a) $3x^2 - 5xy + 2y^2$ (b) $6a^2 - 27ab + 30b^2$
(c) $15t^2 - 33ht - 36h^2$

4 (a) 7% (b) 7% (c) 7.19% (d) 7.23%

5 (a) $(x-2)^2$ (b) $(x+3)(x-3)$
(c) $(x+7)^2$ (d) $25(2x+1)(2x-1)$
(e) $2x(2y^2+y-4)$ (f) $4(x+4)^2$

6 (a) $-\frac{8x}{35}$ (b) $\frac{5a^3}{8b^2}$

7 (a) $\frac{x-5}{3}$ (b) $\frac{11x+43}{30}$

8 (a) $-x^2 - 5x + 66$ (b) $-x^2 - 2x + 24$
(c) $2x^2 + 5x - 12$ (d) $7y^2 - 5y - 2$

9 (a) \$5516.35 (b) \$5530.10 (c) \$5533.23

10 (a) $(x-9)(x+3)$ (b) $(x-5)(x-2)$ (c) $(3x-2)(2x+1)$

11 166.7 g/L

12 (a) 301.59 cm^3 (b) 164.64 cm^3

13 (a) $25x^2 - 10x + 1$ (b) $16x - 24x + 9$
(c) $64 - 48x + 9x^2$ (d) $36 - 60x + 25x^2$

14 Option A (\$6248.58 after 3 years) is better than Option B (\$6127.96 after 3 years)

15 (a) \$920 (b) \$1200.61

16 384 cm^2

17 (a) 81 cm^2 (b) 66.21 cm^2 (c) 37.12 cm^2

18 (a) 24.5 m^2 (b) 34.1 m

19 (a) $(x+5)(x+3)$ (b) $x^2 + 8x + 15$
(c) (i) 16 m^2 (ii) 63 m^2 (d) \$504

20 20 mL

Challenge A (p. 131)

1 $1000 - 1 + 3 - 5 + 7 \ldots$
$- 197 + 199 - 201 = 1000 + 50 \times 2 - 201 = 899$

2 $18x + 23y + 52z + 69w = 733$, $x + y + z + w = 20$. One of x and w must be odd, as must one of x and z. Use trial and error or a spreadsheet to obtain $x = 3$, $y = 10$, $z = 2$, $w = 5$. That is, 3 gifts at \$18, 10 gifts at \$23, 2 gifts at \$52 and 5 gifts at \$69.

3 B

4 D

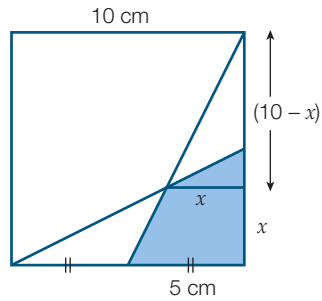
$35x - 35 < 75 < 21x + 84$. $35x < 110$, $x < 3\frac{1}{7}$ and $-9 < 21x$,

$x > -\frac{3}{7}$. $-\frac{3}{7} < x < 3\frac{1}{7}$, $x = 0, 1, 2, 3$.

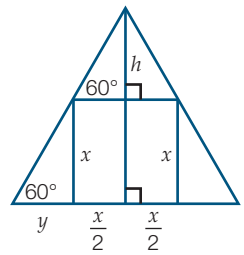
5 D

6 $\frac{5}{10} = \frac{x}{10-x}$, $50 - 5x = 10x$, $3x = 10$, $x = \frac{10}{3}$

Shaded area $= x^2 + x \times (5 - x) = 5x = \frac{50}{3} = 16\frac{2}{3}$ cm^2



7



$\frac{x}{y} = \tan 60°$, $\frac{x}{y} = \sqrt{3}$, $y = \frac{x}{\sqrt{3}}$

Base of equilateral triangle $= x + 2y = x + \frac{2x}{\sqrt{3}} = \frac{(3+2\sqrt{3})x}{3}$

$\frac{h}{\frac{x}{2}} = \tan 60°$, $h = \frac{x \tan 60°}{2} = \frac{x\sqrt{3}}{2}$

Base of equilateral triangle $= x + 2y = x + \frac{2x}{\sqrt{3}} = \frac{(2+\sqrt{3})x}{2}$

Height of equilateral triangle $= x + h = x + \frac{\sqrt{3}x}{2} = \frac{(2+\sqrt{3}x)x}{2}$

Area of square $= x^2$

Area of triangle $= \frac{1}{2} \times \frac{(3+2\sqrt{3})x}{3} \times \frac{(2+\sqrt{3})x}{2}$

$= \frac{(12+7\sqrt{3})x^2}{12}$

Square as fraction of triangle $= x^2 \div \frac{(12+7\sqrt{3})x^2}{12}$

$= \frac{12}{12+7\sqrt{3}}$, 49.7%

OR, using similar triangles, $\frac{h}{\frac{x}{2}} = \frac{x}{y}$ so $h = \frac{x^2}{2y}$

Base of triangle $= x + 2y$

Height of triangle $= h + x = \frac{x^2}{2y} + x = \frac{x(x+2y)}{2y}$

Area of triangle $= \frac{1}{2} \times (x+2y) \times \frac{x(x+2y)^2}{2y} = \frac{x(x+2y)^2}{4y}$

Square as fraction of triangle $= x^2 \div \frac{x(x+2y)^2}{4y} = \frac{4xy}{(x+2y)^2}$

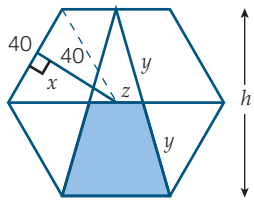
Now $\frac{x}{y} = \tan 60°$ so $y = \frac{x}{\sqrt{3}}$ (or $y = 0.5774x$)

Square as fraction of triangle $= \frac{4xy}{(x+2y)^2} = \frac{4x \times \frac{x}{\sqrt{3}}}{\left(x + \frac{2x}{\sqrt{3}}\right)^2}$

$= \frac{4x^2}{\sqrt{3}x^2\left(1 + \frac{2}{\sqrt{3}}\right)^2} = \frac{4\sqrt{3}}{(\sqrt{3}+2)^2} = \frac{4\sqrt{3}}{(7+4\sqrt{3})}$

$= 0.497$, hence 49.7%

8



$x^2 = 40^2 - 20^2 = 1200.\ x = 20\sqrt{3}$

$h = 2x = 40\sqrt{3}$

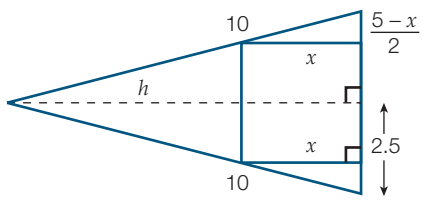
$4y^2 = 20^2 + 1600 \times 3 = 5200.\ y^2 = 1300.\ y = 10\sqrt{13}$

$z^2 = y^2 - x^2 = 1300 - 1200 = 100.\ z = 10$

Shaded area $= \dfrac{20+40}{2} \times 20\sqrt{3} = 600\sqrt{3}\ \text{cm}^2$

9 D

10



$h^2 = 10^2 - 2.5^2 = 93.75,\ h = \sqrt{\dfrac{375}{4}} = \dfrac{5}{2}\sqrt{15}$

From similar right-angled triangles, $\dfrac{h}{2.5} = \dfrac{x}{\frac{5-x}{2}}$

$h = \dfrac{5x}{5-x}$

$5h - hx = 5x$

$(5+h)x = 5h$

$x = \dfrac{5h}{5+h}$

$= \dfrac{5 \times \frac{5\sqrt{15}}{2}}{5 + \frac{5\sqrt{15}}{2}}$

$= \dfrac{5\sqrt{15}}{\sqrt{15}+2} = \dfrac{5\sqrt{15}(\sqrt{15}-2)}{15-4}$

$= \dfrac{5}{11}(15 - 2\sqrt{15})\ \text{cm}$

11 Radius = 15 cm, Area of base $= 3 \times \dfrac{5}{6} \times \pi \times 15^2 + 15$

$\times 15 \tan 60° = \dfrac{1125\pi}{2} + 225\sqrt{3} \approx 2157\ \text{cm}^2$

Volume $= \left(\dfrac{1125\pi}{2} + 225\sqrt{3}\right) \div 10\,000 \times 6 \approx 1.294\ \text{m}^3$

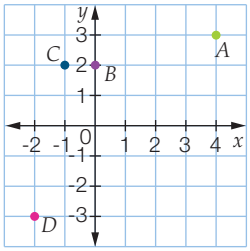
Chapter 4

Recall 4 (p. 132)

1 **(a)** $P = -7$ **(b)** $P = 3$ **(c)** $P = -15$ **(d)** $P = -3$

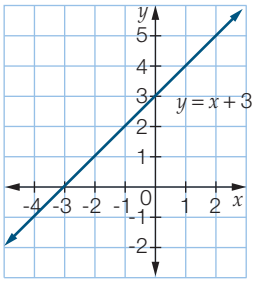
2 **(a)** $a = 7$ **(b)** $b = 24$ **(c)** $c = 10$ **(d)** $d = 7$

3



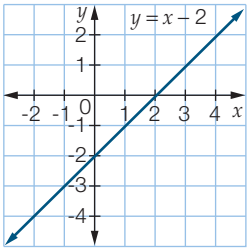
4 **(a)**

x	-2	-1	0	1	2
y	1	2	3	4	5



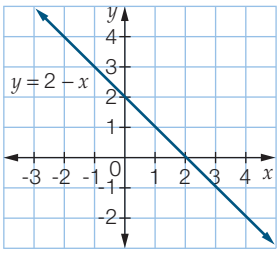
(b)

x	-2	-1	0	1	2
y	-4	-3	-2	-1	0



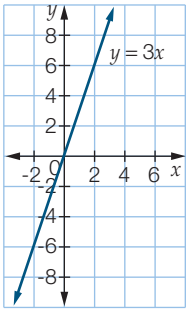
(c)

x	-2	-1	0	1	2
y	4	3	2	1	0

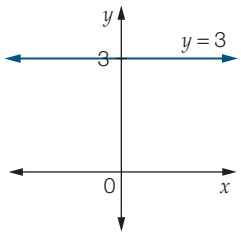


(d)

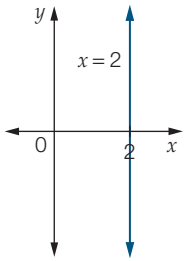
x	-2	-1	0	1	2
y	-6	-3	0	3	6



5 (a)



(b)



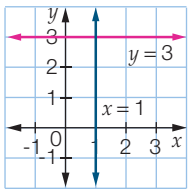
6 (a) (i) 3 (ii) 5

(b) (i) -2 (ii) 3

(c) (i) $-\frac{2}{3}$ (ii) $\frac{4}{3}$

7 (a) $-\frac{5}{3}$ (b) 1

8 (a)



(b) (1, 3)

Exercise 4.1 (p. 136)

1 (a) $x = 3$ (b $x = 8$ (c) $c = -3$ (d) $a = 4$

(e) $x = -6$ (f) $d = -3$ (g) $m = 2$ (h) $n = 1$

(i) $x = -1$ (j) $f = 2$ (k) $e = -2$ (l) $p = \frac{1}{5}$

(m) $x = 0$ (n) $g = 21$ (o) $h = 16$ (p) $k = 3$

(q) $x = 3$ (r) $y = 0$ (s) $x = \frac{5}{4}$ (t) $x = 3$

(u) $x = 7$

2 (a) $a = 1$ (b) $d = -1$ (c) $e = \frac{5}{6}$

(d) $c = 2$ (e) $b = 4$ (f) $f = \frac{13}{6}$ or $2\frac{1}{6}$

(g) $g = 8$ (h) $h = \frac{4}{5}$ (i) $k = -1$

(j) $x = -1$ (k) $x = 1$ (l) $x = -\frac{7}{3}$ or $-2\frac{1}{3}$

(m) $x = \frac{3}{2}$ or $1\frac{1}{2}$ (n) $x = \frac{1}{4}$ (o) $x = \frac{1}{3}$

(p) $m = -\frac{11}{2}$ or $-5\frac{1}{2}$ (q) $d = \frac{17}{4}$ or $4\frac{1}{4}$ (r) $d = 7$

(s) $a = 8$ (t) $a = -\frac{16}{3}$ or $-5\frac{1}{3}$ (u) $g = \frac{8}{7}$ or $1\frac{1}{7}$

(v) $t = 0$ (w) $h = \frac{11}{2}$ or $5\frac{1}{2}$ (x) $j = -\frac{3}{11}$

3 (a) $a = -22$ (b) $b = 8$ (c) $c = 9$

(d) $d = -1$ (e) $x = -2$ (f) $y = 1$

(g) $m = \frac{3}{4}$ (h) $k = \frac{3}{2}$ or $1\frac{1}{2}$ (i) $t = 2$

(j) $x = 14$ (k) $x = 12$ (l) $x = 2.1$

(m) $a = -25$ (n) $p = 15$ (o) $m = -27$

4 (a) $x + 6 = 13, x = 7$ (b) $y - 7 = -2, y = 5$

(c) $2x = 17, x = 8\frac{1}{2}$ (d) $\frac{x}{3} = 12, x = 36$

(e) $3x + 2 = 23, x = 7$ (f) $4x - 5 = -3, x = \frac{1}{2}$

5 (a) C (b) D (c) D

6 (a) B (b) B (c) C

7 8, 9, 10, 11 8 40, 42, 44

9 Nikki 7, Sam 14, Minh 21 10 98 m

11 (a) $p + 7$ (b) $2(p + 7)$ (c) 17

12 length 100 m, breadth 50 m 13 walk 1.5 km, train 9 km

14 15 years old 15 45

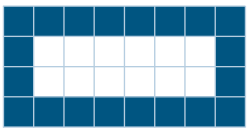
16 24 years old 17 43 km/h and 53 km/h

18 Students to check own answers but examples would be:

$\frac{x+8}{6} = -\frac{x}{2}$, $5x - 2 = -10 + x$ and $-4x - 11 = \frac{3x}{2}$

19 (a) blue tiles $= 2l + 10$ or $2(l + 5)$

(b) (i)



(ii) 20 blue tiles needed

(iii) $b = 2t + 8$ or $b = 2(t + 4)$

(c) For 5×3 white tiles, $n = 20$

Exercise 4.2 (p. 140)

1 (a) $p = 5$ (b) $n = 2$ (c) $t = -18$

(d) $n = 9$ (e) $m = \frac{13}{22}$ (f) $k = -\frac{7}{4}$ or $-1\frac{3}{4}$

(g) $k = 7$ (h) $p = -7$ (i) $x = \frac{5}{6}$

2 D 3 -5

4 Jane 10 km, Tom 8 km, Reena 6 km

5 Jet did not change the sign when he removed the second set of brackets. The lines should read:

$$\frac{3n-1}{4} - \frac{2n+5}{3} = 2$$

$$3(3n - 1) - 4(2n + 5) = 12 \times 2$$

$$9n - 3 - 8n - 20 = 24$$

$$n - 23 = 24$$

$$n = 47$$

6 Students' own answers, e.g. $\frac{3x-8}{4} - \frac{4x+2}{3} = -5$; solution: $x = 4$

Exercise 4.3 (p. 143)

1 (a) $m = \frac{8}{5}$ (b) $m = -\frac{1}{2}$ (c) $m = \frac{3}{8}$ (d) $m = -2$

2 (a) $y = 2x - 1$ (b) $y = -3x + 2$
(c) $y = 6x + 2$ (d) $y = -\frac{3}{4}x - 3$

3 (a) $m = 5$ (b) $m = -3$ (c) $m = 6$
(d) $m = 2$ (e) $m = 4$ (f) $m = -5$

4 D 5 B

6 (a) $m = \frac{4}{3}$ (b) $m = -\frac{3}{2}$ (c) $m = -\frac{5}{4}$
(d) $m = -\frac{2}{5}$ (e) $m = -3$ (f) $m = -\frac{1}{5}$

7 The second ramp is steeper with a gradient of 0.08.The first ramp has a gradient of 0.0777…

8 Students' own answers. For example, (1, 7.5) and (4, 0) lie on $y = -2.5x + 10$.

Exercise 4.4 (p. 147)

1 (a) $\frac{1}{2}$ (b) -2 (c) 0
(d) $\frac{3}{0}$ is undefined

2 (a) $y = 6x - 3$ (b) $y = -2x + 7$
(c) $y = 3x + 9$ (d) $y = -5x + 8$

3 (a) $y = 2x - 1$ (b) $y = 3x + 2$ (c) $y = 5x - 7$
(d) $y = -3x + 10$ (e) $y = -2x + 18$ (f) $y = -x - 1$
(g) $y = \frac{5}{3}x - 5$ (h) $y = \frac{1}{2}x + 8$ (i) $y = -\frac{10}{3}x + 18$

4 (a) positive (b) negative (c) zero
(d) undefined

5 (a) $\frac{1}{3}$ (b) 2 (c) -4 (d) $-\frac{3}{2}$
(e) $-\frac{5}{4}$ (f) -2 (g) 3 (h) 2
(i) -1 (j) $-\frac{2}{3}$ (k) 0 (l) 0
(m) $-\frac{2}{5}$ (n) $\frac{1}{7}$ (o) undefined

6 (a) C (b) B (c) D

7 (a) $5x - y + 38 = 0$ (b) $6x - y + 7 = 0$
(c) $4x - y + 9 = 0$ (d) $5x - 3y - 15 = 0$
(e) $x - 2y + 16 = 0$ (f) $10x + 3y - 54 = 0$

8 (a) (i) (-3, 2) and (2, 2) or (5, -4) and (-7, -4)
(ii) (2, 5) and (2, 7)
(iii) (4, 10) and (5, 6)
(iv) (-1, -1) and (6, 0)
(b) (0, 3) and (2, 7) parallel to (2, 1) and (-2, -7)

9 (a) 800 m (b) 600 m (c) 120
(d) 120 m/min (e) 440 m (f) $d = 200 + 120t$
(g) Alex is walking at a constant speed because the graph is a straight line.

10 (a) 4800 grains (b) 1600 grains
(c) -1600 (d) 1600 grains/min
(e) decrease (f) $S = 4800 - 1600t$
(g) When $t = 2$, $s = 4800 - 3200$
$= 1600$
(h) $0 \le t \le 3$

11 (a) (i) $y = 2.5x - 1$ (ii) $y = 2.5x - 1$
(b) Yes, either point can be used as (x_1, y_1) to find the equation of the line.

12 $a = 17$ 13 $-\frac{9}{35}$

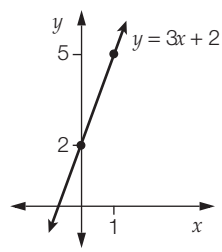
14 Gradient of the line joining the points (-1, -5) and (2, 4) = 3 and the gradient of the line joining the points (2, 4) and (5, 13) = 3. As the two lines have a point in common, (2, 4) and the same gradient, the points are collinear.

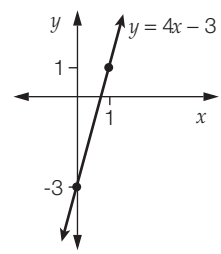
15 (a) (1, 9), (2, 6), (3, 3) (b) (-2, 0), (-1, -1), (1, -3)
(c) (0, 4), (2, 3), (4, 2) (d) (2, -5), (3, -3), (4, -1)
(e) (1, 0), (2, 5), (3, 10)

16 $m = -3$, (1, 4) and (4, -5)

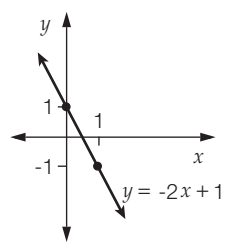
Exercise 4.5 (p. 152)

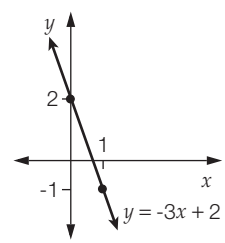
1 (a) $b = 2$, $m = 3$ (b) $b = -3$, $m = 4$



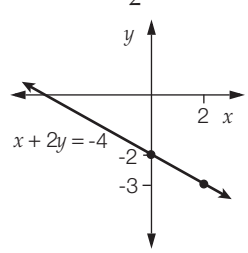


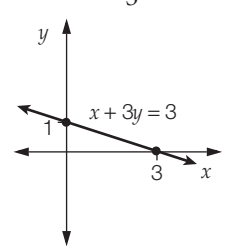
(c) $b = 1$, $m = -2$ (d) $b = 2$, $m = -3$



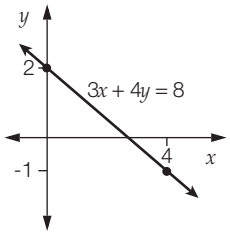


2 (a) $b = -2$, $m = -\frac{1}{2}$ (b) $b = 1$, $m = -\frac{1}{3}$

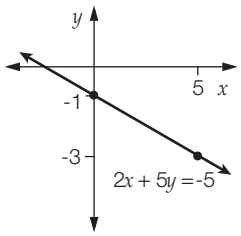




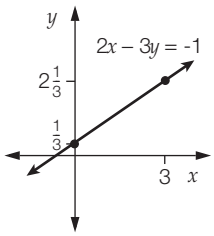
(c) $b = 2, m = -\frac{3}{4}$



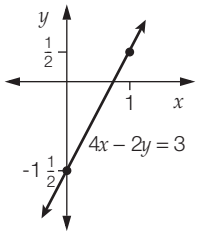
(d) $b = -1, m = -\frac{2}{5}$



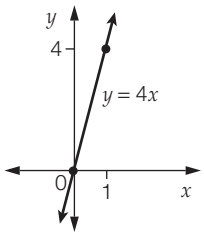
(e) $b = \frac{1}{3}, m = \frac{2}{3}$



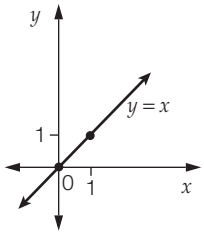
(f) $b = -1\frac{1}{2}, m = 2$



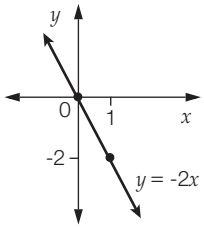
3 (a)



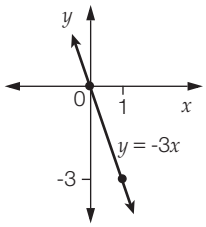
(b)



(c)



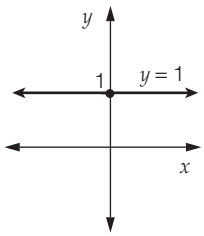
(d)



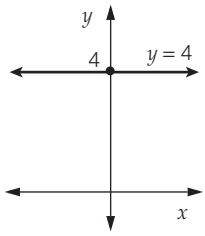
4 (a) D (b) B (c) A

5 (a) C (b) C (c) B

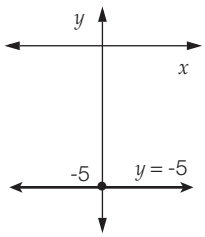
6 (a)



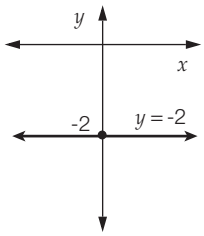
(b)



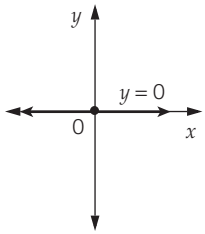
(c)



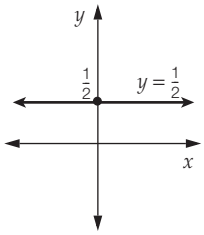
(d)



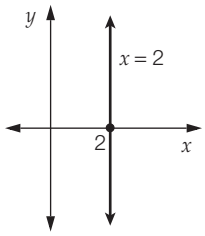
(e)



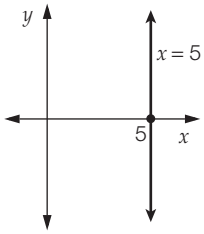
(f)



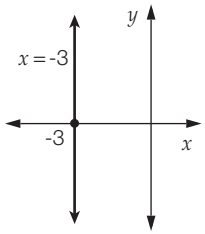
(g)



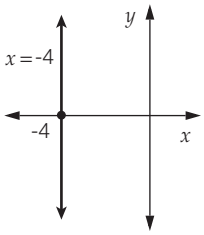
(h)



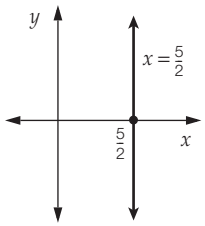
(i)



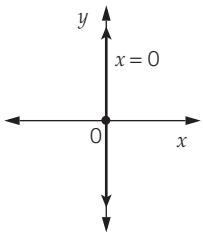
(j)



(k)



(l)



7 (a) D (b) C (c) D

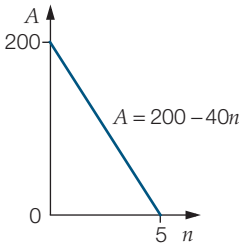
8 (a) 100 km (b) 80 (c) $D = 100 + 80T$

(d) 80 km/h (e) $D = 250 + 100T$ (f) 93 km/h

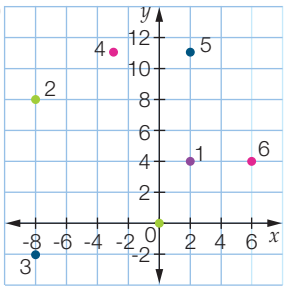
9 (a) \$80 (b) $A = 200 - 40n$

(c) \$200 (d) 5 weeks

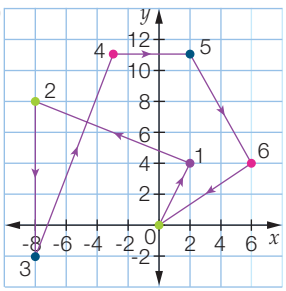
(e)



10 (a)



(b)



(c) (i) $y = 2x$

(ii) $y = -\frac{2}{5}x + 4\frac{4}{5}$ or $5y + 2x = 24$

(iii) $x = -8$

(iv) $y = \frac{13}{5}x + 18\frac{4}{5}$ or $5y - 13x = 94$

(v) $y = 11$

(vi) $y = -\frac{7}{4}x + 14\frac{1}{2}$ or $4y + 7x = 58$

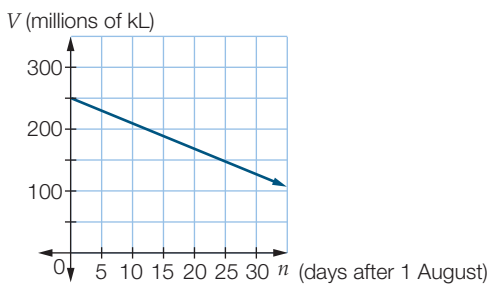
(vii) $y = \frac{2}{3}x$

(d) Yes. It will be on the direct path between the 5th and 6th buoy.

11 (a) $V = 225 - 0.75T$ (b) 5 min

12 (a) $V = 248 - 4n$

(b) Maximum capacity = 248 million kL



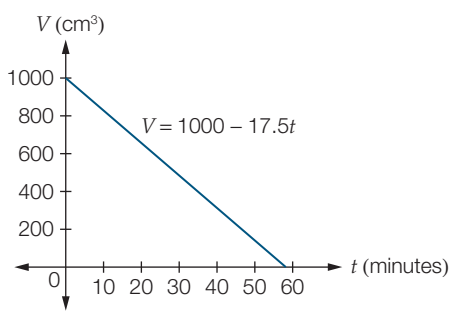
(c) It does not make sense to extend the time and volume into negative values.

13 (a) 1000 cm^3 (b) $V = 1000 - kt$

(c) 125 cm^3 (d) $k = 17.5$

(e) The volume is decreasing. (f) $V = 1000 - 17.5t$

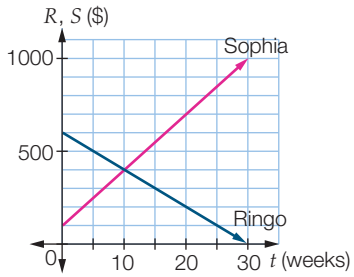
(g)



The ice block has completely melted after 57.15 minutes.

14 (a) Ringo: $R = 600 - 20t$, Sophia: $S = 100 + 30t$

(b) 15.7 weeks



15 $y = 2x - 4$, $y = 2x - 1$, $y = 2x + 2$. The coefficient of x in all three equations is the same, indicating that they have the same gradient.

16 (1) The pink line extending from (0, 0).

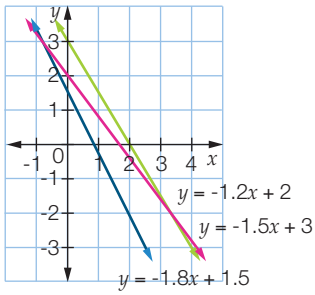
(2) The purple line (gradient is half of the pink one above it).

(3) The blue line (negative gradient shows the reduction in remaining credit).

(4) The black line starts part way up the y-axis representing connection fees.

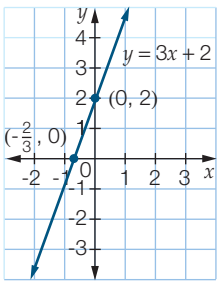
(5) The green line.

17

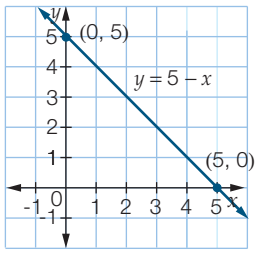


Half-time 4 (p. 156)

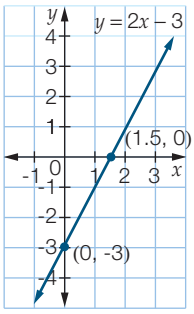
1 (a)



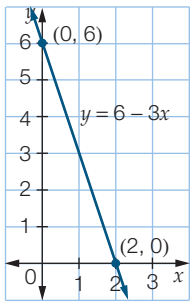
(b)



(c)



(d)



2 Using the points (32, 0) and (212, 100):

$$m = \frac{100-0}{212-32} = \frac{100}{180} = \frac{5}{9}$$

Hence, $C = \frac{5}{9}F + c$. Substituting in (32, 0):

$$0 = \frac{5}{9} \times 32 + c$$

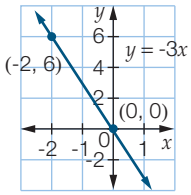
$$0 = \frac{160}{9} + c$$

$$c = -\frac{160}{9}$$

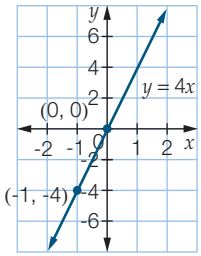
Therefore, $C = \frac{5}{9}F - \frac{160}{9}$

$$C = \frac{5}{9}(F - 32)$$

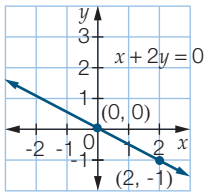
3 (a)



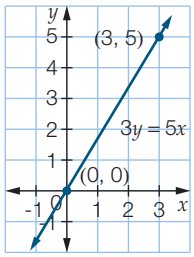
(b)



(c)



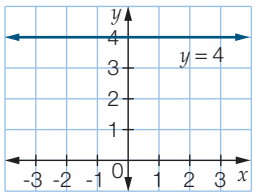
(d)



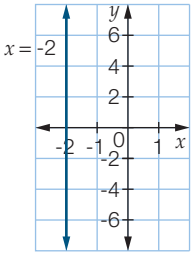
4 (a) $x = 2$ (b) $x = 2$ (c) $x = 19$

(d) $x = 3$ (e) $x = 18$ (f) $x = 3$

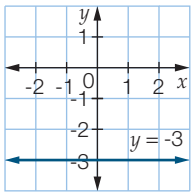
5 (a)



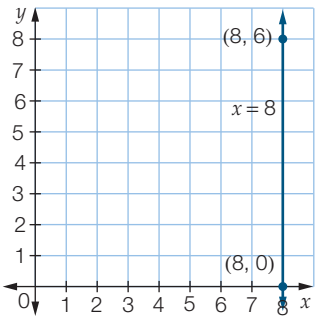
(b)



(c)



(d)



6 (a) $m = -3$ (b) $y = -3x + 2$

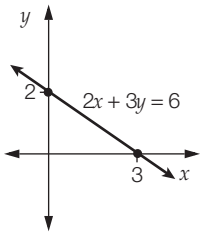
7 (a) $x = 4$ (b) $p = 3$ (c) $m = -5$ (d) $k = 7$

8 (a) $y = 4x + 9$ (b) $y = -x - 4$

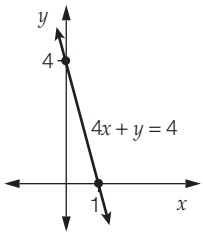
9 (a) $m = 0.5$ (b) $m = -2$ (c) $m = 0.5$ (d) $m = -8$

Exercise 4.6 (p. 158)

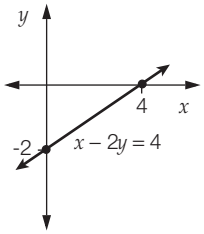
1 (a)



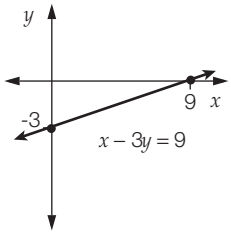
(b)



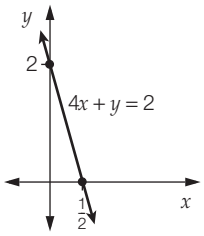
(c)



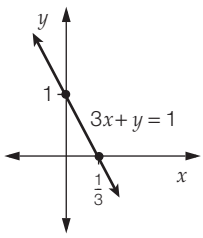
(d)



(e)

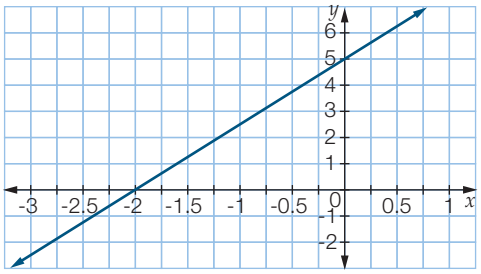


(f)

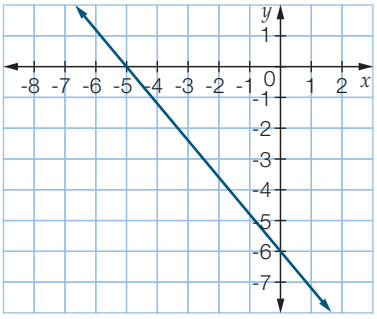


2 C

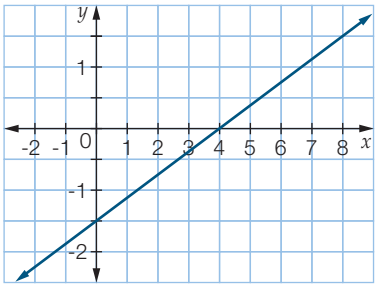
3 (a) x-intercept = -2, y-intercept = 5



(b) x-intercept = -5, y-intercept = -6

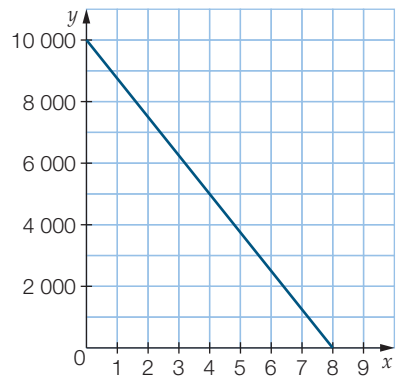


(c) x-intercept = 4, y-intercept = -1.5



4 (a) $125x + y - 10\,000 = 0$

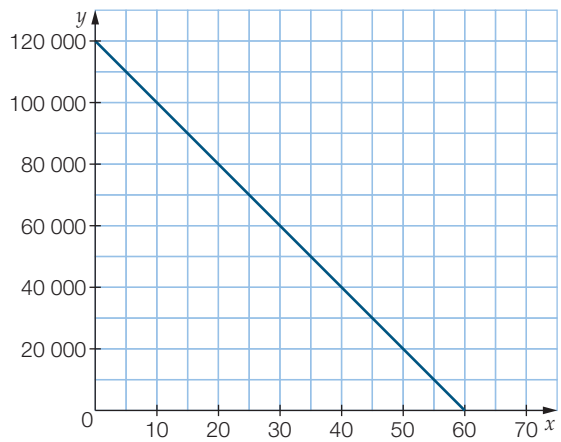
(b)



(c) 3750 litres

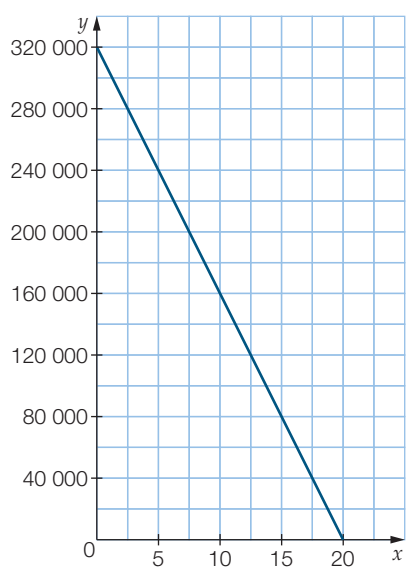
5 (a) $2000x + y - 120\,000 = 0$

(b)



(c) $\frac{25}{18}$ L/min (d) 12 000 L

6 (a)



(b) \$320 000 (c) \$256 000 (d) 13.75 years

7 Suggested answers:

(a) $4x + 3y + 12 = 0$ (b) $4x - 3y + 12 = 0$

Exercise 4.7 (p. 160)

1 (a) The gradient of both lines is 2.5. Hence, they are parallel.

(b) The gradient of both lines is 0.5. Hence, they are parallel.

2 (a) The gradient of $4x + 3y = 7$ is $-\frac{4}{3}$ and the gradient of $8y = 6x - 2$ is $\frac{3}{4}$. $-\frac{4}{3} \times \frac{3}{4} = -1$. Hence, the two lines are perpendicular.

(b) The gradient of $2x - 5y - 1 = 0$ is $\frac{2}{5}$ and the gradient of $4y + 10x = 3$ is $-\frac{5}{2}$. $\frac{2}{5} \times -\frac{5}{2} = -1$. Hence, the two lines are perpendicular.

3 (a) $m_1 = 3$, $m_2 = 3$; lines are parallel.

(b) $m_1 = 2$, $m_2 = -2$; lines are not parallel.

(c) $m_1 = 0.5$, $m_2 = 0.5$; lines are parallel.

(d) $m_1 = -4$, $m_2 = 4$; lines are not parallel.

(e) $m_1 = 5$, $m_2 = 9$; lines are not parallel.

(f) $m_1 = 2$, $m_2 = 2$; lines are parallel.

(g) $m_1 = -3$, $m_2 = -3$; lines are parallel.

(h) $m_1 = -3$, $m_2 = -4$; lines are not parallel.

(i) $m_1 = 4$, $m_2 = 4$; lines are parallel.

(j) $m_1 = -\frac{5}{3}$, $m_2 = \frac{1}{5}$ lines are not parallel.

4 (a) $m_1 = 2$, $m_2 = -\frac{1}{2}$, $2 \times -\frac{1}{2} = -1$; lines are perpendicular.

(b) $m_1 = 4$, $m_2 = -\frac{1}{2}$, $4 \times -\frac{1}{2} = -2$; lines are not perpendicular.

(c) $m_1 = \frac{1}{3}$, $m_2 = -\frac{1}{3}$, $\frac{1}{3} \times -\frac{1}{3} = -\frac{1}{9}$; lines are not perpendicular.

(d) $m_1 = \frac{1}{2}$, $m_2 = -2$, $\frac{1}{2} \times -2 = -1$; lines are perpendicular.

(e) $m_1 = \frac{3}{4}$, $m_2 = -\frac{3}{2}$, $\frac{3}{4} \times -\frac{3}{2} = -\frac{9}{8}$; lines are not perpendicular.

(f) $m_1 = \frac{4}{5}$, $m_2 = -\frac{5}{8}$, $\frac{4}{5} \times -\frac{5}{8} = -\frac{1}{2}$; lines are not perpendicular.

5 C 6 D

7 Gradient of $y = x + 10$ is 1. Gradient between the two points is $\frac{8-2}{4-(-2)} = \frac{6}{6} = 1$. Therefore, the lines are parallel.

8 Gradient of $2x + y - 10 = 0$ is -2. Gradient between the two points is $\frac{-7-5}{4-(-2)} = \frac{-12}{6} = -2$. Therefore, the lines are parallel.

9 Gradient of $y = -\frac{4}{3}x - 2$ is $-\frac{4}{3}$. Gradient between the two points is $\frac{4-1}{0-(-4)} = \frac{3}{4}$. $\frac{3}{4} \times -\frac{4}{3} = -1$. Therefore, the lines are perpendicular.

10 Gradient of $y = -\frac{5}{3}x + 10$ is $-\frac{5}{3}$. Gradient between the two points is $\frac{8-5}{9-4} = \frac{3}{5}$. $-\frac{5}{3} \times \frac{3}{5} = -1$. Therefore, the lines are perpendicular.

11 (a) (i) and (iii) (b) (ii) and (v)

12 $y = 3x - 9$

13 Parallel line is $y = -2x + 5$, perpendicular line is $y = 0.5x + 5$.

14 $m_{AB} = -\frac{3}{2}$, $m_{BC} = 5$, $m_{CA} = \frac{2}{3}$. $-\frac{3}{2} \times \frac{2}{3} = -1$. Therefore, AB is perpendicular to CA and ABC must be a right-angled triangle.

15 $m_{AB} = 1$, $m_{BC} = -\frac{1}{2}$, $m_{DC} = 1$, $m_{AD} = -\frac{1}{2}$.

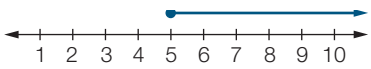
$AB \parallel DC$ and $BC \parallel AD$. Therefore $ABCD$ is a parallelogram.

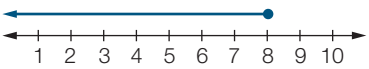
16 $m_{AB} = -\frac{5}{2}$, $m_{BC} = \frac{2}{5}$, $m_{DC} = -\frac{5}{2}$, $m_{AD} = \frac{2}{5}$. $-\frac{5}{2} \times \frac{2}{5} = -1$, therefore $AB \perp BC$, $BC \perp CD$, $CD \perp DA$ and $DA \perp AB$. Therefore $ABCD$ is a quadrilateral with four right angles. $|AB| = |BC| = |CD| = |DA| = 29$. All four sides are of equal length, therefore $ABCD$ is a square.

17 (a) $y = 0.5x + 4$, $2y = x + 3$, $2y - x = 10$
(all have gradients of 0.5)

(b) $y = -3x - 1$, $3x + y = 10$, $3x = 5 - y$
(all have gradients of -3)

Exercise 4.8 (p. 164)

1 (a)


(b)


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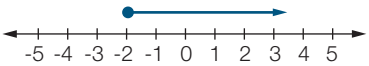

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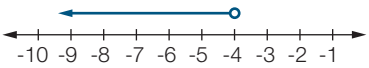
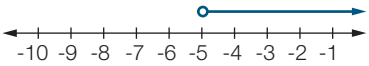
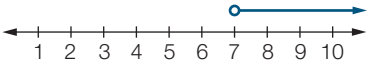
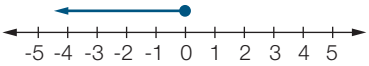
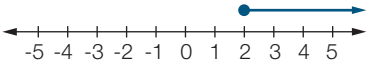
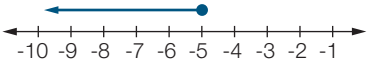


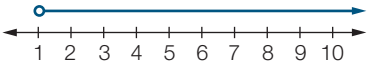
(e)


(f)


(g)


(h)


(i)


(j)


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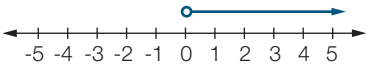
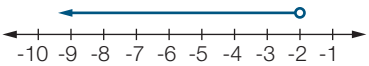
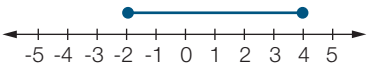
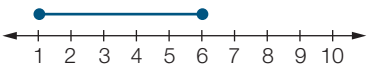
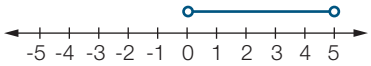
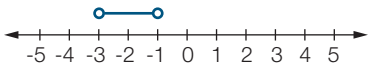


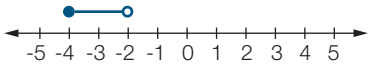
(l)


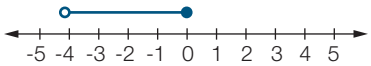
(m)


(n)


(o)


(p)


(q)


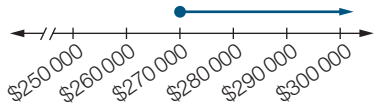
(r)


2 (a) $x > -10$ (b) $x < -5$ (c) $x \le -10$
(d) $x \le 35$ (e) $x \ge -9$ (f) $x > -28$
(g) $x < -\frac{16}{3}$ (h) $x > -\frac{27}{5}$ (i) $x \le -9$
(j) $x \ge 8$ (k) $x \le -2$ (l) $x > -8$
(m) $x > 4$ (n) $x < 11$ (o) $x \le 4$
(p) $x \le -8$ (q) $x > 11$ (r) $x > 10$

3 (a) $y \le -5$ (b) $y \ge -6$ (c) $y \ge -5$
(d) $y > 12$ (e) $y < 21$ (f) $y < 12$
(g) $y \ge -8$ (h) $y \le 2$ (i) $y \ge -4$
(j) $y \le -\frac{1}{15}$ (k) $y \ge \frac{1}{2}$ (l) $y \le \frac{19}{6}$
(m) $y < \frac{13}{3}$ (n) $y > \frac{23}{6}$ (o) $y < 4$

4 (a) B (b) D (c) A (d) D

5 (a) A (b) D (c) D (d) D

6 (a)


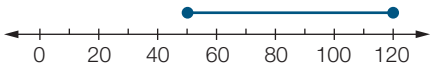
(b) $P \ge 270\,000$

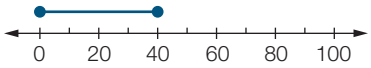
(c) $270\,000 \le P < 320\,000$

7 $x \le 100$ **8** $n \ge 5$ **9** $j > 55$

10 (a) $x \ge \frac{2}{3}$ (b) $x \le -\frac{3}{11}$ (c) $x < 0$
(d) $x > -2$ (e) $x \le 4$ (f) $x \ge 12$
(g) $x > -25$ (h) $x < \frac{34}{7}$ (i) $x \ge 2$

11 (a) $y \ge -1$ (b) $y < 3$ (c) $y \le -\frac{7}{3}$
(d) $y > \frac{1}{3}$ (e) $y \ge 30$ (f) $y \le 36$
(g) $y > -12$ (h) $y < \frac{6}{7}$ (i) $y < \frac{61}{13}$

12 (a) Let f be the number of cartons of full cream milk.
Let l be the number of cartons of low fat milk.
$50 \le f \le 120$


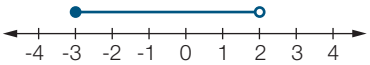
$0 \le l \le 40$


(b) $l + f \le 120$

13 $20p > 428$, $p > 21.4$. Therefore, Tony should sell the calculators for any price higher than \$21.40 each.

14 Let n be the number of liquorice straps. $0.4n \le 1.55$, $n \le 3.875$. As n must be an integer $n \le 3$.

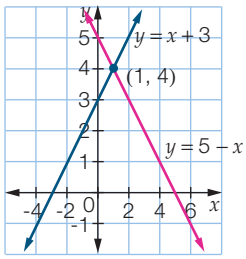
15



16 **(a)** $x - 10 \ge 3x - 8$ **(b)** $13x - 4 \ge 9x + 8$

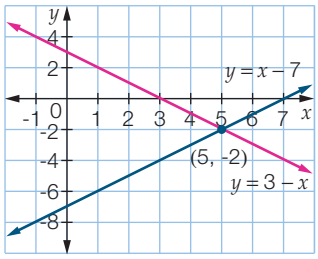
Exercise 4.9 (p. 171)

1 **(a)** **(i)**



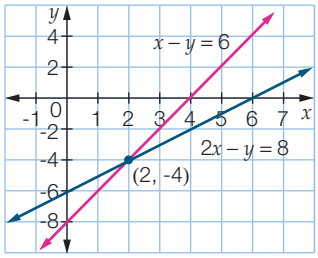
(ii) $x = 1$, $y = 4$

(b) **(i)**



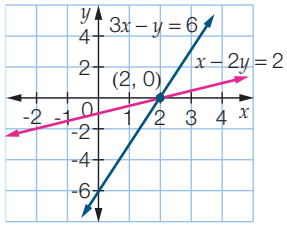
(ii) $x = 5$, $y = -2$

(c) **(i)**



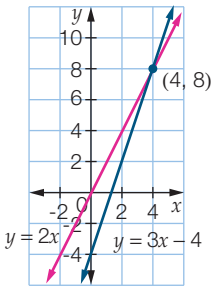
(ii) $x = 2$, $y = -4$

(d) **(i)**



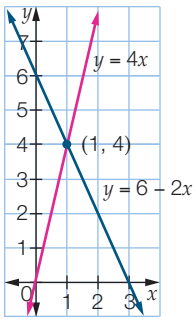
(ii) $x = 2$, $y = 0$

(e) **(i)**



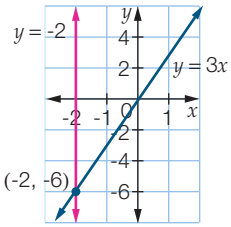
(ii) $x = 4$, $y = 8$

(f) **(i)**



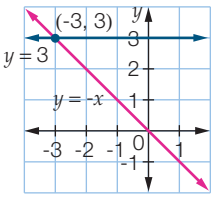
(ii) $x = 1$, $y = 4$

(g) **(i)**



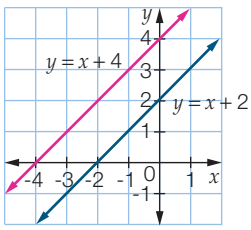
(ii) $x = -2$, $y = -6$

(h) **(i)**



(ii) $x = -3$, $y = 3$

(i) **(i)**



(ii) no solution

2 **(a)** (4, 1) **(b)** (5, -4) **(c)** (3, 1)
(d) (5, -1) **(e)** (1, 3) **(f)** (2, -6)
(g) (-2, 3) **(h)** (2, 2) **(i)** (2, 3)

3 **(a)** (1, -1) **(b)** (-3, 4) **(c)** (-1, 1)
(d) (2, -1) **(e)** (1, -1) **(f)** (-3, 1)
(g) (3, 1) **(h)** (-3, 4) **(i)** (1, 1)
(j) (4, -2) **(k)** (5, 12) **(l)** (9, 4)
(m) (-5, -10) **(n)** (-3, -8) **(o)** (-6, 5)

4 **(a)** (2, 4) **(b)** (-3, -9) **(c)** (4, -2)
(d) (1, 5) **(e)** (3, 2) **(f)** (3, -2)
(g) (-2, 5) **(h)** (9, -1) **(i)** (-1, 9)

5 **(a)** C **(b)** D

6 **(a)** D **(b)** B

7 **(a)** A **(b)** C

8 **(a)** D **(b)** D

9 a pineapple costs \$2.60, a kiwi fruit costs 35 cents

10 a salad roll costs \$3.60, a pie costs \$2.44

11 $x = 7$, $y = 5$

12 7 and 31

13 128 pages and 372 pages

14 30 m by 40 m

15 three 5-cent coins, fifteen 10-cent coins

16 Michelle is 18 years old, Yesmin is 31 years old

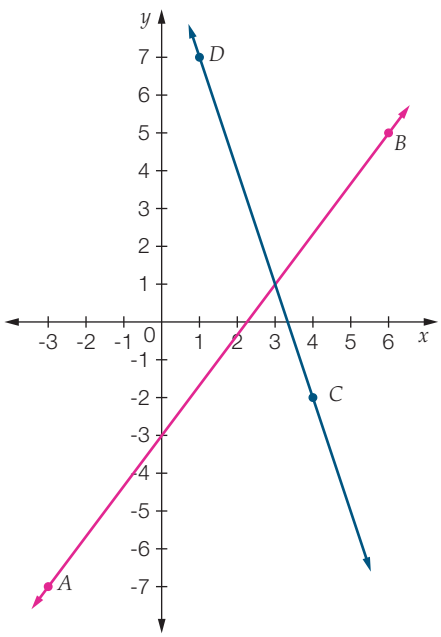
17 **(a)** Let D = distance in km from Wes' house and t = time taken in hours. Wes: $D = 80t$, Tony: $D = 360 - 100t$.

(b) 2 hours

18 Speed of barge is 12 km/h, speed of current is 3 km/h

19 **(a)** $A(-3, -7)$, $B(6, 5)$, $C(4, -2)$, $D(1, 7)$

(b)



(c) (3, 1)

(d) $y_{AB} = \frac{4}{3}x - 3$ (1)

$y_{CD} = -3x + 10$ (2)

substitute (1) into (2):

$$\frac{4}{3}x - 3 = -3x + 10$$

$$4x - 9 = -9x + 30$$

$$13x - 9 = 30$$

$$13x = 39$$

$$x = 3$$

substitute $x = 3$ into (2): $y = -9 + 10 = 1$

Therefore, the treasure is at (3, 1).

20 $x + y = 7$ and $x - y = -1$, $2x - y = 2$ and $x + 2y = 11$, $5x + 2y = 23$ and $3x - y = 5$

21 3, 8 and 14. Method example: Let x, y and z be the three numbers.

$x + y = 11$ (1)
$x + z = 17$ (2)
$y + z = 22$ (3)

Using elimination, (1) – (2):

$$\begin{array}{rl} & x + y = 11 \\ - & x + z = 17 \\ \hline & y - z = -6 \end{array}$$

$y = z - 6$ (4)

Substitute (4) into (3):

$$z - 6 + z = 22$$
$$z = 14$$

Substitute $z = 14$ into (2):

$$x + 14 = 17$$
$$x = 3$$

Substitute $x = 3$ into (1):

$$3 + y = 11$$
$$y = 8$$

Chapter review 4 (p. 175)

1 **(a)** $c = 8$ **(b)** $x = -12.5$ **(c)** $v = 3$

2 **(a)** $a = 1$ **(b)** $k = -13$

3 $y = 2x - 1$ **4** D

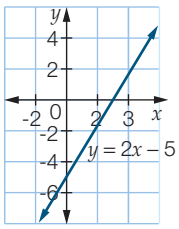
5 **(a)** $m = 5$, y-intercept = (0, 2)

(b) $m = -2$, y-intercept = (0, 0)

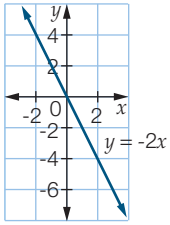
(c) $m = \frac{2}{5}$, y-intercept = (0, -2)

(d) $m = -4$, y-intercept = (0, 8)

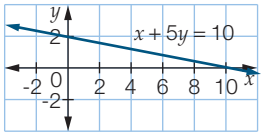
6 **(a)**



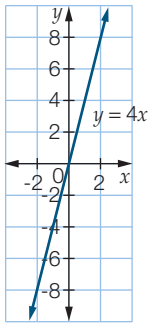
(b)



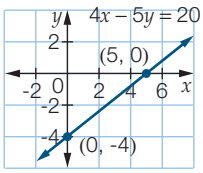
(c)



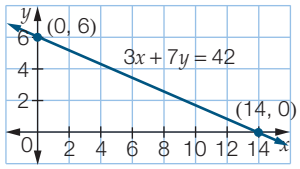
(d)



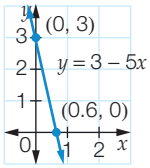
7 (a)



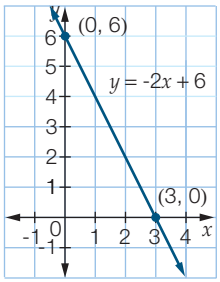
(b)



(c)



(d)



8 (a) Both have gradients of -0.5, therefore they are parallel.

(b) With different gradients of 1 and 1.5, they are not parallel.

(c) Both have gradients of -4, therefore they are parallel.

(d) Both have gradients of 4 so they are parallel.

9 (a) The lines have gradients of -1 and -2. $-1 \times -2 = 2$. Therefore, they are not perpendicular.

(b) The lines have gradients of 1 and -1. $1 \times -1 = -1$. Therefore, they are perpendicular.

(c) The lines have gradients of 0.5 and -2. $0.5 \times -2 = -1$. Therefore, they are perpendicular.

(d) The lines have gradients of $-\frac{3}{2}$ and $\frac{2}{3}$. Their product is equal to -1 so they are perpendicular.

10 (a) $x \geq \frac{7}{3}$ (b) $x > -\frac{35}{3}$ (c) $x > \frac{5}{6}$

11 (4, 6)

12 (-1, 3)

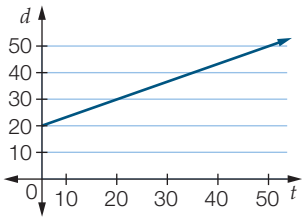
13 18 wombats, 9 emus

14 4 hours

15 (a) 8 hours (b) 1500 L (c) 1500 (d) $V = 1500T$

16 (a) $d = 0.6t + 20$

(b)



(c) Every minute of travel takes Manny 0.6 km further away from Sydney.

(d) He began 20 km away from Sydney.

(e) 28.4 km (f) 46 min and 40 s

17 $y = -x + 9$

18 $y = 5x - 42$

19 (a) $x < 2\frac{4}{5}$ (b) $x \leq -2\frac{2}{3}$

20 6 and -5

21 $a = 9$, $b = -3$

22 (a) $570\,000 \leq p \leq 620\,000$ (b) $57\,000 \leq d \leq 62\,000$

(c) $p > 670\,000$ (d) $p < 670\,000$

23 $d = 165$ km, $s = 82$ km/h

Chapter 5

Recall 5 (p. 178)

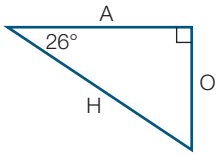
1 (a) $x = 8.17$ (b) $x = 15.7$ (c) $x = 5.20$

(d) $x = 46.377$

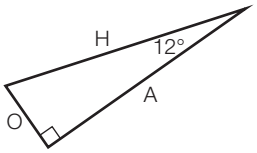
2 (a) 3 cm (b) 15 mm

3 (a) 26 (b) 33.23 (c) 9.95

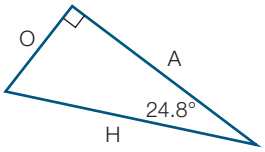
4 (a)



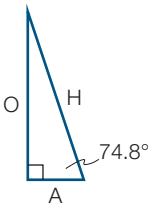
(b)



(c)

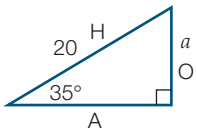


(d)



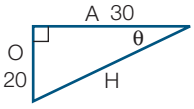
Exercise 5.1 (p. 182)

1 (a)



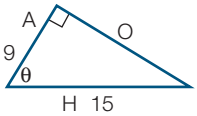
$\sin 35° = \frac{a}{20}$

(b)



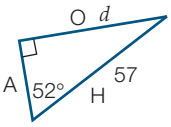
$\tan \theta = \frac{20}{30}$

(c)



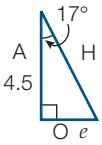
$\cos \theta = \frac{9}{15}$

(d)



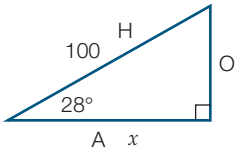
$\sin 52° = \frac{d}{57}$

(e)



$\tan 17° = \frac{e}{4.5}$

(f)



$\cos 28° = \frac{x}{100}$

2 (a) 0.3907 (b) 0.9563 (c) 0.8098

(d) 0.9962 (e) 0.6561 (f) 0.6561

(g) 0.3581 (h) 0.9171 (i) 0.9540

3 (a) 15.50° (b) 35.27° (c) 23.17° (d) 76.80°

(e) 44.99° (f) 8.75°

4 (a) 0.4350 (b) 38.1885 (c) 0.4339 (d) 0.7403

(e) 0.3490 (f) 0.1437

5 **(a)** C **(b)** D

6 **(a)** **(i)** 1.5 **(ii)** 1.5 **(iii)** 1.5

(b) 0.5 **(c)** 0.5

(d) Yes, because the triangles are similar.

(e) $\propto = \beta = 30°$ **(f)** yes **(g)** yes

(h) The triangles are similar so the ratio between the matching sides of the triangles is constant. This means that the ratio of a pair of sides within one triangle is the same as the ratio of the matching pair of sides within the other triangle.

7 **(a)** **(i)** 0.5736 **(ii)** 0.8192 **(iii)** 0.7002 **(iv)** 0.7002

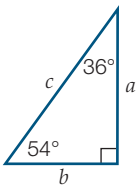
(b) The ratio of sin 35° and cos 35° is the same as tan 35°.

8 1.151 **9** B

10 **(a)** **(i)** 0.722 **(ii)** 0.722

(b) Ratios use the same side lengths $\left(\frac{4.6}{6.37}\right)$.

11 **(a)**



(b) $\sin 54° = \frac{a}{c}$, $\cos 54° = \frac{b}{c}$

(c) $\sin 54° = 0.8090$, $\cos 54° = 0.5878$

(d) $(\sin 54°)^2 + (\cos 54°)^2$
$= 0.8090^2 + 0.5878^2$
$= 0.6545 + 0.3455$
$= 1$

(e) $(\sin 54°)^2 + (\cos 54°)^2$

$= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$= \frac{a^2 + b^2}{c^2}$

$= \frac{c^2}{c^2}$ (Pythagoras' Theorem $a^2 + b^2 = c^2$)

$= 1$

12 An isosceles triangle has two sides equal and two equal angles. In a right-angled isosceles triangle, these two sides are the opposite and adjacent of the two base angles, which are each 45°. So:

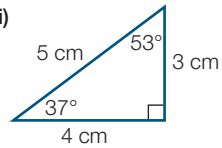
$\tan\theta = \frac{O}{A}$

$= \frac{x}{x}$

$= 1$

13 The value of the tan ratio can be greater than 1. The values of the cos and sin ratios can't be greater than 1, as to calculate them, you need to divide either the adjacent or opposite side by the hypotenuse, and by definition, the hypotenuse will always be larger than the other two sides.

14 **(a)** **(i)** **(ii)**



(iii) $\sin 37° = \frac{3}{5}$ $\sin 53° = \frac{4}{5}$

$= 0.60$ $= 0.80$

$\cos 37° = \frac{4}{5}$ $\cos 53° = \frac{3}{5}$

$= 0.80$ $= 0.60$

$\tan 37° = \frac{3}{4}$ $\tan 53° = \frac{4}{3}$

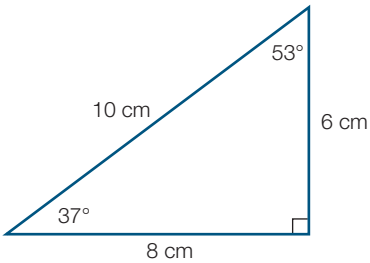
$= 0.75$ $= 1.33$

(iv) $\sin 37° = \cos 53°$

$\cos 37° = \sin 53°$

$\tan 37° = \frac{1}{\tan 53°}$

(b)

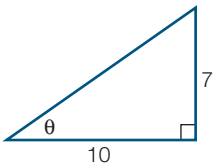


$\sin 37° = 0.60$ $\sin 53° = 0.80$
$\cos 37° = 0.80$ $\cos 53° = 0.60$
$\tan 37° = 0.75$ $\tan 53° = 1.33$

(c) Values of sin, cos and tan are identical as the angles in each triangle are the same.

(d) Measured values agree with the calculator values, rounded to two decimal places.

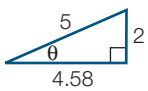
15 **(a)** hypotenuse = 12.2 cm

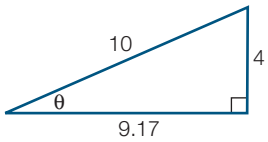


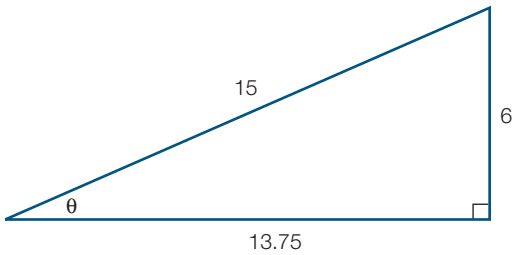
(b) $\sin\theta = \frac{7}{12.2}$ $\cos\theta = \frac{10}{12.2}$

$= 0.5738$ $= 0.8197$

16







17 **(a)** $\theta = 20°$ and $\varphi = 70°$

(b) $\theta = 90 - \varphi$ or $\varphi = 90 - \theta$

Exercise 5.2 (p. 187)

1 **(a)** $a = 167.71$ **(b)** $b = 12.44$ **(c)** $c = 1.05$

(d) $x = 11.53$ **(e)** $x = 11.22$ **(f)** $x = 13.29$

(g) $x = 148.84$ **(h)** $a = 20.58$ **(i)** $b = 42.56$

(j) $d = 32.09$ **(k)** $e = 7.40$ **(l)** $f = 41.08$

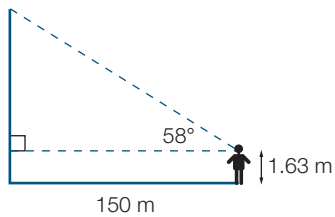
(m) $d = 95.83$ **(n)** $c = 244.74$ **(o)** $e = 6.91$

2 193 cm **3** 3.74 m

4 **(a)** B **(b)** B

5 9.89 m

6 **(a)**



(b) 241.7 m

7 15.37 m

8 325.87 m **9** 39.95 cm

10 **(a)** 19.73 m

(b) dropped from 9.86 m along rope, falls 28.7 m

11 **(a)** trapezium 10.74 m^2, rectangle 12 m^2

(b) rectangular room

12 For a pitch angle of 15°:

(a) 1.21 m **(b)** 19.53 m

For a pitch angle of 25°:

(a) 2.1 m **(b)** 21.03 m

13 16 cm, 16.5 cm, 17 cm

14 Taking 'fly over' to be a height of 8.1 m, two possible angles are 40° and 35°.

$$\tan 40° = \frac{8.1 - 1.2}{x}$$

$$x = \frac{6.9}{\tan 40°} = 8.22 \text{ m}$$

$$\tan 35° = \frac{8.1 - 1.2}{x}$$

$$x = \frac{6.9}{\tan 35°} = 9.9 \text{ m}$$

Exercise 5.3 (p. 192)

1 **(a)** **(i)** $a = 81.15°$ **(ii)** $a = 81°9'10''$

(b) **(i)** $b = 42.21°$ **(ii)** $b = 42°12'50''$

(c) **(i)** $c = 66.58°$ **(ii)** $c = 66°34'43''$

(d) **(i)** $k = 56.22°$ **(ii)** $k = 56°13'13''$

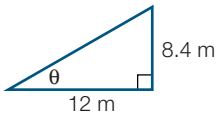
(e) **(i)** $m = 77.96°$ **(ii)** $m = 77°57'22''$

(f) **(i)** $p = 76.29°$ **(ii)** $p = 76°17'17''$

2 **(a)** 12°22′25″ **(b)** 25°42′36″ **(c)** 47°9′23″

(d) 14°33′5″ **(e)** 24°11′40″ **(f)** 54°35′10″

3 **(a)**



(b) 35°

4 **(a)** B **(b)** C

5 0.40° **6** 48° **7** 53°36′56″ **8** B

9 **(a)** 33° **(b)** 14 cm

10 **(a)** 147° **(b)** 25.7 m

11 **(a)** 11.54° to 36.87°

(b) The ladder is too long for the 2 m window. It would be placed 9.8 m away, at an angle of 11.5°.

12 30 cm (22°), 40 cm (30°), 50 cm (39°), 60 cm (49°), 70 cm (61°)

13 Angle of Bilal's shadow with the sun is $\theta = \tan^{-1}\frac{185}{94}$ = 63.06°. For example, take the height of the lamp post as 5 m and the height of the tree as 12 m. Then the length of the lamp post's shadow is $\frac{5}{\tan 63.06°} = 2.54$ m and the length of the tree's shadow is $\frac{12}{\tan 63.06°} = 6.10$ m

Exercise 5.4 (p. 195)

1 14° **2** 506 m **3** 53°

4 39 m **5** 8.55 m

6 **(a)** 209 m **(b)** 134 m

7 **(a)** A **(b)** B

8 24 m **9** 28.84 m **10** 4°

11 **(a)** 177 cm **(b)** 20.2°

12 **(a)** 27.7° **(b)** 12.25 m

13 **(a)** 46°, 50°, 54°

(b) 12.95 m (46°), 13.79 m (50°), 14.56 m (54°)

14 0.95° (after 20 min), 0.48° (after 44 min)

Half-time 5 (p. 198)

1 6.4° 2 B 3 B

4 $\theta = 50.36°$, height = 23.23 m

5 D 6 320 m

Exercise 5.5 (p. 201)

1 (a) (i) N21°W (ii) 339°T

(b) (i) S73°E (ii) 107°T

(c) (i) N58°E (ii) 058°T

(d) (i) S53°W (ii) 233°T

(e) (i) S51°E (ii) 129°T

(f) (i) N73°W (ii) 287°T

2 115 km

3 (a) True bearing: 152°T Compass bearing: S28°E

(b) True bearing: 147°T Compass bearing: S33°E

4 (a) B (b) C

5 75.74 km 6 260 km

7 (a) 056°T

(b) (i) 288 km (ii) 236°T

8 (a) 2.11 km

(b) (i) 5.87 km (ii) 7.98 km

(c) (i) 1.25 km (ii) 5.78 km

9 (a) 1155 m (b) South 227 m

10 30.39 m

11 (a) (i) 115 m (ii) 164 m

(b) 027°T (c) 1004 m

12 (a) 115°T (b) 3.32 km

13 (a) S23°E (b) 674 km (c) 257 km south

(d) N69°W (e) 721 km

14 If $\theta = 30°$, then:

(a) 150° T, S30° E (b) 330° T, N30° W

(c) 210° T, S30° W

15 Students' own answers.

Exercise 5.6 (p. 206)

1 (a) $x = 16.41$, $y = 27.27$ (b) $x = 31.40$, $y = 28.23$

(c) $x = 25.02$, $y = 3.32$ (d) $x = 28.87$, $y = 22.91$

(e) $x = 66.75$, $y = 84.57$ (f) $x = 63.23$, $y = 78.50$

2 65.80 m

3 (a) $x = 14.4$ (b) $x = 25.87$ (c) $x = 37.14$ (d) $x = 10.51$

4 (a) $a = 34.5°$, $x = 123.4$ (b) $k = 57.6°$, $x = 9.5$

(c) $m = 58.5°$, $x = 69.0$ (d) $a = 31.1°$, $x = 30.2$

5 A 6 D 7 $24 000

8 31°23′ 9 53 m 10 154 km 11 62°

12 45°38′31″ 13 180.78 m 14 3.80 cm

15 16.83 mm 16 11.92 m

17 (a) 3.46 cm (b) 38 cm^2 (c) 4.53 cm

18 Sample answers: $a = 20$, $\theta = 10°$ so $c = 20.3$; $b = 5$, $\varphi = 20°$ so $d = 14.6$

19 Sample answer: Let $x = 13$ and $y = 11$, then $\theta = 134.5°$.

Chapter review 5 (p. 210)

1 (a) 0.5736 (b) 1.2482 (c) 0.9944

(d) 0.5715 (e) 0.2823 (f) 0.2905

2 (a) 22.26 cm (b) 11.72 m

3 (a) (i) 42.27° (ii) 42°16′7″

(b) (i) 69.29° (ii) 69°17′23″

(c) (i) 27.45° (ii) 27°26′43″

4 62.45 m 5 N64°W

6 $a = 26.7°$, $x = 21.2$

7 418 cm 8 D 9 D

10 17.16 m 11 19.3° 12 101 m

13 23.56 m 14 72 km, N80°25′

15 No, the angle of the wedge and the vertical height of its end do not match the gap size.

16 84 km

17 Draw a line from the top of the triangle to the midpoint of the base to create a right-angled triangle. Now either use the sine ratio with one side of the original triangle as the hypotenuse and the height as the opposite side, or use the tangent ratio with half of the base as the adjacent side and the height as the opposite side. In both cases the angle will be 60°.

18 34.22 cm

Chapter 6

Recall 6 (p. 212)

1 (a) 3.72 (b) 46.05 (c) 3.77

2 (a) 53°8′ (b) 67°30′ (c) 77°44′

3 (a) 8.95 (b) 107.57

4 20.32 m

5 (a) 10.34 km (b) 288.74°T

Exercise 6.1 (p. 215)

1 (a) 66° (b) 12.81 cm

2 4.05 m^2

3 (a) D (b) B

4 (a) 90° (b) 40.6° (c) 9.22 mm

(d) 11.58 mm (e) 31.2°

5 8.66 cm

6 (a) 147.52 m (b) 134.76 m

(c) 17.12 m/min (d) 6.66°

(e) gradient of AB is 0.45, gradient of AC is 0.12, so AB is steeper.

7 (a) 185.22 cm (b) 2°47′7″

8 (a) 43.46 m (b) 78.41 m (c) 93.00 m (d) 122.5°T

9 (a) 5.37 cm (b) 5.13 cm

10 14 289 m

11 Yes. The distance from A to B is 49.75 cm.

12 248 m

13 (a) 67.71 m (b) 75.12 m

14 1941 cm^2

15 Yes. The angle is 75°. 16 223.07 m

17 The ratio is always 0.07.

18 Two example values: 20° and 30°

(a) $BF = 4.45$ cm, 6.5 cm

(b) $DF = 12.22$ cm, 11.26 cm

$EF = 9.23$ cm, 7.92 cm

$BC = 9.15$ cm, 10.31 cm

(c) 84.50 cm^2, 81.66 cm^2

(d) As the value of θ increased, the area of $ABCD$ decreased.

(e) 234.97 cm^2, 247.98 cm^2

(f) Less plastic was used when θ was smaller and $ABCD$ had a larger area.

19 (a) $h = a\sin\theta$ and $h = b\sin\phi$ so $\frac{a}{b} = \frac{\sin\phi}{\sin\theta}$.

(b) Any of the following values:

$\theta = 21°, \phi = 46°$
$\theta = 22°, \phi = 49°$
$\theta = 23°, \phi = 51°$
$\theta = 24°, \phi = 54°$
$\theta = 25°, \phi = 58°$

Exercise 6.2 (p. 225)

1 (a) 4 (b) 3 (c) 2 (d) 2 (e) 3 (f) 1
(g) 3 (h) 2 (i) 4 (j) 4 (k) 2 (l) 3

2 (a) sin 45° (b) -cos 44° (c) -tan 69° (d) -sin 84°
(e) cos 39° (f) -tan 57° (g) -sin 83° (h) -cos 83°
(i) tan 6° (j) -sin 18° (k) -cos 14° (l) tan 27°

3 (a) $-\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{2}$
(d) 1 (e) $-\frac{1}{2}$ (f) $-\frac{1}{2}$
(g) $\frac{1}{\sqrt{2}}$ (h) $-\frac{\sqrt{3}}{2}$ (i) $\sqrt{3}$
(j) $-\frac{1}{\sqrt{2}}$ (k) $-\frac{1}{\sqrt{3}}$ (l) $-\frac{\sqrt{3}}{2}$

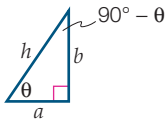
4 B 5 D 6 C

7 For 30°: 150°, 210°, 330°
For 45°: 135°, 225°, 315°
For 60°: 120°, 240°, 300°

8 (a) $\sin\theta = \frac{b}{h}$

$\cos(90 - \theta) = \frac{b}{h}$

$\therefore \sin\theta = \cos(90 - \theta)$



(b) $\cos\theta = \frac{a}{h}$

$\sin(90 - \theta) = \frac{a}{h}$

$\therefore \cos\theta = \sin(90 - \theta)$

9 Use 24° and 204°

sin	+	–
cos	+	–
tan	+	+

10 (a) If θ is in quadrant 1, -θ is in quadrant 4 (a clockwise rotation of θ). So cos (-θ) and cos θ are positive and of equal magnitude.

If θ is in quadrant 2, -θ is in quadrant 3 so both cos (-θ) and cos (θ) are negative and of equal magnitude.

If θ is in quadrant 3, -θ is in quadrant 2 so both cos (-θ) and cos (θ) are negative and of equal magnitude.

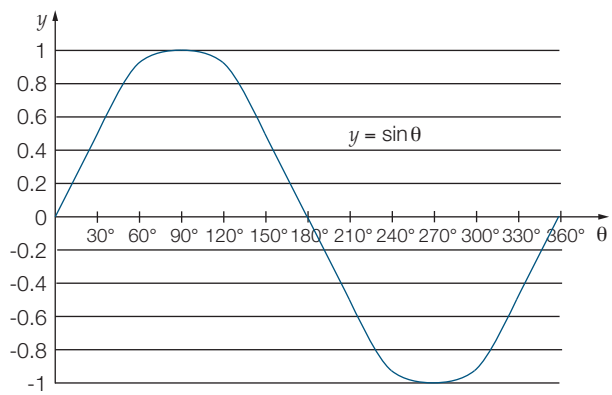
If θ is in quadrant 4, -θ is in quadrant 1 so both cos (-θ) and cos (θ) are positive and of equal magnitude.

(b) Using a similar method as in (a), sin θ is positive in quadrants 1 and 2, and negative in quadrants 3 and 4.

(c) $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$

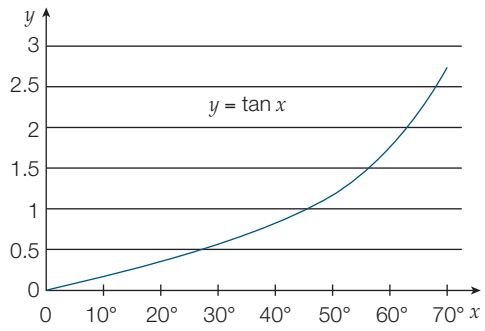
Exercise 6.3 (p. 229)

1



2 C

3 (a)



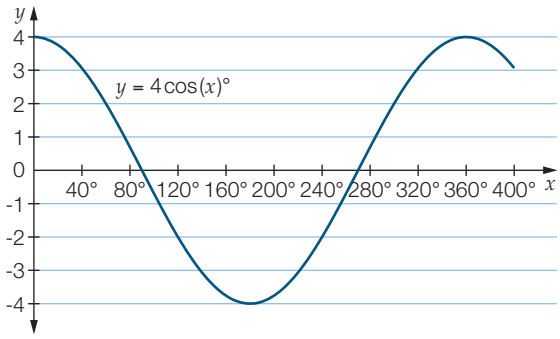
(b) The value of tan x becomes larger at a faster rate and approaches infinity as x approaches 90°.

4 (a) (i) 360° (ii) 360° (iii) 360°

(b) (i) 1 (ii) 2 (iii) 3

(c) The value of a tells you the amplitude.

(d) A graph like the following should have been sketched.



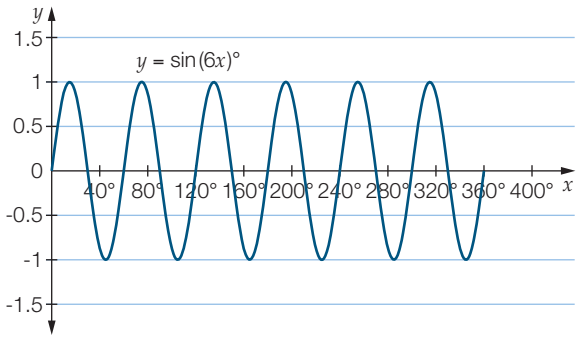
5 (a) (i) 360° (ii) 180° (iii) 120°

(b) (i) 1 (ii) 1 (iii) 1

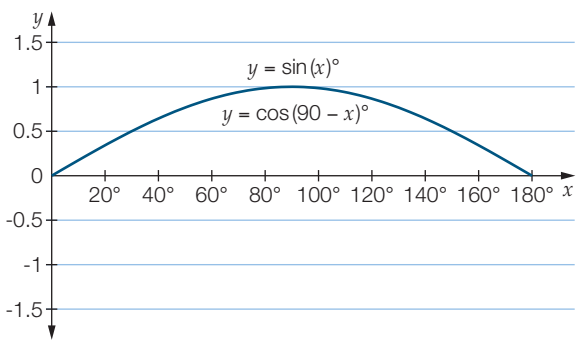
(c) The value of n changes the period.

(In fact, the period is $\frac{360°}{n}$)

(d) A graph like the following should have been sketched.



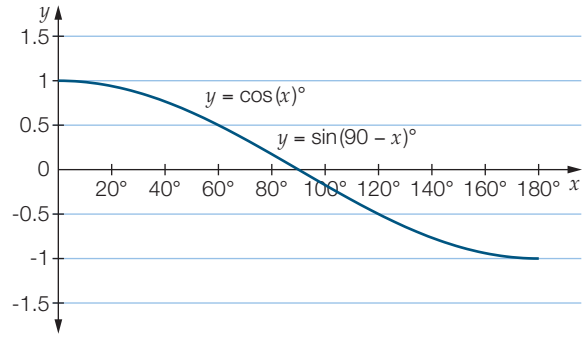
6 (a)



(b) The graphs are identical.

(c) $\sin(x)° = \cos(90 - x)°$

7 (a)



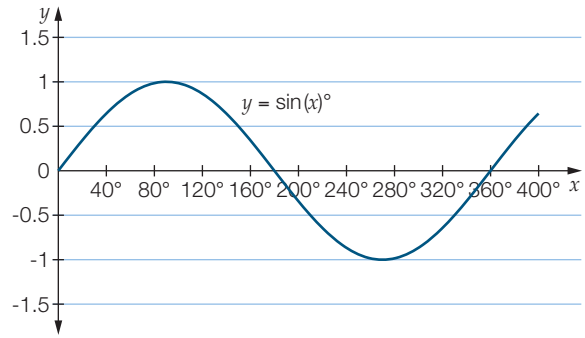
(b) The graphs are identical.

(c) $\cos(x)° = \sin(90 - x)°$

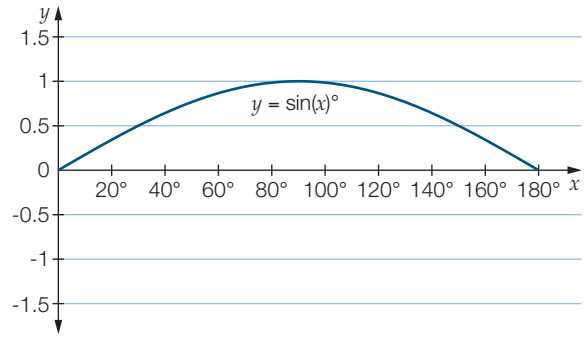
8 It will be 156°, the equivalent angle in quadrant 2.

9 Students' own answers, e.g.

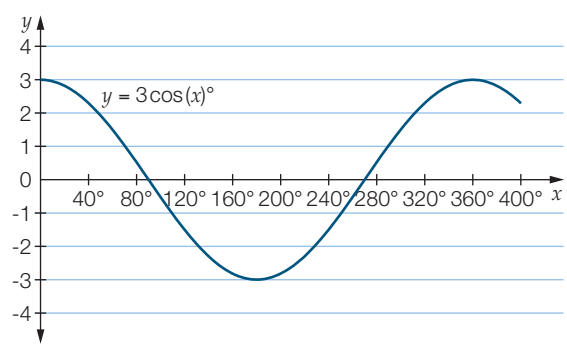
(a)



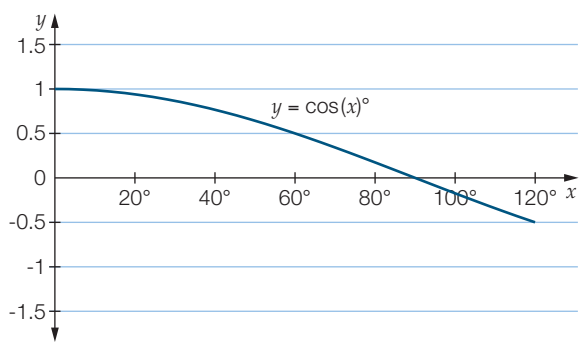
(b)



(c)



(d)



Exercise 6.4 (p. 233)

1 (a) $x = 120°, x = 240°$ (b) $x = 30°, x = 150°$

2 $x = (120 \pm 360n)°, n = 0, 1, 2 \ldots$ and
$x = (240 \pm 360n)°, n = 0, 1, 2 \ldots$

3 $x = (14.48 \pm 360n)°, n = 0, 1, 2 \ldots$ and
$x = (165.52 \pm 360n)°, n = 0, 1, 2 \ldots$

4 B

5 (a) 48.59° and 131.41° (b) 158.20° and 338.20°
(c) 53.13° and 306.87° (d) 41.81° and 138.19°

6 9.10 p.m. 7 C 8 C

9 (a) If the denominator becomes equal to 0.
(b) $x = 210°$ and 330°

10 $x = 0°, 180°$ and 360°

11 $5 \sin x + 2 = 0; x = 203.58°, 336.45°$
$7 \tan x - 3 = 0; x = 23.20°, 203.20°$
$4 \cos x - 3 = 0; x = 41.41°, 318.59°$

12 (a) Students' own answers.
(b) In (i) it had to be a positive tan equation and in (ii) it had to be a negative sin equation.

Half-time 6 (p. 235)

1 (a) 17.10° (b) 27.20 cm

2 (a) $x = 41.8°, 138.2°$ (b) $x = 21.8°, 201.8°$

3 (a) sin 45° (b) -cos 82° (c) tan 20°
(d) -sin 67° (e) cos 60° (f) -tan 21°

4 (a) 65° (b) 13.6 cm
(c) (i) decrease (ii) 9°

5 C

6 (a) $x = (60 \pm 360n)°$ and $x = (300 \pm 360n)°, n = 1, 2, 3, \ldots$
(b) $x = (19.47 \pm 360n)°$ and $x = (160.53 \pm 360n)°, n = 1, 2, 3, \ldots$

Exercise 6.5 (p. 240)

1 (a) 6.56 (b) 5.89 (c) 48.42

2 (a) 82.16° (b) 40.45° (c) 68.03°

3 (a) 56.23°, 123.77° (b) 45.77°, 134.23°
(c) 68.03°, 111.97°

4 (a) 9.01 (b) 51.50 (c) 42.46

5 (a) 54.90° (b) 110.52° (c) 26.05°

6 39° 7 6.78 units

8 $c^2 = a^2 + b^2 - 2ab\cos\theta$ and $d^2 = 4b^2 + 4a^2 - 8ab\cos\theta$
$d^2 = 4c^2 \Rightarrow d = 2c$

9 113.62°

10 $\angle ABC = \theta - \phi$, so $\dfrac{x}{\sin(\theta - \phi)} = \dfrac{y}{\sin\phi}$

$x = \dfrac{y\sin(\theta - \phi)}{\sin\phi}$

11 D 12 60.99 13 C

14–16 Students' own answers.

17 (a) $c^2 = a^2 + b^2 - 2ab\cos C$
$k^2a^2 = a^2 + a^2 - 2a^2\cos C$
$k^2 = 2 - 2\cos C$
$2 - k^2 = 2\cos C$
$\cos C = \dfrac{2 - k^2}{2}$
(b) $k = -1, \cos\theta = 0.5$
$k = 1.5, \cos\theta = -0.125$
(c) $-2 \le k \le 2$

18 If $b = 4a$ and $c = 4a$
$\theta = 82.82°$ ($\cos^{-1}\theta = \frac{1}{8}$)

Exercise 6.6 (p. 245)

1 4.92 m 2 10.60 km 3 32.76 m

4 61.47 cm 5 68.40 m

6 Yes, because the angle is 104.478°.

7 Tim 8.22 km, Zoe 6.45 km; so Zoe is closer.

8 46.55° 9 254 m 10 A

11 037°T and 235°T

12 (a) 47.55 m (b) 123.46 m

13 27.41 mm

14 16.33 cm or 21.50 cm

15 B 16 4885 m

17 Students' own answers.

18 $\theta = 45°, x = 16$ m, $y = 22.63$ m
$\theta = 60°, x = 27.71$ m, $y = 32$ m

Exercise 6.7 (p. 250)

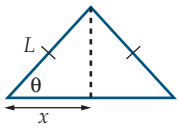
1 (a) 175.96 cm^2 (b) 1269.64 m^2 (c) 330.90 mm^2

2 38°

3 (a) 365.01 cm^2 (b) 115.43 m^2 (c) 1718.63 mm^2

4 1292 km^2 5 5.85 mm^2 6 C

7 (a) 36.38 cm^2 or 431.01 cm^2 (b) 278.60 m^2
(c) 583.30 km^2



8 (a) $x = L\cos\theta$
Length of base $= 2L\cos\theta$
Area of triangle $= \frac{1}{2}L \times 2L\cos\theta \times \sin\theta$
$= L^2\cos\theta\sin\theta$
(b) 15.68 km^2

9 (a) Area $= \frac{1}{2}a(b+c)\sin\theta - \frac{1}{2}ab\sin\theta$

$= \frac{1}{2}ab\sin\theta + \frac{1}{2}ac\sin\theta - \frac{1}{2}ab\sin\theta$

$= \frac{1}{2}ac\sin\theta$

(b) 90°, $\frac{1}{2}ac$ mm^2

10 \$1426

11 (a) $0° < \theta < 180°$

(b) The area cannot be negative.

12 22 cm

13 Any triangle where the length of the equal sides is a square number and the included side is 30° (e.g. 36, 36, 30° or 16, 16, 30°).

Chapter review 6 (p. 253)

1 (a) 24.74 cm (b) 118.27 m

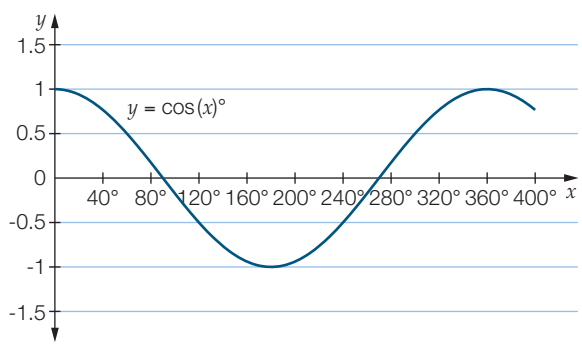
2 26.31 m

3 (a) 1 (b) 4 (c) 2 (d) 3

4 (a) -cos 7° (b) -sin 74° (c) -tan 48°

5 B

6



7 135°, 315°

8 $x = (75.52 \pm 360n)°$ and $x = (284.48 \pm 360n)°$, $n = 0, 1, 2 \ldots$

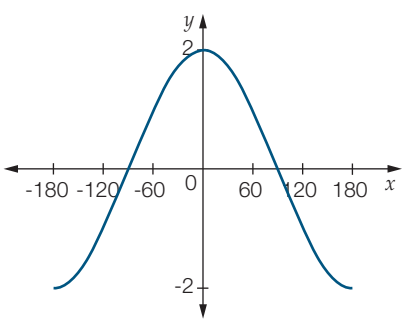
9 C 10 12.81 11 73° 12 207.70 m^2

13 D

14 (a) 180°, 7 (b) 72°, 6 (c) 45°, no amplitude

15 B

16



17 (a) quadrant 1 (b) 320°

18 122° 19 46.03 cm^2 20 39.55 m

21 (a) 62.7° (b) 13.95 cm

22 (a) (i) 332.86 m (ii) 495.02 m

(b) 596.52 m (c) 056°T

23 multiples of 90°

24 (a) (i) Find the third angle α and use the sine formula

$\frac{x}{\sin\theta} = \frac{a}{\sin\alpha}$, where $\alpha = 180° - \theta - \phi$

(ii) 13.88

(b) (i) Each of the two other angles is $90° - \frac{1}{2}\theta$.

Use the sine formula $\frac{x}{\sin\theta} = \frac{a}{\sin\left(90° - \frac{1}{2}\theta\right)}$.

(ii) 14.51

25 D

26 Students' own answers.

Mixed review B (p. 256)

1 (a) \$788.20 (b) \$796.82 (c) \$801.22

2 4.4%

3 (a) $3x^2 + 2x - 8$ (b) $6x^2 + 13x - 5$ (c) $8x^2 - 26x + 15$

4 (a) $(3x-7)(2x+1)$ (b) $4x(3x-2)(2x+3)$

5 (a) $(x-4)(x+1)$ (b) $(x-5)(x-9)$

(c) $(x-4)(x-3)$

6 (a) $\frac{-5}{3x}$ (b) $\frac{3x^2-5x-2}{x^2-1}$ (c) 3

7 (a) 75.40 mm^2 (b) 382.08 cm^2

8 319.41 cm^2 9 720 cm^3 10 6.4 g/cm^3

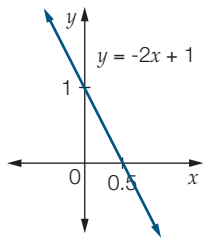
11 (a) $x = 9$ (b) $x = \frac{7}{3}$ (c) $x = 2$ (d) $x = 2$

(e) $x = 2$ (f) $x = 5$ (g) $x = -12$ (h) $x = 5$

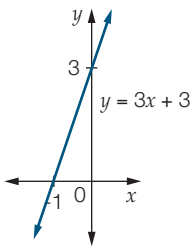
12 (a) 1.5 (b) 3 (c) -2 (d) 2

13 (a) (-2, -1) (b) (1, -1) (c) (-8, 0)

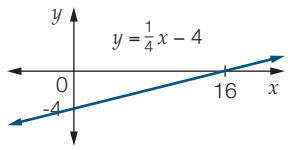
14 (a)



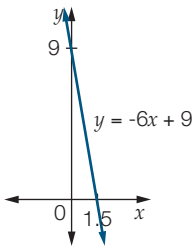
(b)



(c)



(d)



15 **(a)** Yes
$m_1 = -\frac{1}{2}$, $m_2 = 2$, $m_1m_2 = -1$

(b) Yes
$m_1 = 2$, $m_2 = -\frac{1}{2}$, $m_1m_2 = -1$

(c) Yes
$m_1 = -\frac{5}{2}$, $m_2 = \frac{2}{5}$, $m_1m_2 = -1$

(d) No
$m_1 = \frac{4}{3}$, $m_2 = \frac{3}{4}$, $m_1m_2 = -1$

16 **(a)** No
$m_1 = 2$, $m_2 = 1$, $m_1 \neq m_2$,

(b) No
$m_1 = 1$, $m_2 = -1$, $m_1 \neq m_2$

(c) No
$m_1 = -\frac{2}{5}$, $m_2 = \frac{2}{5}$, $m_1 \neq m_2$

(d) Yes
$m_1 = \frac{3}{10}$, $m_2 = \frac{3}{10}$, $m_1 = m_2$

17 **(a)** $x \geq -\frac{3}{2}$ **(b)** $x < -17$

18 **(a)** $x = -1$, $y = -2$ **(b)** $x = 7$, $y = -1$

19 **(a)** $h = 20.22$ **(b)** $h = 15.60$ **(c)** $a = 22.85$

20 63.43° **21** 7.79 km **22** 73 km

23 **(a)** 61.9° **(b)** 13.2 cm

24 **(a)** $x = 36.96$ m **(b)** $x = 39.47$ mm

(c) $x = 12.85°$ or $x = 167.15°$

25 **(a)** 141.42 m^2 **(b)** 513.07 mm^2 **(c)** 233.18 cm^2

26 22 cm **27** 14 m × 10 m **28** Yes

29 49.62 cm **30** D

31 AB has a gradient of $\frac{-2}{3}$, BC has a gradient of $\frac{1}{5}$, CA has a gradient of -5.

BC and CA are perpendicular. Length of BC = length of $CA = \sqrt{26}$. Therefore, ABC is a right-angled isosceles triangle.

32 AB has a gradient of $-\frac{2}{3}$, BC has a gradient of $\frac{3}{2}$, CD has a gradient of $-\frac{2}{3}$, DA has a gradient of $\frac{3}{2}$. AB and BC are perpendicular, BC are CD are perpendicular, CD and DA are perpendicular and DA and AB are perpendicular. Therefore, $ABCD$ is a rectangle.

33 59°, 121°, 115° and 65° **34** $5y = 3x + 21$

35 **(a)** 67.86° **(b)** 756 cm

36 41.71 m

Challenge B (p. 260)

1 B

2 $a = 15 - 4 = 11$, $b = 13 - 3 = 10$. $a - b = 1$

3 D

4 **(a)** $R(0, 2)$, BR: $2x + 9y - 18 = 0$

(b) $P(3, 0)$, PC: $2x + y - 6 = 0$

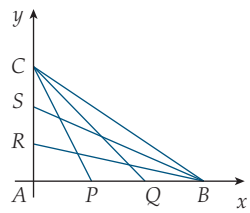
(c) X: $8y - 12 = 0$, $y = 1.5$, $x = 3.6$. $X(2.25, 1.5)$

$Q(6, 0)$, QC: $x + y - 6 = 0$

$S(0, 4)$, SB: $4x + 9y - 36 = 0$

Y: $5y - 12 = 0$. $y = 2.4$, $x = 3.6$. $Y(3.6, 2.4)$

XY: $2x - 3y = 0$. This passes through $A(0, 0)$ so X, Y and A are collinear.



5 **(a)** -1

(b) CE: $y = -x + 7$ or $x + y - 7 = 0$. $D(0, 7)$

(c) AE: $y - 4 = -3(x - 3)$, $3x + y - 13 = 0$. $A\ 4\frac{1}{3}, 0$. $B(0, 13)$

(d) Shaded area = area ΔODC + area ΔBDE = 24.5 + 9 = 33.5 $units^2$.

6 D $P(z, 5z + 3)$, $Q(3, -2)$. $M\ \frac{z+3}{2}, \frac{5z+1}{2}$. Hence $x = \frac{z+3}{2}$,

$y = \frac{5z+1}{2}$, $z = 2x - 3$, $y = \frac{5(2x-3)+1}{2}$, $y = 5x - 7$

7 B **8** 59°

9 $n = 14$, area = 84 cm^2

10 $\cos(\angle MQP) = 0.28$

11 4.58 units

12 13.55 units

13 AB = 3.2 m. Length of timber = 16.9 m, $\angle BAX = 22°$, $\angle BYX = 44°$

Chapter 7

Recall 7 (p. 262)

1 **(a)** mean = 3.2; median = 4; mode = 4; range = 4

(b) mean = 4.7; median = 5.0; mode = 5.0; range = 2

2 **(a)** discrete **(b)** continuous **(c)** nominal

3 **(a)**

STEM	LEAF	
16	1 3	
16	5 5 5 6 7 7 7 9 9	Median
17	0 1 2 2 3	
17	8 9	
18	0 3	

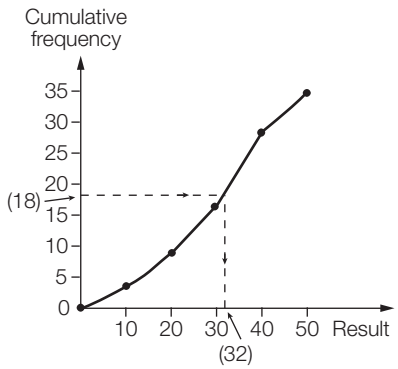
(b) median = 169 cm

4 **(a)** 6.4875 **(b)** 6.50–<6.75

Exercise 7.1 (p. 265)

1 (a)

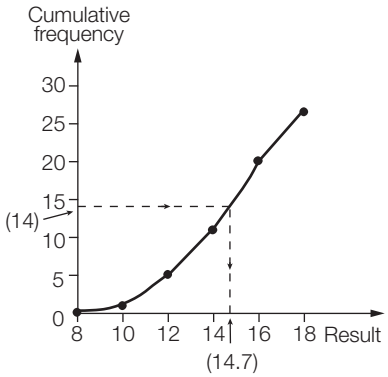
Result	Cumulative frequency
<0	0
<10	4
<20	9
<30	17
<40	28
<50	35



median = 18th term ≈ 32

(b)

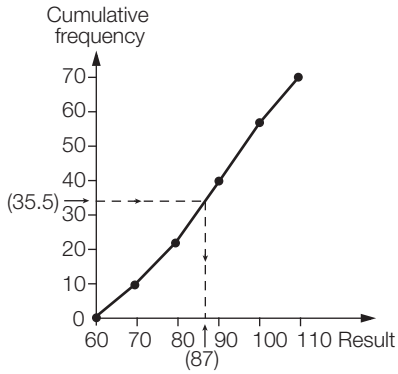
Result	Cumulative frequency
<8	0
<10	1
<12	5
<14	11
<16	20
<18	27



median = 14th term ≈ 14.7

(c)

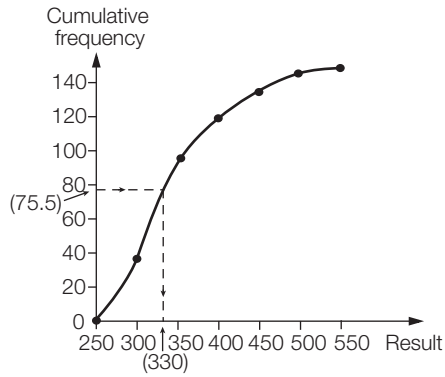
Result	Cumulative frequency
<60	0
<70	9
<80	21
<90	40
<100	57
<110	70



median = 35.5th term ≈ 87

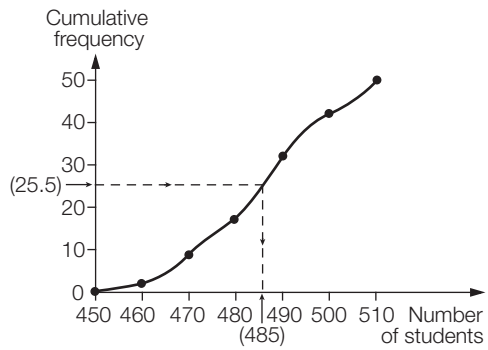
(d)

Result	Cumulative frequency
<250	0
<300	37
<350	95
<400	121
<450	137
<500	146
<550	150



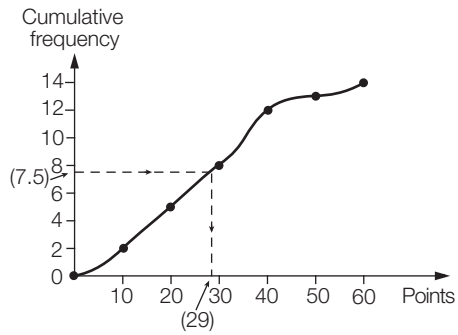
median = 75.5th term ≈ 330

2 (a)



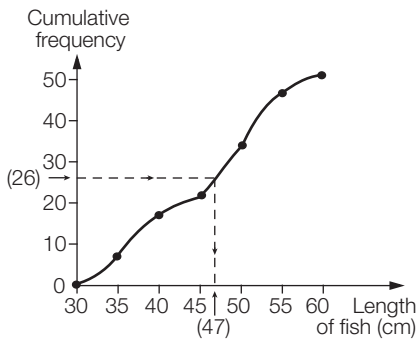
median = 25.5th term ≈ 485 students

(b)



median = 7.5th term ≈ 29 points

3



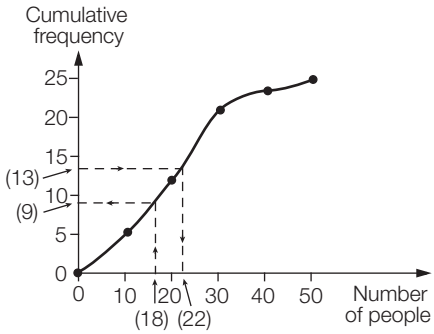
median = 26th term ≈ 47 cm

4 (a) A (b) C (c) B

5 (a) median = 21 people

(b)

No. of people	Cumulative frequency
≤0	0
≤10	5
≤20	12
≤30	21
≤40	23
≤50	25



median = 13th term ≈ 22 people

(c) from raw data 9 trains, from cumulative frequency curve 9 trains

6 (a) 14.7 (b) same

(c) 485.7 students (d) basically the same

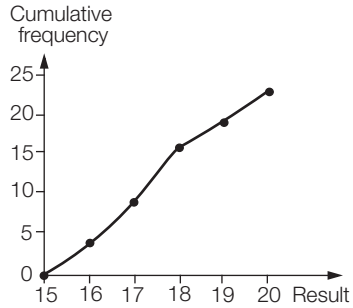
(e) If the data is given in frequency table form, the new method is much quicker and seems to give approximately the same results. If the data is given in cumulative frequency graph form, the new method does not seem to have much to offer. The new method assumes that the results are evenly spread throughout the class interval, but this is not necessarily true. However, this is basically the same assumption made by connecting the points on a cumulative frequency graph with straight line segments.

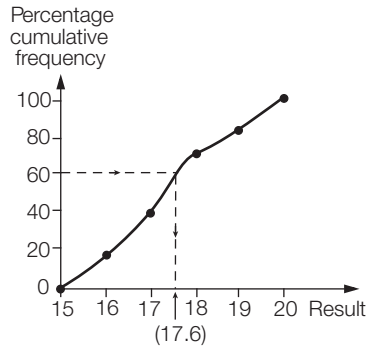
7 Students' own answers.

Exercise 7.2 (p. 273)

1 (a)

Result	Cumulative frequency	Percentage cumulative frequency
<15	0	0
<16	4	17
<17	9	39
<18	16	70
<19	19	83
<20	23	100

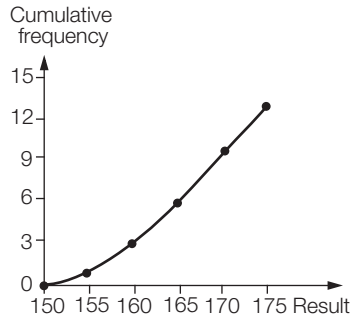


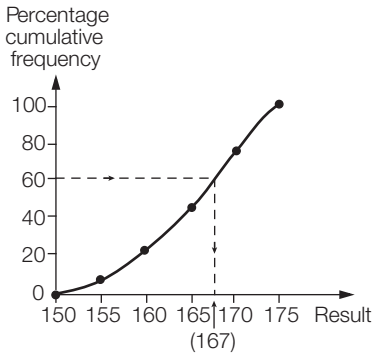


60th percentile is 17.6

(b)

Result	Cumulative frequency	Percentage cumulative frequency
<150	0	0
<155	1	8
<160	3	23
<165	6	46
<170	10	77
<175	13	100



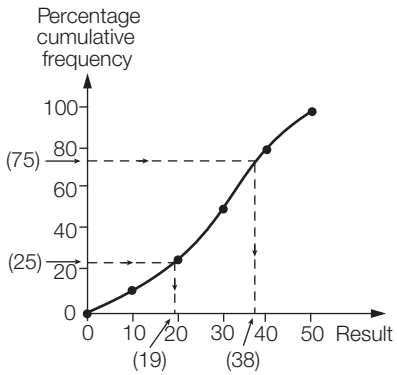


60th percentile is 167

2 **(a)** **(i)** Range = 50 – 0 = 50

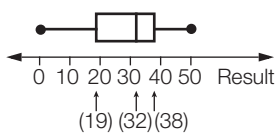
(ii)

Result	Cumulative frequency	Percentage cumulative frequency
<0	0	0
<10	4	11
<20	9	26
<30	17	49
<40	28	80
<50	35	100



IQR = 38 – 19 = 19

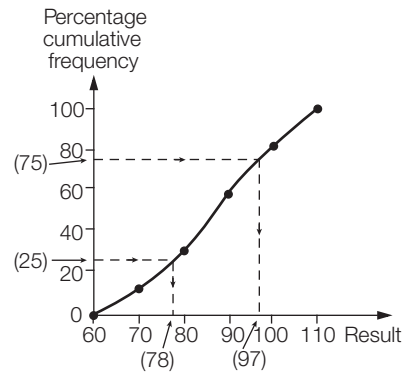
(iii)



(b) **(i)** Range = 109 – 60 = 49

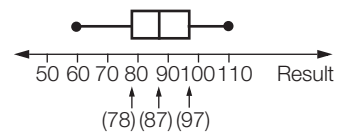
(ii)

Result	Cumulative frequency	Percentage cumulative frequency
<60	0	0
<70	9	13
<80	21	30
<90	40	57
<100	57	81
<110	70	100



IQR = 97 – 78 = 19

(iii)



3 IQR = 14.5, $Q_L = 16.5$ and $Q_U = 31$. The score of 54 is an outlier. Even though 8 seems quite a small total, it is well within the lower bound for outliers. The high score may have been the result of a low-quality opposition.

4 **(a)** 65

(b) $Q_L = 159.5$; $Q_U = 179.5$; IQR = 20

(c) lower than 129.5 or higher than 209.5

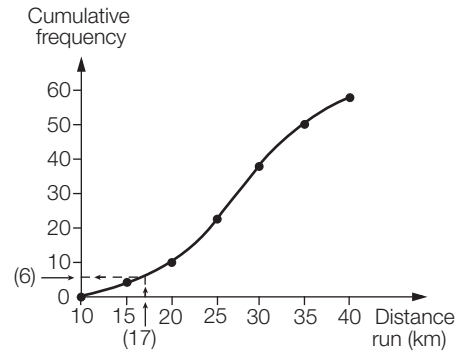
(d) There are no outliers at either end of the data set.

5 **(a)** C **(b)** D **(c)** B

6 D

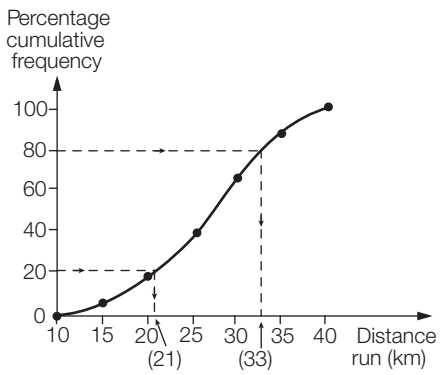
7 **(a)**

Distance run (km)	Cumulative frequency	Percentage cumulative frequency
<10	0	0
<15	4	7
<20	10	17
<25	23	39
<30	39	66
<35	50	85
<40	59	100



(b) Six distances run were less than 17 km.

(c)

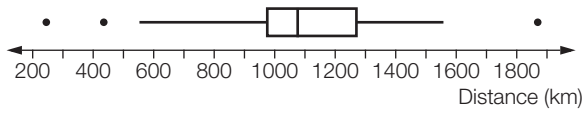


(d) 20th percentile is 21; 0.8 quantile is 33.

8 (a) 1635

(b) $Q_L = 972$; $Q_U = 1275$; IQR = 303

(c)



9 (a) Yes, as long as the outlier is more than 1.5 × IQR above Q_U or more than 1.5 × IQR below Q_L.

(b) Students' own answer.

10 (a) (i) less than -5 and greater than 115

(ii) no

(b) (i) less than 32.5 and greater than 52.5

(ii) yes, at both ends

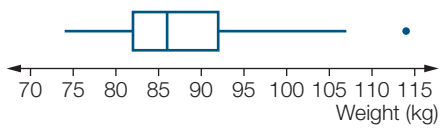
11 (a) minimum: 74; lower quartile: 82; median: 86; upper quartile: 92; maximum: 114

(b) 23.4% (c) 21.3% (d) 25%

(e) This is quite a good match and the slight variation can be explained by the multiple values that occur in the lower (3 values) and upper (2 values) quartiles.

(f) 15; any weight above 107 kg is an outlier, so 114 kg is an outlier.

(g)



12 (a) (i) As an example: 2 2 2 2 2 2 4 4 5 5
5 5 5 5 5 5 6 6 7 7

(ii) The lowest value is the same as the lower quartile so there needs to be at least six 2s in the list, assuming we are using exactly 20 values. The median does not exist in a discrete fashion so must be the same as either the lower or upper quartile. In this case, the upper quartile was chosen. If the lower quartile was chosen, then there would have to be at least eleven 2s.

(b) (i) As an example: 3 3 3 3 3 3 4 5 5 6
6 6 6 6 7 7 7 7 7 7

(ii) In this case, there are no whiskers so we need at least six 3s and six 7s. Place the 6s in the middle to ensure that 6 is the median and then fill in the gaps.

13 (a) mean = \$46 829.27, median = \$45 000

(b) range = \$50 000, IQR = \$10 104.17 (using the linear approximation method)

(c) Yes, values above \$66 093.76, so at the very least, the three highest values are outliers and perhaps one or both of the values in the \$65 000–< \$70 000 class interval.

(d) Removing only the three highest values gives mean = \$45 664.56, median = \$44 791.67 (using the linear approximation method), range = \$35 000, IQR = \$9687.50 (using the linear approximation method).

(e) The mean and range are more affected by the removal of the outliers than are the median and the IQR. This needs to be considered when analysing data sets that contain outliers—the mean and range will need to be treated with some caution.

14 (a) Students' own graph.

(b) Students' own graph.

15 (a) Example: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 34

IQR = 11, $Q_L = 5.5$, $Q_U = 16.5$, so any value above 16.5 + 16.5 = 33 will be an outlier.

(b) Example: 36 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69

IQR = 11, $Q_L = 53.5$, $Q_U = 64.5$, so any value below 53.5 – 16.5 = 37 will be an outlier.

Exercise 7.3 (p. 280)

1 (a) Set A:

(i) 30.74 (ii) 30 (iii) 18 (iv) 10

Set B:

(i) 30.21 (ii) 30 (iii) 28 (iv) 10

(b) Set A:

(i) 51.22 (ii) 51.5 (iii) 16 (iv) 8

Set B:

(i) 51.22 (ii) 51.5 (iii) 56 (iv) 8

(c) Set A:

(i) 16.82 (ii) 15.5 (iii) 15 (iv) 8

Set B:

(i) 16.64 (ii) 14.5 (iii) 15 (iv) 8

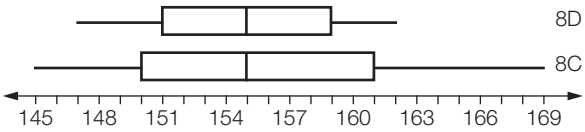
(d) The mean is affected by all changes. The only time the mean did not change in these samples was when the lowest number and the highest number were adjusted by the same amount in opposite directions. The median is very stable and was only affected slightly when data in the middle of the data set was changed. The range fluctuates wildly and is severely affected by any change in the extreme values. The IQR is also quite stable.

2 (a) 8C: 155 cm; 8D: 155 cm

(b) 8C: 24; 8D: 15

(c) 8C: 11; 8D: 8

(d)



(e) 8C has a much greater range and IQR than 8D. They have the same median value. The height of students in 8D is much more consistent than for the students in 8C.

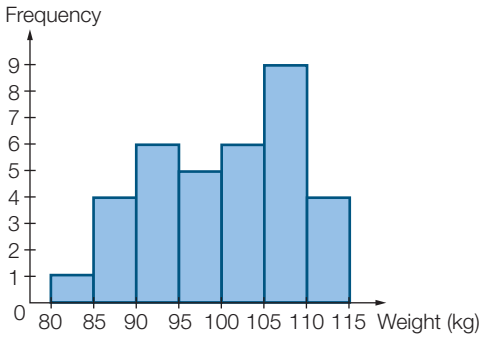
3 (a) (i) Set A: 45; Set B: 50

(ii) Set A: 35; Set B: 35

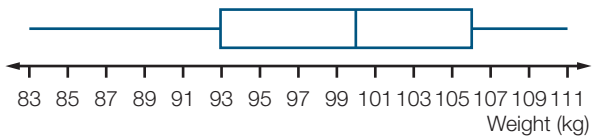
(iii) Set A: 10; Set B: 20

(b) Although they have the same range, the two data sets are quite different. Set B has a much bigger IQR, whereas Set A has a much longer right-hand whisker. About 75% of Set A are shorter than the median of Set B.

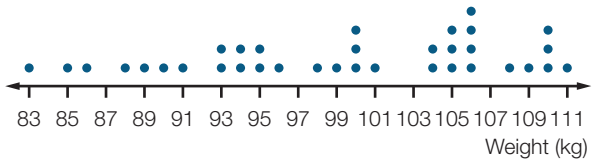
4 (a)



(b) min: 83; lower quartile: 93; median: 100; upper quartile: 106; max: 111



(c)



(d) The dot plot shows that the data set is relatively evenly spread out with a slight skew to the right (negative skew). The box plot shows that the median is close to the middle, but slightly to the right. The grouping in the histogram can mask some trends but it is also relatively even with the highest frequency occurring near the right-hand end.

5 (a) B (b) B

6 The huge range, and relatively big IQR, for Class A means there were some students who did not perform very well on the test as well as some who did extremely well. The fact that some students did well dragged up the mean. Class B was the most consistent class. It depends on what you mean by 'best' as you could argue Class B did best because of its even results. However, the very highest results are unlikely to be found in Class B. The high IQR for Class C indicates that the middle results were very spread out but the other results were not far outside the middle box.

7 (a) (i) 3 (ii) 1

(b) (i) 21 (ii) 1

(c) The median and IQR give a better feel for the data set as a whole but do not give even a hint that we have two extreme outliers.

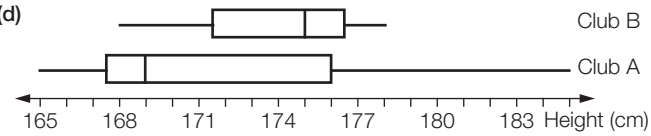
8 (a) Club B seems to be taller.

(b) Mean heights: Club A is 171.75 cm and Club B is 174.1 cm

Median heights: Club A is 169 cm and Club B is 175 cm

(c) Range: Club A is 20 and Club B is 10; IQR: Club A is 8.5 and Club B is 5.

(d)



(e) The bunching of the dots for Club B gives us a clue that it will have a small IQR. The parallel dot plot clearly indicates that the range for Club A is going to be much greater.

(f) Yes. Even though Club B does not have the tallest player, it has more consistently tall players, shown by the smaller range and IQR, spread over higher values.

9 (a) The distances range from 2 to 40 km with a median distance of 12 km. This means about 50% of the students travel up to 12 km to school. The IQR lies between 8 and 26 km.

(b) Students to give their own answers, but it should be greater than 12 km.

(c) Students to give their own answer, but a possible answer would be based on:

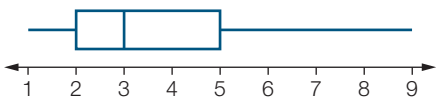
Distance (km)	Frequency
0–<4	10
4–<8	30
8–<12	40
12–<16	10
16–<20	15
20–<24	10
24–<28	15
28–<32	10
32–<36	10
36–<40	10

This data set has a mean of 16.5 km.

10 **(a)** **(i)** positive skew

(ii) minimum: 1; lower quartile: 2; median: 3; upper quartile: 5; maximum: 9

(iii)

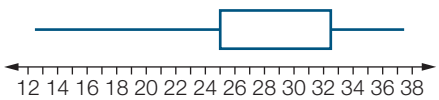


(iv) More of the values are to the right of the median, so the right half of the box and the right whisker are bigger than the other 'half' of the plot.

(b) **(i)** negative skew

(ii) minimum: 12.5; lower quartile: 25; median: 32.5; upper quartile: 32.5; maximum: 37.5

(iii)



(iv) In this case the upper quartile is the same as the median. So, the left 'half' of the box plot is much longer than the right 'half'. The two values will not always coincide but the relationship between sizes will be a feature of negative skew.

11 **(a)** Mean heights: Adelaide 187.77 cm and Essendon 187.83 cm

Mean weights: Adelaide 86.11 kg and Essendon 86.54 kg

(b) Median heights: Adelaide 188 cm and Essendon 187 cm

Median weights: Adelaide 85 kg and Essendon 86 kg

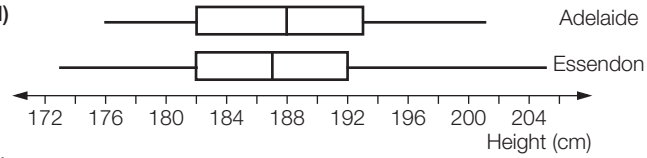
(c) Heights range: Adelaide 25 cm and Essendon 32 cm

Heights IQR: Adelaide 11 cm and Essendon 10 cm

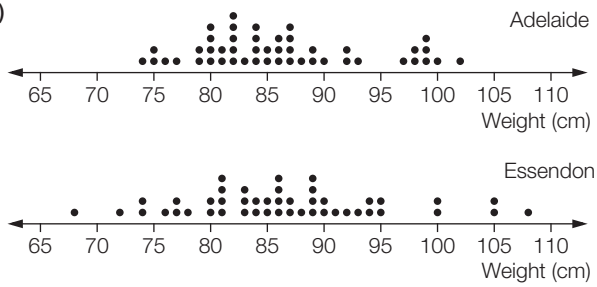
Weights range: Adelaide 28 kg and Essendon 40 kg

Weights IQR: Adelaide 9 kg and Essendon 10 kg

(d)



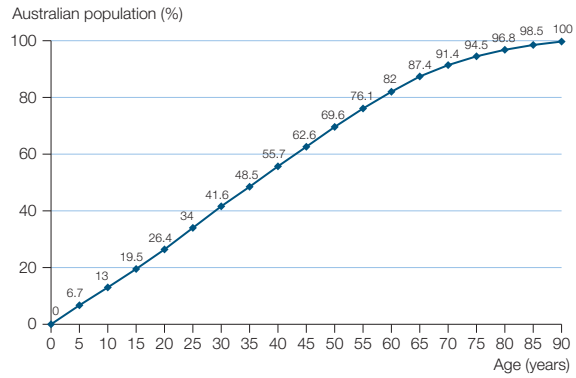
(e)

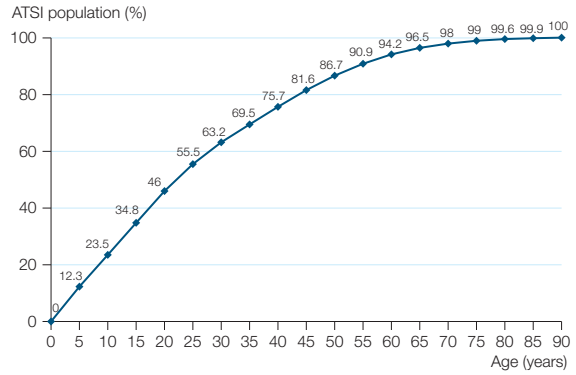


(f) Putting together all of the information collected, the final comments would be that Essendon has a greater range for height than Adelaide but essentially, the two clubs are extremely close in terms of height.

(g) Sample comment: Essendon has the two lightest players but also the two heaviest. Essendon seems to be spread out more evenly, whereas Adelaide is clumped in a couple of places. The weights are quite even but Essendon may be marginally heavier.

12 **(a)**

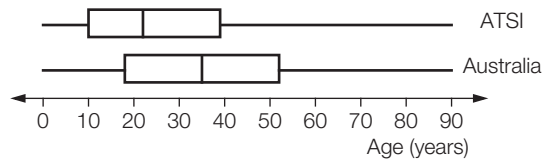




(b) Australia: min = 0; $Q_L = 18$; median = 35; $Q_U = 52$; max = 90

ATSI: min = 0; $Q_L = 10$; median = 22; $Q_U = 39$; max = 90

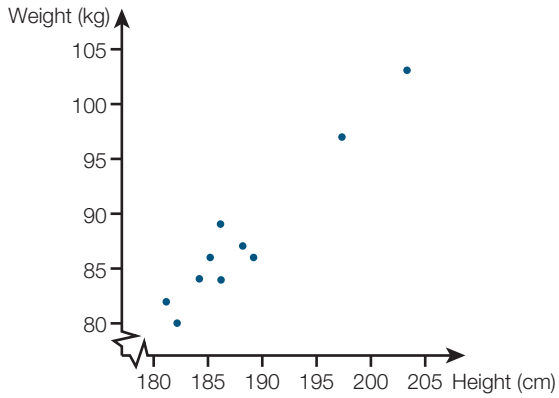
(c)



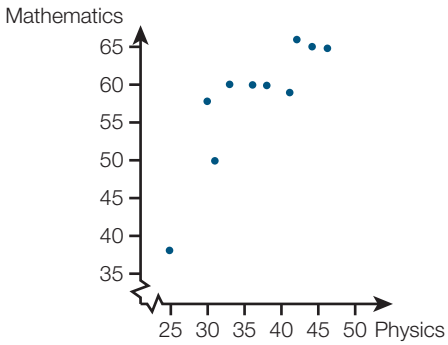
(d) The cumulative frequency curves start to tell the story with their very different shapes. The rapid rise at the beginning of the ATSI graph indicates that a great percentage of the population are young. This is confirmed by the box plot, which shows the median age to 22 and the lower quartile to be 10. This means 25% of the ATSI population is less than 10 years of age, compared to an age of 18 for the Australian population as a whole. The medians are different by 13 years. The upper quartile for the ATSI population is barely bigger than the median for the Australian population.

13 Students' own answers.

Exercise 7.4 (p. 290)

1 

As height increases so it seems does weight. However, this relationship is not perfect so we would probably say there is a strong positive relationship between the two variables.

2 (a) 

(b) For the lower physics marks, it seems that as the mark increased so did the mathematics mark. However, this trend did not continue with the higher physics marks where the mathematics mark seemed to flatten out and not increase at the same rate as the physics mark. Overall, there is a moderate positive relationship.

3 (a) This is a very strong negative linear relationship.

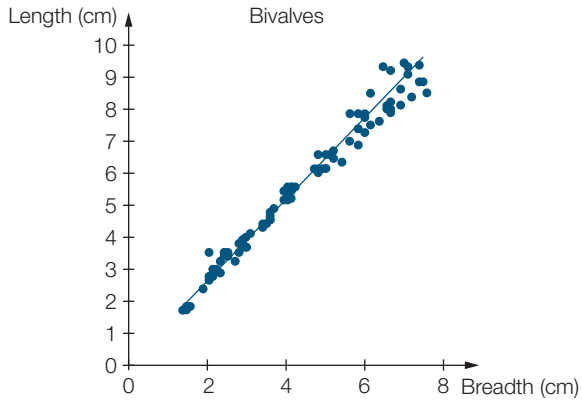
(b) This is a strong positive linear relationship.

(c) This looks like a non-linear relationship; it has a curve associated with it. However, it does display a negative relationship.

(d) This looks like there is no relationship between the variables.

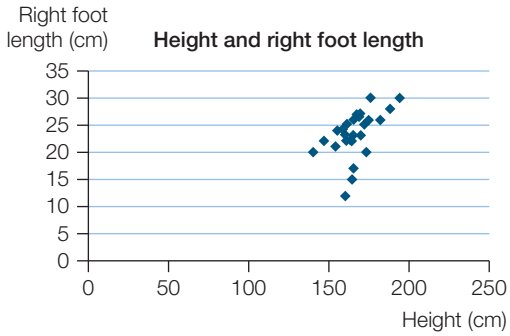
(e) This looks like a non-linear relationship. The points go up and down as you move across the horizontal axis.

(f) This looks like a weak negative linear relationship.

4 (a) 

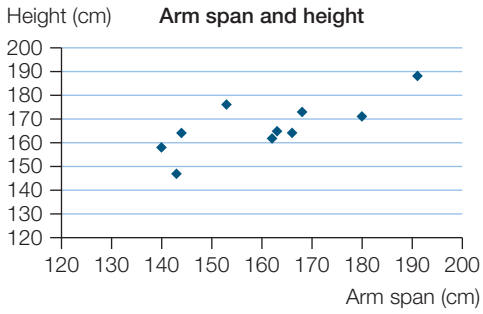
(b) As the breadth of the bivalve increases so does the length. These two variables seem to have a very strong positive linear relationship.

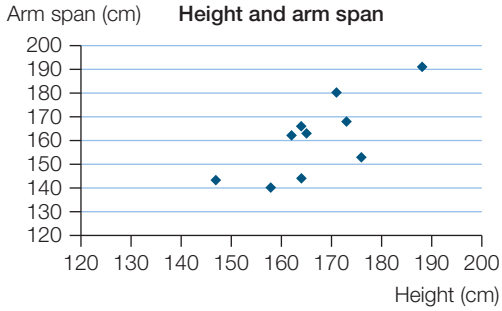
5 (a) Sample answer: Certainly a positive relationship, probably strong.

(b) 

(c) The scatter plot supports a strong positive relationship, although there are a couple of doubtful points.

(d) The right foot lengths of 12, 17 and 15 seem quite small. However, there is no evidence to reject them so they are plotted like the rest.

6 (a) 

(b) 

(c) The strength of the relationship remains the same but the slope of the points has changed.

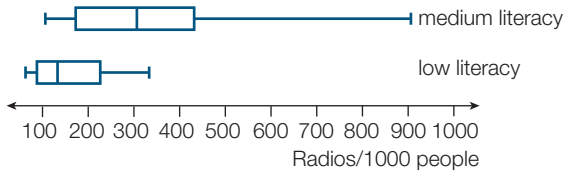
7 (a) secondary

(b) There seems to be a strong positive linear connection between the two variables. Male rates are possibly a little higher than female rates.

(c) No—a different population group entirely.

8 (a) The countries with a higher literacy rate have a higher number of radios.

(b)



(c) The range for medium literacy countries is much higher but the lowest value for the medium countries is higher than the lower quartile for the low countries. The highest literacy value for the low countries is only just higher than the median value for the medium countries. The medium literacy countries have a very long right-hand whisker; it may be that there are one or two outliers that have made the comparisons inaccurate. The numbers of radios are clearly different between these two types of groups of countries. However, it would be incorrect to conclude that a country's literacy rate determines the number of radios or vice-versa.

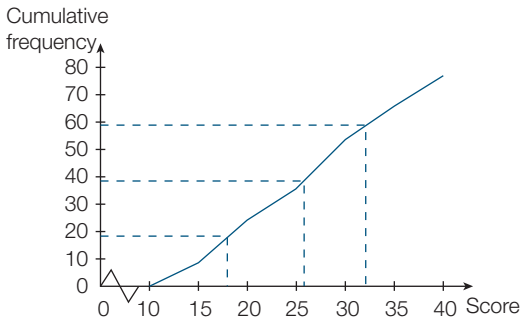
(d) 907 radios per 1000 people in a medium literacy rate country.

9 Students' own answers.

Half-time 7 (p. 293)

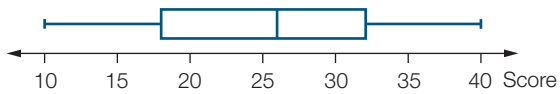
1 (a) Range: 40 – 10 = 30

(b)



From the cumulative frequency graph we get $Q_L = 18$ and $Q_U = 32$, which gives an IQR of 32 – 18 = 14.

(c)

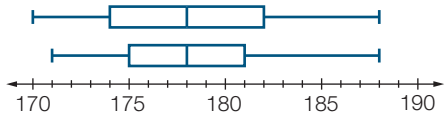


2 (a) Club B appears to be taller but this is mainly based on the fact that it does not have as many players at the lower end of the height scale.

(b) Club A: mean = 178.47, median = 178, range = 18, IQR = 8

Club B: mean = 178.4, median = 178, range = 17, IQR = 6

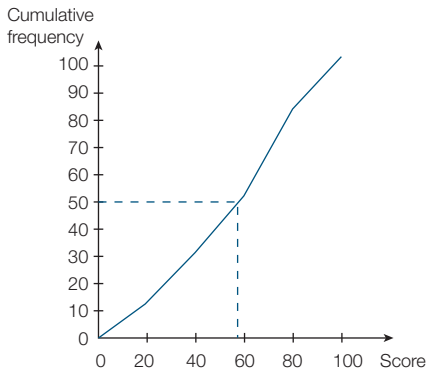
(c)



(d) Club A has a very slightly larger average height, however, they are essentially the same. The IQR in Club A is also slightly larger, which shows a slightly larger spread of heights compared to Club B.

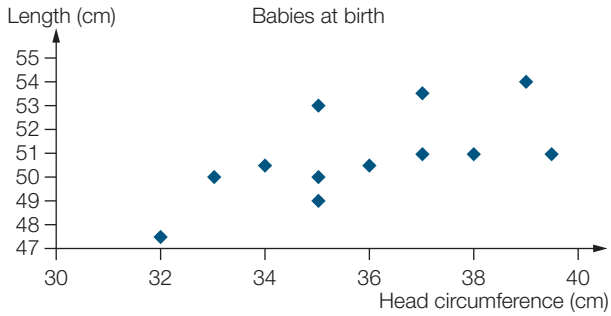
(e) The IQR, as this statistic has the greatest difference and shows how the spread of the data varies between the two clubs.

3



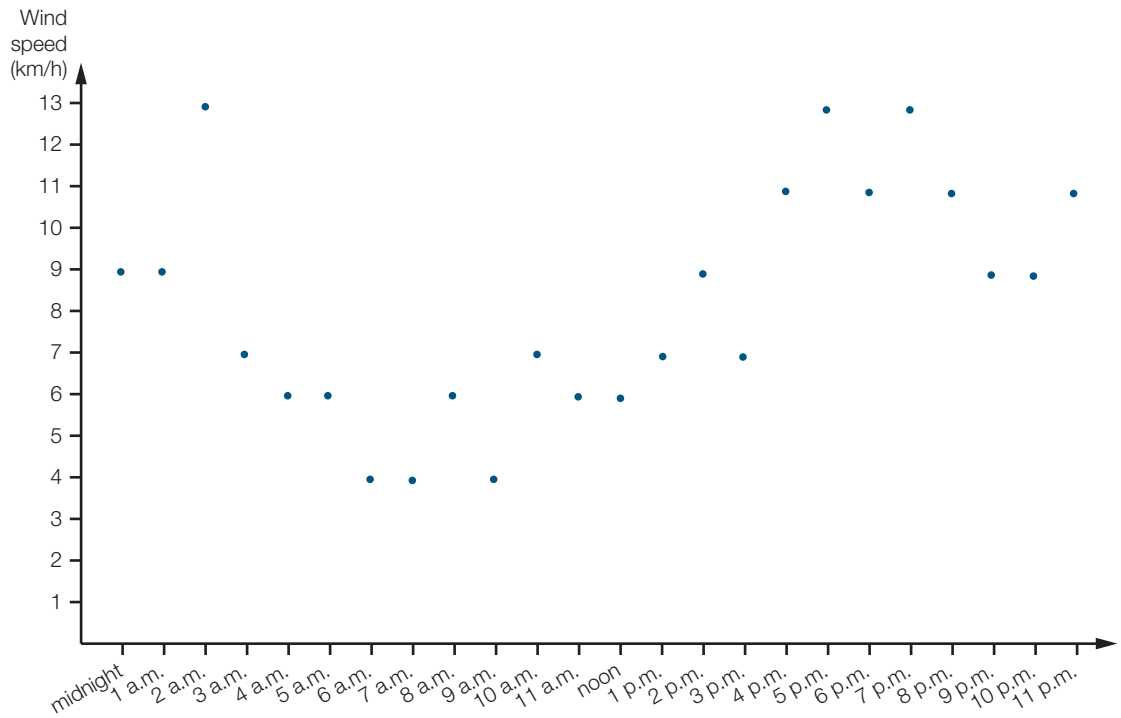
Median = 58

4 There appears to be a moderate positive linear relationship between head circumference and length in babies at birth.



Exercise 7.5 (p. 295)

1 (a)



(b) It is much higher than the readings around it.

(c) The wind speed fluctuates between 11 km/h and 13 km/h.

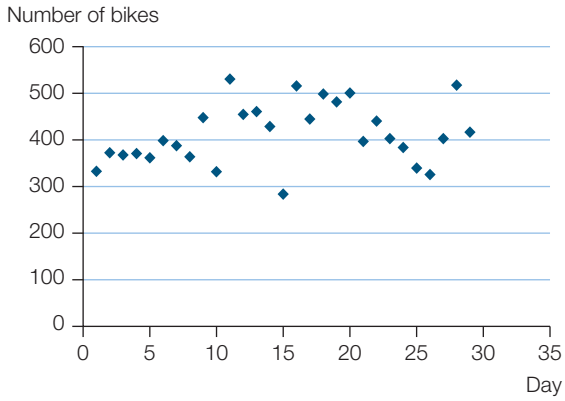
2 (a) Hours spent studying; this is the variable under our control.

(b) Age; this is the variable that is always changing—height does not necessarily change.

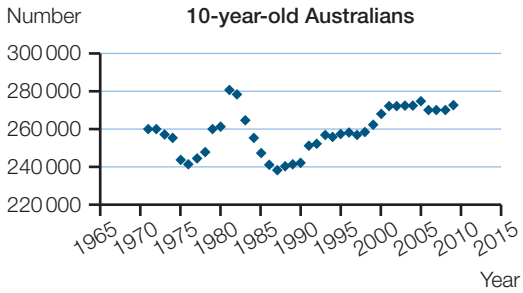
(c) Number of cigarettes; this is what has an impact on age at death.

(d) Number of people; change the number of people and you would expect a change in the amount of rubbish.

3 There does not appear to be a pattern in the number of bicycles. This is perhaps not an unexpected result because the weather could play a significant role in the number of cyclists on any particular day.

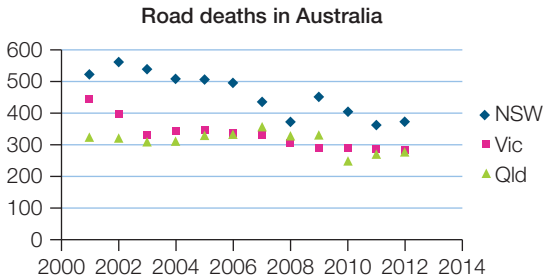


4 (a)



(b) The number of 10-year-old Australians has fluctuated a little over this time period but is now about the same as it was at the beginning of the period. There was a peak in the early 1980s, followed by a trough in the mid-1980s when the lowest number was recorded, but the number has steadily increased since then. The number appears to be becoming more stable.

5 (a)



(b) The NSW figures have fluctuated the most but part of this can be explained by the fact that the NSW numbers are always the biggest for each year. NSW seems to be trending downwards although there was a slight uptick in 2012 that may be an indication of a new trend. Victoria is trending downwards but the figure has been quite steady over the last 4 years shown. The figures for Queensland have been relatively steady but seem to be trending upwards at the moment.

(c) It would make a better comparison if we had these figures given as some sort of rate. This might be something like 'deaths per 100 000 of population'.

6 (a) The population has increased consistently over this period of time. The last 40–50 years have seen the greatest rate of increase.

(b) This inclusion would have minimal impact except perhaps in the very early years shown on the graph. The graph would have started at a higher value, clearly off the axis. Depending on how rapidly the Aboriginal population declined would determine whether or not their inclusion would be detectable.

(c) The population was very small, but the scale needs to be as is to fit today's population into a reasonably sized graph.

(d) In the very early years by counting every individual. Records of arrivals, departures, births and deaths would also have been used.

(e) The very early years may actually be more reliable than the later ones because the later ones have some element of estimation built into them.

(f) There is a census every 5 years and population models are based on the data collected.

7 The number of 90-year-olds is increasing at a fairly rapid rate. The average lifespan is increasing in parallel with improvements in medical care. For younger people, the same trend would probably not be seen.

8 The graph lacks a title and axes labels. There also needs to be a key to identify which colour represents which model. The vertical scale could also be stretched a little to show the differences more clearly.

9 Students' own answers.

10 Students' own answers.

Exercise 7.6 (p. 300)

1 (a) (i) 9% to 15% (ii) 48% to 56%

(b) (i) 39% to 45% (ii) 86% to 90%

2 (a) This type of survey is not reliable as it is a self-selecting sample where people with vested interests may vote multiple times. If the same question was posed on a different day, a completely different result might be obtained.

(b) The question is not clear as it may be about the uniform itself or the way the uniform looks on Miranda Kerr.

(c) The number of results is far too low for the responses to be taken seriously. According to the Essential Media Communications table the margin of error is in the vicinity of 9%.

(d) It is hard to imagine anybody being interested in a broad question such as this. Its appeal seems to be in the newspaper being able to report that Australia is not a racist country.

(e) The results probably give the newspapers the opportunity to write sensational headlines.

3 (a) It would seem to exclude people who do not have regular access to the internet.

(b) Students are required to express their own opinion.

(c) These questions are used to help balance the participants in the survey. If the quota for older males has already been reached then any further such respondents are exited from the survey.

(d) Students are required to express their own opinion.

4 (a) The minimum margin of error would be 6% according to the Essential Media Communications table. The fact that no actual percentages are given means this report should probably just be dismissed out of hand.

(b) At this rate the range would be 43% to 63%.

(c) Students are required to express their own opinion.

5 (a) Over 700 cabin crew were surveyed.

(b) It appears to be an attempt at humour or to catch the eye of the casual reader.

(c) It would be relevant to look at the survey instrument as a whole to see if the reported grievances were offered to the respondents or if the instrument was open-ended.

(d) Students are required to express their own opinion. However, it may be directed towards the statement immediately above it about cabin crew preferring economy class passengers. Statements such as these in articles are not really helpful to the reader.

(e) The percentages add to 100%. This indicates a list may have been provided and the requirement was to choose the biggest gripe from that list.

6 (a) The general thrust is that workers are being ripped off by employers.

(b) The language is emotive and is used to sensationalise the information provided.

(c) Over 1000 workers.

(d) 142 workers

(e) It sounds better to refer to a survey of over 1000 people than one involving only 142. It is a pity this confusion seems to have been deliberate when the findings are of such a concerning nature.

Exercise 7.7 (p. 307)

1 5.72

2 English: mean = 34.33 and standard deviation = 9.22

Mathematics: mean = 34.67 and standard deviation = 7.12

Although the means are effectively the same the standard deviation for mathematics is smaller indicating that the results in that subject are more tightly packed around the mean. In turn, this indicates that the more extreme results will be found in the English marks.

3 mean = 3.69 and standard deviation = 2.52

4 **(a)** Australia: 249; New Zealand: 597; Brazil: 940; Sri Lanka: 628

(b) Australia: 77.73; New Zealand: 190.74; Brazil: 250.39; Sri Lanka: 165.97

(c) As the range increases so does the standard deviation except for Sri Lanka. Its standard deviation is less than New Zealand's despite its range being slightly higher than New Zealand's. This is probably due to the highest peak in Sri Lanka being quite a bit higher than the next value. It may be an outlier.

5 **(a)** **(i)** 4.81 **(ii)** 8.77 **(iii)** 6.12 **(iv)** 9.66

(b) Introducing an outlier at either end has a similar effect as increasing the standard deviation (one of them was further away) and introducing two outliers at either end stretches the standard deviation even further.

6 B

7 **(a)** **(i)** 2.66 **(ii)** 7.99 **(iii)** 2.66 **(iv)** 2.66

(b) If every value in the data set is multiplied by the same number, then the standard deviation is also multiplied by that number. If the same number is added to every value in the data set then the standard deviation remains the same.

8 **(a)** mean: 5.5; standard deviation: 2.33

(b) The mean stays the same but the standard deviation becomes slightly smaller (2.22).

(c) The mean is reduced to 5.18 and the standard deviation is increased to 2.44.

(d) The mean is increased to 5.82 and the standard deviation remains the same.

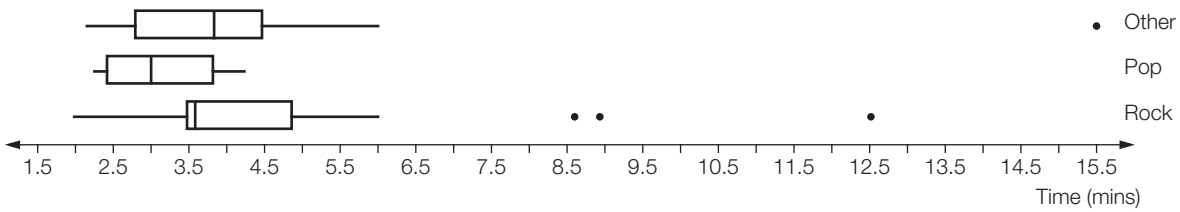
9 C

10 **(a)** mean = 4:22 and standard deviation = 2:12

(b) mean = 3:06 and standard deviation = 0:41

(c) mean = 4:12 and standard deviation = 2:36

(d)



(e) From the evidence presented, you could conclude that pop songs are more formulaic than either of the other categories. The standard deviation was very small, the box has no outliers and the range is very small. The longest pop song is within the third quarter of each of the other box plots. This means at least 25% of each of the other categories are longer than the longest pop song. The number of pop songs was the smallest, which may have had an effect on the data. The Other category has one wild outlier. It is 9.5 minutes longer than the next longest song. On the other hand, there were three outliers in the Rock category.

11 **(a)** Students' own answers.

(b)–(c) The range condition means the data set cannot be mostly one number with just a little tinkering. The first step would be to calculate what the sum of the data set would be, number of values × mean. Then, make a start on the data set so it adds up to this value. Some adjustments may be required to make the standard deviation work.

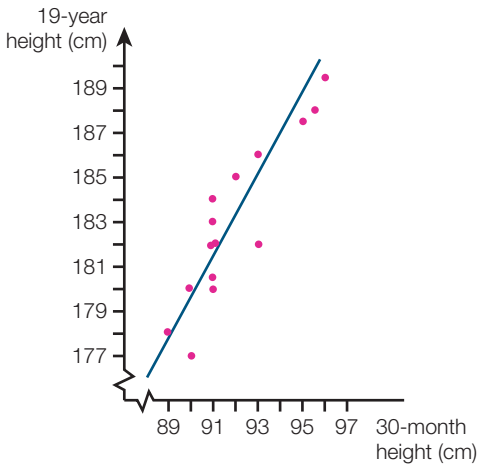
12 Students' own answers. The first step is to identify the five-number summary: min = 12, $Q_L = 13$, median = 16.5, $Q_U = 18$ and max = 20. Construct the data set around these values and then find the mean and the standard deviation.

Exercise 7.8 (p. 312)

1 (a) $y = 12.107x + 4.536$ (b) $y = 15.497x - 14.552$

(c) $y = -4.261x + 46.931$ (d) $y = -4.863xx + 39.999$

2 (a) and (b)



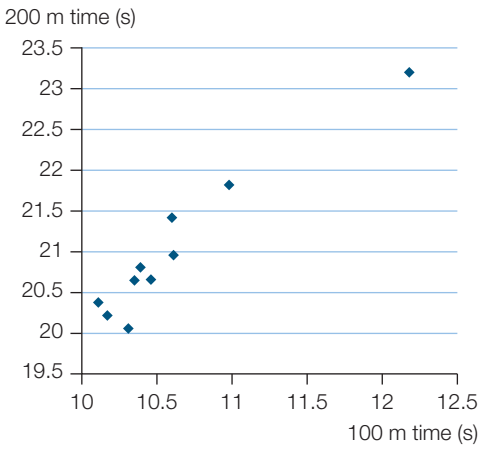
(c) The 30-month height does seem to be a reasonably good predictor of adult height.

(d) approximately 187 cm (will vary depending on the line of best fit)

3 (a) C

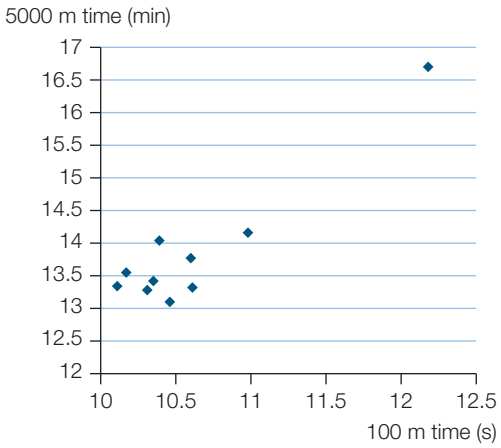
(b) (i) $q = 2.69 + 1.16p$ (ii) A

4 (a)



(b) There does appear to be a strong positive linear connection between these two times. In general, as one value increases, so does the other. It would seem appropriate to fit a straight line for these points.

(c)



(d) There does not appear to be a strong connection between these two times. It would not seem appropriate to fit a straight line to these points.

(e) There is a strong probability that the same person will hold the two sprint records but it will be a different person who holds the other record. The two sprint times are therefore closely linked but no such link is established for a sprint and a middle distance run.

5 (a) The value of r is 0.9987 so it is appropriate to find the equation of a linear line of best fit. It is $F = 4.90m - 0.15$.

(b) This ratio is the gradient of the line of best fit, 4.90.

(c) (i) 19.45 N (ii) 53.75 N (iii) 90.50 N

(d) (i) 244.85 N (ii) 979.85 N (iii) 4899.85 N

6 (a) Solubility = 0.05 × Temperature + 35.05

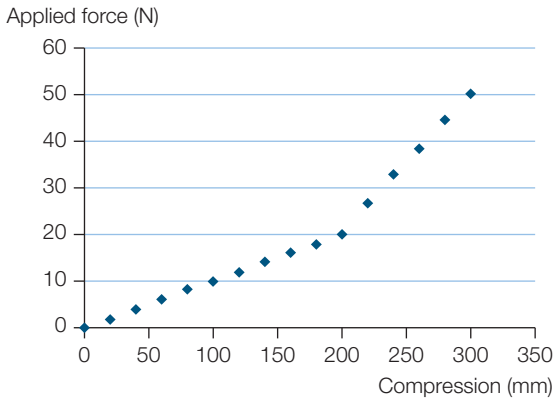
(b) (i) 36.30 units (ii) 38.95 units (iii) 41.05 units

(c) Temperature = 20.02 × Solubility − 701.40

(d) (i) 19.32°C (ii) 69.37°C (iii) 199.50°C

(e) Both of these are extrapolated values; we would need to assume that the connection between the variables remains the same.

7 (a)



The first group ends at the point (200, 20)

(b) $F = 0.100\Delta x - 0.04$, $0 \le \Delta x \le 200$ and
$F = 0.294\Delta x - 37.75$, $200 < \Delta x \le 300$.

(c) (i) 7.5 N (ii) 15.0 N (iii) 28.4 N

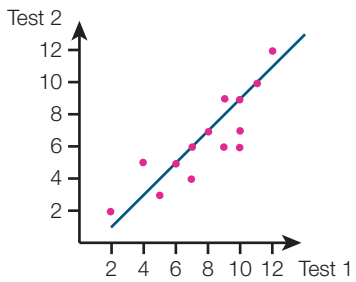
(d) $\Delta x = 9.965F + 0.440$, $0 \le F \le 20$ and
$\Delta x = 3.406F + 128.667$, $20 < F < 50.2$

(e) (i) 50 mm (ii) 231 mm (iii) 282 mm

(f) If we substitute 500 mm into the appropriate equation, we get 110.4 N, which is much greater than the stated value, so we would assume the equation is not valid in that region. Without any other information, the best we could do is assume the data is linear between (300, 50.2) and (500, 90). This would give the equation $F = 0.199\Delta x - 9.5$.

8 (a) Emily has just drawn the line of best fit as connecting the first and last points. It should be drawn to represent the general trend of the points with approximately equal numbers of values either side of the line.

(b)



9 Students' own answers.

Exercise 7.9 (p. 317)

1 (a) Number the students from 1 to 1200. Then use a random number generator to randomly select 60 values. (If a number is generated twice just choose another number to replace it.)

(b) Years 7–9: 9 students each year level;
Year 10: 11 students; Year 11: 12 students;
Year 12: 10 students

2 (a) (i) 26.4% (ii) 51.6%

(b) The headline is not justified as the result could just as easily be switched for Labor and the Coalition. The best result for Labor would be 47.6% and the worst for the Coalition would be 43.4%, which clearly has Labor in the lead.

3 B

4 (a) 200 seedlings (b) 250 seedlings (c) 500 seedlings

5 Expect 13 black, 39 pink, 20 green, 26 blue, and 33 orange. Note that due to rounding, this adds to 131.

6 (a) This indicates there are more females.

(b) 88.68 males per 100 females

(c) For this age range, at least, the claim would seem to be incorrect.

(d) People without a telephone would be excluded. These people are less likely to use social media so this could have serious implications for the survey results.

7 (a) Under 18s would not usually make a significant contribution to household financial comfort levels.

(b) If findings are to be useful for the general population, then the sample needs to mirror the population as closely as possible.

(c) To achieve the balance required the exact number was not used.

(d) There were at least five groupings: 'couples with young children', 'couples with older children', 'single parents', and presumably at least 'singles' and 'couples without children'.

(e) Government policy has an impact but its impact may not be permanent.

8 (a) This may have something to do with cost, or it could be related to access to these numbers.

(b) If the numbers are taken from the telephone directory then those with a silent number could not be selected.

(c) Students' own answer.

(d) This would presumably be a cost issue. It is much more expensive than telephone polling.

(e) There would not be interpreters available for non-English speakers.

9 (a) 15 565 households

(b) 20 426

(c) The accompanying information indicates that more than one person in a household could have been approached for information. There could be only one adult per household but there could also be a child included. Presumably many of the households did not include children.

(d) Government bodies at all levels would be interested in the results, as would health agencies both private and public. Health insurance companies would also be very interested in the results.

(e) This gives some statistical basis to the conclusions drawn. It is a warning that not every result is as clear-cut as it may appear.

(f) This probably means that there was a limited amount of time available to conduct the survey. The more people involved in the survey, the less detailed the information collected would be. So, fewer people means greater detail.

(g) For hotels and motels, the random selection may have too high a probability of choosing an overseas visitor. Alternately, privacy concerns may mean this would be intrusive. For hospitals and nursing homes there would be an expectation of health problems, or else the person would not be in the facility. Short-stay caravan parks might give too high a chance of an overseas visitor as well.

(h) The states and territories vary in population size so this would explain part of the variation in numbers selected. However, this is not in direct ratio, so there may have been a minimum number required from each state or territory.

(i) This means an attempt was being made to see if particular health problems were specific to particular groups within the wider Australian population. Perhaps people from lower socioeconomic circumstances are more likely to have particular health issues, so more money should be directed to these areas to attack those problems.

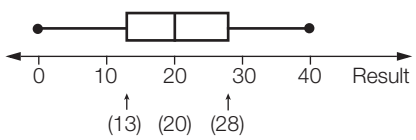
(j) One interpretation of these results would be that government and health agencies need to direct funds to supporting people with these health issues and that, perhaps more importantly, research funding should be directed to finding ways to avoid these particular health problems in the first place. It is better to avoid a problem that to have to treat it.

10 Activity-based question.

Chapter review 7 (p. 322)

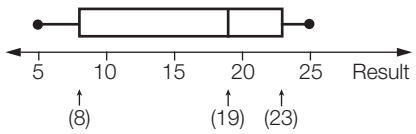
1 (a) (i) median = 20.0 (ii) IQR = 28 – 13 = 15

(iii)



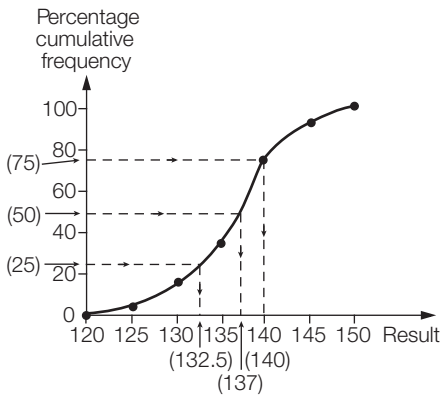
(b) (i) median = 19 (ii) IQR = 23 – 8 = 15

(iii)



2 (a)

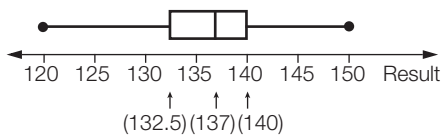
Result	Frequency	Percentage cumulative frequency
<120	0	0
<125	5	3
<130	17	15
<135	32	37
<140	54	74
<145	24	91
<150	13	100



median = 137

(b) IQR = $Q_U - Q_L$ = 140 – 132.5 = 7.5

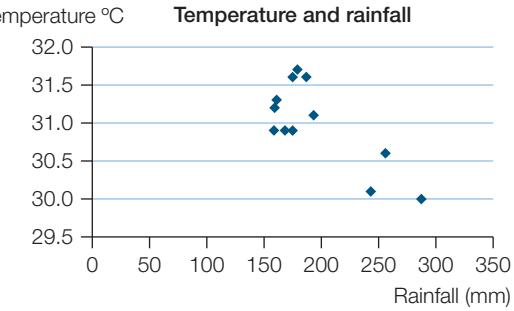
(c)



3 less than 11 cm or greater than 27 cm

4 Team A has been far more consistent than Team B showing lower values for the range and the IQR. However, consistency in scores is not a predictor of success in team-based sports. It looks as if Team B would have lost the game when it scored 4 goals and won the game when it scored 24 goals but other than this it is difficult to make any firm statements. Both teams have the same median number of goals. If pushed you might lean towards saying Team B has been more successful. You definitely need to have access to the raw data.

5 (a)



(b) The scatter plot shows there is a negative linear relationship between rainfall and maximum daily temperature at Changi. The strength of the relationship appears to be moderate.

6 (a) The number of hours spent training. The race time depends on the time spent training, but the race time does not affect the hours spent in training.

(b) The amount of fertiliser. The fertiliser affects the yield. The yield does not affect the amount of fertiliser.

(c) The average amount of space/hen. The amount of space affects the cost of eggs. The cost does not affect the amount of space/hen.

(d) The number of people in the household. The number of people in the household affects the amount of water used. The water use does not affect the number of people in the household.

7 (a) Not reliable. Not everyone watches TV or that particular station. Only a small proportion of viewers will respond anyway.

(b) Not clear. Which roads would benefit? What kind of public transport would be targeted? What type of improvements are envisioned?

(c) The number of votes is too low to give a valid conclusion.

(d) Workers and employers in manufacturing may be interested but governments and economists would need better data before they reached any conclusions.

(e) It makes the TV station appear to be in touch with the issues in Australia's economy facing their viewers.

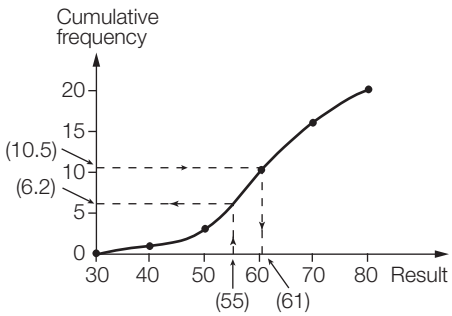
8 (a) 3.77 (b) 1.08 (c) 5.35

9 (a) $y = 5.536x + 2.536$ (b) $y = -2.080x + 19.428$

10 (a) 109 (b) 83 (c) 123

11 (a)

Class interval	Frequency	Cumulative frequency
20–<30	0	0
30–<40	1	1
40–<50	2	3
50–<60	7	10
60–<70	6	16
70–<80	4	20



(b) min = 34.5; Q_1 = 53.55; median = 60.85; Q_3 = 67.75; max = 73.6

(c) From cumulative frequency curve,

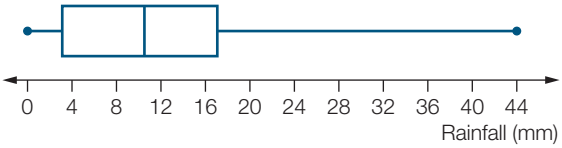
Median = 10.5th term = 61, compared with

Median = $\frac{59.9 + 61.8}{2}$ = 60.85 from the raw data, so they are basically the same.

12 (a) 44 – 0 = 44

(b) min = 0; Q_1 = 3; median = 10.5; Q_3 = 17; max = 44

(c)



13 (a) (i) less than 8.5 and greater than 20.5

(ii) yes (at least one result greater than 20.5)

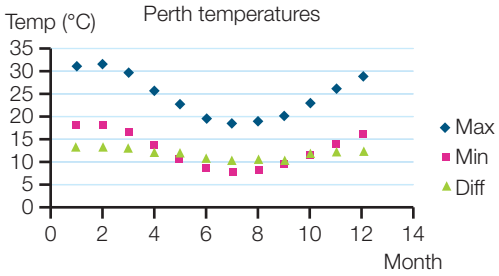
(b) (i) less than 24 and greater than 40

(ii) no (the lowest result is right on the margin)

14 (a)

Jan	Feb	Mar	Apr	May	Jun
13.0	13.3	13.0	12.1	11.9	10.9
Jul	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
10.5	10.7	10.5	11.7	12.1	12.6

(b)



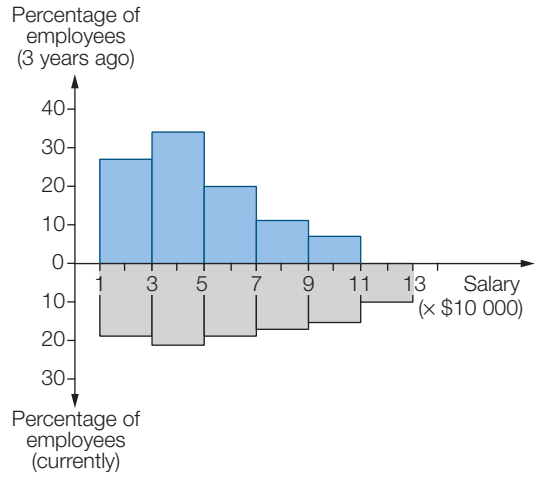
(c) Perth follows a typical temperature pattern for the southern part of Australia—warm to hot in the summer and cool to cold in the winter. As might be expected the difference graph shows the most consistency of the three, but does appear to be remarkably consistent. The smallest differences are seen in the winter months.

(d) No—temperature patterns can be quite local, and temperatures inland vary from coastal temperatures.

(e) The mean minimum for June 2010 was 7.1°C compared to a long-term mean of 8.4°C, whereas the mean maximum was 19.0°C compared to the long-term mean of 19.3°C. June 2010 was slightly more varied in temperature than normal with the minimum being lower and the maximum (slightly) higher. More data would need to be collated before any conclusions could be drawn about changing weather patterns.

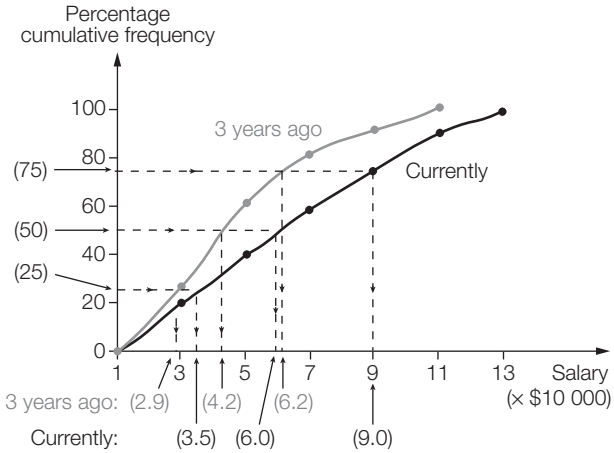
15 (a)

Salary (× $10 000)	Percentage of employees (3 years ago)	Percentage of employees (currently)
1–<3	27	19
3–<5	34	21
5–<7	20	19
7–<9	11	17
9–<11	7	15
11–<13	0	10



(b)

Salary (× $10 000)	Cumu. freq. (3 years ago)	% cumu. freq. (3 years ago)	Cumu. freq. (current)	% cumu. freq. (current)
<1	0	0	0	0
<3	12	27	9	19
<5	27	61	19	40
<7	36	82	28	58
<9	41	93	36	75
<11	44	100	43	90
<13	44	100	48	100



The following values are in $(× 10 000).

x (salary × $10 000)	f (3 years ago)	xf	f (currently)	xf
2	12	24	9	18
4	15	60	10	40
6	9	54	9	54
8	5	40	8	64
10	3	30	7	70
12	0	0	5	60
	44	208	48	306

3 years ago

Mean = 4.73

Median = 4.2

IQR $= Q_U - Q_L$
$= 6.2 - 2.9$
$= 3.3$

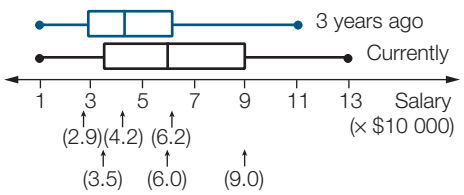
Currently

Mean = 6.38

Median = 6.0

IQR $= Q_U - Q_L$
$= 9.0 - 3.5$
$= 5.5$

(c)



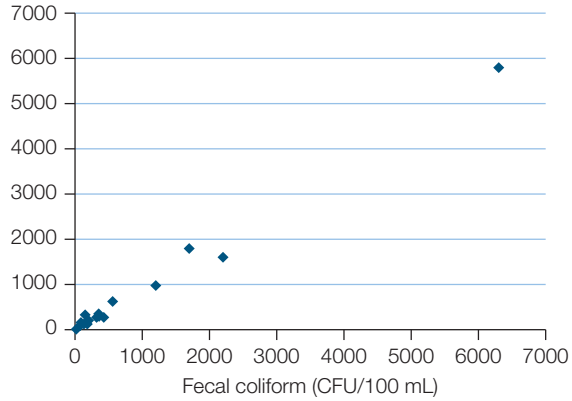
(d) Salaries have increased on average by $15 000 and there is a much greater variation in the salaries paid.
mean difference = $16 500
median difference = $18 000

(e) Total salary bill 3 years ago = $2 080 000
Total salary bill currently = $3 060 000
i.e. $980 000 increase

(f) about 47%

16 (a) 3

(b) *E. coli* (CFU/100 mL)

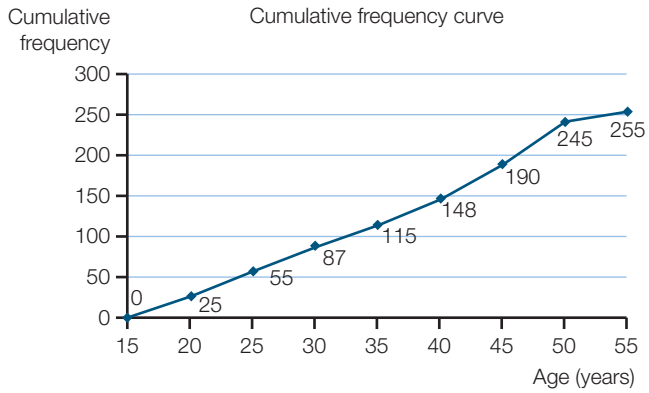


(c) $y = 0.91x + 8.02$

17 (a) There are 255 employees. The data set has most people in the range 41–50 but this is balanced by very few in 51–55. Other than this, it is relatively uniform so the mean will be near the median, say about 38.

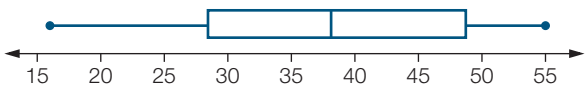
(b) mean = 36.04 and standard deviation = 10.57

(c)



min = 16, $Q_L = 28$, median = 38, $Q_U = 48$, max = 55

(d)



(e) (i) approx 100 (ii) approx 35

18 (a) This does not really seem to be a survey at all. It seems to be the collation of information provided to a site dedicated to the review of different consumer products. It is unlikely to be an unbiased survey as the participants are self-selecting, so there is no guarantee the results represent the wider community.

(b) The size of the bank doesn't really seem to have anything to do with the results reported except there are more customers for the bigger banks, so there is more chance their customers will have visited the site.

(c) Students' own answers.

(d) These comments do not seem to have any bearing on the rest of the article.

Chapter 8

Recall 8 (p. 328)

1 (a) $a = 23°$ (b) $a = 69°$ (c) $x = 15°$

2 (a) $a = 46°$; $b = 46°$; $c = 46°$

3 (a) equilateral (b) right-angled
(c) scalene (d) isosceles
(e) right-angled and isosceles (f) scalene

4 (a) E (b) B (c) A (d) D (e) C

Exercise 8.1 (p. 332)

1 (a) $XY = PQ$ (both 9 cm), $YZ = QR$ (both 16 cm), $ZX = RP$ (both 8 cm); SSS

(b) $\angle BAC = \angle EFD$ (both 65°), $\angle BCA = \angle FED$ (both 50°), $AC = FE$ (both 10); AAS

(c) $\angle JKL = \angle MPN$ (both 90°), $JL = NM$ (both 5), $KJ = PN$ (both 3); RHS

(d) $XZ = RT$ (both 10), $YZ = ST$ (both 10), $\angle XZY = \angle STR$ (both 40°); SAS

(e) $\angle HGJ = MLK$ (both 56°), $JHG = \angle KML$ (both 44°), $HJ = MK$ (both 6); AAS

(f) $\angle PQR = \angle YXW$ (both 90°), $PR = YW$ (both 8), $RQ = WX$ (both 5); RHS

(g) $\angle CBD = \angle FHG$ (both 28°), $CB = FH$ (both 6), $DB = GH$ (both 10); SAS

(h) $JL = AC$ (both 4), $LK = CB$, $JK = AB$ (both 9); SSS

2 (a) $x = 14$ (b) $x = 8$

(c) $x = 7.5$ (d) $x = 20$

(e) $x = \frac{50}{3} = 16\frac{2}{3}$ (f) $x = \frac{51}{5} = 10\frac{1}{5}$

3 (b) and (d) SSS; (c) and (f) AAS; (a) and (e) SAS

4 D 5 C

6 (a) B (b) A

7 (a) A (b) C

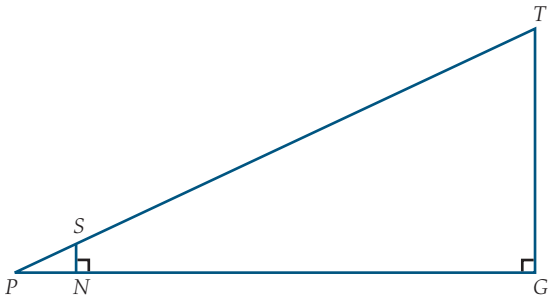
8 (a) $x = 10$ (b) $x = 6$ (c) $x = 5\frac{5}{9}$ (d) $x = 26.4$

9 5.25 m 10 316.67 cm 11 7.16 m

12 (a) ΔDFA ΔBGI, ΔCHJ, ΔDKM, ΔELN, ΔFMO

(b) ΔBKN and ΔCLO; ΔAJG and ΔKNB

13 (a)

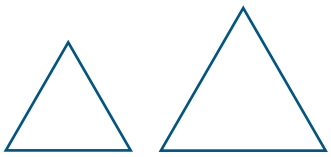


Measure SN, PN and NG

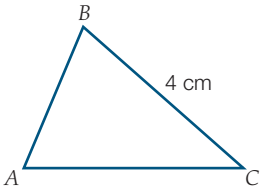
$\frac{TG}{PG} = \frac{SN}{PN}$ so, $TG = \frac{SN \times PG}{PN}$ where $PG = PN + NG$

(b) Need to measure accurately; can't really get your head level with the ground.

14 AAA does not guarantee any sides are equal in length. If it did every equilateral triangle would be the same size.



15 Draw any triangle in which $BC = 4$ cm and then measure the other two sides.



In this triangle $AB = 2.6$ cm and $AC = 5$ cm. So, in ΔDEF, $DE = 2.6 \times 1.5 = 3.9$ cm; $AC = 5 \times 1.5 = 7.5$ cm and $EF = 4 \times 1.5 = 6$ cm

Exercise 8.2 (p. 338)

1 (a) $\Delta ABC \equiv \Delta CDA$

$BC = DA$	(given)
$BA = DC$	(given)
$AC = CA$	(same side)
$\Delta ABC \equiv \Delta CDA$	(SSS)
$\angle BAC = \angle DCA$, so $x = 55°$	

(b) $\Delta ACD \equiv \Delta ACB$

$\angle ACD = \angle ACB$	(given)
$\angle ADC = \angle ABC$	(given)
$AC = AC$	(same side)
$\Delta ACD \equiv \Delta ACB$	(AAS)
$AD = AB$, so $p = 19$ cm	

(c) $\Delta ABE \equiv \Delta CDE$

$AB = CD$	(given)
$\angle ABE = \angle CDE$	(given)
$\angle AEB = \angle CED$	(vertically opposite)
$\Delta ABE \equiv \Delta CDE$	(AAS)
$AE = CE$, so $x = 23$ cm	

(d) $\Delta AOB \equiv \Delta COB$

$AO = CO$	(radius of circle)
$\angle OBA = \angle OBC$	(given)
$\angle OAB = \angle OCB$	(isosceles triangle)
$\Delta AOB \equiv \Delta COB$	(AAS)
$\angle OAC = \angle OCB$, so $y = 26°$	

(e) $\Delta ABO \equiv \Delta CDO$

$AB = CD$	(given)
$AO = CO$	(radius of circle)
$BO = DO$	(radius of circle)
$\Delta ABO \equiv \Delta CDO$	(SSS)
$AO = CO$, so $y = 9$ m	

(f) $\Delta ABE \equiv \Delta DCE$

$\angle AEB = \angle DEC$	(vertically opposite)
$\angle BAE = \angle CDE$	(given)
$AB = DC$	(given)
$\Delta ABE \equiv \Delta CDE$	(AAS)
$AE = ED$, so $p = 6$ cm	

2 (a) $\frac{AB}{DE} = \frac{3}{6} = \frac{1}{2}$; $\frac{BC}{EF} = \frac{4}{8} = \frac{1}{2}$; $\frac{AC}{DF} = \frac{5}{10} = \frac{1}{2}$;

$\Delta ABC \parallel\!| \Delta DEF$	(SSS)

(b)

$OM = OP$	(given)
$ON = OQ$	(given)
$\angle MON = \angle POQ$	(vertically opposite)
$\Delta MON \equiv \Delta POQ$	

Since every pair of congruent triangles are also similar we can say $\Delta MON \parallel\!| \Delta POQ$

(c)

$\angle RVS = \angle RUT$	(given)
$\angle SRV = \angle TRU$	(same angle)
$\angle RSV = \angle RTU$	(angle sum of the triangle)
$\Delta RSV \parallel\!| \Delta RTU$	(AAA)

(d)

$\angle YWX = \angle VWU$	(vertically opposite)
$\angle WYX = \angle UVW$	(alternate angles between parallel lines)
$\angle WXY = \angle WUV$	(alternate angles between parallel lines)
$\Delta WYX \parallel\!| \Delta WVU$	(AAA)

3 C 4 B

5 (a) $\angle RUT = \angle RST$ (given—right angle)
$RT = RT$ (same side—hypotenuse)
$UT = ST$ (given)
$\Delta RUT \equiv \Delta RST$ (RHS)

(b) $SP = RQ$ (given)
$\angle SPQ = \angle RQP$ (given)
$PQ = PQ$ (same side)
$\Delta SPQ \equiv \Delta RQP$ (SAS)

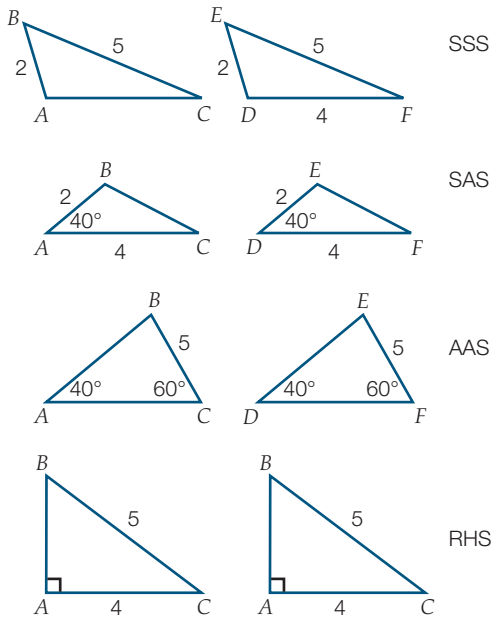
6 (a) $\angle QRS = \angle PRT$ (same angle)
$\angle RQS = \angle RPT$ (corresponding angles on parallel lines)
$\angle RSQ = \angle RTP$ (corresponding angles on parallel lines)
$\Delta RQS \parallel\!| \Delta RPT$ (AAA)

(b) $\angle ABC = \angle BDC$ (given—right angle)
$\angle ACB = \angle BCD$ (same angle)
$\angle BAC = \angle DBC$ (third angle in triangle)
$\Delta ABC \parallel\!| \Delta BDC$ (AAA)

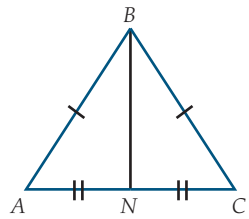
$\angle ADB = \angle ABC$ (given—right angle)
$\angle BAD = \angle CAB$ (same angle)
$\angle ABD = \angle ACB$ (third angle in triangle)
$\Delta ABC \parallel\!| \Delta ADB$ (AAA)
$\Delta BDC \parallel\!| \Delta ADB$ (AAA)

7 (a) (i) not necessarily similar, angles same but sides might not be proportional
(ii) all matching sides in same ratio
(iii) not necessarily similar, angles between equal sides might not be equal
(iv) all angles the same (AAA)
(v) all angles the same (AAA)
(vi) radius is only measure so all distances will be in the same ratio

(b) (i) not similar (ii) side length the same
(iii) not similar (iv) side length the same
(v) one matching side length the same
(vi) radius the same

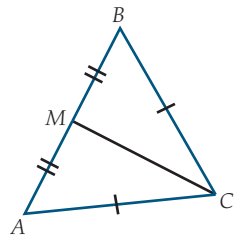
8


Exercise 8.3 (p. 341)

1 (a)


With this construction:

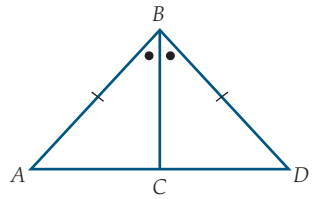
$\Delta ABN \equiv \Delta CBN$
$AN = CN$ (given by construction)
$BN = BN$ (same side)
$AB = CB$ (given)
$\Delta ABN \equiv \Delta CBN$ (SSS)
So, $\angle BAC = \angle BCA$



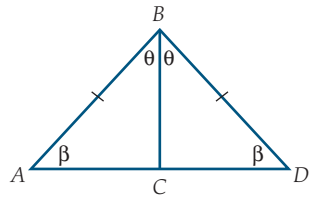
With this construction:

$\Delta BCM \equiv \Delta ACM$
$AM = BM$ (given by construction)
$CM = CM$ (same side)
$CB = CA$ (given)
$\Delta BCM \equiv \Delta ACM$ (SSS)
So, $\angle CBA = \angle CAB$
But, $\angle CAB = \angle BCA$ from the first construction, so
$\angle CBA = \angle CAB = \angle BCA$

(b) All angles are the same (60°)

2 (a)


To show: $AC = CD$
$\angle BAC = \angle BDC$ (isosceles triangle)
$\angle ABC = \angle DBC$ (by construction)
$BC = BC$
So, $\Delta BAC \equiv \Delta BDC$ (AAS)
$\therefore$ so $AC = DC$ (matching sides of congruent triangles)

(b)


To show: $\angle BCA = \angle BCD = 90°$
$2 \times \theta + 2 \times \beta = 180$ (angle sum of triangle)
$\theta + \beta = 90°$
So, $\angle BCA = 180 - (\theta + \beta) = 90°$

3 $\angle CDE + \angle EDF = 180°$ (straight angle)
$\angle EFG + \angle EFD = 180°$ (straight angle)
$\angle EDF = \angle EFD$ (equal angles of isosceles triangle)

$\angle CDE = 180 - \angle EDF$

$\angle EFG = 180 - \angle EFD$
$= 180 - \angle EDF$
$= \angle CDE$

4 A **5** B

6 $AC^2 = AB^2 + BC^2$ (Pythagoras)
$BC^2 = BD^2 + DC^2$ (Pythagoras)
So, $AC^2 = AB^2 + BD^2 + DC^2$ (1)
Also, $AB^2 = DB^2 + AD^2$ (Pythagoras)
Substituting into (1):

$AC^2 = DB^2 + AD^2 + BD^2 + DC^2$
$= DC^2 + 2BD^2 + AD^2$

7 $\Delta ABF \equiv \Delta BCD$
$AB = BC$ (given)
$\angle FAB = \angle DBC$ (corresponding angles, $FA \parallel BD$)
$\angle FBA = \angle DCB$ (corresponding angles, $FB \parallel DC$)
$\Delta ABF \equiv \Delta BCD$ (AAS)

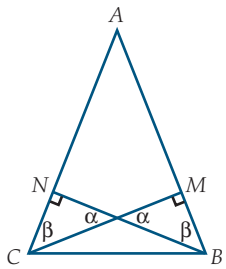
8 To show: $\angle RPT = \angle RTP$

$\angle QPU = \angle STU$ as both these angles are the third angle in a triangle where the other two angles are the same.

So in ΔRPT, $RP = RT$ (opposite equal sides)

$\therefore$ ΔRPT is isosceles.

9



$\angle CNB = \angle BMC$ (90°, given)
$\angle NCB = \angle MBC$ (angles opposite equal sides)
ΔCNB and ΔBMC have a common side, BC

So, $\Delta CNB \equiv \Delta BMC$ (AAS)
So, $NC = MB$ (matching sides)
$AC = AN + NC$
$AB = AM + MB$
$AB = AC$ (given)
So $AN = AM$ (equal sides AC and AB; NC and NB)

10 Let $EC = DB = x$

$\therefore$ $ED = x + DC = BC$

In ΔEFD and ΔBAC

$ED = BC$ (above)

$\angle EFD = \angle BAC$ (given)

$\angle FED = \angle ABC$ (given)

$\therefore$ $\Delta ABC \equiv \Delta FED$ (AAS)

11 Let $BF = EC = x$

$\therefore$ $BC = x + FC = FE$

In ΔABC and ΔDEF

$BC = EF$ (above)

$\angle ABC = \angle DEF$ (given)

$\angle ACB = \angle DFE$ (given)

$\therefore$ $\Delta ABC \equiv \Delta DEF$ (AAS)

12 **(a)** $\Delta ABE \equiv \Delta ACD$
$\angle ABE = \angle ACE$ (given)
$BE = CD$ ($BC = EC$ and DE is common)
$AB = AC$ (given)
$\Delta ABE \equiv \Delta ACD$ (SAS)

(b) $\Delta ABE \equiv \Delta CBD$
$\angle BAE = \angle BCD$ (given)
$\angle ABE = \angle CBD$ ($\angle ABD = \angle CBE$ and $\angle EBD$ is common)
$AB = BC$ (isosceles triangle)
$\Delta ABE \equiv \Delta CBD$ (AAS)

(c) $\Delta ABD \equiv \Delta CBE$
$AD = CE$ ($AD = AE - DE$ and $CE = CD - DE$; DE common)
$\angle BAD = \angle BCE$ (given)
$AB = CB$ (isosceles triangle)
$\Delta ABD \equiv \Delta CBE$ (SAS)

(d) $\Delta ABD \equiv \Delta CBE$
$AB = CB$ (given)
$AD = CE$ ($AD = AE - DE$ and $CE = CD - DE$; DE common)
$\angle BAD = \angle BCE$ (isosceles triangle)
$\Delta ABD \equiv \Delta CBE$ (SAS)

13 $\Delta ADF \equiv \Delta AEF$
$DF = EF$ (given)
$\angle ADF = \angle AEF$ (both 90° as $FD \perp AB$ and $FE \perp AC$)
$AF = AF$ (common side)
$\Delta ADF \equiv \Delta AEF$ (RHS)

14 $\Delta ABC \equiv \Delta CDE$
$\angle ABC = \angle CDE$ (given—right angle)
$AC = CE$ (given C is the midpoint of AE hypotenuse)
$AB = CD$ (given)
$\Delta ABC \equiv \Delta CDE$ (RHS)

15 $\angle OEF = \angle OFE$ (isosceles triangle; call it α)
$\angle OED = \angle ODE$ (isosceles triangle; call it β)
$\angle EDO + \angle OFE + \angle FED = 180°$ (angle sum of triangle)

So, $\beta + \alpha + (\alpha + \beta) = 180°$
$2(\alpha + \beta) = 180°$
$\alpha + \beta = 90°$

As $\angle DFE = \alpha + \beta = 90°$ the angle in a semicircle is a right angle.

16 **(a)** $\angle ABC = \angle ACB$ (opposite equal angles)
$\angle AXY = \angle ABC$ (corresponding angles $XZ \parallel BC$)
$\angle AZY = \angle ACB$ (corresponding angles $XZ \parallel BC$)
$\therefore \angle AXY = \angle AZY$
$\therefore AX = AZ$ (sides opposite equal angles)

(b) $\Delta AXY \equiv \Delta AZY$
$AX = AZ$ (from **(a)**)
$XY = ZY$ (given)
$\angle AXY = \angle AZX$ (from **(a)**)
$\Delta AXY \equiv \Delta AZY$ (SAS)

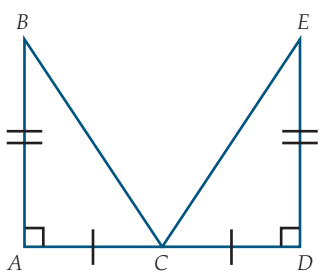
(c) $\Delta BXY \equiv \Delta CZY$
$BX = CZ$ ($AB = AC$ and $AX = AZ$)
$XY = ZY$ (given)
$\angle BXY = \angle CZY$ (Each $(180 - b)°$)
$\Delta BXY \equiv \Delta CZY$ (SAS)

(d) $a + b + 90 = 180$ (angle sum in triangle)
$b = 90 - a$

$\angle ACY = c$ and $\angle CAY = a$ but these are equal since ΔACY is isosceles.

So, $c = a$
$c + d = b$ (corresponding angles on parallel lines)
$a + d = 90 - a$
$d = 90 - 2a$

17



$\Delta ABC \equiv \Delta DEC$

Exercise 8.4 (p. 347)

1 **(a)** To prove: $\Delta ABC \parallel\!| \Delta ADE$
$\angle ADE = \angle ABC$ (corresponding angles, $DE \parallel BC$)
$\angle DAE = \angle BAC$ (same angle)
$\angle AED = \angle ACB$ (corresponding angles, $DE \parallel BC$)
$\Delta ABC \parallel\!| \Delta ADE$ (AAA)

(b) To prove: $AE = EC$
$\Delta ABC \parallel\!| \Delta ADE$ (from **(a)**)

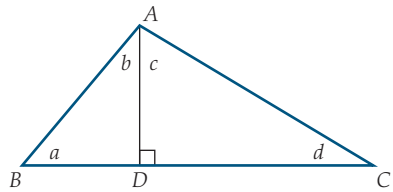
So, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$

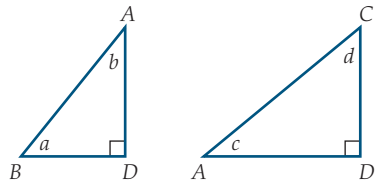
So, $\dfrac{AE}{AC} = \dfrac{1}{2}$

So, $2AE = AC$
So, $AE = EC$

2 To prove: $\Delta ADB \parallel\!| \Delta DCA$

The orientation of the two triangles must be as shown below as AD cannot be in the same position in both triangles.





$a + b = 90°$ (rest of angle sum of 180° for triangle)
$c + d = 90°$ (rest of angle sum of 180° for triangle)
$b + c = 90°$ (right angle in original triangle)

$a + b = b + c$, so $a = c$
$b + c = c + d$, so $b = d$
$\Delta ADB \parallel\!| \Delta DCA$ (AAA)

To prove: $\Delta DCA \parallel\!| \Delta CAB$
$\angle ADC = \angle CAB$ (both 90°)
$\angle ACD = \angle ACB$ (common angle)
$\angle ABD = \angle ABC$ (common angle)
$\Delta DCA \parallel\!| \Delta CAB$ (AAA)

3 To prove: $AD^2 = BD \times DC$
$\Delta ADB \parallel\!| \Delta DCA$ (from Question **2**)

So, $\dfrac{AD}{DC} = \dfrac{BD}{AD}$ (matching sides of similar triangles)

So, $AD \times AD = BD \times DC$
$AD^2 = BD \times DC$

4 To prove: $BA^2 = BD \times BC$ and $CA^2 = CD \times CB$
$\Delta ADB \parallel\!| \Delta CAB$ (from Question **2**)

$\dfrac{BA}{BC} = \dfrac{BD}{BA}$ (matching sides of similar triangles)

So, $BA^2 = BC \times BD$

To prove: $CA^2 = CD \times CB$
$\Delta ADC \parallel\!| \Delta BAC$ (from Question **2**)

$\dfrac{CA}{BC} = \dfrac{CD}{AC}$ (matching sides of similar triangles)

$CA^2 = CD \times BC$

5 $AB^2 = BC \times BD$ (from Question **4**)
$AC^2 = CD \times BC$ (from Question **4**)

$$\begin{aligned} AB^2 + AC^2 &= BC \times BD + CD \times BC \\ &= BC(BD + CD) \\ &= BC \times BC \\ &= BC^2 \end{aligned}$$

6 C

7 **(a)** To show: $\Delta ABO \parallel\!| \Delta DCO$

$\angle ABD = \angle BDC$ (alternate angles, $AB \parallel DC$)

$\angle BAC = \angle ACD$ (alternate angles, $AB \parallel DC$)

$\angle AOB = \angle COD$ (vertically opposite angles)

$\Delta ABO \parallel\!| \Delta DCO$ (AAA)

Now, $\frac{OA}{OC} = \frac{OB}{OD} = \frac{AB}{DC}$ (matching sides of similar triangles)

So, $OA : OC = OB : OD$

(b) We have $\frac{AB}{CD} = \frac{1}{2}$

So, $\frac{AO}{OC} = \frac{1}{2}$ (matching sides of similar triangles are in same ratio)

So, $2AO = OC$

Similarly, $2OB = OD$

$AC = AO + OC$
$= AO + 2AO$
$= 3AO$

So, $AO : AC = 1 : 3$ and similarly $BO : BD = 1 : 3$

8 To prove: $\frac{JM}{MK} = \frac{JN}{NL}$

We will show $\Delta JMN \parallel\!| \Delta JKL$

$\angle MJN = \angle KJL$ (common angle)

$\angle JMN = \angle JKL$ (corresponding angles, $MN \parallel KL$)

$\angle JNM = \angle JLK$ (corresponding angles, $MN \parallel KL$)

$\Delta JMN \parallel\!| \angle JKL$ (AAA)

So, $\frac{JK}{JM} = \frac{JL}{JN}$ (matching sides of similar triangles are proportional)

$\frac{JK}{JM} - 1 = \frac{JL}{JN} - 1$

$\frac{JK - JM}{JM} = \frac{JL - JN}{JN}$

$\frac{MK}{JM} = \frac{NL}{JN}$ ($JK = JM + MK$)

($JL = JN + NL$)

So, $\frac{JM}{MK} = \frac{JN}{NL}$ (reciprocal)

9 To prove: $GH \parallel EF$

We will show $\Delta DGH \parallel\!| \Delta DEF$

$\angle GDH = \angle EDF$ (common angle)

$\frac{DG}{GE} = \frac{DH}{HF}$ (given)

$\frac{GE}{DG} = \frac{HF}{DH}$ (reciprocal)

$\frac{GE}{DG} + 1 = \frac{HF}{DH} + 1$

$\frac{GE + DG}{DG} = \frac{HF + DH}{DH}$

$\frac{DE}{DG} = \frac{DF}{DH}$

Corresponding sides are in the same proportion so

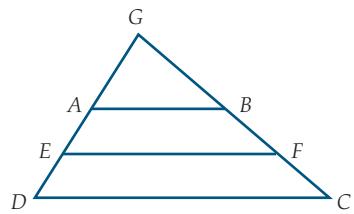
$\Delta DGH \parallel\!| \Delta DEF$ (SAS)

So, $\angle DGH = \angle DEF$ (matching angles in similar triangles)

And $\angle DHG = \angle DFE$ (matching angles in similar triangles)

So, $GH \parallel EF$ (corresponding angles equal on parallel lines)

10 Following the hint we have this diagram:



In ΔGAB and ΔGDC

$\angle AGB = \angle DGC$ (common angle)

$\angle GAB = \angle GDC$ (corresponding angles, $AB \parallel DC$)

$\therefore \Delta GAB \parallel\!| \Delta GDC$ (AAA)

$\therefore$ matching sides are proportional

Let $GA = x$, $AE = y$, $ED = y$

$\therefore GB = mx$, $BC = 2my$, $BF = my$, $FC = my$

$\frac{GA}{GE} = \frac{x}{x + y}$

$\frac{GB}{GF} = \frac{mx}{mx + my} = \frac{x}{x + y}$

now LG is common

$\therefore \Delta GAB \parallel\!| \Delta GEF$ (SAS)

$\therefore \angle GAB = \angle GEF$ (matching angles)

However, these angles are equal corresponding angles

$\therefore AB \parallel EF$

11 To prove: $\Delta BXY \parallel\!| \Delta BYC$

Hint: Prove $\Delta BXY \equiv \Delta CZY$

(See Exercise 8.3 Question 16 for the proof of this; it is not repeated here.)

$BY = CY$ (congruent triangles)

We know $\frac{BX}{BY} = \frac{XY}{YC}$ (as $BX = XY$ and $BY = YC$)

Need to show: $\angle BXY = \angle BYC$

Now, $\angle XBY = \angle XYB$ (isosceles triangle and call it b)

So, $\angle BXY = (180 - 2b)°$

But, $\angle ZYC = b$ (congruent triangle)

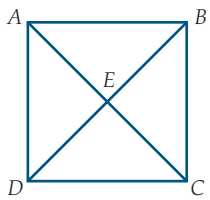
And, $\angle XYB + \angle BYC + \angle ZYC = 180°$ (straight angle)

$b + \angle BYC + b = 180°$

$\angle BYC = (180 - 2b)°$

So, $\angle BXY = \angle BYC$

So, $\Delta BXY \parallel\!| \Delta BYC$ (SAS)

12 

To prove: $\Delta AED \parallel\!| \Delta ADC$

The diagonals of a square bisect each other at right angles

So, $\frac{AE}{AD} = \frac{ED}{DC}$ since $AE = ED$ and $AD = DC$

$\angle AED = \angle ADC$	(both right angles)
So, $\Delta AED \parallel\!| \Delta ADC$	(SAS)

Exercise 8.5 (p. 352)

1 (a) (i) Opposite sides are parallel and one pair of adjacent sides is equal

(ii) rhombus

(b) (i) Two pairs of adjacent sides equal.

(ii) kite

(c) (i) Two pairs of opposite sides parallel; one pair of adjacent sides equal; one right angle.

(ii) square

2 (a) To prove: $\Delta DAB \equiv \Delta DCB$

$\angle DBA = \angle BDC$	(alternate angles, $AB \parallel CD$)
$\angle BDA = \angle CBD$	(alternate angles, $AD \parallel BC$)
$DB = DB$	(common side)
$\Delta DAB \equiv \Delta DCB$	(AAS)

(b) $\angle DAB = \angle BCD$ (matching angles, congruent triangles from (a))

$\angle ABC$ is supplement of $\angle DAB$ (co-interior angles, $AB \parallel CD$)

$\angle ADC$ is supplement of $\angle DCB$ (co-interior angles, $AD \parallel BC$)

$\therefore \angle ABC = \angle ADC$ (supplements of equal angles)

(c) Opposite angles in a parallelogram are equal

(d) To prove: $\Delta DAE \equiv \Delta BCE$

$AD = BC$	(matching sides in congruent triangles)
$\angle AED = \angle BEC$	(vertically opposite)
$\angle ADE = \angle CBE$	(alternate angles, $AD \parallel BC$)
$\Delta DAE \equiv \Delta BCE$	(AAS)

(e) These are matching sides in congruent triangles, so $AE = CE$ and $DE = BE$

(f) The diagonals of a parallelogram bisect each other

3 $AC = DB$ and bisect each other

$\therefore AX = XC = DX = XB$

$AC \perp BD$ $\therefore$ all angles at centre are 90°

The four triangles are congruent (SAS)

$\therefore$ all hypotenuses are equal (matching sides)

acute angles of each triangle are 45° (isosceles triangle, angle sum of triangle)

$\therefore \angle ABC = \angle BCD = \angle CDA = \angle DAB = 90°$

$\therefore$ quadrilateral is a square (all sides equal, angles are right angles)

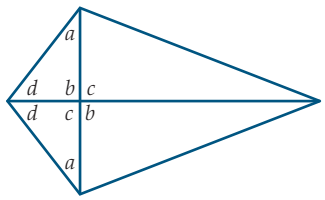
4 (a) To prove: $\angle KLM = \angle KNM$

Will show: $\Delta KLM \equiv \Delta KNM$

$KL = KN$	(given)
$LM = NM$	(given)
$KM = KM$	(common side)
$\Delta KLM \equiv \Delta KNM$	(SSS)

$\angle KLM = \angle KNM$ as these are matching angles in congruent triangles

(b) To prove: $\angle KOL = 90°$



We know the angles marked a are equal as they are in an isosceles triangle.

We know the angles marked d are equal because they are matching angles in congruent triangles.

The angles marked b and c are pairs of vertically opposite angles.

So, $a + b = a + c$	(supplements of d)
So, $b = c$	
But, $b + c = 180°$	(straight angle)
So, $b = c = 90°$	
So, $\angle KOL = 90°$	

5 B 6 C

7 (a) To prove: $PQ = RS$

Will show: $\Delta PQS \equiv \Delta RSQ$

$SQ = SQ$	(common side)
$\angle PSQ = \angle RQS$	(alternate angles, $PS \parallel RQ$)
$\angle PQS = \angle QSR$	(alternate angles, $PQ \parallel RS$)
$\Delta PQS \equiv \Delta RSQ$	(AAS)

So, $PQ = RS$ as they are matching sides on congruent triangles

(b) Opposite sides of a rectangle are equal

8 (a) To show: $\Delta ACD \equiv \Delta ACB$

$AD = BC$	(given)
$DC = AB$	(given)
$AC = AC$	(given)
$\Delta ACD \equiv \Delta ACB$	(SSS)
$\angle BAC = \angle BCA = 45°$	(isosceles triangles)
$\angle DAC = \angle DCA = 45°$	(isosceles triangles)

$\angle BAD = \angle BAC + \angle DAC$

$= 45° + 45°$

$= 90°$

So, $\angle BAD = 90°$, so we have a square

(b) To show: $\Delta ABD \equiv \Delta CBD$

$AB = BC$	(given)
$AD = DC$	(given)
$DB = DB$	(given)
$\Delta ABD \equiv \Delta CBD$	(SSS)

So, $\angle ABD = \angle BDC$ as these are matching angles in congruent triangles. But, these are alternate angles, so the lines must be parallel.

Then, do the same for the other pair of sides

9 (a) Will show: $\Delta ABD \equiv \Delta BDC$

$AD = BC$	(given)
$\angle ADB = \angle DBC$	(alternate angles on parallel lines)
$\angle ABD = \angle BDC$	(alternate angles on parallel lines)
$\Delta ABD \equiv \Delta BDC$	(AAS)
So, $AB = DC$	(matching sides in congruent triangles)
And, $\angle DAB = \angle DCB$	(matching angles in congruent triangles)

So, we have a parallelogram

(b) Will show: $\Delta AED \equiv \Delta CEB$

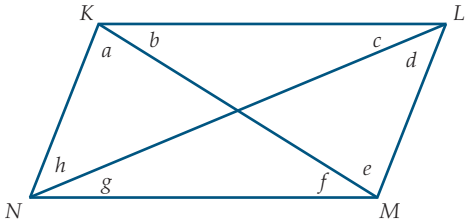
$AE = CE$	(given)
$DE = BE$	(given)
$\angle AED = \angle BEC$	(vertically opposite)
$\Delta AED \equiv \Delta CEB$	(SAS)
So, $AD = BC$	(matching sides in congruent triangles)

In a similar way show $\Delta AEB \equiv \Delta DEC$ and so $AB = DC$

Thus, we have two pairs of opposite sides equal and parallel.

So the quadrilateral is a parallelogram.

(c) Consider the following diagram:



A number of relationships can be established from the diagram

$a + b + c + h = 180°$	(angles in triangle)
$e + f + d + g = 180°$	(angles in triangle)
but $e + f = a + b$	(given)

So, $a + b + c + h = e + f + d + g$

becomes $a + b + c + h = a + b + d + g$

so, $c + h = d + g$

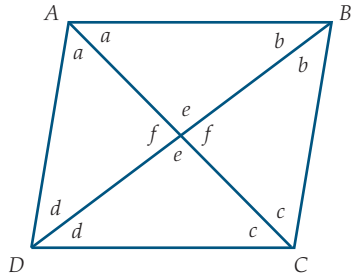
but $c + d = h + g$	(given)

so, $d = h$ and $g = c$

These are alternate angles so $KL \parallel NM$

Then do a similar thing to show $KN \parallel LM$

(d) Consider this diagram:



$2a + b + d = 2c + b + d$	(sum of angles in a triangle = 180°)

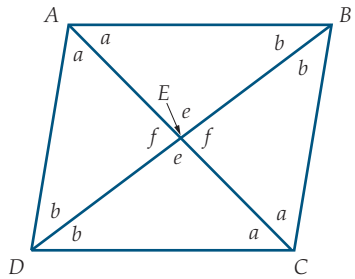
so, $a = c$

$2b + a + c = 2d + a + c$	(sum of angles in a triangle = 180°)

so, $b = d$

similarly $a = c$

Redraw the diagram:



$a + b + e = a + b + f$	(sum of angles in a triangle = 180°)

so, $e = f$

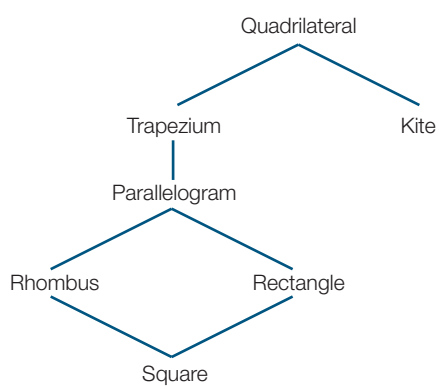
$AD = AB$	(opposite equal angles ΔABD)
$BC = DC$	(opposite equal angles ΔBCD)
$AD = DC$	(opposite equal angles ΔADC)
$\therefore AB = BC = CD = DA$	
$\therefore ABCD$ is a rhombus	(all sides equal)

10 A

11 To show: $\Delta WMZ \equiv \Delta YNZ$

$ZM = NZ$	(given)
$WZ = YZ$	(given)
$\angle ZWM = \angle ZYN$	(both 90°, given)
$\Delta WMZ \equiv \Delta YNZ$	(RHS)

12 The words link in the following way:



13 To show: $\Delta AKD \equiv \Delta CJB$
As $AJ = CK$ and $AB = DC$ (opposite sides of parallelogram) then $JB = DK$
So, $JB = DK$

$AD = BC$	(parallelogram)
$\angle ADK = \angle JBC$	(opposite angles in parallelogram)
$\Delta AKD \equiv \Delta CJB$	(SAS)

14 A square has four right angles and two pairs of opposite sides equal and parallel with all sides the same length, while a rhombus has two pairs of opposite sides equal and parallel with all sides the same length. So, the square has an extra condition.

15 A square has four right angles and two pairs of opposite sides equal and parallel with all sides the same length, whereas a rectangle has four right angles and two pairs of opposite sides equal and parallel. So, the square has an extra condition.

16 For the condition *at least one pair of parallel sides* the shape could be: trapezium, square, rectangle, rhombus or parallelogram

For the condition *at least one pair of equal sides* the shape could be: square, rectangle, parallelogram, rhombus or kite

For the condition *at least one pair of equal angles* the shape could be: square, rectangle, rhombus, parallelogram or kite

17 (a) A rhombus has four equal sides but is not a square, because it has no right angle.

(b) Every square has four right angles. Every shape with four right angles is a square. False—it could be a rectangle.

Every kite has two pairs of adjacent sides equal. Every shape with two pairs of adjacent sides equal is a kite. False—it could be a square or a rhombus.

18 (a) Students' own diagrams.

(b) Students to draw and measure from their own diagrams.

(c) The diagonals of a rhombus intersect at right angles.

(d) A demonstration involving measurement is never sufficient as a proof.

19 A square, rhombus and kite all have diagonals that bisect at right angles, so George is correct.

Half-time 8 (p. 357)

1 (a) To prove: $\Delta ABC \equiv \Delta EDC$

$\angle ABC = \angle CDE$	(both right angles)
$\angle BCA = \angle ECD$	(vertically opposite)
So, $\Delta ABC \parallel\!| \Delta EDC$	(AAA)
$BC = DC$	(AAS)
$\therefore \Delta ABC \equiv \Delta EDC$	

(b) AB and ED are corresponding sides in congruent triangles so must be equal.

2 (a) RHS (b) SAS

3 (a)

$OA = OD$	(radii)
$OB = OC$	(radii)
$AB = CD$	(given)
So, $\Delta AOB \equiv \Delta DOC$	(SSS)
$p = 15$	

(b) In ΔURT and ΔSTR

$RT = RT$	(common side)
$\angle URT = \angle STR$	(alternate angles, $RU \parallel TS$)
$\angle RTU = \angle SRT$	(alternate angles, $RS \parallel UT$)
$\therefore \Delta URT \equiv \Delta STR$	(AAS)
$\therefore a = 11$ cm	(matching sides in congruent triangles)

4 (a) $x = 12$ (b) $x = 15$

5 2.4 cm

6 To prove: $\Delta ADF \equiv \Delta AEF$

$\angle ADF = \angle AEF$	(both right angles)
$AF = AF$	(common side)
$DF = EF$	(given)
$\Delta ADF \equiv \Delta AEF$	(RHS)

7 To prove: $\Delta ABC \equiv \Delta DEC$

$\angle ACB = \angle ECD$	(vertically opposite)
$\angle CBA = \angle CDE$	(alternate angles, $AB \parallel DE$)
$BC = CD$	(C is the midpoint of DB)
$\therefore \Delta ABC \equiv \Delta DEC$	(AAS)

8 $\Delta ABC \parallel\!| \Delta APQ$

$\angle ABC = \angle APQ$	(given)
$\angle CAB = \angle QAP$	(common angle)
$\angle AQP = \angle ACB$	(third angle in triangle)
$\Delta ABC \parallel\!| \Delta APQ$	(AAA)

9 To prove: $\Delta LYZ \parallel\!| \Delta MXZ$

$\angle LYZ = \angle MXZ$	(both right angles)
$\angle YZL = \angle XZM$	(vertically opposite)
$\angle YLZ = \angle XMZ$	(third angle in triangle)
$\Delta LYZ \parallel\!| \Delta MXZ$	(AAA)

10 In ΔACX and ΔBCX

$AC = BC$	(given)
$AX = BX$	(given)
$CX = CX$	(common side)
$\therefore \Delta ACX \equiv \Delta BCX$	(SSS)

11 To prove: $XY = XZ$
We will prove $\Delta YAX \equiv \Delta ZBX$

$\angle YAX = \angle ZBX$	(isosceles ΔABC)
$YA = ZB$	(halves of equal sides)
$XA = XB$	(given)
$\Delta YAX \equiv \Delta ZBX$	(SAS)

XY and XZ are corresponding sides in congruent triangles, so $XY = XZ$.

12 (a) $ABCD$ is a rhombus if $AB = BC = CD = DA$.

ΔBCD is isosceles.	(base angles equal)
$\therefore BC = CD$	
but $BC = DA$	(opposite sides of parallelogram)
and $AB = CD$	(opposite sides of parallelogram)

So $ABCD$ is a rhombus.

(b) $DC = BC$ ($ABCD$ is a rhombus)

CX is common

$\angle DCA = \angle CAB$ (alternate angles on parallel lines)

$\angle CAB = \angle BCA$ (base angles of isoseles triangles)

$\angle DCA = \angle BCA$ (both $= \angle CAB$)

$\therefore \Delta DCX = \Delta BCX$ (SAS)

$DX = BX$

$\angle DXC = \angle BXC$

and $\angle DXC + \angle BXC = 180°$ (angles on a straight line)

So $\angle DXC = \angle BXC = 90°$

Similarly, $AX = CX$ and
$\angle AXB = \angle BXC = 90°$ (SAS)

$\therefore$ diagonals of a rhombus bisect at right angles

13 5 m

Exercise 8.6 (p. 367)

1 Join GO and extend it to H.

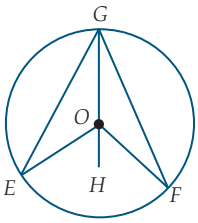
In ΔOGE, $OG = OE$ (radii of a circle)

$\therefore \Delta OGE$ is isosceles

$\therefore \angle OGE = \angle OEG$ (base angles of isosceles triangles)

Hence $\angle HOE = \angle OGE + \angle OEG$ (exterior angle of triangle is sum of interior opposite angles)

$\therefore \angle HOE = 2\angle OGE$



Repeating these steps in ΔOGF we obtain $\angle HOF = 2\angle OGF$.
Now $\angle EOF = \angle HOE + \angle HOF$.

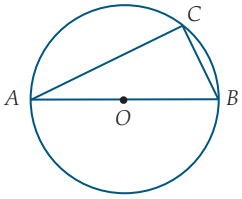
$\therefore \angle EOF = 2\angle OGE + 2\angle OGF$
$= 2(\angle OGE + \angle OGF)$

$\therefore \angle EOF = 2\angle EGF$

2 $\angle AOB = 2\angle ACB$ (angle at the centre is twice the angle at the circumference standing on the same arc)

But $\angle AOB = 180°$ (straight angle)

$\therefore \angle ACB = 90°$



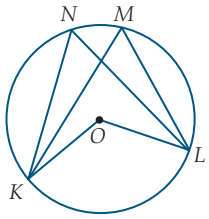
3 Join KO and LO.

$\angle KOL = 2\angle KNL$ (angle at the centre is twice the angle at the circumference standing on the same arc)

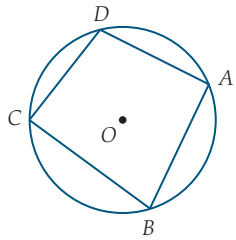
Similarly, $\angle KOL = 2\angle KML$

Hence $2\angle KNL = 2\angle KML$

So $\angle KML = \angle KNL$



4 $\angle COA = 2\angle CBA$ (angle at the centre is twice the angle at the circumference standing on the same arc)



Reflex $\angle COA = 2\angle CDA$ (angle at the centre is twice the angle at the circumference standing on the same arc)

But $\angle COA$ + reflex $\angle COA = 360°$ (angles at a point)

So $2\angle CBA + 2\angle CDA = 360°$

Hence $\angle CBA + \angle CDA = 180°$

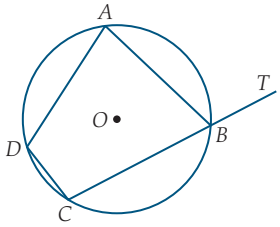
5 $\angle ADC + \angle ABC = 180°$ (opposite angles of a cyclic quadrilateral are supplementary)

$\therefore ADC = 180° - \angle ABC$

but $\angle ABT + \angle ABC = 180°$ (angles on a straight line)

$\therefore \angle ABT = 180° - \angle ABC$

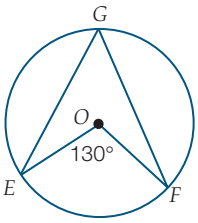
$\therefore \angle ADC = \angle ABT$



6 $\angle EOF = 2\angle EGF$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\therefore 130° = 2\angle EGF$

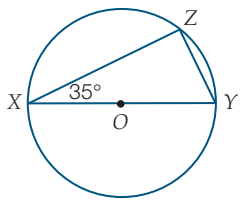
$\angle EGF = 65°$



7 $\angle XZY = 90°$ (angle in a semicircle)

But $\angle ZXY = 35°$

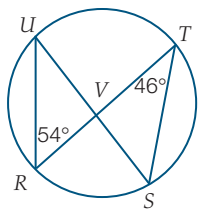
So $\angle ZYX = 55°$ (angle sum of triangle is 180°)



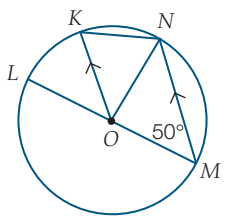
8 $\angle RUS = 46°$ (angles in the same segment)

$\angle UVR = 80°$ (angle sum of ΔUVR)

$\angle TVS = 80°$ (vertically opposite angles)



9 $\angle LOK = 50°$ (corresponding angles, $OK \parallel MN$)



$\angle LON = 2\angle LMN = 100°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\angle OMN = \angle ONM$ (opposite equal sides (radii) of ΔONM)

$\angle KON = \angle ONM$ (alternate angles, $OK \parallel MN$)

Hence $\angle KON = \angle LON - \angle LOK = 100° - 50° = 50°$

$OK = ON$ (radii of a circle)

Hence ΔOKN is isosceles and $\angle OKN = \angle ONK$.

$50° + \angle OKN + \angle ONK = 180°$ (angle sum of ΔOKN)

$\therefore 2\angle OKN = 130°$

$\therefore \angle OKN = 65°$

10 (a) $x = 126°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $y = 70°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $x = 115°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc)

11 (a) $\angle ACD = 90°$ (angle in a semicircle), $z = 46°$ (angle sum of ΔACD)

(b) $\angle ECG = 90°$ (angle in a semicircle), $\angle CGE = 40°$ (angle sum of ΔECG), $x + 40° = 95°$ (exterior angle of triangle is sum of interior opposite angles), $x = 55°$

(c) $\angle OGH = y$ (isosceles triangle, $OG = OH$), $2y = 70°$ (exterior angle of triangle is sum of interior opposite angles), $y = 35°$

12 (a) $\angle BDC = 43°$ (angles in the same segment), $x = 107°$ (angle sum of ΔEDC)

(b) $x = 47°$ (angles in the same segment), $\angle ABD = 90°$ (angle in a semicircle), $y = 43°$, $z = 43°$ (angles in the same segment)

(c) $y = 50°$ (angles in the same segment), $ABD = 90°$ (angle in a semicircle), $\angle ABC = 40°$, $z + 40° = 95°$ (exterior angle of triangle is sum of interior opposite angles), $z = 55°$

(d) $n = 32°$, $m = 32°$ (angles in the same segment)

(e) $\angle BDC = 35°$ (angles in the same segment), $e = 51°$ (angle sum of ΔADC)

(f) $\angle DCA = 44°$ (angles in the same segment), $k = 25°$ (exterior angle of triangle is sum of interior opposite angles)

13 (a) $x = 35°$ (angle at the centre is twice the angle at the circumference standing on the same arc), $\angle FOE = 80°$ (angle at the centre is twice the angle at the circumference standing on the same arc), $2y + 80° = 180°$ (angle sum of isosceles triangle FOE), $y = 50°$

(b) $\angle CDB = 49°$ (angle sum of ΔBDC), $\angle ACD = 49°$ (alternate angles, $AC \parallel DB$), $z = 98°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $\angle MNQ = \angle MQN = 72°$ (base angles of isosceles ΔMNQ), $\angle NQP = 90°$ (angle in a semicircle), $w = 18°$

14 $\angle BOD = 2y$ (angle at centre is twice angle at circumference standing on the same arc)

$OB = OD$ (radii)

$\angle OBD = \angle ODB = x°$ (angles opposite equal sides of a triangle)

$\therefore x + x + 2y = 180$ (angle sum of a triangle)

$\therefore x + y = 90$

15 ΔBDC is isosceles, so $\angle DBC = \angle DCB = 64°$

$\angle ABC = 90°$ (angle in a semicircle) $\therefore \angle ADE = 38°$

$\angle ABD = 26°$ $\therefore \angle ACB = 38°$

$\angle ACD = 26°$ (angles in the same segment)

16 $\angle DGE = 20°$ (angles in the same segment)

$\angle FDE = 48°$ (angles in the same segment)

$\angle GFD = 42°$ (angles in the same segment)

$\angle GDF = 70°$ (angle sum of triangle)

$\angle GEF = 70°$ (angles in the same segment)

$\angle FGD = \angle FGE + \angle DGE = 68°$

$\angle GDE = \angle GDF + \angle FDE = 118°$

$\angle DEF = \angle GED + \angle GEF = 112°$

$\angle EFG = \angle DFE + \angle GFD = 62°$

$\angle GHF = 90°$ (angle sum of a triangle)

$\therefore$ The diagonals GE and DF are perpendicular.

17 $\angle OZY = \angle OYZ = 44°$ (base angles of isosceles triangle OYZ)

$\angle OZX = 62° - 44° = 18°$

$\angle OXZ = 18°$ (base angles of isosceles triangle OXZ)

18 (a) $\angle ODF = 30°$ (isosceles triangle), $\angle FOD = 120°$ (angle sum of triangle), $y = 60°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $\angle ODF = 33°$ (isosceles triangle), $\angle FOD = 114°$ (angle sum of triangle), reflex $\angle FOD = 246°$ (angles at a point), $\angle FBD = 123°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc), $x = 22°$ (angle sum of triangle)

(c) $\angle AOD = 2x$ (angle at the centre is twice the angle at the circumference standing on the same arc), reflex $\angle AOD = 4x°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc), $6x = 360°$ (angles at a point), $x = 60°$

19 Join KO, LO, NO

Let $\angle LNO = x$

$\therefore \angle OLN = \angle ONL = x$ (angles opposite equal sides of a triangle)

$\therefore \angle LON = (180 - 2x)°$ (angles sum of a triangle)

In ΔOKL and ΔONL, $OK = ON$ (radii)

$KL = NL$ (given)

LO is common

$\therefore \Delta OKL \equiv \Delta ONL$

$\therefore \angle OLK = \angle OLN = x$ (corresponding angles in congruent triangles)

$\therefore \angle KLM = (180 - 2x)°$ (MKL is a straight line)

$\therefore \angle KLM = \angle LON$ (both $(180 - 2x)°$)

20 (a) In Δs KPM and QNM,

$\angle KPQ = \angle QNK$ (angles in the same segment)

$\angle PKN = \angle NQP$ (angles in same segment)

$\angle KMP = \angle QMN$ (vertically opposite angles)

$\therefore \Delta KPM \equiv \Delta QNM$ (AAA)

(b) $KM = MQ$ (given)

$\angle PKM = \angle NQM$ (matching angles in similar triangles)

$\angle MNQ = \angle MPK$ (matching angles in similar triangles)

$\therefore \Delta KMP \equiv \Delta QMN$ (AAS)

$\therefore MP = MN$ (matching sides in congruent triangles)

21 In Δs XWU and YZU,

$\angle XWY = \angle XZY$ (angles in the same segment)

$\angle WXZ = \angle WYZ$ (angles in same segment)

But $\angle XWU = \angle WXU$ (opposite equal sides (radii) ΔXWU)

So $\angle XWU = \angle UYZ$ (both equal to $\angle XWY$)

But these are a pair of equal alternate angles between XW and YZ

$\therefore XW \parallel YZ$

22 $\angle CAF = 90°$ (angle in a semicircle)

$\angle DAF = 90°$ (angle in a semicircle)

$\therefore \angle CAF + \angle DAF = 180°$

$\therefore \angle CAD = 180°$, so C, A and D are collinear

23 $\angle SVT = \angle SUT = 45°$ (angle at centre is twice angle at circumference standing on the same arc)

$\angle UTV = \angle TVS = 45°$ (alternate angles equal, $VS \parallel UT$)

$\therefore \angle TWU = 90°$ (angle sum of ΔTWU)

24 $\angle ACD = \angle ADC$ (base angles of isosceles triangle ACD)

$\angle AFB = \angle ACB$ (angles in the same segment standing on AB)

$\angle AFE = \angle ADE$ (angles in the same segment standing on AE)

$\therefore \angle AFB = \angle AFE$ (from line 1)

$\therefore AF$ bisects $\angle EFB$

25 $\angle KML = 90°$ (angle in a semicircle on diameter KL)

$\angle ONL = 90°$ (angle in a semicircle on diameter OL)

$\therefore \angle KML = \angle ONL = 90°$

$\therefore KM \parallel ON$ (pair of corresponding angles are equal)

26 (a) (i) $\angle DAC = 30°$ (ii) $\angle ADB = 70°$

(iii) $\angle ACB = 70°$

(b) Students' own answers.

(c) (i) ΔADC is isosceles because $\angle DAC = \angle DCA = 30°$

(ii) $\angle ABC = 60°$

$\angle ADC = 120°$

(d) (i) ΔADB is isosceles because $\angle ADB = \angle ABD = 70°$

(ii) $\angle BCD = 140°$

$\angle ADB = 40°$

27 Neither is correct. It is just a quadrilateral with a pair of opposite angles each equal to 90°.

$\angle KJM = \angle KLM = 90°$ (angle in a semicircle)

Nothing is known about the lengths of the sides. To be a kite, you would need $JM = LM$. To be a square, you would need $JM = LM = LK = KJ$.

Exercise 8.7 (p. 377)

1 (a) In Δs WOX and YOZ, $WX = YZ$ (given)

$WO = YO$ (radii)

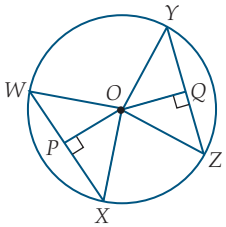
$OX = OZ$ (radii)

$\therefore \Delta WOX \equiv \Delta YOZ$ (SSS)

$\therefore \angle WOX = \angle YOZ$ (matching angles in congruent triangles)

(b) $\angle WXO = \angle YZO$ (matching angles in congruent triangles)

In Δs OXP and OZQ



$\angle OPX = \angle OQZ = 90°$ (given)

$\angle PXO = \angle QZO$ (from above)

$OX = OZ$ (radii)

$\therefore \Delta OXP \equiv \Delta OZQ$ (AAS)

$\therefore OP = OQ$ (matching sides in congruent triangles)

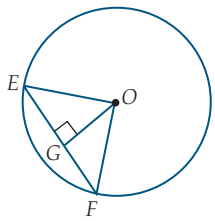
2 In Δs OGE and OGF, $\angle OGE = \angle OGF = 90°$ (given)

$OE = OF$ (radii)

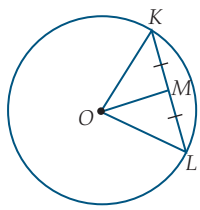
OG is a common side

$\therefore \Delta OGE \equiv \Delta OGF$ (RHS)

$\therefore EG = GF$ (matching sides in congruent triangles)



3 In Δs OMK and OML, $KM = ML$ (given)



$OK = OL$ (radii)

OM is a common side

$\therefore \Delta OKM \equiv \Delta OLM$ (SSS)

$\therefore \angle OMK = \angle OML$ (matching angles in congruent triangles)

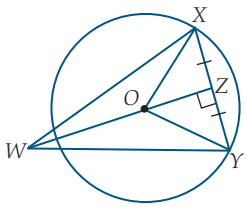
But $\angle OMK + \angle OML = 180°$, so $\angle OMK = \angle OML = 90°$. Hence, $OM \perp KL$.

4 $WX = WY$ as all points on the perpendicular bisector of the interval XY are equidistant from the end points of the interval.

$OX = OY$ (radii of the circle centre O on which X and Y lie)

Because O is equidistant from X and Y, it must lie on the perpendicular bisector of XY.

$\therefore O$ lies on WZ.



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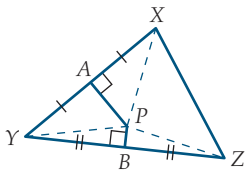


Join PX, PY and PZ.

$AX = AY$ (given)

$\angle PAX = \angle PAY$ (both equal 90°)

PA is common

$\Delta XPA \equiv \Delta YPA$ (SAS)

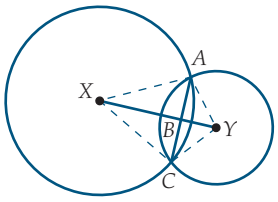
$XP = YP$

Similarly, $\Delta YPB \equiv \Delta ZPB$ (SAS)

$YP = ZP$

$\therefore P$ is the centre of a circle through X, Y and Z, B ($XP = YP = ZP =$ radii)

6



Join AX, AY, CX and CY.

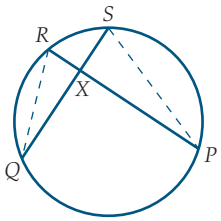
$AX = CX$ (radii of circle with centre X)

$AY = CY$ (radii of circle with centre Y)

$\therefore$ Quadrilateral $AXCY$ is a kite.

$AC \perp XY$ and $AB = CB$ (diagonal joining the non-equal angles of a kite bisect the other diagonal at right angles)

7



Join RQ and SP.

$\angle RQX = \angle SPX$ and $\angle QRX = \angle PSX$ (angles subtended by the same arc at the circumference are equal)

$\Delta RQX \parallel\!| \Delta SPX$ (AAA)

$$\frac{RX}{QX} = \frac{SX}{PX}$$

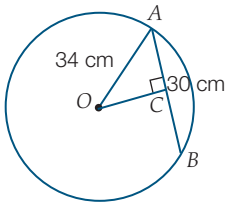
$RX \times PX = SX \times QX$

8 $WX = XY$ (given)

Hence, $x = 70$ (equal chords subtend equal angles at the centre)

Because $\angle WOX = \angle YOZ = 70°$, $y = 2$ (equal angles at the centre are subtended by equal chords)

9 Draw OC perpendicular to AB.



$AC = CB = 15$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

$34^2 = 15^2 + OC^2$

$OC^2 = 34^2 - 15^2 = 1156 - 225 = 931$

$OC = \sqrt{931} \approx 30.5$ cm

10 **(a)** $AB = 5$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $AB = 2 \times AC = 20$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(c) $AB = 15$ cm (Pythagoras' Theorem)

11 **(a)** $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

(b) $\angle AOB = 80°$ (angle sum of isosceles triangle is 180°)

(c) $\angle OFG = 50°$ (base angles of isosceles triangle), $\angle FOG = 80°$ (angle sum of triangle), $\angle AOB = 80°$ (equal chords subtend equal angles at the centre)

12 **(a)** $AE = EB = 24$ cm (E is midpoint of AB, given)

$\angle OEA = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$OA^2 = 7^2 + 24^2$ (Pythagoras' Theorem)

$OA = 25$ cm

(b) $OG^2 = 20^2 + 21^2 = 841$ (Pythagoras' Theorem), $OG = 29$, $OA = 29$ cm

(c) ΔOGF is equilateral; $OA = OG = 10$ cm

13 **(a)** $CE^2 = 53^2 - 28^2 = 2025$ (Pythagoras' Theorem), $CE = 45$, $CD = 2 \times 45 = 90$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $GH^2 = 73^2 - 48^2 = 3025$ (Pythagoras' Theorem), $GH = 55$, $GF = 2 \times 55 = 110$ cm (perpendicular from the centre of a circle to a chord, bisects the chord); $FD = GF = 110$ cm (given), $CD = FD = 110$ cm (equal angles at the centre are subtended by equal chords)

(c) $GH^2 = 85^2 - 77^2 = 1296$ (Pythagoras' Theorem), $GH = 36$, $GF = 2 \times 36 = 72$ cm (perpendicular from the centre of a circle to a chord, bisects the chord), $CD = GF = 72$ cm (equal angles at the centre are subtended by equal chords)

14 In Δs DOE and BDC, $DO = OB$ (radii)

$DE = EC$ (E is midpoint of CD)

$\therefore \dfrac{DO}{DB} = \dfrac{DE}{DC} = \dfrac{1}{2}$

$\angle ODE = \angle BDC$ (same angle)

$\Delta DOE \parallel\!| \Delta DCB$ (common angle with the sides containing it proportional)

$\angle DEO = \angle DCB$ (matching angles of similar triangles)

$\therefore OE \parallel BC$ (pair of corresponding angles are equal)

But $\angle DEO = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle BCD = 90°$

15 Join OD

$\angle OFC = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle OFC + \angle FCD = 180°$ ($\angle FCD = 90°$, given)

But these are a pair of supplementary co-interior angles

$\therefore OF \parallel DC$

$\angle ODC = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle ODC + \angle FCD = 180°$ ($\angle FCD = 90°$, given)

But these are a pair of supplementary co-interior angles)

$\therefore OD \parallel FC$

$\therefore OFCD$ is a parallelogram

$\therefore OF = DC$ (opposite sides of a parallelogram are equal)

But $BC = 2DC$ (D is midpoint of BC)

$\therefore BC = 2OF$

16 $\angle OKF = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\angle QJF = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle OKF + \angle QJF = 180°$

$\angle OKJ$ and $\angle QJK$ are a pair of supplementary co-interior angles contained between OK and QJ.

$\therefore OK \parallel QJ$

$\therefore \angle KOQ + \angle OQJ = 180°$ (co-interior angles, $OK \parallel QJ$)

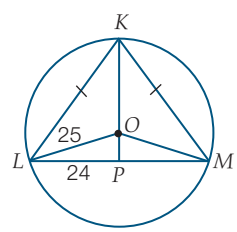
17 Let O be the centre of the circular arch.
By symmetry, the point C must be in the middle of AB, so $AC = 2$ m.

$OC^2 = 2.5^2 - 2^2$ (Pythagoras' Theorem)

$OC = 1.5$ m

$CH = OH - OC = 2.5 - 1.5 = 1$ m

18 Let O be the centre of the circle and P the midpoint of LM.



$\angle OPL = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$OP^2 = 25^2 - 24^2$ (Pythagoras' Theorem)

$OP = 7$ mm

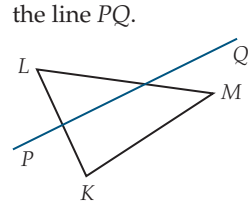
$KP = 25 + 7 = 32$ mm ($OK = 25$ mm, radius)

$KL^2 = 32^2 + 24^2$ (Pythagoras' Theorem)

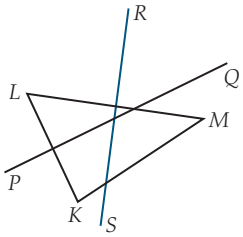
$KL = 40$ mm

19 Students' own answers.

20 The centres of all the circles passing through K and L lie on the line PQ.

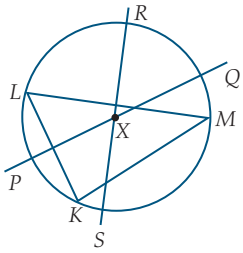


The centres of all the circles passing through L and M lie on the line RS.

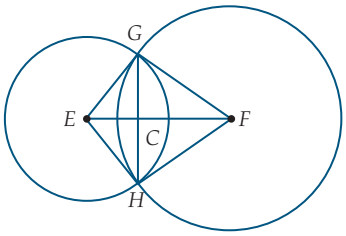


The point of intersection of PQ and RS must be the centre of the only circle that passes through K, L and M.

Label the point of intersection of PQ and RS, X, and draw the circle centre X and radius XL. It will pass through all the points K, L and M.



21 **(i)**



In Δs EGF and EFH,
$EG = EH$ (radii of circle centre E)
$GF = HF$ (radii of circle centre F)

EF is a common side $\therefore \Delta EGF \equiv \Delta EFH$ (SSS)

$\angle GEF = \angle HEF$ (corresponding angles in congruent triangles)

In Δs EGC and EHC, $EG = EH$ (radii of circle centre E)

$\angle GEC = \angle HEC$ (corresponding angles in congruent triangles)

EC is a common side.

$\therefore \Delta EGC = \Delta EHC$ (SAS)

$GC = CH$ (corresponding sides in congruent triangles)

$\therefore EF$ biscets GH

(i.e. the line of centres bisects the common chord).

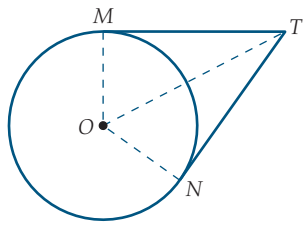
(ii) CE is the line joining the centre of the chord GH to the centre E of the circle.

$\therefore CE \perp GH$

(i.e. the line of centres is perpendicular to the common chord).

Exercise 8.8 (p. 385)

1



Join OAM, ON and OT where O is the centre of the circle.
$\angle OMT = \angle ONT = 90°$ (angle between tangent and chord = 90°)

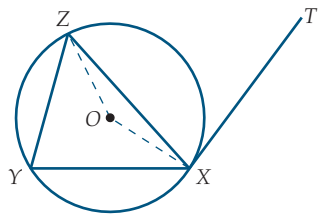
OT is common

$OM = ON$ (both radii of circle)

$\Delta OMT \equiv \Delta ONT$ (RHS congruency)

$MT = NT$

2



Join OA, OC where O is the centre of the circle.

$\angle XOZ + \angle OXZ + \angle OZX = 180°$ (angles in a triangle = 180°)

$\angle OZX = \angle OXZ$ (base angles of isosceles triangle are equal)

$\angle XOZ + 2\angle OXZ = 180°$

$\angle XOZ = 2\angle XYZ$ (angle at centre = 2 × angle at circumference)

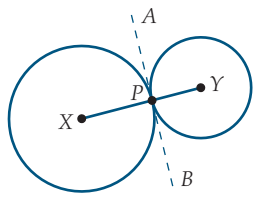
$2\angle XYZ + 2\angle OXZ = 180°$

$\angle XYZ + \angle OXZ = 90°$

$\angle TXZ + \angle OXZ = 90°$ (angle between radius and tangent = 90°)

$\angle XYZ = \angle TXZ$

3



Draw APB a tangent to both circles at point of contact P.

Join XP and YP.

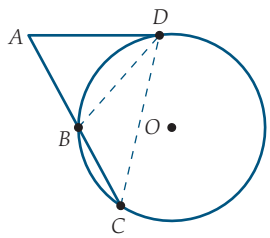
$\angle XPA = 90°$ (angle between radius and tangent = 90°)

$\angle YPA = 90°$ (angle between radius and tangent = 90°)

$\angle XPA + \angle YPA = 180°$

XPA is a straight line (angles on a straight line are supplementary)

4



Join BD and CD.

In ΔADB and ΔACD:

$\angle ADB = \angle ACD$ (angle between tangent and chord = angle in alternate segment)

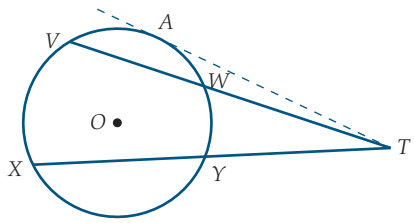
$\angle CAD$ is common to ΔADB and ΔACD

$\Delta ADB \parallel\!| \Delta ACD$ (AAA)

$\frac{AD}{AB} = \frac{AC}{AD}$

$\therefore AD^2 = AB \times AC$

5



Draw tangent TA.

$TA^2 = TW \times TV$ (the square on the tangent is equal to the product of the intercepts of a secant)

and

$TA^2 = TY \times TX$ (the square on the tangent is equal to the product of the intercepts of a secant)

$\therefore TW \times TV = TY \times TX$ (both equal to TA^2)

6 $\angle ABC = 60°$ (angle in alternate segment = angle between tangent and chord)

$BAC = 60°$ (base angles of isosceles triangle are equal)

ΔABC is equilateral (all angles equal 60°)

7 B

8 $YZ = 5$ m

9 $\angle ADC = 74°$

10 70°

11 $XY = 7$ cm

12 $MP = MX$ (both radii)

$PY = XY$ (tangents from an external point to a circle are equal)

$PMXY$ is a kite.

$\angle MYX = \angle MYP$ (diagonal of a kite joining non-equal angles bisects these angles)

Similarly, $\angle NYZ = \angle NYP$

$\angle XYP + \angle ZYP = 180°$ (180° on a straight line)

$2\angle MYP + 2\angle NYP = 180°$

$\angle MYP + \angle NYP = 90°$

$\angle MYN = 90°$

13 $DE = 4$ m

14 $AF^2 = AB \times AE$

$16 = 2 \times AE$

$AE = 8$

$BE = AE - AB$

$= 8 - 2$

$= 6$ cm

$DE = DB - EB$

$= 10 - 6$

$= 4$

$DB \times DE = DC \times DF$

$10 \times 4 = 5 \times DF$

$DF = 8$

$FC = FD - CD$

$= 8 - 5$

$= 3$ cm

$CF = \frac{1}{2} BE$

15 $TB = 9$ cm

16 Students' own answers, e.g.

25 cm, 9 cm

80 cm, 75 cm

100 cm, 96 cm

Chapter review 8 (p. 388)

1 (a) SSS (b) SAS

2 B

3 (a) 5 (b) 9

4 (a) In ΔAOB and ΔDOC

$\angle AOB = \angle DOC$ (vertically opposite angles)

$AO = DO$ (radii in the same circle)

$BO = CO$ (radii in the same circle)

$\therefore \Delta AOB = \Delta DOC$ (SAS)

So $AB = DC$ (corresponding sides in congruent triangles)

$\therefore p = 6$ m

(b) In ΔDEH and ΔGEF

$\angle DEH = \angle GEF$ (vertically opposite angles)

$DH = GF$ (given)

$\angle EDH = \angle EGF$ (alternate angles, $DH \parallel GF$)

$\therefore \Delta DEH \equiv \Delta GEF$ (AAS)

So $HE = FE$ (matching sides in congruent triangles)

$\therefore r = 9$ cm

5 (a) In ΔABC and ΔADE,

$\angle CAB = \angle EAB$ (common angle)

$\angle ABC = \angle ADE$ (given)

$\therefore \angle ACB = \angle AED$ (third angle in a triangle)

$\therefore \Delta ABC \parallel\!| \Delta ADE$ (AAA)

(b) In ΔABC and ΔADE,

$\angle BAC = \angle DAE$ (vertically opposite angles)

$\angle BCA = \angle AED$ (alternate angles and $BC \parallel DE$)

$\angle CBA = \angle ADE$ (alternate angles and $BC \parallel DE$)

$\therefore \Delta ABC \parallel\!| \Delta ADE$ (AAA)

6 In ΔAXC and ΔBXC,

$\angle AXC = \angle BXC = 90°$ (given that $CX \perp AB$)

$\angle ACX = \angle BCX = \frac{1}{2} \times \angle ACB$ (given that CX bisects $\angle ACB$)

Now $CX = CX$ (common side in ΔAXC and ΔBXC)

$\therefore \Delta AXC \equiv \Delta BXC$ and so $AX = BX$ (matching sides in congruent triangles)

7 In Δs DGH and DEF

$\angle GDH = \angle EDF$ (common angle)

$\angle DHG = \angle DFE$ (corresponding angles equal, $GH \parallel EF$)

$\angle DGH = \angle DEF$ (corresponding angles equal, $GH \parallel EF$)

$\therefore \Delta DGH \equiv \Delta DEF$ (AAA)

$$\frac{DE}{DG} = \frac{DF}{DH}$$

$$\frac{DG + GE}{DG} = \frac{DH + HF}{DH}$$

$$\frac{DG}{DG} + \frac{GE}{DG} = \frac{DH}{DH} + \frac{HF}{DG}$$

$$1 + \frac{GE}{DG} = 1 + \frac{HF}{DH}$$

$$\therefore \frac{GE}{DG} = \frac{HF}{DH}$$

8 In ΔABC and ΔCDA

$AB = CD$ (given)

$BC = DA$ (given)

$AC = CA$ (common side)

$\therefore \Delta ABC \equiv \Delta CDA$ (SSS)

So $\angle BAC = \angle DCA$ (matching angles in congruent triangles)

But $\angle BAC = \angle DCA$ are alternate angles,

$\therefore AB \parallel CD$

And $\angle BCA = \angle DAC$ (matching angles in congruent triangles)

But $\angle BCA = \angle DAC$ are alternate angles $\therefore AD \parallel CB$

The quadrilateral is a parallelogram because both pairs of opposite sides are parallel.

Have proved that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

9 In ΔPTQ, ΔRTQ, ΔRTS and ΔPTS

$PT = RT$ (given)

$ST = QT$ (given)

$\angle PTQ = \angle RTQ = \angle RTS = \angle PTS = 90°$ (given)

$\therefore \Delta PTQ \equiv \Delta RTQ \equiv \Delta RTS \equiv \Delta PTS$ (SAS)

So $PQ = RQ$ (corresponding sides in congruent triangles)

ΔPTQ is isosceles so $\angle TPQ = \angle TQP$

$\angle TPQ + \angle TQP + \angle PTQ = 180°$

$2\angle TQP + 90° = 180°$

$2\angle TQP = 90°$

$\angle TQP = 45°$

But $\angle TQP = \angle TQR = \angle TRQ = \angle TRS = \angle TSR = \angle TSP = \angle TPS = \angle TPQ$ (corresponding angles in congruent triangles)

So $\angle TQR = 45°$

$\angle PQR = \angle TQR + \angle TQP = 90°$

$\angle QRS = \angle TRQ + \angle TRS = 90°$

$\angle SPQ = \angle TPQ + \angle TPS = 90°$

$\angle PQR$ and $\angle QRS$ are co-interior angles and they add to 180°.

$\therefore PQ \parallel SR$

$\angle PQR$ and $\angle SPQ$ are co-interior angles and they add to 180°.

$\therefore PS \parallel QR$

Have shown $PQ = RQ$, $\angle PQR = 90°$, $PQ \parallel SR$ and $PS \parallel QR$.

$PQRS$ is a square as it has two adjacent sides, one right angle and both pairs of opposite sides parallel.

Have proved that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then quadrilateral is a square.

10 (a) $y = 59°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $x = 125°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $x = 38°$ (vertically opposite angles, angle sum of triangle)

(d) $x = 83°$ (angles in the same segment, angle sum of triangle)

(e) $k = 39°$ (angles in the same segment, exterior angle of a triangle)

(f) $x = 33°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$y = 90 - 36 = 54°$ (angle in a semicircle, isosceles triangle)

11 (a) $AB = 8$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $AB = 16$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(c) $AE = 40$ mm (line joining midpoint of chord to centre of circle $\perp$ chord, Pythagoras' Theorem), $AB = 80$ mm

12 (a) $\angle AOB = 65°$ (equal chords subtend equal angles at the centre)

(b) $\angle OFG = 55°$ (base angles of isosceles triangle), $\angle GOF = 70°$ (angle sum of triangle), $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

(c) $\angle GOH = 35°$ (perpendicular bisector of chord bisects angle at centre), $\angle GOF = 70°$, $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

13 **(a)** $\angle OXP = 90°$ (angle between radius and tangent)

$\angle OYP = 90°$ (angle between radius and tangent)

$\angle OXP + \angle OYP = 180°$

$PXOY$ is a cyclic quadrilateral (opposite angles of a cyclic quadrilateral are supplementary)

(b) $x + 140° = 180°$ (opposite angles of a cyclic quadrilateral are supplementary)

$x = 40°$

14 12.6 m

15 **(a)** In ΔBAD and ΔBCD

$\angle BDA = \angle BDC = 90°$ (by construction)

$BA = BC$ (given)

$BD = BD$ (common side)

$\therefore \Delta BAD \equiv \Delta BDC$ (RHS)

So $\angle BAD = \angle BCD$ (matching angles in congruent triangles)

$\therefore \angle BAC = \angle BCA$

(b) Have proved that the angles opposite the equal sides in an isosceles triangle are also equal.

16 In ΔABZ and ΔACX,

$AB = AC$ (given)

$AZ = AX$ (given)

$\angle BAZ = \angle CAX$ (common angle)

$\therefore \Delta ABZ \equiv \Delta ACX$ (SAS) (and so $BZ = CX$)

Now consider ΔBYC

$\angle YBC = \angle YCB$ (given)

$BY = CY$ (If two angles in a triangle are equal then the sides opposite these angles are equal—the triangle is isosceles.)

Consider ΔBYX and ΔCYZ

$\angle BYX = \angle CYZ$ (vertically opposite angles)

$BX = CZ$ (as $AB = AC$ and $AZ = AX$)

$XY = ZY$ (as $BY = CY$ and $BZ = CX$)

$\therefore \Delta BXY \equiv \Delta CZY$ (SSS)

17 **(a)** Let $\angle DFE = x$

$\therefore \angle GEF = (90 - x)$ (angle sum of ΔGEF is 180°)

$\therefore \angle GED = x$ ($\angle DEF = 90°$)

In Δs DEF and DEG,

$\angle DFE = \angle DEG = x$

$\angle DEF = \angle DGE = 90°$

$\therefore \Delta DEF \parallel\!| \Delta DGE$ (AAA)

$\frac{EF}{EG} = \frac{DF}{DE} = \frac{DE}{DG}$ (1) (matching sides of similar triangles are proportional)

In Δs DEF and EGF,

$\angle DFE = \angle GFE = x$

$\angle DEF = \angle FGE = 90°$

$\therefore \Delta DEF \parallel\!| \Delta EGF$ (AAA)

(b) $\frac{EF}{GF} = \frac{DF}{EF} = \frac{DE}{EG}$ (2) (matching sides of similar triangles are proportional)

(1) gives $EF = \frac{EG \times DE}{DG}$

(2) gives $EF = \frac{GF \times DE}{EG}$

$\therefore \frac{EG \times DE}{DG} = \frac{GF \times DE}{EG}$

$\therefore EG^2 = DG \times GF$

18 **(a)** In ΔEAB and ΔECD

$\angle ABE = \angle CDE$ (alternate angles, $AB \parallel CD$)

$\angle AEB = \angle CED$ (vertically opposite)

$AB = CD$ (given)

$\therefore \Delta EAB = \Delta ECD$ (by ASA)

$AE = CE$ (matching sides in congruent triangles)

In ΔEAB and ΔECB

$AE = CE$ (found earlier)

$AB = CB$ (given)

$BE = BE$ (common side)

$\therefore \Delta EAB = \Delta ECB$ (by SSS)

$\therefore \angle AEB = \angle CEB$ (matching angles in congruent triangles)

But $\angle AEB + \angle CEB = 180°$ (angles on a straight line, supplementary)

$\therefore \angle AEB = 90°$ and $\angle CEB = 90°$

$\therefore AC \perp BD$

(b) Have just proved that the diagonals of a rhombus intersect at right angles.

19 **(a)** $x = 60°$, $y = 50°$ ($\angle$ at centre, $\angle$s in same segment)

(b) $x = 100°$, $y = 80°$ (co-interior angles, opposite angles in cyclic quadrilateral)

(c) $x = 73°$, $y = 13°$, $z = 107°$ (straight angle, opposite angles in cyclic quadrilateral)

20 $\angle TSR + \angle TSU = 180°$ (angles on a straight line)

$\angle TUS + \angle TUV = 180°$ (angles on a straight line)

$\therefore \angle TSR + \angle TSU + \angle TUS + \angle TUV = 180° + 180° = 360°$

$\angle TSU + \angle TUS + \angle UTS = 180°$ (angles in a triangle)

$\angle TSU + \angle TUS + 90° = 180°$

$\angle TSU + \angle TUS = 90°$

$\angle TSR + \angle TSU + \angle TUS + \angle TUV = 360°$ (found earlier)

$\angle TSR + 90° + \angle TUV = 360°$

$\therefore \angle TSR + \angle TUV = 270°$

21 $a + b = 180°$ (angles on a straight line)

$c + d = 180°$ (angles on a straight line)

$\therefore a + b + c + d = 180° + 180° = 360°$

$\angle GWD = \angle AWB = a$ (vertically opposite angles)

$\angle FXA = \angle DXC = d$ (vertically opposite angles)

$\angle BZE = \angle BWD = b$ (corresponding angles, $AD \parallel HE$)

$\angle CYH = \angle CXA = c$ (corresponding angles, $AD \parallel HE$)

$\angle GWD + \angle BZE + \angle FXA + \angle CYH$

$= a + b + c + d$

$= 360°$

So the interior angles of a trapezium add to 360°.

22 To show: $\Delta LYZ \parallel\!| \Delta MXZ$

$\angle LZY = \angle MZX$	(vertically opposite)
$\angle LYZ = \angle MXZ$	(both 90°)
$\angle ZMX = \angle ZLY$	(third angle in a triangle)
$\Delta LYZ \parallel\!| \Delta MXZ$	(AAA)

23 In ΔUXY and ΔVWX

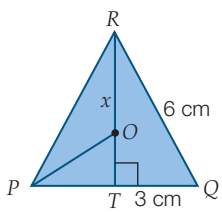
$YX = XW$ (given)

$UY = VX$ (given)

$\angle UYX = \angle VXW$ (corresponding angles, $UY \parallel VX$)

$\therefore \Delta UXY = \Delta VWX$ (SAS)

24 $RT = \sqrt{36 - 9} = \sqrt{27} = 3\sqrt{3}$ (Pythagoras' Theorem in ΔRTQ)



Let radius = x. Hence $OP = x$.

Use Pythagoras' Theorem in ΔPOT:

$x^2 = 3^2 + (\sqrt{27} - x)^2$

$x^2 = 9 + 27 - 2\sqrt{27}x + x^2$

$2\sqrt{27}x = 36$

$6\sqrt{3}x = 36$

$x = \dfrac{36}{6\sqrt{3}} = 2\sqrt{3}$

25 $\angle TWU = 90°$ ($SU \perp VT$)

$\angle WUT = 43°$ (angle sum of triangle)

$\angle SVT = \angle SUT = 43°$ (angles in the same segment)

$\angle SOT = 86°$ (angle at the centre twice the angle at the circumference standing on the same arc)

26 $TO = 15$ cm

$TX = 12$ cm (Pythagoras' Theorem)

$TX^2 = TZ \times TW$

$144 = 9 \times TW$

$TW = 16$ cm

$WZ = 7$ cm

Chapter 9

Recall 9 (p. 393)

1 **(a)** **(i)** 4 **(ii)** 5 **(iii)** 10

(b) **(i)** 10 **(ii)** 18 **(iii)** 0

2 **(a)** $x(x - 7)$ **(b)** $2x(x + 5)$ **(c)** $(x + 2)(x - 2)$

3 **(a)** $3x^2 + 12x$ **(b)** $x^2 + 7x + 10$ **(c)** $x^2 - 4x - 21$

(d) $x^2 + 16x + 64$ **(e)** $x^2 - 4x + 3$ **(f)** $x^2 - 16$

4 **(a)** $(x + 5)(x - 5)$ **(b)** $(3x - 4)(3x + 4)$

(c) $(x + \sqrt{5})(x - \sqrt{5})$ **(d)** $(x - 4)^2$

(e) $(x + 7)^2$ **(f)** $3(x - 11)^2$

5 **(a)** $(x + 5)(x + 2)$ **(b)** $(x - 3)(x - 2)$ **(c)** $(x - 14)(x + 1)$

(d) $(x + 8)(x - 3)$ **(e)** $(x + 9)(x - 4)$ **(f)** $(x + 1)(x + 6)$

(g) $-(x + 2)(x + 1)$ **(h)** $-(x - 5)(x - 2)$ **(i)** $-2(x - 4)(x + 3)$

6 **(a)** 27 **(b)** $\frac{1}{4}$ **(c)** $3\frac{1}{25}$

7 **(a)** $x = -3, 1$ **(b)** $x = 1$ **(c)** $x = 0, 5$

8 **(a)** $x = 3$ **(b)** $x = -4$ **(c)** $x = \pm\frac{6}{11}$

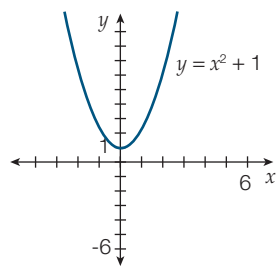
9 **(a)** $(x + 2 - \sqrt{2})(x + 2 + \sqrt{2})$

(b) $(x - 3 + \sqrt{3})(x - 3 - \sqrt{3})$

(c) $(x - \frac{3}{2} + \frac{\sqrt{29}}{2})(x - \frac{3}{2} - \frac{\sqrt{29}}{2})$

Exercise 9.1 (p. 397)

1 **(a)**



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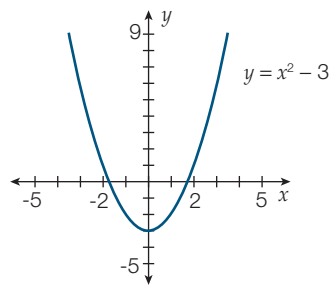

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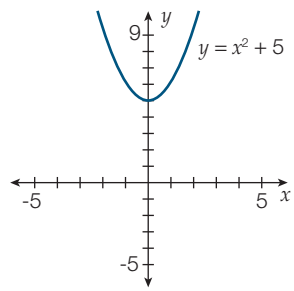

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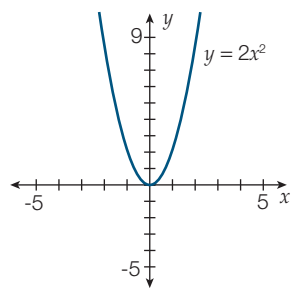

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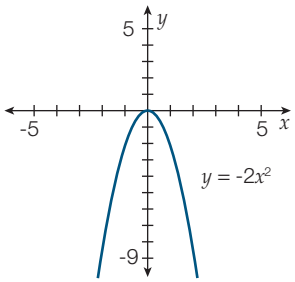

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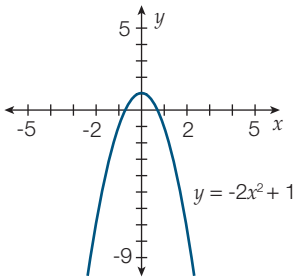

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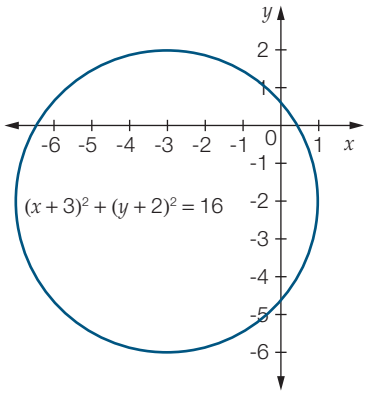

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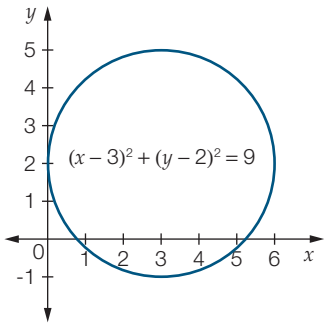

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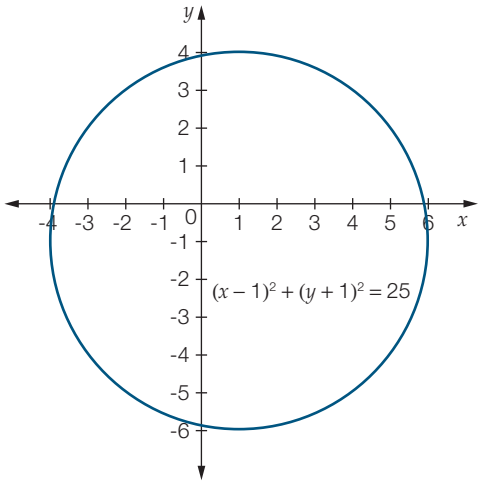

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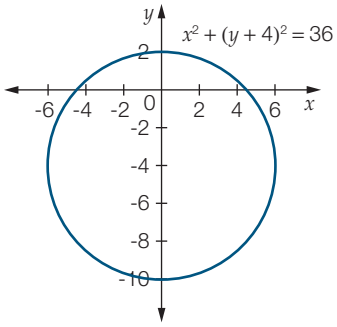

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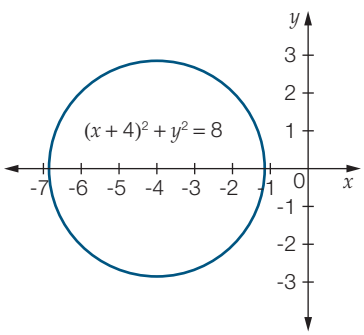

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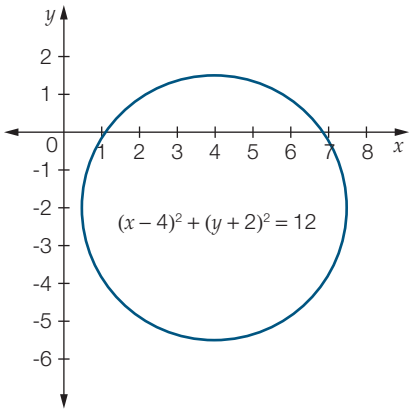
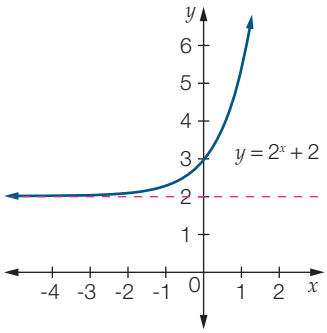
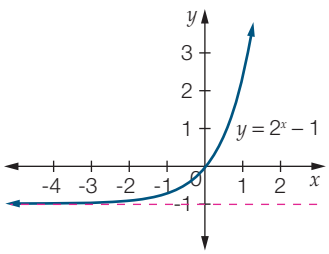


3 (a)



(b)



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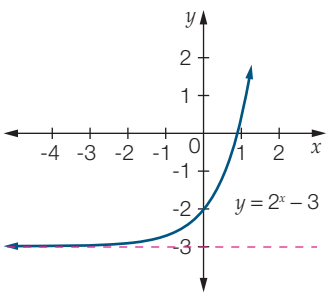

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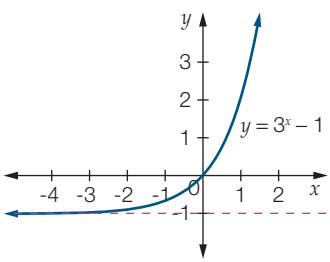

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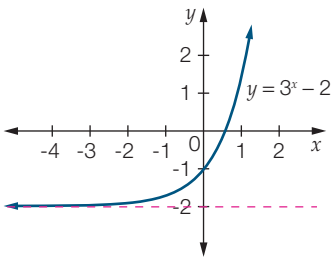

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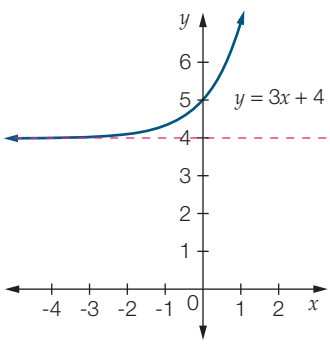
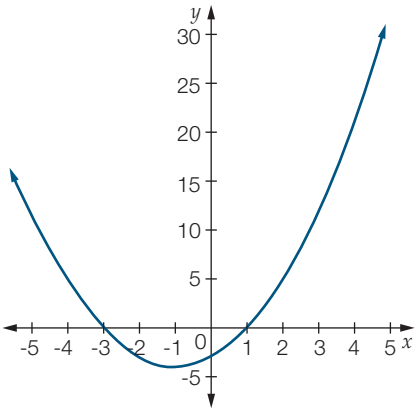


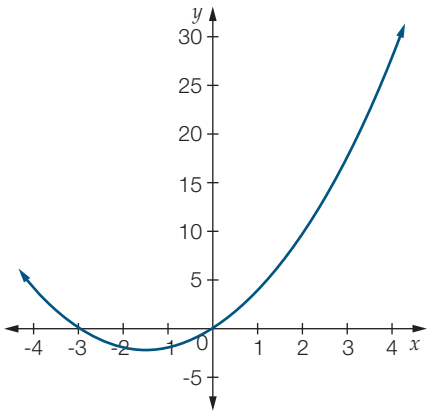
4 (a) (i)

x	-4	-3	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5	12	21



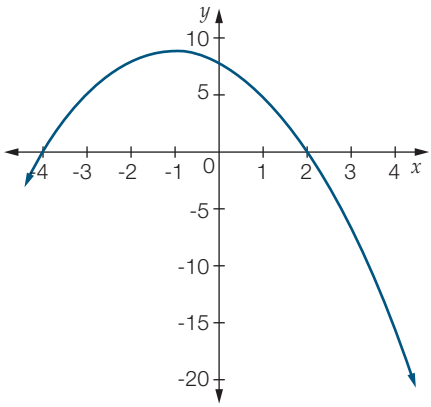
(ii)

x	-4	-3	-2	-1	0	1	2	3	4
y	4	0	-2	-2	0	4	10	18	28



(iii)

x	-4	-3	-2	-1	0	1	2	3	4
y	0	5	8	9	8	5	0	-7	-16

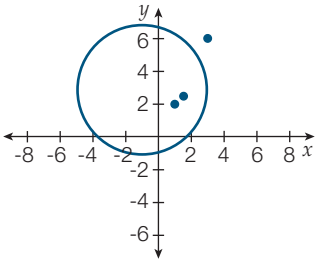


(b) (i) (-1, 4), minimum (ii) (-1.5, -2.25), minimum

(iii) (-1, 9), maximum

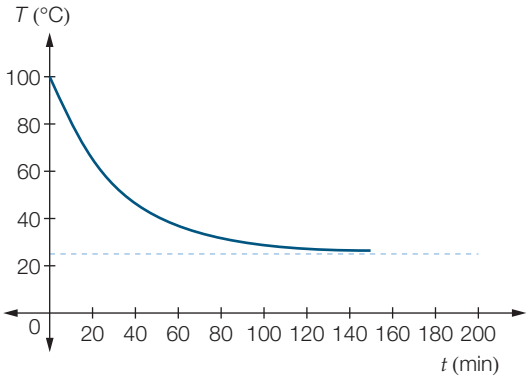
5 C

6 (a)



(b) (1, 2) and (1.5, 2.5) are inside and (3, 6) is outside.

7 (a)



(b) 100°C

(c) (i) 79°C (ii) 65°C (iii) 46°C (iv) 40°C

(d) The first 10 minutes. The coffee lost 21°C in the first 10 minutes, which is the greatest loss in any 10-minute period. This is the steepest part of the graph.

(e) After 107 minutes.

(f) After 3 hours the temperature of the coffee will be approximately 25°.

8 (a) $V = \pi r^2 h$

$= \pi r^2\left(\frac{3}{\pi}\right)$

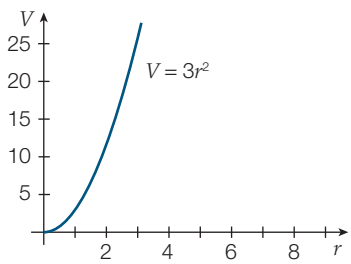
$= 3r^2$

(b) It is a quadratic because the highest power of r is 2.

(c)

r	0	1	2	3	4	5
r^2	0	1	4	9	16	25
V	0	3	12	27	48	75

(d)



(e) (i) 18.8 m^3 (ii) 43.3 m^3

(f) (i) 3.2 m (ii) 4.7 m

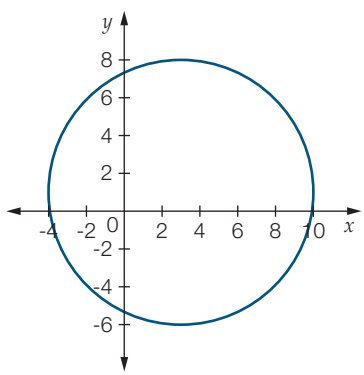
(g) 30 m^3

9 (a) (i) $(x-3)^2 + (y-1)^2 = 49$

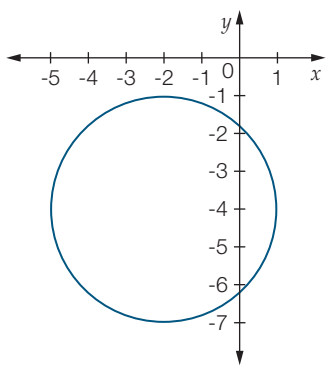
(ii) $(x+2)^2 + (y+4)^2 = 9$

(iii) $(x-2)^2 + (y+5)^2 = 4$

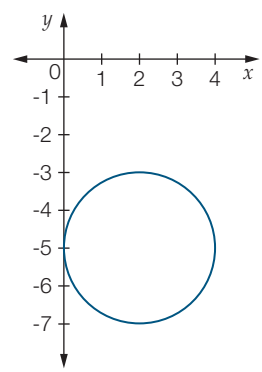
(b) (i)



(ii)



(iii)

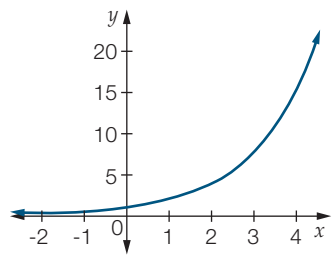


10 D

11 (a)

x	-2	-1	0	1	2	3	4
y	0.25	0.5	1	2	4	8	16

(b)



(c) All of these answers are approximate.

(i) 2.8 (ii) 1.7 (iii) 0.7

(iv) 0.4 (v) 1.4 (vi) 2.8

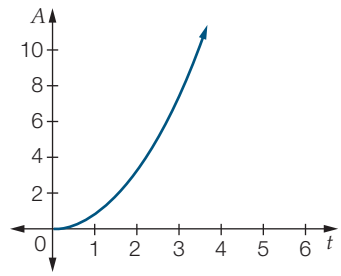
(d) Students are to check **(c)** using a calculator.

(e) All of these answers are approximate.

(i) 2.3 (ii) 3.6 (iii) 3.9

12 (a) $A = l(l - 0.4)$ m^2

(b)



(c) 2 m × 1.6 m

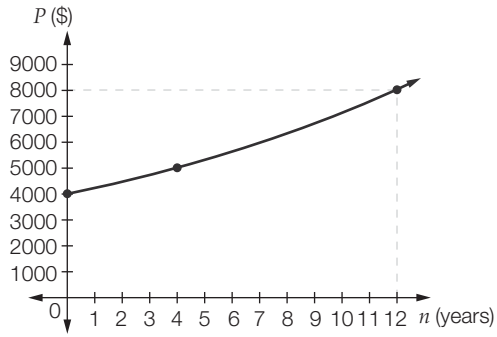
13 (a) 4240

(b)

n	0	1	2	3	4	5
P	4000	4240	4492	4764	5050	5353

(c) $P = 4000(1.06)^n$

(d)



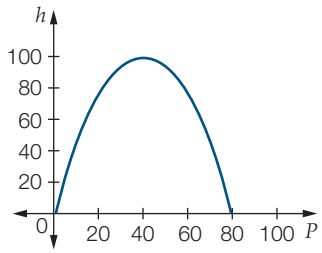
(e) 12 years

(f) $P = 4000(1.06)^{12}$

$= 8048.78$

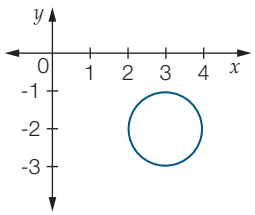
(g) Future time and population can only be positive numbers.

14 (a)



(b) 100 m (c) 80 m (d) 20 106.19 m^2

15 $(x-3)^2 + (y+2)^2 = 1$



The point (2.4, -2.7) is inside the circle, so a tree could not be planted there. If we substitute (2.4, -2.7) into the LHS of the equation we get 0.85. Any value less than 1 is inside the circle, exactly 1 is on the perimeter, and greater than 1 is outside the circle.

16 (a) Students to provide their own equation, but it must have at least $(x \pm h)^2$ and/or $(y \pm k)^2$ in it.

(b) For a maximum turning point the coefficient of x^2 must be negative.

(c) The equation will need to have -1 at the end.

17 (a) Students to provide their own sketch.

(b) The graph should start at the origin, be symmetrical in shape with a maximum turning point. This means the turning point is midway between the two x-intercepts.

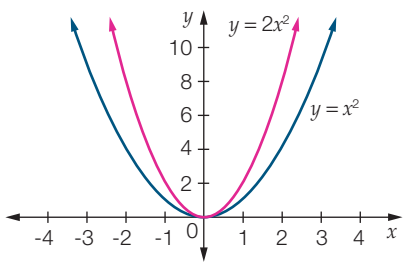
18 $(x+3)^2 + (y+1)^2 = 25$

$(x-4)^2 + (y+2)^2 = 25$

Exercise 9.2 (p. 407)

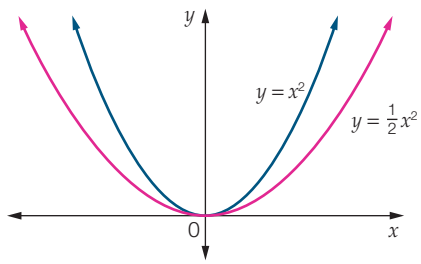
1 (a) (i) 2 (ii) narrower

(iii)

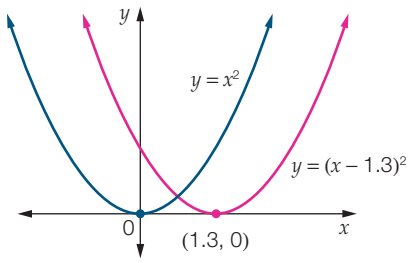


(b) (i) 0.5 or $\frac{1}{2}$ (ii) wider

(iii)

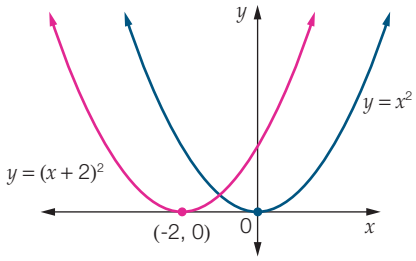


2 (a) (i)



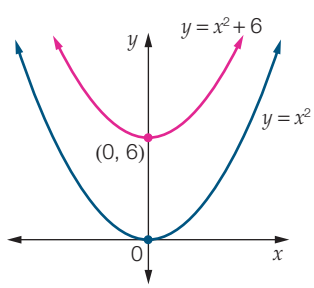
(ii) 1.3 units to the right

(b) (i)



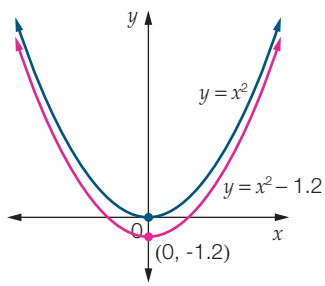
(ii) 2 units to the left

3 (a) (i)



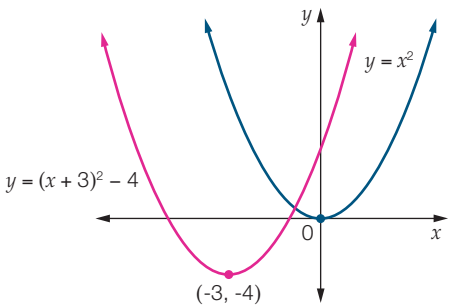
(ii) 6 units up

(b) (i)

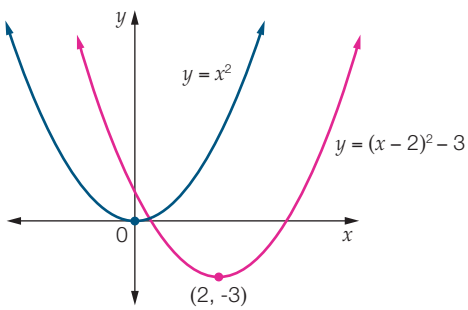


(ii) 1.2 units down

4 (a)



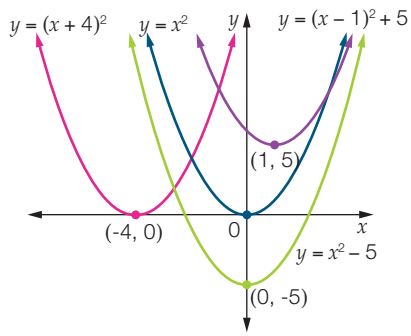
(b)



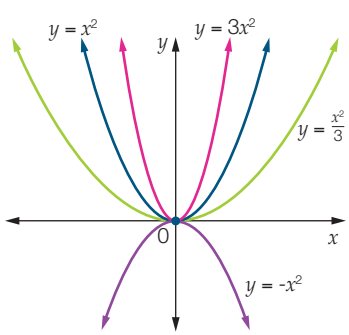
5 $y = (x - 2)^2 - 5$

6 B

7 (a)–(d)

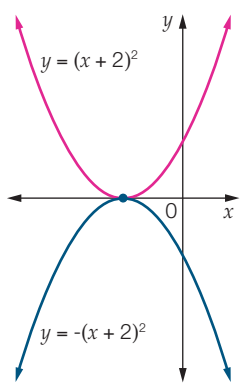


8 (a)–(d)

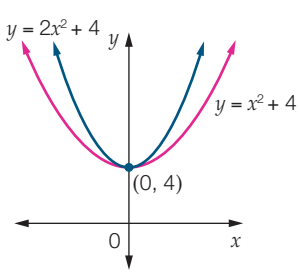


9 D

10 (a) and (b)

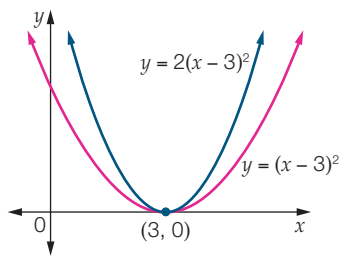


11 (a) and (b)



(c) $y = 2x^2 + 4$

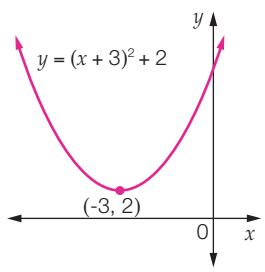
12 (a) and (b)



(c) $y = 2(x - 3)^2$

13 (a) $y = (x + 3)^2 + 2$

(b)



14 $y = (x + 2)^2 + 0.5$

15 (0, 0), (1, 5), (2, 8), (3, 9), (4, 8), (5, 5), (6, 0)

16 (a) The equation will be in the form $y = x^2 + k$.

(b) The equation will be in the form $y = (x - h)^2$.

17 When dealing with brackets, you are looking for the value of x that will make the bracket equal zero. So, $(x + 6)$ requires x to equal -6 to make the bracket 0. This means the turning point will be in line with $x = -6$. Syd is correct in his thinking about the vertical shift.

Exercise 9.3 (p. 412)

1 (a) $x = 0$ or $x = 11$ (b) $x = 0$ or $x = -6$
(c) $x = 0$ or $x = -4$ (d) $x = 0$ or $x = 1$
(e) $x = 0$ or $x = 1$ (f) $x = 0$ or $x = 6$
(g) $x = 3$ or $x = -7$ (h) $x = 4$ or $x = -1$
(i) $x = 2$ or $x = 6$ (j) $x = 4$ or $x = 5$
(k) $x = 7$ or $x = -2$ (l) $x = 3$ or $x = -10$

2 (a) $x = 0$ (b) $x = 0$ (c) $x = 0$
(d) $x = -7$ (e) $x = -1$ (f) $x = 9$

3 (a) $x = 0$ or $x = 4$ (b) $x = 0$ or $x = -1$
(c) $x = 0$ or $x = 6$ (d) $x = 0$ or $x = 5$
(e) $x = 0$ or $x = \frac{1}{2}$ (f) $x = 0$ or $x = -3$

4 (a) $x = -1$ or $x = -2$ (b) $x = -3$ or $x = -6$
(c) $x = 4$ or $x = 7$ (d) $x = 1$ or $x = 9$
(e) $x = -5$ or $x = 6$ (f) $x = -4$ or $x = 2$
(g) $x = 4$ or $x = 8$ (h) $x = -5$ or $x = 3$
(i) $x = -2$ or $x = 8$ (j) $x = -6$ or $x = 2$
(k) $x = -1$ or $x = 2$ (l) $x = -1$ or $x = 3$

5 (a) $x = -4$ or $x = -3$ (b) $x = -6$ or $x = -2$

(c) $x = -7$ or $x = 9$ (d) $x = -8$ or $x = 3$

(e) $x = -3$ or $x = -2$ (f) $x = -3$ or $x = 5$

(g) $x = -10$ or $x = 5$ (h) $x = -3$ or $x = 9$

(i) $x = 6$ or $x = 11$ (j) $x = 1$ or $x = 5$

(k) $x = -8$ or $x = 2$ (l) $x = -7$ or $x = 3$

6 (a) D (b) A

7 (a) $x = -5$ or $x = 2$ (b) $x = -1$ or $x = 3$

(c) $x = -4$ or $x = 7$ (d) $x = -7$ or $x = -1$

(e) $x = -6$ or $x = -1$ (f) $x = -7$ or $x = 6$

8 (a) A (b) D (c) B (d) B (e) B

9 $t = 0$ seconds and $t = 50$ seconds

10 $x = -1$ and $x = 4$

11 (a) $x = -5$ and $x = 8$ (b) $x = 0$ and $x = -4$

(c) $x = 12$ and $x = -1$

12 (a) 5°C (b) $t = 2$ and $t = 8$

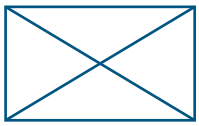
(c) 10 a.m. and 4 p.m.

13 (a) $(x + 4)(x + 6)$ m^2 (b) length 8 m, breadth 6 m

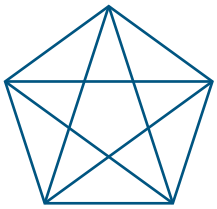
14 60 m

15 (a) $n - 3 = 0$ so $D = 0$

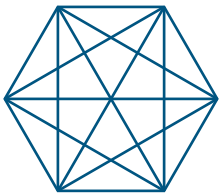
(b) quadrilateral: 2 diagonals



pentagon: 5 diagonals



hexagon: 9 diagonals



(c) (i) 20 (ii) 35 (iii) 54

(d) (i) 20 (ii) 26

(e) (i) 14 (ii) 42

16 Students are to produce their own equations; they could be of the form $(x + a)(x + b) = 0$.

17 Students are to produce their own equations; the easiest way to get the second equation is to make it the negative of the first.

18 Possible equations include: $x^2 + 2x - 2 = 0$; $6x^2 - 13x - 5 = 0$; $x^2 - x - 2 = 0$

Exercise 9.4 (p. 417)

1 (a) $x = 5$ or $x = -5$ (b) $x = 9$ or $x = -9$

(c) $x = 2$ or $x = -2$ (d) $x = \frac{5}{4}$ or $x = -\frac{5}{4}$

(e) $x = \frac{7}{3}$ or $x = -\frac{7}{3}$ (f) $x = \frac{9}{10}$ or $x = -\frac{9}{10}$

(g) $x = 5$ or $x = -5$ (h) $x = 1$ or $x = -1$

(i) $x = 2$ or $x = -2$ (j) $x = 7$ or $x = -7$

(k) $x = 9$ or $x = -9$ (l) $x = 11$ or $x = -11$

(m) $x = \sqrt{13}$ or $x = -\sqrt{13}$

(n) $x = \sqrt{24}$ or $x = -\sqrt{24}$ (or $x = 2\sqrt{6}$ or $x = -2\sqrt{6}$)

(o) $x = \sqrt{32}$ or $x = -\sqrt{32}$ (or $x = 4\sqrt{2}$ or $x = -4\sqrt{2}$)

2 (a) $x = 7$ (b) $x = -4$ (c) $x = -8$

(d) $x = 2$ (e) $x = -1$ (f) $x = 3$

(g) $x = 5$ (h) $x = 1$ (i) $x = -2$

(j) $x = 3$ (k) $x = 5$ (l) $x = 4$

3 (a) D (b) A

4 (a) $x = -4$ or $x = 4$ (b) $x = -3$ or $x = 3$ (c) $x = 0$ or $x = 12$

5 C

6 (a) $x = -12$ or $x = 12$ (b) $x = 6$

7 (a) $x = -1$, $x = 1$, $x = -3$, $x = 3$ (b) $x = -2$, $x = 2$, $x = -5$, $x = 5$

8 4 m by 2.4 m 9 154 m^2

10 Students are to produce their own equations; they could be of the form $(x + a)^2 = 0$.

Exercise 9.5 (p. 420)

1 (a) (i) $(x + 1 - \sqrt{6})(x + 1 + \sqrt{6})$ (ii) $x = -1 \pm \sqrt{6}$

(b) (i) $(x + 2 - \sqrt{11})(x + 2 + \sqrt{11})$ (ii) $x = -2 \pm \sqrt{11}$

(c) (i) $(x + 4 - \sqrt{11})(x + 4 + \sqrt{11})$ (ii) $x = -4 \pm \sqrt{11}$

(d) (i) $\left(x + \frac{3}{2} - \frac{\sqrt{13}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{13}}{2}\right)$ (ii) $x = \frac{-3 \pm \sqrt{13}}{2}$

(e) (i) $\left(x + \frac{5}{2} - \frac{\sqrt{13}}{2}\right)\left(x + \frac{5}{2} + \frac{\sqrt{13}}{2}\right)$ (ii) $x = \frac{-5 \pm \sqrt{13}}{2}$

(f) (i) $\left(x + \frac{3}{2} - \frac{\sqrt{29}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{29}}{2}\right)$ (ii) $x = \frac{-3 \pm \sqrt{29}}{2}$

2 (a) $x = 4 \pm \sqrt{10}$ (b) $x = -3 \pm \sqrt{5}$ (c) $x = \frac{3 \pm \sqrt{5}}{2}$

(d) $x = \frac{5 \pm \sqrt{33}}{2}$ (e) $x = -2 \pm \sqrt{7}$

(f) $x = 10 \pm \sqrt{112}$ (or $x = 10 \pm 4\sqrt{7}$)

(g) $x = 2 \pm \sqrt{10}$ (h) $x = -2 \pm \sqrt{3}$ (i) $-2 \pm \sqrt{7}$

3 (a) C (b) B (c) D

4 (a) $x = 2 \pm \sqrt{6}$ (b) $x = 3 \pm \sqrt{7}$

(c) $x = 2 \pm \sqrt{7}$ (d) $x = \frac{3 \pm \sqrt{29}}{2}$

(e) $x = \frac{5 \pm \sqrt{17}}{2}$ (f) $x = \frac{-7 \pm \sqrt{69}}{2}$

5 (a) $x^2 + 4x + 7$
$= x^2 + 4x + 4 + 3$
$= (x + 2)^2 + 3$

This cannot be factorised.

(b) You get $(x + 2)^2 + 3$, which is not the difference of two squares.

(c) The graph does not cut the x-axis.

6 (a) breadth = $-2 + 2\sqrt{21}$ and length = $2 + 2\sqrt{21}$

(b) breadth = 7.2 cm and length = 11.2 cm

7 $y = -2 + \sqrt{28} \approx 3.3$ cm

8 (a) $x = \frac{1 + \sqrt{801}}{2}$ m (b) $x = 14.65$ m

9 (a) length = $l + 4$, breadth = $l + 2$

(b) $(l + 4)(l + 2) = 215$

(c) $l = 6\sqrt{6} - 3$, $w = 6\sqrt{6} - 5$

(d) 15.7×13.7 cm

10 Students' own equations.

11 Students' own equations that do not result in the difference of two squares.

12 $x = -2 \pm \sqrt{3}$, $x^2 + 4x + 1 = 0$

Exercise 9.6 (p. 424)

1 (a) $x = -\frac{2}{3}$ or 2 (b) $x = -\frac{2}{3}$ or -2

(c) $x = 1$ or 1.2 (d) $x = -2$ or 1.25

2 (a) $x = 1$ or 2.4 (b) $x = -1\frac{2}{3}$ or -1

(c) $x = -0.5$ or $\frac{2}{3}$ (d) $x = -1.5$ or 0.75

3 (a) $x = 0.5$ or 2 (b) $x = \frac{-5 \pm \sqrt{22}}{3}$

(c) $x = \frac{1 \pm \sqrt{31}}{5}$ (d) $x = \frac{-9 \pm \sqrt{129}}{12}$

4 (a) (i) $x = \frac{5 \pm \sqrt{73}}{6}$ (ii) $x = -0.59$ or 2.26

(b) (i) $x = \frac{-5 \pm 2\sqrt{5}}{5}$ (ii) $x = -1.89$ or -0.11

(c) (i) $x = \frac{11 \pm \sqrt{73}}{4}$ (ii) $x = 0.61$ or 4.89

(d) (i) $x = \frac{-5 \pm \sqrt{37}}{2}$ (ii) $x = -5.54$ or 0.54

5 (a) (i) $a = 1, b = 5, c = 2$ (ii) $x = \frac{-5 \pm \sqrt{17}}{2}$

(b) (i) $a = 2, b = -3, c = 1$ (ii) $x = \frac{1}{2}, 1$

(c) (i) $a = -3, b = -1, c = 3$ (ii) $x = \frac{-1 \pm \sqrt{37}}{6}$

(d) (i) $a = 5, b = 4, c = 0$ (ii) $x = -0.8, 0$

6 C

7 (a) $x = \frac{2}{3}$ or 1 (b) $x = \frac{-5 \pm \sqrt{15}}{5}$

(c) $x = -1.5$ or 1.5 (d) $x = \frac{-7 \pm \sqrt{85}}{2}$

(e) $x = -\frac{5}{6}$ or 2 (f) $x = -0.5$

(g) $x = \pm\sqrt{\frac{6}{7}}$ (h) $x = \frac{-3 \pm \sqrt{41}}{8}$

(i) $x = \frac{5}{6}$ (j) $x = \frac{11 \pm \sqrt{37}}{6}$

8 B 9 C

10 $4t^2 - 31t + 60 = 0$, $t = 3\frac{3}{4}$ or 4. After $3\frac{3}{4}$ min and 4 min.

11 (a) after 7 hours

(b) $2h^2 - 11h + 5 = 0$, After 30 mins and after 5 hours.

12 (a) 192 m (b) 6.2 seconds

13 (a) $x = \frac{3 \pm \sqrt{-7}}{4}$, no real solution

(b) $x = \frac{-5 \pm \sqrt{-15}}{10}$, no real solution

(c) $x = \frac{6 \pm \sqrt{-108}}{8}$, no real solution

(d) $x = \frac{-7 \pm \sqrt{-23}}{4}$, no real solution

14 (a) $3t^2 - 17t - 70 = 0$, $t = \frac{17 + \sqrt{1129}}{6} = 8.4$ min

(b) Try to solve $-3t^2 + 17t + 70 = 100$, $3t^2 - 17t + 30 = 0$, $t = \frac{17 \pm \sqrt{-71}}{6}$ (no real solutions). Driver does not reach 100 km/h speed limit.

15 (a) Let kennel be x m wide, hence $2x$ m long.
Total painted area = $4x^2 + 6x - 0.5$

(b) $4x^2 + 6x - 0.5 = 15.7$, $8x^2 + 12x - 1 = 31.4$,

$20x^2 + 30x - 81 = 0$, $x = \frac{-30 + \sqrt{7380}}{40} \approx 1.4$.

breadth 1.4 m, length 2.8 m, height 1 m

16 (a) Let x = side length (cm) of squares cut out from the corners.

(b) base length = $30 - 2x$

(c) base breadth = $25 - 2x$

(d) Tray 1: length of square = 3.7 cm

Tray 2: length of square = 4.3 cm

Tray 3: length of square = 5 cm

(e) Tray 1: 3.7 cm × 17.7 cm × 22.7 cm

Tray 2: 4.3 cm × 16.4 cm × 21.4 cm

Tray 3: 5 cm × 15 cm × 20 cm

17 $\frac{x}{2}(x+7) = 60$, $x^2 + 7x - 120 = 0$, $x = 8$.

hypotenuse = $\sqrt{8^2 + 15^2} = 17$ cm.
perimeter = 40 cm

18 $3n^2 + n - 1610 = 0$, $(3n + 70)(n - 23) = 0$, $n = 23$ terms

19 (a) $x \neq 0$; $2x^2 - x - 21 = 0$. $x = -3$ or 3.5

(b) $x \neq 0$; $x^2 - 9x + 14 = 0$, $x = 2$ or 7

(c) $x \neq 5$; $x^2 - 6x + 9 = 0$, $x = 3$

(d) $x \neq \pm 1$; $x^2 - 4x - 3 = 0$, $x = 2 \pm \sqrt{7}$

(e) $x \neq -2, 0$; $x^2 - 6x - 6 = 0$, $x = 3 \pm \sqrt{15}$

(f) $x \neq 0$; $6x^2 - x - 3 = 0$, $x = \frac{1 \pm \sqrt{73}}{12}$

20 $2x^2 + x - 15 = 0$ and multiples of this equation

21 $a = 0$; $4x^2 - 5 = 0$
$a = 1$; $x^2 + x - 1 = 0$ etc.

22 (a) $x^2 + x + 1 = 0$

(b) $4x^2 + 20x + 25 = 0$, $x = -2.5$

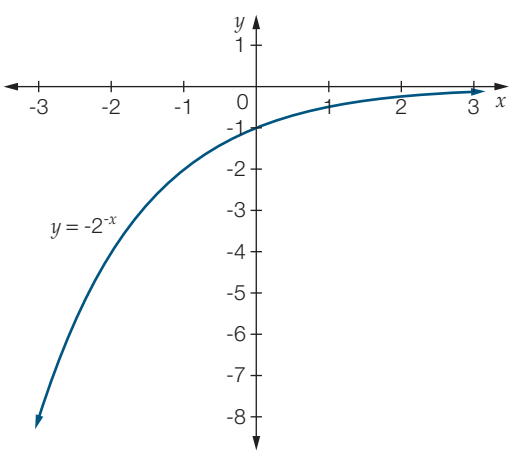
(c) $12x^2 + 7x - 10 = 0$, $x = \frac{2}{3}$ or -1.25

Half-time 9 (p. 428)

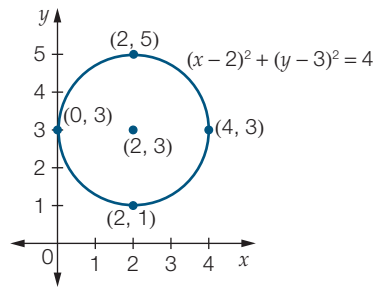
1 (a) $x = -2$ (b) $x = -1, 6$ (c) $x = 8, 1$
(d) $x = 5, -1$ (e) $x = -2, -1$ (f) $x = -5, 2$

2 radius 2 and centre (-1, 2)

3 (a)

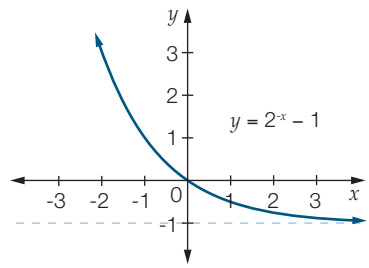


(b)



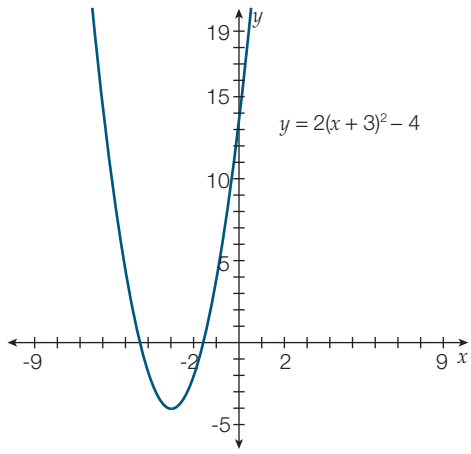
4 (a) $x = -5, -1$ (b) $x = 0, 9$ (c) $x = -5, 2$
(d) $x = -3, 2$ (e) $x = 1, 3$ (f) $x = 3$

5 (a)

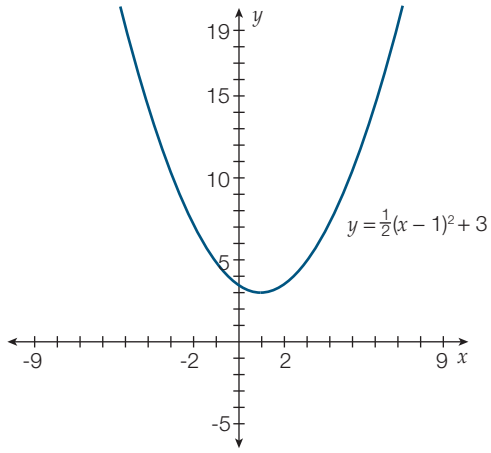


(b) $y = -1$ (c) (0, 0)

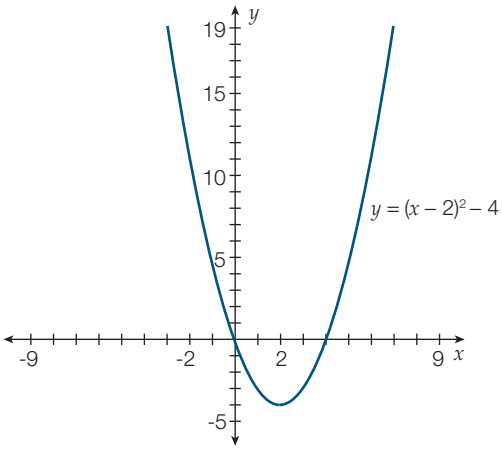
6 (a) dilated by a factor of 2, translated 3 units left and 4 units down



(b) dilated by a factor of a half, translated right 1 unit and up 3 units



(c) translated 2 units right and 4 units down

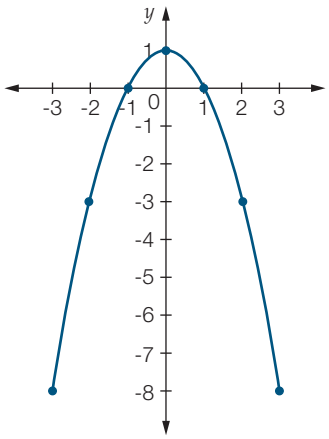


7 $y = (x - 3)^2 + 2$

8 (a)

x	-3	-2	-1	0	1	2	3
y	-8	-3	0	1	0	-3	-8

(b)



(c) (0, 1) (d) $x = 0$

9 (a) $x = 8, -8$ (b) $x = -1$ (c) $x = 4, 0$

10 (a) $x = -1 \pm \sqrt{7}$ (b) $x = \dfrac{7 \pm \sqrt{37}}{2}$ (c) $x = -2 \pm \sqrt{6}$

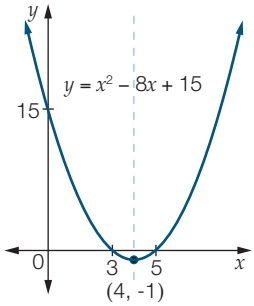
11 (a) $x = \dfrac{5 \pm \sqrt{33}}{4}$ (b) $x = \dfrac{-9 \pm \sqrt{57}}{6}$ (c) $x = \dfrac{1 \pm \sqrt{37}}{6}$

Exercise 9.7 (p. 432)

1 (a) (i) parabola, min T.P. (ii) (0, 15)

(iii) (3, 0), (5, 0) (iv) (4, -1)

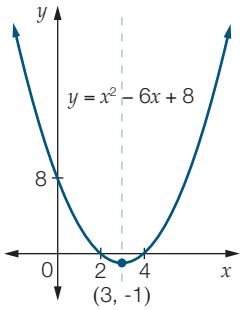
(v) $x = 4$



(b) (i) parabola, min T.P. (ii) (0, 8)

(iii) (2, 0), (4, 0) (iv) (3, -1)

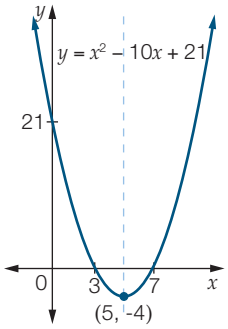
(v) $x = 3$



(c) (i) parabola, min T.P. (ii) (0, 21)

(iii) (3, 0), (7, 0) (iv) (5, -4)

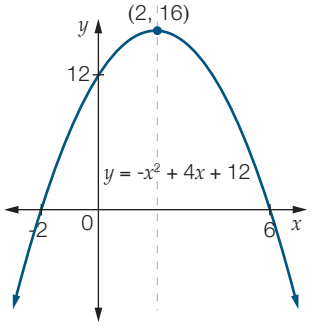
(v) $x = 5$



(d) (i) parabola, max T.P. (ii) (0, 12)

(iii) (-2, 0), (6, 0) (iv) (2, 16)

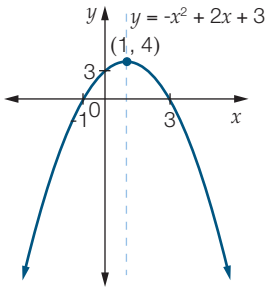
(v) $x = 2$



(e) (i) parabola, max T.P. (ii) (0, 3)

(iii) (-1, 0), (3, 0) (iv) (1, 4)

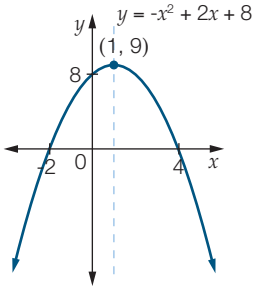
(v) $x = 1$



(f) **(i)** parabola, max T.P. **(ii)** (0, 8)

(iii) (-2, 0), (4, 0) **(iv)** (1, 9)

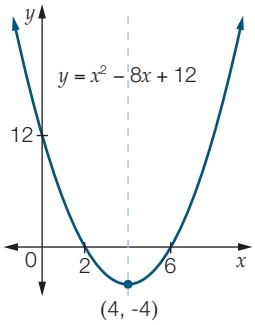
(v) $x = 1$



(g) **(i)** parabola, min T.P. **(ii)** (0, 12)

(iii) (2, 0), (6, 0) **(iv)** (4, -4)

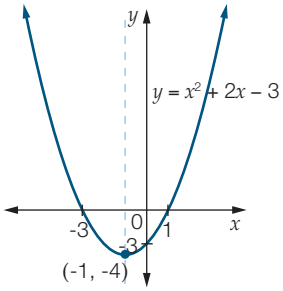
(v) $x = 4$



(h) **(i)** parabola, min T.P. **(ii)** (0, 3)

(iii) (-3, 0), (1, 0) **(iv)** (-1, -4)

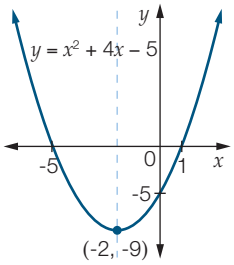
(v) $x = -1$



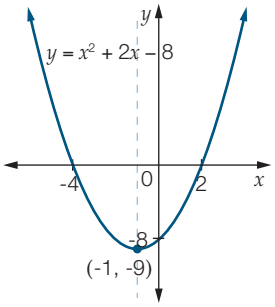
(i) **(i)** parabola, min T.P. **(ii)** (0, -5)

(iii) (-5, 0), (1, 0) **(iv)** (-2, -9)

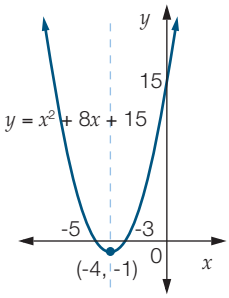
(v) $x = -2$



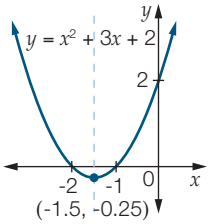
2 **(a)**



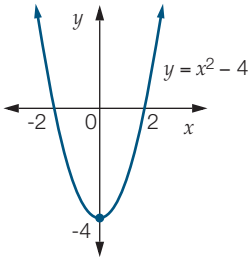
(b)



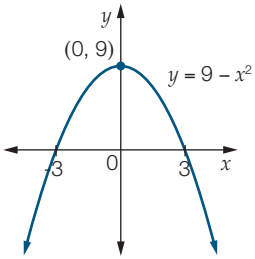
(c)



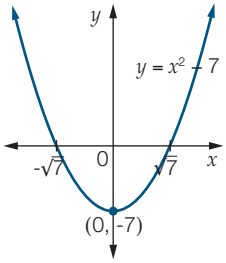
(d)



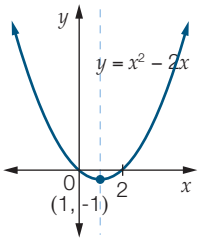
(e)



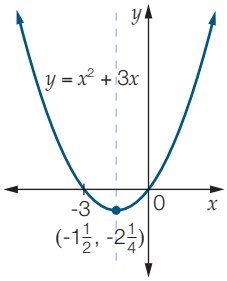
(f)



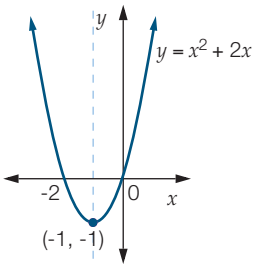
(g)



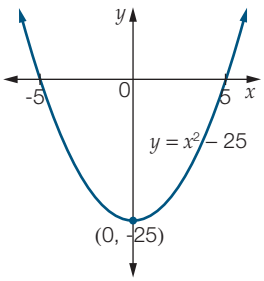
(h)



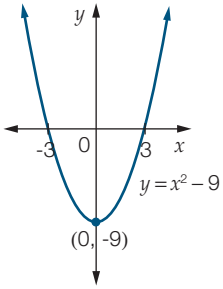
(i)



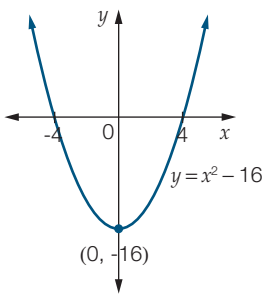
3 **(a)**



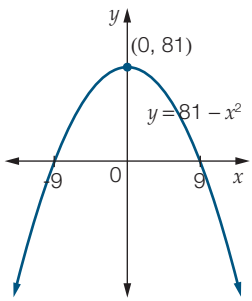
(b)



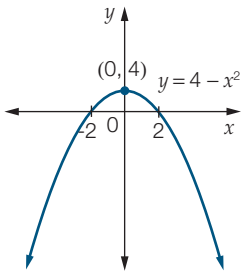
(c)



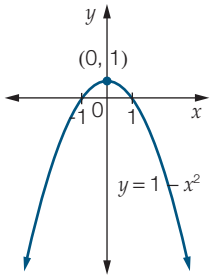
(d)



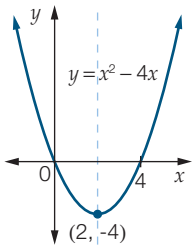
(e)



(f)



(g)



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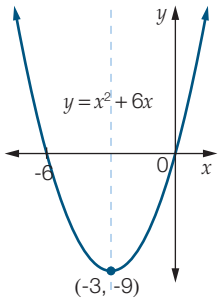
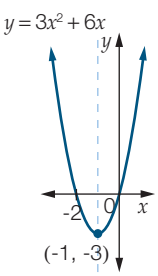
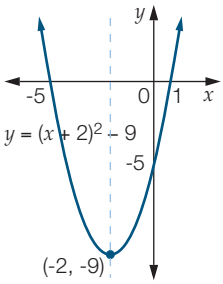


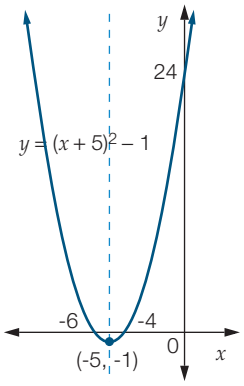
(i)



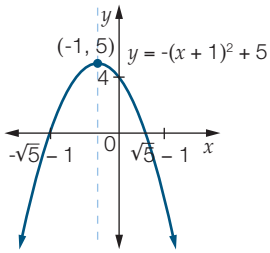
4 (a)



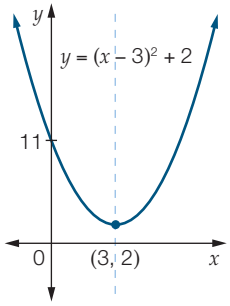
(b)



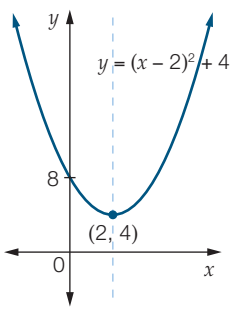
(c)



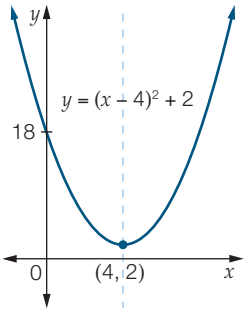
(d)



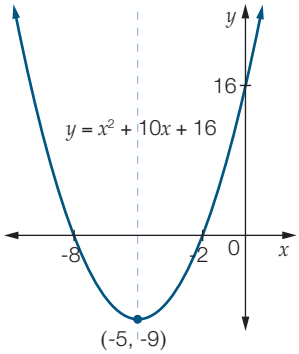
(e)



(f)



5 (a)



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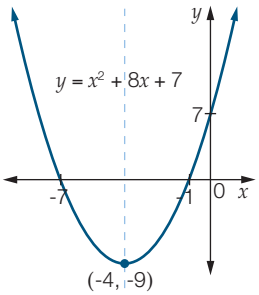

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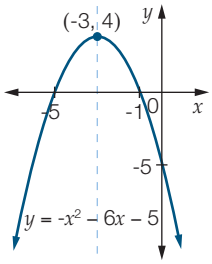

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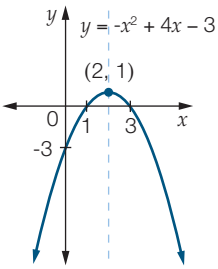
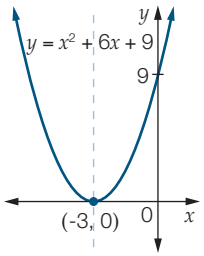


(e)



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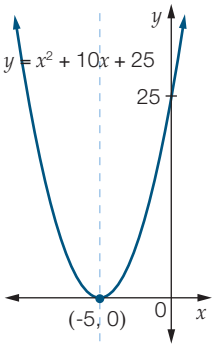

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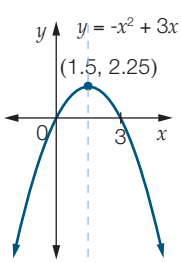
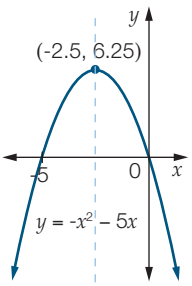
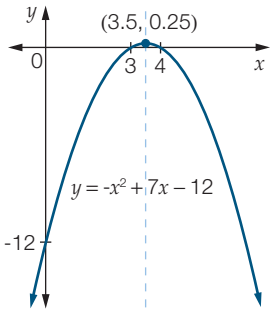


(h)



(i)

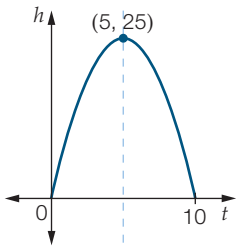


6 B

7 (a) D (b) B

8 (a) C (b) B

9

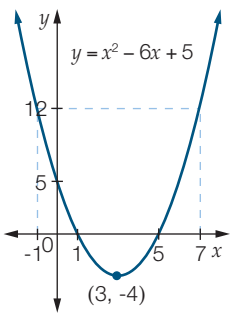


(a) 10 seconds (b) 25 m

10 (a) B (b) B

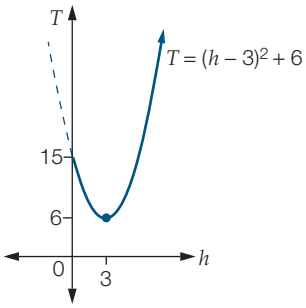
11 (a) D (b) (4, 2)

12 (a)



(b) $x = -1$ or $x = 7$

13 (a) $T = (h - 3)^2 + 6$



(b) 15°C (c) decreasing (d) increasing

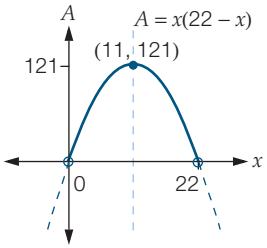
(e) 6°C (f) after 3 hours

(g) No, the temperature of the room is unlikely to increase, and certainly not at this rate. Using the equation, the temperature after 24 hours would be 447°C.

14 $y = (x + 3)(x - 5)$, $y = x^2 - 2x - 15$ or $y = (x - 1)^2 - 16$

15 (a) $A = x(22 - x)$

(b)



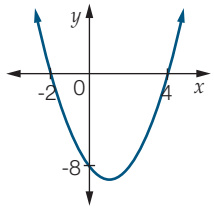
(c) $x > 0$ m (d) $x < 22$ m (e) 121 m^2

(f) square of length 11 m

16 If the equation is in factorised form then it will always have x-intercepts. If it is in turning point form, you can tell by inspection. If the TP is above the x-axis and it is a valley then no x-intercepts and if the TP is below the x-axis and it is a hill then no x-intercepts. If it is in standard form, use completing the square to put it into turning point form and check it from there.

17 (a) The equation factorises to $y = (x - 4)(x + 2)$. Archie may have just found factors of 8 and ignored the sign. As it is negative, one of the factors will be positive and the other negative. Archie needs to be more careful with his factorising.

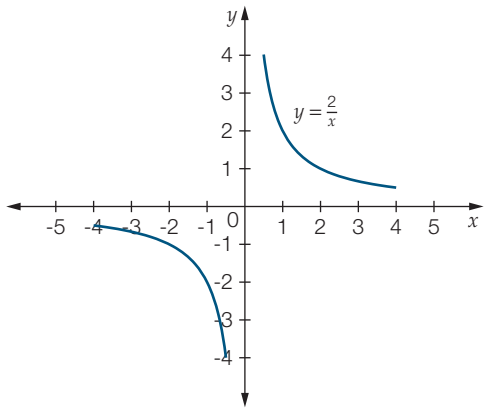
(b)



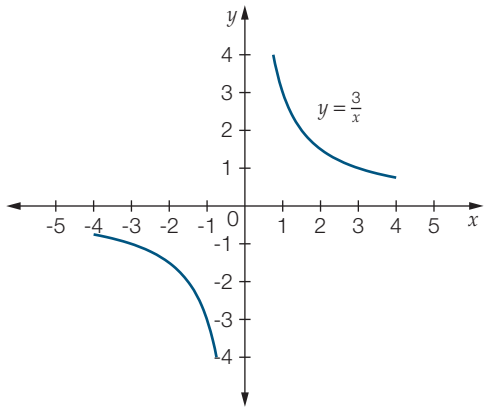
18 Students need to produce their own equations but using factorised form will make it easier. As an example, $(x - 4)(x - 6)$ has x-intercepts at 4 and 6. So the x-coordinate of the turning point is $x = 5$.

Exercise 9.8 (p. 440)

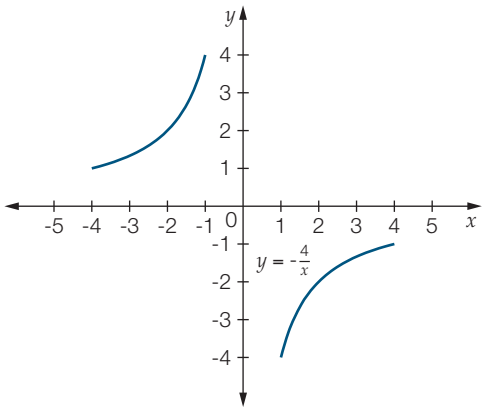
1 (a)



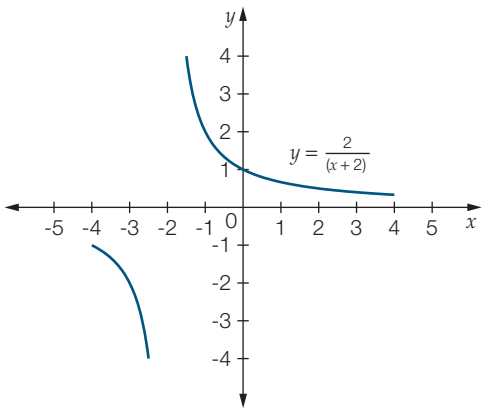
(b)



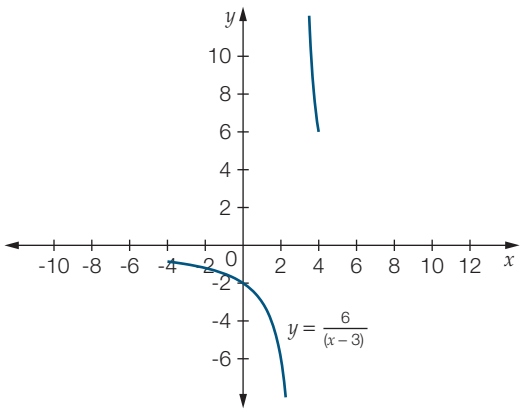
(c)


$y = -\frac{4}{x}$

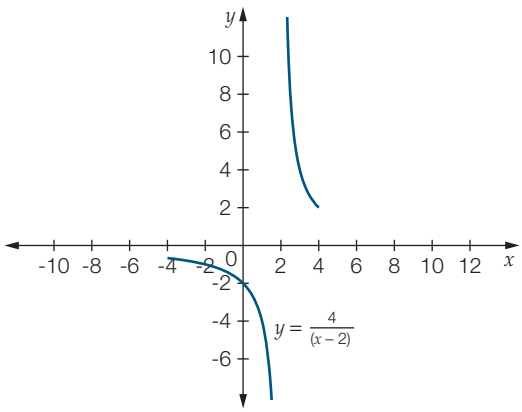
(d)


$y = \frac{2}{(x+2)}$

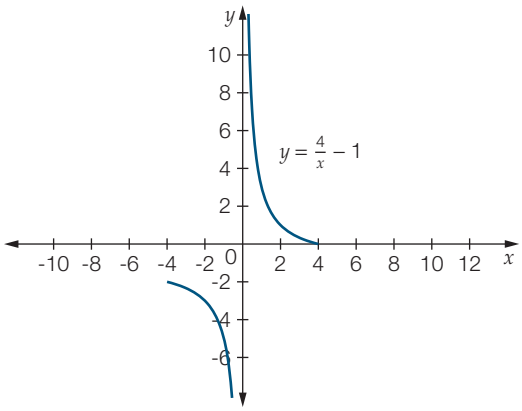
(e)


$y = \frac{6}{(x-3)}$

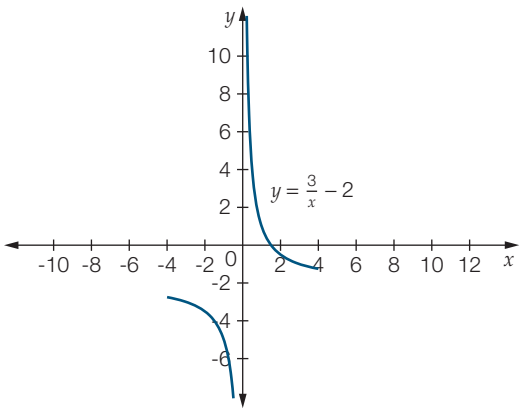
(f)


$y = \frac{4}{(x-2)}$

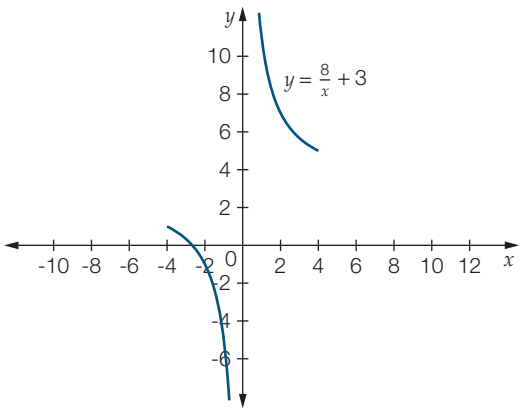
(g)


$y = \frac{4}{x} - 1$

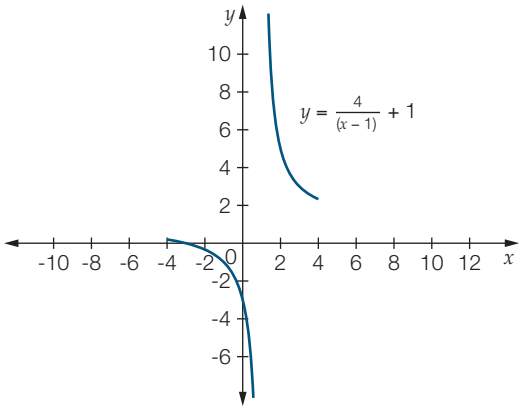
(h)


$y = \frac{3}{x} - 2$

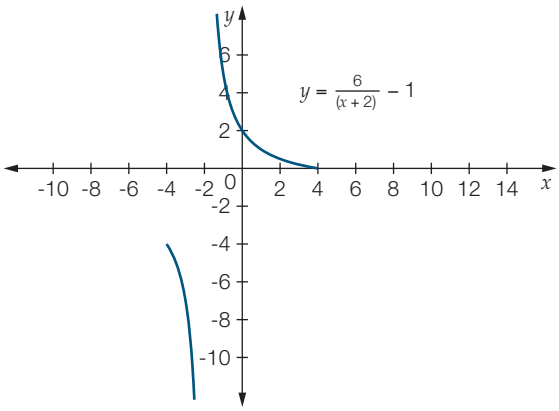
(i)


$y = \frac{8}{x} + 3$

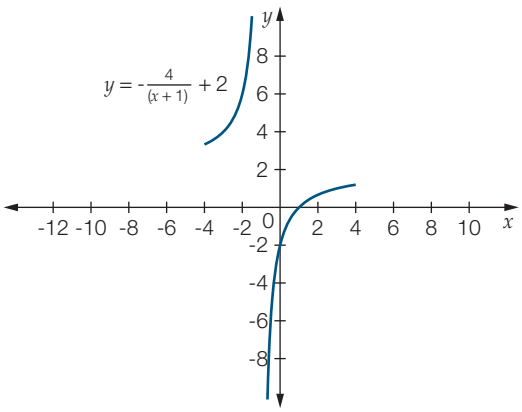
2 (a)


$y = \frac{4}{(x-1)} + 1$

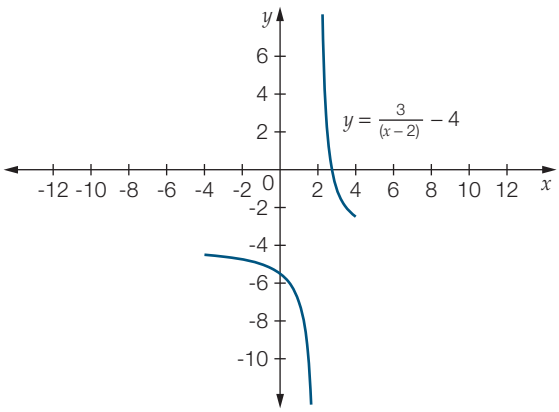
(b)



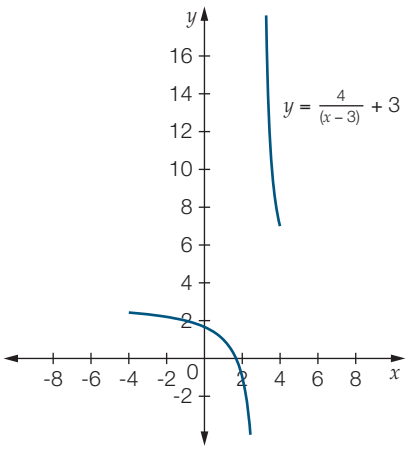
(c)



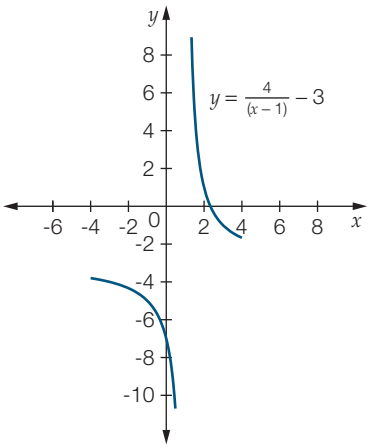
(d)



(e)



(f)



3 (a) $y = \dfrac{1}{x+3} - 1$ (b) $y = \dfrac{1}{x-2} - 3$

4 (a) could be inverse proportion

(b) not inverse proportion (fluctuates)

(c) could be inverse proportion

(d) could be inverse proportion

5 (a) $b = \dfrac{40}{a}$ (b) $d = \dfrac{60}{c}$ (c) $f = \dfrac{216}{e}$

(d) $h = \dfrac{144}{g}$ (e) $l = \dfrac{240}{k^2}$ (f) $n = \dfrac{144}{m^2}$

6 D **7** B

8 (a) $p = \dfrac{k}{q}$ (b) $y = \dfrac{k}{x^2}$ (c) $a = \dfrac{k}{\sqrt{b}}$ (d) $t = \dfrac{k}{w}$

(e) $V = \dfrac{k}{h^2}$ (f) $f = \dfrac{k}{g^3}$ (g) $L = \dfrac{k}{\sqrt{m}}$ (h) $a = \dfrac{k}{\sqrt{d}}$

9 (a) C (b) D (c) D

10 (a) 7 (b) 6.3 (c) 12 (d) 0.97

(e) 5 (f) 3 (g) $\dfrac{1}{2}$ (h) $\dfrac{1}{5}$

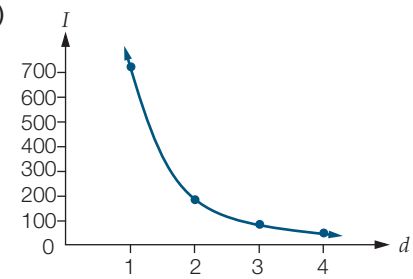
(i) $\dfrac{1}{0.4} = 2.5$ (j) $\dfrac{1}{1.6} = \dfrac{5}{8}$

(k) $\dfrac{1}{0.08} = 12.5$ (l) $\dfrac{1}{1.25} = 0.8$

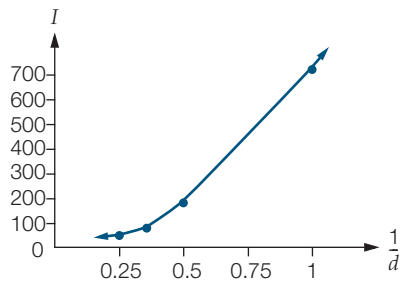
11 (a) $\dfrac{128}{125}$ (b) 4

12 5 hours **13** 2×10^{-8} m **14** 20 km

15 (a)



(b)



(c) No, because it is not a linear relationship.

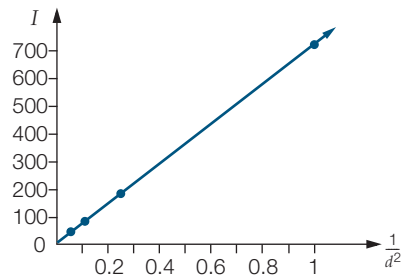
(d)

d	1	2	3	4	6	8	12
d^2	1	4	9	16	36	64	144
I	720	180	80	45	20	11.25	5

(e) 720; yes

(f) $I = \dfrac{720}{d^2}$

(g)



It is a linear relationship: $I \propto \dfrac{1}{d^2}$.

(h) I is reduced by 4 times.

(i) I is reduced by 9 times.

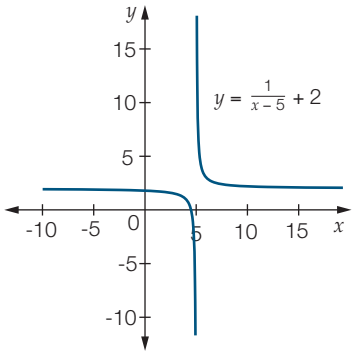
(j) I is increased by 4 times.

16 Students' own answers, e.g. $y = \dfrac{1}{x+2} + 6$.

17 Students' own answers, e.g.

$h = 5, k = 2$

$y = \dfrac{1}{x-5} + 2$

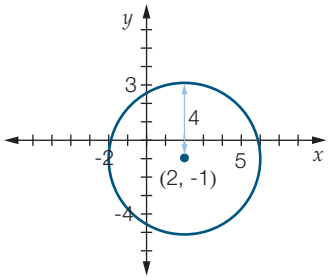


18 possible solution $y = \dfrac{16}{x}$, (1, 16) (2, 8) (4, 4);

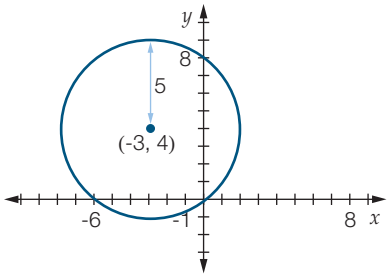
$y = \dfrac{16}{x^2}$, (1, 16) (2, 4) (4, 1); $y = \dfrac{16}{\sqrt{x}}$, (1, 16), (4, 8) (16, 4)

Exercise 9.9 (p. 450)

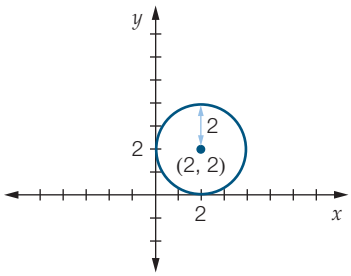
1 (a) $(x-2)^2 + (y+1)^2 = 16$



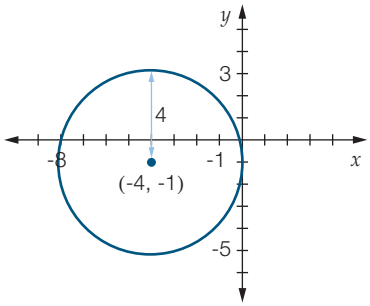
(b) $(x+3)^2 + (y-4)^2 = 25$



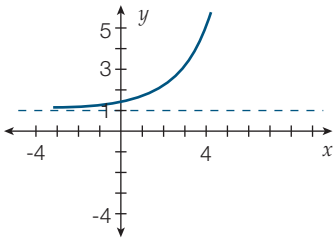
(c) $(x-2)^2 + (y-2)^2 = 4$



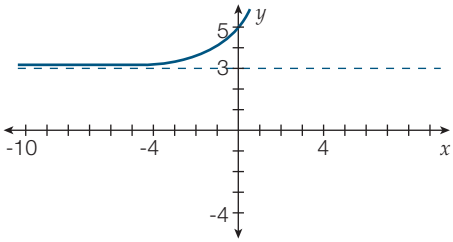
(d) $(x+4)^2 + (y+1)^2 = 16$



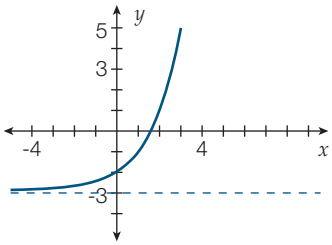
2 (a) $y = 2^{x-2} + 1$



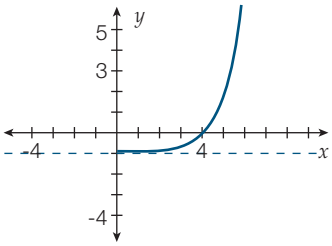
(b) $y = 2^{x+1} + 3$



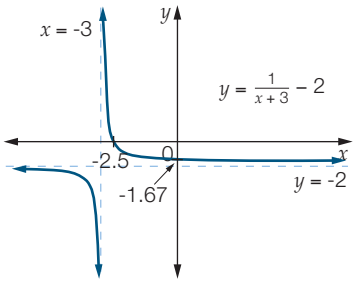
(c) $y = 2^x - 3$



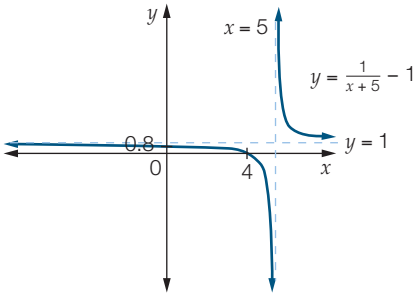
(d) $y = 3^{x-4} - 1$



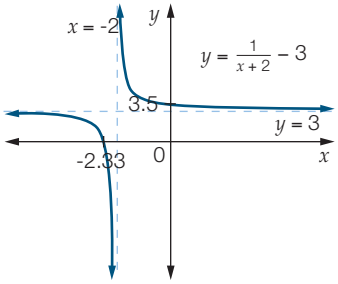
3 (a)



(b)



(c)

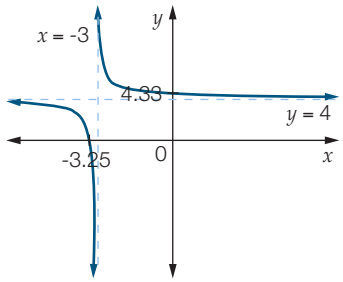


4 (a) $x^2 + (y+1)^2 = 4$ (b) $(x+1)^2 + (y+1)^2 = 9$

(c) $(x-2)^2 + (y-3)^2 = 16$ (d) $(x+5)^2 + (y+3)^2 = 4$

5 (a) $y = \dfrac{1}{x+3} + 4$

(b)



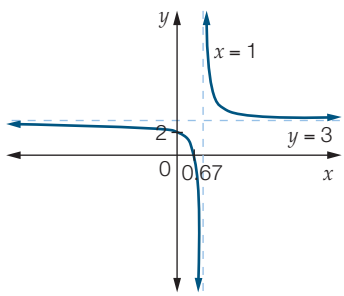
6 C

7 (a) $(x-1)^2 + (y+7)^2 = 25$ (b) $(x-3)^2 + (y+4)^2 = 64$

8 (a) $\dfrac{1}{x-1} + 3 = \dfrac{1}{x-1} + \dfrac{3}{x-1} = \dfrac{1+3(x-1)}{x-1} = \dfrac{1+3x-3}{x-1}$

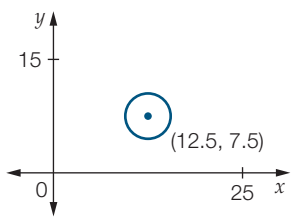
$= \dfrac{3x-2}{x-1} = \text{RHS}$

(b)



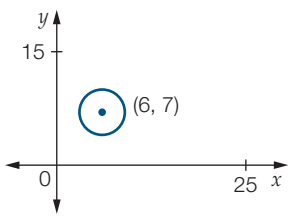
9 (a) (12.5, 7.5) (b) $(x-12.5)^2 + (y-7.5)^2 = 9$

(c)



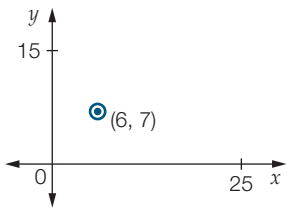
(d) (i) $(x-6)^2 + (y-7)^2 = 9$

(ii)



(e) (i) $(x-6)^2 + (y-7)^2 = 4$

(ii)



10 (a) Something in the form $y = \dfrac{1}{x-h} + k$.

(b) $x = h$ and $y = k$

(c) Students to sketch their own graph.

11 As an example $(x-2)^2 + (y-1)^2 = 1$

Exercise 9.10 (p. 459)

1 (a) (0, 2), (-3, -2) (b) (-2, -1), (-3, -1)

(c) (0.3, 5.5), (6.7, 37.5)

2 (a) (0, -3), (3, 0) (b) (-1, 2), (-2, -1)

(c) (1.38, 1.76), (-0.58, -2.16)

3 (a) (1, -3), $(\frac{1}{4}, 2)$ (b) (3, -1), $(-\frac{3}{2}, -10)$

(c) (1.41, 0.41), (-1.41, -2.41)

4 (a) $x^2 - 4x + 6 = 2x - 3$

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

$x = 3$

As there is only 1 solution, the line is a tangent to the parabola at the point (3, 3).

(b) $x^2 + 14x + 18 = 4x - 7$

$x^2 + 10x + 25 = 0$

$(x + 5)^2 = 0$

$x = -5$

As there is only 1 solution, the line is a tangent to the parabola at the point (-5, -27).

(c) $x^2 + 5x - 3 = x - 7$

$x^2 + 4x + 4 = 0$

$(x + 2)^2 = 0$

$x = -2$

As there is only 1 solution, the line is a tangent to the parabola at the point (-2, -9).

5 (a) $x = \dfrac{-6 \pm \sqrt{36 - 40}}{4}$; no solution

(b) $x = \dfrac{30 \pm \sqrt{900 - 920}}{20}$; no solution

(c) $x = \dfrac{20 \pm \sqrt{400 - 440}}{10}$; no solution

6 (0.31, 0.62), (4, 8)

7 A

8 $(x + 2)^2 + y^2 = 1$

$x^2 + 4x + 4 + (x + 3)^2 = 1$

$x^2 + 4x + 4 + x^2 + 6x + 9 = 1$

$2x^2 + 10x + 12 = 0$

$x^2 + 5x + 6 = 0$

$(x + 2)(x + 3) = 0$

$x = -2, -3$

Intersects at 2 points (-2, 10), (-3, 0).

9 $y = 2x$

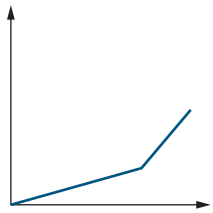
10 Students' own answers;

e.g. for $x^2 + y^2 = 10$ and $y = 3x$

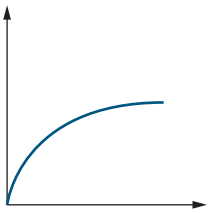
$x = \pm 1$; and points of intersection (-1, -3), (1, 3).

Exercise 9.11 (p. 464)

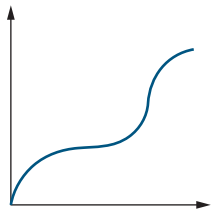
1 (a)



(b)



(c)



2 (a) constant (b) variable (c) variable

(d) variable (e) constant

2 (a) constant (b) constant (c) variable (d) variable

4 Pete runs 2 kilometres to the bus stop for 10 minutes. He then travels 7 kilometres on the bus for 10 minutes, gets off the bus and stays at the bus stop, possibly waiting for a friend. He walks another kilometre to school at a constant rate for 5 minutes. He stays at his locker for 10 minutes.

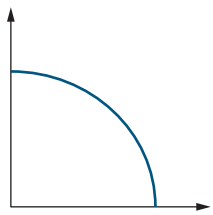
5 \$22/h 6 120 km/h 7 B

8 Yes, if a customer does not exceed the cap. However, it is variable if the cap is exceeded.

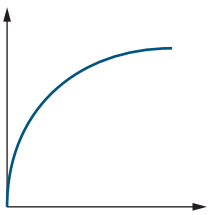
9 First company \$494.10, second company \$535.50; so first company is cheaper for this amount of consumption.

10 Graphs may vary, but should follow the trends of the following.

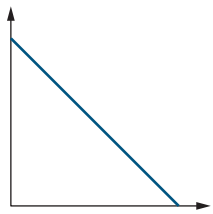
(a)



(b)



(c)



11 (a) decreasing at a decreasing rate

(b) increasing at a constant rate

(c) increasing at a decreasing rate

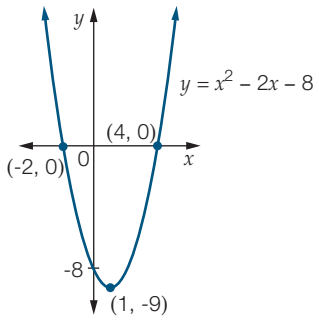
12 Emma is correct. It is impossible to travel at a constant rate, even using cruise control the variation may only be slight but it will still occur. Tim is confusing a constant rate with an average rate over the time period.

Chapter review 9 (p. 466)

1 (a) exponential; $y = 3$; asymptotes $x = 2$ and $y = 3$

(b) circle; centre (-1, 3) and $r = 3$

2



(a) x-intercepts: (-2, 0) and (4, 0); y-intercept: (0, -8)

(b) (1, -9) minimum **(c)** $x = 1$ **(d)** $y = -5$

(e) $x = 5$ and -3 **(f)** 5 and -3 **(g)** -2 and 4

3 **(a)** A **(b)** A **(c)** B

4 **(a)** $x = 0$ or 6 **(b)** $x = -2$ or 7 **(c)** $x = 0$ or 8

(d) $x = 0$ or -4 **(e)** $x = -1$ or 7 **(f)** $x = -16$ or 1

5 **(a)** $x = -7$ or -5 **(b)** $x = -3$ or 2 **(c)** $x = 1$ or -9

(d) $x = 1$ or 8 **(e)** $x = 1$ or -4 **(f)** $x = 3$ or 1

6 **(a)** $x = -1$ or 5 **(b)** $x = -1$ or 6 **(c)** $x = -4$ or -3

(d) $x = 9$ **(e)** $x = -8$ or 4 **(f)** $x = -3$ or 4

7 **(a)** $x = \pm\frac{9}{2}$ **(b)** $x = -3$ **(c)** $x = 6$

8 **(a)** $x = 2 \pm \sqrt{2}$ **(b)** $x = -4 \pm \sqrt{7}$ **(c)** $x = -1 \pm \sqrt{5}$

9 **(a)** $x = \pm 3$ **(b)** $(2x + 1)(x - 5) = 0$, $x = -0.5$ or 5

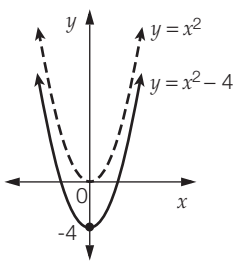
(c) $x = \dfrac{5 \pm \sqrt{3}}{2}$ **(d)** $x = \dfrac{-5 \pm \sqrt{17}}{4}$

(e) $x = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{3 \pm \sqrt{3}}{3}$ **(f)** $x = \dfrac{-7 \pm \sqrt{205}}{26}$

10 **(a)** **(i)** Move 4 units down. **(ii)** (0, -4)

(iii) (0, -4), (-2, 0), (2, 0)

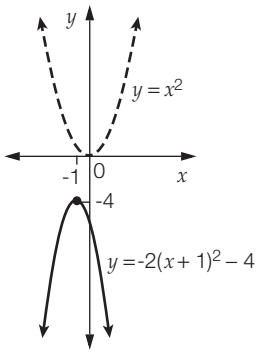
(iv)



(b) **(i)** Reflect in x-axis, dilate by a factor of 2 (make narrower) and translate 1 unit left and 4 units down.

(ii) (-1, -4) **(iii)** (0, -6)

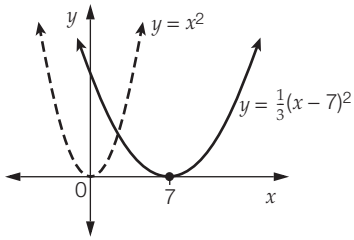
(iv)



(c) **(i)** Dilate by a factor of $\frac{1}{3}$ (make wider) and move 7 units right.

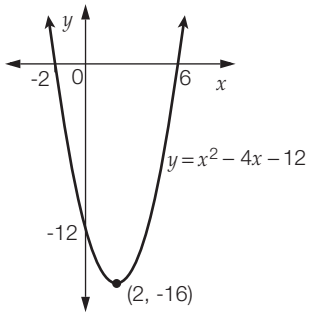
(ii) (7, 0) **(iii)** $(0, \frac{49}{3})$ (7, 0)

(iv)



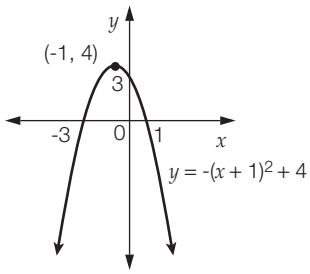
11 **(a)** **(i)** (0, -12) **(ii)** (-2, 0), (6, 0) **(iii)** (2, -16)

(iv)



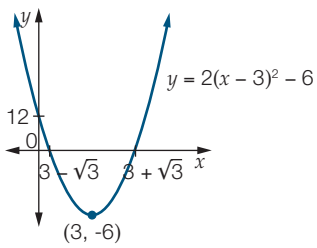
(b) **(i)** (0, 3) **(ii)** (-3, 0), (1, 0) **(iii)** (-1, 4)

(iv)



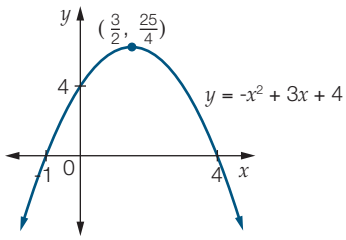
(c) **(i)** (0, 12) **(ii)** $(3 \pm \sqrt{3}, 0)$ **(iii)** (3, -6)

(iv)

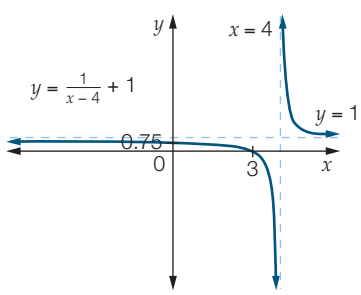


(d) **(i)** (0, 4) **(ii)** (-1, 0), (4, 0) **(iii)** $(\frac{3}{2}, \frac{25}{4})$

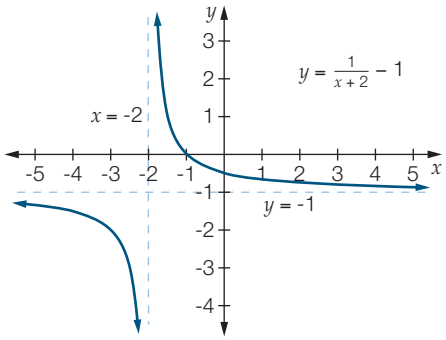
(iv)



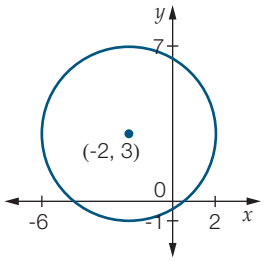
12 **(a)**



(b)

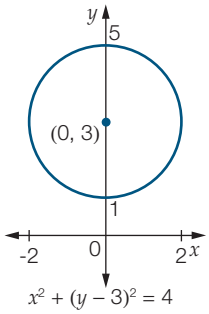


13 (a)



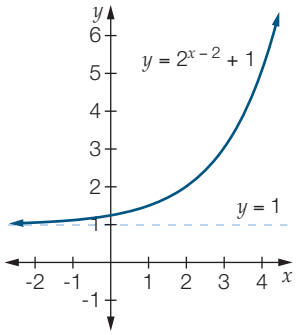
$(x + 2)^2 + (y - 3)^2 = 16$

(b)

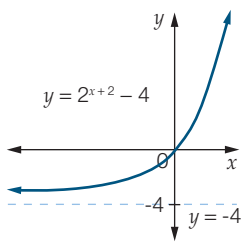


$x^2 + (y - 3)^2 = 4$

(c)



(d)



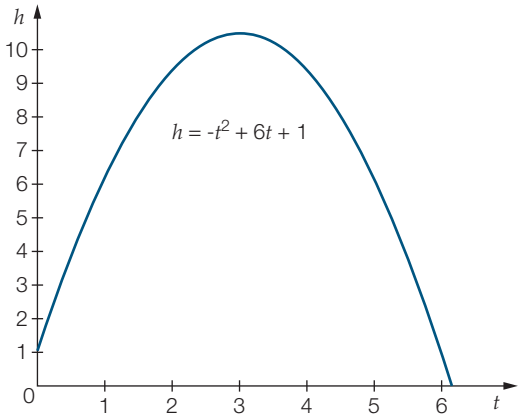
14 (2, 13), (-1, 4)

15 (a) constant (b) variable (c) constant

16 (a)

t	0	1	2	3	4	5	6	7
h	1	6	9	10	9	6	1	-6

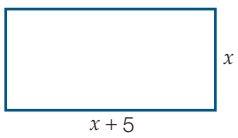
(b)



(c) 9 m (d) 1.27 seconds (e) $2 < t < 4$

(f) 10 m (g) 5.9 seconds

17 (a)



(b) $(x + 5)$ cm

(c) $x(x + 5)$ cm^2 (d) $x(x + 5) = 24$

(e) $x = -8$ or 3 (f) length 8 cm, breadth 3 cm

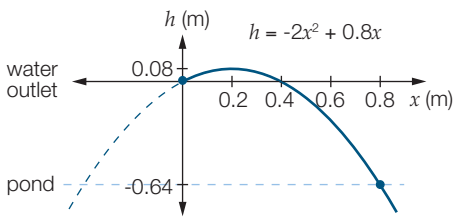
(g) $x = -8$ is not a feasible solution as a distance measurement must be positive.

18 (a) A horizontal distance of 0.4 m from the outlet.

(b) 0.64 m below the outlet

(c) 72 cm or 0.72 m

(d)

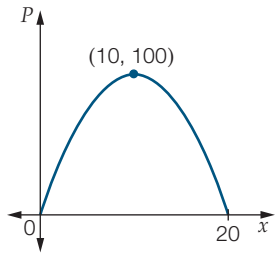


19 (a) 6 cm (b) 14 cm by 10 cm (c) 36 cm^2

(d) 104 cm^2 (e) 64 cm^2

20 (a) $x + y = 20$ (b) $y = 20 - x$ (c) $x(20 - x) = P$

(d)



The maximum product is 100 and it is obtained by using $x = y = 10$.

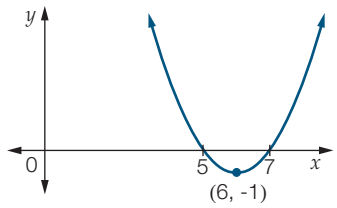
21 (a) 0.57 m

(b) The ball is 0.972 m high at the net, which is more than the height of the net (0.913 m).

(c) The ball lands 30 m from the fence, which is just outside the court.

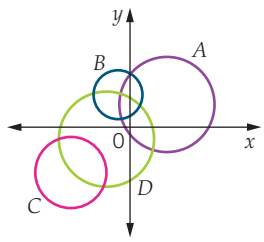
22 (a) (6, -1) (b) $y = (x - 6)^2 - 1$

(c)



23 (a) A and D

(b)



D intersects the other three graphs.

24 $x \neq 0$; $40x^2 - 6x - 1 = 0$, $x = -0.1, 0.25$

25 $m^2 - m - 12 = 0$, $(m - 4)(m + 3) = 0$, $m = 4$ or -3, $x^2 = 4$, $x = \pm 2$

26 $\frac{n}{2}(10 + 4(n - 1)) = 860$, $2n^2 + 3n - 860 = 0$, $(2n + 43)(n - 20) = 0$.
$n = 20$

Mixed review C (p. 471)

1 \$20.23

2 **(a)** $(a-9)(a+4)$ **(b)** $(a-6)(a-2)$

(c) $(2a+5)^2$

3 **(a)** 104 cm^2 **(b)** 37 m^2 **(c)** 103.5 cm^2

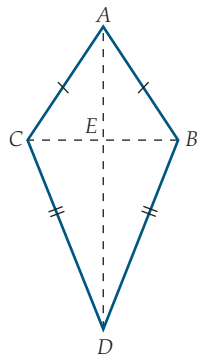
4 **(a)** -6 **(b)** $-\frac{15}{16}$ **(c)** $\frac{11}{6}$

5 **(a)** 42.7° **(b)** 23.9° **(c)** 50.5°

6 4.85 cm

7 **(a)** Range = 30, IQR ≈ 18 **(b)** Range = 30, IQR ≈ 13

8 The proof makes use of the following diagram, and its named vertices.

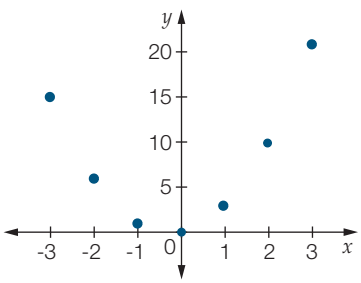


To show: $\Delta ACD \equiv \Delta ABD$
$AC = AB$ (given)
$CD = BD$ (given)
$AD = AD$ (shared)
So, $\Delta ACD \equiv \Delta ABD$
So, $\angle BAE = \angle CAE$
(matching angles in congruent triangles)
Now to show: $\Delta ABE \equiv \Delta ACE$
$\angle BAE = \angle CAE$ (from above)
$AC = AB$ (given)
$AE = AE$ (shared)
So, $\Delta ABE \equiv \Delta ACE$
So, $\angle AEB = \angle AEC$
(corresponding angles in congruent triangles)
But, $\angle AEB + \angle AEC = 180°$ (straight angle)
So, $\angle AEB = \angle AEC = 90°$
So, the diagonals of a kite intersect at right angles.

9 **(a)**

x	-3	-2	-1	0	1	2	3
y	15	6	1	0	3	10	21

(b)



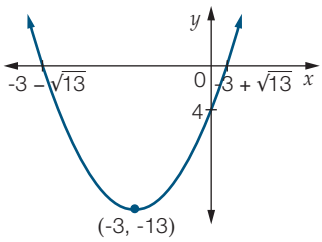
(c) (-0.25, -0.125)

(d) x-intercepts: -0.5 and 0; y-intercept: 0

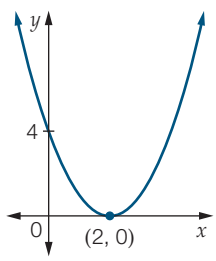
10 $x = -1$ and $y = -3$

11 **(a)** $x = \pm 5$ **(b)** $x = \pm 10$ **(c)** $x = 7$

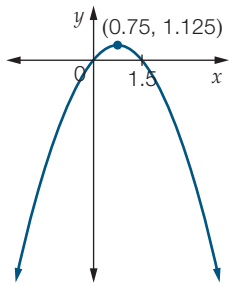
12 **(a)**



(b)



(c)



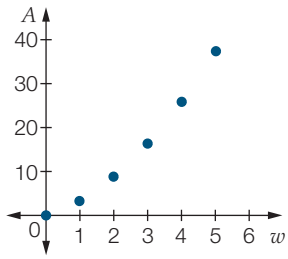
13 **(a)** $x = -7$ and -6 **(b)** $x = 0$ and -0.5 **(c)** $x = 2$ and 3

14 **(a)** (2, 3) **(b)** (-4, -1) **(c)** (7, -6)

15 $(x-4)^2 + (y-2)^2 = 25$ **16** C **17** C

18 **(a)** $A = w(w + 2.5)$ cm^2

(b)



(c) $w = 5$ from the graph so the dimensions are 5 cm by 7.5 cm

19 **(a)** $x = 5$

(b) $y = (x-4)(x-6)$
$y = x^2 - 10x + 24$

Challenge C (p. 473)

1 C **2** B

3 ΔADC and ΔDBC are equal in area. They have the same height, the perpendicular distance from C to AB.

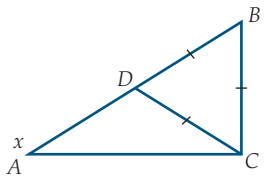
Hence their bases must be the same, so $AD = DB$.

ΔADC is isosceles.

ΔDBC is equilateral so $\angle BDC = 60°$

Hence, $\Delta ADC = 120°$ and $\angle DCA = \angle DAC = 30°$

$\therefore x = 150°$. (You can now say that ΔABC is right angled at C.)



4 B $AE = 25$, $AD = 24$, $ED = 7$, by Pythagoras' Theorem

$AFD = 90°$ (angle in a semicircle)

$AB = DF$, $BC = FE$ (matching sides of similar triangles)

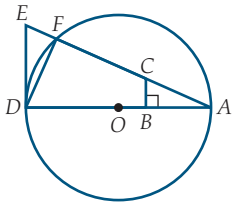
$AC = ED = 7$, $CB = EF$

$FC = 25 - (EF + 7)$

$DB = 24 - AB$

Perimeter of $CBDF = DB + BC + CF + FD$

$= 24 - FD + EF + 25 - (EF + 7) + FD = 49 - 7 = 42$ units



5 45°

6 (a) $x^2 - 3x - 18 = -14$, $x^2 - 3x - 4 = 0$,
$(x - 4)(x + 1) = 0$. $x = -1, 4$

(b) $(x^2 - 3x)^2 - 3(x^2 - 3x) - 4 = 0$. $x^2 - 3x = -1$, $(x^2 - 3x) = 4$.

$x^2 - 3x + 1 = 0$, $x^2 - 3x - 4 = 0$. $x = \dfrac{3 \pm \sqrt{9 - 4}}{2}$,

$(x - 4)(x + 1) = 0$. $x = \dfrac{3 + \sqrt{5}}{2}, \dfrac{3 - \sqrt{5}}{2}, -1, 4$.

7 (a) $(x - 7)(x + 1) = 0$. $x = -1, 7$. Sum $= 6$.

(b) $(x - 2)(x + 2) = 0$. $x = -2, 2$. Product $= -4$.

(c) $x(x^2 - 6x + 5) = 0$. $x(x - 1)(x - 5) = 0$. $x = 0, 1, 5$. Average $= 2$

8 (a) $7 = (c - 2)(c - 8) + 7$
$(c - 2)(c - 8) = 0$
$c = 2$ or $c = 8$

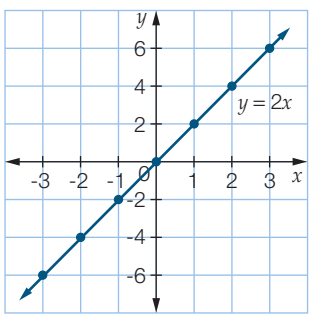
(b) vertex $(5, -2)$

9 (a) $y = -3, 2$ (b) $(-\sqrt{5}, 2), (0, -3), (\sqrt{5}, 2)$

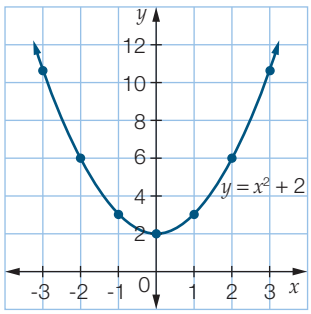
Chapter 10

Recall 10 (p. 474)

1 (a)



(b)

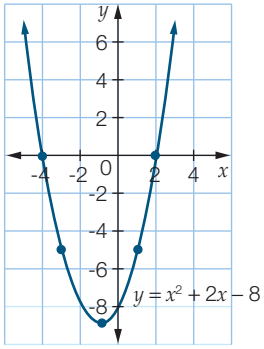


2 (a) neither (b) linear (c) quadratic

3 (a)

x	-5	-4	-3	-2	-1	0	1	2	3
y	7	0	-5	-8	-9	-8	-5	0	7

(b)



(c) (i) -8 (ii) -4, 2 (iii) (-1, -9)

(d) minimum (e) $x = -1$

4 (a) $(x - 6)(x + 1)$ (b) $(2x + 9)(2x - 9)$ (c) $3(x - 3)(x + 4)$

(d) $(x - 7)^2$ (e) $-(x - 10)(x + 2)$

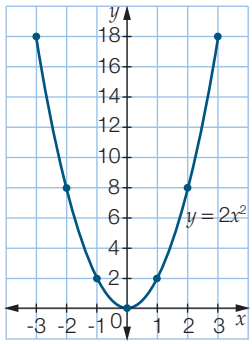
(f) $(x - 3 - 2\sqrt{2})(x - 3 + 2\sqrt{2})$

5 (a) $x = -2$ or 1 (b) $x = \pm 9$ (c) $x = 0$ or 2.5

Exercise 10.1 (p. 478)

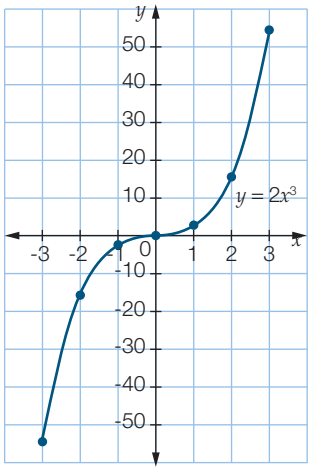
1 (a)

x	-3	-2	-1	0	1	2	3
y	18	8	2	0	2	8	18



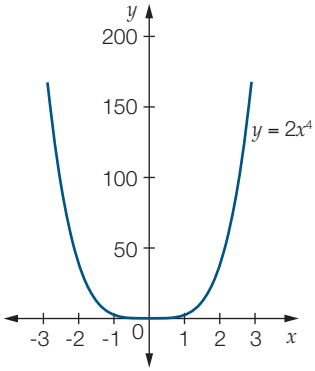
(b)

x	-3	-2	-1	0	1	2	3
y	-54	-16	-2	0	2	16	54



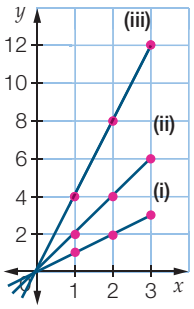
(c)

x	-3	-2	-1	0	1	2	3
y	162	32	2	0	2	32	162



2 (a) U (b) Z (c) W (d) Y (e) X (f) V

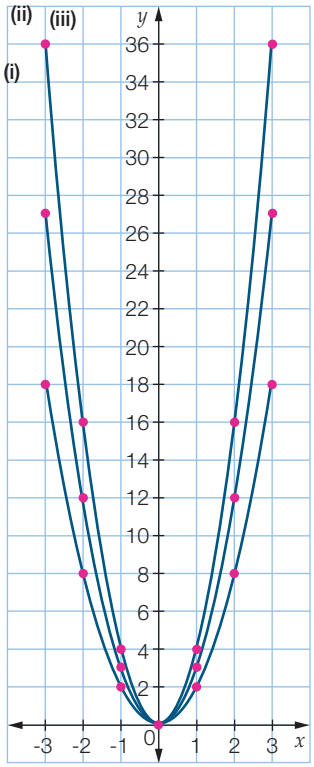
3 (a)



(b) The graph becomes steeper as the coefficient gets larger.

(c) No, graphs with a negative gradient become steeper as the coefficient of x decreases.

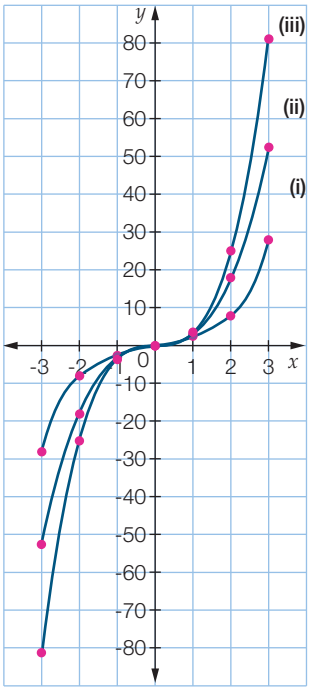
4 (a)



(b) The parabola becomes narrower about the y-axis as the coefficient increases.

(c) No, parabolas with a negative gradient become narrower as the coefficient of x decreases.

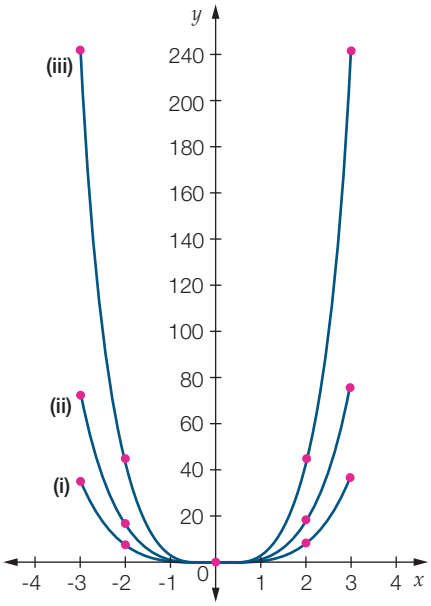
5 (a)



(b) The cubic gets steeper (gets closer to the y-axis) as the coefficient increases.

(c) No, cubics with a negative gradient become steeper as the coefficient of x decreases.

6 (a)

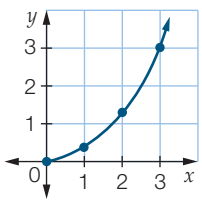


(b) The quartic becomes narrower about the y-axis as the coefficient increases.

(c) No, quartics with a negative gradient become narrower as the coefficient of x decreases.

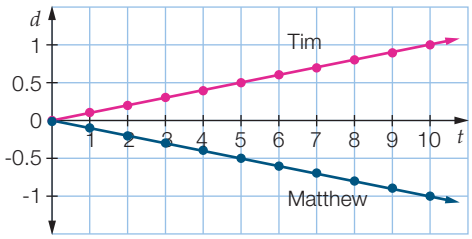
7 B 8 D

9 **(a)**



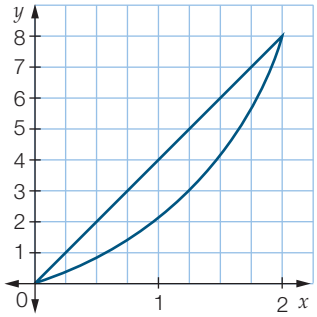
(b) $y = 0.75x$

10 **(a)**



(b) 2 km **(c)** $d = 0.12t$

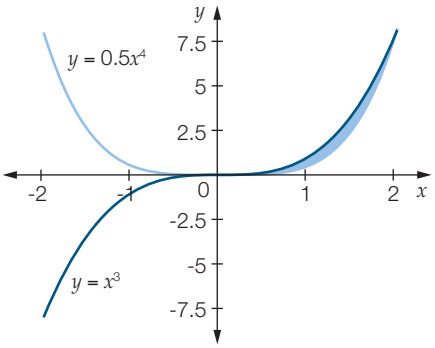
11 **(a)**



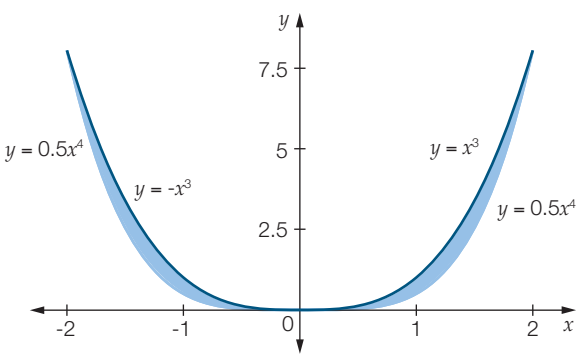
(b) $2\frac{2}{3}$ m^2

(c) $y = 4x$, $y = -2x^2$, $x = -2$ to 0

12 **(a), (b)**



(c)



(d) $y = -x^3$ for $-2 \le x \le 0$

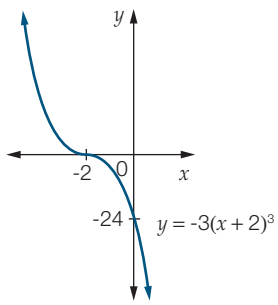
13 **(a)** $c = 10, 11$

(b) Gradient is undefined, the line is vertical.

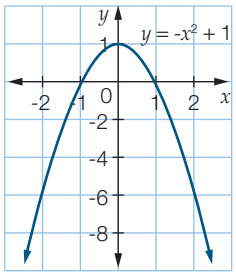
14 The graphs will intersect at $(-1, a)$ and $(1, a)$. This is true for both positive and negative values of a.

Exercise 10.2 (p. 484)

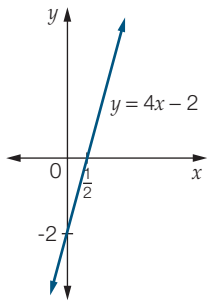
1



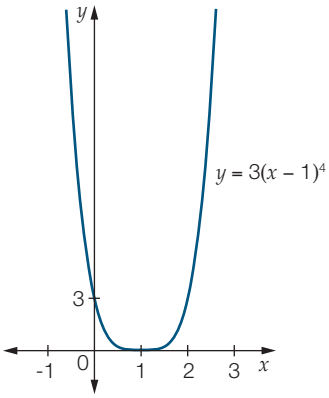
2



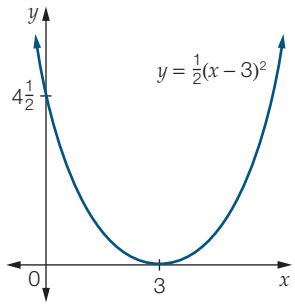
3



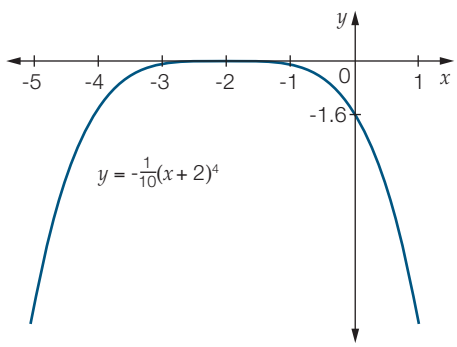
4



5



6



7 D **8** C **9** C

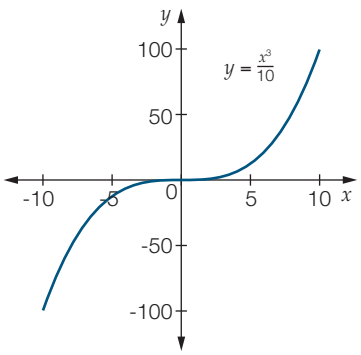
10 **(a)** $h = 1, k = 2$

(b) Negative, the turning point is a maximum.

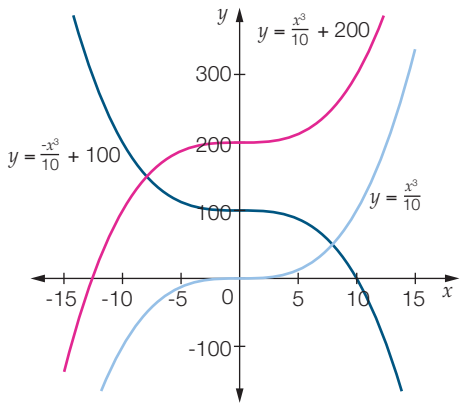
(c) $a = -3$

11 **(a)** $y = \frac{x^3}{10}$

(b)



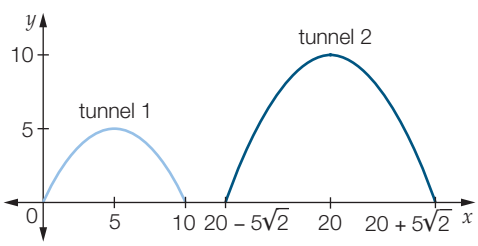
(c)



(d) The snake has moved up vertically 100 cm, the shape of the graph is exactly the same, but it is a mirror image of its initial shape. It is reflected in the line $y = 50$.

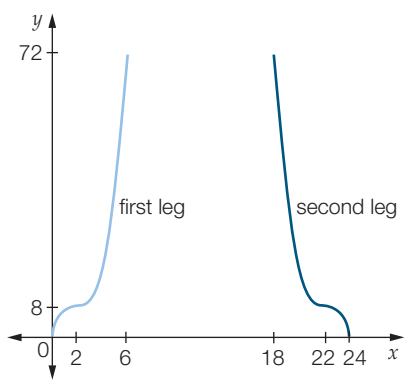
12 (a) $y = 10 - 0.2(x - 20)^2$

(b)



(c) 15.8 m

13 (a), (c)



(b) $y = -(x - 22)^3 + 8$ (d) 12 cm

14 $y = -3(x + 2)^3 + 4$, $y = -5(x + 1)^3 - 8$

Exercise 10.3 (p. 492)

1 (a) degree 3, leading coefficient 2, constant term 2

(b) degree 4, leading coefficient -4, constant term 5

(c) degree 1, leading coefficient 1, constant term 0

(d) degree 2, leading coefficient -1, constant term -6

2 (a) 62 (b) -142 (c) -28 (d) 8

3 (a) $5x^3 + 11x^2 - 13x - 3$ (b) $10x^4 - 3x^3 - 6x^2 + 29x + 6$

4 $2x^5 - 3x^4 - x^3 + 7x^2 + 15x - 20$

5 (a) $148\frac{15}{25}$ or $148\frac{3}{5}$ (b) $338\frac{5}{12}$

(c) $348\frac{8}{17}$ (d) $106\frac{4}{19}$

(e) $510\frac{7}{29}$ (f) $88\frac{122}{123}$

6 (a) quotient is $x + 3$, remainder is 1

(b) quotient is $x + 5$, remainder is 9

7 (a) quotient is $x^2 - 2x + 10$, remainder is -15

(b) quotient is $x^3 + 3x^2 + 6x + 18$, remainder is 55

8 $12x^3 - 55x^2 + 30x - 20$

9 (a) $x + 4$, 1 (b) $2x + 1$, 5

(c) $2x - 5$, 6 (d) $3x^2 + 6x + 14$, 24

10 A

11 $4x^2 + x - 3 = (4x - 3)(x + 1)$ so remainder is zero as it is a factor.

12 D **13** $x^3 - x^2 - 5x - 25 - \frac{120}{x - 5}$

14 (a) $R(x) = 16x^3 - 6x^2 + 4$; 108

(b) $R(x) = 3x^3 - 16x^2 - 17x + 22$; -152

(c) $R(x) = 4x^3 - 26x^2 - 28x + 36$; 34

15 (a) $x^4 - 3x - 6$ (b) 4

(c) $A(2) = 4$, $B(2) = 1$, $\frac{A(2)}{B(2)} = 4$, which is the same as the answer for **(b)**.

16 (a) $4x^3 + x + 8$

(b) (i) $4x^2 + 1 + \frac{8}{x}$ (ii) if $x = 0$, $\frac{8}{x}$ is undefined.

17 (a) $4x^2 - 10x + 6$

(b) $8x^3 - 30x^2 - 27x$

(c) (i) $8x - 22 - \frac{49}{x - 1}$

(ii) $x = 8$ cm, $x - 1 = 7$ cm $\Rightarrow$ 35 small rectangles

18 $2x^5 - x^4 - 3x^3 - 1$

19 $3x^3 - 14x^2 + 13x + 6$

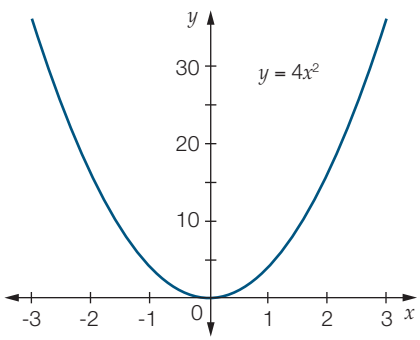
20 **(a)** $1, x+\frac{1}{x+1}, x^2+1, x^3+x+\frac{1}{x+1}, x^4+x^2+1$

(b) If $P_n = x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$ then for n odd, answers are $1, x^2+1, x^4+x^2+1$, so $\frac{P_n(x)}{x+1} = Q_{n-1}$, where $Q_{n-1}(x) = x^{n-1} + x^{n-3} + \ldots$ which are even powers of x when n odd.

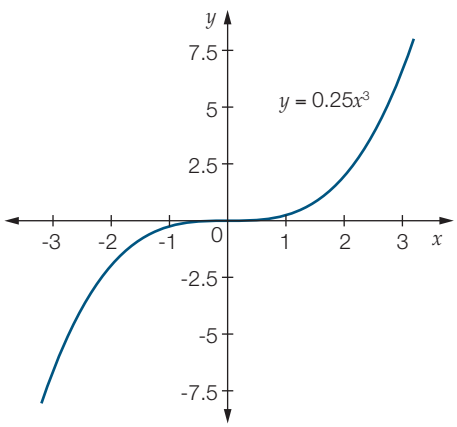
(c) If $P_n = x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$ then for n even, answers are $x+\frac{1}{x+1}, x^3+x+\frac{1}{x+1}$, so $\frac{P_n(x)}{x+1} = Q_{n-1} + \frac{1}{x+1}$, where $Q_{n-1}(x) = x^{n-1} + x^{n-3} + \ldots$ which are odd powers of x when n is even.

Half-time 10 (p. 495)

1 **(a)**



(b)

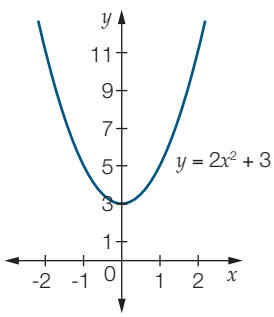


2 **(a)** $3x^4 + 4x^2 - 2x - 5$

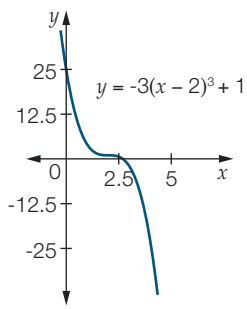
(b) 0

(c) $3x^3 + 3x^2 + 7x + 3$ rem 2

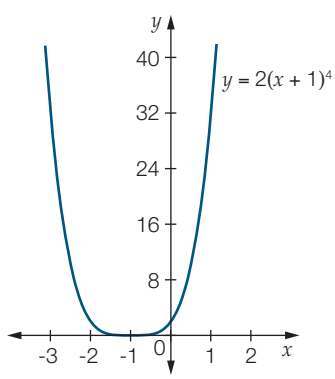
3 **(a)**



(b)

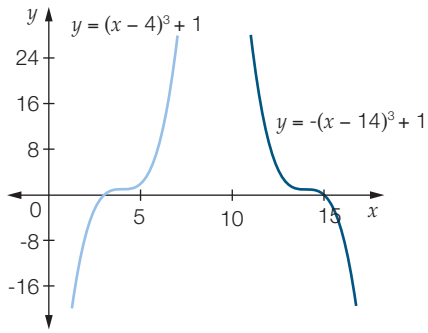


(c)



4 **(a)** $P(3) = 0$ **(b)** $x^2 + x - 6$

5 **(a)**



(b) $x = 9$

6 **(a)** $2x^3 + 11x^2 + 11x$

(b) $8x^5 + 6x^4 - 27x^3 + 10x^2 + x - 2$

Exercise 10.4 (p. 499)

1 $x^2 + 6x + 7 + \frac{17}{x-2}$, $P(2) = 17$. The remainder $= P(2)$.

2 **(a)** $P(2) = 1$ **(b)** $P(-3) = 104$

3 $(x-1)(x-2)(x-3)$

4 $(x-1)(x+1)(x+2)(x-3)$

5 **(a)** 44 **(b)** -23 **(c)** $-\frac{21}{16}$

(d) $-\frac{31}{32}$ **(e)** 550 **(f)** -140

6 **(a)** $(x-1)(x-2)(x+4)$ **(b)** $(x+1)(x-3)^2$

(c) $(x+1)(x-2)(x-5)$ **(d)** $(x-1)(x-2)(x+6)$

(e) $(x+2)^3$ **(f)** $(x+1)^2(x-5)$

7 **(a)** -86 **(b)** -5 **(c)** -12 **(d)** -31

8 **(a)** $(x-1)(x+1)(x+2)(x+4)$ **(b)** $x(x-1)(x-2)(x-4)$

(c) $(x-1)^2(x-2)(x+3)$ **(d)** $(x-1)^3(x+5)$

(e) $(x-1)(x+1)(x-2)^2$ **(f)** $(x-1)(x+2)(x-3)(x+4)$

9 B **10** D **11** -6

12 **(a)** $(x-1)(x-2)(x-3) = 0$, $x = 1, 2$ or 3

(b) $(x+1)(x-2)(x+2) = 0$, $x = -2, -1$ or 2

(c) $(x+2)(x+3)(x+4) = 0$, $x = -2, -3$ or -4

(d) $(x-1)(x+1)(x-2)(x+3) = 0$, $x = -1, -3, 1$ or 2

(e) $(x-1)(x+1)(x-2)(x+2) = 0$, $x = \pm 1, \pm 2$

(f) $x(x-1)(x-\sqrt{6})(x+\sqrt{6}) = 0$, $x = \pm\sqrt{6}$, 0 or 1

13 **(a)** $(x-1)^2(x-2)^2$ **(b)** 4 cm **(c)** 1 s, 2 s

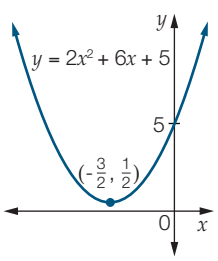
(d) The top is moving away from the fixed point.

14 $x^3 - 3x^2 - x + 1$, $x^3 - 2x^2 - 3x - 2$

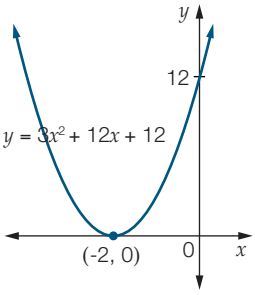
15 $x^4 - 4x^3 + x^2 + 6x = 0$ and multiples of the LHS.

Exercise 10.5 (p. 508)

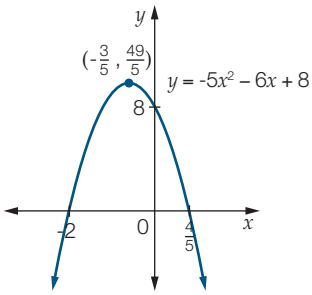
1 **(a)**



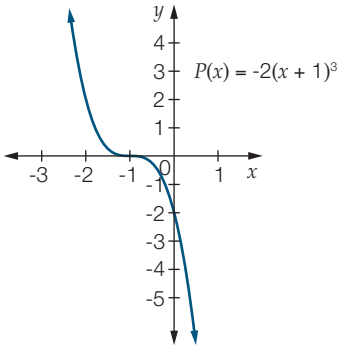
(b)



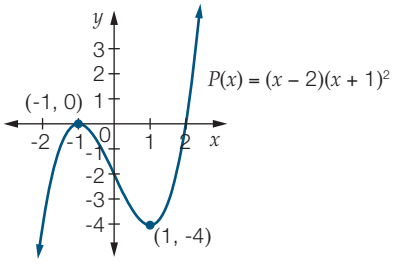
(c)



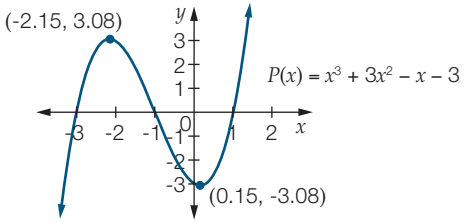
2 **(a)** (-1, 0)



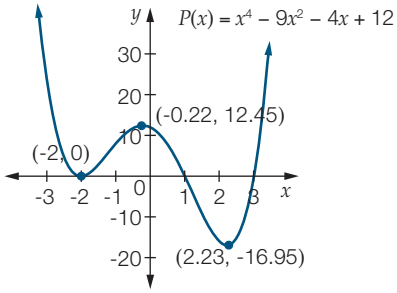
(b) (-1, 0) and (2,0)



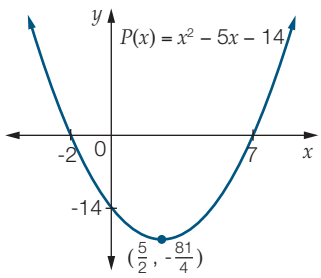
(c) (-3, 0), (-1, 0), (1, 0)



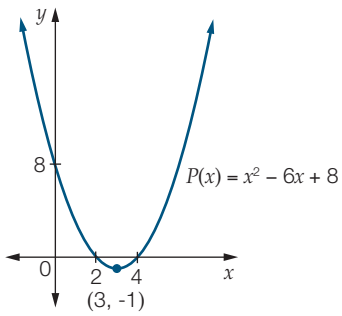
3



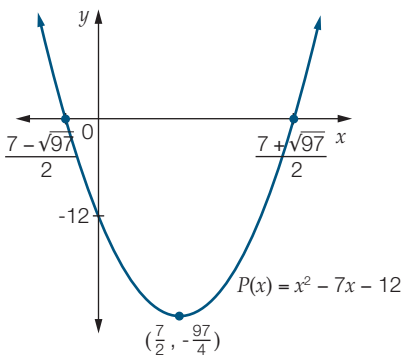
4 **(a)**



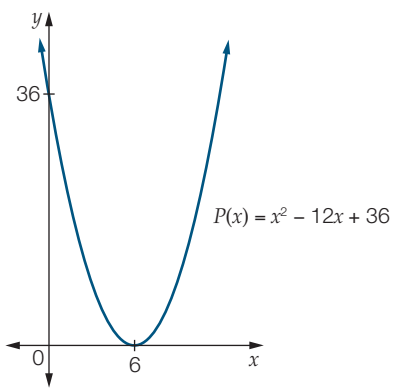
(b)



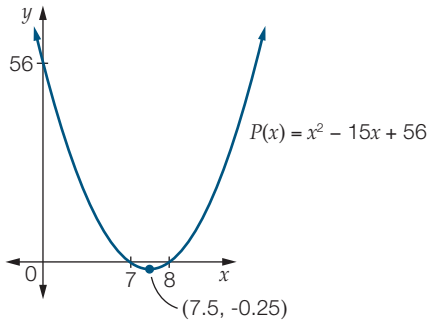
(c)



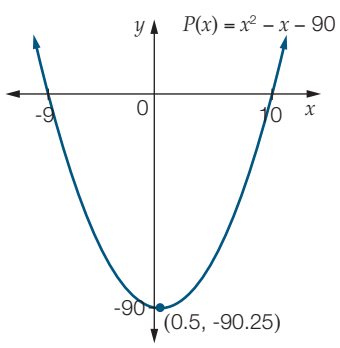
(d)



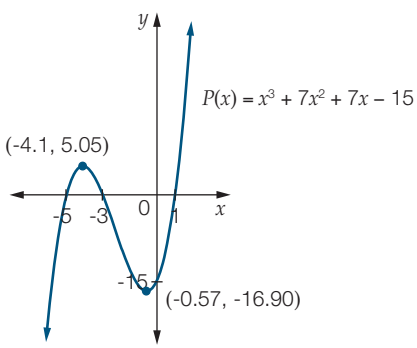
(e)



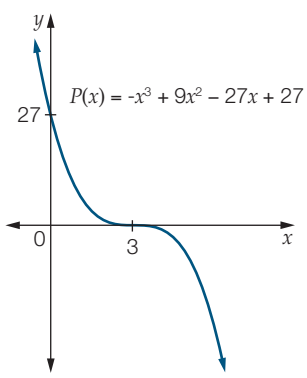
(f)



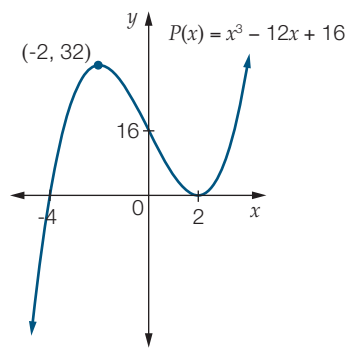
5 (a)



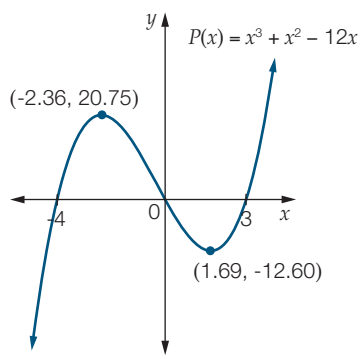
(b)



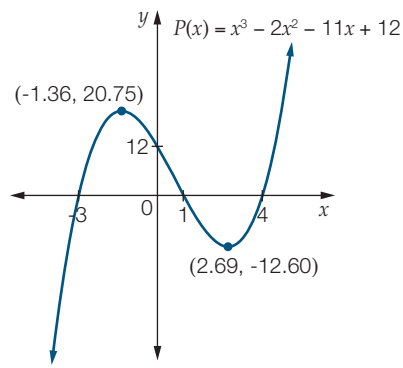
(c)



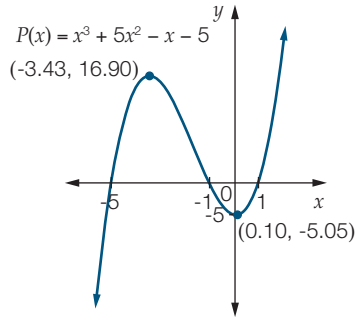
(d)



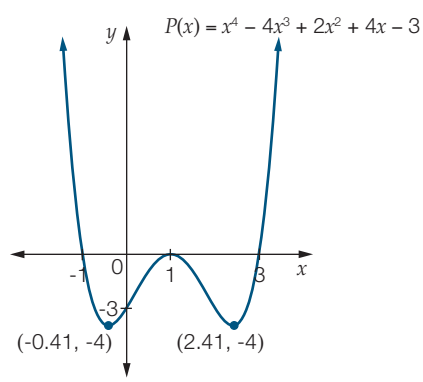
(e)



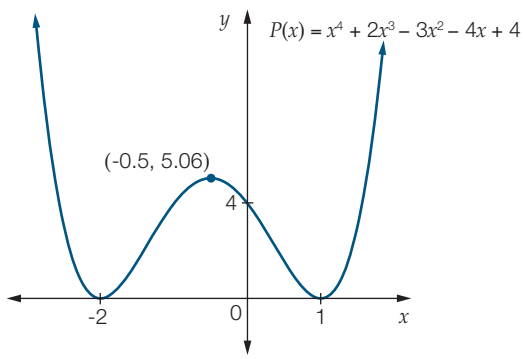
(f)



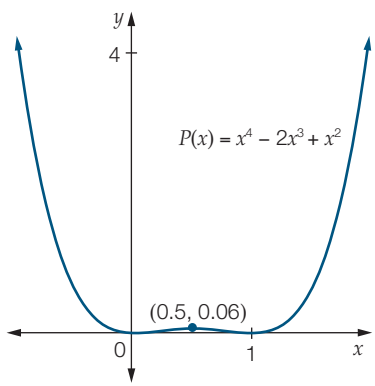
6 (a)



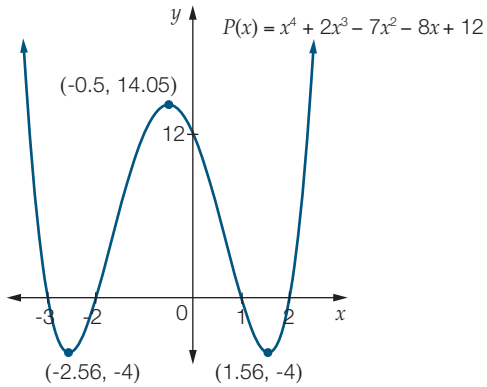
(b)



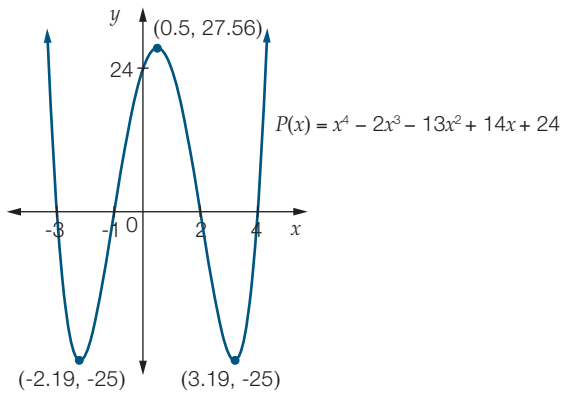
(c)



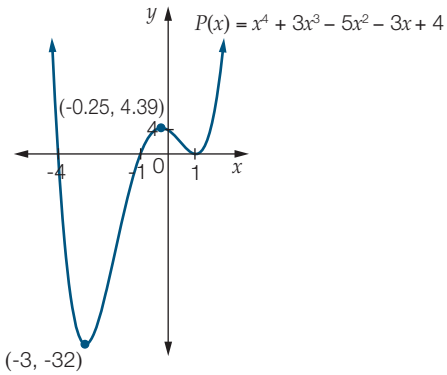
(d)



(e)

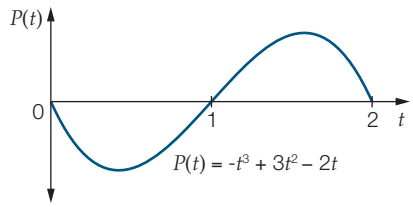


(f)



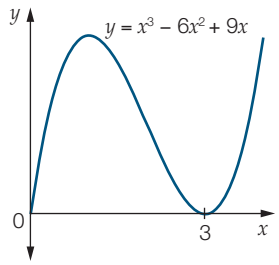
7 D **8** D **9** $x = 5, x = -7$ **10** $x = 1, x = -2$

11 **(a)** $P(t) = -t(t-1)(t-2)$



(b) 1 s, 0.39 m **(c)** 1 s, 0.39 m

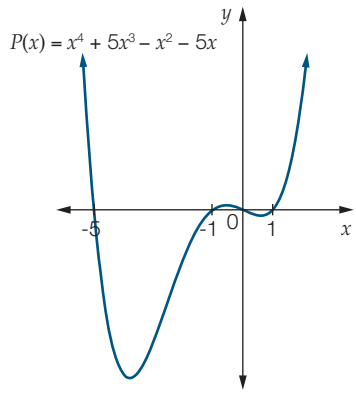
12 **(a)** $y = x(x-3)^2$



(b) (3, 0) **(c)** $y = -x(x-3)$

(d) Yes, as (2.5, 1.2) is not on the new parabola.

13 **(a)** $P(x) = x(x-1)(x+1)(x+5)$

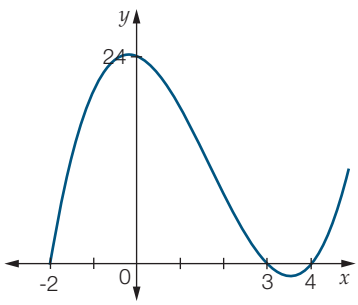


(b) negative pulse: $-5 < x < -1$ and $0 < x < 1$.
positive pulse: $-1 < x < 0$.

(c) **(i)** width 400 m, depth 61 m **(ii)** No

14 **(a)** $(x+2)(x-3)(x-4)$

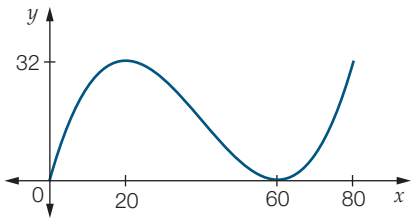
(b)



(c) 600 m

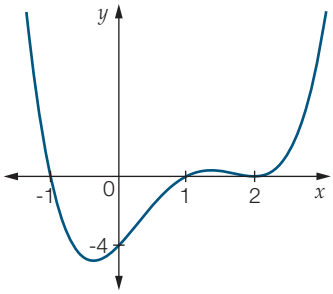
15 **(a)** $y = 0.001x(x-60)^2$

(b)



(c) 0.36 min

16 $P(x) = (x+1)(x-2)(x-2)(x-1)$



17 **(a)** $P_1(x) = (x+1)^2(x-3)^2$, $P_2(x) = (x+1)(x-3)^3$

(b) $P_1(x) = (x+1)^2(x-3)(x-4)$, $P_2(x) = (x+1)(x-3)^2(x-5)$

(c) $P_1(x) = x(x+1)(x-3)(x-4)$,
$P_2(x) = (x+1)(x-3)(x-5)(x+6)$

(d) $P_1(x) = 5(x+1)^3(x-3)$, $P_2(x) = 0.5(x+1)(x-3)^3$

Exercise 10.6 (p. 516)

1 **(a)** Domain 2, 3, 5; Range -1, 4, 11

(b) Domain -6, -4, -2; Range 2, 4, 6

(c) Domain 0, 3, 6; Range -6, -2, 4

2 **(a)** $(-\infty, 6)$ **(b)** $(-5, 7]$ **(c)** $[3, \infty)$

(d) $(1, 6)$ **(e)** $(-\infty, 9]$ **(f)** $[-2, 4]$

3 **(a)** Domain [0, 10]; Range [-8, 2]

(b) Domain $(-\infty, \infty)$; Range $[4, \infty)$

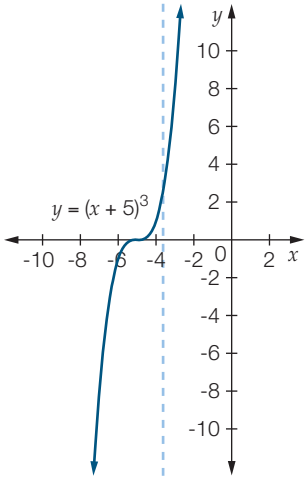
(c) Domain $(-\infty, \infty)$; Range $(-\infty, \infty)$

4 **(a)** one-to-one **(b)** one-to-one

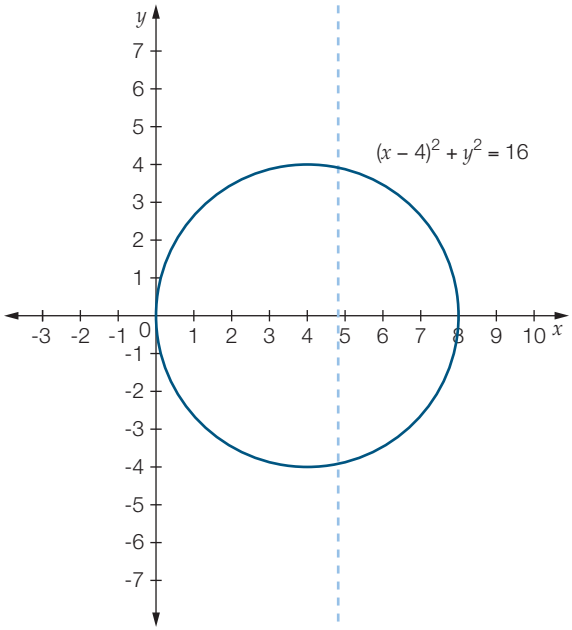
(c) many-to-one **(d)** one-to-one

(e) many-to-many **(f)** one-to-many

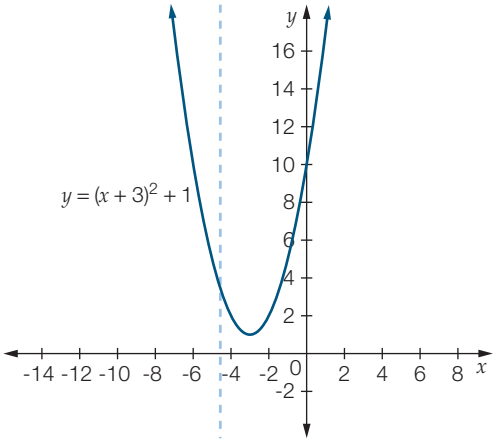
5 **(a)** function



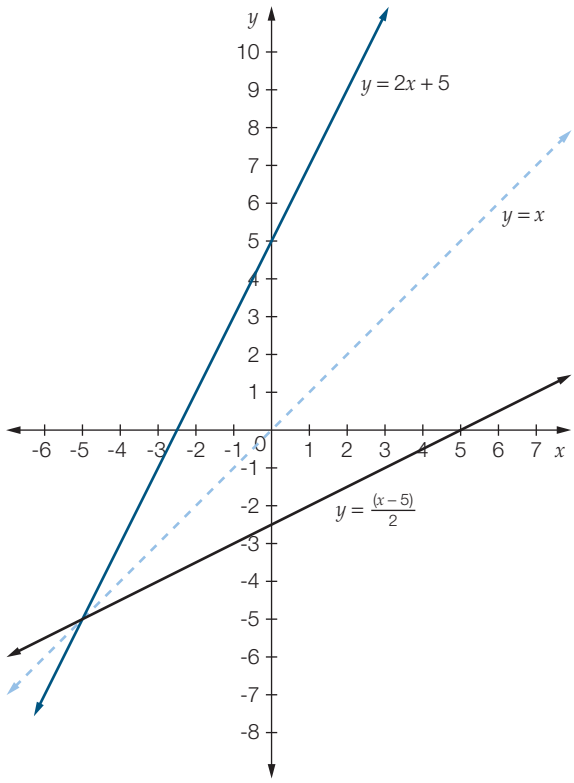
(b) relation



(c) function



6 $f^{-1}(x) = \frac{x-5}{2}$



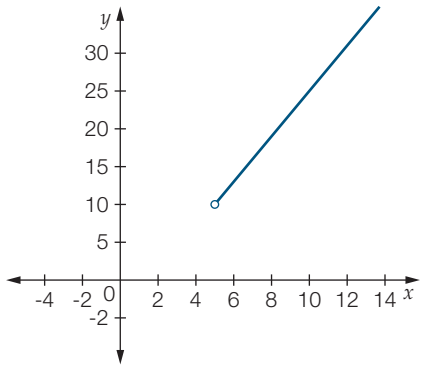
Domain $f(x)$ $(-\infty, \infty)$; Range $f(x)$ $(-\infty, \infty)$

Domain $f^{-1}(x)$ $(-\infty, \infty)$; Range $f^{-1}(x)$ $(-\infty, \infty)$

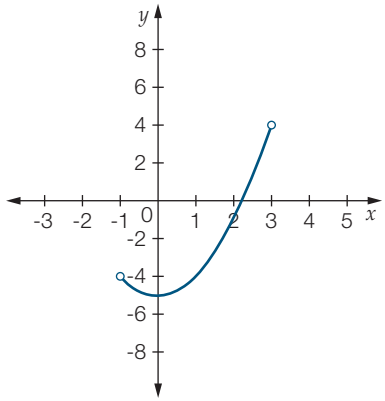
7 (a) $f(3) = 5$ (b) $f(0) = -7$ (c) $f(-1) = -11$

(d) $f(10) = 33$ (e) $f(6) = 17$ (f) $f(-5) = -27$

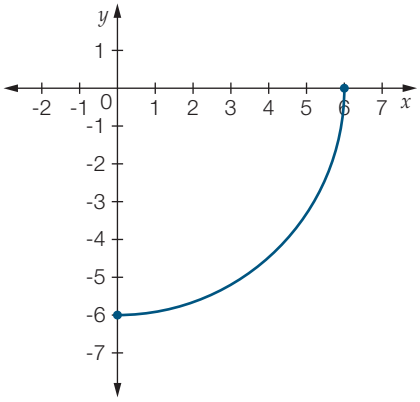
8 (a) function



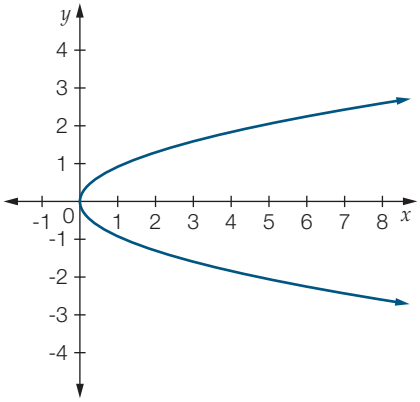
(b) function



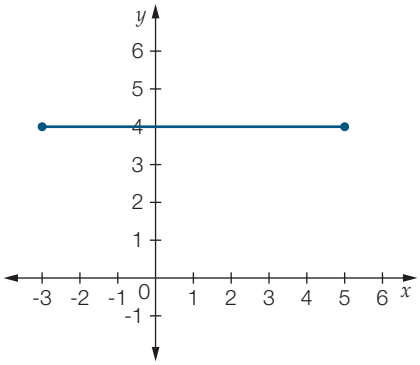
(c) function



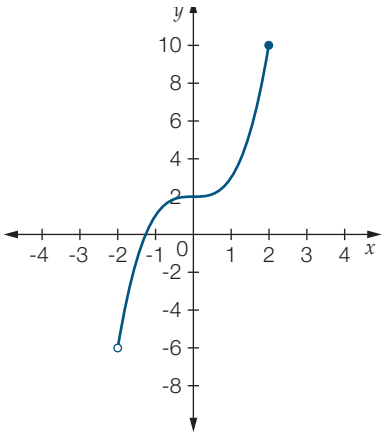
(d) relation



(e) function



(f) function



9 (a) Domain $[-1, \infty)$; Range $[0, \infty)$

(b) Domain $(-\infty, \infty)$; Range $(-2, \infty)$

(c) Domain $[-2, 2]$; Range $[-2, 0]$

10 (a) $x^2 + 4x + 5$

(b) $4x^2 + 4$

(c) $x^2 - 8x + 20$

(d) $\frac{x^2}{4} + 4$

11 (a) $1 - \frac{1}{x+1}$ (b) $1 - \frac{1}{2x}$ (c) $1 - \frac{1}{4-x}$ (d) $1 - x$

12 $y = \sqrt{4 - x^2}$; Range $[0, 2]$

$y = -\sqrt{4 - x^2}$; Range $[-2, 0]$

13 $f(x) = (x - 2)^2$; Domain $[0, \infty)$

$f(x) = (x - 2)^2$; Domain $(-\infty, 0]$

14 (a) $y = -3 + \sqrt{x + 2}$; Domain $[2, \infty)$

(b) $\sqrt{25 - x^2}$; Domain $[0, 5]$

15 $(1, 1)$

16 No. Domain $f(x)$ $(-\infty, 2]$ and domain $f^{-1}(x)[3, \infty)$, so the graphs will not intersect.

17 (a) Students' own answers, e.g. $f(x) = x^2 - 4$ $[0, \infty)$.

(b) $f^{-1}(x) = \sqrt{x + 4}$; Domain $[-4, \infty)$, Range $[0, \infty)$

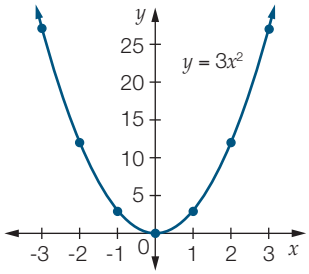
18 No, Sol is incorrect. $(x - 3)^2 + (y - 1)^2 = 4$ $[3, 5]$ would graph as a semicircle, the RHS of a circle with centre (3, 1) and radius 2. It would fail the vertical line test.

19 Dominic is correct. Linear equations in the form of $x = a$ are not functions. They do not pass the vertical line test.

Chapter review 10 (p. 518)

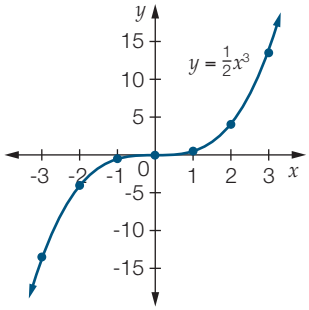
1 (a)

x	-3	-2	-1	0	1	2	3
y	27	12	3	0	3	12	27



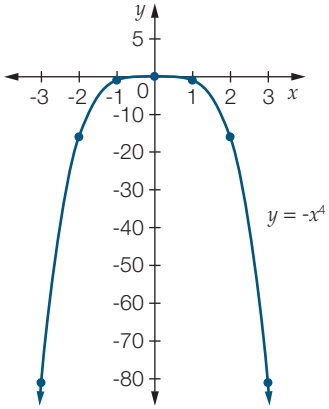
(b)

x	-3	-2	-1	0	1	2	3
y	-13.5	-4	-0.5	0	0.5	4	13.5

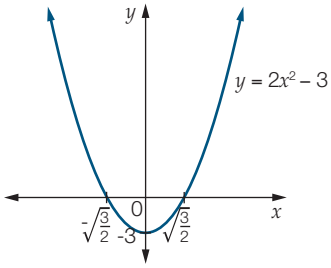


(c)

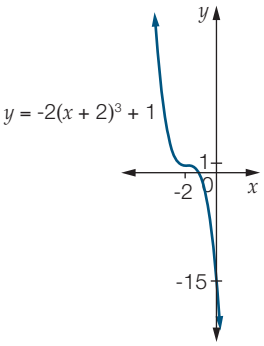
x	-3	-2	-1	0	1	2	3
y	-81	-16	-1	0	-1	-16	-81



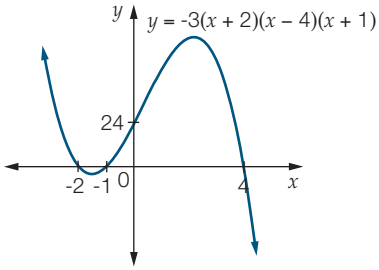
2 (a)



(b)



(c)



3 (a) (i) 13 (ii) 16 (iii) -42

(b) (i) $9x^2 + 5x$ (ii) $12x^3 - 13x^2 + 11x - 6$

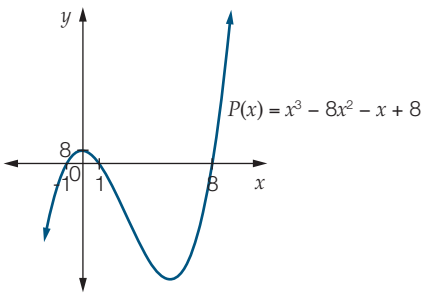
(iii) $3x^2 - 5x + 5$

4 $x^2 + x + 6, 1$

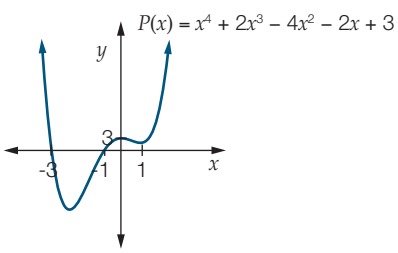
5 (a) -2 (b) $-\frac{5}{16}$

6 (a) $(x + 1)(x - 2)(x + 7)$ (b) $(x + 1)(x - 1)(x + 2)(x - 2)$

7 **(a)**



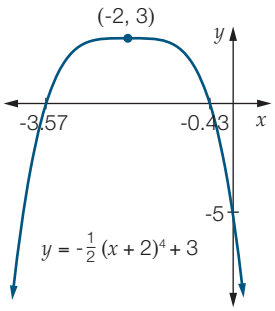
(b)



8 **(a)** $(-\infty, 6]$ **(b)** $[0, 10]$ **(c)** $(-4, \infty)$

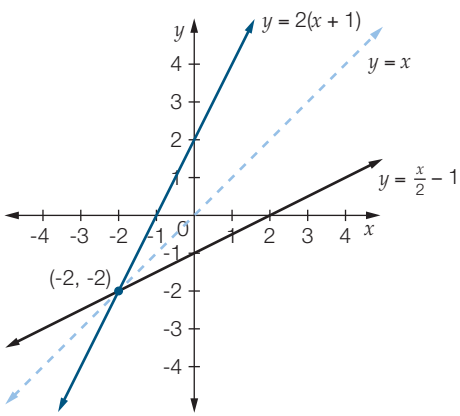
9 **(a)** $y = -\frac{1}{2}(x+2)^4 + 3$

(b)



10 remainder = -7

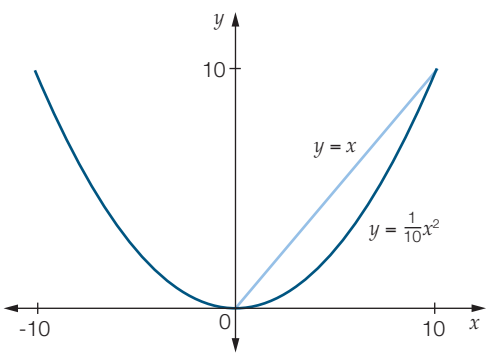
11



Domain $f(x)$ $(-\infty, \infty)$, Range $f(x)$ $(-\infty, \infty)$

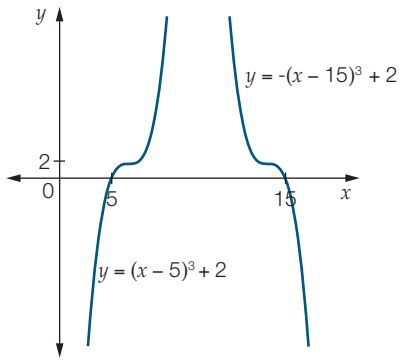
Domain $f^{-1}(x)$ $(-\infty, \infty)$, Range $f^{-1}(x)$ $(-\infty, \infty)$

12 **(a)**



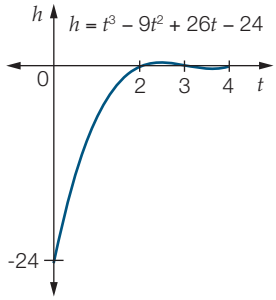
(b) 10 cm

13 **(a)**



(b) $x = 10$

14 **(a)** $h = (t-2)(t-3)(t-4)$



(b) The capsule is released at a depth of 24 m. It breaks the surface after 2 seconds and submerges 1 second later. It stays just below the surface for another second before breaking the surface again.

Chapter 11

Recall 11 (p. 520)

1 **(a)** $\frac{1}{5}$ **(b)** $\frac{4}{13}$

2 **(a)** $\frac{7}{8}$ **(b)** $\frac{11}{15}$ **(c)** $\frac{2}{5}$ **(d)** $\frac{3}{10}$ **(e)** 0.4 **(f)** 0.01

3 **(a)** boy, girl; boy, boy; girl, boy; girl, girl

(b) head, tail

4 **(a)** **(i)** 25 **(ii)** 25 **(iii)** 6 **(iv)** 9 **(v)** 15

(b) **(i)** 1, 2, 3, 4, 5, 6 **(ii)** 3

Exercise 11.1 (p. 526)

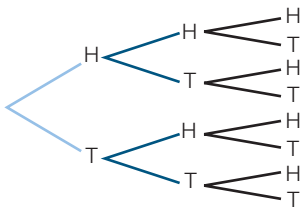
1 (a) $\frac{2}{7}$ (b) $\frac{5}{7}$

(c) We assume that the probability of the car being hard to start is the same every day. Really we are assuming that it is not dependent on the weather.

2 (a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{3}{8}$ (d) $\frac{7}{8}$

3 (a) $\frac{1}{12}$ (b) $\frac{1}{2}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$ (e) $\frac{1}{6}$ (f) $\frac{1}{2}$

4 (a)

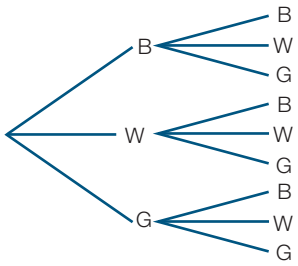


(b) (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)

(c) (i) $\frac{1}{8}$ (ii) $\frac{1}{8}$ (iii) $\frac{1}{8}$ (iv) $\frac{1}{4}$ (v) $\frac{1}{2}$

5 (a) $\frac{7}{12}$ (b) $\frac{5}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{24}$ (e) $\frac{1}{12}$ (f) $\frac{11}{12}$

6 (a)



(b) Each sock has an equal probability of being selected from each drawer, so the outcomes for the new event are also equally likely.

(c) (i) $\frac{1}{9}$ (ii) $\frac{2}{9}$ (iii) $\frac{1}{3}$ (iv) $\frac{2}{3}$

7 (a) $\frac{1}{12}$ (b) $\frac{11}{12}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$

8 (a) B (b) D (c) C

9 (a) C (b) B (c) B

10 $\frac{4}{9}$

11 (a) A (b) C

12 $\frac{7}{9}$

13 (a) $\frac{1}{45}$ (b) $\frac{1}{5}$ (c $\frac{22}{45}$ (d) $\frac{23}{45}$ (e) 0 (f) $\frac{11}{45}$

14 (a) Possible totals

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(b) 36

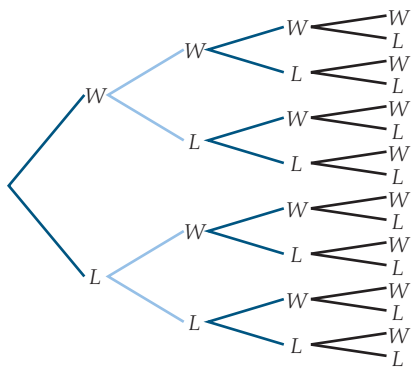
(c) Each die individually has an equal chance of producing 1, 2, 3, 4, 5, 6 so together the dice have an equal chance of producing each of the 36 possible outcomes.

(d) $\frac{1}{9}$ (e) $\frac{5}{12}$ (f) $\frac{1}{6}$

15 No, as some friends would be closer than others, so more likely to be contacted by Skye.

16 (a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$ (e) $\frac{1}{2}$ (f) $\frac{2}{3}$

17 (a)



(b) (*W*, *W*, *W*, *W*), (*W*, *W*, *W*, *L*), (*W*, *W*, *L*, *W*), (*W*, *W*, *L*, *L*), (*W*, *L*, *W*, *W*), (*W*, *L*, *W*, *L*), (*W*, *L*, *L*, *W*), (*W*, *L*, *L*, *L*), (*L*, *W*, *W*, *W*), (*L*, *W*, *W*, *L*), (*L*, *W*, *L*, *W*), (*L*, *W*, *L*, *L*), (*L*, *L*, *W*, *W*), (*L*, *L*, *W*, *L*), (*L*, *L*, *L*, *W*), (*L*, *L*, *L*, *L*); 16 possible outcomes

(c) (i) $\frac{1}{16}$ (ii) $\frac{3}{8}$ (iii) $\frac{5}{16}$

(iv) $\frac{1}{2}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{4}$

18 No, the existing cars in the court have no influence over what the new residents drive.

19 (a) Sample space

		Die 2					
		2	4	6	8	10	12
Die 1	1	(1, 2)	(1, 4)	(1, 6)	(1, 8)	(1, 10)	(1, 12)
	1	(1, 2)	(1, 4)	(1, 6)	(1, 8)	(1, 10)	(1, 12)
	2	(2, 2)	(2, 4)	(2, 6)	(2, 8)	(2, 10)	(2, 12)
	2	(2, 2)	(2, 4)	(2, 6)	(2, 8)	(2, 10)	(2, 12)
	3	(3, 2)	(3, 4)	(3, 6)	(3, 8)	(3, 10)	(3, 12)
	3	(3, 2)	(3, 4)	(3, 6)	(3, 8)	(3, 10)	(3, 12)

(b) Possible totals

		Die 2					
		2	4	6	8	10	12
Die 1	1	3	5	7	9	11	13
	1	3	5	7	9	11	13
	2	4	6	8	10	12	14
	2	4	6	8	10	12	14
	3	5	7	9	11	13	15
	3	5	7	9	11	13	15

(c) (i) $\frac{1}{3}$ (ii) $\frac{2}{9}$ (iii) $\frac{7}{18}$ (iv) $\frac{7}{18}$

(d) You cannot just look at one diagonal to find the totals; you have to check all over the lattice. However, it is not much more difficult in this case due to the nature of the numbers involved.

(e) Possible products

		Die 2					
		2	4	6	8	10	12
Die 1	1	2	4	6	8	10	12
	1	2	4	6	8	10	12
	2	4	8	12	16	20	24
	2	4	8	12	16	20	24
	3	6	12	18	24	30	36
	3	6	12	18	24	30	36

(f)

Product	2	4	6	8	10	12	16	18	20	24	30	36
Probability	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{18}$

20 (a) Even though the past long-term trend is $\frac{19}{20}$, short-term variation is possible. It is also possible that Aiden has switched jobs.

(b) The probability does not change. The balls do not have memories so they don't know if they have been drawn before.

21 (a) A is the event rolling a 1 or 2

(b) B is the event rolling a multiple of 3

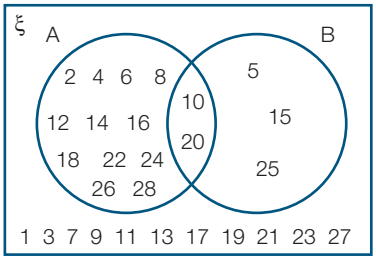
(c) C is the event rolling a number greater then 7

22 4W/1L, 8W/2L, 12W/3L

23 The correct answer is $\frac{7}{8}$. Sharanpreet has calculated the probability of exactly 1 boy, not at least 1 boy.

Exercise 11.2 (p. 532)

1 (a)



(b) (i) $\frac{1}{2}$ (ii) $\frac{1}{14}$ (iii) $\frac{23}{28}$

(c)

	B	not A
A	$\frac{1}{14}$	$\frac{3}{7}$
not A	$\frac{3}{28}$	$\frac{11}{28}$

2 (a) $\frac{3}{16}$ (b) $\frac{7}{16}$ (c) $\frac{1}{16}$ (d) $\frac{5}{16}$ (e) $\frac{1}{2}$ (f) $\frac{1}{4}$

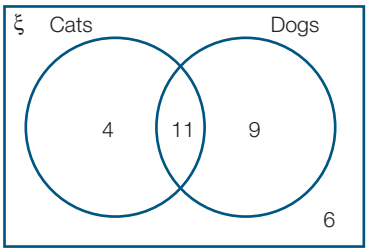
3 (a)

	Piano	not Piano	
Singing	4	2	6
not Singing	4	10	14
	8	12	20

(b) 4 take both lessons

(c) (i) $\frac{1}{10}$ (ii) $\frac{1}{5}$

4 (a)



(b) (i) $\frac{2}{15}$ (ii) $\frac{3}{10}$ (iii) $\frac{11}{30}$

5 (a)

	B	not B
A	$\frac{1}{8}$	$\frac{5}{16}$
not A	$\frac{1}{16}$	$\frac{1}{2}$

(b)

	C	not C
D	0.2	0.5
not D	0.2	0.1

6 (a) 1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20

(b) 1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20

(c) 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 19, 20

(d) 3, 7, 12, 17, 20

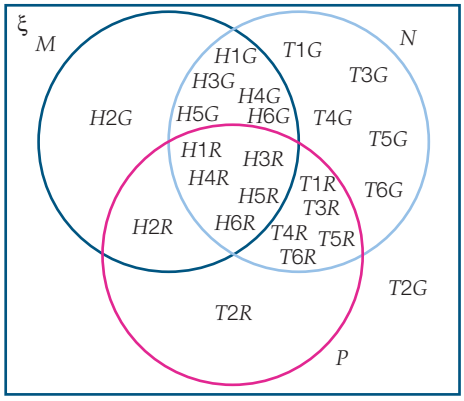
(e) 2, 7, 9, 12

(f) 5, 7, 12, 14

(g) 7, 12

(h) 6, 11, 16, 19

7 (a)



(b) not M = {T1R, T1G, T2R, T2G, T3R, T3G, T4R, T4G, T5R, T5G, T6R, T6G}

not N = {H2R, H2G, T2R, T2G}

not P = {H1G, H2G, H3G, H4G, H5G, H6G, T1G, T2G, T3G, T4G, T5G, T6G}

(c) No, because there are 4 red balls and only 3 green balls so there is a greater chance of H1R occurring than H1G.

8 (a)

	TV	no TV	
internet	236	1200	**1436**
no internet	554	10	**564**
	790	**1210**	**2000**

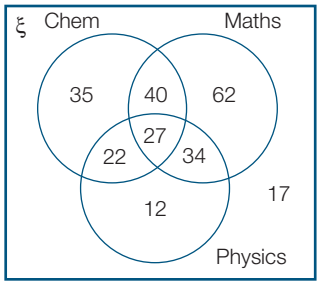
(b) $\frac{59}{500}$

9 (a)

	Twitter	not Twitter	
Facebook	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
not Facebook	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
	$\frac{3}{8}$	$\frac{5}{8}$	1

(b) (i) $\frac{1}{8}$ (ii) $\frac{5}{8}$

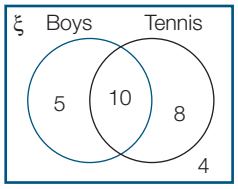
10 (a)



(b) 62 (c) 12 (d) 35 (e) 249

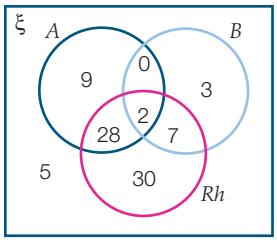
(f) $\frac{163}{249}$, it may be similar but it is unlikely to be exactly the same.

11 (a)



(b) (i) $\frac{8}{27}$ (ii) $\frac{5}{27}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{3}$

12 (a)



(b) 84 (c) $\frac{1}{2}$ (d) $\frac{5}{12}$

(e) A+: $\frac{1}{3}$ = 33.3%; A–: $\frac{3}{28}$ = 10.7%

B+: $\frac{1}{12}$ = 8.3%; B –: $\frac{1}{28}$ = 3.6%

AB+: $\frac{1}{42}$ = 2.4%; AB–: 0%

O+: $\frac{5}{14}$ = 35.7%; O–: $\frac{5}{84}$ = 6%

(f) The A group is larger than expected and the O group is smaller than expected.

13 Mert has not taken into account students who may get their news from either the TV or the internet, but neither has taken into account there may be students who get their news from neither.

14 The existing figures add up to over 200 people because we are told both 36 and 44 people have never received a speeding fine.

Exercise 11.3 (p. 537)

1 (a) $\frac{1}{4}$ (b) $\frac{3}{8}$

2 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

3 (a) yes (b) no (c) yes

4 (a) D (b) A (c) D

5 (a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$ (d) $\frac{3}{5}$ (e) $\frac{17}{20}$

6 (a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) $\frac{2}{5}$ (d) $\frac{11}{20}$ (e) $\frac{11}{20}$

(f) $\frac{9}{10}$ (g) $\frac{1}{4}$ (h) $\frac{3}{20}$

7 (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{13}$ (d) $\frac{1}{52}$ (e) $\frac{27}{52}$

(f) $\frac{7}{26}$ (g) $\frac{1}{26}$ (h) $\frac{2}{13}$ (i) $\frac{9}{52}$ (j) $\frac{3}{26}$

(k) $\frac{11}{13}$ (l) $\frac{1}{4}$

8 (a) Six pairs: AB, AC, AD, BC, BD, CD

(b) AC and BC are mutually exclusive.

(c) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{3}$, $P(D) = \frac{1}{3}$

(d) $P(A \text{ or } B) = \frac{1}{2}$, $P(A \text{ or } C) = \frac{5}{6}$, $P(A \text{ or } D) = \frac{2}{3}$,

$P(B \text{ or } C) = \frac{2}{3}$, $P(B \text{ or } D) = \frac{1}{2}$, $P(C \text{ or } D) = \frac{1}{2}$

(e) $P(A \text{ and } B) = \frac{1}{3}$, $P(A \text{ and } C) = 0$, $P(A \text{ and } D) = \frac{1}{6}$,

$P(B \text{ and } C) = 0$, $P(B \text{ and } D) = \frac{1}{6}$, $P(C \text{ and } D) = \frac{1}{6}$

(f) $P(A) + P(B) = \frac{5}{6}$

$P(A) + P(C) = \frac{5}{6}$

$P(A) + P(D) = \frac{5}{6}$

$P(B) + P(C) = \frac{2}{3}$

$P(B) + P(D) = \frac{2}{3}$

$P(C) + P(D) = \frac{2}{3}$

(g) In general, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. However, if A and B are mutually exclusive $P(A \text{ and } B) = 0$, and so $P(A \text{ or } B) = P(A) + P(B)$.

9 (a) The vehicle types listed are not mutually exclusive, so there could be arguments as to where a particular vehicle should be placed; or a particular vehicle may be registered in more than one category, thereby making later calculations difficult.

(b) The categories have to be clearly mutually exclusive but also cover every type of vehicle that may go past. Including the category 'other' would cover the second requirement.

10 'multiple of 5' and 'multiple of 7'; 'multiple of 4' and 'multiple of 9'; 'even number' and 'odd number'; 'composite number' and 'prime number' are some of the pairs that could be listed

Half-time 11 (p. 539)

1 (a)

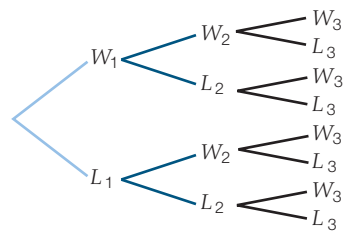
	iPhone	not iPhone	
iPad	60	15	**75**
not iPad	150	175	**325**
	210	**190**	**400**

(b) (i) $\frac{3}{8}$ (ii) $\frac{3}{80}$

2 (a) $\frac{2}{10} = \frac{1}{5}$ (b) $\frac{1}{2}$ (c) $\frac{6}{10} = \frac{3}{5}$ (d) $\frac{3}{10}$

3 0.09

4 (a) If W = Suzanne wins and L = Suzanne loses:



$W_1W_2W_3$, $W_1W_2L_3$, $W_1L_2W_3$, $W_1L_2L_3$, $L_1W_2W_3$, $L_1W_2L_3$, $L_1L_2W_3$, $L_1L_2L_3$

(b) 8

(c) (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$

(iv) $\frac{1}{2}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{4}$

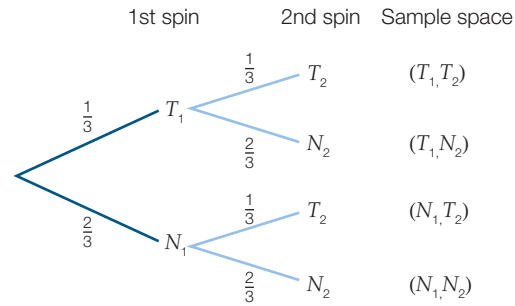
5 (a) A and B; A and C; B and C

(b) $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{2}$; $P(C) = \frac{1}{8}$

(c) A and C are mutually exclusive.

Exercise 11.4 (p. 542)

1 (a)



Probability of each outcome

$P(T_1, T_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

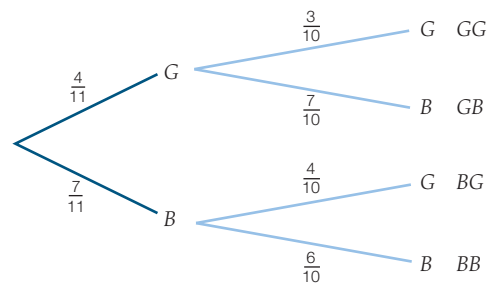
$P(T_1, N_2) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

$P(N_1, T_2) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

$P(N_1, N_2) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

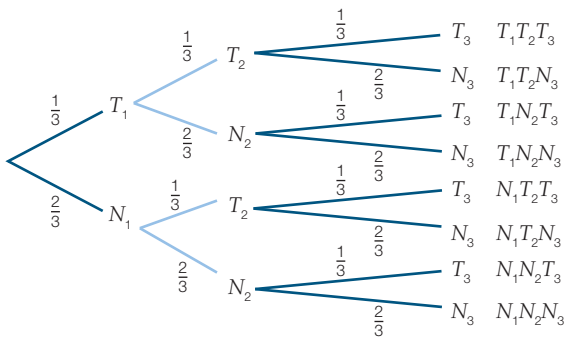
(b) (i) $\frac{1}{9}$ (ii) $\frac{4}{9}$ (iii) $\frac{4}{9}$ (iv) $\frac{5}{9}$

2



(a) $\frac{42}{110} = \frac{21}{55}$ (b) $\frac{98}{110} = \frac{49}{55}$

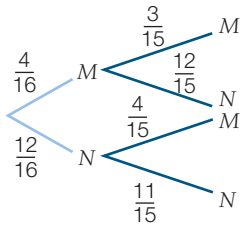
3 (a)



(b) $\frac{1}{27}$ **(c)** $\frac{8}{27}$

4 (a) B **(b)** D **(c)** B **(d)** C **(e)** B

5 (a)

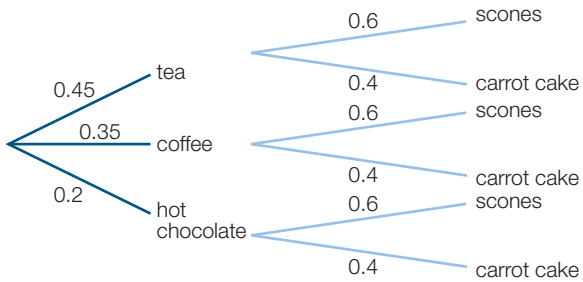


(b) MM, MN, NM, NN

(c) $\frac{1}{20}$ **(d)** $\frac{2}{5}$ **(e)** $\frac{11}{20}$

6 (a) 0.2 **(b)** 0.4

(c)



(d) **(i)** 0.27 **(ii)** 0.08 **(iii)** 0.35 **(iv)** 0.79

7 (a) **(ii)** $P(G_1G_2) = 0.64$, $P(G_1D_2) = 0.16$, $P(D_1G_2) = 0.16$, $P(D_1D_2) = 0.04$

(iii) 1

(b) **(ii)** $P(F_1F_2) = \frac{1}{36}$, $P(F_1N_2) = \frac{5}{36}$, $P(N_1F_2) = \frac{5}{36}$,

$P(N_1N_2) = \frac{25}{36}$

(iii) 1

(c) **(ii)** $P(HA) = \frac{1}{60}$, $P(H \text{ not } A) = \frac{1}{12}$, $P(A \text{ not } H) = \frac{3}{20}$,

$P(\text{not } H \text{ not } A) = \frac{3}{4}$

(iii) 1

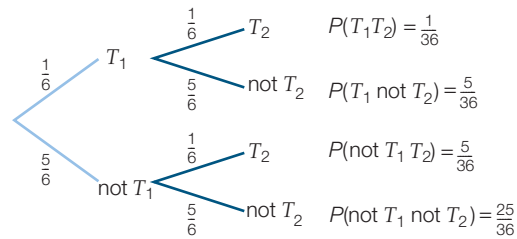
(d) **(ii)** $P(B_1B_2B_3) = 0.064$, $P(B_1B_2C_3) = 0.096$,
$P(B_1C_2B_3) = 0.096$, $P(B_1C_2C_3) = 0.144$,
$P(C_1B_2B_3) = 0.096$, $P(C_1B_2C_3) = 0.144$,
$P(C_1C_2B_3) = 0.144$, $P(C_1C_2C_3) = 0.216$

(iii) 1

8 (a) 0.008 **(b)** 0.096 **(c)** 0.488

9 (a) not T = a 3 is not rolled with a die

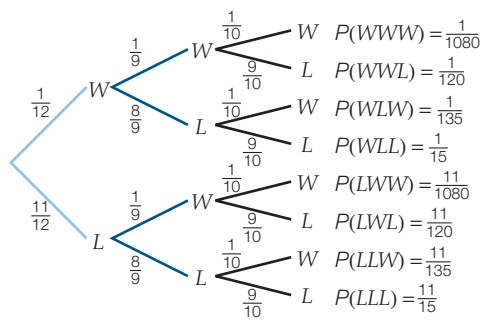
(b)



(c) **(i)** $\frac{1}{36}$ **(ii)** $\frac{25}{36}$ **(iii)** $\frac{5}{18}$ **(iv)** $\frac{11}{36}$ **(v)** $\frac{35}{36}$

10 (a) $\frac{1}{169}$ **(b)** $\frac{1}{8}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{1}{2}$

11 (a)



(b) **(i)** $\frac{1}{1080}$ **(ii)** $\frac{259}{1080}$ **(iii)** $\frac{288}{1080} = \frac{4}{15}$

12

Total	0	1	2
Probability	0.327 68	0.4096	0.2048

Total	3	4	5
Probability	0.0512	0.0064	0.000 32

13 (a)

Number of rolls	1	2	3
Probability	$0.1\dot{6}$	$0.13\dot{8}$	$0.115\,\dot{7}4\dot{0}$

Number of rolls	4	5	6
Probability	0.096 45	0.080 376	0.066 98

(b) approximately 0.665

(c) This means that it could still take more than six rolls to get the first 6. In fact, it could (theoretically) take an infinite number of rolls.

14 (a) $\frac{7}{29}$ **(b)** $\frac{3}{11}$

(c) R3 = $\frac{5}{16}$, R4 = $\frac{4}{11}$, R5 = $\frac{3}{7}$, R6 = $\frac{1}{2}$, R7 = $\frac{1}{2}$

(d) The probabilities add to more than one. This is because we assumed that elimination had not taken place before calculating the probability for the next round. This means the events did not all stem from the same tree but a new path was started after each round.

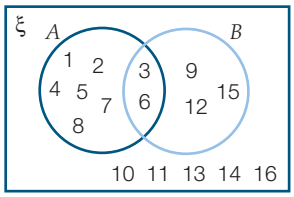
15 The P(leading team winning) would be greater than 0.5, an estimate would be 0.9. Leading 15 to 0 means this team only requires another 10 points whereas the trailing team still requires 25 points. Also, the result of the first 15 points challenges our assumption of the teams being equal in ability.

16 Students will produce their own diagrams and explanations.

17 The two events are not mutually exclusive, in fact, playing one of the sports increases the possibility of playing the other, as the person has a demonstrated interest in sport. Armapreet has treated them as mutually exclusive.

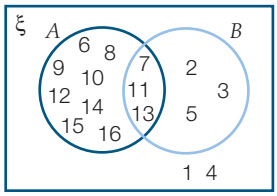
Exercise 11.5 (p. 549)

1 (a) (i)



(ii) $\frac{2}{5}$ **(iii)** $\frac{1}{4}$

(b) (i)



(ii) $\frac{1}{2}$ **(iii)** $\frac{3}{11}$

2 0.5746

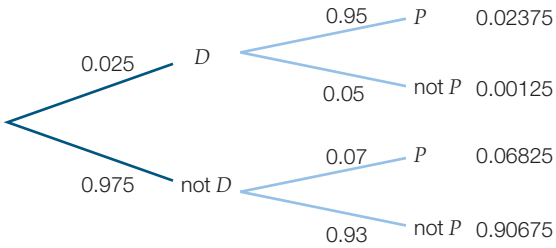
3 (a) (i) $\frac{2}{3}$ **(ii)** $\frac{3}{7}$

(b) (i) $0.1\dot{5}$ **(ii)** $0.2414 = \frac{7}{29}$

4 (a) C **(b)** D

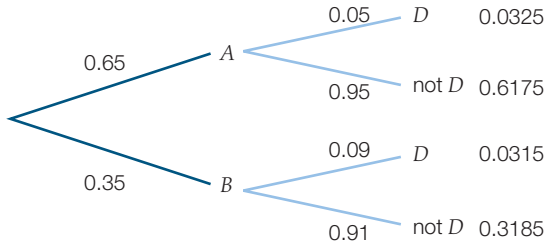
5 (a) $\frac{1}{13}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{1}{13}$ **(d)** $\frac{1}{2}$

6 (a) D = 'is infected with the disease'
P = 'tests positive for the disease'



(b) 0.2582 **(c)** 0.9986

7 (a) D = 'shoe is defective'



(b) 0.5078 **(c)** 0.3403

8 (a) 0.9375 **(b)** 0.7895 **(c)** 0.8

9 (a) 0.784 **(b)** 0.91 **(c)** 0.7

(d) 0.83692 **(e)** 0.973

10 $0.\dot{5}$

11 (a) Not enough information has been provided to answer the question.

(b) Given $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \text{ and } B) = 0.1$, find $P(A \text{ given } B)$. $P(A \text{ given } B) = 0.25$

Exercise 11.6 (p. 553)

1 $\frac{1}{10}$

2 (a) $\frac{1}{78}$ **(b)** $\frac{9}{39} = \frac{3}{13}$

3 (a) $\frac{10}{27}$ **(b)** $\frac{1}{9}$ **(c)** $\frac{2}{9}$ **(d)** $\frac{16}{27}$

4 (a) $\frac{3}{40}$ **(b)** $\frac{3}{40}$ **(c)** $\frac{27}{40}$

5 (a) I **(b)** D **(c)** I **(d)** D **(e)** D **(f)** D

6 (a) $\frac{1}{35}$ **(b)** $\frac{2}{35}$ **(c)** $\frac{4}{15}$ **(d)** $\frac{2}{35}$

(e) $\frac{16}{105}$ **(f)** $\frac{16}{105}$ **(g)** $\frac{37}{105}$ **(h)** $\frac{68}{105}$

7 (a) A **(b)** D **(c)** A

8 (a) $\frac{3}{7}$ **(b)** $\frac{4}{35}$ **(c)** $\frac{2}{7}$

9 (a) $\frac{5}{51}$ **(b)** $\frac{28}{153}$ **(c)** $\frac{2}{51}$ **(d)** $\frac{49}{153}$

(e) $\frac{104}{153}$ **(f)** $\frac{8}{17}$ **(g)** $\frac{29}{51}$ **(h)** $\frac{91}{153}$

10 (a) $\frac{1}{8}$ **(b)** $\frac{1}{64}$ **(c)** $\frac{3}{64}$ **(d)** $\frac{1}{4}$

11 (a) 0.35 **(b)** 0.5

12 (a) 0.3 **(b)** 0.49 **(c)** 0.027 **(d)** 0.0081

13 (a) $\frac{4}{91}$ **(b)** $\frac{24}{91}$

14 (a) (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

(b) $P(A) = \frac{1}{6}$; $P(B) = \frac{1}{6}$; $P(C) = \frac{5}{36}$; $P(D) = \frac{1}{12}$

(c) 6 pairs: AB, AC, AD, BC, BD, CD

(d) See if $P(A) \times P(B) = P(A \text{ and } B)$. If it is then the events are independent.

(e) Only AB are independent.

15 (a) 0.5

(b) Use $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.65$

16 If independent $P(J) \times P(M) = P(J \text{ and } M)$, where J stands for 'has a part-time job' and M stands for 'studies mathematics'. $P(J) = \frac{475}{900}$,

$P(M) = \frac{870}{900}$, $P(J \text{ and } M) = \frac{470}{900}$, $P(J) \times P(M) = \frac{551}{1080}$.

So, $P(J) \times P(M) \neq P(J \text{ and } M)$.

17 (a) 0 (b) 0.1575

18 Students could define events such as 'rolling a 3 on the first roll', 'rolling a 5 on the second roll', 'rolling at least one multiple of 3', 'rolling a total of 6 or more'. Some questions then might be: 'Find the probability of each event', 'Are any of the pairs of events independent?' and so on.

19 (a) Drawing cards from a pack without replacement. Many pairs of events could be formed from this. Rolling a pair of dice and looking at a pair of events where one event is to do with a particular number on the first die and the second is to do with the total of the two dice.

(b) Pairs of simple events based on separate tosses of a coin, such as head on each, and similar pairs of events based on rolling two dice.

Chapter review 11 (p. 556)

1 (a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$

(d) $\frac{2}{5}$ (e) $\frac{3}{5}$ (f) $\frac{4}{5}$

2 (a) $\frac{3}{15} = \frac{1}{5}$ (b) $\frac{8}{15}$ (c) $\frac{10}{15} = \frac{2}{3}$

3 $\frac{4}{6} = \frac{2}{3}$

4 (a) Possible outcomes

		Spinner 2			
		1	2	3	4
Spinner 1	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)

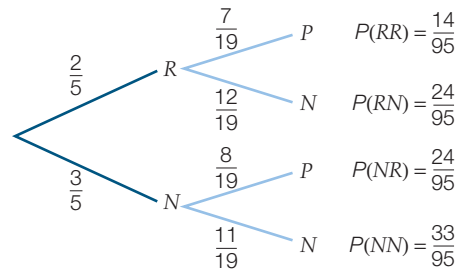
(b) (i) $\frac{1}{16}$ (ii) $\frac{3}{8}$ (iii) $\frac{9}{16}$ (iv) $\frac{1}{4}$

5 $\frac{1}{4}$

6 (a) 18 (b) 8 (c) 13 (d) 4 (e) 10 (f) 3

7 (a) D (b) B

8 (a)



(b) $\frac{14}{95}$ (c) $\frac{33}{95}$ (d) $\frac{62}{95}$

9 (a) (i) 0.4286 (ii) 0.3333

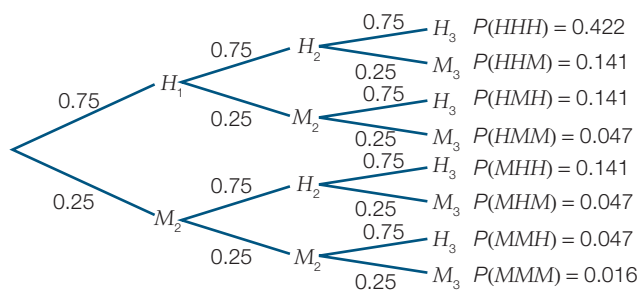
(b) (i) 0.4706 (ii) 0.2759

10 $\frac{6}{99} = \frac{2}{33}$

11 (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{3}{7}$ (d) $\frac{6}{7}$ (e) $\frac{2}{7}$ (f) $\frac{3}{7}$

12 (a) 0.25

(b) If H = hit and M = miss:



(c) (i) 0.422 (ii) 0.141 (iii) 0.984

13 (a)

	lemonade	not lemonade	
cola	12	18	30
not cola	28	2	30
	40	20	60

(b) (i) $\frac{2}{3}$ (ii) $\frac{7}{15}$ (iii) $\frac{1}{2}$ (iv) $\frac{3}{10}$

(v) $\frac{1}{5}$ (vi) $\frac{1}{3}$ (vii) $\frac{29}{30}$ (viii) $\frac{1}{30}$

14 (a) $\frac{7}{16}$ (b) $\frac{9}{16}$ (c) $\frac{13}{16}$ (d) $\frac{3}{8}$

15 (a) 0.35 (b) 0.275

(c) 0.015 (d) 0.012

16 (a) (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{3}{8}$ (iv) $\frac{1}{8}$

(b) 1. Yes, this is an expected result as these outcomes cover the sample space.

(c) $P(0 \text{ tails}) = P(3 \text{ heads})$; $P(1 \text{ tail}) = P(2 \text{ heads})$ etc.

(d) (i) $\frac{1}{8}$ (ii) $\frac{1}{8}$ (iii) $\frac{1}{8}$

(e) Any outcome with exactly 1 tail has exactly the same probability as any other outcome with exactly 1 tail. The order does not matter.

17 (a) $\frac{16}{81}$ (b) $\frac{344}{1215}$ (c) $\frac{296}{1215}$

18 (a) 0.015 (b) 0.16 (c) 0.032 (d) 0.12

(e) Sydney and Canberra are relatively close and if it rains in one city, the probability of rain in both cities will be higher. The two events are not independent.

Chapter 12

Recall 12 (p. 560)

1 (a) $\pm\sqrt{3}$ (b) $\sqrt[3]{5}$ (c) $\pm\sqrt{7}$ (d) $\sqrt[3]{6}$ (e) $\pm\sqrt[4]{10}$

2 (a) 2 (b) 6 (c) 3 (d) 1 (e) 2
(f) 4 (g) -3 (h) 1 (i) 2 (j) -1

3 (a) 3.32 (b) -2.65 (c) 2.47 (d) 1156 (e) 19 683

4 (a) $3x + 12$ (b) $-8y + 12$

5 (a) m^5 (b) $12k^3p^9$ (c) x^7
(d) 3 (e) $m^{12}n^4$ (f) x^6y^{-4} $(= \frac{x^6}{y^4})$

6 (a) $x^2 - y^2$ (b) $c^2 - 81$ (c) $4p^2 - 9$

Exercise 12.1 (p. 563)

1 (a) irrational (b) rational (c) rational
(d) irrational (e) rational (f) irrational
(g) irrational (h) rational (i) irrational
(j) irrational (k) irrational (l) rational

2 (a) 8 (b) -5 (c) -6 (d) 9 (e) 42 (f) -28
(g) 13 (h) 7 (i) 69 (j) 30 (k) 16 (l) 73

3 (a) 3.146 (b) 0.592 (c) -0.960
(d) 5.916 (e) -1.732 (f) 2.571
(g) 2.154 (h) 2.449 (i) -4.160

4 (a) $\frac{7}{9}$ (b) $\frac{6}{11}$ (c) $\frac{59}{90}$
(d) $\frac{2}{90}$ (e) $\frac{27}{370}$ (f) $\frac{88}{13875}$
(g) $\frac{1}{150}$ (h) $\frac{905}{1111}$ (i) $\frac{343}{60}$ or $5\frac{43}{60}$

5 (a) C (b) D (c) D (d) D

6 (a) C (b) A

7 (a) B (b) D (c) B

8 (a) 2.236 (b) 1.732 (c) 8.061
(d) 7.071 (e) 1.596 (f) -5.482
(g) -2.410 (h) -6.557 (i) 3.464
(j) 13.416 (k) -9.950 (l) 67.082

9 (a) 5 m (b) 6 cm (c) $\sqrt{46}$ mm (d) $\sqrt{17}$ m

10 (a) $\sqrt{60}$ m (b) 31 m

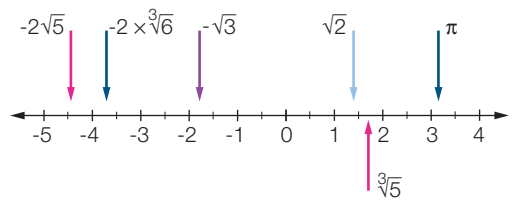
(c) The precision of measuring devices is important as far as accuracy is concerned. In this situation, a measurement to the nearest metre is therefore the most appropriate.

11 (a) 2 m (b) 3 m (c) $\sqrt[3]{35}$ cm (d) $\sqrt[3]{15}$ cm

12 (a) $\sqrt[3]{7}$ cm (b) 19 mm

13 sample answers: (a) $\sqrt{2}, \pi, \sqrt[3]{5}$ (b) $-\sqrt{3}, -2\sqrt{5}, -2\sqrt[3]{6}$

(c)



14 Lie the metre ruler on a flat surface and rule a 1 m line with the pencil. Place an end of the ruler on the line and rule another metre line at right angles to the first. Run the string from one end of a metre line to the other (as if forming a triangle). This length is $\sqrt{2}$ metres.

Exercise 12.2 (p. 568)

1 (a) $\sqrt{6}$ (b) $\sqrt{55}$ (c) $\sqrt{70}$ (d) 7 (e) 4
(f) 8 (g) $6\sqrt{2}$ (h) $5\sqrt{7}$ (i) $3\sqrt{2}$ (j) $10\sqrt{6}$
(k) $4\sqrt{33}$ (l) $40\sqrt{2}$ (m) 35 (n) 24 (o) 40
(p) 42 (q) $15\sqrt{6}$ (r) $144\sqrt{2}$

2 (a) 3 (b) 4 (c) 5 (d) 2
(e) 6 (f) 3 (g) 1 (h) 2
(i) $\frac{5}{3}$ (j) $\frac{10}{7}$ (k) $\frac{2}{9}$ (l) $\frac{6}{11}$
(m) $\frac{1}{4}$ (n) $\frac{1}{3}$ (o) 4 (p) 5
(q) 1 (r) $\frac{1}{3}$ (s) 30 (t) 10
(u) $6\sqrt{6}$ (v) $\frac{\sqrt{5}}{5}$ (w) $\frac{2\sqrt{7}}{5}$ (x) $\frac{2\sqrt{3}}{3}$

3 (a) $3\sqrt{3}$ (b) $6\sqrt{3}$ (c) $9\sqrt{2}$ (d) $4\sqrt{6}$
(e) $3\sqrt{11}$ (f) $10\sqrt{5}$ (g) $9\sqrt{6}$ (h) $9\sqrt{5}$
(i) $10\sqrt{2}$ (j) $-8\sqrt{6}$ (k) $-15\sqrt{7}$ (l) $36\sqrt{3}$

4 (a) $\sqrt{12}$ (b) $\sqrt{63}$ (c) $\sqrt{72}$ (d) $\sqrt{27}$
(e) $-\sqrt{600}$ (f) $-\sqrt{40}$ (g) $-\sqrt{192}$ (h) $\sqrt{175}$
(i) $\sqrt{128}$ (j) $\sqrt{245}$ (k) $-\sqrt{108}$ (l) $-\sqrt{275}$

5 (a) C (b) B (c) C (d) D (e) B

6 (a) $45\sqrt{2}$ (b) $16\sqrt{7}$ (c) $-28\sqrt{15}$ (d) $-32\sqrt{5}$

(e) $-15\sqrt{5}$ (f) 120 (g) 48 (h) $36\sqrt{6}$

(i) $168\sqrt{21}$ (j) $100\sqrt{15}$ (k) $216\sqrt{14}$ (l) $54\sqrt{3}$

7 (a) $2\sqrt{2}$ (b) $3\sqrt{3}$ (c) $9\sqrt{5}$ (d) $-12\sqrt{6}$

(e) $\frac{3\sqrt{3}}{2}$ (f) $\frac{4\sqrt{2}}{5}$ (g) $6\sqrt{2}$ (h) $3\sqrt{7}$

(i) $\frac{5\sqrt{5}}{6}$ (j) $-\frac{2\sqrt{11}}{3}$ (k) $-\frac{3\sqrt{3}}{5}$ (l) $\frac{7\sqrt{3}}{10}$

8 (a) 75 m^2 (b) 16 m^2 (c) 12 km^2 (d) $9\sqrt{6}$ cm^2

9 One solution is $b = 9$, $c = 2$

10 (a) $2\sqrt{10}$ mm (b) $6\sqrt{5}\pi$ m

11 (a) $\sqrt{0.2825}$ m or $\sqrt{2825}$ cm (b) 6 windows

12 $\sqrt{24}, \frac{\sqrt{96}}{2}, \frac{\sqrt{216}}{3}$

13 (a) $\sqrt{5} \times \sqrt{5}$ (b) $\frac{8\sqrt{2}}{\sqrt{8}}$

Exercise 12.3 (p. 572)

1 (a) $10\sqrt{2}$ (b) $3\sqrt{5}$ (c) $5\sqrt{3}$

(d) $8\sqrt{7}$ (e) $-4\sqrt{3}$ (f) $-23\sqrt{11}$

(g) $11\sqrt{2}$ (h) 0 (i) $-3\sqrt{5}$

(j) $5\sqrt{6} - 5\sqrt{3}$ (k) $7\sqrt{2} - 5\sqrt{5}$ (l) $-\sqrt{3} - 4\sqrt{11}$

2 (a) $4\sqrt{2}$ (b) $\sqrt{7}$ (c) $3\sqrt{5}$

(d) $36\sqrt{2}$ (e) $\sqrt{7}$ (f) $-4\sqrt{10}$

(g) $9\sqrt{10}$ (h) $27\sqrt{2}$ (i) $8\sqrt{6}$

(j) $49\sqrt{3}$ (k) $-30\sqrt{5}$ (l) $19\sqrt{2}$

(m) $2\sqrt{7}$ (n) 0

3 (a) B (b) B (c) D

4 (a) C (b) D

5 (a) $8\sqrt{2}$ (b) $20\sqrt{3}$

(c) $11\sqrt{2}$ (d) $8\sqrt{10} + 12\sqrt{7}$

6 (a) $x = 4\sqrt{5}$ (b) $y = \sqrt{6}$

7 5π cm^2

8 $\frac{3\pi}{4}$ cm^2

9 (a) $6\sqrt{5} + 4\sqrt{3}$ m (b) $6\sqrt{15}$ m^2 (c) no

10 $7\sqrt{5} - 4\sqrt{5} = 3\sqrt{5}$; $-10\sqrt{5} + 13\sqrt{5} = 3\sqrt{5}$; $8\sqrt{5} - 5\sqrt{5} = 3\sqrt{5}$

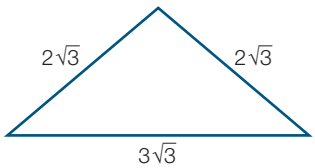
11 length = $2\sqrt{6}$ cm, breadth = 2 cm, perimeter = $4\sqrt{6} + 4$ cm;
length = $2\sqrt{3}$ cm, breadth = $2\sqrt{2}$ cm,
perimeter = $4\sqrt{3} + 4\sqrt{2}$ cm

12 (a) (i) Ruby has added the numbers underneath the square root.

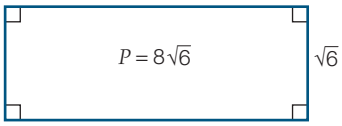
(ii) When subtracting one surd from another, Ruby has forgotten to include the surd part of the solution.

(b) (i) $7\sqrt{3}$ (ii) $3\sqrt{6}$

(c) (i)



(ii)



Exercise 12.4 (p. 576)

1 (a) $3\sqrt{7} + 6$ (b) $-4\sqrt{11} - 4\sqrt{3}$

(c) $18\sqrt{5} - 9\sqrt{2}$ (d) $3\sqrt{35} - 5\sqrt{14}$

(e) $8\sqrt{10} + 24\sqrt{6}$ (f) $3\sqrt{10} - 21\sqrt{15} + 9\sqrt{35}$

(g) $3 + 2\sqrt{3}$ (h) $15\sqrt{14} - 70$

(i) $24\sqrt{15} - 6\sqrt{5} - 15$ (j) $6\sqrt{7} - 12$

(k) $12 - 12\sqrt{2} + 20\sqrt{3}$ (l) $40 - 4\sqrt{5} - 20\sqrt{3}$

(m) $\sqrt{10} + 3\sqrt{5} + 4\sqrt{2} + 12$ (n) $3 + \sqrt{21} - \sqrt{15} - \sqrt{35}$

(o) $5 + 2\sqrt{6}$ (p) $88 - 16\sqrt{10}$

(q) 25 (r) -53

2 (a) B (b) D (c) C (d) C

3 (a) $\pi(19 - 6\sqrt{10})$ m^2 (b) $(2\sqrt{35} + 12\sqrt{7})$ cm^2

4 $\pi(21 + 8\sqrt{5})$ cm^2

5 (a) 21 km^2 (b) $(14 + 3\sqrt{2} + \sqrt{10})$ km

(c) 21 405 m

6 (a) (i) $(18\sqrt{2} - 12)$ m (ii) $(12\sqrt{2} - 8)$ m

(iii) $(15\sqrt{2} - 10)$ m (iv) $(21\sqrt{2} - 14)$ m

(b) (i) 13.46 m (ii) 8.97 m

(iii) 11.21 m (iv) 15.70 m

7 (a) 7π cm^2 (b) $(5 + 2\sqrt{7})$ cm by $(3 + 2\sqrt{7})$ cm

(c) $(43 + 16\sqrt{7})$ cm^2 (d) $(43 + 16\sqrt{7} - 7\pi)$ cm^2

(e) 63.34 cm^2

8 (a) 47.7 cm (b) 22 752.9 cm^2

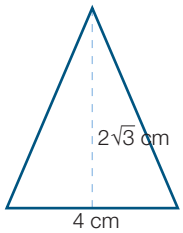
(c) $(12\,110 + 1440\sqrt{55})$ cm^2 (d) 22 789.3 cm^2

(e) part (d)

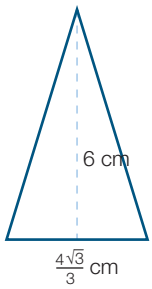
9 (a) $\sqrt{18}$ or $3\sqrt{2}$ cm (b) $150\sqrt{2}$ cm (c) 212.1 cm

(d) 4.2 cm (e) 210 cm (f) part (c)

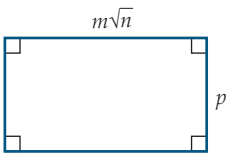
10 base 4 cm, height $2\sqrt{3}$ cm



base $\frac{4\sqrt{3}}{3}$ cm, height 6 cm



11 (a)



$P = 2(m\sqrt{n} + p)$
If $m = 10$, $n = 3$ and $p = 4$
Then $P = 2(10\sqrt{3} + 4) = 42.641$ cm

(b) $A = m\sqrt{n} \times p = mp\sqrt{n}$

(c) If $m = 2$, $n = 2$ and $p = 3$ then $A = 8.485\,\text{m}^2$

Exercise 12.5 (p. 581)

1 (a) $\frac{\sqrt{7}}{7}$ (b) $\frac{4\sqrt{3}}{3}$ (c) $-2\sqrt{6}$ (d) $\frac{\sqrt{6}}{2}$

(e) $\sqrt{55}$ (f) $\frac{\sqrt{30}}{2}$ (g) $-\frac{3\sqrt{7}}{28}$ (h) $\frac{2\sqrt{15}}{7}$

2 (a) $\frac{3+\sqrt{7}}{2}$ (b) $8 - 4\sqrt{3}$

(c) $\frac{\sqrt{15}+\sqrt{6}}{3}$ (d) $\frac{4\sqrt{7}-10}{3}$

(e) $\frac{24+15\sqrt{3}}{3} = 8 + 5\sqrt{3}$ (f) $\frac{22+10\sqrt{5}}{4} = \frac{11+5\sqrt{5}}{2}$

(g) $\frac{6\sqrt{3}-\sqrt{42}-3\sqrt{2}+\sqrt{7}}{11}$ (h) $\frac{5\sqrt{6}-8\sqrt{3}}{6}$

3 (a) D (b) A (c) B

4 (a) B (b) D

5 (a) $\frac{2\sqrt{3}+\sqrt{5}+1}{4}$ (b) $\frac{\sqrt{7}+2\sqrt{6}+5}{6}$

(c) $\frac{5\sqrt{6}-5\sqrt{3}-12-3\sqrt{11}}{15}$ (d) $\frac{3\sqrt{5}-3\sqrt{3}-2\sqrt{7}-4}{6}$

6 (a) $\frac{5\sqrt{3}}{2}$ (b) 4 shelves (c) $(5\sqrt{6} - 8\sqrt{2})$ m

7 (a) $15\,000\sqrt{6}$ cm^2 (b) $50\sqrt{6}$ tiles

(c) As $122 < 50\sqrt{6} < 123$, the minimum number of tiles is 123, but maybe buy 130 tiles to allow for wastage.

(d) $15\,000\sqrt{6} + 6000\sqrt{15} + 600\sqrt{35} + 1500\sqrt{14}$ cm^2

8 $\frac{112\sqrt{3}-294}{249}$ or $\frac{14(8\sqrt{3}-21)}{249}$

9 Many answers are possible, just multiply the surd by a form of 1. As examples: $\frac{4\sqrt{6}}{3}$, $\frac{8}{\sqrt{6}}$, $\frac{4\sqrt{10}}{\sqrt{15}}$

10 (a) $\frac{\sqrt{2}}{3}, \frac{2}{3}$ (b) $\frac{2\sqrt{7}}{7}, \frac{4\sqrt{7}}{7}$

(c) $\frac{\sqrt{15}}{5}, \frac{3\sqrt{15}}{5}$ (d) $\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{2}$

11 $2 - \sqrt{3}$ and $4 - 2\sqrt{3}$

12 When two surds that are the same are multiplied, the result is a whole number, e.g. $\sqrt{6} \times \sqrt{6} = 6$. Multiplying by the conjugate will remove the square roots from the denominator, resulting in a rationalised denominator. It is also like the difference of two squares.

Half-time 12 (p. 583)

1 (a) 6 (b) 88 (c) $\frac{9}{4}$

2 $5\sqrt{6} + 3\sqrt{2}$ m

3 (a) $\frac{\sqrt{30}}{6}$ (b) $\frac{5\sqrt{3}}{6}$ (c) $\frac{12+3\sqrt{11}}{5}$

4 $3\sqrt{5}$ m

5 (a) rational (b) rational (c) irrational (d) rational

6 $(18 + 12\sqrt{15})$ m^2

7 (a) $6\sqrt{2}$ (b) 360 (c) $\frac{3}{5}$

8 (a) -15 (b) 17 (c) 34

Exercise 12.6 (p. 585)

1 (a) $12x^5$ (b) $\frac{a^5}{b^3}$ (c) $\frac{p}{q^3}$

(d) x^2 (e) $a^{20}b^9$ (f) $6m^{10}n$

(g) $\frac{1}{q^7}$ (h) $\frac{1}{c^9d^5}$ (i) 2

(j) $\frac{a^2}{b^2}$ (k) $\frac{m^5}{n^5}$ (l) 1

(m) q (n) $\frac{3k^3}{10}$ (o) $\frac{2a^6b^4}{3}$

2 (a) k^7p^6 (b) x^5y^5 (c) $\frac{6m^6n^5}{5}$

(d) $\frac{2q^{11}}{p^4}$ (e) $\frac{7y^{10}}{x^4}$ (f) $\frac{1}{m^6n^2}$

(g) $\frac{1}{k^7p^4}$ (h) $\frac{2a^5}{3b}$ (i) $\frac{5fg^4}{2}$

(j) x^2y^4 (k) $\frac{2m^3}{3}$ (l) $\frac{q^2}{5}$

3 (a) B (b) A (c) C

4 (a) x^9y^4 (b) $\frac{a^{12}}{b^{11}}$ (c) $\frac{4}{k^{18}p^2}$

(d) $9m^{20}n^2$ (e) $\frac{x^{27}}{y^{11}}$ (f) $\frac{m^6}{n^{20}}$

(g) $\frac{1}{q^{22}}$ (h) f^6g^{11} (i) $\frac{a}{b^7}$

(j) $\frac{m^{14}}{n^{11}}$ (k) $\frac{4x^{13}}{y^{18}}$ (l) $\frac{12d^{15}}{c^3}$

5 (a) x^2 (b) $\frac{a^3}{b^5}$ (c) pq (d) d^4f^2

(e) $\frac{1}{y^2}$ (f) $\frac{1}{a^3}$ (g) $\frac{n}{m^2}$ (h) $\frac{q^2}{p^7}$

(i) $\frac{b^9}{a^{16}}$ (j) $\frac{x^{13}}{y^7}$ (k) $\frac{12q^4}{p^{17}}$ (l) $\frac{9y^7}{8x^{14}}$

6 (a) $5^3, 5^2, 5^1, 5^0$ (b) $5^{-1}, 5^{-2}, 5^{-3}, 5^{-4}, 5^{-5}$

(c) $125, 25, 5, 1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}$

7 (a) $\frac{m^9}{n^2}$ (b) $\frac{b^3}{a}$ (c) $\frac{x^5}{y^4}$ (d) $\frac{y^{19}}{x^6}$ (e) $\frac{9}{a^2b^7}$

(f) $\frac{8}{d^5f^7}$ (g) $\frac{10b}{a^9}$ (h) $\frac{n^3}{m^8}$ (i) $\frac{1}{x^2}$

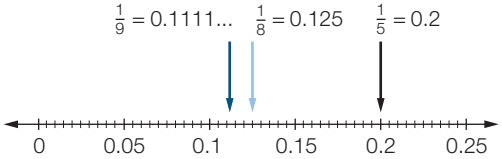
8 (a) 1 divided by the original number (b) $192a$

9 (a) $(6m^2n^3)^2 \times n^2$; $9m^3n^4 \times 4mn^4$

(b) $\frac{12m^7n^3}{15m^3n^9}$; $24m^4 \div 30n^6$

10 $a = 2, b = 3$; $a = 3, b = 2$; $a = 5, b = 1$

These give the fractions: $\frac{1}{8}, \frac{1}{9}$ and $\frac{1}{5}$



11 (a) Murray just said $-2 \times -4 = 8$ and possibly put it as the power in the denominator because he thought the negatives needed to be accounted for. He is wrong in at least two ways.

(b) Raphael knows that indices are added when numbers are multiplied. $\frac{1}{a^6}$ may be Raphael's answer, which would be correct.

(c) When numbers with indices and the same base are multiplied, keep the base the same and add the indices.

$a^{-2} \times a^{-4} = a^{-6} = \frac{1}{a^6}$.

Exercise 12.7 (p. 590)

1 (a) $x^{2/3}$ (b) $a^{5/6}$ (c) $k^{8/15}$ (d) $m^{1/4}$

(e) $p^{1/6}$ (f) $y^{1/4}$ (g) $5p^2$ (h) $7x^3$

(i) $2a^{1/3}b$ (j) $x^2y^{1/2}$ (k) $k^{1/2}n^4$ (l) $\frac{p^2}{q^5}$

2 (a) 14 (b) 23 (c) 19 (d) 8 (e) 6 (f) 8

(g) 5 (h) 4 (i) 7.94 (j) 14.42 (k) 2.15 (l) 2.29

3 $4^{-1}, 6^{-\frac{1}{5}}, 2^{\frac{1}{2}}, 3^{\frac{1}{3}}$

4 (a) D (b) B (c) B (d) C

5 (a) a (b) d^3 (c) $\frac{x}{y}$ (d) $\frac{1}{m^2n^4}$

(e) $\frac{1}{q^2}$ (f) $\frac{1}{b^4}$ (g) $\frac{m^{1/3}}{2n^{1/6}}$ (h) $\frac{y^{7/6}}{5x^{1/3}}$

6 (a) B (b) A

7 (a) 0.5 (b) 0.437 (c) 0.164 (d) 0.196

(e) 0.369 (f) 0.343 (g) 0.398 (h) 0.359

8 (a) $x = \sqrt[3]{3}$ (b) $x = 4$ (c) $x = \frac{1}{2}$

9 (a) x^3 (b) a^3 (c) $2m^2n$

10 (a) $x = \pm 2$ (b) $x = 3$ (c) $x = -2$ (d) $x = 2$

11 (a) 7 (b) $x = 7, y = 5$ (c) 256

(d) $d = 256, e = \frac{3}{4}$

12 $1476.106^{\frac{4}{5}}, 7^3, 1.793^{10}$

13 $\left(n^{\frac{1}{4}}m^{\frac{1}{5}}\right)^3, (\sqrt[4]{n} \times \sqrt[5]{m})^3; (-\sqrt[4]{n^3})(-\sqrt[5]{m^3})$

Exercise 12.8 (p. 594)

1 (a) $\log_2 32 = 5$ (b) $\log_2 256 = 8$

(c) $\log_2 4096 = 12$ (d) $\log_3 27 = 3$

(e) $\log_3 243 = 5$ (f) $\log_3 6561 = 8$

(g) $\log_{10} 100 = 2$ (h) $\log_{10}(1\,000\,000) = 6$

(i) $\log_{10}(100\,000\,000) = 8$ (j) $\log_2\left(\frac{1}{4}\right) = -2$

(k) $\log_2\left(\frac{1}{8}\right) = -3$ (l) $\log_2\left(\frac{1}{16}\right) = -4$

2 (a) $10^2 = 100$ (b) $10^3 = 1000$

(c) $10^5 = 100\,000$ (d) $2^4 = 16$

(e) $2^7 = 128$ (f) $2^9 = 512$

(g) $10^{-3} = \frac{1}{1000}$ (h) $10^{-4} = \frac{1}{10\,000}$

(i) $10^{-6} = \frac{1}{1\,000\,000}$ (j) $2^{-3} = \frac{1}{8}$

(k) $2^{-5} = \frac{1}{32}$ (l) $2^{-7} = \frac{1}{128}$

3 (a) C (b) D

4 (a) (i) $3^x = 3^4$ (b) (i) $x = 4$

(ii) $3^x = 3^8$ (ii) $x = 8$

(iii) $3^x = 3^9$ (iii) $x = 9$

(iv) $5^x = 5^{-2}$ (iv) $x = -2$

(v) $5^x = 5^{-3}$ (v) $x = -3$

(vi) $5^x = 5^{-6}$ (vi) $x = -6$

(vii) $6^{-x} = 6^3$ (vii) $x = -3$

(viii) $6^{-x} = 6^5$ (viii) $x = -5$

(ix) $6^{-x} = 6^{-4}$ (ix) $x = 4$

5 (a) (i) $x = 2^4$ (b) (i) $x = 16$

(ii) $x = 2^6$ (ii) $x = 64$

(iii) $x = 2^7$ (iii) $x = 128$

(iv) $\frac{1}{81} = 3^x$ (iv) $x = -4$

(v) $\frac{1}{243} = 3^x$ (v) $x = -5$

(vi) $\frac{1}{6561} = 3^x$ (vi) $x = -8$

(vii) $625 = x^4$ (vii) $x = 5$

(viii) $343 = x^3$ (viii) $x = 7$

(ix) $\frac{1}{3125} = x^{-5}$ (ix) $x = 5$

6 (a) (i) 10 (ii) 5 (iii) 7

(b) $\log_a(a) = 1$

7 (a) (i) $x = 1$ (ii) $x = 1$ (iii) $x = 1$

(b) $\log_a 1 = 0$

8 We cannot find the square root of a negative number.

9 (a) Students to verify that both equations are true.

(b) 6

(c) $2^{12} = 4096$ is the index equivalent of the first statement and $4^y = 4096$ is the second. But, this could be written as $(2^2)^y = 4096$ or $2^{2y} = 4096$, so $2y = 12$ and $y = 6$.

10 $\log_9(531\,441) = 6$ and $\log_{27}(531\,441) = 4$

11 Effectively, Lee is saying $\log_6(-36) = -2$, which would mean that $6^{-2} = -36$, which is clearly incorrect. Lee should have divided both sides by -1 to get $(6)^2 = 36$, which has the equivalent form $\log_6(36) = 2$.

Exercise 12.9 (p. 598)

1 (a) $\log_2 32$ (b) $\log_2 64$ (c) $\log_2 256$

(d) $\log_{10} 30$ (e) $\log_{10} 21$ (f) $\log_{10} 36$

2 (a) $\log_2 2$ (b) $\log_2 5$ (c) $\log_2 5$

(d) $\log_{10} 14$ (e) $\log_{10} 5$ (f) $\log_{10} 6$

3 (a) $4\log_2 5$ (b) $3\log_2 6$ (c) $4\log_2 7$

(d) $3\log_{10} 7$ (e) $4\log_{10} 8$ (f) $5\log_{10} 11$

4 (a) 5 (b) 7 (c) 8 (d) 2 (e) 1 (f) 1

5 (a) A (b) C

6 (a) 3 (b) 11 (c) 1 (d) 2 (e) 8 (f) -9

7 (a) 2 (b) $\frac{5}{6}$ (c) 3 (d) 2 (e) $\frac{3}{2}$ (f) $\frac{4}{3}$

8 (a) $\frac{1}{4}$ (b) 2 (c) $\frac{2}{3}$

9 (a) $y = 4x^3$ (b) $y = 11x^5$

10 $5\log_2(y) = \log_2(x-1) + 2$

$$\log_2\left(\frac{y^5}{x-1}\right) = 2$$

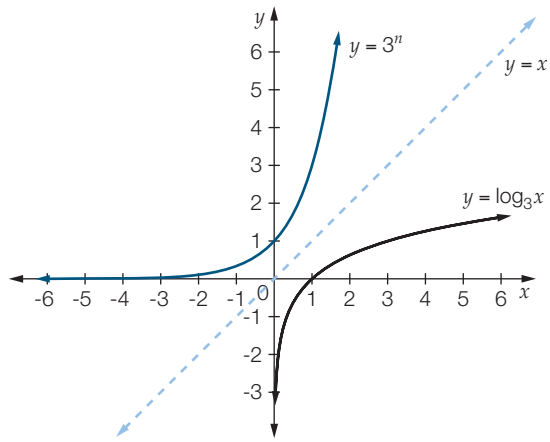
$$2^2 = \frac{y^5}{x-1}$$

$$4(x-1) = y^5$$

11 No, the correct answer is 2. The mistake Danielle made was when she took out the $\frac{1}{2}$. This was really $\frac{1}{\frac{1}{2}}$ which is the same as 2.

Exercise 12.10 (p. 603)

1



2 (a) $x = \frac{1}{2}$ (b) $x = \frac{2}{3}$ (c) $x = -\frac{1}{3}$

(d) $x = 6$ (e) $x = \frac{1}{4}$ (f) $x = \frac{1}{5}$

(g) $x = 216$ (h) $x = \frac{1}{8}$ (i) $x = \sqrt{7}$

3 (a) $x = -2$ (b) $x = 4$ (c) $x = 3$

(d) $x = \frac{5}{2}$ (e) $x = \frac{5}{3}$ (f) $x = -\frac{1}{2}$

(g) $x = -\frac{1}{4}$ (h) $x = -\frac{1}{2}$ (i) $x = -\frac{3}{4}$

4 D 5 C 6 $a_{H^+} = 10^{-7}$

7 (a) (i) $S = 10^4$ (ii) $S = 10^5$

(b) 10 times stronger (c) $10^{0.2} \approx 1.585$

(d) $M = 6.9$

8 (a) 70 dB (b) 140 dB

(c) twice as loud (d) $I = 6.3 \times 10^{-3}$ W/m^2

(e) 110 dB (f) rock concert

(g) pneumatic drill at 130 dB

(h) (i) Home, just cause discomfort.

(ii) Pneumatic drill and jet aircraft cause pain.

9 Students' own answers, e.g. $64 = 2^x$; $x = 6$.

10 Students' own answers, e.g. $-3 = \log_3 x$; $x = \frac{1}{27}$.

Chapter review 12 (p. 605)

1 (a) 4.123 (b) 2.924 (c) 34.641 (d) 2.280

2 (a) $12\sqrt{30}$ (b) 70 (c) $\frac{8}{5}$ or $1\frac{3}{5}$ (d) $9\sqrt{2}$

(e) $2\sqrt{10}$ (f) $10\sqrt{5}$ (g) $30\sqrt{5}$ (h) $5\sqrt{3}$

3 (a) $\sqrt{72}$ (b) $-\sqrt{63}$

4 (a) $\sqrt{2}$ (b) $4\sqrt{7} - \sqrt{3}$

(c) $29\sqrt{3}$ (d) $46\sqrt{2} - \sqrt{6}$

5 (a) $4\sqrt{15} - 10 + \sqrt{10}$ (b) $12 - 18\sqrt{14} - 6\sqrt{2}$

(c) $5\sqrt{5} - 8$ (d) $95 + 24\sqrt{14}$

6 (a) C (b) D

7 (a) $\frac{5\sqrt{2}}{2}$ (b) $2\sqrt{2}$ (c) $\frac{7\sqrt{5}}{10}$

8 (a) $\frac{21 + 7\sqrt{5}}{4}$ (b) $\frac{2\sqrt{6} - \sqrt{14} - 4\sqrt{3} + 2\sqrt{7}}{5}$

9 (a) $9a^2b^{10}$ (b) c^2d^{14} (c) u^2t

(d) $2xy^{10}$ (e) $m^{12}n^9$ (f) $a^{12}b$

10 (a) $\frac{6}{a^6}$ (b) $\frac{27x^4}{16y}$ (c) $\frac{n^6}{2}$

11 (a) $4mn^{1/3}$ (b) $p^3q^{1/2}$ (c) $a^{1/2}$

(d) $\frac{b}{a^9}$ (e) $\frac{2n^{4/3}}{m^{1/3}}$ (f) $\frac{729n^7}{m^{19}}$

12 (a) $k^{5/2}$ (b) $x^{3/4}$ (c) $m^{-2/3}$

13 (a) $\log_4(16\,384) = 7$ (b) $\log_2(1024) = 10$

(c) $\log_{10}\frac{1}{1000} = -3$

14 (a) $2^{14} = 16\,384$ (b) $3^5 = 243$ (c) $10^{-5} = \frac{1}{100\,000}$

15 (a) 9 (b) -2 (c) $\log_3(50)$ (d) $\log_5(10)$

16 (a) $x = 6$ (b) $x = -3$ (c) $x = 5$ (d) $x = -5$

17 (a) $2\sqrt{58}$ cm (b) 15.23 cm

18 (a) length 25.3 cm, height 20.2 cm

(b) 1749.3 cm^2

(c) $(180\sqrt{6} + 108\sqrt{7} + 160\sqrt{15} + 48\sqrt{70})$ cm^2

(d) 1747.9 cm^2 (e) (b) was larger by 1.4 cm^2.

19 (a) 5 (b) 7 (c) -6 (d) $\frac{1}{16}$ (e) 8 (f) -4

20 (a) 3 (b) 3 (c) 2 (d) $\frac{3}{4}$

21 (a) $5\sqrt{2}$ cm (b) $\sqrt{\frac{160}{\sqrt{3}}}$ cm (c) $10\pi\sqrt{3}$ m

22 $-\frac{17 + 136\sqrt{5}}{319}$

23 (a) $y = 7x^3$ (b) $y = \frac{5}{x^2}$

Mixed review D (p. 608)

1 \$20.23

2 area = 36.2 cm^2, perimeter = 24.7 cm

3 (a) $2\sqrt{6}$ (b) 18 (c) 6

(d) $40\sqrt{2}$ (e) $12\sqrt{15}$ (f) $126\sqrt{2}$

4 (a) $4\sqrt{5}$ (b) $12\sqrt{15}$ (c) $35\sqrt{21}$

5 (a) $a = -6$ (b) $-\frac{15}{16}$ (c) $a = \frac{11}{6}$

6 (a) $\frac{4\sqrt{15}}{5}$ (b) $\frac{\sqrt{3} + 5}{11}$ (c) $\frac{\sqrt{10} - \sqrt{6}}{2}$

7 (a) $(k - 1)(k - 3)$ (b) $(3t - 5)(t + 2)$

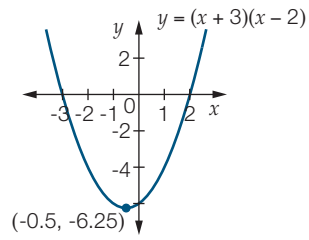
8 (a) $\frac{2a^6b^4}{3}$ (b) $a^{10}b^7$ (c) $\frac{1}{n^3k^{17}}$

9 (a) $a = 82.45°$ (b) 69.12° (c) 31.54°

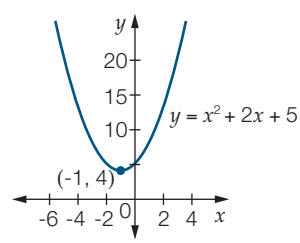
10 (a) $\frac{7}{3}$ (b) -2

11 (a) 24.3 (b) 5.78

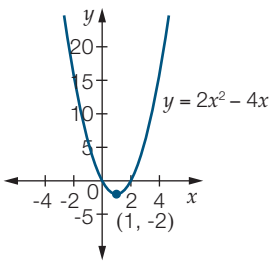
12 (a)



(b)



(c)



13 (a) $(8\sqrt{5} + 2\sqrt{3})$ cm (b) $(15 + \sqrt{15})$ cm^2

14 (a) A rectangle is a quadrilateral in which all angles are right angles.

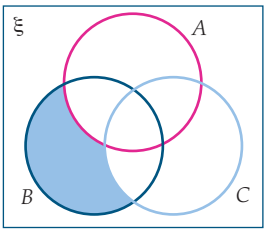
(b) A rhombus is a quadrilateral with all sides equal in length.

(c) A kite is a quadrilateral with two pairs of adjacent sides equal in length.

15 (a) $\frac{1}{12}$ (b) $\frac{5}{36}$ (c) $\frac{1}{6}$

16 (a) $3^{-5} < 5^{-3}$ (b) $9^{\frac{1}{2}} > 9^{-1}$ (c) $8^{-\frac{1}{3}} = \frac{2^{-3}}{2^{-2}}$

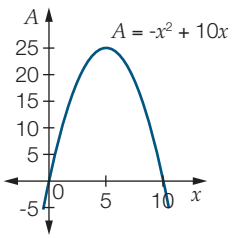
17



18 (a) $2x^2 - 4x + 20$ (b) $6x^2 + 11x - 35$

19 (a) $10 - x$ (b) $A = x(10 - x) = -x^2 + 10x$

(c)



(d) 5 m by 5 m (e) 25 m^2

20 $8\sqrt{2}$ cm

21 (a) Check students' diagrams carefully. Equal side lengths should be approximately 11.2 cm in length.

(b) Draw a line from the top of the triangle to the midpoint of the base to create two congruent right-angled triangles. You can now use Pythagoras' Theorem to calculate the equal side lengths of the isosceles triangle:

$l^2 = 5^2 + 10^2$

$l^2 = 125$

$l = \sqrt{125} = 5\sqrt{5}$ cm

Challenge D (p. 610)

1 D.

There are 36 possible combinations from two rolls, 9 of which score a 3 or less: (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)

2 (a) $\frac{1}{6}$; the second spin is an independent event from the first.

(b) (i)

Prize	Frequency	Relative frequency
\$0	8	0.16
\$5	7	0.14
\$10	9	0.18
\$20	10	0.20
\$30	8	0.16
\$60	8	0.16
Total	**50**	**1**

(ii) \$1045; $5 \times 7 + 10 \times 9 + 20 \times 10 + 30 \times 8 + 60 \times 8 = 1045$

(c) (i) You would make a profit of \$40:

$5 \times 8 + 10 \times 8 + 20 \times 8 + 30 \times 8 + 60 \times 8 = 1000$

$48 \times 20 = 960$

$1000 - 960 = 40$

(ii) You would make a \$45 profit.

3 B

Side length of base $= 12\sqrt{10}$, so each triangular face a height of $\frac{1680}{12\sqrt{10}} = 14\sqrt{10}$. Using Pythagoras' Theorem, the height of the pyramid, h, is given by $h^2 = (14\sqrt{10})^2 - (6\sqrt{10})^2$, so the height $= \sqrt{1600} = 40$.

4 A

Area of four semicircles $= 2\pi$, area of inner square $= 4$, radius of inner circle $= \sqrt{2}$, area of inner circle $= 2\pi$.

Area of shaded region $= 4 + 2\pi - 2\pi = 4$.

5 (a) $\frac{1}{x} = \frac{1}{3} + \frac{1}{4}$, $x = \frac{12}{7}$

(b) $x + 3 = 5$, $x = 2$

(c) $2^x = 2 \times 2^{48} + 2 \times 2^{48} = 2 \times 2^{49} = 2^{50}$, $x = 50$

(d) $2^{2x-4} = 2^3$, $x = 3.5$

(e) $2^{4x} = 2^{x+4}(2 - 1)$

$2^{4x} = 2^{x+4}$

$4x = x + 4$

$x = \frac{4}{3}$

Glossary and index

addition law for mutually exclusive events $P(A \text{ or } B) = P(A) + P(B)$; can only be used when A and B are mutually exclusive (p. 536)

adjacent side the side in a right-angled triangle next to the given angle that is not the hypotenuse (p. 179)

adjusted value the value of an item after depreciation (p. 28)

algebraic fractions expressions written in fraction form that contain at least one variable (p. 70)

amount the total value after interest has been added to the principal (p. 3)

amplitude half the distance between the minimum and maximum y-values of the graph of a trigonometric function (p. 226)

angle at the centre an angle that has its vertex at the centre of a circle (p. 359)

angle at the circumference an angle formed by the intersection of two chords at the circumference of a circle (p. 359)

angle in a semicircle an angle formed by joining the ends of the diameter of a circle to the circumference to form a right angle (p. 360)

angle of depression the angle between the horizontal and the line of sight when you look down (p. 194)

angle of elevation the angle between the horizontal and the line of sight when you look up (p. 194)

angles in the same segment angles subtended by the same arc of a circle at the circumference are equal (p. 360)

annulus the area between two circles that have the same centre but different radii (p. 91)

appreciation the increase in the cost or value of something over a period of time (p. 4)

arc a part of the circumference of a circle (p. 359)

asymptote a line representing a value on a graph that a curve approaches but never cuts or crosses (pp. 227, 394)

axis of symmetry an imaginary line along which a mirror could be placed so that the reflection would be indistinguishable from the original (p. 394)

bias a preference or belief that is not supported by evidence (p. 299)

binomial product the product of two expressions that each have two terms; e.g. $(x + 1)(x + 2)$ (p. 39)

bisect when a point is placed on a line interval such that the distance from one end to the point is the same as the distance from the other end to the point (p. 345)

bivariate data a data set that contains values for two variables (p. 286)

box plot a visual representation of the five-number summary (p. 269)

capacity the volume of gas or liquid that a container can hold, often measured in litres (L) (p. 101)

chord a line segment connecting any two points on a circle (p. 359)

coincident two or more lines that lie on top of each other (p. 167)

compass bearing a direction given in terms of turning a number of degrees east or west of north or south (p. 199)

complementary event the event that is the opposite of the event under consideration (p. 521)

completing the square a method for factorising quadratic expressions that combines the use of the Perfect Square rule and the Difference of Two Squares rule (p. 62)

composite shape a shape or solid made up of more than one basic shape or solid (p. 90)

compound interest interest that is added, on a regular basis, to an investment or a loan, thus increasing the amount of the investment or loan (p. 4)

concentration the amount of a substance dissolved in a solution per unit quantity of the solution (p. 120)

conditional statement extra information is given in a probability situation that changes the universal set such as 'the probability of A given that B also happens' (p. 547)

cone a solid with a circular base that tapers through the height to a single vertex (p. 107)

congruent two figures that are exactly the same shape and size (p. 329)

conjugate surds two surd expressions such as $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{a} - \sqrt{b})$ that differ only in that one has an addition sign between the two terms and the other has a subtraction sign (p. 579)

convenience sampling sampling by selecting in a convenient way (p. 316)

cosine one of the three basic trigonometric ratios; the ratio between the adjacent side and the hypotenuse in a right-angled triangle (p. 179)

cosine rule a rule to find side c if side lengths a and b and the included angle C are known: $c^2 = a^2 + b^2 - 2ab\cos C$ (p. 237)

coterminal angles have a vertex at the origin of a Cartesian plane, start from the positive direction of the x-axis and share a common terminal side (p. 232)

cubic an expression or equation where the highest power of a variable is three (p. 475)

cumulative depreciation the total loss in value of an item over a number of years (p. 28)

cumulative frequency curve a graph that shows the number of results less than a given value in grouped data (p. 263)

cyclic quadrilateral a quadrilateral where all four vertices are points that lie on the circumference of a circle (p. 360)

cylinder a solid with a circular base and a uniform cross-section (p. 97)

definition the minimum amount of information needed to identify a particular figure (p. 340)

degree in a polynomial expression, the highest power of the variable (p. 487)

density the mass per unit volume of a substance (p. 120)

dependent events in probability, when the outcome of one event affects the outcome of another event (p. 551)

dependent variable the variable within a bivariate pair that varies because of changes in the other (independent) variable (pp. 133, 294)

depreciation the decrease over time of the value of an item (p. 28)

depreciation value the loss in dollar value of an item over the course of one year (p. 28)

diameter the distance from one side of a circle to the other side through the centre (p. 359)

difference of two squares rule useful for factorising and expanding binomial expressions as $a^2 - b^2 = (a + b)(a - b)$ (p. 43)

dilation a transformation that makes the graph of a quadratic or quartic equation wider or narrower. For linear or cubic equations, varying the dilation factor makes the graph steeper or flatter (p. 401)

dilation factor the factor by which a graph has been made narrower or wider. For graphs with equations of the form $y = ax^n$, a is the dilation factor (p. 401)

disjoint sets sets that are mutually exclusive. In a Venn diagram, sets that do not overlap (p. 536)

distributive law a rule used to expand brackets: $a(b + c) = ab + ac$ (pp. 39, 575)

dividend a number being divided by another number; e.g. in 300 ÷ 15, 300 is the dividend (p. 488)

divisor the number being divided into another number; e.g. in 300 ÷ 15, 15 is the divisor (p. 488)

domain all the x-values given by a relation (p. 511)

effective interest rate the interest rate on a loan that takes into account the number of compounding periods per year. It is expressed as the percentage of the principal that is earned in the first year. Because the effect of compounding is taken into account, it may differ from the nominal interest rate (p. 24)

element member of a set (p. 511)

entire surd a surd expression where the entire number is under the root sign; e.g. $\sqrt{50}$ rather than $5\sqrt{2}$ (p. 566)

equidistant at equal distances from a point (p. 371)

equivalent angle an angle in the first quadrant that has the same sin, cos or tan value as an angle in another quadrant (p. 221)

event an outcome you are interested in when conducting a probability experiment (p. 521)

expected life the period of time for which an item (such as machinery or a computer) is expected to operate effectively, expressed as units of time or units of work output (p. 30)

exponential a relationship between variables where one of the variables is an exponent, such as $N = 2^t$ (p. 394)

extrapolation making predictions from a graph outside the range of known values (p. 310)

factor theorem if, after dividing a polynomial by an expression, the remainder is zero, then the expression is a factor of the polynomial (p. 496)

factorise by grouping in pairs factorise an expression with four terms by grouping terms 'two and two' or 'three and one' (p. 47)

factorising writing an expression so that it is the product of two or more factors (p. 47)

five-number summary the five key values used to draw a box plot: minimum, lower quartile, median, upper quartile and maximum (p. 269)

flat rate a rate of interest used to calculate the simple interest on an amount borrowed or invested (p. 4)

general form for a linear equation, equations in the form of $ax + by + c = 0$, where the gradient $= -\frac{a}{b}$ and the y-intercept $= (0, -\frac{c}{b})$ (p. 157)

gradient a measure of the steepness or slope of a line joining two points, calculated as:

$$\frac{\text{change in vertical position}}{\text{change in horizontal position}} \quad \text{or,}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or,}$$

$$m = \frac{rise}{run} \quad \text{(p. 141)}$$

highest common factor (HCF) the highest factor that is common to all parts of an expression (p. 47)

hypotenuse the longest side of a right-angled triangle, opposite the right angle (p. 179)

implied (maximal) domain the largest possible domain of a given relation (p. 511)

included angle the angle between a pair of sides (p. 330)

independent events in probability, two or more events where the outcome of one event has no effect or influence on the others (p. 551)

independent variable the variable within a bivariate pair that causes change in the other (pp. 133, 294)

inflation rate the rate of increase in prices within an economy, usually expressed as a percentage per year (p. 4)

initial cost the original purchase price of an item, also called prime cost (p. 28)

interest money paid by the lender for the use of money borrowed, or by the bank for the use of money invested (p. 3)

interval notation notation used to write the interval for the domain and range of a function: (a, b), $[a, b)$, $(a\ b]$, $[a, b]$ (p. 511)

interpolation making predictions from a graph within the range of known values (p. 310)

interquartile range the difference between the upper quartile and the lower quartile of a data set; the range in which the middle 50% of data values lie (p. 268)

inverse proportion a relationship between two variables in which one increases by a factor when the other decreases by the same factor (p. 436)

irrational numbers numbers that cannot be written in fractional form; infinite non-recurring decimals (p. 561)

leading coefficient the coefficient of the leading term in a polynomial (p. 487)

leading term the first term of a polynomial expression, with the highest power (p. 487)

line of best fit a line drawn to approximate the relationship between two variables that have been plotted as a scatter plot, usually resulting from the collection of experimental or observational data (p. 310)

linear equation an equation that has 1 as the highest power of any variable (p. 133)

linear graph a graph that is a straight line (p. 133)

linear inequality a statement that involves an algebraic expression and one of the four inequality symbols, $>$, $\geq$, $<$, $\leq$; e.g. $x \geq 7$ is read as 'x is greater than or equal to 7'. $3 < x < 8$ is read as 'x is greater than 3 and less than 8' (p. 162)

linear relationship a relationship between two variables that, when graphed, results in a straight line (p. 133)

logarithm another word for index number, or exponent. Logarithms are used to write expressions involving indices in another way; i.e. $2^3 = 8$ is written as $\log_2(8) = 3$ (p. 593)

long-run proportion the fraction of trials that would be expected to perform in a particular way if a very large number of trials were held (p. 521)

major arc an arc that covers more than half the circumference of a circle (p. 359)

major segment a segment that covers more than half the area of a circle (p. 359)

many-to-many correspondence many elements of one set correspond to many elements of another set (p. 512)

many-to-one correspondence many elements of one set correspond to one element of another set (p. 512)

minor arc an arc that covers less than half the circumference of a circle (p. 359)

minor segment a segment that covers less than half the area of a circle (p. 359)

minute (angle measure) one-sixtieth of a degree (p. 179)

monic quadratic a quadratic expression where the coefficient of the squared value is 1 (p. 51)

mutually exclusive impossible to occur at the same time (p. 536)

net a series of connected 2D shapes that can be folded together to form a 3D solid (p. 97)

nominal interest rate an interest rate stated as 'per annum' by a bank that does not take into account the effect of compounding throughout the year. The actual amount of interest earned may be slightly higher (*see also* effective interest rate) (p. 24)

non-monic quadratic a quadratic expression where the coefficient of the squared variable is not 1 (p. 66)

Null Factor Law if the product of two factors is zero, then at least one of the factors is zero. If $ab = 0$, then $a = 0$ or $b = 0$ (p. 410)

ogive another name for a cumulative frequency curve (p. 263)

one-to-many correspondence one element of one set corresponds to many elements of another set (p. 512)

one-to-one correspondence one element of one set corresponds to one element of another set (p. 512)

opposite side the side in a right-angled triangle opposite the given angle (p. 179)

outlier a value that lies outside a certain range; a value more than 1.5 × IQR below the lower quartile, or more than 1.5 × IQR above the upper quartile, of a data set (p. 269)

parabola the symmetrical, curved shape of the graph of a quadratic equation (p. 394)

parallel box plot two or more box plots drawn on the same scale; useful for comparing data sets (p. 277)

parallel dot plot two or more dot plots drawn for comparative purposes; it is common to include a scale for each dot plot to serve as a dividing line between the plots (p. 277)

per annum 'for each year' in the Latin language (p. 3)

percentage cumulative frequency curve a cumulative frequency curve where the vertical axis (cumulative frequency) has been converted to percentages (p. 268)

percentile the value below which the stated percentage of results lie (p. 268)

perfect square rules useful in factorising and expanding binomial expressions as $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ (p. 43)

period the horizontal distance covered by one full cycle of the graph of a trigonometric function (p. 226)

perpendicular two lines that intersect or meet at right angles. The product of their gradients is -1: $m_1 \times m_2 = -1$ (p. 159)

perpendicular bisector a line that intersects another line at a right angle and cuts the other line in half (p. 349)

point of inflection a point on a curve where the shape of the curve changes direction. The curve can change from concave to convex, or vice versa. The points of inflection in cubics and quartics have a zero gradient (p. 476)

polynomial an expression involving a sum of terms raised to different powers (p. 487)

prime cost *see* initial cost (p. 28)

principal the original amount of money that was borrowed or loaned (p. 3)

prism a solid that has a regular shape, uniform cross-section and faces that are all polygons (p. 97)

probability the study of chance, or likelihood (p. 521)

probability tree diagram a tree diagram showing the probabilities of all outcomes as branches (p. 540)

property a fact about a particular figure that can be proved from its definition (p. 340)

prove to demonstrate that a theorem is true in all cases (p. 336)

pyramid a solid made up of a polygonal base and triangular sides that all meet at a vertex (p. 107)

quadrant one quarter of a circle, formed when two perpendicular diameters are drawn in a circle (p. 220)

quadratic expression an expression in which the highest power of a pronumeral is 2 (p. 51)

quadratic trinomial an expression consisting of three terms in which the highest power of a pronumeral is 2 (p. 51)

quantile a percentile expressed in decimal form (p. 268)

quartic an expression or equation where the highest power of the variable is 4 (p. 475)

quartile 25% of a data set; the lower quartile, Q_L, is the smallest 25% of the data while the upper quartile, Q_U, is the largest 25% of the data (p. 268)

quotient the result of dividing one quantity by another (p. 488)

random sampling sampling in which every element in the population being sampled has an equal chance of being selected (p. 316)

range (algebraic) all the y-values generated by a relation for a given domain (p. 511)

range (statistics) the difference between the largest result and the smallest result in a data set; one of the measures of spread (p. 268)

rational numbers numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ (p. 561)

rationalising the denominator the process of changing a fractional expression that has a surd component in the denominator into an equivalent form that has a rational denominator (p. 579)

real numbers all rational and irrational numbers together (p. 561)

rectangular hyperbola a graph of an inverse relationship in which one variable decreases as the other increases (p. 435)

reducing balance depreciation when items depreciate by smaller and smaller amounts each year (p. 29)

reflection a type of transformation where a shape or graph is inverted or turned upside down over an axis of symmetry (e.g. the x-axis) (p. 401)

regression line a least squares line of best fit (p. 310)

relation a set of ordered pairs that may be connected by a rule (p. 511)

reliability of data the ability to get the same general results if a survey is repeated under the same conditions (p. 299)

remainder any amount remaining after a division process has been completed (p. 488)

remainder theorem if the polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$. If $P(x)$ is divided by $(ax - b)$, then the remainder is $P\left(\frac{b}{a}\right)$ (p. 496)

residual value the value of an item (such as a piece of machinery) at the end of its useful life (p. 30)

salvage value *see* residual value (p. 30)

sample space the full list of possible outcomes in a probability experiment (p. 521)

scatter plot a graph of bivariate data; each data pair is represented by a point on the graph (p. 286)

scrap value *see* residual value (p. 30)

secant a line drawn from an external point that intersects with a circle or any other curve at two points (p. 381)

second (angle measure) one-sixtieth of a minute (angle measure) (p. 179)

sector of a circle the area formed between an arc and two radii drawn from its endpoints (p. 359)

segment the area formed between a chord and the circumference of a circle (p. 359)

set a collection of items (called elements) (p. 511)

similar figures that are the same shape but not the same size (p. 330)

simple interest interest calculated on the original principal of the investment or loan. It does not take into account increases in the principal over time as interest is added (p. 3)

simultaneous equations two, or more, equations that are true for the same pair of x- and y-values (p. 167)

sine one of the three basic trigonometric ratios; the ratio between the opposite side and the hypotenuse in a right-angled triangle (p. 179)

sine rule a rule to find an unknown angle when two sides and one angle opposite one of these sides are known: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where A, B and C are the three angles and a, b, and c are the sides opposite the angles, respectively (p. 236)

sketch a drawing or graph that is not absolutely accurate, but shows all the key features required for the purpose (p. 150)

sphere a ball-shaped solid (p. 108)

standard deviation one of the measures of spread, related to the mean (p. 304)

straight-line depreciation decrease in value by a fixed amount each year (p. 28)

stratified random sampling random sampling from a population but sampled in proportion to strata in the population (p. 316)

subtend to form an angle at a point (p. 359)

subtended supported at the circumference or centre of a circle by an arc; describes an angle (p. 359)

surds irrational numbers that can only be expressed exactly using the radical or root sign (p. 561)

surface area the sum of the areas of all the external surfaces of a solid (p. 97)

tangent (geometry) a line drawn from an external point that intersects with a circle at one point only, or that intersects with any curve at a point on that curve with the same gradient as the tangent line (p. 381)

tangent (trigonometry) one of the three basic trigonometric ratios; the ratio between the opposite side and the adjacent side in a right-angled triangle (p. 179)

tapered solids solids in which the cross-section gets smaller as you move from the base to the top (p. 107)

theorem a proven statement about a general result or relationship (pp. 340, 345)

total possible depreciation the prime cost of an item, less any resale value or scrap value (p. 30)

transformation a change of shape or position for an object or graph (p. 401)

translation a transformation that involves sliding the shape, or graph, across the page (p. 401)

tree diagram a branching diagram that assists in identifying the sample space of each outcome in a multiple event (p. 522)

trigonometric ratio a ratio of the lengths of two sides in a right-angled triangle (p. 179)

true bearing a direction given in terms of the number of degrees turned in a clockwise direction from north (p. 199)

turning point the point on a curve where the graph changes direction, also known as the vertex. The turning point of a parabola is either a minimum or maximum value (p. 394)

turning point form the form of a quadratic equation that enables the turning point to be read off directly; $y = a(x - b)^2 + c$ has turning point (b, c) (p. 429)

two-way table in probability, a way of presenting information so that outcomes for one event are listed across the top and outcomes for a second event are listed down the side. The individual cells of the table represent the intersection of two outcomes (pp. 522, 530)

uniform cross-section a cross-section of a solid that is consistent from the base to the top (p. 97)

unique equations give non-parallel and non-coincident graphs, have only one point of intersection and one unique solution (p. 167)

unit circle a circle with a radius of one unit, used to define the trigonometric functions (p. 220)

unit cost depreciation the amount of depreciation, in dollars, per unit of an item's expected life (p. 30)

universal set the full list of outcomes under consideration (p. 530)

validity the rational quality of the conclusions drawn from the data collected, e.g. whether it is reasonable, reliable and correct that one variable causes the other to occur (p. 299)

variance a statistical measure of spread; it is the mean of the squares of the deviations from the mean value (p. 304)

Venn diagram in probability, a way of presenting information. A rectangle representing the universal set is drawn and one or more areas are drawn inside it. Areas may be disjoint (no overlap), intersect (partially overlap) or one may be entirely inside another (subset or complete overlap) (p. 530)

vertex another name for the turning point of a parabola (p. 394); or, a point where lines meet

vertical line test if a vertical line can be drawn through any part of a graph of a relation and intersects with the graph at one point only, then the relation is a function (p. 512)

written-down value *see* adjusted value (p. 28)