NEW FORMAT

AJ Sadler

Mathematics Methods

3

Student Book

Unit 3

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Mathematics Methods Unit 3
1st revised Edition
A J Sadler

Publishing editor: Robert Yen
Project editor: Alan Stewart
Cover design: Chris Starr (MakeWork)
Text designers: Sarah Anderson, Nicole Melbourne, Danielle Maccarone
Permissions researcher: Helen Mammides
Answer checker: George Dimitriadis
Production controller: Erin Dowling
Typeset by: Nikki M Group Pty Ltd

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National Library of Australia Cataloguing-in-Publication Data
Sadler, A. J., author.
Mathematics methods. Unit 3 / A J Sadler.

1st revised edition
9780170395137 (paperback)
For secondary school age.

Mathematics--Study and teaching (Secondary)
Mathematics--Textbooks.

Cengage Learning Australia
Level 7, 80 Dorcas Street
South Melbourne, Victoria Australia 3205

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road, Albany, North Shore 0632, NZ

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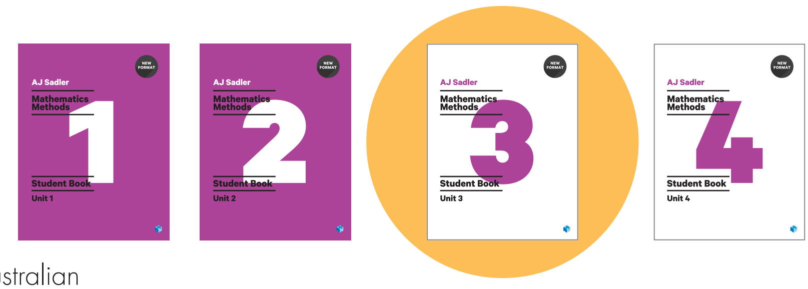
Printed in China by China Translation & Printing Services.
1 2 3 4 5 6 7 21 20 19 18 17

PREFACE

This text targets Unit Three of the West Australian course *Mathematics Methods*, a course that is organised into four units, units one and two for year eleven and units three and four for year twelve.



The West Australian course, *Mathematics Methods*, is based on the Australian Curriculum Senior Secondary course *Mathematical Methods*. Apart from some small changes, mainly to the wording, the unit threes of these courses are closely aligned. Hence this book would also be suitable for students following unit three of the Australian Curriculum course *Mathematical Methods*.

The book contains text, examples and exercises containing many carefully graded questions. A student who studies the appropriate text and relevant examples should make good progress with the exercise that follows.

The book commences with a section entitled **Preliminary work**. This section briefly outlines work of particular relevance to this unit that students should either already have some familiarity with from the mathematics studied in earlier years, or for which the brief outline included in the section may be sufficient to bring the understanding of the concept up to the necessary level.

As students progress through the book they will encounter questions involving this preliminary work in the **Miscellaneous exercises** that feature at the end of each chapter. These miscellaneous exercises also include questions involving work from preceding chapters to encourage the continual revision needed throughout the unit.

Some chapters commence with a '**Situation**' or two for students to consider, either individually or as a group. In this way students are encouraged to think and discuss a situation, which they are able to tackle using their existing knowledge, but which acts as a fore-runner and stimulus for the ideas that follow. Students should be encouraged to discuss their solutions and answers to these situations and perhaps to present their method of solution to others. For this reason answers to these situations are generally not included in the book.

At times in this series of books I have found it appropriate to go a little outside the confines of the syllabus for the unit involved. In this regard readers will find in this text that when applying product, quotient and chain rules to functions of the form $f(ax - b)$ I go beyond the '$ax - b$' to include functions involving other linear combinations of x^n.

I take a similar approach when considering $\int f(ax - b)\,dx$ and consider more general forms of $\int f'(x) f(x)\,dx$. When considering small changes I include mention of small percentage change and marginal rates of change. I introduce the concept of 'e' through a consideration of continuous compounding.

Alan Sadler

ISBN 9780170395137

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ISBN 9780170395137

ISBN 9780170395137

IMPORTANT NOTE

This series of texts has been written based on my interpretation of the appropriate *Mathematics Methods* syllabus documents as they stand at the time of writing. It is likely that as time progresses some points of interpretation will become clarified and perhaps even some changes could be made to the original syllabus. I urge teachers of the *Mathematics Methods* course, and students following the course, to check with the appropriate curriculum authority to make themselves aware of the latest version of the syllabus current at the time they are studying the course.

Acknowledgements

As with all of my previous books I am again indebted to my wife, Rosemary, for her assistance, encouragement and help at every stage.

To my three beautiful daughters, Rosalyn, Jennifer and Donelle, thank you for the continued understanding you show when I am 'still doing sums' and for the love and belief you show.

To the delightfully supportive team at Cengage – I thank you all.

Alan Sadler

ISBN 9780170395137

PRELIMINARY WORK

This book assumes that you are already familiar with a number of mathematical ideas from your mathematical studies in earlier years.

This section outlines the ideas which are of particular relevance to Unit Three of the *Mathematical Methods* course and for which familiarity will be assumed, or for which the brief explanation given here may be sufficient to bring your understanding of the concept up to the necessary level.

Read this 'Preliminary work' section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of those concepts up to an appropriate level for this unit. (If you do understand the work but feel somewhat 'rusty' with regards to applying the ideas some of the chapters afford further opportunities for revision as do some of the questions in the miscellaneous exercises at the end of chapters.)

- Chapters in this book will continue some of the topics from this preliminary work by building on the assumed familiarity with the work.
- The miscellaneous exercises that feature at the end of each chapter may include questions requiring an understanding of the topics briefly explained here.

Number

It is assumed that you are familiar with, and competent in the use of, positive and negative numbers, recurring decimals, square roots and cube roots and that you are able to choose levels of accuracy to suit contexts and distinguish between exact values, approximations and estimates.

Numbers expressed with positive, negative and fractional powers should also be familiar to you as should be the following index laws:

$$a^n \times a^m = a^{n+m} \qquad a^n \div a^m = a^{n-m} \qquad a^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad (a^n)^m = a^{n \times m}$$

$$(ab)^n = a^n \times b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

An ability to simplify expressions involving square roots is also assumed.

Note: The set of numbers that you are currently familiar with is called the set of **real numbers**. We use the symbol $\mathbb{R}$ for this set.

$\mathbb{R}$ contains many subsets of numbers such as the whole numbers, the integers, the prime numbers etc. (If you are also a student of *Mathematics Specialist* you will also have encountered numbers beyond this real system. Such considerations are beyond the scope of this unit.)

The absolute value

The absolute value of a number is the distance on the number line that the number is from the origin. The absolute value of x is written $|x|$ and equals x when x is positive, and equals $-x$ when x is negative. Thus $|3| = 3, |-3| = 3, |4| = 4, |-4| = 4$.

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Compounding

One of the situations featuring at the start of one chapter in this book involves the idea of compound interest. You are probably aware that if you were to invest \$1000 into a savings account it can earn interest. If this interest is, say 6% compounded annually, then after one year the account will be worth \$1060 (= \$1000 × 1.06).

However, if the compounding were to occur every six months, i.e. 3% every six months, then the interest earned at the end of the first six months would itself earn interest in the second six months.

Thus, with compounding every six months:

$$\begin{aligned}\text{Amount in account after one year} &= \$1000 \times 1.03 \times 1.03 \\ &= \$1000 \times 1.03^2 \\ &= \$1060.90\end{aligned}$$

With compounding every quarter year:

$$\begin{aligned}\text{Amount in account after one year} &= \$1000 \times \left(1 + \frac{0.06}{4}\right)^4 \\ &= \$1000 \times 1.015^4 \\ &= \$1061.36 \text{ (nearest cent).}\end{aligned}$$

With compounding every month:

$$\begin{aligned}\text{Amount in account after one year} &= \$1000 \times \left(1 + \frac{0.06}{12}\right)^{12} \\ &= \$1000 \times 1.005^{12} \\ &= \$1061.68 \text{ (nearest cent).}\end{aligned}$$

With compounding every day:

$$\begin{aligned}\text{Amount in account after one year} &= \$1000 \times \left(1 + \frac{0.06}{365}\right)^{365} \\ &= \$1061.83 \text{ (nearest cent).}\end{aligned}$$

Measures of central tendency

The **mean**, the **median** and the **mode** are all measures used to summarise a set of scores. The mean and the median each indicate a 'central score'. The mode is often included in these 'averages' but there is no guarantee that the mode is a 'central' measure.

The mean, or common average, of a set of scores is found by summing the scores and then dividing by the number of scores.

The median is found by listing the scores in order of size and locating the middle score or, for an even number of scores, the mean of the middle two.

The mode is the most common score. If there are two scores that are equally 'most common' we say the set of scores is bimodal because it has two modes. We do not find the mean of the two modes.

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Measures of spread (or dispersion)

The **range** of a set of scores is the difference between the highest score and the lowest score and gives a simple measure of how widely the scores are spread. Whilst the range is easy to calculate it is determined using just two of the scores. For this reason it is of limited use.

The measurements of **variance** and **standard deviation** are more commonly used measures of dispersion. The variance is found by finding how much each of the scores differs from the mean, squaring these values and finding the average of the squared values. The standard deviation is the square root of the variance.

Consider the eight scores listed below for which the mean is 18.

Scores:	12	15	16	16	18	20	22	25
Deviation from mean:	−6	−3	−2	−2	0	+2	+4	+7

$$\text{Variance of scores} = \frac{(-6)^2+(-3)^2+(-2)^2+(-2)^2+(0)^2+(2)^2+(4)^2+(7)^2}{8}$$

$$= 15.25$$

$$\text{Standard deviation} = \sqrt{15.25} \text{ i.e. } 3.91 \text{ (correct to two decimal places)}$$

Many calculators can determine the standard deviation, and other statistical information for a set of scores:

$\bar{x} = 18$	⟵	The mean of the scores.
$\sum x = 144$	⟵	The sum of the scores.
$\sum x^2 = 2714$	⟵	The sum of the squares of the scores.
$x\sigma_n = 3.90512483$	⟵	The standard deviation of the scores.
$x\sigma_{n-1} = 4.17475405$	⟵	A different standard deviation – see second note below.
$n = 8$	⟵	The number of scores.

Note

- The standard deviation, σ, is a measure of spread. For most distributions very few, if any, of the scores would be more than three standard deviations from the mean, i.e. the vast majority of the scores (and probably all of them) would lie between $(\bar{x} - 3\sigma)$ and $(\bar{x} + 3\sigma)$.
- The calculator display shown has two different standard deviations:
  σ_n is the standard deviation of the eight scores.
  σ_{n-1} gives an answer a little bigger than σ_n by dividing the sum of the squared deviations by $(n - 1)$ rather than n. This would be used if the eight scores were a sample taken from a larger population and we wanted to use the standard deviation of the sample to estimate the standard deviation of the whole population. Division by $(n - 1)$ rather than n compensates for the fact that there is usually less variation in a small sample than there is in the population itself. If the sample is large then n will be large and there will be little difference between σ_n and σ_{n-1}.

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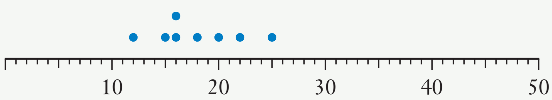
Change of origin and change of scale

Consider again the set of eight scores:

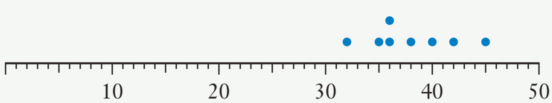
12 15 16 16 18 20 22 25

Showing the scores as a dot frequency diagram, and some summary statistics:



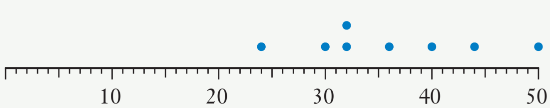
$\bar{x}$ = 18
$\sum x$ = 144
$\sum x^2$ = 2714
$x\sigma_n$ = 3.90512483
$x\sigma_{n-1}$ = 4.17475405
n = 8

Now suppose we increase all of the scores by 20. This will see them all move 20 places to the right on the dot frequency diagram. (We refer to this sort of transformation as a *change of origin*.) With all of the scores increased by 20 we would expect the mean to increase by 20. However, the points are no more, or less, spread out, than they were before. Hence the standard deviation should be unchanged.



$\bar{x}$ = 38
$\sum x$ = 304
$\sum x^2$ = 11674
$x\sigma_n$ = 3.90512483
$x\sigma_{n-1}$ = 4.17475405
n = 8

Suppose instead we were to multiply all of the original scores by 2. (We refer to this sort of transformation as a *change of scale*.) The scores would again all increase in value but would also become more spread out than the original set. We would expect the mean and the standard deviation of this new set of scores to be twice the mean and standard deviation of the original set.



$\bar{x}$ = 36
$\sum x$ = 288
$\sum x^2$ = 10856
$x\sigma_n$ = 7.81024967
$x\sigma_{n-1}$ = 8.34950811
n = 8

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Probability

The probability of something happening is a measure of the likelihood of it happening and is given as a number between zero (no chance of happening) to 1 (certain to happen).

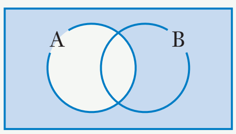
With activities such as rolling a die or flipping a coin, whilst we are unable to consistently predict the outcome of a particular die roll or coin flip, when these activities are repeated a large number of times each has a predictable long run pattern. For less predictable events the **long term relative frequency** with which an event occurs is then our best guess at the probability of the event occurring. Probability based on experimental or observed data like this is called **empirical probability**.

An event occurring and it not occurring are **complementary events**.

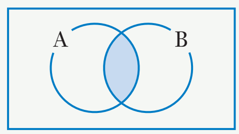
If P(event occurring) = a then P(event not occurring) = $1 - a$.

Venn diagrams can be a useful form of display for probability questions.

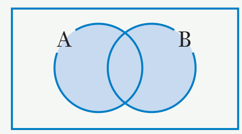
'Not A' is A′, the complement of A.



'A and B' is $A \cap B$, the intersection of A and B.



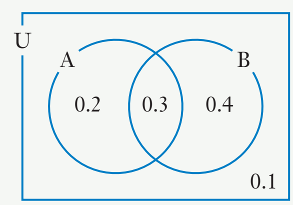
'A or B' is $A \cup B$, the union of A and B.



In some situations we may be given some extra piece of information or condition that allows us to restrict our attention to only certain members of the sample space. This is called **conditional probability**.

For the probability of A **given** B we write $P(A|B)$.

Hence, if the Venn diagram on the right shows the probabilities of the events A and B occurring then:



$$P(A|B) = \frac{0.3}{0.7} = \frac{3}{7} \qquad P(B|A) = \frac{0.3}{0.5} = \frac{3}{5}$$

Counting

It is assumed that you are familiar with the notation ^nC_r for the number of combinations of r different objects taken from a set containing n different objects.

There are ^nC_r combinations of r objects chosen from n different objects where

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

- ^nC_r may also be written as $\binom{n}{r}$. For example $\binom{7}{2} = {}^7C_2$
- ^nC_r can be thought of as 'from n **choose** r'.

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Algebra

It is assumed that you are already familiar with manipulating algebraic expressions, in particular:

- Expanding and simplifying:

For example, $4(x + 3) - 3(x + 2)$ expands to $4x + 12 - 3x - 6$ which simplifies to $x + 6$

$(x - 7)(x + 1)$ expands to $x^2 + 1x - 7x - 7$ which simplifies to $x^2 - 6x - 7$

$(2x - 7)^2$, i.e. $(2x - 7)(2x - 7)$ expands to $4x^2 - 28x + 49$

- Factorising:

For example,

$21x + 7$ factorises to $7(3x + 1)$

$15apy + 12pyz - 6apq$ factorises to $3p(5ay + 4yz - 2aq)$

$x^2 - 6x - 7$ factorises to $(x - 7)(x + 1)$

$x^2 - 9$ factorises to $(x - 3)(x + 3)$

the last one being an example of the *difference of two squares* result:

$x^2 - y^2$ factorises to $(x - y)(x + y)$

- Solving equations.

In particular, linear equations, simultaneous equations, quadratic equations, exponential equations (e.g. $2^x + 3 = 35$), trigonometrical equations (e.g. $\sin x = 0.5$ for $0 \le x \le 360°$), and in the use of your calculator to solve equations.

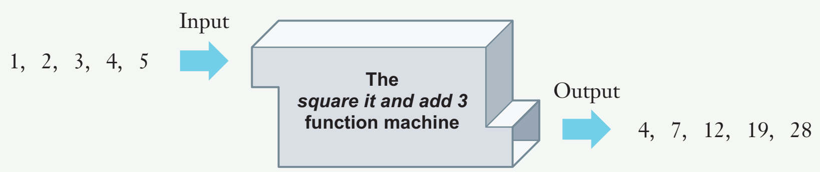
Function

It is assumed that you are familiar with the idea that in mathematics any rule that takes any input value that it can cope with and assigns to it a particular output value is called a **function**.

Familiarity with the function notation $f(x)$ is also assumed.

It can be useful at times to consider a function as a machine. A box of numbers (the **domain**) is fed into the machine, a certain rule is applied to each number, and the resulting output forms a new box of numbers, the **range**.

In this way $f(x) = x^2 + 3$, with domain {1, 2, 3, 4, 5}, could be 'pictured' as follows:



If we are not given a specific domain we assume it to be all the numbers that the function can cope with.

Thus the function $f(x) = \sqrt{x - 3}$ has a domain of all the real numbers greater than or equal to 3. I.e., $\{x \in \mathbb{R}: x \ge 3\}$

For this domain the function can put out all the real numbers greater than or equal to zero. Thus the range of the function will be all real numbers greater than or equal to 0.

ISBN 9780170395137

It is assumed you are particularly familiar with linear and quadratic functions, their characteristic equations and their graphs, and have some familiarity with the graphs of $y = x^3$, $y = \sqrt{x}$ and $y = \dfrac{1}{x}$.

It is further assumed that the effect altering the values of a, b, c and d have on the graph of $y = af\,[b(x - c)] + d$ is something you have previously considered for various functions.

Remember that linear and quadratic functions are members of the larger family of functions called **polynomial functions**. These are functions of the form

$$f(x) = a_n x^n + a_{n-1}\,x^{n-1} + a_{n-2}\,x^{n-2} + \ldots + a_2\,x^2 + a_1\,x + a_0$$

where n is a non-negative integer and a_n, a_{n-1}, a_{n-2}, … are all numbers, called the **coefficients** of x^n, x^{n-1}, x^{n-2} etc.

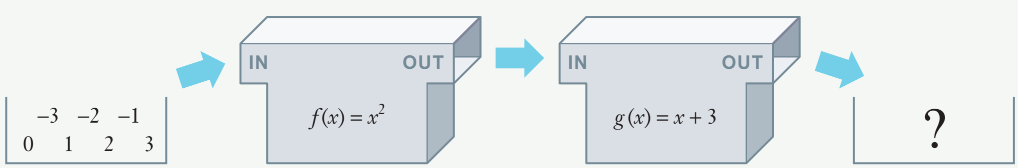
The highest power of x is the **order** of the polynomial.

Thus linear functions, $y = mx + c$, are polynomials of order 1,
quadratic functions, $y = ax^2 + bx + c$, are polynomials of order 2,
cubic functions, $y = ax^3 + bx^2 + cx + d$, are polynomials of order 3, etc.

Though not an idea you would necessarily be familiar with, but one that should seem reasonable, is that of using the output from one function as the input of a second function. In this way we form a **composite function**, also referred to as a **function of a function**.

Suppose that $f(x) = x^2$ and $g(x) = x + 3$.

If we feed the set of numbers {–3, –2, –1, 0, 1, 2, 3} into f and then feed the output into g what numbers will g output?



With the domain stated, combining the functions f and g in this way will give a final output of {3, 4, 7, 12}:

{–3, –2, –1, 0, 1, 2, 3} $\xrightarrow{f(x)}$ {0, 1, 4, 9} $\xrightarrow{g(x)}$ {3, 4, 7, 12}

We write this combined function as $g[f(x)]$
or as $g\,o\,f(x)$ or $g \circ f(x)$ for 'g of f of x'
or as $gf(x)$.

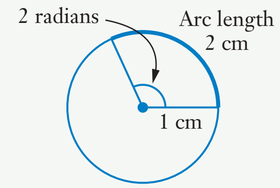
Note that though our 'machine diagram' above shows the 'f function' first we write the combined function as $g\,f(x)$. This is to show that the 'f function', being closest to the '(x)', operates on the x values first.

Radian measure

A particularly useful concept involved with angle measurement is the radian.

An arc of length 1 unit, in a circle of unit radius, subtends an angle of 1 radian at the centre of the circle.

An arc of length 2 units, in a circle of unit radius, subtends an angle of 2 radians at the centre of the circle, and so on.



Thus an arc of length 2π units, in a circle of unit radius, will subtend an angle of 2π radians at the centre of the circle. However, if the radius is 1 unit an arc of 2π(1) is the full circumference of the circle and will subtend an angle of 360° at the centre.

Thus $\quad 2\pi$ radians = 360°
I.e. $\quad \pi$ radians = 180°

(Thus, correct to one decimal place, 1 radian is equivalent to 57.3°.)

Using radian measure to determine arc length, sector area and segment area using the following formulae should also be familiar.

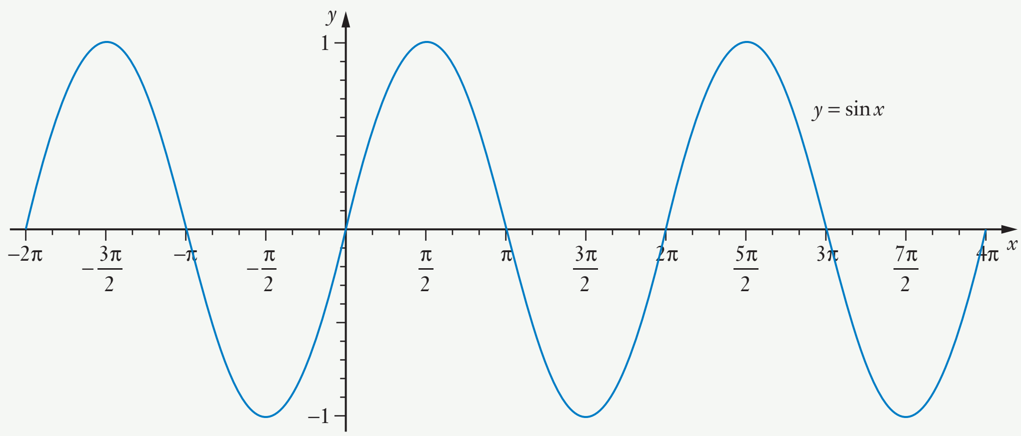
$$\text{Arc length} = r\theta$$

$$\text{Sector area} = \frac{1}{2}r^2\theta$$

$$\text{Segment area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

Trigonometric functions

You should be familiar with the trigonometrical ratios of sine, cosine and tangent as functions, the graphs of these functions, use of the terms amplitude, cycle, period and phase. For example, the graph of $y = \sin x$ for $-2\pi \le x \le 4\pi$ is shown below. It has an amplitude of 1 and a period of 2π.



A knowledge of the exact values of sine, cosine and tangent for various angles is assumed, for example:

$$\sin 30° = \frac{1}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sin 120° = \frac{\sqrt{3}}{2} \qquad \cos(3\pi) = -1,$$

$$\tan\left(\frac{11\pi}{6}\right) = -\frac{1}{\sqrt{3}} \text{ (Or, expressed with a rational denominator, } -\frac{\sqrt{3}}{3}.\text{)}$$

ISBN 9780170395137

The following trigonometric identities should also be familiar to you:

$$\sin^2 A + \cos^2 A = 1$$

(Remember, we write powers of trigonometric functions, e.g. $(\sin A)^2$, as $\sin^2 A$.)

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

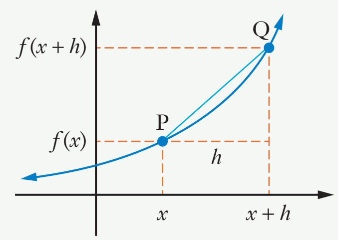
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

You should be able to solve equations involving trigonometric functions using technology and, in straightforward cases, algebraically.

For example, asked to solve the equation $\sin 3x = 0.5$, for $0 \le x \le \pi$, you should be able to determine that $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$.

Differentiation

It is assumed that you are familiar with the idea of the **gradient**, or *slope*, of a line and in particular that whilst a straight line has the same gradient everywhere, the gradient of a curve varies as we move along the curve.



To find the gradient at a particular point, P, on a curve $y = f(x)$ we choose some other point, Q, on the curve whose x-coordinate is a little more than that of point P.

Suppose P has an x-coordinate of x and Q has an x-coordinate of $(x + h)$.

The corresponding y-coordinates of P and Q will then be $f(x)$ and $f(x + h)$.

Thus the gradient of PQ $= \dfrac{f(x+h) - f(x)}{h}$.

We then bring Q closer and closer to P, i.e. we allow h to tend to zero, and we determine the limiting value of the gradient of PQ.

i.e. Gradient at P = limit of $\dfrac{f(x+h) - f(x)}{h}$ as h tends to zero.

This gives us the **instantaneous rate of change** of the function at P.

The process of determining the **gradient formula** or **gradient function** of a curve is called **differentiation**.

Writing h, the small increase, or *increment*, in the x coordinate, as δx, where 'δ' is a Greek letter pronounced 'delta', and $f(x + h) - f(x)$, the small increment in the y coordinate as δy, we have:

$$\text{Gradient function} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This **derivative** is written as $\dfrac{dy}{dx}$ and pronounced 'dee y by dee x'.

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This 'limiting chord process' gives the following results:

$$\text{If} \quad y = x^2 \quad \text{then} \quad \frac{dy}{dx} = 2x.$$

$$\text{If} \quad y = x^3 \quad \text{then} \quad \frac{dy}{dx} = 3x^2.$$

$$\text{If} \quad y = x^4 \quad \text{then} \quad \frac{dy}{dx} = 4x^3.$$

$$\text{If} \quad y = x^5 \quad \text{then} \quad \frac{dy}{dx} = 5x^4.$$

The general statement being:

$$\text{If} \quad y = ax^n \quad \text{then} \quad \frac{dy}{dx} = anx^{n-1}.$$

You should also be familiar with the following points:

- If $y = f(x)$ then the derivative of y with respect to x can be written as $\frac{dy}{dx}$, $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$.
- A shorthand notation using a 'dash' may be used for differentiation with respect to x. Thus if $y = f(x)$ we can write $\frac{dy}{dx}$ as $f'(x)$ or simply y' or f'.
- If $y = f(x) \pm g(x)$ then $\frac{dy}{dx} = f'(x) \pm g'(x)$. (The sum and difference rules.)

Whenever we are faced with the task of finding the gradient formula, gradient function or derivative of some 'new' function, for which we do not already have a rule, for example if we wanted to determine the gradient function for $y = \sin x$, we simply go back to the basic principle:

$$\text{Gradient at } P(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Antidifferentiation

Antidifferentiation is, as its name suggests, the opposite of differentiation. Given the **derivative**, or **gradient function**, $\frac{dy}{dx}$, antidifferentiation returns us to the function, or **primitive**.

However there are a many functions that differentiate to $2x$, for example:

If $y = x^2$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 + 1$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 - 3$ then $\frac{dy}{dx} = 2x$. Etc.

Thus we say that the antiderivative of $2x$ is $x^2 + c$ where c is some constant. Given further information it may be possible to determine the value of this constant.

The general statement is:

$$\text{If} \quad \frac{dy}{dx} = ax^n \quad \text{then} \quad y = \frac{ax^{n+1}}{n+1} + c$$

remembered as: 'Increase the power by one and divide by the new power.'

(Clearly this rule cannot apply for $n = -1$. Such a situation is beyond the scope of this unit.)

Hence the antiderivative of $6x^2 + 7$ is $\frac{6x^3}{3} + \frac{7x^1}{1} + c$

i.e. $2x^3 + 7x + c$

It is also assumed that you are familiar with the fact that antidifferentiation is also known as **integration**. Instead of being asked to find the antiderivative of $6x^2 + 7$ we could be asked to **integrate** $6x^2 + 7$.

Integration uses the symbol $\int$.

Hence the fact that the antiderivative of $6x^2 + 7$ is $2x^3 + 7x + c$

could be written as $\int (6x^2 + 7)\, dx = 2x^3 + 7x + c$

the 'dx' indicating that the antidifferentiation, or integration, is with respect to the variable x.

Our general rule for antidifferentiating ax^n could then be written:

$$\int an^x \, dx = \frac{ax^{n+1}}{n+1} + c$$

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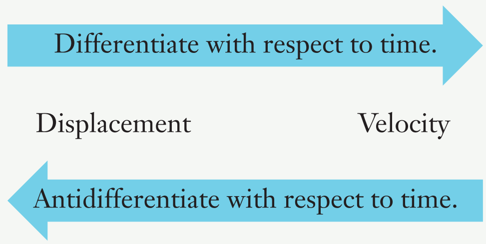
Rectilinear motion

One of the commonest rates of change that concerns us is the rate at which we change our location. If we measure our location as a **displacement** from some fixed point or origin, then the rate at which we change our displacement is our **velocity**. With differentiation giving us the rate of change of one variable with respect to another, it follows that if we differentiate displacement with respect to time we obtain velocity.

Thus:



An object moving in a straight line with velocity v has **speed** $|v|$.

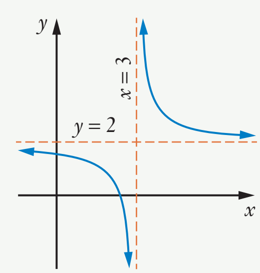
The graphs of functions

Whilst the gradient function allows us to determine the gradient of a curve at points on the curve (and hence locate *turning points*, *points of inflection* and investigate *concavity*) the gradient is not the only noteworthy feature of the graph of a function. We may also be interested in any *intercepts with the axes*, any *asymptotes* and whether the curve shows any *symmetry*.

Read through the following to refresh your memory of the meaning of these terms.

The graph on the right is the graph of

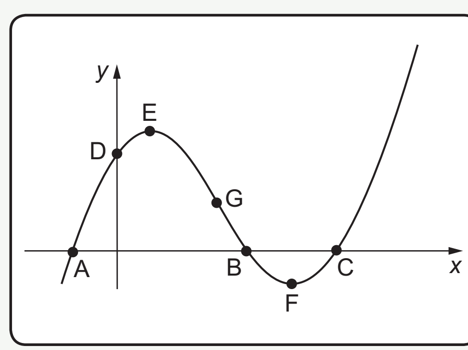
$$y = \frac{2x - 5}{x - 3}$$



Notice that the graph gets closer and closer to the lines $x = 3$ and $y = 2$ without ever quite touching these lines. (Perhaps you could have predicted this behaviour from the given equation.) The lines $x = 3$ and $y = 2$ are **asymptotes** to the curve.

The graph shown on the right does not appear to have any asymptotes but some other noteworthy features are:

- The x-axis intercepts.
  Points A, B and C in the diagram.
- The y-axis intercept.
  Point D in the diagram.



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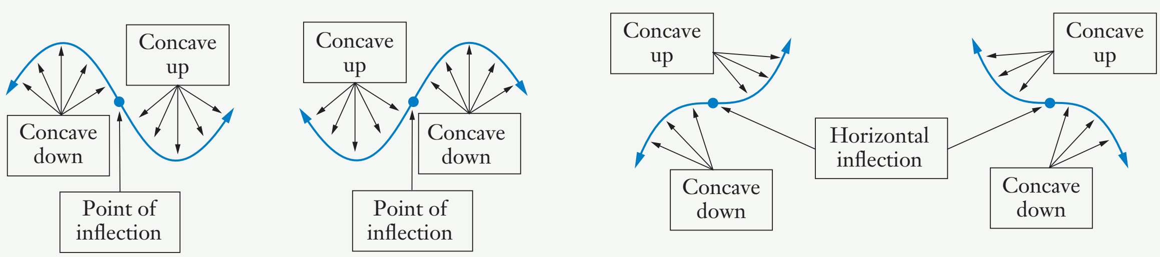
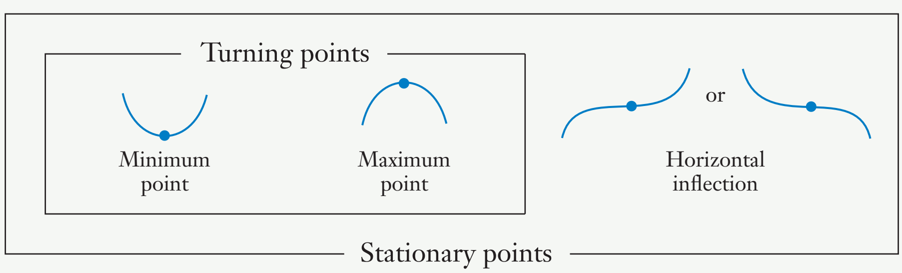
- Any **turning points** the graph has.
  Points E and F in the diagram.
  E is a **maximum turning point** and F is a **minimum turning point**.
  Technically these are **local** maximum and **local** minimum points. There may be locations on the graph that are 'higher' or 'lower' than these points but *in their locality* they are the highest or lowest points.
- If $y = f(x)$ is shaped $\cap$ (or part of $\cap$) we say that it is **concave down**.
  The previous graph appears to be concave down to the left of point G.
- If $y = f(x)$ is shaped $\cup$ (or part of $\cup$) we say that it is **concave up**.
  The previous graph appears to be concave up to the right of point G.
- The points on a curve where it changes from being concave down to concave up, or from concave up to concave down, are called **points of inflection**. Point G in the previous diagram is a point of inflection.
  If, at a point of inflection, the graph is momentarily horizontal then the point is a point of **horizontal inflection**.



- Maximum and minimum points are sometimes referred to as **turning points**.
  Maximum points, minimum points and points of horizontal inflection are sometimes referred to as **stationary points**.



At all stationary points the gradient is zero.

Also remember:

- Some graphs possess **symmetry**.
  E.g. $y = x^2$ has line symmetry, $y = x^3$ has rotational symmetry.
- Some functions are undefined for certain values of x.
  E.g. $y = \sqrt{x}$ is undefined for $x < 0$.
- Some functions have regions on the graph where the function cannot exist.
  E.g. $xy = 1$ cannot exist where the x and y coordinates are of different sign. (Also $x \neq 0$ and $y \neq 0$.)

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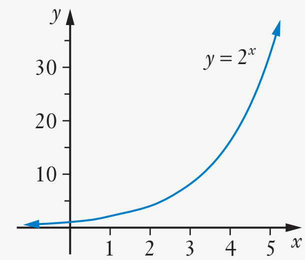
It is also assumed that you are familiar with the basic shape of the graphs of **exponential functions**, i.e. functions of the form $f(x) = a^x$ (for $a > 0$), for example that of $y = 2^x$ shown on the right, and how these graphs will differ for different values of a.



Use of differentiation to locate stationary points

It is anticipated that from your study on Unit 2 of *Mathematics Methods* you are familiar with using differentiation to locate stationary points of a function and in applying this technique to determine the optimal values for various situations. This process will be further explored in chapter two of this text.

Use of technology

You are encouraged to use your calculator, computer programs and the internet during this unit.

Define $f(x) = x^4 + x$

$\lim_{h \to 0}\left(\frac{f(3+h) - f(3)}{h}\right)$

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$\frac{d}{dt}(5t^2 + 6t)$

$10 \cdot t + 6$

$\frac{d}{da}(a^3 - 3a^2 + 5)|a = 3$

9

$\int (6 \cdot x^2 + 7)\,dx$

$2 \cdot x^3 + 7 \cdot x$

$\int (12 \cdot x^2)\,dx$

$4 \cdot x^3$

$\int (12 \cdot x^2 + 2 \cdot x - 3)\,dx$

$4 \cdot x^3 + x^2 - 3 \cdot x$

However you should make sure that you can also perform the basic processes such as solving equations, sketching graphs, differentiation etc., without the assistance of such technology when required to do so.

Note: The illustrations of calculator displays shown in the book may not exactly match the display from your calculator. The illustrations are not meant to show you exactly what your calculator will necessarily display but are included more to inform you that at that moment the use of a calculator could well be appropriate.

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1.

Differentiation

- Second (and higher order) derivatives
- The product rule
- The quotient rule
- The chain rule
- Miscellaneous exercise one

Examples 1, 2 and 3 that follow revise the application of the rule:

WS Basic differentiation

$$\text{If} \quad y = ax^n \quad \text{then} \quad \frac{dy}{dx} = anx^{n-1}.$$

The rule does not just apply to n taking non-negative integer values but is also true for n taking fractional and negative values as well.

Also remember that if $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x). \qquad \text{(The sum and difference rules.)}$$

EXAMPLE 1

Determine the gradient function, $\frac{dy}{dx}$, for each of the following.

a $y = 7x^5$ **b** $y = 3x^2 + 2x - 5$ **c** $y = \frac{3}{x^2}$

d $y = 5\sqrt{x}$ **e** $y = (x^2 + 1)(2x - 3)$

Solution

a If $y = 7x^5$

$$\frac{dy}{dx} = 7(5)x^{5-1} = 35x^4$$

b If $y = 3x^2 + 2x - 5 \ (= 3x^2 + 2x^1 - 5x^0)$

$$\frac{dy}{dx} = 6x + 2$$

c If $y = \frac{3}{x^2} = 3x^{-2}$

$$\frac{dy}{dx} = -6x^{-3} = -\frac{6}{x^3}$$

d If $y = 5\sqrt{x} = 5x^{\frac{1}{2}}$

$$\frac{dy}{dx} = 5\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \frac{5}{2\sqrt{x}}$$

e If $y = (x^2 + 1)(2x - 3)$ expanding gives $y = 2x^3 - 3x^2 + 2x - 3$

Hence $\frac{dy}{dx} = 6x^2 - 6x + 2$

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These same answers can be obtained from some calculators.

$$\frac{d}{dx}(7x^5) \quad 35\cdot x^4$$

$$\frac{d}{dx}(3x^2+2x-5) \quad 6\cdot x+2$$

$$\frac{d}{dx}\left(\frac{3}{x^2}\right) \quad \frac{-6}{x^3}$$

$$\frac{d}{dx}(5\sqrt{x}) \quad \frac{5}{2\cdot\sqrt{x}}$$

$$\frac{d}{dx}((x^2+1)(2x-3)) \quad 6\cdot x^2-6\cdot x+2$$

EXAMPLE 2

Determine the gradient of the curve $y = x^2 + 3\sqrt{x}$ at the point (4, 22).

Solution

If $y = x^2 + 3\sqrt{x}$

$= x^2 + 3x^{\frac{1}{2}}$

then $\frac{dy}{dx} = 2x + \frac{3}{2\sqrt{x}}$.

At the point (4, 22), $x = 4$

and so $\frac{dy}{dx} = 2(4) + \frac{3}{2\sqrt{4}}$.

$= 8.75$

$$\frac{d}{dx}(x^2+3\sqrt{x})\Big|x=4 \quad \frac{35}{4}$$

The gradient of the curve $y = x^2 + 3\sqrt{x}$ at the point (4, 22) is 8.75.

EXAMPLE 3

Determine the equation of the tangent to the curve $y = \frac{4}{x}$ at the point (4, 1).

Solution

If $y = \frac{4}{x}$ i.e. $y = 4x^{-1}$

then $\frac{dy}{dx} = -4x^{-2}$

Thus at the point (4, 1), $\frac{dy}{dx} = -0.25$.

At (4, 1) the tangent will have a gradient of −0.25.
Thus the tangent will have an equation of the form

$y = -0.25x + c$.

$x = 4, y = 1$ must 'fit': $1 = -0.25(4) + c$

$\therefore$ $c = 2$

The required equation is $y = -0.25x + 2$.

$$\text{tangentLine}\left(\frac{4}{x}, x=4\right) \quad 2-\frac{x}{4}$$

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Second (and higher order) derivatives

If $y = 2x^5$ then the gradient function, $\frac{dy}{dx}$, equals $10x^4$.

Differentiating again gives 'the gradient function of the gradient function'.

We call this the *second derivative* of y with respect to x and write it as $\frac{d^2y}{dx^2}$.

Thus with $y = 2x^5$, $\frac{dy}{dx} = 10x^4$ and $\frac{d^2y}{dx^2} = 40x^3$

Continuing this process: $\frac{d^3y}{dx^3} = 120x^2$, $\frac{d^4y}{dx^4} = 240x$, etc.

Alternatively, using the dash notation,

With $f(x) = 2x^5$, $f'(x) = 10x^4$, $f''(x) = 40x^3$, $f'''(x) = 120x^2$, etc.

EXAMPLE 4

Find the coordinates of any points on the curve $y = 2x^3$ where the second derivative has a value of 24.

Solution

Either algebraically:

If $y = 2x^3$ then $\frac{dy}{dx} = 6x^2$

and $\frac{d^2y}{dx^2} = 12x$

Thus we require points for which $12x = 24$

i.e. $x = 2$

Or, by calculator:

$$\text{solve}\left(\frac{d^2}{dx^2}(2 \cdot x^3) = 24, x\right)$$

$\{x = 2\}$

If $x = 2$, $y = 2(2)^3$
$= 16$

Thus $y = 2x^3$ has a second derivative of 24 at (2, 16).

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Exercise 1A

(Whilst you are encouraged to explore the ability of your calculator to determine expressions for the derivative, to determine its value at particular points on a curve and to find the equation of tangents to curves, it is suggested that you do most of the following questions algebraically to ensure that you can follow the basic processes without a calculator.)

Determine the gradient function $\frac{dy}{dx}$ for each of the following.

1 $y = 5x + 17$

2 $y = 3x^2 - 2x$

3 $y = 2x^3 - x^2$

4 $y = 15 - 2x$

5 $y = \frac{x}{5}$

6 $y = \frac{5}{x}$

7 $y = 3x^2 - \frac{3}{x^2}$

8 $y = 10\sqrt{x}$

9 $y = 10 + 4\sqrt{x}$

10 $y = \frac{8}{\sqrt{x}}$

11 $y = \sqrt[3]{x}$

12 $y = \frac{5x^2 - 8x}{x}$

13 $y = 6 + \frac{1}{x}$

14 $y = 5(7x^2 - 2)$

15 $y = (x^2 - 1)(3x + 2)$

Determine $\frac{d^2y}{dx^2}$ for each of the following.

16 $y = x^2$

17 $y = x^3$

18 $y = 3x^2 + x$

19 $y = 2x^3 + 2x - 34$

20 $y = 2x^2 - x - 3$

21 $y = 4x^3 + 3x^2 + 2x$

22 $y = \sqrt{x}$

23 $y = 8\sqrt{x}$

24 $y = \frac{1}{x}$

25 $y = \frac{x}{5} + 7$

26 $y = \frac{5}{x} + 7$

27 $y = x^2 + \frac{4}{x^2}$

Determine $f'(x)$ for each of the following.

28 $f(x) = 3x - \frac{1}{x}$

29 $f(x) = 5x^2 + 8\sqrt{x}$

30 $f(x) = \frac{4x^2}{\sqrt{x}}$

Determine $f''(x)$ for each of the following.

31 $f(x) = 3x^4 + 4x^3$

32 $f(x) = \frac{3}{2x^3}$

33 $f(x) = 5x^3 - \frac{1}{x^2}$

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34 Find the gradient of $y = 2x^3 - 2x + 1$ at the point (1, 1).

35 Find the gradient of $y = 8 - \frac{5}{x}$ at the point (–1, 13).

36 Find the gradient of $y = 3x^2 - \frac{1}{x^2}$ at the point (–1, 2).

37 Find the value of $f''(-3)$ for $f(x) = 2x^3 - 3x^2 + 4x + 2$.

38 If $f(x) = 5x - 2x^3$ find

a $f'(x)$, **b** $f'(2)$, **c** $f''(x)$, **d** $f''(-2)$.

39 Find the equation of the tangent to the curve $y = 5x^2$ at the point (–2, 20).

40 Find the equation of the tangent to the curve $y = x + \frac{6}{x}$ at the point (2, 5).

41 Find the equation of the tangent to the curve $y = \frac{x^3 + 2\sqrt{x}}{x}$ at the point (1, 3).

42 Find the coordinates of the point(s) on the following curves where the derivative is as stated.

a $y = 2x^3 + 6x^2 - 8x + 4$. $\frac{dy}{dx} = 10$.

b $y = 5 + 6\sqrt{x}$. $\frac{dy}{dx} = 5$.

43 Find the coordinates of the point(s) on the following curves where the second derivative is as stated.

a $y = \frac{x^3}{12}$. $\frac{d^2y}{dx^2} = 1.5$.

b $y = x^3 - 2x^2$. $\frac{d^2y}{dx^2} = 2$.

44 The curve $y = ax^3 + bx^2 + cx + 5$ passes through the point P(–1, 4) and at the point P the first and second derivatives of the curve are 8 and –24 respectively.

Find the values of the constants a, b and c.

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WS
The product rule

The product rule

Consider the function $y = x(x + 3)$.

To determine $\frac{dy}{dx}$ we could simply expand the bracket to obtain $y = x^2 + 3x$

and then differentiate to give $\frac{dy}{dx} = 2x + 3$

Could we obtain this same answer without having to first expand $x(x + 3)$?

I.e. can we develop a rule for differentiating the product of two functions?

$$y = f(x) \times g(x)$$

Note: Initially this 'product rule' for differentiating $f(x) \times g(x)$ may seem to be of limited use because expanding the expression, and then differentiating, is likely to be reasonably straightforward in many cases anyway. However such straightforward expansion may not always be the case and then the product rule can prove to be very useful.

Work through the following investigation and see if you can discover the rule for differentiating products.

INVESTIGATION

If $y = (x + 3)(x + 2)$ then expansion gives $y = x^2 + 5x + 6$

thus $\frac{dy}{dx} = 2x + 5$

Now $(x + 3) + (x + 2) = 2x + 5$!! Could we simply differentiate a product by summing the two parts? Clearly we need to investigate further before we can state a rule with any confidence. Copy and complete the table below and see if you can determine the rule for differentiating $y = f(x)\, g(x)$.

Function as a product	Expanded	$\frac{dy}{dx}$
$y = (x + 3)(x + 2)$	$y = x^2 + 5x + 6$	$2x + 5$
$y = (x + 7)(x + 2)$		
$y = (x + 5)(x - 3)$		
$y = (x + 5)(2x - 1)$		
$y = (x + 2)(2x - 4)$		
$y = (2x + 3)(x - 1)$		
$y = (3x + 1)(x - 1)$		
$y = (2x + 1)(3x + 2)$		
$y = (5x + 1)(2x + 3)$		
$y = (x^2 + 1)(x + 3)$		
$y = (x^2 + 3)(2x^2 + 1)$		

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Did you discover a rule for differentiating a product? Well done if you did.

The product rule can be stated as follows:

If $y = f(x)\,g(x)$ then $\dfrac{dy}{dx} = g(x)\,f'(x) + f(x)\,g'(x)$

Alternatively, if we use u and v to represent the two functions $f(x)$ and $g(x)$:

If $y = uv$ then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

i.e:

$$\frac{dy}{dx} = (\text{2nd function} \times \text{derivative of 1st}) + (\text{1st function} \times \text{derivative of 2nd})$$

As addition is commutative, i.e. $a + b = b + a$, this could alternatively be written:

$$\frac{dy}{dx} = (\text{1st function} \times \text{derivative of 2nd}) + (\text{2nd function} \times \text{derivative of 1st})$$

EXAMPLE 5

Differentiate **a** $y = (5x - 1)(2x + 3)$ **b** $y = (3x - 5)(x^2 + 5x - 7)$

Solution

a $y = (5x - 1)(2x + 3)$ is of the form $y = uv$ where $u = 5x - 1$ and $v = 2x + 3$.

Using the product rule

$$\begin{aligned}\frac{dy}{dx} &= (2x+3)(5) + (5x-1)(2)\\ &= 10x + 15 + 10x - 2\\ &= 20x + 13\end{aligned}$$

b $y = (3x - 5)(x^2 + 5x - 7)$ is of the form $y = uv$ where $u = 3x - 5$ and $v = x^2 + 5x - 7$.

Using the product rule

$$\begin{aligned}\frac{dy}{dx} &= (x^2 + 5x - 7)(3) + (3x - 5)(2x + 5)\\ &= 3x^2 + 15x - 21 + 6x^2 + 5x - 25\\ &= 9x^2 + 20x - 46\end{aligned}$$

$$\frac{d}{dx}((5x-1)(2x+3))$$

$$20 \cdot x + 13$$

$$\frac{d}{dx}((3x-5)(x^2+5x-7))$$

$$9 \cdot x^2 + 20 \cdot x - 46$$

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Exercise 1B

In this exercise many of the questions require you to 'use the product rule'. In such cases your method should clearly show your use of the rule. For questions that do not have such a requirement use your calculator if you wish.

1 By writing x^3 as $(x)(x^2)$ differentiate $y = x^3$ using the product rule.

Use the product rule to differentiate each of the following with respect to x.

2 $y = (x + 6)(x + 1)$

3 $y = (x + 7)(x - 3)$

4 $y = (3x + 1)(x + 4)$

5 $y = (x + 1)(3x + 4)$

6 $y = (2x + 3)(5x + 1)$

7 $y = (6x + 5)(2x + 3)$

8 $y = (x + 4)(x^2 + 2)$

9 $y = (x + 5)(x^2 - 3)$

10 $y = (x + 7)(x^2 + 1)$

11 $y = (x - 10)(x^2 + 8)$

12 $y = (2x - 1)(x^2 + 7x - 2)$

13 $y = (3x + 4)(x^2 - 3x + 4)$

14 $y = (2x - 3)(x^2 + 5x - 1)$

15 $y = (3x + 1)(x^2 - 7x + 1)$

Use the product rule to determine the gradient of each of the following at the given point.

16 $y = (x + 3)(x - 2)$ at $(3, 6)$.

17 $y = (3x + 1)(x - 5)$ at $(3, -20)$.

18 $y = (3x - 2)(2x + 1)$ at $(1, 3)$.

19 $y = (x - 4)(x^2 - 1)$ at $(2, -6)$.

20 Find the equation of the tangent to $y = (3x - 5)(x + 2)$ at the point $(2, 4)$.

21 Find the equation of the tangent to $y = (1 + 2x)(5x - 1)$ at the point $(1, 12)$.

First solve questions **22** and **23** without the assistance of your calculator then try the questions again using the ability of your calculator to determine derivatives and to solve equations.

22 Find the coordinates of any points on the curve $y = (2x - 1)(3x + 4)$ where the gradient is -1.

23 Find the coordinates of any points on the curve $y = (x - 3)(2x^2 - 11)$ where the gradient is 37.

24 Determine the coordinates of any points on the curve

$$y = (x - 3)(x^2 - 8)$$

where the gradient is the same as that of the straight line $y = x$.

25 **a** Use the product rule to differentiate $\sqrt{x^3} \times (2x + 1)$.

b Differentiate $\sqrt{x^3} \times (2x + 1)$ by first expanding the bracket and then differentiating each term.

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The quotient rule

WS The quotient rule

To differentiate $y = \frac{u}{v}$ where u and v are each functions of x, we use the **quotient rule**:

$$\text{If} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLE 6

Differentiate with respect to x **a** $y = \frac{3x-5}{5x-7}$ **b** $y = \frac{3x}{x^2+3}$.

Solution

a $y = \frac{3x-5}{5x-7}$ is of the form $y = \frac{u}{v}$ with $u = 3x - 5$ and $v = 5x - 7$.

Using the quotient rule

$$\frac{dy}{dx} = \frac{(5x-7)(3)-(3x-5)(5)}{(5x-7)^2} = \frac{4}{(5x-7)^2}$$

b $y = \frac{3x}{x^2+3}$ is of the form $y = \frac{u}{v}$ with $u = 3x$ and $v = x^2 + 3$.

Using the quotient rule

$$\frac{dy}{dx} = \frac{(x^2+3)(3)-(3x)(2x)}{(x^2+3)^2} = \frac{3x^2+9-6x^2}{(x^2+3)^2} = \frac{9-3x^2}{(x^2+3)^2}$$

$$\frac{d}{dx}\left(\frac{3x-5}{5x-7}\right) \qquad \frac{4}{(5\cdot x-7)^2}$$

$$\frac{d}{dx}\left(\frac{3x}{x^2+3}\right) \qquad \frac{-3\cdot(x^2-3)}{(x^2+3)^2}$$

Exercise 1C

In this exercise many of the questions require you to 'use the quotient rule'. In such cases your method should show your use of the rule. For questions that do not have such a requirement use your calculator if you wish.

1 By writing x^2 as $\frac{x^5}{x^3}$ differentiate $y = x^2$ using the quotient rule.

2 Rather than differentiating $y = \frac{1}{x^n}$ by writing it as $y = x^{-n}$, use the quotient rule instead.

Use the quotient rule to differentiate each of the following with respect to x.

3 $\frac{2x}{x+3}$

4 $\frac{3x}{5x-1}$

5 $\frac{6x}{4x-3}$

6 $\frac{7x}{1-2x}$

7 $\frac{5x+1}{2x+3}$

8 $\frac{5x+1}{2x-3}$

9 $\frac{6x-1}{5x+2}$

10 $\frac{3x-1}{2x-1}$

11 $\frac{1-3x}{3x+1}$

12 $\frac{5x}{x^2+1}$

13 $\frac{2x^2}{x^3+1}$

14 $\frac{3x^2}{x^5+3}$

15 Clearly showing your use of the quotient rule, determine the gradient of the curve $y = \frac{3x}{x-2}$ at the point (4, 6).

16 Determine the gradient of the curve $y = \frac{4x}{x^2-1}$ at the point (3, 1.5).

17 Find the equation of the tangent to $y = \frac{3x+5}{x-3}$ at the point (5, 10).

18 Determine the coordinates of any points on the curve $y = \frac{2x-1}{5-4x}$ where the gradient is equal to 6.

19 **a** Differentiate $\frac{2x-3}{x}$ using the quotient rule.

b By writing $\frac{2x-3}{x}$ as $(2x-3)(x^{-1})$ differentiate $\frac{2x-3}{x}$ using the product rule and express your answer as a single fraction.

c Use the fact that
$$\frac{2x-3}{x} = \frac{2x}{x} - \frac{3}{x} = 2 - \frac{3}{x}$$
to differentiate $\frac{2x-3}{x}$.

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The chain rule

The chain rule

Mixed differentiation problems

Higher derivatives

If we are told that $y = 3x^2 + 4$ we know that $\frac{dy}{dx}$, the gradient function, is $6x$.

However, suppose we are not given y directly in terms of x but instead are given y in terms of some other variable, say u, and given this other variable, u, in terms of x. Can we find $\frac{dy}{dx}$?

For example if $y = 4u + 3$ and $u = x^2 - 4$ can we find $\frac{dy}{dx}$?

We could substitute for u, from $u = x^2 - 4$, into

$$\begin{aligned} y &= 4u + 3 \text{ to give} \\ y &= 4(x^2 - 4) + 3 \\ &= 4x^2 - 16 + 3 \\ \text{i.e} \quad y &= 4x^2 - 13 \\ \text{and so} \quad \frac{dy}{dx} &= 8x \end{aligned}$$

However it is possible to determine $\frac{dy}{dx}$ in terms of x, without having to first substitute for u, by using a rule called **the chain rule**:

$$\text{If} \quad y = f(u) \quad \text{and} \quad u = g(x) \quad \text{then} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

If $y = 4u + 3$ and $u = x^2 - 4$

then $\frac{dy}{du} = 4$ and $\frac{du}{dx} = 2x$

Then, by the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (4)(2x) \\ &= 8x, \qquad \text{as before.} \end{aligned}$$

Note: The chain rule can be remembered by imagining the 'du's cancelling:

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}}\frac{\cancel{du}}{dx}.$$

However, as we were reminded in the *Preliminary work* at the start of this book, the terms $\frac{dy}{du}$ and $\frac{du}{dx}$ are *not* fractions, they are *limits of fractions*, $\frac{dy}{du} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ and $\frac{du}{dx} = \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$.

Whilst such 'cancelling' is useful for recalling the rule it cannot really be carried out.

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EXAMPLE 7

Find $\frac{dy}{dx}$, in terms of x, given that $y = u^2 - 5u$ and $u = 7x - 3$.

Solution

If $\quad y = u^2 - 5u \quad$ and $\quad u = 7x - 3$

then $\quad \frac{dy}{du} = 2u - 5 \quad$ and $\quad \frac{du}{dx} = 7.$

Using the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (2u - 5)\,7 \\ &= 7(14x - 11)\end{aligned}$$

Note:
- More 'links' can be put into the chain as required. (See example 8.)
- In the above example y is a function of u and u is a function of x. Thus we have **a function of a function** or **a composite function** as encountered in the *Preliminary work* section.

$$x \rightarrow \boxed{u = 7x - 3} \rightarrow \boxed{y = u^2 - 5u} \rightarrow y$$

For example:

$$x = 2 \rightarrow \boxed{7(2) - 3 = 11} \rightarrow \boxed{(11)^2 - 5(11)} \rightarrow y = 66$$

EXAMPLE 8

Find $\frac{dy}{dx}$, in terms of x, given that $y = 3t^2$, $t = 5p - 2$ and $p = 6x + 1$.

Solution

If $\quad y = 3t^2, \quad t = 5p - 2 \quad$ and $\quad p = 6x + 1$

then $\quad \frac{dy}{dt} = 6t \quad \frac{dt}{dp} = 5 \quad$ and $\quad \frac{dp}{dx} = 6.$

Using the chain rule
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dp}\frac{dp}{dx} \\ &= (6t)(5)(6) \\ &= 180(5p - 2) \\ &= 180(5(6x + 1) - 2) \\ &= 540(10x + 1)\end{aligned}$$

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The chain rule proves to be most useful when finding $\frac{dy}{dx}$ for certain functions in which y **is** given directly in terms of x but for which we choose to introduce a third variable, thus allowing the chain rule to be employed. This technique is demonstrated in the next example.

EXAMPLE 9

Differentiate **a** $y = (2x - 3)^4$ **b** $y = (3x^2 + 4)^5$

Solution

a To differentiate $y = (2x - 3)^4$ let $u = 2x - 3$

then $y = u^4$.

Thus $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2$.

By the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (4u^3)(2)\\ &= 4(2x-3)^3(2)\\ &= 8(2x-3)^3\end{aligned}$$

b To differentiate $y = (3x^2 + 4)^5$ let $u = 3x^2 + 4$

then $y = u^5$.

Thus $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = 6x$.

By the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (5u^4)(6x)\\ &= 5(3x^2+4)^4(6x)\\ &= 30x(3x^2+4)^4\end{aligned}$$

$$\frac{d}{dx}((2x-3)^4) \quad 8\cdot(2\cdot x-3)^3$$

$$\frac{d}{dx}((3x^2+4)^5) \quad 30\cdot x\cdot(3\cdot x^2+4)^4$$

Points to note

- Consider how long the previous example would have taken if we had to differentiate each part by first expanding the initial expressions (without the assistance of a calculator) and then differentiate each term!
- The final answers in the previous example are given in terms of the variable x, given in the question, and not in terms of the variable u which we introduced to help us differentiate.
- With practice you should be able to differentiate expressions like those of the previous example without having to write down the full process. (See the next example.)
- Considering the general case: If $y = [f(x)]^n$, then by letting $u = f(x)$ and using the chain rule, we obtain the following result.

$$\text{If} \quad y = [f(x)]^n \quad \text{then} \quad \frac{dy}{dx} = n\,[f(x)]^{n-1} f'(x)$$

EXAMPLE 10

Differentiate **a** $y = (7 + 2x)^3$ **b** $y = (x^2 + 3x + 1)^6$ **c** $y = \dfrac{1}{x^3 + 1}$

Solution

a If
$$\begin{aligned} y &= (7 + 2x)^3 \\ \frac{dy}{dx} &= 3(7 + 2x)^2(2) \\ &= 6(7 + 2x)^2 \end{aligned}$$

b If
$$\begin{aligned} y &= (x^2 + 3x + 1)^6 \\ \frac{dy}{dx} &= 6(x^2 + 3x + 1)^5(2x + 3) \\ &= 6(2x + 3)(x^2 + 3x + 1)^5 \end{aligned}$$

c If
$$\begin{aligned} y &= (x^3 + 1)^{-1} \\ \frac{dy}{dx} &= -1(x^3 + 1)^{-2}3x^2 \\ &= -\frac{3x^2}{(x^3 + 1)^2} \end{aligned}$$

The reader should confirm that applying the quotient rule for part **c**, instead of the chain rule, gives the same answer.

$$\frac{d}{dx}((7+2x)^3)$$
$$6\cdot(2\cdot x+7)^2$$
$$\frac{d}{dx}((x^2+3x+1)^6)$$
$$6\cdot(x^2+3\cdot x+1)^5\cdot(2\cdot x+3)$$
$$\frac{d}{dx}\left(\frac{1}{x^3+1}\right)$$
$$\frac{-3\cdot x^2}{(x^3+1)^2}$$

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EXAMPLE 11

Determine the gradient of the curve $y = (x^2 - 7)^4$ at the point (3, 16).

Solution

Either algebraically

If $y = (x^2 - 7)^4$

$$\frac{dy}{dx} = 4(x^2 - 7)^3(2x)$$

$$= 8x(x^2 - 7)^3$$

At (3, 16), $x = 3$

and $\frac{dy}{dx} = 24(3^2 - 7)^3$

$$= 24 \times 8$$

$$= 192.$$

or by calculator

$\frac{d}{dx}((x^2 - 7)^4)|x = 3$

192

The gradient of the curve $y = (x^2 - 7)^4$ at the point (3, 16) is 192.

Exercise 1D

1 Find $\frac{dy}{dx}$, in terms of x, given that $y = 7u - 3$ and $u = 2x^2 + 5x - 3$.

2 Find $\frac{dp}{dt}$, in terms of t, given that $p = 3s^2$ and $s = 2t + 1$.

3 Find $\frac{dh}{dr}$, in terms of r, given that $h = 5p^2 - 3$ and $p = 1 - 2r^2$.

4 Find $\frac{dy}{dx}$, in terms of x, given that $y = u^2 + 3$, $u = 4p - 3$ and $p = 3x + 2$.

5 Differentiate $y = (3x + 2)^5$ by letting $u = 3x + 2$ and using the chain rule. Show your working fully and give your answer in terms of x.

6 Differentiate $y = (x^2 + 2)^3$ by letting $u = x^2 + 2$ and using the chain rule. Show your working fully and give your answer in terms of x.

7 Differentiate $y = \frac{1}{(8x - 3)}$ by letting $u = 8x - 3$ and using the chain rule. Show your working fully and give your answer in terms of x.

8 Differentiate $y = \sqrt{2x + 3}$ by letting $u = 2x + 3$ and using the chain rule. Show your working fully and give your answer in terms of x.

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9 Differentiate $y = \dfrac{1}{\sqrt{6x+1}}$ by letting $u = 6x + 1$ and using the chain rule. Show your working fully and give your answer in terms of x.

10 Differentiate $y = \dfrac{1}{(3x^2 + 2x + 1)^2}$ by letting $u = 3x^2 + 2x + 1$ and using the chain rule. Show your working fully and give your answer in terms of x.

Find the gradient function $\dfrac{dy}{dx}$ for each of the following. Do each one without the assistance of a calculator and then check your answer with your calculator.

11 $y = (5x + 2)^4$

12 $y = (7x - 3)^3$

13 $y = (2 - 3x)^3$

14 $y = (4 + 7x)^2$

15 $y = (3x^2 + 5)^3$

16 $y = (2x^3 + 1)^6$

17 $y = (x + 2)^{-3}$

18 $y = (2x + 5)^{-1}$

19 $y = \dfrac{1}{(x+2)}$

20 $y = \dfrac{1}{(7x-3)^2}$

21 $y = 3x + (2x + 3)^5$

22 $y = \sqrt{x+1}$

Determine the gradient of each of the following at the given point without the assistance of your calculator.

23 $y = (10x + 1)^5$ at (0, 1).

24 $y = (6x - 1)^3$ at (1, 125).

25 $y = (1 + x^4)^3$ at (–1, 8).

26 $y = (2x - 3)^{-4}$ at (2, 1).

27 $y = \dfrac{1}{(2x^2+1)^3}$ at (0, 1).

28 $y = x^2 + (x - 1)^5$ at (2, 5).

Use your calculator to determine the gradient of each of the following at the given point.

29 $y = \dfrac{1}{3 + 2x + 3x^2}$ at (1, 0.125).

30 $y = \dfrac{40}{\sqrt{1+x}}$ at (3, 20).

31 $y = \dfrac{36}{1+\sqrt{x}}$ at (4, 12).

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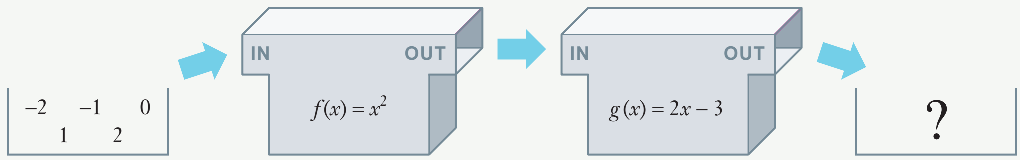
Miscellaneous exercise one

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the Preliminary work section at the beginning of the book.

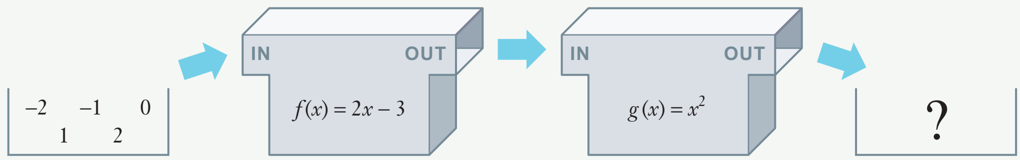
1 Each of the following diagrams show a composite function $gf(x)$.

With the domain as shown determine the range of $gf(x)$.

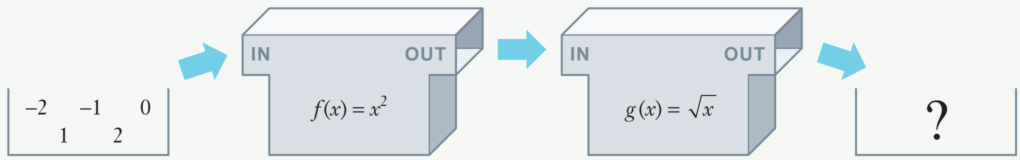
a



b



c



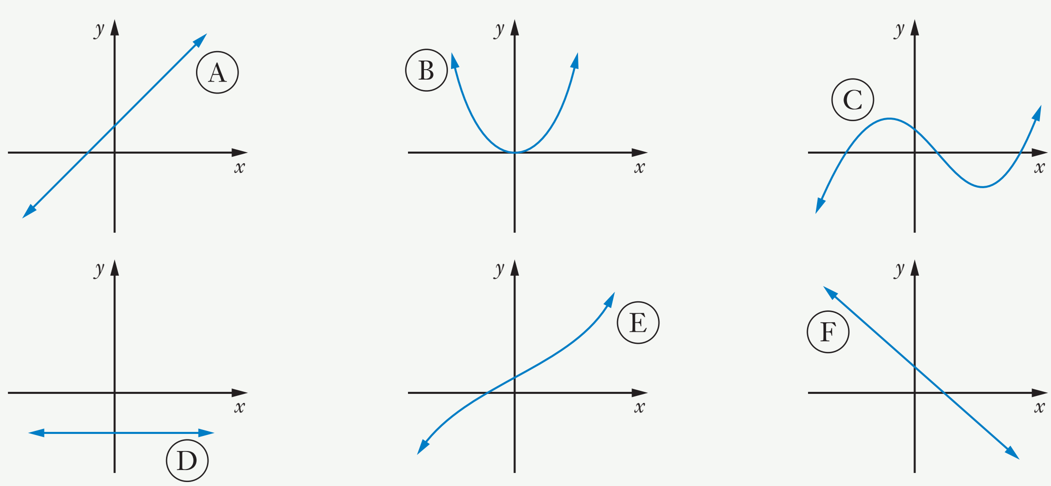
2 For the graphs A to F shown below state which have

a $\dfrac{dy}{dx}$ always positive,

b $\dfrac{dy}{dx}$ always negative,

c $\dfrac{dy}{dx}$ never negative,

d $\dfrac{dy}{dx}$ independent of x.



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3 (You should be able to do this question mentally and simply write the answer.)

If $y = 5 - 7x^2$ determine $\dfrac{d^2y}{dx^2}$.

4 Find $\dfrac{dy}{dx}$ for

a $y = 5x^2$ **b** $y = 3 + 5x^2$ **c** $y = (3 + 5x)^2$

5 Clearly showing the use of the product rule,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

determine $\dfrac{dy}{dx}$ for each of the following:

a $y = (x + 1)(x - 3)$ **b** $y = (2x - 1)(5x + 4)$

c $y = (2x + 3)^2$ **d** $y = (x^2 - 4)(3x + 5)$

6 Find the gradient of $y = 2(x^2 - 5)^7$ at the point $(-2, -2)$.

7 Differentiate $\dfrac{x^3 - 3x^2}{x}$. (Hint: You do not need the quotient rule for this one.)

8 Find the gradient of $y = \dfrac{4}{2x + 3}$ at the point $(-1, 4)$.

9 Find the equation of the tangent to $y = \dfrac{2x - 3}{x + 1}$ at $(3, 0.75)$, giving your answer in the form $ay = bx + c$, with a, b and c taking integer values.

10 With the assistance of your calculator:

Find the coordinates of the points where the curve

$$y = \frac{13x + 1}{2x + 2}$$

cuts the line $y = x + 2$.

Find the gradient of the curve at each of these points.

11 **a** Use the product rule to obtain the derivative of $(x + 4)(2x - 1)$.

b *Hence*, and without the assistance of a calculator, determine the derivative of $(3x - 1)(x + 4)(2x - 1)$.

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2.

Applications of differentiation

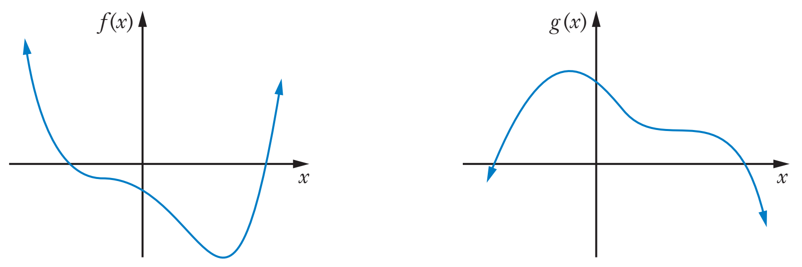
- Examining the second derivative
- Locating turning points and points of inflection
- Sketching graphs
- Rates of change
- Acceleration
- Optimisation
- Small changes
- Small percentage changes
- Marginal rates of change
- Miscellaneous exercise two

In your study of Unit 2 of *Mathematics Methods* you would have seen how differentiation could be used to locate any stationary points on functions, and hence be useful in determining local maximum and local minimum values of functions. We will now see how the second derivative can be of use in this **optimisation** process.

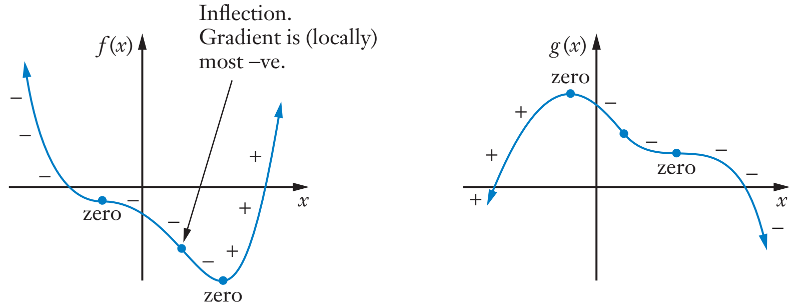
Examining the second derivative

WS Concavity

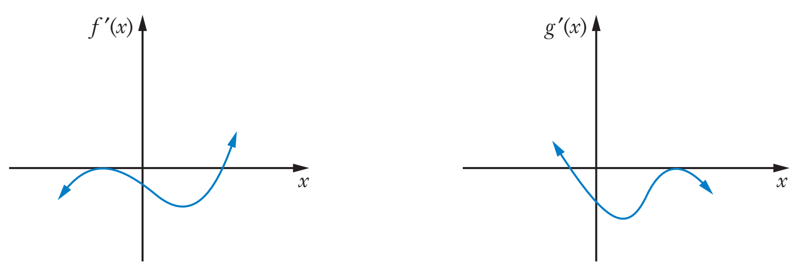
The diagrams below show the graphs of two functions $f(x)$ and $g(x)$.



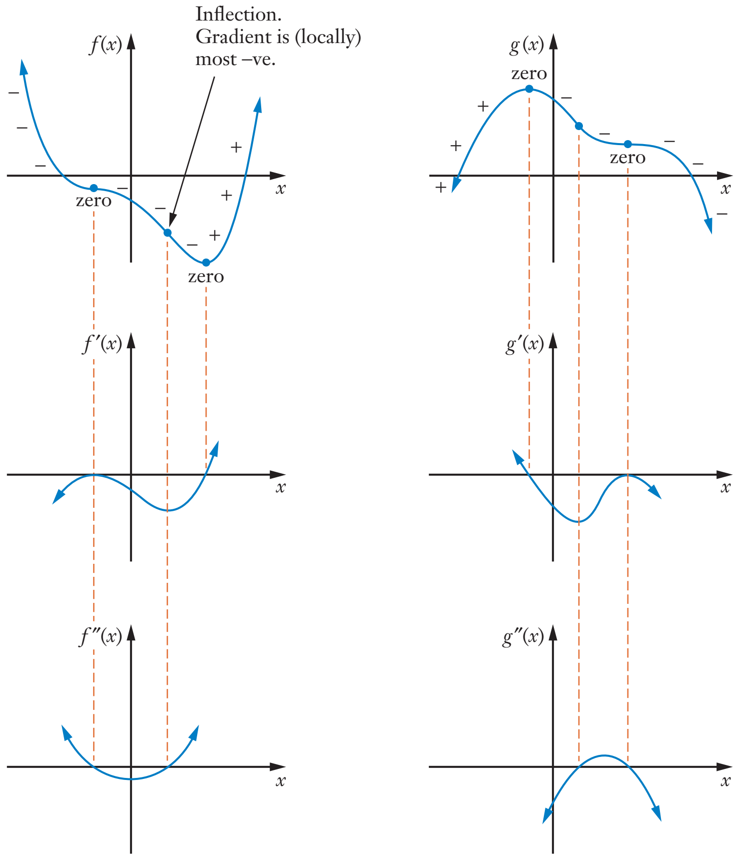
These functions are shown again below but now with the gradients at various places on the curves marked as positive, negative or zero.



This allows a sketch of the first derivatives $f'(x)$ and $g'(x)$ to be made:



The next page shows this process continued from $f(x)$ and $g(x)$, through $f'(x)$ and $g'(x)$ to $f''(x)$ and $g''(x)$.



Take particular notice of the following:

- Wherever $f''(x) < 0$ then $f(x)$ is concave down.
- Wherever $f''(x) > 0$ then $f(x)$ is concave up.
- At all of the points of inflection $f''(x)$ is zero.

Note: Care needs to be taken with the third dot point above. Whilst it is true that at all points of inflection the second derivative is zero, we cannot assume that if the second derivative is zero we necessarily have a point of inflection. Consider for example the function $y = x^4$. At the point (0, 0) the second derivative is zero, but on $y = x^4$ the point (0, 0) is a minimum point.

Locating turning points and points of inflection

The properties of the second derivative stated above can be useful if we wish to determine the nature of any turning points on a curve, as the next example demonstrates.

The example also reminds us that the nature and location of any turning points can also be determined:

- by examining the sign of the gradient on either side of the turning point, (the sign test),

or

- from a calculator.

(Techniques you would already be familiar with from studying Unit 2 of this course.)

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EXAMPLE 1

Clearly showing your use of calculus, determine the coordinates of any stationary points on the curve

$$y = x^3 - 12x^2 + 36x - 15$$

and state the nature of each.

Solution

$$\begin{aligned}
&\text{If} & y &= x^3 - 12x^2 + 36x - 15 \\
&\text{then} & \frac{dy}{dx} &= 3x^2 - 24x + 36 \\
&\text{For stationary points:} & \frac{dy}{dx} &= 0, \\
&\text{i.e.} & 3x^2 - 24x + 36 &= 0 \\
&\therefore & x^2 - 8x + 12 &= 0 \\
& & (x-2)(x-6) &= 0 \\
&\text{Giving} & x &= 2 \quad \text{or} \quad 6 \\
&\text{When } x = 2 & y &= 2^3 - 12(2)^2 + 36(2) - 15 \\
& & &= 17 \\
&\text{When } x = 6 & y &= 6^3 - 12(6)^2 + 36(6) - 15 \\
& & &= -15
\end{aligned}$$

Thus $y = x^3 - 12x^2 + 36x - 15$ has two stationary points, one at (2, 17) and the other at (6, –15).

Determining the nature of the stationary points using the sign test:

Consider the sign of the gradient of the function either side of $x = 2$:

	$x = 1.9$	$x = 2$	$x = 2.1$
$\frac{dy}{dx} = 3(x-2)(x-6)$	+ve	zero	–ve
	/	—	\

Thus (2, 17) is the local maximum.

Similar working shows (6, –15) is a local minimum.

Determining the nature of the stationary points using the second derivative test:

With $\frac{dy}{dx} = 3x^2 - 24x + 36$ it follows that $\frac{d^2y}{dx^2} = 6x - 24$.

If $x = 2$, $\frac{d^2y}{dx^2} = -\text{ve}$

$\therefore$ concave down, a maximum.

If $x = 6$, $\frac{d^2y}{dx^2} = +\text{ve}$

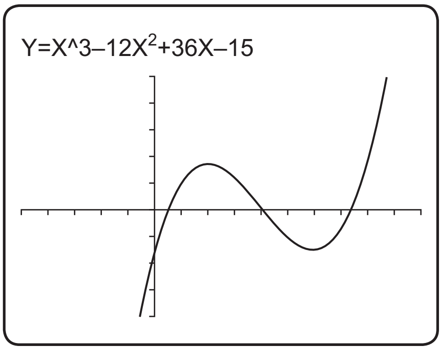
$\therefore$ concave up, a minimum.

Thus, as before, (2, 17) is a local maximum and (6, –15) is a local minimum.

Note

- The second derivative test uses the facts stated earlier, i.e.:
  - ➔ if $f''(x) < 0$, i.e. negative, then $f(x)$ is concave down. Hence **with $f'(x) = 0$ and $f''(x)$ negative we have a maximum** point.
  - ➔ if $f''(x) > 0$, i.e. positive, then $f(x)$ is concave up. Hence **with $f'(x) = 0$ and $f''(x)$ positive we have a minimum** point.
- **If $f'(x) = 0$ and $f''(x) = 0$ then we could have maximum, minimum or inflection and would need to investigate further using the sign test.**
- The nature of each turning point could alternatively have been determined by viewing a graphical display of the function.

  From this it can be seen that the turning point at (2, 17) is a local maximum and the one at (6, –15) is a local minimum.



- The whole task could be completed using a calculator but some explanation and method would need to be shown to ensure that you met the requirement to *clearly show your use of calculus*.

Define $f(x) = x^3 - 12 \cdot x^2 + 36x - 15$

done

solve$\left(\frac{d}{dx}(f(x)) = 0, x\right)$

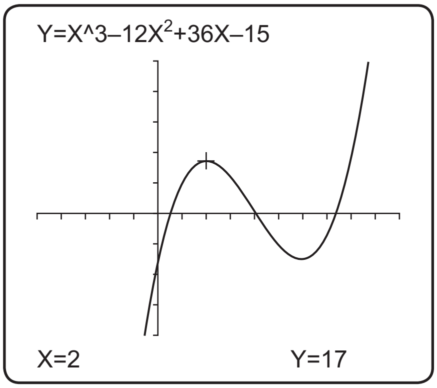
$\{x = 2, x = 6\}$

$f(2)$

17

$f(6)$

–15



- What we found in the last example was the **local** maximum (and **local** minimum) i.e. the point which is a maximum point compared to others in that locality. In some cases we may be concerned with the maximum or minimum value a function can take for some interval $a \le x \le b$. We are then concerned with the **global** maxima, which may or may not coincide with the local maxima.

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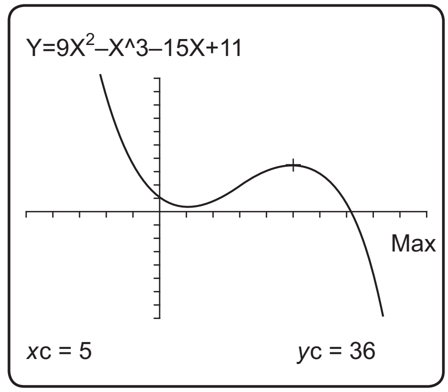
For example consider the graph of

$$f(x) = 9x^2 - x^3 - 15x + 11$$

with the local maximum at (5, 36), as shown on the right.

If we were asked for the maximum value of this function in the interval, $-2 \le x \le 7$ we see that $f(-2)$ will give this greatest value. With $f(-2) = 85$ we say that the global maximum for the interval $-2 \le x \le 7$ is 85.



EXAMPLE 2

Clearly showing your use of calculus, determine the *exact* coordinates of any stationary points on the curve $y = 2x + \frac{6}{x}$ and use the second derivative test to determine the nature of each.

Solution

If $\quad y = 2x + \frac{6}{x}\ (= 2x + 6x^{-1}) \quad$ then $\quad \frac{dy}{dx} = 2 - \frac{6}{x^2}$

Stationary points will occur where $\quad \frac{dy}{dx} = 0, \quad$ i.e. $\quad 2 - \frac{6}{x^2} = 0$

$\therefore \quad x^2 = 3$

giving $\quad x = \sqrt{3} \quad$ or $\quad -\sqrt{3}$

Thus there are two stationary points, when $x = \sqrt{3}$ and when $x = -\sqrt{3}$.

When $x = \sqrt{3}$

$$\begin{aligned} y &= 2\sqrt{3} + \frac{6}{\sqrt{3}} = 2\sqrt{3} + \frac{6}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} \\ &= 2\sqrt{3} + 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

$2\sqrt{3} + \frac{6}{\sqrt{3}}$

$4\cdot\sqrt{3}$

Thus a stationary point exists at $(\sqrt{3}, 4\sqrt{3})$.

Similarly, when $x = -\sqrt{3}$, $y = -4\sqrt{3}$. The other stationary point is at $(-\sqrt{3}, -4\sqrt{3})$.

Second derivative test

With $\quad \frac{dy}{dx} = 2 - \frac{6}{x^2} \quad (= 2 - 6x^{-2}), \quad$ it follows that $\quad \frac{d^2y}{dx^2} = \frac{12}{x^3}$.

If $\quad x = \sqrt{3}$

$\frac{d^2y}{dx^2} = +\text{ve}$

$\therefore$ minimum

If $\quad x = -\sqrt{3}$

$\frac{d^2y}{dx^2} = -\text{ve}$

$\therefore$ maximum

The point $(\sqrt{3}, 4\sqrt{3})$ is a local minimum and $(-\sqrt{3}, -4\sqrt{3})$ a local maximum.

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WS

Curve sketching with derivatives

Sketching graphs

The ability to determine the location and nature of any turning points on a graph, together with an ability to determine any intercepts with the axes, asymptotes, any symmetry that may exist and the behaviour of the function as $x \to \pm\infty$, all help in the production of a sketch of the graph of a function.

- Whilst we would not expect to read values from a sketch graph with any great accuracy the sketch should be neatly drawn and should show the noteworthy features of the graph.
- The reader should already be familiar with sketching graphs from studying Unit Two of this course. However, we now have the second derivative test that we can use and the ability to differentiate more complicated functions.

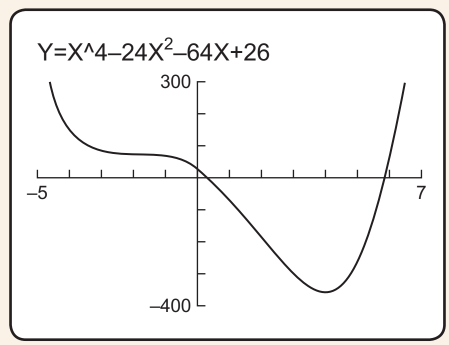
EXAMPLE 3 **(Sketching *with* the assistance of a graphic calculator.)**

With the aid of a graphic calculator produce a sketch of $y = x^4 - 24x^2 - 64x + 26$ indicating on your sketch the location of any stationary points, intercepts with the axes and points of inflection that appear in the interval $-5 \le x \le 7$. (If any rounding is necessary give answers correct to 2 decimal places.)

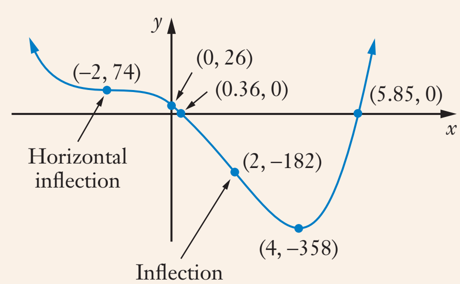
Solution

By substituting $x = 0$ we can determine that the curve cuts the y-axis at (0, 26).

Displaying the graph (see diagram on the right) indicates that the curve cuts the x-axis at approximately (0.5, 0) and (6, 0), has a minimum point at or near $x = 4$, changes concavity at or near $x = -2$ (horizontal(?) inflection) and at or near $x = 2$ (inflection).



Using the calculator facilities to determine the coordinates of these points allows a sketch to be made:



EXAMPLE 4 **(Sketching *without* the assistance of a graphic calculator.)**

For the function $y = x^3 + 3x^2 - 24x + 20$ and *without* the assistance of a calculator determine:

a the coordinates of the y-axis intercept,

b the behaviour of the function as $x \to \pm\infty$,

c the location and nature of any turning points,

d the coordinates of any points for which $\frac{d^2y}{dx^2} = 0$.

Show your answers to the previous parts on a sketch of the graph.

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Solution

a $y = x^3 + 3x^2 - 24x + 20$

When $x = 0$, $y = 20$

The y-axis intercept has coordinates (0, 20).

b For $x \to \pm\infty$ the x^3 term will dominate.

Thus as $x \to +\infty$, $y \to +\infty$ (and faster than x does).

and as $x \to -\infty$, $y \to -\infty$ (and faster than x does).

c With

$$\begin{aligned} y &= x^3 + 3x^2 - 24x + 20 \\ \frac{dy}{dx} &= 3x^2 + 6x - 24 \\ &= 3(x^2 + 2x - 8) \\ &= 3(x + 4)(x - 2) \end{aligned} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 6x + 6$$

Thus when $x = -4$ $y' = 0$ and $y'' = -\text{ve}$. A maximum point.

For this value of x,

$$\begin{aligned} y &= (-4)^3 + 3(-4)^2 - 24(-4) + 20 \\ &= -64 + 48 + 96 + 20 \\ &= 100 \end{aligned}$$

The function has a maximum turning point at (–4, 100).

Also when $x = 2$ $y' = 0$ and $y'' = +\text{ve}$. A minimum point.

For this value of x

$$\begin{aligned} y &= (2)^3 + 3(2)^2 - 24(2) + 20 \\ &= 8 + 12 - 48 + 20 \\ &= -8 \end{aligned}$$

The function has a minimum turning point at (2, –8).

d $\dfrac{d^2y}{dx^2} = 0$ when $6x + 6 = 0$

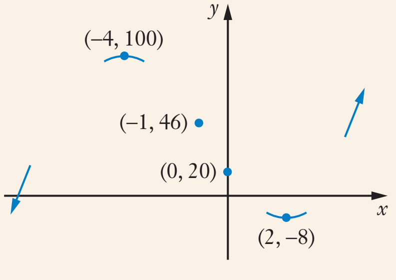
i.e. when $x = -1$

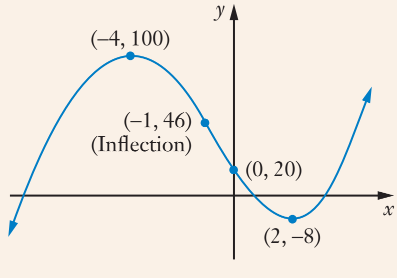
For this value of x

$$\begin{aligned} y &= (-1)^3 + 3(-1)^2 - 24(-1) + 20 \\ &= -1 + 3 + 24 + 20 \\ &= 46. \end{aligned}$$

The second derivative is zero at the point (–1, 46).

Placing the above information on a graph, below left, a sketch of the function can be completed, as shown below right.





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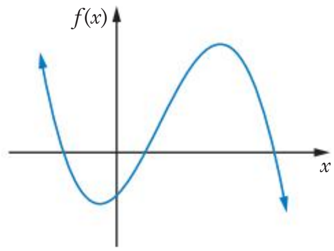
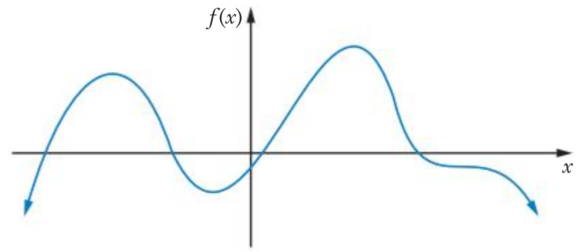
Exercise 2A

1 Copy the following graphs and then draw $f'(x)$ and $f''(x)$ for each one.

a



b



Use calculus techniques to determine the exact coordinates of any stationary points on the following curves, and use the second derivative test (and the sign test if necessary) to determine whether maximum, minimum or horizontal inflection.

2 $y = x^2 - 12x + 40$

3 $y = 5 + 8x - x^2$

4 $y = x^3 - 9x$

5 $y = x^3 - 9x^2 - 21x + 60$

6 $y = (x-1)^4 + 2$

7 $y = x + \dfrac{4}{x+3}$

8 $y = x + \dfrac{5}{x}$

9 $y = (2x-1)^5 + 1$

10 With the aid of a graphic calculator produce a sketch of

$$y = x^3 - 9x^2 + 15x + 74$$

indicating on your sketch the location of any stationary points, intercepts with the axes and points of inflection.

11 For the function $y = x^3 - 6x^2 - 15x + 30$ and *without* the assistance of a calculator determine:

a the coordinates of the y-axis intercept,

b the behaviour of the function as $x \to \pm\infty$,

c the location and nature of any turning points,

d the coordinates of any points for which $\dfrac{d^2y}{dx^2} = 0$.

Show your answers to the previous parts on a sketch of the graph of the function.

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12 For the function $y = x^4 - 4x^3 + 1$ and *without* the assistance of a calculator determine:

a the coordinates of the y-axis intercept,

b the behaviour of the function as $x \to \pm\infty$,

c the location and nature of any turning points,

d the coordinates of any points for which $\frac{d^2y}{dx^2} = 0$.

Show your answers to the previous parts on a sketch of the graph of the function.

13 Use the ability of your calculator to differentiate algebraic expressions to determine $\frac{dy}{dx}$ given that $y = (x - 3)^3(3x + 7)$, giving your answer in factorised form.

See if you can obtain this same factorised answer using the product rule and chain rule and without the assistance of your calculator.

Without the assistance of your calculator, determine the coordinates of any stationary points on the curve

$$y = (x - 3)^3(3x + 7)$$

and use the sign test and/or the second derivative to determine the nature of each.

14 The graph of

$$f(x) = \frac{x^3}{8} - x^2 + 2x + 1$$

has a local maximum at $\left(\frac{4}{3}, 2\frac{5}{27}\right)$ and a local minimum at (4, 1).

Using your calculator purely to assist with the arithmetic, if necessary, determine the maximum value of $f(x)$ for

a $0 \le x \le 5$,

b $0 \le x \le 6$.

For questions **15** to **17**

a use calculus to determine the coordinates of any points where $f''(x) = 0$,

b by checking a graphic calculator display of the graph of $f(x)$ state whether each point from **a** is a point of inflection or not. If yes, determine whether it is horizontal inflection.

15 $f(x) = x^3 - 12x$

16 $f(x) = 8x^3 - x^4$

17 $f(x) = x^4$

18 For $f(x) = x^3 - 3x - 2$ determine $f'(x)$,
$f''(x)$,
and find a such that $f''(a) = 0$.

By considering the sign of $f''(x)$ on either side of $x = a$, to see if the concavity changes, state whether $(a, f(a))$ is a point of inflection or not.

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Rates of change

Given y in terms of x the process of differentiation gives us $\frac{dy}{dx}$, the rate of change of y with respect to x.

If instead of y we use V, where V represents volume, and in place of x we use t, where t represents time, we can use differentiation to determine $\frac{dV}{dt}$, the rate of change of volume with respect to time.

Similarly if we are told a rule relating A m^2, the area of a particular algal bloom, to T, the temperature of the surroundings in degrees Celsius, then differentiation can be used to determine $\frac{dA}{dT}$, the rate of change of A with respect to T, in m^2/°C.

EXAMPLE 5

A colony of bacteria is increasing in such a way that the number of bacteria present after t hours is given by N where $N = 250 + 100t + 50t^3$.

Find **a** the number of bacteria present when $t = 5$,

b an expression for the instantaneous rate of change of N with respect to t,

c the rate the colony is increasing, in bacteria/hour, when **i** $t = 2$, **ii** $t = 10$.

Solution

a

$$N = 250 + 100t + 50t^3$$

When $t = 5$ $\quad N = 250 + 100(5) + 50(5)^3 = 7000$

When $t = 5$ there are 7000 bacteria present.

b

$$N = 250 + 100t + 50t^3$$

$$\therefore \quad \frac{dN}{dt} = 100 + 150t^2$$

The instantaneous rate of change of N with respect to t is

$(100 + 150t^2)$ bacteria/h.

c **i**

$$\frac{dN}{dt} = 100 + 150t^2$$

when $t = 2$ $\quad \frac{dN}{dt} = 700$

When $t = 2$, the colony is increasing at 700 bacteria/hr.

ii When $t = 10$ $\quad \frac{dN}{dt} = 15\,100$

When $t = 10$, the colony is increasing at 15 100 bacteria/h.

Define $N(t) = 250 + 100t + 50t^3$	done
$N(5)$	7000
$\frac{d}{dt}(N(t))$	$150t^2 + 100$
$\frac{d}{dt}(N(t))\|t = 2)$	700
$\frac{d}{dt}(N(t))\|t = 10)$	15100

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Exercise 2B

1 If $P = 2a^3 + 3a - 7$ find an expression for the rate of change of P with respect to a.

2 If $Y = p - 5p^2 + 2p^3$ find an expression for the rate of change of Y with respect to p.

3 If $Q = (2t - 1)^3$ find an expression for the rate of change of Q with respect to t.

4 If $A = \dfrac{3x - 2}{2x + 5}$ find an expression for the rate of change of A with respect to x.

5 If $P = (2q - 5)(3q^2 + 1)$ find an expression for the rate of change of P with respect to q.

6 If $V = (1 + 0.5t)^3$ find the rate of change of V (cm^3) with respect to t (seconds) when

a $t = 2$, **b** $t = 6$, **c** $t = 10$.

7 A colony of flying insects takes over a nesting site and the population of the colony grows such that N, the number of insects present, t days after taking over the site, is given by $N = 500 - 5t^2 + 10t^3$.

Find **a** an expression for the rate of change of N with respect to t,

b the rate at which the population is changing when

i $t = 1$, **ii** $t = 5$, **iii** $t = 10$.

8 For the first 20 seconds of its motion the height of a particular rocket above the Earth's surface, t seconds after launch, is h metres where $h = 5t(1 + 2t)$.

Find both the height and the rate of change of height with respect to time when

a $t = 1$, **b** $t = 5$, **c** $t = 20$.

9 A colony of bacteria is increasing in such a way that the number of bacteria present after t hours is given by N where $N = 5(2t + 1)^3$.

Find **a** the number of bacteria present initially (i.e. when $t = 0$),

b the number of bacteria present when $t = 5$,

c an expression for the instantaneous rate of change of N with respect to t,

d the rate the colony is increasing, in bacteria/hour, when

i $t = 2$, **ii** $t = 5$, **iii** $t = 10$.

10 A charity group launches a new appeal to raise \$15 000 for a particular project. A computer analysis of previous appeals indicates that for this appeal, \$$R$, the amount by which the funds raised fall below the target of \$15 000, and w, the number of weeks after the much publicised launch, will approximately follow the rule

$$R = 15\,000 - 5000\sqrt{w} - \frac{800}{w + 1}$$

According to this model

a How many weeks after the launch could the organisers expect the target to be reached? (Hint: Use a graphic calculator.)

b Find an expression for the rate of change of R with respect to w and evaluate this expression (to nearest \$100) for $w = 1$, $w = 3$ and $w = 8$.

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WS Higher derivatives

Acceleration

The *Preliminary work* section at the beginning of this book reminded us that one of the commonest rates of change that concerns us is the rate at which we change our location. If we measure our location as a **displacement** from some fixed point or origin, then the rate at which we change our displacement is our **velocity**.

Thus if the displacement is x [$= f(t)$] then v, the velocity as a function of time, is given by:

$$v = \frac{dx}{dt}$$

Similarly the rate of change of velocity with respect to time, $\frac{dv}{dt}$, gives **acceleration**, a.

$$\text{Thus} \quad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

For example, if $\quad x = 5t^3 + 6t^2 + 7t + 1$

then $\quad v = \frac{dx}{dt} = 15t^2 + 12t + 7$

and $\quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 30t + 12.$

EXAMPLE 6

A body moves in a straight line such that its displacement from an origin O, at time t seconds, is x metres where $x = t^3 + 6t + 5$.

Find the acceleration of the body when $t = 3$.

Solution

If $\quad x = t^3 + 6t + 5$

then $\quad v = \frac{dx}{dt} = 3t^2 + 6$

and $\quad a = \frac{dv}{dt} = 6t.$

Thus, when $t = 3$, $\quad a = 6(3)$
$\quad = 18$

When $t = 3$ the acceleration is 18 m/s^2.

$$\frac{d^2}{dt^2}(t^3 + 6t + 5)\Big| t = 3$$

18

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EXAMPLE 7

A body moves in a straight line such that its displacement from an origin O, at time t seconds, is x metres where $x = 5t^2 + 7t + 3$.
Find the acceleration of the body when $t = 0$ (i.e. the *initial* acceleration of the body).

Solution

If $\quad x = 5t^2 + 7t + 3$

then $\quad v = \dfrac{dx}{dt} = 10t + 7$

and $\quad a = \dfrac{dv}{dt} = 10.$

(I.e., the acceleration is a constant 10 m/s^2.)

Thus the initial acceleration is 10 m/s^2.

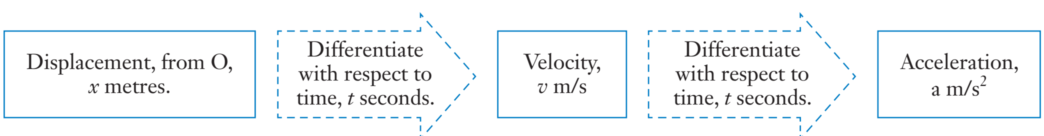
$$\frac{d^2}{dt^2}(5t^2+7t+3)\Big|t=0$$

10

Note: • We have already seen that we write y' for $\dfrac{dy}{dx}$ and y'' for $\dfrac{d^2y}{dx^2}$.

For differentiation with respect to *time* we tend to use a dot notation rather than a dash.
Thus $\dfrac{dy}{dt}$ is sometimes written as $\dot{y}$ and $\dfrac{d^2y}{dt^2}$ is sometimes written as $\ddot{y}$.

• The relationship between displacement, velocity and acceleration is shown below:



(Different units of displacement and time, e.g. km and hours, would give different units for velocity, km/h, and acceleration, km/h^2.)

• Displacement, velocity and acceleration are **vector** quantities, i.e. they have size and direction. This section will only consider rectilinear motion, i.e. motion in a straight line. For such motion there are only two directions possible and these are distinguished by use of positive and negative.

• If we take positive as being ‘to the right’, a body with positive acceleration will either be moving to the right and increasing its speed or moving to the left and decreasing its speed.

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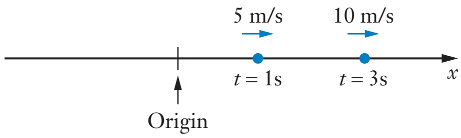
Exercise 2C

1 Each of the following diagrams show the velocity and location of a body at two times. Given that the body is experiencing constant acceleration state whether this acceleration is positive or negative.

a



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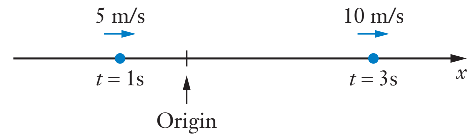
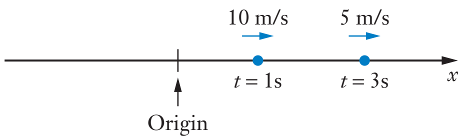
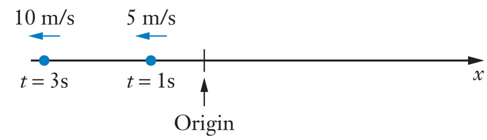


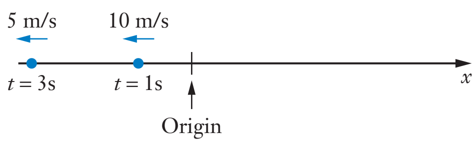
c



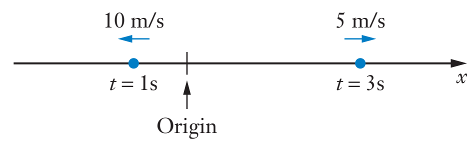
d



e



f



Questions **2** and **3** involve a body moving in a straight line with its displacement, x metres, given as a function of the time, t seconds.

2 If $x = 5t^2 + 6t$ find

a the velocity when $t = 2$,

b the acceleration when $t = 3$.

3 If $x = \dfrac{(2t+1)^3}{10}$ find

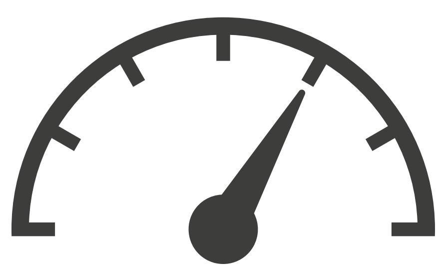
a the velocity when $t = 2$,

b the acceleration when $t = 2$.

Questions **4** and **5** involve a body moving in a straight line with its velocity, v metres/second, given as a function of the time, t seconds.

4 If $v = \dfrac{2t+3}{t+1}$ find the acceleration when $t = 4$.

5 If $v = (2t-1)^5$ find the acceleration when $t = 1.5$.



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6 For each of the following, x metres is the displacement of a body from an origin O, at time t seconds. Find the instantaneous acceleration of the body for the given value of t.

a $x = 2t^3 + 4, t = 2.$

b $x = 7t, t = 3.$

c $x = \dfrac{27}{2t+1}, t = 1.$

d $x = \sqrt{2t+1}, t = 4.$

e $x = (9 - 2t)^4, t = 4.$

f $x = 2t(1 + 5t)^3, t = 0.4.$

7 The displacement of a body from an origin O, at time t seconds, is x metres where

$$x = t^2 - 11t + 3, t \geq 0.$$

a Find the initial velocity of the body.

b Find the acceleration of the body.

c Find the value of t for which the body has a velocity of 5 m/s.

d Find the values of t for which the body has a speed of 5 m/s.

Note: To have a *speed* of 5 m/s the velocity can be 5 m/s or –5 m/s.

8 The displacement of a body from an origin O, at time t seconds, is x metres where

$$x = 27t + 3t^2 - \frac{t^3}{3} - 90, t \geq 0.$$

Find the displacement of the body from O when the acceleration is zero.

Optimisation

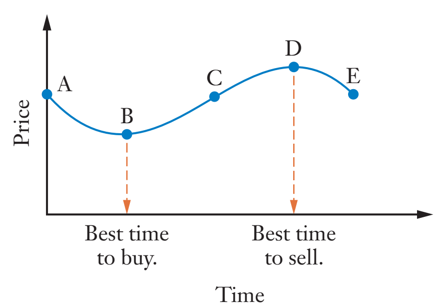
WS Optimisation

WS Optimisation – Area and volume

WS Optimisation – Cost and revenue

WS Optimisation – Mixed problems

Let us suppose that the selling price of housing units in a large block of such units varies over a period of time in the manner indicated in the diagram on the right.



For someone buying a unit the most desirable or **optimum** time to buy will be at the time given by point B. This is the time during the period under consideration when the price is at its lowest.

For someone selling a unit the most desirable or optimum time to sell will be at the time given by point D. This is the time during the period under consideration when the price is at its highest.

The techniques used earlier in this chapter to determine the location and nature of any maximum or minimum points on a curve can be used to determine the most desirable or optimum situation for an event to occur. As the *Preliminary work* section at the beginning of this text stated, this application of differentiation to determine an optimal solution should not be new to you. However now, from the earlier work of this chapter, you also have the **second derivative test** available to use when attempting to determine the nature of any stationary points on the function involved.

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The following list should serve to remind you of the steps to follow when locating stationary points to solve applied optimisation problems.

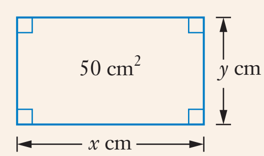
- If a diagram is not given then draw one if it helps.
- Identify the variable that is to be maximised, or minimised. If this variable is, say, C then you must find an equation with C as the subject, i.e. C = ???.
- If this equation for C involves two variables (other than C) find another equation that will allow us to substitute for one of the variables.
- When you have C in terms of one variable, say x, then you could view the function on your calculator and locate any turning points, or, if the use of calculus is to be demonstrated, find the values of x for which $\frac{dC}{dx} = 0$.
- Use the second derivative or the sign test or a graphic calculator display to determine whether maximum or minimum.
- Check that the value of x for the required maximum, or minimum, is within the values that the situation allows x to lie and check that it gives the global maximum, or minimum.

EXAMPLE 8

The area of a rectangle is to be 50 cm^2. Find the dimensions of the rectangle if its perimeter is to be a minimum.

Solution

With the perimeter of the rectangle as P cm and x and y as shown in the diagram on the right, it follows that



$$xy = 50 \qquad \text{and} \qquad P = 2x + 2y$$

$$\text{Thus} \quad y = \frac{50}{x} \qquad \text{and so} \qquad P = 2x + \frac{100}{x}$$

$$= 2x + 100x^{-1}$$

$$\therefore \quad \frac{dP}{dx} = 2 - 100x^{-2}$$

$$= 2 - \frac{100}{x^2}$$

$$\text{and} \quad \frac{d^2P}{dx^2} = 200x^{-3}$$

$$= \frac{200}{x^3}$$

If $\frac{dP}{dx} = 0$ then $2 - \frac{100}{x^2} = 0$ giving $x = \sqrt{50}$ $(-\sqrt{50}$ not applicable).

$$= 5\sqrt{2}.$$

For this value of x, $\frac{d^2P}{dx^2}$ is positive and $y = 5\sqrt{2}$.

Hence the dimensions for the perimeter to be a minimum are $5\sqrt{2}$ cm $\times$ $5\sqrt{2}$ cm.

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Exercise 2D

For each of the following questions use your calculator when appropriate but do make sure that you demonstrate *your* use and understanding of the calculus processes involved and, in particular, your use of the second derivative test when appropriate.

1 The total profit, $\$P$, generated from the production and marketing of x items of a certain product is given by

$$P = 25x^2 + 5000x - x^3.$$

Find the value of x that gives maximum profit and determine what this maximum profit will be.

2 The total profit, $\$P$, generated from the production and marketing of x items of a certain product is given by

$$P = 10\,000x - x^3 + 275x^2 - 1\,000\,000.$$

How many items should be made for maximum profit?

What would this maximum profit will be?

3 The profit, $\$P$, made by a company producing and marketing x items of a certain product is given by:

$$P = -\frac{x^3}{3} + 20x^2 + 2100x - 25\,000.$$

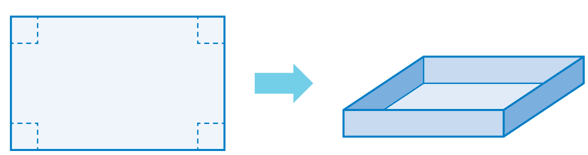
Use calculus methods, and showing full reasoning, find the value of x for maximum profit and determine this maximum profit (to the nearest \$1000).

4 Let us suppose that for an orchard involving a particular type of fruit tree, the average weight of fruit, w kg, produced per tree in a year depends on N, the number of trees per 100 m^2, according to the rule

$$w = (600 - 15N), \qquad \text{for } 10 \le N \le 25.$$

Using calculus, and clearly justifying that your value would indeed give a maximum, determine the value of N that gives the maximum total weight of this fruit that would be produced per 100 m^2 for $10 \le N \le 25$.

5 An open box is to be made by cutting squares from the corners of a rectangular sheet of card and folding up the resulting 'flaps' to form the sides of the box.



Use calculus ideas to determine the maximum capacity of such a box if the original card is

a 25 cm by 40 cm, **b** 33 cm by 40 cm.

6 The cost, $\$C$, for the production of x units of a certain product is given by

$$C = 0.025x^2 + 2x + 1000, x > 0.$$

Use calculus to find the value of x for which the **average cost per unit** is a minimum, using the second derivative test to confirm a minimum, and find this minimum average cost.

7 1000 cm^3 of metal is to be cast as a rectangular block with square ends.

Use calculus to show that for the least surface area the rectangular block needs to be a cube.

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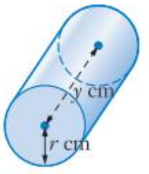
8 A company makes and sells two products which we will call an A and a B.
Each A that the company makes gives a profit of \$5600.
Each B that the company makes gives a profit of \$200.
The company has the capacity to produce 20 As each month if it makes no Bs.
The company has the capacity to produce 400 Bs each month if it makes no As.
If the company chooses to make x As in a month, $0 \le x \le 20$, then it can only produce $(400 - x^2)$ Bs.
Use calculus and the second derivative test to determine how many of each the company should make each month to maximise its profit.
What would this maximum profit be?

9 The diagram shows a cylindrical container with end radius r cm and length y cm.

If the total of the circumference of a circular end and the length of the cylinder is to be 120 cm, find the values of r and y that will maximise the capacity of the container.



10 Fencing is to be used to construct an enclosed rectangular region. The area of the region is to be 8000 m^2. Fencing costing \$16 per metre is to be used for three sides and fencing costing \$24 per metre used for the fourth side.

Find the dimensions of the rectangle that will minimise the cost of fencing, and find this minimum cost.

11 The probability of a patient recovering from a particular disease is given by

$$\frac{15x}{64 + x^2} \text{ for } 0 \le x \le 20$$

where x is the number of units of drug X that is administered.

Use calculus to determine the value of x that gives the greatest probability of recovery, viewing the function on a graphic calculator to confirm that your value of x does indeed give a maximum, and determine this maximum.

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12 Scientists monitor the spread of a particular disease through a population of native animals. They notice that whilst initially the proportion who have the disease increases, some of the population do not get the disease, possibly due to some of the animals carrying a natural immunity. They also find that with treatment the proportion of the population with the disease eventually decreases.

In an attempt to model P, the proportion of the animals with the disease, t months after it is thought to have started, the following rule is suggested:

$$P = \frac{18t}{t^2 + 5t + 100}.$$

For this suggested rule, use calculus to determine the maximum value of P, the value of t for which it occurs, and use either the sign test or the second derivative test (whichever you deem most appropriate) to confirm that the P value is indeed a maximum.

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13 The cost, \$$C$, for the production of x units of a certain product is given by

$$C = (x + 10)^3, \quad x > 0.$$

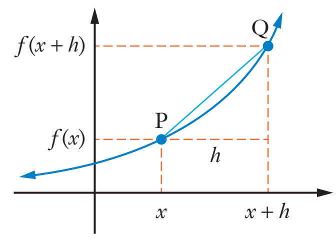
Find the value of x for which the **average cost per unit** is a minimum and find this minimum average cost.

14 Vehicle A is 25 km due east of vehicle B.
A moves due west at 60 km/h and B moves due north at 80 km/h.
Find an expression for the distance of separation for the two vehicles t hours later.
After how many minutes is this separation distance a minimum and what is this minimum distance?

15 Performing any differentiation with a calculator that can determine gradient functions, show that for an isosceles triangle of fixed perimeter to have a maximum area the triangle needs to be equilateral.

Small changes

The *Preliminary work* reminded us of the fact that differentiation is based on the idea that for the curve on the right:



$$\text{Gradient at P} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Writing δx in place of h, the small increase, or increment, in the x coordinate,
and δy in place of $f(x + h) - f(x)$, the small increment in the y coordinate,

we arrived at

$$\text{Gradient function} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad \text{We wrote this as } \frac{dy}{dx}.$$

Thus if δx, the change in x, is small then

$$\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$$

Hence δy, the small change in y, caused by δx, the small change in x, can be approximately determined using

$$\delta y \approx \frac{dy}{dx} \delta x$$

In this way, differentiation allows us to determine an approximation for the small change in one variable given the small change in another, related, variable.

If δx, the change in x, is small then $\dfrac{dy}{dx} \approx \dfrac{\delta y}{\delta x}$

and so $\delta y \approx \dfrac{dy}{dx} \delta x$

Note: It was mentioned in the *Preliminary work* section that the symbol δ is a Greek letter pronounced *delta*. The capital of this letter is written Δ and in this calculus context δx, the small change in x, is sometimes written as Δx.

EXAMPLE 9

If $y = 3x^3 + 2x - 1$ use differentiation to find the approximate change in y when x changes from 2 to 2.01.

Solution

If $y = 3x^3 + 2x - 1$ then $\frac{dy}{dx} = 9x^2 + 2$

Thus for δx a small change in x, $\frac{\delta y}{\delta x} \approx 9x^2 + 2$

In this case $x = 2$ and $\delta x = 0.01$, thus

$$\delta y \approx (9(2)^2 + 2)(0.01) = 0.38$$

When x changes from 2 to 2.01 the change in y is approximately 0.38.

The reader should compare this approximate value for the change in y with the exact answer given by $(3(2.01)^3 + 2(2.01) - 1) - (3(2)^3 + 2(2) - 1)$.

EXAMPLE 10

Find the approximate change in the area of a square when expansion causes the sides to increase from 20 cm to 20.25 cm.

Solution

The area, A, of a square is related to the side length, x, by the rule $A = x^2$.

If $A = x^2$ then $\frac{dA}{dx} = 2x$

Thus for δx a small change in x, $\frac{\delta A}{\delta x} \approx 2x$

In this case $x = 20$ and $\delta x = 0.25$, thus

$$\delta A \approx 2(20)(0.25) = 10$$

When the side length changes from 20 cm to 20.25 cm the approximate change in the area is 10 cm^2.

Again the reader should compare this approximate value for the change in area with that of $(20.25)^2 - (20)^2$.

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EXAMPLE 11

Find, correct to four decimal places, the radius of a sphere of volume 1000 cm^3.

Use calculus to determine the approximate change necessary in the radius of the sphere to cause the volume to change from 1000 cm^3 to 1010 cm^3.

Solution

The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

If the volume is 1000 cm^3 then $1000 = \frac{4}{3}\pi r^3$.

Thus $r^3 = \frac{3000}{4\pi}$

Giving $r = 6.2035$ (to 4 decimal places).

With $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dr} = 4\pi r^2$

For δV a small change in V, $\frac{\delta V}{\delta r} \approx 4\pi r^2$

$\therefore \quad \frac{\delta r}{\delta V} \approx \frac{1}{4\pi r^2}$

In this case $r \approx 6.2035$ and $\delta V = 10$, thus

$$\delta r \approx \frac{10}{4\pi(6.2035)^2} \approx 0.02$$

The radius of a sphere of volume 1000 cm^3 is 6.2035 cm, correct to four decimal places, and an increase of approximately 0.02 cm is required in the radius to increase the volume to 1010 cm^3.

Small percentage changes

EXAMPLE 12

If $V = 2x^3$ use differentiation to find the approximate percentage change in V when x changes by 2%.

Solution

In this question we are given $\frac{\delta x}{x} = \frac{2}{100}$ and are required to find $\frac{\delta V}{V}$.

If $V = 2x^3$ then $\frac{dV}{dx} = 6x^2$.

Thus for δx a small change in x, $\frac{\delta V}{\delta x} \approx 6x^2$.

Therefore

$$\frac{\delta V}{V} \approx \frac{6x^2\delta x}{V} = \frac{6x^2\delta x}{2x^3} = 3\frac{\delta x}{x} = \frac{6}{100}$$

When x changes by 2%, V changes by approximately 6%.

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Marginal rates of change

Suppose a firm produces x units of a particular commodity. There are three important functions of x that will interest the firm.

- The cost function, $C(x)$, the cost of producing the x units.
- The revenue function, $R(x)$, the income from selling the x units.
- The profit function, $P(x) = R(x) - C(x)$.

Considering the cost function, if the number of units produced increases by 1, i.e. $\delta x = 1$,

then $$\frac{\delta C}{1} \approx \frac{dC}{dx}.$$

In this way $\frac{dC}{dx}$, called the **marginal cost**, gives the approximate cost of producing one more unit at the stage of production that has just seen the xth unit produced.

Similarly:

$\frac{dR}{dx}$, the **marginal revenue**, gives the approximate extra revenue brought in by the sale of one more item after the xth item has been sold.

$\frac{dP}{dx}$, the **marginal profit**, gives the approximate extra profit produced by the sale of one more item after the xth item has been sold.

EXAMPLE 13

A manufacturing firm produces and subsequently sells x items of a certain product. The total cost of producing these x items is \$$C$, with C given by

$$C(x) = 6x + 10\sqrt{x} + 500.$$

Use differentiation to determine the approximate cost of producing one more item at the stage in the production when $x = 100$.

Solution

With $C(x) = 6x + 10\sqrt{x} + 500$ then $\quad \frac{dC}{dx} = 6 + \frac{5}{\sqrt{x}}$

Thus $\quad \frac{\delta C}{\delta x} \approx 6 + \frac{5}{\sqrt{x}}$

With $\delta x = 1$ and $x = 100$ $\quad \delta C \approx 6 + \frac{5}{\sqrt{100}}$

$$= 6.5$$

It will cost approximately \$6.50 to produce one more item at the stage in the production when $x = 100$.

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Exercise 2E

1 If $f(x) = x^2 + 4x$ use differentiation to find the approximate change in the value of the function when x changes from 4 to 4.02.

Compare your answer to $f(4.02) - f(4)$.

2 If $f(x) = 2x^2 - 5x$ use differentiation to find the approximate change in the value of the function when x changes from 3 to 3.01.

Compare your answer to $f(3.01) - f(3)$.

3 If $y = x^3 + 4$ use differentiation to find the approximate change in y when x changes from 1 to 1.05.

4 If $y = 2x^3 - 4x$ use differentiation to find the approximate change in y when x changes from 5 to 5.01.

5 If $y = t^3 + 3t^2 - 6t + 4$ use differentiation to find the approximate change in y when t changes from 2 to 2.01.

6 If $y = \dfrac{1}{t+1}$ use differentiation to find the approximate change in y when t changes from 4 to 4.1.

7 If $y = \sqrt{t}$ use differentiation to find the approximate change in y when t changes from 25 to 26.

8 If $y = 3x^2$ use differentiation to determine the approximate percentage change in y when x increases by 5%.

9 If $y = t^3$ use differentiation to determine the approximate percentage change in y when t increases by 2%.

10 Use differentiation to determine the approximate change in the area of a circle when the radius changes from 10 cm to 10.1 cm.

11 Find, correct to three decimal places, the radius of a circle of area 120 cm^2. Use differentiation to determine the approximate change in the radius when the area changes from 120 cm^2 to 121 cm^2.

12 The cost of producing n units of a particular product is given by

$C = n^3 - 45n^2 + 800n + 1000$ (in dollars).

Use differentiation to determine the approximate cost of producing one more item at the stage in the production when $n = 20$.

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13 The total revenue, $R, from the sale of x items is given by

$$R = 25x - 0.001x^2.$$

Use differentiation to determine the approximate revenue increase from the sale of one more item at the stage in the production when $x = 200$.

14 A temperature change causes the radius of a metal sphere to contract from 10 cm to 9.99 cm. Use differentiation to find the approximate change in the surface area.

15 Let us suppose that a person's body surface area, A, is related to the person's weight according to the rule $A = kW^{0.4}$ for some constant k.

Use differentiation to find the approximate percentage gain in A when W increases by 2%.

16 Use differentiation to find the approximate increase in the surface area of a spherical soap bubble if its radius changes from 2.5 cm to 2.6 cm.

17 The profit, $P, a company makes by the production and sale of x items of a certain product is given by

$$P = 20x^2 - 4000 - \frac{x^3}{12}.$$

Use differentiation to determine the approximate profit increase from the sale of one more item at the stage in the production when $x = 100$.

18 A company has the task of producing spheres of volume 288π cm^3 ± 5 cm^3. Use differentiation to determine the radius of the spheres giving your answer in the form a cm ± b cm with b rounded to 3 decimal places.

19 An oil slick is approximately circular of radius r m and thickness 5 cm. Find an expression for the volume of oil in the slick in m^3.

Use differentiation to determine the approximate increase in the radius if a further 1 m^3 of oil leaks into the slick when the radius is 20 m, the thickness of the slick remaining at 5 cm.

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20 A hollow metal cube has an exterior edge length of 10 cm.

Use differentiation to find the approximate volume of metal required to make the cube if the walls are 2 mm thick.

21 The time period T for a simple pendulum of length l is given by $T = 2\pi\frac{\sqrt{l}}{\sqrt{g}}$ where g is a constant.

Find the percentage change in T when l changes by 6%.

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Miscellaneous exercise two

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Express each of the following in the form $a\sqrt{2}$.

a $\sqrt{8}$ **b** $\sqrt{32}$ **c** $\sqrt{50}$

d $\sqrt{18}$ **e** $\sqrt{98}+3\sqrt{2}$ **f** $\sqrt{200}-\sqrt{72}$

g $\frac{4}{\sqrt{2}}$ **h** $5\sqrt{2}-\frac{2}{\sqrt{2}}$ **i** $\frac{20}{\sqrt{2}}-\sqrt{128}$

2 By writing x^5 as $(x^3)(x^2)$ differentiate $y = x^5$ using the product rule.

3 By writing x^5 as $\frac{x^7}{x^2}$ differentiate $y = x^5$ using the quotient rule.

For questions **4** to **15**, differentiate with respect to x.

4 $10x^2$ **5** $5x$ **6** $2x+1$

7 $2x^3-3x^2$ **8** $5x^3+7x^2-6x+1$ **9** -9

10 $(2x-1)(3x+2)$ **11** $3x^2(2x-1)$ **12** $(2x+5)^2$

13 $(2x-1)^7$ **14** $\frac{5x+1}{2x-3}$ **15** $\frac{5x+1}{3x^2-1}$

16 If $x = 5t^3+3t+6$ find an expression for $\frac{d^2x}{dt^2}$ and evaluate this for $t = 3$.

17 If $x = 3t^2-\frac{2}{t}$ find an expression for $\frac{d^2x}{dt^2}$ and evaluate this for $t = 2$.

18 If $x = (2t+3)^5$ find an expression for $\frac{d^2x}{dt^2}$ and evaluate this for $t = 1$.

19 If $x = t^3+20t^2-500\sqrt{t}$ find an expression for $\frac{d^2x}{dt^2}$ and evaluate this for $t = 25$.

20 If $A = 3x^2y$ and $y = (25-2x)^5$ show that $\frac{dA}{dx} = 6x(25-2x)^4(25-7x)$.

Hence determine the values of x for which A has stationary points and use either the second derivative test or the sign test, whichever you consider the more appropriate, to determine the nature of each.

21 Find the equation of the tangent to $y = (x - 1)(x^2 - x - 5)$ at the point $(-1, 6)$.

22 The sensitivity of a drug is given by $\frac{dR}{dq}$ where R is the reaction to an injection of q units of the drug. If a particular drug is such that

$$R = \frac{q^2(400 - q)}{2}$$

find the sensitivity of the drug when

a $q = 50$,

b $q = 100$.

23 a Use the quotient rule to determine $\frac{dy}{dx}$ for $y = \frac{x^2 + 2}{x - 1}$.

b Using your calculator if you wish, determine the exact coordinates of any stationary points on the curve

$$y = \frac{x^2 + 2}{x - 1}.$$

c By viewing the graph of $y = \frac{x^2 + 2}{x - 1}$ on a graphic calculator determine the nature of each stationary point.

24 The curve $y = ax^2 + b$ passes through the points A (3, 39) and B $(-2, c)$.
The tangent to the curve at A is parallel to the line $y - 30x + 7 = 0$.
Find the values of the constants a, b and c and the equation of the tangent to the curve at B.

25 If $y = \frac{20}{x}$ use differentiation to determine the approximate percentage change in y when x increases by 2%.

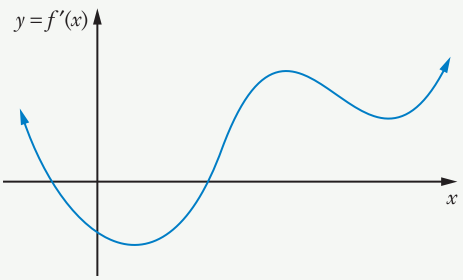
26 The diagram on the right shows the graph of

$$y = f'(x).$$

Sketch a possible graph for

a $y = f''(x)$

b $y = f(x)$



27 A function $y = f(x)$ is such that $f(3) = 1$
and $f''(3) = 0$.
Can we conclude that the graph of $y = f(x)$ has an inflection point at (3, 1)?
(Explain your answer.)

 ISBN 9780170395137

3.

Antidifferentiation

- Antidifferentiation
- Antidifferentiating powers of x
- Further antidifferentiation
- Rectilinear motion
- Miscellaneous exercise three

As mentioned in the *Preliminary work* section at the beginning of this text:

The illustrations of calculator displays shown in this text may not exactly match the displays from your calculator. The illustrations are not included to show what your calculator will necessarily display but are more to inform you that, at that moment, the use of your calculator could well be appropriate.

Antidifferentiation

As we were reminded in the *Preliminary work* section at the beginning of this text, **antidifferentiation** (also called **integration** for a reason we will see later in this book) is the opposite of differentiation.

The derivative of x^2 is $2x$.

The antiderivative of $2x$ is $x^2 + c$.

Using the integration symbol we could write this as:

$$\int 2x\,dx = x^2 + c$$

The '+ c' is needed because all functions of the form $x^2 + c$ differentiate to $2x$, for example:

The derivative of $x^2 + 1$ is $2x$. The derivative of $x^2 + 3$ is $2x$.

The derivative of $x^2 - 3$ is $2x$. The derivative of $x^2 - 7$ is $2x$.

We say that the *family* of curves $y = x^2 + c$ all have the same gradient function.

Given further information it may be possible to determine the value of the constant, c, as we will see in example 2.

Antidifferentiating powers of *x*

The *Preliminary work* section also reminded us of the rule:

$$\text{If} \quad \frac{dy}{dx} = ax^n \quad \text{then} \quad y = \frac{ax^{n+1}}{n+1} + c$$

This can be remembered as:

'Increase the power by one and divide by the new power.'

Using the integration symbol this rule is written:

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

Because integrals of this form involve an unknown constant they are called **indefinite integrals**.

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EXAMPLE 1

Find the antiderivative of each of the following.

a x^4 **b** $6x^3$ **c** 7 **d** $8x + \sqrt{x}$

Solution

a If $\frac{dy}{dx} = x^4$

then $y = \frac{x^5}{5} + c$

The antiderivative is $\frac{x^5}{5} + c$.

b If $\frac{dy}{dx} = 6x^3$

then $y = \frac{6x^4}{4} + c$

The antiderivative is $\frac{3x^4}{2} + c$.

c If $\frac{dy}{dx} = 7$ (i.e. $7x^0$)

then $y = \frac{7x^1}{1} + c$

The antiderivative is $7x + c$.

d If $\frac{dy}{dx} = 8x + x^{\frac{1}{2}}$

then $y = \frac{8x^2}{2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

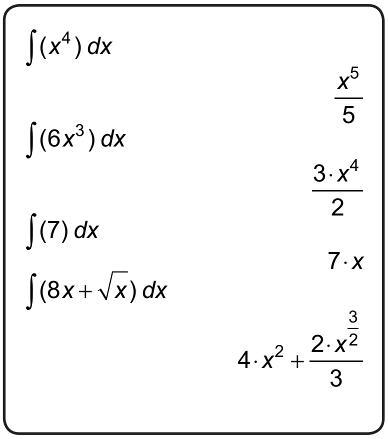
The antiderivative is $4x^2 + \frac{2}{3}x^{\frac{3}{2}} + c$.

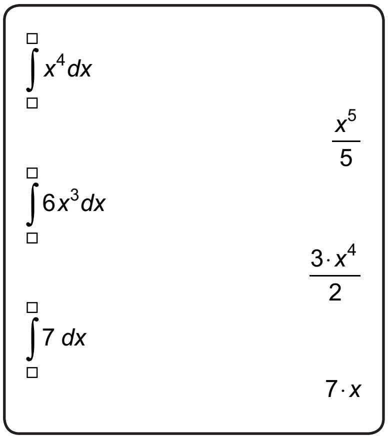
The reader should

- confirm that differentiating each of the above answers does give the required gradient function

and

- confirm these answers using a calculator. Remember, calculators tend not to include the '+ c' (we need to remember to include the constant ourselves when writing an answer obtained from a calculator) and the displays may feature spaces for entries to be made above and below the integral sign, as shown below right. This is for *definite integrals*, a concept we will meet later in this book. For the moment simply leave such entries empty.





Notice that the results for example 1 support the following statements:

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

and for constant, a,

$$\int af(x)\,dx = a\int f(x)\,dx$$

These statements together define the **linearity property** of antidifferentiation.

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EXAMPLE 2

If $\frac{dy}{dx} = 4x - \frac{2}{x^2}$ and, when $x = -1, y = 2$, find

a y in terms of x,

b y, when $x = 0.5$.

Solution

a If

$$\begin{aligned} \frac{dy}{dx} &= 4x - 2x^{-2} \\ y &= 4\frac{x^2}{2} - \frac{2x^{-1}}{-1} + c \\ &= 2x^2 + \frac{2}{x} + c \end{aligned}$$

We are told that when $x = -1, y = 2$.

Thus $2 = 2(-1)^2 + \frac{2}{-1} + c$

i.e. $2 = c$

$\therefore$ $y = 2x^2 + \frac{2}{x} + 2$

b If $x = 0.5$,

$$\begin{aligned} y &= 2(0.5)^2 + \frac{2}{0.5} + 2 \\ &= 6.5 \end{aligned}$$

When $x = 0.5$, $y = 6.5$.

Exercise 3A

Find the antiderivative of each of the following.

1 x^6

2 x^3

3 $10x^4$

4 $7x^2$

5 $8x$

6 8

7 $\sqrt{x}$

8 $\sqrt[3]{x}$

9 $x^{\frac{5}{2}}$

10 $6x^{\frac{3}{2}}$

11 $4x^{-\frac{1}{2}}$

12 $\frac{4}{\sqrt{x}}$

13 $\frac{10}{x^4}$

14 $-\frac{9}{x^2}$

15 $-\frac{16}{\sqrt{x}}$

16 $6x^2 - 4x + 3$

17 $12x^3 + 3$

18 $x^3 + 3x^2 + 2x$

19 $1 + 4x + 18x^2$

20 $3\sqrt{x} + 6x$

21 $3x^2 + 14x + 8$

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22 $(3x+2)(x+4)$

23 $(x-2)(x+6)$

24 $(3x-2)(3x+2)$

25 $4x(3x^2+3)$

26 $\dfrac{4x^2+5x}{x}$

27 $\dfrac{2x+1}{x^3}$

28 $\dfrac{6x+4}{\sqrt{x}}$

29 $\dfrac{1-x^2}{\sqrt{x}}$

30 $\dfrac{\sqrt{x}}{x}+1$

31 Find y in terms of x given that $\dfrac{dy}{dx}=6x^2+1$ and $y=13$ when $x=2$.

32 Find y in terms of x given that $\dfrac{dy}{dx}=4x-3$ and $y=29$ when $x=-3$.

33 Find A in terms of t given that $\dfrac{dA}{dt}=1-\dfrac{6}{t^2}$ and $A=-2$ when $t=2$.

34 Find v in terms of x given that $\dfrac{dv}{dx}=x+\dfrac{1}{\sqrt{x}}$ and $v=2$ when $x=4$.

35 If $f'(x)=\dfrac{6x^2}{5}-\dfrac{5}{6x^2}$ and $f(5)=51$ find **a** $f(x)$, **b** $f(1)$, **c** $f(-1)$.

Further antidifferentiation

Suppose we are asked to find the antiderivative of $(x+3)^5$.

We could expand the bracket and then antidifferentiate each term but this would be a tedious process to do 'by hand'.

Using a calculator, and remembering to include the '+ c', we find that:

$$\int (x+3)^5\,dx \qquad \frac{(x+3)^6}{6}$$

$$\int (2\cdot x+3)^5\,dx \qquad \frac{(2\cdot x+3)^6}{12}$$

The antiderivative of $(x+3)^5$ is $\dfrac{(x+3)^6}{6}+c$.

Can we simply apply the rule '*Increase the power by one and divide by the new power*'?

Were we to apply this rule in an attempt to antidifferentiate $(2x+3)^5$ it would suggest an

answer of $\dfrac{(2x+3)^6}{6}+c$.

However, as the display shows, this is not the correct answer. Instead the antiderivative of

$(2x+3)^5$ is $\dfrac{(2x+3)^6}{12}+c$.

Why did '*Increase the power by one and divide by the new power*' work for the first case but not the second?

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To answer this question consider what happens when we differentiate $(2x+3)^6$.

$$\text{If} \quad y = (2x+3)^6 \quad \text{then} \quad \frac{dy}{dx} = 6(2x+3)^5 \times 2 = 12(2x+3)^5$$

Notice that the derivative of $(2x + 3)$ also appears in the answer because, as we saw when using the chain rule:

$$\text{If } y = [f(x)]^n \text{ then } \frac{dy}{dx} = n[f(x)]^{n-1}\boldsymbol{f'(x)}$$

From this it follows that:

$$\text{If} \quad \frac{dy}{dx} = f'(x)[f(x)]^n \quad \text{then} \quad y = \frac{[f(x)]^{n+1}}{n+1} + c$$

This is very similar to the rule for antidifferentiating ax^n, i.e. '*Increase the power by one and divide by the new power*', the important difference being that we need the derivative of the function that is raised to a power to be present too.

Using the integration symbol this rule is written:

$$\int f'(x)[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Note

- In most of the following examples two methods of solution are shown.

  In 'method one' the approach is to make an intelligent first attempt at the antiderivative, differentiate it, and then use the result to adjust the first attempt appropriately. If we are attempting to antidifferentiate an expression that is of the form

  $$f'(x)[f(x)]^n$$

  or some scalar multiple thereof, our initial attempt should be of the form $f(x)^{n+1}$.

  In 'method two' the given expression is first manipulated so that the task becomes that of determining

  $$a\int f'(x)[f(x)]^n\,dx$$

  from which the answer, $a\dfrac{[f(x)]^{n+1}}{n+1} + c$, follows.

  The reader should be able to follow both methods but is advised to 'adopt' whichever one they prefer.

- Do not expect all of the questions to involve expressions of the form

  $$f'(x)[f(x)]^n.$$

  Example 4 part **a**, $x(6x-1)^2$, is not of this form for example.

EXAMPLE 3

Antidifferentiate **a** $(2x+1)^4$, **b** $\dfrac{12}{(1-3x)^2}$.

Solution

a **Method one.** (Making an intelligent guess then adjusting.)

Noticing that $(2x+1)^4$ is of the form $f'(x)[f(x)]^n$, except for a suitable scalar multiple, we try $(2x+1)$ to 'the next power up'.

If $\quad y = (2x+1)^5$

then $\quad \dfrac{dy}{dx} = 5(2x+1)^4(2)$

$\quad\quad = 10(2x+1)^4$

Our initial trial needs to be divided by ten.

The required antiderivative is:

$$\frac{1}{10}(2x+1)^5 + c.$$

Method two. (Rearranging.)

(First note that the derivative of $2x+1$ is 2.)

Rearranging $(2x+1)^4$ into the form $a\,f'(x)[f(x)]^n$:

$$\begin{aligned}\int (2x+1)^4\,dx &= \int \frac{1}{2}\times 2\times(2x+1)^4\,dx\\ &= \frac{1}{2}\times\int 2\times(2x+1)^4\,dx\\ &= \frac{1}{2}\times\frac{(2x+1)^5}{5}+c\\ &= \frac{(2x+1)^5}{10}+c\end{aligned}$$

The reader should also ensure they can obtain this same antiderivative using a calculator.

$\int (2\cdot x+1)^4 dx$

$\dfrac{(2\cdot x+1)^5}{10}$

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b **Method one.** (Making an intelligent guess then adjusting.)

Noticing that $12(1-3x)^{-2}$ is of the form $f'(x)[f(x)]^n$, except for a suitable scalar multiple, we try $(1-3x)$ to 'the next power up'.

I.e. given $\frac{dy}{dx} = 12(1-3x)^{-2}$ we try $y = (1-3x)^{-1}$

If $y = (1-3x)^{-1}$

then
$$\frac{dy}{dx} = (-1)(1-3x)^{-2}(-3)$$
$$= 3(1-3x)^{-2}$$

Our initial trial needs to be multiplied 4.

The required antiderivative is

$$\frac{4}{1-3x} + c.$$

Method two. (Rearranging.)

(First note that the derivative of $1-3x$ is -3.)

Rearranging $12(1-3x)^{-2}$ into the form $a\,f'(x)[f(x)]^n$:

$$\begin{aligned}\int 12(1-3x)^{-2}\,dx &= \int (-4)\times(-3)\times(1-3x)^{-2}\,dx \\ &= -4\times\int(-3)\times(1-3x)^{-2}\,dx \\ &= -4\times\frac{(1-3x)^{-1}}{-1}+c \\ &= \frac{4}{1-3x}+c\end{aligned}$$

Again the reader should ensure they can obtain this same antiderivative using a calculator.

$$\int \frac{12}{(1-3\cdot x)^2}\,dx \qquad \frac{-4}{3\cdot x-1}$$

EXAMPLE 4

Antidifferentiate **a** $x(6x-1)^2$, **b** $x(6x^2-1)^2$.

Solution

a The derivative of $6x-1$ is 6. Hence $x(6x-1)^2$ is **not** a scalar multiple of something of the form $f'(x)[f(x)]^n$.

In this case we expand the expression and then antidifferentiate.

If
$$\frac{dy}{dx} = x(6x-1)^2 = x(36x^2-12x+1) = 36x^3-12x^2+x$$

then
$$y = 9x^4-4x^3+\frac{x^2}{2}+c$$

The required antiderivative is $9x^4-4x^3+\dfrac{x^2}{2}+c$.

b We could again choose to expand $x(6x^2-1)^2$ and antidifferentiate each term or, noticing that we have an expression of the form $f'(x)[f(x)]^n$, except for a suitable scalar multiple, we could proceed as in example 3.

Method one. (Making an intelligent guess then adjusting.)

Given $\dfrac{dy}{dx} = x(6x^2-1)^2$ we try $y = (6x^2-1)^3$

If $y = (6x^2-1)^3$

then
$$\frac{dy}{dx} = (3)(6x^2-1)^2(12x) = 36x(6x^2-1)^2$$

We need to divide our initial trial by 36.

The required antiderivative is $\dfrac{(6x^2-1)^3}{36}+c$.

$$\int x\cdot(6\cdot x-1)^2\,dx \qquad 9\cdot x^4-4\cdot x^3+\frac{x^2}{2}$$

$$\int x\cdot(6\cdot x^2-1)^2\,dx \qquad \frac{(6\cdot x^2-1)^3}{36}$$

Method two. (Rearranging.)

$$\int x(6x^2-1)^2\,dx = \int \frac{1}{12}\times(12x)\times(6x^2-1)^2\,dx$$
$$= \frac{1}{12}\times\int (12x)\times(6x^2-1)^2\,dx$$
$$= \frac{1}{12}\times\frac{(6x^2-1)^3}{3}+c$$
$$= \frac{(6x^2-1)^3}{36}+c$$

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EXAMPLE 5

If $\dfrac{dA}{dt} = 2(5+4t)^3$ find A in terms of t given that when $t = -1, A = 1$.

Solution

Method one. (Making an intelligent guess then adjusting.)

Noticing that $2(5+4t)^3$ is of the form $f'(x)[f(x)]^n$, except for a suitable scalar multiple, we try $(5+4t)$ to 'the next power up'.

If $\quad A = (5+4t)^4$

then $\quad \dfrac{dA}{dt} = 4(5+4t)^3(4)$

$\quad = 16(5+4t)^3$

Our initial trial must be divided by 8. $\quad \therefore \quad A = \dfrac{(5+4t)^4}{8} + c.$

Now when $t = -1, A = 1.$

$\therefore \quad 1 = \dfrac{(5-4)^4}{8} + c \quad$ giving $\quad c = \dfrac{7}{8}.$

Thus $\quad A = \dfrac{(5+4t)^4}{8} + \dfrac{7}{8}.$

Method two. (Rearranging.)

$$\int 2(5+4t)^3\,dt = \int \frac{1}{2} \times 4 \times (5+4t)^3\,dt$$

$$= \frac{1}{2} \times \int 4 \times (5+4t)^3\,dt$$

$$= \frac{1}{2} \times \frac{(5+4t)^4}{4} + c$$

$$\therefore \quad A = \frac{(5+4t)^4}{8} + c$$

$$\int 2\cdot(5+4\cdot t)^3\,dt \qquad \frac{(4\cdot t+5)^4}{8}$$

The constant is found as in method 1, i.e.:

When $t = -1, A = 1.$

$\therefore \quad 1 = \dfrac{(5-4)^4}{8} + c \quad$ giving $\quad c = \dfrac{7}{8}.$

Thus $\quad A = \dfrac{(5+4t)^4}{8} + \dfrac{7}{8}.$

EXAMPLE 6

Find y in terms of x given that $\frac{dy}{dx} = 72x(3x^2 - 1)^5$ and $y = 100$ when $x = 1$.

Solution

Method one. (Making an intelligent guess then adjusting.)

Noticing that $72x(3x^2 - 1)^5$ is of the form $f'(x)[f(x)]^n$, except for a suitable scalar multiple, we try $(3x^2 - 1)^6$.

If $\quad y = (3x^2 - 1)^6$

then $\quad \frac{dy}{dx} = 6(3x^2 - 1)^5(6x)$

$\quad = 36x(3x^2 - 1)^5$

Our initial trial must be multiplied by 2.

$\therefore \quad y = 2(3x^2 - 1)^6 + c.$

Now when $x = 1, y = 100.$

$\therefore \quad 100 = 2(2)^6 + c \quad$ giving $\quad c = -28.$

Thus $\quad y = 2(3x^2 - 1)^6 - 28.$

Method two. (Rearranging.)

$$\int 72x(3x^2 - 1)^5\,dx = \int 12 \times 6x \times (3x^2 - 1)^5\,dx$$

$$= 12 \times \int 6x \times (3x^2 - 1)^5\,dx$$

$$= 12 \times \frac{(3x^2 - 1)^6}{6} + c$$

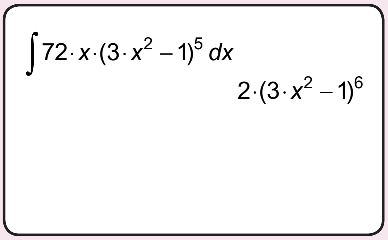
$\therefore \quad y = 2(3x^2 - 1)^6 + c$

The constant is found as in method 1, i.e.:

When $x = 1, y = 100.$

$\therefore \quad 100 = 2(2)^6 + c \quad$ giving $\quad c = -28.$

Thus $\quad y = 2(3x^2 - 1)^6 - 28.$



Note: With practice you may find you are able to write the antiderivative directly, without formally presenting either method.

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Exercise 3B

Find the antiderivative of each of the following.
(Try to do most without the assistance of a calculator.)

1 $(3x+2)^3$

2 $(3x+2)^4$

3 $x(3x+2)$

4 $(1+5x)^4$

5 $(1-5x)^3$

6 $10x(x^2+5)^4$

7 $20x(x^2-7)^4$

8 $x(1+5x)^2$

9 $(2x+1)^2$

10 $x(2x+1)^2$

11 $(5x+1)^3$

12 $21(5-7x)^3$

13 $16(2x+1)^3$

14 $45(3x-2)^4$

15 $(2x-1)(x^2-x+3)^4$

16 $48(6x+1)^3$

17 $2(5x+1)^3$

18 $150(x-1)(3x^2-6x+1)^4$

19 $5(3x-1)^4$

20 $3(9x+1)^2$

21 $x(3x+4)$

22 $2(3x-1)^2$

23 $2x(x-1)^2$

24 $(x+1)(x-1)$

25 $(1+x)^3$

26 $(1-x)^3$

27 $x(1+x)$

28 $2x(1+x)^2$

29 $12x(1+x^2)^2$

30 $2x(1+x^2)^6$

31 $-24(1-2x)^3$

32 $54(2x-1)^8$

33 $15(5-6x)^4$

34 $(3-2x)^3$

35 $6(2x-3)^8$

36 $12(5-6x)^3$

37 $(2x+1)(x^2+x+3)^4$

38 $20x(5x^2+3)^7$

39 $(1-2x)(x^2-x+3)^4$

40 $\dfrac{1}{(x+2)^4}$

41 $\dfrac{5}{(x+1)^2}$

42 $(1-x)(x^2-2x+1)^3$

43 $\dfrac{2}{(x+3)^3}$

44 $\dfrac{18x}{(x^2-3)^4}$

45 $\dfrac{1}{(x-2)^2}$

46 $\dfrac{1}{(2x-1)^2}$

47 $\dfrac{20}{(3-2x)^3}$

48 $10(6x-1)(3x^2-x+1)^4$

49 $-\dfrac{1}{(x-2)^3}$

50 $\dfrac{12}{(3x-1)^2}$

51 $\dfrac{20}{(1-5x)^3}$

52 $\sqrt{3x+2}$

53 $12\sqrt{2x-5}$

54 $\dfrac{6}{\sqrt{1+2x}}$

55 $1+(1-5x)^2$

56 $12\sqrt[3]{3x-2}$

57 $1+x(1-5x)^2$

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58 $\dfrac{12}{(2x-3)^4}$

59 $12(2x+1)^2 + 9(3x-2)^2$

60 $\sqrt{x+3} + \sqrt{x+1}$

61 $\dfrac{10x+15}{\sqrt{x^2+3x-1}}$

62 Find A in terms of p given that $\dfrac{dA}{dp} = 6(p+1)^2$ and $A = 21$ when $p = 1$.

63 Find y in terms of x given that $\dfrac{dy}{dx} = 20(2x+1)^4$ and $y = 25$ when $x = 0$.

64 Find $f(x)$ given that $f'(x) = 32(3-2x)^3$ and $f(2) = 1$.

65 Find y in terms of x given that $\dfrac{dy}{dx} = 15x(5x^2-1)^2$ and $y = 40$ when $x = 1$.

66 Find v in terms of t given that $\dfrac{dv}{dt} = \dfrac{100t}{(t^2+1)^3}$ and when $t = 2$, $v = 7$.

67 Find x in terms of t given that $\dfrac{dx}{dt} = -\dfrac{10}{(2t+1)^2}$ and $x = 2$ when $t = -1$.

68 If $\dfrac{dy}{dx} = 24(2x-1)^3$ and $y = 5$ when $x = 0$ find

a y in terms of x,

b y, when $x = 1$,

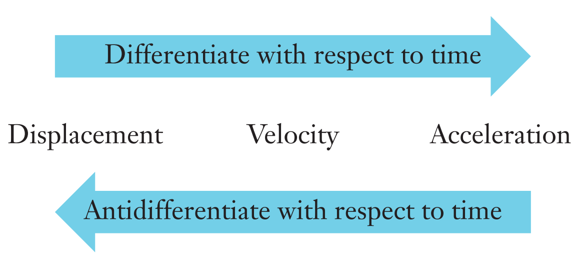
c x, when $y = 245$.

Rectilinear motion

From our understanding of antidifferentiation as the opposite of differentiation it follows that antidifferentiating, or integrating, velocity with respect to time gives displacement and that antidifferentiating, or integrating, acceleration with respect to time gives velocity.

i.e. $x = \int v\,dt$ and $v = \int a\,dt$

These facts are summarised in the following diagram:



Remember that each antidifferentiation will introduce a constant which, given sufficient information, may be determined.

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EXAMPLE 7

A particle travels along a straight line with its velocity at time t seconds given by v m/sec where $v = 3t^2 + 2$.
The initial displacement of the particle from a point O on the line is ten metres.
Find **a** the acceleration when $t = 4$,

b the displacement from O when $t = 5$.

Solution

a

$$v = 3t^2 + 2$$

Now $a = \frac{dv}{dt}$

$= 6t$

When $t = 4$ the acceleration is 24 m/s^2.

b

$$x = \int v\,dt = \int (3t^2 + 2)\,dt = t^3 + 2t + c$$

We know that initially, i.e. when $t = 0$, $x = 10$.

$\therefore \quad 10 = (0)^3 + 2(0) + c \qquad$ i.e. $c = 10$.

Thus $\quad x = t^3 + 2t + 10 \qquad \therefore$ When $t = 5$, $x = 145$.

When $t = 5$ the displacement from O is 145 metres.

Exercise 3C

Questions **1** to **9** all involve rectilinear motion with x metres, v m/s and a m/s^2 being the displacement, velocity and acceleration of a body respectively, with respect to an origin O, at time t seconds.

1 If $v = 6t^2 + 4$ find **a** the acceleration when $t = 4$,
b the displacement when $t = 2$ given that for $t = 1$, $x = 5$.

2 If $a = 6t - 2$ and when $t = 0$ the velocity is 1 m/s and the displacement is 5 m, find
a the acceleration when $t = 1$,
b the velocity when $t = 2$,
c the displacement when $t = 3$.

3 If $a = 2t(5 - 6t)$ and the body is initially at O and moving with velocity 2 m/s find
a the velocity when $t = 2$,
b the speed when $t = 2$,
c the displacement when $t = 3$.

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4 A body has an initial displacement of 5 m and velocity 2 m/s. Find the displacement and velocity after 4 seconds given that $a = \dfrac{6}{(t+1)^3}$.

5 If $v = \dfrac{1}{(t+1)^2}$ and when $t = 0$, $x = 3$, find x when $t = 4$.

6 A body is initially at rest at O and moves such that $a = 2 + \sqrt{t}$.

Find **a** the velocity when $t = 9$,

b the displacement when $t = 9$.

7 If $x = 5t + \dfrac{4}{t}$, $t > 0$, find the displacement when the velocity is 4 m/s.

8 If $a = 8 - t$, $t \geq 0$, find the displacement when the velocity is 2 m/s given that when $t = 0$, $x = 16$ and $v = 20$.

9 If $a = \dfrac{48(2t+1)^2}{5}$ and when $t = 1$, $v = 44$ and $x = 19$, find expressions for v and x as functions of t.

10 A body is initially at an origin, O, and at that instant the body has a velocity of 14 m/s. The acceleration, t seconds later, is $(3t - 11)$ m/s^2. Find the velocity of the body when it is *next* at O.

11 A body is initially at rest at an origin, O, and moves in a straight line such that its acceleration, t seconds later, is $(18 - 6t)$ m/s^2.

Find **a** the value of t when the body is next at rest and the displacement at this time,

b the distance the body moves from $t = 5$ to $t = 7$.

12 A train leaves a station and accelerates from rest at a constant 0.25 m/s^2 for two minutes.

a How far does the train travel in this time?

b What is the velocity of the train at the end of the two minutes?

13 A body is moving in a straight line with constant acceleration a m/s^2. Use calculus to obtain expressions for v, the velocity of the body in m/s, and s, the displacement of the body in metres, at time t, given that when $t = 0$, $s = 0$ and $v = u$.

14 A particle travels along a straight line such that its acceleration at time t seconds is equal to $(6t + 1)$ m/s^2. When $t = 2$ the displacement is 12 metres and when $t = 3$ the displacement is 34 metres. Find the displacement and velocity when $t = 4$.

15 A particle travels along a straight line with its acceleration at time t seconds equal to $(3t + 2)$ m/s^2. The particle has an initial positive velocity and travels 30 m in the fourth second. Find the velocity of the body when $t = 5$.

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Miscellaneous exercise three

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

Use the product rule to determine $\frac{dy}{dx}$ for each of the following.

1 $y = (x + 3)(x^2 + 1)$

2 $y = (x - 5)(x^2 - 7)$

3 $y = (x + 1)(x^2 + x + 1)$

4 $y = (2x - 3)(x^2 + 5)$

5 $y = (3x - 2)(3x^2 + x - 1)$

6 $y = (4x + 1)(x^2 - 5x + 1)$

7 If $f(x) = (2x - 3)^5$ find, without the assistance of your calculator:

a $f'(x)$ **b** $f'(2)$ **c** $f''(x)$ **d** $f''(2)$

8 Given that $y = ax^3 + x^2 + bx + 3$, $y'(2) = 50$ and $y''(1) = 23$ find the values of the constants a and b.

9 The displacement of a body from an origin O, at time t seconds, is x metres where

$$x = 2t^3 - 9t^2 + 5, t \geq 0.$$

Find the displacement and the velocity of the body from O when the acceleration is zero.

10 Find the coordinates of the points on the graphs of the given functions where the gradient is as stated.

a $y = 3x^2 + 2x$, gradient $= -10$.

b $y = x^3 - 5x$, gradient $= 43$.

c $y = \frac{5 - x}{3x + 1}$, gradient $= -1$.

11 A rocket is launched from its pad at ground level and moves vertically upwards with its engines causing it to accelerate at $(28 - 0.2t)$ m/s^2 for the first two minutes, t being the time, in seconds, since the launch.

At the end of the two minutes the engines reduce the thrust they produce to a level just sufficient to maintain the vertical velocity achieved at that time.

How high is the rocket after **a** 1 minute? **b** 2 minutes? **c** 3 minutes?

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12 (Without the use of your calculator.)

The cost function for a particular item is given by $\$C$ where C is given by

$$C = \frac{1}{12}n^3 - 12n^2 + 800n + 1000$$

and n is the number of such items produced.

Find the marginal cost for $n = 100$ and explain what this tells you about the cost of producing one more item at this level of production.

13 The owner of a garden centre wishes to fence a rectangular area of 300 m^2. She wishes to fence three sides with fencing that costs \$9 per metre and the fourth side with fencing costing \$15 per metre. Find the dimensions of the rectangular area that will minimise her fencing costs.

14 A purification unit takes used coolant and removes the impurities that have become dissolved in it. The cost, $\$C$, of removing p% of the impurities from one tonne of coolant is given by $C = \dfrac{1000p}{100 - p}$.

a Find to the nearest dollar the cost of removing, from 1 tonne of coolant,

i 1% of the impurities, **ii** 5% of the impurities,

iii 50% of the impurities, **iv** 95% of the impurities.

b Find an expression for the rate of change of C with respect to p.

15 A manufacturing firm produces and sells x items of a certain product. The total cost of producing these x items is $\$C$, with C given by

$$C = 5x + 120\sqrt{x} + 5000.$$

a Find the total cost of producing **i** 16 items,

ii 400 items.

b Find the average production cost per item when **i** $x = 16$,

ii $x = 400$.

c Find an expression for the marginal cost, $\dfrac{dC}{dx}$.

d Use your answer to **c** to determine the approximate cost of producing one more item at the stage in the production when **i** $x = 16$,

ii $x = 400$.

e Compare your answers for **d** part **i** to $C(17) - C(16)$

and for **d** part **ii** to $C(401) - C(400)$.

16 Clearly showing your use of calculus, and in particular the second derivative test, determine the coordinates and nature of any stationary points on the graph of

$$y = (2x + 5)^3 + \frac{54}{x}.$$

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4.

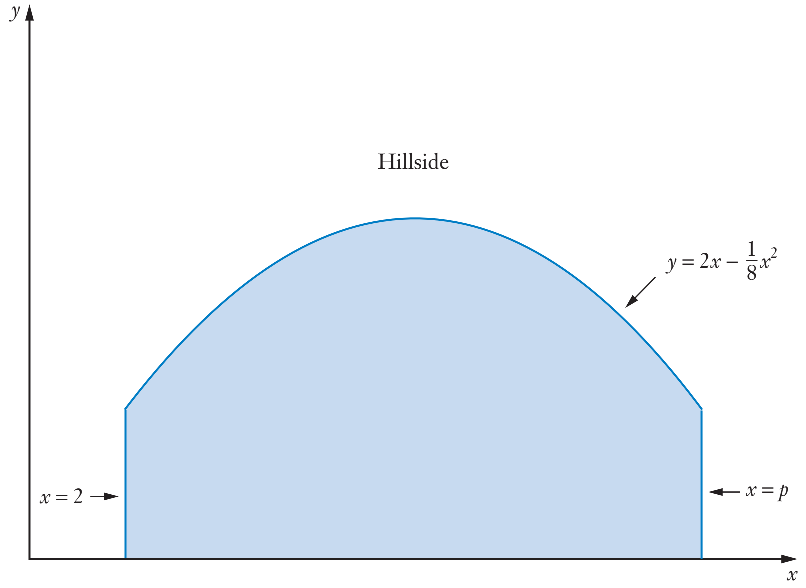
Area under a curve

- Area under a curve
- Definite integrals
- Area under a curve – further examples
- Regions that are wholly or partly below the x-axis
- Area between curves
- Using definite integrals to find total change from rate of change
- Miscellaneous exercise four

Situation

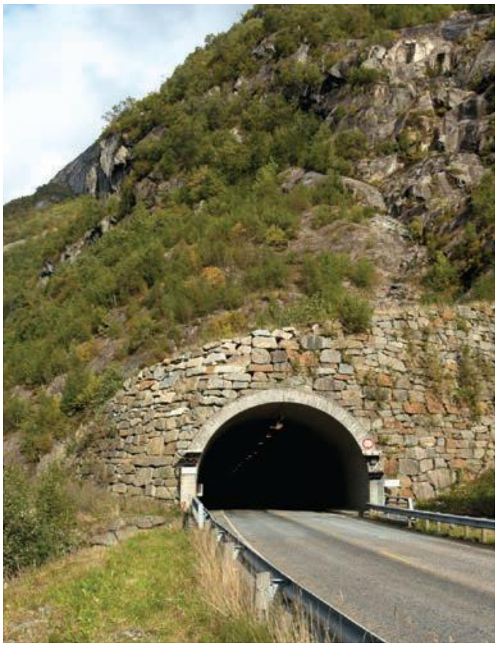
Making a tunnel

A road tunnel is to be made in a hillside. The proposed scheme is as shown in the diagram below, with the symmetrical cross-section of the tunnel shown shaded. All units are in metres.



In the model above the tunnel is bounded by the x-axis, the lines $x = 2$, $x = p$ and the curve $y = 2x - \dfrac{1}{8}x^2$.

- Find the value of p.
- What will be the maximum height of the tunnel?
- Suppose that a wide rectangular load wanted to go through this tunnel.
  If the load was 6.4 metres wide calculate the greatest height it could be and still go through the tunnel.
- On graph paper draw an accurate version of the above diagram and use it to estimate the shaded area. Hence answer the following question:
  If the tunnel is to be 400 metres long what is the least volume of earth that must be removed for it to be made?



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If we consider the various shapes that your mathematical studies to date allow you to find the area of, we are probably limited to:

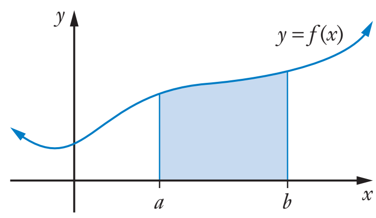
Triangles,
squares, rectangles, parallelograms, trapeziums,
circles, various parts of circles,
and any shapes that are composites of the above.

In real life many shapes that we might want to determine the area of, for example the wing of an aircraft, a canoe paddle, a ship's rudder, an insect wing, a sail, a surfboard, the blade of a fan, etc., are not made up of the above shapes (as we saw in the situation on the previous page involving the cross sectional area of a tunnel). However we can obtain a reasonable approximation for such areas if we divide them up into small squares, or perhaps rectangular strips, and sum the areas of such squares or strips - the approximation becoming more accurate the more squares or strips we divide the shape into.

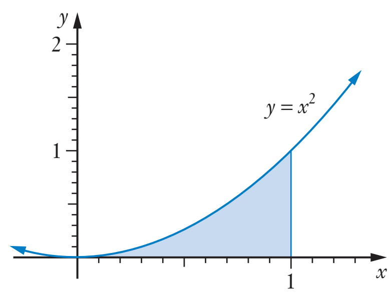
Area under a curve

Suppose we wish to determine the area between some curve, $y = f(x)$, and the x-axis, from $x = a$ to $x = b$, as shown in the diagram on the right.

(We refer to this as the area *under* the curve from $x = a$ to $x = b$.)



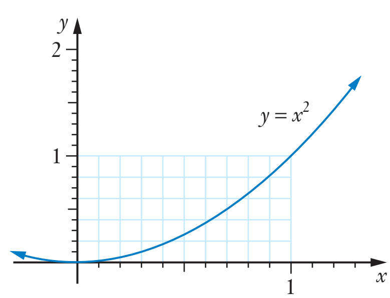
Let us start by considering the area between the curve $y = x^2$ and the x-axis, from $x = 0$ to $x = 1$, as shown in the diagram.



By drawing the graph accurately and by simply counting squares an approximate answer can be obtained.

In the diagram on the right one square unit consists of 50 of the little squares. Approximately 17 of these little squares lie in the required region so the area of this region is approximately:

$$\frac{17}{50} = 0.34 \text{ units}^2.$$



Alternatively we could estimate the area by dividing the region up into a number of strips, say 4, approximate each of these strips to a rectangle and sum the areas of these rectangles. We could do this in two ways, one which would overestimate the area (when the rectangles *circumscribe* the area) and one which would underestimate it (when the rectangles *inscribe* the area).

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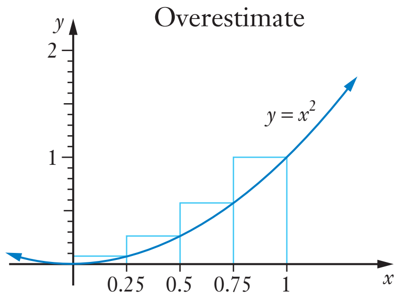
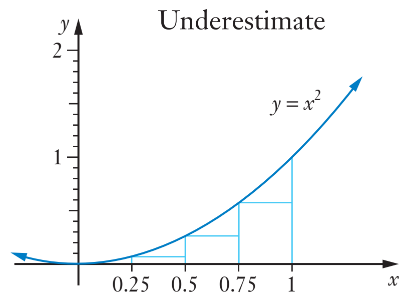




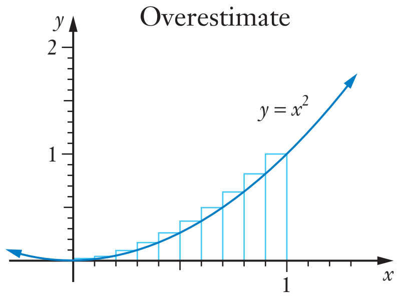
Area of 1st rect	$= 0.25(0.25)^2$	$\approx 0{\cdot}02$
Area of 2nd rect	$= 0.25(0.5)^2$	≈ 0.06
Area of 3rd rect	$= 0.25(0.75)^2$	≈ 0.14
Area of 4th rect	$= 0.25(1)^2$	$= 0{\cdot}25$
Total area (overestimated)		≈ 0.47

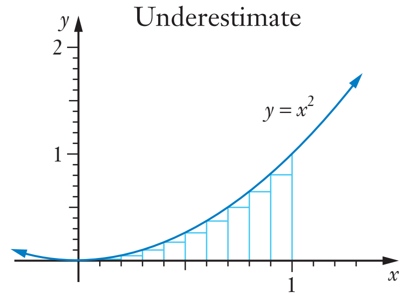
Area of 1st rect	$= 0.25(0)^2$	$= 0$
Area of 2nd rect	$= 0.25(0{\cdot}25)^2$	$\approx 0{\cdot}02$
Area of 3rd rect	$= 0.25(0.5)^2$	≈ 0.06
Area of 4th rect	$= 0.25(075)^2$	≈ 0.14
Total area (underestimated)		≈ 0.22

A reasonable approximation would be the mean of the two estimates.

The area is approximately 0.34 units^2.

A better approximation would occur if we divided the region into more strips. Division into ten equal width strips is shown below.





Area of 1st rect	$= 0.1(0.1)^2$	$= 0.001$
Area of 2nd rect	$= 0.1(0.2)^2$	$= 0.004$
Area of 3rd rect	$= 0.1(0.3)^2$	$= 0.009$
Area of 4th rect	$= 0.1(0.4)^2$	$= 0.016$
Area of 5th rect	$= 0.1(0.5)^2$	$= 0.025$
Area of 6th rect	$= 0.1(0.6)^2$	$= 0.036$
Area of 7th rect	$= 0.1(0.7)^2$	$= 0.049$
Area of 8th rect	$= 0.1(0.8)^2$	$= 0.064$
Area of 9th rect	$= 0.1(0.9)^2$	$= 0.081$
Area of 10th rect	$= 0.1(1)^2$	$= 0.1$
Total area (overestimated)		$= 0.385$

Area of 1st rect	$= 0.1(0)^2$	$= 0$
Area of 2nd rect	$= 0.1(0.1)^2$	$= 0.001$
Area of 3rd rect	$= 0.1(0.2)^2$	$= 0.004$
Area of 4th rect	$= 0.1(0.3)^2$	$= 0.009$
Area of 5th rect	$= 0.1(0.4)^2$	$= 0.016$
Area of 6th rect	$= 0.1(0.5)^2$	$= 0.025$
Area of 7th rect	$= 0.1(0.6)^2$	$= 0.036$
Area of 8th rect	$= 0.1(0.7)^2$	$= 0.049$
Area of 9th rect	$= 0.1(0.8)^2$	$= 0.064$
Area of 10th rect	$= 0.1(0.9)^2$	$= 0.081$
Total area (underestimated)		$= 0.285$

Approximate area $= \dfrac{0.385 + 0.285}{2} = 0.335$ units^2 i.e approx. $\dfrac{1}{3}$ units^2.

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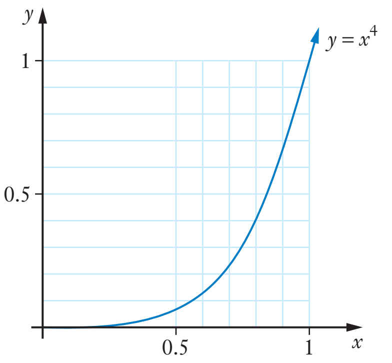
Exercise 4A

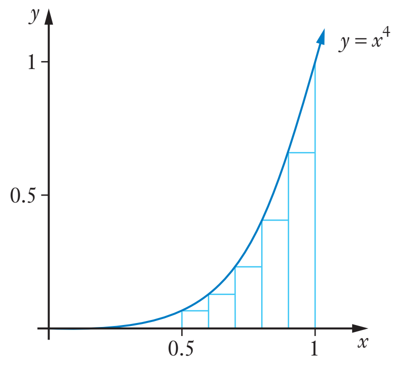
1 **a** By counting squares on the diagram on the right estimate the area under $y = x^4$ from $x = 0.5$ to $x = 1$.

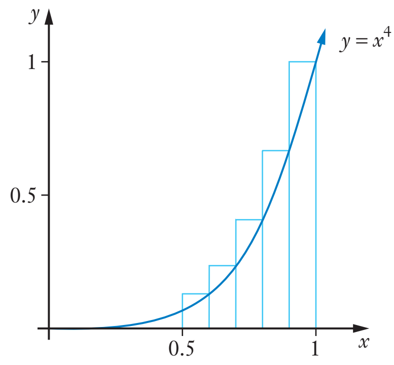


b Using rectangles, as shown below, find an underestimate and an overestimate for the area under

$$y = x^4 \text{ from } x = 0.5 \text{ to } x = 1$$

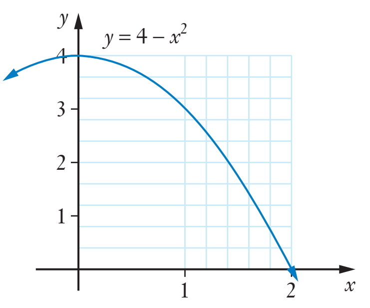
and then determine the mean of these two figures.



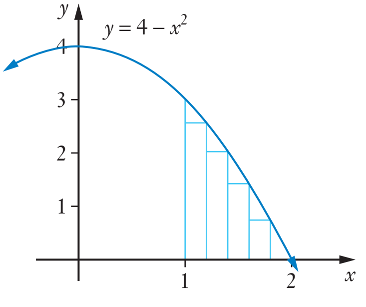


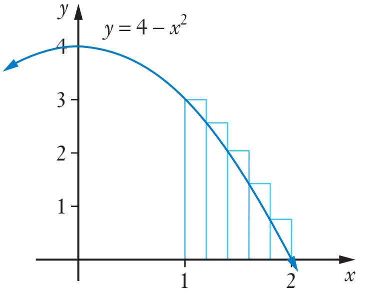
2 **a** By counting squares on the diagram on the right estimate the area under

$$y = 4 - x^2 \text{ from } x = 1 \text{ to } x = 2.$$



b Using rectangles, as shown below, find an underestimate and an overestimate for the area under $y = 4 - x^2$ from $x = 1$ to $x = 2$ and then determine the mean of these two figures.





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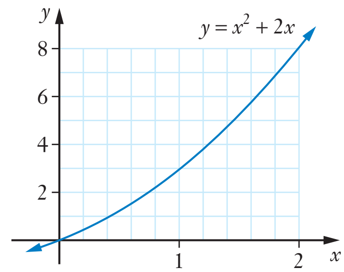
3 **a** By counting squares on the diagram on the right estimate the area under

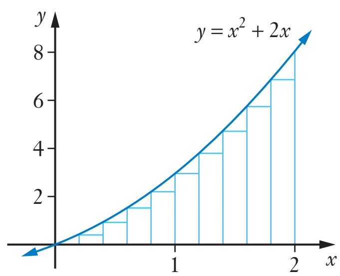
$$y = x^2 + 2x \text{ from } x = 0 \text{ to } x = 2.$$

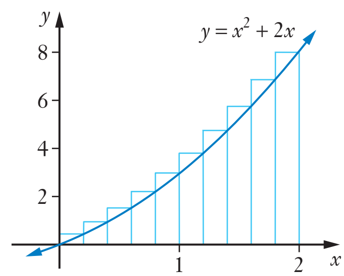


b Using rectangles, as shown below, find an underestimate and an overestimate for the area under

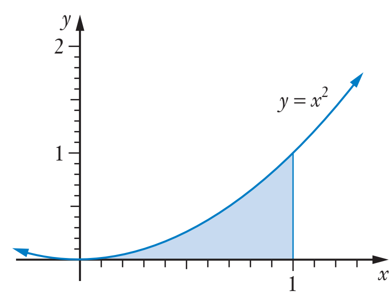
$$y = x^2 + 2x \text{ from } x = 0 \text{ to } x = 2$$

and then determine the mean of these two figures.





Earlier in this chapter, using this process of summing rectangles, we obtained an estimate for the area under $y = x^2$, from $x = 0$ to $x = 1$, of $\frac{1}{3}$ units^2.



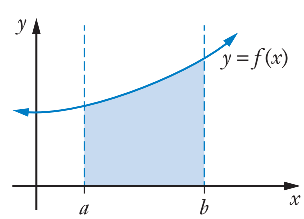
If we were to repeat this process to find estimates for the area under $y = x^2$, from $x = 0$ to $x = a$, with 'a' taking the values 1, 2, 3, 4, 5 and 6 this process would give us the following table:

Area between the x-axis and $y = x^2$ from $x = 0$ to $x = a$							
Value of a	0	1	2	3	4	5	6
Area	0	$\frac{1}{3}$	$\frac{8}{3}$	$\frac{27}{3}$	$\frac{64}{3}$	$\frac{125}{3}$	$\frac{216}{3}$

However, whilst we will reconsider these values soon, let us now move away from the particular function $y = x^2$, and consider the more general idea of the area *under*

$$y = f(x) \text{ from } x = a \text{ to } x = b,$$

shown shaded in the diagram on the right.



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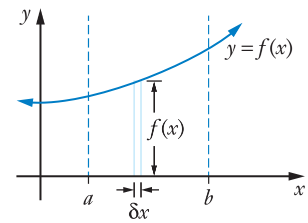
To determine the area under $y = f(x)$ from $x = a$ to $x = b$ we divide the area into elementary strips of width δx, as shown in the diagram on the right.



Each strip will approximate to a rectangle of height $f(x)$ and width δx.

Summing the areas of these rectangles will give an approximate value for the area under $y = f(x)$ from $x = a$ to $x = b$, with the approximation getting closer and closer to the exact answer as we increase the number of rectangles involved, i.e. as we allow δx to tend to zero. The value this sum 'seems to be heading towards' is the limiting value and will equal the required area.

In mathematics we use the Greek capital letter, sigma, Σ, to represent a summation. Values placed below and above the summation sign indicate what the summation goes 'from' and 'to'.

Thus the area between $y = f(x)$ and the x-axis, from $x = a$ to $x = b$, is given by:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$$

We define such a **limit of a sum** as **integration** and with a **s**ummation involved we use the 'stretched S' symbol that you have already encountered

$$\int$$

and write $\displaystyle\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x$ as $\displaystyle\int_a^b f(x)\,dx$.

However you do not need to be alarmed that we already use the term *integration*, and the above symbol, to mean *antidifferentiate* because it turns out that by antidifferentiating $f(x)$ we obtain the same answer as the limiting sum process would give. (As indeed you may have noticed in the table on the previous page for the area under $y = x^2$, or if you did not, glance back at that table now to confirm this.)

Hence to find the area under a curve, the limit of a sum we obtain by considering rectangles can be evaluated using antidifferentiation, an easier process than summing the areas of many rectangles (and a fact we will consider more formally in chapter 5).

Thus whilst integration is indeed 'a limit of a sum', it can easily be evaluated by antidifferentiating and so we use the term integration and the 'stretched S symbol' for antidifferentiation, as we have already become accustomed to.

To evaluate $\displaystyle\int_a^b f(x)\,dx$:

(1) Antidifferentiate $f(x)$ with respect to x (and omit the '$+ c$').

(2) Substitute b into your answer from (1).

(3) Substitute a into your answer from (1).

(4) Calculate: (Part (2) answer) – (Part (3) answer).

I.e.

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

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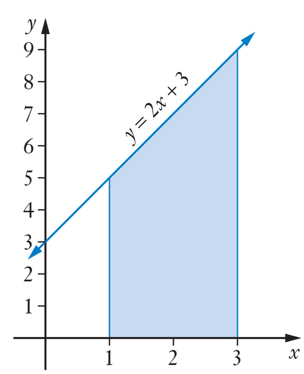
So does antidifferentiation really give us a way of finding the area under a curve without actually having to sum lots of little rectangles?

How about we see if it works for something we know the area of.

Such as?

Well how about the area under $y = 2x + 3$ from $x = 1$ to $x = 3$. That will just be a trapezium and we can find the area of that without 'limiting sums' or antidifferentiating and see if we get the same answer using this calculus stuff.

In the diagram on the right the shaded area is the area under $y = 2x + 3$ from $x = 1$ to $x = 3$.



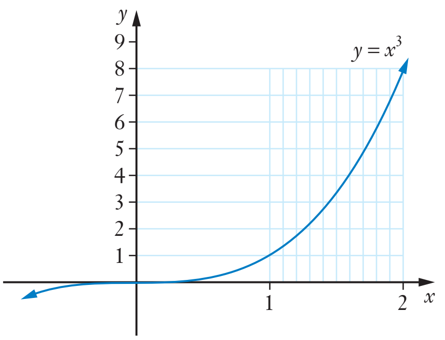
The required area is a trapezium.

$$\text{Area of this trapezium} = 2\left(\frac{9+5}{2}\right) = 14 \text{ units}^2$$

Using calculus:

$$\begin{aligned}\text{Area} &= \int_1^3 (2x+3)\,dx \\ &= \left[x^2 + 3x\right]_1^3 \\ &= (3^2 + 3(3)) - (1^2 + 3(1)) \\ &= (9+9) - (1+3) \\ &= 14 \text{ units}^2\end{aligned}$$

As a further check consider the area under $y = x^3$ from $x = 1$ to $x = 2$.



Using calculus:

$$\begin{aligned} \text{Area} &= \int_1^2 x^3\,dx \\ &= \left[\frac{x^4}{4}\right]_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= 3.75 \text{ units}^2 \end{aligned}$$

The reader should confirm that this answer is consistent with the approximate answer that can be obtained by counting the little rectangles on the given diagram.

Note The limit as δx tends to zero, of summations of the form

$$\sum_{x=a}^{x=b} f(x)\,\delta x$$

have particular importance in many mathematical applications including calculating areas, calculating volumes, locating centres of gravity, and in such fields as economics, science, engineering, psychology and others. Thus it is particularly useful that antidifferentiation provides us with a way of evaluating expressions of the form:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$$

Definite integrals

WS Integration of power functions

EXAMPLE 1

Find **a** $\int 6x^2\,dx$, **b** $\int \frac{1}{\sqrt{x}}\,dx$,

c $\int_1^2 (3x^2 + 4)\,dx$, **d** $\int_0^1 6(2x+1)^3\,dx$.

Solution

a $$\begin{aligned} \int 6x^2\,dx &= \frac{6x^3}{3} + c \\ &= 2x^3 + c \end{aligned}$$

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b $\displaystyle\int \frac{1}{\sqrt{x}}\,dx = \int x^{-0.5}\,dx$

$\displaystyle = \frac{x^{0.5}}{0.5} + c$

$= 2\sqrt{x} + c$

c $\displaystyle\int_1^2 (3x^2+4)\,dx = \left[x^3+4x\right]_1^2$

$= (2^3 + 4 \times 2) - (1^3 + 4 \times 1)$

$= (16) - (5)$

$= 11$

d $\displaystyle\int_0^1 6(2x+1)^3\,dx = 3 \times \int_0^1 2(2x+1)^3\,dx$

$\displaystyle = \left[\frac{3(2x+1)^4}{4}\right]_0^1$

$\displaystyle = \frac{3(2+1)^4}{4} - \frac{3(0+1)^4}{4}$

$\displaystyle = \frac{243}{4} - \frac{3}{4}$

$= 60$

Notice that in the previous example the answers to parts **a** and **b** included the necessary constant. This constant is not necessary in part **c** because, were we to include it, it would eventually cancel itself out as shown below.

Writing the solution to **c** as follows $\displaystyle\int_1^2 (3x^2+4)\,dx = \left[x^3+4x+c\right]_1^2$

$= (16+c) - (5+c)$

$= 11$ as before.

Integrals of the form $\int f(x)\,dx$ will involve the constant of integration and are called **indefinite integrals**.

Integrals of the form $\int_a^b f(x)\,dx$ will not involve the constant of integration and are called **definite integrals**.

Make sure you can also use your calculator to evaluate definite integrals.

$\int_1^2 (3x^2+4)\,dx$ 11

$\int_0^1 6(2x+1)^3\,dx$ 60

Note the *linearity property* of definite integrals:

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and

$$\int_a^b [k \times f(x)]\,dx = k \times \int_a^b f(x)\,dx.$$

Also note that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx, \qquad \int_a^a f(x)\,dx = 0, \qquad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

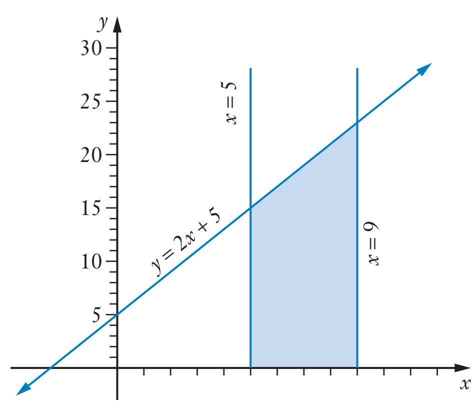
We still need to consider further the use of definite integrals to determine areas under graphs but first work through the next exercise to gain practice in evaluating definite integrals.

Exercise 4B

1 In the diagram on the right the area under $y = 2x + 5$ from $x = 5$ to $x = 9$ is shaded.

a Use a method that does not use calculus to determine the shaded area.

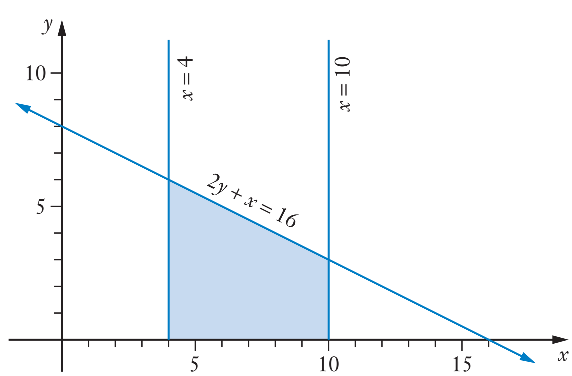
b Determine the shaded area by evaluating $\int_5^9 (2x+5)\,dx$.



2 In the diagram on the right the area bounded by the x-axis and the lines $x = 4$, $x = 10$ and $2y + x = 16$ is shown shaded.

a Use a method that does not use calculus to determine the shaded area.

b Determine the shaded area by evaluating a suitable definite integral.



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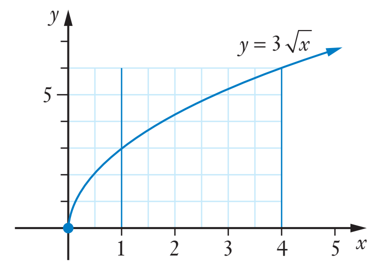
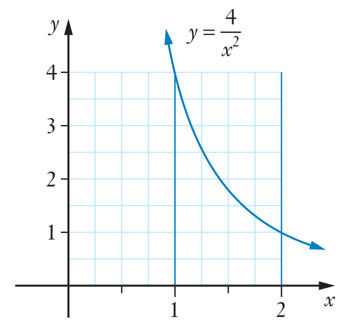
3 **a** Evaluate $\int_1^4 3\sqrt{x}\,dx$.

b Use the graph shown on the right to estimate the area under $y = 3\sqrt{x}$ from $x = 1$ to $x = 4$ by counting squares and check that your answer is consistent with the answer you obtained for part **a**.



4 **a** Evaluate $\int_1^2 \frac{4}{x^2}\,dx$.

b Use the graph shown on the right to estimate the area under $y = \frac{4}{x^2}$ from $x = 1$ to $x = 2$ by counting squares and check that your answer is consistent with the answer you obtained for part **a**.



Evaluate the following definite integrals ‘by hand’ and then use your calculator to check your answers.

5 $\int_0^2 \frac{x^2}{4}\,dx$

6 $\int_2^4 \frac{x^2}{4}\,dx$

7 $\int_1^3 10x\,dx$

8 $\int_{-1}^1 (4x+5)\,dx$

9 $\int_2^2 (4-x^2)\,dx$

10 $\int_2^3 3x^2\,dx$

11 $\int_{-1}^2 (6x^2+7)\,dx$

12 $\int_0^3 (1+x^2)\,dx$

13 $\int_3^6 x(1+x)\,dx$

14 $\int_2^3 (9-x^2)\,dx$

15 $\int_0^1 (2+x)^4\,dx$

16 $\int_0^1 (2+5x)^4\,dx$

17 $\int_0^1 12x(1+x^2)^2\,dx$

18 $\int_3^3 (4+x^2)^2\,dx$

19 $\int_{-1}^1 (1+x^2)^2\,dx$

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20 Evaluate **a** $\int_0^1 x^2\,dx$, **b** $\int_1^3 x^2\,dx$, **c** $\int_0^3 x^2\,dx$.

21 Evaluate **a** $\int_0^4 (4x - x^2)\,dx$, **b** $\int_4^5 (4x - x^2)\,dx$, **c** $\int_0^5 (4x - x^2)\,dx$.

22 Evaluate **a** $\int_1^3 (3x^2 + 2x)\,dx$, **b** $\int_3^1 (3x^2 + 2x)\,dx$.

23 Evaluate **a** $\int_0^3 x^2\,dx$, **b** $\int_0^3 3x^2\,dx$, **c** $\int_0^3 4x^2\,dx$.

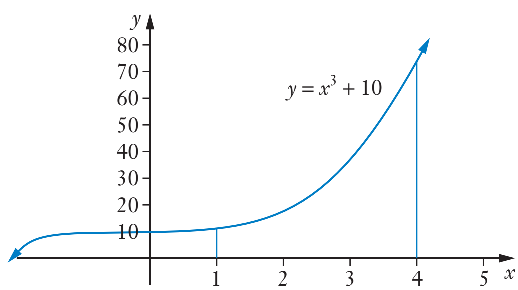
Determine each of the following giving your answers as exact values.

24 $\int_{-\pi}^{\pi} (2x + 3)\,dx$

25 $\int_{\sqrt{2}}^{2} (2x + 6x^2)\,dx$

Area under a curve – further examples

Suppose we are asked to determine the area between $y = x^3 + 10$ and the x-axis from $x = 1$ to $x = 4$, see diagram on right.



There are a number of ways that we could proceed:

- We could (but we wouldn't want to!) approximate the area to a number of equal width rectangles, obtain an estimate for the area, increase the number of rectangles, improve our estimate, etc., etc.
- We could evaluate $\int_1^4 (x^3 + 10)\,dx$, either algebraically or by calculator:

$$\begin{aligned}\int_1^4 (x^3 + 10)\,dx &= \left[\frac{x^4}{4} + 10x\right]_1^4 \\ &= (64 + 40) - \left(\frac{1}{4} + 10\right) \\ &= 93\frac{3}{4}.\end{aligned}$$

$\int_1^4 (x^3 + 10)dx$

93.75

The required area is $93\frac{3}{4}$ units^2.

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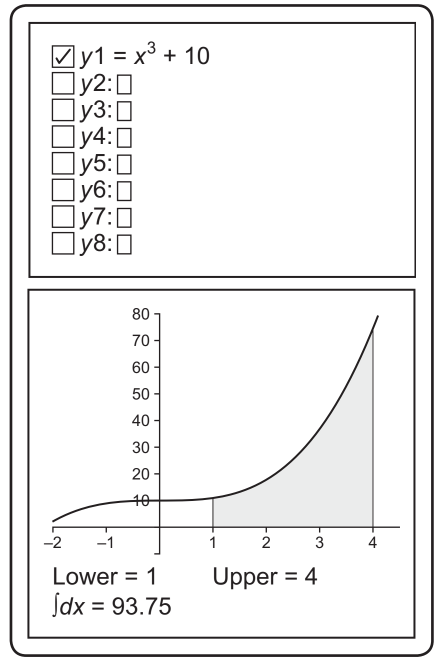
- We could use the ability of some calculators to display the graph of

$$y = x^3 + 10,$$

to show the required region shaded and to state its area.



Regions that are wholly or partly below the x-axis

Particular care needs to be taken when the area we are determining lies wholly or partly below the x-axis, as the next example shows.

WS Calculating physical areas

EXAMPLE 2

Find the area between $y = 3x - x^2$ and the x-axis from

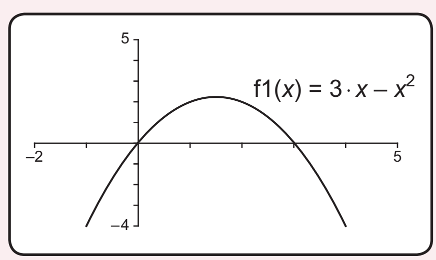
a $x = 0$ to $x = 3$,
b $x = 3$ to $x = 4$,
c $x = 0$ to $x = 4$.

Solution

First view the graph of $y = 3x - x^2$ on a graphic calculator:

Or, recognising the function is quadratic, produce a sketch on paper.

(Note that the curve cuts the x-axis at (0, 0) and (3, 0).)



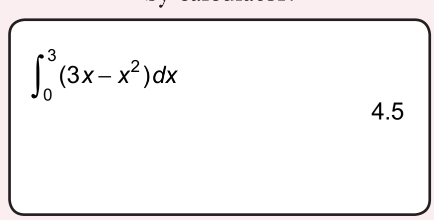
a Algebraically or by calculator:

$$\int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0)$$

$$= 4.5$$

The required area is 4.5 units².



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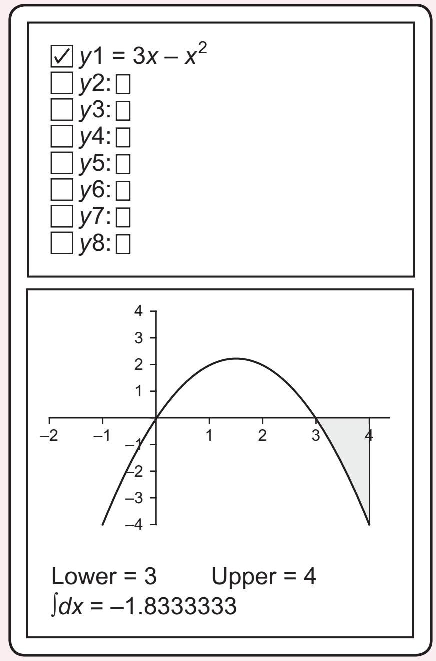
b Similarly $\int_3^4 (3x - x^2)\,dx = -1\frac{5}{6}$

$$\int_3^4 (3x - x^2)dx \qquad -\frac{11}{6}$$



The answer is negative because the curve lies *below* the x-axis for $3 < x < 4$.

However, area cannot be negative so the required area is $1\frac{5}{6}$ units².

Note that while $\int_3^4 (3x - x^2)\,dx = -1\frac{5}{6}$ the required area is $1\frac{5}{6}$ units².

Hence: $\int_a^b f(x)\,dx$ gives the **signed** area between $f(x)$ and the x-axis from $x = a$ to $x = b$.

c From **a** area from $x = 0$ to $x = 3$ is $4\frac{1}{2}$ units².

From **b** area from $x = 3$ to $x = 4$ is $1\frac{5}{6}$ units².

Thus the area from $x = 0$ to $x = 4$ is $6\frac{1}{3}$ units².

In part **c** above had we simply evaluated

$$\int_0^4 (3x - x^2)\,dx$$

we would have obtained an answer of

$$4\frac{1}{2} + \left(-1\frac{5}{6}\right), \text{ i.e. } 2\frac{2}{3}.$$

$$\int_0^4 (3x - x^2)dx \qquad \frac{8}{3}$$

This is the correct evaluation of $\int_0^4 (3x - x^2)\,dx$ but it is *not* the required area.

This shows the importance of using a graphic calculator to view the situation for these 'area under a curve' questions.

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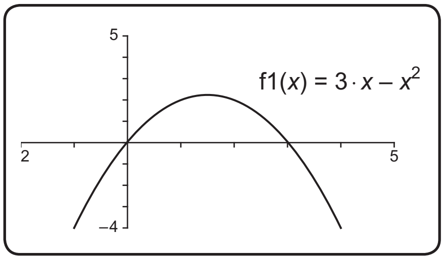
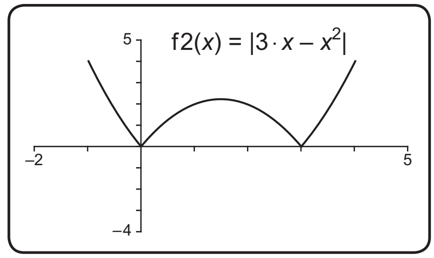
Alternatively we could use the fact that the graph of $y = |3x - x^2|$ (i.e. the absolute value of $3x - x^2$) will be that of $y = 3x - x^2$ with all those parts that lie below the x-axis reflected up above the x-axis:





Thus part **c** of the last example could be determined using the ability of some calculators to evaluate

$$\int_0^4 |3x - x^2|\, dx.$$

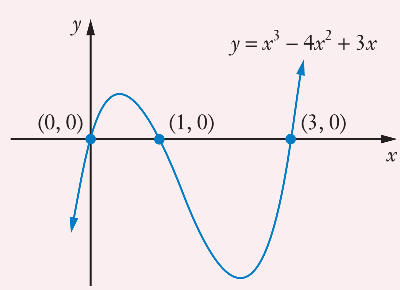
$$\int_0^4 \text{abs}\,(3x - x^2)\,dx \qquad \frac{19}{3}$$

EXAMPLE 3

Find the area enclosed between $y = x^3 - 4x^2 + 3x$ and the x-axis.

Solution

First use your graphic calculator to view the situation. (Remember you already have a good idea what a cubic function can look like.)



Note that the function cuts the x-axis at (0, 0), (1, 0) and (3, 0).

Either algebraically or with the assistance of a calculator:

$$\int_0^1 (x^3 - 4x^2 + 3x)\,dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_0^1 = \frac{5}{12}$$

$$\int_1^3 (x^3 - 4x^2 + 3x)\,dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right]_1^3 = -\frac{8}{3}$$

The required area is $\frac{5}{12}$ units^2 + $\frac{8}{3}$ units^2 = $\frac{37}{12}$ units^2.

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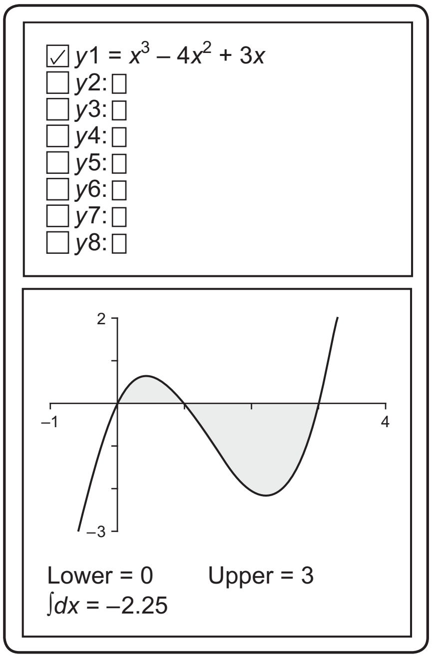
$$\int_0^3 \left| x^3 - 4x^2 + 3x \right| dx \qquad \frac{37}{12}$$

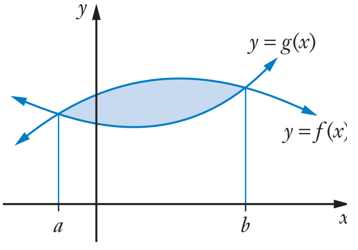


- Get to know the capabilities of your calculator.
- Practise both algebraic and calculator approaches for determining definite integrals.

WS Calculating areas between curves

Area between curves

The diagram on the right shows two curves, $y = f(x)$ and $y = g(x)$, intersecting at $x = a$ and $x = b$.



$$\text{Shaded area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

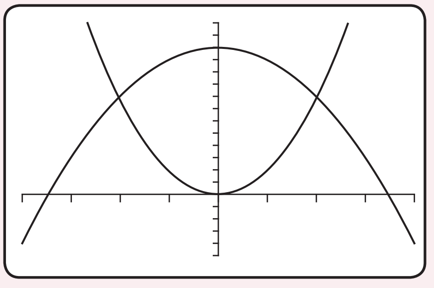
$$= \int_a^b \left[f(x) - g(x) \right] dx$$

EXAMPLE 4

Find the area enclosed between $y = 2x^2$ and $y = 12 - x^2$.

Solution

First view the situation on your calculator.



Either algebraically or from the calculator we determine that the graphs intersect when $x = 2$ and again when $x = -2$.

$$\text{Required area} = \int_{-2}^{2} (12 - x^2)\,dx - \int_{-2}^{2} 2x^2\,dx$$

$$= \int_{-2}^{2} (12 - x^2 - 2x^2)\,dx$$

$$= \int_{-2}^{2} (12 - 3x^2)\,dx$$

$$= 32$$

The area enclosed between $y = 2x^2$ and $y = 12 - x^2$ is 32 units^2.

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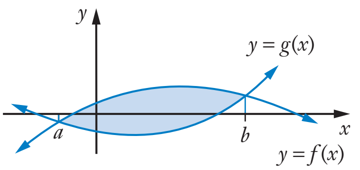


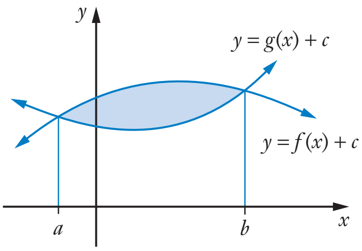
Suppose part of the required area between two curves lies below the x-axis? (See diagram.)

Question: Do we have to do anything different in such cases?

Answer: No. The area will still be given by

$$\int_a^b [f(x) - g(x)]\, dx$$

(provided $y = f(x)$ is the 'top' function for $a < x < b$).



This can be seen if a constant, c, is added to each function (as in the second diagram.)

The area between the two curves is clearly the same as before and, if we use calculus:

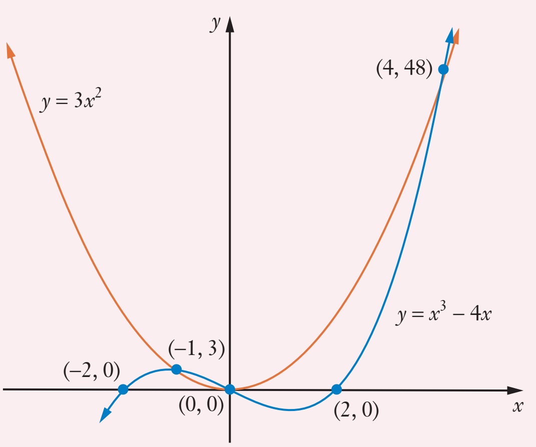
$$\begin{aligned} \text{Shaded area} &= \int_a^b [f(x) + c]\, dx - \int_a^b [g(x) + c]\, dx \\ &= \int_a^b [f(x) + c - g(x) - c]\, dx \\ &= \int_a^b [f(x) - g(x)]\, dx \text{ as before.} \end{aligned}$$

EXAMPLE 5

Make a sketch showing the graphs of $y = x^3 - 4x$ and $y = 3x^2$ indicating clearly on your sketch the coordinates of any points where the curves intersect the axes and each other. Find the area enclosed between $y = x^3 - 4x$ and $y = 3x^2$.

Solution

Either algebraically, or from a calculator, the necessary information can be obtained for the sketch to be made.



Noting carefully which function is 'above' the other in each of the two regions the required total area will be

$$\left(\int_{-1}^{0} (x^3 - 4x)\, dx - \int_{-1}^{0} 3x^2 dx\right)$$

$$+ \left(\int_0^4 3x^2\, dx - \int_0^4 (x^3 - 4x)\, dx\right)$$

$$= \int_{-1}^{0} (x^3 - 4x - 3x^2)\, dx + \int_0^4 (3x^2 - x^3 + 4x)\, dx$$

Evaluating this expression gives 32.75. The required area is 32.75 units^2.

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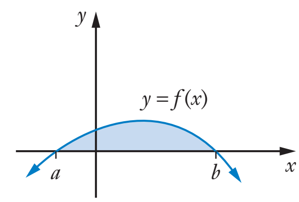
Alternatively, in the previous example, defining $f(x)$ as $x^3 - 4x$ and $g(x)$ as $3x^2$ we could use our calculator to evaluate $\int_{-1}^{4} |f(x) - g(x)|\, dx$ and thus avoid having to decide which function is above and which below.

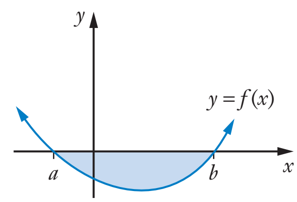
Exercise 4C

1 For which of the graphs drawn below is the shaded area equal to $\int_a^b f(x)\, dx$?

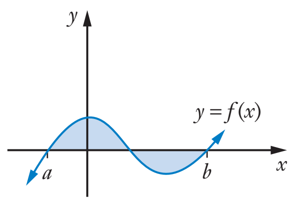
a



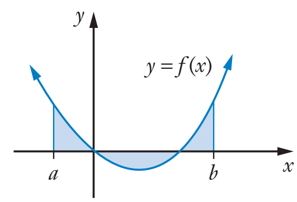
b



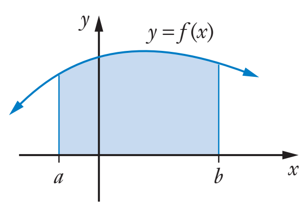
c



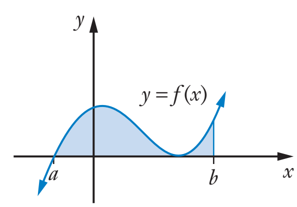
d



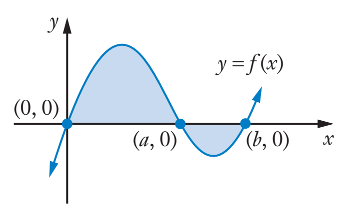
e



f



2 For each of the following state whether evaluating the given expression would give the total area shown shaded on the right.



a $\int_0^a f(x)\, dx + \int_a^b f(x)\, dx$

b $\int_0^b f(x)\, dx$

c $\int_0^a f(x)\, dx - \int_a^b f(x)\, dx$

d $\left|\int_0^b f(x)\, dx\right|$

e $\int_0^b |f(x)|\, dx$

f $\int_0^a |f(x)|\, dx + \int_a^b |f(x)|\, dx$

g $\left|\int_0^a f(x)\, dx\right| + \left|\int_a^b f(x)\, dx\right|$

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3 Use calculus to determine the area between $y = 2x + 1$ and the x-axis from $x = 0$ to $x = 2$. Check your answer using area formulae.

4 Find the area between $y = \frac{x^4}{4}$ and the x-axis from $x = 2$ to $x = 4$.

5 Find the area between $y = (x - 2)^2 + 3$ and the x-axis from $x = 1$ to $x = 3$.

6 Find the area bounded by $y = 8 - 2x^2$ and the x-axis.

7 Find the area between $y = 1 - x^3$ and the x-axis from $x = 0$ to $x = 1$.

8 Find the area between $y = (x + 1)^3 + 1$ and the x-axis from $x = -2$ to $x = 0$.

9 Find the area between $y = x^2 - 1$ and the x-axis from $x = 0$ to $x = 2$.

10 Find the area between $y = 1 - x^3$ and the x-axis from $x = 0$ to $x = 2$.

11 Find the area between $y = (x + 1)^3$ and the x-axis from $x = -2$ to $x = 0$.

12 Find the area between $y = 12x(1 - x^2)^3$ and the x-axis from $x = -1$ to $x = 1$.

13 Find the *exact* area enclosed between $y = 2 - x^2$ and the x-axis.

14 Make a sketch showing the graphs of $y = x^2$ and $y = 3 - 2x$ indicating clearly on your sketch the coordinates of any points where the functions intersect the axes and each other.

Find the area enclosed between $y = x^2$ and $y = 3 - 2x$.

15 Make a sketch showing the graphs of $y = (x - 3)^2$ and $y = x - 1$ indicating clearly on your sketch the coordinates of any stationary points and of any points where the functions intersect the axes and each other.

Find the area enclosed between $y = (x - 3)^2$ and $y = x - 1$.

16 Make a sketch showing the graphs of $y = x^2 - 2x + 3$ and $y = 2x^2$ indicating clearly on your sketch the coordinates of any stationary points and of any points where the functions intersect the axes and each other.

Find the area enclosed between $y = x^2 - 2x + 3$ and $y = 2x^2$.

17 Make a sketch showing the graphs of $y = x$ and $y = x^3$ indicating clearly on your sketch the coordinates of any stationary points and of any points where the functions intersect the axes and each other.

Find the area enclosed between $y = x$ and $y = x^3$.

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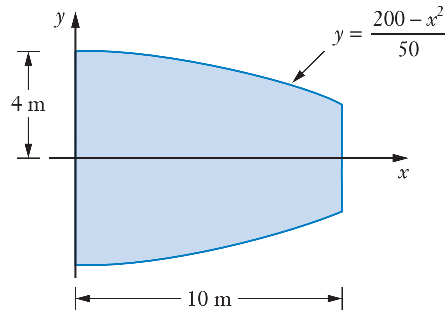
18 Make a sketch showing the graphs of $y = 2x^3 - 3x$ and $y = 7x$ indicating clearly on your sketch the *exact* coordinates of any points where the functions intersect the axes and each other.

Showing full algebraic reasoning determine the area enclosed between the curve with equation $y = 2x^3 - 3x$ and the line with equation $y = 7x$.

19 Part of a flat deck of a ship is shaped as shown shaded in the diagram on the right. The area is symmetrical with the x-axis the line of symmetry. (1 metre = 1 unit on each axis).

This part of the deck is to be given a special fire resistant coating.

The firm carrying out this process quotes a price of \$45 per square metre. How much will the job cost?



Using definite integrals to find total change from rate of change

WS
Total change

Suppose that some quantity, Q, changes with respect to some other variable, r, such that

$$\frac{dQ}{dr} = 3r^2 + 2r + 3.$$

Integration gives $$Q = r^3 + r^2 + 3r + c.$$

Without further information we cannot determine c, the constant of integration. However, if we require the change in Q when r changes, say from 5 to 10, this can be determined without knowing c.

$$\begin{aligned} Q(10) &= 10^3 + 10^2 + 3(10) + c \\ &= 1130 + c \\ Q(5) &= 5^3 + 5^2 + 3(5) + c \\ &= 165 + c \\ Q(10) - Q(5) &= (1130 + c) - (165 + c) \\ &= 965 \end{aligned}$$

When r changes from 5 to 10, Q increases by 965.

What we have found here is the definite integral:

$$\int_5^{10} (3r^2 + 2r + 3)\, dr.$$

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Thus we could set our method out as follows:

Change in Q as r changes from 5 to 10 $= \int_5^{10} (3r^2 + 2r + 3)\, dr$

$= \left[r^3 + r^2 + 3r \right]_5^{10}$

$=$ 965 as before.

$\int_5^{10} (3r^2 + 2r + 3) dr$

965

Thus $\int_a^b \frac{dQ}{dr}\, dr$ gives the total change in Q when r changes from a to b.

EXAMPLE 6

An oil tank is drained of oil such that if V kL of oil are in the tank t minutes after draining commences then $\frac{dV}{dt} = 2t - 20$.

The initially full tank is emptied in 5 minutes.

a Write an expression which, if evaluated, would give the number of kilolitres of oil drained from the tank in the first minute.

b Determine the number of kilolitres of oil drained from the tank in the fourth minute.

Solution

a Total change in V in the first minute is given by: $\int_0^1 (2t - 20)\, dt$.

V is decreasing so this expression will give a negative answer.

Thus the number of kilolitres drained in the 1st minute is $-\int_0^1 (2t - 20)\, dt$.

b Algebraically or by calculator.

$$\int_3^4 (2t - 20)\, dt = -13$$

Thus 13 kL are drained from the tank in the fourth minute.

$\int_3^4 (2t^2 - 20) dt$

-13

Exercise 4D

1 $C is the cost of producing x tonnes of a certain product where C is such that:

$$\frac{dC}{dx} = 3x^2 - 60x + 500.$$

Find the extra cost of producing

a 20 tonnes rather than 10 tonnes,

b 50 tonnes rather than 40 tonnes.

2 $C is the cost of producing x units of a certain product where C is such that:

$$\frac{dC}{dx} = \frac{250}{\sqrt{x}}.$$

Find the extra cost incurred by producing 100 units rather than 25.

3 $C is the cost of producing x units of a certain product where C is such that:

$$C'(x) = \frac{400}{x+1}.$$

Using your calculator to evaluate the appropriate definite integral, find, to the nearest dollar, the extra cost incurred by producing

a 20 units rather than 10,

b 40 units rather than 20.

4 To test the safety of various designs of hot air balloons, models of each design are constructed in the laboratory and then each is punctured and their deflation is monitored. For one such model the rate of change in the volume of air in the balloon with respect to time, from the time of puncture ($t = 0$) to total deflation, is approximately given by the rule:

$$\frac{dV}{dt} = 40(t - 25) \text{ cm}^3\text{/sec}$$

iStock.com/MariuszBlach

a Write a definite integral which, if evaluated, would give the number of cubic centimetres of air escaping from the balloon from $t = 5$ to $t = 8$. (You may assume that the balloon takes longer than 8 seconds to deflate.)

b Evaluate your expression from **a**.

5 In a particular chemical reaction the temperature of the reagents, T°C, increases such that $\frac{dT}{dt} = \frac{t^{0.1}}{2}$ where the t is the time in seconds from the start of the reaction.

a Write a definite integral which, if evaluated, would give the number of °C by which the temperature of the reagents rises from $t = 5$ to $t = 10$.

b Evaluate your expression from **a** giving your answer correct to 1 decimal place.

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6 The population of a particular country (P million people) was changing such that t years after records were first kept

$$\frac{dP}{dt} \approx 5.1 + 0.04t$$

If it is now 20 years since records were first kept use the above rule to determine an approximate value, to the nearest half million, for the increase in the population in the next eight years.

7 An oil tank is drained of oil such that if V kL of oil are in the tank t minutes after draining commences then $\frac{dV}{dt} = 0.15t^2 - 20$.

The initially full tank is emptied in 10 minutes.

For each of the following **i** write a definite integral which, if evaluated, would determine the required answer in kL,

and **ii** determine the answer (to the nearest kL).

a How much oil was in the full tank?

b How much oil was drained from the tank in the first minute?

c How much oil was drained from the tank in the tenth minute?

8 The total sales, N, of a new product, t weeks after its launch is such that:

$$\frac{dN}{dt} = 600 + \frac{600}{(t+1)^2}$$

Find the number of sales in **a** the first 4 weeks,

b the fifth week.

9 A motor car manufacturer introduces a new sports model. The total number, N, that the company has produced t weeks after production commenced is such that:

$$\frac{dN}{dt} = 150 - \frac{600}{(t+2)^2}.$$

Find the number produced in **a** the first 4 weeks,

b the second 4 weeks,

c the second week,

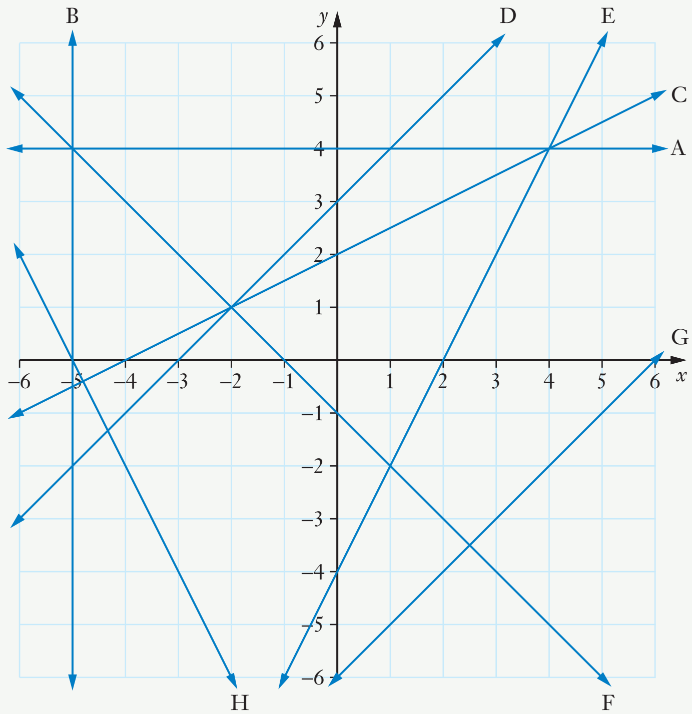
d the fourth week.

iStock.com/Vladimiroquai

Miscellaneous exercise four

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Determine the equation of each of the straight lines A to H shown below.



2 If $f(x) = 2x - 3x^3$ find

a $f'(x)$ **b** $f'(5)$ **c** $f''(x)$ **d** $f''(-5)$

3 Evaluate the following definite integrals.

a $\int_1^3 2x\,dx$ **b** $\int_1^4 \sqrt{x}\,dx$

Use the product rule to determine $\frac{dy}{dx}$ for each of the following.

4 $y = (x + 5)(x - 3)$

5 $y = (x + 5)(3 - x)$

6 $y = (2x + 1)(x + 5)$

7 $y = (5 - 2x)(2x + 1)$

8 $y = (x + 1)^2(2x + 7)$

9 $y = (2x + 5)^3(5x + 6)$

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10 Find the equation of the tangent to each of the following at the point indicated.

a $y = (3x + 1)^2$ at $(-1, 4)$

b $y = (4x)^{-1}$ at $(0.25, 1)$

c $y = (3x - 5)^4$ at $(2, 1)$

d $y = \dfrac{2x - 3}{x - 3}$ at $(4, 7)$

11 Determine the coordinates of any points on the curve

$$y = (2x - 3)(x^2 - 1)$$

where the gradient is equal to -2.

12 The curve $y = ax^3 + bx^2 + cx + d$, for constant a, b, c and d, cuts the y-axis at $(0, 30)$ and the gradient of the curve at this point is -1.

The gradient at the point on this curve where $x = 1$, is -17, and at this point the second derivative with respect to x is -10.

With the assistance of a calculator if you wish, find the coordinates of all those points where $y = ax^3 + bx^2 + cx + d$ cuts the x-axis.

13 Find the following indefinite integrals.

a $\int 20x^3\,dx$

b $\int 6\sqrt{x}\,dx$

c $\int (x + 3)^4\,dx$

d $\int (2x + 3)^4\,dx$

e $\int 60x^2(1 + x^3)^4\,dx$

f $\int (1 + x^2)^2\,dx$

14 A firm produces and sells x units of a particular item with the cost and revenue functions given in dollars by, respectively, $C(x) = 6000 + 18x$,

and $R(x) = 25.5x$.

Find

a an expression for $P(x)$, the profit function in dollars,

b how many units must be produced and sold for the firm to break even with this product,

c the marginal cost, marginal revenue and marginal profit.

15 A particle travels along a straight line with its acceleration at time t seconds equal to

$$6(t + 1) \text{ m/s}^2.$$

When $t = 1$ the displacement is 3 metres. When $t = 2$ the displacement is 19 metres.

Find the displacement and velocity when $t = 3$.

16 Find A in terms of p in each of the following cases:

a $\dfrac{dA}{dp} = (2p - 1)^3$ and $A = 0.5$ when $p = 0$.

b $\dfrac{dA}{dp} = 8p(p^2 - 1)^3$ and $A = 45$ when $p = 2$.

17 **a** Find the area between $y = -3x^2$ and the x-axis from $x = 0$ to $x = 2$.

b Find the area between $y = 3 - 3x^2$ and the x-axis from $x = 0$ to $x = 2$.

18 Without the assistance of a calculator

a find $\int \frac{3x+1}{\sqrt{x}}\,dx$,

and **b** show that $\int_4^5 \frac{3x+1}{\sqrt{x}}\,dx = 12\sqrt{5} - 20$.

19 Without the assistance of your calculator:

a Find the coordinates of any points where $y = x^3 - 5x^2 - 6x$
and $y = x^2 - 9x - 10$
cut the axes.

b Show algebraically that the curve $y = x^3 - 5x^2 - 6x$
cuts the curve $y = x^2 - 9x - 10$
at the points $(-1, a)$, $(2, b)$ and $(5, c)$ and determine a, b and c.

c Show the points from **a** and **b** on a sketch of the two curves.
(Coordinates of turning points need not be given on the sketch.)

With the assistance of your calculator:

d Find the area enclosed between $y = x^3 - 5x^2 - 6x$
and $y = x^2 - 9x - 10$.

20 **a** The cost, \$$C$, for the production of x units of a certain product is given by

$$C = ax^3 - bx^2 + cx$$

for a, b and c positive constants.

Show that for the value of x that makes the average cost per unit a minimum then:
average cost per unit = marginal cost.

b The cost, \$$C$, for the production of x units of a certain product is given by $C = f(x)$.

Show that the task of finding the value of x for which the average cost per unit is minimised, if such a minimum exists, involves solving the equation

$$xf'(x) = f(x),\ x \neq 0.$$

Show that this same equation results from attempting to find the value of x for which average cost per unit = marginal cost.

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5.

The fundamental theorem of calculus

- The fundamental theorem of calculus
- Miscellaneous exercise five

In everyday language the word integration means the bringing together, or combining, of the parts, to make a unified whole. We might talk of:

a group integrating well into society,

the integration of the year sevens into high school,

the integration of three companies into one unified company,

a group of animals of a particular species being released from captivity and integrating with other animals, of the same species, already in the area,

etc.

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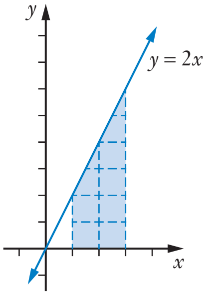
Hence the use of the word integration in mathematics when we are summing strips of area to make a unified whole. However, so useful is the fact that this *limit of a sum*, that we call integration, can be determined by antidifferentiating, that when asked to determine a definite integral we usually simply antidifferentiate automatically without thinking of the area context at all. For example, when question **3** part **a** of the previous Miscellaneous exercise asked you to evaluate

$$\int_1^3 2x\,dx$$

it is likely that you proceeded algebraically, as shown below left, and not by considering areas, as shown below right.

$$\begin{aligned}\int_1^3 2x\,dx &= \left[x^2\right]_1^3\\ &= 3^2 - 1^2\\ &= 8\end{aligned}$$



Thus, asked to determine $\int_1^2 (3x^2 + 4)\,dx$ we do not need to consider limiting sums but can instead simply use our ability to antidifferentiate $3x^2 + 4$:

$$\begin{aligned}\int_1^2 (3x^2 + 4)\,dx &= \left[x^3 + 4x\right]_1^2\\ &= [(2)^3 + 4(2)] - [(1)^3 + 4(1)]\\ &= 16 - 5\\ &= 11\end{aligned}$$

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Note

- Remember that we do not need to include the '+ c' in evaluating a *definite* integral because, were we to include it, it would eventually cancel itself out.

  However, if we are asked to determine $\int (3x^2 + 4)\,dx$, we now have an *indefinite* integral and the $+\,c$ would need to be included.

  Thus whilst

  $$\begin{aligned}\int_1^2 (3x^2 + 4)\,dx &= \left[x^3 + 4x\right]_1^2 \\ &= 16 - 5 \\ &= 11,\end{aligned}$$

  $$\int (3x^2 + 4)\,dx = x^3 + 4x + c.$$

- If we write $\int_a^b f(x)\,dx$, the answer is dependent only on the function and the values of a and b. The letter x is a '*dummy*' variable. We could replace it with a different letter and still obtain the same answer.

  For example, whilst we found $\int_1^2 (3x^2 + 4)\,dx = 11$ it is also the case that

  $$\int_1^2 (3t^2 + 4)\,dt = 11 \qquad \int_1^2 (3u^2 + 4)\,du = 11 \qquad \int_1^2 (3p^2 + 4)\,dp = 11$$

Exercise 5A

For questions **1** to **3** first answer each question without the assistance of your calculator then, if you wish, confirm your answers with a calculator.

1 **a** Determine $\int (12t^2 + 6t)\,dt$

b Determine $\int_1^x (12t^2 + 6t)\,dt$

c Determine $\frac{d}{dx}\left(\int_1^x (12t^2 + 6t)\,dt\right)$

2 **a** Determine $\int \left(1 - \frac{1}{t^2}\right)dt$

b Determine $\int_3^x \left(1 - \frac{1}{t^2}\right)dt$

c Determine $\frac{d}{dx}\left(\int_3^x \left(1 - \frac{1}{t^2}\right)dt\right)$

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3 a Determine $\int 2t(t^2+3)^4\,dt$

b Determine $\int_{-2}^{x} 2t(t^2+3)^4\,dt$

c Determine $\frac{d}{dx}\left(\int_{-2}^{x} 2t(t^2+3)^4\,dt\right)$

4 Use your answers to questions **1**, **2** and **3** to complete the following:

$$\frac{d}{dx}\left(\int_{a}^{x} f(t)\,dt\right) = ????$$

Use the result from question **4** to determine each of the following, then check your answer on your calculator if you wish.

5 $\frac{d}{dx}\left(\int_{1}^{x} 4t\,dt\right)$

6 $\frac{d}{dx}\left(\int_{0}^{x} 5t^2\,dt\right)$

7 $\frac{d}{dx}\left(\int_{5}^{x} 2t^3\,dt\right)$

8 $\frac{d}{dx}\left(\int_{0}^{x} \frac{2t}{5-t}\,dt\right)$

9 $\frac{d}{dx}\left(\int_{5}^{x} (t+3)^4\,dt\right)$

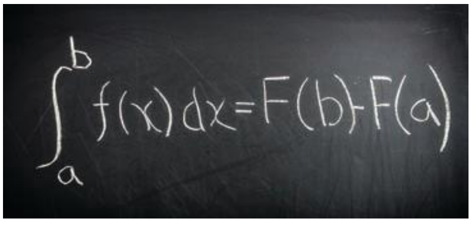
10 $\frac{d}{dx}\left(\int_{0}^{x} 16t(t^2+3)^4\,dt\right)$

The fundamental theorem of calculus

The fact that we can determine definite integrals using antidifferentiation, rather than referring back to their definition as a limiting sum, is very useful, as we found when determining the area under a curve.

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a)$$ where $F(x)$ is the antiderivative of $f(x)$.

Indeed so important is the fact that the limit of a sum, which we call a definite integral, can be determined using antidifferentiation it is referred to as **the fundamental theorem of calculus**.



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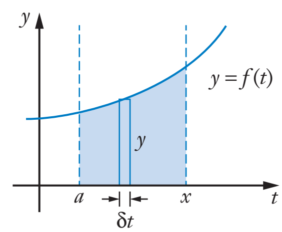
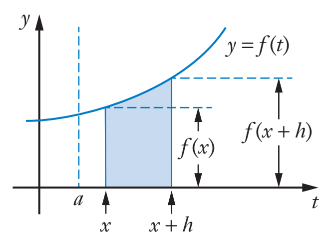
Justification of the fact that the area under a curve, i.e. the limit of the sum of many rectangles, can be obtained using antidifferentiation, is given below.



Consider the area under a curve from some fixed left hand boundary to a variable right hand boundary. The area, A, under the curve will then be a function of the variable right hand boundary. Suppose the curve is $y = f(t)$, the left hand boundary is $t = a$ and the right hand variable boundary is $t = x$.

$$A(x) = \lim_{\delta t \to 0} \sum_{t=a}^{t=x} y\,\delta t$$

$$A(x) = \int_a^x f(t)\,dt \qquad [1]$$



For the diagram on the right:

The shaded area $= A(x+h) - A(x)$

But $hf(x) < A(x+h) - A(x) < hf(x+h)$

Thus $f(x) < \dfrac{A(x+h) - A(x)}{h} < f(x+h)$

I.e., the expression $\dfrac{A(x+h) - A(x)}{h}$ lies between $f(x)$ and $f(x+h)$.

Thus, as $h \to 0$ $\quad \dfrac{A(x+h) - A(x)}{h} \to f(x)$, $\quad$ i.e., $\displaystyle\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$

Therefore $\quad A'(x) = f(x) \qquad [2]$

Hence $\quad A(x) = F(x) + c \qquad$ where $F(x)$ is an antiderivative of $f(x)$.

But the area from $x = a$ to $x = a$ must be zero thus:

$0 = F(a) + c \qquad$ giving $\quad c = -F(a)$

and so $\quad A(x) = F(x) - F(a)$

∴ Area from a to b is $\quad A(b) = F(b) - F(a)$.

Thus

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where } F(x) \text{ is the antiderivative of } f(x).$$

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Alternatively the previous boxed result can be written:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Equation [1] on the previous page stated that $A(x) = \int_a^x f(t)\,dt.$

Equation [2] on the previous page stated that $A'(x) = f(x).$

From these it follows that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

(A statement you may already be familiar with from Exercise 5A question **4**.)

Thus $\int_a^b f'(x)\,dx = f(b) - f(a)$ and $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

From the rule above left we see that integrating the derivative of a function 'gives us the function back'.

and from the rule above right, differentiating the integral of a function 'gives us the function back'.

Hence the limit of a sum, which we call a definite integral, can be evaluated using antidifferentiation because integration and differentiation are opposite processes. This is what the **fundamental theorem of calculus** is all about.

The two boxed results above show the opposite nature of this relationship between the definite integral and differentiation. They are the two parts of the **fundamental theorem of calculus**. You used the rule above right to determine the answers to questions **5** to **10** of Exercise 5A.

Whilst the definite integral is the limit of a sum the fact that this integration process is the opposite of differentiation means that we tend use the integration symbol,

$$\int$$

without any values for a and b, to mean 'find the antiderivative of' or, simply, 'integrate', as you are already accustomed to doing.

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EXAMPLE 1

Find $\frac{d}{dx}\left(\int_0^x \frac{1+t^2}{2}\,dt\right)$.

Solution

With differentiation and integration being opposite processes we do not actually have to perform the integration, substitute in the values and then differentiate.

The fundamental theorem states that $$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Therefore $$\frac{d}{dx}\left(\int_0^x \frac{1+t^2}{2}\,dt\right) = \frac{1+x^2}{2}$$

$$\frac{d}{dx}\left(\int_0^x \frac{1+t^2}{2}dt\right)$$

$$\frac{x^2+1}{2}$$

EXAMPLE 2

Integrate $\frac{d}{dx}(x^3+5x-1)$ with respect to x.

Solution

With integration and differentiation being opposite processes we do not actually have to perform the differentiation as we would then only integrate our answer, and thereby obtain the initial function back again. However we do have to remember that an indefinite integral will give us a '+ c'.

Thus $$\int\left(\frac{d}{dx}(x^3+5x-1)\right)dx = x^3+5x+c$$

$$\int\left(\frac{d}{dx}(x^3+5x-1)\right)dx$$

$$x^3+5x$$

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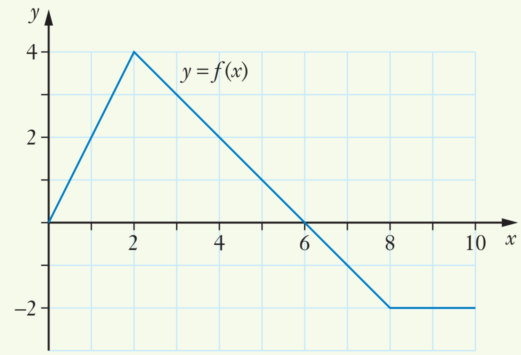
EXAMPLE 3

For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right.



Determine:

a $f(2)$ **b** $f(7)$

c $\int_0^4 f(x)\,dx$ **d** $\int_2^{10} f(x)\,dx$

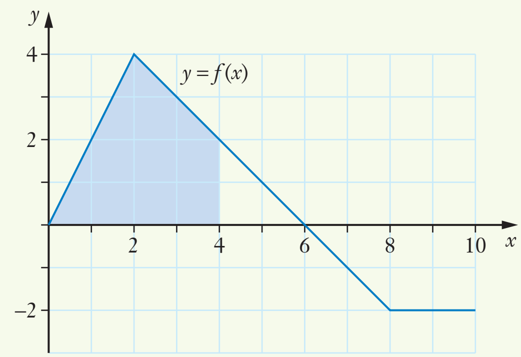
Solution

a When $x = 2, y = 4$. Thus $f(2) = 4$.

b When $x = 7, y = -1$. Thus $f(7) = -1$.

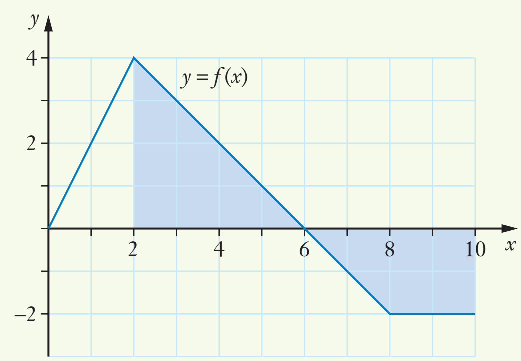
c Interpreting the definite integral as the signed area under a curve:

$$\int_0^4 f(x)\,dx = 10$$



d Similarly

$$\begin{aligned}\int_2^{10} f(x)\,dx &= (8) + (-6) \\ &= 2\end{aligned}$$



Exercise 5B

1 Find $\frac{d}{dx}\left(\int_0^x (2t + 3t^2)\,dt\right)$

2 Find $\frac{d}{dx}\left(\int_1^x (t^4 + 5)\,dt\right)$

3 Integrate the following with respect to x.

a $\frac{d}{dx}(x^2+5)$

b $\frac{d}{dx}(6x^3-4x^2+2x+1)$

4 For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right.

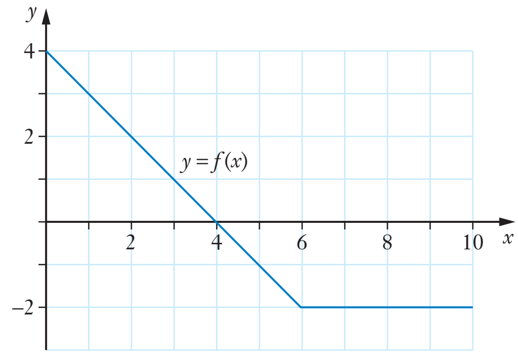
Determine:

a $f(3)$

b $f(6)$

c $\int_0^4 f(x)\,dx$

d $\int_0^{10} f(x)\,dx$



5 For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right.

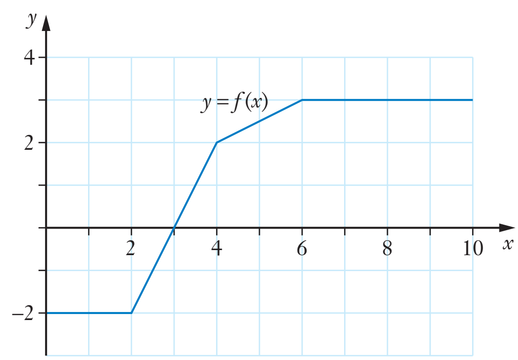
Determine:

a $f(1)$

b $f(7)$

c $\int_4^6 f(x)\,dx$

d $\int_1^8 f(x)\,dx$



6 For $0 \le x \le 10$ the function $y = f(x)$ is as defined by the graph on the right.

Determine:

a $\int_0^2 f(x)\,dx$

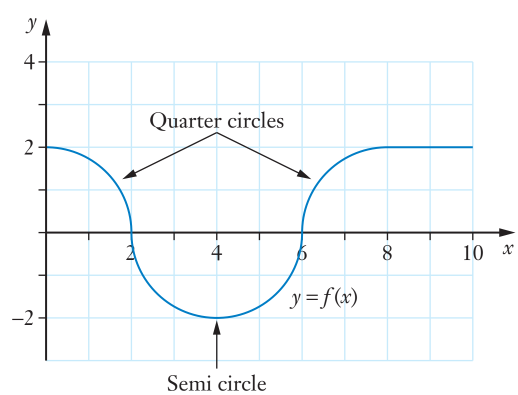
b $\int_2^{10} f(x)\,dx$

c For what values of a, $0 \le a \le 10$, is

$$\int_0^a f(x)\,dx = 0$$

d For what values of a, $0 \le a \le 10$, is

$$\int_0^a f(x)\,dx < 0.$$



7 Find $\frac{dy}{dx}$ given that $y = x^2\sqrt{x} + \int_0^x (1+3t^2)^4\,dt$

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Miscellaneous exercise five

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

For numbers **1** to **8** differentiate the given expression with respect to x.

1 $2x^3 + \sqrt{x}$

2 $(x + 5)(x - 3)$

3 $(3x - 1)^2$

4 $(3x - 1)^5$

5 $(5x - 1)(2x^3 - 3)$

6 $(5x - 1)(2x - 3)^3$

7 $\dfrac{2x + 3}{x - 1}$

8 $\dfrac{x - 1}{2x + 3}$

9 Clearly showing your use of the product rule, determine the gradient of the curve

$$y = (x - 1)(x^2 - 2)$$

at the point (0, 2).

10 (Without the assistance of a calculator.)

Use calculus techniques to determine the exact coordinates, and the nature, of any stationary points on the curve $y = x + \dfrac{6}{x}$.

11 **a** Describe what each of the responses 3, 28, 5 and 2 shown in the display on the right is informing us about the function $f(x)$ making sure that one of your statements includes the phrase

the average rate of change of $f(x)$

and another includes the phrase

the instantaneous rate of change of $f(x)$.

b If the $f(x)$ that gave rise to the display on the right is a quadratic function,

i.e. $f(x) = ax^2 + bx + c,\ a \neq 0$,

find $f(x)$.

c If instead the $f(x)$ that gave rise to the display shown is a cubic function,

i.e. $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$,

find a possible $f(x)$.

$f(0)$	3
$f(5)$	28
$\dfrac{f(5) - f(0)}{5}$	5
$\lim\limits_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)\Bigg\vert x = 1$	2

12 Find y in terms of x given that $\frac{dy}{dx} = 50(2x+1)^4$ and $y = 7$ when $x = 0$.

13 Find $f(x)$ given that $f''(x) = 144(2x-1)^2$, $f'(1) = 26$ and $f(1) = 6$.

14 (Without the use of your calculator.)

Find an expression for the marginal revenue if the total revenue, $\$R$, from the sale of x items is given by:

$$R = 30x - 0.02x^2.$$

Find the marginal revenue when $x = 100$.

By approximately how much will the total revenue increase due to the sale of the 101st item?

15 Without the assistance of your calculator:

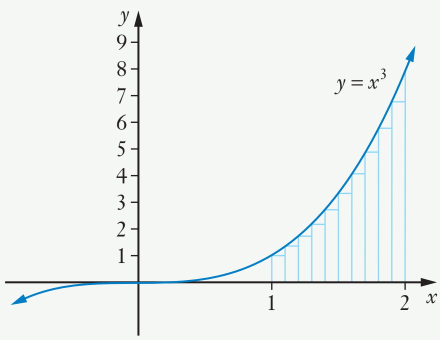
a Produce a sketch of $y = (x+1)(x-2)^2$ showing the coordinates of any places where the curve cuts or touches the axes and of all stationary points and points of inflection.

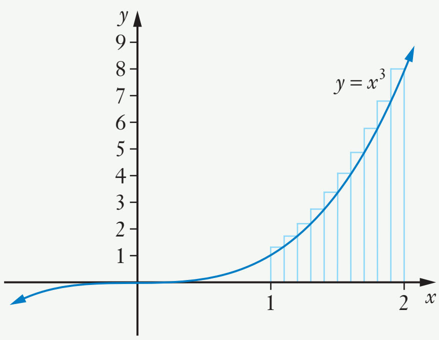
b Find the area enclosed by the x-axis and the curve $y = (x+1)(x-2)^2$.

16 Using ten rectangles in each case, as shown below, find an underestimate (using inscribed rectangles) and an overestimate (using circumscribed rectangles) for the area under

$$y = x^3 \text{ from } x = 1 \text{ to } x = 2$$

and determine the mean of these two estimates.





17 The radius of a sphere is measured as 5 cm ± 0.1 cm. Use differentiation to find the volume of the sphere in the form V cm^3 ± b cm^3 giving V and b to the nearest integer.

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6.

The exponential function

- Growth and decay
- The derivative of e^x
- More on growth and decay
- Integrating exponential functions
- Miscellaneous exercise six

Situation

The three Ripoff brothers, Muscles, Brains and Sneaky, run 'The Concrete Boot Loan Company'. They charge 100% per annum on all loans, with the interest compounded annually!

If you borrow $100 for one year then at the end of the year Muscles comes to collect what you owe. How much would that be?

> '*Hey*,' said Brains, '*I think we need to raise our interest rate. We're being too easy on the customers.*'
> '*Yeah*,' agreed Sneaky, '*But we need to do it without them realising we've done it.*'
>
> '*Ow you gonna do that Sneaky?*' asked Brains.
>
> '*Well, what if we still charge 100% per annum but compound it every six months. That way we charge 50% per six months which sounds the same but in fact we'll get more.*'

How much would Muscles be sent to collect after one year if you have borrowed $100 for the year under this new scheme?

> '*But if that's gonna get us more, why not charge 100% per annum compounded quarterly (i.e. four times per year)?' chipped in Muscles.*
>
> Brains and Sneaky stared at Muscles in amazement. He was never one to put forward sensible suggestions but what he said seemed to have some merit.

How much would you owe at the end of the $100 one year loan under this new scheme?

> Sneaky had been thinking.
>
> '*Well if Muscles' idea is okay, why not compound monthly, or weekly, or daily, or even by the hour, or the minute!! A customer borrowing $100 would still be thinking they'd be paying 100% per annum but if we compound weekly or daily or hourly they'd owe us a fortune at the end of the year!*'
>
> '*If only we could compound continuously*,' said Sneaky, '*we'd make heaps.*'

How does it affect what you owe at the end of a $100 one-year loan as this compounding period is reduced? Investigate and write a brief report of your findings.

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The situation on the previous page involved the growth of a loan. Did you find that the number 2.718 28 (approximately), played an important part?

You should have found that

- if $\$P$ is borrowed for one year at 100% per annum, compounded n times in the year, the amount owed at the end of the year is $P\left(1+\frac{1}{n}\right)^n$.

and that

- as $n \to \infty$ then $\left(1+\frac{1}{n}\right)^n \to 2.71828$, correct to five decimal places.

Hence if we compound an infinite number of times in the year, i.e. **continuous** compounding, then the amount owed at the end of the year will be $\$P \times 2.71828$.

If the loan continues for another year the amount owed would be $\$P \times (2.71828)^2$.

A third year and the amount owed would be $\$P \times (2.71828)^3$ etc.

Of course most financial institutions would not charge 100% interest per annum! However, this number, 2.718 28, also arises when other rates are charged.

If the rate was 7% per annum it can be shown that with continuous compounding the amount owed at the end of 1 year would be $\$P \times (2.71828)^{0.07}$. After 2 years it would be $\$P \times (2.71828^2)^{0.07}$, after 3 years $\$P \times (2.71828^3)^{0.07}$ etc.

This number, 2.718 28, came from consideration of $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]$.

We call this limiting value '*e*':

> e is defined to be $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]$ and is approximately 2.718 28.

- Many calculators have an e^x button. Use your calculator to confirm that

$$e^1 \approx 2.71828, \quad e^2 \approx 7.38906, \quad e^{-0.5} \approx 0.60653.$$

- Investigate $\lim_{n\to\infty}\left[\left(1+\frac{a}{n}\right)^n\right]$ for $a \neq 1$.

The repetitive multiplication by some constant gives rise to expressions in which the variable appears as an index, power or *exponent*, e.g. 2^x, 3^{n-1} etc.

In this chapter we are particularly interested in **exponential expressions** having a **base** of e, for example e^x, e^{2x}, e^{2t}, $e^{-0.4t}$, etc.

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The constant e (≈ 2.71828) allows us to describe mathematically many situations involving that amazing phenomenon – *growth*.

- Population, spread of disease, investments, demand for a resource such as oil are all examples in which the growth can be exponential.
- Radioactivity, the temperature of an object placed in cooler surroundings, the concentration of a drug in the bloodstream are all examples in which the decay can be exponential.

The number e can often be used to describe these situations mathematically.

Growth and decay

Many growth and decay situations involve some variable, say A, growing or decaying continuously, according to a rule of the form $A = A_0e^{kt}$

where A is the amount present at time t,

A_0 is the initial amount (i.e. the amount present at $t = 0$),

and k is some constant dependent on the situation.

EXAMPLE 1

A certain culture of bacteria grows in such a way that t days after observation commences the number of bacteria present, N, is given by:

$$N \approx 2000e^{0.75t}.$$

Determine the number of bacteria present

a when observation commenced,

b three days after observation commenced,

c ten days after observation commenced.

Solution

a If $t = 0$ $\quad N \approx 2000e^{0.75 \times 0}$
$\quad = 2000$

When observation commenced there were approximately 2000 bacteria.

b If $t = 3$ $\quad N \approx 2000e^{0.75 \times 3}$
$\quad \approx 18975$

Three days after observation commenced there were approximately 19 000 bacteria present.

$2000e^{0.75\times3}$
18975.47167
$2000e^{0.75\times10}$
3616084.829

c If $t = 10$ $\quad N \approx 2000e^{0.75 \times 10}$
$\quad \approx 3616085$

Ten days after observation commenced there were approximately 3 600 000 bacteria present.

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EXAMPLE 2

If $1000 is invested at 12% per annum interest, compounded continuously, the investment will be worth $$A$ after t years where

$$A = 1000e^{0.12t}.$$

Find the value of t, correct to one decimal place, for which the value of the investment is $8000.

Solution

We are given $\quad A = 1000e^{0.12t}$

If $A = 8000$ then $\quad 8000 = 1000e^{0.12t}$

Using the solve facility on a calculator

$t = 17.3$ (correct to 1 decimal place)

Thus the value of the investment is $8000 when $t = 17.3$ (correct to 1 decimal place).

solve $(8 = e^{0.12 \times t}, t)$

$\{t = 17.32867951\}$

Note: When solving the equation of the previous question, some calculators, if set to give exact answers, may give the answer as 25 ln (2). This exact form uses the idea of a 'logarithm', a concept we will meet in the next unit of *Mathematics Methods*. For now simply have your calculator output the decimal answer.

Exercise 6A

1 If $1000 is invested at 12% per annum interest, compounded continuously, the investment will be worth $$A$ after t years where

$$A = 1000e^{0.12t}.$$

Find the value of this investment after

a 5 years,

b 10 years,

c 25 years.

2 If $$P$ is invested at 8% per annum interest, compounded continuously, the investment grows to $$Pe^{0.08t}$ after t years.

If the investment is worth $27 819.26 after ten years, find P.

3 A scientific experiment starts with 100 grams of a particular radioactive element. The element decays such that the amount present t hours later is $100e^{-0.03t}$.

How many grams of the element will have decayed after ten hours?

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4 Sales of a particular chocolate bar increase whilst an advertising campaign is in progress. At a time of t weeks after the campaign ceases, the sales have fallen to S bars per week, where $S \approx 2\,000\,000e^{-0.15t}$.

Determine the number of bars sold per week

a when the campaign ceases,

b 2 weeks after the campaign ceases,

c 4 weeks after the campaign ceases,

d 6 weeks after the campaign ceases.

5 A freefalling object falls such that its downward speed, t seconds after release, is given by v m/s where

$$v = 75(1 - e^{-0.13t}) \text{ m/s.}$$

Find the downward speed of the body after

a 5 seconds, **b** 20 seconds, **c** 40 seconds.

6 If $Y = 20 + \dfrac{40}{e^{0.05x}}$ find x (correct to two decimal places if necessary) given that

a $Y = 60$, **b** $Y = 30$, **c** $Y = 21$.

7 A disease is spreading through a particular community of people such that N, the number of people infected t days after the first reported case, is given by

$$N = \frac{3000}{1 + 2999e^{-0.4t}}.$$

After how many days should it be expected that 1000 people in this community are infected with the disease?

8 If a payment of $\$P$ is made every year into an account that attracts a fixed interest rate of r% per annum compounded continuously and the account is closed t years later the balance due will be:

$$\frac{P(e^{0.01rt} - 1)}{1 - e^{-0.01r}}.$$

a Find the balance due after 10 years if \$2000 is invested each year and the interest rate is a fixed 10% per annum compounded continuously.

b Find the number of years the scheme must run if an investor wants to invest \$3000 per year and close the account when the balance reaches \$154000, assuming a constant interest rate of 8% per year compounded continuously.

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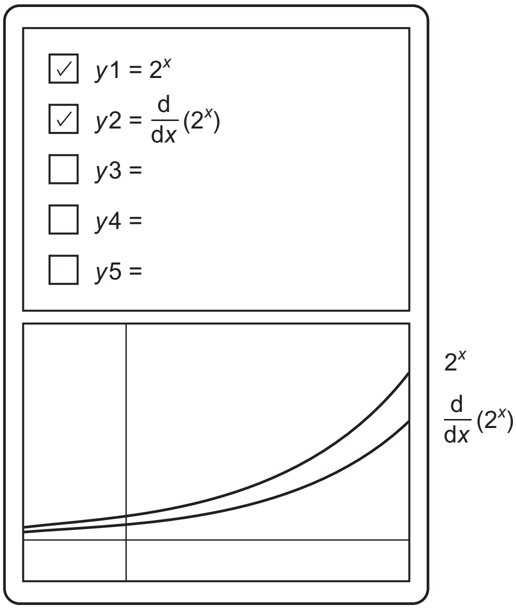
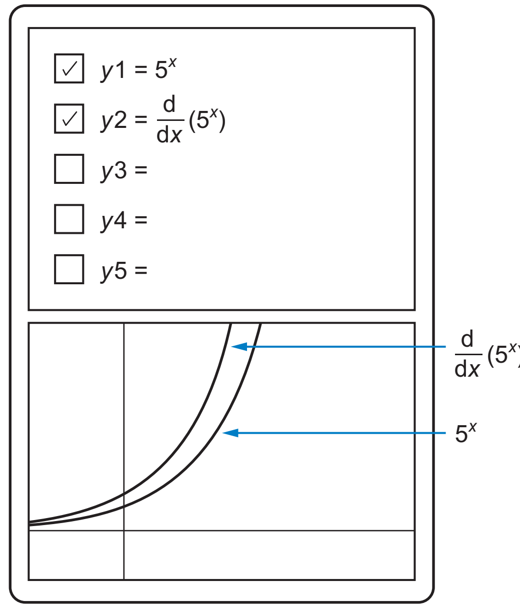
The derivative of e^x

The display below left shows both $y = 2^x$ and $y = \frac{d}{dx}(2^x)$. Notice that the graph of the derivative also appears to be an exponential function but it lies 'below' that of $y = 2^x$.

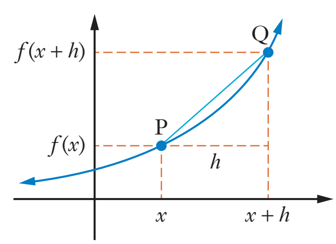
The display below right shows both $y = 5^x$ and $y = \frac{d}{dx}(5^x)$. Notice that the graph of the derivative again appears to be an exponential function but this time the graph of the derivative is 'above' that of $y = 5^x$.





This suggests that for some value of a, between $a = 2$ and $a = 5$, the graphs of $y = a^x$ and $y = \frac{d}{dx}(a^x)$ coincide. I.e. for some value of a between $a = 2$ and $a = 5$, $\frac{d}{dx}(a^x) = a^x$.

However, as the *Preliminary work* mentioned, if we want to differentiate a function for which we do not already have a rule, for example $f(x) = a^x$, we go back to the basic 'limiting chord process'.



i.e. Gradient at P, see diagram, $= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

For $f(x) = a^x$,

$$\frac{d}{dx}(a^x) = \lim_{h \to 0}\left[\frac{a^{x+h} - a^x}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{a^x(a^h - 1)}{h}\right]$$

$$= a^x \lim_{h \to 0}\left[\frac{a^h - 1}{h}\right] \qquad [1]$$

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Examining values of $\frac{a^h - 1}{h}$, shown below correct to 7 decimal places, for various values of a, and as h gets smaller and smaller:

	$\frac{a^h - 1}{h}$		
h	$a = 2$	$a = 2.5$	$a = 3$
1	1.000 000 0	1.500 000 0	2.000 000 0
0.1	0.717 734 6	0.959 582 3	1.161 231 7
0.01	0.695 555 0	0.920 501 5	1.104 669 2
0.001	0.693 387 5	0.916 710 7	1.099 216 0
0.0001	0.693 171 2	0.916 332 7	1.098 672 6
0.000 01	0.693 149 6	0.916 294 9	1.098 618 3
0.000 001	0.693 147 4	0.916 291 2	1.098 612 9
0.000 000 1	0.693 147 2	0.916 290 8	1.098 612 3

Thus, once again, as the graphs on the previous page suggested, there exists a value of a between 2 and 5, and indeed now between 2.5 and 3, for which

$$\lim_{h \to 0}\left[\frac{a^h - 1}{h}\right] = 1$$

and hence for this value of a, from equation [1] on the previous page,

$$\frac{d}{dx}(a^x) = a^x.$$

Given that the above discussion follows on from pages introducing 'e', it probably didn't take you long to realise that the value of a for which the derivative of a^x is itself, is the number e.

Consider $\lim_{h \to 0}\left[\frac{e^h - 1}{h}\right]$:

If $h = 1$ $\quad \frac{e^h - 1}{h} \approx 1.718\,28$

If $h = 0.1$ $\quad \frac{e^h - 1}{h} \approx 1.051\,71$

If $h = 0.01$ $\quad \frac{e^h - 1}{h} \approx 1.005\,02$

If $h = 0.001$ $\quad \frac{e^h - 1}{h} \approx 1.000\,50$

If $h = 0.0001$ $\quad \frac{e^h - 1}{h} \approx 1.000\,05$

The above figures suggest that $\lim_{h \to 0}\left[\frac{e^h - 1}{h}\right] = 1$ and so, from [1], $\frac{d}{dx}(e^x) = e^x$.

Thus if $y = e^x$ then $\frac{dy}{dx} = e^x$. The exponential function, e^x, differentiates to itself!

If $y = e^x$ then $\frac{dy}{dx} = e^x$.

EXAMPLE 3

Differentiate

a $x^3 + e^x$

b $5e^x$

c $e^{x^2 - 5x + 1}$

Solution

a If $y = x^3 + e^x$

$$\frac{dy}{dx} = 3x^2 + e^x$$

b If $y = 5e^x$

$$\frac{dy}{dx} = 5e^x$$

c If $y = e^{x^2 - 5x + 1}$

Let $u = x^2 - 5x + 1$ then $y = e^u$.

$\frac{du}{dx} = 2x - 5$ and $\frac{dy}{du} = e^u$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{(Chain rule)}$$

$$= e^u(2x - 5)$$

$$= (2x - 5)e^{x^2 - 5x + 1}$$

$\frac{d}{dx}(x^3 + e^x)$ $\quad e^x + 3\cdot x^2$

$\frac{d}{dx}(5e^x)$ $\quad 5\cdot e^x$

$\frac{d}{dx}(e^{x^2-5x+1})$ $\quad (2\cdot x - 5)e^{x^2-5\cdot x+1}$

The general statement of example 3 part **c** is:

If $y = e^{f(x)}$ then, by the chain rule, $\frac{dy}{dx} = f'(x)e^{f(x)}$.

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EXAMPLE 4

Differentiate **a** e^{5x-2} **b** e^{x^2+x} **c** x^2e^x.

Solution

a If $y = e^{5x-2}$

$$\frac{dy}{dx} = 5e^{5x-2}$$

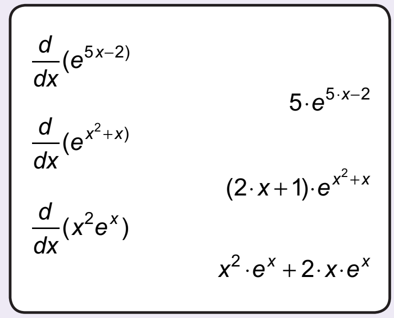
b If $y = e^{x^2+x}$

$$\frac{dy}{dx} = (2x+1)\,e^{x^2+x}$$

c If $y = x^2e^x$

$$\frac{dy}{dx} = e^x(2x) + x^2(e^x)$$

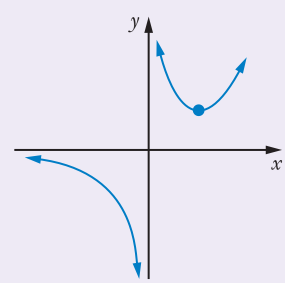
$$= xe^x(2+x)$$



EXAMPLE 5

The sketch on the right shows part of the curve $y = \frac{e^x}{x}$.

Use calculus to prove that the local minimum shown is the only stationary point on the curve and to determine its exact location.



Solution

If $y = \frac{e^x}{x}$ then, using the quotient rule,

$$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$$

$$= \frac{e^x(x-1)}{x^2}$$

Thus $\frac{dy}{dx} = 0$ for $x = 1$ (and *only* for $x = 1$).

Now if $x = 1$ then $y = e$.

The minimum point shown in the diagram is the only stationary point on the curve and it has coordinates $(1, e)$.

ISBN 9780170395137

Exercise 6B

Differentiate each of the following with respect to x.

1 e^x	**2** $7e^x$	**3** $3e^x$
4 $6e^x$	**5** $9e^x$	**6** $-8e^x$
7 e^{5x}	**8** e^{7x}	**9** e^{-2x}
10 $5e^{3x}$	**11** $4e^{0.5x}$	**12** $-2e^{-0.5x}$
13 $6e^x + 2x^3 + 3x^2$	**14** $2e^x + \sqrt{x}$	**15** $e^{5x} + e^{2x}$
16 $2e^{4x}$	**17** $2e^{3x} + 3e^{2x}$	**18** $5e^{3x} + x^4$
19 e^{3x-1}	**20** e^{x^2+3}	**21** e^{5x-1}
22 e^{3x^2+2x-1}	**23** e^{x^3}	**24** xe^{2x}
25 x^3e^x	**26** $e^x\sqrt{x}$	**27** $\dfrac{e^x}{2x}$
28 $e^x(1+2x)^3$	**29** $e^x(1-2x)^5$	**30** $\dfrac{1}{e^{3x}}$

31 Find the exact gradient of $y = e^{2x} + x^2$ at the point $(1, e^2 + 1)$.

32 Find the exact gradient of $y = xe^x$ at the point $(1, e)$.

33 Find the equation of the tangent to $y = 5e^{2x}$ at the point $(0, 5)$.

34 If \$100 is invested at 8% per annum, compounded continuously the account grows to $\$100e^{0.08t}$ after t years. What is the instantaneous rate of growth, in dollars per year correct to two decimal places, when

a $t = 1$? **b** $t = 10$? **c** $t = 20$? **d** $t = 40$?

35 Damage to a poorly-maintained grain store causes the marketable weight of grain in the store to fall from its initial amount of A_0 tonnes to an amount A_t, t weeks later, according to the rule:

$$A_t = 100e^{-0.1t} \text{ tonnes}$$

a Determine A_0.

b What is the marketable weight of grain in the store when $t = 5$? (Answer to the nearest tonne.)

At what rate is A falling, in tonnes per week, when

c $t = 2$? **d** $t = 5$? **e** $t = 8$?

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More on growth and decay

Notice that if $y = Ae^{kt}$ then $\frac{dy}{dt} = kAe^{kt}$

i.e. $\frac{dy}{dt} = ky.$ [1]

Thus, from equation [1], in functions of the form $y = Ae^{kt}$ the rate of change of y with respect to t is proportional to y itself. This sentence is repeated below. Read it again carefully to take in what it means:

In functions of the form $y = Ae^{kt}$ the rate of change of y with respect to t is proportional to y itself.

What this sentence is telling us explains why functions of the form $y = Ae^{kt}$ describe growth or decay situations. A population will tend to reproduce itself at a rate proportional to its size and will continue this constant proportion unless some special factors are introduced that may stimulate or inhibit growth. If country A has a larger population than country B then we would expect the number of babies born in country A in one year to be more than the number of babies born in country B in that year, all other factors being equal.

It is the fact that functions of the form $y = Ae^{kt}$ are such that $\frac{dy}{dt} = ky$ that makes them suitable functions for describing many growth and decay situations.

Any growth or decay situation in which the rate of change of the population is proportional to the population itself, i.e. $\frac{dP}{dt} = kP$, can be modelled by an equation of the form $P = P_0e^{kt}$ where P_0 is the population at time $t = 0$.

Thus: If $\frac{dP}{dt} = kP$ then $P = P_0e^{kt}$ where P_0 is the value of P when $t = 0$.

Or, in terms of x and y:

If $\frac{dy}{dx} = ky$ then $y = y_0e^{kx}$ where y_0 is the value of y when $x = 0$.

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EXAMPLE 6

Demographers monitored a particular country's population growth over a 30-year period from 1985, when the population was 2 000 000. They found that the population was continuously growing with the instantaneous rate of increase in the population per year, $\frac{dP}{dt}$, always close to $\frac{P}{20}$.

a Estimate the population of this country at the end of the 30-year period.

b If this pattern of growth continues estimate the population in the years 2025, 2040 and 2065.

Solution

a Let the population t years after 1985 be P.

> **Remember**
>
> If $\frac{dP}{dt} = kP$ then $P = P_0e^{kt}$

We are told that $\frac{dP}{dt} \approx 0.05P$

Hence $P = P_0e^{0.05t}$

Taking $t = 0$ at 1985 then $P_0 = 2\,000\,000$, the $t = 0$ population.

Thus $P = 2\,000\,000e^{0.05t}$

When $t = 30$ $P = 2\,000\,000e^{0.05(30)}$

$\approx 8\,960\,000$

The population of this country at the end of the 30-year period was approximately nine million.

b By 2025, $t = 40$ and so $P = 2\,000\,000e^{0.05(40)}$

$\approx 15\,000\,000$

By 2040, $t = 55$ and so $P = 2\,000\,000e^{0.05(55)}$

$\approx 31\,000\,000$

By 2065, $t = 80$ and so $P = 2\,000\,000e^{0.05(80)}$

$\approx 109\,000\,000$

Assuming the pattern of growth continues the population estimates for 2025, 2040 and 2065 would be 15 million, 31 million and 109 million respectively.

If the situation involves a quantity **decaying** rather than growing then the rate of change of the quantity with respect to time will be negative, rather than positive. (See the next example.)

ISBN 9780170395137

EXAMPLE 7

A particular radioactive isotope decays continuously at a rate of 9% per year. One kilogram of this isotope is produced in a particular industrial process. How much remains undecayed after 20 years?

Solution

If A kg remains undecayed after t years then $\frac{dA}{dt} = -0.09A$

This is of the form $\frac{dA}{dt} = kA$ and so $A = A_0e^{-0.09t}$

When $t = 0$, $A = 1$. Thus $A = 1e^{-0.09t}$

When $t = 20$ $A = e^{-0.09 \times 20}$

≈ 0.165

Approximately 165 grams remain undecayed after 20 years.

EXAMPLE 8

A savings account is opened with a deposit of \$400 and attracts interest at a rate of 8% per annum compounded continuously.

a If the interest rate is maintained for five years what will be the balance of the account at the end of this time?

b How many years (correct to one decimal place) will it take for the balance in the account to be treble the initial deposit?

Solution

a The principal grows continuously at 8% p.a. $\therefore$ $\frac{dP}{dt} = 0.08P$.

This is of the form $\frac{dP}{dt} = kP$ and so $P = P_0e^{0.08t}$

When $t = 0$, $P = 400$. Thus $P = 400e^{0.08t}$

When $t = 5$ $P = 400e^{0.08 \times 5}$

$\approx \$596.73$

After five years the account balance will be \$596.73.

b If $P = 1200$ then $1200 = 400e^{0.08t}$

i.e. $3 = e^{0.08t}$

Solving with a calculator gives $t = 13.7$ (1 decimal place)

The initial deposit will treble after approximately 13.7 years.

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Exercise 6C

For this exercise use the fact that if $\frac{dP}{dt} = kP$ then $P = P_0e^{kt}$.

1 If $\frac{dA}{dt} = 2.5A$ and $A = 50$ when $t = 0$, find A when **a** $t = 1$, **b** $t = 3$.

2 If $\frac{dP}{dt} = 0.01P$ and $P = 2000$ when $t = 0$, find P when **a** $t = 10$, **b** $t = 50$.

3 If $\frac{dQ}{dt} = \frac{3Q}{100}$ and $Q = 150$ when $t = 0$, find Q when **a** $t = 2$, **b** $t = 25$.

4 If $\frac{dA}{dt} = -0.1A$ and $A = 20\,000$ when $t = 0$, find A when **a** $t = 10$, **b** $t = 20$.

5 If $\frac{dX}{dt} = \frac{X}{2}$ and $X = 6$ million when $t = 5$, find X when **a** $t = 10$, **b** $t = 20$.

6 If $\frac{dP}{dt} = 0.025P$ and $P = 2000$ when $t = 10$, find P when **a** $t = 11$, **b** $t = 20$.

7 A particular country has a population of 250 million. Records indicate that the population growth rate is 3% per year, i.e. $\frac{dP}{dt} = 0.03P$.

Estimate the population of the country after a further **a** 10 years, **b** 50 years.

8 Repeat question 7 but now for a growth rate of 2.5%.

9 A particular radioactive isotope decays continuously at a rate of 12% per year. Three kilograms of this isotope are produced in a particular industrial process. How much remains undecayed after 20 years?

10 A 30-year-old person makes a 'one-off' payment of $5000 to a savings plan with the intention of leaving it untouched for 25 years. If the investment attracts a fixed guaranteed interest rate of 11% per annum, compounded continuously, find the value of the investment at the end of the 25 years.

11 How much does a person need to deposit in an account attracting a constant interest rate of 12% per annum, compounded continuously, for it to have grown to $20 000 after 20 years?

12 Let us suppose that the cost of goods is rising continuously at 5% per annum. The rate of change in the cost of an article costing \$P would then be such that

$$\frac{dP}{dt} = 0.05P.$$

Under these conditions what would be the cost in 100 years of a chocolate bar now costing 80 cents?

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13 Repeat question **12** if costs rise continuously at 8% per year rather than 5%.

14 The instantaneous rate of decline of a particular species of frog is 5% per month.
If the current population is 10 000 what will it be in

a 5 months? **b** 10 months?

15 The population of a particular country varies with time. The rate of change of the population from the year 2000 onwards was found to be proportional to the population itself, i.e. $\frac{dP}{dt} = kP$.
P is the population of the country t years after 2000 and k maintains a roughly constant value.

a If the instantaneous growth rate was always roughly 2% per year find k.

b If the population in 2000 was 20 million, and assuming the growth rate remains constant, when will the population reach 50 million?

16 N, the number of people in the world known to be suffering from a particular disease, was thought to be increasing such that $\frac{dN}{dt} = 0.05N$, with t the time in years.

If the number of known sufferers was 1 500 000 in 2000, and nothing is done to alter the rate of increase, estimate the number of known sufferers in the year

a 2025, **b** 2050.

17 A colony of bacteria grows such that $\frac{dP}{dt} = 1.2P$ where P is the number of organisms present t hours after observation commenced.
When observation commenced $P \approx 1000$.
After approximately how many hours will the population be

a 1 million? **b** 2 million?

c What is the 'doubling' time for this population?

d What is the 'quadrupling' time for this population?

18 The instantaneous rate of decline in the number of rabbits on a particular property is 25% per month. If there were 2000 rabbits on the property two months ago how many will there be in two months time?

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19 A company finds that S, the number of sales per week of a particular product, falls such that $\frac{dS}{dt} = -0.24S$, where t is the number of weeks since the end of the company's promotion and advertising campaign. They decide to repeat the campaign when weekly sales fall to $0.45S_0$, where S_0 is the weekly sales at the end of the first campaign. How many weeks after the end of the first campaign should they expect to have to launch the repeat campaign?

ISBN 9780170395137

WS Finding definite integrals

WS Calculating physical areas

Integrating exponential functions

We now know that if $y = e^x$ then $\frac{dy}{dx} = e^x$.

Thus

$$\int e^x \, dx = e^x + c$$

Also, if $y = e^{f(x)}$ we let $u = f(x)$ and so $y = e^u$

Then, by the chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= e^u \times f'(x) \\ &= f'(x)e^{f(x)} \end{aligned}$$

Thus

$$\int f'(x)e^{f(x)} \, dx = e^{f(x)} + c$$

EXAMPLE 9

Find **a** $\int e^{6x} \, dx$ **b** $\int 10xe^{x^2} \, dx$ **c** $\int_0^1 8e^{2x} \, dx$.

Solution

a **Method one** (Making an intelligent guess then adjusting.)

Try $y = e^{6x}$

Then $\frac{dy}{dx} = 6e^{6x}$

Thus our initial trial needs to be divided by 6.

$$\therefore \int e^{6x} \, dx = \frac{e^{6x}}{6} + c.$$

Method two (Rearranging to set up $\int f'(x)e^{f(x)} \, dx$.)

$$\begin{aligned} \int e^{6x} \, dx &= \int \frac{1}{6} \times 6e^{6x} \, dx \\ &= \frac{1}{6} \times \int 6e^{6x} \, dx \\ &= \frac{1}{6} \times e^{6x} + c \\ &= \frac{e^{6x}}{6} + c \end{aligned}$$

$\int e^{6x} dx$

$\frac{e^{6 \cdot x}}{6}$

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b **Method one** (Making an intelligent guess then adjusting.)

Try $\quad y = e^{x^2}$

Then $\quad \dfrac{dy}{dx} = 2xe^{x^2}$

Thus our initial trial needs to be multiplied by 5.

$$\therefore \int 10xe^{x^2}\,dx = 5e^{x^2} + c.$$

Method two (Rearranging.)

$$\begin{aligned}\int 10xe^{x^2}\,dx &= \int 5 \times 2xe^{x^2}\,dx \\ &= 5 \times \int 2xe^{x^2}\,dx \\ &= 5e^{x^2} + c\end{aligned}$$

c

$$\begin{aligned}\int_0^1 8e^{2x}\,dx &= \left[4e^{2x}\right]_0^1 \\ &= 4e^2 - 4 \\ &= 4(e^2 - 1).\end{aligned}$$

$\int 10xe^{x^2}dx$	$5 \cdot e^{x^2}$
$\int_0^1 8e^{2x}dx$	$4 \cdot (e^2 - 1)$

Exercise 6D

Attempt each question without the assistance of your calculator, then use your calculator to check your answer if you wish.

Find the following indefinite integrals.

1 $\int 6e^{3x}\,dx$

2 $\int 6e^{2x}\,dx$

3 $\int e^{5x}\,dx$

4 $\int 3e^{9x}\,dx$

5 $\int 5e^{3x}\,dx$

6 $\int \frac{5}{e^x}\,dx$

7 $\int 4\sqrt{e^x}\,dx$

8 $\int \frac{1}{e^{2x}}\,dx$

9 $\int (4e^{2x} + 2x)\,dx$

10 $\int (e^{3x} + e^{2x})\,dx$

11 $\int 3e^{-2x}\,dx$

12 $\int \left(\frac{4}{e^{2x}} + \frac{e^{2x}}{4}\right)dx$

13 $\int 2xe^{x^2}\,dx$

14 $\int 6e^{2x+1}\,dx$

15 $\int (8xe^{x^2+5})\,dx$

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Evaluate the following definite integrals, giving *exact* answers.

16 $\int_0^2 5e^x\,dx$

17 $\int_0^1 e^{5x}\,dx$

18 $\int_1^2 (e^x + 4e^{2x})\,dx$

19 $\int_0^2 2(x + e^{2x})\,dx$

20 $\int_{-1}^0 \frac{1}{e^x}\,dx$

21 $\int_0^2 6(\sqrt{e^x} + x^2)\,dx$

22 If $\frac{dA}{dt} = 5e^{2t}$, and $A = 3$ when $t = 0$, find

a A in terms of t,

b the exact value of A, when $t = 0.5$.

23 If $f'(x) = 6(x^2 - 2e^{3x})$, and $f(0) = 1$, find

a $f(x)$,

b $f(2)$ as an exact value.

24 a Find the area between $y = e^x$ and the x-axis from $x = 0$ to $x = 3$ giving your answer correct to one decimal place.

b Find the area between $y = e^x - e$ and the x-axis from $x = 0$ to $x = 3$ giving your answer as an exact value.

Miscellaneous exercise six

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Clearly showing your use of the product rule, find the equation of the tangent to

$$y = (2x - 1)(3x + 2)$$

at the point (1, 5).

2 Without the assistance of your calculator, find $\frac{dy}{dx}$ for each of the following.

a $y = (x + 2)^5$

b $y = (2x + 1)^5$

c $y = \frac{x - 5}{x + 5}$

d $y = \frac{5x - 1}{x + 5}$

e $y = 4x^3 - e^x + 5$

f $y = e^{5x} + 5x$

3 The tangent to the curve $y = ax^3$ at the point $(5, b)$ has a gradient of 30. Find the values of the constants a and b.

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4 A particle is performing rectilinear motion with x metres and v m/s, the displacement and velocity of the particle respectively, with respect to an origin O, at time t seconds. The motion is such that $v = e^{0.1t}$ and when $t = 0$, $x = 12$.

Find **a** the initial acceleration of the particle,

b the acceleration of the particle when $t = 20$,

c the displacement of the particle when $t = 10$, to the nearest centimetre.

5 Use first principles, i.e. $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, to show that if $y = x^2 + 3x$ then $\dfrac{dy}{dx} = 2x + 3$.

6 The curve $y = x^3 + 3x^2 - 10x$ cuts the x-axis in three places. Find the coordinates of each of these points and determine the gradient of the curve at each one.
Find the total area of any regions enclosed by the curve and the x-axis.

7 The tangent to the curve $y = ax^2 + 5$ at the point $(-1, b)$ is perpendicular to the line $2y = -x + 8$. Given that if a line with gradient m_1 is perpendicular to a line with gradient m_2 then $m_1 m_2 = -1$, find the values of the constants a and b.

8 Evaluate the following definite integrals without the assistance of your calculator.

a $\displaystyle\int_2^{10} x\,dx$ **b** $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

d $\displaystyle\int_0^1 6e^{2x}\,dx$ **e** $\displaystyle\int_{-1}^2 (3x^2 + 4x)\,dx$ **f** $\displaystyle\int_2^3 \frac{4x}{(x^2-3)^2}\,dx$

9 Differentiate each of the following with respect to x, without the assistance of your calculator, and then use your calculator to check your answers.

a $2x^3 + 4\sqrt{x}$ **b** $x^3 + e^x$ **c** $\dfrac{2x-1}{x+3}$

d $x^4 e^x$ **e** $(2x^3 + 4\sqrt{x})^5$ **f** $\displaystyle\int_x^5 \frac{e^{5t}}{t}\,dt$

10 The velocity-time graph for the motion of a train along a straight track is shown below right.

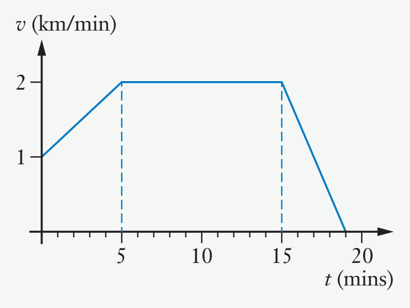
a When $t = 0$ the train is at position A and when $t = 5$ the train is at B. Find the distance from A to B.

b When $t = 15$ the train is at a position C. Find the distance from A to C.

c When $t = 19$ the train is at a station D. Find the distance from A to D.

Hint

What do we get if we integrate velocity with respect to time?
So what will the area 'under' a velocity–time graph give?



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11 A falling object does not keep accelerating indefinitely but, due to air resistance, reaches a terminal speed. Suppose that the speed of such an object, t seconds after the fall commences is v m/s where

$$v = \frac{200}{3}(1 - e^{-0.15t}).$$

Find the speed of the object after five seconds.
What is the terminal speed?

12 For δx, a small change in x, then δy, the associated small change in y, can be determined using $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$.
Use the above statement to determine the approximate change in the exterior surface area of a closed cylindrical tin when the base radius changes from 10 cm to 10.2 cm with the height remaining unchanged at 20 cm.

13 An initial 'one-off' investment of \$500 grows continuously in such a way that

$$\frac{dP}{dt} = 0.08P$$

where \$$P$ is in the account t years after the investment was opened.
How much is the investment worth after **a** 5 years? **b** 15 years?

14 A metal bar of temperature 120°C is placed in an environment with temperature 25°C. The temperature of the bar, t minutes later, is approximately T°C where

$$T = 25 + 95e^{-0.3t}.$$

Find the rate at which the temperature of the bar is falling, in °C/minute correct to one decimal place, after

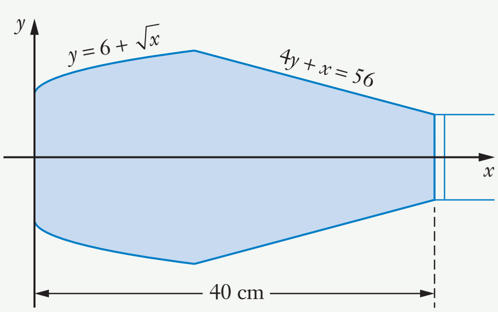
a 1 minute, **b** 3 minutes, **c** 15 minutes.

15 Without the assistance of a graphic calculator produce a sketch of each of the following, clearly indicating on your sketch:

- the coordinates of any points where the graph cuts the axes,
- the exact coordinates of all turning points,
- the behaviour of the curve as $x \to \pm\infty$.

a $y = x^2e^x$ **b** $y = \frac{e^x}{x^2}$ **c** $y = \frac{1}{1+e^x}$

16 A new rowing oar is being investigated. The shape of the blade is as shown shaded on the right. The y-axis forms the left hand boundary, the x-axis is a line of symmetry and 1 cm = 1 unit on each axis. Find the shaded area, giving your answer correct to the nearest square cm.



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7.

Calculus of trigonometric functions

- $\lim_{h \to 0} \frac{\sin h}{h}$
- $\lim_{h \to 0} \frac{1 - \cos h}{h}$
- Differentiation of sine and cosine
- Antidifferentiation of functions involving sine and cosine
- Miscellaneous exercise seven

In the previous chapter, when wanting to determine the derivative of e^x, we returned to the basic definition:

$$\text{If} \quad y = f(x) \quad \text{then} \quad \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Indeed, question **5** of Miscellaneous exercise six also required us to return to this **first principles** approach to show that if $y = x^2 + 3x$ then $\frac{dy}{dx} = 2x + 3$.

Rather than use this first principles approach every time, we tend to use it to identify patterns and establish general rules, and then apply these rules to determine derivatives as required. Let us now use this approach to determine the derivative of the trigonometric function $y = \sin x$.

Note: We will use the following identity which the *Preliminary work* reminded us of:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

Using the first principles definition, and the fact, not proved here, that

$$\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x):$$

If $y = \sin x$ then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \sin h}{h} - \lim_{h \to 0} \frac{\sin x - \sin x \cos h}{h} \\
&= \cos x \lim_{h \to 0} \frac{\sin h}{h} - \sin x \lim_{h \to 0} \frac{1 - \cos h}{h} \qquad \bigstar
\end{aligned}$$

Thus to determine $\frac{d}{dx}(\sin x)$ we need to investigate $\lim_{h \to 0} \frac{\sin h}{h}$

and $\lim_{h \to 0} \frac{1 - \cos h}{h}$.

Such investigations follow on the next few pages and then we will return to statement ★ above and apply what we find out about the two limits.

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$\lim_{h \to 0} \frac{\sin h}{h}$

To determine some limits as $h \to 0$, for example $\lim_{h \to 0}(2x + 3 + h)$, we simply substitute $h = 0$ into the expression to obtain the limit (which in the example would give $2x + 3$).

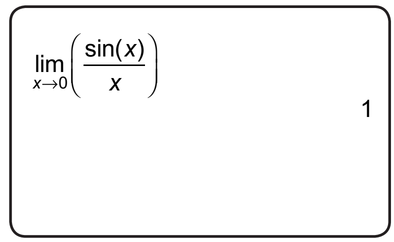
However, in this case, substitution of $h = 0$ into $\frac{\sin h}{h}$ gives $\frac{0}{0}$, which is undefined, so to determine the limit, further investigation is needed.

We could:

- Ask our calculator, see the display on the right (for x in radians).
- View the graph of

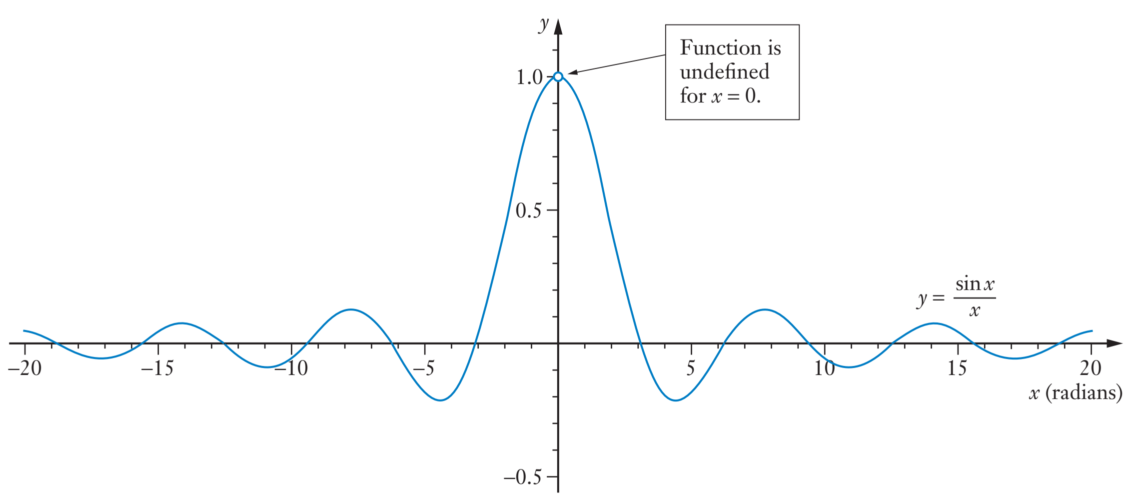
  $$y = \frac{\sin x}{x}$$

  and see what seems to be happening as x approaches zero.
- Consider a table of values for $\frac{\sin x}{x}$ as x approaches zero.



Viewing the graph

The graph of $y = \frac{\sin x}{x}$, for x in radians, is shown below. Notice that as we move along the graph, getting closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 1.



Hence the graph of $y = \frac{\sin x}{x}$ supports the calculator statement, i.e. that $\lim_{x \to 0} \frac{\sin x}{x} = 1$,

(and hence that $\lim_{h \to 0} \frac{\sin h}{h} = 1$).

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Note: The graph shows an 'open circle' at $x = 0$ because $\frac{\sin x}{x}$ is undefined there. However this does not stop us investigating the behaviour of $\frac{\sin x}{x}$ as x gets closer and closer to zero.

Tables of values

Considering x approaching zero 'from the left' and x approaching zero 'from the right' the following tables of values can be created (calculated values shown rounded to ten decimal places):

x approaching 0 from the left.

x	$\frac{\sin x}{x}$ (values from calculator)
−0.1	0.998 334 166 5
−0.01	0.999 983 333 4
−0.001	0.999 999 833 3
−0.0001	0.999 999 998 3
−0.000 01	1.000 000 000 0
−0.000 001	1.000 000 000 0
−0.000 000 1	1.000 000 000 0

x approaching 0 from the right.

x	$\frac{\sin x}{x}$ (values from calculator)
0.1	0.998 334 166 5
0.01	0.999 983 333 4
0.001	0.999 999 833 3
0.0001	0.999 999 998 3
0.000 01	1.000 000 000 0
0.000 001	1.000 000 000 0
0.000 000 1	1.000 000 000 0

Once again the statement

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

appears reasonable.

Note
- The fact that as x approaches zero, $\frac{\sin x}{x}$ approaches 1 means that **for small angles, measured in radians, $\sin x \approx x$.**

  For example: $\sin 0.01 = 0.009\,999\,833\,3 \approx 0.01.$

  $\sin 0.025 = 0.024\,997\,395\,9 \approx 0.025.$

  $\sin 0.0084 = 0.008\,399\,901\,2 \approx 0.0084.$

  (The reader should confirm these values on a calculator.)
- A proof of $\lim_{x \to 0} \frac{\sin x}{x} = 1$ is shown on the next page.

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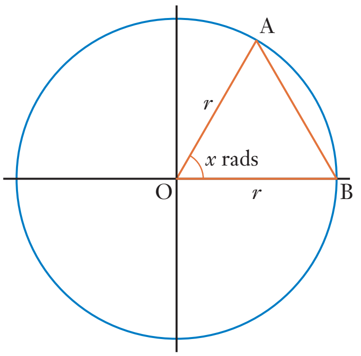
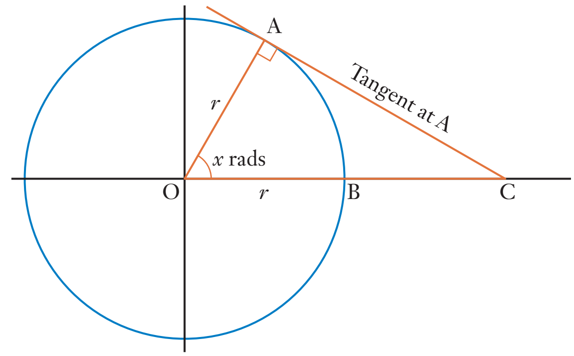
Proof

Consider the following circles of radius r, with $0 < x < \frac{\pi}{2}$.





$$\begin{aligned} \text{Area } \triangle OAB &< \text{Area sector OAB} \\ \frac{1}{2}r^2 \sin x &< \frac{1}{2}r^2 x \\ \sin x &< x \\ \frac{\sin x}{x} &< 1 \qquad [1] \end{aligned}$$

$$\begin{aligned} \text{Area sector OAB} &< \text{Area } \triangle OAC \\ \frac{1}{2}r^2 x &< \frac{1}{2}(OA)(AC) \\ \frac{1}{2}r^2 x &< \frac{1}{2}r(r \tan x) \\ x &< \frac{\sin x}{\cos x} \\ \cos x &< \frac{\sin x}{x} \qquad [2] \end{aligned}$$

Combining [1] and [2]: $\cos x < \frac{\sin x}{x} < 1$

Writing $x \to 0^+$ for x tending towards zero from the positive side:

As $x \to 0^+$ $\quad \cos x \to \cos 0 = 1$

Thus as $x \to 0^+$, $\frac{\sin x}{x}$ is 'sandwiched' between $\cos x$, which is approaching 1, and 1 itself.

Therefore $\frac{\sin x}{x}$ must also approach 1 as $x \to 0^+$.

Similar reasoning can be used to show that $\frac{\sin x}{x} \to 1$ as $x \to 0^-$.

Thus $\lim_{x \to 0} \frac{\sin x}{x} = 1$, as required.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

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$\lim_{h \to 0} \frac{1 - \cos h}{h}$

Substitution of $h = 0$ into $\frac{1 - \cos h}{h}$ gives $\frac{0}{0}$ and so, as before, to determine the limit we need to investigate further.

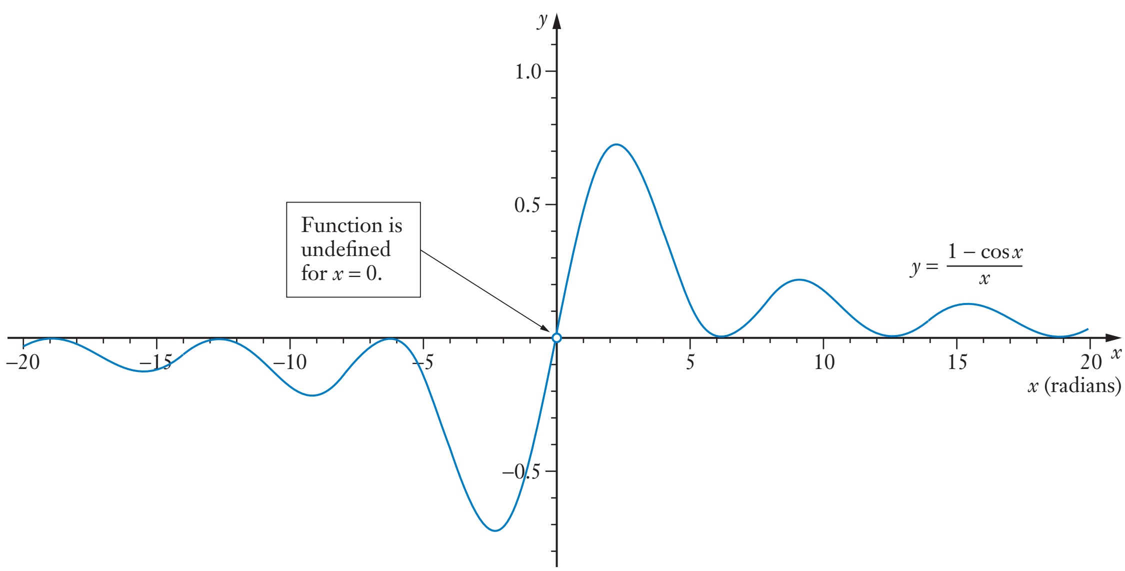
According to the display on the right

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

$$\lim_{x \to 0} \left(\frac{1 - \cos(x)}{x} \right) \qquad 0$$

Viewing the graph

As we move along the graph of $y = \frac{1 - \cos x}{x}$, shown below, and get closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 0.



Hence the graph of $y = \frac{1 - \cos x}{x}$ supports the calculator statement, i.e. that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

(and hence that $\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$).

Note again that whilst the function is undefined for $x = 0$ this does not stop us investigating the behaviour of the function as x gets closer and closer to zero.

Tables of values

Copy and complete the following tables.

x approaching 0 from the left.

x	$\dfrac{1-\cos x}{x}$ (values from calculator)
–0.1	
–0.01	
–0.001	
–0.0001	
–0.000 01	
–0.000 001	

x approaching 0 from the right.

x	$\dfrac{1-\cos x}{x}$ (values from calculator)
0.1	
0.01	
0.001	
0.0001	
0.000 01	
0.000 001	

Do your completed tables support the statement

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0 \qquad ?$$

Proof

This second useful trigonometric limit is proved below. The proof uses the fact, not proved here, that we can write $\lim\limits_{x\to a}(f(x)\times g(x))$ as $\lim\limits_{x\to a} f(x)\times \lim\limits_{x\to a} g(x)$.

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\left(\frac{1-\cos x}{x}\,\frac{1+\cos x}{1+\cos x}\right)\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{\sin x}{(1+\cos x)}\right)\\
&= \lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}\\
&= 1 \times \frac{\sin 0}{1+\cos 0}\\
&= 0
\end{aligned}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

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Differentiation of sine and cosine

Let us now return to statement ★ made on the first page (page 131) of this chapter:

$$\text{If} \quad y = \sin x \quad \text{then} \quad \frac{dy}{dx} = \cos x \lim_{h\to 0} \frac{\sin h}{h} - \sin x \lim_{h\to 0} \frac{1-\cos h}{h}. \qquad ★$$

Applying our two trigonometric limits:

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0} \frac{1-\cos h}{h} = 0,$$

statement ★ becomes:

$$\text{If} \quad y = \sin x \quad \text{then} \quad \frac{dy}{dx} = \cos x\,(1) - \sin x\,(0) = \cos x$$

$$\begin{aligned}
\text{Further, if} \quad y = \cos x \quad \text{then} \quad \frac{dy}{dx} &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= -\cos x \lim_{h\to 0} \frac{1-\cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -\cos x(0) - \sin x(1) \\
&= -\sin x
\end{aligned}$$

Thus

If	y	=	$\sin x$
then	$\frac{dy}{dx}$	=	$\cos x$

and

If	y	=	$\cos x$
then	$\frac{dy}{dx}$	=	$-\sin x$

Remember

The limit facts used to obtain the above results are true for angles in radians. Thus the above facts again assume radian measure.

The two boxed results above, together with an ability to use

the sum and difference rules, If $y = u \pm v$, $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$

the product rule, If $y = u \times v$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

the quotient rule, If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

and the chain rule, If $y = f(u)$ and $u = g(x)$ $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

which gives us, If $y = [f(x)]^n$ $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

Differentiating trigonometric functions

Higher derivatives

allow us to differentiate many functions that involve the trigonometric ratios, as the following examples demonstrate.

In each of examples 1 to 6, part **a** reminds you of the rule applied to an expression that does not involve trigonometric functions, and then part **b** applies the rule to a trigonometric expression.

EXAMPLE 1 (Sum and difference rules)

Differentiate **a** $x^2 - x^3$ **b** $x^2 + \sin x$

Solution

a If
$$y = x^2 - x^3$$
$$\frac{dy}{dx} = 2x - 3x^2$$

b If
$$y = x^2 + \sin x$$
$$\frac{dy}{dx} = 2x + \cos x$$

EXAMPLE 2 (Product rule)

Differentiate **a** $3x^2$ **b** $5\sin x$

Solution

a If
$$\begin{aligned} y &= 3x^2 \\ \frac{dy}{dx} &= x^2 \times 0 + 3 \times 2x \\ &= 6x \end{aligned}$$

b If
$$\begin{aligned} y &= 5\sin x \\ \frac{dy}{dx} &= \sin x \times 0 + 5 \times \cos x \\ &= 5\cos x \end{aligned}$$

EXAMPLE 3 (Product rule)

Differentiate **a** $(3x - 4)(5x^2 + 3)$ **b** $(2 - \cos x)(1 + \sin x)$

Solution

a If
$$\begin{aligned} y &= (3x - 4)(5x^2 + 3) \\ \frac{dy}{dx} &= (5x^2 + 3) \times 3 + (3x - 4) \times 10x \\ &= 15x^2 + 9 + 30x^2 - 40x \\ &= 45x^2 - 40x + 9 \end{aligned}$$

b If
$$\begin{aligned} y &= (2 - \cos x)(1 + \sin x) \\ \frac{dy}{dx} &= (1 + \sin x) \times (\sin x) + (2 - \cos x) \times \cos x \\ &= \sin x + \sin^2 x + 2\cos x - \cos^2 x \end{aligned}$$

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EXAMPLE 4 **(Quotient rule)**

Differentiate **a** $\dfrac{x^2-1}{2x-3}$ **b** $\dfrac{\sin x}{\cos x}$

Solution

a If
$$y = \frac{x^2-1}{2x-3}$$
$$\frac{dy}{dx} = \frac{(2x-3)(2x)-(x^2-1)(2)}{(2x-3)^2}$$
$$= \frac{2x^2-6x+2}{(2x-3)^2}$$

b If
$$y = \frac{\sin x}{\cos x}$$
$$\frac{dy}{dx} = \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$

Note

- From the result above it follows that if $y = \tan x$, $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$.
- Writing $\dfrac{1}{\cos x}$ as $\sec x$ we could write: If $y = \tan x$, $\dfrac{dy}{dx} = \sec^2 x$.

  However, whilst the term '$\sec x$' (and $\operatorname{cosec} x$ and $\cot x$) will be familiar to students who followed Unit Two of *Mathematics Specialist* it is not required knowledge for this unit. That being said, students who are following both Mathematics Methods and Mathematics Specialist may like to now consider derivatives for $\sec x$, $\operatorname{cosec} x$ and $\cot x$.

EXAMPLE 5 (Chain rule)

a By letting $u = 2x - 7$ determine the derivative of $3(2x-7)^5$ using the chain rule.

b By letting $u = 2x + 1$ determine the derivative of $\sin(2x+1)$ using the chain rule.

Solution

a If $y = 3(2x-7)^5$ then with $u = (2x-7)$, $y = 3u^5$.

Hence $\dfrac{du}{dx} = 2$ and $\dfrac{dy}{du} = 15u^4$

Thus, by the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= 15u^4 \times 2 \\ &= 30u^4 \\ &= 30(2x-7)^4\end{aligned}$$

b If $y = \sin(2x+1)$ then with $u = (2x+1)$, $y = \sin u$.

Hence $\dfrac{du}{dx} = 2$ and $\dfrac{dy}{du} = \cos u$

Thus, by the chain rule

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u) \times 2 \\ &= 2\cos u \\ &= 2\cos(2x+1)\end{aligned}$$

As is probably already the case with non-trigonometric expressions, the reader will, with practice, be able to differentiate expressions like $3(2x-7)^5$ and $\sin(2x+1)$ directly, without formally using the chain rule, as in the next examples.

EXAMPLE 6 (Chain rule)

Differentiate **a** $(2x+3)^4$ **b** $\sin^4 x$

Solution

a If

$$\begin{aligned}y &= (2x+3)^4 \\ \frac{dy}{dx} &= 4(2x+3)^3(2) \\ &= 8(2x+3)^3\end{aligned}$$

b If

$$\begin{aligned}y &= \sin^4 x \\ \frac{dy}{dx} &= 4(\sin x)^3 \cos x \\ &= 4\sin^3 x \cos x\end{aligned}$$

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EXAMPLE 7 (Chain rule)

Differentiate **a** $\cos(2x+3)$ **b** $\sin(5-2x)$

Solution

a If $y = \cos(2x+3)$

$$\frac{dy}{dx} = -2\sin(2x+3)$$

Thoughts

'cos' differentiates to '– sin'.

$2x + 3$ differentiates to 2.

Answer: $(2)(-\sin(2x+3))$

b If $y = \sin(5-2x)$

$$\frac{dy}{dx} = -2\cos(5-2x)$$

Thoughts

'sin' differentiates to 'cos'.

$5 - 2x$ differentiates to –2.

Answer: $(-2)(\cos(5-2x))$

Note: You are already familiar with using your calculator to obtain derivatives. Whilst you are encouraged to use this facility when appropriate, make sure you can differentiate trigonometrical functions without the assistance of a calculator when required to do so. Also be aware that because of the various trigonometrical identities that exist, and because of the method of display, the calculator answer may sometimes, at first glance, appear different to the expression you obtain.

For example, asked to determine

$$\frac{d}{dx}(\sin x \cos x),$$

for which we might use the product rule and write the answer as

$$\cos x \cos x + \sin x(-\sin x)$$

i.e. $$\cos^2 x - \sin^2 x$$

a calculator may give an equally correct, but rather different-looking answer, as shown at right.

$\frac{d}{dx}(\sin(x)\cdot\cos(x))$

$2\cdot(\cos(x))^2 - 1$

Exercise 7A

Find the derivatives, with respect to x, of each of the following

Sum and difference rules

1 $x^5 - x^2$

2 $3 + x^3$

3 $5 - \cos x$

4 $\sin x - \cos x$

5 $\cos x - \sin x$

6 $x - \tan x$

Product rule

7 $(x + 1)(2x - 3)$

8 $5x^2(1 - 5x)$

9 $6\sin x$

10 $4\cos x$

11 $x\sin x$

12 $x^2 \cos x$

Quotient rule

13 $\dfrac{x}{3x^2 - 1}$

14 $\dfrac{x^2 + 1}{x^2 - 1}$

15 $\dfrac{\cos x}{x}$

16 $\dfrac{\sin x}{x}$

17 $\dfrac{x}{\sin x}$

18 $\dfrac{x}{\cos x}$

Chain rule

Find $\dfrac{dy}{dx}$, in terms of x, for each of the following.

19 $y = 3u^2 - 5$ and $u = x^2 + 1$.

20 $y = \sqrt{u}$ and $u = x^2 - 1$.

21 $y = \sin u$ and $u = 6x$.

22 $y = \cos u$ and $u = 2x + 3$.

23 $y = \sin^2 x$

24 $y = \sin^3 x$

25 $y = \cos^5 x$

26 $y = \cos 3x$

27 $y = \sin(3x - 7)$

28 $y = \cos(2x + 5)$

Miscellaneous

Differentiate each of the following with respect to x.

29 $2 - 3\cos x$

30 $3x + 2\cos x$

31 $\sin 2x$

32 $x^2 - \cos x$

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33 $\frac{1+\sin x}{x^2}$

34 $3\sin x - 2\cos x$

35 $\cos 3x$

36 $\cos 9x$

37 $3\cos 2x$

38 $5\sin 3x$

39 $2\sin 3x + 3\cos 2x$

40 $\sin^5 x$

41 $5\cos^2 x$

42 $\sqrt{\sin x}$

Determine $f'(x)$ for each of the following.

43 $f(x) = \sin 7x$

44 $f(x) = \sin 8x$

45 $f(x) = \sin 4x + \cos 4x$

46 $f(x) = 2\sin(3x - 1)$

47 $f(x) = 4\cos(4x + 3)$

48 $f(x) = 2\sin^3 x$

49 $f(x) = 3\cos^2 x$

50 $f(x) = x\cos x$

51 $f(x) = x^2\cos x$

52 $f(x) = 2x\sin x$

53 $f(x) = \frac{2\sin x}{\cos x}$

54 $f(x) = 2\tan x$

Determine the gradient of each of the following at the given point.

55 $y = \sin x$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

56 $y = \cos 2x$ at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

57 $y = 2\sin x\cos x$ at the point $(0, 0)$.

58 $y = 3\sin^2 x$ at the point $(\pi, 0)$.

Determine $\frac{d^2y}{dx^2}$ for each of the following.

59 $y = \sin x$

60 $y = \cos 5x$

61 $y = 3\sin 2x$

62 $y = \sin x + \cos x$

63 Find the equation of the tangent to the curve $y = x\sin x$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

64 Find the equation of the tangent to $y = x + 3\cos 2x$ at the point $(0, 3)$.

65 If $f(x) = \sin 2x$ find exact values for

a $f'\left(\frac{\pi}{6}\right)$

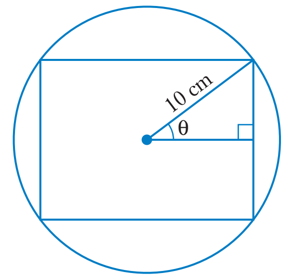
b $f''\left(\frac{\pi}{6}\right)$.

66 If $y = \sin x°$, find $\dfrac{dy}{dx}$.

Hint

$1° = \dfrac{\pi}{180}$ radians.

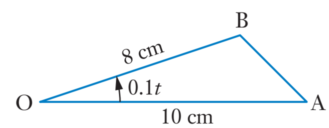
67 With θ as shown in the diagram, and $0 < \theta < \dfrac{\pi}{2}$, show that the area of a rectangle drawn with all four vertices touching a circle of radius 10 cm, is A cm^2 where



$$A = 400 \sin\theta \cos\theta.$$

Use calculus to prove that the rectangle drawn in this way, and having maximum area, will be a square, and determine its area and side length.

68 Triangle OAB has OA = 10 cm, OB = 8 cm and $\angle$BOA = $0.1t$ radians, where t is the time in seconds. Thus when $t = 0$, OB lies along OA and as t increases $\angle$AOB 'opens'.



Find an expression in terms of t for the rate of change of the area of ΔOAB with respect to time.

Determine the instantaneous rate of change in the area of ΔOAB with respect to time when

a $t = 1$,

b $t = 5$,

c $t = 10$,

d $t = 20$.

(Give answers in cm^2/s and correct to two decimal places.)

69 Given that $x = 5\sin(3t), \quad t \geq 0$:

a find the maximum value of x and the smallest value of t for which it occurs

b find the three smallest values of t for which $x = 2.5$

c find, correct to 1 decimal place, $\dfrac{dx}{dt}$ when $t = 0.6$

d prove that $\dfrac{d^2x}{dt^2} = kx$, and find k.

70 Use calculus to determine, correct to 4 decimal places, the value of θ, for $0 \leq \theta \leq \dfrac{\pi}{2}$, which maximises

$$3\sin\theta + 4\cos\theta,$$

and find this maximum value.

(Note: Students who followed *Unit Two* of *Mathematics Specialist*, may remember solving this sort of optimisation question there without using calculus.)

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Antidifferentiation of functions involving sine and cosine

Finding indefinite integrals 1

Finding indefinite integrals 2

Finding definite integrals

Displacement, velocity and acceleration

Our ability to differentiate expressions involving trigonometric functions means that our antidifferentiation, or integration, can now also involve functions of this type. The following examples, and the exercise that follows, involve determining antiderivatives of trigonometrical functions.

EXAMPLE 8

Antidifferentiate **a** $\cos x$, **b** $\sin x$.

Solution

a We know that if $y = \sin x$

then $\dfrac{dy}{dx} = \cos x$

Thus $\displaystyle\int \cos x \, dx = \sin x + c$

The antiderivative is $\sin x + c$.

b We know that if $y = \cos x$

then $\dfrac{dy}{dx} = -\sin x$

Thus $\displaystyle\int \sin x \, dx = -\cos x + c$

The antiderivative is $-\cos x + c$.

$$\int \cos(x)dx \qquad \sin(x)$$

$$\int \sin(x)dx \qquad -\cos(x)$$

Note

- If $y = \sin f(x)$ then, by the chain rule, $\dfrac{dy}{dx} = f'(x)\cos f(x)$.

  Hence
  $$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

  Similarly
  $$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

- In the following examples, some show the method of making an intelligent first guess and then making suitable adjustments and some show that of making an initial rearrangement to aid the antidifferentiation process. With practice some of the antiderivatives can be written directly by 'mentally juggling' an intelligent first guess.
- The antiderivative of $\tan x$ is beyond the requirements of this unit.
- In some cases, rearranging the given expression using one of the trigonometric identities can be useful: see example 12.

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EXAMPLE 9

Find the antiderivative of

a $8\sin 2x$, **b** $8\cos(2x+1)$, **c** $\cos 5x + \sin 2x$.

Solution

a Try $y = \cos 2x$

then, by the chain rule, $\frac{dy}{dx} = (2)(-\sin 2x)$

$= -2\sin 2x$

Hence if $y = -4\cos 2x$

then $\frac{dy}{dx} = 8\sin 2x$, as required.

The required antiderivative is $-4\cos 2x + c$.

b Try $y = \sin(2x+1)$

then, by the chain rule $\frac{dy}{dx} = 2\cos(2x+1)$

Hence the required antiderivative is $4\sin(2x+1) + c$.

c Try $y = \sin 5x - \cos 2x$

then $\frac{dy}{dx} = 5\cos 5x + 2\sin 2x$

Hence the required antiderivative is $\frac{1}{5}\sin 5x - \frac{1}{2}\cos 2x + c$.

EXAMPLE 10

Find the antiderivative of

a $3\sin x$, **b** $15\cos 5x$.

Solution

a

$$\begin{aligned} \int 3\sin x\, dx &= 3 \times \int \sin x\, dx \\ &= 3 \times (-\cos x) + c \\ &= -3\cos x + c. \end{aligned}$$

The required antiderivative is $-3\cos x + c$.

b (First note that the derivative of $5x$ is 5.)

$$\begin{aligned} \int 15\cos 5x\, dx &= 3 \times \int 5 \times \cos 5x\, dx \\ &= 3 \times \sin 5x + c \\ &= 3\sin 5x + c \end{aligned}$$

The required antiderivative is $3\sin 5x + c$.

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EXAMPLE 11

Find the antiderivative of **a** $\cos^4 x \sin x$, **b** $3\sin^3 4x \cos 4x$.

Solution

a Try $y = \cos^5 x$

then
$$\frac{dy}{dx} = 5\cos^4 x(-\sin x)$$
$$= -5\cos^4 x \sin x$$

Thus if $y = -\frac{1}{5}\cos^5 x$

$$\frac{dy}{dx} = \cos^4 x \sin x \text{ as required.}$$

The required antiderivative is $-\frac{1}{5}\cos^5 x + c$.

b (Note that the derivative of $\sin 4x$ is $4\cos 4x$. By the rearrangement approach we attempt to set up an expression of the form $f'(x)[f(x)]^n$).

$$\int 3\sin^3 4x \cos 4x \, dx = \frac{3}{4} \times \int (4\cos 4x) \times \sin^3 4x \, dx$$
$$= \frac{3}{4} \times \frac{\sin^4 4x}{4} + c$$
$$= \frac{3\sin^4 4x}{16} + c$$

The required antiderivative is $\frac{3}{16}\sin^4 4x + c$.

EXAMPLE 12

Find the antiderivative of $\sin 4x \cos 3x + \cos 4x \sin 3x$.

Solution

Remembering that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ it follows that

$$\sin 4x \cos 3x + \cos 4x \sin 3x = \sin(4x + 3x)$$
$$= \sin 7x$$

Hence
$$\int (\sin 4x \cos 3x + \cos 4x \sin 3x)\, dx = \int \sin 7x \, dx$$
$$= -\frac{1}{7}\cos 7x + c$$

The required antiderivative is $-\frac{1}{7}\cos 7x + c$.

Exercise 7B

Find the antiderivative of each of the following.

1 $5\cos x$

2 $2\sin x$

3 $-10\sin x$

4 $-2\cos x$

5 $6\cos 2x$

6 $2\cos 6x$

7 $12\sin 4x$

8 $-\sin 3x$

9 $-8\cos 10x$

10 $\sin\dfrac{x}{2}$

11 $\cos\dfrac{3x}{2}$

12 $-6\sin\dfrac{2x}{3}$

13 $6\sin(2x+3)$

14 $3\cos(2x-3)$

15 $\cos\left(2x+\dfrac{2\pi}{3}\right)$

16 $\sin(-x)$

17 $\dfrac{4}{\cos^2 x}$ (Hint: What is the derivative of $\tan x$?)

18 $6\cos 2x + 6\sin 3x$

19 $\cos 8x - 4\sin 2x$

20 $2x + 4\cos x + 6\cos 2x$

21 $3 + 4x - 6x^2 + 10\cos 5x - 2\sin 4x$

22 $\cos^3 x\sin x$

23 $30\cos^5 x\sin x$

24 $\sin 5x\cos 2x + \cos 5x\sin 2x$

25 $\sin 3x\cos x - \cos 3x\sin x$

26 $\cos 5x\cos 2x - \sin 5x\sin 2x$

27 $\cos 5x\cos x + \sin 5x\sin x$

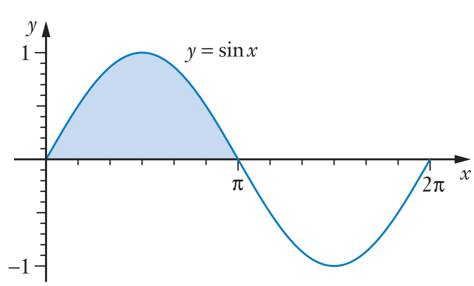
28 Evaluate $\int_0^{\frac{\pi}{2}} \sin x\, dx$

29 Evaluate $\int_0^{\frac{\pi}{2}} \cos x\, dx$

30 Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos\dfrac{x}{2}\, dx$

31 Evaluate **a** $\int_0^{\frac{\pi}{4}} \sin x\, dx$, **b** $\int_{\frac{\pi}{4}}^{0} \sin x\, dx$.

32 Find the area enclosed by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.



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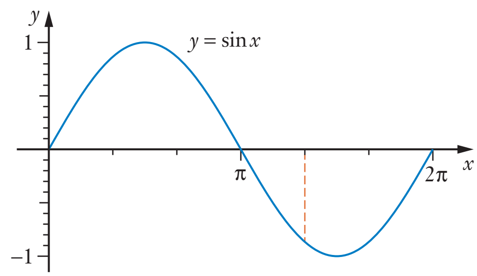
33 Find the area between $y = \sin x$ and the x-axis from:

a $x = \pi$ to $x = \dfrac{4\pi}{3}$,

b $x = 0$ to $x = \dfrac{4\pi}{3}$.



34 A particle moves along a straight line such that its velocity, at time t seconds, is given by v metres per second, where

$$v = 2\cos 2t.$$

When $t = 0$ the displacement of the particle from an origin O is 5 metres.

Find **a** the greatest speed of the particle,

b an expression for the displacement of the particle from O at time t,

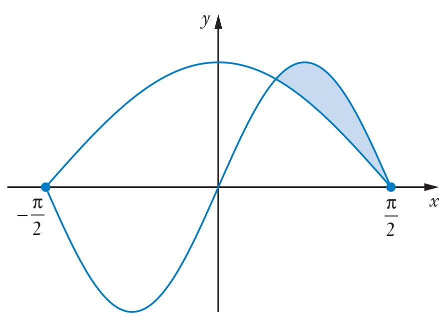
c the least distance the particle is from the origin,

d an expression for the acceleration of the particle at time t.

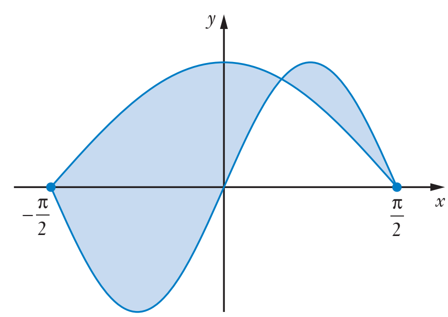
35 Each of the graphs below show $y = \sin 2x$ and $y = \cos x$, each for $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

Use your calculator to determine the area shaded in each case.

a



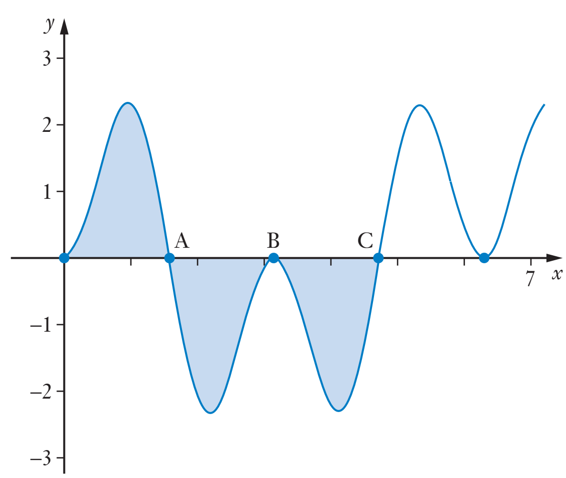
b



36 The graph of $y = 6\cos x \sin^2 x$ is shown on the right for $0 \le x \le 7$.

a Find the exact coordinates of points A, B and C, all of which lie on the x-axis.

b With the assistance of your calculator if you wish, find the total shaded area.



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Miscellaneous exercise seven

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

Differentiate the following with respect to x.

1 e^x

2 $2e^x$

3 $8e^x$

4 $e^x + \sin x$

5 $e^{\cos x}$

6 $e^{\sin 2x}$

7 $e^{2\sin x}$

8 $e^x + \frac{1}{x}$

9 $4(\sqrt{x}) + e^{3x}$

10 $e^x\sqrt{x}$

11 $e^x \sin x$

12 $e^x \cos 2x$

13 $e^x \sin^2 x$

14 e^{3x^2+2}

15 $e^{x^2+\sin x}$

16 If $T = (2r+3)^3$ find, without the assistance of a calculator, an expression for the rate of change of T with respect to r.

17 Evaluate the following definite integrals 'by hand' and then use your calculator to check your answers.

a $\int_0^2 4e^{2x}\, dx$

b $\int_2^5 \frac{1}{x^2}\, dx$

c $\int_1^2 30(2x-3)^4\, dx$

18 Find the exact values of x, for $-2\pi \le x \le 2\pi$, for which the gradient of the curve

$$y = e^x \sin x$$

is zero.

19 If $y = e^{-x}\sin x$, find $\frac{dy}{dx}$ when $x = \pi$.

20 Write down an expression of the form ax^n for $\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$.

(This should not require much, if any, working.)

21 It is expected that 3 000 000 tonnes of a particular resource is to be available for use this year but the amount available in t years' time is expected to be A tonnes where

$$A = 3\,000\,000e^{-0.1t}.$$

Is A increasing or decreasing as t increases?

Find the value of $\frac{dA}{dt}$, in tonnes/year, when

a $t = 2$,

b $t = 5$,

c $t = 10$.

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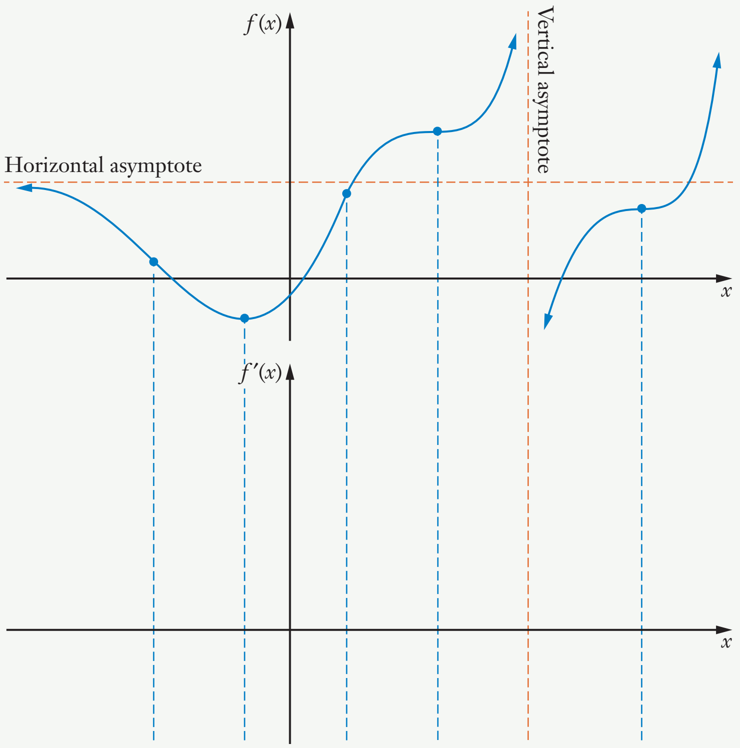
22 Copy the following graph of $y = f(x)$ and complete the graph of $f'(x)$. (Pay particular regard to the points marked •).

Remember: An **asymptote** is a line that a curve gets closer and closer to without ever quite touching.



23 The velocity-time graph shown below is for a particle moving in a straight line, from rest at A, through B to C and then back to rest at B.

a What is the velocity of the particle 8 seconds after leaving A?

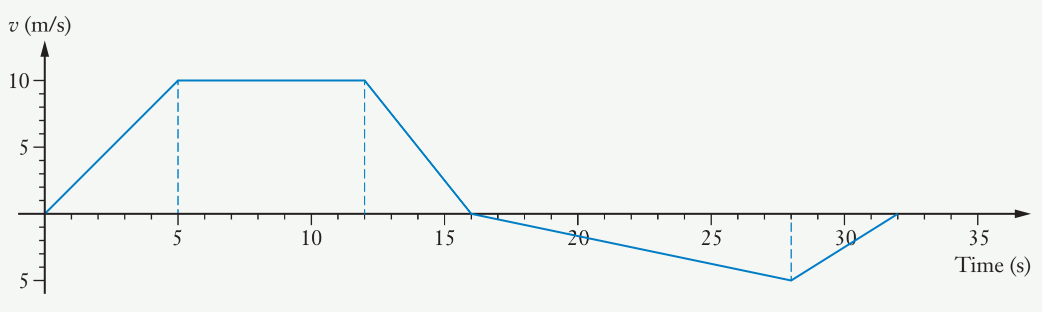
b What is the acceleration of the particle 13 seconds after leaving A?

c How far does the particle move in the first 28 seconds?

d What is the particle's displacement from A 28 seconds after leaving A?

How far is **e** C from A?

f B from A?



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24 One suggestion for solving the water shortage in certain parts of the world involved towing icebergs from the polar regions to other parts of the world. Clearly some ice would melt on the journey but enough might be left to make the journey worthwhile, especially if the iceberg was 'lagged' in some way. To investigate this situation mathematically we could consider the iceberg to be roughly spherical with an initial radius of 100 m and assume that the initial radius reduces by 3 m each day of travel.

a Find a formula for the volume, V m^3 ($V \geq 0$), of the iceberg after x days of travel.

b On which day will the volume have reduced to half the original amount?

c Find a formula that will allow the rate at which the volume is changing to be determined (in m^3/day) given x.

d What is the rate of volume loss, in m^3/day, when $x = 5$?

e The idea that the radius would reduce by 3 metres per day was not felt to be realistic because, as the journey progressed, the iceberg would be towed into increasingly warmer surroundings. An alternative formula could be:

$$V = \frac{4}{3}\pi(100 - 2x - x^2)^3 \qquad \text{for } V \geq 0.$$

Find a formula for the instantaneous rate of volume loss now and again evaluate it for $x = 5$.

25 An object placed in a particular fluid sinks such that, t seconds after release, its downward velocity is v m/s where $v = 2(1 - e^{-0.2t})$ m/s.

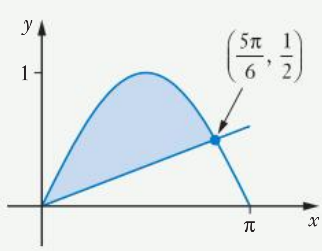
Find, correct to two decimal places,

a the speed of the object 2 seconds after release,

b the downward acceleration of the object 2 seconds after release,

c the downward acceleration of the object 10 seconds after release.

26 The diagram shows the line $y = \frac{3x}{5\pi}$ and the curve $y = \sin x$ for $0 \leq x \leq \pi$.



a Find as an exact value the enclosed area shown shaded in the diagram.

b View the line and the curve on your calculator and hence write an exact answer for the total area *enclosed* between the line and the curve for unrestricted values of x.

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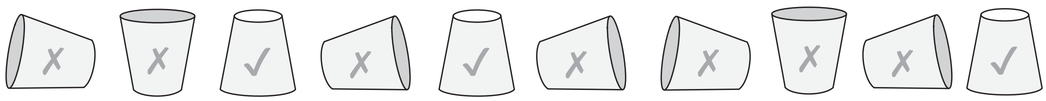
8.

Discrete random variables

- Discrete random variables
- Mean or expected value of a discrete random variable
- The standard deviation of a discrete random variable
- Miscellaneous exercise eight

Situation One

Suppose we consider it a 'success' if, when we drop an empty plastic cup onto a surface, it lands 'base up'. If you dropped ten such plastic cups onto a surface, how many 'successes' would you expect to achieve? Investigate.



Situation Two

Kai and Tenielle are playing a board game. In order to start each of their pieces out from the starting point they must roll a six on a normal, fair, six-sided die. They begin to wonder how many rolls of the die it takes, on average, to get a six. Clearly a six could be achieved in one roll, but it could take a lot more than just one roll.

Ten such results for 'rolling a die until a six results' are shown below:

2	2	1	3	6							5 rolls required.
3	2	2	6								4 rolls required.
3	4	1	6								4 rolls required.
5	4	5	2	6							5 rolls required.
5	1	4	2	1	3	6					7 rolls required.
5	3	5	2	3	2	4	4	2	3	6	11 rolls required.
5	1	3	5	4	4	2	4	6			9 rolls required.
1	2	1	4	2	2	2	6				8 rolls required.
6											1 roll required.
5	4	5	3	6							5 rolls required.

Tabulating these results:

Number of rolls required	1	2	3	4	5	6	7	8	9	10	11
Frequency	1	0	0	2	3	0	1	1	1	0	1

We could determine the mean and the standard deviation for this data:

$$\frac{1\times 1 + 2\times 4 + 3\times 5 + 1\times 7 + 1\times 8 + 1\times 9 + 1\times 11}{10}$$

$$\text{Mean} = 5.9$$

$$\sqrt{\frac{1\times(1-5.9)^2 + 2\times(4-5.9)^2 + 3\times(5-5.9)^2 + 1\times(7-5.9)^2 + 1\times(8-5.9)^2 + 1\times(9-5.9)^2 + 1\times(11-5.9)^2}{10}}$$

Standard deviation = 2.737 (correct to 3 decimal places).

Carry out this activity yourself for twenty events of 'rolling a die until a six results'.

Produce a table of results for your twenty occasions like the table shown above and calculate the mean and standard deviation of your results.

Discrete random variables

Situation One on the previous page asked:

If you dropped ten plastic cups onto a surface, how many 'successes' would you expect to achieve? (With a success as defined on the previous page.)

The situation really requires us to carry out the dropping of the ten plastic cups a number of times, to gain some idea of the relative frequencies with which the various possibilities, from zero successes to 10 successes, might occur.

If we use X to represent the number of 'successes' obtained in such an experiment, then X can take the values 0, 1, 2, 3, … 10.

In Situation Two, Kai and Tenielle wondered

How many rolls of a die it takes, on average, to get a six.

If we use X to represent the number of rolls it takes to get a six, then X can take the values 1, 2, 3, 4, …

In each of the above cases the X referred to can take a number of possible integer values, with the value taken dependent on the outcome of a random process. We call X a **discrete random variable**. The word **discrete** means 'separate' or 'individually distinct' which is the case here because X can only take the distinct values 0, 1, 2, 3, … 10 in Situation One and 1, 2, 3, … etc., in Situation Two. (Contrast this with a situation involving measuring the time a randomly chosen able-bodied person takes to run 100 metres. The variable, time in this case, could take any value, between realistic limits. In such a case the random variable would be **continuous**.)

In some cases the frequencies with which the possible values of the random variable may occur are determined empirically, i.e. by actually carrying out the experiment and observing the results, as the previous situations required of you. We can then use the results to estimate probabilities associated with each value of the random variable, or perhaps to estimate the mean value (this would be a **point estimate** because the estimate is a single value, rather than a range of values in which we would expect the mean to lie).

Sometimes the symmetry of the situation, for example the '50-50' nature of a coin flip, allows the theoretical probability of each value occurring to be determined.

For example, for Situation Two on the previous page, theoretical probabilities could be determined as follows:

$$\text{P(a 6 in 1 roll)} = \frac{1}{6} = \frac{1}{6}$$

$$\text{P(first 6 on 2nd roll)} = \frac{5}{6}\times\frac{1}{6} = \frac{5}{36}$$

$$\text{P(first 6 on 3rd roll)} = \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \frac{25}{216}$$

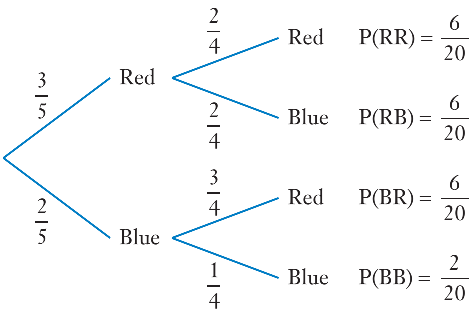
$$\text{P(first 6 on 4th roll)} = \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6} = \frac{125}{1296}$$

Note: The above probabilities form a *geometric progression* with a first term of $\frac{1}{6}$ and a common ratio of $\frac{5}{6}$. What is the *sum to infinity* of such a geometric sequence?

ISBN 9780170395137

Consider the situation of a bag containing 5 discs, 3 red and 2 blue, indistinguishable except for their colour. Suppose that two discs are randomly selected from the bag, one after the other, the first disc not being returned to the bag before the second is chosen.

The tree diagram for this situation is shown on the right.



If we are interested in the number of red discs this process is likely to select the following table of probabilities would be useful:

Number of red discs	0	1	2
Probability	0.1	0.6	0.3

If we use X to represent the number of reds selected, then X can only take distinct values, in this case the integer values 0, 1 or 2. The value X takes depends on a random process. Thus X is a **discrete random variable** and the table gives the probability associated with each value X can take.

We write $P(X = 0) = 0.1$,
$P(X = 1) = 0.6$,
$P(X = 2) = 0.3$.

- The possible values of a random variable must be numerical. (This requirement allows us to consider such things as mean value and standard deviation.)
- Some examples of discrete random variables are given below:

Activity	**A possible discrete random variable**	**Values variable can take**
Rolling a normal die.	Number on uppermost face.	1, 2, 3, 4, 5, 6.
Rolling two normal dice.	Sum of 2 uppermost numbers.	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.
Flipping a coin n times.	Number of tails obtained.	0, 1, 2, 3, 4, 5, …, n.
Cars passing a checkpoint.	Number of cars passing in 5 minutes.	0, 1, 2, 3, 4, 5, 6, 7, …
Having 5 children.	Number of boys.	0, 1, 2, 3, 4, 5.
Repeatedly flipping coin.	Number of flips until a tail is obtained.	1, 2, 3, 4, 5, …
Temperature investigation.	Number of days in January with temp ≥ 35°C.	0, 1, 2, 3, …, 31.
Subtracting integers.	$a - b$ for $a \in \{1, 2, 3, 4\}$ and $b \in \{1, 2, 3\}$.	−2, −1, 0, 1, 2, 3.
Attempting 10 questions in a multiple choice test.	Number of questions correct.	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

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EXAMPLE 1

Suppose that a fair coin is flipped three times. If X is the discrete random variable 'Number of heads obtained', copy and complete the following table.

x	0	1	2	3
P($X = x$)				

Hence determine **a** $P(X < 1)$, **b** $P(X \geq 2)$.

Solution

Considering the eight equiprobable events:

TTT | HTT THT TTH | THH HTH HHT | HHH

$X = 0$ | $X = 1$ | $X = 2$ | $X = 3$

$P(X = 0) = \frac{1}{8}$ $P(X = 1) = \frac{3}{8}$ $P(X = 2) = \frac{3}{8}$ $P(X = 3) = \frac{1}{8}$

The table can then be completed:

x	0	1	2	3
P($X = x$)	0.125	0.375	0.375	0.125

a $P(X < 1) = 0.125$

b $P(X \geq 2) = 0.375 + 0.125$
$= 0.5$

Note

- The table of probabilities completed in the previous example shows how the total probability of 1 is distributed amongst the possible values the random variable X can take. The table gives the **probability distribution** for the random variable X.
- The possible values the random variable can take must together cover all eventualities without overlap. We say they must be **exhaustive** and **mutually exclusive**.
- The sum of the probabilities in a probability distribution must be 1.
- From our understanding of probability it also follows that for each value of x, $0 \leq P(X = x) \leq 1$.
- If the various possible values that the discrete random variable X can take all have the same probability of occurring then we say that the discrete random variable is **uniform**. For the activity of rolling a normal die, if X is the number shown on the uppermost face then X is a **uniform discrete random variable**.

  For a uniform discrete random variable with n possible values, the probability of each value occurring is $\frac{1}{n}$.

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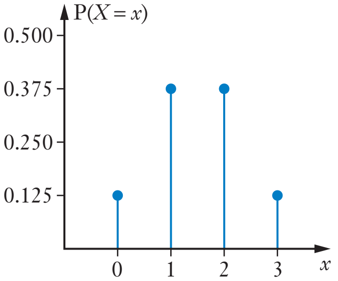
- For each value the discrete random variable, X, can take, the table assigns the corresponding probability of X taking that value. In mathematics we call a rule or relationship that assigns to each element of one set an element from a second set, a **function**. In the previous example the pairs of values in the table show the **probability function** for the random variable X. We frequently use the notation $f(x)$ to represent a function so we will sometimes use $f(x)$ for $P(X = x)$.
- The table below shows the **cumulative probabilities** for example 1. Check that you agree with each of the probabilities.

x	0	1	2	3
$P(X \leq x)$	0.125	0.5	0.875	1

  Note that in a cumulative probability table each probability must be at least as big as the one before it and the final probability must be 1.

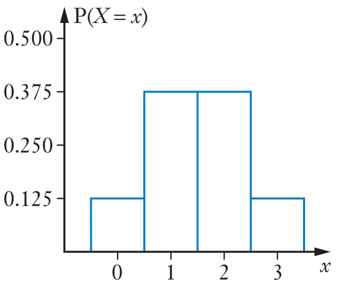
- The probability function may be presented as a table showing the pairs of values $(x, P(X = x))$, as in the previous example, and sometimes it may be possible to express the function as a rule, as in the next example.
- Probability functions can also be presented graphically. One way this may be done is to use vertical lines, as shown on the right.

  This type of display, with its separated vertical lines, emphasises that we are dealing with a random variable that takes separate values, i.e. a *discrete* random variable.

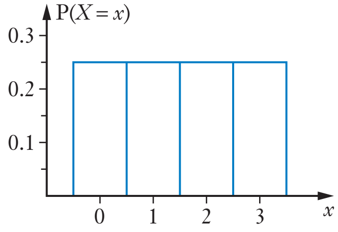


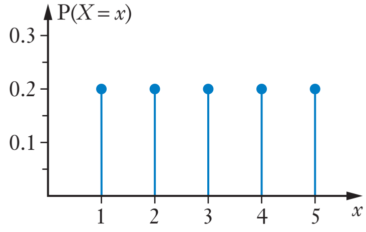
An alternative form of display that is commonly used shows the distribution as columns of equal width, as shown in the second diagram on the right.

This second form is sometimes referred to as the *probability histogram*.



The diagrams below are for **uniform random variables** with the one below left having four outcomes and below right having five outcomes.





EXAMPLE 2

The probability function for a discrete random variable X is given by:

$$P(X = x) = \begin{cases} k(6 - x) & \text{for } x = 1, 2, 3, 4, 5. \\ 0 & \text{for all other values of } x. \end{cases}$$

Copy and complete the following probability distribution for X, giving the probabilities as numbers (i.e. k should be evaluated).

x	1	2	3	4	5
$P(X = x)$					

Determine **a** $P(X = \text{even})$ **b** $P(X > 3)$ **c** $P(X = 4 \mid X > 3)$

Solution

$P(X = 1) = k(6 - 1) = 5k$

$P(X = 2) = k(6 - 2) = 4k$

$P(X = 3) = k(6 - 3) = 3k$

$P(X = 4) = k(6 - 4) = 2k$

$P(X = 5) = k(6 - 5) = k$

But these probabilities must add up to 1. Thus $15k = 1$

$\therefore k = \dfrac{1}{15}$

The table can then be completed:

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

a $P(X = \text{even}) = \dfrac{4}{15} + \dfrac{2}{15} = \dfrac{2}{5}$

b $P(X > 3) = \dfrac{2}{15} + \dfrac{1}{15} = \dfrac{1}{5}$

c $P(X = 4 \mid X > 3) = \dfrac{2}{15} \div \dfrac{1}{5} = \dfrac{2}{3}$

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EXAMPLE 3

A spinner shows the numbers 1, 2, 3.

For each spin of this spinner the probability associated with each outcome, 1, 2 or 3, is as shown in the table.

x	1	2	3
$P(X = x)$	0.5	0.3	k

a Determine the value of k.

The spinner is spun twice. Determine the probability of getting

b a 2 and then a 3,

c a 2 and a 3 in any order,

d the same number twice,

e a total of 4 when the two numbers obtained are added together,

f a 2 on the second spin given that the two spins give a total of 4.

Solution

a The probabilities must sum to 1.

$$\therefore \quad 0.5 + 0.3 + k = 1$$

$$\text{Thus} \quad k = 0.2$$

b
$$\begin{aligned} P(2 \text{ then } 3) &= P(2 \text{ on 1st spin}) \times P(3 \text{ on 2nd spin} \mid 2 \text{ on 1st}) \\ &= 0.3 \times 0.2 \\ &= 0.06 \end{aligned}$$

c
$$\begin{aligned} P(2 \text{ and } 3 \text{ in any order}) &= P(2 \text{ then } 3) + P(3 \text{ then } 2) \\ &= 0.3 \times 0.2 + 0.2 \times 0.3 \\ &= 0.12 \end{aligned}$$

d
$$\begin{aligned} P(\text{same number twice}) &= P(1 \text{ then } 1) + P(2 \text{ then } 2) + P(3 \text{ then } 3) \\ &= 0.5 \times 0.5 + 0.3 \times 0.3 + 0.2 \times 0.2 \\ &= 0.38 \end{aligned}$$

e
$$\begin{aligned} P(\text{total of } 4) &= P(1 \text{ then } 3) + P(3 \text{ then } 1) + P(2 \text{ then } 2) \\ &= 0.5 \times 0.2 + 0.2 \times 0.5 + 0.3 \times 0.3 \\ &= 0.29 \end{aligned}$$

f
$$\begin{aligned} P(2 \text{ second} \mid \text{total of } 4) &= \frac{P(2 \text{ then } 2)}{P(1 \text{ then } 3) + P(3 \text{ then } 1) + P(2 \text{ then } 2)} \\ &= \frac{0.09}{0.29} \\ &= \frac{9}{29} \end{aligned}$$

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EXAMPLE 4

A batch of 100 components include 5 that are faulty. Four components are randomly selected from the batch without replacement. If X is the number of faulty components in the selection determine the probability distribution for X.

Solution

Using ✓ for not faulty and ✗ for faulty:

$$\begin{aligned} P(X=0) &= P(✓✓✓✓) \\ &= \frac{95}{100}\times\frac{94}{99}\times\frac{93}{98}\times\frac{92}{97} \\ &\approx 0.811\,875 \end{aligned}$$

$$\begin{aligned} P(X=1) &= P(✓✓✓✗)+P(✓✓✗✓)+P(✓✗✓✓)+P(✗✓✓✓) \\ &= \frac{95}{100}\times\frac{94}{99}\times\frac{93}{98}\times\frac{5}{97}\times 4 \\ &\approx 0.176\,495 \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(✓✓✗✗)+P(✓✗✓✗)+P(✓✗✗✓)+P(✗✓✓✗)+P(✗✓✗✓)+P(✗✗✓✓) \\ &= \frac{95}{100}\times\frac{94}{99}\times\frac{5}{98}\times\frac{4}{97}\times 6 \\ &\approx 0.011\,387 \end{aligned}$$

$$\begin{aligned} P(X=3) &= P(✓✗✗✗)+P(✗✓✗✗)+P(✗✗✓✗)+P(✗✗✗✓) \\ &= \frac{95}{100}\times\frac{5}{99}\times\frac{4}{98}\times\frac{3}{97}\times 4 \\ &\approx 0.000\,242 \end{aligned}$$

$$\begin{aligned} P(X=4) &= P(✗✗✗✗) \\ &= \frac{5}{100}\times\frac{4}{99}\times\frac{3}{98}\times\frac{2}{97} \\ &\approx 0.000\,001 \end{aligned}$$

These probabilities are shown tabulated below:

x	0	1	2	3	4
P($X = x$)	0.811 875	0.176 495	0.011 387	0.000 242	0.000 001

Alternatively these probabilities can be determined using the notation ^nC_r to represent the number of combinations of r objects chosen from n different objects, as shown on the next page.

As the *Preliminary work* reminded us:

^nC_r is also written $\binom{n}{r}$, it equals $\dfrac{n!}{(n-r)!r!}$ and can be thought of as 'from n choose r'.

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Number of samples of 4 $= {}^{100}C_4$

Number with none faulty $= {}^{5}C_0 \times {}^{95}C_4$

Thus $\quad P(X = 0) = \dfrac{{}^{5}C_0 \times {}^{95}C_4}{{}^{100}C_4}$

$\approx 0.811\,875$

	Faulty	Okay
From	5	95
Choose	0	4

Number with one faulty $= {}^{5}C_1 \times {}^{95}C_3$

Thus $\quad P(X = 1) = \dfrac{{}^{5}C_1 \times {}^{95}C_3}{{}^{100}C_4}$

$\approx 0.176\,495$

	Faulty	Okay
From	5	95
Choose	1	3

Number with two faulty $= {}^{5}C_2 \times {}^{95}C_2$

Thus $\quad P(X = 2) = \dfrac{{}^{5}C_2 \times {}^{95}C_2}{{}^{100}C_4}$

$\approx 0.011\,387$

	Faulty	Okay
From	5	95
Choose	2	2

Number with three faulty $= {}^{5}C_3 \times {}^{95}C_1$

Thus $\quad P(X = 3) = \dfrac{{}^{5}C_3 \times {}^{95}C_1}{{}^{100}C_4}$

$\approx 0.000\,242$

	Faulty	Okay
From	5	95
Choose	3	1

Number with four faulty $= {}^{5}C_4 \times {}^{95}C_0$

Thus $\quad P(X = 4) = \dfrac{{}^{5}C_4 \times {}^{95}C_0}{{}^{100}C_4}$

$\approx 0.000\,001$

	Faulty	Okay
From	5	95
Choose	4	0

Exercise 8A

1 For each of the following state whether the variable is a discrete variable or a continuous variable.

- **a** The heights of students in a year eight class.
- **b** The number of heads obtained in ten flips of a coin.
- **c** The weight of breakfast cereal in packets that each state ‘contains approximately 500 grams’.
- **d** The number of coins in a piggy bank.
- **e** The number of students in a school.
- **f** The individual weights of 50 dogs seen at a vet’s surgery one week.
- **g** The time taken to complete a task.

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For each of the following state whether the table of values could represent a probability function for a discrete variable with possible values 0, 1, 2, 3, 4, 5.

2

x	0	1	2	3	4	5
$f(x)$	0.1	0.2	0.25	0.25	0.1	0.3

3

x	0	1	2	3	4	5
$f(x)$	0.2	0.1	0.1	0.2	0.1	0.1

4

x	0	1	2	3	4	5
$f(x)$	−0.2	0.1	0.3	0.5	0.2	0.1

5

x	0	1	2	3	4	5
$f(x)$	0.2	0.1	0.4	0.1	0.1	0.1

Each of the following tables show probability distributions for the random variable X, with X able to take the values 0, 1, 2, 3, 4. Determine k in each case.

6

x	0	1	2	3	4
$P(X = x)$	0.1	0.1	0.2	0.2	k

7

x	0	1	2	3	4
$P(X = x)$	0.25	k	0.25	0.1	0.35

8

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$2k$	$3k$	$2k$

9

x	0	1	2	3	4
$P(X = x)$	$-0.25k$	$k + 0.9$	$k + 1$	$-0.5k$	$k + 0.9$

10 Suppose that a fair coin is flipped twice. If X is the discrete random variable 'Number of tails obtained', construct a table showing the probability distribution for X.

11 The probability distribution for the random variable X is as shown below:

x	0	1	2	3	4	5
$P(X = x)$	0.2	0.4	0.1	0.1	0.1	0.1

Determine

a $P(X = 0)$ **b** $P(X \geq 1)$ **c** $P(2 < X \leq 4)$

d $P(X = 1 | X \geq 1)$ **e** $P(X > 4 | X \geq 2)$ **f** $P(X \leq 4 | X \geq 2)$

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12 The probability distribution for the random variable X is as shown below:

x	0	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.3	0.2	0.1	0.1

Determine

a $P(X > 2)$ **b** $P(X \geq 3)$ **c** $P(1 < X < 4)$

d $P(X = 3 | X > 2)$ **e** $P(X = 5 | X \geq 3)$ **f** $P(X < 4 | X \geq 3)$

13 The discrete random variable X has the following probability distribution:

x	0	1	2	3	4	5	6	7	8	9	10
$P(X = x)$	0.005	0.01	0.04	0.12	0.2	0.25	0.2	0.12	0.04	0.01	0.005

Copy and complete the following table showing cumulative probabilities.

x	0	1	2	3	4	5	6	7	8	9	10
$P(X \leq x)$											

14 The table below shows the cumulative probabilities for the random variable X.

x	0	1	2	3	4	5
$P(X \leq x)$	0.04	0.2	0.5	0.8	0.96	1

Given that the possible values X can take are 0, 1, 2, 3, 4, 5, copy and complete the following table.

x	0	1	2	3	4	5
$P(X = x)$						

15 Suppose that for each flip of a biased coin the probability of getting a head is 0.4. If this coin is flipped twice, and X is the number of tails obtained, copy and complete the following probability distribution table.

x			
$P(X = x)$			

16 Suppose that for each flip of a biased coin the probability of getting a head is $\frac{1}{3}$. If this coin is flipped three times, and X is the number of heads obtained, copy and complete the following probability distribution.

x				
$P(X = x)$				

17 A bag contains 5 marbles, 3 red and 2 blue. Suppose that three marbles are randomly selected from the bag, one after the other, each selected marble not being returned to the bag after selection. Create a probability distribution table for the random variable X, the number of reds this random selection process produces.

18 The probability function for a discrete random variable X is given by

$$P(X = x) = \begin{cases} kx & \text{for} \quad x = 1, 2, 3, 4, 5. \\ 0 & \text{for} \quad \text{all other values of } x \end{cases}$$

Copy and complete the following probability distribution for X, giving the probabilities as numbers (i.e. k should be evaluated).

x	0	1	2	3	4	5
$P(X = x)$						

Determine

a $P(X = \text{even})$ **b** $P(X < 2)$ **c** $P(X > 2)$

19 The probability function for a discrete random variable X is given by

$$P(X = x) = \begin{cases} k(5 - x) & \text{for} \quad x = 1, 2, 3, 4. \\ 0 & \text{for} \quad \text{all other values of } x \end{cases}$$

Copy and complete the following probability distribution for X, giving the probabilities as numbers (i.e. k should be evaluated).

x	0	1	2	3	4
$P(X = x)$					

Determine

a $P(X = \text{even})$ **b** $P(X \leq 2)$ **c** $P(X \geq 2)$

20 A spinner shows the numbers 1, 2, 3, 4. For each spin of this spinner the probability associated with each outcome, 1, 2, 3 or 4, is as given in the table below.

x	1	2	3	4
$P(X = x)$	0.2	0.4	0.1	k

a Determine the value of k.

The spinner is spun twice. Determine the probability of getting

b a 2 and then a 4,

c a 2 and a 4 in any order,

d a total of 6 when the two numbers obtained are added together,

e a 4 on the second spin given that the two spins gave a total of 6.

The spinner is spun three times. Determine the probability of getting

f a 4 and then a 3 and then a 2,

g a 4, 3 and 2 in any order,

h a total of ten when the three numbers obtained are added together,

i the same number three times.

21 A batch of 50 components include 5 that are faulty. Four components are randomly selected from the batch. If X is the number of faulty components in the selection, determine the probability distribution for X, giving probabilities correct to five decimal places.

ISBN 9780170395137

Mean or expected value of a discrete random variable

WS
Expected values using a calculator

Consider the probability distribution for rolling a normal, fair, six-sided die.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

If we were to roll such a die 6000 times we would expect to obtain roughly 1000 of each of the six possible numbers. Hence we would expect our mean to be approximately:

$$\frac{1000 \times 1 + 1000 \times 2 + 1000 \times 3 + 1000 \times 4 + 1000 \times 5 + 1000 \times 6}{6000} = 3.5$$

Writing the above calculation of the mean as follows

$$\frac{1000 \times 1}{6000} + \frac{1000 \times 2}{6000} + \frac{1000 \times 3}{6000} + \frac{1000 \times 4}{6000} + \frac{1000 \times 5}{6000} + \frac{1000 \times 6}{6000}$$

this is the same as

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}.$$

Thus we could obtain the mean value by summing the products:

(value of the random variable) × (probability of that value occurring).

When working with random variables the mean value is sometimes referred to as the **expected value**. For the random variable X, the mean or expected value is sometimes written as $\mathrm{E}(X)$.

Note

Do not be misled by the use of the word 'expected'. Clearly 3.5 is not the outcome we 'expect' from one roll of a normal die. (Just as 'two and a bit children' is not the number of children we expect to find in a randomly chosen family.) Hence the expected value is not the value we expect to get with one roll of the die but is instead the number we expect our long term average to be close to.

For the following probability distribution for the random variable X,

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.1	0.2	0.2	0.4	0.1

$$\begin{aligned} \text{the mean or expected value} &= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.4 + 5 \times 0.1 \\ &= 3.2 \end{aligned}$$

If the discrete random variable, X, has possible values x_i, with $\mathrm{P}(X = x_i) = \mathrm{p}_i$ then

$$\mathrm{E}(X) = \Sigma(x_i \mathrm{p}_i)$$

the summation being carried out over all of the possible values x_i.

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EXAMPLE 5

A game involves a player paying money to roll two dice.

If the total of the two numbers on the uppermost faces is
- less than 6, then the player receives \$5,
- more than 9, then the player receives \$8,
- any other score, then the player receives nothing.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

a What is the amount the player could expect to receive, on average, per game (i.e. what is the mean or expected payout per game)?

b If the cost for each roll of the two dice is \$3, is a player likely to win, lose or break even in the long term?

Solution

a If X is the sum of the two uppermost numbers the probability distribution for X is as follows:

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
	Player receives \$5.				Player receives nothing.				Player receives \$8.		

Hence if Y is the number of dollars the player receives the probability distribution for Y is as follows:

y (dollars received)	0	5	8
$P(Y = y)$	$\frac{20}{36}\left(=\frac{5}{9}\right)$	$\frac{10}{36}\left(=\frac{5}{18}\right)$	$\frac{6}{36}\left(=\frac{1}{6}\right)$

$$\begin{aligned} \text{Expected payout per game} &= \$0 \times \frac{5}{9} + \$5 \times \frac{5}{18} + \$8 \times \frac{1}{6} \\ &= \$2.72 \text{ (to the nearest cent)} \end{aligned}$$

b If each game costs \$3 to play and the average payout is \$2.72 per game the player is likely to lose in the long term. (For example, playing 100 games would cost the player \$300 for an expected return of \$272 – a loss of \$28.)

Note

The game would be regarded as being 'fair' if the cost to play equalled the expected payout per game.

ISBN 9780170395137

The standard deviation of a discrete random variable

Variance and standard deviation

Remember that to determine the standard deviation of a set of scores we find the deviation of each score from the mean, find the average of the squares of these deviations (this gives us the variance of the scores) and then find the square root of our answer.

Thus, as the *Preliminary work* reminded us, for the following 8 scores (for which the mean is 18):

	12	15	16	16	18	20	22	25
Deviation from mean:	−6	−3	−2	−2	0	+2	+4	+7

$$\text{Variance of scores} = \frac{(-6)^2+(-3)^2+(-2)^2+(-2)^2+(0)^2+(2)^2+(4)^2+(7)^2}{8}$$

$$= 15.25$$

$$\text{Standard deviation} = \sqrt{15.25} \quad \text{i.e. } 3.91 \text{ (correct to two decimal places)}$$

Similarly, if the discrete random variable X has possible values x_i with $P(X = x_i) = p_i$ and a mean, or expected value $E(X)$, of μ then

The variance, sometimes written $\text{Var}(X)$, $= \Sigma[p_i(x_i - \mu)^2]$ and the standard deviation is the square root of the variance.

For the dice-rolling situation:

x	1	2	3	4	5	6
P($X = x$)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

As determined earlier, $E(X) = 3.5$. Hence the variance of the distribution is given by

$$\frac{1}{6}\times(1-3.5)^2+\frac{1}{6}\times(2-3.5)^2+\frac{1}{6}\times(3-3.5)^2+\frac{1}{6}\times(4-3.5)^2+\frac{1}{6}\times(5-3.5)^2+\frac{1}{6}\times(6-3.5)^2$$

$$= \frac{35}{12}$$

The standard deviation will then be $\sqrt{\text{Var}(X)}$, i.e. 1.7078, correct to 4 decimal places.

The standard deviation of X is sometimes written $SD(X)$.

Of course, rather than performing these calculations to determine the mean, or expected value, and the standard deviation of a discrete random variable ‘by hand’, we can use the statistical capability of a calculator. Considering the probability as the frequency of occurrence, use the statistics facility of a calculator to output the expected value and the standard deviation for the dice-rolling probability distribution given above.

ISBN 9780170395137

EXAMPLE 6

The discrete random variable X can take the values 1, 2, 3 and 4 with the probability distribution as given in the table at right.

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.4	0.3

a Find E(X), the expected value of X, and Var(X), the variance of X.

b If $Y = 3X$ find E(Y), the expected value of Y, and Var(Y), the variance of Y.

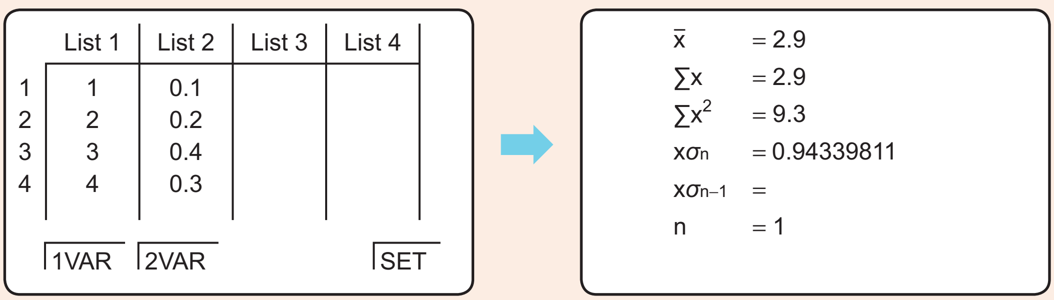
c If $Z = 3X + 4$ find E(Z), the expected value of Z, and Var(Z), the variance of Z.

Solution

a Either by using the statistical functions of a calculator, or by calculation, as shown below,

$$\begin{aligned} E(X) &= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.4 + 4 \times 0.3 \\ &= 2.9 \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= 0.1 \times (1-2.9)^2 + 0.2 \times (2-2.9)^2 + 0.4 \times (3-2.9)^2 + 0.3 \times (4-2.9)^2 \\ &= 0.89 \quad \text{(And hence standard deviation} = \sqrt{0.89} = 0.9434 \text{ to 4 decimal places.)} \end{aligned}$$



b With $Y = 3X$ we now have the distribution:

y	3	6	9	12
$P(Y = y)$	0.1	0.2	0.4	0.3

We have a *change of scale*, as referred to in the *Preliminary work*.

This will multiply the mean value by 3 and the standard deviation by 3 (and hence the variance by 3^2).

Thus $\begin{aligned} E(Y) &= 3 \times E(X) \\ &= 3 \times 2.9 \\ &= 8.7 \end{aligned}$ and $\begin{aligned} \text{Var}(Y) &= 3^2 \times \text{Var}(X) \\ &= 9 \times 0.89 \\ &= 8.01 \end{aligned}$

c With $Z = 3X + 4$ we have both a change of scale and a change of origin. The change of scale alters both the mean and the standard deviation whereas the change of origin does not alter the standard deviation (because it does not alter the spread).

Thus $\begin{aligned} E(Z) &= 3 \times E(X) + 4 \\ &= 3 \times 2.9 + 4 \\ &= 12.7 \end{aligned}$ and $\begin{aligned} \text{Var}(Z) &= 3^2 \times \text{Var}(X) \\ &= 9 \times 0.89 \\ &= 8.01 \end{aligned}$

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Note

(For interest.) Prior to the ready availability of calculators with statistical capabilities, calculating the variance, and the standard deviation, of a probability distribution could be a tedious process, especially if E(X) was not an integer. Subtracting each value from E(X), in order to calculate variance, was a chore. In such cases, the alternative formula $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$ could be used. The reader might like to check that applying this formula to the distribution for X given in the previous example also gives Var(X) = 0.89.

Exercise 8B

The tables in number **1** to **4** show discrete probability distributions for the discrete random variable X. Find k and the mean (or expected) value of X in each case.

1

x	1	2	3	4	5
P($X = x$)	k	0.35	0.35	0.15	0.05

2

x	0	5	10	15	20	25
P($X = x$)	0.1	0.1	0.1	0.1	k	$2k$

3

x	1	2	3	4	5	6	7	8
P($X = x$)	k	k	$2k$	k	$2k$	$3k$	$4k$	$6k$

4

x	1	2	5	8	10	20	25	50	100
P($X = x$)	0.05	0.15	k	0.25	0.15	0.10	0.05	0.03	0.02

5 The expected value of the discrete probability distribution given below is 2.7. Find the values of p and q and hence determine Var(X), the variance of X.

x	1	2	3	4	5
P($X = x$)	0.3	p	0.2	q	0.1

6 The expected value of the discrete probability distribution given below is $\frac{52}{9}$. Find the values of p and q.

x	0	1	2	3	4	5	6	7	8
P($X = x$)	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	p	q

7 If the discrete random variable X has an expected value, $E(X)$, of 10 and a standard deviation of 1.5, find

a $E(X + 5)$,

b the standard deviation of $(X + 5)$,

c $E(3X - 4)$,

d the standard deviation of $(3X - 4)$.

8 The probability distribution of the discrete random variable X is as shown below.

x	10	20	30	40	50
$P(X = x)$	0.3	0.2	0.2	0.2	0.1

a Find $E(X)$, the expected value of X, and $Var(X)$, the variance of X.

Hence write the value of each of the following:

b $E(X + 3)$ **c** $E(2X)$ **d** $E(2X + 3)$

e $Var(X + 3)$ **f** $Var(2X)$ **g** $Var(2X + 3)$

9 A *uniform* discrete random variable, X, can take the values 1, 2, 3, 4, 5.

Find $E(X)$ and $Var(X)$.

10 Sue and Bob are developing a gambling game. Each 'play' of the game involves the rolling of two normal fair six sided dice and the two numbers on the uppermost faces being added together.

- Any double, i.e. two 1s, two 2s, two 3s etc., pays \$30.
- A total of 7 pays \$15.
- Anything other than the above two outcomes pays nothing.

If the discrete random variable, \$$X$, is the amount paid out on a single play, copy and complete the following probability distribution table for X.

x (\$ paid out)	0	15	30
$P(X = x)$			

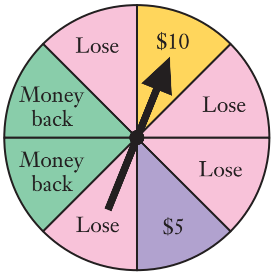
In the long run, Sue and Bob want to make an average of \$0.50 profit per game played. How much should they charge for each 'play'?

11 A game is being devised based on the spinner shown on the right. The spinner features eight equal-size sectors such that each sector is equally likely to be the one the arrow finally points to.

A player pays an amount to play the game and each play involves one spin of the spinner.

Each sector indicates the amount received should the arrow end up pointing to that sector, with 'lose' meaning nothing is received and 'money back' meaning the cost of the game is paid back.

How much should the organisers charge for each game so that in the long-run they should at least break even (or be very close to break even)?



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12 A fair eight sided die featuring the numbers 1, 2, 3, 4, 5, 6, 7 and 8 with one number on each face, is rolled.

a The discrete random variable X is the number shown on the uppermost face. Find the mean value of X.

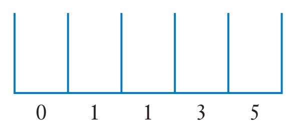
b The discrete random variable Y is the square of the number on the uppermost face. Find the mean value of Y.

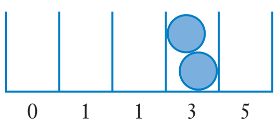
c The discrete random variable Z is $\dfrac{1}{\text{the number on the uppermost face}}$.

Find the mean value of Z.

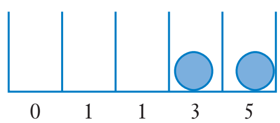
13 A game involves two balls being released, one after the other, rolling down a slope and each ending up in one of five numbered slots, as shown below on the right. The random release mechanism is such that each slot has an equal chance of receiving each ball.

A player pays \$5 for one release of the two balls. The total of the scores achieved by the two balls is the player's score for the game and the player receives that number of dollars as a prize.

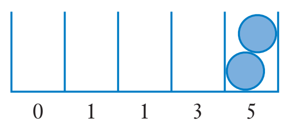




\$6



\$8



\$10

a What percentage of games would you expect to result in a prize of more than \$6 being awarded (i.e. a profit to the player of more than \$1)?

b What total score would we expect 40% of the plays to exceed?

c Approximately how much should the organisers expect to be 'up' after 100 plays of the game?

14 Bill is offered a part-time job as a salesman of new cars. He is told that an analysis of the data from other salespeople indicates that if X is the number of new car sales he can expect to achieve in a fortnight, when working the proposed number of hours, the probability distribution for X tends to approximately follow the pattern:

x	0	1	2	3	4	5
$P(X = x)$	0.15	0.25	0.35	0.15	0.05	0.05

He is offered two schemes of reward:

[1] \$500 per fortnight plus \$250 for each new car sold,

or [2] \$nil per fortnight plus \$475 per new car sold.

Based on expected earnings which scheme should he prefer? (Justify your answer.)

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15 **a** A financial analyst estimates that, for a particular investment scheme, if $\$X$ is the value after three years of an initial investment of \$1000 then X has a probability distribution:

x	500	800	1000	2000	3000
$P(X = x)$	0.2	0.3	0.1	0.3	0.1

Determine the mean, or expected, value of the initial \$1000 at the end of the three years.

b An alternative investment offers $\$Y$ as the final value after three years of an initial investment of \$1000 with the same analyst estimating the probability distribution for Y as:

y	800	1000	1200	1500
$P(Y = y)$	0.1	0.2	0.2	0.5

Determine the mean, or expected, value of the initial \$1000 at the end of the three years.

c Which of the two schemes would you advise is the better and why?

Miscellaneous exercise eight

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 A medical research team monitoring the spread of a particular notifiable disease in a certain country town surveyed the number of cases that the authorities were notified about over a period of time. The total number of notified cases, t weeks after the survey commenced was N where:

$$N \approx \frac{100\,000}{1 + 499e^{-0.08t}}.$$

Determine
a the number of recorded cases when the survey commenced,
b the number of recorded cases when $t = 5$,
c the number of recorded cases when $t = 10$,
d what happens as $t \to \infty$ if no action occurs to inhibit the spread of the disease.

2 Differentiate each of the following expressions with respect to x.

a $\dfrac{6}{x}$

b $\dfrac{6}{\sqrt{x}}$

c $5x^2 - e^x$

d $e^{3x^2} + 1$

e e^{3x^2+1}

f $(2x - 3)(2x + 1)^5$

g $10 \sin x$

h $\sin 10x$

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3 Determine $\frac{d}{dx}\left(\int_0^x (3t^2 - 5)\, dt\right)$.

4 For each of the following state whether the variable X is a uniform discrete random variable and explain your reasoning for each one.

a X is the number of sixes obtained when two normal fair dice are rolled.

b X is the number on the uppermost face of a normal fair die when it is rolled.

c X is the height of a randomly chosen male aged between 20 and 25.

5 Find the coordinates of any points where the curve $y = \frac{5x^2}{x-1}$

cuts the line $y = 5x + 3$.

Find the gradient of the curve at these points.

6 Find the exact gradient of $y = x^2e^{2x}$ at the point $(1, e^2)$.

7 Find the equation of the normal to $y = \frac{x^2}{x-2}$ at the point (3, 9).

Note:

- The *normal* to a curve, at some point A on the curve, is the line through A perpendicular to the tangent to the curve at point A.
- The gradients of two perpendicular lines have a product of −1.

8 Evaluate the following definite integrals without the assistance of your calculator.

a $\int_0^2 10x^4\, dx$ **b** $\int_2^4 2\, dx$ **c** $\int_2^3 (2 + 6x)\, dx$

9 Find the area between $y = \sin x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{2}$.

10 Let us suppose that a particular freefalling object, under specific conditions of air resistance etc., falls such that its downward velocity, t seconds after release, is given by v m/s, where $v = 75(1 - e^{-0.13t})$ m/s.

a An object in free fall does not keep accelerating indefinitely but, due to air resistance, reaches a terminal velocity. What is the terminal velocity of this free falling object?

b Find, in m/s^2 and correct to two decimal places, the acceleration of the object

i 5 seconds after release,

ii 20 seconds after release.

11 Find $f(x)$ given that $f''(x) = 20(3 - x)^3 + 6x - 6$, $f'(1) = -83$ and $f(1) = 28$.

12 By writing x^5 as $\frac{x^3 \times x^4}{x^2}$ differentiate $y = x^5$ using the product and quotient rules.

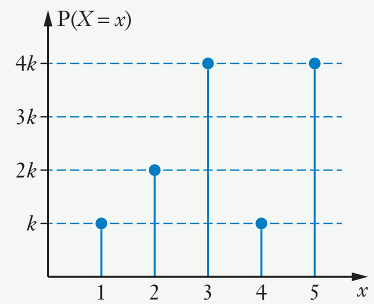
13 Find the exact coordinates of any points on $y = \frac{e^x}{2x}$ where the gradient is 0.

14 Find the exact values of x, $-2\pi \le x \le 2\pi$, for which the curve $y = e^x \cos x$ has a gradient of zero.

15 Integrate the following with respect to x without the assistance of your calculator.

a $4x$ **b** $6e^{2x}$ **c** $\frac{d}{dx}(x^2 + e^x)$ **d** $\frac{d}{dx}(x^2 e^x)$

16 The discrete random variable X can take the values 1, 2, 3, 4, 5. The probability distribution for X is shown graphically on the right.



Determine **a** k,

b $P(X = 3)$,

c $P(X > 3)$,

d $P(X \ge 3)$,

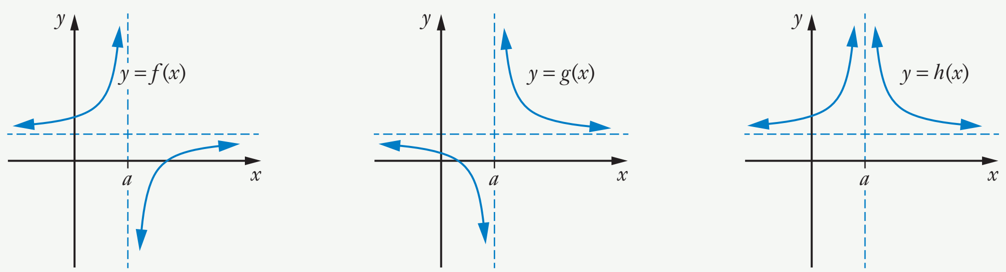
e $P(X = 3 \mid X > 3)$,

f $P(X = 3 \mid X \ge 3)$.

g Without the use of your calculator, determine $E(X)$, the expected value of X.

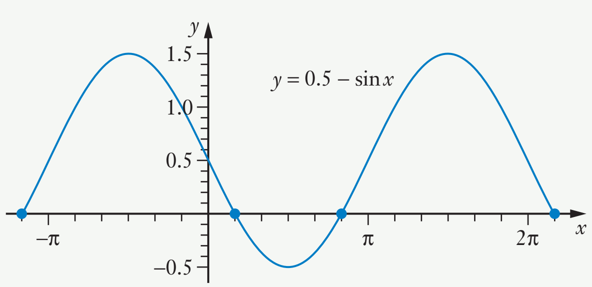
h With the assistance of your calculator, determine the standard deviation of X, giving your answer correct to two decimal places.

17 The graphs below are of $y = f(x)$, $y = g(x)$ and $y = h(x)$.



Produce sketches showing the graphs of $y = f'(x)$, $y = g'(x)$ and $y = h'(x)$.

18 Find as exact values



a $\int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx$

b $\left| \int_0^{\frac{5\pi}{6}} (0.5 - \sin x)\,dx \right|$

c the area between $y = 0.5 - \sin x$ and the x-axis from $x = 0$ to $x = \frac{5\pi}{6}$.

ISBN 9780170395137

9.

Bernoulli and binomial distributions

Situation

It is April 1st and Ms Hamish, a teacher of mathematics, decides to see if she can trick her year 12 class into thinking they have a surprise test.

She walks into class brandishing some sheets of paper and announces:

'Okay separate the desks we have a spot test on probability today.'

'Twenty questions, multiple choice, no talking'

The surprised, and not a little alarmed, students express their concerns:

'You didn't tell us we had a test.' *'Can we use our calculators?'*

'Does the mark count towards our assessment?'

'Are notes allowed?' *'That's not fair.'* *'Does it count?'*

'Are other classes doing the same test?'

'Come on, Miss, give us a break.' *'What did you say it was on?'*

'Is this an April Fool joke, Miss?'

Ignoring all such comments Ms Hamish walks around the class giving out a response sheet to each student. Each sheet shows the numbers 1 to 20 each with five possible responses (a) to (e). Part of such a sheet is shown below:

PROBABILITY TEST Name: ______________________

For each question circle one response out of (a), (b), (c), (d) or (e).

1. (a)	**(b)**	**(c)**	**(d)**	**(e)**
2. (a)	**(b)**	**(c)**	**(d)**	**(e)**
3. (a)	**(b)**	**(c)**	**(d)**	**(e)**

'You have one minute to make your twenty choices.' says Ms Hamish.

'Start now.'

'This is an April Fool trick, isn't it, Miss? asks one student.

'Well, yes it is, actually, and there are no questions to go with the responses, but I want you all to do it anyway because we will discuss the responses later.'

At the end of the minute all students have made their twenty random guesses, as has Ms Hamish on a sheet which she then proclaims as 'the answer sheet'!

As she reads out the 'correct' responses according to the random guesses she made, the students mark their response sheets.

Given that there were 21 students in the class and each student could have a final score of 0, 1, 2, 3, …. 20 roughly how many students would you expect to get each score? Make a table of your estimates and then compare and discuss your table with others in your class.

ISBN 9780170395137

The Bernoulli distribution

Binomial probability experiments

Using the binomial probability distribution

Binomial probability – Mean and standard deviation

Bernoulli distributions

Considering the situation from the previous page, with the possible final scores being the discrete values 0, 1, 2, 3, …, 20, and there being 21 students in the class would we expect the following uniform distribution?

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Number of students	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Such a distribution would be somewhat unlikely, to say the least.

If we consider the situation of the previous page to involve 20 *trials*, with each trial being that of answering a question, then for each trial there are two possibilities, which we will call success and failure. If we call the failure a zero and the success a 1, we have a discrete random variable with the two possible values, 0 and 1, with probabilities 0.8 and 0.2 respectively.

x	0	1
$\mathrm{P}(X = x)$	0.8	0.2

A trial which can be considered to have just two mutually exclusive and exhaustive outcomes, sometimes referred to as *success* and *failure*, is called a **Bernoulli trial**, named after one of the famous Bernoulli family of Swiss mathematicians. The associated random variable, with its two possible values, 0 and 1, is called a **Bernoulli random variable**. If the probability of success is p then the probability of failure will be $(1 - p)$.

	Failure	**Success**
X	0	1
$\mathrm{P}(X = x)$	$1 - p$	p

We say that the Bernoulli random variable has **parameter** p, the probability of obtaining a 1. The parameter is a constant *characteristic* of the situation. Each time the Bernoulli trial is carried out the probability of success is p. Knowing p allows us to determine the probabilities associated with each value of the Bernoulli random variable.

If we roll a normal six-sided die there are six possible outcomes, 1, 2, 3, 4, 5 and 6. We have a uniform discrete random variable with probability distribution:

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

However, if we are only concerned with 'getting a 6', which we will call a success, or 'not getting a 6', which we will call a failure, we have a Bernoulli random variable, with probability distribution:

	Failure	**Success**
x	0	1
$\mathrm{P}(X = x)$	$\frac{5}{6}$	$\frac{1}{6}$

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Some examples of Bernoulli random variables:

- Flipping a coin with obtaining a head being considered a success.

	Tail	Head
x	0	1
P($X = x$)	$\frac{1}{2}$	$\frac{1}{2}$

- Guessing the answer to a multiple-choice situation in which there are four answers to choose from, only one of which is correct.

	Wrong answer	Correct answer
x	0	1
P($X = x$)	$\frac{3}{4}$	$\frac{1}{4}$

- A randomly-selected seed germinating (success) or not,

	Not germinating	Germinating
x	0	1
P($X = x$)	$1 - p$	p

with the value of p being estimated by experiment.

Applying $\quad E(X) = \Sigma(x_i p_i)$,

and $\quad Var(X) = \Sigma[p_i(x_i - E(X))^2] \quad$ or $\quad Var(X) = E(X^2) - [E(X)]^2$

to the **Bernoulli distribution** with parameter p:

x	0	1
P($X = x$)	$1 - p$	p

$$\begin{aligned} E(X) &= 0 \times (1 - p) + 1 \times p \\ &= p \\ Var(X) &= (1 - p) \times (0 - p)^2 + p \times (1 - p)^2 \\ &= p(1 - p)(p + 1 - p) \\ &= p(1 - p) \end{aligned}$$

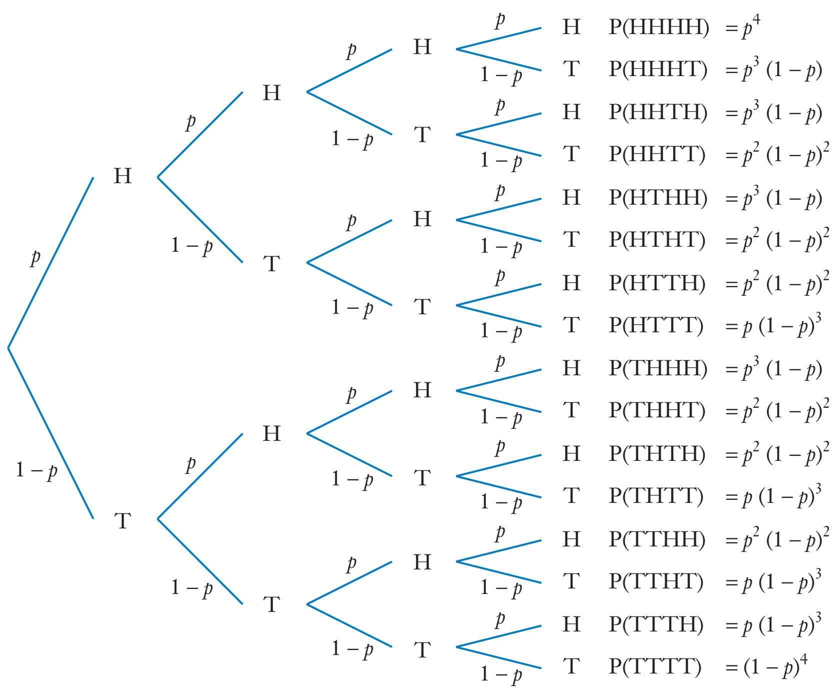
- The long-term mean, or expected value E(X), of a Bernoulli distribution with parameter p is p.
- The variance of a Bernoulli distribution with parameter p is $p(1 - p)$.
- A Bernoulli random variable can be used as the probability model for situations involving two mutually exclusive outcomes.

Binomial distributions

If a Bernoulli trial is performed repeatedly, with the probability of success in a trial occurring with constant probability, i.e. the trials are **independent**, the distribution that arises by considering the number of successes is called a **binomial distribution**.

For example, suppose that a biased coin is flipped four times and that on each flip the probability of the result being a head, which we will call a success, is p. Each flip of the coin is a Bernoulli trial with P(success) $= p$ and P(failure) $= 1 - p$.

The probability tree diagram for this coin flipping is as follows:



Notice that '2 heads and 2 tails in any order' occur on 6 of the final outcomes, and each with probability $p^2(1-p)^2$.

If X is the number of heads this procedure produces, then

$$P(X = 2) \quad = \quad 6p^2(1-p)^2$$

The complete probability distribution for X is as follows:

x	0	1	2	3	4
P($X = x$)	$(1-p)^4$	$4p(1-p)^3$	$6p^2(1-p)^2$	$4p^3(1-p)$	p^4

Suppose instead that this coin were flipped eight times and we again use X for the number of heads. What would $P(X = 2)$ be now?

The tree diagram would be large and tedious to construct so instead we 'think it through' as explained on the next page.

ISBN 9780170395137

From the start of such a tree diagram, to a final outcome involving '2 heads and 6 tails' we travel along eight branches, 2 of which have probability p, and the other 6 having probability $(1 - p)$.

Question: How many such branches are there?

Answer: The same number as there are ways of choosing which 2 of the 8 branches will be the 'p branches', i.e. ${}^8C_2$ (= 28).

Thus
$$\begin{aligned} P(X=2) &= {}^8C_2\, p^2(1-p)^6 \\ &= 28p^2(1-p)^6 \end{aligned}$$

Similarly
$$\begin{aligned} P(X=3) &= {}^8C_3\, p^3(1-p)^5 \\ &= 56p^3(1-p)^5 \end{aligned}$$

$$\begin{aligned} P(X=4) &= {}^8C_4\, p^4(1-p)^4 \\ &= 70p^4(1-p)^4 \end{aligned}$$
etc.

To generalise:

Suppose that an event or trial is repeated n times, and in each trial we consider only two outcomes, A and A′. Further suppose that in each trial the probability of event A occurring is p (from which it follows that P(A′) = $1 - p$).

In the n trials, the probability of A occurring x times ($x \le n$), is:

$${}^nC_x\, p^x(1-p)^{n-x}$$

or using the form $\binom{n}{x}$ for ${}^nC_x$
$$\binom{n}{x} p^x(1-p)^{n-x}$$

EXAMPLE 1

A normal fair die is rolled ten times.

Determine the probability of obtaining exactly **a** three sixes **b** five sixes.

Solution

If the discrete random variable X is the number of sixes in ten rolls of the die then

$$P(X=x) = {}^{10}C_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}$$

a $$\begin{aligned} P(X=3) &= {}^{10}C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \\ &\approx 0.155 \end{aligned}$$

In 10 rolls of a normal die the probability of obtaining exactly three sixes is approximately 0.155.

b $$\begin{aligned} P(X=5) &= {}^{10}C_5 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^5 \\ &\approx 0.013 \end{aligned}$$

In 10 rolls of a normal die the probability of obtaining exactly five sixes is approximately 0.013.

EXAMPLE 2

Suppose that each time a particular soccer player takes a penalty kick the probability of scoring a goal is 0.6. Determine the probability that if the player takes eight penalty kicks they will score

a exactly five times,

b at least five times.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Solution

a Defining the random variable X as the number of goals scored in the eight attempts, X can take the values 0, 1, 2, 3, 4, 5, 6, 7, 8.

$P(X=x) = {}^8C_x(0.6)^x(0.4)^{8-x}$ Thus $P(X=5) = {}^8C_5(0.6)^5(0.4)^3 \approx 0.279$

The probability of the player scoring exactly five goals in eight penalty attempts is approximately 0.28.

b
$$\begin{aligned} P(X \geq 5) &= P(X=5) + P(X=6) + P(X=7) + P(X=8) \\ &= {}^8C_5(0.6)^5(0.4)^3 + {}^8C_6(0.6)^6(0.4)^2 + {}^8C_7(0.6)^7(0.4)^1 + {}^8C_8(0.6)^8(0.4)^0 \\ &\approx 0.2787 + 0.2090 + 0.0896 + 0.0168 \\ &= 0.594 \text{ correct to three decimal places.} \end{aligned}$$

The probability of the player scoring at least five goals in eight penalty attempts is approximately 0.59.

Note

- If a Bernoulli trial is performed n times, and the probability of success in each trial is p, the probability of exactly x successes in the n trials is

$$^nC_x\, p^x(1-p)^{n-x}$$

- The number of trials, n, and the probability of success on each trial, p, are called the **parameters** of the distribution. If we know that a random variable is binomially distributed and the parameters n and p are known, the probability distribution can be completely determined.
- If the discrete random variable X is binomially distributed with parameters n and p this is sometimes written as:

$$X \sim b(n, p), \quad X \sim B(n, p), \quad X \sim \text{bin}(n, p) \quad \text{or} \quad X \sim \text{Bin}(n, p).$$

- For a binomial distribution involving n trials, with p the probability of success on each trial:

Mean, i.e. E(X), $= \boldsymbol{np}$

and **Var(X)** $= \boldsymbol{np(1-p)}$ or npq, where $q = 1 - p$, the probability of failure on each trial.

Hence **SD(X)** $= \boldsymbol{\sqrt{np(1-p)}}$ or $\sqrt{npq}$ where $q = (1 - p)$.

ISBN 9780170395137

- Considering the number of marbles obtained of a certain colour when marbles are selected from a bag, with each marble replaced before the next is drawn, will involve a binomial distribution. If replacement does not occur the probability of 'success' is not the same for each trial and the distribution would not be binomial. However if the bag contains a very large number of marbles, and the sample size is comparatively small, the probability of success is almost constant and the binomial distribution could be used to model the situation. This is often the situation in an opinion poll when a small sample is chosen from a large population.

EXAMPLE 3

A Bernoulli trial has P(success) = p and P(failure) = $(1 - p)$. The trial is carried out five times, with each trial outcome independent of the outcomes of the other trials. If the random variable X is the number of successes achieved in these five trials show the probability distribution of X in terms of p.

Solution

In this case $X \sim \text{Bin}(5, p)$ and so: $\quad P(X = x) = \binom{5}{x} p^x (1-p)^{5-x}$

Thus

$$P(X = 0) = \binom{5}{0} p^0 (1-p)^5 \qquad P(X = 1) = \binom{5}{1} p^1 (1-p)^4$$

$$P(X = 2) = \binom{5}{2} p^2 (1-p)^3 \qquad P(X = 3) = \binom{5}{3} p^3 (1-p)^2$$

$$P(X = 4) = \binom{5}{4} p^4 (1-p)^1 \qquad P(X = 5) = \binom{5}{5} p^5 (1-p)^0$$

The complete distribution would be:

x	0	1	2	3	4	5
P($X = x$)	$(1-p)^5$	$5p(1-p)^4$	$10p^2(1-p)^3$	$10p^3(1-p)^2$	$5p^4(1-p)$	p^5

Notice that if we write $(1 - p)$ as q, this becomes

x	0	1	2	3	4	5
P($X = x$)	q^5	$5pq^4$	$10p^2q^3$	$10p^3q^2$	$5p^4q$	p^5

the same terms as are obtained in the expansion of $(q + p)^5$:

$$(q+p)^5 = q^5 + 5pq^4 + 10p^2q^3 + 10p^3q^2 + 5p^4q + p^5$$

Why should this be?

Well, when determining $P(X = 2)$, the ${}^5C_x$ ways (= 10) arises when we consider the number of ways the two 'p branches' could be chosen from the five branches. In the expansion of $(q + p)^5$, the p^2q^3 term involves with the number of ways the two brackets supplying the p could be chosen from the five brackets. Once again ${}^5C_x$ is involved.

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EXAMPLE 4

Each question of a multiple-choice test paper offers five answers, one of which is correct. A student answers 7 questions by simply guessing which response is correct each time. If we define the random variable X as how many of these seven questions the student gets correct determine the probability distribution for X, giving probabilities correct to 3 decimal places.

Solution

No. of trials = 7. P(success, i.e. gets question correct) = 0.2. $X \sim \text{Bin}(7, 0.2)$.

$P(X=0) = {}^7C_0\ 0.2^0\ 0.8^7 \approx 0.2097$

$P(X=1) = {}^7C_1\ 0.2^1\ 0.8^6 \approx 0.3670$

$P(X=2) = {}^7C_2\ 0.2^2\ 0.8^5 \approx 0.2753$

$P(X=3) = {}^7C_3\ 0.2^3\ 0.8^4 \approx 0.1147$

$P(X=4) = {}^7C_4\ 0.2^4\ 0.8^3 \approx 0.0287$

$P(X=5) = {}^7C_5\ 0.2^5\ 0.8^2 \approx 0.0043$

$P(X=6) = {}^7C_6\ 0.2^6\ 0.8^1 \approx 0.0004$

$P(X=7) = {}^7C_7\ 0.2^7\ 0.8^0 \approx 0.0000$

The complete probability distribution is, correct to three decimal places:

x	0	1	2	3	4	5	6	7
$P(X = x)$	0.210	0.367	0.275	0.115	0.029	0.004	0.000	0.000

EXAMPLE 5

When driving to work a motorist encounters 8 sets of traffic lights.

Let us suppose that for each of these the probability that the motorist has to stop at the lights is a constant 0.4. Find the probability that in the journey to work the motorist has to stop at

a exactly six of the eight sets of lights,

b at least six of the eight sets of lights.

Solution

If X is the number of lights the motorist stops at then $X \sim \text{Bin}(8, 0.4)$

a
$$\begin{aligned}\text{P(stop at exactly six)} &= P(X=6)\\ &= {}^8C_6\ 0.4^6\ 0.6^2\\ &\approx 0.041\end{aligned}$$

b
$$\begin{aligned}\text{P(stop at at least six)} &= \text{P(stop at 6)} + \text{P(stop at 7)} + \text{P(stop at 8)}\\ &= P(X=6) + P(X=7) + P(X=8)\\ &= {}^8C_6\ 0.4^6\ 0.6^2 + {}^8C_7\ 0.4^7\ 0.6^1 + {}^8C_8\ 0.4^8\ 0.6^0\\ &\approx 0.050\end{aligned}$$

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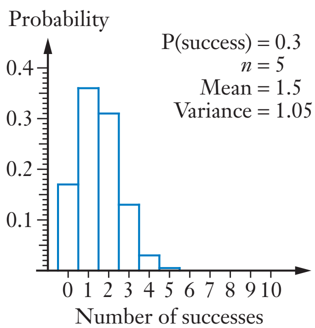
Graphs of binomial distributions

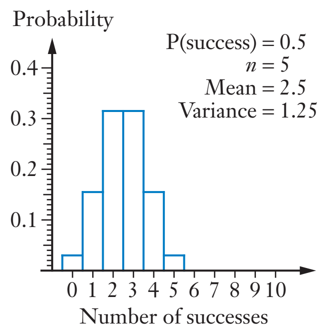
The graphs for the binomial distributions with

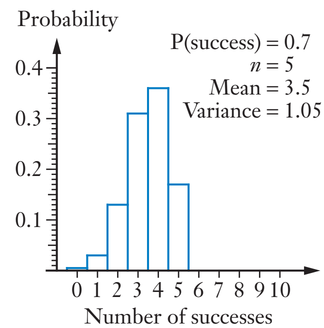
$$p = 0.3, \quad p = 0.5 \quad \text{and} \quad p = 0.7$$

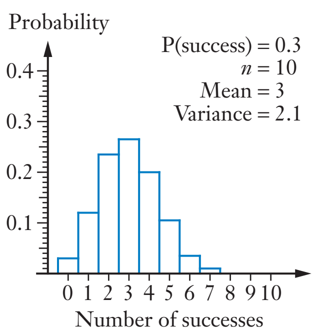
are shown below for

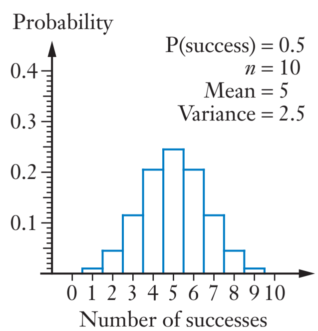
$$n = 5, \quad n = 10 \quad \text{and} \quad n = 15$$

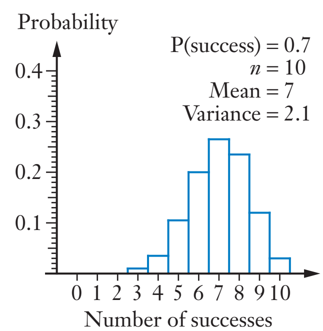


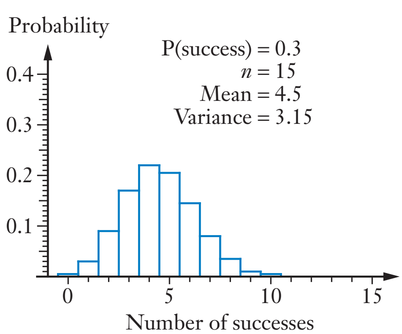


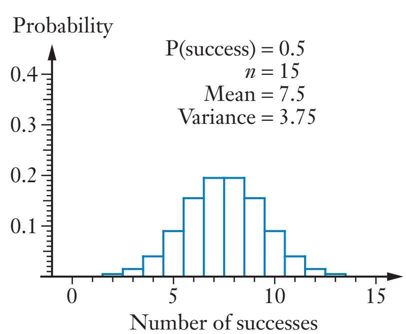


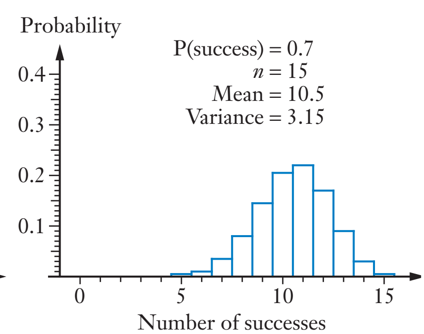












Note
- the symmetrical nature of the graphs for which P(success) = 0.5.
- the skewed nature of the graphs for which P(success) $\neq$ 0.5.
- the graphs of P(success) = k and P(success) = $1 - k$ are mirror images.
- the graphs for P(success) $\neq$ 0.5 appear to move towards a more symmetrical distribution as n increases.

ISBN 9780170395137

INVESTIGATION

Get your calculator or computer spreadsheet to output random numbers between 0.000 00 and 0.999 99 and record the first five digits after the decimal point.

For example, for:	0.776 41	0.213 91	0.358 90	0.536 52	0.791 46	0.499 46
record:	77 641	21 391	35 890	53 652	79 146	49 946

Note how many digits in each group of five are 1s, 2s or 3s:

77 641	21 391	35 890	53 652	79 146	49 946
1	4	1	2	1	0

Do this for one hundred sets of five digits and tabulate your results:

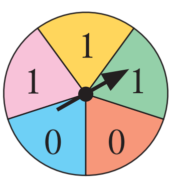
No. of 1s, 2s and 3s in set of 5	0	1	2	3	4	5
Tally	/	///	/		/	
Frequency						

Compare your results with those suggested by modelling this activity using a binomial distribution with $n = 5$

and $\text{P(success)} = \text{P(digit being 1, 2 or 3)} = 0.3$.

Exercise 9A

1 Find the mean, or expected value, and the variance of a Bernoulli distribution with parameter 0.6.

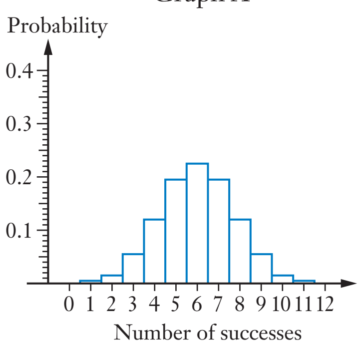


2 The three graphs below show binomial distributions for

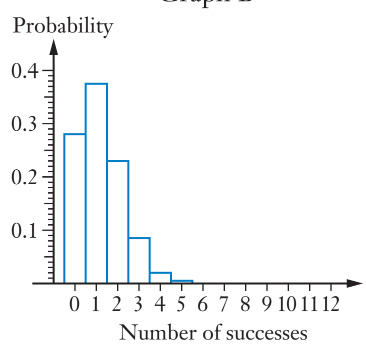
$n = 12$ and P(success) = 0.1, 0.5 and 0.8.

Which graph has which P(success) value?

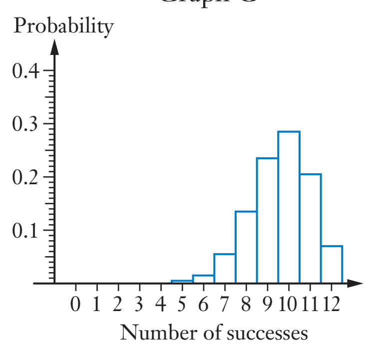
Graph A



Graph B



Graph C



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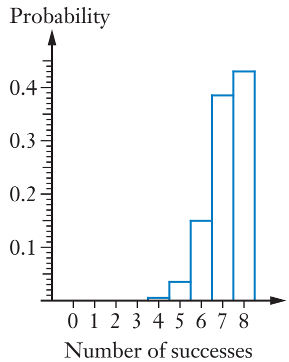
3 The three graphs below show binomial distributions for

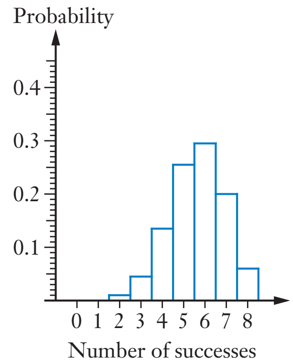
$$n = 8 \text{ and P(success)} = 0.5, 0.7 \text{ and } 0.9.$$

Which graph has which P(success) value?

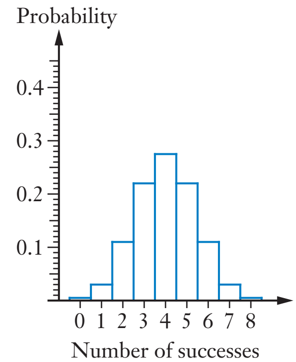
Graph A



Graph B



Graph C



4 Find the mean and standard deviation of a binomial distribution with n, the number of trials, equal to 12 and p, the probability of success in each trial equal to 0.25.

5 A binomial distribution has a mean of 9.6 and a standard deviation of 2.4. Find n, the number of trials and p, the probability of success in each trial.

6 The discrete random variable X is binomially distributed with parameters $n = 8$ and $p = 0.25$. The probability distribution for X is shown below:

x	0	1	2	3	4	5	6	7	8
P($X = x$) Rounded to 4 dp	0.1001	0.2670	0.3115	a	b	0.0231	0.0038	0.0004	0.0000

a Find the values of a and b.

b Find the mean, μ, and the standard deviation, σ, of X.

c Find $P(\mu - \sigma \leq X \leq \mu + \sigma)$ giving your answer rounded to 3 decimal places.

7 The discrete random variable X is binomially distributed with parameters $n = 9$ and $p = 0.6$.

Find

a $P(X = 8)$ **b** $P(X = 9)$ **c** $P(X \geq 8)$ **d** $P(X < 8)$

8 If $X \sim \text{Bin}(6, 0.7)$ determine:

a $P(X = 5)$ **b** $P(X = 6)$ **c** $P(X \geq 5)$ **d** $P(X < 5)$

9 A normal fair die is rolled eight times.

Determine the probability of obtaining exactly **a** two sixes **b** six sixes.

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10 The coach of a hockey team feels that for each short corner the team takes, the probability of a goal resulting is 0.3.

Assuming this probability is correct, what is the probability that when this team takes nine short corners, four goals will result?

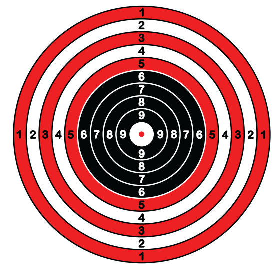
Alamy Stock Photo/Simon Balson

11 Suppose that each time I shoot at a target the probability of my shot scoring a 'bull' is 0.7.

Assuming that this probability remains constant, determine the probability that in ten such shots I will score

a 6 bulls **b** 8 bulls

c more than 8 bulls **d** at least 8 bulls.



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12 Each question of a multiple-choice test paper offers four answers, one of which is correct. A student answers 20 questions by randomly guessing which response is correct each time. If we define the random variable X as how many of these twenty questions the student gets correct, determine correct to 3 decimal places

a $P(X = 5)$ **b** $P(X = 10)$ **c** $P(8 < X \leq 10)$

13 A particular disease is fatal if not treated. However the treatment itself is particularly risky. For anyone with the disease and given the treatment the probability that the treatment will work is 0.4. If six people with the disease are given the treatment what is the probability that the treatment will work for more than half of these six?

Values from tables and calculators

Nowadays individual probabilities and cumulative probabilities for various distributions can be obtained directly from many calculators without having to input complicated expressions.

For example, the display on the right shows that for

$$X \sim \text{Bin}(4, 0.3),$$
$$P(X = 2) = 0.2646$$

and

$$P(X \leq 2) = 0.9163$$

```
binomial PDf(2, 4, 0.3)
                         0.2646
binomial CDf(0, 2, 4, 0.3)
                         0.9163
```

(The letters PDf in the display stands for *probability density function*. CDf indicates *cumulative* probabilities are being given.)

Some calculators are also able to display the graph of the distribution as well.

Not all calculators require the same order of input suggested by the display shown above. Get to know your calculator with regard to its ability to display values from binomial probability distributions.

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Prior to the ready availability of such calculators, binomial probabilities were obtained either by use of the formula, as we did in the previous exercise, or from books displaying tables of probabilities. Part of a typical table display of binomial probabilities is shown below. (Other pages of the book would show cumulative probabilities.)

Can you use the table shown below to find the probability given on the previous page,

i.e. that for $X \sim \text{Bin}(4, 0.3)$,

$P(X = 2) = 0.2646$?

n	x	p										
		0.1	**0.2**	**0.25**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.75**	**0.8**	**0.9**
2	**0**	.8100	.6400	.5625	.4900	.3600	.2500	.1600	.0900	.0625	.0400	.0100
	1	.1800	.3200	.3750	.4200	.4800	.5000	.4800	.4200	.3750	.3200	.1800
	2	.0100	.0400	.0625	.0900	.1600	.2500	.3600	.4900	.5625	.6400	.8100
3	**0**	.7290	.5120	.4219	.3430	.2160	.1250	.0640	.0270	.0156	.0080	.0010
	1	.2430	.3840	.4219	.4410	.4320	.3750	.2880	.1890	.1406	.0960	.0270
	2	.0270	.0960	.1406	.1890	.2880	.3750	.4320	.4410	.4219	.3840	.2430
	3	.0010	.0080	.0156	.0270	.0640	.1250	.2160	.3430	.4219	.5120	.7290
4	**0**	.6561	.4096	.3164	.2401	.1296	.0625	.0256	.0081	.0039	.0016	.0001
	1	.2916	.4096	.4219	.4116	.3456	.2500	.1536	.0756	.0469	.0256	.0036
	2	.0486	.1536	.2109	.2646	.3456	.3750	.3456	.2646	.2109	.1536	.0486
	3	.0036	.0256	.0469	.0756	.1536	.2500	.3456	.4116	.4219	.4096	.2916
	4	.0001	.0016	.0039	.0081	.0256	.0625	.1296	.2401	.3164	.4096	.6561

EXAMPLE 6

Given that $X \sim \text{Bin}(10, 0.1)$, find **a** $P(X = 3)$ **b** $P(X \leq 3)$ **c** $P(X = 3 \mid X \leq 3)$

Solution

Using tables, or a calculator:

a $P(X = 3) = 0.0574$

b $P(X \leq 3) = 0.9872$

c $P(X = 3 \mid X \leq 3) = \dfrac{0.0574}{0.9872}$

≈ 0.058

```
binomPdf(10, 0.1, 3)
                          0.057396
binomCdf(10, 0.1 0, 3)
                          0.987205
```

Assessing improvement using a binomial model

To test the effectiveness of their two-hour

'Improve Your Golf Swing'

course the organisers arrange for ten attendees to play a shot before attending the course and then play a similar shot after the course. For each shot the organisers measured how far each person's shot finished from the target flag.

Suppose eight of the ten finished closer to the flag with their second attempt, i.e. after attending the course, than before it. Could we conclude that their attendance at the course was responsible for the apparent improvement?

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One aspect we could investigate (as well as considering such things as just how similar the two shots were, for example, if the weather conditions were the same) would be to see how likely it is that eight, or more, of the ten would improve purely by chance, and not by attending the course.

Let us suppose that the probability of someone attempting this shot and improving on their second attempt 'by pure luck', and not by attending the course, is 0.5. The chance that out of ten people, eight or more will improve at the second attempt is then $P(X \geq 8)$ with $X \sim \text{Bin}(10, 0.5)$.

In this case $P(X \geq 8) = 0.0547$.

Thus there is only a 5 or 6 percent probability that eight or more of the golfers would improve 'by chance' (if our binomial model and the 0.5 probability is appropriate).

Hence the fact that eight did improve suggests that the improvement could be for some reason, rather than pure chance, and that could be their attendance at the course. However further investigation would be advisable before making too many claims about the ability of the course to bring such improvement.

Exercise 9B

1 The discrete random variable X is binomially distributed with parameters $n = 8$ and $p = 0.2$. Use your calculator or a book of tables to determine:

a $P(X = 4)$ **b** $P(X = 6)$ **c** $P(X \leq 6)$ **d** $P(X < 7)$

2 The discrete random variable X has a binomial distribution with parameters $n = 20$ and $p = 0.6$. Use your calculator or a book of tables to determine:

a $P(X = 10)$ **b** $P(X = 14)$ **c** $P(X \leq 14)$ **d** $P(X < 15)$

3 Given that $X \sim \text{Bin}(9, 0.4)$, find:

a $P(X = 2)$ **b** $P(X \leq 3)$ **c** $P(X = 2 \mid X \leq 3)$

4 Given that $X \sim \text{Bin}(20, 0.7)$, find:

a $P(X = 15)$ **b** $P(X \geq 15)$ **c** $P(X = 15 \mid X \geq 15)$

5 Given that $X \sim \text{Bin}(15, 0.8)$, find:

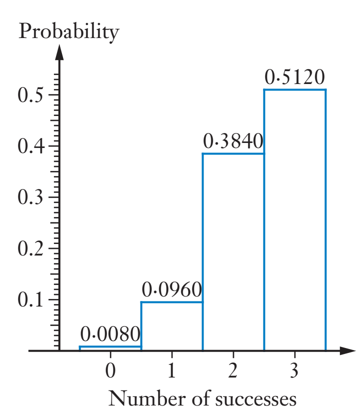
a $P(X \geq 7 \mid X \leq 10)$ **b** $P(X \leq 10 \mid X \geq 7)$

6 Given that $X \sim \text{Bin}(12, 0.3)$, find:

a $P(X \leq 5 \mid X \geq 3)$ **b** $P(X \geq 3 \mid X \leq 5)$

7 The graph on the right shows a binomial probability distribution.

Use the table on the previous page to determine the probability of success on each of the Bernoulli trials involved in this distribution.



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8 A biased coin is such that on each flip the probability of getting a head is 0.4. The coin is flipped 20 times. Find the probability of obtaining

a exactly 12 heads, **b** no more than 12 heads, **c** at least 12 heads.

9 A gardener sells punnets containing eight seedlings. In an attempt to minimise the number of punnets that cannot be sold because they contain less than eight seedlings, the gardener plants ten seeds in each punnet and, if more than eight germinate he takes the extra ones out. If each seed has a probability of 0.9 of germinating what is the probability that of the ten seeds placed in a punnet

a exactly eight will germinate?

b at least eight will germinate?

c less than eight will germinate?

10 At each of the fifteen fences in an equestrian cross country event, Kerry can either go clear or incur penalty points. If the probability of her incurring penalty points at any fence is a constant 0.1, find the probability that in the fifteen-fence event she incurs penalty points at

a exactly three of the fifteen fences,

b less than three of the fifteen fences,

c more than three of the fifteen fences.

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11 Each question of a multiple-choice test paper offers five answers, one of which is correct. A student answers 20 questions by randomly guessing which response is correct each time. If we define X as the number of questions the student gets correct, determine correct to 3 decimal places

a $P(X = 5)$, **b** $P(X = 10)$, **c** $P(X \geq 10)$, **d** $P(3 \leq X \leq 7)$.

12 Suppose the probability of a particular hereditary characteristic being passed from a ewe to each lamb she gives birth to is 0.25. If, over a period of time, the ewe gives birth to six lambs determine, correct to four decimal places, the probability that

a none of the six will have the characteristic,

b all six will have the characteristic,

c exactly three will have the characteristic,

d at least three will have the characteristic.

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13 The probability of any randomly chosen component being faulty is 0.01. If ten of these components are randomly selected what is the probability that at least one will be defective?

14 Two fair dice are rolled and the two numbers on the uppermost faces are added together.

a In one roll of these dice what is the probability of obtaining a total of 7?

If these dice are rolled ten times and the total obtained is noted each time, what is the probability of obtaining a total of 7

b at least once?

c on less than three of the ten occasions?

d on at least three of the ten occasions?

15 A multiple-choice test contains 12 questions each offering 4 answers, only one of which is correct each time. A student knows the correct answers to 7 of the questions but randomly guesses the answers to the remaining 5. What is the probability that the student will get at least 10 out of the 12 correct?

16 Matt and Joel play soccer and both are strikers. Matt tends to have more shots on goal during a match than Joel does but, for each shot on goal, Joel seems to have a better chance of scoring.

Let us suppose that for each shot on goal that Matt has, the probability of the shot scoring is 0.2, and for each shot on goal that Joel has, the probability of the shot scoring is 0.4.

If Joel has three attempts on goal and Matt has six, which of these two strikers has the greater probability of scoring at least one goal?

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17 If I flip a fair coin twice, then P(≤ 2 heads) = 1.

If I flip a fair coin three times, P(≤ 2 heads) = 0.875.

If I flip a fair coin four times, P(≤ 2 heads) = 0.6875.

What is the greatest number of times I can flip a fair coin and still have the value of P(≤ 2 heads) above 0.2?

18 Let us suppose that each time a particular basketball player shoots a three-point attempt the probability of scoring is 0.4. How many three-point attempts does this player need to make for the probability of at least three successes to exceed 0.75?

19 Prior to the arrival of a specialist shooting coach for a weekend workshop, the twenty members of a basketball club took part in a shooting drill in which they took one shot at basket from each of fifteen places on a basketball court. The number of successes out of fifteen was recorded for each player.

At the end of the weekend workshop the twenty members again took one shot from each of the same fifteen places and scores out of fifteen were again recorded.

Sixteen of the twenty members recorded higher scores in the 'after the workshop test' than the 'before the workshop' test.

Comment on this improvement.

20 Fifty students sat a multiple-choice test which involved fifteen questions.

On each question the students had to select one of four answers a, b, c or d.

After the test the students complained that whilst two of the questions involved work they had been taught, the other thirteen were on things they had been given no prior information about and they were left having to guess answers for those.

The scores the students obtained were as follows:

Score	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No. of students	0	0	0	6	9	16	8	4	5	0	1	0	1	0	0	0

Do you believe the complaint made by the students? (Justify your answer.)

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Miscellaneous exercise nine

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 An object with a temperature of 80°C is placed in an environment with temperature 15°C. The temperature of the object, t seconds later, is T°C, where T approximately follows the mathematical rule

$$T = 15 + 65e^{-0.004t}.$$

Find the temperature of the object after **a** 5 minutes, **b** 10 minutes.

Differentiate each of the following with respect to x.

2 $(x+3)(x+2)$

3 $(2x+1)^3$

4 $(3-2x)^2$

5 $\dfrac{x}{x+1}$

6 $(2x+1)(6x^3-5)$

7 $x(2-3x)^3$

8 $(2x+1)(4+7x)^4$

9 e^x

10 $5x^2+e^x$

11 $4e^{3x}+x^4-2$

12 e^{2x-4}

13 e^{3x+1}

14 x^2e^x

15 $x+xe^x$

16 $\sin x$

17 $\cos 3x$

18 $\sin(3x-5)$

19 $e^{2x}\sin 4x$

20 Given that $X \sim \text{Bin}(20, 0.25)$, find $P(X = 12)$. (Round your answer to 6 decimal places.)

21 We are interested in the number of successes a person is likely to have when they attempt to guess the outcome of a normal fair die being rolled eight times.

Explain why a binomial distribution would be a suitable model for this situation.

22 Differentiate each of the following with respect to x without the assistance of your calculator, and then use your calculator to check your answers.

a $\dfrac{1}{\sqrt{3x-1}}$ **b** $\dfrac{5x^2+1}{x}$ **c** $\displaystyle\int_3^x \frac{t-1}{t^3}\,dt$ **d** $\displaystyle\int_1^x \frac{1+t^3}{\sqrt{t}}\,dt$

23 If $f(x) = 2(2x-1)^3$ find **a** $f(2)$, **b** $f(0.5)$, **c** $f'(x)$, **d** $f'(3)$.

24 Find the exact gradient of $y = 3x^2 + e^{2x} + 3$ at the point $(1, 6+e^2)$.

25 Integrate the following with respect to x without the assistance of your calculator, and then use your calculator to check your answers (but remember that your answers should include '+ c').

a $15x^4$ **b** $6x^2-4x+6$ **c** $\dfrac{x+3}{\sqrt{x}}$ **d** $(2x+3)^5$

e $(5x-2)^3$ **f** $\sin x$ **g** $\cos 2x$ **h** $\sin(2x-1)$

i $4\sin 3x$ **j** $4x(x^2+3)^4$ **k** $\dfrac{d}{dx}(x^5-7x)$ **l** $\dfrac{d}{dx}(e^x\sqrt{x}-7x)$

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26 If $A = \dfrac{6x+3}{x-1}$ find an expression for the rate of change of A with respect to x.

27 If $T = 3\sqrt{p}$ find the rate of change of T with respect to p when

a $p = 16$, **b** $p = 25$, **c** $p = 36$.

28 Determine the gradient of the curve $y = (2x+3)(x^2+3)$ at the point $(-1, 4)$.

29 Find y in terms of x given that $\dfrac{dy}{dx} = 2x - 5$ and when $x = 1$, $y = 7$.

30 Find y as a function of x given that $\dfrac{dy}{dx} = 10x - 6$ and $y = 9$ when $x = 2$.

31 Find $f(x)$ given that $f'(x) = 12(8-2x)^2$ and $f(4) = 6$.

32 Find x in terms of t given that $\dfrac{dx}{dt} = -\dfrac{18}{(3t+1)^2}$ and $x = 4.5$ when $t = 1$.

33 If $\dfrac{dy}{dx} = 5(2x+1)^4$ and $y = 125$ when $x = 1$ find

a y in terms of x, **b** y, when $x = 0$, **c** x, when $y = 19.5$ ($x \in \mathbb{R}$).

34 Use calculus to determine the area between $y = 2x + 1$ and the x-axis from $x = -2$ to $x = 2$. Check your answer using area formulae.

35 Find y as a function of x given that $\dfrac{d^2y}{dx^2} = 30x - 14$, $y = 1$ when $x = 1$ and $y = -9$ when $x = -1$.

36 If \$500 is invested at 12% per annum compounded continuously, the account grows to $\$500e^{0.12t}$ after t years. What is the instantaneous rate of growth, in dollars per year correct to two decimal places, when

a $t = 1$? **b** $t = 5$? **c** $t = 10$? **d** $t = 25$?

37 Manuel estimates that each time he throws a dart at the dartboard, aiming at treble 20, the chance of his dart successfully scoring treble 20 is 0.1. What is the probability that in ten such attempts he will successfully score treble 20 at least once? (Give your answer rounded to two decimal places.)

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38 If $f(x) = 3x^2 + x$ use differentiation to find the approximate change in the value of the function when x changes from 5 to 5.04.

Compare your answer to $f(5.04) - f(5)$.

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39 If $V = 5x^3$ use differentiation to find the approximate percentage change in V when x changes by 3%.

40 Find the area enclosed between $y = 2\sin x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

41 An engineer requires a function of the form $f(x) = \frac{2x+3}{2x+a}$, for constant a, to be such that $f'(3) = -16$.

With the assistance of your calculator if you wish, find the possible value(s) of a.

42 A charitable organisation sets up a fundraising craft stall in the centre of a city, selling items made by members. The council allows the group to operate the stall from 9 a.m. to midday one Saturday morning. The group finds that the amount in their cash box grows during this time from the original amount they started with as a cash float to \$$A$ where $\frac{dA}{dt} = 1.2e^{0.01t}$, t being the number of minutes past 9 a.m. ($0 \le t \le 180$).

How much did they raise during the three hours (to the nearest dollar)?

43 An object moves such that its displacement, x metres, from an origin, O, at time t seconds is given by $x = e^{\cos t}$.

Find an expression for the velocity of the object at time t seconds and determine the velocity of the object when $t = \frac{\pi}{2}$.

44 A fair six-sided die has its faces numbered 1, 1, 3, 3, 3, 6.

The die is thrown twice and the number on the uppermost face is noted each time and the two numbers are then added together.

Find **a** the probability that the total obtained is 6,

b the expected mean value of the total if this activity were to be carried out many times.

45 If $y = \frac{1+\sin x}{1-\sin x}$, determine $\frac{dy}{dx}$.

46 A normal fair six-sided die is rolled 4 times.

In theory this could result in 0, 1, 2, 3 or 4 sixes.

a Which of these numbers is the most likely number of sixes to occur?

b Is it more likely that this number of sixes will result or more likely that it will not result?

c If this '4-roll event' was repeated many times what would you expect the long-term average number of sixes obtained per event to be?

47 Determine the exact gradient of $y = \frac{\cos x}{x}$ at the point $\left(\frac{\pi}{3}, \frac{3}{2\pi}\right)$.

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48 Use calculus to determine the nature and exact coordinates of any turning points on the curve $y = \sqrt{3}\sin x + \cos x$ for $-2\pi < x < 2\pi$.

49 A *uniform* discrete random variable, X, can take the integer values 1, 2, 3, 4, 5, …, n.

Find $E(X)$ in terms of n.

Remember: The sum to n terms of the arithmetic progression with first term a and common difference d is $\frac{n}{2}(2a + (n-1)d)$.

50 The point $(-3, a)$ lies on the curve $y = \frac{5x-7}{2x+10}$ and the tangent to the curve at this point is parallel to $y = bx + 3$, and cuts the y-axis at $(0, c)$.

Determine **a** the values of the constants a, b and c,

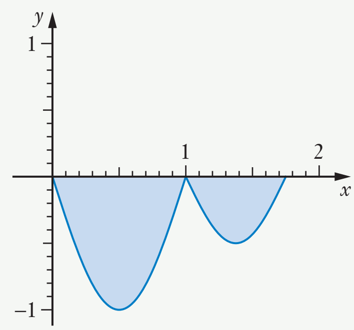
b the coordinates of the other point on the curve where the tangent is parallel to $y = bx + 3$.

51 A yacht designer is investigating possible shapes for fins attached to the keel of a yacht. One possibility involves parts of the following curves and is shown on the right.

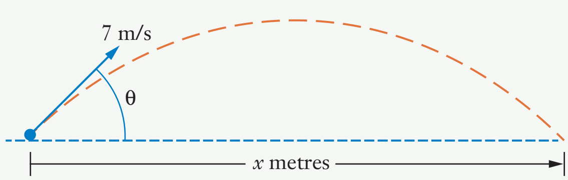
$$y = -\sin(\pi x)$$

$$y = -\frac{1}{2}\sin\left(\frac{4\pi}{3}(x-1)\right)$$

Find the total area shaded.



52 If an object is projected from a point on horizontal ground, with an initial speed of 7 m/s and at an angle of θ to the horizontal, the distance from the point of projection to the point of landing is x metres, where $x = 10\sin\theta\cos\theta$.



a Use the product rule to find an expression for $\frac{dx}{d\theta}$, for θ in radians.

b Use calculus to show that for x to be a maximum the angle of projection needs to be 45° to the horizontal.

Justify that this would indeed give a maximum value for x (as opposed to a minimum or horizontal inflection) and find this maximum value.

c Given the trigonometric identity:

$$2\sin\theta\cos\theta = \sin 2\theta$$

express x in terms of $\sin 2\theta$ and then use your knowledge of the amplitude of $\sin A$, and the value of A for which it occurs, to confirm your answers to part **b**.

ISBN 9780170395137

ANSWERS

Exercise 1A PAGE 6

1 5

2 $6x - 2$

3 $6x^2 - 2x$

4 -2

5 $\frac{1}{5}$

6 $-\frac{5}{x^2}$

7 $6x + \frac{6}{x^3}$

8 $\frac{5}{\sqrt{x}}$

9 $\frac{2}{\sqrt{x}}$

10 $-\frac{4}{x^{\frac{3}{2}}}$

11 $\frac{1}{3x^{\frac{2}{3}}}$

12 5

13 $-\frac{1}{x^2}$

14 $70x$

15 $9x^2 + 4x - 3$

16 2

17 $6x$

18 6

19 $12x$

20 4

21 $24x + 6$

22 $-\frac{1}{4x^{\frac{3}{2}}}$

23 $-\frac{2}{x^{\frac{3}{2}}}$

24 $\frac{2}{x^3}$

25 0

26 $\frac{10}{x^3}$

27 $2 + \frac{24}{x^4}$

28 $3 + \frac{1}{x^2}$

29 $10x + \frac{4}{\sqrt{x}}$

30 $6\sqrt{x}$

31 $36x^2 + 24x$

32 $\frac{18}{x^5}$

33 $30x - \frac{6}{x^4}$

34 4

35 5

36 -8

37 -42

38 **a** $5 - 6x^2$ **b** -19 **c** $-12x$ **d** 24

39 $y = -20x - 20$

40 $y = -0.5x + 6$

41 $y = x + 2$

42 **a** $(-3, 28)$ and $(1, 4)$ **b** $(0.36, 8.6)$

43 **a** $(3, 2.25)$ **b** $(1, -1)$

44 $a = 5, b = 3, c = -1$

Exercise 1B PAGE 10

1 $3x^2$ (as expected)

2 $2x + 7$

3 $2x + 4$

4 $6x + 13$

5 $6x + 7$

6 $120x + 17$

7 $24x + 28$

8 $3x^2 + 8x + 2$

9 $3x^2 + 10x - 3$

10 $3x^2 + 14x + 1$

11 $3x^2 - 20x + 8$

12 $6x^2 + 26x - 11$

13 $9x^2 - 10x$

14 $6x^2 + 14x - 17$

15 $9x^2 - 40x - 4$

16 7

17 4

18 11

19 -5

20 $y = 13x - 22$

21 $y = 23x - 11$

22 $(-0.5, -5)$

23 $(-2, 15), (4, 21)$

24 $(3, 0), (-1, 28)$

25 **a** $5x^{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{2}$ **b** $5x^{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{2}$

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Exercise 1C PAGE 12

1 $2x$ (as expected)
2 $-nx^{-n-1}$ (as expected)
3 $\dfrac{6}{(x+3)^2}$
4 $-\dfrac{3}{(5x-1)^2}$
5 $-\dfrac{18}{(4x-3)^2}$
6 $\dfrac{7}{(1-2x)^2}$
7 $\dfrac{13}{(2x+3)^2}$
8 $-\dfrac{17}{(2x-3)^2}$
9 $\dfrac{17}{(5x+2)^2}$
10 $-\dfrac{1}{(2x-1)^2}$
11 $-\dfrac{6}{(3x+1)^2}$
12 $\dfrac{5(1-x^2)}{(x^2+1)^2}$
13 $\dfrac{2x(2-x^3)}{(x^3+1)^2}$
14 $\dfrac{9x(2-x^5)}{(x^5+3)^2}$
15 -1.5
16 -0.625
17 $y=-3.5x+27.5$
18 (1, 1) and (1.5, −2)
19 **a** $\dfrac{3}{x^2}$ **b** $\dfrac{3}{x^2}$ **c** $\dfrac{3}{x^2}$

Exercise 1D PAGE 17

1 $7(4x+5)$
2 $12(2t+1)$
3 $40r(2r^2-1)$
4 $24(12x+5)$
5 $15(3x+2)^4$
6 $6x(x^2+2)^2$
7 $-\dfrac{8}{(8x-3)^2}$
8 $\dfrac{1}{\sqrt{2x+3}}$
9 $-\dfrac{3}{\sqrt{(6x+1)^3}}$
10 $-\dfrac{4(3x+1)}{(3x^2+2x+1)^3}$
11 $20(5x+2)^3$
12 $21(7x-3)^2$
13 $-9(2-3x)^2$
14 $14(4+7x)$
15 $18x(3x^2+5)^2$
16 $36x^2(2x^3+1)^5$
17 $-\dfrac{3}{(x+2)^4}$
18 $-\dfrac{2}{(2x+5)^2}$
19 $-\dfrac{1}{(x+2)^2}$
20 $-\dfrac{14}{(7x-3)^3}$
21 $3+10(2x+3)^4$
22 $\dfrac{1}{2\sqrt{x+1}}$
23 50
24 450
25 −48
26 −8
27 0
28 9
29 −0.125
30 −2.5
31 −1

Miscellaneous exercise one PAGE 19

1 **a** {−3, −1, 5} **b** {1, 9, 25, 49} **c** {0, 1, 2}
2 **a** A, E **b** F **c** A, D, E **d** A, D, F
3 −14
4 **a** $10x$ **b** $10x$ **c** $30+50x$
5 **a** $2x-2$ **b** $20x+3$ **c** $8x+12$ **d** $9x^2+10x-12$
6 −56
7 $2x-3$
8 −8
9 $16y=5x-3$
10 (0.5, 2.5), gradient $\dfrac{8}{3}$. (3, 5), gradient $\dfrac{3}{8}$.
11 **a** $4x+7$ **b** $18x^2+38x-19$

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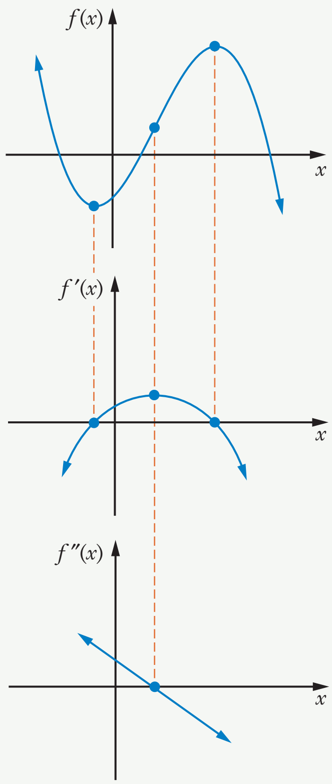
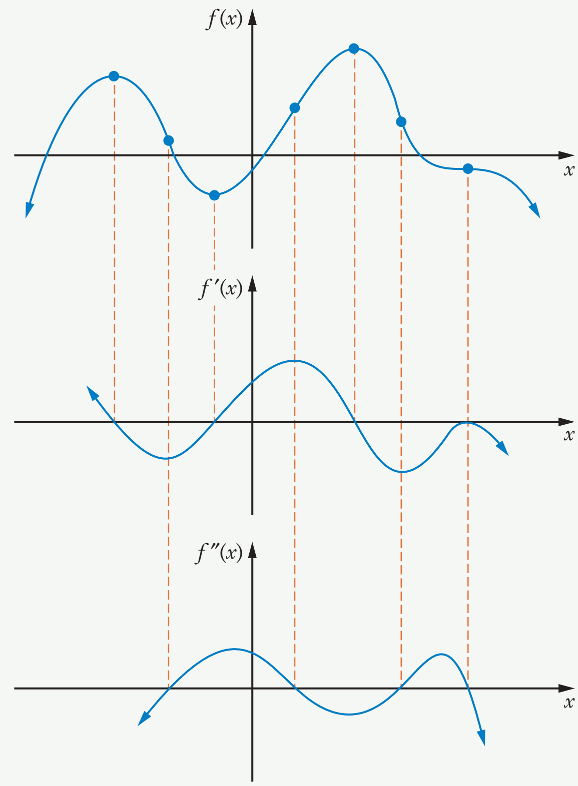
Exercise 2A PAGE 30

1 **a**



b



2 Minimum at (6, 4).

3 Maximum at (4, 21).

4 Maximum at $(-\sqrt{3}, 6\sqrt{3})$, minimum at $(\sqrt{3}, -6\sqrt{3})$.

5 Maximum at (–1, 71), minimum at (7, –185).

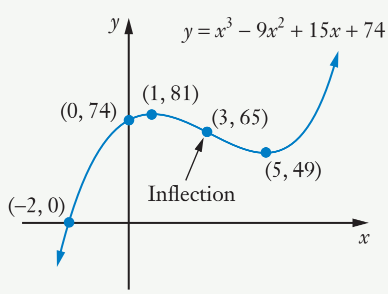
6 Minimum at (1, 2).

7 Maximum at (–5, –7), minimum at (–1, 1).

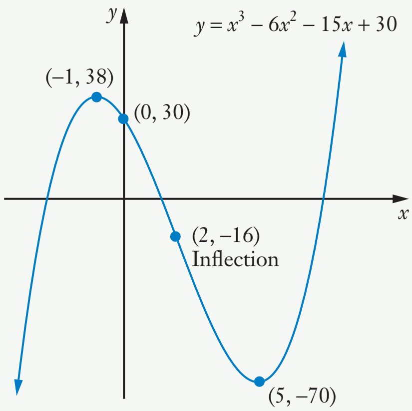
8 Maximum at $(-\sqrt{5}, -2\sqrt{5})$, minimum at $(\sqrt{5}, 2\sqrt{5})$.

9 Horizontal inflection at (0.5, 1).

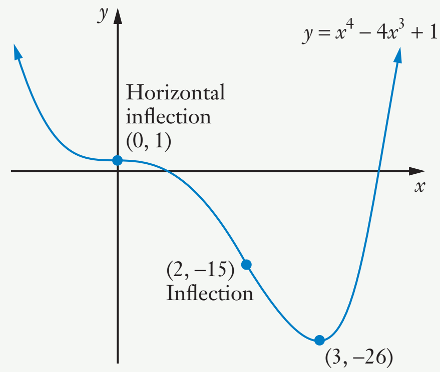
10



11 **a** (0, 30)
b As $x \to \infty, y \to \infty$ (faster than x does).
As $x \to -\infty, y \to -\infty$ (faster than x does).
c Maximum at (–1, 38), minimum at (5, –70).
d (2, –16)



12 **a** (0, 1)
b As $x \to \infty, y \to \infty$ (faster than x does).
As $x \to -\infty, y \to \infty$ (faster than $x \to -\infty$).
c Minimum at (3, –26)
d (0, 1) and (2, –15)



13 $\frac{dy}{dx} = 12(x+1)(x-3)^2$. Min at (–1, –256), horizontal inflection at (3, 0).

14 **a** $2\frac{5}{27}$ **b** 4

15 **a** (0, 0) **b** Inflection at (0, 0), not horizontal inflection.

16 **a** (0, 0), (4, 256) **b** Horizontal inflection at (0, 0), Inflection (not horizontal) at (4, 256)

17 **a** (0, 0) **b** Not a point of inflection. [(0, 0) is a minimum point.]

18 $3x^2 - 3$, $6x$, $a = 0$. $(a, f(a))$, i.e. (0, –2), is a point of inflection (not horizontal).

Exercise 2B PAGE 33

1 $6a^2 + 3$ **2** $1 - 10p + 6p^2$ **3** $6(2t-1)^2$ **4** $\frac{19}{(2x+5)^2}$ **5** $18q^2 - 30q + 2$

6 **a** 6 cm^3/sec **b** 24 cm^3/sec **c** 54 cm^3/sec

7 **a** $30t^2 - 10t$ **b** **i** 20 insects/day **ii** 700 insects/day **iii** 2900 insects/day

8 **a** 15 m, 25 m/s **b** 275 m, 105 m/s **c** 4100 m, 405 m/s

9 **a** 5 **b** 6655 **c** $30(2t+1)^2$ bacteria/hour
d **i** 750 bacteria/hour **ii** 3630 bacteria/hour **iii** 13 230 bacteria/hour

10 **a** 9 weeks (8.9) **b** $\frac{800}{(w+1)^2} - \frac{2500}{\sqrt{w}}$, –2300 \$/week, –1400 \$/week, –900 \$/week.

Exercise 2C PAGE 36

1 **a** positive **b** positive **c** negative **d** negative **e** positive **f** positive

2 **a** 26 m/s **b** 10 m/s^2 **3** **a** 15 m/s **b** 12 m/s^2 **4** –0.04 m/s^2 **5** 160 m/s^2

6 **a** 24 m/s^2 **b** 0 m/s^2 **c** 8 m/s^2 **d** $-\frac{1}{27}$ m/s^2 **e** 48 m/s^2 **f** 900 m/s^2

7 **a** –11 m/s **b** 2 m/s^2 **c** 8 **d** 3, 8

8 9 m

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Exercise 2D PAGE 39

1 For maximum profit $x = 50$. The max profit is then \$187 500.

2 200 items should be made for a maximum profit of \$4 000 000.

3 For maximum profit $x = 70$. The max profit is then \$106 000 (nearest \$1000).

4 Maximum total weight of fruit per 100 m^2 produced when $N = 20$.

5 **a** Maximum capacity is 2250 cm^3. **b** Maximum capacity is 3528 cm^3.

6 When $x = 200$ the average cost per unit has its minimum value of \$12.

8 The company should make 14 As and 204 Bs for a maximum profit of \$119 200.

9 $r = \frac{40}{\pi}$, $y = 40$.

10 The dimensions should be 80 m × 100 m with one of the 80 m sides fenced with the \$24 per metre fencing. The minimum cost will then be \$6400.

11 $x = 8$ gives the greatest probability of recovery, this maximum being 0.9375.

12 Maximum P is 0.72 (i.e. 72%) when $t = 10$.

13 When $x = 5$ the average cost per unit is a minimum. The minimum average cost is \$675/unit.

14 Separation distance t hours later $= \sqrt{625 - 3000t + 10\,000t^2}$
The separation distance is a minimum after 9 minutes. This minimum separation distance is 20 km.

Exercise 2E PAGE 45

1 0.24, $f(4.02) - f(4) = 0.2404$
2 0.07, $f(3.01) - f(3) = 0.0702$
3 0.15
4 1.46
5 0.18
6 –0.004 (i.e. a decrease)
7 0.1
8 y increases by approximately10%
9 y increases by approximately 6%
10 2π cm^2
11 6.180 cm, 0.026 cm
12 \$200
13 \$21
14 Surface area decreases by 0.8π cm^2
15 0.8%
16 2π cm^2
17 \$1500
18 (6 ± 0.011) cm
19 $\frac{\pi r^2}{20}$ m^3, $\frac{1}{2\pi}$ m (≈ 16 cm)
20 120 cm^3
21 3%

Miscellaneous exercise two PAGE 47

1 **a** $2\sqrt{2}$ **b** $4\sqrt{2}$ **c** $5\sqrt{2}$ **d** $3\sqrt{2}$ **e** $10\sqrt{2}$ **f** $4\sqrt{2}$
g $2\sqrt{2}$ **h** $4\sqrt{2}$ **i** $2\sqrt{2}$

2 $5x^4$ (as expected)
3 $5x^4$ (as expected)
4 $20x$
5 5
6 2
7 $6x^2 - 6x$
8 $15x^2 + 14x - 6$
9 0
10 $12x + 1$
11 $18x^2 - 6x$
12 $8x + 20$
13 $14(2x - 1)^6$
14 $-\frac{17}{(2x-3)^2}$
15 $-\frac{15x^2 + 6x + 5}{(3x^2 - 1)^2}$
16 $30t$, 90
17 $6 - \frac{4}{t^3}$, 5.5
18 $80(2t + 3)^3$, 10 000
19 $6t + 40 + 125t^{-\frac{3}{2}}$, 191

20 Stationary points occur for $x = 0$ (local minimum), $x = \frac{25}{7}$ (local maximum), $x = 12.5$ (horizontal inflection).

21 $y = 3x + 9$

22 **a** 16 250 **b** 25 000

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23 **a** $\frac{dy}{dx} = \frac{x^2 - 2x - 2}{(x-1)^2}$ **b** $(1-\sqrt{3}, 2-2\sqrt{3}),\ (1+\sqrt{3}, 2+2\sqrt{3})$ **c** Maximum at $(1-\sqrt{3}, 2-2\sqrt{3})$, Minimum at $(1+\sqrt{3}, 2+2\sqrt{3})$.

24 $a = 5, b = -6, c = 14, y = -20x - 26$

25 y decreases by approximately 2%

26 Compare your answers with those of others in your class.

27 No we cannot conclude that (3, 1) is a point of inflection.
At all points of inflection, provided $f''(x)$ exists, $f''(x) = 0$ but a point where $f''(x) = 0$ does not have to be an inflection point. Consider $f(x) = (x-3)^4 + 1$ or $f(x) = (x-3)^4 + x - 2$ for example.

Exercise 3A PAGE 53

1 $\frac{x^7}{7} + c$ **2** $\frac{x^4}{4} + c$ **3** $2x^5 + c$ **4** $\frac{7x^3}{3} + c$

5 $4x^2 + c$ **6** $8x + c$ **7** $\frac{2}{3}x^{\frac{3}{2}} + c$ **8** $\frac{3}{4}x^{\frac{4}{3}} + c$

9 $\frac{2}{7}x^{\frac{7}{2}} + c$ **10** $\frac{12}{5}x^{\frac{5}{2}} + c$ **11** $8x^{\frac{1}{2}} + c$ **12** $8\sqrt{x} + c$

13 $-\frac{10}{3x^3} + c$ **14** $\frac{9}{x} + c$ **15** $-32\sqrt{x} + c$ **16** $2x^3 - 2x^2 + 3x + c$

17 $3x^4 + 3x + c$ **18** $\frac{x^4}{4} + x^3 + x^2 + c$ **19** $x + 2x^2 + 6x^3 + c$ **20** $2x^{\frac{3}{2}} + 3x^2 + c$

21 $x^3 + 7x^2 + 8x + c$ **22** $x^3 + 7x^2 + 8x + c$ **23** $\frac{x^3}{3} + 2x^2 - 12x + c$ **24** $3x^3 - 4x + c$

25 $3x^4 + 6x^2 + c$ **26** $2x^2 + 5x + c$ **27** $-\frac{2}{x} - \frac{1}{2x^2} + c$ **28** $4x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + c$

29 $2\sqrt{x} - \frac{2}{5}x^{\frac{5}{2}} + c$ **30** $2\sqrt{x} + x + c$ **31** $y = 2x^3 + x - 5$ **32** $y = 2x^2 - 3x + 2$

33 $A = t + \frac{6}{t} - 7$ **34** $V = \frac{x^2}{2} + 2\sqrt{x} - 10$

35 **a** $\frac{2x^3}{5} + \frac{5}{6x} + \frac{5}{6}$ **b** $\frac{31}{15}$ **c** $-\frac{2}{5}$

Exercise 3B PAGE 61

1 $\frac{1}{12}(3x+2)^4 + c$ **2** $\frac{1}{15}(3x+2)^5 + c$ **3** $x^3 + x^2 + c$ **4** $\frac{1}{25}(1+5x)^5 + c$

5 $-\frac{1}{20}(1-5x)^4 + c$ **6** $(x^2+5)^5 + c$ **7** $2(x^2-7)^5 + c$ **8** $\frac{1}{2}x^2 + \frac{10}{3}x^3 + \frac{25}{4}x^4 + c$

9 $\frac{1}{6}(2x+1)^3 + c$ **10** $x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 + c$ **11** $\frac{1}{20}(5x+1)^4 + c$ **12** $-\frac{3(5-7x)^4}{4} + c$

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13 $2(2x+1)^4+c$

14 $3(3x-2)^5+c$

15 $\frac{(x^2-x+3)^5}{5}+c$

16 $2(6x+1)^4+c$

17 $\frac{1}{10}(5x+1)^4+c$

18 $5(3x^2-6x+1)^5+c$

19 $\frac{1}{3}(3x-1)^5+c$

20 $\frac{1}{9}(9x+1)^3+c$

21 x^3+2x^2+c

22 $\frac{2}{9}(3x-1)^3+c$

23 $\frac{1}{2}x^4-\frac{4}{3}x^3+x^2+c$

24 $\frac{1}{3}x^3-x+c$

25 $\frac{1}{4}(1+x)^4+c$

26 $-\frac{1}{4}(1-x)^4+c$

27 $\frac{1}{2}x^2+\frac{1}{3}x^3+c$

28 $\frac{x^4}{2}+\frac{4x^3}{3}+x^2+c$

29 $2(1+x^2)^3+c$

30 $\frac{(1+x^2)^7}{7}+c$

31 $3(1-2x)^4+c$

32 $3(2x-1)^9+c$

33 $-\frac{1}{2}(5-6x)^5+c$

34 $-\frac{1}{8}(3-2x)^4+c$

35 $\frac{1}{3}(2x-3)^9+c$

36 $-\frac{1}{2}(5-6x)^4+c$

37 $\frac{(x^2+x+3)^5}{5}+c$

38 $\frac{(5x^2+3)^8}{4}+c$

39 $-\frac{(x^2-x+3)^5}{5}+c$

40 $-\frac{1}{3(x+2)^3}+c$

41 $-\frac{5}{(x+1)}+c$

42 $-\frac{(x^2-2x+1)^4}{8}+c$

43 $-\frac{1}{(x+3)^2}+c$

44 $-\frac{3}{(x^2-3)^3}+c$

45 $\frac{1}{2-x}+c$

46 $\frac{1}{2(1-2x)}+c$

47 $\frac{5}{(3-2x)^2}+c$

48 $2(3x^2-x+1)^5+c$

49 $\frac{1}{2(x-2)^2}+c$

50 $\frac{4}{1-3x}+c$

51 $\frac{2}{(1-5x)^2}+c$

52 $\frac{2}{9}(3x+2)^{\frac{3}{2}}+c$

53 $4(2x-5)^{\frac{3}{2}}+c$

54 $6\sqrt{1+2x}+c$

55 $x-\frac{1}{15}(1-5x)^3+c$

56 $3(3x-2)^{\frac{4}{3}}+c$

57 $x+\frac{1}{2}x^2-\frac{10}{3}x^3+\frac{25}{4}x^4+c$

58 $-\frac{2}{(2x-3)^3}+c$

59 $2(2x+1)^3+(3x-2)^3+c$

60 $\frac{2}{3}(x+3)^{\frac{3}{2}}+\frac{2}{3}(x+1)^{\frac{3}{2}}+c$

61 $10\sqrt{x^2+3x-1}+c$

62 $A=2(p+1)^3+5$

63 $y=2(2x+1)^5+23$

64 $f(x)=5-4(3-2x)^4$

65 $y=\frac{(5x^2-1)^3}{2}+8$

66 $v=8-\frac{25}{(t^2+1)^2}$

67 $x=\frac{5}{2t+1}+7$

68 **a** $y=3(2x-1)^4+2$ **b** 5 **c** −1 or 2

Exercise 3C PAGE 63

1 **a** 48 m/s^2 **b** 23 m

2 **a** 4 m/s^2 **b** 9 m/s **c** 26 m

3 **a** −10 m/s **b** 10 m/s **c** −30 m

4 22.6 m, 4.88 m/s

5 3.8 m

6 **a** 36 m/s **b** 145.8 m

7 12 m

8 700 m

9 $v=\frac{8}{5}(2t+1)^3+0.8,\quad x=\frac{(2t+1)^4}{5}+0.8t+2.$

10 −6 m/s

11 **a** 6, 108 m **b** 18 m

12 **a** 1.8 km **b** 30 m/s

13 $v=u+at, s=ut+\frac{1}{2}at^2$

14 75 m, 52.5 m/s

15 52 m/s

Miscellaneous exercise three PAGE 65

1 $3x^2 + 6x + 1$ **2** $3x^2 - 10x - 7$ **3** $3x^2 + 4x + 2$ **4** $6x^2 - 6x + 10$

5 $27x^2 - 6x - 5$ **6** $12x^2 - 38x - 1$

7 **a** $10(2x-3)^4$ **b** 10 **c** $80(2x-3)^3$ **d** 80

8 $a = 3.5, b = 4$ **9** −8.5 m, −13.5 m/s

10 **a** (−2, 8) **b** (−4, −44) and (4, 44). **c** (1, 1) and $\left(-\frac{5}{3}, -\frac{5}{3}\right)$.

11 **a** 43.2 km **b** 144 km **c** 259.2 km

12 $900 per unit. It will cost approximately $900 to produce the 101st item.

13 15 m by 20 m with one of the 15 m sides having the $15 fencing.

14 **a** **i** $10 **ii** $53 **iii** $1000 **iv** $19 000 **b** $\frac{100\,000}{(100-p)^2}$

15 **a** **i** $5560 **ii** $9400 **b** **i** $347.50 **ii** $23.50

c $5 + \frac{60}{\sqrt{x}}$ **d** **i** $20 **ii** $8

e $C(17) - C(16) \approx \$19.77$, $C(401) - C(400) \approx \$8.00$

16 Maximum at (−3, −19), minimum at (−1.5, −28), maximum at (−1, −27), minimum at (0.5, 324).

Exercise 4A PAGE 72

1 **a** Approximately 0.2 units^2.

b Underestimate: 0.149 79. Overestimate: 0.243 54. Mean: 0.196 665, i.e. approx. 0.197.

2 **a** Approximately 1.7 units^2. **b** Underestimate: 1.36. Overestimate: 1.96. Mean: 1.66.

3 **a** Approximately 6.6 units^2. **b** Underestimate: 5.88. Overestimate: 7.48. Mean: 6.68.

Exercise 4B PAGE 78

1 **a** 76 units^2 **b** 76 units^2 **2** **a** 27 units^2 **b** 27 units^2

3 **a** 14 **4** **a** 2

5 $\frac{2}{3}$ **6** $4\frac{2}{3}$ **7** 40 **8** 10 **9** 0 **10** 19

11 39 **12** 12 **13** 76.5 **14** $2\frac{2}{3}$ **15** 42.2 **16** 671

17 14 **18** 0 **19** $3\frac{11}{15}$ **20** **a** $\frac{1}{3}$ **b** $8\frac{2}{3}$ **c** 9

21 **a** $10\frac{2}{3}$ **b** $-2\frac{1}{3}$ **c** $8\frac{1}{3}$ **22** **a** 34 **b** −34

23 **a** 9 **b** 27 **c** 36 **24** 6π **25** $18 - 4\sqrt{2}$

Exercise 4C PAGE 86

1 a, e, f

2 **a** No **b** No **c** Yes **d** No **e** Yes **f** Yes **g** Yes

3 6 units^2 **4** 49.6 units^2 **5** $6\frac{2}{3}$ units^2 **6** $21\frac{1}{3}$ units^2 **7** 0.75 units^2 **8** 2 units^2

9 2 units^2 **10** 3.5 units^2 **11** 0.5 units^2 **12** 3 units^2 **13** $\frac{8\sqrt{2}}{3}$ units^2

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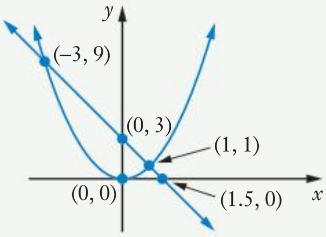
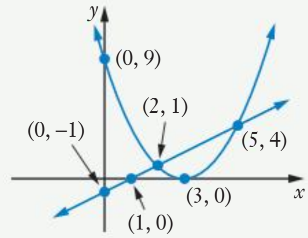
14



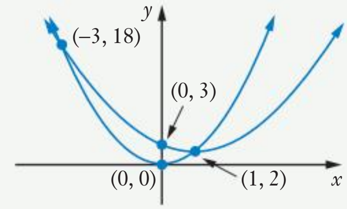
$10\frac{2}{3}$ units^2

15



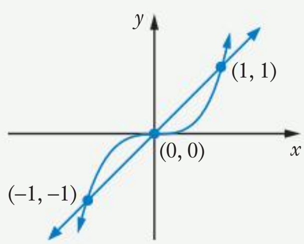
4.5 units^2

16



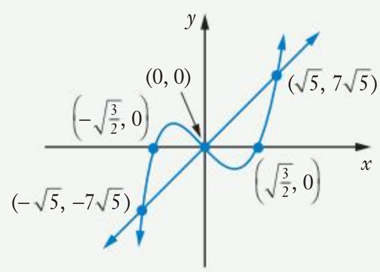
$10\frac{2}{3}$ units^2

17



0.5 units^2

18



25 units^2

19 \$3000

Exercise 4D PAGE 90

1 **a** \$3000 **b** \$39 000

2 \$2500

3 **a** \$259 **b** \$268

4 **a** $\int_5^8 40(25-t)\,dt$ **b** 2220

5 **a** $\int_5^{10} \frac{t^{0.1}}{2}\,dt$ **b** 3.1

6 48.5 million

7 **a** **i** $\int_0^{10} (20-0.15t^2)\,dt$ **ii** 150 kL

b **i** $\int_0^{1} (20-0.15t^2)\,dt$ **ii** 20 kL

c **i** $\int_9^{10} (20-0.15t^2)\,dt$ **ii** 6 kL

8 **a** 2880 **b** 620

9 **a** 400 **b** 560 **c** 100 **d** 130

Miscellaneous exercise four PAGE 92

1 A: $y=4$ B: $x=-5$ C: $y=0.5x+2$ D: $y=x+3$
E: $y=2x-4$ F: $y=-x-1$ G: $y=x-6$ H: $y=-2x-10$

2 **a** $2-9x^2$ **b** -223 **c** $-18x$ **d** 90

3 **a** 8 **b** $\frac{14}{3}$

4 $2x+2$

5 $-2x-2$

6 $4x+11$

7 $8-8x$

8 $2(x+1)(3x+8)$

9 $(2x+5)^2(40x+61)$

10 **a** $y=-12x-8$ **b** $y=-4x+2$ **c** $y=12x-23$ **d** $y=-5x+27$

11 (0, 3), (1, 0)

12 (–1.5, 0), (2, 0), (5, 0)

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13 a $5x^4 + c$ **b** $4x^{\frac{3}{2}} + c$ **c** $\frac{(x+3)^5}{5} + c$ **d** $\frac{(2x+3)^5}{10} + c$

e $4(1+x^3)^5 + c$ **f** $x + \frac{2x^3}{3} + \frac{x^5}{5} + c$

14 a $7.5x - 6000$ **b** 800 **c** \$18 per unit, \$25.50 per unit, \$7.50 per unit

15 53 m, 45 m/s

16 a $A = \frac{(2p-1)^4}{8} + \frac{3}{8}$ **b** $A = (p^2-1)^4 - 36$

17 a 8 units^2 **b** 6 units^2

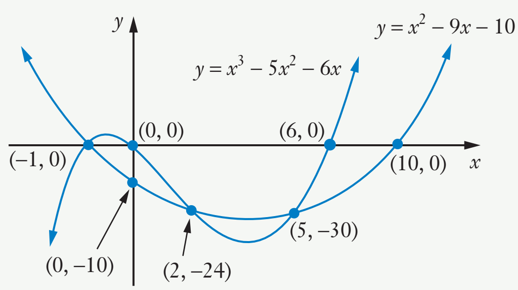
18 a $2x^{\frac{3}{2}} + 2\sqrt{x} + c$

19 a $y = x^3 - 5x^2 - 6x$ cuts the axes at (–1, 0), (0, 0) and (6, 0).

$y = x^2 - 9x - 10$ cuts the axes at (–1, 0), (0, –10) and (10, 0).

b $a = 0, b = -24, c = -30$.

c



d 40.5 units^2

Exercise 5A PAGE 98

1 a $4t^3 + 3t^2 + c$ **b** $4x^3 + 3x^2 - 7$ **c** $12x^2 + 6x$

2 a $t + \frac{1}{t} + c$ **b** $x + \frac{1}{x} - \frac{10}{3}$ **c** $1 - \frac{1}{x^2}$

3 a $\frac{(t^2+3)^5}{5} + c$ **b** $\frac{(x^2+3)^5}{5} - \frac{16\,807}{5}$ **c** $2x(x^2+3)^4$

4 $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ **5** $4x$ **6** $5x^2$ **7** $2x^3$

8 $\frac{2x}{5-x}$ **9** $(x+3)^4$ **10** $16x(x^2+3)^4$

Exercise 5B PAGE 103

1 $2x + 3x^2$ **2** $x^4 + 5$ **3 a** $x^2 + c$ **b** $6x^3 - 4x^2 + 2x + c$

4 a 1 **b** –2 **c** 8 **d** –2

5 a –2 **b** 3 **c** 5 **d** 9

6 a π **b** $4 - \pi$ **c** 0, 4 and 8 **d** $4 < a < 8$

7 $\frac{5}{2}x^{\frac{3}{2}} + (1+3x^2)^4$

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Miscellaneous exercise five PAGE 105

1 $6x^2 + \frac{1}{2\sqrt{x}}$ **2** $2x + 2$ **3** $6(3x - 1)$ **4** $15(3x - 1)^4$

5 $40x^3 - 6x^2 - 15$ **6** $(2x - 3)^2(40x - 21)$ **7** $-\frac{5}{(x-1)^2}$ **8** $\frac{5}{(2x+3)^2}$

9 -2 **10** Max at $(-\sqrt{6}, -2\sqrt{6})$, min at $(\sqrt{6}, 2\sqrt{6})$.

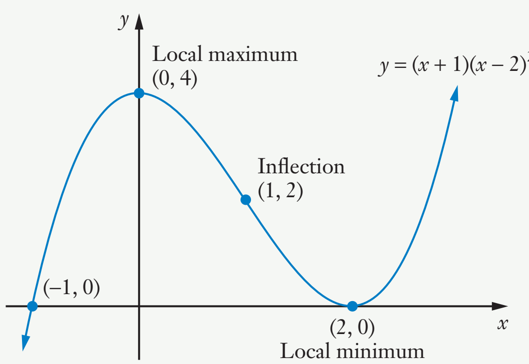
11 **a** The function $f(x)$ has a value of 3 when $x = 0$.
The function $f(x)$ has a value of 28 when $x = 5$.
The average rate of change of $f(x)$, from $x = 0$ to $x = 5$, is 5 units per unit change in x.
The instantaneous rate of change of $f(x)$ when $x = 1$ is 2.

b $f(x) = x^2 + 3$

c Many possible cubics but all must have $d = 3$, $25a + 5b + c = 5$ and $3a + 2b + c = 2$.
One possibility is $f(x) = 3x^3 - 21x^2 + 35x + 3$.

12 $y = 5(2x + 1)^5 + 2$ **13** $f(x) = 3(2x - 1)^4 + 2x + 1$ **14** $30 - 0.04x$, \$26 per unit, \$26

15 **a**



b 6.75 units^2

16 Underestimate: 3.4075. Overestimate: 4.1075. Mean: 3.7575. **17** (524 ± 31) cm^3

Exercise 6A PAGE 112

1 **a** \$1822.12 **b** \$3320.12 **c** \$20 085.54

2 12 500 **3** ~26 g

4 **a** ~2 000 000 **b** ~1 500 000 **c** ~1 100 000 **d** ~800 000

5 **a** 35.8 m/sec (1 dp) **b** 69.4 m/sec (1 dp) **c** 74.6 m/sec (1 dp)

6 **a** 0 **b** 27.73 **c** 73.78

7 ~18

8 **a** \$36 112.55 **b** ~20 years ($t = 19.98$ to 2 dp)

Exercise 6B PAGE 118

1 e^x **2** $7e^x$ **3** $3e^x$ **4** $6e^x$

5 $9e^x$ **6** $-8e^x$ **7** $5e^{5x}$ **8** $7e^{7x}$

9 $-2e^{-2x}$ **10** $15e^{3x}$ **11** $2e^{0.5x}$ **12** $e^{-0.5x}$

13 $6e^x + 6x^2 + 6x$ **14** $2e^x + \frac{1}{2\sqrt{x}}$ **15** $5e^{5x} + 2e^{2x}$ **16** $8e^{4x}$

17 $6e^{3x} + 6e^{2x}$ **18** $15e^{3x} + 4x^3$ **19** $3e^{3x-1}$ **20** $2xe^{x^2+3}$

21 $5e^{5x-1}$ **22** $(6x+2)e^{3x^2+2x-1}$ **23** $3x^2e^{x^3}$ **24** $e^{2x}(1+2x)$

25 $x^2e^x(3+x)$ **26** $\dfrac{e^x(1+2x)}{2\sqrt{x}}$ **27** $\dfrac{e^x(x-1)}{2x^2}$ **28** $e^x(1+2x)^2(7+2x)$

29 $-e^x(1-2x)^4(2x+9)$ **30** $-\dfrac{3}{e^{3x}}$ **31** $2(e^2+1)$ **32** $2e$

33 $y=10x+5$

34 **a** \$8.67 per year **b** \$17.80 per year **c** \$39.62 per year **d** \$196.26 per year

35 **a** 100 **b** 61 tonnes **c** 8.19 tonnes/week **d** 6.07 tonnes/week **e** 4.49 tonnes/week

Exercise 6C PAGE 122

1 **a** ~609 **b** ~90 400 **2** **a** ~2210 **b** ~3297

3 **a** ~159 **b** ~318 **4** **a** ~7358 **b** ~2707

5 **a** $\sim 7.309\times 10^7$ **b** $\sim 1.085\times 10^{10}$ **6** **a** ~2050 **b** ~2570

7 **a** ~340 million **b** ~1120 million **8** **a** ~320 million **b** ~870 million

9 ~272 g **10** \$78 213.16 **11** \$1814.36 **12** \$118.73

13 \$2384.77 **14** **a** ~7800 **b** ~6100

15 **a** 0.02 **b** ~2046 **16** **a** ~5.2 million **b** ~18 million

17 **a** ~5.8 **b** ~6.3 **c** ~0.58 hours **d** ~1.16 hours

18 ~740 **19** ~3.3

Exercise 6D PAGE 125

1 $2e^{3x}+c$ **2** $3e^{2x}+c$ **3** $\frac{1}{5}e^{5x}+c$ **4** $\frac{1}{3}e^{9x}+c$

5 $\frac{5}{3}e^{3x}+c$ **6** $-\dfrac{5}{e^x}+c$ **7** $8\sqrt{e^x}+c$ **8** $-\dfrac{1}{2e^{2x}}+c$

9 $2e^{2x}+x^2+c$ **10** $\frac{1}{3}e^{3x}+\frac{1}{2}e^{2x}+c$ **11** $-\dfrac{3}{2e^{2x}}+c$ **12** $-\dfrac{2}{e^{2x}}+\dfrac{e^{2x}}{8}+c$

13 $e^{x^2}+c$ **14** $3e^{2x+1}+c$ **15** $4e^{x^2+5}+c$ **16** $5(e^2-1)$

17 $\dfrac{e^5-1}{5}$ **18** $2e^4-e^2-e$ **19** $3+e^4$ **20** $e-1$

21 $12e+4$ **22** **a** $\dfrac{1+5e^{2t}}{2}$ **b** $\dfrac{1+5e}{2}$

23 **a** $2x^3-4e^{3x}+5$ **b** $21-4e^6$ **24** **a** 19.1 units^2 **b** (e^3-3e+1) units^2

Miscellaneous exercise six PAGE 126

1 $y=13x-8$

2 **a** $5(x+2)^4$ **b** $10(2x+1)^4$ **c** $\dfrac{10}{(x+5)^2}$ **d** $\dfrac{26}{(x+5)^2}$ **e** $12x^2-e^x$ **f** $5e^{5x}+5$

3 $a=0.4$, $b=50$

4 **a** 0.1 m/s^2 **b** $0.1e^2$ m/s^2 (≈ 0.739 m/s^2) **c** $(10e+2)$ m (≈ 29.18 m)

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6 (−5, 0) gradient 35, (0, 0) gradient −10, (2, 0) gradient 14.
Total area enclosed by curve and x-axis is 101.75 units^2 (= 93.75 units^2 + 8 units^2).

7 $a = -1, b = 4$

8 **a** 48 **b** 0.5 **c** $e - 1$ **d** $3e^2 - 3$ **e** 15 **f** $\frac{5}{3}$

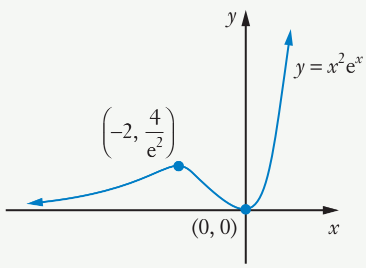
9 Answers to be checked with calculator.

10 **a** 7.5 km **b** 27.5 km **c** 31.5 km

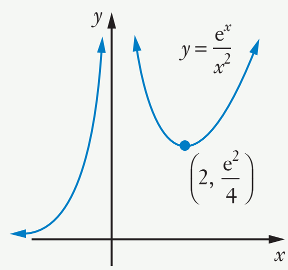
11 35.2 m/s (rounded to 1 dp), $\frac{200}{3}$ m/s. **12** 16π cm^2

13 **a** \$745.91 **b** \$1660.06 **14** **a** 21.1 °C/min **b** 11.6 °C/min **c** 0.3 °C/min

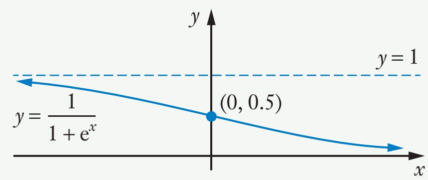
15 **a**



b



c



16 613 cm^2

Exercise 7A PAGE 142

1 $5x^4 - 2x$ **2** $3x^2$ **3** $\sin x$ **4** $\cos x + \sin x$

5 $-\sin x - \cos x$ **6** $1 - \frac{1}{\cos^2 x}$ (i.e. $-\tan^2 x$) **7** $4x - 1$ **8** $10x - 75x^2$

9 $6\cos x$ **10** $-4\sin x$ **11** $\sin x + x\cos x$ **12** $2x\cos x - x^2 \sin x$

13 $-\frac{3x^2 + 1}{(3x^2 - 1)^2}$ **14** $-\frac{4x}{(x^2 - 1)^2}$ **15** $-\frac{x\sin x + \cos x}{x^2}$ **16** $\frac{x\cos x - \sin x}{x^2}$

17 $\frac{\sin x - x\cos x}{\sin^2 x}$ **18** $\frac{\cos x + x\sin x}{\cos^2 x}$ **19** $12x(x^2 + 1)$ **20** $\frac{x}{\sqrt{x^2 - 1}}$

21 $6\cos 6x$ **22** $-2\sin(2x + 3)$ **23** $2\sin x\cos x$ **24** $3\sin^2 x\cos x$

25 $-5\cos^4 x\sin x$ **26** $-3\sin 3x$ **27** $3\cos(3x - 7)$ **28** $-2\sin(2x + 5)$

29 $3\sin x$ **30** $3 - 2\sin x$ **31** $2\cos 2x$ **32** $2x + \sin x$

33 $\frac{x\cos x - 2\sin x - 2}{x^3}$ **34** $3\cos x + 2\sin x$ **35** $-3\sin 3x$ **36** $-9\sin 9x$

37 $-6\sin 2x$ **38** $15\cos 3x$ **39** $6\cos 3x - 6\sin 2x$ **40** $5\sin^4 x\cos x$

41 $-10\cos x\sin x$ **42** $\frac{\cos x}{2\sqrt{\sin x}}$ **43** $7\cos 7x$ **44** $8\cos 8x$

45 $4\cos 4x - 4\sin 4x$ **46** $6\cos(3x - 1)$ **47** $-16\sin(4x + 3)$ **48** $6\sin^2 x\cos x$

49 $-6\cos x\sin x$ **50** $\cos x - x\sin x$ **51** $2x\cos x - x^2\sin x$ **52** $2\sin x + 2x\cos x$

53 $\frac{2}{\cos^2 x}$ **54** $\frac{2}{\cos^2 x}$ **55** $\frac{\sqrt{3}}{2}$ **56** $-\sqrt{3}$

57 2 **58** 0 **59** $-\sin x$ **60** $-25\cos 5x$

61 $-12\sin 2x$ **62** $-\sin x - \cos x$ **63** $y = x$ **64** $y = x + 3$

65 **a** 1 **b** $-2\sqrt{3}$ **66** $\frac{\pi}{180}\cos x°$ **67** 200 cm^2, $10\sqrt{2}$ cm

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68 $4\cos(0.1t)$ cm^2/s

a 3.98 cm^2/s **b** 3.51 cm^2/s **c** 2.16 cm^2/s **d** −1.66 cm^2/s

69 **a** $5, \frac{\pi}{6}$ **b** $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}$ **c** −3.4 **d** −9

70 0.6435, 5

Exercise 7B PAGE 148

1 $5\sin x + c$ **2** $-2\cos x + c$ **3** $10\cos x + c$ **4** $-2\sin x + c$

5 $3\sin 2x + c$ **6** $\frac{1}{3}\sin 6x + c$ **7** $-3\cos 4x + c$ **8** $\frac{1}{3}\cos 3x + c$

9 $-\frac{4}{5}\sin 10x + c$ **10** $-2\cos\frac{x}{2} + c$ **11** $\frac{2}{3}\sin\frac{3x}{2} + c$ **12** $9\cos\frac{2x}{3} + c$

13 $-3\cos(2x+3) + c$ **14** $\frac{3}{2}\sin(2x-3) + c$ **15** $\frac{1}{2}\sin\left(2x + \frac{2\pi}{3}\right) + c$ **16** $\cos(-x) + c$, i.e. $\cos x + c$

17 $4\tan x + c$ **18** $3\sin 2x - 2\cos 3x + c$ **19** $\frac{1}{8}\sin 8x + 2\cos 2x + c$ **20** $x^2 + 4\sin x + 3\sin 2x + c$

21 $3x + 2x^2 - 2x^3 + 2\sin 5x + \frac{1}{2}\cos 4x + c$ **22** $-\frac{1}{4}\cos^4 x + c$ **23** $-5\cos^6 x + c$

24 $-\frac{1}{7}\cos 7x + c$ **25** $-\frac{1}{2}\cos 2x + c$ **26** $\frac{1}{7}\sin 7x + c$ **27** $\frac{1}{4}\sin 4x + c$

28 1 **29** 1 **30** $2 - \sqrt{2}$

31 **a** $1 - \frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}} - 1$

32 2 units^2

33 **a** 0.5 units^2 **b** 2.5 units^2

34 (Could use calculus to determine the greatest speed and the least distance but easier to use the facts that $-1 \le \sin 2t \le 1$ and $-1 \le \cos 2t \le 1$.)

a 2 m/s **b** $(5 + \sin 2t)$ metres **c** 4 metres **d** $-4\sin 2t$ m/s^2

35 **a** 0.25 units^2 **b** 2.5 units^2

36 **a** $A\left(\frac{\pi}{2}, 0\right)$, $B(\pi, 0)$, $C\left(\frac{3\pi}{2}, 0\right)$ **b** 6 units^2

Miscellaneous exercise seven PAGE 150

1 e^x **2** $2e^x$ **3** $8e^x$ **4** $e^x + \cos x$

5 $-\sin x\, e^{\cos x}$ **6** $2\cos 2x\, e^{\sin 2x}$ **7** $2\cos x\, e^{2\sin x}$ **8** $e^x - \frac{1}{x^2}$

9 $\frac{2}{\sqrt{x}} + 3e^{3x}$ **10** $\frac{e^x(2x+1)}{2\sqrt{x}}$ **11** $e^x(\cos x + \sin x)$ **12** $e^x(\cos 2x - 2\sin 2x)$

13 $e^x \sin x(\sin x + 2\cos x)$ **14** $6x\, e^{3x^2+2}$ **15** $(2x + \cos x)e^{x^2 + \sin x}$ **16** $6(2r+3)^2$

17 **a** $2(e^4 - 1)$ **b** $\frac{3}{10}$ **c** 6

18 $-\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ **19** $-e^{-\pi}$ **20** $0.5x^{-0.5}$

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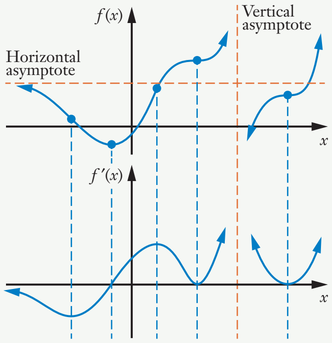
21 A is decreasing as t increases.

a –245 619 tonnes per year (nearest 1 tonne/year).

b –181 959 tonnes per year (nearest 1 tonne/year).

c –110 364 tonnes per year (nearest 1 tonne/year).

22



23 **a** 10 m/s **b** –2.5 m/s^2 **c** 145 m **d** 85 m **e** 115 m **f** 75 m

24 **a** $V = \frac{4}{3}\pi(100 - 3x)^3$ for $V \geq 0$. **b** On the 7th day. ($x \approx 6.88$).

c Decreasing at $12\pi(100 - 3x)^2$ m^3/day **d** ~270 000 m^3/day

e $8\pi(1 + x)(100 - 2x - x^2)^2$ m^3/day, ~640 000 m^3/day

25 **a** 0.66 m/s **b** 0.27 m/s^2 **c** 0.05 m/s^2

26 **a** $\left(1 + \frac{\sqrt{3}}{2} - \frac{5\pi}{24}\right)$ units^2 **b** $\left(2 + \sqrt{3} - \frac{5\pi}{12}\right)$ units^2

Exercise 8A PAGE 163

1 **a** Continuous **b** Discrete **c** Continuous **d** Discrete **e** Discrete **f** Continuous
g Continuous

2 No **3** No **4** No **5** Yes **6** 0.4 **7** 0.05

8 0.1 **9** –0.8

10

x	0	1	2
$P(X = x)$	0.25	0.5	0.25

11 **a** 0.2 **b** 0.8 **c** 0.2 **d** 0.5 **e** 0.25 **f** 0.75

12 **a** 0.4 **b** 0.4 **c** 0.5 **d** 0.5 **e** 0.25 **f** 0.5

13

x	0	1	2	3	4	5	6	7	8	9	10
$P(X \leq x)$	0.005	0.015	0.055	0.175	0.375	0.625	0.825	0.945	0.985	0.995	1

14

x	0	1	2	3	4	5
$P(X = x)$	0.04	0.16	0.3	0.3	0.16	0.04

15

x	0	1	2
$P(X = x)$	0.16	0.48	0.36

16

x	0	1	2	3
$P(X = x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

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17

x	1	2	3
$P(X = x)$	0.3	0.6	0.1

18

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{1}{3}$

a $\frac{2}{5}$ **b** $\frac{1}{15}$ **c** $\frac{4}{5}$

19

x	1	2	3	4
$P(X = x)$	0.4	0.3	0.2	0.1

a 0.4 **b** 0.7 **c** 0.6

20 **a** 0.3 **b** 0.12 **c** 0.24 **d** 0.25 **e** 0.48 **f** 0.012
g 0.072 **h** 0.117 **i** 0.1

21

x	0	1	2	3	4
$P(X = x)$	0.646 96	0.308 08	0.042 99	0.001 95	0.000 02

Exercise 8B PAGE 171

1 $k = 0.1$, $E(X) = 2.7$ **2** $k = 0.2$, $E(X) = 17$ **3** $k = 0.05$, $E(X) = 5.85$ **4** $k = 0.2$, $E(X) = 11.6$

5 $p = 0.15$, $q = 0.25$, $\text{Var}(X) = 1.91$ **6** $p = \frac{5}{18}$, $q = \frac{2}{9}$

7 **a** 15 **b** 1.5 **c** 26 **d** 4.5

8 **a** $E(X) = 26$, $\text{Var}(X) = 184$ **b** 29 **c** 52 **d** 55
e 184 **f** 736 **g** 736

9 $E(X) = 3$, $\text{Var}(X) = 2$.

10

x (\$ paid out)	0	15	30
$P(X = x)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

For the desired long term average profit per game they should charge \$8 per play.

11 To at least break even in the long run (or be very close to break even) the organisers need to charge at least \$2.50 per game.

12 **a** $E(X) = 4.5$ **b** $E(Y) = 25.5$ **c** $E(Z) = \frac{761}{2240}$

13 **a** Would expect approximately 12% of the games to result in prize of more than \$6.
b Would expect 40% of the scores to exceed a total score of 4.
c Organisers should expect to be 'up' by approximately \$100 after 100 plays of the game.

14 Scheme [1] has expected fortnightly earnings (i.e. long term average fortnightly earnings) of \$962.50.
Scheme [2] has expected fortnightly earnings (i.e. long term average fortnightly earnings) of \$878.75.
Based on expected earnings he should prefer scheme [1].

15 **a** \$1340 **b** \$1270
c Compare your answers and justifications with those of others in your class but note that consideration of the probability distributions should suggest that it is perhaps not as straightforward as simply choosing the scheme with the higher expected return. Hence your response should discuss more than just choosing the higher expected return.

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Miscellaneous exercise eight PAGE 174

1 **a** ~200 **b** ~298 **c** ~444 **d** As $t \to \infty$, $N \to 100\,000$.

2 **a** $-\frac{6}{x^2}$ **b** $-\frac{3}{\sqrt{x^3}}$ **c** $10x - e^x$ **d** $6xe^{3x^2}$

e $6xe^{3x^2+1}$ **f** $4(2x+1)^4(6x-7)$ **g** $10\cos x$ **h** $10\cos 10x$

3 $3x^2 - 5$

4 **a** X is not a uniform discrete random variable because whilst the possible values of X are discrete values (0, 1 and 2), the probabilities $P(X = x)$ are not the same for all values of x. In this case $P(X = 0) = \frac{25}{36}$, $P(X = 1) = \frac{5}{18}$, $P(X = 2) = \frac{1}{36}$.

b X is a uniform discrete random variable because the possible values of X are discrete values (1, 2, 3, 4, 5, 6) and in each case the probability of occurrence is the same, i.e $\frac{1}{6}$.

c X is not a uniform discrete random variable because the variable involved is continuous, not discrete.

5 (−1.5, −4.5) grad 4.2 **6** $4e^2$ **7** $3y = x + 24$

8 **a** 64 **b** 4 **c** 17

9 1 unit^2 **10** **a** 75 m/s **b** **i** 5.09 m/s^2 **ii** 0.72 m/s^2

11 $f(x) = (3 - x)^5 + x^3 - 3x^2 - 2$ **12** $5x^4$ (as expected). **13** $(1, 0.5e)$ **14** $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

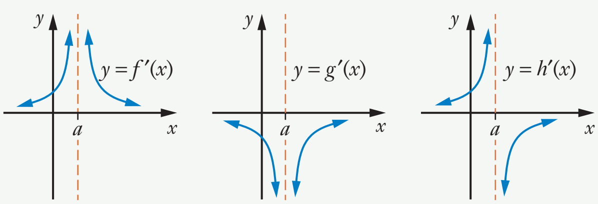
15 **a** $2x^2 + c$ **b** $3e^{2x} + c$ **c** $x^2 + e^x + c$ **d** $x^2e^x + c$

16 **a** $\frac{1}{12}$ **b** $\frac{1}{3}$ **c** $\frac{5}{12}$ **d** $\frac{3}{4}$

e 0 **f** $\frac{4}{9}$ **g** $\frac{41}{12}$

h 1.32 (Note: The exact value, not requested, would be $\frac{\sqrt{251}}{12}$.)

17



18 **a** $\frac{5\pi - 6\sqrt{3} - 12}{12}$ **b** $\frac{12 + 6\sqrt{3} - 5\pi}{12}$ **c** $\frac{6\sqrt{3} - \pi - 4}{4}$ units^2

Exercise 9A PAGE 188

1 Mean = 0.6, variance = 0.24.

2 P(success) = 0.1 is graph B, P(success) = 0.5 is graph A, P(success) = 0.8 is graph C.

3 P(success) = 0.5 is graph C, P(success) = 0.7 is graph B, P(success) = 0.9 is graph A.

4 Mean = 3, standard deviation = 1.5

5 $n = 24$, $p = 0.4$

6 **a** $a = 0.2076$, $b = 0.0865$

b $\mu = 2$, $\sigma = 0.5\sqrt{6}$ i.e. 1.225 rounded to 3 dp

c $P(\mu - \sigma \le X \le \mu + \sigma) = 0.786$ rounded to 3 dp

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(For numbers **7** to **13** answers are given rounded to 4 dp unless question requests otherwise.)

7 **a** 0.0605 **b** 0.0101 **c** 0.0705 **d** 0.9295

8 **a** 0.3025 **b** 0.1176 **c** 0.4202 **d** 0.5798

9 **a** 0.2605 **b** 0.0004

10 0.1715

11 **a** 0.2001 **b** 0.2335 **c** 0.1493 **d** 0.3828

12 **a** 0.202 **b** 0.010 **c** 0.037

13 0.1792

Exercise 9B PAGE 192

Unless requested otherwise answers tend to be given rounded to 4 dp or, if answer may have been obtained using previous 4 dp answers, then answer might be given rounded to 3 dp.

1 **a** 0.0459 **b** 0.0011 **c** 0.9999 **d** 0.9999

2 **a** 0.1171 **b** 0.1244 **c** 0.8744 **d** 0.8744

3 **a** 0.1612 **b** 0.4826 **c** 0.334

4 **a** 0.1789 **b** 0.4164 **c** 0.430

5 **a** 0.9952 **b** 0.1636

6 **a** 0.8423 **b** 0.7134

7 0.8

8 **a** 0.0355 **b** 0.9790 **c** 0.0565

9 **a** 0.1937 **b** 0.9298 **c** 0.0702

10 **a** 0.1285 **b** 0.8159 **c** 0.0556

11 **a** 0.175 **b** 0.002 **c** 0.003 **d** 0.762

12 **a** 0.1780 **b** 0.0002 **c** 0.1318 **d** 0.1694

13 0.0956

14 **a** $\frac{1}{6}$ **b** 0.8385 **c** 0.7752 **d** 0.2248

15 0.1035 **16** Joel **17** 7 **18** At least 9 attempts.

19 Compare your comments with those of others in your class.

20 Compare your reasoning with that of others in your class.

Miscellaneous exercise nine PAGE 195

1 **a** ~35°C **b** ~21°C **2** $2x + 5$ **3** $6(2x + 1)^2$

4 $4(2x - 3)$ **5** $\frac{1}{(x + 1)^2}$ **6** $48x^3 + 18x^2 - 10$ **7** $2(2 - 3x)^2(1 - 6x)$

8 $2(4 + 7x)^3(35x + 18)$ **9** e^x **10** $10x + e^x$ **11** $12e^{3x} + 4x^3$

12 $2e^{2x - 4}$ **13** $3e^{3x + 1}$ **14** $xe^x(x + 2)$ **15** $1 + e^x + xe^x$

16 $\cos x$ **17** $-3 \sin 3x$ **18** $3 \cos(3x - 5)$ **19** $2e^{2x}(2 \cos 4x + \sin 4x)$

20 0.000 752 (correct to six decimal places).

21 On each roll there can be considered to be two outcomes, that of guessing correctly (success) and that of guessing incorrectly (failure). Each time the probability of success remains the same $\left(\frac{1}{6}\right)$.

Thus we have repeated, independent, Bernoulli trials and we are considering the number of successes. Hence a binomial distribution would be an appropriate model.

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22 Answers to be checked with calculator.

23 **a** 54 **b** 0 **c** $12(2x-1)^2$ **d** 300

24 $6+2e^2$

25 Answers to be checked with calculator.

26 $-\dfrac{9}{(x-1)^2}$

27 **a** 0.375 **b** 0.3 **c** 0.25

28 6

29 $y=x^2-5x+11$

30 $y=5x^2-6x+1$

31 $f(x)=6-2(8-2x)^3$

32 $x=\dfrac{6}{3t+1}+3$

33 **a** $y=\dfrac{(2x+1)^5}{2}+\dfrac{7}{2}$ **b** 4 **c** 0.5

34 8.5 units^2

35 $y=5x^3-7x^2+3$

36 **a** \$67.65/year **b** \$109.33/year **c** \$199.21/year **d** \$1205.13/year

37 0.65

38 Using calculus the small change in the function is approximately 1.24.
$f(5.04)-f(5)=1.2448$.
The value of 1.24 is a good approximation.

39 V changes by approximately 9%.

40 1 units^2

41 –5, –7.125

42 \$606

43 Velocity of object at time t is $-\sin t\times e^{\cos t}$ m/sec. When $t=\dfrac{\pi}{2}$, velocity = –1 m/sec.

44 **a** $\dfrac{1}{4}$ **b** $\dfrac{17}{3}$

45 $\dfrac{2\cos x}{(1-\sin x)^2}$

46 **a** Zero. (P(Zero sixes) = 0.482.)
b More likely to get a number other than zero sixes than to get zero sixes. P(zero sixes) = 0.482, P(not zero sixes) = 0.518.
c Long term average number of sixes would be close to $\dfrac{2}{3}$.

47 $\dfrac{-3(\pi\sqrt{3}+3)}{2\pi^2}$

48 Maximum turning point at $\left(-\dfrac{5\pi}{3},2\right)$. Minimum turning point at $\left(-\dfrac{2\pi}{3},-2\right)$.

Maximum turning point at $\left(\dfrac{\pi}{3},2\right)$. Minimum turning point at $\left(\dfrac{4\pi}{3},-2\right)$.

49 $0.5(n+1)$

50 **a** $a=-5.5$, $b=4$, $c=6.5$ **b** (–7, 10.5)

51 $\dfrac{11}{4\pi}$ units^2

52 **a** $\dfrac{dx}{d\theta}=10\cos^2\theta-10\sin^2\theta\ \ (=10-20\sin^2\theta)$

b (Justify *maximum* either by considering the sign of $\dfrac{dx}{d\theta}$ either side of 45° or by consideration of second derivative.)
$x_{max}=5$.

c $x=5\sin 2\theta$
Maximum value of $\sin A$ is 1, when angle A is $\dfrac{\pi}{2}$ radians, or 90°.
Thus maximum value of $5\sin 2\theta$ is 5, when 2θ is $\dfrac{\pi}{2}$. This maximum x value is 5 when angle of projection is $\dfrac{\pi}{4}$ radians, or 45°, thus confirming part **b** answers.

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ISBN 9780170395137

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