CATHERINE SMITH | NICK SIMPSON | RAYMOND ROZEN

JACARANDA MATHS QUEST

SPECIALIST MATHEMATICS 11

FOR QUEENSLAND

UNITS 1&2

studyon

jacaranda
A Wiley Brand

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CATHERINE SMITH | NICK SIMPSON | RAYMOND ROZEN

CONTRIBUTING AUTHORS

Pauline Holland | Paula Evans | Steven Morris | Margaret Swale

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ABOUT THIS RESOURCE

Jacaranda Maths Quest 11 Specialist Mathematics Units 1 & 2 for Queensland is expertly tailored to comprehensively address the intent and structure of the new syllabus. The ***Jacaranda Maths Quest for Queensland*** series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

Preparing students for exam success

Chapter openers place mathematics in real-world contexts to drive engagement.

FREE access to studyON — our study, revision and exam practice tool — is included with every title. studyON allows you to revise at the concept, chapter, curriculum topic or unit level.

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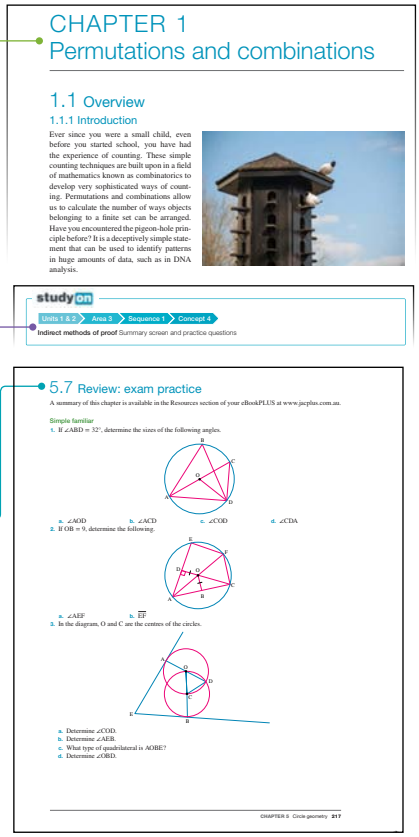

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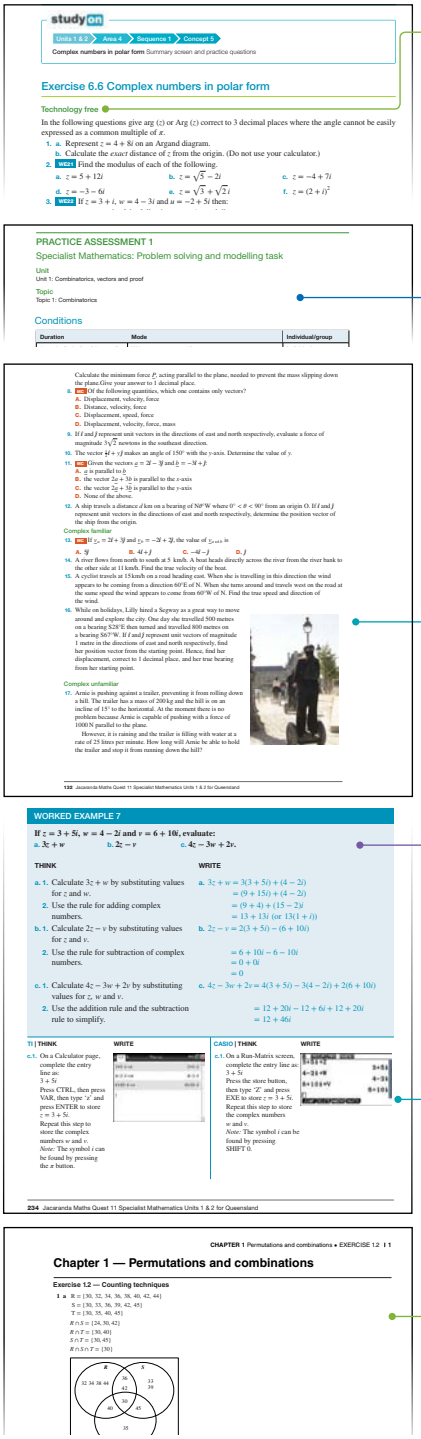
Two complete sets of practice assessments modelled on QCAA guidelines — a set for student revision and a quarantined set for teachers — are included. Exemplary responses and worked solutions are provided for teachers.

Chapter questions and activities are aligned with Marzano and Kenall's taxonomy of cognitive process — retrieval, comprehension, analysis and knowledge utilisation.

Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Selected worked examples demonstrate the use of non-CAS calculators.

Free fully worked solutions are provided, enabling students to get help where they need it, whether at home or in the classroom — help at the point of learning is critical. Answers are provided at the end of each chapter in the print and offline PDF.



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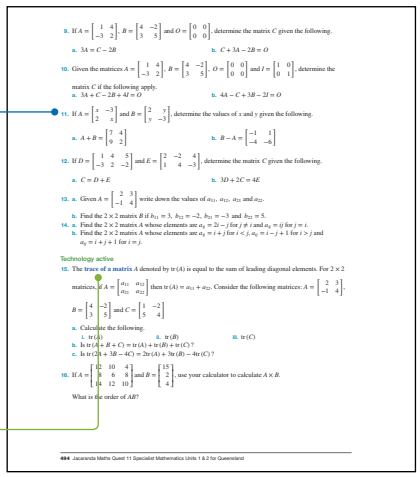
An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.



eBookPLUS features

Fully worked solutions for every question

Digital documents: downloadable SkillSHEETS to support skill development and SpreadSHEETS to explore mathematical relationships and concepts

Chapter summaries in downloadable format to assist in study and exam preparation

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Interactivities and video eLessons placed at the point of learning to enhance understanding and correct common misconceptions

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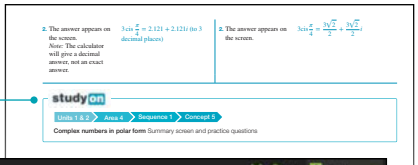

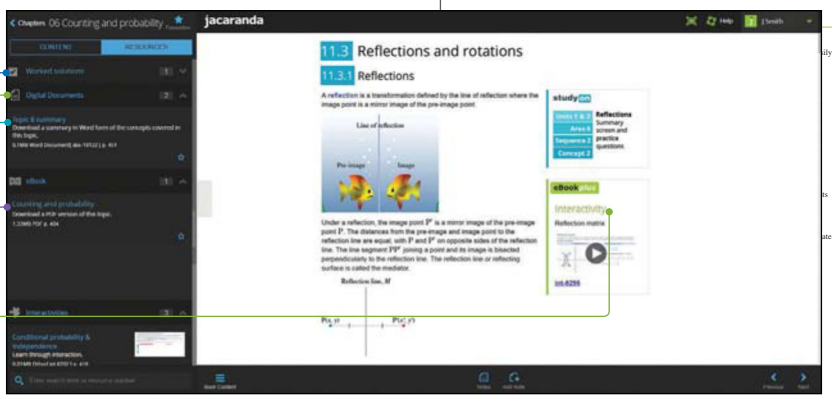

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A downloadable PDF of the entire solutions manual, containing worked solutions for every question in the text

A set of four practice assessments: a problem solving and modelling task and three examination-style assessments

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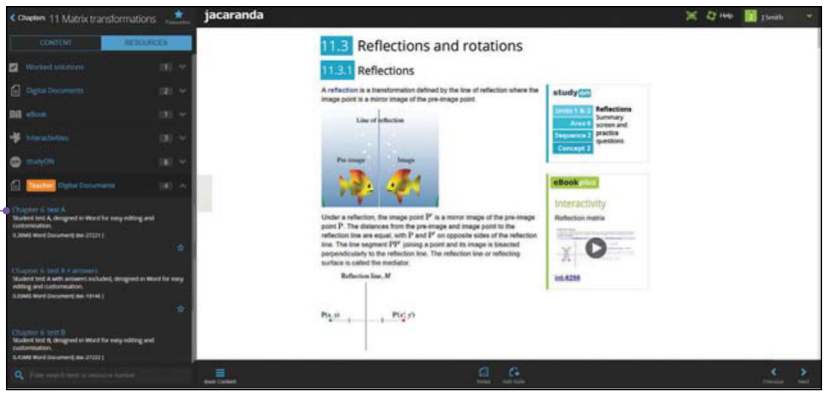
In the Resources tab of every chapter there are two topic tests in downloadable, customisable Word format with worked solutions.



In the Prelims section of the eGuidePLUS

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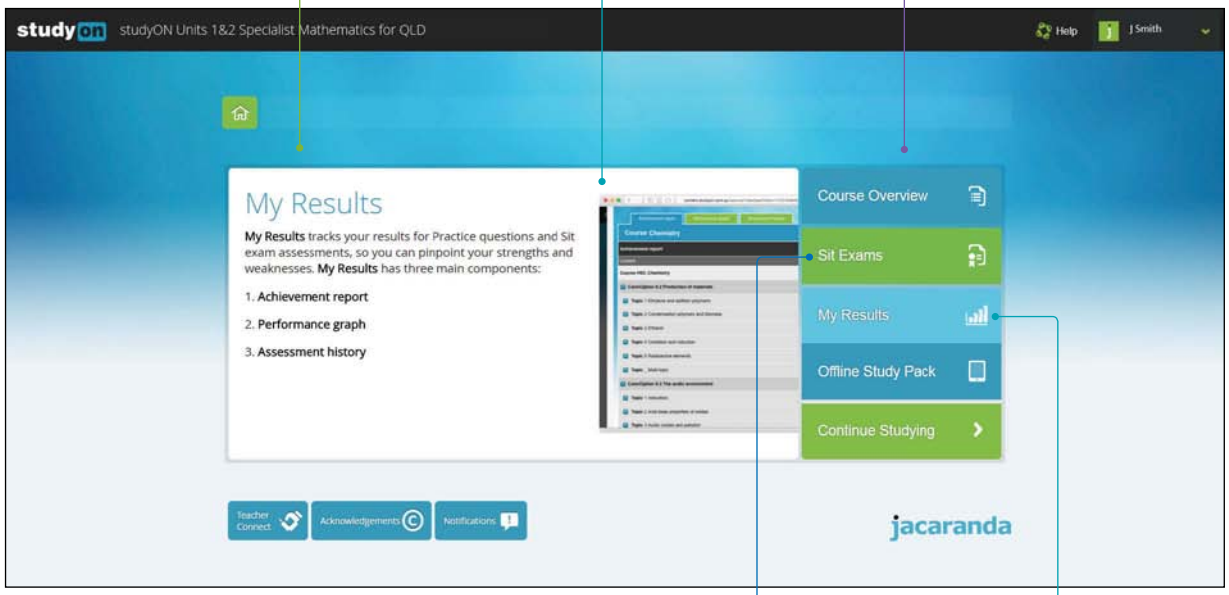
studyON provides a complete study solution. An interactive and highly visual online study, revision and exam practice tool, it is designed to help students and teachers maximise exam results.

Concept summary screens and interactivities summarise key concepts and help prevent misconceptions.

Direct links from the eBookPLUS help scaffold students' understanding and study practices.

The studyON question hierarchy allows students in the *Continue Studying* feature to revise across the entire course, or to drill down to concept level for a more granular set of questions.

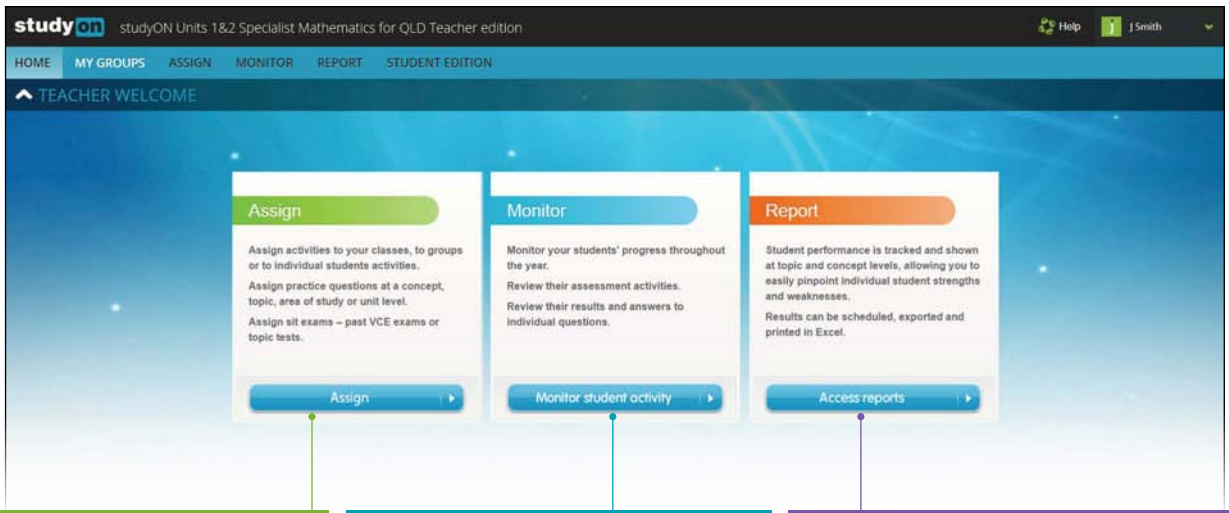


studyON prepares students for actual exams:

- The *Sit Exams* feature allows students to sit timed practice exams.
- Exam-style questions have been authored by our team of highly qualified teachers.
- From 2020, offcial past QCAA exam questions will be available for Units 3 & 4 with exemplary worked solutions to provide feedback for every question.

studyON's built-in progress tracker enables self-diagnosis of strengths and weaknesses at a topic and concept level, so students know exactly what needs extra revision and can sit their exams with confidence.

studyON Teacher edition is a powerful diagnostic tool



Enables teachers to assign activities for extra revision and practice, and track progress at an individual, group and classroom level

Allows teachers to monitor students' activities and results to pinpoint strengths and weaknesses. Armed with evidence-based insights, teachers can intervene at the right time.

Alignment with the Jacaranda text helps with planning, and instant feedback saves marking time. Built-in reporting functionality lets teachers easily schedule, export and print reports in Excel.

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studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access exam questions** that have been written by experienced examiners for practice at a concept, area or entire course level, and receive immediate feedback. **From 2020, QCAA questions** will be included with exemplary worked solutions.
- **Sit past QCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

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CHAPTER 1
Permutations and combinations

1.1 Overview

1.1.1 Introduction

Ever since you were a small child, even before you started school, you have had the experience of counting. These simple counting techniques are built upon to develop very sophisticated ways of counting, in a field of mathematics known as combinatorics. Permutations and combinations allow us to calculate the number of ways objects belonging to a finite set can be arranged.

Have you encountered the pigeon-hole principle before? It is a deceptively simple statement that can be used to identify patterns in huge amounts of data, such as in DNA analysis.

Have you encountered Pascal's triangle before? It spans the mathematical fields of combinations, probability, the binomial theorem, Fibonacci numbers and the bell-shaped normal distribution. In developing his triangle, Blaise Pascal (1623–1662) made a fundamental contribution to the field of combinatorics.

In this chapter you will apply combinatorics to determine, for example, the number of ways a team of 5 players can be chosen from a group of 10. Consider its usefulness in developing rosters for staff or flow charts for projects. Just as Pascal's triangle spans mathematical fields, combinatorics spans industries as varied as gambling, internet information transfer and security, communication networks, computer chip architecture, logistics and DNA modelling. It has applications in any field where different choices mean different efficiencies.

LEARNING SEQUENCE

1.1 Overview
1.2 Counting techniques
1.3 Factorials and permutations
1.4 Permutations with restrictions
1.5 Combinations
1.6 Applications of permutations and combinations
1.7 Pascal's triangle and the pigeon-hole principle
1.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Counting techniques

1.2.1 Review of set notation

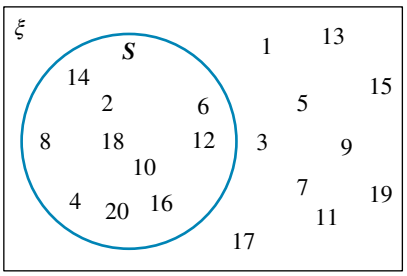
A **set**, S, is a collection of objects. The objects in a set are referred to as the **elements** of the set.

A set can be written in a variety of ways. Consider the following. Let the sample space be the set of numbers between 1 and 20, that is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Let the set S be the set of even numbers between 1 and 20 inclusive.

S can be:

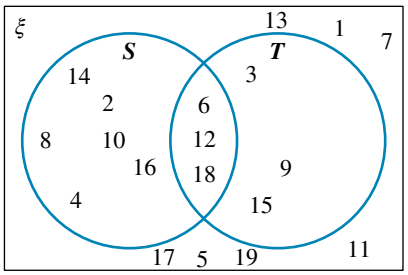
- written as a list: $S = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ (in any order)
- written as a rule: $S = \{n: n = 2r \text{ for } 1 \leq r \leq 10\}$
- shown in a Venn diagram.



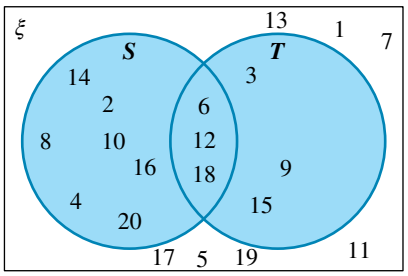
The complement of S, written as S', is the set of all things NOT in S. In this example, $S' = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

The complete set of objects being considered is called the **universal set**, ξ. It is represented by the rectangle in the Venn diagram, and is abbreviated with Greek letter ξ (pronounced 'ksi'). In this example, $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Now consider a second set, T, which is the set of numbers that are multiples of 3 between 1 and 20; that is, $T = \{3, 6, 9, 12, 15, 18\}$. The sets S and T can be combined in various ways.

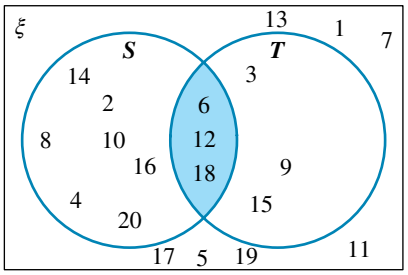


The **union** of S and T is all the elements in either S or T or both. It is shown as follows.



$S \cup T = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$

The **intersection** of S and T is all the elements that are in both S and T.



$S \cap T = \{6, 12, 18\}$

1.2.2 The inclusion–exclusion principle

Continuing the example from subsection 1.2.1, we have the two sets $S = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and $T = \{3, 6, 9, 12, 15, 18\}$. If we find the number of elements in the two sets, we obtain $n(S) = 10$ and $n(T) = 6$.

In general, for two sets S and T, $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

This is known as the **inclusion–exclusion principle**.

$n(S \cup T)$ is the number of elements in the union of S and T (i.e. the number of elements in S or T or both). This is equal to $n(S)$ (the number of elements in S) plus $n(T)$ (the number of elements in T) minus $n(S \cap T)$ (the number of elements in both S and T, as these have already been counted in sets S and T).

For our example above:

$$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$

$$n(S) = 10, n(T) = 6, n(S \cap T) = 3$$

So, $n(S \cup T) = 13$, which can be confirmed by counting all the elements which occur in S, T or both.

If three sets are involved, the inclusion–exclusion principle becomes:

$$\boldsymbol{n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T) - n(T \cap R) - n(S \cap R) + n(S \cap T \cap R)}$$

An example of the inclusion–exclusion principle with three sets is shown in Worked example 1.

WORKED EXAMPLE 1

Let R be the set of natural numbers between 15 and 30 inclusive that are divisible by 2.
Let S be the set of natural numbers between 15 and 30 inclusive that are divisible by 3.
Let T be the set of natural numbers between 15 and 30 inclusive that are divisible by 5.

a. Construct a Venn diagram to represent R, S and T.

b. Use this diagram to evaluate $n(R \cup S \cup T)$.

c. Recall the inclusion–exclusion principle to compute $n(R \cup S \cup T)$.

THINK

a. 1. R is the set of natural numbers between 15 and 30 divisible by 2, so:
$R = \{16, 18, 20, \ldots 30\}$

2. S is the set of natural numbers between 15 and 30 divisible by 3, so:
$S = \{15, 18, 21, 24, 27, 30\}$
$R \cap S$ are the numbers which occur in both S and R, so:
$R \cap S = \{18, 24, 30\}$

3. T is the set of natural numbers between 15 and 30 divisible by 5, so:
$T = \{15, 20, 25, 30\}$
$R \cap T = \{20, 30\}$
$S \cap T = \{15, 30\}$
$R \cap S \cap T = \{30\}$

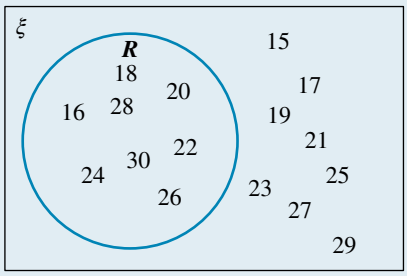
b. $n(R \cup S \cup T)$ is the number of elements in sets R, S and T. Count the number of elements within each portion of each circle.

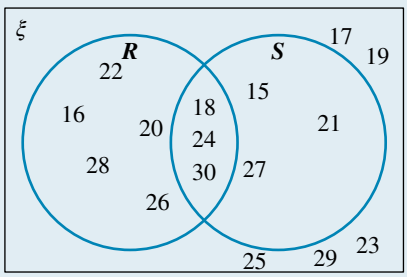
c. 1. Recall the inclusion-exclusion principle formula.

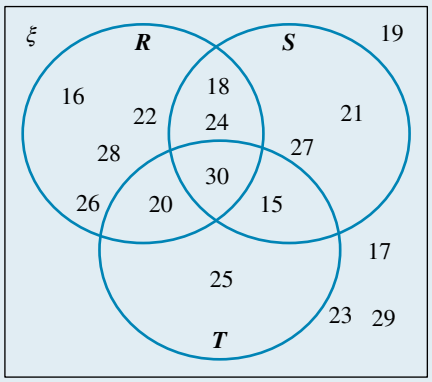
2. The number of elements in each set can be substituted into the formula.

WRITE

a.







b. $n(R \cup S \cup T) = 4 + 2 + 1 + 2 + 1 + 1 + 1$
$= 12$

c. 1. $n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T)$
$- n(T \cap R) - n(S \cap R) + n(S \cap T \cap R)$

2. $n(S \cup T \cup R) = 6 + 4 + 8 - 2 - 2 - 3 + 1$
$= 12$
This agrees with the calculation from the Venn diagram in part **b**.

1.2.3 Types of counting techniques

Counting techniques allow us to determine the number of ways an activity can occur. This in turn allows us to calculate the **probability** of an event. Recall from your earlier probability studies that the probability of event A, $\mathrm{P}(A)$, can be determined by counting the number of elements in A and dividing by the total number in the sample space, ξ, according to the formula

$$\mathrm{P}(A) = \frac{n(A)}{n(\xi)}.$$

Different types of counting techniques are employed depending on whether order is important. When order is important, this is called an **arrangement** or a **permutation**; when it is not important, it is called a **selection** or a **combination**. Permutations and combinations are defined more formally in sections 1.3 and 1.5.

1.2.4 The addition and multiplication principles

To count the number of ways in which an activity can occur, first make a list. Let each outcome be represented by a letter and then systematically list all the possibilities.

Consider the following question:

In driving from Brisbane to Rockhampton I can take any one of 4 different roads *and* in driving from Rockhampton to Townsville there are 3 different roads I can take. How many different routes can I take in driving from Brisbane to Townsville?

To answer this, let R_1, R_2, R_3, R_4 stand for the 4 roads from Brisbane to Rockhampton and T_1, T_2, T_3 stand for the 3 roads from Rockhampton to Townsville.

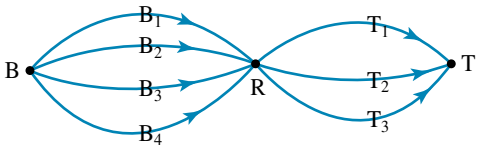
Use the figure to systematically list the roads:

$\mathrm{R}_1\mathrm{T}_1$, $\mathrm{R}_1\mathrm{T}_2$, $\mathrm{R}_1\mathrm{T}_3$

$\mathrm{R}_2\mathrm{T}_1$, $\mathrm{R}_2\mathrm{T}_2$, $\mathrm{R}_2\mathrm{T}_3$

$\mathrm{R}_3\mathrm{T}_1$, $\mathrm{R}_3\mathrm{T}_2$, $\mathrm{R}_3\mathrm{T}_3$

$\mathrm{R}_4\mathrm{T}_1$, $\mathrm{R}_4\mathrm{T}_2$, $\mathrm{R}_4\mathrm{T}_3$



Hence, there are 12 different ways I can drive from Brisbane to Townsville.

In the above example it can be argued logically that if there are 4 ways of getting from Brisbane to Rockhampton and 3 ways of getting from Rockhampton to Townsville then there are 4×3 ways of getting from Brisbane to Townsville.

This idea is formalised in the **multiplication principle**.

The multiplication principle should be used when there are operations or events (say, A and B), where one event is followed by the other – that is, when order is important.

The multiplication principle:
If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B in the order AB.

***Note:* In this case '*and*' means to multiply.**

A useful technique for solving problems based on the multiplication principle is to use boxes. In the example above we would write

1st	2nd
4	3

The value in the '1st' column represents the number of ways the first operation — the trip from Brisbane to Rockhampton — can be performed.

The value in the '2nd' column stands for the number of ways the second operation — the trip from Rockhampton to Townsville — can be performed.

To apply the multiplication principle you multiply the numbers in the lower row of boxes.

WORKED EXAMPLE 2

Two letters are to be chosen from the set of 5 letters. A, B, C, D and E, where order is important.

a. Recall how to list all the different ways that this may be done.

b. Use the multiplication principle to calculate the number of ways that this may be done.

c. Determine the probability the first letter will be a C.

THINK	WRITE
a 1. Begin with A in first place and make a list of each of the possible pairs.	**a** AB AC AD AE
2. Make a list of each of the possible pairs with B in the first position.	BA BC BD BE
3. Make a list of each of the possible pairs with C in the first position.	CA CB CD CE
4. Make a list of each of the possible pairs with D in the first position.	DA DB DC DE
5. Make a list of each of the possible pairs with E in the first position. *Note:* AB and BA need to be listed separately as order is important.	EA EB EC ED
b The multiplication principle could have been used to determine the number of ordered pairs.	**b** 5 | 4
1. Rule up two boxes which represent the pair.	
2. Write down the number of letters which may be selected for the first box. That is, in first place any of the 5 letters may be used.	
3. Write down the number of letters which may be selected for the second box. That is, in second place, any of the 4 letters may be used. *Note:* One less letter is used to avoid repetition.	
4. Evaluate.	$5 \times 4 = 20$ ways
5. Answer the question.	There are 20 ways in which 2 letters may be selected from a group of 5 where order is important.

c 1. Recall the probability formula.The total number in the set $n(\xi)$ was determined in part **b**.

c $P(A) = \dfrac{n(A)}{n(\xi)}$

$n(\xi) = 20$

2. Let A be the event that the pair starts with a C. Draw a table showing the requirement imposed by the first letter to be C.

1	

3. Complete the table. Once the first letter has been completed, there are 4 choices for the second letter. Use the multiplication principle to determine the number of combinations starting with C.

1	4

There are $1 \times 4 = 4$ possible combinations beginning with C.
So, $n(A) = 4$.

4. Use the probability formula to answer the question.

$$P(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{4}{20}$$
$$= \frac{1}{5}$$

This is confirmed by examining the answer to part **a**.

WORKED EXAMPLE 3

a. Use the multiplication principle to calculate how many ways an arrangement of 5 numbers can be chosen from {1, 2, 3, 4, 5, 6}.
b. Determine the probability of the number ending with 4.

THINK **WRITE**

a 1. Instead of listing all possibilities, draw 5 boxes to represent the 5 numbers chosen.
Label each box on the top row as 1st, 2nd, 3rd, 4th and 5th.
Note: The word arrangement implies order is important.

2. Fill in each of the boxes showing the number of ways a number may be chosen.
a. In the 1st box there are 6 choices for the first number.
b. In the 2nd box there are 5 choices for the second number as 1 number has already been used.
c. In the 3rd box there are 4 choices for the third number as 2 numbers have already been used.
d. Continue this process until each of the 5 boxes is filled.

1st	2nd	3rd	4th	5th
6	5	4	3	2

3. Use the multiplication principle as this is an *'and'* situation.
4. Answer the question.

No. of ways $= 6 \times 5 \times 4 \times 3 \times 2$
$= 720$
An arrangement of 5 numbers may be chosen 720 ways.

b. 1. Recall the probability formula.
The total number of arrangements, $n(\xi)$, was determined in part **a**.

b. $P(A) = \dfrac{n(A)}{n(\xi)}$
$n(\xi) = 720$

2. Let A be the event that the number ends with 4.
Draw a table showing the requirement imposed by the last letter to be 4.

1st	2nd	3rd	4th	5th
				1

3. Complete the table. Once the last number has been completed, there are 5 choices for the number in the first position, 4 choices for the next number. Continue this process until each of the 5 columns has been filled.
Use the multiplication principle to determine the number of combinations ending with 4.

1st	2nd	3rd	4th	5th
5	4	3	2	1

There are
$5 \times 4 \times 3 \times 2 \times 1 = 120$ possible combinations ending with 4.
So, $n(A) = 120$.

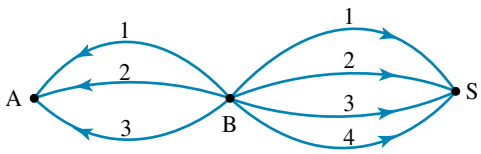
4. Use the probability formula to answer the question.

$P(A) = \dfrac{n(A)}{n(\xi)}$
$= \dfrac{120}{720}$
$= \dfrac{1}{6}$
This is confirmed by examining the answer to part **a**.

Now consider a different situation, one in which the two operations do not occur one after the other.

I am going to travel from Brisbane to either Sydney *or* Adelaide. There are 4 ways of travelling from Brisbane to Sydney and 3 ways of travelling from Brisbane to Adelaide.

How many different ways can I travel to either Sydney *or* Adelaide?



It can be seen from the figure that there are $4 + 3 = 7$ ways of completing the journey. This idea is summarised in the **addition principle**.

The addition principle should be used when two distinct operations or events occur in which one event is not followed by another — that is, when the events are mutually exclusive.

The addition principle:
If there are *n* ways of performing operation A and *m* ways of performing operation B, then there are *n* + *m* ways of performing A *or* B.

Note: **In this case 'or' means to add.**

WORKED EXAMPLE 4

One or two letters are to be chosen from the set of 6 letters A, B, C, D, E, F. Assuming order is important, use the multiplication principle and the addition principle to calculate:

a. the number of ways to choose 2 letters.

b. the number of ways to choose 1or 2 letters.

THINK	WRITE
a 1. Determine the number of ways of choosing 1 letter.	Number of ways of choosing 1 letter = 6.
2. Rule up two boxes for the first and second letters.	<table><tr><td>1st</td><td>2nd</td></tr><tr><td>6</td><td>5</td></tr></table>
3 Determine the number of ways of choosing 2 letters from 6. In the 1st box there are 6 choices for the first letter. In the 2nd box there are 5 choices for the second letter as 1 letter has already been used.	
4. Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of ways of choosing 2 letters from 6.	Number of ways of choosing 2 letters $= 6 \times 5$ $= 30$
5. Answer the questions.	There are 30 ways of choosing 2 letters.
b 1. Determine the number of ways of choosing 1 or 2 letters from 6 letters. Use the addition principle as this is an '*or*' situation.	The number of ways of choosing 1 or 2 letters is $6 + 30 = 36$.
2. Answer the question.	There are 36 ways of choosing 1 or 2 letters from 6.

The multiplication and addition principles can be used to count the number of elements in the union of two or three sets in the same way as for one set. Remember, the multiplication principle is used with '*and*' situations, and the addition principle is used with '*or*' situations.

WORKED EXAMPLE 5

Oscar's cafe offers a choice of 3 starters, 9 main courses and 4 desserts.

a. How many choices of 3-course meals (starter, main, dessert) are available?

b. How many choices of starter and main course meals are offered?

c. How many choices of meals comprising a main course and dessert are offered?

d. How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?

e. If one of the starter options is chicken wings and one of the mains is grilled fish, determine the probability of choosing chicken wings and grilled fish in a 3-course meal.

THINK	WRITE
a 1. Consider each course as separate sets and rule up 3 boxes to represent each course — starter, main, dessert. Label each box on the top row as S, M and D.	**a**

S	M	D
3	9	4

2. Determine the number of ways of choosing each meal: starter = 3, main = 9, dessert = 4.	
3. Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of choices of 3-course meals.	Number of choices $= 3 \times 9 \times 4$ $= 108$
4. Answer the question.	There are 108 choices of 3-course meals.
b 1. Rule up 2 boxes to represent each course — starter, main. Label each box on the top row as S and M.	**b**

S	M
3	9

2. Determine the number of ways of choosing each meal: starter = 3, main = 9.	
3. Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of choices of starter and main courses.	Number of choices $= 3 \times 9$ $= 27$
4. Answer the question.	There are 27 choices of starter and main course.
c 1. Rule up 2 boxes to represent each course — main and dessert. Label each box on the top row as M and D.	**c**

M	D
9	4

2. Determine the number of ways of choosing each meal: main = 9, dessert = 4.	
3. Use the multiplication principle (as this is an '*and*' situation) to evaluate the number of choices of main course and dessert.	Number of choices $= 9 \times 4$ $= 36$
4. Answer the question.	There are 36 choices of main course and dessert.
d 1. Determine the number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered. Use the addition principle as this is an '*or*' situation.	**d** The number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered, is: $108 + 27 + 36 = 171$
2. Answer the question.	There are 171 ways of choosing 2- or 3-course meals, assuming that a main course is always ordered.

e **1.** Recall the probability formula.
The total number in the set $n(\xi)$ was determined in part **a**.

$$P(A) = \frac{n(A)}{n(\xi)}$$
$$n(\xi) = 108$$

2. Let A be the event that the meal contains chicken wings and grilled fish.
Draw a table showing the requirement imposed that the starter and main be these 2 dishes.

S	M	D
1	1	

3. Complete the table — there are 4 dessert options. Use the multiplication principle to determine the number of combinations of the meal.

S	M	D
1	1	4

There are $1 \times 1 \times 4 = 4$ possible combinations for the meal.
So, $n(A) = 4$.

4. Use the probability formula to answer the question.

$$P(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{4}{108}$$
$$= \frac{1}{27}$$

The probability of choosing chicken wings and grilled fish as part of the 3-course meal is $\frac{1}{27}$.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concepts 1 & 2

Set notation Summary screen and practice questions
Counting techniques Summary screen and practice questions

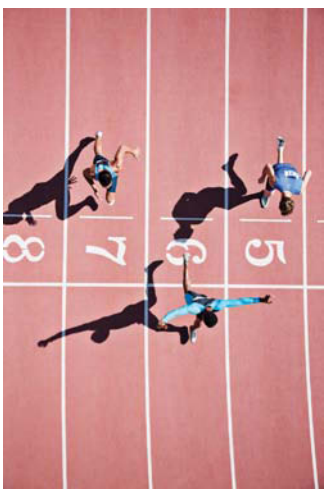
Exercise 1.2 Counting techniques

Technology free

1. **WE1** Let R be the set of natural numbers between 30 and 45 inclusive that are divisible by 2.
Let S be the set of natural numbers between 30 and 45 inclusive that are divisible by 3.
Let T be the set of natural numbers between 30 and 45 inclusive that are divisible by 5.
a. Construct a Venn diagram to represent R, S and T.
b. Use this diagram to evaluate $n(R \cup S \cup T)$.
c. Recall the inclusion–exclusion principle to compute $n(R \cup S \cup T)$.

2. Recall the inclusion–exclusion principle to calculate the number of cards in a deck of 52 that are either red or even or a 4.
3. Student Services has the following data on Year 11 students and their sport commitments:
   - 18 play no sport.
   - 16 play netball (and possibly other sports).
   - 24 play football.
   - 20 are involved in a gym program.
   - 7 play netball and football.
   - 6 play netball and are in the gym program.
   - 15 play football and are involved in the gym program.
   - 5 students do all three activities.

   How many students are there in Year 11?
4. **WE2** Two letters are to be chosen from A, B and C, where order is important.
   a. Recall how to list all the different ways that this may be done.
   b. Use multiplication principle to calculate the number of ways that this may be done.
   c. Determine the probability the last letter will be a B.
5. List all the different arrangements possible for a group of 2 colours to be chosen from B (blue), G (green), Y (yellow) and R (red).
6. List all the different arrangements possible for a group of 3 letters to be chosen from A, B and C.
7. a. **WE3** Use the multiplication principle to calculate how many ways can an arrangement of 2 letters be chosen from A, B, C, D, E, F and G?
   b. In how many ways can an arrangement of 3 letters be chosen from 7 different letters?
   c. In how many ways can an arrangement of 4 letters be chosen from 7 different letters?
   d. How many different arrangements of 5 letters can be made from 7 letters?
   e. Determine the probability of the letters starting with an E.
8. a. A teddy bear's wardrobe consists of 3 different hats, 4 different shirts and 2 different trousers. How many different outfits can the teddy bear wear?
   b. A surfboard is to have 1 colour on its top and a different colour on its bottom. The 3 possible colours are red, blue and green. In how many different ways can the surfboard be coloured?
   c. A new phone comes with a choice of 3 cases, 2 different sized screens and 2 different storage capacities. With these choices, determine how many different arrangements are possible.
   d. Messages can be sent by placing 3 different coloured flags in order on a pole. If the flags come in 4 colours, determine how many different messages can be sent.
9. a. **WE4** One or 2 letters are to be chosen in order from the letters A, B, C, D, E, F and G. Use the multiplication principle and the addition principle to calculate the number of ways can this be done.
   b. Two or 3 letters are to be chosen in order from the letters A, B, C, D, E, F and G. In how many ways can this be done?



10. Manish is in a race with 7 other runners. If we are concerned only with the first, second and third placings, in how many ways can Manish finish first or second or third?
11. **WE5** Hani and Mary's restaurant offers its patrons a choice of 4 entrees, 10 main courses and 5 desserts.
    a. How many choices of 3-course meals (entree, main, dessert) are available?
    b. How many choices of entree and main course are offered?

c. How many choices of meals comprising a main course and dessert are offered?
d. How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?
e. Determine the probability of choosing vegetable soup for entree and roast for main in a 3-course meals.

12. Jake is able to choose his work outfits from the following items of clothing: 3 jackets, 7 shirts, 6 ties, 5 pairs of trousers, 7 pairs of socks and 3 pairs of shoes.
a. Calculate how many different outfits are possible if he wears one of each of the above items. (He wears matching socks and matching shoes.)
b. If Jake has the option of wearing a jacket or not, but he must wear one of each of the above items, determine how many different outfits are possible. Justify your answer.

13. MC There are 12 people on the committee at the local football club. In how many ways can a president and a secretary be chosen from this committee?
A. 2 **B.** 23 **C.** 132 **D.** 144

14. MC A TV station runs a cricket competition called *Classic Catches*. Six catches, A to F, are chosen and viewers are asked to rank them in the same order as the judges. The number of ways in which the six catches can be ranked:
A. 1 **B.** 6 **C.** 30 **D.** 720

Technology active

15. The local soccer team sells 'doubles' at each of their games to raise money. A 'double' is a card with 2 digits on it representing the score at full time. The card with the actual full time score on it wins a prize. If the digits on the cards run from 00 to 99, how many different tickets are there?

16. Jasmin has a phone that has a 4-digit security code. She remembers that the first number in the code was 9 and that the others were 3, 4 and 7 but forgets the order of the last 3 digits. How many different trials must she make to be sure of unlocking the phone?

17. Julia has a banking app that has two 4-digit codes. She remembers that she used the digits 1, 3, 5 and 7 on the first code and 2, 4, 6 and 8 on the second code, but cannot remember the order. What is the maximum number of trials she would need to make before she has opened both codes? (Assume that she can try an unlimited number of times and once the first code is correct, she can try the second code.)

18. How many different 4-digit numbers can be made from the numbers 1, 3, 5 and 7 if the numbers can be repeated (that is 3355 and 7777 are valid)?

19. How many 4-digit numbers can be made from the numbers 1, 3, 5, 7, 9 and 0 if the numbers can be repeated? (Remember — a number cannot start with 0.)

20. A combination lock has 3 digits each from 0 to 9.
a. How many combinations are possible?
The lock mechanism becomes loose and will open if the digits are within one either side of the correct digit. For example if the true combination is 382 then the lock will open on 271, 272, 371, 493 and so on.
b. How many combinations would unlock the safe?
c. List the possible combinations that would open the lock if the true combination is 382.

1.3 Factorials and permutations

1.3.1 Factorials

The Physical Education department is to display 5 new trophies along a shelf in the school foyer and wishes to know in how many ways this can be done.

Using the multiplication principle from the previous section, the display may be done in the following way:

Position 1	Position 2	Position 3	Position 4	Position 5
5	4	3	2	1

That is, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Depending on the number of items we have, this method could become quite time consuming.

In general when we need to multiply each of the integers from a particular number, n, down to 1, we write $n!$, which is read as n **factorial**.

Hence:

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \\ 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\,320 \end{aligned}$$

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

The number of ways n distinct objects may be arranged is $n!$(n factorial) where:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

That is, $n!$ is the product of each of the integers from n down to 1.
A special case of the factorial function is: $0! = 1$.

WORKED EXAMPLE 6

Evaluate the following factorials.

a. $7!$ **b.** $13!$ **c.** $\frac{8!}{5!}$ **d.** $\frac{(n-1)!}{(n-3)!}$

THINK	WRITE
a. 1. Write 7! in its expanded form and evaluate.	**a.** $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 5040$
2. Verify the answer obtained using the factorial function on a calculator.	
b. 1. Write 13! in its expanded form and evaluate.	**b.** $13! = 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 6\,227\,020\,800$

2. Verify the answer obtained using the factorial function on a calculator.

c. 1. Write each factorial term in its expanded form.

c. $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$

2. Cancel down like terms.

$= 8 \times 7 \times 6$

3. Evaluate.

$= 336$

4. Verify the answer obtained using the factorial function on a calculator.

d. 1. Write each factorial term in its expanded form.

d. $\frac{(n-1)!}{(n-3)!} = \frac{(n-1)(n-2)(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}{(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}$

2. Cancel like terms.

$= (n-1)(n-2)$

TI | THINK

c.1. On a Calculator page, complete the entry line as:
$\frac{8!}{5!}$
then press ENTER.
Note: The factorial symbol can be found by pressing CTRL, then the Catalogue button.

2. The answer appears on the screen.

WRITE

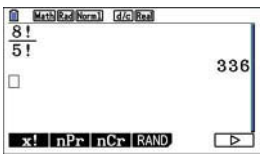
$\frac{8!}{5!} = 336$

CASIO | THINK

c.1. On a Run-Matrix screen, complete the entry line as:
$\frac{8!}{5!}$
then press EXE.
Note: To find the factorial symbol, press OPTN, then press F6, and then select PROB and x! by pressing F3, then F1.

2. The answer appears on the screen.

WRITE



$\frac{8!}{5!} = 336$

In parts **c** and **d** of Worked example 6, there was no need to fully expand each factorial term.

The factorial $\frac{8!}{5!}$ could have first been simplified to $\frac{8 \times 7 \times 6 \times 5!}{5!}$ and then the 5! terms cancelled.

The factorial $\frac{(n-1)!}{(n-3)!}$ could have first been simplified to $\frac{(n-1)(n-2)(n-3)!}{(n-3)!}$ and then the $(n-3)!$ terms cancelled.

1.3.2 Permutations

The term permutation is often used instead of the term arrangement, and in this section we begin by giving a formal definition of permutation.

Previously, we learned that if you select 3 letters from 7 where *order is important*, the number of possible arrangement is:

1st	2nd	3rd
7	6	5

The number of arrangements $= 7 \times 6 \times 5$
$= 210$

This value may also be expressed in factorial form: $7 \times 6 \times 5 = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = \dfrac{7!}{4!}$

Using more formal terminology we say that in choosing 3 things from 7 things where order is important, the number of permutations is ${}^{7}\mathrm{P}_{3} = 7 \times 6 \times 5$. The letter P is used to remind us that we are finding permutations.

The number of ways of choosing r things from n distinct things is given by the rule:

$$
\begin{aligned}
{}^{n}\mathrm{P}_{r} &= n \times (n-1) \times \ldots \times (n-r+1) \\
&= \frac{n \times (n-1) \times \ldots \times (n-r+1)(n-r)!}{(n-r)!} \\
{}^{n}\mathrm{P}_{r} &= \frac{n!}{(n-r)!}
\end{aligned}
$$

The definition of ${}^{n}\mathrm{P}_{r}$ may be extended to the cases of ${}^{n}\mathrm{P}_{n}$ and ${}^{n}\mathrm{P}_{0}$.
${}^{n}\mathrm{P}_{n}$ represents the number of ways of choosing n objects from n distinct things.

$$
\begin{aligned}
{}^{n}\mathrm{P}_{n} &= n \times (n-1) \times (n-2) \times \ldots \times (n-n+1) \\
&= n \times (n-1) \times (n-2) \times \ldots \times 1 \\
&= n!
\end{aligned}
$$

From the definition:

$$
\begin{aligned}
{}^{n}\mathrm{P}_{n} &= \frac{n!}{(n-n)!} \\
&= \frac{n!}{0!}
\end{aligned}
$$

Therefore, equating both sides, we obtain: $n! = \dfrac{n!}{0!}$.
This can occur only if $0! = 1$.

$$
\begin{aligned}
{}^{n}\mathrm{P}_{0} &= \frac{n!}{(n-0)!} \\
&= \frac{n!}{n!} \\
&= 1
\end{aligned}
$$

The two special cases are:

$$
{}^{n}\mathrm{P}_{n} = n!
$$

$$
{}^{n}\mathrm{P}_{0} = 1
$$

WORKED EXAMPLE 7

a. Calculate the number of permutations for ${}^6P_4$ by expressing it in expanded form.
b. Write ${}^8P_3$ as a quotient of factorials and hence evaluate.

THINK	WRITE
a. 1. Write down the first 4 terms beginning with 6.	**a.** ${}^6P_4 = 6 \times 5 \times 4 \times 3$
2. Evaluate.	$= 360$
b. 1. Recall the rule for permutations.	**b.** ${}^nP_r = \dfrac{n!}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	${}^8P_3 = \dfrac{8!}{(8-3)!}$
	$= \dfrac{8!}{5!}$
3. Use a calculator to evaluate 8! and 5!	$= \dfrac{40\,320}{120}$
4. Evaluate.	$= 336$

TI | THINK

a.1. On a Calculator page, press MENU, then select:
5: Probability
2: Permutations.
Complete the entry line as:
nPr (6, 4)
then press ENTER.

2. The answer appears on the screen.

WRITE

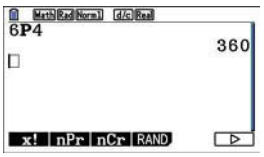
${}^6P_4 = 360$

CASIO | THINK

a.1. On a Run-Matrix screen, press OPTN, then F6. Select PROB by pressing F3, then select nPr by pressing F2. Complete the entry line as:
6**P**4
then press EXE.

2. The answer appears on the screen.

WRITE



${}^6P_4 = 360$

WORKED EXAMPLE 8

The netball club needs to appoint a president, secretary and treasurer. From the committee 7 people have volunteered for these positions. Each of the 7 nominees is happy to fill any one of the 3 positions.
a. Determine how many different ways these positions can be filled.
b. For three years, the same 7 people volunteer for these positions. Determine the probability one of them is president 3 years in a row.

THINK	WRITE
a. 1. Recall the rule for permutations. *Note:* Order is important, so use permutations.	**a.** ${}^nP_r = \dfrac{n!}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	${}^7P_3 = \dfrac{7!}{(7-3)!}$
	$= \dfrac{7!}{4!}$

3. Use a calculator to evaluate 7! and 4!
4. Evaluate.
5. Answer the question.

$= \frac{5040}{24}$
$= 210$

There are 210 different ways of filling the positions of president, secretary and treasurer.

b. 1. In three years of the president's position, each year this could be awarded to one of the 7 people.

b.

7	7	7

The total number of ways the president's position could be filled is $7 \times 7 \times 7$.
$n(\xi) = 7 \times 7 \times 7$
$= 343$

2. Let A be the event that the same person fills the position three years in a row.

There are 7 choices of the same person to fill the positions three years in a row.
$n(A) = 7$.

3. Calculate the probability that the same person fills the president's position three years in a row.

$P(A) = \frac{n(A)}{n(\xi)}$
$= \frac{7}{7 \times 7 \times 7}$
$= \frac{1}{49}$

The probability that one of the 7 volunteers, fills the president's position three years in a row is $\frac{1}{49}$.

1.3.3 Arrangements in a circle

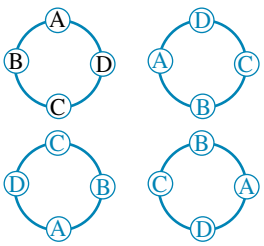
Consider this problem: In how many different ways can 7 people be seated, 4 at a time, on a bench?

By now you should quickly see the answer: ${}^7P_4 = 840$.

Let us change the problem slightly: In how many different ways can 7 people be seated, 4 at a time, at a circular table?

The solution must recognise that when people are seated on a bench, each of the following represents a different arrangement:

ABCD BCDA CDAB DABC



However, when sitting in a circle, each represents the *same* arrangement. It is important to note that in a circle arrangement we do not consider positions on the circle as different — it does not matter where the circle starts.

In each case B has A on the left and C on the right.

We conclude that the number ${}^7P_4$ gives 4 times the number of arrangements of 7 people in a circle 4 at a time. Therefore, the number of arrangements is $\frac{{}^7P_4}{4} = 210$.

In general, the number of different ways *n* objects can be arranged, *r* at a time, in a circle is:

$$\frac{{}^nP_r}{r}$$

WORKED EXAMPLE 9

a. By recalling the appropriate formula, give an expression for the number of different arrangements if, from a group of 8 people, 5 are to be seated at a round table.
b. Evaluate this expression.
c. Each table receives one lucky door prize and one lucky seat prize. Determine the probability of the same person at one table winning both.

THINK	WRITE
a. 1. Write down the rule for the number of arrangements in a circle.	**a.** $\frac{{}^{n}\mathrm{P}_{r}}{r}$
2. Substitute the given values of n and r into the formula.	$= \frac{{}^{8}\mathrm{P}_{5}}{5}$
3. Answer the question.	The number of ways of seating 5 people from a group of 8 people at a round table is given by the expression $\frac{{}^{8}\mathrm{P}_{5}}{5}$.
b. 1. Use a calculator to evaluate ${}^{8}\mathrm{P}_{5}$.	**b.** ${}^{8}\mathrm{P}_{5} = \frac{6720}{5}$
2. Evaluate.	$= 1344$
3. Answer the question.	The number of ways of seating 5 from a group of 8 people at a round table is 1344.
c. 1. There are 2 prizes, and each prize can be won by any one of the 5 people.	**c.** $\boxed{5}\boxed{5}$ The total number of ways the prizes could be won is 5×5. $n(\xi) = 5 \times 5$ $= 25$
2. Let A be the event that the same person wins both prizes.	There are 5 choices of the same person to win both prizes. $n(A) = 5$
3. Calculate the probability that the same person wins both.	$\mathrm{P}(A) = \frac{n(A)}{n(\xi)}$ $= \frac{5}{5 \times 5}$ $= \frac{1}{5}$ The probability of the same person winning both prizes is $\frac{1}{5}$.

Resources

Digital document: SkillSHEET Calculating ${}^{n}P_{r}$ (doc-26824)
Digital document: SpreadSHEET Permutations (doc-26825)

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 3

Factorials Summary screen and practice questions

Exercise 1.3 Factorials and permutations

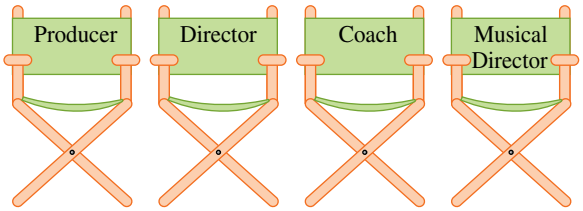
Technology free

1. Recall the definition of $n!$ and write each of the following in expanded form.
   a. $4!$ b. $5!$ c. $6!$ d. $7!$
2. **WE6c** Evaluate the following factorials.
   a. $\frac{9!}{5!}$ b. $\frac{10!}{4!}$ c. $\frac{7!}{3!}$ d. $\frac{6!}{0!}$
3. **WE6d** Evaluate the following factorials.
   a. $\frac{n!}{(n-5!)}$ b. $\frac{(n+3)!}{(n+1)!}$ c. $\frac{(n-3)!}{n!}$ d. $\frac{(n-2)!}{(n+2)!}$

Technology active

4. **WE6a, b** Evaluate the following factorials.
   a. $4!$ b. $5!$ c. $6!$ d. $10!$
   e. $14!$ f. $9!$ g. $7!$ h. $3!$
5. **WE7a** Calculate each of the following by expressing it in expanded form.
   a. ${}^{8}P_{2}$ b. ${}^{7}P_{5}$ c. ${}^{8}P_{7}$
6. **WE7b** Write each of the following as a quotient of factorials and hence evaluate.
   a. ${}^{9}P_{6}$ b. ${}^{5}P_{2}$ c. ${}^{18}P_{5}$
7. Use your calculator to determine the value of:
   a. ${}^{20}P_{6}$ b. ${}^{800}P_{2}$ c. ${}^{18}P_{5}$
8. **WE8** A soccer club will appoint a president and a vice-president. Eight people have volunteered for either of the two positions.
   a. In how many different ways can these positions be filled?
   b. For three years the same 8 people volunteered for these positions. What is the probability one of them is president for both years?
9. There are 26 players in an online game. How many different results for 1st, 2nd, 3rd and 4th can occur?
10. A rowing crew consists of 4 rowers who sit in a definite order. How many different crews are possible if 5 people try out for selection?

11. The school musical needs a producer, director, musical director and script coach. Nine people have volunteered for any of these positions. In how many different ways can the positions be filled? (*Note:* One person cannot take on more than 1 position.)



12. There are 14 swimmers in a race. In how many different ways can the 1st, 2nd and 3rd positions be filled?

13. WE9 **a.** By recalling the appropriate formula, give an expression for the number of different arrangements if, from a group of 15 people, 4 are to be seated at a round table.

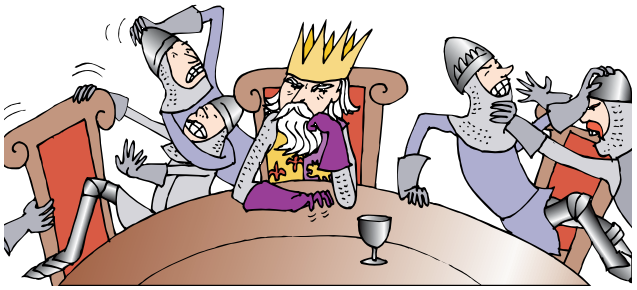
b. Evaluate this expression.

c. Each table receives a lucky door prize, a lucky seat prize and a best-dressed prize. Determine the probability of the same person at one table winning all 3 prizes.

14. A round table seats 6 people. From a group of 8 people, give an expression for, and hence calculate, the number of ways 6 people can be seated at the table.

15. At a dinner party for 10 people all the guests were seated at a circular table. How many different arrangements were possible?

16. At one stage in the court of Camelot, King Arthur and 12 knights would sit at the round table. If each person could sit anywhere determine how many different arrangements were possible.



17. MC Which one of the following permutations cannot be calculated?

A. ${}^{1000}P_{100}$ **B.** ${}^{1}P_{0}$ **C.** ${}^{8}P_{8}$ **D.** ${}^{4}P_{8}$

18. MC The result of 100! is greater than 94!.
Which of the following gives the best comparison between these two numbers?

A. 100! is 6 more than 94! **B.** 100! is 6 times bigger than 94!

C. 100! is about 10 000 more than 94! **D.** 100! is ${}^{100}P_{6}$ times bigger than 94!

19. In how many ways can the letters of the word TODAY be arranged if they are used once only and taken:

a. 3 at a time? **b.** 4 at a time? **c.** 5 at a time?

Show your answers in the form ${}^{n}P_{r}$ and then evaluate.

1.4 Permutations with restrictions

1.4.1 Like objects

A 5-letter word is to be made from 3 As and 2 Bs. How many different permutations or arrangements can be made?

If the 5 letters were all different, it would be easy to calculate the number of arrangements. It would be $5! = 120$. Perhaps you can see that when letters are repeated, the number of different arrangements will be less than 120. To analyse the situation let us imagine that we can distinguish one A from another. We will write A_1, A_2, A_3, B_1 and B_2 to represent the 5 letters.

As we list some of the possible arrangements we notice that some are actually the same, as shown in the table.

$A_1A_2B_1A_3B_2$ $A_1A_3B_1A_2B_2$ $A_2A_1B_1A_3B_2$ $A_2A_3B_1A_1B_2$ $A_3A_1B_1A_2B_2$ $A_3A_2B_1A_1B_2$	$A_1A_2B_2A_3B_1$ $A_1A_3B_2A_2B_1$ $A_2A_1B_2A_3B_1$ $A_2A_3B_2A_1B_1$ $A_3A_1B_2A_2B_1$ $A_3A_2B_2A_1B_1$	Each of these 12 arrangements is the same — AABAB — if $A_1 = A_2 = A_3$ and $B_1 = B_2$.
$B_2A_1A_2B_1A_3$ $B_2A_1A_3B_1A_2$ $B_2A_2A_1B_1A_3$ $B_2A_2A_3B_1A_1$ $B_2A_3A_1B_1A_2$ $B_2A_3A_2B_1A_1$	$B_1A_1A_2B_2A_3$ $B_1A_1A_3B_2A_2$ $B_1A_2A_1B_2A_3$ $B_1A_2A_3B_2A_1$ $B_1A_3A_1B_2A_2$ $B_1A_3A_2B_2A_1$	Each of these 12 arrangements is the same — BAABA — if $A_1 = A_2 = A_3$ and $B_1 = B_2$.

The number of repetitions is 3! for the As and 2! for the Bs. Thus, the number of different arrangements in choosing 5 letters from 3 As and 2 Bs is $\frac{5!}{3! \times 2!}$.

The number of different ways of arranging n objects made up of groups of indistinguishable objects, n_1 in the first group, n_2 in the second group and so on, is:

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!}.$$

Note: If there are elements of the group which are not duplicated, then they can be considered as a group of 1. It is not usual to divide by 1!; it is more common to show only those groups which have duplications.

WORKED EXAMPLE 10

Determine how many different permutations of 7 counters can be made from 4 black and 3 white counters.

THINK	WRITE
1. Write down the total number of counters.	There are 7 counters in all; therefore, $n = 7$.
2. Write down the number of times any of the coloured counters are repeated.	There are 3 white counters; therefore, $n_1 = 3$. There are 4 black counters; therefore, $n_2 = 4$.
3. Write down the rule for arranging groups of like things.	$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!}$.
4. Substitute the values of n, n_1 and n_2 into the rule.	$= \frac{7!}{3! \times 4!}$
5. Expand each of the factorials.	$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$
6. Simplify the fraction.	$= \frac{7 \times 6 \times 5}{6}$
7. Evaluate.	$= 35$

8. Answer the question.	Thirty-five different arrangements can be made from 7 counters, of which 3 are white and 4 are black.

1.4.2 Restrictions

Sometimes restrictions are introduced so that a smaller number of objects from the original group need to be considered. This results in limiting the numbers of possible permutations.

WORKED EXAMPLE 11

A rowing crew of 4 rowers is to be selected, in order from the first seat to the fourth seat, from 8 candidates. Determine how many different arrangements are possible if:

a. there are no restrictions

b. Jason or Kris must row in the first seat

c. Jason must be in the crew, but he can row anywhere in the boat

d. Jason is not in the crew.

THINK

a. 1. Write down the permutation formula. *Note:* 4 rowers are to be selected from 8 and the order is important.

2. Substitute the given values of n and r into the permutation formula.

3. Expand the factorials or use a calculator to evaluate 8! and 4!.

4. Evaluate.

5. Answer the question.

b. 1. Apply the multiplication principle since two events will follow each other; that is, Jason will fill the first seat and the remaining 3 seats will be filled in $7 \times 6 \times 5$ ways or Kris will fill the first seat and the remaining 3 seats will be filled in $7 \times 6 \times 5$ ways.

J	7	6	5

or

K	7	6	5

WRITE

a. ${}^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!}$

$${}^{8}\mathrm{P}_{4} = \frac{8!}{(8-4)!}$$

$$= \frac{8!}{4!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$= 8 \times 7 \times 6 \times 5$$

$$= 1680$$

There are 1680 ways of arranging 4 rowers from a group of 8.

b. No. of arrangements
= no. of ways of filling the first seat × no. of ways of filling the remaining 3 seats.
$= 2 \times {}^{n}\mathrm{P}_{r}$

2. Substitute the values of n and r into the formula and evaluate.

$= 2 \times {}^7P_3$
$= 2 \times 210$
$= 420$

3. Answer the question.

There are 420 ways of arranging the 4 rowers if Jason or Kris must row in the first seat.

c. 1. Apply the addition principle, since Jason must be in either the first, second, third or fourth seat. The remaining 3 seats will be filled in $7 \times 6 \times 5$ ways each time.

J	7	6	5	+	7	J	6	5
7	6	J	5	+	7	6	5	J

c. No. of arrangements
= no. of arrangements with Jason in seat 1
+ No. of arrangements with Jason in seat 2
+ No. of arrangements with Jason in seat 3
+ No. of arrangements with Jason in seat 4.

2. Substitute the values of n and r into the formula.

No. of arrangements
$= 1 \times {}^7P_3 + 1 \times {}^7P_3 + 1 \times {}^7P_3 + 1 \times {}^7P_3$
$= 4 \times {}^7P_3$
$= 4 \times 210$

3. Evaluate.

$= 840$

4. Answer the question.

There are 840 ways of arranging the 4 rowers if Jason must be in the crew of 4.

d. As Jason is not in the crew, there are only 7 candidates. Four rowers are to be chosen from 7 and order is important.

d. ${}^7P_4 = \dfrac{7!}{(7-4)!}$
$= \dfrac{7!}{3!}$
$= \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}$
$= 7 \times 6 \times 5 \times 4$
$= 840$

There are 840 ways of arranging the crew when Jason is not included.

WORKED EXAMPLE 12

a. Calculate the number of permutations of the letters in the word COUNTER.
b. In how many of these do the letters C and N appear side by side?
c. In how many permutations do the letters C and N appear apart?
d. Determine the probability of the letters C and N appearing side by side.

THINK

a. 1. Count the number of letters in the given word.

2. Determine the number of ways the 7 letters may be arranged.

WRITE

a. There are 7 letters in the word COUNTER.

The 7 letters may be arranged $7! = 5040$ ways.

3. Answer the question.

There are 5040 permutations of letters in the word COUNTER.

b. 1. Imagine the C and N are 'tied' together and are therefore considered as 1 unit. Determine the number of ways C and N may be arranged: CN and NC.

b. Let C and N represent 1 unit.
They may be arranged $2! = 2$ ways.

2. Determine the number of ways 6 things can be arranged.
Note: There are now 6 letters: the 'CN' unit along with O, U, T, E and R.

Six things may be arranged $6! = 720$ ways.

3. Determine the number of permutations in which the letters C and N appear together.

The number of permutations $= 2 \times 6!$
$= 2 \times 720$
$= 1440$

4. Answer the question.

There are 1440 permutations in which the letters C and N appear together.

c. 1. Determine the total number of arrangements of the 7 letters.

c. Total number of arrangements $= 7!$
$= 5040$

2. Write down the number of arrangements in which the letters C and N appear together, as obtained in **a**.

Arrangements with C and N together $= 1440$

3. Determine the difference between the values obtained in steps 1 and 2.
Note: The number of arrangements in which C and N are apart is the total number of arrangements less the number of times they are together.

The number of arrangements $= 5040 - 1440$
$= 3600$

4. Answer the question.

The letters C and N appear apart 3600 times.

d. 1. Recall the probability formula.

d. $P(A) = \dfrac{n(\xi)}{n(A)}$

2. State the number of elements in the set ξ, that is $n(\xi)$.

From part **a**, there are 5040 permutations of letters in the word COUNTER.
$n(\xi) = 5040$

3. Determine the number of elements in the set A, that is the number of arrangements in which the letters C and N appear side by side.

From part **b**, there are 1440 permutations in which the letters C and N appear side by side.
$n(A) = 1440$

4. Calculate the answer.

$$P(A) = \frac{n(A)}{n(\xi)}$$
$$= \frac{1440}{5040}$$
$$= \frac{2}{7}$$

The probability of the letters C and N appearing side by side is $\frac{2}{7}$.

WORKED EXAMPLE 13

Consider the two words 'PARALLEL' and 'LINES'.

a. How many arrangements of the letters of the word LINES have the vowels grouped together?

b. How many arrangements of the letters of the word LINES have the vowels separated?

c. How many arrangements of the letters of the word PARALLEL are possible?

d. Determine the probability that in a randomly chosen arrangement of the word PARALLEL, the letters A are together.

THINK	WRITE
a. 1. Group the required letters together.	**a.** There are two vowels in the word LINES. Treat these letters, I and E, as one unit.
2. Arrange the unit of letters together with the remaining letters.	Now there are four groups to arrange: (IE), L, N, S. These arrange in 4! ways.
3. Use the multiplication principle to allow for any internal rearrangements.	The unit (IE) can internally rearrange in 2! ways. Hence, the total number of arrangements is: $4! \times 2! = 24 \times 2$ $= 48$
b. 1. State the method of approach to the problem.	**b.** The number of arrangements with the vowels separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.
2. State the total number of arrangements.	The five letters of the word LINES can be arranged in $5! = 120$ ways.
3. Calculate the answer.	From part **a**, there are 48 arrangements with the two vowels together. Therefore, there are $120 - 48 = 72$ arrangements in which the two vowels are separated.
c. 1. Count the letters, stating any identical letters.	The word PARALLEL contains 8 letters of which there are 2 As and 3 Ls.
2. Recall the rule $\frac{n!}{n_1!\,n_2!\ldots}$ and state the number of distinct arrangements.	There are $\frac{8!}{2! \times 3!}$ arrangements of the word PARALLEL.
3. Calculate the answer.	$\frac{8!}{2! \times 3!} = \frac{8 \times 7 \times 6 \times 5 \times \cancel{4}^{2} \times \cancel{3!}}{\cancel{2} \times \cancel{3!}}$ $= 3360$ There are 3360 arrangements.
d. 1. State the number of elements in the sample space.	**d.** There are 3360 total arrangements of the word PARALLEL, so $n(\xi) = 3360$ or $\frac{8!}{2! \times 3!}$.
2. Group the required letters together.	For the letters A to be together, treat these two letters as one unit. This creates seven groups: (AA), P, R, L, L, E, L, of which three are identical Ls.

3. Calculate the number of elements in the event.

The seven groups arrange in $\frac{7!}{3!}$ ways. As the unit (AA) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account. Hence, $\frac{7!}{3!}$ is the number of elements in the event.

4. Calculate the required probability.
Note: It helps to use factorial notation in the calculations.

The probability that the As are together

$$= \frac{\text{number of arrangements with the As together}}{\text{total number of arrangements}}$$

$$= \frac{7!}{3!} \div \frac{8!}{2! \times 3!}$$

$$= \frac{7!}{3!} \times \frac{2! \times 3!}{8 \times 7!}$$

$$= \frac{2}{8}$$

$$= \frac{1}{4}$$

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Units 1 & 2 > Area 1 > Sequence 1 > Concept 4

Permutations Summary screen and practice questions

Exercise 1.4 Permutations with restrictions

Technology active

1. Recall the appropriate formula and calculate the number of different arrangements can be made using the 6 letters of the word NEWTON, assuming:
   a. the first N is distinct from the second N
   b. there is no distinction between the 2 Ns.
2. How many different permutations can be made using the 11 letters of the word ABRACADABRA?
3. **WE10** Determine how many different arrangements of 5 counters can be made using 3 red and 2 blue counters.
4. Determine how many different arrangements of 9 counters can be made using 4 black, 3 red and 2 blue counters.
5. A collection of 12 books is to be arranged on a shelf. The books consist of 3 copies of *Great Expectations*, 5 copies of *Catcher in the Rye* and 4 copies of *Huntin', Fishin' and Shootin'*. How many different arrangements of these books are possible?

6. A shelf holding 24 cans of dog food is to be stacked using 9 cans of Yummy and 15 cans of Ruff for Dogs. Determine how many different ways the shelf can be stocked.
7. **WE11** A cricket team of 11 players is to be selected, in batting order, from 15. Determine how many different arrangements are possible if:
   a. there are no restrictions
   b. Arjun must be in the team at number 1
   c. Arjun must be in the team but he can be anywhere from 1 to 11
   d. Arjun is not in the team.
8. The Student Council needs to fill the positions of president, secretary and treasurer from 6 candidates. Each candidate can fill only one of the positions. Determine how many ways can this be done if:
   a. there are no restrictions
   b. Tan must be secretary
   c. Tan must have one of the 3 positions
   d. Tan is not in any of the positions.
9. The starting 5 in a basketball team is to be picked, in order, from the 10 players in the squad. Determine how many ways can this be done if:
   a. there are no restrictions
   b. Jamahl needs to be player number 5
   c. Jamahl and Anfernee must be in the first 5 players (starting 5)
   d. Jamahl is not in the team.
10. **WE12** a. Calculate the number of permutations of the letters in the word MATHS.
   b. In how many of these do the letters M and A appear together?
   c. In how many permutations do the letters M and A appear apart?
   d. Determine the probability of the letters M and A appearing apart.
11. A rowing team of 4 rowers is to be selected in order from 8 rowers.
   a. In how many different ways can this be done?
   b. In how many of these ways do 2 rowers, Jane and Lee, sit together in the boat?
   c. In how many ways can the crew be formed without using Jane or Lee?
   d. In how many ways can the crew be formed if it does not contain Jane?

12. A decathlon has 12 runners.
   a. In how many ways can 1st, 2nd and 3rd be filled?
   b. In how many ways can 1st, 2nd and 3rd be filled if Najim finishes first?
13. **WE13** Consider the words SIMULTANEOUS and EQUATIONS.
   a. How many arrangements of the letters of the word EQUATIONS have the letters Q and U grouped together.
   b. How many arrangements of the letters of the word EQUATIONS have the letters Q and U separated?
   c. How many arrangements of the letters of the word SIMULTANEOUS are possible?
   d. Determine the probability that in a randomly chosen arrangement of the word SIMULTANEOUS, both the letters U are together.
14. **MC** If the answer is 10, which of the following options best matches this answer?
   A. The number of ways 1st and 2nd can occur in a race with 5 entrants
   B. The number of distinct arrangements of the letters in NANNA
   C. The number of permutations of the letters in POCKET where P and O are together
   D. ${}^{10}P_2 \div {}^{4}P_2$

15. MC If the answer is 480, which of the following options best matches this answer?

A. The number of ways 1st and 2nd can occur in a race with 5 entrants

B. The number of distinct arrangements of the letters in NANNA

C. The number of permutations of the letters in POCKET where P and O are apart

D. ${}^{10}P_2 \div {}^{4}P_2$

16. The clue in a crossword puzzle says that a particular answer is an anagram of STOREY. An anagram is another word that can be obtained by rearranging the letters of the given word.

a. Determine the number of possible arrangements of the letters of STOREY.

b. The other words in the crossword puzzle indicate that the correct answer is O__T__.How many arrangements are now possible? Can you see the word?

1.5 Combinations

1.5.1 When order does not matter

A group of things chosen from a larger group where order is not important is called a combination. In previous sections we performed calculations of the number of ways a task could be done where order is important — permutations or arrangements. We now examine situations where *order does not matter*.

Suppose 5 people have nominated for a committee consisting of 3 members. It does not matter in what order the candidates are placed on the committee, it matters only whether they are there or not. If order was important we know there would be ${}^{5}P_3$, or 60, ways in which this could be done. Here are the possibilities:

ABC	ACB	BAC	BCA	CAB	CBA
ABD	ADB	BAD	BDA	DAB	DBA
ABE	AEB	BAE	BEA	EAB	ABA
ACE	AEC	CAE	CEA	EAC	ECA
ACD	ADC	CAD	CDA	DAC	DCA
ADE	AED	DAE	DEA	EAD	EDA
BCD	BDC	CBD	CDB	DBC	DCB
BCE	BEC	CBE	CEB	EBC	ECB
BDE	BED	DBE	DEB	EBD	EDB
CDE	CED	DCE	DEC	ECD	EDC

The 60 arrangements are different only if we take order into account; that is, ABC is different from CAB and so on. You will notice in this table that there are 10 distinct committees corresponding to the 10 distinct rows. Each row merely repeats, in a different order, the committee in the first column. This result (10 distinct committees) can be arrived at logically:

1. There are ${}^{5}P_3$ ways of choosing or selecting 3 from 5 in order.
2. Each choice of 3 is repeated 3! times.
3. The number of distinct selections or combinations is ${}^{5}P_3 \div 3! = 10$.

This leads to the general rule of selecting r objects from n objects.

The number of ways of choosing or selecting r objects from n distinct objects, where order is not important, is given by ${}^{n}C_r$:

$$ {}^{n}C_r = \frac{{}^{n}P_r}{r!} $$

C is used to represent combinations.

WORKED EXAMPLE 14

Write these combinations as statements involving permutations, then calculate them.

a. ${}^{7}C_{2}$ **b.** ${}^{20}C_{3}$

THINK	WRITE
a. 1. Recall the rule for ${}^{n}C_{r}$.	**a.** ${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!}$
2. Substitute the given values of n and r into the combination formula.	${}^{7}C_{2} = \frac{{}^{7}P_{2}}{2!}$
3. Simplify the fraction.	$= \frac{\left(\frac{7!}{5!}\right)}{2!}$
	$= \frac{7!}{5!} \div 2!$
	$= \frac{7!}{5!} \times \frac{1}{2!}$
	$= \frac{7 \times 6 \times 5!}{5! \times 2 \times 1}$
4. Evaluate.	$= \frac{7 \times 6}{2 \times 1}$
	$= \frac{42}{2}$
	$= 21$
b. 1. Write down the rule for ${}^{n}C_{r}$.	**b.** ${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!}$
2. Substitute the values of n and r into the formula.	${}^{20}C_{3} = \frac{{}^{20}P_{3}}{3!}$
3. Simplify the fraction.	$= \frac{\left(\frac{20!}{17!}\right)}{3!}$
	$= \frac{20!}{17!} \div 3!$
	$= \frac{20!}{17!} \times \frac{1}{3!}$
	$= \frac{20 \times 19 \times 18 \times 17!}{17! \times 3 \times 2 \times 1}$
4. Evaluate.	$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1}$
	$= \frac{6840}{6}$
	$= 1140$

WORKED EXAMPLE 15

Apply the concept of ${}^nC_r$ to calculate the number ways a basketball team of 5 players be selected from a squad of 9 if the order in which they are selected does not matter.

THINK

1. Recall the rule for ${}^nC_r$.
   Note: Since order does not matter, use the ${}^nC_r$ rule.
2. Substitute the values of n and r into the formula.
3. Simplify the fraction.
4. Evaluate.

WRITE

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$${}^9C_5 = \frac{{}^9P_5}{5!}$$

$$= \frac{\left(\frac{9!}{4!}\right)}{5!}$$

$$= \frac{9!}{4!} \div 5!$$

$$= \frac{9!}{4!} \times \frac{1}{5!}$$

$$= \frac{9!}{4!\,5!}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$$

$$= \frac{3024}{24}$$

$$= 126$$

The formula we use to determine the number of ways of selecting r objects from n distinct objects, where order is not important, is useful but needs to be simplified.

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$= \frac{\frac{n!}{(n-r)!}}{r!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

$$^nC_r = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

$0 \leq r \leq n$ **where** r **and** n **are non-negative integers.**

The formula for ^nC_r is exactly that for the binomial coefficients used in the binomial theorem, which is explored with reference to Pascal's triangle in section 1.7.

on Resources

Interactivity: Counting techniques (int-6293)

WORKED EXAMPLE 16

Determine the value of the following.

a. $^{12}C_5$ **b.** $\binom{10}{2}$

THINK

a. 1. Recall the formula for ^nC_r.

2. Substitute the given values of n and r into the combination formula.

3. Simplify the fraction.

4. Evaluate.

b. 1. Recall the rule for $\binom{n}{r}$.

2. Substitute the given values of n and r into the combination formula.

3. Simplify the fraction.

WRITE

a. $^nC_r = \frac{n!}{(n-r)!\,r!}$

$^{12}C_5 = \frac{12!}{(12-5)!\,5!}$

$= \frac{12!}{7!\,5!}$

$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{7! \times 5 \times 4 \times 3 \times 2 \times 1}$

$= \frac{12 \times 11 \times \cancel{10}^{2} \times \cancel{9}^{3} \times \cancel{8}^{2}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$

$= 12 \times 11 \times 3 \times 2$

$= 792$

b. $\binom{n}{r} = {}^nC_r$

$= \frac{n!}{(n-r)!\,r!}$

$\binom{10}{2} = \frac{10!}{(10-2)!\,2!}$

$= \frac{10!}{8!\,2!}$

$= \frac{10 \times 9 \times 8!}{8! \times 2 \times 1}$

$= \frac{10 \times 9}{2 \times 1}$

4. Evaluate.

$$= \frac{90}{2}$$
$$= 45$$

1.5.2 Probability calculations

The combination formula is always used in selection problems. Most calculators have a ${}^{n}C_{r}$ key to assist with the evaluation when the figures become large.

Both the multiplication and addition principles apply and are used in the same way as for permutations.

The calculation of probabilities from the rule $P(A) = \frac{n(A)}{n(\xi)}$ requires that the same counting technique is used for the numerator and denominator. We have seen for permutations that it can assist calculation to express numerator and denominator in terms of factorials and then simplify. Similarly for combinations, express the numerator and denominator in terms of the appropriate combinatoric coefficients and then carry out the calculations.

WORKED EXAMPLE 17

A committee of 5 students is to be chosen from 7 boys and 4 girls. Use ${}^{n}C_{r}$ and the multiplication and addition principles to answer the following.

a. Calculate how many committees can be formed.

b. Calculate how many of the committees contain exactly 2 boys and 3 girls.

c. Calculate how many committees have at least 3 girls.

d. Determine the probability of the oldest and youngest students both being on the committee in part a.

THINK

a. 1. As there is no restriction, choose the committee from the total number of students.

2. Use the formula ${}^{n}C_{r} = \frac{n!}{r! \times (n-r)!}$ to calculate the answer.

WRITE

a. There are 11 students in total from whom 5 students are to be chosen. This can be done in ${}^{11}C_{5}$ ways.

$${}^{11}C_{5} = \frac{11!}{5! \times (11-5)!}$$
$$= \frac{11!}{5! \times 6!}$$
$$= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5! \times 6!}$$
$$= \frac{11 \times \cancel{10}^{2} \times \cancel{9} \times \cancel{8}^{3} \times 7}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$
$$= 462$$

There are 462 possible committees.

b. 1. Select the committee to satisfy the given restriction.

2. Use the multiplication principle to form the total number of committees.
Note: The upper numbers on the combinatoric coefficients sum to the total available, $7 + 4 = 11$, while the lower numbers sum to the number that must be on the committee, $2 + 3 = 5$.

3. Calculate the answer.

b. The 2 boys can be chosen from the 7 boys available in ${}^7C_2$ ways.The 3 girls can be chosen from the 4 girls available in ${}^4C_3$ ways.

The total number of committees which contain two boys and three girls is ${}^7C_2 \times {}^4C_3$.

$$\begin{aligned} {}^7C_2 \times {}^4C_3 &= \frac{7!}{2! \times 5!} \times 4 \\ &= \frac{7 \times 6}{2!} \times 4 \\ &= 21 \times 4 \\ &= 84 \end{aligned}$$

There are 84 committees possible with the given restriction.

c. 1. List the possible committees which satisfy the given restriction.

2. Write the number of committees in terms of combinatoric coefficients.

3. Use the addition principle to state the total number of committees.

4. Calculate the answer.

c. As there are 4 girls available, at least 3 girls means either 3 or 4 girls.The committees of 5 students which satisfy this restriction have either 3 girls and 2 boys, or they have 4 girls and 1 boy.

3 girls and 2 boys are chosen in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy are chosen in ${}^4C_4 \times {}^7C_1$ ways.

The number of committees with at least three girls is ${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$.

$$\begin{aligned} {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 &= 84 + 1 \times 7 \\ &= 91 \end{aligned}$$

There are 91 committees with at least 3 girls.

d. 1. State the number in the sample space.

2. Form the number of ways the given event can occur.

3. State the probability in terms of combinatoric coefficients.

d. The total number of committees of 5 students is ${}^{11}C_5 = 462$ from part **a**.

Each committee must have 5 students. If the oldest and youngest students are placed on the committee, then 3 more students need to be selected from the remaining 9 students to form the committee of 5. This can be done in ${}^9C_3$ ways.

Let A be the event the oldest and the youngest students are on the committee.

$$\begin{aligned} \mathrm{P}(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{{}^9C_3}{{}^{11}C_5} \end{aligned}$$

4. Calculate the answer.

$$P(A) = \frac{9!}{3! \times 6!} \div \frac{11!}{5! \times 6!}$$
$$= \frac{9!}{3! \times 6!} \times \frac{5! \times 6!}{11!}$$
$$= \frac{1}{3!} \times \frac{5!}{11 \times 10}$$
$$= \frac{5 \times 4}{110}$$
$$= \frac{2}{11}$$

The probability of the committee containing the youngest and the oldest students is $\frac{2}{11}$.

WORKED EXAMPLE 18

Evaluate the following using your calculator and comment on your results.

a. ${}^9C_3$ **b.** ${}^9C_6$ **c.** ${}^{15}C_5$ **d.** ${}^{15}C_{10}$ **e.** ${}^{12}C_7$ **f.** ${}^{12}C_5$

THINK

a–f. Use your calculator to evaluate the listed combinations.

Comment on your results.

WRITE

a. ${}^9C_3 = 84$
b. ${}^9C_6 = 84$
c. ${}^{15}C_5 = 3003$
d. ${}^{15}C_{10} = 3003$
e. ${}^{12}C_7 = 792$
f. ${}^{12}C_5 = 792$

So ${}^9C_3 = {}^9C_6$, ${}^{15}C_5 = {}^{15}C_{10}$ and ${}^{12}C_7 = {}^{12}C_5$.
Its appear that when, for example, ${}^{12}C_p = {}^{12}C_q$
$p + q = 12$.

TI | THINK

a.1. On a Calculator page, press MENU, then select:
5: Probability
3: Combinations.
Complete the entry line as:
nCr(9, 3)
then press ENTER.

2. The answer appears on the screen.

WRITE

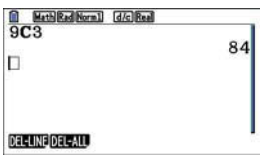
${}^9C_3 = 84$

CASIO | THINK

a.1. On a Run-Matrix screen, press OPTN then F6. Select PROB by pressing F3, then select nCr by pressing F3. Complete the entry line as:
9C3
then press EXE.

2. The answer appears on the screen.

WRITE



${}^9C_3 = 84$

For each of the preceding examples, it can be seen that ${}^{n}\mathrm{C}_{r} = {}^{n}\mathrm{C}_{n-r}$. This may be derived algebraically:

$$\begin{aligned}
{}^{n}\mathrm{C}_{n-r} &= \frac{{}^{n}\mathrm{P}_{n-r}}{(n-r)\,!} \\
&= \frac{\left(\frac{n!}{[n-(n-r)]!}\right)}{(n-r)} \\
&= \frac{\left(\frac{n!}{r!}\right)}{(n-r)\,!} \\
&= \frac{n!}{r!} \times \frac{1}{(n-r)\,!} \\
&= \frac{n!}{r!\,(n-r)\,!} \\
&= \frac{n!}{(n-r)\,!\,r!} \\
&= \frac{{}^{n}\mathrm{P}_{r}}{r!} \\
&= {}^{n}\mathrm{C}_{r}
\end{aligned}$$

Drawing on our understanding of combinations, we have:

- ${}^{n}C_{r} = {}^{n}C_{n-r}$, as choosing r objects must leave behind $(n - r)$ objects and vice versa
- ${}^{n}C_{0} = 1 = {}^{n}C_{n}$, as there is only one way to choose none or all of the n objects
- ${}^{n}C_{1} = n$, as there are n ways of choosing 1 object from a group of n objects.

on Resources

Digital document: SkillSHEET Listing possibilities (doc-26826)
Digital document: SpreadSHEET Combinations (doc-26827)

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 5

Combinations Summary screen and practice questions

Exercise 1.5 Combinations

Technology free

1. **WE14** Write each of the following as statements in terms of permutations.
   a. ${}^{8}\mathrm{C}_{3}$ b. ${}^{19}\mathrm{C}_{2}$ c. ${}^{1}\mathrm{C}_{1}$ d. ${}^{5}\mathrm{C}_{0}$
2. Write each of the following using the notation ${}^{n}\mathrm{C}_{r}$.
   a. $\frac{{}^{8}\mathrm{P}_{2}}{2!}$ b. $\frac{{}^{9}\mathrm{P}_{3}}{3!}$ c. $\frac{{}^{8}\mathrm{P}_{0}}{0!}$ d. $\frac{{}^{10}\mathrm{P}_{4}}{4!}$
3. **WE15** Apply the concept of ${}^{n}\mathrm{C}_{r}$ to calculate the number of ways three types of ice-cream can be chosen in any order from a supermarket freezer if the freezer contains:
   a. 3 types b. 6 types c. 10 types d. 12 types.
4. A mixed netball team must have 3 women and 4 men in the side. If the squad has 6 women and 5 men wanting to play, determine how many different teams are possible.

5. A *quinella* is a bet made on a horse race which pays a win if the punter selects the first 2 horses in any order. Determine how many different quinellas are possible in a race that has:
   a. 8 horses b. 16 horses.
6. MC At a party there are 40 guests and they decide to have a toast. Each guest 'clinks' glasses with every other guest. How many clinks are there in all?
   A. 39 **B.** 40 **C.** 40! **D.** 780
7. MC On a bookshelf there are 15 books — 7 geography books and 8 law books. Abena selects 5 books from the shelf — 2 geography books and 3 law books. How many different ways can she make this selection?
   A. ${}^{15}C_2 \times {}^{15}C_3$ **B.** ${}^{15}C_7 \times {}^{15}C_8$ **C.** ${}^{7}C_2 \times {}^{8}C_3$ **D.** ${}^{7}C_2 + {}^{8}C_3$

Technology active

8. A cricket team of 11 players is to be chosen from a squad of 15 players. Determine how many ways can this be done.
9. A basketball team of 5 players is to be chosen from a squad of 10 players. Determine how many ways can this be done.
10. WE16 Determine the value of the following:
    a. ${}^{12}C_4$ b. ${}^{11}C_1$ c. ${}^{12}C_{12}$
    d. $\binom{21}{15}$ e. $\binom{100}{1}$ f. $\binom{17}{14}$
11. From a pack of 52 cards, a hand of 5 cards is dealt.
    a. How many different hands are there?
    b. How many of these hands contain only red cards?
    c. How many of these hands contain only black cards?
    d. How many of these hands contain at least one red and at least one black card?
12. WE17 A committee of 5 students is to be chosen from 6 boys and 8 girls. Use ${}^{n}C_r$ and the multiplication principle to answer the following.
    a. Calculate how many committees can be formed.
    b. Calculate how many of the committees contain exactly 2 boys and 3 girls.
    c. Calculate how many committees have at least 4 boys.
    d. Determine the probability of neither the oldest nor the youngest student being on the committee.
13. A rugby union squad has 12 forwards and 10 backs in training. A team consists of 8 forwards and 7 backs. Determine how many different teams can be chosen from the squad.
14. A music collection contains 32 albums. Determine how many ways 5 albums can be chosen from the collection.

Questions **15**, **16** and **17** refer to the following information. The Maryborough Tennis Championships involve 16 players. The organisers plan to use 3 courts and assume that each match will last on average 2 hours and that no more than 4 matches will be played on any court per day.

15. In a 'round robin' each player plays every other player once.
    a. If the organisers use a round robin format, determine how many games will be played in all.
    b. For how many days would the tournament last?
16. The organisers split the 16 players into two pools of 8 players each. After a 'round robin' within each pool, a final is played between the winners of each pool.
    a. Determine how many matches are played in the tournament.
    b. How long does the tournament last?

17. A 'knock out' format is one in which the loser of every match drops out and the winners proceed to the next round until there is only one winner left.

a. If the game starts with 16 players, determine how many matches are needed before a winner is obtained.

b. How long would the tournament last?

18. Lotto is a gambling game played by choosing 6 numbers from 45. Gamblers try to match their choice with those numbers chosen at the official draw. No number can be drawn more than once and the order in which the numbers are selected does not matter.

a. Calculate how many different selections of 6 numbers can be made from 45.

b. Suppose the first numbers drawn at the official draw are 42, 3 and 18. How many selections of 6 numbers will contain these 3 numbers?

Note: This question ignores supplementary numbers. Lotto is discussed further in the next section.

19. **a.** **WE18** Calculate the value of:

i. ${}^{12}C_3$ and ${}^{12}C_9$ **ii.** ${}^{15}C_8$ and ${}^{15}C_7$ **iii.** ${}^{10}C_1$ and ${}^{10}C_9$ **iv.** ${}^{8}C_3$ and ${}^{8}C_5$

b. What do you notice? Give your answer as a general statement such as 'The value of ${}^{n}C_r$ is ...'.

1.6 Applications of permutations and combinations

1.6.1 Permutations and combinations in the real world

Counting techniques, particularly those involving permutations and combinations, can be applied in gambling, logistics and various forms of market research. In this section we investigate when to use permutations and when to use combinations as well as examining problems associated with these techniques.

Permutations are used to count when order is important. Some examples are:

- the number of ways the positions of president, secretary and treasurer can be filled
- the number of ways a team can be chosen from a squad *in distinctly different positions*
- the number of ways the first three positions of a race can be filled.

Combinations are used to count when order is not important. Some examples are:

- the number of ways a committee can be chosen
- the number of ways a team can be chosen from a squad
- the number of ways a hand of 5 cards can be dealt from a deck.

These relatively simple applications of permutations and combinations are explored in the worked examples that follow. However, it is important to be mindful that the modern world relies on combinatorial algorithms. These algorithms are important for any system that benefits from finding the fastest ways to operate. Examples include communication networks, molecular biology, enhancing security and protecting privacy in internet information transfer, data base queries and data mining, computer chip design, simulations and scheduling.

WORKED EXAMPLE 19

a. Ten points are marked on a page and no three of these points are in a straight line. Determine how many triangles can be drawn joining these points.

b. Determine how many different 3-digit numbers can be made using the digits 1, 3, 5, 7 and 9 without repetition.

THINK	WRITE
a. 1. *Note:* A triangle is made by choosing 3 points. It does not matter in what order the points are chosen, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**a** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. Substitute the given values of n and r into the combination formula.	${}^{10}C_3 = \frac{10!}{(10-3)!\,3!}$
	$= \frac{10!}{7!\,3!}$
3. Simplify the fraction.	$= \frac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1}$
	$= \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 10 \times 3 \times 4$
	$= 120$
5. Answer the question.	120 triangles may be drawn by joining 3 points.
6. Verify the answer obtained by using the combination function on a calculator.	
b. 1. *Note:* Order is important here. Recall the rule for ${}^nP_r$.	**b** ${}^nP_r = \frac{n}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	${}^5P_3 = \frac{5!}{(5-3)}$
	$= \frac{5!}{2!}$
3. Evaluate.	$= \frac{5 \times 4 \times 3 \times 2}{2!}$
	$= 5 \times 4 \times 3$
	$= 60$
4. Answer the question.	Sixty 3-digit numbers can be made without repetition from a group of 5 numbers.
5. Verify the answer obtained by using the permutation function on a calculator.	

WORKED EXAMPLE 20

Jade and Kelly are 2 of the 10 members of a basketball squad. Calculate how many ways can a team of 5 be chosen if:

a. both Jade and Kelly are in the 5

b. neither Jade nor Kelly is in the 5

c. Jade is in the 5 but Kelly is not.

THINK	WRITE
a. 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**a** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade and Kelly are included then there are 3 positions to be filled from the remaining 8 players. Substitute the given values of n and r into the combination formula.	${}^8C_3 = \frac{8!}{(8-3)!\,3!}$ $= \frac{8!}{5!\,3!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5}{5! \times 3 \times 2 \times 1}$ $= \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 8 \times 7$ $= 56$
5. Answer the question.	If Jade and Kelly are included, then there are 56 ways to fill the remaining 3 positions.
b. 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**b** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade and Kelly are not included then there are 5 positions to be filled from 8 players. Substitute the given values of n and r into the combination formula.	${}^8C_5 = \frac{8!}{(8-5)!\,5!}$ $= \frac{8!}{3!\,5!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$ $= \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 8 \times 7$ $= 56$
5. Answer the question.	If Jade and Kelly are not included, then there are 56 ways to fill the 5 positions.
c. 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**c** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade is included and Kelly is not then there are 4 positions to be filled from 8 players. Substitute the given values of n and r into the combination formula.	${}^8C_4 = \frac{8!}{(8-4)!\,4!}$ $= \frac{8!}{4!\,4!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!}$ $= \frac{\cancel{8} \times 7 \times \cancel{6}^2 \times 5}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$

4. Evaluate.

$= 7 \times 2 \times 5$
$= 70$

5. Answer the question.

If Jade is included and Kelly is not, then there are 70 ways to fill the 4 positions.

6. Verify each of the answers obtained by using the combination function on a calculator.

1.6.2 Lotto systems

An interesting application of combinations as a technique of counting is a game that Australians spend many millions of dollars on each week — lotteries. There are many varieties of lottery games in Australia. To play Saturday Gold Lotto in Queensland, a player selects 6 numbers from 45 numbers. The official draw chooses 6 numbers and 2 supplementary numbers. Depending on how the player's choice of 6 numbers matches the official draw, prizes are awarded in different divisions.

Division 1: 6 winning numbers
Division 2: 5 winning numbers and one of the supplementary numbers
Division 3: 5 winning numbers
Division 4: 4 winning numbers
Division 5: 3 winning numbers and one of the supplementary numbers
If the official draw was:

Winning numbers						Supplementaries	
13	42	6	8	20	12	2	34

A player who chose:

8 34 13 12 20 45

would win a Division 4 prize and a player who chose:

8 34 13 12 22 45

would win a Division 5 prize.

A player may have 7 lucky numbers 4, 7, 12, 21, 30, 38 and 45, and may wish to include all possible combinations of these 7 numbers in a 6 numbers lotto entry.

This can be done as follows:

4 7 12 21 30 38
4 7 12 21 30 45
4 7 12 21 38 45
4 7 12 30 38 45
4 7 21 30 38 45
4 12 21 30 38 45
7 12 21 30 38 45

The player does not have to fill out 7 separate entries to enter all combinations of these 7 numbers 6 at a time but rather can complete a 'System 7' entry by marking 7 numbers on the entry form.

A System 9 consists of all entries of 6 numbers from the chosen 9 numbers.

WORKED EXAMPLE 21

Use the information on lottery systems given above.
A player uses a System 8 entry with the numbers 4, 7, 9, 12, 22, 29, 32 and 36.
The official draw for this game was 4, 8, 12, 15, 22, 36 with supplementaries 20 and 29.
a. How many single entries are equivalent to a System 8?
b. List 3 of the player's entries that would have won Division 4.
c. Determine how many of the player's entries would have won Division 4.

THINK	WRITE
a 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**a** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* A System 8 consists of all entries consisting of 6 numbers chosen from 8. Substitute the given values of n and r into the combination formula.	${}^8C_6 = \frac{8!}{(8-6)!\,6!}$
	$= \frac{8!}{2!\,6!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6!}{2 \times 7 \times 6!}$
	$= \frac{\cancel{8}^4 \times 7}{\cancel{2} \times 1}$
4. Evaluate.	$= 4 \times 7$
	$= 28$
5. Answer the question.	A System 8 is equivalent to 28 single entries.
6. Verify each of the answers obtained by using the combination function on a calculator.	
b *Note:* Division 4 requires 4 winning numbers. The player's winning numbers are 4, 12, 22 and 36. Any of the other 4 numbers can fill the remaining 2 places. List 3 of the player's entries that would have won Division 4.	**b** Some of the possibilities are: 4 12 22 36 7 9 4 12 22 36 7 29 4 12 22 36 7 32
c 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**c** ${}^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* To win Division 4 the numbers 4, 12, 22 and 36 must be included in the entry. The other 2 spaces can be filled with any of the other 4 numbers in any order. Substitute the given values of n and r into the combination formula.	${}^4C_2 = \frac{4!}{(4-2)!\,2!}$
	$= \frac{4!}{2!\,2!}$
3. Simplify the fraction.	$= \frac{4 \times 3 \times 1!}{\cancel{2} \times 1 \times \cancel{2}!}$
	$= \frac{\cancel{4}^2 \times 3}{\cancel{2} \times 1}$
4. Evaluate.	$= 2 \times 3$
	$= 6$

5. Answer the question.

Six of the player's entries would have won Division 4.

6. Verify each of the answers obtained by using the combination function on a calculator.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 6

Applications of permutations and combinations Summary screen and practice questions

Exercise 1.6 Applications of permutations and combinations

Technology active

1. **WE19** Determine how many ways there are:
   a. to draw a line segment between 2 points on a page with 10 points on it
   b. to make a 4-digit number using the digits 2, 4, 6, 8 and 1 without repetition
   c. to choose a committee of 4 people from 10 people
   d. for a party of 15 people to shake hands with one another.
2. Determine how many ways there are:
   a. for 10 horses to fill 1st, 2nd and 3rd positions
   b. to choose a team of 3 cyclists from a squad of 5
   c. to choose 1st, 2nd and 3rd speakers for a debating team from 6 candidates
   d. for 20 students to seat themselves in a row of 20 desks.
3. The French flag is known as a tricolour flag because it is composed of the 3 bands of colour. Determine how many different tricolour flags can be made from the colours red, white, blue and green if each colour can be used only once in one of the 3 bands and order is important.

4. In a taste test a market research company has asked people to taste 4 samples of coffee and try to identify each as one of four brands. Subjects are told that no 2 samples are the same brand. Determine how many different ways the samples can be matched to the brands.
5. **WE20** A volleyball team of 6 players is to be chosen from a squad of 10 players. Calculate how many ways can this be done if:
   a. there are no restrictions
   b. Stephanie is to be in the team
   c. Stephanie is not in the team
   d. two players, Stephanie and Alison, are not both in the team together.

6. A cross-country team of 4 runners is to be chosen from a squad of 9 runners. Determine how many ways this can be done if:
   a. there are no restrictions
   b. Cecily is to be one of the 4
   c. Cecily and Michael are in the team
   d. either Cecily or Michael but not both are in the team.

7. **MC** A netball team consists of 7 different positions: goal defence, goal keeper, wing defence, centre, wing attack, goal attack and goal shooter. The number of ways a squad of 10 players can be allocated to these positions is:

   A. 10! **B.** 7! **C.** $\frac{10!}{7!}$ **D.** ${}^{10}P_7$

8. **WE21** Use the information on lotteries given on page 41.
   A player uses a System 8 entry with the numbers 9, 12, 14, 17, 27, 34, 37 and 41. The official draw for this game was 9, 13, 17, 20, 27, 41 with supplementaries 25 and 34.
   a. How many single entries are equivalent to a System 8?
   b. List 3 of the player's entries that would have won Division 4.
   c. Determine how many of the player's entries would have won Division 4.
9. Use the information on lotteries given on page 41.
   A player uses a System 9 entry with the numbers 7, 10, 12, 15, 25, 32, 35, 37 and 41. The official draw for this game was 7, 11, 15, 18, 25, 39 with supplementaries 23 and 32.
   a. To how many single entries is a System 9 equivalent?
   b. List 3 of the player's entries that would have won Division 5.
   c. How many of the player's entries would have won Division 5?
10. In the gambling game roulette, if a gambler puts $1 on the winning number he will win $35. Suppose a gambler wishes to place five $1 bets on 5 different numbers in one spin of the roulette wheel. If there are 36 numbers in all, determine how many ways the five bets can be placed.
11. A soccer team of 11 players is to be chosen from a squad of 17. If one of the squad is selected as goalkeeper and any of the remaining players can be selected in any of the positions, determine how many ways can this be done if:
    a. there are no other restrictions
    b. Karl is to be chosen
    c. Karl and Andrew refuse to play in the same team
    d. Karl and Andrew are either both in or both out.

12. **MC** A secret chemical formula requires the mixing of 3 chemicals. A researcher does not remember the 3 chemicals but has a shortlist of 10 from which to choose. Each time she mixes 3 chemicals and tests the result she takes 15 minutes.
    How long does the researcher need, to be absolutely sure of getting the right combination?

    A. 120 hours **B.** 7.5 hours
    C. 15 hours **D.** 30 hours

Questions **13** and **14** refer to the following information: Keno is a popular game in clubs and pubs around Australia. In each round a machine randomly generates 20 winning numbers from 1 to 80. In one entry a player can select up to 15 numbers.

13. Suppose a player selects an entry of 6 numbers.
- **a.** Determine how many ways an entry of 6 numbers can contain 6 winning numbers.
  Suppose an entry of 6 numbers has exactly 3 winning numbers in it.
- **b.** In how many ways can the 3 winning numbers be chosen?
- **c.** In how many ways can the 3 losing numbers be chosen?
- **d.** How many entries of 6 numbers contain 3 winning numbers and 3 losing numbers?

14. Suppose a player selects an entry of 20 numbers.
- **a.** Determine how many ways an entry of 20 numbers can contain 20 winning numbers.
- **b.** Suppose an entry of 20 numbers has exactly 14 winning numbers in it.
  - **i.** In how many ways can the 14 winning numbers be chosen?
  - **ii.** In how many ways can the 6 losing numbers be chosen?
  - **iii.** How many entries of 20 numbers contain 14 winning numbers and 6 losing numbers?
  - **iv.** How many entries of 20 numbers contain no winning numbers?

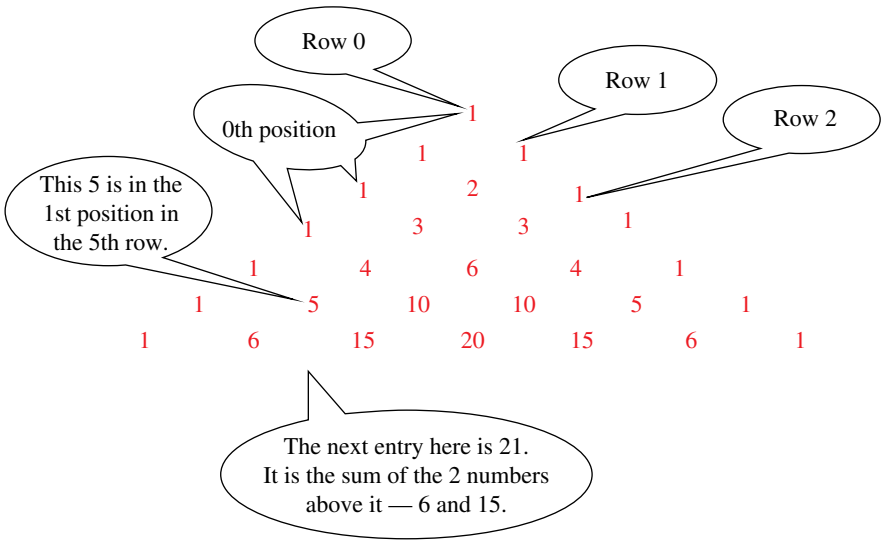
1.7 Pascal's triangle and the pigeon-hole principle

1.7.1 Pascal's triangle

Combinations are useful in other areas of mathematics, such as probability and binomial expansions. If we analyse the ${}^nC_r$ values closely, we notice that they produce the elements of any row in **Pascal's triangle** or each of the coefficients of a particular binomial expansion.

The triangle shown was named after the French mathematician Blaise Pascal. He was honoured for his application of the triangle to his studies in the area of probability.

Each new row in Pascal's triangle is obtained by first placing a 1 at the beginning and end of the row and then adding adjacent entries from the previous row.



Each element in Pascal's triangle can be calculated using combinations. For example, 10 is the 2nd element in the 5th row of Pascal's triangle; that is, ${}^5C_2 = 10$ (assuming 1 is the zeroth (0th) element). Hence, the triangle can be written using $\binom{n}{r}$ or ${}^nC_r$ notation.

$n = 0$:							1											${}^0C_0$			
$n = 1$:						1		1								${}^1C_0$		${}^1C_1$			
$n = 2$:					1		2		1						${}^2C_0$		${}^2C_1$		${}^2C_2$		
$n = 3$:				1		3		3		1				${}^3C_0$		${}^3C_1$		${}^3C_2$		${}^3C_3$	
$n = 4$:			1		4		6		4		1		${}^4C_0$		${}^4C_1$		${}^4C_2$		${}^4C_3$		${}^4C_4$
$n = 5$:	1		5		10		10		5		1	${}^5C_0$		${}^5C_1$		${}^5C_2$		${}^5C_3$		${}^5C_4$		${}^5C_5$

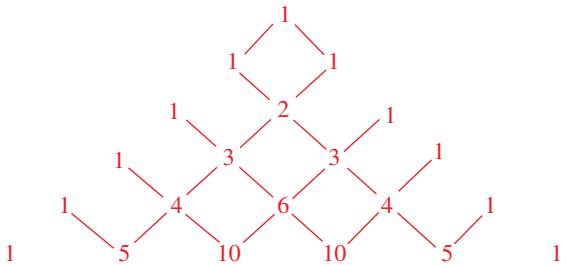
Note that the first and last number in each row is always 1.
Each coefficient is obtained by adding the two coefficients immediately above it.
The binomial expansion can therefore be generalised using combinations.

Pascal's triangle shows that the *r*th element of the *n*th row of Pascal's triangle is given by ${}^nC_r$.
It is assumed that the 1 at the beginning of each row is the 0th element.
This gives Pascal's identity:

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \text{ for } 0 < r < n $$

The relationship between Pascal's triangle and combinations can be extended to the **binomial theorem**. This theorem gives a rule for expanding an expression such as $(a + b)^n$. Expanding expressions such as this may become quite difficult and time consuming using the usual methods of algebra. Consider the coefficients of the binomial expansion.

$(x + y)^0 = 1$
$(x + y)^1 = x + y$
$(x + y)^2 = x^2 + 2xy + y^2$
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$



The coefficients in the binomial expansion are equal to the numbers in Pascal's triangle.
These relationships are summarised in the following table.

$(x + y)^0 = 1$	Coefficient: 1	${}^0C_0$
$(x + y)^1 = x + y$	Coefficients: 1 1	${}^1C_0$, ${}^1C_1$,
$(x + y)^2 = x^2 + 2xy + y^2$	Coefficients: 1 2 1	${}^2C_0$, ${}^2C_1$, ${}^2C_2$
$(y + x)^3 = y^3 + 3x^2y + 3xy^2 + x^3$	Coefficients: 1 3 3 1	${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$	Coefficients: 1 4 6 4 1	${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$, ${}^4C_4$
$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$	Coefficients: 1 5 10 10 5 1	${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$, ${}^5C_5$

The binomial expansion can therefore be generated using combinations:

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n$$
$$= x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + y^n$$

Resources

Interactivity: Pascal's triangle and binomial coefficients (int-2554)
Digital document: SpreadSHEET Pascal's triangle (doc-26828)

WORKED EXAMPLE 22

Refer to Pascal's triangle above and answer the following questions.
a. What number is in the 4th position in the 6th row?
b. Complete the 7th row in Pascal's triangle.
c. The numbers 7 and 21 occur side by side in the 7th row. What element in the 8th row occurs below and in between these numbers?

THINK

a 1. Locate the 6th row and the 4th position.
Note: Remember the 0th row is 1 and the first row is 1 1. In the 6th row the 1 on the left is in the 0th position.

2. Answer the question.

b 1. Write down the elements of the 6th row.

2. Obtain the 7th row.
a. Place the number 1 at the beginning of the row.
b. Add the first 2 adjacent numbers from the 6th row (1 and 6).
c. Place this value next to the 1 on the new row and align the value so that it is in the middle of the 2 numbers (directly above) which created it.
d. Repeat this process with the next 2 adjacent numbers from the 6th row (6 and 15).
e. Once the sums of all adjacent pairs from the sixth row have been added, place a 1 at the end of the row.

3. Answer the question.

c 1. Add the numbers 7 and 21 in order to obtain the element in the 8th row which occurs below and in between these numbers.

WRITE

a 6th row ⇒ 1 6 15 20 **15** 6 1

The number in the 4th position in the 6th row is 15.

b. 6th row ⇒ 1 6 15 20 15 6 1
7th row ⇒ 1 7 21 35 35 21 7 1

The 7th row is
1 7 21 35 35 21 7 1.

c 7 21
28

2. Answer the question.

The element in the 8th row which occurs below and in between 7 and 21 is 28.

WORKED EXAMPLE 23

Use combinations to calculate the number in the 5th position in the 9th row of Pascal's triangle.

THINK	WRITE
1. Write down the combination rule.	${}^nC_r$
2. Substitute the values for n and r into the rule. *Note:* The row is represented by $n = 9$. The position is represented by $r = 5$.	${}^9C_5 = 126$
3. Evaluate using a calculator.	
4. Answer the question.	The value of the number in the 5th position in the 9th row is 126.

WORKED EXAMPLE 24

Use the binomial theorem to expand $(a + 2)^4$.

THINK	WRITE
1. Recall the rule for the binomial theorem.	$(x + y)^n = x^n + {}^nC_1x^{n-1}y^1 + \ldots {}^nC_rx^{n-r}y^r + \ldots y^n$
2. Substitute the values for a, b and n into the rule: $x = a, y = 2$ and $n = 4$.	$(a+2)^4 = a^4 + {}^4C_1a^32^1 + {}^4C_2a^22^2 + {}^4C_3a^12^3 + 2^4$
3. Simplify.	$= a^4 + 4 \times a^3 \times 2 + 6 \times a^2 \times 4 + 4 \times a \times 8 + 16$ $= a^4 + 8a^3 + 24a^2 + 32a + 16$

WORKED EXAMPLE 25

What is the 4th term in the expansion of $(x + y)^7$?

THINK	WRITE
1. Recall that the rule for the 4th term can be obtained from the binomial theorem: $(x + y)^n = x^n + {}^nC_1x^{n-1}y^1 + \ldots {}^nC_rx^{n-r}y^r + \ldots y^n$ Write down the rule for the r th term.	r th term $= {}^nC_ra^{n-r}b^r$
2. Substitute the values for x, y, n and r into the rule: $a = x, b = y, n = 7$ and $r = 4$.	${}^nC_rx^{n-r}y^r = {}^7C_4x^{7-4}y^4$
3. Simplify. *Note:* The 0th term corresponds to the first element of the expansion.	$= 35x^3y^4$
4. Answer the question.	The 4th term is equal to $35x^3y^4$.

1.7.2 Pigeon-hole principle

Henri Poincaré, a famous mathematician, once described mathematics as 'the art of giving the same name to different things'. Consider three phenomena, which on the surface appear different — population growth, the value of investments and radioactive decay. Each can be described by one mathematical concept: exponential change. The mathematician gives three seemingly different things the same name.

The **pigeon-hole principle** is a good example of how mathematics gives the same name to different things.

The pigeon-hole principle:
If there are $(n + 1)$ pigeons to be placed in n pigeon-holes, then there is at least one pigeon-hole with at least two pigeons in it.

Notes:

- Note the precise use of language in this statement, in particular the importance of the phrase 'at least'.
- Some may view the pigeon-hole principle as an obvious statement, but used cleverly it is a powerful problem-solving tool.

WORKED EXAMPLE 26

In a group of 13 people show that there are at least 2 whose birthday falls in the same month.

THINK	WRITE
1. Think of each person as a pigeon and each month as a pigeon-hole.	There are 12 months and 13 people.
2. If there are 13 pigeons to be placed in 12 holes at least one hole must contain at least two pigeons.	Using the pigeon-hole principle: 13 people to be assigned to 12 months. At least one month must contain at least two people. That is, at least two people have birthdays falling in the same month.

Generalised pigeon-hole principle:
If there are $(nk + 1)$ pigeons to be placed in n pigeon-holes, then there is at least one pigeon-hole with at least $(k + 1)$ pigeons in it.

WORKED EXAMPLE 27

In a group of 37 people show that there are at least 4 whose birthdays lie in the same month.

THINK	WRITE
1. Think of each person as a pigeon and each month as a pigeon-hole.	There are 12 months and 37 people.
2. Recall the generalised pigeon-hole principle.	Using the generalised pigeon-hole principle: 37 people to be assigned to 12 months.

3. $(nk + 1)$ pigeons to be allocated to n holes; $n = 12 \rightarrow k = 3$	The value of n is 12 and k is 3. So at least one month has at least $(k + 1)$ or 4 people in it. That is, at least 4 people have birthdays falling in the same month.

WORKED EXAMPLE 28

On resuming school after the Christmas vacation, many of the 22 teachers of Eastern High School exchanged handshakes. Mr Yisit, the Social Science teacher, said, 'Isn't that unusual — with all the handshaking, no two people shook hands the same number of times'.

Not wanting to spoil the fun, the Mathematics teacher, Mrs Pigeon, said respectfully, 'I am afraid you must have counted incorrectly. What you say is not possible'

How can Mrs Pigeon make this statement?

THINK	WRITE
1. Think of the possible number of handshakes by a person as a pigeon-hole.	For each person there are 22 possible numbers of handshakes; that is, 0 to 21.
2. If two or more people have 0 handshakes, the problem is solved. Consider the cases where there is 1 person with 0 handshakes or 0 persons with 0 handshakes.	1 person with 0 handshakes: If there is 1 person with 0 handshakes, there can be no person with 21 handshakes. Thus, there are 21 people to be assigned to 20 pigeon-holes. Therefore, there must be at least one pigeon-hole with at least two people in it. 0 people with 0 handshakes: If there is no person with 0 handshakes, there are 22 people to be assigned to 21 pigeon-holes. Therefore, there must be at least one pigeon-hole with at least two people in it (at least two people have made the same number of handshakes).
3. Conclude using a sentence.	Thus, there are at least two people who have made the same number of handshakes.

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Units 1 & 2 > Area 1 > Sequence 1 > Concepts 7 & 8

Pascal's triangle Summary screen and practice questions
The pigeon-hole principle Summary screen and practice questions

Exercise 1.7 Pascal's triangle and the pigeon-hole principle

Technology free

1. Write the first 8 rows in Pascal's triangle.
2. **WE22** Refer to Pascal's triangle on page 45 and answer the following questions.
   a. What number is in the 4th position in the 8th row?
   b. Complete the 9th row in Pascal's triangle.
   c. If 9 and 36 occur side by side in the 9th row, what element in the 10th row occurs below and in between these numbers?

Technology active

3. **WE23** Use combinations to:
   a. calculate the number in the 7th position of the 8th row of Pascal's triangle
   b. calculate the number in the 9th position of the 12th row of Pascal's triangle
   c. generate the 10th row of Pascal's triangle.
4. **WE24** Use the binomial theorem to expand:
   a. $(x+y)^2$ b. $(n+m)^3$ c. $(a+3)^4$
5. **WE25** a. What is the 4th term in the expansion of $(x+2)^5$?
   b. What is the 3rd term in the expansion of $(p+q)^8$?
   c. What is the 7th term in the expansion of $(x+2)^9$?
6. **MC** A row of Pascal's triangle is given below. What number is located at position x?

$$\underset{x}{}$$
$$1 \;\; 9 \;\; 36 \;\; 84 \;\; 126 \;\; 126 \;\; 84 \;\; 36 \;\; 9 \;\; 1$$

   A. 8 **B.** 28 **C.** 45 **D.** 120
7. **MC** $16x^3$ is definitely a term in the binomial expansion of:
   A. $(x+2)^3$ **B.** $(x+4)^3$ **C.** $(x+2)^4$ **D.** $(x+4)^4$
8. a. In Pascal's triangle, calculate the sum of all elements in the:
      i. 0th row ii. 1st row iii. 2nd row
      iv. 3rd row v. 4th row vi. 5th row
   b. i. What do you notice?
      ii. Complete the statement: 'The sum of the elements in the nth row of Pascal's triangle is ...'
9. Use the result from question 8 to deduce a simple way of calculating:

$${}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$$

10. **WE26** In a cricket team consisting of 11 players, show that there are at least 2 whose phone numbers have the same last digit.
11. **WE27** A squad of 10 netballers is asked to nominate when they can attend training. They can choose Tuesday only, Thursday only or Tuesday and Thursday. Show that there is at least one group of at least 3 players who agree with one of these options.
12. J&L lollies come in five great colours — green, red, brown, yellow and blue. How many J&Ls do I need to select to be sure I have 6 of the same colour?

13. The new model WBM roadster comes in burgundy, blue or yellow with white or black trim. That is, the vehicle can be burgundy with white or burgundy with black and so on. How many vehicles need to be chosen to ensure at least 3 have the same colour combination?
14. Is it possible to show that in a group of 13 people, there are at least 2 whose birthdays fall in February?
15. **WE28** Nineteen netball teams entered the annual state championships. However, it rained frequently and not all games were completed. No team played the same team more than once. Mrs Organisit complained that the carnival was ruined and that no two teams had played the same number of games. Show that she is incorrect in at least part of her statement and that at least two teams played the same number of games.

16. Prove that in any group of 6 people either at least 3 are mutual friends or at least 3 are strangers.

1.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. a. State the inclusion–exclusion principle for 3 sets, R, S and T.
   b. Use the inclusion–exclusion principle to calculate the number of cards in a deck of 52 cards that are either red, a jack or a court card (king, queen or jack).
2. a. State the multiplication principle.
   b. One or two letters are to be chosen from the letters A, B, C, D, E and F. In how many different ways can this be done without replacement, if order is important?
3. a. State the definition of ${}^{n}\mathrm{P}_{r}$.
   b. Without a calculator, compute the value of ${}^{10}\mathrm{P}_{3}$.
   c. Prove that ${}^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!}$.
4. A free-style snowboard competition has 15 entrants. In how many ways can the first, second and third places be filled? You may wish to use technology to answer this question.
5. a. Suppose 5 people are to be seated. Explain why there are fewer ways of seating 5 people at a circular table compared with seating the group on a straight bench.
   b. How many ways can 5 people be seated at a round table?
6. The main cricket ground in Brisbane is called the Gabba. It is short for Woolloongabba. How many different arrangements of letters can be made from the word WOOLLOONGABBA? You may wish to use technology to answer the question.
7. Apply the concept of ${}^{n}\mathrm{C}_{r}$ to calculate the number of ways 12 different ingredients can be chosen from a box of 30 different ingredients. What can you conclude about the ingredients left behind? Do not use algebra to explain this.
8. A committee of 5 men and 5 women is to be chosen from 8 men and 9 women. In how many ways can this be done?
9. A netball team of 7 players is to be chosen from a squad of 11 players. Suppose any squad member can play any position. In how many ways can this be done:
   a. if each player is chosen to play a particular position
   b. If players have no particular position?

   You may choose to use technology to answer this question.

10. A ward in a city hospital has 15 nurses due to work Friday. There are 3 shifts needed to be staffed by 5 nurses on each shift. How many ways can this be done assuming each nurse works only one shift? (The order of the nurses on each shift and the order of each shift are not important.)

11. a. In your own words explain how to construct Pascal's triangle.

b. Where in Pascal's triangle would you find the number 56, which is equal to ${}^{8}C_{3}$?

12. a. Give an example of the pigeon-hole principle.

b. Jock has black, blue and beige socks. How many socks does he need to have to be sure that at least one pair are the same colour? Justify your answer.

Complex familiar

13. Assuming that car number plates are sequenced as follows: DLV334 → DLV335 → … DLV999 → DLW000 and so on. Using this sequence, how many number plates are there between DLV334 and DNU211 inclusive?

14. Jane, Laura and Steff are in a netball squad of 13 players. The team only takes 9 players to games. Jane, Laura and Steff's parents share the driving if they can: if 2 or more of the girls are playing, one parent drives them to the game and another parent picks them up. The parents do not share the driving if only one of the girls is playing. For what proportion of games would you predict the girls' parents are able to share the driving?

15. a. Show algebraically that ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r-1}$.

b. Interpret this result in terms of Pascal's triangle.

16. Consider the expansion $(x + y + z)^5 = x^5 + 5x^4y + 5x^4z + \ldots$
Use your knowledge of permutations involving identical objects to determine the coefficient of the term x^2yz^2. Recognize that this term could be written as $xxyzz$, $xzyzx$ and so on.

Complex unfamiliar

17. a. A school uses identification (ID) cards that consist of two letters from A to D followed by 3 digits chosen from 0 to 9. Each digit may be repeated but letters cannot be repeated. If the school receives about 800 new students each year, after how many years will they run out of unique ID numbers?

b. For a scene in a movie, five boy–girl couples are needed. If they are to be selected from 10 boys and 12 girls, in how many ways can this be done? (Assume the order of the couples does not matter.)

18. Juan has a bike lock with a 4-digit key but has forgotten the code. He knows that he used either the year he bought the bike, 2016, or the current year, 2019. Also, he jumbled the digits, so 2016 and 2019 are not possible keys. How many different possible keys are there to unlock the bike?

19. a. Consider the first three rows of Pascal's triangle.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

The sum of the entries of the first three rows of Pascal's triangle is 7 (= 1 + 1 + 1 + 1 + 2 + 1). How many rows of the triangle are needed for the sum to exceed 20 000? Justify your result.

b. The inclusion–exclusion principle for three sets R, S and T states:
$n(R \cup S \cup T) = n(R) + n(S) + n(T) - n(R \cap S) - n(R \cap T) - n(S \cap T) + n(R \cap S \cap T)$
Generalise this principle to four sets, Q, R, S and T.

20. A plane is covered in points at 1-unit spacing. Each point on the plane is coloured red, blue or white. Show there are three points of the same colour at a maximum distance $2\sqrt{2}$ from each other.

studyon

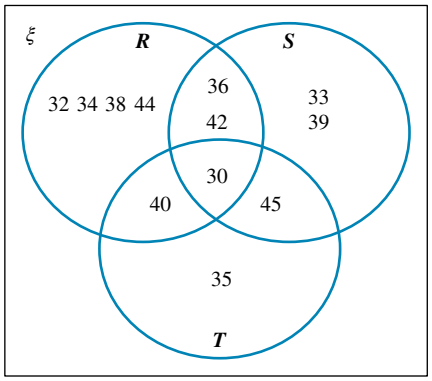
Units 1 & 2 Sit chapter test

Answers

Chapter 1 Permutations and combinations

Exercise 1.2 Counting techniques

1. a.



b. $n(R \cup S \cup T) = 12$
c. $n(R \cup S \cup T) = 12$

2. $n(\text{red, even or 4}) = 36$

3. 55 students

4. a. AB BA CA
AC BC CB **b.** 6 **c.** $\frac{1}{3}$

5. BG GB YB RB
BY GY YG RG
BR GR YR RY

6. ACB BAC CAB
ABC BCA CBA

7. a. 42 **b.** 210 **c.** 840 **d.** 2520 **e.** $\frac{1}{7}$

8. a. 24 **b.** 6 **c.** 12 **d.** 24

9. a. 49 **b.** 252

10. 126

11. a. 200 **b.** 40 **c.** 50 **d.** 290 **e.** $\frac{1}{40}$

12. a. 13 230
b. 17 640
Jake may wear 13 230 outfits with a jacket or 4410 outfits without a jacket. Therefore he has a total of 17 640 outfits to choose from. The assumption made with this problem is that no item of clothing is exactly the same; that is, none of the 7 shirts are exactly the same.

13. C

14. D

15. 100

16. 6

17. 48

18. 256

19. 1080

20. a. 1000
b. 27
c.

271	371	471
272	372	472
273	373	473
281	381	481
282	382	482
283	383	483
291	391	491
292	392	492
293	393	493

Exercise 1.3 Factorials and permutations

1. a. $4 \times 3 \times 2 \times 1$
b. $5 \times 4 \times 3 \times 2 \times 1$
c. $6 \times 5 \times 4 \times 3 \times 2 \times 1$
d. $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

2. a. 3024 **b.** 151 200 **c.** 840 **d.** 720

3. a. $n(n-1)(n-2)(n-3)(n-4)$
b. $(n+3)(n+2)$
c. $\dfrac{1}{n(n-1)(n-2)}$
d. $\dfrac{1}{(n+2)(n+1)n(n-1)}$

4. a. 24 **b.** 120
c. 720 **d.** 3 628 800
e. $8.717\,829\,12 \times 10^{10}$ **f.** 362 880
g. 5040 **h.** 6

5. a. $8 \times 7 = 56$
b. $7 \times 6 \times 5 \times 4 \times 3 = 2520$
c. $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 40\,320$

6. a. $\dfrac{9!}{3!} = 60\,480$ **b.** $\dfrac{5!}{3!} = 20$ **c.** $\dfrac{18!}{13!} = 1\,028\,160$

7. a. 27 907 200 **b.** 639 200 **c.** 1 028 160

8. a. 56 **b.** $\frac{1}{8}$

9. 358 800

10. 120

11. 3024

12. 2184

13. a. $\dfrac{{}^{15}P_4}{4}$ **b.** 8190 **c.** $\frac{1}{16}$

14. 3360

15. 362 880

16. 479 001 600

17. D

18. D

19. a. ${}^5P_3 = 60$ **b.** ${}^5P_4 = 120$ **c.** ${}^5P_5 = 120$

Exercise 1.4 Permutations with restrictions

1. a. ${}^6P_6 = 720$ **b.** $\dfrac{{}^6P_6}{2} = 360$

2. 83 160

3. 10

4. 1260

5. 27 720

6. 1 307 504

7. **a.** 5.45×10^{10} **b.** 3.63×10^{9}
   c. 4.00×10^{10} **d.** 1.45×10^{10}

8. **a.** 120 **b.** 20 **c.** 60 **d.** 60

9. **a.** 30 240 **b.** 3024 **c.** 6720 **d.** 15 120

10. **a.** 120 **b.** 48 **c.** 72 **d.** $\frac{3}{5}$

11. **a.** 1680 **b.** 180 **c.** 360 **d.** 840

12. **a.** 1320 **b.** 110

13. **a.** 80 640 **b.** 282 240
    c. 119 750 440 **d.** $\frac{1}{6}$

14. B

15. C

16. **a.** 720 **b.** 24, OYSTER

Exercise 1.5 Combinations

1. **a.** $\frac{{}^{8}P_{3}}{3!}$ **b.** $\frac{{}^{19}P_{2}}{2!}$ **c.** $\frac{{}^{1}P_{1}}{1!}$ **d.** $\frac{{}^{5}P_{0}}{0!}$

2. **a.** ${}^{8}C_{2}$ **b.** ${}^{9}C_{3}$ **c.** ${}^{8}C_{0}$ **d.** ${}^{10}C_{4}$

3. **a.** 1 **b.** 20 **c.** 120 **d.** 220

4. 100

5. **a.** 28 **b.** 120

6. D

7. C

8. 1365

9. 252

10. **a.** 495 **b.** 11 **c.** 1
    d. 54 264 **e.** 100 **f.** 680

11. **a.** 2 598 960 **b.** 65 780
    c. 65 780 **d.** 2 467 400

12. **a.** 2002 **b.** 840 **c.** 126 **d.** $\frac{36}{91}$

13. 59 400

14. 201 376

15. **a.** 120 **b.** 10 days

16. **a.** 57 **b.** 4 days 6 hours

17. **a.** 15 **b.** 1 day 4 hours

18. **a.** 8 145 060 **b.** 11 480

19. **a.** **i.** 220, 220 **ii.** 6435, 6435
    iii. 10, 10 **iv.** 56, 56
    b. The value of ${}^{n}C_{r}$ is the same as ${}^{n}C_{n-r}$.

Exercise 1.6 Applications of permutations and combinations

1. **a.** 45 **b.** 120 **c.** 210 **d.** 105

2. **a.** 720 **b.** 10 **c.** 120 **d.** 2.4×10^{18}

3. 24

4. 24

5. **a.** 210 **b.** 126
   c. 84 **d.** 140

6. **a.** 126 **b.** 56 **c.** 21 **d.** 70

7. D

8. **a.** 28
   b. Sample responses include: 9 17 27 41 12 14
   9 17 27 41 12 37 9 17 27 41 12 34
   c. 6

9. **a.** 84
   b. Sample responses include: 7 15 25 32 10 12
   7 15 25 32 10 35 7 15 25 32 10 37
   c. 10

10. 376 992

11. **a.** 8008 **b.** 5005 **c.** 5005 **d.** 4004

12. D

13. **a.** 38 760 **b.** 1140 **c.** 34 220 **d.** 39 010 800

14. **a.** 1
    b. **i.** 38 760 **ii.** 50 063 860
    iii. 1 940 475 213 600 **iv.** 4 191 844 505 805 495

Exercise 1.7 — Pascal's triangle, and the pigeon-hole principle

1. **Row**

0							1							
1						1		1						
2					1		2		1					
3				1		3		3		1				
4			1		4		6		4		1			
5		1		5		10		10		5		1		
6	1		6		15		20		15		6		1	
7		7		21		35		35		21		7		1
8	1		8		28		56		70		56		28		8		1

2. **a.** 70
   b. 1 9 36 84 126 126 84 36 9 1
   c. 45

3. **a.** 8
   b. 220
   c. ${}^{10}C_{0}$ ${}^{10}C_{1}$ ${}^{10}C_{2}$ ${}^{10}C_{3}$ ${}^{10}C_{4}$ ${}^{10}C_{5}$ ${}^{10}C_{6}$ ${}^{10}C_{7}$ ${}^{10}C_{8}$ ${}^{10}C_{9}$ ${}^{10}C_{10}$
   1 10 45 120 210 252 210 120 45 10 1

4. a. $x^2 + 2xy + y^2$

b. $n^3 + 3n^2m + 3nm^2 + m^3$

c. $a^4 + 12a^3 + 54a^2 + 108a + 81$

5. a. $80x$ **b.** $56p^5q^3$ **c.** $4608x^2$

6. B

7. D

8. a. i. 1 **ii.** 2 **iii.** 4
iv. 8 **v.** 16 **vi.** 32

b. i. The sum of the elements in each row of Pascal's triangle is a power of 2:

Row	Sum
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$
5	$2^5 = 32$

ii. The sum of the elements in the nth row of Pascal's triangle is 2^n.

9. $2^6 = 64$

10, 11. Sample responses can be found in the worked solutions in the online resources.

12. 26

13. 13

14–16. Sample responses can be found in the worked solutions in the online resources.

1.8 Review: exam practice

1. a. $n(R \cup S \cup T) = n(R) + n(S) + n(T) - n(R \cap S) - n(R \cap T) - n(S \cap T) + n(R \cap S \cap T)$

b. n (red, court card or jack) = 32

2. a. If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B in the order AB.

b. 64 ways

3. a. ${}^nP_r$ is the number of ways of choosing r objects from n distinct things, when order is important. ${}^nP_r = \dfrac{n!}{(n-r)!}$

b. ${}^{10}P_3 = 720$

c. ${}^nP_r = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$

$$= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

4. ${}^{15}P_3 = 2730$

5. a. When people are sitting in a circle, we cannot tell the difference between arrangements such as between {A, B, C, D, E} and {C, D, E, A, B}. In a circle, these represent the same arrangement. Therefore, there are fewer ways to arrange people in a circle than in a straight line.

b. 24

6. 32 432 400

7. ${}^{30}C_{12} = {}^{30}C_{30-12} = {}^{30}C_{18} = 86\,493\,225$
Each separate time we choose r things from n distinct things, we also leave $n - r$ objects. (We can interchange taking and leaving.) Hence, ${}^nC_r = {}^nC_{n-r}$

8. 7056

9. a. 1 663 200 **b.** 330

10. 756 756

11. a. In Pascal's triangle, the first two rows (the 0 row and the 1 row) contain only 1s. In subsequent rows, each row begins and ends with a 1, and each other term is the sum of the two terms above it. Row 2 is 1 2 1.

b. Row 8, position 3 (and 5)

12. a. Sample response: In a club with 367 members, there will be at least 2 people with the same birthday (assuming 366 days in a year for leap years).

b. 4 socks. Jock has socks of three different colours. If he selects 4 socks, which is one more than the number of types of socks he has, he would, by the pigeon-hole principle, have at least one matching pair.

13. 50 878

14. 540 out of 715

15. Sample responses can be found in the worked solutions in the online resources.

16. 30

17. a. 15 years **b.** 199 584

18. 46 possible combinations

19. a. 15 rows

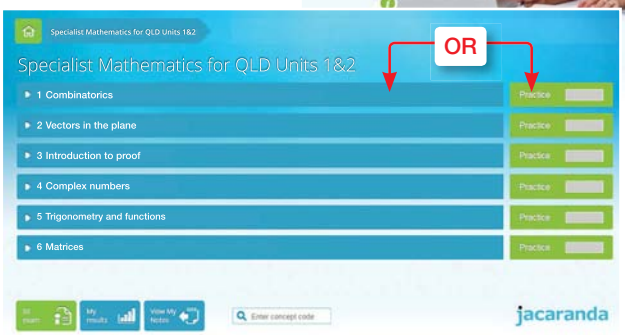
b. $n(Q \cup R \cup S \cup T)$
$= n(Q) + n(R) + n(S) + n(T) - n(Q \cap R) - n(Q \cap s)$
$-n(Q \cap T) - n(R \cap S) - n(R \cap T) - n(S \cap T)$
$+ n(Q \cap R \cap S) + n(Q \cap R \cap T) + n(Q \cap S \cap T)$
$+ n(R \cap S \cap T) - n(Q \cap R \cap S \cap T)$

20. Sample responses can be found in the worked solutions in the online resources.

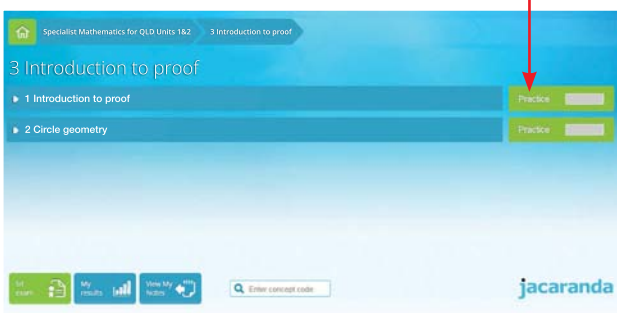
TOPIC 1 Combinatorics

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

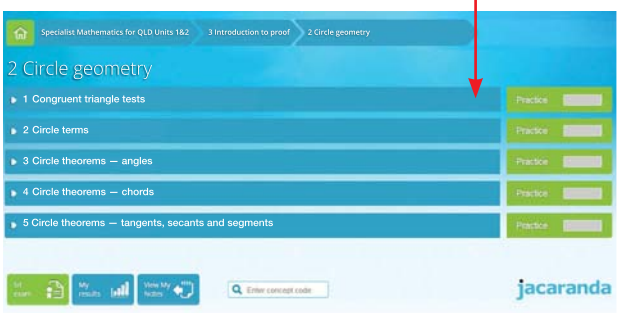
- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



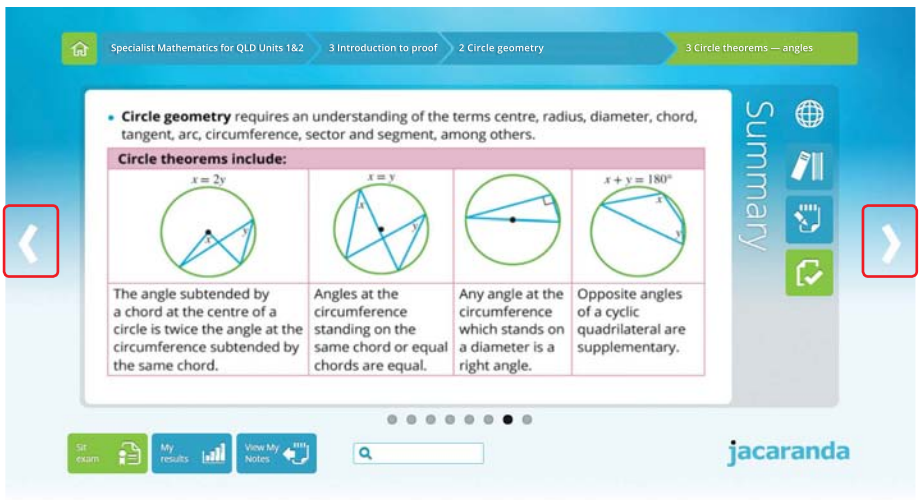
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



CHAPTER 2
Vectors in the plane

2.1 Overview

2.1.1 Introduction

A vector is a directed line segment that is described using both magnitude and direction. The length of the line segment is the magnitude of the vector and the direction is determined by a reference to the angle from a fixed line. Wind is a vector as it is described using magnitude and direction, for example a north-easterly wind of 20 km/h.

Consider the two yachts shown in the photograph. Could they be in danger of a collision? One yacht is travelling at a speed of 15 km/h and the other at a speed of 12 km/h. Their speeds give no indication of their direction. Their respective directions are as important as their speed for a correct analysis their path.

Vectors are used in physics to describe and analyse any quantity that has both magnitude and direction, such as motion and force. Vectors can be used by coaches in many sporting areas to illustrate the importance of the angle of contact with the ball or the optimum position to kick a goal. For example, by watching film clips with vectors superimposed over their golf swings, professional golfers can determine the perfect angle to hold their club to ensure the best outcome.

LEARNING SEQUENCE

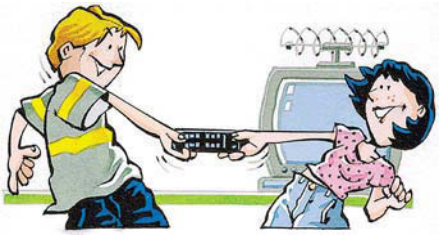
2.1 Overview
2.2 Vectors and scalars
2.3 Position vectors in the plane
2.4 Scalar multiplication
2.5 The scalar (dot) product
2.6 The projection of vectors — scalar and vector resolutes
2.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Vectors and scalars

2.2.1 Introduction

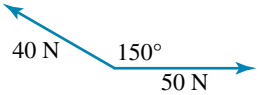
In mathematics, one of the important distinctions is between **scalar** quantities and **vector** quantities. Scalar quantities have **magnitude** only; vector quantities have *direction as well as magnitude*. Most of the quantities that we use are scalar, and include such measurements as time (for example 1.2 s; 15 min), mass (3.4 kg; 200 t) and area (3 cm^2; 400 ha). For some quantities it is very important that we know both magnitude and direction.



Consider the force involved in Daniel and Anna fighting over who gets to use the television remote control.

Daniel exerts a force of 40 N and Anna exerts a force of 50 N and they apply these forces as shown.

In what direction will the remote control move and what is the force in that direction? That is, what is the resultant force?



The resultant force depends not only on the size of each force but the direction in which the forces are applied.

> **A vector is a quantity that has magnitude and direction.**
> **A scalar is a real number only.**

2.2.2 Vector notation

A vector is shown graphically as a line, with a *tail* (start) and *head* (end). The length of the line indicates the magnitude and the orientation of the line indicates its direction.

In the figure, the head of the vector is at point B (indicated with an arrow), and the tail is at point A.

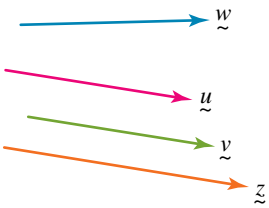
When writing this vector, we can use the points A and B to indicate the start and end points with a special arrow to indicate that it is a vector: $\overrightarrow{AB}$. Some textbooks use a single letter, in bold, such as $\boldsymbol{w}$, but this is difficult to write using pen and paper, so $\underset{\sim}{w}$ can also be used. The symbol (~) is called a *tilde*.

2.2.3 Equality of vectors

Vectors are defined by *both* magnitude and direction.

> **Two vectors are equal if *both* their magnitude and direction are equal.**

In the figure, the following statements can be made:



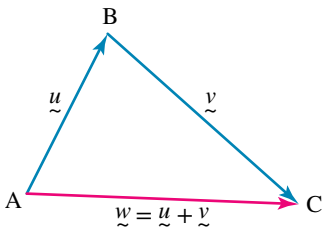
$\underset{\sim}{u} = \underset{\sim}{v}$

$\underset{\sim}{u} \neq \underset{\sim}{w}$ (directions are not equal)

$\underset{\sim}{u} \neq \underset{\sim}{z}$ (magnitudes are not equal).

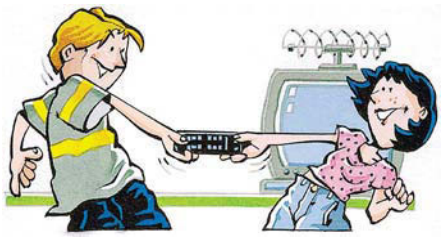
2.2.4 Addition of vectors — The triangle rule

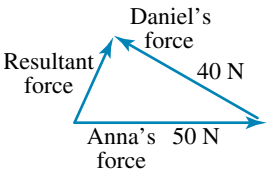
Consider a vector, $\underset{\sim}{u}$, that measures the travel from A to B and another vector, $\underset{\sim}{v}$, that measures the subsequent travel from B to C. The net result is as if the person travelled directly from A to C (vector $\underset{\sim}{w}$). Therefore, we can say that $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v}$.



To add two vectors, take the tail of one vector and join it to the head of another. The result of this addition is the vector from the *tail of the first vector* to the *head of the second vector*.

Returning to Daniel and Anna who are fighting over the television remote control, we see that the forces they apply to the remote control unit can be represented as a sum of two vectors.

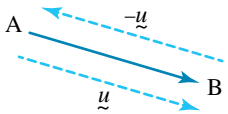




From this figure we are able to get a rough idea of the magnitude and direction of the resultant force. In the following sections, we will learn techniques for calculating the resultant magnitude and direction accurately.

2.2.5 The negative of a vector

If $\underset{\sim}{u}$ is the vector from A to B, then $-\underset{\sim}{u}$ is the vector from B to A.

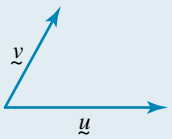


We can subtract vectors by adding the *negative* of the second vector to the first vector.

WORKED EXAMPLE 1

Using the vectors shown, draw the results of:

a. $\underset{\sim}{u} + \underset{\sim}{v}$ **b.** $-\underset{\sim}{u}$ **c.** $\underset{\sim}{u} - \underset{\sim}{v}$ **d.** $\underset{\sim}{v} - \underset{\sim}{u}$.



THINK

a. 1. Move $\underset{\sim}{v}$ so that its tail is at the head of $\underset{\sim}{u}$.

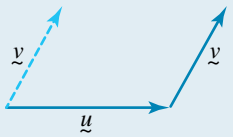
2. Join the tail of $\underset{\sim}{u}$ to the head of $\underset{\sim}{v}$ to find $\underset{\sim}{u} + \underset{\sim}{v}$.

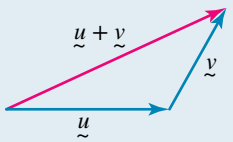
b. Reverse the arrow on $\underset{\sim}{u}$ to obtain $-\underset{\sim}{u}$.

c. 1. Reverse $\underset{\sim}{v}$ to get $-\underset{\sim}{v}$.

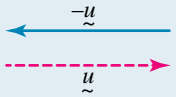
WRITE

a.

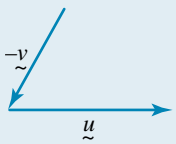




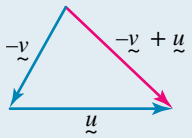
b.



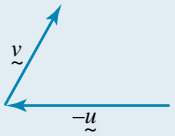
c.



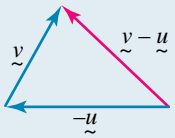
2. Join the tail of $-\underset{\sim}{v}$ to the head of $\underset{\sim}{u}$ to get $-\underset{\sim}{v} + \underset{\sim}{u}$, which is the same as $\underset{\sim}{u} - \underset{\sim}{v}$ or $\underset{\sim}{u} + (-\underset{\sim}{v})$.



d. 1. Reverse $\underset{\sim}{u}$ to get $-\underset{\sim}{u}$. The vectors are now 'aligned properly' with the head of $-\underset{\sim}{u}$ joining the tail of $\underset{\sim}{v}$.



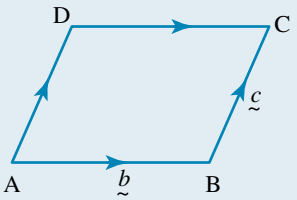
2. Join the tail of $-\underset{\sim}{u}$ to the head of $\underset{\sim}{v}$ to get $\underset{\sim}{v} - \underset{\sim}{u}$. Note that this is the same as $(-\underset{\sim}{u} + \underset{\sim}{v})$



WORKED EXAMPLE 2

The parallelogram ABCD can be defined by the two vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$. In terms of these vectors, find:

a. the vector from A to D
b. the vector from C to D
c. the vector from D to B.



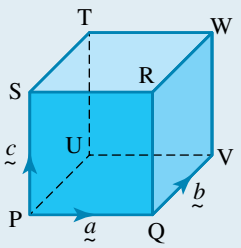
THINK	WRITE
a. The vector from A to D is equal to the vector from B to C since ABCD is a parallelogram.	**a.** $\overrightarrow{AD} = \underset{\sim}{c}$
b. The vector from C to D is equal to the vector from B to A and is the reverse of B to A, which is $\underset{\sim}{b}$.	**b.** $\overrightarrow{CD} = -\underset{\sim}{b}$
c. The vector from D to B is obtained by adding the vector from D to A to the vector from A to B.	**c.** $\overrightarrow{DB} = -\underset{\sim}{c} + \underset{\sim}{b}$ $= \underset{\sim}{b} - \underset{\sim}{c}$

WORKED EXAMPLE 3

A cube PQRSTUVW can be defined by the three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ as shown.

Express in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$:

a. the vector joining P to V
b. the vector joining P to W
c. the vector joining U to Q
d. the vector joining S to W
e. the vector joining Q to T.



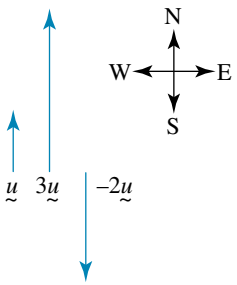
THINK	WRITE
All of the opposite sides in a cube are equal in length and parallel. Therefore all opposite sides can be expressed as the same vector.	
a. The vector from P to V is obtained by adding the vector from P to Q to the vector from Q to V.	**a.** $\overrightarrow{PV} = \overrightarrow{PQ} + \overrightarrow{QV}$ $\overrightarrow{PV} = \underset{\sim}{a} + \underset{\sim}{b}$
b. The vector from P to W is obtained by adding the vectors P to V and V to W.	**b.** $\overrightarrow{PW} = \overrightarrow{PV} + \overrightarrow{VW}$ $\overrightarrow{PW} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$
c. The vector from U to Q is obtained by adding the vectors U to P and P to Q.	**c.** $\overrightarrow{UQ} = \overrightarrow{UP} + \overrightarrow{PQ}$ $\overrightarrow{UQ} = -\underset{\sim}{b} + \underset{\sim}{a}$ $= \underset{\sim}{a} - \underset{\sim}{b}$
d. The vector from S to W is obtained by adding the vectors S to R and R to W.	**d.** $\overrightarrow{SW} = \overrightarrow{SR} + \overrightarrow{RW}$ $\overrightarrow{SW} = \underset{\sim}{a} + \underset{\sim}{b}$
e. The vector from Q to T is obtained by adding the vectors Q to P, P to S and S to T.	**e.** $\overrightarrow{QT} = \overrightarrow{QP} + \overrightarrow{PS} + \overrightarrow{ST}$ $\overrightarrow{QT} = -\underset{\sim}{a} + \underset{\sim}{c} + \underset{\sim}{b}$ $= \underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}$

2.2.6 Multiplying a vector by a scalar

Multiplication of a vector by a *positive* number (scalar) affects only the *magnitude* of the vector, not the direction. For example, if a vector $\underset{\sim}{u}$ has a direction of north and a magnitude of 10, then the vector $3\underset{\sim}{u}$ has a direction of north and magnitude of 30.

If the scalar is *negative*, then the direction is reversed. Therefore, $-2\underset{\sim}{u}$ has a direction of south and a magnitude of 20.

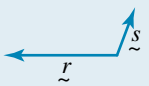
Scalar multiples of a vector are all parallel as multiplication by a positive scalar only affects the magnitude of a vector, and multiplication by a negative scalar only affects the vector's magnitude and reverses its direction.



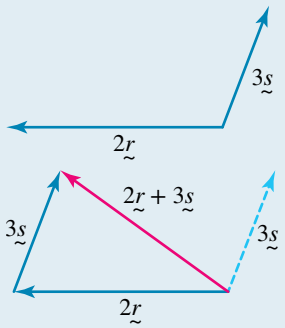
WORKED EXAMPLE 4

Use the vectors shown at right to draw the result of:

a. $2\underset{\sim}{r} + 3\underset{\sim}{s}$ **b.** $2\underset{\sim}{s} - 4\underset{\sim}{r}$.



THINK	WRITE
a. 1. Increase the magnitude of $\underset{\sim}{r}$ by a factor of 2 and $\underset{\sim}{s}$ by a factor of 3.	**a.**
2. Move the tail of $3\underset{\sim}{s}$ to the head of $2\underset{\sim}{r}$. Then join the tail of $2\underset{\sim}{r}$ to the head of $3\underset{\sim}{s}$ to get $2r + 3\underset{\sim}{s}$.	

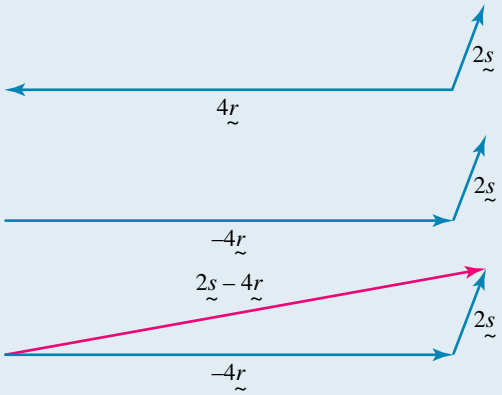


b. 1. Increase the magnitude of $\underset{\sim}{s}$ by a factor of 2 and $\underset{\sim}{r}$ by a factor of 4.

2. Reverse the arrow on $4\underset{\sim}{r}$ to get $-4\underset{\sim}{r}$.

3. Join the tail of $-4\underset{\sim}{r}$ to the head of $2\underset{\sim}{s}$.

b.



Vectors can be used to solve real-world problems involving movement and direction. Direction is typically referenced using a compass and expressed as the degrees from north. This is known as a bearing. We can then use Pythagoras' theorem and trigonometry to solve displacement and bearing problems through vector addition.

WORKED EXAMPLE 5

A boat travels 30 km north and then 40 km west.
a. Make a vector drawing of the path of the boat.
b. Draw the vector that represents the net displacement of the boat.
c. Determine the magnitude of the net displacement.
d. Calculate the bearing (from true north) of this net displacement vector.

THINK

a. 1. Set up vectors (tail to head), one pointing north, the other west.

2. Indicate the distances as 30 km and 40 km respectively.

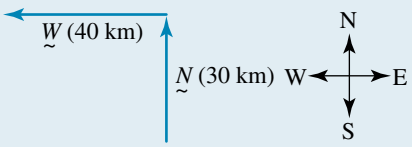
b. Join the tail of the $\underset{\sim}{N}$ vector with the head of the $\underset{\sim}{W}$ vector.

c. 1. Let R km = length of $\underset{\sim}{N} + \underset{\sim}{W}$.

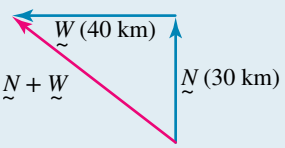
2. The length (magnitude) of $\underset{\sim}{R}$ can be calculated using Pythagoras' theorem.

WRITE

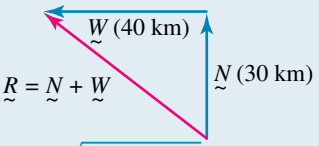
a.



b.



c.



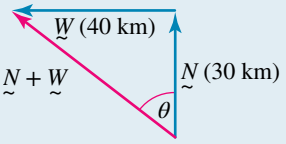
$R = \sqrt{30^2 + 40^2}$

$= \sqrt{900 + 1600}$

$= 50$ km

d. 1. Indicate the angle between $\underset{\sim}{N}$ and $\underset{\sim}{N} + \underset{\sim}{W}$ as θ.

d.



2. Use trigonometry to find θ, where the magnitude of the opposite side is 40. The hypotenuse, $\underset{\sim}{R}$, was determined in part **c** as 50.

$$\sin\theta = \frac{40}{50}$$
$$= 0.8$$
$$\theta = 53.13°$$

3. The true bearing is 360° minus 53.13°.

Therefore the true bearing is:
$360° - 53.13° = 306.87°$

Resources

Digital document: SkillSHEET Bearings (doc-26829)

studyon

Units 1 & 2 > Area 2 > Sequence 1 > Concepts 1 & 2

Introduction to vectors Summary screen and practice questions
Operations on vectors Summary screen and practice questions

Exercise 2.2 Vectors and scalars

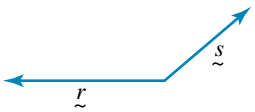
Technology free

1. a. WE1 Draw the result of:

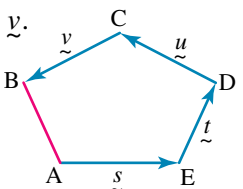
i. $\underset{\sim}{r} + \underset{\sim}{s}$ ii. $\underset{\sim}{r} - \underset{\sim}{s}$ iii. $\underset{\sim}{s} - \underset{\sim}{r}$

b. WE4 Draw the result of:

i. $2\underset{\sim}{r} + 2\underset{\sim}{s}$ ii. $2\underset{\sim}{r} - 2\underset{\sim}{s}$ iii. $3\underset{\sim}{s} - 4\underset{\sim}{r}$



2. WE2 The pentagon ABCDE at right can be defined by the four vectors, $\underset{\sim}{s}$, $\underset{\sim}{t}$, $\underset{\sim}{u}$ and $\underset{\sim}{v}$. Find in terms of these 4 vectors:

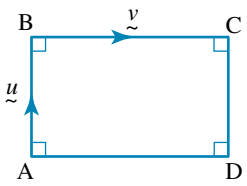


a. the vector from A to D **b.** the vector from A to B
c. the vector from D to A **d.** the vector from B to E
e. the vector from C to A.

3. MC A girl travels 4 km north and then 2 km south. What is the net displacement vector?

A. 6 km north **B.** 6 km south
C. 2 km north **D.** −2 km north

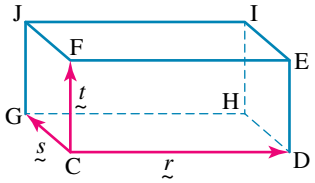
4. In the rectangle ABCD, the vector joining A to B is denoted by $\underset{\sim}{u}$ and the vector joining B to C is $\underset{\sim}{v}$. Which pairs of points are joined by:



   a. $\underset{\sim}{u} + \underset{\sim}{v}$?
   b. $\underset{\sim}{u} - \underset{\sim}{v}$?
   c. $\underset{\sim}{v} - \underset{\sim}{u}$?
   d. $3\underset{\sim}{u} + 2\underset{\sim}{v} - 2\underset{\sim}{u} - \underset{\sim}{v}$?

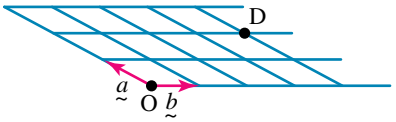
5. MC Consider the following relationships between vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$.
   $\underset{\sim}{u} = 2\underset{\sim}{v} + \underset{\sim}{w}$
   $\underset{\sim}{w} = \underset{\sim}{v} - \underset{\sim}{u}$
   Which of the following statements is true?
   A. $\underset{\sim}{u} = \underset{\sim}{w}$ **B.** $\underset{\sim}{u} = \underset{\sim}{v}$ **C.** $\underset{\sim}{u} = \frac{2}{3}\underset{\sim}{v}$ **D.** $\underset{\sim}{u} = \frac{3}{2}\underset{\sim}{v}$

6. WE3 A rectangular prism (box) CDEFGHIJ can be defined by three vectors $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$ as shown at right.

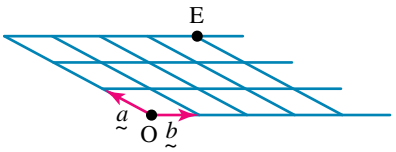


   Express in terms of $\underset{\sim}{r}$, $\underset{\sim}{s}$ and $\underset{\sim}{t}$:
   a. the vector joining C to H
   b. the vector joining C to J
   c. the vector joining G to D
   d. the vector joining F to I
   e. the vector joining H to E
   f. the vector joining D to J
   g. the vector joining C to I
   h. the vector joining J to C.

7. In terms of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the figure, define the vector joining O to D.



8. In terms of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, define the vector joining E to O.



9. Which of the following are vector quantities? Justify your answer.
   speed velocity displacement force volume angle
10. Which of the following are scalar quantities? Justify your answer.
    speed time acceleration velocity length displacement
11. A 2-dimensional vector can be determined by its length and its angle with respect to (say) true north. What quantities could be used to represent a 3-dimensional vector?

Technology active

12. A pilot plans to fly 300 km north then 400 km east.
    a. Make a vector drawing of her flight plan.
    b. Show the resulting net displacement vector.
    c. Calculate the length (magnitude) of this net displacement vector.
    d. Calculate the bearing (from true north) of this net displacement vecto

13. Another pilot plans to travel 300 km east, then 300 km north-east. Show that the resultant bearing is 67.5 degrees. Determine how far east of its starting point the plane has travelled, to 1 decimal place
14. An aeroplane travels 400 km west, then 600 km north. How far is the aeroplane from its starting point? Determine the bearing of the resultant displacement. Give your answers to 1 decimal place.

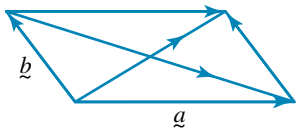
15. Using technology or a piece of graph paper, draw a vector, $\underset{\sim}{a}$, that is 3 units east and 5 units north of the origin. Draw another vector, $\underset{\sim}{b}$, that is 5 units east and 3 units north of the origin. On the same graph, draw the following vectors.

a. $\underset{\sim}{a} + \underset{\sim}{b}$ **b.** $\underset{\sim}{a} + 3\underset{\sim}{b}$ **c.** $\underset{\sim}{a} - \underset{\sim}{b}$ **d.** $\underset{\sim}{b} - \underset{\sim}{a}$ **e.** $3\underset{\sim}{b} - 4\underset{\sim}{a}$

f. $0.5\underset{\sim}{a} + 2.5\underset{\sim}{b}$ **g.** $\underset{\sim}{a} - 2.5\underset{\sim}{b}$ **h.** $4\underset{\sim}{a}$ **i.** $2.5\underset{\sim}{a} - 1.5\underset{\sim}{b}$ **j.** $\underset{\sim}{b} - 2.5\underset{\sim}{a}$

16. Find the direction and magnitude of a vector joining point A to point B, where B is 10 m east and 4 m north of A.

17. Consider a parallelogram defined by the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and its associated diagonals, as shown.



Show that the vector sum of the diagonal vectors is $2\underset{\sim}{a}$.

18. Show, *by construction*, that for any vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$:

$$3(\underset{\sim}{u} + \underset{\sim}{v}) = 3\underset{\sim}{u} + 3\underset{\sim}{v}$$

(This is called the *Distributive Law*.)

19. Show, *by construction*, that for any three vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$:

$$(\underset{\sim}{a} + \underset{\sim}{b}) + \underset{\sim}{c} = \underset{\sim}{a} + (\underset{\sim}{b} + \underset{\sim}{c})$$

(This is called the *Associative Law*.)

20. Show, *by construction*, that for any two vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$:

$$3\underset{\sim}{r} - \underset{\sim}{s} = -(\underset{\sim}{s} - 3\underset{\sim}{r})$$

21. A girl walks the following route: 400 m north — 300 m east — 200 m north — 500 m west — 600 m south — 200 m east.

Make a vector drawing of these six paths. Determine the net displacement vector.

22. As you will learn shortly, vectors can be represented by two values: the horizontal (or x) component and the vertical (or y) component.

Consider the vector $\underset{\sim}{w}$, defined by joining the origin to the point (4, 5), and the vector $\underset{\sim}{v}$, defined by joining the origin to (2, 3). Determine the horizontal and vertical components of each vector.

Demonstrate *graphically* that the sum $\underset{\sim}{w} + \underset{\sim}{v}$ has an x-component of 6 (that is, 4 + 2), and a y-component of 8 (that is, 5 + 3).

23. Using the same vectors, $\underset{\sim}{w}$ and $\underset{\sim}{v}$, as in question **22**, demonstrate *graphically* that the difference vector, $\underset{\sim}{w} - \underset{\sim}{v}$, has an x-component of 2 and a y-component of 2.

24. Using the same vectors, $\underset{\sim}{w}$ and $\underset{\sim}{v}$, as in question **22**, demonstrate *graphically* that:

a. the vector $4\underset{\sim}{w}$ has an x-component of 16 and a y-component of 20

b. the vector $-2\underset{\sim}{v}$ has an x-component of -4 and a y-component of -6.

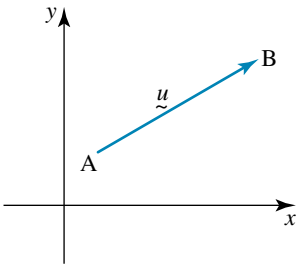
25. Using the results from questions **22**, **23** and **24**, what can you deduce about an *algebraic* method (as opposed to a graphical method) of addition, subtraction and multiplication of vectors?

2.3 Position vectors in the plane

2.3.1 Cartesian form of a vector

As a vector has both magnitude and direction, it can be represented in 2-dimensional planes.

In the figure, the vector $\underset{\sim}{u}$ joins the point A to point B.

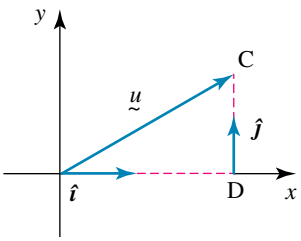


Using the Cartesian plane, an **identical vector** can be considered to join the origin with the point C.

It is easy to see that $\underset{\sim}{u}$ is made up of two components: one along the x-axis and one parallel to the y-axis. Let $\hat{\boldsymbol{\imath}}$ be a vector along the x-axis with magnitude 1. Similarly, let $\hat{\boldsymbol{\jmath}}$ be a vector along the y-axis with magnitude 1. Vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are known as **unit vectors**, and are discussed in section 2.3.5.

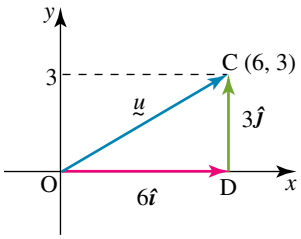
We can say the vector $\underset{\sim}{u}$ is the **position vector** of point C relative to the origin.

Note: Unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ may also be shown as $\underset{\sim}{i}$ and $\underset{\sim}{j}$ respectively.



With vectors, it is equivalent to travel along $\underset{\sim}{u}$ from the origin directly to C, or to travel first along the x-axis to D and then parallel to the y-axis to C. In either case we started at the origin and ended up at C. Clearly, then, $\underset{\sim}{u}$ is made up of some multiple of $\hat{\boldsymbol{\imath}}$ in the x-direction and some multiple of $\hat{\boldsymbol{\jmath}}$ in the y-direction.

For example, if the point C has coordinates (6, 3) then $\underset{\sim}{u} = 6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$. This is the **Cartesian form of a vector**.



The Cartesian form of a vector from the origin to the point (x, y) is given by:

$$\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$$

2.3.2 Ordered pair notation and column vector notation

The vector $\underset{\sim}{u} = 6\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ can be expressed as an ordered pair (6, 3) or in column vector notation as $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$.

Similarly, the vector $\underset{\sim}{u} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ can be expressed as an ordered pair (x, y) or in column vector notation as $\begin{bmatrix} x \\ y \end{bmatrix}$.

2.3.3 The magnitude of a vector

By using Pythagoras' theorem on a position vector, we can find its length, or *magnitude*.

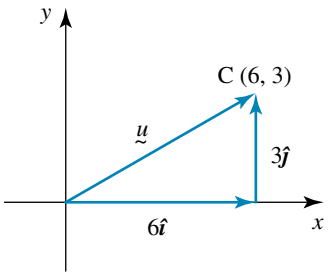
Consider the vector $\underset{\sim}{u}$ shown.

The magnitude of $\underset{\sim}{u}$, denoted as $|\underset{\sim}{u}|$ or r, is given by:

$$|\underset{\sim}{u}| = \sqrt{6^2 + 3^2}$$

$$|\underset{\sim}{u}| = \sqrt{45}$$

$$= 3\sqrt{5}$$

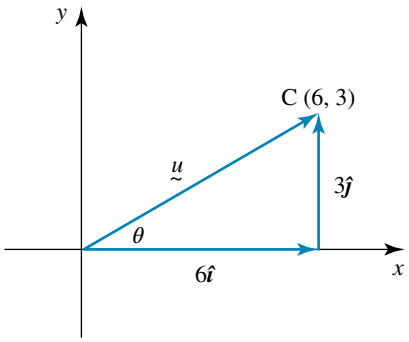


The magnitude of a vector, $\underset{\sim}{u} = x\hat{i} + y\hat{j}$, is given by:

$$r = |\underset{\sim}{u}| = \sqrt{x^2 + y^2}$$

2.3.4 The polar form of a vector

From what we already know about trigonometry, we can work out the angle (θ) that $\underset{\sim}{u}$ makes with the positive x-axis (that is, anticlockwise from the positive x-axis). This gives us the direction of $\underset{\sim}{u}$.



The angle θ can be calculated as:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{3}{6}\right)\\ &= \tan^{-1} 0.5\\ &= 0.464 \text{ radians}\\ &= 26.6°\end{aligned}$$

The result obtained by this method needs to be adjusted if the angle is in the 2nd, 3rd, or 4th quadrants. This allows us to express the vector in **polar form** as $\left[3\sqrt{5},\ 26.6°\right]$.

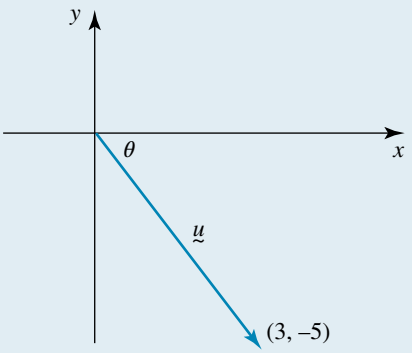
The *direction* of a vector, $\underset{\sim}{u} = x\hat{i} + y\hat{j}$, is given by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
The polar form of vector $\underset{\sim}{u}$ is $[r, \theta]$,

where $\underset{\sim}{u}$ is a vector of magnitude r in the direction of θ from the positive direction of the x-axis.

WORKED EXAMPLE 6

Using the vector shown, determine:

a. **the magnitude of $\underset{\sim}{u}$**
b. **the direction of $\underset{\sim}{u}$ (express the angle with respect to the positive x-axis)**
c. **the true bearing of $\underset{\sim}{u}$.**
d. **the expression of $\underset{\sim}{u}$ in polar form.**



THINK

a. 1. Use Pythagoras' theorem or the rule for magnitude of a vector with the x- and y-components 3 and -5 respectively.
2. Simplify the surd.

WRITE

a. $|\underset{\sim}{u}| = \sqrt{3^2 + (-5)^2}$.

$|\underset{\sim}{u}| = \sqrt{9 + 25}$

$= \sqrt{34}$ (= 5.831 to 3 decimal places)

b. 1. The angle is in the 4th quadrant since $x = 3$ and $y = -5$. Use trigonometry to find the angle θ from the x- and y-component values, recalling $\tan\theta = \frac{\text{opp}}{\text{adj}}$.

b. $\theta = \tan^{-1}\left(\frac{-5}{3}\right)$

2. Use a calculator to simplify.

$\theta = -59°$

c. The negative sign implies that the direction is 59° clockwise from the x-axis. The true bearing from north is the angle measurement from the positive y-axis to the vector $\underset{\sim}{u}$.

c. True bearing $= 90° + 59°$
$= 149°$

d. 1. Recall the polar form of a vector is $[r, \theta]$.

d. $[r, \theta]$

2. Write $\underset{\sim}{u}$ in polar form.

$\underset{\sim}{u}$ is a vector of magnitude $\sqrt{34}$ in a direction of $-59°$ from the positive direction of the x-axis. In polar form, $\underset{\sim}{u} = \left[\sqrt{34}, -59°\right]$.

TI | THINK

d. 1. Put the calculator in Degree mode.
On a Calculator page, complete the entry line as:
$\begin{bmatrix} 3 & -5 \end{bmatrix}$
Press MENU, then select:
7: Matrix & Vector
C: Vector
4: Convert to Polar
then press ENTER.
Note: The matrix template can be found by pressing the Templates button.

2. The answer appears on the screen.
Note: The calculator will give decimal values, not exact values.

WRITE

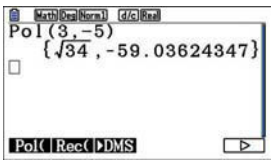
$\underset{\sim}{u} = (5.831, -59°)$

CASIO | THINK

d. 1. Put the calculator in Degree mode.
On the Run-Matrix screen, press OPTN, then press F6 to scroll across to more menu options. Select ANGLE by pressing F5, press F6 to scroll across to more menu options, then select Pol(by pressing F1. Complete the entry line as:
Pol (3, −5)
then press EXE.

2. The answer appears on the screen.

WRITE



$\underset{\sim}{u} = (\sqrt{34}, -59°)$

2.3.5 Unit vectors

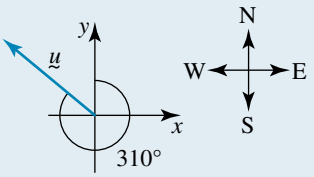
As we have seen, any vector $\underset{\sim}{u}$ is composed of x- and y- components denoted by $x\hat{\boldsymbol{\imath}}$, $y\hat{\boldsymbol{\jmath}}$. The vectors, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$ are called unit vectors, as they each have a magnitude of 1. This allows us to resolve a vector into its components.

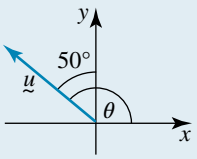
If a 2-dimensional vector $\underset{\sim}{u}$ makes an angle of θ with the positive x-axis and it has a magnitude of r, then we can find its x- and y-components using the formulas:

$$x = |r| \cos\theta$$
$$y = |r| \sin\theta$$

WORKED EXAMPLE 7

Consider the vector $\underset{\sim}{u}$, whose magnitude is 30 and whose bearing (from N) is 310°. Determine its x- and y-components and write $\underset{\sim}{u}$ in terms of $\hat{i}$ and $\hat{j}$, that is, in Cartesian form.



THINK	WRITE
1. Change the bearing into an angle with respect to the positive x-axis (θ).	
2. The angle between $\underset{\sim}{u}$ and the positive y-axis is $360° - 310° = 50°$.	
3. Calculate θ.	$\theta = 90° + 50°$ $= 140°$
4. Find the x- and y-components using trigonometry.	$x = \lvert r \rvert \cos\theta$ $\quad y = \lvert r \rvert \sin\theta$ $= 30 \cos 140°$ $\quad = 30 \sin 140°$ $= -22.98$ $\quad = 19.28$
5. Express $\underset{\sim}{u}$ as a vector in Cartesian form.	$\underset{\sim}{u} = -22.98\hat{i} + 19.28\hat{j}$

TI | THINK

1. Determine the angle that $\underset{\sim}{u}$ makes with the positive x-axis.
2. Put the calculator in Degree mode.
   On a Calculator page, complete the entry line as: $(30\angle 140)$
   Press MENU, then select:
   7: Matrix & Vector
   C: Vector
   5: Convert to Rectangular
   then press ENTER.
   Note: The $\angle$ symbol can be found by pressing CTRL and the Catalogue button.
3. The answer appears on the screen. The first value is the x-component and the second value is the y-component.

WRITE

$\theta = 50° + 90° = 140°$

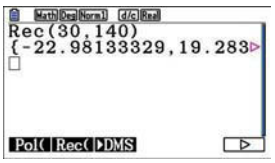
$\underset{\sim}{u} = -22.98\hat{i} + 19.28\hat{j}$

CASIO | THINK

1. Determine the angle that $\underset{\sim}{u}$ makes with the positive x-axis.
2. Put the calculator in Degree mode.
   On the Run-Matrix screen, press OPTN, then press F6 to scroll across to more menu options. Select ANGLE by pressing F5, press F6 to scroll across to more menu options, then select Rec(by pressing F2.
   Complete the entry line as: Rec (30, 140)
   then press EXE.
3. The answer appears on the screen. The first value is the x-component and the second value is the y-component.

WRITE

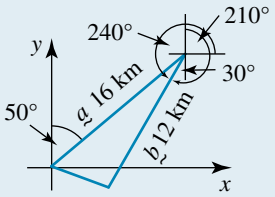
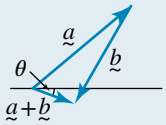
$\theta = 50° + 90° = 140°$



$\underset{\sim}{u} = -22.98\hat{i} + 19.28\hat{j}$

WORKED EXAMPLE 8

A bushwalker walks 16 km in a direction of bearing 050°, then walks 12 km in a direction of bearing 210°. Determine the resulting position of the hiker giving magnitude and direction from the starting point. State the resultant vector in polar form.

THINK	WRITE
1. Draw a clear diagram to represent the situation.	
2. Resolve the position vectors into their x- and y-components by recalling the component formulas, $x = \lvert r\rvert \cos\theta$ and $y = \lvert r\rvert \sin\theta$, where θ is the angle from the x-axis.	$\underset{\sim}{a} = 16\cos 40°\hat{\boldsymbol{\imath}} + 16\sin 40°\hat{\boldsymbol{\jmath}}$ $\underset{\sim}{b} = 12\cos 240°\hat{\boldsymbol{\imath}} + 12\sin 240°\hat{\boldsymbol{\jmath}}$
3. Simplify position vectors.	$\underset{\sim}{a} = 12.2567\hat{\boldsymbol{\imath}} + 10.2846\hat{\boldsymbol{\jmath}}$ $\underset{\sim}{b} = -6\hat{\boldsymbol{\imath}} - 10.3923\hat{\boldsymbol{\jmath}}$
4. Use the triangle rule of addition of vectors. 	$\underset{\sim}{a} + \underset{\sim}{b} = (12.2567 - 6)\hat{\boldsymbol{\imath}} + (10.2846 - 10.3923)\hat{\boldsymbol{\jmath}}$ $= 6.2567\hat{\boldsymbol{\imath}} - 0.1077\hat{\boldsymbol{\jmath}}$
5. Find the angle θ by recalling the direction formula, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.	$\theta = \tan^{-1}\left(\frac{-0.1077}{6.2567}\right)$ $= -0.986°$
6. Find the magnitude.	$\lvert\underset{\sim}{a} + \underset{\sim}{b}\rvert = \sqrt{x^2 + y^2}$ $= \sqrt{6.26^2 + (-0.11)^2}$ $= \sqrt{39.2}$ $= 6.26$
7. State the resultant vector polar form.	The bushwalker's final position is 6.26 km at an angle of $-0.986°$ from the starting point: $[6.26, -0.986°]$.

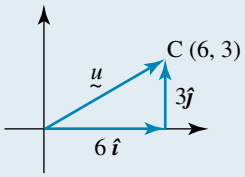
Unit vectors can also be found in the direction of *any* vector. This is merely the original vector divided by its magnitude.

The unit vector of any vector $\underset{\sim}{u}$ in the direction of $\underset{\sim}{u}$, denoted by $\hat{\underset{\sim}{u}}$, is:

$$\hat{\underset{\sim}{u}} = \frac{\underset{\sim}{u}}{\lvert\underset{\sim}{u}\rvert} = \frac{\underset{\sim}{u}}{r}$$

WORKED EXAMPLE 9

Determine the unit vector in the direction of $\underset{\sim}{u}$.
Confirm the unit vector has a magnitude of 1.



THINK	WRITE
1. Express the vector in component form.	$\underset{\sim}{u} = 6\hat{i} + 3\hat{j}$
2. Calculate the magnitude of the vector $\underset{\sim}{u}$ using Pythagoras' theorem.	$r = \sqrt{6^2 + 3^2}$ $= \sqrt{45}$ $= 3\sqrt{5}$
3. Divide each component of the original vector by the magnitude to get $\hat{\underset{\sim}{u}}$.	$\hat{\underset{\sim}{u}} = \frac{6}{3\sqrt{5}}\hat{i} + \frac{3}{3\sqrt{5}}\hat{j}$ $= \frac{2\sqrt{5}}{5}\hat{i} + \frac{\sqrt{5}}{5}\hat{j}$
4. Confirm that $\hat{\underset{\sim}{u}}$ has a magnitude of 1.	$\lvert\hat{\underset{\sim}{u}}\rvert = \sqrt{x^2 + y^2}$ $= \sqrt{\frac{20}{25} + \frac{5}{25}}$ $= \sqrt{\frac{25}{25}}$ $= 1$

TI | THINK

1. On a Calculator page, press MENU, then select:
   7: Matrix & Vector
   C: Vector
   1: Unit Vector
   Complete the entry line as:
   unitV $\left(\begin{bmatrix} 6 & 3 \end{bmatrix}\right)$
   then press ENTER.
   Note: The matrix template can be found by pressing the templates button.
2. The answer appears on the screen.
   Note: The calculator will give decimal values, not exact values.

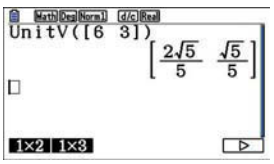
WRITE

$\hat{\underset{\sim}{u}} = 0.894\hat{i} + 0.447\hat{j}$

CASIO | THINK

1. On the Run-Matrix screen, press OPTN then select MAT/VCT by pressing F2. Press F6 twice to scroll across to more menu options, then select UnitV(by pressing F5.
   Press EXIT twice to return to the main menu, then select MATH by pressing F4. Select MAT/VCT by pressing F1, then press F6 to scroll across and select 1×2 by pressing F1.
   Complete the entry line as:
   UnitV $\left(\begin{bmatrix} 6 & 3 \end{bmatrix}\right)$
   then press EXE.
2. The answer appears on the screen.

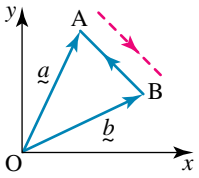
WRITE



$\hat{\underset{\sim}{u}} = \frac{2\sqrt{5}}{5}\hat{i} + \frac{\sqrt{5}}{5}\hat{j}$

2.3.6 A vector between two points

In the figure $\underset{\sim}{a}$ is the position vector of point A, $\overrightarrow{OA}$, and $\underset{\sim}{b}$ is the position vector of point B, $\overrightarrow{OB}$, relative to the origin.



The vector describing the location of A relative to B, $\overrightarrow{BA}$, is easily found using vector addition as $-\underset{\sim}{b} + \underset{\sim}{a}$ or $\underset{\sim}{a} - \underset{\sim}{b}$.

Similarly, the vector describing the location of B relative to A, $\overrightarrow{AB}$, is $\underset{\sim}{b} - \underset{\sim}{a}$.

If *A* and *B* are points defined by position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, then

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

WORKED EXAMPLE 10

a. Determine the position vector locating point B (3, −3) from point A (2, 5).
b. Calculate the length of this vector.

THINK

a. 1. Express the point A as a position vector $\underset{\sim}{a}$.

2. Express the point B as a position vector $\underset{\sim}{b}$.

3. The location of B relative to A, $(\overrightarrow{AB})$, is $\underset{\sim}{b} - \underset{\sim}{a}$.

b. The length of $(\overrightarrow{AB})$ is $|\underset{\sim}{b} - \underset{\sim}{a}|$.

WRITE

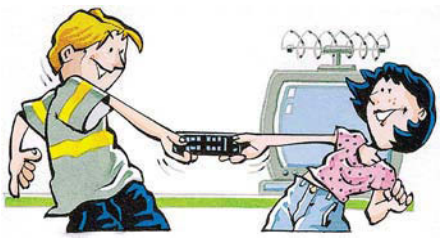
a. Let $\overrightarrow{OA} = \underset{\sim}{a} = 2\hat{\imath} + 5\hat{\jmath}$

Let $\overrightarrow{OB} = \underset{\sim}{b} = 3\hat{\imath} - 3\hat{\jmath}$

$$\begin{aligned}\overrightarrow{AB} &= \underset{\sim}{b} - \underset{\sim}{a} \\ &= 3\hat{\imath} - 3\hat{\jmath} - (2\hat{\imath} + 5\hat{\jmath}) \\ &= \hat{\imath} - 8\hat{\jmath}\end{aligned}$$

b.
$$\begin{aligned}\left|\overrightarrow{AB}\right| &= |\underset{\sim}{b} - \underset{\sim}{a}| \\ &= \sqrt{1^2 + (-8)^2} \\ &= \sqrt{65} \text{ (or } 8.06)\end{aligned}$$

We are now in a position to resolve the problem of finding, accurately, the resultant force acting on the television remote control when Daniel and Anna are pulling on it.



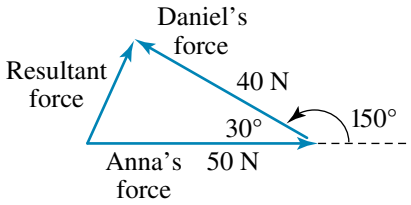
First redraw the diagram to show the addition of vectors.
Taking the direction of Anna's force as the $\hat{\imath}$ direction:
Anna's force, $\underset{\sim}{a} = 50\hat{\imath} + 0\hat{\jmath}$

Daniel's force, $\underset{\sim}{b} = 40\cos 150°\hat{\imath} + 40\sin 150°\hat{\jmath}$
$= -34.6410\hat{\imath} + 20\hat{\jmath}$

Resultant force, $\underset{\sim}{a} + \underset{\sim}{b} = (50 - 34.6410)\hat{\imath} + 20\hat{\jmath}$
$= 15.3590\hat{\imath} + 20\hat{\jmath}$

Magnitude $= |\underset{\sim}{a} + \underset{\sim}{b}|$
$= 25.2\text{N}$

Direction $\theta = \tan^{-1}\dfrac{20}{15.3590}$
$= 52.48°$



Resources

Digital document: SpreadSHEET Position vector (doc-26830)

studyon

Units 1 & 2 > Area 2 > Sequence 1 > Concepts 3, 4 and 5

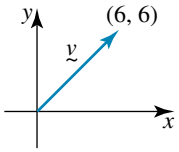
The magnitude of a vector and polar form Summary screen and practice questions
Unit vectors Summary screen and practice questions
Resolution of vectors Summary screen and practice questions

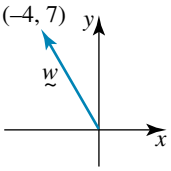
Exercise 2.3 Position vectors in the plane

Technology active

1. State the x, y components of the following vectors.
a. $3\hat{\imath} + 4\hat{\jmath}$ **b.** $6\hat{\imath} - 3\hat{\jmath}$ **c.** $3.4\hat{\imath} + \sqrt{2}\hat{\jmath}$

2. WE6a, b For each of the following, determine:
i. the magnitude of the vector
ii. the direction of each vector. (Express the direction with respect to the positive x-axis.)

a.


b.


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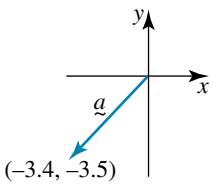

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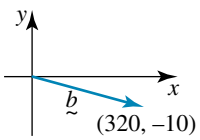
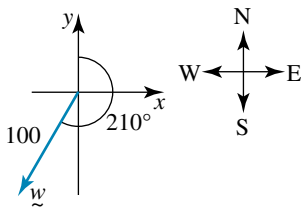


3. **i.** WE6c, d Determine the true bearing of each vector in question **2**.
ii. Express each vector in question **2** in polar form.

4. **WE7** Consider the vector $\underset{\sim}{w}$ shown. Its magnitude is 100 and its bearing is 210° True.



Find the x- and y-components of $\underset{\sim}{w}$, and express them as *exact* values (surds). State the answer in the form $\underset{\sim}{w} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.

5. **MC** A vector with a bearing of 60 degrees from N and a magnitude of 10 has:

A. x-component $= \frac{\sqrt{3}}{2}$, y-component $= \frac{1}{2}$ **B.** x-component $= \frac{1}{2}$, y-component $= \frac{\sqrt{3}}{2}$

C. x-component $= 5\sqrt{3}$, y-component $= 5$ **D.** x-component $= 5$, y-component $= 5\sqrt{3}$

6. An aeroplane travels on a bearing of 147 degrees for 457 km. Express its position as a vector in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (to 1 decimal place).

7. A ship travels on a bearing of 331 degrees for 125 km. Express its position as a vector in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (to 1 decimal place).

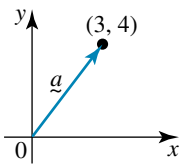
8. **WE8** A pilot flies 420 km in a direction 45° south of east and then 200 km in a direction 60° south of east. Calculate the resultant displacement from the starting position giving both magnitude and direction. State the final vector in polar form to 1 decimal place.

9. The instructions to Black-eye the Pirate's hidden treasure say: 'Take 20 steps in a north-easterly direction and then 30 steps in a south-easterly direction.' However, a rockfall blocks the first part of the route in the north-easterly direction. How could you head directly to the treasure?

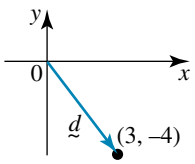
10. Two scouts are in contact with home base. Scout A is 15 km from home base in a direction 30° north of east. Scout B is 12 km from home base in a direction 40° west of north. Determine how far is scout B from scout A.

11. **WE9** Determine unit vectors in the direction of the given vector for the following:

a.



b.



c. $\underset{\sim}{b} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ **d.** $\underset{\sim}{e} = -4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$

e. $\underset{\sim}{c} = \hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{\jmath}}$

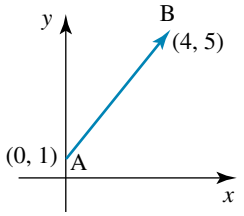
12. **MC** A unit vector in the direction of $3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ is:

A. $\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}}$ **B.** $\frac{3}{5}\hat{\boldsymbol{\imath}} - \frac{4}{5}\hat{\boldsymbol{\jmath}}$ **C.** $\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ **D.** $\frac{3}{25}\hat{\boldsymbol{\imath}} - \frac{4}{25}\hat{\boldsymbol{\jmath}}$

13. Not all unit vectors are smaller than the original vectors. Consider the vector $\underset{\sim}{v} = 0.3\hat{\boldsymbol{\imath}} + 0.4\hat{\boldsymbol{\jmath}}$. Show that the unit vector in the direction of $\underset{\sim}{v}$ is twice as long as $\underset{\sim}{v}$.

14. Find the unit vector in the direction of $\underset{\sim}{w} = -0.1\hat{\boldsymbol{\imath}} - 0.02\hat{\boldsymbol{\jmath}}$.

15. Consider the points A $(0, 1)$ and B $(4, 5)$ in the figure. A vector joining A to B can be drawn.



- **a.** Show that an equivalent position vector is given by: $4\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.
- **b.** Similarly, show that an equivalent position vector joining B to A is given by: $-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$.

16. **WE10** For each of the following pairs of points determine:

- **i.** the position vectors locating the second point from the first point
- **ii.** the length of this vector as an exact value.

a. $(0, 2)$, $(4, -5)$ **b.** $(2, 3)$, $(5, 4)$
c. $(4, -5)$, $(0, 2)$ **d.** $(5, 4)$, $(2, 3)$
e. $(3, 7)$, $(5, 7)$ **f.** $(7, -3)$, $(3, -3)$

17. Determine the position vectors from question **16**, by going from the second point to the first.

18. Determine unit vectors in the direction of the position vectors for each of the vectors of question **16**.

19. Let $u = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\underset{\sim}{e} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

- **a.** Determine:
  - **i.** $|\underset{\sim}{u}|$ **ii.** $|\underset{\sim}{e}|$ **iii.** $\hat{\underset{\sim}{u}}$ **iv.** $\hat{\underset{\sim}{e}}$ **v.** $\underset{\sim}{u} + \underset{\sim}{e}$ **vi.** $|\underset{\sim}{u} + \underset{\sim}{e}|$
- **b.** Confirm or reject the statement that $|\underset{\sim}{u}| + |\underset{\sim}{e}| = |\underset{\sim}{u} + \underset{\sim}{e}|$

20. Let $\underset{\sim}{u} = -3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $e = 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

- **a.** Determine:
  - **i.** $|\underset{\sim}{u}|$ **ii.** $|\underset{\sim}{e}|$ **iii.** $\hat{\underset{\sim}{u}}$ **iv.** $\hat{\underset{\sim}{e}}$ **v.** $\underset{\sim}{u} + \underset{\sim}{e}$ **vi.** $|\underset{\sim}{u} + \underset{\sim}{e}|$
- **b.** Confirm or reject the statement that $|\underset{\sim}{u}| + |\underset{\sim}{e}| = |\underset{\sim}{u} + \underset{\sim}{e}|$.

21. To calculate the *distance* between two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, simply find $|\underset{\sim}{a} - \underset{\sim}{b}|$.
Calculate the distance between these pairs of vectors:

- **a.** $3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$
- **b.** $5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

22. A river flows through the jungle from west to east at a speed of 3 km/h. An explorer wishes to cross the river by boat, and attempts this by travelling at 5 km/h due north.
Determine:

- **a.** the vector representing the velocity of the river
- **b.** the vector representing the velocity of the boat
- **c.** the resultant (net) vector of the boat's journey
- **d.** the bearing of the boat's journey
- **e.** the magnitude of the net vector.

23. Consider the data from question **22**. At what bearing should the boat travel so that it arrives at the opposite bank of the river due north of the starting position?

24. A boat travels east at 20 km/h, while another boat travels south at 15 km/h. Determine:

- **a.** a vector representing each boat and the difference between the boats
- **b.** the magnitude of the difference vector
- **c.** the bearing of the difference vector.

25. Consider the vector $\underset{\sim}{u} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and the vector $\underset{\sim}{v} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$. Find the angles of each of these vectors with respect to the x-axis. Show that these two vectors are perpendicular to each other. Also show that the products of each vector's corresponding x- and y-components add up to 0. Can you confirm that this is a pattern for all perpendicular vectors?
26. A river has a current of 4 km/h westward. A boat which is capable of travelling at 12 km/h is attempting to cross the river by travelling due north. Determine:
    a. the vector representing the net velocity of the boat
    b. the bearing of the actual motion of the boat
    c. how long it takes to cross the river, if the river is 500 m wide (from north to south). (*Hint:* The maximum 'speed' of the boat is still 12 km/h.)

2.4 Scalar multiplication of vectors

2.4.1 Equality of two vectors

When a vector in $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ form is multiplied by a scalar, each coefficient is multiplied by the scalar.

> If $\underset{\sim}{a} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, then $k\underset{\sim}{a} = kx\hat{\boldsymbol{\imath}} + ky\hat{\boldsymbol{\jmath}}$.

For example, if $\underset{\sim}{a} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$, then $2\underset{\sim}{a} = 2\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ and $-\underset{\sim}{a} = -\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$.

Given the vectors $\overrightarrow{OA} = \underset{\sim}{a} = x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}}$ and $\overrightarrow{OB} = \underset{\sim}{b} = x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}}$, the two vectors are equal, $\underset{\sim}{a} = \underset{\sim}{b}$, if and only if $x_1 = x_2$ and $y_1 = y_2$.

WORKED EXAMPLE 11

If $\underset{\sim}{a} = x\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 4\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}}$, determine the value of x if the vector $\underset{\sim}{c} = 2\underset{\sim}{a} - 3\underset{\sim}{b}$ is parallel to the y-axis.

THINK	WRITE
1. Substitute the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ into $\underset{\sim}{c} = 2\underset{\sim}{a} - 3\underset{\sim}{b}$.	$\underset{\sim}{c} = 2\underset{\sim}{a} - 3\underset{\sim}{b}$ $= 2(x\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}) - 3(4\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}})$

2. Multiply the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ by the scalars 2 and −3.
$\underset{\sim}{c} = (2x\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}}) - (12\hat{\boldsymbol{\imath}} - 15\hat{\boldsymbol{\jmath}})$

3. Simplify
$\underset{\sim}{c} = (2x - 12)\hat{\boldsymbol{\imath}} + (-6 + 15)\hat{\boldsymbol{\jmath}}$
$\underset{\sim}{c} = (2x - 12)\hat{\boldsymbol{\imath}} + 9\hat{\boldsymbol{\jmath}}$

4. As vector $\underset{\sim}{c}$ is parallel to the y-axis, its $\hat{\boldsymbol{\imath}}$ component must be zero. Equate the $\hat{\boldsymbol{\imath}}$ component to zero.
$2x - 12 = 0$

5. Solve for x.
$2x = 12$
$x = 6$

WORKED EXAMPLE 12

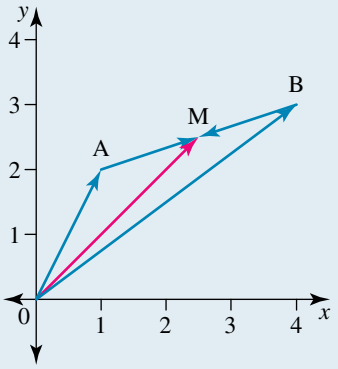
Determine the vector representing the midpoint of the line segment AB if A = (1, 2) and B = (4, 3).

THINK

1. Draw a diagram to represent this situation.

WRITE

Let M be the midpoint of AB.



2. Write $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in component form.
$\overrightarrow{OA} = \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$
$\overrightarrow{OB} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$

3. Write $\overrightarrow{AB}$ in component form.
$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$
$= \overrightarrow{OB} - \overrightarrow{OA}$
$= 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$
$= 3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$

4. Write $\overrightarrow{OM}$ in component form.
$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$
$= \hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \frac{1}{2}(3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$
$= \frac{5}{2}\hat{\boldsymbol{\imath}} + \frac{5}{2}\hat{\boldsymbol{\jmath}}$

Further consideration of Worked example 12 demonstrates that the vector representing the midpoint, M, of the line segment AB can be determined using the formula:

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

2.4.2 Solving vector problems

When solving vector problems, we may often need to solve simultaneous, linear or even non-linear equations to solve for the unknown in each case.

WORKED EXAMPLE 13

Given the vectors $\underset{\sim}{a} = 2\hat{i} - 4\hat{j}$, $\underset{\sim}{b} = 3\hat{i} - 8\hat{j}$ and $\underset{\sim}{c} = \hat{i} + 4\hat{j}$, determine the values of the scalars m and n if $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

THINK	WRITE
1. Substitute the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ into $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.	$\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$ $\hat{i} + 4\hat{j} = m(2\hat{i} - 4\hat{j}) + n(3\hat{i} - 8\hat{j})$
2. Multiply the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ by the scalars m and n.	$\hat{i} + 4\hat{j} = (2m\hat{i} - 4m\hat{j}) + (3n\hat{i} - 8n\hat{j})$
3. Simplify.	$\hat{i} + 4\hat{j} = (2m + 3n)\hat{i} - (4m + 8n)\hat{j}$
4. Equate the components.	Since the $\hat{i}$ components are equal: $1 = 2m + 3n$ [1] Since the $\hat{j}$ components are equal: $4 = -4m - 8n$ [2]
5. Solve the simultaneous equations.	$2 \times [1]$ $2 = 4m + 6n$ $4 = -4m - 8n$ [2] Adding gives $6 = -2n \Rightarrow n = -3$
6. Substitute into [1] to solve for m.	$2m = 1 - 3n$ $2m = 1 + 9 = 10$ $m = 5$
7. Write the answer.	$\underset{\sim}{c} = 5\underset{\sim}{a} - 3\underset{\sim}{b}$

2.4.3 Parallel vectors

Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if $\underset{\sim}{a} = k\underset{\sim}{b}$, where $k \in R$. Conversely, if $\underset{\sim}{a} = k\underset{\sim}{b}$, then $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$.

WORKED EXAMPLE 14

Given the vectors $\underset{\sim}{r} = x\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{s} = 5\hat{\imath} - 6\hat{\jmath}$, calculate the value of x in each case if:
a. the length of the vector $\underset{\sim}{r}$ is 7
b. the vector $\underset{\sim}{r}$ is parallel to the vector $\underset{\sim}{s}$.

THINK	WRITE
a. 1. Determine the magnitude of the vector in terms of the unknown value.	a. $\underset{\sim}{r} = x\hat{\imath} + 3\hat{\jmath}$ $\lvert\underset{\sim}{r}\rvert = \sqrt{x^2 + 3^2}$ $= \sqrt{x^2 + 9}$
2. Equate the length of the vector to the given value.	Since $\lvert\underset{\sim}{r}\rvert = 7$: $\sqrt{x^2 + 9} = 7$
3. Square both sides and solve for the unknown value. Both answers are acceptable values.	$x^2 + 9 = 49$ $x^2 = 40$ $x = \pm\sqrt{40}$ $= \pm\sqrt{4 \times 10}$ $= \pm 2\sqrt{10}$
b. 1. If two vectors are parallel, then one is a scalar multiple of the other. Substitute for the given vectors and expand.	b. $\underset{\sim}{r} = k\underset{\sim}{s}$ $x\hat{\imath} + 3\hat{\jmath} = k(5\hat{\imath} - 6\hat{\jmath})$ $= 5k\hat{\imath} - 6k\hat{\jmath}$
2. For the two vectors to be equal, both components must both be equal.	From the $\hat{\jmath}$ component, $3 = -6k$, so $k = -\frac{1}{2}$. From the $\hat{\imath}$ component, $x = 5k$.
3. Solve for the unknown value in this case.	$x = 5k$ but $k = -\frac{1}{2}$, so $x = -\frac{5}{2}$.

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Units 1 & 2 > Area 2 > Sequence 1 > Concept 6

Scalar multiplication of vectors Summary screen and practice questions

Exercise 2.4 Scalar multiplication

Technology free

1. WE11 If $\underset{\sim}{a} = 4\hat{\imath} - 5\hat{\jmath}$ and $\underset{\sim}{b} = 3\hat{\imath} + y\hat{\jmath}$, determine the value of y if the vector $\underset{\sim}{c} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$ is parallel to the x-axis.
2. The vector $\overrightarrow{\text{CD}} = 7\hat{\imath} - 5\hat{\jmath}$, the coordinates of point C are $(x, -3)$ and the coordinates of point D are $(4, y)$. Determine the values of x and y.

3. **WE12** Determine the vector representing the midpoint of the line segment AB if A = (3, 4) and B = (7, 8).
4. **a.** Determine the value of x if the vector $x\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ is parallel to the vector $5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$.
   b. Determine the value of y if the vector $-4\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ is parallel to the vector $6\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.
5. **a.** For the vectors $\underset{\sim}{a} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, show that the vector $2\underset{\sim}{a} + 3\underset{\sim}{b}$ is parallel to the x-axis.
   b. For the vectors $\underset{\sim}{c} = 3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\underset{\sim}{d} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, show that the vector $4\underset{\sim}{c} - 3\underset{\sim}{d}$ is parallel to the y-axis.
   c. Given the vectors $\underset{\sim}{a} = x\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, find the value of x if the vector $3\underset{\sim}{a} + 4\underset{\sim}{b}$ is parallel to the y-axis.
   d. If $\underset{\sim}{c} = 5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\underset{\sim}{d} = 4\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, find the value of y if the vector $5\underset{\sim}{c} + 7\underset{\sim}{d}$ is parallel to the x-axis.
6. **a.** Show that the points A (3, −2), B (4, −5) and C (1, 4) are collinear.
   b. Show that the points (5, −3), (2, 1) and (8, −7) are collinear.

Technology active

7. **a.** Given the points A (5, −2) and B (−1, 3), determine:
      i. the vector $\overrightarrow{\text{AB}}$
      ii. the distance between the points A and B
      iii. a unit vector parallel to $\overrightarrow{\text{AB}}$.
   b. Two points C and D are given by (−4, 3), (2, −5) respectively. Determine:
      i. the vector $\overrightarrow{\text{CD}}$
      ii. the distance between the points C and D
      iii. a unit vector parallel to $\overrightarrow{\text{DC}}$.
8. **WE13** Given the vectors $\underset{\sim}{a} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\underset{\sim}{b} = -3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\underset{\sim}{c} = \hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}}$, determine the values of the scalars m and n if $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.
9. Given the vectors $\underset{\sim}{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and $\underset{\sim}{s} = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, determine the values of x and y if $2\underset{\sim}{r} + 3\underset{\sim}{s} = 8\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.
10. **WE14** Given the vectors $\underset{\sim}{r} = x\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\underset{\sim}{s} = 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$, calculate the value of x if:
    a. the length of the vector $\underset{\sim}{r}$ is 6
    b. the vector $\underset{\sim}{r}$ is parallel to the vector $\underset{\sim}{s}$.
11. Calculate the value of y for the vectors $\underset{\sim}{a} = 4\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ if:
    a. the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are equal in length
    b. the vector $\underset{\sim}{a}$ is parallel to the vector $\underset{\sim}{b}$.
12. **a.** Given the vectors $\underset{\sim}{a} = x\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = -2\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, and also given that $3\underset{\sim}{a} + 4\underset{\sim}{b} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, determine the values of x and y.
    b. For the vectors $\underset{\sim}{a} = 4\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$, $\underset{\sim}{b} = -7\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $m\underset{\sim}{a} + n\underset{\sim}{b} = 8\hat{\boldsymbol{i}} + 13\hat{\boldsymbol{j}}$, determine the values of m and n.
    c. If $\underset{\sim}{a} = 5\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$, $\underset{\sim}{b} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $m\underset{\sim}{a} + n\underset{\sim}{b} = 2\hat{\boldsymbol{j}}$, determine the values of m and n.

2.5 The scalar (dot) product

2.5.1 Calculating the dot product

In a previous section we studied the result of multiplying a vector by a scalar. In this section we study the multiplication of two vectors.

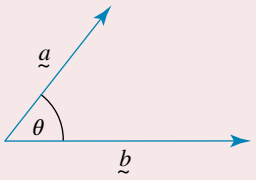
The scalar or *dot* product of two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, is denoted by $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

There are two ways of calculating the **dot product**. The first method follows from its definition. (The second method is shown later.) Consider the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

By definition, the dot product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is given by:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$$

where θ is the angle between (the positive directions of) $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Note: The vectors are not aligned as for addition or subtraction, but their two tails are joined.

Properties of the dot product

- The dot product is a scalar. It is the result of multiplying three scalar quantities: the magnitudes of the two vectors and the cosine of the angle between them.
- The order of multiplication is unimportant (*commutative* property); thus,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$$

- The dot product is *distributive*; thus,

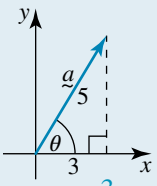
$$\underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{b}$$

- Since the angle between $\underset{\sim}{a}$ and itself is 0°

$$\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$$

WORKED EXAMPLE 15

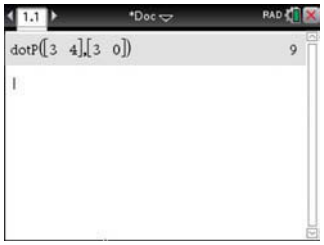
Let $\underset{\sim}{a} = 3\hat{i} + 4\hat{j}$ and $\underset{\sim}{b} = 3\hat{i}$. Determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

THINK	WRITE
1. Find the magnitudes of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\|\underset{\sim}{a}\| = \sqrt{3^2 + 4^2}$ $= 5$ $\|\underset{\sim}{b}\| = \sqrt{3^2}$ $= 3$
2. Draw a right-angled triangle showing the angle that $\underset{\sim}{a}$ makes with the positive x-axis since $\underset{\sim}{b}$ is along the x-axis.	
3. Find $\cos\theta$, knowing that $a = 5$ and the x-component of $\underset{\sim}{a}$ is 3.	$\cos\theta = \frac{3}{5}$
4. Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ using the equation with the angle between 2 vectors	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\| \times \|\underset{\sim}{b}\| \times \cos\theta$ $= 5 \times 3 \times \frac{3}{5}$
5. Simplify.	$= 9$

TI | THINK

1. On a Calculator page, press MENU, then select:
   7: Matrix & Vector
   C: Vector
   3: Dot Product
   Complete the entry line as:
   dotP $\left(\begin{bmatrix}3 & 4\end{bmatrix}\ \begin{bmatrix}3 & 0\end{bmatrix}\right)$
   then press ENTER.
   Note: The matrix template can be found by pressing the templates button.

WRITE



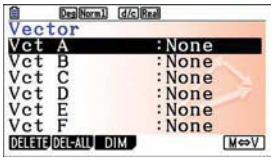
2. The answer appears on the screen.

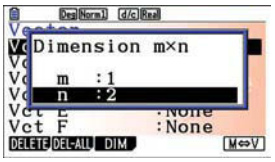
$\underset{\sim}{a} \cdot \underset{\sim}{b} = 9$

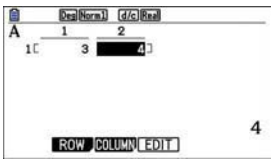
CASIO | THINK

1. On the Run-Matrix screen, select MAT/VCT by pressing F3.
   Select M ⇔ V by pressing F6 to switch from the Matrix screen to the Vector screen.
   To define $\underset{\sim}{a}$ as Vector A, highlight VCT A and press EXE. Change the dimension to 1 × 2, then press EXE.
   Enter the values 3 and 4 into the matrix template, then press EXIT.
   Repeat these steps to define $\underset{\sim}{b}$ as Vector B.

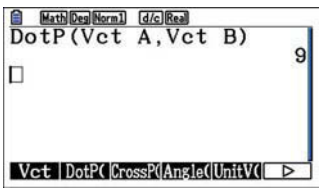
WRITE







2. On the Run-Matrix screen, press OPTN, then select MAT/VCT by pressing F2. Press F6 twice to scroll across, then select DotP(by pressing F2.
   Complete the entry line as:
   DotP(VctA,Vct B)
   then press EXE.
   Note: Press F1 for 'Vct'.



3. The answer appears on the screen.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 9$

Note: An easier method for finding the dot product will now be shown.

2.5.2 The scalar product of vectors expressed in component form

Consider the dot product of the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$. Firstly, consider $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}$ in detail. By definition, $|\hat{\boldsymbol{\imath}}| = 1$ and, since the angle between them is 0°, $\cos\theta = 1$, thus $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = 1$. To summarise these results:

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = 1 \ (\text{since } \theta = 0°)$$

$$\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} = 1 \ (\text{since } \theta = 0°)$$

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} = 0 \ (\text{since } \theta = 90°)$$

Using this information, we can develop another way to calculate the dot product of any vector.
Let $\underset{\sim}{a} = x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}}$ where x_1, y_1, x_2, y_2, are constants. Then we can write $\underset{\sim}{a} \cdot \underset{\sim}{b}$ as:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}}) \cdot (x_2\hat{\boldsymbol{\imath}} + y_2\hat{\boldsymbol{\jmath}})$$

$$= x_1x_2(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}) + x_1y_2(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}) + y_1x_2(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}}) + y_1y_2(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}})$$

Considering the various unit vector dot products (in brackets), the 'like' products ($\hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}}$) are 1; the rest are 0. Therefore:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$$

This is a very important result.

We only need to multiply the corresponding *x*- and *y*-components of two vectors to find their dot product.

WORKED EXAMPLE 16

Let $\underset{\sim}{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. Determine $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

THINK	WRITE
1. Write down $\underset{\sim}{a} \cdot \underset{\sim}{b}$ using the equation $\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) \cdot (6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}})$
2. Multiply the corresponding components.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3 \times 6 + 4 \times -4$
3. Simplify.	$= 18 - 16$ $= 2$

2.5.3 Finding the angle between two vectors

We now have two formulas for calculating the dot product:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \theta$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$$

Combining these two formulas allows us to determine the angle between the vectors.
Rearranging the two equations, we obtain the result:

$$\cos \theta = \frac{x_1x_2 + y_1y_2}{|\underset{\sim}{a}|\,|\underset{\sim}{b}|}$$

Note: The angle will always be between 0° and 180° as 180° is the maximum angle between two vectors.

WORKED EXAMPLE 17

Let $\underset{\sim}{a} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$. Calculate the angle between them to the nearest degree.

THINK	WRITE
1. Determine the dot product using the equation $\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \cdot (2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}})$ $= 4 \times 2 + 3 \times -3$
2. Simplify.	$= -1$

3. Find the magnitude of each vector.

$$|\underset{\sim}{a}| = \sqrt{4^2 + 3^2}$$
$$= \sqrt{25}$$
$$= 5$$
$$|\underset{\sim}{b}| = \sqrt{2^2 + (-3)^2}$$
$$= \sqrt{13}$$

4. Substitute results into the equation $\cos\theta = \dfrac{x_1x_2 + y_1y_2}{|\underset{\sim}{a}||\underset{\sim}{b}|}$.

$$\cos\theta = \frac{-1}{5\sqrt{13}}$$

5. Simplify the result for $\cos\theta$.

$$= -0.05547$$

6. Take $\cos^{-1}$ of both sides to obtain θ and round the answer to the nearest degree.

$$\theta = \cos^{-1}(-0.05547)$$
$$= 93°$$

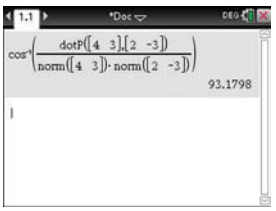
TI | THINK

1. On a Calculator page, complete the entry line as

$$\cos^{-1}\left(\frac{\text{dotP}([4\ \ 3],[2\ \ -3])}{\text{norm}([4\ \ 3]) \times \text{norm}([2\ \ -3])}\right)$$

then press ENTER.
Note: 'dotP(' can be found by pressing MENU and then selecting
7: Matrix & Vector
C: Vector
3: Dot Product.
Similarly, 'norm' can be found by pressing MENU and then selecting
7: Matrix & Vector
7: Norms
1: Norm.

WRITE

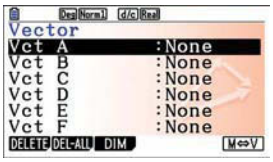


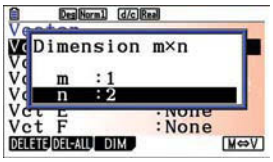
2. The answer appears on the screen. The angle is 93°.

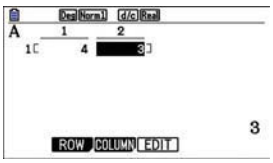
CASIO | THINK

1. On the Run-Matrix screen, select MAT/VCT by pressing F3.
Select M ⇔ V by pressing F6 to switch from the Matrix screen to the Vector screen.
To define $\underset{\sim}{a}$ as Vector A, highlight VCT A and press EXE. Change the dimension to 1×2, then press EXE.
Enter the values 4 and 3 into the matrix template, then press EXIT.
Repeat these steps to define $\underset{\sim}{b}$ as Vector B.

WRITE







2. On the Run-Matrix screen, press OPTN, then select MAT/VCT by pressing F2.
Press F6 twice to scroll across, then select Angle(by pressing F4.
Complete the entry line as: Angle(VctA,Vct B)
then press EXE.
Note: Press F1 for 'Vct'.

3. The answer appears on the screen. The angle is 93°.

2.5.4 Special results of the dot product

Perpendicular vectors

If two vectors are perpendicular then the angle between them is 90° and the equation $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$ becomes:

$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= |\underset{\sim}{a}||\underset{\sim}{b}| \cos 90^\circ \\ &= |\underset{\sim}{a}||\underset{\sim}{b}| \times 0 \qquad (\text{Since } \cos 90^\circ = 0) \\ &= 0 \end{aligned}$$

If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

WORKED EXAMPLE 18

Find the constant m if the vectors $\underset{\sim}{a} = 4\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{b} = -3\hat{\imath} + m\hat{\jmath}$ are perpendicular.

THINK	WRITE
1. Find the dot product using the equation $\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (4\hat{\imath} + 3\hat{\jmath}) \cdot (-3\hat{\imath} + m\hat{\jmath})$
2. Simplify.	$= -12 + 3m$
3. Set $\underset{\sim}{a} \cdot \underset{\sim}{b}$ equal to zero since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = -12 + 3m$ $= 0$
4. Solve the equation for m.	$m = 4$

Parallel vectors

If vector $\underset{\sim}{a}$ is parallel to vector $\underset{\sim}{b}$, then $\underset{\sim}{a} = k\underset{\sim}{b}$ where $k \in R$.
Note: When applying the dot product to parallel vectors, θ (the angle between them) may be either 0° or 180° depending on whether the vectors are in the same or opposite directions.

WORKED EXAMPLE 19

Let $\underset{\sim}{a} = 5\hat{\imath} + 2\hat{\jmath}$. Find a vector parallel to $\underset{\sim}{a}$ such that the dot product is 87.

THINK	WRITE
1. Let the required vector $\underset{\sim}{b} = k\underset{\sim}{a}$.	Let $\underset{\sim}{b} = k(5\hat{\imath} + 2\hat{\jmath})$ $= 5k\hat{\imath} + 2k\hat{\jmath}$
2. Find the dot product of $\underset{\sim}{a} \cdot \underset{\sim}{b}$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (5\hat{\imath} + 2\hat{\jmath}) \cdot (5k\hat{\imath} + 2k\hat{\jmath})$
3. Simplify.	$= 25k + 4k$ $= 29k$
4. Equate the result to the given dot product 87.	$29k = 87$
5. Solve for k.	$k = 3$
6. Substitute $k = 3$ into vector $\underset{\sim}{b}$.	$\underset{\sim}{b} = 15\hat{\imath} + 6\hat{\jmath}$

studyon

Units 1 & 2 > Area 2 > Sequence 1 > Concept 7

The scalar (dot) product Summary screen and practice question

Exercise 2.5 The scalar (dot) product

Technology free

1. **WE15** Determine the dot product of the vectors $3\hat{\imath}+3\hat{\jmath}$ and $6\hat{\imath}+2\hat{\jmath}$ using the equation $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$.
2. Compare the result from question 1 with that obtained by finding the dot product using the equation $\underset{\sim}{a}\cdot\underset{\sim}{b}=x_1x_2+y_1y_2$. Which is probably the most accurate?
3. **WE16** Determine $\underset{\sim}{a}\cdot\underset{\sim}{b}$ in each of the following cases.
   a. $\underset{\sim}{a}=2\hat{\imath}+3\hat{\jmath}$, $\underset{\sim}{b}=3\hat{\imath}+3\hat{\jmath}$
   b. $\underset{\sim}{a}=4\hat{\imath}-2\hat{\jmath}$, $\underset{\sim}{b}=5\hat{\imath}+\hat{\jmath}$
   c. $\underset{\sim}{a}=-\hat{\imath}+4\hat{\jmath}$, $\underset{\sim}{b}=3\hat{\imath}-7\hat{\jmath}$
   d. $\underset{\sim}{a}=5\hat{\imath}+9\hat{\jmath}$, $\underset{\sim}{b}=2\hat{\imath}-4\hat{\jmath}$
   e. $\underset{\sim}{a}=-3\hat{\imath}+\hat{\jmath}$, $\underset{\sim}{b}=\hat{\imath}+4k$
   f. $\underset{\sim}{a}=10\hat{\imath}$, $\underset{\sim}{b}=-2\hat{\imath}$
   g. $\underset{\sim}{a}=3\hat{\imath}+5\hat{\jmath}$, $\underset{\sim}{b}=\hat{\imath}$
   h. $\underset{\sim}{a}=6\hat{\imath}-2\hat{\jmath}$, $\underset{\sim}{b}=-\hat{\imath}-4\hat{\jmath}$
4. **MC** The dot product of $\underset{\sim}{a}=3\hat{\imath}-3\hat{\jmath}$ and $\underset{\sim}{b}=\hat{\imath}-2\hat{\jmath}$ is:
   A. 3 **B.** 12 **C.** 21 **D.** 9
5. Let $\underset{\sim}{a}=x\hat{\imath}+y\hat{\jmath}$. Show that $\underset{\sim}{a}\cdot\underset{\sim}{a}=x^2+y^2$.
6. Let $\underset{\sim}{a}=2\hat{\imath}-5\hat{\jmath}$ and let $\underset{\sim}{b}=-\hat{\imath}-2\hat{\jmath}$. Determine their dot product.
7. Let $\underset{\sim}{a}=3\hat{\imath}+2\hat{\jmath}$, $\underset{\sim}{b}=\hat{\imath}-2\hat{\jmath}$ and $\underset{\sim}{c}=5\hat{\imath}-2\hat{\jmath}$. Demonstrate, using these vectors, the property:

$$\underset{\sim}{c}\cdot(\underset{\sim}{a}-\underset{\sim}{b})=\underset{\sim}{c}\cdot\underset{\sim}{a}-\underset{\sim}{c}\cdot\underset{\sim}{b}$$

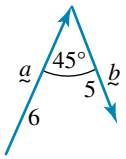
   Formally, this means that vectors are *distributive* over subtraction.
8. Repeat question 7 for the property:

$$\underset{\sim}{c}\cdot(\underset{\sim}{a}+\underset{\sim}{b})=\underset{\sim}{c}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{b}$$

   Formally, this means that vectors are *distributive* over addition.
9. **MC** If $\underset{\sim}{a}=5i+4j$ which of the following is perpendicular to u?
   A. $-5\hat{\imath}-4\hat{\jmath}$ **B.** $3\hat{\imath}+4\hat{\jmath}$ **C.** $-5\hat{\imath}$ **D.** $-4\hat{\imath}+5\hat{\jmath}$
10. **MC** If $(\underset{\sim}{a}-\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b})=0$, then:
    A. $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$
    B. $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have equal magnitudes
    C. $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$
    D. $\underset{\sim}{a}$ is a multiple of $\underset{\sim}{b}$
11. **MC** If $(\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})=|\underset{\sim}{b}^2|$, then:
    A. $\underset{\sim}{a}=\underset{\sim}{b}$
    B. $\underset{\sim}{a}$ must be equal to the zero vector, $\underset{\sim}{0}$
    C. $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$
    D. $|\underset{\sim}{a}|$ must be equal to $\sqrt{2}\,|\underset{\sim}{b}|$

Technology active

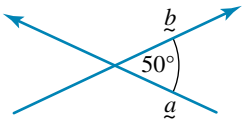
12. MC Consider the two vectors shown.



Their dot product is:

A. 30
B. 21.2
C. -21.2
D. There is insufficient data to determine the dot product.

13. Consider the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Their magnitudes are 7 and 8 respectively. Calculate $\underset{\sim}{a} \cdot \underset{\sim}{b}$, to the nearest whole number.

14. Determine the dot product of the following pairs of vectors.

a. $4\hat{\imath} - 3\hat{\jmath}$ and $7\hat{\imath} + 4\hat{\jmath}$
b. $\hat{\imath} + 2\hat{\jmath}$ and $-9\hat{\imath} + 4\hat{\jmath}$
c. $8\hat{\imath} + 3\hat{\jmath}$ and $-2\hat{\imath} - 3\hat{\jmath}$
d. $5\hat{\imath} - 5\hat{\jmath}$ and $5\hat{\imath} + 5\hat{\jmath}$

15. WE17 Calculate the angle between each pair of vectors in question **14** to the nearest degree.

16. MC The angle between the vectors $2\hat{\imath} + 3\hat{\jmath}$ and $2\hat{\imath} - 3\hat{\jmath}$ is closest to:

A. 0° **B.** 67° **C.** 90° **D.** 113°

17. MC The angle between the vectors $2\hat{\imath} - 3\hat{\jmath}$ and $-4\hat{\imath} + 6\hat{\jmath}$ is closest to:

A. 0° **B.** 69° **C.** 111° **D.** 180°

18. WE18 Determine the constant m, if the vectors $\underset{\sim}{b} = m\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{a} = 6\hat{\imath} - 2\hat{\jmath}$ are perpendicular.

19. Determine the constant m, such that $\underset{\sim}{b} = m\hat{\imath} - 2m\hat{\jmath}$ is perpendicular to $\underset{\sim}{a} = 4\hat{\imath} - 3\hat{\jmath}$.

20. WE19 Let $\underset{\sim}{a} = 2\hat{\imath} + 4\hat{\jmath}$. Find a vector parallel to $\underset{\sim}{a}$ such that their dot product is 40.

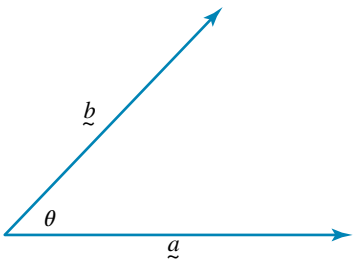
21. Let $\underset{\sim}{a} = 4\hat{\imath} - 3\hat{\jmath}$. Find a vector parallel to $\underset{\sim}{a}$ such that their dot product is 80.

2.6 The projection of vectors — scalar and vector resolutes

2.6.1 Introduction

Previously we have resolved a vector parallel and perpendicular to the x- and y-axes. In this section we consider a generalisation of this process where we resolve one vector parallel and perpendicular to another vector.

Consider the two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, shown. The angle between them, as for a dot product, is given by θ. It can be shown that $\underset{\sim}{b}$ is made up of a **projection** acting in the direction of $\underset{\sim}{a}$ and another projection acting perpendicular to $\underset{\sim}{a}$.



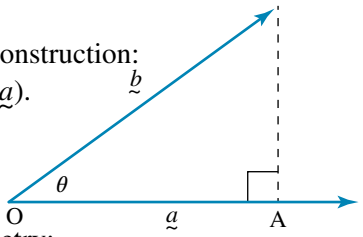
Firstly we wish to find the projection in the direction of $\underset{\sim}{a}$.

2.6.2 The scalar resolute

To obtain the projection of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$, we perform the following construction:

1. Drop a perpendicular line from the head of $\underset{\sim}{b}$ to $\underset{\sim}{a}$ (this is perpendicular to $\underset{\sim}{a}$). This line joins $\underset{\sim}{a}$ at point A.
2. We wish to find the length of the line OA.

This construction is shown.



Let the length of $\underset{\sim}{b}$ (its magnitude) be denoted by $|\underset{\sim}{b}|$. Then, from trigonometry:

$$\text{OA} = |\underset{\sim}{b}| \cos\theta$$

But from the definition of the dot product:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta \quad \text{(from the equation } \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta\text{)}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\text{OA}$$

Therefore, solving for OA:

$$\text{OA} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}|}$$

$$= \left(\frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}\right) \underset{\sim}{b}$$

But we know that $\dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \hat{\underset{\sim}{a}}$, the unit vector in the direction of $\underset{\sim}{a}$, and therefore

$$\text{OA} = \hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}$$

This quantity, the length OA, is called the **scalar resolute** of $\underset{\sim}{b}$ on $\underset{\sim}{a}$. It effectively indicates 'how much' of $\underset{\sim}{b}$ is in the direction of $\underset{\sim}{a}$.

The scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$ is given by $\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}$, where $\hat{\underset{\sim}{a}}$ is the unit vector in the direction of $\underset{\sim}{a}$.

WORKED EXAMPLE 20

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} - 2\underset{\sim}{j}$. Determine:

a. the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$

b. the scalar resolute of on $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

THINK

a. 1. Find the magnitude of $\underset{\sim}{a}$.

WRITE

a $|\underset{\sim}{a}| = \sqrt{3^2 + 4^2}$

$= 5$

2. Find $\hat{\underset{\sim}{a}}$ by dividing $\underset{\sim}{a}$ by $|\underset{\sim}{a}|$.

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$$

$$= \frac{\underset{\sim}{a}}{5}$$

3. Simplify.

$$= \frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$$

$$= \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$$

4. Find the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ using $\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}$.

5. Simplify.

$$\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b} = \left(\frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}\right) \cdot (6\underset{\sim}{i} - 2\underset{\sim}{j})$$

$$= \frac{18}{5} - \frac{8}{5}$$

$$= \frac{10}{5}$$

$$= 2$$

b. 1. Find the magnitude of $\underset{\sim}{b}$.

b $|\underset{\sim}{b}| = \sqrt{6^2 + (-2)^2}$

$$= \sqrt{40}$$

2. Find $\hat{\underset{\sim}{b}}$ by dividing $\underset{\sim}{b}$ by $|\underset{\sim}{b}|$.

$$\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{1}{\sqrt{40}}(6\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}})$$

3. Find the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$ using $\hat{\underset{\sim}{b}} \cdot \underset{\sim}{a}$.

$$\hat{\underset{\sim}{b}} \cdot \underset{\sim}{a} = \frac{1}{\sqrt{40}}(6\underset{\sim}{i} - 2\underset{\sim}{j}) \cdot (3\underset{\sim}{i} + 4\underset{\sim}{j})$$

4. Simplify.

$$= \frac{1}{\sqrt{40}}(18 - 8)$$

$$= \frac{10}{\sqrt{40}}$$

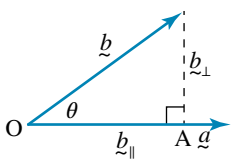
$$= \frac{\sqrt{10}}{2}$$

Notes:

- The two scalar resolutes are not equal.
- The scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$ can easily be evaluated as $\frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|}$.

2.6.3 Vector resolutes

Consider, now, the vector joining O to A at right. Its magnitude is just the scalar resolute $(\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})$, while its direction is the same as $\underset{\sim}{a}$, that is $\hat{\underset{\sim}{a}}$. This quantity is called the **vector resolute** of $\underset{\sim}{b}$ parallel to $\underset{\sim}{a}$ and is denoted by the symbol $\underset{\sim}{b}_{\parallel}$.



The vector resolute of $\underset{\sim}{b}$ parallel to $\underset{\sim}{a}$ is given by:

$$\underset{\sim}{b}_{\parallel} = (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\hat{\underset{\sim}{a}}$$

Consider the geometry of the above figure. The original vector can be seen to be the sum of two other vectors, namely $\underset{\sim}{b}_{\parallel}$ and $\underset{\sim}{b}_{\perp}$. This second vector is called the vector resolute of $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$ and can be computed simply as follows:

$\underset{\sim}{b} = \underset{\sim}{b}_{\parallel} + \underset{\sim}{b}_{\perp}$ (by addition of vectors)
$\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - \underset{\sim}{b}_{\parallel}$ (by rearranging the vector equation)

By substitution for $\underset{\sim}{b}_{\parallel}$ from the equation $b_{\parallel} = (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\hat{\underset{\sim}{a}}$, we can determine the vector resolute of $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$:

The vector resolute of $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$ is given by:

$$\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\hat{\underset{\sim}{a}}$$

In practice, once $\underset{\sim}{b}_{\parallel}$ has been calculated, simply subtract it from $\underset{\sim}{b}$ to get $\underset{\sim}{b}_{\perp}$.

WORKED EXAMPLE 21

Let $\underset{\sim}{a} = -2\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{b} = 4\hat{\imath} + 2\hat{\jmath}$. Find:
a. the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$
b. the vector resolute of $\underset{\sim}{b}$ parallel to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\parallel}$
c. the vector resolute of $\underset{\sim}{b}$ perpendicular to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\perp}$.

THINK	WRITE
a. 1. Calculate the magnitude of $\underset{\sim}{a}$.	a. $\lvert\underset{\sim}{a}\rvert = \sqrt{(-2)^2 + 3^2}$
	$= \sqrt{13}$
2. Determine $\hat{\underset{\sim}{a}}$.	$\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}$
	$= \dfrac{1}{\sqrt{13}}(-2\hat{\imath} + 3\hat{\jmath})$
3. Determine the scalar resolute using $\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}$.	$\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b} = \dfrac{1}{\sqrt{13}}(-2\hat{\imath} + 3\hat{\jmath}) \cdot (4\hat{\imath} + 2\hat{\jmath})$
4. Simplify.	$= \dfrac{1}{\sqrt{13}}(-8 + 6)$
	$= -\dfrac{2}{\sqrt{13}}$

b. 1. Determine $\underset{\sim}{b}_{\parallel}$ using equation $b_{\parallel} = (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\,\hat{\underset{\sim}{a}}$

2. Simplify.

b. $\underset{\sim}{b}_{\parallel} = (\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b})\,\hat{\underset{\sim}{a}}$

$= \left(-\frac{2}{\sqrt{13}}\right)\left[\frac{1}{\sqrt{13}}(-2\hat{\imath} + 3\hat{\jmath})\right]$

$= -\frac{2}{13}(-2\hat{\imath} + 3\hat{\jmath})$

$= \frac{4}{13}\hat{\imath} - \frac{6}{13}\hat{\jmath}$

c. 1. Determine $\underset{\sim}{b}_{\perp}$ by subtraction of $\underset{\sim}{b}_{\parallel}$ from $\underset{\sim}{b}$ as in equation $\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - \underset{\sim}{b}_{\parallel}$.

2. Simplify by subtracting $\hat{\imath}$, $\hat{\jmath}$, and $\underset{\sim}{k}$ components.

c. $\underset{\sim}{b}_{\perp} = \underset{\sim}{b} - \underset{\sim}{b}_{\parallel}$

$= 4\hat{\imath} + 2\hat{\jmath} - \left(\frac{4}{13}\hat{\imath} - \frac{6}{13}\hat{\jmath}\right)$

$\underset{\sim}{b}_{\perp} = \frac{48}{13}\hat{\imath} - \frac{20}{13}\hat{\jmath}$

studyon

Units 1 & 2 > Area 2 > Sequence 1 > Concept 8

Scalar and vector resolutes Summary screen and practice

Exercise 2.6 The projections of vectors — scalar and vector resolutes

Technology free

1. **WE20** For each of the following pairs of vectors, determine:

i. the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{u}$.

ii. the scalar resolute of $\underset{\sim}{u}$ on $\underset{\sim}{a}$.

a. $\underset{\sim}{u} = 2\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{a} = 4\hat{\imath} + 5\hat{\jmath}$

b. $\underset{\sim}{u} = 5\hat{\imath} - 2\hat{\jmath}$ and $\underset{\sim}{a} = 3\hat{\imath} - \hat{\jmath}$

c. $\underset{\sim}{u} = -2\hat{\imath} + 6\hat{\jmath}$ and $\underset{\sim}{a} = \hat{\imath} - 4\hat{\jmath}$

d. $\underset{\sim}{u} = 3\hat{\imath} - 2\hat{\jmath}$ and $\underset{\sim}{a} = -4\hat{\imath} - 3\hat{\jmath}$

e. $\underset{\sim}{u} = 8\hat{\imath} - 6\hat{\jmath}$ and $\underset{\sim}{a} = -5\hat{\imath} + \hat{\jmath}$

2. **WE21** For each pair of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, determine:

i. the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$.

ii. the vector resolute of $\underset{\sim}{b}$, parallel to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\parallel}$.

iii. the vector resolute of $\underset{\sim}{b}$, perpendicular to $\underset{\sim}{a}$, namely $\underset{\sim}{b}_{\perp}$.

a. $\underset{\sim}{a} = 3\hat{\imath} - \hat{\jmath}$; $\underset{\sim}{b} = 2\hat{\imath} + 5\hat{\jmath}$

b. $\underset{\sim}{a} = 4\hat{\imath} + 5\hat{\jmath}$; $\underset{\sim}{b} = 8\hat{\imath} + 10\hat{\jmath}$

c. $\underset{\sim}{a} = 4\hat{\imath} + 3\hat{\jmath}$; $\underset{\sim}{b} = -3\hat{\imath} + 4\hat{\jmath}$

d. $\underset{\sim}{a} = \hat{\imath} + \hat{\jmath}$; $\underset{\sim}{b} = 2\hat{\imath} + \hat{\jmath}$

e. $\underset{\sim}{a} = 2\hat{\imath} + 3\hat{\jmath}$; $\underset{\sim}{b} = 2\hat{\imath} - 3\hat{\jmath}$

f. $\underset{\sim}{a} = 3\hat{\imath} + \hat{\jmath}$; $\underset{\sim}{b} = 2\hat{\jmath}$

Technology active

3. An injured bushwalker is located at a position relative to a camp given by the vector $2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$. A searcher heads off from the camp in a direction parallel to the vector $3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$. All measurements are in kilometres.
   a. How far is the searcher from the camp when closest to the bushwalker?
   b. What is the minimum distance between the searcher and the bushwalker?
4. A distressed yacht is located at a position given by the vector $5\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ relative to a cruiser. A rescue boat is sent off from the cruiser and travels in a direction parallel to the vector $3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$. If all measurements are in kilometres calculate, to the nearest metre, how close the rescue boat gets to the yacht.
5. Given the two vectors $\underset{\sim}{a} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$, determine:
   a. the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
   b. the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
   c. the vector resolute of $\underset{\sim}{a}$ perpendicular $\underset{\sim}{b}$.
6. Given the two vectors $\underset{\sim}{r} = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{s} = 3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$, determine:
   a. the vector component of $\underset{\sim}{r}$ parallel to $\underset{\sim}{s}$
   b. the vector component of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s}$.
7. Two points A and B are given by (5, 2) and (4, −3) respectively. Let E be the point on $\overrightarrow{\text{OB}}$ such that $\overrightarrow{\text{OE}}$ is the vector resolute of $\overrightarrow{\text{OA}}$ onto $\overrightarrow{\text{OB}}$.
   a. Determine $\left|\overrightarrow{\text{OE}}\right|$.
   b. Show that the vector $\overrightarrow{\text{OC}} = -3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ is perpendicular and equal in length to the vector $\overrightarrow{\text{OB}}$.
   c. Point D is placed so that OCDE is a rectangle. Show that the area of this rectangle is equal to $\overrightarrow{\text{OA}} \,.\, \overrightarrow{\text{OB}}$.
8. Two points A and B are given by (−1, 5) and (3, −2) respectively. Let C be the point on $\overrightarrow{\text{OB}}$ such that $\overrightarrow{\text{OC}}$ is the vector resolute of $\overrightarrow{\text{OA}}$ onto $\overrightarrow{\text{OB}}$.
   a. Determine the scalar resolute of $\overrightarrow{\text{OA}}$ onto $\overrightarrow{\text{OB}}$.
   b. Show that the vector $\overrightarrow{\text{OE}} = -2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ is perpendicular and equal in length to the vector $\overrightarrow{\text{OB}}$.
   c. Point D is placed so that OCDE is a rectangle. Show that the area of this rectangle is equal to $\overrightarrow{\text{OA}} \,.\, \overrightarrow{\text{OB}}$.
9. a. Given the two vectors $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. determine:
      i. a unit vector parallel to $\underset{\sim}{b}$
      ii. the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
      iii. the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
      iv. the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.
   b. For the two vectors $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ determine the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.
   c. If $\underset{\sim}{r} = 4\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{s} = -3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$, determine the vector resolute of $\underset{\sim}{r}$ perpendicular to $\underset{\sim}{s}$.

10. a. If $|\underset{\sim}{a}| = 4$, $|\underset{\sim}{b}| = 3$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, determine $|\underset{\sim}{a} + \underset{\sim}{b}|$ and $|\underset{\sim}{a} - \underset{\sim}{b}|$.

b. If $|\underset{\sim}{a}| = 5$, $|\underset{\sim}{b}| = 12$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, determine $|\underset{\sim}{a} + \underset{\sim}{b}|$ and $|\underset{\sim}{a} - \underset{\sim}{b}|$.

c. If $|\underset{\sim}{r}| = 7$, $|\underset{\sim}{s}| = 24$ and $\underset{\sim}{r} \cdot \underset{\sim}{s} = 0$, determine $|\underset{\sim}{r} + \underset{\sim}{s}|$ and $|\underset{\sim}{r} - \underset{\sim}{s}|$.

d. Given that $|\underset{\sim}{a}| = a$, $|\underset{\sim}{b}| = b$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, deduce $|\underset{\sim}{a} + \underset{\sim}{b}|$ and $|\underset{\sim}{a} - \underset{\sim}{b}|$.

11. a. If $|\underset{\sim}{a}| = 4$, $|\underset{\sim}{b}| = 3$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2$, calculate:

i. $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$ **ii.** $(\underset{\sim}{a} + 2\underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$ **iii.** $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - 2\underset{\sim}{b})$

b. If $|\underset{\sim}{a}| = 6$, $|\underset{\sim}{b}| = 7$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = -4$, calculate:

i. $|\underset{\sim}{a} + \underset{\sim}{b}|$ **ii.** $|\underset{\sim}{a} - \underset{\sim}{b}|$ **iii.** $|3\underset{\sim}{a}, -, 2\underset{\sim}{b}|$

12. a. If the angle between the vectors $2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ is 45°, determine the value of y.

b. If the angle between the vectors $x\hat{\boldsymbol{\imath}} - 2\sqrt{3}\hat{\boldsymbol{\jmath}}$ and $-3\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}}$ is 150°, determine the value of x.

13. a. Determine a unit vector in the xy plane perpendicular to $3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$.

b. Determine a unit vector in the xy plane perpendicular to $-5\hat{\boldsymbol{\imath}} + 12\hat{\boldsymbol{\jmath}}$.

c. Determine a unit vector in the xy plane perpendicular to $7\hat{\boldsymbol{\imath}} + 24\hat{\boldsymbol{\jmath}}$.

d. Deduce a unit vector in the xy plane perpendicular to $a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}$.

2.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. If $\underset{\sim}{a} = 4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ then evaluate (without technology) $4\underset{\sim}{a} - 2.5\underset{\sim}{b}$.

2. A boat sails 5 km due east from H, turns northward at a bearing of 45° (N45°E) for a distance of 10 km and then travels due north for a further 5 km to point X.

a. Determine the position vector from H to X.

b. Calculate the distance from H to X (correct to 2 decimal places).

3. Calculate the angle between $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 16\hat{\boldsymbol{\imath}} + 24\hat{\boldsymbol{\jmath}}$.

4. MC A unit vector perpendicular to: $3\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ is:

A. $4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ **B.** $0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}$

C. $0.8\hat{\boldsymbol{\imath}} + 0.6\hat{\boldsymbol{\jmath}}$ **D.** $4\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$

5. If $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} + a\hat{\boldsymbol{\jmath}}$, $\underset{\sim}{b} = 2a\hat{\boldsymbol{\imath}} - a\hat{\boldsymbol{\jmath}}$, and it is known that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, calculate a.

6. Let $\underset{\sim}{a} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, $\underset{\sim}{b} = -\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$. Calculate the angle between the two vectors, in radians, to 4 decimal places.

7. Let $\underset{\sim}{a} = 3\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = -4\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$. Determine:

a. $\underset{\sim}{a} + \underset{\sim}{b}$ **b.** $\underset{\sim}{a} - \underset{\sim}{b}$ **c.** $\underset{\sim}{a} \cdot \underset{\sim}{b}$

d. $\hat{\underset{\sim}{a}}$ **e.** the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

8. Find value(s) of p such that $p\hat{\boldsymbol{\imath}} + 2(1 - 3p)\hat{\boldsymbol{\jmath}}$ is perpendicular to $2p\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$.

Using the vectors $\underset{\sim}{a} = \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ answer questions **9** and **10**.

9. Determine the scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.

10. Determine the vector resolute of $\underset{\sim}{b}$ parallel to $\underset{\sim}{a}$.

11. a. Given the points A (2, 5) and B (−4, 7), find the vector $\overrightarrow{OM}$ where M is the midpoint of AB.

b. Show that the points (−2, −6), (1, −3) and (5, 1) are collinear.

c. Given the points A (3, −5) and B (−3, 4), find the vector $\overrightarrow{OP}$ where P is a point on AB such that $\overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB}$.

12. a. Demonstrate that the points (−6, 4), (2, 0) and (−2, −4) form an isosceles triangle.

b. Demonstrate that the points $\left(2\sqrt{3}, 4\sqrt{3}\right)$, (4, −2) and (−4, 2) form an equilateral triangle.

c. Demonstrate that the points (−7, 5), (1, −3) and (6, 2) form a right-angled triangle.

Complex familiar

13. a. Given the points A (3, −4), B (x, y) and C (7, 8), calculate the values of x and y if $\overrightarrow{AB} = \overrightarrow{BC}$.

b. Given the points A (x, 3), B (2, −1), C (7, −2) and D (2, −4), calculate the value of x if $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

c. Given the points A (3, 5), B (−1, y), C (2, 7) and D (−6, 3), calculate the value of y if $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

14. The parallelogram OXYZ has O at the origin. The vector joining O to Z is given by $5\hat{\boldsymbol{\imath}}$ while the vector joining O to X is given by $2\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$. You may wish to use technology to answer the following.

a. Sketch the parallelogram, labelling all vertices.

b. State the vectors joining Z to Y and Y to X.

c. State the vectors which represent the diagonals of the parallelogram.

d. Determine the cosine of the angle between the diagonals. Express your answer in simplest surd form.

e. Calculate the angle that OX makes with the x-axis.

f. State the vector resolute of the vector joining O to X in the direction of OZ.

g. Let P be a point on the extended line of XY, such that the vector joining P to Z is perpendicular to OY. Determine the coordinates of P.

h. Calculate the area of the parallelogram.

15. If $\underset{\sim}{a} = 5\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = x\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, calculate the value of x if:

a. the two vectors are equal in length

b. the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular

c. the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel

d. the scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is equal to $\frac{2}{\sqrt{61}}$.

16. Given the two vectors $\underset{\sim}{a} = 4\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$, determine:

a. a unit vector parallel to $\underset{\sim}{b}$

b. the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$

c. the angle between the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$

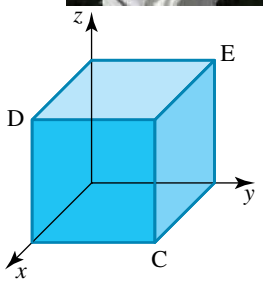
d. the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$

e. the vector resolute of $\underset{\sim}{a}$ perpendicular $\underset{\sim}{b}$.

Complex unfamiliar

17. a. OAB is a triangle in which M is the midpoint of AB. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Express OM and AM in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and hence show that $\left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2 = 2\left|\overrightarrow{AM}\right|^2 + 2\left|\overrightarrow{OM}\right|^2$.
This result is known as Apollonius' theorem. Apollonius of Perga (262–190 BCE) was a Greek astronomer. He is also noted for naming the conic sections. There is a crater on the moon named after him.

b. If $\underset{\sim}{a} = \cos(A)\,\hat{\boldsymbol{\imath}} + \sin(A)\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = \cos(B)\,\hat{\boldsymbol{\imath}} + \sin(B)\hat{\boldsymbol{\jmath}}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and hence show that $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$.



18. Consider the box shown. You may wish to use technology of you choice to answer the following.
The coordinates (in cm) of point D are (3, 0, 4), and the coordinates of E are (0, 5.5, 4).

a. Determine the coordinates of point C.

b. Express the line joining C to E as a vector.

c. Show that the two diagonals in the same plane as CE intersect with an angle of 73.7°.

d. Calculate the volume of the box in litres.

e. Express the longest diagonal, from the origin, as a vector.

f. Determine the length of this diagonal.

g. Determine the angle that this diagonal makes with the other long diagonal.

19. Given the vectors $\underset{\sim}{a} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, find the values of x and y if the length of the vector $\underset{\sim}{a}$ is $\sqrt{34}$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 9$.

20. If $\underset{\sim}{a} = x\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$, $\underset{\sim}{b} = -3\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, $|2\underset{\sim}{a} + 3\underset{\sim}{b}| = \sqrt{65}$ and $|3\underset{\sim}{a} + 2\underset{\sim}{b}| = \sqrt{85}$, determine the values of x and y.

studyon

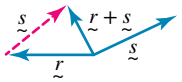
Unit 1 Sit chapter test

Answers

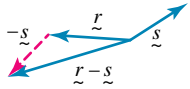
Chapter 2 Vectors in the plane

Exercise 2.2 Vectors and scalars

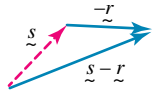
1. a. i.



ii.



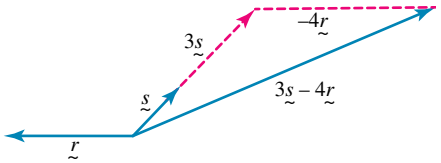
iii.



b. i. Same as **1 a i** except scaled by a factor of 2

ii. Same as **1 a ii** except scaled by a factor of 2

iii.



2. a. $\underset{\sim}{s} + \underset{\sim}{t}$

b. $\underset{\sim}{s} + \underset{\sim}{t} + \underset{\sim}{u} + \underset{\sim}{v}$

c. $-\underset{\sim}{s} - \underset{\sim}{t}$

d. $-\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{t}$

e. $-\underset{\sim}{u} - \underset{\sim}{t} - \underset{\sim}{s}$

3. C

4. a. A to C

b. D to B

c. B to D

d. A to C

5. D

6. a. $\underset{\sim}{r} + \underset{\sim}{s}$

b. $\underset{\sim}{s} + \underset{\sim}{t}$

c. $\underset{\sim}{r} - \underset{\sim}{s}$

d. $\underset{\sim}{r} - \underset{\sim}{s}$

e. $\underset{\sim}{t} - \underset{\sim}{s}$

f. $\underset{\sim}{s} + \underset{\sim}{t} - \underset{\sim}{r}$

g. $\underset{\sim}{r} + \underset{\sim}{s} + \underset{\sim}{t}$

h. $-\underset{\sim}{s} - \underset{\sim}{t}$

7. $2\underset{\sim}{a} + 4\underset{\sim}{b}$

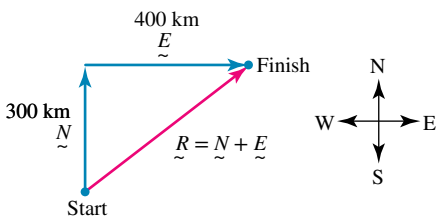
8. $-3\underset{\sim}{a} - 4\underset{\sim}{b}$

9. Displacement, velocity, force

10. Speed, time, length

11. 1 magnitude and 2 angles

12. a, b



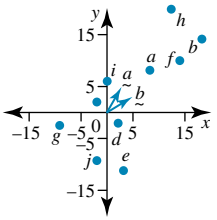
c. 500 km

d. 53.1° clockwise from N

13. 512.1 km; find bearing using trigonometry

14. 721.1 km, 326.3° (clockwise from N)

15. Each part of the answer has coordinates as shown in the diagram *a*, *b*, ... *j*. The original vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are also drawn.



16. Magnitude = 10.77, direction 68.2° True.

17–20. Sample responses can be found in the worked solutions in the online resources.

21. $\underset{\sim}{Q}$

22–24. Sample responses can be found in the worked solutions in the online resources.

25. One can deduce that *x* and *y* components can be added/subtracted/multiplied separately.

Exercise 2.3 Position vectors in the plane

1. a. 3, 4 **b.** 6, −3 **c.** 3.4, $\sqrt{2}$

2. a. i. $6\sqrt{2}$ **ii.** 45°

b. i. $\sqrt{65}$ **ii.** 119.7°

c. i. 4.88 **ii.** 225.8°

d. i. 320.16 **ii.** 358.2°

3. i. a. 045° **ii. a.** $\left[6\sqrt{2}, 45°\right]$

b. 330.3° **b.** $\left[\sqrt{65}, -60.3°\right]$

c. 224.2° **c.** [4.88, 225.8°]

d. 091.8° **d.** [320.16, 358.2°]

4. $\underset{\sim}{w} = -50\hat{\boldsymbol{\imath}} - 50\sqrt{3}\hat{\boldsymbol{\jmath}}$

5. C

6. $248.9\hat{\boldsymbol{\imath}} - 383.3\hat{\boldsymbol{\jmath}}$

7. $60.6\hat{\boldsymbol{\imath}} + 109.3\hat{\boldsymbol{\jmath}}$

8. [615.4, −49.8]

9. 36 steps 11.3° south of east

10. 20.8 km

11. a. $\frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}}$ **b.** $\frac{3}{5}\hat{\boldsymbol{\imath}} - \frac{4}{5}\hat{\boldsymbol{\jmath}}$ **c.** $\frac{4}{5}\hat{\boldsymbol{\imath}} + \frac{3}{5}\hat{\boldsymbol{\jmath}}$

d. $-\frac{4}{5}\hat{\boldsymbol{\imath}} + \frac{3}{5}\hat{\boldsymbol{\jmath}}$ **e.** $\frac{1}{\sqrt{3}}\hat{\boldsymbol{\imath}} - \frac{\sqrt{2}}{\sqrt{3}}\hat{\boldsymbol{\jmath}}$

12. B

13. A sample response can be found in the worked solutions in the online resources.

14. $-0.98\hat{\boldsymbol{\imath}} - 0.20\hat{\boldsymbol{\jmath}}$

15. Sample responses can be found in the worked solutions in the online resources.

16. a. i. $4\hat{\boldsymbol{\imath}} - 7\hat{\boldsymbol{\jmath}}$ **ii.** $\sqrt{65}$

b. i. $3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ **ii.** $\sqrt{10}$

c. i. $-4\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$ **ii.** $\sqrt{65}$

d. i. $-3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ **ii.** $\sqrt{10}$

e. i. $2\hat{\boldsymbol{\imath}}$ **ii.** 2

f. i. $-4\hat{\boldsymbol{\imath}}$ **ii.** 4

17. a. $-4\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$ **b.** $-3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ **c.** $4\hat{\boldsymbol{\imath}} - 7\hat{\boldsymbol{\jmath}}$

d. $3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ **e.** $-2\hat{\boldsymbol{\imath}}$ **f.** $4\hat{\boldsymbol{\imath}}$

18. **a.** $\frac{4}{\sqrt{65}}\hat{\imath} - \frac{7}{\sqrt{65}}\hat{\jmath}$ **b.** $\frac{3}{\sqrt{10}}\hat{\imath} + \frac{1}{\sqrt{10}}\hat{\jmath}$
c. $-\frac{4}{\sqrt{65}}\hat{\imath} + \frac{7}{\sqrt{65}}\hat{\jmath}$ **d.** $-\frac{3}{\sqrt{10}}\hat{\imath} - \frac{1}{\sqrt{10}}\hat{\jmath}$
e. $\hat{\imath}$ **f.** $-\hat{\imath}$

19. **a.** **i.** $\sqrt{29}$ **ii.** $\sqrt{13}$
iii. $\frac{5}{\sqrt{29}}\hat{\imath} - \frac{2}{\sqrt{29}}\hat{\jmath}$ **iv.** $-\frac{2}{\sqrt{13}}\hat{\imath} + \frac{3}{\sqrt{13}}\hat{\jmath}$
v. $3\hat{\imath} + \hat{\jmath}$ **vi.** $\sqrt{10}$
b. Reject, because magnitude is different.

20. **a.** **i.** 5 **ii.** $\sqrt{26}$ **iii.** $-\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{\jmath}$
iv. $\frac{5}{\sqrt{26}}\hat{\imath} - \frac{1}{\sqrt{26}}\hat{\jmath}$ **v.** $2\hat{\imath} + 3\hat{\jmath}$ **vi.** $\sqrt{13}$
b. Reject, because the magnitude is different.

21. **a.** $\sqrt{2}$ **b.** $\sqrt{58}$

22. **a.** $3\hat{\imath}$
b. $5\hat{\jmath}$
c. $3\hat{\imath} + 5\hat{\jmath}$
d. 031.0°T
e. $\sqrt{34}$ km/h

23. 329.0°

24. **a.** $\underset{\sim}{a} = 20\hat{\imath}$, $\underset{\sim}{b} = -15\hat{\jmath}$, $20\hat{\jmath} + 15\hat{\jmath}$
b. 25
c. 053.1°

25. 53.1°, −36.9° Difference = 90°; sample responses can be found in the worked solutions in the online resources.

26. **a.** $-4\hat{\imath} + 12\hat{\jmath}$
b. 341.6°
c. 0.0417 h or 2.5 minutes

Exercise 2.4 Scalar multiplication

1. $\frac{15}{2}$

2. $x = -3, y = -8$

3. $\overrightarrow{OM} = 5\hat{\imath} + 6\hat{\jmath}$

4. **a.** $-\frac{5}{2}$ **b.** $-\frac{15}{2}$

5. **a, b.** Sample responses can be found in the worked solutions in the online resources.
c. $-\frac{20}{3}$ **d.** $\frac{15}{7}$

6. Sample responses can be found in the worked solutions in the online resources.

7. **a.** **i.** $-6\hat{\imath} + 5\hat{\jmath}$
ii. $\sqrt{61}$
iii. $\frac{1}{\sqrt{61}}(-6\hat{\imath} + 5\hat{\jmath})$
b. **i.** $6\hat{\imath} - 8\hat{\jmath}$ **ii.** 10 **iii.** $\frac{1}{5}(-3\hat{\imath} + 4\hat{\jmath})$

8. $m = 2, n = 3$

9. $x = -2, y = 5$

10. **a.** $\pm 2\sqrt{5}$ **b.** $-\frac{12}{5}$

11. **a.** $\pm\sqrt{13}$ **b.** $-\frac{5}{2}$

12. **a.** $x = 4, y = -2$
b. $m = -5, n = -4$
c. $m = \frac{1}{2}, n = \frac{5}{4}$

Exercise 2.5 The scalar (dot) product

1. 23.99

2. Dot product = 24; more accurate, since no angle needed

3. **a.** 15 **b.** 18 **c.** −31 **d.** −26
e. 1 **f.** −20 **g.** 3 **h.** 2

4. D.

5. Sample responses can be found in the worked solutions in the online resources.

6. 8

7, 8. Sample responses can be found in the worked solutions in the online resources.

9. D

10. B

11. D

12. C

13. −36

14. **a.** 16 **b.** −1 **c.** −25 **d.** 0

15. **a.** 67° **b.** 93° **c.** 77° **d.** 90°

16. D

17. D

18. $m = 1$

19. $m = \frac{3}{2}$

20. $4\hat{\imath} + 8\hat{\jmath}$

21. $\frac{64}{5}\hat{\imath} - \frac{48}{5}\hat{\jmath}$

Exercise 2.6 The projections of vectors — scalar and vector resolutes

1. **a.** **i.** $\frac{23\sqrt{13}}{13}$ **ii.** $\frac{23\sqrt{41}}{41}$
b. **i.** $\frac{17\sqrt{29}}{29}$ **ii.** $\frac{17\sqrt{10}}{10}$
c. **i.** $-\frac{13\sqrt{10}}{10}$ **ii.** $-\frac{26\sqrt{17}}{17}$
d. **i.** $-\frac{6\sqrt{13}}{13}$ **ii.** $-\frac{6}{5}$
e. **i.** $-\frac{23}{5}$ **ii.** $\frac{23\sqrt{26}}{13}$

2. **a.** **i.** $\frac{1}{\sqrt{10}}$
ii. $\underset{\sim}{b}_{\parallel} = \frac{3}{10}\hat{\imath} - \frac{1}{10}\hat{\jmath}$
iii. $\underset{\sim}{b}_{\perp} = \frac{17}{10}\hat{\imath} + \frac{51}{10}\hat{\jmath}$
b. **i.** $2\sqrt{41}$
ii. $\underset{\sim}{b}_{\parallel} = 8\hat{\imath} + 10\hat{\jmath}$
iii. $\underset{\sim}{b}_{\perp} = \underset{\sim}{0}$
c. **i.** 0
ii. $\underset{\sim}{b}_{\perp} = \underset{\sim}{0}$
iii. $\underset{\sim}{b}_{\perp} = -3\hat{\imath} + 4\hat{\jmath}$
d. **i.** $\frac{3}{\sqrt{2}}$
ii. $\underset{\sim}{b}_{\parallel} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$
iii. $\underset{\sim}{b}_{\perp} = \frac{1}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}$
e. **i.** $-\frac{5}{\sqrt{13}}$
ii. $\underset{\sim}{b}_{\parallel} = \frac{-10}{13}\hat{\imath} - \frac{15}{13}\hat{\jmath}$
iii. $\underset{\sim}{b}_{\perp} = \frac{36}{13}\hat{\imath} - \frac{24}{13}\hat{\jmath}$
f. **i.** $\frac{2}{\sqrt{10}}$
ii. $b_{\parallel} = \frac{3}{5}\hat{\imath} + \frac{1}{5}\hat{\jmath}$
iii. $\underset{\sim}{b}_{\perp} = -\frac{3}{5}\hat{\imath} + \frac{9}{5}\hat{\jmath}$

3. **a.** 3.6 km **b.** 0.2 km or 200 metres

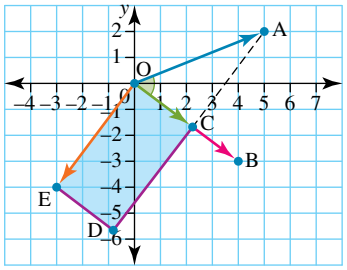
4. 316 metres

5. **a.** $\frac{10}{\sqrt{5}}$ **b.** $2(\hat{\imath} - 2\hat{\jmath})$ **c.** $2\hat{\imath} + \hat{\jmath}$

6. **a.** $\frac{18}{25}(3\hat{\imath} - 4\hat{\jmath})$ **b.** $-\frac{1}{25}(4\hat{\imath} + 3\hat{\jmath})$

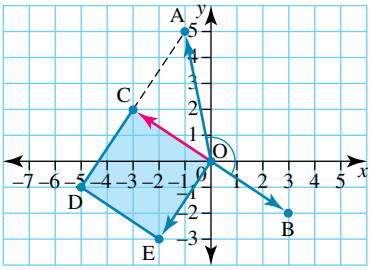
7. **a.** 2.8

b. Sample responses can be found in the worked solutions in the online resources.



8. a. $-\sqrt{13}$

b, c. Sample responses can be found in the worked solutions in the online resources.



9. a. i. $\frac{1}{\sqrt{2}}(\hat{\imath}+\hat{\jmath})$ **ii.** $\sqrt{2}$

iii. $\hat{\imath}+\hat{\jmath}$ **iv.** $2\hat{\imath}-2\hat{\jmath}$

b. $\frac{2}{5}(2\hat{\imath}-\hat{\jmath})$

c. $\frac{1}{25}(76\hat{\imath}+57\hat{\jmath})$

10. a. 5, 5 **b.** 13, 13

c. 25, 25 **d.** $\sqrt{a^2+b^2}, \sqrt{a^2+b^2}$

11. a. i. 7 **ii.** 0 **iii.** -4

b. i. $\sqrt{77}$ **ii.** $\sqrt{93}$ **iii.** $2\sqrt{142}$

12. a. $3, -\frac{1}{3}$ **b.** 2

13. a. $\pm\frac{1}{5}(4\hat{\imath}+3\hat{\jmath})$ **b.** $\pm\frac{1}{13}(12\hat{\imath}+5\hat{\jmath})$

c. $\pm\frac{1}{25}(24\hat{\imath}-7\hat{\jmath})$ **d.** $\pm\frac{1}{\sqrt{a^2+b^2}}(b\hat{\imath}-a\hat{\jmath})$

2.7 Review: exam practice

1. $11\hat{\imath}-22\hat{\jmath}$

2. a. $(5+5\sqrt{2})\hat{\imath}+(5+5\sqrt{2})\hat{\jmath}$

b. 17.07 km

3. 90°

4. C

5. 6

6. 1.3909

7. a. $-\hat{\imath}-4\hat{\jmath}$ **b.** $7\hat{\imath}-6\hat{\jmath}$ **c.** -17

d. $\frac{3}{\sqrt{34}}\hat{\imath}-\frac{5}{\sqrt{34}}\hat{\jmath}$ **e.** 135°

8. $\frac{9\pm\sqrt{69}}{2}$

9. $\frac{-4}{(\sqrt{3})}$

10. $-\frac{4}{5}(i-2j)$

11. a. $-\hat{\imath}+6\hat{\jmath}$

b. Sample responses can be found in the worked solutions in the online resources.

c. $\hat{\imath}-2\hat{\jmath}$

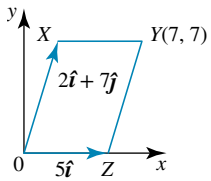
12. Sample responses can be found in the worked solutions in the online resources.

13. a. $x=5, y=2$

b. $x=12$

c. $y=3$

14. a.



b. $2\hat{\imath}+7\hat{\jmath}, -5\hat{\imath}$

c. $7\hat{\imath}+7\hat{\jmath}, -3\hat{\imath}+7\hat{\jmath}$

d. $\frac{2\sqrt{29}}{29}$

e. 74.1°

f. $2\hat{\imath}$

g. $(-2, 7)$

h. 35 square units

15. a. $\pm 2\sqrt{13}$ **b.** $\frac{18}{5}$ **c.** $-\frac{5}{2}$ **d.** 4

16. a. $\frac{1}{\sqrt{5}}(2\hat{\imath}-\hat{\jmath})$ **b.** $\frac{13}{\sqrt{5}}$ **c.** 24.775°

d. $\frac{13}{5}(2\hat{\imath}-\hat{\jmath})$ **e.** $-\frac{6}{5}(\hat{\imath}+2\hat{\jmath})$

17. Sample responses can be found in the worked solutions in the online resources.

18. a. (3, 5.5, 0)

b. $-3\hat{\imath}+4\underset{\sim}{k}$

c. Sample responses can be found in the worked solutions in the online resources.

d. 0.066 litres

e. $3\hat{\imath}+5.5\hat{\jmath}+4\underset{\sim}{k}$

f. 7.43 cm

g. 84.5°

19. $x=-3, y=5$ or $x=\frac{75}{13}, y=-\frac{11}{13}$

20. $x=5, y=4$ or $x=3.24, y=-1.22$

CHAPTER 3
Applications of vectors in the plane

3.1 Overview

3.1.1 Introduction

In Chapter 2, we discussed the theory of vectors. As vectors allow us to consider motion and force in three dimensions, they are the ideal mathematical tool for dealing with the motion of objects and the forces acting on them. In this chapter, we will consider the applications of vectors — displacement, velocity, forces in equilibrium and relative velocity.

An understanding of the causes of motion is fundamental to our understanding of the world. All around us objects are in motion, from tiny particles to planets.

Sir Isaac Newton was an English mathematician and physicist in the seventeenth century. He is considered one of the most influential scientists of all time. In 1665, when Cambridge University was closed due to an outbreak of the bubonic plague, Newton was sent home to his family's farm at Woolsthorpe Manor. There he spent time formulating ideas that would lead to revolutionary advances in mathematics, optics, physics and astronomy. (The event of an apple falling from the tree and inspiring his theories on gravity is thought to have occurred during this time.)

The publication of Newton's *Philosophiae Naturalis Principia Mathematica* in 1687 was one of the keys to unlocking the mysteries of the physical world. His three simple laws accurately describe the motion of large objects, and with his mathematical descriptions of mechanics and gravity, lunar and planetary motion can be explained. The mathematical models embodied in his First and Second Laws of Motion are based on the effect that forces have on the acceleration of objects treated as point particles.

LEARNING SEQUENCE

3.1 Overview
3.2 Displacement and velocity
3.3 Force and the triangle of forces
3.4 Force and the state of equilibrium
3.5 Relative velocity
3.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 Displacement and velocity

3.2.1 Applications of vector addition

Technology will be required throughout this chapter where trigonometric ratios for angles other than the angles 30°, 45° and 60° need to be determined.

Vectors have applications in surveying, navigation, orienteering and many other areas where problems need to be solved that involve displacement and velocity.

When considering a vector problem, draw a diagram and use the properties of two-dimensional vectors to help solve the problem.

WORKED EXAMPLE 1

A man walks 4 km due north, then 3 km due west. If $\hat{i}$ and $\hat{j}$ represent unit vectors of 1 kilometre in the directions of east and north respectively, determine:

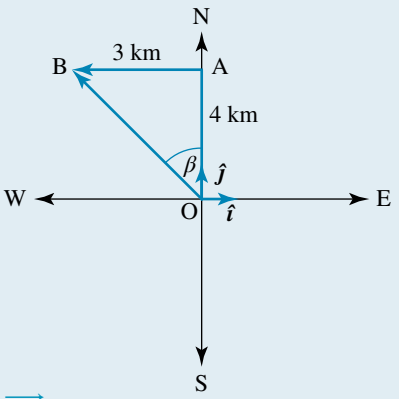
a. his position vector

b. his displacement and true bearing from his starting point.

THINK

1. Let the man start at point O. He then walks 4 km north to point A, then 3 km west to point B.

WRITE



2. Find the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$. As $\hat{i}$ represents a unit vector in an easterly direction, the vector $\overrightarrow{AB}$ in a westerly direction will be negative.

$\overrightarrow{OA} = 4\hat{j}$

$\overrightarrow{AB} = -3\hat{i}$

3. Use vector addition, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$, to find his position vector.

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\overrightarrow{OB} = -3\hat{i} + 4\hat{j}$

4. The distance OB is the magnitude of this vector. It can be abbreviated to $d(\text{OB})$.

$d(\text{OB}) = \sqrt{(-3)^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5 \text{ km}$

5. Use direction cosines to find the bearing.

$$\cos(\beta) = \frac{4}{5}$$
$$\beta = \cos^{-1}\left(\frac{4}{5}\right)$$
$$\beta = 36.87°$$
$$\beta = 36°52'$$

Alternatively, $360 - 36°52' = 323°8'$ as a true bearing.

6. State the final answer in a sentence.

The man's displacment is 5 km on a bearing 323°8′T.

Although the next two worked examples could be solved using the sine and cosine rules, they can be now solved using the method of resolution of vectors, which is usually an easier method.

WORKED EXAMPLE 2

A group of hikers walk 3 km due south, then turn and walk 2 km on a bearing S25°W. If $\hat{i}$ and $\hat{j}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:

a. the position vector of the hikers from their initial starting point

b. the hikers' distance and bearing from their starting point.

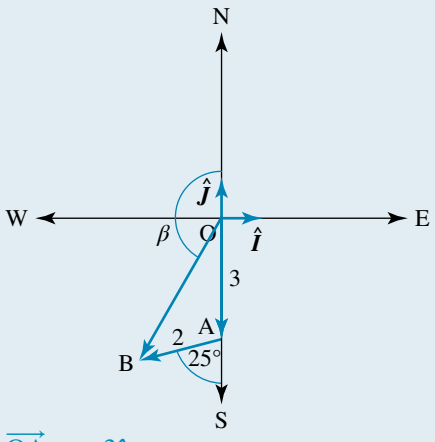
Give your answers correct to 3 decimal places.

THINK

WRITE

a. 1. Draw a diagram, labelling the starting point as the origin, O, the first point as A and their final point as B.

a.



2. The hikers walk 3 km south from O to the point A. As $\hat{j}$ represents a unit vector in a northerly direction, the southerly vector $\overrightarrow{OA}$ will be negative.

$$\overrightarrow{OA} = -3\hat{j}$$

3. They walk from A to B, a distance of 2 km on a bearing S25°W. Resolve the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = -2\sin(25°)\hat{i} - 2\cos(25°)\hat{j}$$

4. Use vector addition to find $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$, which represents their position vector, and group like components.
5. Use a calculator to find the position vector correct to 3 decimal places.

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= -3\hat{j} + (-2\sin(25°)\,\hat{i} - 2\cos(25°)\hat{j}) \\ &= -2\sin(25°)\,\hat{i} - (3 + 2\cos(25°))\hat{j}\end{aligned}$$

$$\overrightarrow{OB} = -0.845\hat{i} - 4.813\hat{j}$$

b. 1. The distance from O to B is the magnitude of this vector.

b. $$\begin{aligned}d(\text{OB}) &= \left|\overrightarrow{OB}\right| \\ &= \sqrt{(-0.845)^2 + (-4.813)^2} \\ &= \sqrt{23.88} \\ &= 4.886\end{aligned}$$

2. The bearing is the angle that the vector $\overrightarrow{OB}$ makes with the y-axis or the north direction.

$$\begin{aligned}\cos(\beta) &= \frac{-4.813}{4.8857} \\ \beta &= \cos^{-1}(0.2016) \\ &= 170.04° \\ &= 170°2'\end{aligned}$$

3. Alternative bearings are possible.

$360° - 170°2' = 189°58'$ as a true bearing.

4. State the final answer in a sentence.

The hikers are 4.89 km from their starting point on a bearing 189°58′T.

3.2.2 Applications using vector subtraction

Sometimes when solving vector problems, we may need to subtract two vectors.

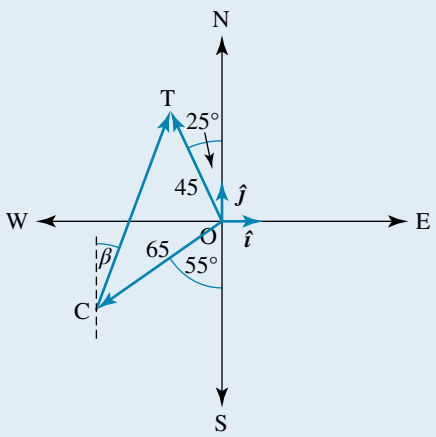
WORKED EXAMPLE 3

A train leaves a point O and travels a distance of 45 km on a bearing N25°W. At the same time a car leaves the same point O and travels a distance of 65 km on a bearing S55°W. If $\hat{i}$ and $\hat{j}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine the position vector of the train from the car. Hence, calculate the final distance correct to 2 decimal places and the true bearing of the train from the car.

THINK

1. Draw a diagram, labelling the origin, O, the train at point T and the car at the point C. The vectors $\overrightarrow{OT}$ and $\overrightarrow{OC}$ represent the position vectors of the train and car respectively.

WRITE



2. By resolving the vector $\overrightarrow{OT}$, write down the position vector of the train.	$\overrightarrow{OT} = -45\sin(25°)\hat{\imath} + 45\cos(25°)\hat{\jmath}$
3. By resolving the vector $\overrightarrow{OC}$, write down the position vector of the car.	$\overrightarrow{OC} = -65\sin(55°)\hat{\imath} - 65\cos(55°)\hat{\jmath}$
4. The position vector of the train from the car is given by $\overrightarrow{CT}$. This vector is found using subtraction of the two vectors $\overrightarrow{OT}$ and $\overrightarrow{OC}$.	$\overrightarrow{OT} = \overrightarrow{OC} + \overrightarrow{CT}$ $\overrightarrow{CT} = \overrightarrow{OT} - \overrightarrow{OC}$
5. Substitute for the two vectors.	$\overrightarrow{CT} = (-45\sin(25°)\hat{\imath} + 45\cos(25°)\hat{\jmath})$ $-(-65\sin(55°)\hat{\imath} - 65\cos(55°)\hat{\jmath})$
6. Use the rules for subtraction of vectors.	$\overrightarrow{CT} = (65\sin(55°) - 45\sin(25°))\hat{\imath}$ $+ (45\cos(25°) + 65\cos(55°))\hat{\jmath}$
7. Using a calculator, give the position vector correct to 3 decimal places.	$\therefore \overrightarrow{CT} = 34.227\hat{\jmath} + 78.066\hat{\jmath}$
8. The distance of the train from the car is the magnitude of this vector.	$\left\lvert\overrightarrow{CT}\right\rvert = \sqrt{34.227^2 + 78.066^2}$ $= \sqrt{7265.788}$ $= 85.24$
9. The bearing is the angle that the vector $\overrightarrow{CT}$ makes with the north direction.	$\cos(\beta) = \frac{78.066}{85.24}$ $\beta = \cos^{-1}(0.9158)$ $= 23.675$ $= 23°40'$
10. State the final answers in a sentence. Note that other possible equivalent bearings are also acceptable.	The train is at a final distance of 85.24 km from the car on a bearing 23°40′T.

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 1

Applications of vector addition and subtraction Summary screen and practice questions

Exercise 3.2 Displacement and velocity

Technology active

1. **WE1** A train leaves Roma Street Station and travels 7 km due east, then turns a corner and travels 24 km south. If $\hat{\imath}$ and $\hat{\jmath}$ represent unit vectors of 1 kilometre in the directions of east and north respectively, determine:
   a. the position vector of the train
   b. the train's displacement from its starting point.

2. A jogger runs 1200 metres due south, then 500 metres due west. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of 1 metre in the directions of east and north respectively, determine:
   a. the position vector of the jogger from her initial starting point
   b. her displacement from her starting point.

3. **WE2** A plane flies 35 km due south, then turns and travels 25 km on a bearing S35°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. the plane's position vector from its initial starting point
   b. the plane's distance (to 2 decimal places) and bearing from its starting point.

4. A ship moves 70 km due west, then turns and moves 240 km on a bearing N53°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. the ship's position vector from its initial starting point
   b. the ship's distance and bearing from its starting point.
5. **WE3** Amanda leaves a point O and runs a distance of 6 km on a bearing N37°E. At the same time Brianna leaves the same point O and walks a distance of 2 km on a bearing S48°E. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. the position vector of Amanda from Brianna
   b. the final distance (correct to 2 decimal places) and the true bearing of Amanda from Brianna.
6. A train leaves a point O and travels at a speed of 100 km/h on a bearing N34°W. At the same time a car leaves the same point O and travels at a speed of 60 km/h on a bearing S50°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. the position vector of the train from the car after 30 minutes
   b. the distance between the car and the train and the true bearing of the train from the car at this time.
7. Petra is an avid horse rider. One day she leaves the stable and rides 3 km due east, then turns and rides 2 km 138°T. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. Petra's position vector from the stable
   b. Petra's distance in kilometres, correct to 2 decimal places, and true bearing from the stable.

8. A yacht moves 1600 metres due south from a buoy, then turns and travels 800 metres on a bearing 158°T. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 metre in the directions of east and north respectively, determine (to 2 decimal places):
   a. the yacht's position vector from the buoy
   b. the yacht's distance, correct to the nearest metre, and true bearing from the buoy.
9. A bus leaves a depot and travels 4 km due west, then turns and travels 6 km on a bearing N58°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
   a. the position vector of the bus from the depot
   b. the bus's distance in kilometres, correct to 2 decimal places, and bearing from the depot.
10. A water skier leaves a point O and travels a distance of 300 metres on a bearing N27°W. At the same time a boat leaves the same point O and travels a distance of 800 metres on a bearing S39°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 metre in the directions of east and north respectively, determine:
    a. the final position vector of the water skier from the boat
    b. the distance in metres, correct to 1 decimal place, and the true bearing of the water skier from the boat in their final position.
11. A plane leaves an airport and travels at 300 km/h on a bearing S48°W. At the same time a helicopter leaves the same airport and travels at 90 km/h on a bearing E26°S. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine:
    a. the position vector of the helicopter from the plane after 30 minutes
    b. the distance, correct to the nearest kilometre, and the true bearing of the helicopter from the plane after 30 minutes.
12. A police car leaves a station and travels a distance of d_1 km due east, then turns and travels a further distance of d_2 on a bearing θ°T, where $0° < \theta < 90°$. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine the position vector and distance of the police car from the station.

13. A taxi leaves a point O and travels a distance of d_1 km on a bearing Eθ_1°N, where $0° < \theta_1 < 90°$. At the same time a motorbike leaves the same point O and travels a distance of d_2 km on a bearing Eθ_2°S, where $0° < \theta_2 < 90°$. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 kilometre in the directions of east and north respectively, determine the final position vector and distance of the taxi from the motorbike.

3.3 Force and the triangle of forces

3.3.1 What is a force?

We all know from experience what a **force** is. It is a 'push' or a 'pull'. The force due to gravity acts on us all the time. A bar magnet repels a second bar magnet: a magnetic force acts here. The force of friction slows down the wheel of a bike when the brakes are applied. Air resistance retards the motion of athletes. In both these cases there is relative motion between two objects. The strings of a tennis racquet when stretched exert a force on a tennis ball while the strings and ball are in contact. In all these examples, objects which have **unbalanced forces** acting on them tend to undergo a change in their motion; objects which have a **balanced set of forces** acting on them maintain their motion. Objects under balanced forces are said to be in **equilibrium**.

The forces that we will discuss in this chapter can be classified as one of three types: field forces, applied forces, and resistive forces.

Field forces

Field forces act without physical contact between objects. The weight force which acts on an object due to the presence of an external gravitational field or the electric force which influences the motion of a charged particle are examples of field forces.

Applied forces

Applied forces are the pushes or pulls exerted on objects due to contact. They are forces with which we have daily experience. The normal contact force acting upwards on a book resting on a table or on us as we stand on the floor or sit on a chair are examples of applied forces. Other examples of applied forces include tensile forces in taut strings and cables (as in the cable used by a crane or rescue helicopter), and compressive forces acting on weight-bearing rods.

Resistive forces

Air drag and friction are examples of **resistive forces**. This type of force occurs when two objects move or attempt to move relative to one another. Air drag has been put to good use in the design of hang glider; it is found also, in the resistance between a moving body like a car and the air. An example of friction is seen in a bicycle that is slowing down on level ground, even without the brakes being applied.

3.3.2 What is a particle?

We are all familiar with the notion of a **particle**, but in Newtonian dynamics a particle is used to model an object and is taken to be a point. That is, the size of the object is not relevant and any internal movements such

as spin and change in shape are not included in the model. A particle can, in principle, model the movement of a bullet, a car, a diver or a planet, provided that we accept the model as an approximation to the motion of the real objects they represent. Any force that acts on a particle is said to act through the *point* that defines the position of the particle.

For example, if we were to describe the various forces acting on a car, we would include the upwardly directed contact forces exerted on the car by the road at each of the four tyres in addition to frictional forces exerted on the tyres by the road. If the car was moving, we would add the drag forces due to the movement of the car through the air. There would be the weight force acting on the car due to the gravitational field of the earth. If we were to treat the car as a particle, we would describe all these forces as acting through a single point. The word *particle* serves to define the position of an object and sets it apart from the rest of the immediate environment.

3.3.3 What assumptions do we make in Newtonian dynamics?

In modelling motion we make the following assumptions so that problems can be solved to give reasonably accurate predictions.

Term	Meaning
Light (body or string)	Object has no mass.
Smooth	No frictional forces are exerted.
Inextensible	Strings or ropes do not stretch.
Rigid	Objects do not change shape when forces are applied to them.
Perfectly elastic	Applied forces do not permanently deform an object (for example, a spring).

In many cases we ignore the presence of forces which would be insignificant, such as air drag on slowly moving objects.

3.3.4 The resultant force, $\underset{\sim}{R}$

Crucial to a good understanding of Newton's laws of motion is the concept of a **net or resultant force** acting on a particle. Force is a vector quantity because a force has not only a magnitude but also a direction. The unit of force is the **newton (N)**. (This is a *derived unit*, which is simply one defined in terms of the standard units, namely distance, time and mass. This will be discussed later.)

Forces can be described in two ways:

1. Using $\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ notation
   For example, $\underset{\sim}{F} = 2\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}}$ N (only coplanar forces will be considered).
2. Using size and direction notation.
   For example, $\underset{\sim}{F} = 200$ N, N45°E is a force of magnitude 200 N directed at an angle 45° from north towards east.

The net or resultant force is simply the vector sum of all real physical forces acting on the particle. It represents the sum or total force acting on a particle representing an object. It is not in itself a real force, only the sum of real forces.

The net or resultant force acting on a particle is the vector sum of all real forces acting on that particle.

Force diagrams

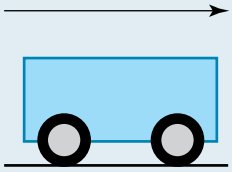
Individual forces are one of three types — field, contact or resistive — and are drawn as vectors which indicate their direction and magnitude.

WORKED EXAMPLE 4

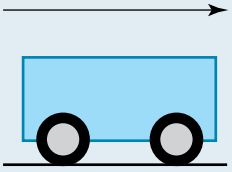
Draw 'vector diagrams' to represent the forces involved in the situations shown below as a set of vectors acting on a point particle. Indicate the relative size of the force by the length of the vector arrows. Further indicate the nature of each of the forces acting by labelling them $\underset{\sim}{W}$ for weight, $\underset{\sim}{N}$ for normal contact, $\underset{\sim}{F}$ for friction, $\underset{\sim}{A}$ for applied force and $\underset{\sim}{D}$ for air drag or air resistance.

a. A stationary person

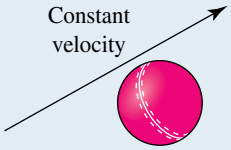
b. Constant velocity



c. Accelerating

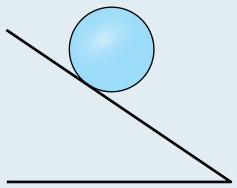


d.



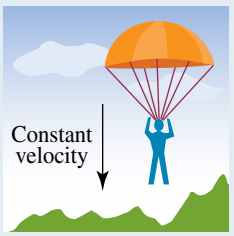
Cricket ball through the air

e.



Ball rolling down a slope

f.



THINK

WRITE

In the force diagrams for **a** to **f**, treat each object as a particle.

a. There are two equal, opposing forces: the weight force down, which is a field force, and the normal contact force up.

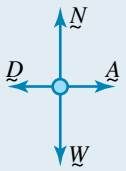
a.

b. 1. A cart moving at constant velocity has balanced horizontal forces.

2. Any resistive forces ($\underset{\sim}{D}$) must be balanced by an applied force ($\underset{\sim}{A}$) to keep the cart moving at constant velocity.

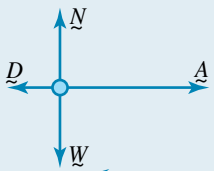
3. The cart will have balanced vertical forces arising from the weight force of the cart and the normal contact force.

b.



c. An accelerating cart will have balanced vertical forces as in **a** and **b** but unbalanced horizontal forces giving rise to an acceleration.

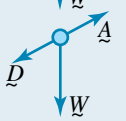
c.



d. 1. A ball moving through the air will have the vertical force of weight and the resistive force of air drag.

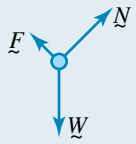
2. As the ball is moving at constant velocity, the drag force will be in the opposite direction to that of the ball and the applied force acting on it to maintain constant velocity.

d.



e. A ball rolling down a slope will have a weight force directed vertically down, a normal contact force perpendicular to the slope and a resistive or frictional force.

e.



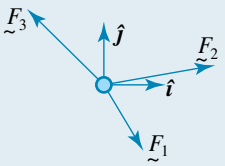
f. The parachutist has a constant velocity (i.e. is not accelerating) so will have two balanced vertical forces: one down due to the weight force and the second upwards due to air drag.

f.

WORKED EXAMPLE 5

Three forces — $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ — act on a ball as shown in the force vector diagram. The three forces are described by the vectors:

$\underset{\sim}{F}_1 = 4\hat{i} - 5\hat{j}$
$\underset{\sim}{F}_2 = 10\hat{i} + 2\hat{j}$
$\underset{\sim}{F}_3 = -6\hat{i} + 7\hat{j}$.



a. Determine the resultant force $\underset{\sim}{R}$, the sum of the three forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$.

b. Determine the magnitude of the resultant force $|\underset{\sim}{R}|$.

c. Determine the angle that the vector $\underset{\sim}{R}$ makes with the $\hat{i}$ vector.

d. The force $\underset{\sim}{F}_1$ is changed so that $\underset{\sim}{R} = 0$; that is, the resultant force equals zero. Determine the force $\underset{\sim}{F}_1$.

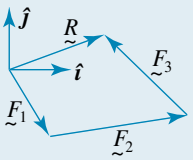
THINK

a. 1. The resultant force $\underset{\sim}{R}$ is the sum of the forces $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$.

2. Evaluate $\underset{\sim}{R}$.

WRITE

a.



$\underset{\sim}{R} = \sum \underset{\sim}{F}$
$= \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$
$= 4\hat{i} - 5\hat{j} + 10\hat{i} - 2\hat{j} - 6\hat{i} + 7\hat{j}$
$= 8\hat{i} + 4\hat{j}$

b. 1. The magnitude of the vector $\underset{\sim}{R}$ is its length.

2. The symbol $|\underset{\sim}{R}|$ or R is used to represent the magnitude of a vector $\underset{\sim}{R}$.

3. Recall that $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 1$ and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = 0$, and evaluate $|\underset{\sim}{R}|$.

b. $|\underset{\sim}{R}| = \sqrt{\underset{\sim}{R} \cdot \underset{\sim}{R}}$

$= \sqrt{(8\hat{\imath} + 4\hat{\jmath}) \cdot (8\hat{\imath} + 4\hat{\jmath})}$

$= \sqrt{8^2 + 4^2}$

$= \sqrt{80}$

$= 4\sqrt{5}$

c. 1. Define θ.

2. Then use the result for the dot product of two vectors $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$.

3. Evaluate θ to the nearest tenth of a degree.

c. Let the angle that $\underset{\sim}{R}$ makes with $\hat{\imath}$ be θ.

Let $\underset{\sim}{R} \cdot \hat{\imath} = |\underset{\sim}{R}|\,|\hat{\imath}| \cos\theta$.

$$\cos\theta = \frac{(8\hat{\imath} + 4\hat{\jmath}) \cdot \hat{\imath}}{4\sqrt{5} \times 1}$$

$$\cos\theta = \frac{8}{4\sqrt{5}}$$

$\approx 26{\cdot}6°$

d. 1. The vector sum of all forces is now equal to zero. Set $\underset{\sim}{R} = 0$.

2. Make $\underset{\sim}{F}_1$ the subject of the equation.

3. Substitute $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ into the equation and simplify to find $\underset{\sim}{F}_1$.

d. $\underset{\sim}{R} = \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3$

$= 0$

$\underset{\sim}{F}_1 = -(\underset{\sim}{F}_2 + \underset{\sim}{F}_3)$

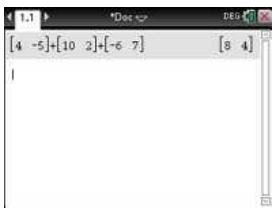
$\underset{\sim}{F}_1 = -(10\hat{\imath} + 2\hat{\jmath} - 6\hat{\imath} + 7\hat{\jmath})$

$= -4\hat{\imath} - 9\hat{\jmath}$

TI | THINK

a.1. On a Calculator page, complete the entry line as: $[4\ -5] + [10\ 2] + [-6\ 7]$ then press ENTER.
Note: The matrix template can be found by pressing the templates button.

WRITE



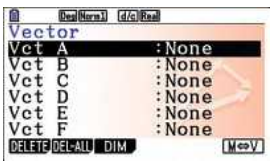
2. The answer appears on the screen.

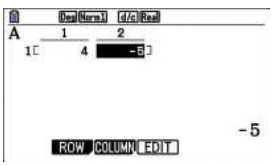
$\underset{\sim}{R} = 8\hat{\imath} + 4\hat{\jmath}$

CASIO | THINK

a.1. On the Run-Matrix screen, select MAT/VCT by pressing F3.
Select M ⇔ V by pressing F6 to switch from the Matrix screen to the Vector screen.
To define $\underset{\sim}{F}_1$ as Vector A, highlight VCT A and press EXE.
Change the dimension to 1×2, then press EXE.
Enter the values 4 and –5 into the matrix template. then press EXIT.
Repeat these steps to define $\underset{\sim}{F}_2$ as Vector B and $\underset{\sim}{F}_3$ as Vector C.

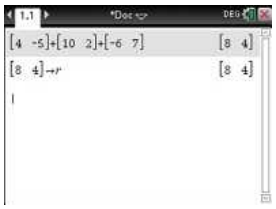
WRITE





2. On the Run-Matrix screen, complete the entry line as:
Vct A+Vct B+VctC
then press EXE.
Note: To find 'Vct', press OPTN, then select MAT/VCT by pressing F2. Press F6 twice to scroll to more menu options, then select Vct (by pressing F1).

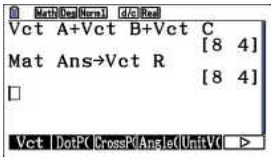
3. Press VAR and type 'r' to store the result as $\underset{\sim}{R}$.



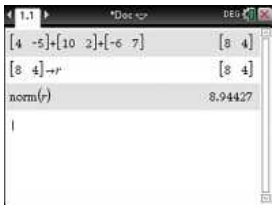
3. The answer appears on the screen.

$\underset{\sim}{R} = 8\hat{\imath} + 4\hat{\jmath}$

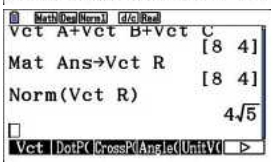
4. Press the 'Store' button, then select Vct (by pressing F1) and type 'R'. This will store the result as $\underset{\sim}{R}$.



b.1. Press MENU, then select:
7: Matrix & Vector
7: Norms
1: Norm.
Complete the entry line as:
norm(r)
then press ENTER.



b.1. Press OPTN, then select MAT/VCT by pressing F2. Press F6 three times to scroll to more menu options, then select Norm (by pressing F1).
Complete the entry line as:
Norm(Vct R)
then press EXE.



2. The answer appears on the screen.
Note: The calculator will give a decimal answer, not an exact answer.

$|\underset{\sim}{R}| = 8.944$

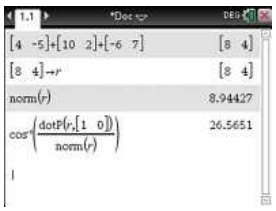
2. The answer appears on the screen.

$|\underset{\sim}{R}| = 4\sqrt{5}$

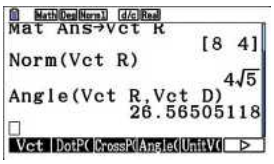
c.1. Complete the next entry line as:

$$\cos^{-1}\left(\frac{\text{dotP}(r, [1 \quad 0])}{\text{norm}(r)}\right)$$

then press ENTER.
Note: 'dotP(' can be found by pressing MENU and then selecting:
7: Matrix & Vector
C: Vector
3: Dot Product.
Similarly, 'norm' can be found by pressing MENU and then selecting:
7: Matrix & Vector
7: Norms
1: Norm.



c.1. Use the steps outlined in **a1** to define the vector $\hat{\imath} + 0\hat{\jmath}$ as Vector D.
Press OPTN, then select MAT/VCT by pressing F2. Press F6 twice to scroll to more menu options, then select Angle (by pressing F4).
Complete the entry line as:
Angle(Vct R, Vct D)then press EXE.



2. The answer appears on the screen.

The angle is 26.6°.

2. The answer appears on the screen.

The angle is 26.6°

d.1. Find an expression for $\underset{\sim}{F}_1$.

$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 0$

$\therefore \underset{\sim}{F}_1 = 0 - \underset{\sim}{F}_2 - \underset{\sim}{F}_3$

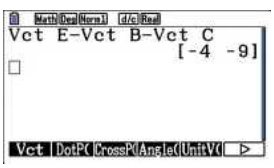
d.1. Find an expression for $\underset{\sim}{F}_1$.

$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 0$

$\therefore \underset{\sim}{F}_1 = 0 - \underset{\sim}{F}_2 - \underset{\sim}{F}_3$

2. Complete the next entry line as:
[0 0] – [10 2] – [−6 7]
then press ENTER.

2. Use the steps outlined in **a1** to define the vector $0\hat{\imath} + 0\hat{\jmath}$ as Vector E.
Complete the next entry line as:
Vct E – Vct B – Vct C
then press EXE.



3. The answer appears on the screen.

$\underset{\sim}{F}_1 = -4\hat{\imath} - 9\hat{\jmath}$

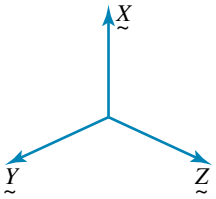
3. The answer appears on the screen.

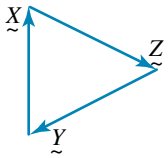
$\underset{\sim}{F}_1 = -4\hat{\imath} - 9\hat{\jmath}$

The triangle of forces

If three non-parallel forces acting on a particle have a resultant force of zero then the three forces can be represented in a triangle since the vector sum of the forces is zero.

From the figure, if $\underset{\sim}{R} = \underset{\sim}{X} + \underset{\sim}{Y} + \underset{\sim}{Z} = 0$, then the three forces $\underset{\sim}{X}$, $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ can be represented in a triangle representing the magnitude and direction of the three forces.

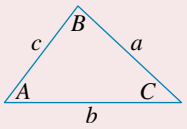




The advantage of representing three forces in a triangle is that the sine rule and/or cosine rule can be used to solve some problems involving three forces whose vector sum is zero. That is:

Sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$



WORKED EXAMPLE 6

Three forces — $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ — act on an object such that the resultant force is zero. The force $\underset{\sim}{A}$ acts at an angle of 150° to the force $\underset{\sim}{B}$ and they have the same magnitude of 20 N.

a. Determine the magnitude of $\underset{\sim}{C}$.

b. Calculate the angle that the force $\underset{\sim}{C}$ makes with $\underset{\sim}{B}$ to the nearest degree.

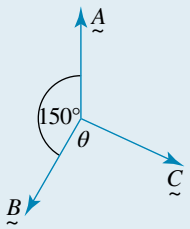
THINK

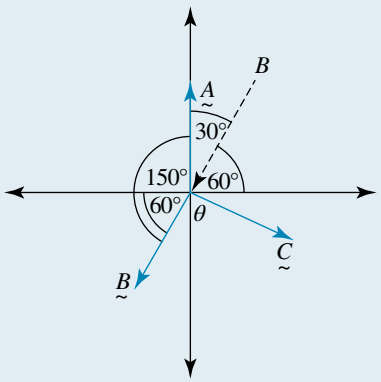
a. 1. Draw a force vector diagram of three forces acting through a point with an angle of 150° between $\underset{\sim}{A}$ and $\underset{\sim}{B}$ and an angle of θ between $\underset{\sim}{B}$ and $\underset{\sim}{C}$.

2. To determine the angle between A and B, draw a sketch.

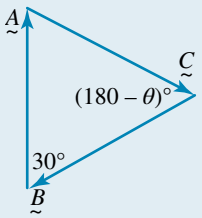
WRITE

a.





3. Since the resultant of the three forces is zero, place them in a triangle of forces.
4. Mark the angle between $\underset{\sim}{A}$ and $\underset{\sim}{B}$ as $180° - 150° = 30°$ and the angle between $\underset{\sim}{B}$ and $\underset{\sim}{C}$ as $180° - \theta$.
5. Recall the cosine rule and substitute $a = C$, $b = 20$, $c = 20$ and $A = 30°$.
6. Solve for C to find the magnitude of $\underset{\sim}{C}$.
7. State the solution.



$a^2 = b^2 + c^2 - 2bc\cos A$

$C^2 = 20^2 + 20^2 - 2 \times 20 \times 20\cos 30°$

$= 107.18$

$C = 10.35$

The magnitude of force $\underset{\sim}{C}$ is approximately 10.35 newtons.

b. 1. Recall the sine rule and substitute $a = 20$, $A = 180° - \theta$, $b = 10.35$ and $B = 30$.
2. Invert both sides of the equation.
3. Make $\sin(180 - \theta)°$ the subject of the equation.
4. Find the value of $180° - \theta$.
5. Solve for θ.
6. State the solution to the nearest degree.

b. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

$\dfrac{20}{\sin(180 - \theta)°} = \dfrac{10.35}{\sin 30°}$

$\dfrac{\sin(180 - \theta)°}{20} = \dfrac{\sin 30°}{10.35}$

$\sin(180 - \theta)° = \dfrac{20\ \sin 30°}{10.35}$

$= 0.9662$

$180° - \theta = \sin^{-1}(0.9662)$

$= 75.06°$

$\theta = 104.94°$

The angle between forces $\underset{\sim}{B}$ and $\underset{\sim}{C}$ is approximately 105°.

Note: The angle between two forces in the 'real' situation and the angle between them in a triangle of three forces are supplementary; that is, they sum to 180°.

on Resources

Digital document: SkillSHEET The sine rule (doc-26831)
Digital document: SkillSHEET The cosine rule (doc-26832)
Digital document: SpreadSHEET Vector addition (doc-26833)

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concepts 2 & 3

Types of forces Summary screen and practice questions
Resultant force and equilibrium Summary screen and practice questions

Exercise 3.3 Force and the triangle of forces

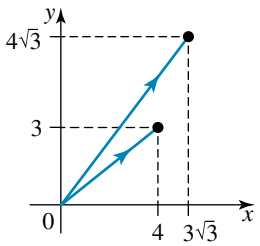
Technology free

1. **WE4** Draw vector diagrams to represent forces which act on the following objects. Indicate the relative size of the force by the length of the vector arrows. Further indicate the nature of each of the forces acting by labelling them: $\underset{\sim}{N}$ for normal contact forces, $\underset{\sim}{W}$ for gravitational forces (that is, weight), $\underset{\sim}{A}$ for applied forces, $\underset{\sim}{D}$ for air resistance (drag) forces, and $\underset{\sim}{F}$ for friction forces.

- **a.** A book sitting on a table.
- **b.** A ball falling vertically through the air at constant speed.
- **c.** A car driving on a horizontal road at a constant speed.
- **d.** A boat drifting through the water at constant speed.
- **e.** A body sliding across a smooth horizontal surface at constant velocity.
- **f.** A car accelerating on a horizontal road.
- **g.** A body at rest on an inclined plane.
- **h.** A body sliding down an inclined plane at constant speed.
- **i.** A ball travelling vertically up (include air resistance).
- **j.** A ball travelling vertically down (include air resistance).

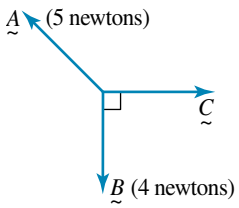
Technology active

2. Refer to the diagram to answer the following questions.



- **a.** Using $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ notation, determine the exact value of the resultant force.
- **b.** Determine the magnitude of a third force for the resultant force to be zero.

3. **MC** The three coplanar forces $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ act in such a way that the resultant force is zero. Use a triangle of forces to assist in answering the following questions.



- **a.** The magnitude of $\underset{\sim}{C}$ is:
  A. 15 N **B.** 21 N **C.** 3 N **D.** −21 N
- **b.** The angle between the forces $\underset{\sim}{A}$ and $\underset{\sim}{C}$, to the nearest degree, is:
  A. $180 - \sin^{-1}(0.8)$ **B.** $\sin^{-1}(0.8)$ **C.** $\cos^{-1}(0.8)$ **D.** $180 - \cos^{-1}(0.8)$

4. **WE5** Two forces — $\underset{\sim}{F}_1$ and $\underset{\sim}{F}_2$ — act on an object. They are described by the vectors:

i. $\underset{\sim}{F}_1 = 13\hat{\boldsymbol{\imath}} - 5\hat{\boldsymbol{\jmath}}, \underset{\sim}{F}_2 = 4\hat{\boldsymbol{\imath}} + 9\hat{\boldsymbol{\jmath}}$

ii. $\underset{\sim}{F}_1 = 8\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}, \underset{\sim}{F}_2 = 14\hat{\boldsymbol{\imath}} + 9\hat{\boldsymbol{\jmath}}$

iii. $\underset{\sim}{F}_1 = 2\sqrt{2}\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}, \underset{\sim}{F}_2 = -3\sqrt{5}\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$.

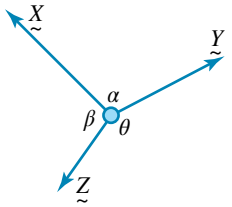
For each of the above, determine:

- **a.** the resultant force, $\underset{\sim}{R}$, acting on the object
- **b.** the magnitude of the resultant force, $|\underset{\sim}{R}|$
- **c.** a third force, $\underset{\sim}{F}_3$, applied to the body so that the resultant force is equal to zero.

5. Three forces — $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ and $\underset{\sim}{F}_3$ — act on an object. They are described by the vectors:
   i. $\underset{\sim}{F}_1 = 3\hat{\imath} - 5\hat{\jmath}$, $\underset{\sim}{F}_2 = 4\hat{\imath} + 9\hat{\jmath}$ and $\underset{\sim}{F}_3 = -6\hat{\imath} - 2\hat{\jmath}$
   ii. $\underset{\sim}{F}_1 = \hat{\imath} - \hat{\jmath}$, $\underset{\sim}{F}_2 = -2\hat{\imath} + 3\hat{\jmath}$ and $\underset{\sim}{F}_3 = \sqrt{2}\,\hat{\imath} - 2\hat{\jmath}$.

   For each of the above, determine:
   a. the resultant force, $\underset{\sim}{R}$, acting on the object
   b. the magnitude of the resultant force, $|\underset{\sim}{R}|$
   c. the angle that the vector $\underset{\sim}{R}$ makes to the $\hat{\imath}$ vector, to 1 decimal place
   d. a fourth force, $\underset{\sim}{F}_4$, applied to the body so that the resultant force is equal to zero.
6. In each of the following cases:
   i. determine the resultant force, $\underset{\sim}{R}$, for the specified forces acting on a body
   ii. deduce the additional force needed to attain equilibrium; that is, $\underset{\sim}{R} = 0$.

   a. $\underset{\sim}{F} = 25$ N at N30°E and $\underset{\sim}{G} = 45$ N at N30°W
   b. $\underset{\sim}{W} = -80\hat{\jmath}$ and $\underset{\sim}{F} = 50\hat{\imath} + 75\hat{\jmath}$
   c. $\underset{\sim}{A} = 26\hat{\imath} - 80\hat{\jmath}$ and $\underset{\sim}{B} = -30\hat{\imath} + 92\hat{\jmath}$
   d. $\underset{\sim}{F} = 10$ N at S50°W, $\underset{\sim}{G} = 7.0$ N at N45°W and $\underset{\sim}{H} = 12$ N at N70°E
7. **WE6** Three forces — $\underset{\sim}{X}$, $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ — act on an object such that the resultant force $R = 0$. The force Y acts at an angle of 120° to the force $\underset{\sim}{X}$ and has the same magnitude as the force $\underset{\sim}{X}$, which is 10 N.
   a. Determine the magnitude of $\underset{\sim}{Z}$.
   b. Calculate the angle that the force $\underset{\sim}{Z}$ makes to $\underset{\sim}{X}$.
8. Three coplanar forces —$\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ — have magnitudes of 1000 N, 1200 N and 1700 N respectively. They act on a body such that the resultant force is zero. Determine the angle between $\underset{\sim}{A}$ and $\underset{\sim}{B}$.
9. Three forces — $\underset{\sim}{X}$, $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ — add to give a resultant force equal to zero.



   a. Calculate β (to the nearest degree) if $\alpha = 135°$, $|\underset{\sim}{Z}| = 200$ N and $|\underset{\sim}{X}| = 150$ N.
   b. Determine $|\underset{\sim}{Z}|$ (to 1 decimal place) if $\alpha = 100°$, $\beta = 135°$ and $|\underset{\sim}{Y}| = 27$ N.
   c. If $|\underset{\sim}{X}| = |\underset{\sim}{Y}|$ and $\alpha = 60°$, demonstrate that $|\underset{\sim}{Z}| = \sqrt{3}\,|\underset{\sim}{X}|$.

3.4 Force and the state of equilibrium

3.4.1 Newton's First Law of Motion

Historically, Newton's First Law of Motion had its origins in some of the work done by Galileo. The law concerns itself with the motion of a particle when the resultant force, $\underset{\sim}{R}$, acting on it is zero. It was a commonly held belief that in order to have motion, an unbalanced force was required. This belief seemed quite natural: clearly, carts need to be pulled by horses in order to move and that balls rolling across a lawn come to rest. However, experiments showed that the force of friction could be used in both cases to explain the apparent necessity of an applied force to maintain constant motion.

It was Newton who stated that in the absence of an unbalanced force (that is, $\underset{\sim}{R} = 0$) acting on a body, the body would continue in a state of uniform motion. The phrase 'uniform motion' is used to describe the motion of a body with zero acceleration; that is, the velocity of the body is a constant.

Today we would write this as:

If $\underset{\sim}{R} = 0$ (the resultant force acting on the body is zero)
then $v = c$ (the velocity of the body is a constant)
and thus $a = 0$ (the acceleration of the body is zero).

If the resultant force acting on a body is zero, then its acceleration is zero.

It is important to recognise that the inference above is equally valid in reverse. That is,

If $a = 0$ (the acceleration of the body is zero)
then $v = c$ (the velocity of the body is a constant)
and thus $\underset{\sim}{R} = 0$ (the resultant force acting on the body is zero).

When the net or resultant force acting on an object is equal to zero, it is commonly said that the body is 'in equilibrium'. Such an object moves at a constant velocity. These types of situation belong to a class of problems called **statics**.

To solve dynamics and related **kinematics** problems there is a clear strategy.

- Read the question carefully.
- Draw a clear diagram that contains all the information.
- Superimpose arrows depicting vectors which act on the body in question. This is called a *force vector* diagram.
- Find the resultant force vector that is the sum of all forces acting on the body.

Mass and weight

We know from experience that some objects are harder to push than others; that is, it is more difficult to modify their motion. We call this property of matter **inertial mass**. The **SI** (Système International d'Unités) unit for mass is the kilogram. It is harder to stop or accelerate a large truck than a small car, the truck having the greater mass.

Weight is a vector quantity. It is a force, equal to the product of the acceleration due to gravity and the mass on which it is acting. We often refer to 'the weight of the object' but the object does not possess weight as an intrinsic quantity; it does, however, possess mass. The SI unit for weight, since weight is a force, is the **newton**. Suppose that an astronaut is somewhere in space where there is a zero gravitational field ($g = 0$ N/kg); he may be 70 kg, but he is said to be 'weightless'. The gravitational field strength $\boldsymbol{g}$ is a measure of the magnitude of the force of gravity acting on a unit mass. Consequently it has the dimensions of force per unit mass. In SI units this is the newton per kilogram (N/kg). The same 70 kg man on the surface of the Earth where the vector g has a magnitude of 9.8 N/kg down would have a weight of 686 N or 70 N downward. That is, the weight $\underset{\sim}{W}$ of a mass m in a gravitational field g is given by the equation:

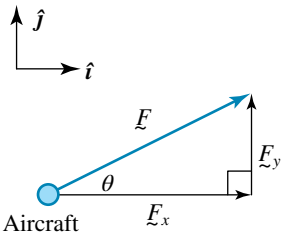
$$\underset{\sim}{W} = m\underset{\sim}{g}$$

Note: The value for g of 9.8 m/s^2 down is used universally in examples and problems in this textbook. Sometimes the unit for g is quoted as N/kg. This is the acceleration due to gravity. For a mass of 1 kg, it becomes equal to the weight force when all other forces are ignored.

3.4.2 Resolving a force into its components

In many cases the size of a force acting in a particular direction needs to be calculated. For example, when a jet aircraft is taking off from a runway the engines provide a **thrust**. Part of the thrust assists in lifting the plane up into the air (the vertical **component**) and the other component exerts a horizontal push making the plane move forward. In the diagram at right, the combined thrust of the jet's engines is represented by a single vector $\underset{\sim}{F}$ acting on a particle representing the aircraft.

If $\underset{\sim}{F}$ is the applied force acting on the plane due to the action of the engines then we can write the force vector as the sum of two vectors $\underset{\sim}{F}_x$ and $\underset{\sim}{F}_y$ that are parallel to the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ respectively.



Thus $\underset{\sim}{F} = \underset{\sim}{F}_x + \underset{\sim}{F}_y$

$\underset{\sim}{F} = \underset{\sim}{F}_x\hat{\boldsymbol{\imath}} + \underset{\sim}{F}_y\hat{\boldsymbol{\jmath}}$

where $\underset{\sim}{F}_x = |F| \cos\theta$

and $\underset{\sim}{F}_y = |\underset{\sim}{F}| \sin\theta.$

The quantities F_x and F_y are referred to as the *components* of the vector force $\underset{\sim}{F}$ in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions respectively.

A force $\underset{\sim}{F}$ can be resolved into perpendicular components:

$$\underset{\sim}{F} = F\cos\theta\hat{\boldsymbol{\imath}} + F\sin\theta\hat{\boldsymbol{\jmath}}$$

where $F = |\underset{\sim}{F}|$ and θ is the angle between $\underset{\sim}{F}$ and $\hat{\boldsymbol{\imath}}$.

For example, if the combined thrust of the engines is 2×10^6 N and the plane during take-off had a 15° elevation angle, then the vertical component would be:

$$\begin{aligned} F_y &= |\underset{\sim}{F}| \sin\theta \\ &= 2 \times 10^6 \times \sin\ 15° \\ &\approx 5.18 \times 10^5 \text{ N} \end{aligned}$$

while the horizontal component would be:

$$\begin{aligned} F_x &= |\underset{\sim}{F}| \cos\theta \\ &= 2 \times 10^6 \times \cos 15° \\ &= 1.93 \times 10^6 \text{ N} \end{aligned}$$

$$\text{or } \underset{\sim}{F} \approx \left(1.93 \times 10^6\right)\hat{\boldsymbol{\imath}} + \left(5.18 \times 10^5\right)\hat{\boldsymbol{\jmath}}.$$

Note: The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components are usually horizontal and vertical components respectively, but they can be rotated to suit a particular problem.

WORKED EXAMPLE 7

A water skier is being pulled along by rope attached to a speed boat across a horizontal lake. The rope makes an angle of 5° to the horizontal and exerts a force of 6000 N on the skier.

a. Calculate the horizontal component of the force exerted on the skier by the rope.

b. Calculate the vertical component of the force exerted on the skier by the rope. The skier is moving with a constant velocity.

c. Calculate the size of the horizontal resistance forces on the skier.

THINK

1. The skier has four forces acting on him: the weight force, $\underset{\sim}{W}$, acting vertically downwards; the normal reaction, $\underset{\sim}{N}$, of the water on the skier, acting vertically upwards; the tension force, $\underset{\sim}{T}$, in the rope, acting 5° to the horizontal; and the resistance forces, $\underset{\sim}{F}$, acting horizontally against the direction of motion.
2. Draw a force vector diagram showing the forces acting on the skier.

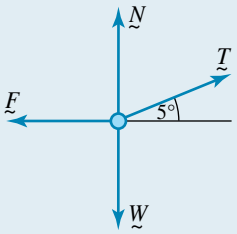
a. Evaluate the magnitude of the horizontal component of $\underset{\sim}{T}$, T_H.

b. Evaluate the magnitude of the vertical component of $\underset{\sim}{T}$, T_V.

c. 1. Constant velocity means the acceleration is zero and in turn the resultant of the horizontal forces is zero.

2. The magnitude of the horizontal resistance forces, F, is equal to the horizontal component of the tension force, $\underset{\sim}{T}$.

WRITE



a. $T_H = 6000 \cos 5°$
$\approx 5997\,\text{N}$

b. $T_V = 6000 \sin 5°$
$\approx 523\,\text{N}$

c. Since acceleration = 0,

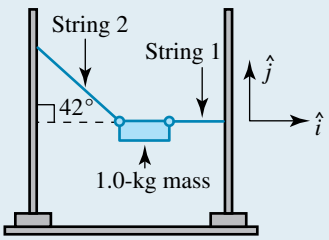
$F = T_H \approx 5997\,\text{N}$

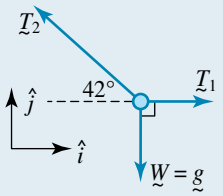
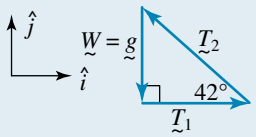
WORKED EXAMPLE 8

In a science laboratory a 1.0-kg mass is suspended by two taut strings as shown. The tension forces in string 1 and string 2 are $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ respectively.

a. Draw a force vector diagram showing all three forces which act on the 1.0 kg mass.

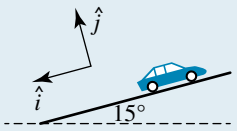
b. By resolving vectors into $\hat{i}$ and $\hat{j}$ components, determine the magnitudes of $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ respectively.



THINK	WRITE
a. 1. The mass has three forces acting on it: the weight force vertically downwards and the two tension forces. One tension force acts horizontally; the second acts at an angle of 42° to the horizontal.	**a.** 
2. Draw the force vector diagram.	
b. 1. The weight vector can be written as $-g\hat{j}$.	**b.** $\underset{\sim}{W} = -g\hat{j}$
2. The first tensile force can be written as $T_1\hat{i}$.	$\underset{\sim}{T}_1 = T_1\hat{i}$
3. The second tensile force can be resolved as $\underset{\sim}{T}_2 = -\underset{\sim}{T}_2 \cos 42°\hat{i} + T_2 \sin 42°\hat{j}$.	$\underset{\sim}{T}_2 = -T_2 \cos 42°\hat{i}$ $+T_2 \sin 42°\hat{j}$
4. Express the resultant force, $\underset{\sim}{R}$, as the sum of the three forces.	$\underset{\sim}{R} = (T_1 - T_2 \cos 42°)\hat{i}$ $+(T_2 \sin 42° - g)\hat{j}$
5. Set the sum of the three vectors to zero in accordance with Newton's First Law of Motion.	But $\underset{\sim}{R} = 0\hat{i} + 0\hat{j}$.
6. Set the $\hat{i}$ component of $\underset{\sim}{R}$ to zero and call it equation [1].	$T_1 - T_2 \cos 42° = 0$ [1]
7. Set the $\hat{j}$ component of $\underset{\sim}{R}$ to zero and call it equation [2].	$T_2 \sin 42° - g = 0$ [2]
8. Solve equation [2] for T_2.	$T_2 = \dfrac{g}{\sin 42°}$ $\approx 14.6\,\text{N}$
9. Solve for T_1 by substituting T_2 into equation [1].	$T_1 = T_2 \cos 42°$ $\approx 10.9\,\text{N}$
Note: Part **b** could also be solved by drawing a triangle of forces and solving using trigonometry.	$W = g = T_1 \tan 42°$ 
	$T_1 = \dfrac{g}{\tan 42°}$ $\approx 10.9\,\text{N}$
	and $W = g = T_2 \sin 42°$ $T_2 = \dfrac{g}{\sin 42°}$ $\approx 14.6\,\text{N}$ (as in part **b** above)

WORKED EXAMPLE 9

A car of mass 800 kg is parked in a street which has an angle of elevation of 15°. The $\hat{\imath}$ direction is parallel down the street and the $\hat{\jmath}$ direction is perpendicular to the street.



The car is subject to three forces, namely its weight, $\underset{\sim}{W}$, the normal contact force, $\underset{\sim}{N}$, of the road acting on the car and the applied force of the brake (this is actually a static friction force) $\underset{\sim}{F}$.

a. Draw a vector diagram indicating the three forces, $\underset{\sim}{W}$, $\underset{\sim}{N}$ and $\underset{\sim}{F}$, acting on the car, taking the car as a particle.

b. Determine the magnitude of the resultant force $\underset{\sim}{R}$.

c. Resolve the weight, $\underset{\sim}{W}$, into its components and express it as a vector using $\hat{\imath} - \hat{\jmath}$ notation.

d. Calculate the magnitude of $\underset{\sim}{N}$, the normal contact force.

e. Calculate the magnitude of the applied force of the brake $\underset{\sim}{F}$.

THINK

a. 1. A stationary car parked on a street will have a vertical weight force, a normal contact force and a static frictional force resisting its sliding or rolling down the street.

2. Draw the force vector diagram.

b. 1. The car is in equilibrium since it is stationary.

2. Apply Newton's First Law of Motion: the resultant force, $\underset{\sim}{R}$, must be zero.

3. Therefore, the magnitude of the resultant force, R, is zero.

c. 1. Draw a diagram showing the resolution of the weight, $\underset{\sim}{W}$, into components parallel to $\hat{\imath}$ and $\hat{\jmath}$.

2. The component of $\underset{\sim}{W}$ parallel to $\hat{\imath}$, W_x, is $W \sin 15°$.

3. Substitute $W = 800g$ and evaluate.

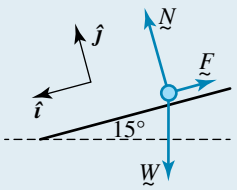
4. The component of W parallel to $\hat{\jmath}$, W_y, is $W \cos 15°$.

5. Substitute $W = 800g$ and evaluate.

6. Express $\underset{\sim}{W}$ in vector notation.

WRITE

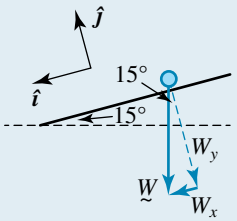
a.



b. $\underset{\sim}{R} = 0$

$R = 0\,\text{N}$

c.



$W_x = 800g \sin 15°$
$\approx 207g$

$W_y = W \cos 15°$
$= -800g \cos 15°$
$\approx -773g$

$\underset{\sim}{W} = W_x\hat{\imath} + W_y\hat{\jmath}$
$\underset{\sim}{W} = 207g\hat{\imath} - 773g\hat{\jmath}$

d. 1. The component of the net force parallel to the $\hat{j}$ vector is zero.

d. $(N - 773g)\hat{j} = 0\hat{j}$

2. Solve for the magnitude of the normal N.

$N = 773g$

e. 1. The component of the net force parallel to the $\hat{i}$ vector is zero.

e. $(207g - F)\hat{i} = 0\hat{i}$

2. Solve for the magnitude of the applied force of the brake, F.

$F = 207g$

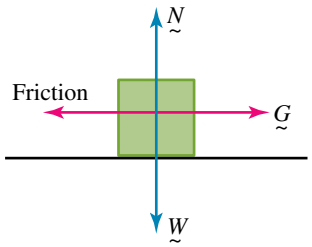
Note: In general, when an object of mass m and hence weight $m\underset{\sim}{g}$ is on an inclined plane with incline angle θ, the weight vector can be resolved into two components:

Component 1: magnitude $mg \sin\theta$, acting down the plane
Component 2: magnitude $mg \cos\theta$, acting perpendicular to the plane.
This is opposed by the normal contact force, $\underset{\sim}{N}$.

3.4.3 Friction

A body resting on a table is acted on by a number of forces. As we have discussed the weight $\underset{\sim}{W}$ acts downwards and is given by the formula

$$\underset{\sim}{W} = \text{mass} \times \underset{\sim}{g}.$$



The normal reaction, $\underset{\sim}{N}$, is the force exerted by the table on the body which opposes and balances the weight. Suppose the body is further acted on by a horizontal force $\underset{\sim}{G}$. If the table is smooth, there is no friction and the mass will move to the right. If the table is not smooth a frictional force will oppose the motion. The frictional force depends on the roughness of the surface and the normal reaction $\underset{\sim}{N}$.

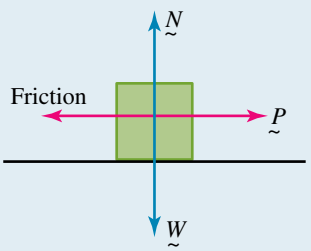
Friction = $\mu \times$ normal reaction, where μ is the coefficient of friction.

$$\underset{\sim}{F} = \mu \times \underset{\sim}{N}$$

Note that this formula gives the maximum value for friction. If, in the diagram, $\underset{\sim}{G}$ is less than $\mu\underset{\sim}{N}$, then the frictional force will just balance the force $\underset{\sim}{G}$.

WORKED EXAMPLE 10

A body of mass 4 kg, at rest on a table, is acted on by a horizontal force, $\underset{\sim}{P}$, as shown in the figure. If the body just begins to move when $\underset{\sim}{P}$ is 12 N, calculate the coefficient of friction between the body and the table.



THINK

1. The weight of the body is $m \times g$.

WRITE

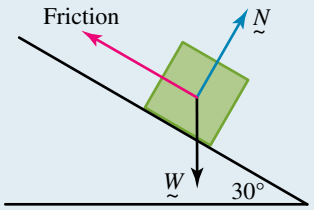
$W = m \times g$
$= 4 \times 9.8$
$= 39.2\ \text{N}$

2. The normal force, N, must balance the weight.	$N = W$ $= 39.2$ N
3. The friction will oppose P and its maximum value is $\mu \times N$.	$F = \mu \times N$ $12 = \mu \times 39.2$ $\mu = 0.3$

WORKED EXAMPLE 11

A body of mass 4 kg rests on a plane inclined at an angle of 30° to the horizontal. It is just prevented from moving by friction. Calculate:

a. the weight, $|\underset{\sim}{W}|$
b. the magnitude of the normal reaction, $\underset{\sim}{N}$
c. the magnitude of the frictional force, $\underset{\sim}{F}$
d. the coefficient of friction, μ.



THINK	WRITE
a. Recall the formula for weight, $W = m \times g$, and substitute the known values: $m = 4$ kg and $g = 9.8$ m/s^2. The units are newtons.	**a.** $\|\underset{\sim}{W}\| = m \times g$ $= 4 \times 9.8$ $= 39.2$ N
b. 1. First resolve each force into its components. Take the $\hat{\boldsymbol{\imath}}$ direction to be down the plane and the $\hat{\boldsymbol{\jmath}}$ direction to be perpendicular to the plane.	**b.** $\underset{\sim}{N} = N\hat{\boldsymbol{\jmath}}$ $\underset{\sim}{W} = 39.2 \sin 30\hat{\boldsymbol{\imath}} + -39.2 \cos 30\hat{\boldsymbol{\jmath}}$ $= 19.6\hat{\boldsymbol{\imath}} + -33.9\hat{\boldsymbol{\jmath}}$ $\underset{\sim}{F} = -\mu N\hat{\boldsymbol{\imath}}$
2. If a body is at rest, the vector sum of forces is 0.	$F + N + W = 0$ $-F\hat{\boldsymbol{\imath}} + N\hat{\boldsymbol{\jmath}} + 19.6\hat{\boldsymbol{\imath}} + -33.9\hat{\boldsymbol{\jmath}} = 0$
3. If $a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + c\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}} = 0$, then $a = -c$ and $b = -d$.	The $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ components add to 0. $-F = -19.6$ $N = 33.9$ $\therefore N = 33.9$ N
c. Take the value of F from part **b3**.	**c.** $F = 19.6$ N
d. The coefficient of friction, μ, can be calculated using $F = \mu N$.	**d.** $F = \mu N$ $19.6 = \mu \times 33.9$ $\mu = 0.58$

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concepts 4, 5 & 6

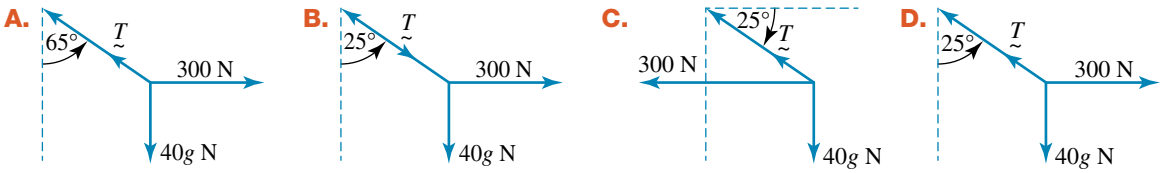
Newton's First Law of Motion Summary screen and practice questions
Friction and tension Summary screen and practice questions
Motion on an inclined plane Summary screen and practice questions

Exercise 3.4 Force and the state of equilibrium

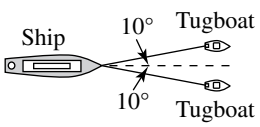
Technology active

Note: Use $\underset{\sim}{g} = 9.8\ \text{m/s}^2$ down for all problems involving weight.

1. **WE7** A girl is pulling along her baby brother in a cart attached to a rope. The cart is on a horizontal path. The rope makes an angle of 20° to the horizontal and exerts a force on the cart of magnitude 25 N.
   a. Calculate the horizontal component of the force exerted on the cart by the rope.
   b. Calculate the vertical component of the force exerted on the cart by the rope.
   The cart moves along at a constant velocity; that is, the acceleration of the cart is zero.
   c. Calculate the size of the horizontal friction force acting on the cart. Give you answer to 1 decimal place.
2. **MC** A child of mass 40 kg is held on a 'swinging rope' at an angle of 25° to the vertical by a horizontal force of 300 N. If $\underset{\sim}{T}$ is the tension force of the rope acting on the child, then:
   a. the force vector diagram which best represents this situation is:

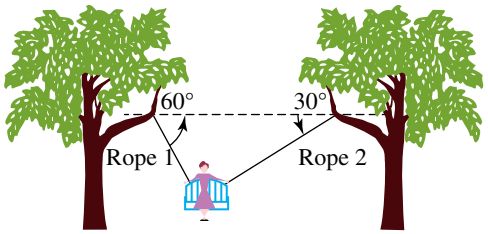


   b. the horizontal component of $\underset{\sim}{T}$ is:
      A. 40g N
      B. 300 N
      C. 127 N
      D. 272 N
   c. the vertical component of $\underset{\sim}{T}$ is:
      A. 300 N
      B. 127 N
      C. 40g N
      D. 200 N
   d. the magnitude of $\underset{\sim}{T}$ is nearest to:
      A. 494 N
      B. 440 N
      C. 477 N
      D. 92 N
3. A ship of mass m is being pulled by two tugboats. It glides through the water at constant velocity. The angle between the two ropes connecting the two tugboats to the front of the ship is 20° and they each support a tension of magnitude 20 000 N.



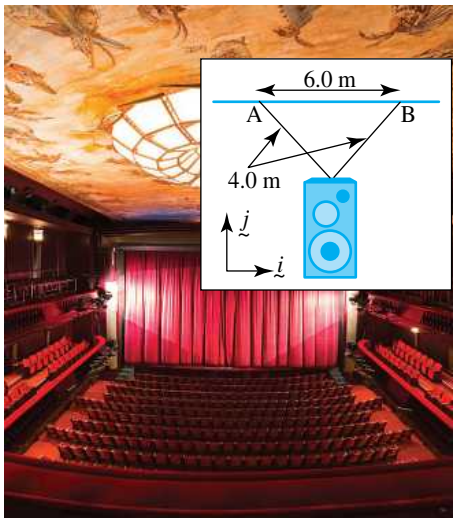
   a. Draw a force vector diagram showing all three coplanar forces which act on the ship.
   b. Determine the magnitude of the resultant force acting on the ship. (Be careful; use Newton's First Law of Motion.)
   c. Calculate the magnitude of the force of friction due to the water acting on the ship, correct to 1 decimal place.

4. **WE8** A swing chair of mass 8 kg is suspended by two taut ropes as shown at right. The tension forces in rope 1 and rope 2 are $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$ respectively.



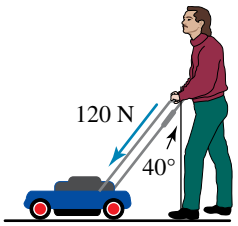
   a. Draw a force vector diagram showing all three forces which act on the swing.
   b. By resolving vectors into $\hat{\boldsymbol{i}}$–$\hat{\boldsymbol{j}}$ components find the exact magnitudes of $\underset{\sim}{T}_1$ and $\underset{\sim}{T}_2$.

5. A speaker in an auditorium has a mass of 50 kg and is suspended from the ceiling by two 4.0-m ropes. The ropes are attached to the ceiling at points A and B whose separation is 6.0 m as shown at right.



   a. Draw a force vector diagram showing all forces which act on the speaker.
   b. Calculate the angle that each rope makes with the ceiling to 1 decimal place.
   c. Determine the vertical component of the tension force in each rope.
   d. Give the vectors for the tensions in the left and right rope respectively using $\hat{\boldsymbol{i}}$–$\hat{\boldsymbol{j}}$ notation.
   e. Calculate the magnitude of the tension in each rope to 1 decimal plate.
   f. The speaker is to be raised by increasing the separation between the points A and B, but the ropes will break if the tension exceeds 4000 N. Find the maximum possible separation to the nearest metre between A and B; that is, when the tensions in the ropes are equal to 4000 N.

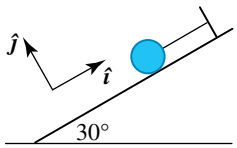
6. Sam earns some extra pocket money by mowing his neighbour's front lawn. When he pushes the lawnmower at a constant velocity he applies a force of 120 N down the shaft of the mower which is angled at 40° to the vertical. The lawnmower has a mass of 40 kg.



   a. Draw a force vector diagram illustrating all four forces acting on the lawnmower. (Treat the lawnmower as a particle.)
   b. Calculate both the vertical and horizontal components of the force that Sam applies to the lawnmower, correct to 1 decimal place.
   c. Using $\hat{\boldsymbol{i}}$–$\hat{\boldsymbol{j}}$ notation, write a vector equation for the resultant force, $\underset{\sim}{R}$, acting on the lawnmower in terms of the four forces acting on the lawnmower.
   d. Determine the magnitude of the force of friction acting on the lawnmower as it moves across the lawn at constant velocity.
   e. Determine the magnitude of the normal contact force.

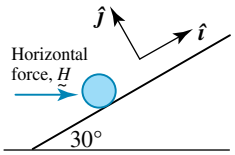
7. **WE9** A 1.5-kg mass is placed on a smooth inclined plane angled at 30° to the horizontal. To stop it from sliding down the plane a string is attached to the upper side as shown. The unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are also shown.

The unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are also shown.



a. Draw a force vector diagram showing the forces which act on the mass. Label the forces $\underset{\sim}{N}$ for the normal contact force, $\underset{\sim}{T}$ for the tension force and $\underset{\sim}{W}$ for the force arising from the effect of gravity.
b. What is the magnitude of the resultant force, $\underset{\sim}{R}$?
c. Determine the weight vector, $\underset{\sim}{W}$, using $\hat{\boldsymbol{\imath}}$–$\hat{\boldsymbol{\jmath}}$ notation.
d. Determine the tension force, $\underset{\sim}{T}$, using $\hat{\boldsymbol{\imath}}$–$\hat{\boldsymbol{\jmath}}$ notation.
e. Determine the magnitude of the normal contact force, N.

8. A 1.5-kg mass is placed on a smooth inclined plane angled at 30° to the horizontal. To stop it from sliding down the plane a horizontal force is applied to the mass as shown . The unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are also shown.



a. Draw a force vector diagram showing the forces which act on the mass. Label the forces $\underset{\sim}{N}$ for the normal contact force, $\underset{\sim}{H}$ for the horizontal force and $\underset{\sim}{W}$ for the force due to gravity.
b. Determine the weight vector, $\underset{\sim}{W}$, using $\hat{\boldsymbol{\imath}}$–$\hat{\boldsymbol{\jmath}}$ notation.
c. Determine the horizontal force, $\underset{\sim}{H}$, using $\hat{\boldsymbol{\imath}}$–$\hat{\boldsymbol{\jmath}}$ notation.
d. Determine the normal contact force, $\underset{\sim}{N}$, using $\hat{\boldsymbol{\imath}}$–$\hat{\boldsymbol{\jmath}}$ notation.
e. Show that $\underset{\sim}{H} = \dfrac{|\underset{\sim}{W}|}{\sqrt{3}} = \dfrac{\sqrt{3}\,|\underset{\sim}{W}|}{3}$.

9. **WE 10** A box containing books has a mass of 40 kg. It requires a force of 300 N to move the box across the floor. Calculate the coefficient of friction between the box and the floor.

10. The maximum deceleration the BMV Tycoon can achieve under braking is 4 m/s^2. If this vehicle has a mass of 1900 kg, calculate:
a. the frictional force on the car
b. the normal reaction
c. the coefficient of friction between the tyres and the road.

11. A force of 200 N is needed to keep a log of mass 300 kg sliding along a horizontal path. Calculate the coefficient of friction between the log and the path.

12. **WE 11** A body of mass 6 kg rests on a plane inclined at an angle of 40° to the horizontal. It is just prevented from moving by friction. Calculate the:
a. weight
b. normal reaction
c. frictional force
d. coefficient of friction.

13. A book lies on a horizontal table. One end of the table is raised until, at an angle of 35°, the book starts to slide. Calculate the coefficient of friction between the book and the table.

14. An object of mass 2 kg rests on a plane inclined at an angle of 40° to the horizontal. If the coefficient of friction between the object and the plane is 0.2, calculate the resultant force down the plane.

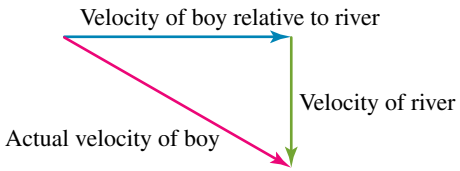
3.5 Relative velocity

3.5.1 Relationship between velocities

As you sit at your desk reading this paragraph it appears that you are not moving significantly. While this may be apparent to someone in the room watching you, an observer with a different point of reference may disagree entirely. As you sit in the room you are moving around the Sun at about 100 000 km/h. Measurement of velocity depends on the frame of reference.

Most people have experienced the sensation of movement while sitting in a stationary train and observing a nearby train moving. The visual messages tell our brain we are moving and, for a moment, we become disoriented.

Consider the situation of a boy rowing a boat across a swiftly flowing river. The boy thinks he is rowing directly towards the bank opposite but to an observer on the shore his velocity is different. The situation can be described using vectors.



In general the relationship between relative velocities is

$$\underset{\sim}{v}_a = \underset{\sim}{v}_{a \text{ rel } b} + \underset{\sim}{v}_b$$

where $\underset{\sim}{v}_{a \text{ rel } b}$ is the velocity of a ($\underset{\sim}{v}_a$) relative to the velocity of b (v_b).

WORKED EXAMPLE 12

The pilot of a boat heads due north at a speed of 12 km/h with respect to the water. The water moves at a velocity of 5 km/h in an easterly direction. Calculate the velocity of the boat as seen by an observer on the shore.

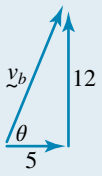
THINK

1. The velocity of the boat relative to the water ($\underset{\sim}{v}_{b \text{ rel } w}$) is 12 m/s north. The velocity of the water ($\underset{\sim}{v}_w$) is 5 km/h east.
   We want to find the velocity of the boat, $\underset{\sim}{v}_b$.
   Take the $\hat{\boldsymbol{i}}$ direction to be easterly and the $\hat{\boldsymbol{j}}$ direction to be northery.
2. Use the relative velocity formula.
3. Write the answer as a speed and direction.

WRITE

$\underset{\sim}{v}_{b \text{ rel } w} = 12\hat{\boldsymbol{j}}$

$\underset{\sim}{v}_w = 5\hat{\boldsymbol{i}}$



$\underset{\sim}{v}_b = \underset{\sim}{v}_{b \text{ rel } w} + \underset{\sim}{v}_w$

$= 12\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{i}}$

$|\underset{\sim}{v}_b| = \sqrt{12^2 + 5^2}$

$= 13$

$\tan\theta = \frac{12}{5}$

$\theta = 67.4°$

An observer on the shore sees the boat moving at 13 km/h at 67.4° north of east.

WORKED EXAMPLE 13

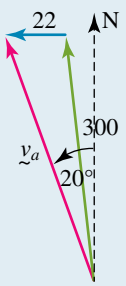
A pilot is to fly a plane to a destination which is 450 km from his present position in a direction N20° W. There is a wind from the east at 22 km/h and the plane has an airspeed of 300 km/h. In what direction should the plane head to reach the destination?

THINK

1. Use the relative velocity formula and draw a diagram.
   The wind velocity is $\underset{\sim}{v}_b$ and the airspeed of the plane is $\underset{\sim}{v}_{a \text{ rel } b}$.
   The speed of the plane, relative to the ground is $\underset{\sim}{v}_a$.

WRITE

$\underset{\sim}{v}_a = \underset{\sim}{v}_{a \text{ rel } b} + \underset{\sim}{v}_b$



2. Let x be the direction the plane needs to head.
Recall the sine rule:

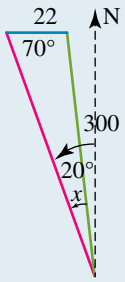
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and substitute

$a = 22, A = x, b = 300$ and

$B = 180° - (90 + 20)°$

$= 70°.$



$$\frac{\sin x}{22} = \frac{\sin 70°}{300}$$

$$\sin x = 0.0689$$

$$x = 3.95°$$

3. Solve for the plane's required direction.

$$\text{Direction} = 20° - 3.95°$$

$$= 16.05°$$

The plane should head 16.05° west of north or N16.05°W

studyon

Units 1 & 2 > Area 2 > Sequence 2 > Concept 7

Relative velocity Summary screen and practice questions

Exercise 3.5 Relative velocity

Technology active

1. **WE12** The pilot of a boat heads due east at a speed of 16 km/h with respect to the water. The water moves south at a velocity of 4 km/h. Calculate the velocity of the boat as seen by an observer on the shore, correct to 1 decimal place.
2. A plane with an airspeed of 300 km/h heads due north. A wind blows from the west at 30 km/h. Determine the velocity of the plane relative to the ground, correct to 1 decimal place.
3. A rowing crew heads across river at 15 m/s. The crew heads in a direction 45° north of west. At the same time the current flows at 3 m/s due north. What is the speed and direction of the boat on the river? Give your answer to 1 decimal place.
4. **WE13** The driver of a cross-river ferry wants to head to a pier directly across the stream, at right angles to the bank. His boat can travel at 20 km/h and there is a current flowing downstream at 6 km/h. At what angle to the bank should the driver head in order to travel directly across the river?

5. A passenger on a cruise ship walks at 6 km/h towards the stern (the back) of the ship. The ship travels at 19 km/h in a north-westerly direction. Determine the speed of the passenger with respect to the water.

6. Copy and complete the following table.

	$\underset{\sim}{v}_a$	$\underset{\sim}{v}_b$	$\underset{\sim}{v}_{a \text{ rel } b}$
a	40 m/s N	20 m/s E	
b		15 m/s S	20 m/s N
c	25 m/s NE	20 m/s E	
d	4 m/s SE		30 m/s W
e	3 m/s 30° S of E	5 m/s 25° W of N	

7. A cyclist rides north at 15 km/h and observes that the wind appears to come to her from the north-east. On the return journey she rides at the same speed in the opposite direction. Now the wind appears to be coming from the south-east. Calculate the true speed and direction of the wind.

8. A cyclist rides north at 15 km/h and observes that the wind appears to come to her from the north-east. On the return journey she rides at the same speed in the opposite direction. Now the wind appears to be coming from 30° south of east. Calculate the true speed and direction of the wind.

9. As a jogger runs along level ground at 9 km/h, the rain appears to be heading directly towards her at an angle of 10° with the vertical. When she turns around and travels at the same speed in the opposite direction the rain still appears to be coming directly towards her but now makes an angle of 5° with the vertical. Calculate the speed and direction of the rain.

3.6 Review: exam practice

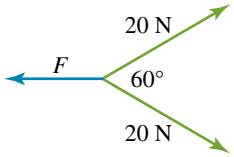
A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

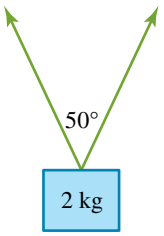
1. A body is in equilibrium under the action of three forces $\underset{\sim}{f}_1, \underset{\sim}{f}_2$ and $\underset{\sim}{f}_3$.
   $\underset{\sim}{f}_1 = -3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$ and $\underset{\sim}{f}_2 = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

   Determine the magnitude of the vector $\underset{\sim}{f}_3$

2. If the three forces shown are in equilibrium, determine the magnitude of the force F.

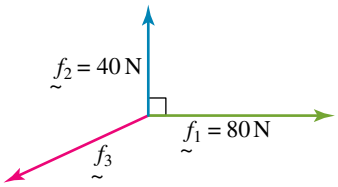


3. Two forces, equal in magnitude, are sufficient to keep the mass shown at right in equilibrium against the force of gravity.



   Calculate the magnitude of each force, to 1 decimal place.

4. An object is in equilibrium under the action of three forces $\underset{\sim}{f}_1, \underset{\sim}{f}_2$ and $\underset{\sim}{f}_3$.

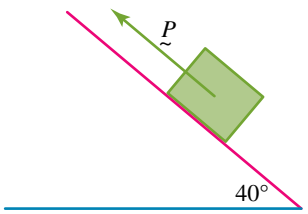


   Calculate the magnitude and direction of $\underset{\sim}{f}_3$ to 1 decimal place.

5. Three coplanar forces have magnitudes of 400 N, 500 N and 600 N. They act on a body such that the resultant force is zero. Find the angle between the 500 N and 600 N forces.

6. A body of mass 6.0 kg is suspended from a ceiling by two ropes. The angle between the two ropes is 90° and they are connected to the ceiling at points A and B respectively. The tension, T_2, in rope 2 is twice the tension, T_1, in rope 1. Give you answers to 1 decimal place where appropriate.
   a. Draw a force vector diagram for the 6.0 kg mass.
   b. Calculate the magnitudes of the tensions T_1 and T_2.
   c. Find the angles that rope 1 and rope 2 make with the horizontal.
   d. If rope 1 has a length x, show that the distance AB is $\dfrac{\sqrt{5x}}{2}$.

7. A mass of 4 kg rests on a plane which has a coefficient of friction of 0.25.



Calculate the minimum force $\underset{\sim}{P}$, acting parallel to the plane, needed to prevent the mass slipping down the plane.Give your answer to 1 decimal place.

8. **MC** Of the following quantities, which one contains only vectors?
   - **A.** Displacement, velocity, force
   - **B.** Distance, velocity, force
   - **C.** Displacement, speed, force
   - **D.** Displacement, velocity, force, mass

9. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors in the directions of east and north respectively, evaluate a force of magnitude $3\sqrt{2}$ newtons in the southeast direction.

10. The vector $\frac{1}{2}\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ makes an angle of 150° with the y-axis. Determine the value of y.

11. **MC** Given the vectors $\underset{\sim}{a} = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{b} = -3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$:
   - **A.** $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$
   - **B.** the vector $2\underset{\sim}{a} + 3\underset{\sim}{b}$ is parallel to the x-axis
   - **C.** the vector $2\underset{\sim}{a} + 3\underset{\sim}{b}$ is parallel to the y-axis
   - **D.** None of the above.

12. A ship travels a distance d km on a bearing of Nθ°W where $0° < \theta < 90°$ from an origin O. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors in the directions of east and north respectively, determine the position vector of the ship from the origin.

Complex familiar

13. **MC** If $\underset{\sim}{v}_a = 2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$ and $\underset{\sim}{v}_b = -2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$, the value of $\underset{\sim}{v}_{a \text{ rel } b}$ is

    A. $5\hat{\boldsymbol{\jmath}}$ **B.** $4\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ **C.** $-4\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}$ **D.** $\hat{\boldsymbol{\jmath}}$

14. A river flows from north to south at 5 km/h. A boat heads directly across the river from the river bank to the other side at 11 km/h. Find the true velocity of the boat.

15. A cyclist travels at 15 km/h on a road heading east. When she is travelling in this direction the wind appears to be coming from a direction 60°E of N. When she turns around and travels west on the road at the same speed the wind appears to come from 60°W of N. Find the true speed and direction of the wind.

16. While on holidays, Lilly hired a Segway as a great way to move around and explore the city. One day she travelled 500 metres on a bearing S28°E then turned and travelled 800 metres on a bearing S67°W. If $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ represent unit vectors of magnitude 1 metre in the directions of east and north respectively, find her position vector from the starting point. Hence, find her displacement, correct to 1 decimal place, and her true bearing from her starting point.

Complex unfamiliar

17. Arnie is pushing against a trailer, preventing it from rolling down a hill. The trailer has a mass of 200 kg and the hill is on an incline of 15° to the horizontal. At the moment there is no problem because Arnie is capable of pushing with a force of 1000 N parallel to the plane.

    However, it is raining and the trailer is filling with water at a rate of 25 litres per minute. How long will Arnie be able to hold the trailer and stop it from running down the hill?

18. When the surf gets big at Kirra there is always a sweep running from south to north.

Jodie is heading out at 8 km/h. If she heads directly out to sea and she wants to get to the take-off area, how far up the beach should she walk before paddling out?

19. A ball of mass 0.20 kg is shot vertically in the air. It decelerates under the action of two forces: the weight force and the force of air resistance. When the ball moves with a speed of 40 m/s, it has an air resistance of 1.0 N. When the ball is stationary, the air resistance force is equal to zero.

a. Determine the magnitude and direction of the resultant force acting on the ball when it is moving upwards at a speed of 40 m/s.

b. Determine the magnitude and direction of the resultant force acting on the ball when it is at its maximum height above the ground.
Later, the ball is travelling toward the ground at 40 m/s.

c. Determine the magnitude and direction of the resultant force acting on the ball when it is moving downwards at a speed of 40 m/s.

20. In a James Bond movie scene, a car of mass 1500 kg rolls in a straight line from rest down a road inclined at 10° from the horizontal. It takes 10 seconds for the car to reach the bottom of the incline at a speed of 43.2 km/h where the road becomes level. The handbrake of the car was on, providing a constant retardation force of $20g$ newtons.

a. Calculate the value of the coefficient of friction correct to 3 decimal places.
At the bottom of the incline the car jolts, and the handbrake is rendered inactive.

b. How much further does the car travel? (Give your answer to the nearest tenth of a metre.)

studyon

Units 1 & 2 Sit chapter test

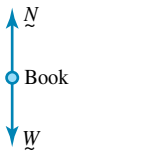
Answers

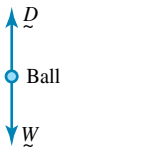
Chapter 3 Applications of vectors in the plane

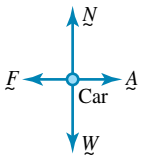
Exercise 3.2 Displacement and velocity

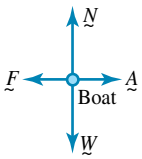
1. a. $7\hat{\imath} - 24\hat{\jmath}$ b. 25 km 163°44′T
2. a. $-500\hat{\imath} - 1200\hat{\jmath}$ b. 1300 m 202°37′T
3. a. $-14.34\hat{\imath} - 55.48\hat{\jmath}$ b. 57.26 km S14°30′W
4. a. $-261.67\hat{\imath} + 144.44\hat{\jmath}$ b. 298.89 km N61°6′W
5. a. $-2.13\hat{\imath} - 6.13\hat{\jmath}$ b. 6.49 km 19°5′T
6. a. $-4.98\hat{\imath} + 60.74\hat{\jmath}$ b. 60.94 km N4°39′W
7. a. $4.34\hat{\imath} - 1.49\hat{\jmath}$ b. 4.59 km 109°4′T
8. a. $299.69\hat{\imath} - 2341.75\hat{\jmath}$ b. 2360.85 m 172°42′T
9. a. $-9.09\hat{\imath} + 3.18\hat{\jmath}$ b. 9.63 km N 70°43′W
10. a. $367.26\hat{\imath} + 889.02\hat{\jmath}$ b. 961.9 m 22°27′T
11. a. $151.92\hat{\imath} + 80.64\hat{\jmath}$ b. 172 km 62°2′T
12. $(d_1 + d_2 \sin(\theta))\hat{\imath} + d_2 \cos(\theta)\hat{\jmath}$, $\sqrt{d_1^2 + d_2^2 + 2d_1 d_2 \sin(\theta)}$
13. $\left(d_1 \cos(\theta_1) - d_2 \cos(\theta_2)\right)\hat{\imath} + \left(d_1 \sin(\theta_1) + d_2 \sin(\theta_2)\right)\hat{\jmath}$, $\sqrt{d_1^2 + d_2^2 - 2d_1 d_2 \cos(\theta_1 + \theta_2)}$

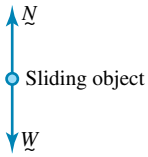
Exercise 3.3 Force and the triangle of forces

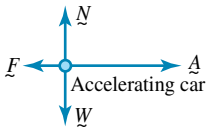
1. a. 

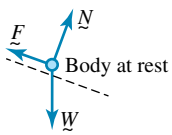
b. 

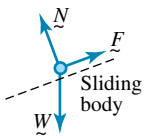
c. 

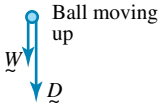
d. 

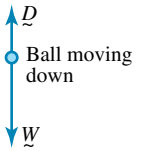
e. 

f. 

g. 

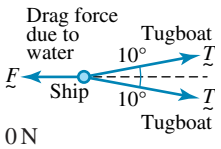
h. 

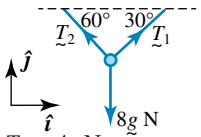
i. 

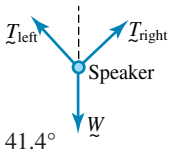
j. 

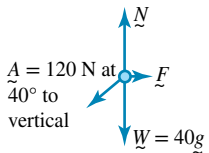
2. a. $(4 + 3\sqrt{3})\hat{\imath} + (3 + 4\sqrt{3})\hat{\jmath}$ b. $\sqrt{100 + 48\sqrt{3}} = 13.53$ N
3. a. C b. A
4. a. i. $9\hat{\imath} + 4\hat{\jmath}$
   ii. $-6\hat{\imath} - 3\hat{\jmath}$
   iii. $(2\sqrt{2} - 3\sqrt{5})\hat{\imath} - \hat{\jmath}$
   b. i. $\sqrt{97}$
   ii. $3\sqrt{5}$
   iii. $\sqrt{54 - 12\sqrt{10}}$
   c. i. $-9\hat{\imath} - 4\hat{\jmath}$
   ii. $6\hat{\imath} + 3\hat{\jmath}$
   iii. $(3\sqrt{5} - 2\sqrt{2})\hat{\imath} + \hat{\jmath}$
5. a. i. $\hat{\imath} + 2\hat{\jmath}$
   ii. $(\sqrt{2} - 1)\hat{\imath}$
   b. i. $\sqrt{5}$
   ii. $\sqrt{2} - 1$
   c. i. 63.4°
   ii. 0°
   d. i. $-\hat{\imath} - 2\hat{\jmath}$
   ii. $(1 - \sqrt{2})\hat{\imath}$
6. a. i. 61.4 N at N9.4°W
   ii. 61.4 N at S9.4°E
   b. i. $50\hat{\imath} - 5\hat{\jmath}$
   ii. $-50\hat{\imath} + 5\hat{\jmath}$
   c. i. $-4\hat{\imath} + 12\hat{\jmath}$
   ii. $4\hat{\imath} - 12\hat{\jmath}$
   d. i. 2.9 N at N26.9°W
   ii. 2.9 N at S26.9°E
7. a. 10 N b. 120°
8. 79.2°
9. a. 77°
   b. 37.6 N
   c. Sample responses can be found in the worked solutions in the online resources.

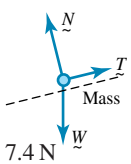
Exercise 3.4 Force and the state of equilibrium

1. a. 23.5 N b. 8.6 N c. 23.5 N
2. a. D b. B c. C d. A
3. a. 

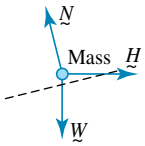
   b. 0 N
   c. 39 392.3 N
4. a. 

   b. $T_1 = 4g$ N;
   $T_2 = 4\sqrt{3}\,g$ N
5. a. 

   b. 41.4°
   c. 245 N
   d. Left rope: $-278\hat{\imath} + 245\hat{\jmath}$ N; right rope: $278\hat{\imath} + 245\hat{\jmath}$ N
   e. 370.6 N
   f. 8.0 m
6. a. 

   b. $A_V = 91.9$ N down
   $A_H = 77.1$ N left
   c. $\underset{\sim}{R} = \left(F_{\text{friction}} - 77.1\right)\hat{\imath} + (N - 483.9)\hat{\jmath} = 0$
   d. 77.1 N
   e. 483.9 N
7. a. 

   b. 0 N c. $-7.4\hat{\imath} - 12.7\hat{\jmath}$ N
   d. 7.4 N e. 12.7 N

8. a. 

   b. $-7.4\hat{\boldsymbol{\imath}} - 12.7\hat{\boldsymbol{\jmath}}$ N
   c. $7.4\hat{\boldsymbol{\imath}} - 4.3\hat{\boldsymbol{\jmath}}$ N
   d. $17\hat{\boldsymbol{\jmath}}$ N
   e. Sample responses can be found in the worked solutions in the online resources.
9. 0.77
10. a. 7600 N b. 18 620 N c. 0.41
11. 0.07
12. a. 58.8 N b. 45.04 N c. 37.8 N d. 0.84
13. 0.7
14. 9.6 N

Exercise 3.5 Relative velocity

1. 16.5 km/h at 14.0° S of E
2. 301.5 km/h at 5.7° E of N
3. 17.3 m/s at 52.1° N of W
4. 72.5°
5. 13 km/h
6. a. 45 m/s at 27° W of N
   b. 5 m/s N
   c. 18 m/s at 8° W of N
   d. 33 m/s at 5° S of E
   e. 8 m at 52° S of E
7. 15 km/h E (i.e. towards the west)
8. 19.4 km/h 78° W of S
9. 68.3 km/s at 2.5° towards the jogger's original direction

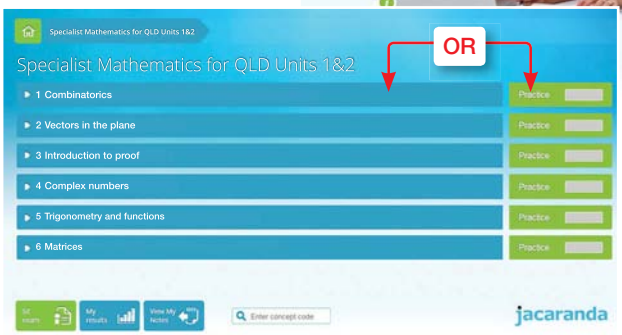
3.6 Review: exam practice

1. $\sqrt{85}$
2. 34.6 N
3. 10.8 N
4. 89.4 N at 153.4° to $\underset{\sim}{f}_1$
5. 138.6°
6. a. Sample responses can be found in the worked solutions in the online resources.
   b. 26.3 N, 52.6 N
   c. 26.6°, 63.4°
   d. Sample responses can be found in the worked solutions in the online resources.
7. 17.7 N
8. A
9. $3\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}}$
10. $-\frac{\sqrt{3}}{2}$
11. D
12. $-d\sin(\theta)\hat{\boldsymbol{\imath}} + d\cos(\theta)\hat{\boldsymbol{\jmath}}$
13. B
14. 12.1 km/h at 65.6° to the bank
15. 8.7 km/h N (i.e. from the north)
16. $-501.7\hat{\boldsymbol{\imath}} - 754.1\hat{\boldsymbol{\jmath}}$; distance 905.7 km, bearing 213°38′T
17. 7 minutes 46 seconds
18. 60 m
19. a. 2.96 N down
   b. 1.96 N down
   c. 0.96 N down
20. a. 0.038 b. 193.3 m

TOPIC 2 Vectors in the plane

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

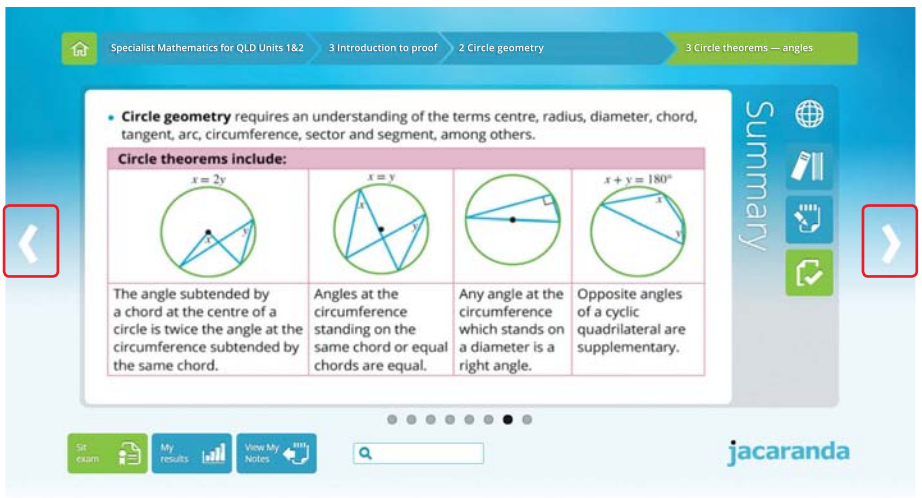
- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



- Select **Practice** at the sequence level to access all questions in the sequence.

- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 1

Specialist Mathematics: Problem solving and modelling task

Unit

Unit 1: Combinatorics, vectors and proof

Topic

Topic 1: Combinatorics

Conditions

Duration	Mode	Individual/group
4 weeks (including 3 hours of class time)	Written report, up to 10 pages (maximum 2000 words) excluding appendix	Individual
Resources available		
The use of technology is required, for example: • computer • internet • spreadsheet program • graphics calculator • other software/technology		

Milestones		
Week 1		
Week 2		
Week 3		
Week 4 (assessment submission)		
Criterion	**Marks allocated**	**Result**
Formulate *Assessment objectives 1, 2, 5	4	
Solve *Assessment objectives 1, 6	7	
Evaluate and verify *Assessment objectives 4, 5	5	
Communicate *Assessment objective 3	4	
Total	**20**	
Scaffolding	The approach to problem-solving and modelling on the following flowchart must be used.	

* Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Context

At the beginning of the 2015/2016 English Premier League (EPL) season, Leicester City were placed at odds of 5000 to 1 of winning the championship. It was considered likely that they would be relegated in the following season, and yet they won the premiership. According to sports writer Posnanski, 'The Premier League is the world's biggest sports league, and it is almost flawlessly designed — or so it seemed — to make sure that a team like Leicester City NEVER wins the Premier League.'[1]

Posnanski was referring to the lack of incentives to help small teams be successful, but there is another potential reason that Leicester City should not have been able to win the competition. There are 38 rounds in the EPL season, compared to 26 in the Australian National Rugby League (NRL) regular season, 16 in the American National Football League (NFL) regular season and 27 in the Australian A-League competition. Having more rounds in the competition should make it less likely for a weaker team to be successful.

Task

In this task, you are going to explore the claim that weaker teams (or players) have a better chance of winning when there are fewer rounds.

You must use mathematical and statistical procedures to determine the validity of the claim. Test your claim using data from a competition or competitions. You may choose to explore the claim from the perspective of the number of rounds in the competition, the number of games required to win a round (for example volleyball, tennis, snooker), or both.

To complete this task, you must:

- present your findings as an investigative report based on the approach to problem-solving and mathematical modelling outlined in the Specialist Mathematics syllabus and on the flow chart on the following page of this instrument
- respond with a range of understanding and skills, such as using mathematical language, appropriate calculations, tables of data, graphs and diagrams
- provide a response that highlights the real-life application of mathematics
- respond using a written report format that can be read and interpreted independently of the instrument task sheet
- develop a unique response.
- use both analytic procedures and technology.

[1]Posnanski, J. 2016. 'In a league of their own: The unpredictable, unrepeatable, unbelievable run of Leicester City', NBC Sports. http://sportsworld.nbcsports.com/leicester-city-premier-league-champions/

Formulate

Design a plan to investigate the claim that weaker teams (or players) have a better chance of winning when there are fewer rounds.

Translate the problem into a mathematically purposeful representation by first determining the applicable mathematical and/or statistical principles, concepts, techniques and technology that are required to make progress with the problem.

Identify and document appropriate assumptions, variables and observations, based on the logic of a proposed solution and model.

Solve

Draw on your knowledge of mathematical and statistical procedures, concepts and techniques to determine the validity of the claim that weaker teams (or players) have a better chance of winning when there are fewer rounds.

Synthesise and refine your model, and generate and test hypotheses with secondary data and information. This process involves an examination and critical analysis of the model being developed to ensure the problem has been solved or can actually be solved.

The use of technology must be demonstrated in the process of solving the problem.

Is it solved?

Evaluate and verify

Once a possible solution has been achieved, consider the reasonableness of the solution and the utility of the model in terms of the problem. Evaluate your results and make a judgment about the solution to the problem in relation to the original claim that weaker teams (or players) have a better chance of winning when there are fewer rounds.

This will involve exploring the strengths and limitations of the solution and/or model. Where necessary, this will require going back through the process to further refine your model to provide a valid solution.

Is the solution verified?

Communicate

Clearly and fully communicate your development of solutions and models to the claim that weaker teams (or players) have a better chance of winning when there are fewer rounds.

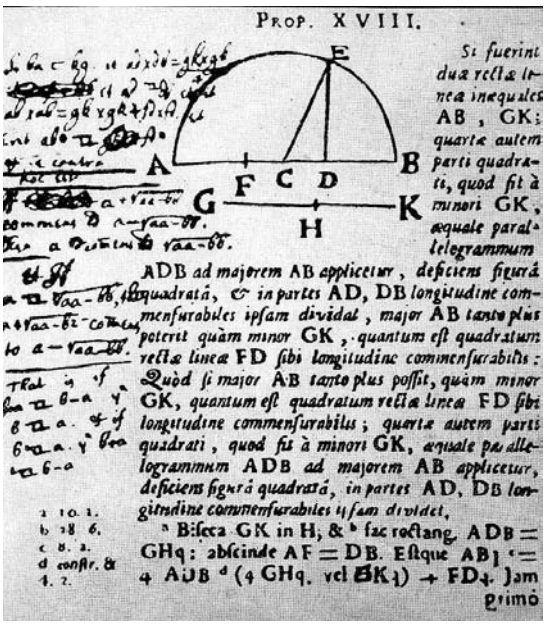
Communicate your findings systematically and concisely using mathematical, statistical and everyday language including through a suitable introduction, body and conclusion.

Draw conclusions by discussing the key results and the strengths and limitations of the solution and/or model. Offer further explanation, justification, and/or recommendations, framed in the context of the initial problem.

CHAPTER 4
Introduction to proof

4.1 Overview

4.1.1 Introduction


PROP. XVIII.

Si fuerint duæ rectæ lineæ inæquales AB, GK; quartæ autem parti quadrati, quod fit à minori GK, æquale parallelogrammum ADB ad majorem AB applicetur, deficiens figurâ quadratâ, & in partes AD, DB longitudine commensurabiles ipsam dividat, major AB tanto plus poterit quàm minor GK, quantum est quadratum rectæ lineæ FD sibi longitudine commensurabilis: Quòd si major AB tanto plus possit, quàm minor GK, quantum est quadratum rectæ lineæ FD sibi longitudine commensurabilis; quartæ autem parti quadrati, quod fit à minori GK, æquale parallelogrammum ADB ad majorem AB applicetur, deficiens figurâ quadratâ, in partes AD, DB longitudine commensurabiles ipsam dividet.

In your study of mathematics, you have already used a number of rules, for example Pythagoras's theorem. Mathematicians like to be certain that a rule is always true before they use it. They use proofs to demonstrate this. Once something is proven, it is certain to be true and it will always be true. Proofs make mathematics different to the sciences, because they are not like theories. (A theory is the best current explanation for our observations, and may be replaced by a better theory sometime in the future.)

In Greece during the 5th century BCE, philosophers developed the idea of proving that a mathematical statement or proposition was true. To do this they needed to agree on the definitions of some basic terms. Also, to have starting points for their arguments, they needed to agree that some basic statements, called axioms, were true. Once a statement was proven true using a rigorous logical argument based on the definitions and axioms, it was called a theorem. Once a theorem was proved, it could be used to prove other more complicated theorems. Euclid (325–263 BCE) collected these definitions, theorems and proofs into a series of 13 books called *The Elements* — a book so influential it has been used for over 2000 years, not only in the study of mathematics, but also the development of logic. The image shows Sir Isaac Newton's copy of Euclid with his own notes in the margin.

However, this certainty was challenged in 1931 by Kurt Gödel (1906–1978), who proved that in any complex mathematical system with a certain number of axioms, there will be some statements that can be neither proved nor disproved using the axioms.

If all theorems have already been proved, why is it important to learn how to construct a proof? The first reason for a proof is to convince someone that a statement is true. The second reason is that it helps us to understand why the statement is true.

LEARNING SEQUENCE

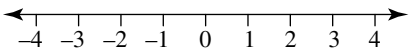
4.1 Overview
4.2 Number systems and writing propositions
4.3 Direct proofs using Euclidean geometry
4.4 Indirect methods of proof
4.5 Proofs with rational and irrational numbers
4.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 Number systems and writing propositions

4.2.1 The real number system

The set of **real numbers** consists of all the numbers that can be thought of as points on a number line. The set of real numbers is represented by the symbol R.



The set of real numbers has a number of subsets.

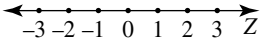
The set of **natural numbers** (or counting numbers) is represented by the symbol N, where $N = \{1, 2, 3, \ldots\}$.

The set of **integers** consists of all the positive and negative natural numbers and 0 (which is neither positive nor negative). Represented by the symbol Z, the set of integers can be divided into the subsets of positive integers (or natural numbers), represented by Z^+ (or N), and negative integers, represented by Z^-. That is:

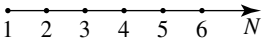
$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$
$$Z^+ = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$Z^- = \{-1, -2, -3, -4, -5, -6, \ldots\}$$

Using set notation, $Z = Z^+ \cup Z^- \cup \{0\}$.

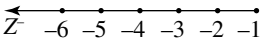
Integers and natural numbers may be represented on the number line as illustrated below.



The set of integers



The set of positive integers or natural numbers



The set of negative integers

Note: Integers on the number line are marked with a solid dot to indicate that they are the only points we are interested in.

When an integer is divided by another integer, the result is a **rational number**. More formally, a rational number can be written as the ratio of two integers, a and b, in the form $\frac{a}{b}$ where $b \neq 0$ and a and b do not have any common factors (except 1). The set of rational numbers is represented by the symbol Q.

In set notation,

$$Q = \left\{ \frac{a}{b},\ a, b \in R,\ b \neq 0,\ (a, b) = 1 \right\}$$

where $(a, b) = 1$ means that the greatest common divisor of a and b is 1.

If $b = 1$, then $\frac{a}{b} = a$. Therefore, integers are a subset of the set of rational numbers.

Some rational numbers may be expressed as terminating decimals (that is, they contain a specific number of digits). For example:

$$\frac{1}{2} = 0.5$$
$$\frac{1}{8} = 0.125$$
$$-\frac{9}{5} = -1.8$$

Other rational numbers may be expressed as recurring decimals (non-terminating or periodic decimals). For example:

$$\frac{1}{3} = 0.333\,333\,3\ldots = 0.\dot{3}$$

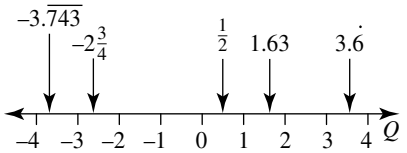
$$\frac{7}{11} = 0.636\,363\,63\ldots = 0.\dot{6}\dot{3} = 0.\overline{63}$$

$$\frac{53}{90} = 0.588\,888\,88\ldots = 0.5\dot{8}$$

$$\frac{15}{7} = 2.142\,857\,142\,857\ldots = 2.\dot{1}42\,85\dot{7} = 2.\overline{142\,857}$$

Note: Recurring decimals with more than one digit can be shown by either a dot over the first digit and the last digit in the recurring sequence, or an overscore over the recurring sequence.

Rational numbers can be represented on a number line as illustrated below.



Any real numbers that are not rational numbers are called **irrational numbers**. These include **surds** (for example $\sqrt{2}$ and $\sqrt{10}$) and all other decimals that neither terminate nor recur, such as π and e. There is no common symbol for irrational numbers, but because they are the set of real numbers excluding the set of rational numbers, they can be represented by $R \setminus Q$.

Irrational numbers may be represented by decimals. For example,

$$\sqrt{0.03} = 0.173\,205\,080\ldots$$

$$\sqrt{18} = 4.242\,640\,68\ldots$$

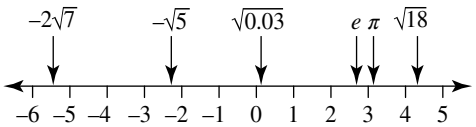
$$-\sqrt{5} = -2.236.6797\ldots$$

$$-2\sqrt{7} = -5.291\,502\,62\ldots$$

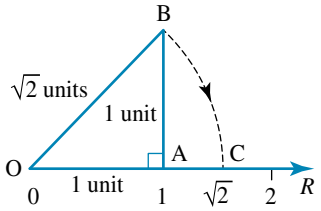
$$\pi = 3.141\,592\,653\ldots$$

$$e = 2.718\,281\,828\ldots$$

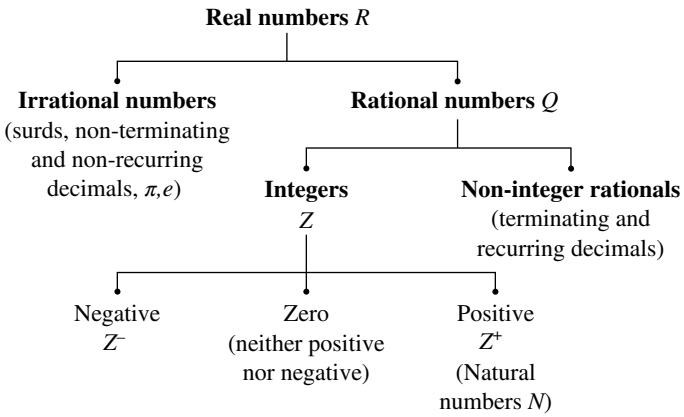
These decimal approximations can be shown on a number line as illustrated below.



Using geometry, Pythagoras' **theorem** and a compass, irrational numbers in surd form can be represented on the number line exactly, as shown.



In summary, the set of real numbers can be divided into two main sets: rational and irrational numbers. These may be divided into further subsets as illustrated below.



Definitions in mathematics are precise descriptions of mathematical terms, such as those given above for real numbers, integers, rational numbers and irrational numbers. Definitions make it easier for us to discuss and apply mathematical concepts.

WORKED EXAMPLE 1

Identify the following numbers as either rational or irrational.

a. 7 **b.** $\frac{3}{11}$ **c.** 0.25

d. $0.010\,110\,111\,\ldots$ **e.** $3.\dot{7}$ **f.** $\sqrt{5}$

THINK	WRITE
a. 7 is an integer. All integers are rational.	**a.** Rational
b. $\frac{3}{11}$ is the quotient of two integers. It is rational.	**b.** Rational
c. 0.25 is a terminating decimal. It is rational.	**c.** Rational
d. 0.010 110 111 … is neither terminating nor eventually repeating. It is irrational.	**d.** Irrational
e. $3.\dot{7}$ is a recurring decimal. It is rational.	**e.** Rational
f. $\sqrt{5}$ is a surd. It is irrational.	**f.** Irrational

4.2.2 Converting between decimal fractions and common fractions

Rational numbers can be expressed as the quotient of two integers or as numbers whose decimal expansions are either terminating or eventually recurring. It is necessary to be able to express each rational number in both formats.

Expressing common fractions as decimal fractions

Consider the following decimal fractions and their common fraction equivalents.

$$0.1 = \frac{1}{10} \qquad 0.37 = \frac{37}{100} \qquad 0.163 = \frac{163}{100}$$

All terminating decimals can be expressed with a power of 10 as the denominator. As the only prime factors of 10 are 2 and 5, any common fraction whose denominator can be written as powers of 2 and/or 5 will be a terminating decimal.

Consider $\frac{1}{40}$. The denominator can be written as powers of 2 and 5: $40 = 2^3 \times 5$. Therefore, the number can be expressed as a terminating decimal. $\frac{1}{40} = \frac{1}{40} \times \frac{25}{25} = \frac{25}{1000} = 0.025$.

Consider $\frac{1}{3}$. As the denominator is not a multiple of either 2 or 5, the number will be a recurring decimal. This decimal can be found by division: $3\overline{)1.^{1}0^{1}0^{1}0}$ $0.333...$. Therefore, $\frac{1}{3} = 0.\dot{3}$.

Likewise, the rational number $\frac{5}{6}$ is a recurring decimal as $\frac{5}{6} = 0.8\dot{3}$, as can be found by division: $6\overline{)5.^{5}0^{2}0^{2}0}$ $0.833...$. Therefore, $\frac{5}{6} = 0.8\dot{3}$.

WORKED EXAMPLE 2

Express the following numbers as decimal fractions.

a. $\frac{1}{50}$ **b.** $\frac{7}{9}$ **c.** $\frac{5}{11}$

THINK

a. 1. The denominator can be written as powers of 2 and 5.

2. Write the denominator as a power of 10. As $50 = 5^2 \times 2$, it needs to be multiplied by 2 in order to be written as a power of 10 $\left(10^2 = 100\right)$.

3. Write the result as the decimal fraction 2 hundredths.

WRITE

a. $50 = 5^2 \times 2$

$$\frac{1}{50} = \frac{1}{5^2 \times 2} \times \frac{2}{2}$$
$$= \frac{1 \times 2}{(5 \times 2)^2}$$
$$= \frac{2}{10^2}$$
$$\frac{1}{50} = \frac{2}{100}$$
$$= 0.02$$

b. 1. The denominator cannot be written as powers of 2 and 5. The decimal will be recurring. Use division to determine the decimal equivalent.

b.
$$9\overline{)7.^{7}0^{7}0^{7}0}\quad 0.\ 7\ 7\ldots$$

2. Write the result as a recurring decimal fraction.

$$\frac{7}{9} = 0.\dot{7}$$

c. 1. The denominator cannot be written as powers of 2 and 5. The decimal will be recurring. Use division to determine the decimal equivalent.

c.
$$11\overline{)5.^{5}0^{6}0^{5}0^{6}0^{5}0}\quad 0.\ 4\ 5\ 4\ 5\ldots$$

2. Write the result as a recurring decimal fraction.

$$\frac{5}{11} = 0.\dot{4}\dot{5}\ (\text{or } 0.\overline{45})$$

TI | THINK

c.1. On a Calculator page, complete the entry line as:
$\frac{5}{11}$
Press MENU then select:
2: Number
1: Convert to Decimal
then press ENTER.
Note: If the Calculation Mode is set to Approximate, simply complete the entry line as $\frac{5}{11}$, then press ENTER.

2. The answer appears on the screen.

WRITE

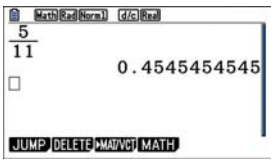
$\frac{5}{11} = 0.\dot{4}\dot{5}$

CASIO | THINK

c.1. On the Run-Matrix screen, complete the entry line as:
$\frac{5}{11}$
then press EXE.
Press the S⇔D button to convert the fraction to a decimal.

2. The answer appears on the screen.

WRITE



$\frac{5}{11} = 0.\dot{4}\dot{5}$

Expressing decimal fractions as common fractions

The fraction 0.04 can be expressed as $\frac{4}{100}$. This simplifies to $\frac{1}{25}$.

For recurring decimals, the process is as follows.

WORKED EXAMPLE 3

Express the following recurring decimals as common fractions.

a. $0.\dot{2}$ **b.** $3.2\dot{1}\dot{5}$

THINK

a. 1. Let variable x equal the recurring decimal.

2. $10x$ and x will have identical recurring decimals.

3. Subtracting x from $10x$ will result in a whole number.

4. Solve for x.

5. Write your concluding statement.

WRITE

a. Let $x = 0.\dot{2}$
$= 0.2222\ldots$
$10x = 2.2222\ldots$
$10x - x = 2.\dot{2} - 0.\dot{2}$
$9x = 2$
$x = \frac{2}{9}$
$0.\dot{2} = \frac{2}{9}$

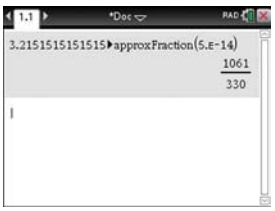
b. 1. Let variable x equal the recurring decimal.	**b.** Let $x = 3.2\dot{1}\dot{5}$.
2. Finding $10x$ will result in a number with only the recurring part after the decimal.	$10x = 32.\dot{1}\dot{5}$
3. $10x$ and $1000x$ will have identical recurring decimals.	$1000x = 3215.\dot{1}\dot{5}$
4. Subtracting $10x$ from $1000x$ will result in a whole number.	$1000x - 10x = 3215.\dot{1}\dot{5} - 32.\dot{1}\dot{5}$
	$990x = 3183$
5. Solve for x and simplify.	$x = \frac{3183}{990}$
	$= \frac{1061}{330}$
6. Write your concluding statement.	$3.2\dot{1}\dot{5} = \frac{1061}{330}$ or $3.2\dot{1}\dot{5} = 3\frac{71}{330}$

TI | THINK

b.1. On a Calculator page, complete the entry line as: 3.215151515151515
Press MENU, then select:
2: Number
2: Approximate to Fraction
then press ENTER.

2. The answer appears on the screen.

WRITE



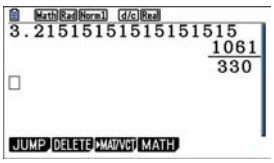
$3.2\dot{1}\dot{5} = \frac{1061}{330}$

CASIO | THINK

b.1. On the Run-Matrix screen, complete the entry line as: 3.215151515151515
then press EXE.
Press the S⇔D button to convert the decimal to a fraction.

2. The answer appears on the screen.

WRITE



$3.2\dot{1}\dot{5} = \frac{1061}{330}$

4.2.3 Propositions

A **proposition** or **mathematical statement** is a sentence that is either true or false. It may contain words and symbols. For example, '3 is a prime number' is a mathematical statement that is true. However, '$\sqrt{7} = 3$' is a false mathematical statement. Propositions may be combined to make larger, more complex propositions (that must be either true or false, but not both). In propositions, some words have very precise meanings that may differ slightly from their everyday use.

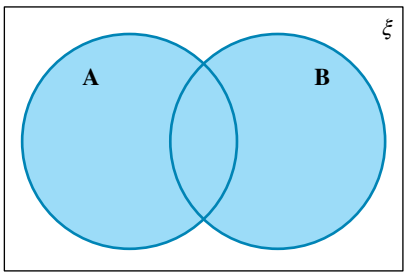
Either/or

In everyday language, 'either A or B' means that one of the options is true, but not both. For example, if you were asked if you would like salad or vegetables with your meal, the wait staff are not expecting you to choose both. However, in mathematics, saying that A or B is true means that at least one of them is true (which also means that they both might be).

A truth table can be used to display all possible true/false combinations. A truth table contains one column for each input variable, and a column showing all possible logically valid results. Each row contains the unique possible configurations of the input values.

A	B	A or B
T	T	T
T	F	T
F	T	T
F	F	F

A or B can also be presented in a Venn diagram, $A \cup B$.



WORKED EXAMPLE 4

From the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}, select the numbers that are either even or prime.

THINK	WRITE
1. Select the even numbers.	Even numbers: {2, 4, 6, 8}
2. Select the prime numbers.	Prime numbers: {2, 3, 5, 7}
3. Numbers that are either even or prime will be in at least one of the sets.	Even or prime: {2, 3, 4, 5, 6, 7, 8}

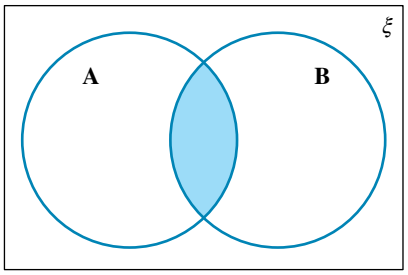
And

The use of the word 'and' in mathematics and everyday language is similar. In mathematics, saying that A and B is true means both of them are true.

A truth table for A and B displays all possible true/false combinations, $A \cap B$.

A	B	A and B
T	T	T
T	F	F
F	T	F
F	F	F

A and B can also be presented in a Venn diagram, A ∩ B.



WORKED EXAMPLE 5

From the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}, select the numbers that are even and prime.

THINK	WRITE
1. Select the even numbers.	Even numbers: {2, 4, 6, 8}
2. Select the prime numbers.	Prime numbers: {2, 3, 5, 7}
3. Numbers that are even and prime will be in both sets.	Even and prime: {2}

Negation

Sometimes it is necessary to know what the opposite of a statement is. This is called 'negating' a statement.

The **negation** of a true statement is false, and the negation of a false statement is true.

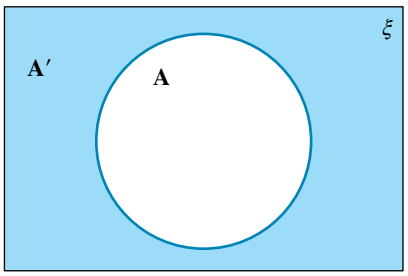
Using symbols, the negation of statement A is written as ¬A.

The negation of the earlier true statement '3 is a prime number' is the false statement '3 is not a prime number'.

The negation of A can be shown in a truth table.

A	¬A
T	F
F	T

¬A can be presented in a Venn diagram, A'.



WORKED EXAMPLE 6

Write the negation of the statement $5 > 7$.

THINK	WRITE
1. The negation of 'larger than' is 'not larger than'.	$5 \not> 7$
2. If a number is not larger than a number, it must be smaller than or equal to the number. This is a better way to write the statement.	$5 \leq 7$

Negation of A or B

In Worked example 4, we looked at numbers that were even numbers or prime numbers. The numbers that weren't included were the ones that were not even and were not prime.

In general, the negation of the statement 'A or B' becomes 'not A and not B'.

By comparing ¬(A or B) with ¬A and ¬B, this can be confirmed.

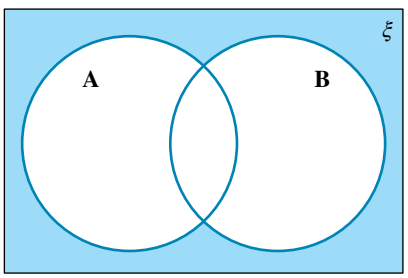
A	B	(A or B)	¬(A or B)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

A	B	¬A	¬B	(¬A and ¬B)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Let us examine the second row of both tables, where A is true and B is false. In the upper table (¬(A or B)), (A or B) is true, because in an 'or' statement, at least one option must be true (but both options do not need to be true). If (A or B) is true, then ¬(A or B) must be false.

Comparing this to the same line in the lower table (¬A and ¬B), ¬A is false and ¬B is true, so (¬A and ¬B) must be false, because in an 'and' statement, both options must be true.

Hence, ¬(A or B) is the same as ¬A and ¬B. It can also be presented on a Venn diagram, $(A \cup B)'$.



The following Worked example is an extension of Worked example 4.

WORKED EXAMPLE 7

From the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}, the set of numbers that are either even or prime is {2, 3, 4, 5, 6, 7, 8}.

a. Select the numbers that are neither even nor prime.

b. Identify another way this set of numbers could be described.

THINK	WRITE
a. The numbers that are neither even nor prime are the numbers that are not in the set {2, 3, 4, 5, 6, 7, 8}.	**a.** Neither even nor prime: {1, 9}
b. 1. Identify the set of numbers that are not even (i.e. the odd numbers). Identify the set of numbers that are not prime.	**b.** Odd numbers: {1, 3, 5, 7, 9} Not prime numbers: {1, 4, 6, 8, 9}
2. The members of the set {1, 9} are members of the set of odd numbers and the set of numbers that are not prime.	The set {1, 9} could also be described as the set of numbers that are odd and not prime.

Negation of A and B

In Worked example 5, we looked at numbers from the set {1, 2, 3, 4, 5, 6, 7, 8, 9} that were even numbers and prime numbers, and the only result was 2. The numbers that were not included were the ones that weren't even or weren't prime.

In general, the negation of the statement 'A and B' becomes 'not A or not B'.

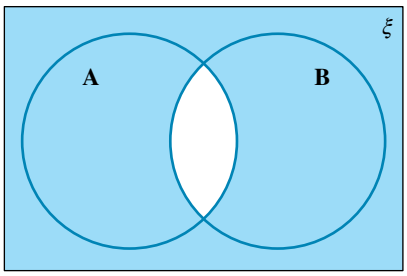
By comparing ¬(A and B) with ¬A or ¬B, this can be confirmed.

A	B	(A and B)	¬(A and B)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

A	B	¬A	¬B	(¬A or ¬B)
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

As above, by examining the second row of both tables, we see that when A is true and B is false, (A and B) is false, so ¬(A and B) is true. (In 'and' statements, both options must be true.) Comparing this to the same line in the lower table (¬A or ¬B), ¬A is false and ¬B is true, so (¬A or ¬B) is true, as only one option must be true in an 'or' statement.

Hence, the negation of A and B, ¬(A and B), is the same as ¬A or ¬B. This can also be presented on a Venn diagram, $(A \cap B)'$.



The following Worked example is an extension of Worked example 5.

WORKED EXAMPLE 8

From the set of numbers {1, 2, 3, 4, 5, 6, 7, 8, 9}, the set of numbers that are even and prime is {2}.

a. Select the numbers that are not (even and prime).

b. Identify another way this set of numbers could be described.

THINK	WRITE
a. The numbers that are not (even and prime) are the numbers that are not in the set {2}.	Not (even and prime): {1, 3, 4, 5, 6, 7, 8, 9}
b. 1. Identify the set of numbers that are not even (that is, odd numbers). Identify the set of numbers that are not prime.	Odd numbers: {1, 3, 5, 7, 9} Not prime numbers: {1, 4, 6, 8, 9}
2. The members of the set {1, 3, 4, 5, 6, 7, 8, 9} are members of the set of odd numbers or the set of non-prime numbers.	This could also be described as the set of numbers that are odd or not prime.

If, then statements (implication)

Statements using **implication** (also called implicative statements) are normally written in the form 'if P, then Q' or 'P implies Q'.

Some examples are:

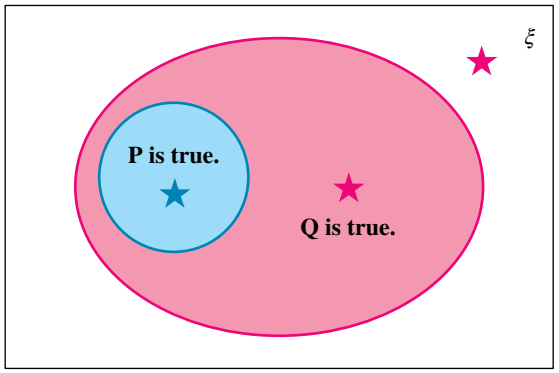
- If $x = 5$, then $x^2 = 25$.
- If $x = 3$, then $x^3 = 27$.
- If a quadrilateral is a rectangle, then its diagonals are congruent and bisect each other.

'If P, then Q' can be written using symbols as P ⇒ Q. The above statements can be rewritten as:

- $x = 5 \Rightarrow x^2 = 25$
- $x = 3 \Rightarrow x^3 = 27$
- A quadrilateral is a rectangle ⇒ its diagonals are congruent and bisect each other.

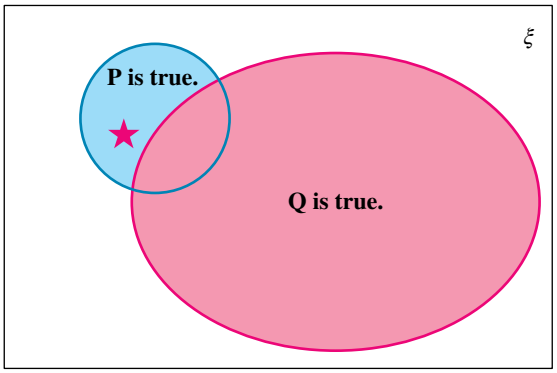
Implicative statements only tell you about what happens if P is true. They do not tell you what happens if Q is true.

An implicative statement can be shown on a Venn diagram. Inside the circle labelled 'P is true', the statement P is true. Outside the circle labelled 'P is true', the statement P is false.



★ Whenever P is true, Q is also true.

★ Whenever P is not true, Q might or might not be true.



P ⇒ Q is a false statement.

★ P is true but Q is not true.

To disprove the implicative statement P ⇒ Q, you need to find a case where P is true but Q is not true. As seen in the truth table for P ⇒ Q, there is only one combination of P and Q that will make the statement false.

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

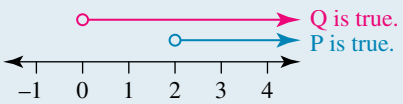
Note: If P is false, the statement P ⇒ Q is said to be 'vacuously true'. That means it is technically true but doesn't really tell us anything. 'If pigs could fly, then … ' (You can put what you like there and it is vacuously true, but pigs can't fly.)

The negation of the statement P ⇒ Q can be considered a false statement, that is, P ⇒ Q is false if and only if (**iff**) Q is false. Hence, the negation of P ⇒ Q is logically P and not Q (P and ¬Q). This is demonstrated in the truth table.

P	Q	¬Q	P and ¬Q
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

WORKED EXAMPLE 9

Determine if the following are true statements.
a. If a shape has four sides, then it is a rectangle.
b. If $x^3 > 8$, then $x > 0$.

THINK	WRITE
a. 1. This is an 'if, then' statement. Identify the propositions.	**a.** P: A shape has four sides. Q: It is a rectangle.
2. Consider if the statement is true. If P is true (the shape has four sides), that means Q is also true (the shape is a rectangle). If the statement is not true, then we should be able to find when P is true (the shape has four sides) and Q is not true (the shape is not a rectangle).	A parallelogram is an example of a shape that has four sides but is not a rectangle.
3. Write the concluding statement.	The statement is false.
b. 1. This is an 'if, then' statement. Identify the propositions.	**b.** P: $x^3 > 8$ Q: $x > 0$
2. Consider if the statement is true. If P is true ($x^3 > 8$), that means Q is also true ($x > 0$). If the statement is not true, then we should be able to find when P is true and Q is not true.	P: $x^3 > 8$ $x > 2$ 
3. Whenever P is true, Q is also true. Write the concluding statement.	The statement is true.

Equivalence

The **converse** of the statement 'if P, then Q' is 'if Q, then P' or $Q \Rightarrow P$. The converse of a statement may or may not be true. The converse of the previous examples can be written as follows.

- If $x^2 = 25$, then $x = 5$. In this case the converse is false, as $x = -5$ is also possible. This means that it is possible for $x^2 = 25$ but $x \neq 5$.
- If $x^3 = 27$, then $x = 3$. In this case the converse is true.
- If the diagonals of a quadrilateral are congruent and bisect each other, then it is a rectangle. In this case the converse is true, remembering a square is also rectangle.

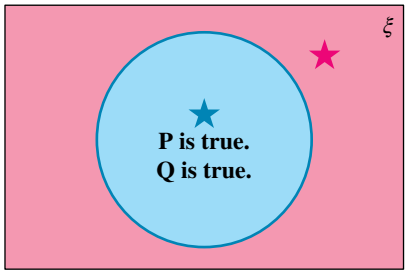
If $P \Rightarrow Q$ and $Q \Rightarrow P$, then P and Q are said to be **equivalent statements**.

Equivalent statements can also be expressed as 'P if and only if Q'. This is written symbolically as $P \Leftrightarrow Q$. Rewriting the first two examples above using symbols gives us:

- $x = 5 \Rightarrow x^2 = 25$
- $x = 3 \Leftrightarrow x^3 = 27$

P $\Leftrightarrow$ Q is true if both P and Q are true. It is also true if P and Q are both false, as illustrated in the truth table for P $\Leftrightarrow$ Q.

P	Q	P $\Leftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T



P $\Leftrightarrow$ Q is a true statement.
★ P is true and Q is true.
★ P is false and Q is false.

WORKED EXAMPLE 10

For each of the following, write the converse statement and identify if implication ($\Rightarrow$) or equivalence ($\Leftrightarrow$) is the more appropriate symbol.

a. If an animal is a kangaroo, then it is a marsupial.

b. If a number, n, is odd, then n^2 is odd.

THINK

a. 1. Identify the two propositions.

2. Write the converse statement, Q $\Rightarrow$ P.

3. Determine if the converse is true.

4. The converse is not true, so implication is more appropriate.

b. 1. Identify the two propositions.

2. Write the converse statement, Q $\Rightarrow$ P.

3. Determine if the converse is true.

4. The converse is true, so equivalence is more appropriate.

WRITE

a. P: An animal is a kangaroo.
Q: The animal is a marsupial.

The converse is 'If an animal is a marsupial, then it is a kangaroo.'

The converse is not true.

The symbol $\Rightarrow$ should be used.
P $\Rightarrow$ Q

b. P: n is odd.
Q: n^2 is odd.

The converse is 'If n^2 is odd, then n is odd.'

The converse is true.

The symbol $\Leftrightarrow$ should be used.
P $\Leftrightarrow$ Q

In determining if P $\Leftrightarrow$ Q is true, it is necessary to confirm that P $\Rightarrow$ Q is true and that Q $\Rightarrow$ P is also true.

4.2.4 Quantifiers

The proposition $9 > 7$ is a true statement, and the proposition $4 > 7$ is a false statement. However, $x > 7$ might be true or false depending on the value of x. A proposition that includes variables is known as a **propositional function**. Because it is a function, we can use function notation to name the function. In this case, we might call it P(x).

Quantifiers are used to give information about the values of the variables in propositional functions so that we can determine if the function is true or false

The universal quantifier

The **universal quantifier**, *for all*, is written with the symbol ∀. This means that all possible values for the variable are considered.

Consider the propositional function $x > 7$. Using the universal quantifier, the statement becomes 'For all real numbers x, $x > 7$.' Written with symbols, the proposition is $\forall x \in R$, $x > 7$. As the proposition is not true for all possible values of x, it is a false statement.

WORKED EXAMPLE 11

Consider the proposition $\forall x \in R$, $x^2 \geq 0$. Write the proposition in words and determine if it is true or false.

THINK	WRITE
1. The symbol ∀ means 'for all'. $x \in R$ means x is a member of the set of real numbers.	For all real numbers x, $x^2 \geq 0$.
2. If $x \geq 0$, $x^2 \geq 0$, and if $x < 0$, $x^2 > 0$. Therefore, the proposition is always true.	The proposition is true.

The existential quantifier

The **existential quantifier**, *there exists*, is written with the symbol ∃. This means that there is a value for the variable that would make the propositional function true.

Let us again consider the propositional function $x > 7$, but this time using the existential quantifier. The statement becomes 'There exists a real number, x, where $x > 7$.' Written in symbols, the proposition is $\exists x \in R$, $x > 7$. In this case, because there are values of x that make the proposition true, it is true.

WORKED EXAMPLE 12

Consider the proposition $\exists x \in N$, $2x + 1$ is a multiple of 3. Write the proposition in words and determine if it is true or false.

THINK	WRITE
1 The symbol ∃ means 'there exists'. $x \in N$ means x is a member of the set of natural numbers.	There exists a natural number x where $2x + 1$ is a multiple of 3.
2 If $x = 1$, then $2x + 1$ is a multiple of 3.	The proposition is true for $x = 1$ and is therefore true.

Negating quantifiers

Consider the statement 'For all even numbers, x, x^2 is even.' This is a true statement. However, it would not be true if you could find one even number, x, where x^2 was not even.

Now consider the false statement 'All prime numbers are odd.' This can be rewritten as 'For all prime numbers, p, p is odd.' The negation of this false statement is the true statement 'There exists a prime number, p, where p is not odd.'

The negation of the statement $\forall x \in X,\ \mathrm{P}(x)$ is the statement $\exists x \in X,\ \neg \mathrm{P}(x)$

This also means that the negation of the statement $\exists x \in X,\ \mathrm{P}(x)$ is the statement $\forall x \in X,\ \neg \mathrm{P}(x)$.

Combining quantifiers

If a propositional function has more than one variable, then quantifiers are needed for all of the variables. If the quantifiers are the same type, the order that they are written in is immaterial. Also, if a quantifier is shown before a group of variables separated by commas, the quantifier applies to all of the variables in the group. For example, the following statements are equivalent.

$$\forall x \in R,\ \forall y \in R$$

$$\forall y \in R,\ \forall x \in R$$

$$\forall x,\ y \in R$$

If the quantifiers are different, then the order becomes important. The statement $\forall x,\ \exists y$ means that for each possible value of x, there exists a value for y that makes a proposition true. The statement $\exists x,\ \forall y$ means that there is a value for x that makes the statement true for any possible value of y.

WORKED EXAMPLE 13

For each of the propositions below, write the proposition in words and determine if it is true or false.

a. $\forall x \in R,\ \exists y \in R, y > x^2$

b. $\exists x \in R,\ \forall y \in R,\ y > x^2$

THINK	WRITE
a. 1. $\forall x,\ \exists y$ means that for each possible value of x, there exists a value for y that makes a proposition true.	**a.** For all real numbers x, there exists a real number y where $y > x^2$.
2. For every real number, there is another real number greater than the square of the original. Therefore, the proposition is true.	True
b. 1. $\exists x,\ \forall y$ means that there is a value for x that makes the statement true for any possible value of y.	**b.** There exists a real number x where for all real numbers y, $y > x^2$.
2. If y is a negative number, the inequality is never true. Therefore, the proposition is false.	False

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Units 1 & 2 | Area 3 | Sequence 1 | Concepts 1 & 2

The real number system Summary screen and practice questions
Propositions Summary screen and practice questions

Exercise 4.2 Number systems and writing propositions

Technology free

1. **WE1** Identify the following numbers as either rational or irrational.
   a. 6.2 b. 0.321 c. 0.25
   d. 3.121 221 222… e. 156 f. $\sqrt{13}$
2. Identify the following numbers as either rational or irrational.
   a. $\sqrt{7}$ b. $\sqrt[3]{9}$ c. $\sqrt{81}$ d. π
3. **WE2** Express the following numbers as decimal fractions.
   a. $\frac{17}{20}$ b. $\frac{107}{125}$ c. $\frac{2}{3}$ d. $\frac{8}{15}$
4. Express the following numbers as decimal fractions.
   a. $1\frac{3}{5}$ b. $\frac{9}{16}$ c. $\frac{22}{7}$ d. $\frac{13}{12}$
5. **WE3** Express the following recurring decimals as common fractions.
   a. $0.\dot{5}$ b. $0.\dot{2}\dot{3}$ c. $3.0\dot{1}$
   d. $6.\dot{1}\dot{6}$ e. $7.1\dot{1}\dot{2}$ f. $0.\dot{7}0\dot{2}$
6. From the set of numbers {0, 3, 6, 9, 12, 15, 18, 21, 24}:
   a. **WE4** select the numbers that are either odd or less than 12
   b. **WE5** select the numbers that are odd and less than 12
   c. **WE6** select the numbers that are the negation of odd
   d. select the numbers that are not less than 12
   e. **WE7** select the numbers that are neither odd nor less than 12. Identify another way that this set of numbers could be described.
   f. **WE8** select the numbers that are not (odd and less than 12). Identify another way that this set of numbers could be described.
7. **WE9** Determine if the following are true statements.
   a. If a polygon has three sides, then it is a triangle.
   b. If a polygon is a square, then it is a rectangle.
   c. If the sides of a square are doubled in length, then the area of the square is also doubled.
   d. If $x^3 > 27$, $x \in N$, then $x > 2$, $x \in N$
   e. If $x^2 > 16$, $x \in N$, then $x > 4$, $x \in N$
   f. If $x^2 < 4$, $x \in N$, then $x < 2$, $x \in N$
8. **WE10** For each of the following, write the converse statement and identify if implication ($\Rightarrow$) or equivalence ($\Leftrightarrow$) is the more appropriate symbol.
   a. If it is a fish, then it lives in water.
   b. If the quadrilateral is a rhombus, then the diagonals bisect each other.
   c. If $n - m > 0$, then $n > m$.
   d. If $n + 1$ is odd, then n is even.
9. **WE11** For each of the propositions below, write the proposition in words and determine if it is true or false.
   a. $\forall x \in N$, $2x$ is even.
   b. $\forall x \in N$, $2x + 1$ is a multiple of 3.
   c. $\forall x \in N$, $x > 0$
   d. $\forall x \in R$, $x > 0$ or $x \leq 0$.
   e. $\forall x \in N$, x is even.
   f. $\forall x \in R$, $x^2 + 1 \geq 0$

10. **WE12** For each of the propositions below, write the proposition in words and determine if it is true or false.
    a. $\exists x \in Z, x + 5 = 7$
    b. $\exists x \in R, x^2 < 0$
    c. $\exists x \in N$, x is even.
    d. $\exists x \in R$, $x^2 + 1 = 0$
    e. $\exists x \in R$, $x^2 - 1 = 0$
    f. $\exists x \in Q$, $\sqrt{x} \in Q$
11. **WE13** For each of the propositions below, write the proposition in words and determine if it is true or false.
    a. $\forall x \in R$, $\exists y \in R$, $y = x$
    b. $\exists x \in R$, $\forall y \in R, y = x$
    c. $\forall x \in R$, $\exists y \in R, xy = 0$
    d. $\exists x \in R$, $\forall y \in R, xy = 0$
12. For each of the propositions below, determine if it is true or false.
    a. $\forall x \in R$, $\exists y \in R$, $x > y$
    b. $\exists x \in R$, $\forall y \in R$, $x > y$
    c. $\forall x \in R$, $\forall y \in R$, $x > y$
    d. $\forall x \in R$, $\forall y \in (x, \infty)$, $x < y$
13. Consider the statement '$\forall x \in N$, $x(x + 1)$ is even.' Decide if this is a true statement and justify your decision.
14. The negation of A and B can be written as 'not A or not B'. Determine another way to write the negation of A and B or C in terms of 'not A', 'not B' and 'not C'. You must provide evidence to support your claim. *Note*: 'A and B or C' can be thought of as '(A and B) or C'.

Technology active

15. Use the graph of $y = \sin x + \cos x$ to complete the true proposition $\forall x \in (\ldots)$, $\sin x + \cos x > 1$
16. Use the graph of $x^2 + y^2 = 9$ to answer the following.
    a. Use quantifiers for x and y to create a true proposition.
    b. Use quantifiers for x and y to create a false proposition.

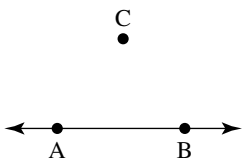
4.3 Direct proofs using Euclidean geometry

4.3.1 Axioms and postulates

As mentioned in Section 4.1.1, in mathematics there are some basic statements that are accepted as true without proving them. In general, when these statements refer to working with numbers, the statements are called **axioms**. When the statements are about geometry, they are called **postulates**.

In *The Elements*, Euclid tried to use only four postulates in his geometric **proofs**. The first four postulates state that:

- a straight line can be drawn from one point to another (that is, any two points will form a straight line)
- it is possible to produce a finite straight line
- a circle can be described with any centre and radius
- all right angles are equal to each other.



However, he found that it was necessary to introduce a fifth postulate:

- if there is a line and a point not on the line, then only one line can be drawn through the point that is parallel to the original line.

Mathematicians have discovered that Euclid's fifth postulate is true when working with flat surfaces but is not true in other instances (for example the geometry of the earth). This means that geometry can be either Euclidean (a flat surface) or non-Euclidean. In this section, we are going to be constructing proofs using Euclidean geometry.

4.3.2 The fundamentals

Consider the following set of statements.

- Penguins are birds.
- Birds can fly.
- Therefore, penguins can fly.
  Or can they? Where does the logic break down?

There are a number of methods of combining relationships to produce new relationships. These techniques assist in the logical construction of proofs. We must also state the postulates that will be the basis for our proofs.

Transitive property of equality

Consider the following:

$$\left.\begin{array}{l} a = b \\ b = c \end{array}\right\} \Rightarrow a = c$$

Because both a and c are equal to b, we can conclude that $a = c$. This technique is called the transitive property of equality.

Substitution

Consider the following:

$$\left.\begin{array}{l} a + b = c \\ b = d \end{array}\right\} \Rightarrow a + d = c$$

Because $b = d$, the statement $a + b = c$ is equivalent to the statement $a + d = c$. This technique is called substitution.

Matching

Consider the following:

$$\left.\begin{array}{l} a + b = c \\ a + d = c \end{array}\right\} \Rightarrow b = d$$

Because the two statements can be rewritten as $b = c - a$ and $d = c - a$, it is clear that $b = d$. This technique is called matching.

Adding equals

Consider the following:

$$\left.\begin{array}{l} a = b \\ c = d \end{array}\right\} \Rightarrow a + c = b + d$$

As $a = b$, $a + c = b + c$.
As $c = d$, $a + c = b + d$.
This technique is called adding equals.

These techniques can be used to reach a logical conclusion that is necessarily true. This is known as deduction.

WORKED EXAMPLE 14

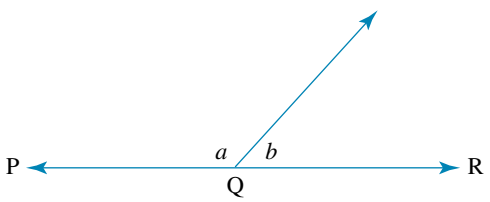
Use techniques of deduction to find the following.
a. If $x = 20$ and $x = y$, find y.
b. If $a + b = 16$ and $b = c$, find $a + c$.
c. If $m + n = 45$ and $p + n = 45$, find the relationship between m and p.
d. If $a = 6$ and $b = 10$, find $a + b$.
e. If $x + y = 56$, $y = a$ and $a = c$, find the relationship between x and c.

THINK	WRITE
a. As x is equal to both y and 20, the transitive property of equals can be used.	**a.** $x = 20$ $x = y$ $\therefore y = 20$
b. As $b = c$, substitution can be used to replace b with c.	**b.** $a + b = 16$ $b = c$ $\therefore a + c = 16$
c. As both statements equal 45, matching can be used to show that m must equal p.	**c.** $m + n = 45$ $p + n = 45$ $\therefore m = p$
d. $a + b$ can be found by adding equals.	**d.** $a = 6$ $b = 10$ $\therefore a + b = 6 + 10$ $= 16$
e. 1. Using substitution, y can be replaced by a.	**e.** $x + y = 56$ $y = a$ $\therefore x + a = 56$
2. Using substitution again, a can be replaced by c.	$a = c$ $\therefore x + c = 56$

4.3.3 Two additional postulates

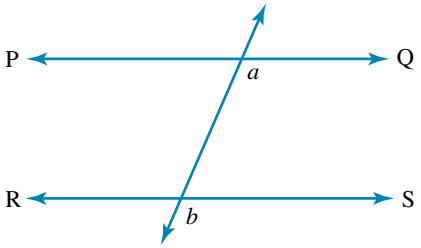
- Adjacent angles on a straight line are supplementary; conversely, supplementary adjacent angles are on a straight line.

$\overleftrightarrow{PQR} \Leftrightarrow a + b = 180°$



- If two lines are parallel, the corresponding angles are congruent; conversely, if corresponding angles are congruent, they are on parallel lines.

PQ ∥ RS $\iff a = b$



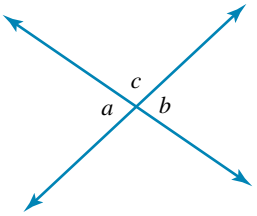
4.3.4 Theorems of Euclidean geometry

The direct proof is the most common form of proof in geometry. In a direct proof, the statements are made in a logical order, beginning with a statement that we know is true and ending with the statement that we wish to prove. The reason that each statement can be made is given beside the statement.

The postulates can be used to prove new theorems. Consider the following proof for vertically opposite angles:

$a + c = 180°$
$c + b = 180°$

$\therefore a = b$ (matching)

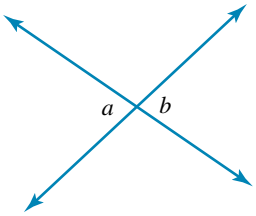


This proves that vertically opposite angles are congruent.

In a similar fashion, the following theorems can be proved in the order given using the given postulates and the previously proven theorems.

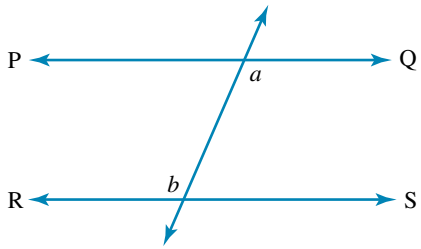
- Vertically opposite angles are congruent.

a, b are vertically opposite $\Rightarrow a = b$



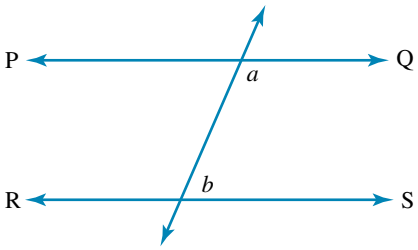
- If two lines are parallel, the alternate angles are congruent; conversely, if alternate angles are congruent, the two lines are parallel.

PQ ∥ RS $\iff a = b$



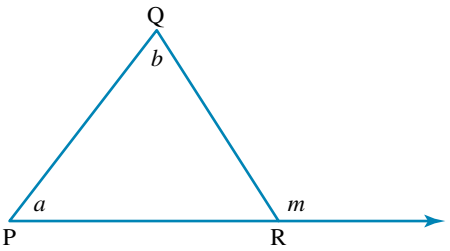
- If two lines are parallel, the co-interior angles are supplementary; conversely, if co-interior angles are supplementary, the two lines are parallel.

$PQ \parallel RS \iff a = b$



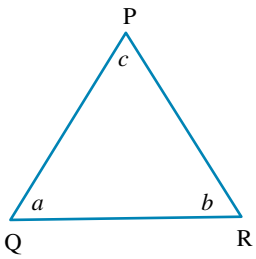
- The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

$m = a + b$



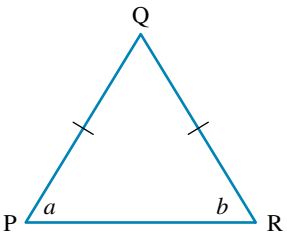
- The sum of the angles of a triangle is 180°.

$a + b + c = 180°$



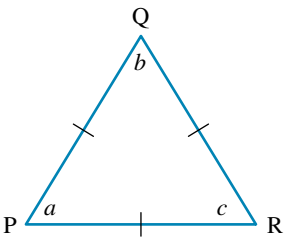
- If two sides of a triangle are congruent (i.e. the triangle is an isosceles triangle), the angles opposite those sides are also congruent. Conversely, if two angles of a triangle are congruent, the sides opposite those angles are congruent.

$\overline{PQ} \cong \overline{PR} \Leftrightarrow a = b$



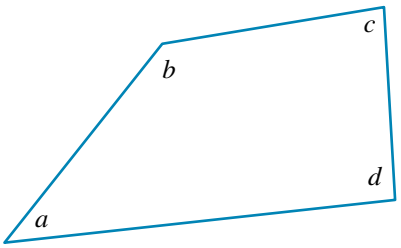
- If three sides of a triangle are congruent (i.e. the triangle is equilateral), the angles of the triangle are also congruent and equal to 60°. Conversely, if three angles of a triangle are congruent, the sides are congruent.

$\overline{PQ} \cong \overline{PR} \cong \overline{QR} \Leftrightarrow a = b = c = 60°$



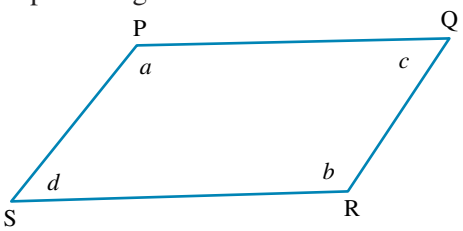
- The sum of the angles of a quadrilateral is 360°.

$a + b + c + d = 360°$



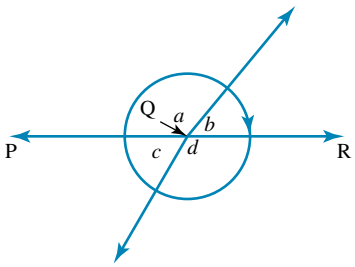
- The opposite angles of a parallelogram are congruent; conversely, if the opposite angles of a quadrilateral are congruent, it is a parallelogram.

PQRS is a parallelogram
$\iff a = b$ and $c = d$



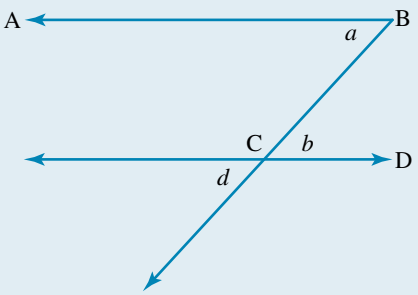
- Angles at a point add to 360°.

$a + b = 180°$
$b + c = 180°$
$a + b + c + d = 360°$



WORKED EXAMPLE 15

Use the diagram shown, the postulates given above and the vertically opposite angles theorem to prove that:
a. if two lines are parallel, the alternate angles are congruent
b. if alternate angles are congruent, the lines are parallel.



THINK	WRITE
a. 1. Assume that $\overline{AB}$ is parallel to $\overline{CD}$ to prove the first half of the proof.	**a.** Assume $\overline{AB} \parallel \overline{CD}$.
2. Angles a and d are corresponding and therefore congruent.	$a = d$
3. Angles b and d are vertically opposite and therefore congruent.	$b = d$
4. Using the transitive property of equals, $a = b$.	$\therefore a = b$

5. Write the concluding statement.

If two lines are parallel, the alternate angles are congruent.

b. 1. Assume that a and b are congruent.

b. Assume that $a = b$.

2. Angles b and d are vertically opposite and therefore congruent.

$b = d$

3. Using the transitive property of equals, $a = d$

$\therefore a = d$

4. Angles a and d are corresponding and congruent. Therefore, $\overline{AB}$ is parallel to $\overline{CD}$.

$\therefore \overline{AB} \parallel \overline{CD}$

5. Write the concluding statement.

If alternate angles are congruent, the lines are parallel.

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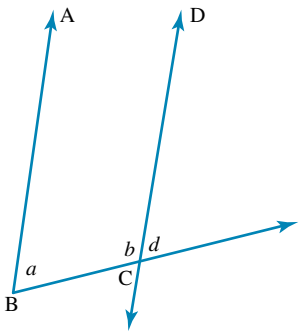
Units 1 & 2 > Area 3 > Sequence 1 > Concept 3

Direct proofs using Euclidean geometry Summary screen and practice questions

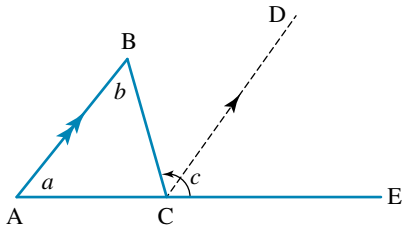
Exercise 4.3 Direct proofs using Euclidean geometry

Technology free

1. **WE14** Use the transitive property of equals to determine the value of a in each of the following.
   a. $a = x$; $x = 17$ **b.** $a = y$; $y = 15$ **c.** $z = -15$; $a = z$ **d.** $p = \frac{2}{3}$; $a = p$
2. Use substitution to determine a relationship between a and b in each of the following.
   a. $a + d = 7$; $d = b$ **b.** $3ax = 17$; $x = b$
   c. $2ac = 5$; $c = b + 1$ **d.** $b^2 - 2bc = 17$; $a = \frac{c}{b}$
3. Use matching to determine a third relationship in each of the following.
   a. $x + y = 15$; $x + z = 15$ **b.** $a + b = 25$; $a + c = 25$
   c. $a - b = 32$; $x - b = 32$ **d.** $a + b = 7$; $b - c = 7$
4. Add equals to determine a value of $x + y$ in each of the following.
   a. $x = 5$; $y = -10$ **b.** $x = 27$; $y = 15$
   c. $x = 2$; $y = 0$ **d.** $x = -7$; $y = 7$
5. Use techniques of deduction to determine the following.
   a. If $x = 4$ and $x = y$, determine y.
   b. If $a + b = 21$ and $b = c$, determine $a + c$.
   c. If $m + n = 19$ and $p + n = 19$, determine the relationship between m and p.
   d. If $a = 8$ and $b = 9$, determine $a + b$.
6. Use techniques of deduction to determine the following.
   a. If $2ab = 50$ and $b = c$, determine the relationship between a and c.
   b. If $x + y = 34$, $x = a$ and $y = b$, determine the relationship between a and b.
7. **WE15** Use the diagram shown and the given postulates to prove the following.
   a. If two lines are parallel, the co-interior angles are supplementary (that is, $a + b = 180$).
   b. If co-interior angles are supplementary, the lines are parallel.



8. Use the diagram, the alternate angle theorem and the given postulates to prove that $c = a + b$.



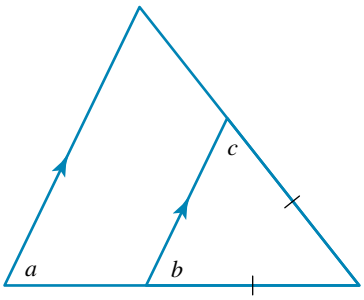
9. Use the external angle of a triangle theorem and the given postulates to prove that the sum of angles in a triangle is 180°.

10. Prove that the sum of the angles in a quadrilateral is 360°. *Hint*: Divide the quadrilateral into two triangles.

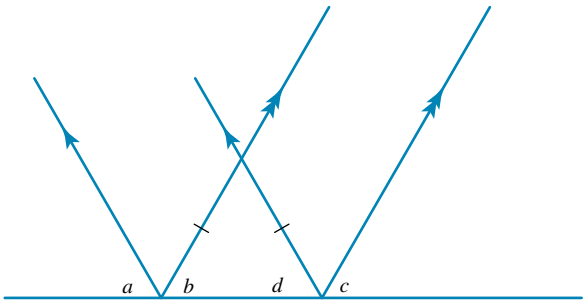
11. Prove that the opposite angles of a parallelogram are congruent, and also prove the converse.

12. Use any of the postulates and theorems given above to prove that $a = c$.

a.

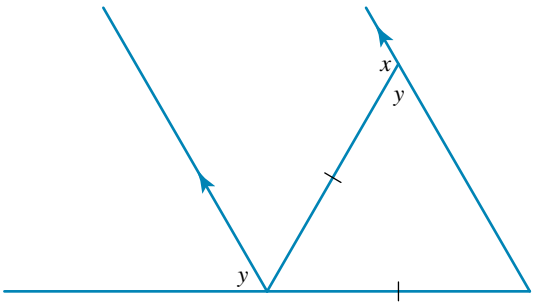


b.

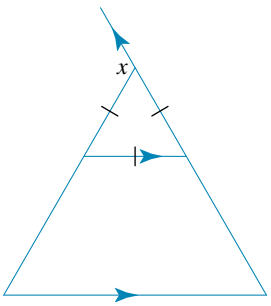


13. Use any of the postulates and theorems given above to prove that $x + y = 180°$.

a.

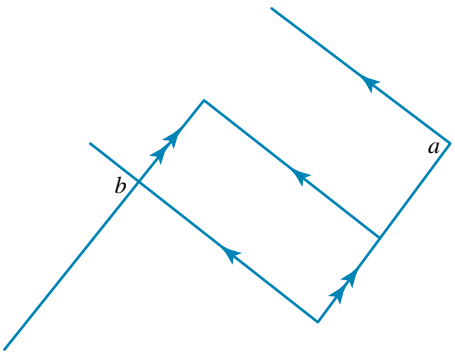


b.

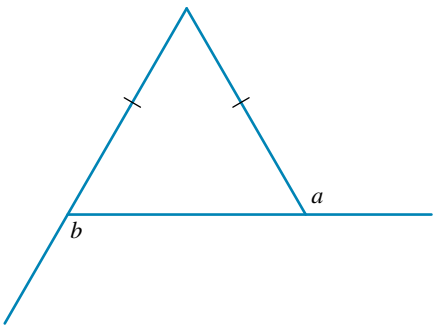


14. Prove that $a = b$ in each of the following.

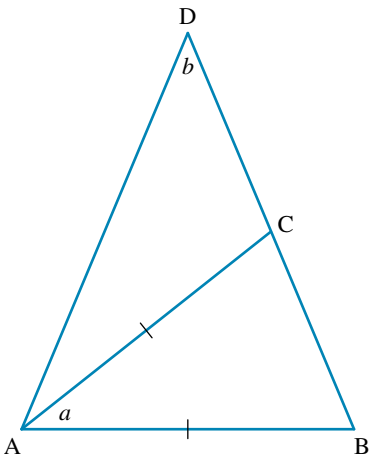
a.



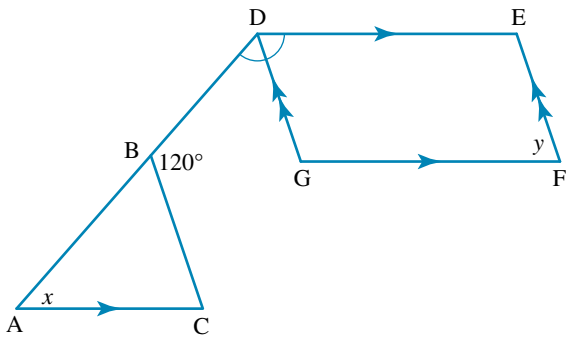
b.



15. If $\overline{AD} \cong \overline{BD}$, prove that $a = b$.



16. If $\angle BDG \cong \angle EDG$, identify a relationship between x and y. You must justify this relationship.



4.4 Indirect methods of proof

When we attempt to construct a proof, sometimes the direct path is not the most efficient.

4.4.1 Disproof by example/proof by counter example

Proofs in mathematics require more than just lots of examples to demonstrate that a conjecture is correct. Examples can help to convince us that a claim is likely to be true, but they cannot prove that a statement is true.

A single example is all that is needed to show that a conjecture is false. This is called a **counter example**.

To prove that $P \Rightarrow Q$ is false, it is necessary to find an example where P is true and Q is not true.

WORKED EXAMPLE 16

Identify a counter example to disprove the conjecture 'All primes are odd numbers.'

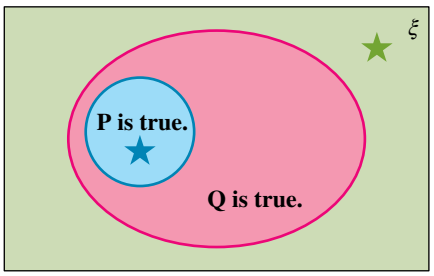
THINK	WRITE
1. We need to find an even number that is also prime.	The number 2 is both prime and even.
2. Write a concluding a statement.	Not all primes are odd numbers.

4.4.2 Contrapositive

The **contrapositive** of the conjecture 'If P, then Q' is 'If not Q, then not P.' Notice that both are P and Q are negated and the order is reversed. The contrapositive of a true statement is also true.

Consider the true proposition 'If x is even, then $x + 1$ is odd.'

The contrapositive would be 'If $x + 1$ is not odd, then x is not even' or 'If $x + 1$ is even, then x is odd'. Notice that these propositions are also true.



If P ⇒ Q is true, then ¬Q ⇒ ¬P is also true.
★ P ⇒ Q is true.
★ Q is not true ⇒ P is not true.

Previously, we used a Venn diagram to demonstrate what P ⇒ Q looks like when it is true. You can see on the diagram that if not Q is true, then not P will also be true. Using symbols, if P ⇒ Q is true, then ¬Q ⇒ ¬P is also true.

This can also be seen by comparing truth tables. P ⇒ Q has the same truth table as ¬Q ⇒ ¬P.

We know that if a proposition is true, the contrapositive is also true. Therefore, if we prove that the contrapositive is true, the proposition is also true.

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	¬Q	¬P	¬Q ⇒ ¬P
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

WORKED EXAMPLE 17

Consider the statement 'If $n + m$ is odd, then $n \neq m$.'
a. Identify the propositions and write the contrapositive statement.
b. Prove the statement by proving the contrapositive.

THINK	WRITE
a. Identify the propositions and their negatives. Write the contrapositive by negating both statements and reversing the order.	a. P: $n + m$ is odd ¬P: $n + m$ is even Q: $n \neq m$ ¬Q: $n = m$ Contrapositive: If $n = m$, then $n + m$ is even.

b. 1. If $n = m$, then substitution can be used.	**b.** If $n = m$, then $n + m = n + n$ $= 2n$
2. Write concluding statements.	If $n = m$, then $n + m$ is even. Therefore, if $n + m$ is odd, then $n \neq m$.

4.4.3 Proof by contradiction

To prove something by **contradiction**, we assume that what we want to prove is not true. The proof continues until the initial assumption is contradicted. As the initial assumption must be false, the alternative is true.

If you were trying to prove that P $\Rightarrow$ Q is true by contradiction, you would try to show the following:

1. Assume the proposition to be proved is false, that is, P $\Rightarrow$ Q is not true (if P is true and Q is not true).
2. Show that the proposition contradicts the initial assumptions, that is, show that if P is true, Q is true.
3. Conclude that as the solution does not meet the initial assumptions, the original assumption must be false, and hence the proposition is true: P $\Rightarrow$ Q.

WORKED EXAMPLE 18

Use proof by contradiction to prove the proposition 'There are no positive integer solutions to the Diophantine equation $x^2 - y^2 = 1$.'
Note: **This could also have been written as 'If x and y are positive integers, then $x^2 - y^2 \neq 1$.'**

THINK	WRITE
1. Assume that the proposition is not true. This means that it is possible a solution exists where x and y are both positive integers.	Assume that $x^2 - y^2 = 1$, $x, y \in Z^+$.
2. Factorise the equation.	$(x + y)(x - y) = 1$
3. As $x, y \in Z^+$, $x + y$ is a positive integer and $x - y$ is an integer.	Two integers, $x + y$ and $x - y$, multiply to give 1. Both integers must equal 1 or -1. But, if $x, y \in Z^+$, then $x + y \in Z^+$. Therefore, $x + y \neq -1$. Thus, $x + y = 1$ and $x - y = 1$.
4. Use simultaneous equations solve the equations.	$x + y = 1$ [1] $x - y = 1$ [2] [1] + [2]: $2x = 2$ $x = 1$ Substituting $x = 1$ into [1] results in $y = 0$.

5. The initial assumption was that $x, y \in Z^+$; $y = 0$ does not fit with this assumption.
6. As the only solution does not meet the original assumption, the original assumption must be false.

$y = 0$ contradicts the initial assumption that $y \in Z^+$.
There are no positive integer solutions to the equation $x^2 - y^2 = 1$.

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Units 1 & 2 > Area 3 > Sequence 1 > Concept 4

Indirect methods of proof Summary screen and practice questions

Exercise 4.4 Indirect methods of proof

Technology free

1. In the following list, prime numbers are marked in red.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

From the list, select counter examples to disprove each of the following statements.
   a. All numbers of the form $6n + 1$ are prime, $n \in N$.
   b. All numbers of the form $2^n + 1$ are prime, $n \in N$.
2. **WE16** Identify counter examples to disprove each of the following conjectures.
   a. If $x^2 = 100$, then $x = 10$.
   b. If a quadrilateral has four congruent sides, then it is a square.
   c. If n is a multiple of 3, then it is odd.
   d. If n is a multiple of 5, then the last digit is 5.
3. Write the contrapositive statement to each of the following propositions:
   a. If today is Monday, then tomorrow is Tuesday.
   b. If A bisects a line segment, then A is the midpoint of the line segment.
4. If proof by contradiction was being used to justify that $\sqrt{ab} \leq \frac{a+b}{2}$, what would the initial assumption be?
5. **WE17** Consider the statement 'If $n > m$, then $n - m > 0$.'
   a. Write the contrapositive statement.
   b. Prove the statement by proving the contrapositive.
6. At the beginning of a proof by contradiction 'that x is a positive number', Ailsa wrote 'Assume x is a negative number.' What has Ailsa overlooked?
7. **WE18** Use proof by contradiction to prove the proposition 'There are no integer solutions to the Diophantine equation $4x^2 - y^2 = 1$.'
8. Use proof by contradiction to prove the proposition 'If $x, y \in Z$, then $2x + 4y \neq 7$.'

9. Analyse the following proof and identify the flaw in the logic.
   Proposition: All integers are the same.
   Assume that there are two integers a and b that are not the same.
   Therefore, $\exists c, a = b + c$.
   Multiplying both sides by $a - b$: $a(a - b) = (b + c)(a - b)$
   $$a^2 - ab = ab - b^2 + ac - bc$$
   Expanding the brackets: $a^2 - ab - ac = ab - b^2 - bc$
   $$a(a - b - c) = b(a - b - c)$$
   Therefore, $a = b$.
   This contradicts the initial assumption.
   Therefore, all integers are the same.
10. Is the proposition 'There are no integer solutions to the equation $25 - y^2 = x^2$' true or false? Provide appropriate evidence to justify your claim.
11. Consider the statement 'If a and b are integers, there is no solution to the equation $a + b = \frac{1}{2}$.'
    a. Write the contrapositive statement.
    b. Prove the statement by proving the contrapositive.
12. Consider the statement 'If $x + y > 5$, then either $x > 2$ or $y > 3$, $x, y \in R$.' This statement can be justified by proving the contrapositive.
    a. Write the contrapositive statement.
    b. Justify the statement by proving the contrapositive.
13. Prove the statement 'There are no integer solutions to $x^2 - y^2 = 2$.'
14. Prove that statement 'For every $x \in Q^+$, there exists $y \in Q^+$ for which $y < x$.'

4.5 Proofs with rational and irrational numbers

Earlier in this chapter, you explored ways to construct proofs. In this section, you will use the skills that you developed to write proofs with rational and irrational numbers.

4.5.1 Proofs with consecutive numbers

If a series of numbers are consecutive, they can be written as n, $n + 1$, $n + 2$, …
Sometimes, it is easier to call the middle number n. so the series would be … $n - 1$, n, $n + 1$.

WORKED EXAMPLE 19

Demonstrate that the sum of three consecutive natural numbers is equal to 3 times the middle number.

THINK	WRITE
1. As the example talks about the middle number, it might be easier to let the middle number be n, where n is a natural number.	Let the middle number be n, $n \in N$. The series is $n - 1$, n, $n + 1$.
2. Find the sum of the three numbers.	Sum $= (n - 1) + n + (n + 1)$ $= 3n$
3. Write the concluding statement.	The sum of three consecutive natural numbers is equal to 3 times the middle number.

4.5.2 Prove that a number is irrational by contradiction

A proof by contradiction can also be used to prove that a number is irrational. Begin by assuming that the number is rational and work to prove that the assumption is false.

WORKED EXAMPLE 20

Prove that the following numbers are irrational.

a. $\sqrt{2}$ **b.** $\log_2 5$

THINK	WRITE
a. 1. Use a proof by contradiction and assume that $\sqrt{2}$ is rational. This means that $\sqrt{2}$ can be written as the ratio of two integers a and b, where $b \neq 0$ and a and b have no common factors.	**a.** Assume $\sqrt{2}$ is rational. $\therefore \sqrt{2} = \frac{a}{b}, a, b \in Z, (a, b) = 1, b \neq 0$
2. Multiply by b and square both sides.	$\sqrt{2}b = a$ [1] $2b^2 = a^2$
3. This means that 2 divides a^2 evenly and therefore 2 divides a evenly. *Note*: $a \mid b$ means that a divides into b evenly, or a is a factor of b.	$\therefore 2 \mid a^2$ $\therefore 2 \mid a$
4. If 2 divides a evenly, then a is a multiple of 2.	Let $a = 2m, m \in Z$. [2]
5. Substitute [2] in [1] and repeat the process to show that 2 will divide b evenly.	Substitute [2] in [1]: $2b^2 = (2m)^2$ $2b^2 = 4m^2$ $b^2 = 2m^2$ $\therefore 2 \mid b^2$ $\therefore 2 \mid b$
6. 2 is a divisor of both a and b.	2 divides both a and b. But $(a, b) = 1$. The assumption is false.
7. Write your concluding statement.	$\sqrt{2}$ is irrational.
b. 1. Use proof by contradiction and assume that $\log_2 5$ is rational.	**b.** Assume $\log_2 5 = \frac{a}{b}$, $a, b \in Z, (a, b) = 1, b \neq 0$.
2. Rearrange the equation.	$2^{\frac{a}{b}} = 5$ $(2^a)^{\frac{1}{b}} = 5$ $2^a = 5^b$
3. The only case in which powers of 2 and 5 are equal is $2^0 = 5^0$.	As $(2, 5) = 1$, the only solution is $a = b = 0$. But $b \neq 0$. The assumption is false.
4. Write your concluding statement.	$\log_2 5$ is irrational.

4.5.3 Proofs with odd and even numbers

If a proof involves odd or even numbers, you can use the fact that for any integer n, $2n$ will be even and $2n + 1$ will be odd.

WORKED EXAMPLE 21

Prove that n^2 is odd if and only if n is odd.

THINK	WRITE
1. 'If and only if' means you need to prove that n^2 is odd if n is odd and also that n is odd if n^2 is odd. Begin by assuming that n is odd. This means that it is 1 more than an even number.	If n is odd, $n = 2a + 1$, $a \in Z$.
2. Find an expression for n^2.	$n^2 = (2a + 1)^2$ $= 4a^2 + 4a + 1$ $= 2\left(2a^2 + 2a\right) + 1$
3. Odd numbers can be expressed as 1 more than an even number.	This is an odd number.
4. Write your concluding statement.	Therefore, if n is odd, then n^2 is odd.
5. Now assume that n^2 is odd, meaning that it can be written as 1 more than an even number.	If n^2 is odd, then $n^2 = 2b + 1$, $b \in Z$.
6. Consider $n^2 - 1$. This expression can be factorised as the difference of two squares.	$n^2 - 1 = (n + 1)(n - 1)$
7. Substitute $n^2 = 2b + 1$.	$(2b + 1) - 1 = (n + 1)(n - 1)$ $2b = (n + 1)(n - 1)$
8. If the product of two numbers is even, then one of the numbers must be even.	Either $n + 1$ is even, which means that n is odd, or $n - 1$ is even, which also means that n is odd.
9. Write your concluding statement.	Therefore, if n^2 is odd, then n is odd. Therefore, n^2 is odd if and only if n is odd.

4.5.4 Prove that a set of numbers is infinite

A proof by contradiction can be used to prove that a set of numbers is infinite (that is, it has an infinite number of members). Begin by assuming that the set is finite and work to prove that this assumption is false.

WORKED EXAMPLE 22

Prove that there are infinitely many prime numbers.

THINK	WRITE
1. Use a proof by contradiction. Begin by assuming that there are a finite number of primes.	Assume that $p_1 \ldots p_n$ are the only prime numbers.

2. There must be a number that is 1 more than the product of all of the primes. Call this number q. *Note:* The notation $\prod_{i=1}^{n} p_i$ indicates the product of all prime numbers between 1 and n.	Let $q = \prod_{i=1}^{n} p_i + 1$.
3. If we divide q by any of the primes, the remainder is 1. This means that q is not a product of any of the primes p_i.	If we divide q by any of the primes, the remainder is 1. Therefore, q is prime, or there is a prime number greater than p_n that is a factor of q.
4. There are more prime numbers than initially assumed.	This means that the assumption is false, as there must be a prime number greater than p_n.
5. Write your concluding statement.	There are infinitely many prime numbers.

4.5.5 Other proofs with real numbers

It can be possible to prove that an expression is larger or smaller than another expression. It can also be possible to prove that an expression is not equal to another expression.

WORKED EXAMPLE 23

a. Prove that for $a, b \in R$, $a^2 + b^2 \geq 2ab$.

b. Prove that $\frac{1}{x} + \frac{1}{y} \neq \frac{1}{x+y}$ for $x, y \neq 0$.

THINK	WRITE
a. 1. The elements a^2, b^2 and $2ab$ are part of a perfect square expansion. A perfect square is always greater than or equal to 0.	**a.** $(a-b)^2 \geq 0$
2. Expand the brackets.	$a^2 - 2ab + b^2 \geq 0$
3. Rearrange the terms to form the proof.	$\therefore a^2 + b^2 \geq 2ab$
b. 1. Use a proof by contradiction and begin by assuming that the equation is true. Rewrite $\frac{1}{x} + \frac{1}{y}$ as a single fraction.	**b.** Assume that $\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y}$ $\frac{y}{xy} + \frac{x}{xy} = \frac{1}{x+y}$ $\frac{y+x}{xy} = \frac{1}{x+y}$
2. Rearrange the equation.	$(x+y)^2 = xy$
3. $(x+y)^2$ will be positive. (It cannot equal zero as $x, y \neq 0$.)	$(x+y)^2 > 0$ $\therefore xy > 0$

4. Expanding $(x + y)^2$ to show an xy term.

$$(x + y)^2 = xy$$
$$x^2 + 2xy + y^2 = xy$$
$$x^2 + y^2 = -xy$$

5. $x^2 + y^2$ will be positive.

$x^2 + y^2 > 0$; therefore, $-xy > 0$
$xy < 0$

6. Identify the contradiction.

But earlier, we found that $xy > 0$.

7. State your conclusions.

Therefore, the assumption is false.

$$\frac{1}{x} + \frac{1}{y} \neq \frac{1}{x + y}$$

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Units 1 & 2 > Area 3 > Sequence 1 > Concept 5

Proofs with rational and irrational numbers Summary screen and practice questions

Exercise 4.5 Proofs with rational and irrational numbers

Technology free

1. **WE19** Demonstrate that the sum of five consecutive natural numbers is equal to 5 times the middle number.
2. Demonstrate that if n and m are multiples of 3, then $n + m$ is a multiple of 3.
3. Demonstrate that if m, n and p are consecutive natural numbers, then $n^2 - mp = 1$.
4. An even number, n, can be written in the form $n = 2a$, $a \in Z$. Demonstrate that if m and n are even numbers, then $m^2 + n^2$ and $m^2 - n^2$ are both divisible by 4.
5. An odd number, n, can be written in the form $n = 2a + 1$, $a \in Z$. Demonstrate that the sum of two consecutive odd numbers is divisible by 4.
6. Consider four consecutive numbers, n_1, n_2, n_3, n_4. Demonstrate that $n_1 + n_2 + n_3 + n_4 = n_3 n_4 - n_1 n_2$.

7. **WE20** **a.** Prove that $\sqrt{5}$ is irrational.
   b. Prove that $\sqrt{3}$ is irrational.
8. **a.** Prove that $\log_3 7$ is irrational.
   b. Prove that $\log_5 11$ is irrational.
9. Prove that for any natural number n, $\frac{n(n+1)}{2}$ is a natural number.
10. Use your understanding of divisibility to show the following. *Note*: $a \mid b$ means that a divides into b evenly, or a is a factor of b.
    a. If $a \mid b$ and $b \mid c$, then $a \mid c$.
    b. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

11. **WE21** Prove that n^2 is even if and only if n is even. *Hint:* When assuming that n^2 is even, consider factorising $n^2 - 1$.

12. Prove that all numbers of the form $n^3 - n$, $n \in Z$ are multiples of 6.

13. Prove that if x is rational and y is irrational, then $x + y$ is irrational.

14. **WE22** **a.** Prove that there are infinitely many integers. (*Hint:* Assume that n is the largest integer.)

b. Prove that there are infinitely many even numbers.

15. **WE23** Prove that for $a > 0$, $a + \frac{1}{a} \geq 2$. *Hint:* Begin by noting that $(a - 1)^2 \geq 0$.

16. Prove that for $a, b \in R^+ \cup \{0\}$, $\frac{1}{2}(a + b) \geq \sqrt{ab}$.

17. Prove that for $a, b \in Z$, if $4 \mid (a^2 + b^2)$, then a and b are not both odd.

18. Two resistors have resistances of a ohms and b ohms. If the resistors are placed in series, the combined resistance is $R_S = a + b$. If the resistors are placed in parallel, the combined resistance is found using $\frac{1}{R_P} = \frac{1}{a} + \frac{1}{b}$. Justify that $R_S \geq R_P$. *Hint:* Use a proof by contradiction.

Technology active

19. Consider $\sqrt{n}$ where n is an integer. There are two irrational numbers, $\sqrt{2}$ and $\sqrt{3}$, between the rational numbers $\sqrt{1} = 1$ and $\sqrt{4} = 2$. There are four irrational numbers between the next rational pair, $\sqrt{4} = 2$ and $\sqrt{9} = 3$. Find a rule for the number of irrational numbers of the form $\sqrt{n}$ between consecutive integers m and $m + 1$. Prove that your rule works.

20. **a.** Arrange the integers 1 to 200 in a table similar to the one shown, marking all the primes and multiples of 6. You may like to use a spreadsheet or similar to assist with this.

1	2	3	4	5	6	7
	8	9	10	11	12	13
	14	15	16	17	18	19
	20	21	22	23	24	25
	26	27	28	29	30	31
	32	33	34	35	36	37
				Primes are in red.	Multiples of 6	Primes are in red.

Note that with the exception of the primes 2 and 3, the other primes are all 1 less than or 1 more than a multiple of 6. This means that primes can be expressed in the form $6n \pm 1$.

b. Sophie Germain primes, p, are prime numbers where $2p + 1$ is also prime. This means that 2 is a Sophie Germain prime, because 5 is also prime. Find the other Sophie Germain primes between 1 and 200. Note that, with the exception of 2 and 3, they are all 1 less than a multiple of 6. Justify why this is so.

4.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Identify the following numbers as either rational or irrational.
   - **a.** $\frac{23}{7}$
   - **b.** 4.32
   - **c.** 3.454 545 454 5 …
   - **d.** 1.010 110 111 011 11 …
   - **e.** $\sqrt{7}$
   - **f.** $\sqrt{\frac{25}{9}}$
2. Determine if the following are true statements.
   - **a.** $\exists x \in N, x^2 < 1$
   - **b.** $\forall x \in R, x > 3$
   - **c.** $x > 4 \Rightarrow x > 3$
   - **d.** $|x| > 5 \Rightarrow x > 5$
   - **e.** $x = 7 \Leftrightarrow x^2 = 49$
   - **f.** $x^2 < 16 \Rightarrow x < 4$
3. For each of the following, write the converse and determine if implication ($\Rightarrow$) or equivalence ($\Leftrightarrow$) is the more appropriate symbol.
   - **a.** If an animal is a monotreme, then it is a mammal.
   - **b.** If a number is prime, then it has exactly two factors.
   - **c.** If $x > 2$, then $x^3 > 8$.
   - **d.** If $x = 9$, then $x^2 = 81$.
4. Write the negation of each of the following statements.
   - **a.** $x < 4, x \in R$
   - **b.** The number is odd or a multiple of 5.
   - **c.** The number is an even perfect square.
   - **d.** If x is an interesting number, then x has exactly 3 factors.
5. Express the following common fractions as decimal fractions.
   - **a.** $\frac{3}{50}$
   - **b.** $\frac{7}{25}$
   - **c.** $\frac{13}{11}$
   - **d.** $\frac{13}{6}$
   - **e.** $\frac{1}{24}$
   - **f.** $\frac{7}{15}$
6. Express the following decimals as simplest common fractions.
   - **a.** 1.252
   - **b.** $0.\dot{6}$
   - **c.** $0.\dot{1}\dot{7}$
   - **d.** $0.0\overline{87}$
   - **e.** $0.7\dot{9}\dot{0}$
   - **f.** $3.\overline{296}$
7. Using a technology of your choice, identify a counter example to disprove the conjecture 'All numbers of the form $2^{2^n} + 1$ are prime, $n \in Z^+ \cup \{0\}$.'
8. If Pravdeep was using proof by contradiction to prove that the diagonals of a trapezium do not bisect each other, what would her initial assumption be?
9. Prove that the following numbers are irrational.
   - **a.** $\sqrt{7}$
   - **b.** $\log_5 7$
10. Prove each of the following propositions by proving the contrapositive.
    - **a.** If n^2 is even, then n is even, $n \in N$.
    - **b.** If n^3 is odd, then n is odd, $n \in N$.
11. Prove that the sum of four consecutive whole numbers is 2 less than a multiple of 4.
12. Prove that when the product of three consecutive numbers is added to the middle number, the result is the cube of the middle number.

Complex familiar

13. Identify what is wrong with each of the following proofs.

a. Proposition: $2 = 1$

Let x and y be two non-zero integers where $x = y$.

$$x = y \qquad [1]$$
$$x^2 = xy$$
$$x^2 - y^2 = xy - y^2$$
$$(x + y)(x - y) = y(x - y)$$
$$x + y = y \qquad [2]$$

Substitute [1] into [2] (for x):

$$y + y = y$$
$$2y = y$$
$$2 = 1$$

b. Proposition: $-6 = -6$

$$9 - 15 = 4 - 10$$
$$9 - 15 + \frac{25}{4} = 4 - 10 + \frac{25}{4}$$
$$\left(3 - \frac{5}{2}\right)^2 = \left(2 - \frac{5}{2}\right)^2$$
$$3 - \frac{5}{2} = 2 - \frac{5}{2}$$
$$3 = 2$$

14. Determine if the following propositions are true.

a. $\forall x, y \in R, x - y \in Z$

b. $\exists y \in Q, \forall x \in Q, \frac{x}{y} \in Z$

c. $\forall y \in Q, \exists x \in Q, \frac{x}{y} \in Z$

d. $\exists x, y \in Q, \frac{x}{y} \in Z$

15. Prove that in a set of any three different natural numbers, there are always two whose sum is divisible by 2.

16. The numbers 1 to 100 are arranged in a 10 by 10 grid. Part of the grid is shown below, with a 2 by 2 square and its 4 corner elements highlighted.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

Show that for any square in the grid:

a. the sum of the numbers inside the square is equal to the sum of the 4 corner numbers.
(In the example, this means that $25 + 26 + 35 + 36 = 14 + 17 + 44 + 47$.)

b. the sum of each diagonal is equal to the sum of the numbers inside the square.
(In the example, this means that $14 + 25 + 36 + 47 = 17 + 26 + 35 + 44 = 25 + 26 + 35 + 36$.)

Complex unfamiliar

17. If n_1 is a 2-digit number where the digit in the tens place is greater than the digit in the units place, the number n_2 is found by interchanging the two digits. Prove that $n_1 - n_2$ is a multiple of 9.

18. Consider the following conjecture: 'The sum of the squares of any pair of consecutive natural numbers is always 1 more than a multiple of 4.' Use your calculator to explore examples and demonstrate that the conjecture is likely to be true. Use algebra to prove the conjecture.

19. If $a = b$ and $c = d$, then $ac = bd$. If it is true that $a > b$ and $c > d$, then show that $ac > bd$ is false. Justify your conclusion.

20. Prove that for $a,\ b \in Z$, $a^2 - 4b \neq 2$.

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Answers

Chapter 4 Introduction to proof

Exercise 4.2 Number systems and writing propositions

1. **a.** Rational **b.** Rational
   c. Rational **d.** Irrational
   e. Rational **f.** Irrational
2. **a.** Irrational **b.** Irrational
   c. Rational **d.** Irrational
3. **a.** 0.85 **b.** 0.856
   c. $0.\dot{6}$ **d.** $0.5\dot{3}$
4. **a.** 1.6 **b.** 0.5625
   c. $3.\dot{1}42\,857\dot{1}$ or $3.\overline{142\,857}$ **d.** $1.08\dot{3}$
5. **a.** $\frac{5}{9}$ **b.** $\frac{23}{99}$ **c.** $\frac{271}{90}$ or $3\frac{1}{90}$
   d. $\frac{610}{99}$ or $6\frac{16}{99}$ **e.** $\frac{2347}{330}$ or $7\frac{37}{330}$ **f.** $\frac{26}{37}$
6. **a.** {0, 3, 6, 9, 15, 21}
   b. {3, 9}
   c. {0, 6, 12, 18, 24}
   d. {12, 15, 18, 21, 24}
   e. {12, 18, 24}; even and greater than or equal to 12
   f. {0, 6, 12, 15, 18, 21, 24}; even or greater than or equal to 12
7. **a.** True **b.** True **c.** False
   d. True **e.** False **f.** True
8. **a.** If it lives in water, then it is a fish. ($\Rightarrow$)
   b. If the diagonals of a quadrilateral bisect each other, it is a rhombus. ($\Rightarrow$)
   c. If $n > m$, then $n - m > 0$. ($\Leftrightarrow$)
   d. If n is even, then $n + 1$ is odd. ($\Leftrightarrow$)
9. **a.** For all natural numbers, x, $2x$ is even. True
   b. For all natural numbers, x, $2x + 1$ is a multiple of 3. False
   c. For all natural numbers, x, $x > 0$. True
   d. For all real numbers, x, either $x > 0$ or $x \leq 0$. True
   e. For all natural numbers, x, x is even. False
   f. For all real numbers, x, $x^2 + 1 \geq 0$. True
10. **a.** There is an integer, x, so that $x + 5 = 7$. True
    b. There is a real number, x, so that $x^2 < 0$. False
    c. There is a natural number, x, that is even. True
    d. There is a real number, x, so that $x^2 + 1 = 0$. False
    e. There is a real number, x, so that $x^2 - 1 = 0$. True
    f. There is a rational number, x, so that $\sqrt{x}$ is also rational. True
11. **a.** For all real numbers, x, there exists a real number, y, so that $x = y$. True
    b. There exists a real number, x, so that for all real numbers, y, $x = y$. False
    c. For all real numbers, x, there exists a real number, y, so that $xy = 0$. True
    d. There exists a real number, x, so that for all real numbers, y, $xy = 0$. True
12. **a.** True **b.** False
    c. False **d.** True
13. True
14. (Not A or not B) and not C; with symbols, (¬A or ¬B) and ¬C
15. One of several solutions: $\left(0, \frac{\pi}{2}\right)$
16. Answers may vary. Sample answers:
    a. $\forall x, y \in (-3, 3)$, $x^2 + y^2 < 9$
    b. $\forall x \in R, \exists y \in R$, $x^2 + y^2 = 9$

Exercise 4.3 Direct proofs using Euclidean geometry

1. **a.** $a = 17$ **b.** $a = 15$
   c. $a = -15$ **d.** $a = \frac{2}{3}$
2. **a.** $a + b = 7$ **b.** $3ab = 17$
   c. $2a(b + 1) = 5$ **d.** $b^2 - 2ab^2 = 17$
3. **a.** $y = z$ **b.** $b = c$
   c. $a = x$ **d.** $a = -c$
4. **a.** -5 **b.** 42
   c. 2 **d.** 0
5. **a.** $y = 4$ **b.** $a + c = 21$
   c. $m = p$ **d.** $a + b = 17$
6. **a.** $ac = 25$ **b.** $a + b = 34$

7–15. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources.

16. $x + 2y = 180°$

Exercise 4.4 Indirect methods of proof

1. **a.** 25 **b.** 9
2. **a.** $x = -10$
   b. Counter examples include: a rhombus
   c. Counter examples include: the numbers 6, 12, 18, …
   d. Counter examples include: the numbers 10, 20, 30, …
3. **a.** If tomorrow is not Tuesday, then today is not Monday.
   b. If A is not the midpoint of a line segment, then A does not bisect the line segment.
4. $\sqrt{ab} > \dfrac{a+b}{2}$
5. **a.** If $n - m \leq 0$, then $n \leq m$.
   b. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources.
6. Ailsa forgot that x could equal 0.

7, 8. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources

9. At the 'Expanding the brackets' step, $a(a - b - c) = b(a - b - c)$ only if $a - b - c = 0$. Therefore, the proof breaks down.
10. False. One solution is $x^2 = 9$ and $y^2 = 16$ (disproof by counter example).
11. **a.** There is a solution to $a + b = \frac{1}{2}$ if a or b is not an integer.
    b. Students will need to write their own proofs. A sample response can be found in the worked solutions in the online resources.

12. a. If $x \leq 2$ and $y \leq 3$, then $x + y \leq 5$.

b. Students will need to write their own proofs.
A sample response can be found in the worked solutions in the online resources.

13, 14. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources.

Exercise 4.5 Proofs with rational and irrational numbers

1–19. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources.

20. a.

1	**2**	**3**	4	**5**	6	**7**
	8	9	10	**11**	12	**13**
	14	15	16	**17**	18	**19**
	20	21	22	**23**	24	25
	26	27	28	**29**	30	**31**
	32	33	34	35	36	**37**
	38	39	40	**41**	42	**43**
	44	45	46	**47**	48	49
	50	51	52	**53**	54	55
	56	57	58	**59**	60	**61**
	62	63	64	65	66	**67**
	68	69	70	**71**	72	**73**
	74	75	76	77	78	**79**
	80	81	82	**83**	84	85
	86	87	88	**89**	90	91
	92	93	94	95	96	**97**
	98	99	100			
				Primes are in **bold**.	Multiples of 6	Primes are in **bold**.

b. The Sophie Germain primes less than 200 are 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179 and 191. A sample response for the proof can be found in the worked solutions in the online resources.

4.6 Review: exam practice

1. a. Rational **b.** Rational **c.** Rational
d. Irrational **e.** Irrational **f.** Rational

2. a. False **b.** False **c.** True
d. False **e.** False **f.** True

3. a. $\Rightarrow$ **b.** $\Leftrightarrow$ **c.** $\Leftrightarrow$ **d.** $\Rightarrow$

4. a. $x \geq 4, x \in R$
b. The number is even and not a multiple of 5.
c. The number is odd or not a perfect square.
d. x is an interesting number and it does not have exactly 3 factors.

5. a. 0.06 **b.** 0.28 **c.** $1.\dot{1}\dot{8}$
d. $2.16\dot{}$ **e.** $0.041\dot{6}$ **f.** $0.4\dot{6}$

6. a. $\frac{313}{250}$ or $1\frac{63}{250}$ **b.** $\frac{2}{3}$ **c.** $\frac{17}{99}$
d. $\frac{29}{330}$ **e.** $\frac{87}{110}$ **f.** $\frac{89}{27}$ or $3\frac{8}{27}$

7. Examples include 4 294 967 297.

8. The diagonals of the trapezium bisect each other.

9–12. Students will need to write their own responses. Sample responses can be found in the worked solutions in the online resources.

13. a. The line $(x + y)(x - y) = y(x - y)$ does not simplify to $x + y = y$, because dividing by $x - y$ would mean dividing by 0 as $x = y$.

b. $\left(3 - \frac{5}{2}\right)^2 = \left(2 - \frac{5}{2}\right)^2$ means that $\pm\left(3 - \frac{5}{2}\right) = \pm\left(2 - \frac{5}{2}\right)$. In this instance, $-\left(3 - \frac{5}{2}\right) = \left(2 - \frac{5}{2}\right)$. That is, the line $\left(3 - \frac{5}{2}\right) = \left(2 - \frac{5}{2}\right)$ is incorrect.

14. a. False **b.** False **c.** True **d.** True

15–18. Students will need to write their own responses. Sample responses can be found in the worked solutions in the online resources.

19. $ac > bd$ is true only if $a, b, c > 0$. The statement is therefore false.

20. Students will need to write their own proofs. Sample responses can be found in the worked solutions in the online resources.

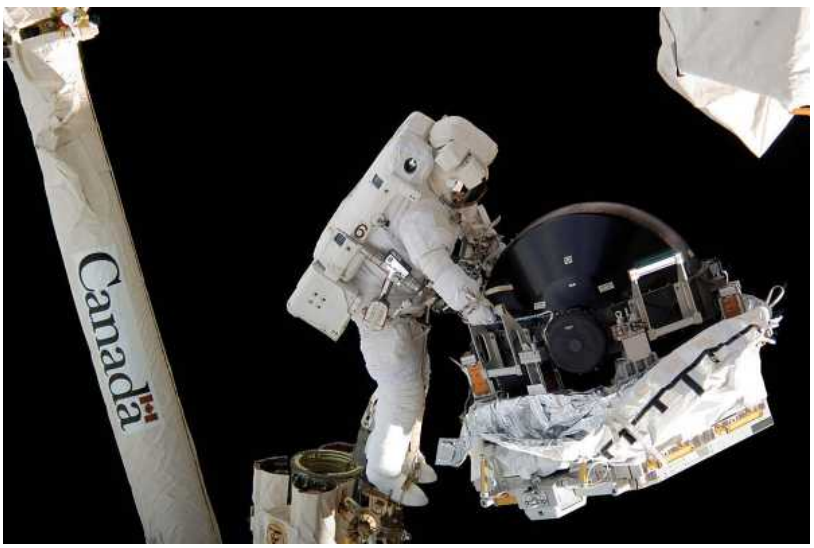
CHAPTER 5
Circle geometry

5.1 Overview

5.1.1 Introduction

Circles (from the Greek word *kirkos*, meaning 'ring') have been studied since ancient times. They are a useful shape because they enclose the largest possible area for a given perimeter (or have the smallest possible perimeter for a given area).

The study of circles has many applications, including the development and use of gyroscopes. A gyroscope is composed of a circular disc that can freely rotate to assume any orientation. This disc is mounted onto an axis in the centre of a larger and more stable wheel. As the disc spins, it wants to stay in the same orientation; any external force applied to change the orientation of the spin axis will be resisted, due to the conservation of angular momentum. Gyroscopes are used to maintain the stability of many machines, from bicycles, motorcycles and ships through to smart phones, virtual reality headsets, robots, aeroplanes and space telescopes.



LEARNING SEQUENCE

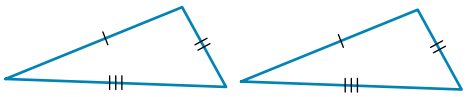
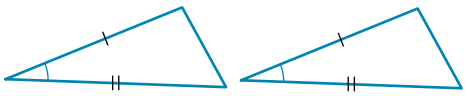
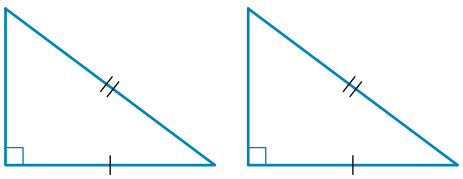
5.1 Overview
5.2 Review of congruent triangle tests
5.3 Circle properties 1 — angles in a circle and chords
5.4 Circle properties 2 — tangents, secants and segments
5.5 Circle properties 3 — cyclic quadrilaterals
5.6 Geometric proofs using vectors
5.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 Review of congruent triangle tests

5.2.1 Congruent triangle tests

You have already learned about tests that can be used to prove that triangles are congruent. These congruency tests are shown in the table below.

Description	Test	Abbreviation
Three sides of one triangle are congruent to the three sides of the other triangle.		SSS (side–side–side)
Two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.		SAS (side–angle–side)
Two angles and a side of one triangle are congruent to two angles and the corresponding side of the other triangle.		AAS (angle–angle–side)
The triangles are right-angled, and the hypotenuse and a side of one triangle are congruent to the hypotenuse and side of the other triangle.		RHS (right angle–hypotenuse–side)

The tests SSS, SAS and AAS are postulates, meaning that they are assumed to be true. The test RHS can be proven by using SSS and Pythagoras' theorem.

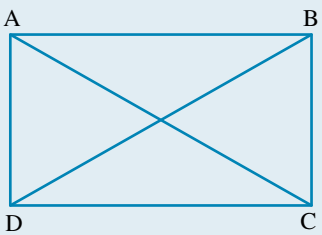
WORKED EXAMPLE 1

Demonstrate that the diagonals of a rectangle are congruent.

THINK

1. Draw a diagram of a rectangle, marking the diagonals and labelling the points.

WRITE



2. We want to show that AC is congruent to BD. We can show ΔABC is congruent to ΔBAD to do this.

Consider ΔABC and ΔBAD.
$\overline{AD} \cong \overline{BC}$ (rectangle)
$\angle BAD = \angle ABC = 90°$ (rectangle)
$\overline{AB}$ is common to both.
ΔABC ≅ ΔBAD (SAS)

3. The triangles are congruent, so the matching sides are congruent.

AC ≅ BD
The diagonals of a rectangle are congruent.

studyon

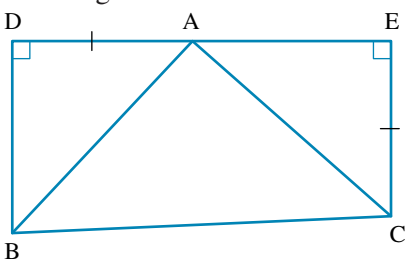
Units 1 & 2 > Area 3 > Sequence 2 > Concept 1

Congruent triangle tests Summary screen and practice questions

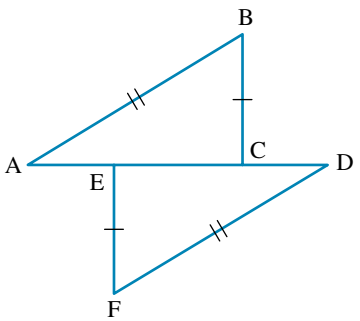
Exercise 5.2 Review of congruent triangle tests

Technology free

1. WE1 Demonstrate that a diagonal of a parallelogram divides it into two congruent triangles.
2. Demonstrate that the diagonals of a rectangle bisect each other.
3. Demonstrate that the opposite sides of a parallelogram are congruent.
4. For a rhombus, demonstrate that:
   a. the diagonals bisect the corner angles
   b. the diagonals bisect each other at right angles.
5. Prove that if a quadrilateral has one pair of opposite sides that are both congruent and parallel, it is a parallelogram.
6. Prove that if the diagonals of a quadrilateral bisect each other at right angles, all of the sides are congruent.
7. Prove that the diagonals of a kite intersect at right angles.
8. Using the congruency test SSS, prove that RHS is also a congruency test.
9. If $\angle ABC = \angle ACB = 45°$, prove that $\overline{DE}$ is a straight line.
   Hint: First prove that two triangles are congruent.



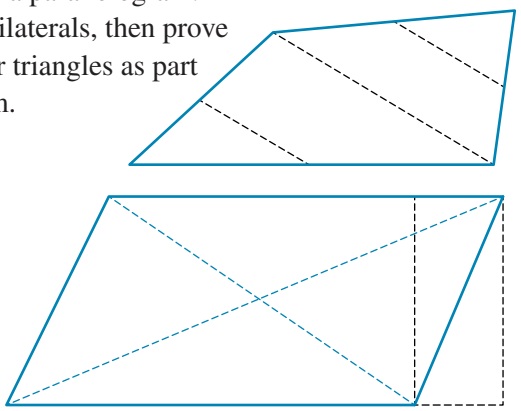
10. Triangle ABC is an equilateral triangle. The points X, Y and Z bisect $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. Prove that ΔXYZ is an equilateral triangle.
11. If $\overline{AE} \cong \overline{CD}$, prove that $\overline{AB}$ is parallel to $\overline{FD}$.



12. Prove that the quadrilateral formed by joining the midpoints of the sides of a square is also a square.

Technology active

13. The midpoints of the sides of any quadrilateral join to form a parallelogram. Use geometry software such as GeoGebra to explore quadrilaterals, then prove that the statement is true. *Hint:* You will need to use similar triangles as part of your proof; the diagram shown may help you to see them.
14. The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides. Use geometry software such as GeoGebra to convince yourself that this is true. Use Pythagoras' theorem to prove that it is true. The diagram shown may help you.

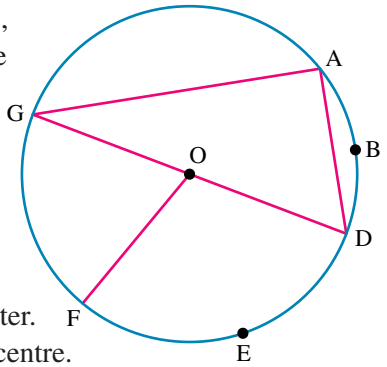


5.3 Circle properties 1 — angles in a circle and chords

5.3.1 Useful definitions

Chord definitions

- **Circumference**: the distance around the circle (the perimeter), that is, the length of the circle if it was opened up and straightened into a line segment. (Circumference may also be used to refer to the edge of the circle.) A, B, D, E, F and G are all **points on the circle**.
- **Centre of the circle**: the middle point, equidistant from all points on the circumference; point O.
- **Chord**: an interval jointing two points on the circle. $\overline{AG}$, $\overline{AD}$ and $\overline{GD}$ are all chords.
- **Diameter**: a chord that passes through the centre. $\overline{GD}$ is also a diameter.
- **Radius**: any straight line joining a point on the circumference to the centre. $\overline{OG}$, $\overline{OD}$ and $\overline{OF}$ are all radii. Notice that every diameter consists of two radii.

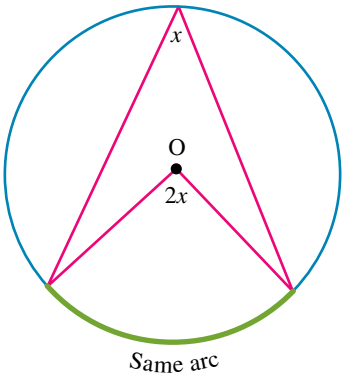
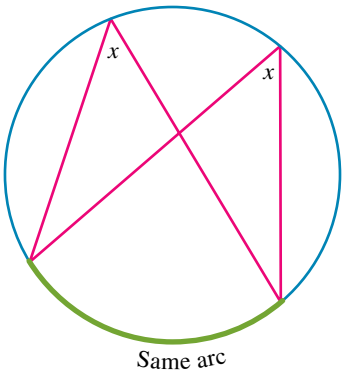
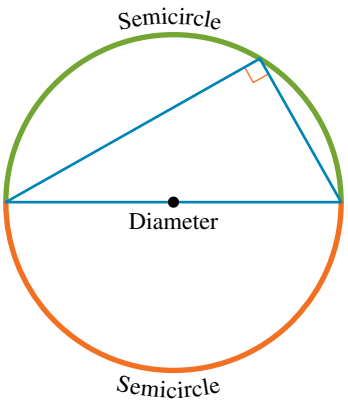


Angle definitions

- **Arc**: part of a circle defined by three points, for example arc $\overset{\frown}{ABD}$. A **minor arc** is smaller than a semicircle; a **major arc** is larger than a semicircle. The arc $\overset{\frown}{ABD}$ is said to **subtend** the angle $\angle AGD$ as G is on the circumference of the circle. $\angle AGD$ is called an **angle at the circumference** subtended by the arc $\overset{\frown}{ABD}$. The same angle is sometimes described as **standing** on chord $\overline{AD}$.
- **Angle at the centre**: the angle made by two radii intersecting the centre of the circle, for example $\angle FOD$, which is subtended by the arc $\overset{\frown}{FED}$.
- **Angle in a semicircle**: $\angle GAD$, as $\overline{GD}$ is a diameter.

5.3.2 Useful theorems for angles

There are a number of theorems involving chords and angles in circles. These can be proved by using the congruent triangle tests from the previous section and any previously proven circle theorems. The proofs for these are explored in the worked examples and the exercises that follow. The theorems can also be used to calculate other angles.

Theorem	Diagram	Symbol
Theorem 1: The angle at the centre of the circle is twice the angle at the circumference subtended on the same arc. The proof of this theorem is shown in Worked example 3.	 If the magnitude of the angle at the circumference is $x°$, then the magnitude of the angle at the centre subtended on the same arc is $2x°$.	
Theorem 2: Angles at the circumference subtended by the same arc are congruent. The proof of this theorem is completed in Exercise 5.3. A sample proof is provided in the worked solutions in your online resources.	 If the magnitude of the angle at the circumference is $x°$, then the magnitude of any angle at the circumference subtended on the same arc is $x°$.	
Theorem 3: The angle in a semicircle is a right angle. The proof of this theorem is completed in Exercise 5.3. A sample proof is provided in the worked solutions in your online resources.	 The angle in the semicircle is 90°.	

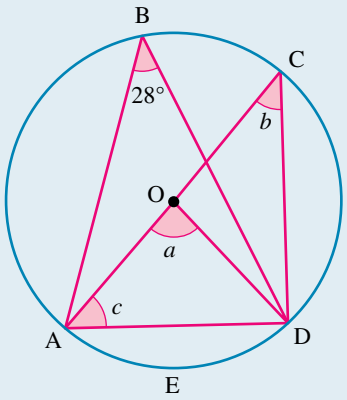
These theorems can be used to calculate the values of unknown angles.

on Resources

- **Interactivity:** Circle theorem 1 (int-6218)
- **Interactivity:** Circle theorem 2 (int-6219)
- **Interactivity:** Circle theorem 3 (int-6220)

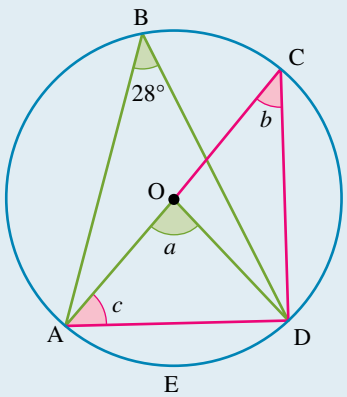
WORKED EXAMPLE 2

Determine the values of a, b and c in the circle shown.



THINK

1. $\angle AOD$ and $\angle ABD$ are both standing on arc $\overset{\frown}{AED}$. By recognising the first theorem above, $\angle AOD$ is twice $\angle ABD$.



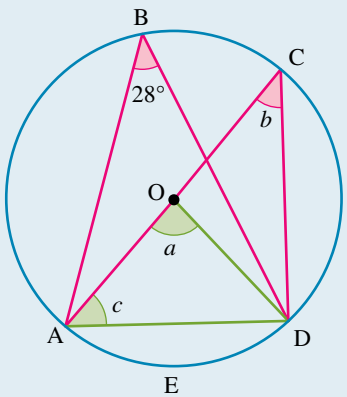
WRITE

$\angle ABD = 28°$

$\therefore a = 2 \times 28°$

$= 56°$

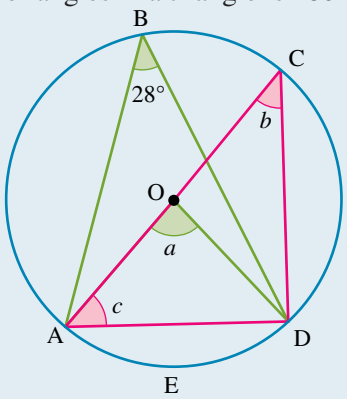
2. $\angle ABD$ and $\angle ACD$ are both standing on arc $\overset{\frown}{AED}$. By recognising the second theorem above, $\angle ABD = \angle ACD$.



$\angle ABD = 28°$

$\therefore b = 28°$

3. ∠ADC is in a semicircle. Recognising the third theorem above, ∠ADC = 90°, and recalling the sum of angles in a triangle is 180°, $b + c = 90°$.



$\angle ADC = 90°$

$b + c = 90°$

$c = 90 - 28°$

$c = 62°$

Worked example 2 uses theorems to calculate unknown values. As introduced in the previous chapter, postulates and previously proven theorems can also be used to prove other theorems.

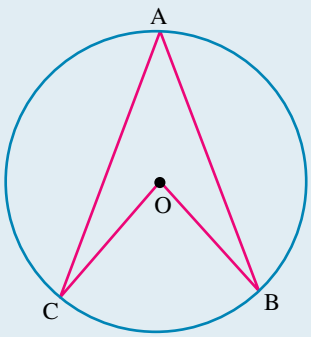
WORKED EXAMPLE 3

Prove theorem 1: the angle at the centre of a circle is twice the angle at the circumference subtended on the same arc.

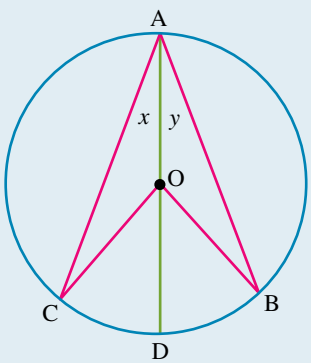
THINK

1. Draw a diagram.

WRITE



2. If a diameter is drawn passing through A and O, two isosceles triangles are formed.



3. As AO and CO are radii, ΔAOC is isosceles. Find the relationship between ∠CAO and ∠COD.

Consider ΔAOC.
Let $\angle CAO = x$.
$\angle ACO \cong \angle CAO$
$\angle ACO = x$

$\angle DOC = \angle ACO + \angle CAO$
$= x + x$
$= 2x$

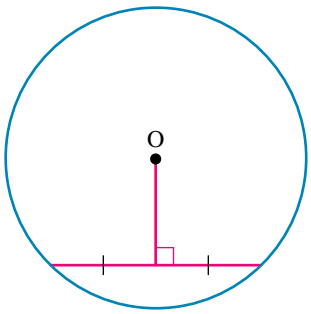
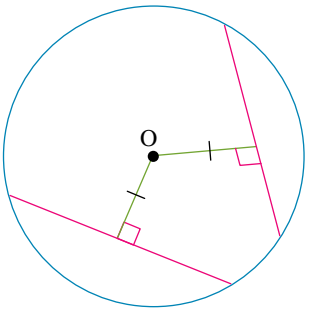
4. The steps are the same for the other triangle.

Similarly, if $\angle BAO = y$, then $\angle DOB = 2y$.

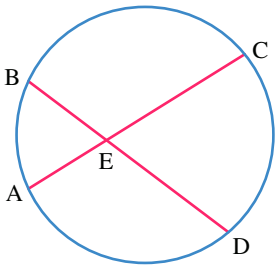
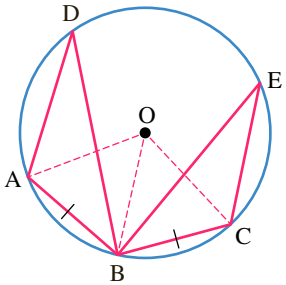
5. Write the concluding statements.

$\angle CAB = x + y$ and $\angle COB = 2x + 2y$
$= 2(x + y)$
Therefore, the angle at the centre is twice the angle at the circumference subtended by the same arc.

5.3.3 Useful theorems for chords

Theorem	Diagram	Symbol
Theorem 4: If a line from the centre bisects a chord, it is perpendicular to the chord. The converse is also true: if a line from the centre is perpendicular to a chord, it bisects the chord. The theorem is proved in Worked example 5. The proof of the converse is left for Exercise 5.3, and a sample proof is provided in the worked solutions in your online resources.	 A line from the centre that bisects the chord is perpendicular to the chord and the converse.	The symbol refers to both the theorem and its converse.
Theorem 5: If chords are congruent, then they are equidistant from the centre of the centre. The converse is also true: chords that are equidistant from the centre are congruent. The proof of this theorem is completed in Exercise 5.3. A sample proof is provided in the worked solutions in your online resources.	 Congruent chords are equidistant from the centre.	The symbol refers to both the theorem and its converse.

Continued

Theorem	Diagram	Symbol
Theorem 6: If two chords intersect inside a circle, then the point of intersection divides each chord into two segments so that the product of the lengths of the segments for both chords is the same. The converse is also true. (This theorem is also known as the intersecting chords theorem.) The proof of this theorem is left for Exercise 5.3. A sample proof is provided in the worked solutions in your online resources.	 $BE \cdot DE = AE \cdot CE$	The symbol refers to both the theorem and its converse.
Theorem 7: If chords are congruent, they subtend equal angles at the centre. The converse is also true: chords subtending equal angles at the centre of a circle have the same length. The proof of this theorem and its converse is left for Exercise 5.3. A sample proof is provided in the worked solutions in your online resources.	 Congruent chords subtend equal angles at the centre. If $\overline{AB} \cong \overline{BC}$, $\angle AOB \cong \angle BOC$.	The symbol refers to both the theorem and its converse.

on Resources

- **Interactivity:** Circle theorem 4 (int-7257)
- **Interactivity:** Circle theorem 5 (int-7258)
- **Interactivity:** Circle theorem 6 (int-6223)

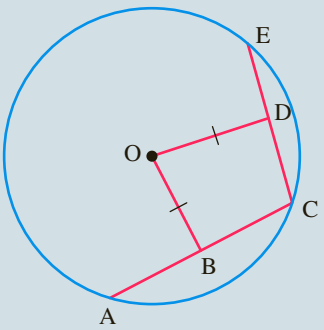
The theorems given above can be used to solve problems involving unknown angles and lengths.

WORKED EXAMPLE 4

If AB = BC = 15, find:

a. $\angle OBC$

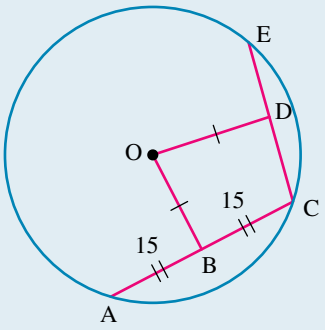
b. EC



THINK

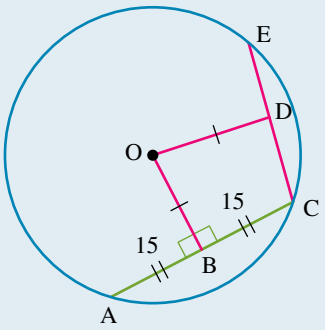
Mark the known information on the diagram.

WRITE



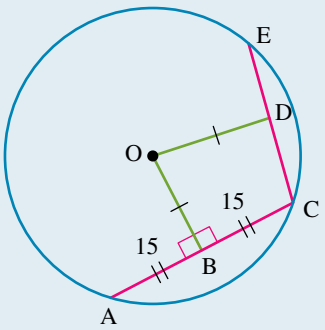
a. Chord $\overline{AC}$ has been bisected. Therefore, the line from the centre, $\overline{OB}$, is perpendicular to the chord.

a.



$\overline{OB}$ bisects $\overline{AC}$; therefore, $\angle OBC = 90°$.

b. Chords $\overline{AC}$ and $\overline{CE}$ are equidistant from the centre and are therefore congruent.



$\overline{OB} \cong \overline{OD}$

$\therefore \overline{AC} \cong \overline{EC}$

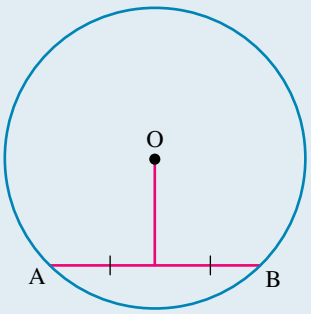
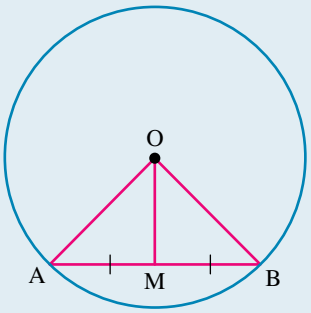
$AC = 15 + 15$

$= 30$

$\therefore EC = 30$

WORKED EXAMPLE 5

Prove theorem 4: if a line from the centre bisects a chord, it is perpendicular to the chord.

THINK	WRITE
1. Draw a diagram of a circle, with a line from the centre bisecting a chord.	
2. Draw in radii $\overline{OA}$ and $\overline{OB}$ so that congruent triangles can be used.	
3. Demonstrate that $\Delta OAM \cong \Delta OBM$.	Consider ΔOAM and ΔOBM. $\overline{OA} \cong \overline{OB}$ (radii) $\overline{AM} \cong \overline{BM}$ (given) $\overline{OM}$ is common to both. Therefore, $\Delta OAM \cong \Delta OBM$ (SSS).
4. Using the congruent triangles, the corresponding angles are also equal.	$\angle OMA \cong \angle OMB$ (congruent triangles) $\angle OMA + \angle OMB = 180°$ Therefore, $\angle OMA = \angle OMB = 90°$.
5. Write the concluding statement.	Therefore, if a line from the centre bisects a chord, it is perpendicular to the chord.

studyon

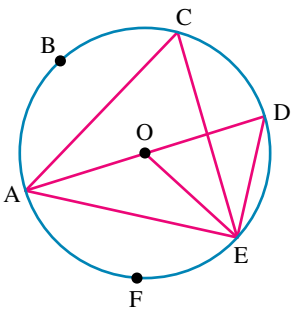
Units 1 & 2 › Area 3 › Sequence 2 › Concepts 2, 3 & 4

Circle terms Summary screen and practice questions
Circle theorems — angles Summary screen and practice questions
Circle theorems — chords Summary screen and practice questions

Exercise 5.3 Circle properties 1 — angles in a circle and chords

Technology free

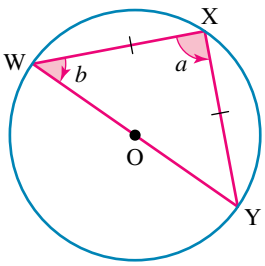
1. Refer to the diagram of the circle shown to answer the following questions.



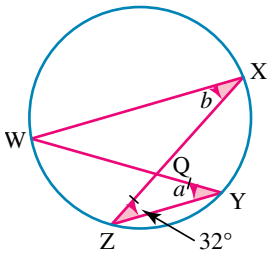
- **a.** Identify the points on the circle.
- **b.** Identify the radii.
- **c.** Identify a major arc
- **d.** Identify the chords.
- **e.** Identify an angle in a semicircle.
- **f.** Identify an angle at the centre
- **g.** Identify two angles standing on arc $\overset{\frown}{AFE}$.

2. **WE2** Determine the values of a and b in each of the circles shown.

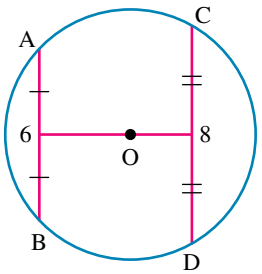
a.



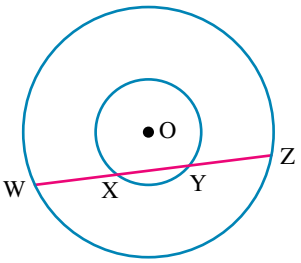
b.



3. **WE4** a. If the radius of the circle shown is 5 units, how far apart are the chords $\overline{AB}$ and $\overline{CD}$?

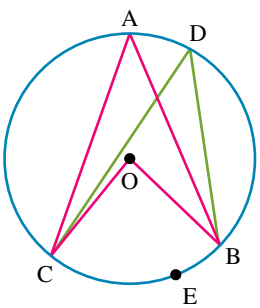


b. If the radii of the circles shown are 13 and 15 units and the distance from the centre to $\overline{WZ}$ is 12 units, determine the length of $\overline{WX}$.

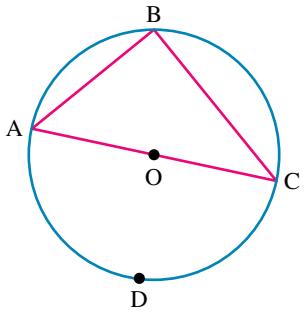


4. Prove the following theorems. You will need to use theorem 1 in your solutions.

- **a.** **WE3** Prove theorem 2: angles at the circumference subtended by the same arc are congruent. This diagram might help you.

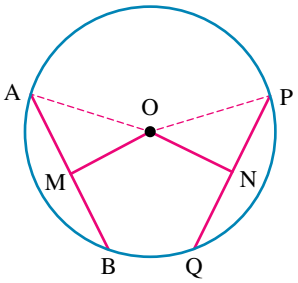


b. Prove theorem 3: the angle subtended by a semicircle is a right angle. This diagram might help you.



5. **WE5** Prove the converse of theorem 4: a line from the centre that is perpendicular to a chord will bisect the chord.

6. **a.** Prove theorem 5: if chords are congruent, they are equidistant from the centre.

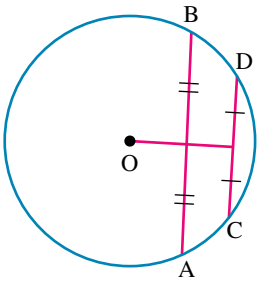


b. Prove the converse of theorem 5: if chords are equidistant from the centre, they are congruent.

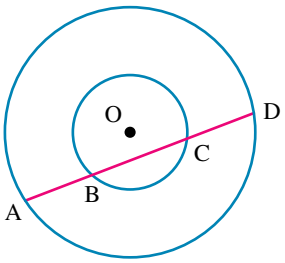
Note: You may use theorem 4 for both proofs.

Hint: The diagram shown may help you. Remember that if ON is the distance from O to the chord $\overline{PQ}$, then $\overline{ON}$ and $\overline{PQ}$ are at right angles to each other.

7. Prove that $\overline{AB}$ is parallel to $\overline{CD}$.



8. Prove that $\overline{AB}$ is congruent to $\overline{CD}$.

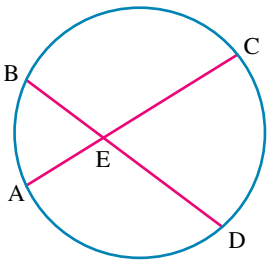


9. Consider the circle shown.

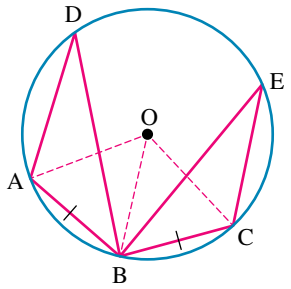
a. Prove that $\Delta ABE \approx \Delta DCE$.

b. Hence, prove that $BE \cdot DE = AE \cdot CE$.

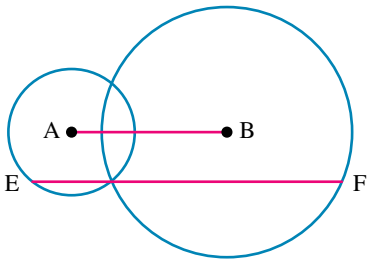
Note: This is known as the intersecting chords theorem: when two chords of a circle intersect, the product of the lengths of the intervals on one chord equals the product of the lengths of the intervals on the other chord (theorem 6, above).

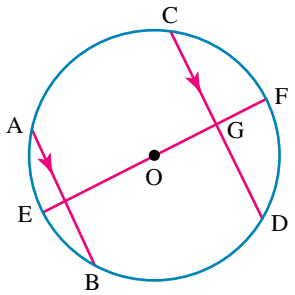


10. Prove each of the following. The diagram may assist you.
    a. Chords of equal length subtend equal angles at the centre and the converse (theorem 7).
    b. Angles at the circumference standing on chords of equal length are congruent.



11. Two congruent circles with centres O and C intersect at A and B. The centre of each circle is a point on the other circle. If $\overline{AD}$ and $\overline{AE}$ are diameters, prove that:
    a. ΔAED is an equilateral triangle
    b. AB is $\sqrt{3}$ times the length of the radius.
12. Question 9 involves proving the intersecting chords theorem. Prove that the converse is also true, that is, if two line segments $\overline{AC}$ and $\overline{BD}$ intersect at E and $BE \cdot DE = AE \cdot CE$, then A, B, C and D are points on a circle.
13. A and B are the centres of two intersecting circles. $\overline{EF}$ passes through one of the points where the circles intersect. If $\overline{AB} \parallel \overline{EF}$, prove that $EF = 2AB$.
14. Prove that $GD^2 = EG \cdot GF$.





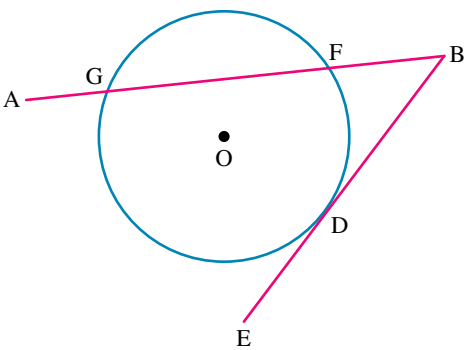
Technology active

15. A regular pentagon has a side of 3 cm. If a circle is drawn so that the vertices of the pentagon lie on the circumference of the circle, what is the radius of the circle? Use suitable technology to verify your answer.
16. A second regular pentagon is formed by bisecting the sides of the pentagon from question 15. A circle is drawn that passes through the vertices of the new pentagon. What is the ratio of the radius of the original circle to the radius of the new circle? Does this ratio remain the same if you continue to create pentagons by bisecting the sides of the previous pentagons? You may use suitable technology to explore your answer.

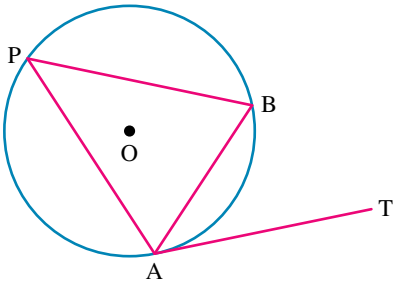
5.4 Circle properties 2 — tangents, secants and segments

5.4.1 Useful definitions

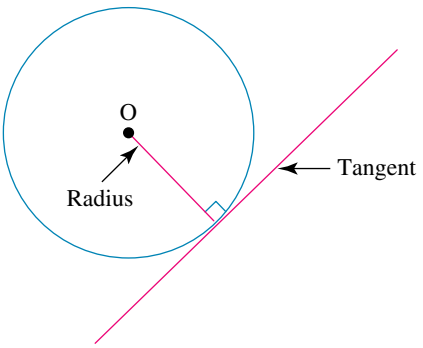
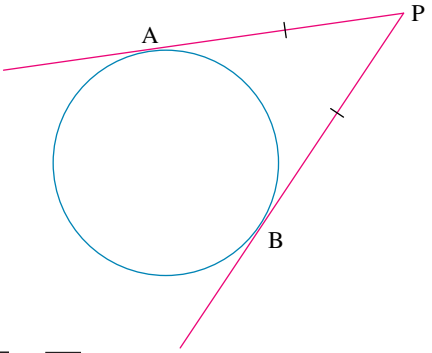
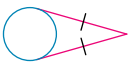
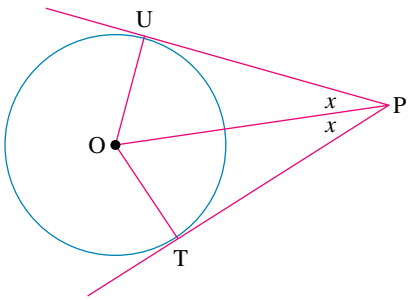
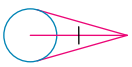
- **External** to the circle: outside the circle; for example points A, B and E
- **Secant**: a line that cuts a circle at two distinct points; for example $\overline{AB}$
- **Tangent**: a line that touches a circle at one point; for example $\overline{EB}$
- **Segment**: a region of a circle cut by a chord; for example, chord $\overline{GF}$ divides the circle into two segments.



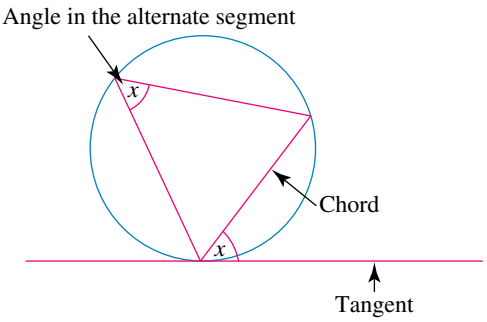
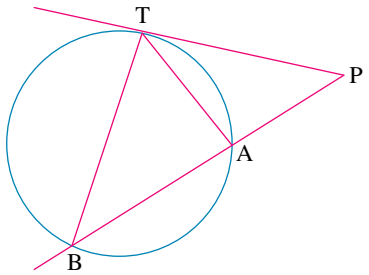
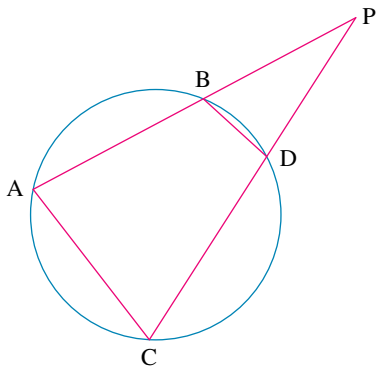
- Chord $\overline{AB}$ divides the circle into two segments. Notice that $\angle BAT$ is in the segment that is closest to the tangent $\overline{AT}$. As $\angle APB$ is in the other segment, it is said to be in the **alternate segment** to $\angle BAT$.



5.4.2 Useful theorems and an axiom

Theorem	Diagram	Symbol
Theorem 8: A tangent and a radius intersect at right angles. The proof of this theorem is completed in Exercise 5.4. A sample proof is provided in the worked solutions in your online resources.	 The tangent to the circle and the radius meet at 90°.	
Theorem 9: Two tangents drawn from an external point to a circle are congruent. The proof of this theorem, which uses theorem 8, is completed in Worked example 7.	 $\overline{PA} \cong \overline{PB}$	
Theorem 10: The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to that external point. The proof of this theorem is completed in Exercise 5.4. A sample proof is provided in the worked solutions in your online resources.	 $\angle UPO = \angle TPO$	

Continued

Theorem	Diagram	Symbol
Theorem 11: The angle formed by a tangent and a chord is congruent to the angle in the alternate segment. This is known as the alternate segment theorem. The proof of this theorem is completed in Exercise 5.4. A sample proof is provided in the worked solutions in your online resources.	 The angle formed by a tangent and a chord is congruent to the angle in the alternate segment.	
Theorem 12: If a secant (meeting the circle at A and B) and a tangent (meeting the circle at T) are drawn from an external point P, the square of the length of the tangent equals the product of the lengths to the circle on the secant. The converse is also true. This theorem is known as the tangent–secant theorem. The proof of this theorem is completed in Exercise 5.4. A sample proof is provided in the worked solutions in your online resources.	 $PT^2 = PA \cdot PB$	The symbol stands for this theorem and its converse.
Theorem 13: If two secants intersect outside the circle as shown, then the product of one internal secant segment length and its external length is equal to the product of the other secant lengths. This theorem is known as the secant–secant theorem. The proof of this theorem is completed in Exercise 5.4. A sample proof is provided in the worked solutions in your online resources.	 $PA \cdot PB = PC \cdot PD$	The symbol stands for this theorem and its converse.

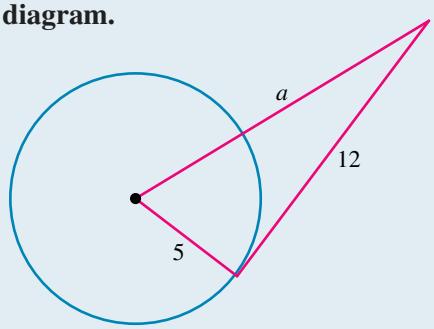
The theorems given above can be used to solve problems involving unknown angles and lengths.

on Resources

- **Interactivity:** Circle theorem 8 (int-7260)
- **Interactivity:** Circle theorem 9 (int-7261)
- **Interactivity:** Circle theorem 10 (int-7262)
- **Interactivity:** Circle theorem 11 (int-7263)
- **Interactivity:** Circle theorem 12 (int-7264)
- **Interactivity:** Circle theorem 13 (int-7265)

WORKED EXAMPLE 6

Determine the value of a in the diagram.

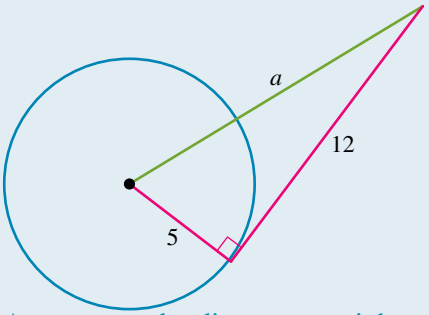


THINK

1. A tangent and radius meet at right angles.

2. Use Pythagoras' theorem to solve for a.

WRITE



A tangent and radius meet at right angles The length of the hypotenuse is a.

$a^2 = 5^2 + 12^2$

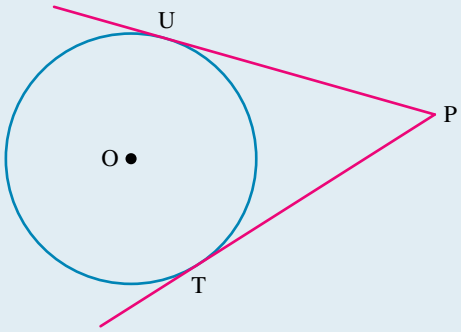
$= 25 + 144$

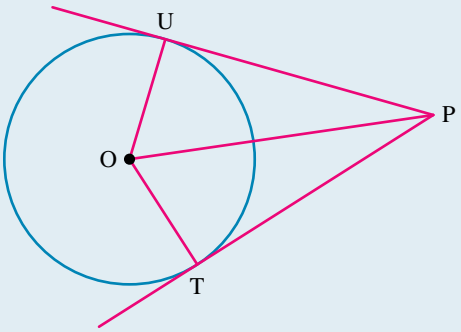
$= 169$

$a = 13$

WORKED EXAMPLE 7

Use the diagram to prove theorem 9: two tangents drawn from an external point to a circle are congruent.



THINK	WRITE
1. The radii need to be drawn in and O and P connected so that congruent triangles can be found.	
2. Identify the congruent sides to begin to demonstrate that ΔOPU ≅ ΔOPT.	Consider ΔOPU and ΔOPT. $\overline{OU} \cong \overline{OT}$ (both radii) $\overline{OP}$ is common to both.
3. Use the theorem that the radius and tangent meet at right angles.	$\angle OUP = \angle OTP = 90°$
4. Conclude that the triangles are congruent using RHS.	Therefore, ΔOUP ≅ ΔOTP (RHS).
5. $\overline{UP}$ and $\overline{TP}$ are corresponding sides and are therefore congruent.	As $\overline{UP}$ and $\overline{TP}$ are corresponding sides, $\overline{UP} \cong \overline{TP}$.
6. Write the concluding statement.	Therefore, tangents drawn from an external point are congruent.

studyon

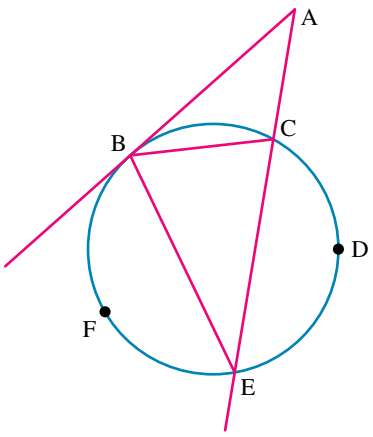
Units 1 & 2 | Area 3 | Sequence 2 | Concepts 5 & 6

Circle theorems — tangents Summary screen and practice questions
Circle theorems — secants and segments Summary screen and practice questions

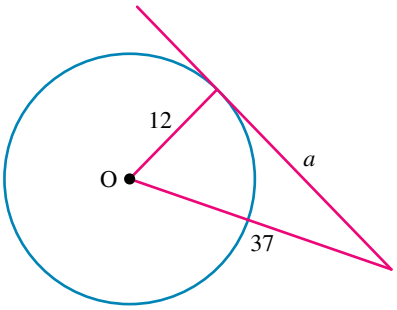
Exercise 5.4 Circle properties 2 — tangents, secants and segments

Technology free

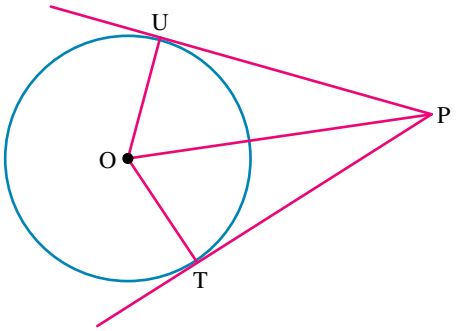
1. Refer to the diagram to identify the following.
   a. Tangent
   b. Secant
   c. Major segment
   d. Angle in the alternate segment to ∠ABC



2. **WE6** Determine the value of a.

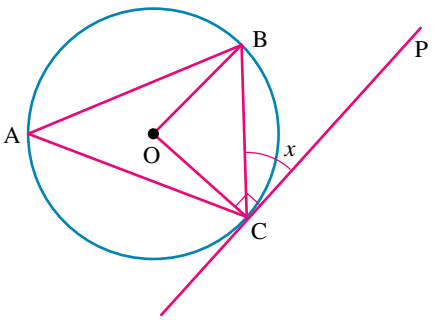


3. **WE7** Use the diagram to prove the following.

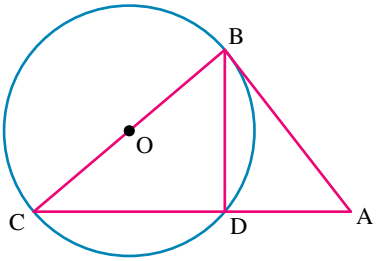


   a. $\overline{OP}$ bisects the angle made between the tangents drawn from P to the circle.
   b. $\overline{OP}$ bisects $\overline{TU}$.

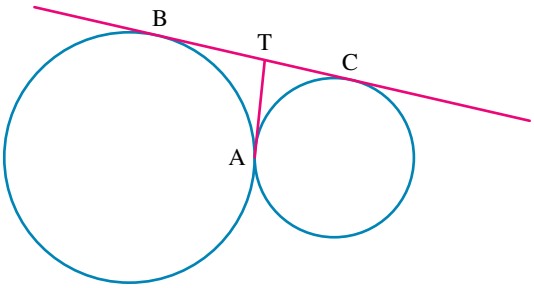
4. Prove the alternate segment theorem (theorem 11): the angle formed by a tangent and a chord is congruent to the angle in the alternate segment. The diagram may assist you.



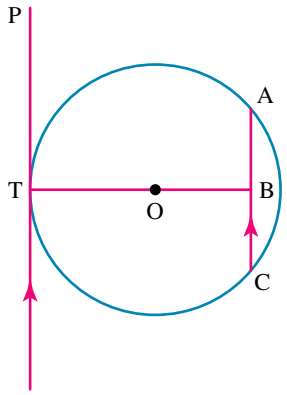
5. If $\overline{AB}$ is a tangent and $\overline{BC}$ is a diameter, prove that $\angle ADB \cong \angle ABC$.



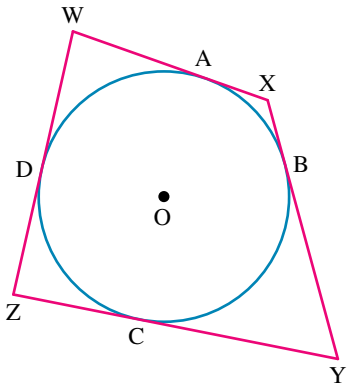
6. $\overline{TA}$ touches both circles at A. $\overline{BC}$ is a tangent to both circles at B and C. Prove that $\overline{BT} \cong \overline{CT}$.



7. Prove that $\overline{AB} \cong \overline{BC}$.

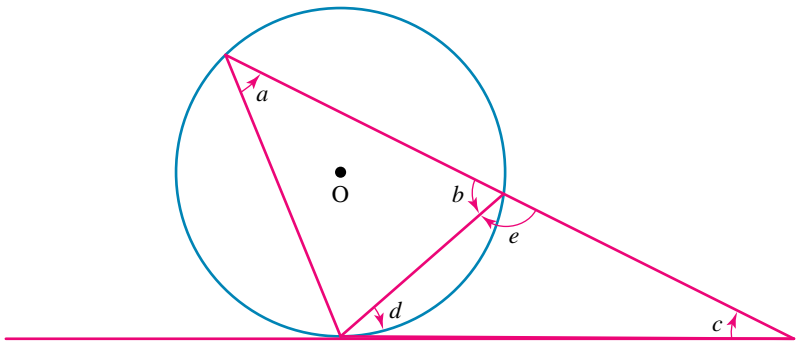


8. Prove that $WX + ZY = WZ + XY$.

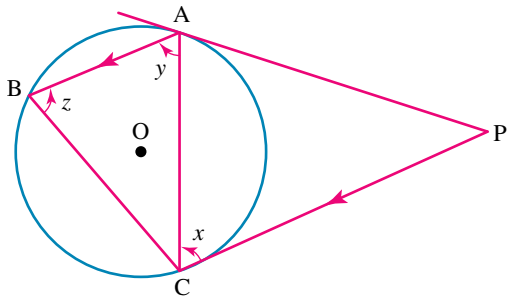


9. Use the alternate segment theorem and any other theorems you know to prove the following.

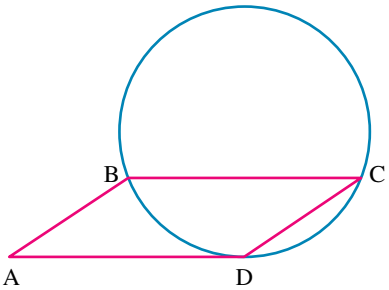
a. $b = a + c$



b. ΔABC is an isosceles triangle.

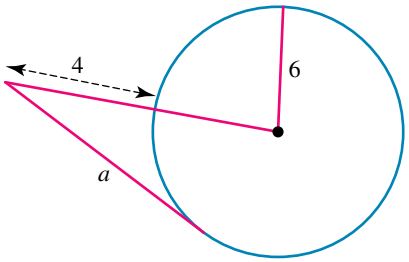


10. ABCD is a parallelogram where $\overline{AD}$ is a tangent to a circle at D, and B and C are points on the circle. Prove that $\angle CBD \cong \angle BAD$.

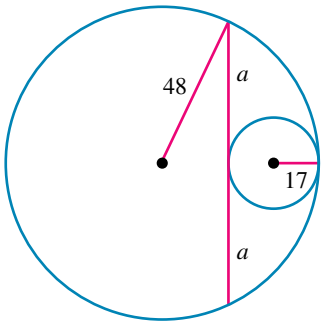


11. Find the value of a in each of the following diagrams.

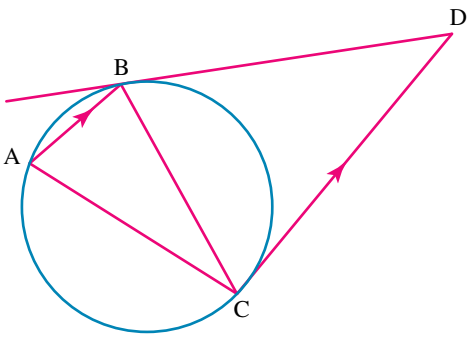
a.



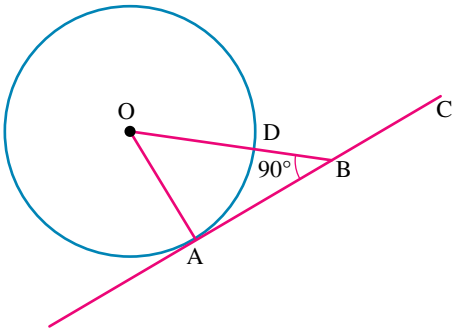
b.



12. Prove that ΔABC is isosceles.



13. A proof by contradiction can be used to prove that the tangent drawn to a circle is perpendicular to the radius at the point of contact (theorem 6).



Let $\overline{OA}$ be the radius of the circle and $\overline{AC}$ be the tangent to the circle at A. Assume that the tangent and radius do not meet at 90°. This means that there must be a line from O that does meet the tangent at 90°. Let $\overline{OB}$ pass through the circle at D and intersect the tangent at point B at 90°.

a. What type of triangle is ΔAOB?

b. According to your answer in part **a**, what is the longest side of the triangle?

c. What do you know about $\overline{OA}$ and $\overline{OD}$? What does that tell you about OA and OB?

d. Use your answers to construct a proof by contradiction to prove that the tangent drawn to a circle is perpendicular to the radius at the point of contact.

14. Two circles intersect at A and B. If $\overline{AM}$ and $\overline{AN}$ are diameters of one of the circles and tangents to the other, prove that M, N and B are collinear.

15. Tangents from a point T touch a circle at A and B. If O is the centre of the circle, prove that $\angle BTA = 2\angle OAB$.

16. Two circles intersect at points A and B. The tangents from point A intersect with the circles at C and D. $\overline{CD}$ also intersects with the circles at P and Q. Prove that $\overline{AP} \cong \overline{AQ}$.

Technology active

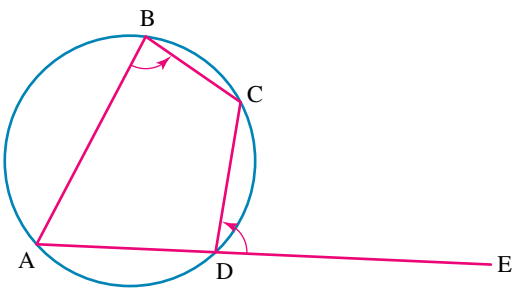
17. Using suitable geometry software, construct a circle with a tangent from P touching the circle at T and a secant from P cutting the circle at A and B. Verify with your software that $PT^2 = PA \cdot PB$. Prove the relationship. *Note:* This is called the tangent–secant theorem (theorem 12).

18. Using suitable geometry software, construct a circle with two secants $\overline{AB}$ and $\overline{CD}$ that intersect at an external point P. Verify with your software that $PA \cdot PB = PC \cdot PD$. Without using the tangent–secant theorem, prove the relationship. *Note:* This is called the secant–secant theorem (theorem 13).

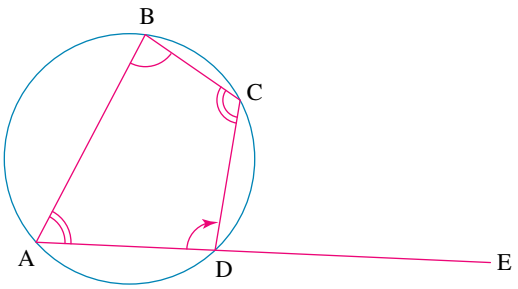
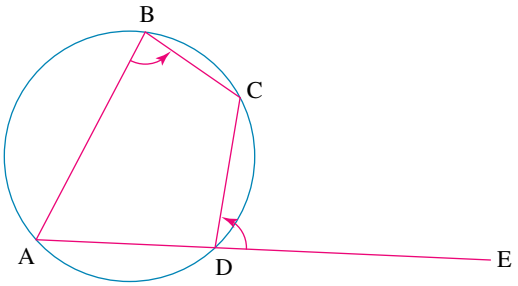
5.5 Circle properties 3 — cyclic quadrilaterals

5.5.1 Useful definitions

- A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle.
- Quadrilateral ABCD is a cyclic quadrilateral as all vertices lie on the circumference of the circle.
  In this example, ∠CDE is an **exterior angle to a cyclic quadrilateral** and ∠ABC is the **interior opposite angle in a cyclic quadrilateral**.



5.5.2 Useful theorems

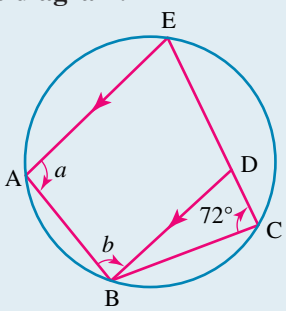
Theorem	Diagram	Symbol
Theorem 14: The opposite angles of a cyclic quadrilateral are supplementary. This theorem is proved in Worked example 9.	 $\angle ABC + \angle CDA = 180°$ and $\angle BCD + \angle DAB = 180°$	
Theorem 15: The exterior angle of a cyclic quadrilateral is congruent to the interior opposite angle. The proof of this theorem is left for Exercise 5.5. A sample proof is provided in the worked solutions in your online resources.	 $\angle CDE = \angle ABC$	

on Resources

- **Interactivity:** Circle theorem 14 (int-7266)
- **Interactivity:** Circle theorem 15 (int-7267)

WORKED EXAMPLE 8

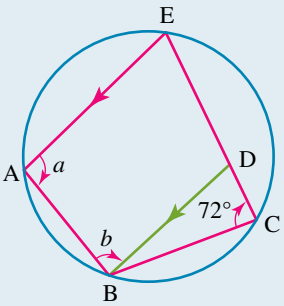
Determine the values of *a* and *b* in the diagram.



THINK

1. ABCE is a cyclic quadrilateral. Opposite angles of a cyclic quadrilateral are supplementary. ∠EAB is opposite ∠ECB.

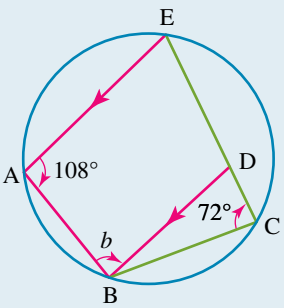
WRITE



ABCE is a cyclic quadrilateral.
Therefore:

$\angle EAB + \angle ECB = 180°$
$a + 72° = 180°$
$a = 108°$

2. Co-interior angles are supplementary.



$\overline{AE} \parallel \overline{BC}$

$\angle EAB + \angle ABD = 180°$
$108° + b = 180°$
$b = 72°$

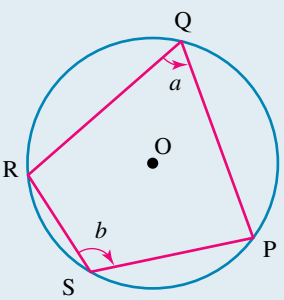
WORKED EXAMPLE 9

Prove theorem 14: the opposite angles of a cyclic quadrilateral are supplementary.

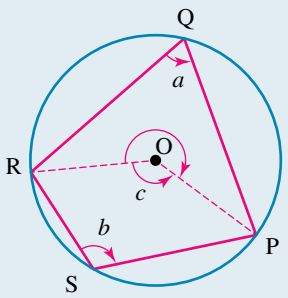
THINK

1. Draw a diagram of a cyclic quadrilateral and mark a pair of opposite angles.

WRITE

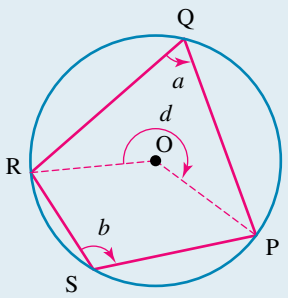


2. $\angle ROP$ and $\angle RQP$ are both subtended by $\overset{\frown}{RSP}$.



Let $\angle RQP = a$.
Let $\angle ROP = c$.
$\therefore c = 2a$

3. Reflex $\angle ROP$ and $\angle RSP$ are both subtended by $\overset{\frown}{RQP}$.



Let $\angle RSP = b$.
Let $\angle ROP = d$ (reflex angle).
$\therefore d = 2b$

4. The angles at the origin must add to 360°.

$c + d = 360°$

5. Use substitution for c and d.

$2a + 2b = 360°$
$a + b = 180°$

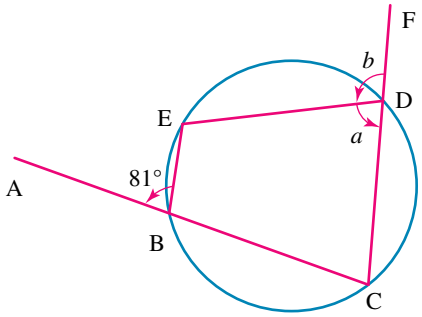
6. Write the concluding statement.

Therefore, the opposite angles of a cyclic quadrilateral are supplementary.

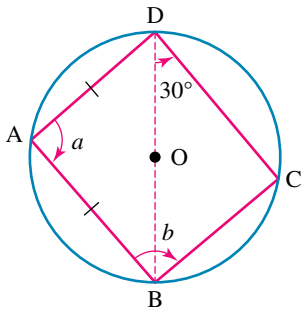
Exercise 5.5 Circle properties 3 — cyclic quadrilaterals

Technology free

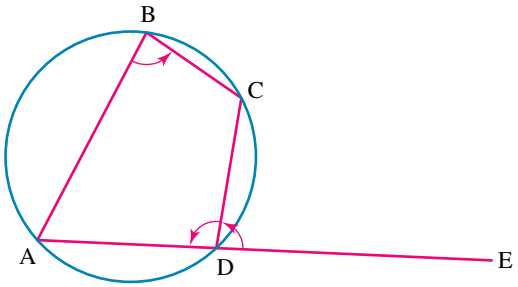
1. **WE8** Find the values of a and b in the diagram shown.



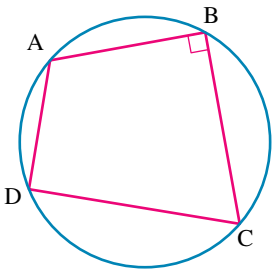
2. Find the values of a and b in the diagram shown.



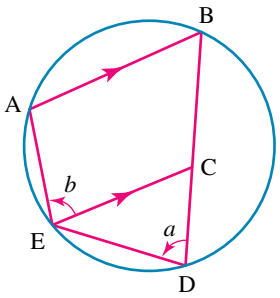
3. **WE9** Prove theorem 15: the exterior angle of a cyclic quadrilateral is congruent to the interior opposite angle. (You may use theorem 14, that opposite angles in a cyclic quadrilateral are supplementary , as part of your proof.)



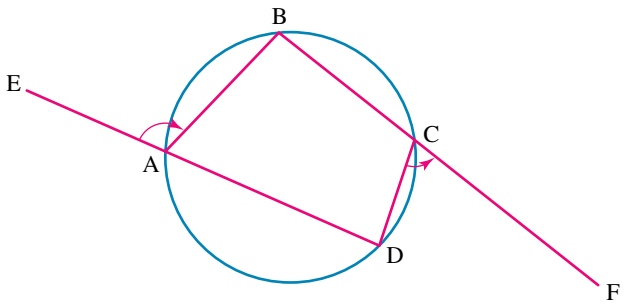
4. If one angle of a cyclic quadrilateral is a right angle, demonstrate that two of the points must be at either end of a diameter.



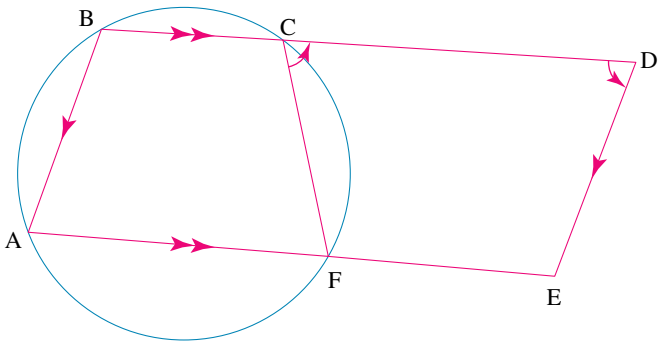
5. Prove that $a = b$.



6. Prove that ∠EAB and ∠DCF are supplementary.



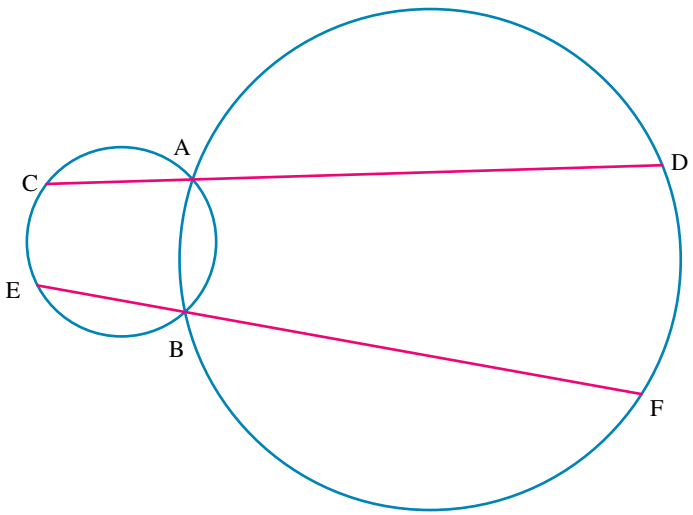
7. Prove that ∠FCD and ∠EDC are congruent.



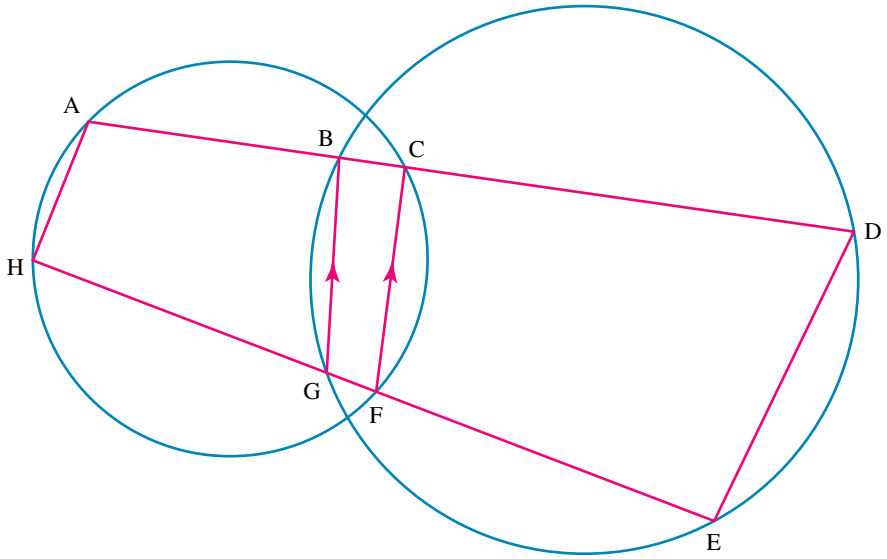
8. PQRS is a parallelogram. A and B lie on $\overline{PS}$ and $\overline{QR}$ respectively, so that PQBA is a cyclic quadrilateral. Prove that RSAB is a cyclic quadrilateral.

9. If ABCD is a cyclic quadrilateral and $\overline{AC}$ bisects ∠BAD, prove that ΔBCD is isosceles.

10. Two circles intersect at A and B. If $\overline{CD}$ passes through A and $\overline{EF}$ passes through B, prove that $\overline{CE} \| \overline{DF}$.



11. In a quadrilateral ABCD, prove that the bisectors of the angles intersect to form a cyclic quadrilateral.
12. If a parallelogram is inscribed inside a circle, prove that it is a rectangle.
13. ABCDEF is a hexagon with vertices on the circumference of a circle.
    a. If $\overline{AB} \| \overline{ED}$ and $\overline{BC} \| \overline{FE}$, prove that $\angle ABC \cong \angle FED$. (*Hint:* Join B and E.)
    b. By proving that $\angle ADC \cong \angle DAF$, demonstrate that $\overline{CD} \| \overline{AF}$.
14. Prove that $\overline{AH} \| \overline{DE}$.



Technology active

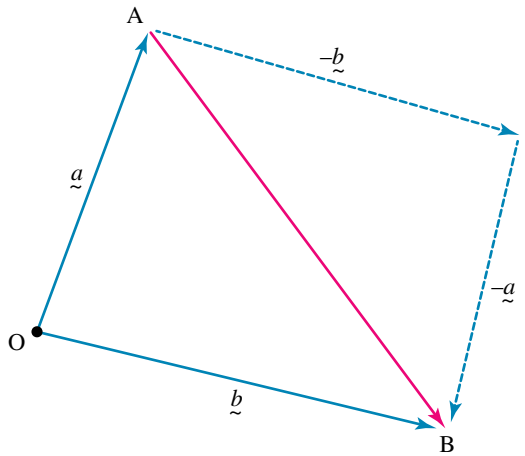
15. Using suitable drawing software, draw a cyclic quadrilateral ABCD. Explore the relationship between $\angle A$, $\angle B$ and $\angle C$ if point D is moved to the centre of the circle. Prove this relationship.
16. Using suitable drawing software, determine the type of quadrilateral formed by the tangents to the points of a cyclic quadrilateral when the cyclic quadrilateral is a rectangle.

5.6 Geometric proofs using vectors

Vectors can be used to prove theorems and other general statements in geometry.

5.6.1 Review of vectors

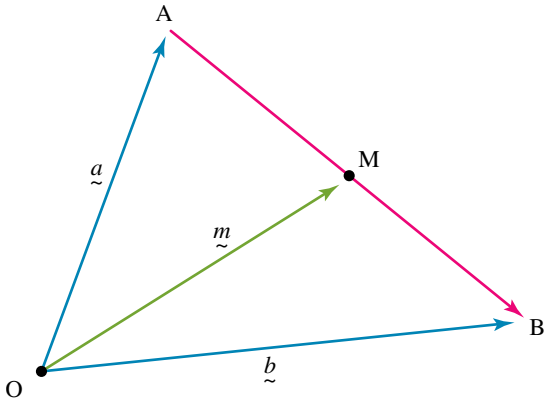
- A **vector** is an entity that has magnitude and direction.
  A **scalar** is an entity that only has magnitude (a number).
- If O is the origin, the vector connecting O to A can be written as $\overrightarrow{OA}$ or $\underset{\sim}{a}$. Similarly, $\overrightarrow{OB} = \underset{\sim}{b}$.
- The vector connecting A to B can be written as $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$.
- If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, then $\overrightarrow{AB} = n\overrightarrow{CD}$, where n is a scalar.
- If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular, then $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$.
- The magnitude of a vector $\underset{\sim}{a}$ can be found using $|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}$, i.e. $|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}$.



- The dot product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ can be found using $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- If two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, then $\overrightarrow{AB} \parallel \overrightarrow{CD}$ and $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$.
- If $\overrightarrow{AB} = n\overrightarrow{BC}$, then A, B and C are collinear.
- If M is the midpoint of $\overline{AB}$, $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$
$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$\underset{\sim}{m} = \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

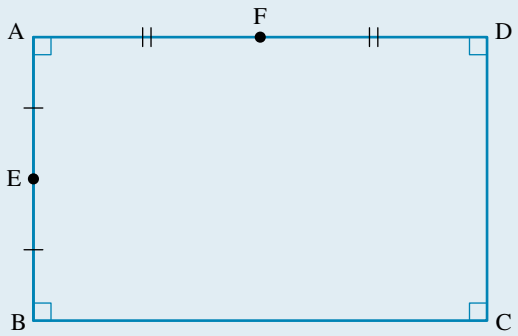
$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$



WORKED EXAMPLE 10

ABCD is a rectangle. Express the following in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

a. $\overrightarrow{AB}$ **b.** $\overrightarrow{CB}$ **c.** $\overrightarrow{CD}$ **d.** $\overrightarrow{AD}$ **e.** $\underset{\sim}{e}$ **f.** $\underset{\sim}{f}$



THINK	WRITE
a. $\overrightarrow{AB}$ means the vector connecting A to B. This can be expressed in terms of a and b.	$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$
b. $\overrightarrow{CB}$ means the vector connecting C to B. This can be expressed in terms of c and b.	$\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$
c. 1. $\overrightarrow{CD}$ means the vector connecting C to D. As ABCD is a rectangle, $\overrightarrow{CD}$ is the reverse of the vector $\overrightarrow{AB}$.	$\overrightarrow{CD} = \overrightarrow{BA}$
2. As $\overrightarrow{CD} = \overrightarrow{BA}$, this is the reverse of $\overrightarrow{AB}$.	$\overrightarrow{CD} = \underset{\sim}{a} - \underset{\sim}{b}$
d. 1. $\overrightarrow{AD}$ means the vector connecting A to D. As ABCD is a rectangle, $\overrightarrow{AD}$ is equal to vector $\overrightarrow{BC}$.	$\overrightarrow{AD} = \overrightarrow{BC}$
2. Vector $\overrightarrow{AD}$ can be expressed in terms of b and c.	$\overrightarrow{AD} = \underset{\sim}{c} - \underset{\sim}{b}$

e. 1. E is the midpoint of $\overrightarrow{AB}$, so vector $\overrightarrow{AE}$ is half the magnitude of $\overrightarrow{AB}$.

$$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB}$$

2. $\overrightarrow{AE}$ can be expressed in terms of $\underset{\sim}{a}$ and $\underset{\sim}{e}$. $\overrightarrow{AB}$ can be expressed in terms of a and b.

$$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AB}$$
$$\underset{\sim}{e} - \underset{\sim}{a} = \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$
$$\underset{\sim}{e} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a} + \underset{\sim}{a}$$

3. Express $\underset{\sim}{e}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$$\underset{\sim}{e} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$$

f. 1. F is the midpoint of $\overrightarrow{AD}$, so vector $\overrightarrow{AF}$ is half the magnitude of $\overrightarrow{AD}$.

$$\overrightarrow{AF} = \frac{1}{2}\overrightarrow{AD}$$

2. $\overrightarrow{AF}$ can be expressed in terms of $\underset{\sim}{a}$ and $\underset{\sim}{f}$. $\overrightarrow{AD}$ can be expressed in terms of b and c (from part **d**).

$$\underset{\sim}{f} - \underset{\sim}{a} = \frac{1}{2}\overrightarrow{BC}$$
$$= \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b})$$

3. Express $\underset{\sim}{f}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

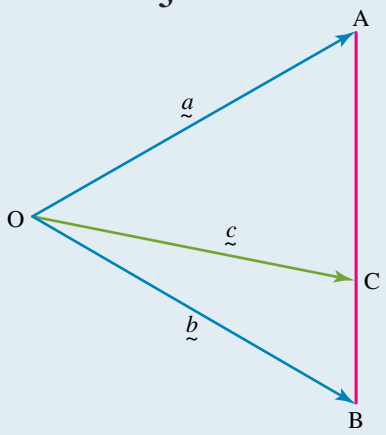
$$\underset{\sim}{f} = \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$$

Using vectors in proofs

In proofs, vectors can be used in different ways. Sometimes it is easier to rewrite the vectors using position vectors, and sometimes it is easier to write them in terms of sums and differences of other vectors.

WORKED EXAMPLE 11

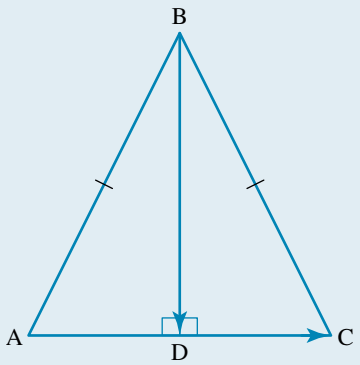
AOB is a triangle. Point C is drawn on $\overline{AB}$ so that $AC = \frac{2}{3}AB$. The position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown on the diagram. Demonstrate that $\underset{\sim}{c} = \frac{1}{3}(2\underset{\sim}{b} + \underset{\sim}{a})$.



THINK	WRITE
1. Express $\overrightarrow{AB}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$
2. Express $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.	$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$
3. Express $\overrightarrow{AC}$ as a multiple of $\overrightarrow{AB}$.	$\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AB}$
4. Replace $\overrightarrow{AC}$ with $\underset{\sim}{c} - \underset{\sim}{a}$ and $\overrightarrow{AB}$ with $\underset{\sim}{b} - \underset{\sim}{a}$, and rearrange for $\underset{\sim}{c}$.	$\underset{\sim}{c} - \underset{\sim}{a} = \frac{2}{3}(\underset{\sim}{b} - \underset{\sim}{a})$ $\underset{\sim}{c} = \frac{2}{3}\underset{\sim}{b} - \frac{2}{3}\underset{\sim}{a} + a$ $= \frac{2}{3}\underset{\sim}{b} + \frac{1}{3}\underset{\sim}{a}$ $= \frac{1}{3}(2\underset{\sim}{b} + \underset{\sim}{a})$

WORKED EXAMPLE 12

In ΔABC, AB = BC. Using the properties of vectors only, show that the line $\overline{BD}$, drawn so that $\overline{BD}$ is perpendicular to $\overline{AC}$, divides $\overline{AC}$ in half. That is, $|\overrightarrow{AD}| = |\overrightarrow{DC}|$.



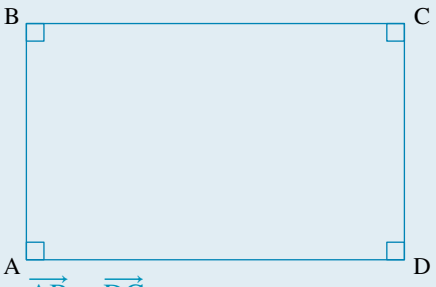
THINK	WRITE
1. Consider ΔABD	$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$
2. Use $\lvert\overrightarrow{AB}\rvert^2 = \overrightarrow{AB} \cdot \overrightarrow{AB}$ and find the dot product of both sides.	$\overrightarrow{AB} \cdot \overrightarrow{AB} = \left(\overrightarrow{AD} + \overrightarrow{DB}\right) \cdot \left(\overrightarrow{AD} + \overrightarrow{DB}\right)$ $\lvert\overrightarrow{AB}\rvert^2 = \overrightarrow{AD} \cdot \overrightarrow{AD} + 2\overrightarrow{AD} \cdot \overrightarrow{DB} + \overrightarrow{DB} \cdot \overrightarrow{DB}$ $= \lvert\overrightarrow{AD}\rvert^2 + \lvert\overrightarrow{DB}\rvert^2 + 2\overrightarrow{AD} \cdot \overrightarrow{DB}$
3. As $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{DB}$, $\overrightarrow{AD} \cdot \overrightarrow{DB} = 0$.	$\overrightarrow{AD} \perp \overrightarrow{DB}$, so $\overrightarrow{AD} \cdot \overrightarrow{DB} = 0$. Therefore, $\lvert\overrightarrow{AB}\rvert^2 = \lvert\overrightarrow{AD}\rvert^2 + \lvert\overrightarrow{DB}\rvert^2$.

4. Repeat using ΔCBD.	$\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$ $\overrightarrow{BC} \cdot \overrightarrow{BC} = (\overrightarrow{BD} + \overrightarrow{DC}) \cdot (\overrightarrow{BD} + \overrightarrow{DC})$ $\left\|\overrightarrow{BC}\right\|^2 = \overrightarrow{BD} \cdot \overrightarrow{BD} + 2\overrightarrow{BD} \cdot \overrightarrow{DC} + \overrightarrow{DC} \cdot \overrightarrow{DC}$ $= \left\|\overrightarrow{BD}\right\|^2 + \left\|\overrightarrow{DC}\right\|^2 + 2\overrightarrow{BD} \cdot \overrightarrow{DC}$
5. ΔABC is isosceles and $\left\|\overrightarrow{AB}\right\|^2 = \left\|\overrightarrow{BC}\right\|^2$.	$\overrightarrow{BD} \perp \overrightarrow{DC}$, so $\overrightarrow{BD} \cdot \overrightarrow{DC} = 0$. Therefore, $\left\|\overrightarrow{BC}\right\|^2 = \left\|\overrightarrow{BD}\right\|^2 + \left\|\overrightarrow{DC}\right\|^2$. As $\left\|\overrightarrow{AB}\right\|^2 = \left\|\overrightarrow{BC}\right\|^2$ and $\left\|\overrightarrow{AB}\right\|^2 = \left\|\overrightarrow{AD}\right\|^2 + \left\|\overrightarrow{BD}\right\|^2$. $\left\|\overrightarrow{AD}\right\| = \left\|\overrightarrow{DC}\right\|$.
6. Write concluding statements.	$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ and $\left\|\overrightarrow{AD}\right\| = \left\|\overrightarrow{DC}\right\|$. Therefore, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$, so D is the midpoint of $\overline{AC}$.

A useful method to demonstrate that lines bisect each other is to let the bisector of one line be one point and the bisector of the other line be a different point, then use position vectors to show that the points are in the same position.

WORKED EXAMPLE 13

Use a vector method to demonstrate that the diagonals of a rectangle bisect each other.

THINK	WRITE
1. Draw a rectangle ABCD.	
2. Observe that $\overrightarrow{AB} = \overrightarrow{DC}$ and write this in terms of the position vectors.	$\overrightarrow{AB} = \overrightarrow{DC}$ $\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{c} - \underset{\sim}{d}$ $\underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{a} + \underset{\sim}{c}$
3. Let E be the midpoint of $\overline{AC}$.	Let E be the midpoint of $\overline{AC}$.
4. Write $\overrightarrow{AC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.	$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$
5. As E is the midpoint of $\overline{AC}$, $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$.	$\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$

6. Replace with position vectors and rearrange to have $\underset{\sim}{e}$ as the subject.

$$\underset{\sim}{e} - \underset{\sim}{a} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$$
$$\underset{\sim}{e} = \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a} + \underset{\sim}{a}$$
$$\underset{\sim}{e} = \frac{1}{2}\underset{\sim}{c} + \frac{1}{2}\underset{\sim}{a}$$
$$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$$

7. Let F be the midpoint of $\overline{BD}$. Repeat steps **3–6** to find an expression for $\underset{\sim}{f}$.

Let F be the midpoint of $\overline{BD}$.
$\overrightarrow{BD} = \underset{\sim}{d} - \underset{\sim}{b}$
$\overrightarrow{BF} = \frac{1}{2}\overrightarrow{BD}$
$\underset{\sim}{f} - \underset{\sim}{b} = \frac{1}{2}(\underset{\sim}{d} - \underset{\sim}{b})$
$\underset{\sim}{f} = \frac{1}{2}\underset{\sim}{d} + \frac{1}{2}\underset{\sim}{b}$
$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{d})$

8. We observed in step **2** that $\underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{a} + \underset{\sim}{c}$. Use this in the expression for $\underset{\sim}{f}$.

As $\underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{a} + \underset{\sim}{c}$, $\underset{\sim}{f} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$.

9. The expressions for $\underset{\sim}{e}$ and $\underset{\sim}{f}$ are equal, so E and F are the same point. Write the concluding statement.

As $\underset{\sim}{e} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$ and $\underset{\sim}{f} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$, E and F are the same point. As E and F are the bisectors of the diagonals, the diagonals bisect each other.

studyon

Units 1 & 2 > Area 3 > Sequence 2 > Concept 7

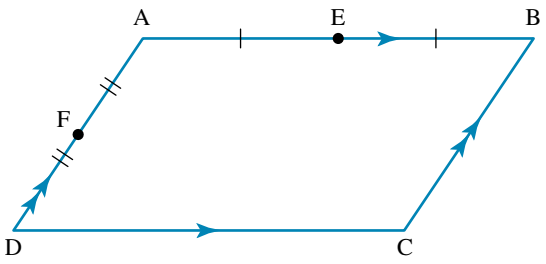
Geometric proofs using vectors Summary screen and practice questions

Exercise 5.6 Geometric proofs using vectors

Technology free

1. **WE10** ABCD is a parallelogram. Express the following in terms of $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

a. $\overrightarrow{AB}$
b. $\overrightarrow{CD}$
c. $\overrightarrow{AD}$
d. $\underset{\sim}{f}$



2. OPQRSTUV is a cube, where O is the origin. Express the following in terms of $\underset{\sim}{s}$, $\underset{\sim}{p}$ and $\underset{\sim}{z}$.

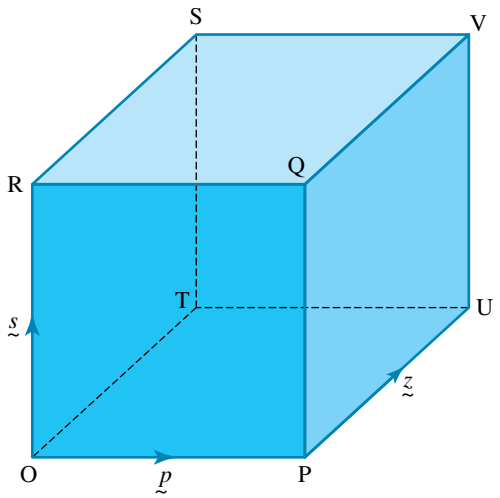
a. $\overrightarrow{OU}$

b. $\overrightarrow{OV}$

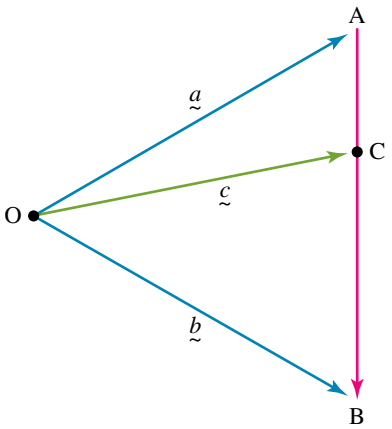
c. $\overrightarrow{TP}$

d. $\overrightarrow{RV}$

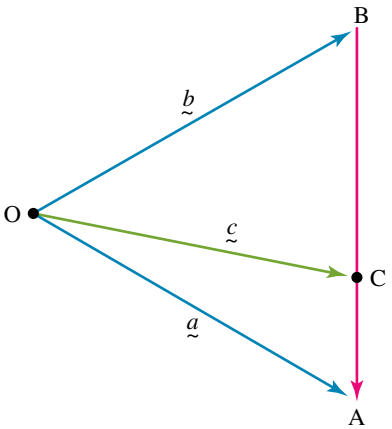
e. $\overrightarrow{PS}$



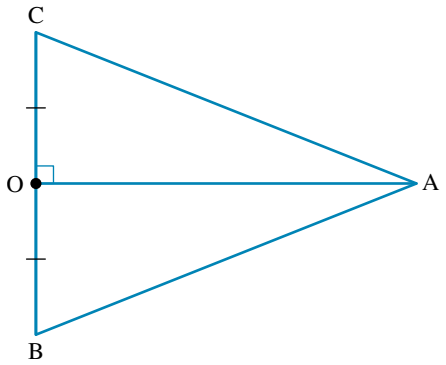
3. **WE11** AOB is a triangle. Point C is drawn on $\overline{AB}$ so that $AC = \frac{1}{4}AB$. The position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown on the diagram. Demonstrate that $\underset{\sim}{c} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$.



4. OAB is a triangle with point C along the line $\overline{AB}$ such that $AC = \frac{1}{n}AB$. The vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown. Demonstrate that $\underset{\sim}{c} = \frac{1}{n}\underset{\sim}{b} + \frac{(n-1)}{n}\underset{\sim}{a}$.



5. In triangle ABC, O is the midpoint of $\overline{CB}$ and $\overline{OA}$ is perpendicular to $\overline{CB}$.



a. If $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$, find in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ only:

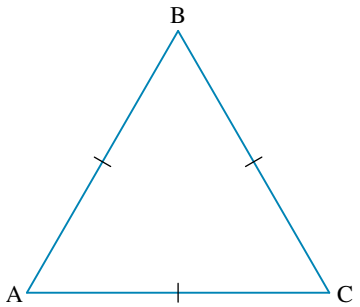
i. $\overrightarrow{OC}$ **ii.** $\overrightarrow{BA}$

iii. $\overrightarrow{CA}$ **iv.** the dot product $\overrightarrow{BA} \cdot \overrightarrow{BA}$.

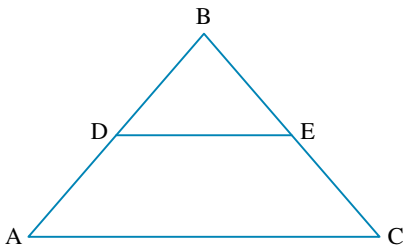
v. the dot product $\overrightarrow{CA} \cdot \overrightarrow{CA}$.

b. Hence, what can you say about AC and AB?

6. **WE12** ABC is an equilateral triangle. Using the properties of vectors only, show that the line $\overline{BD}$, drawn so that $\overline{BD}$ is perpendicular to $\overline{AC}$, divides $\overline{AC}$ in half. That is, $\left|\overrightarrow{AD}\right| = \left|\overrightarrow{DC}\right|$.



7. Consider the triangle ABC. Point D is the midpoint of line $\overline{AB}$ and E is the midpoint of line $\overline{BC}$. Use vectors to demonstrate that line $\overline{DE}$ is parallel to $\overline{AC}$ and $DE = \frac{1}{2}AC$.



8. Use a vector method to demonstrate Pythagoras' theorem.

9. **WE13** Use a vector method to demonstrate that the diagonals of a parallelogram bisect each other.

10. Use vectors to demonstrate that the angle subtended by a diameter of a circle is a right angle.

11. Use vectors to demonstrate that joining the midpoints of the sides of a parallelogram results in a parallelogram.

12. Use vectors to demonstrate that joining the midpoints of the sides of a quadrilateral results in a parallelogram.

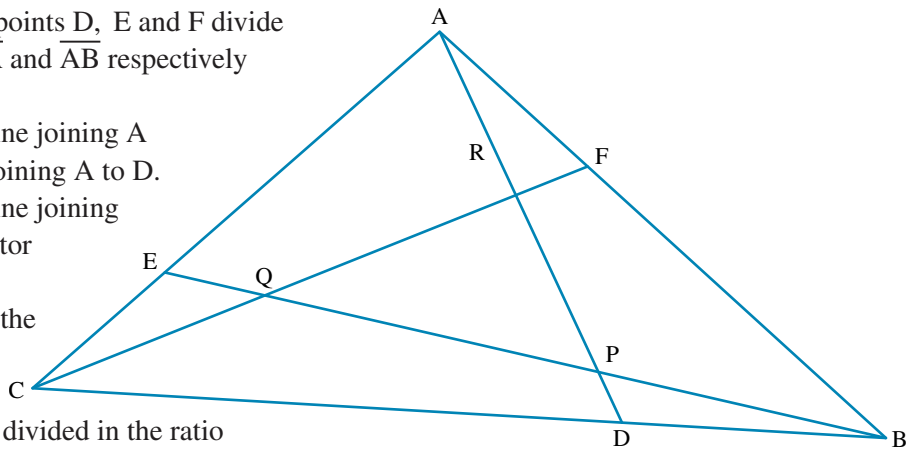
13. Prove that the diagonals of a rhombus intersect at 90°.

14. Prove that the diagonals of a parallelogram meet at 90° if and only if it is a rhombus.

15. Consider any two major diagonals of a cube. Use a vector method to prove that:
    a. the diagonals bisect each other
    b. the acute angle between the diagonals is 70.53°.
16. The median of a triangle is the line drawn from one vertex to the midpoint of the opposite side. Demonstrate that:
    a. the medians of a triangle are concurrent (pass through the same point)
    b. the distance from the vertex to the point of intersection is $\frac{2}{3}$ of the median length.

Technology active

17. In any triangle ABC, the points D, E and F divide the line segments $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively in the ratio 1 : 2.
    a. Demonstrate that the line joining A and P is $\frac{6}{7}$ of the line joining A to D.
    b. Demonstrate that the line joining R and P is $\frac{3}{7}$ of the vector joining A to D.
    c. Determine the ratio of the area of ΔQRP to the area of ΔABC.
    d. If $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ are divided in the ratio 1 : n, determine the ratio of the area of ΔQRP to the area of ΔABC.

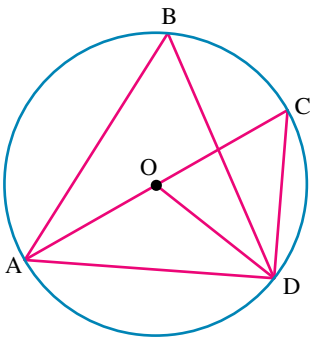


5.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

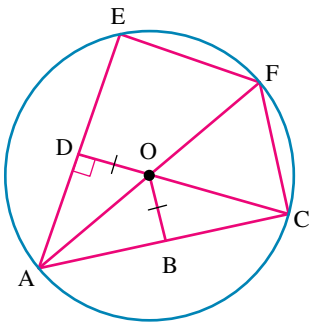
Simple familiar

1. If $\angle ABD = 32°$, determine the sizes of the following angles.



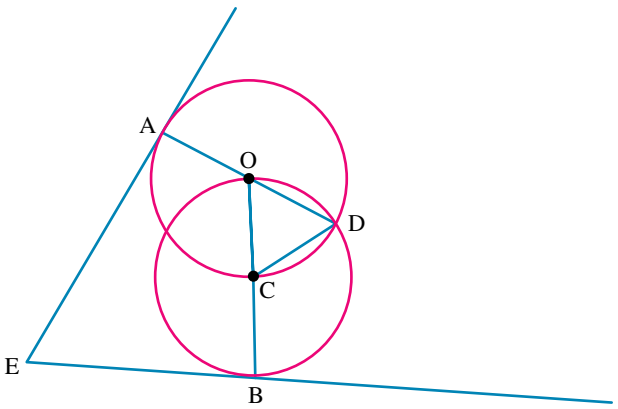
a. $\angle AOD$ **b.** $\angle ACD$ **c.** $\angle COD$ **d.** $\angle CDA$

2. If OB = 9, determine the following.



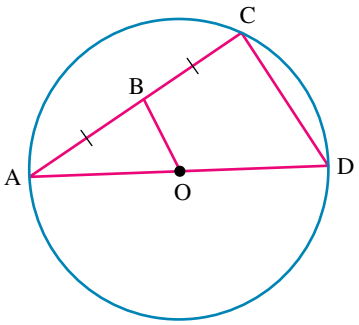
a. $\angle AEF$ **b.** $\overline{EF}$

3. In the diagram, O and C are the centres of the circles.

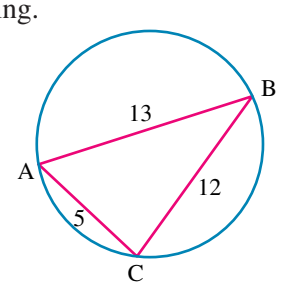


a. Determine $\angle COD$.
b. Determine $\angle AEB$.
c. What type of quadrilateral is AOBE?
d. Determine $\angle OBD$.

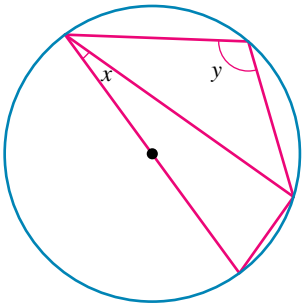
4. Prove that $\overline{OB} \| \overline{CD}$.



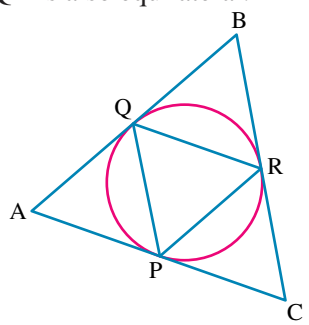
5. Is $\overline{AB}$ a diameter? Explain your reasoning.



6. Find the relationship between x and y.



7. A smaller circle inside a larger circle touches the larger circle at A and passes through the centre of the larger circle. If line ABC passes through the smaller and larger circles at B and C respectively, prove that $\overline{AB} \cong \overline{BC}$.
8. Prove that a line from the centre bisects a chord if and only if it is perpendicular to the chord.
9. If ΔABC is equilateral, prove that ΔPQR is also equilateral.



10. Prove that the angle at the centre of a circle is twice the angle at the circumference.
11. A circle is drawn through the vertices of a square ABCD. If the square has a side of 5 cm, find the area inside the circle but outside the square.
12. The diameter $\overline{AB}$ of a circle is extended through B to T. A tangent is drawn from T, touching the circle at C. $\overline{AC}$ is extended through C to D so that $\angle ATD = 90°$. Prove that $\overline{CT} \cong \overline{TD}$.

Complex familiar

13. Determine the magnitude of the angle at the circumference that is subtended by a chord of the same length as the radius of the circle.

14. Use a vector method to prove that a triangle can be constructed from the medians of a triangle without changing the lengths or slopes of the medians.

15. The point theorem says that if a line from point P cuts a circle at A and D and another line from P cuts the circle at C and B, then $PA \cdot PD = PC \cdot PB$.

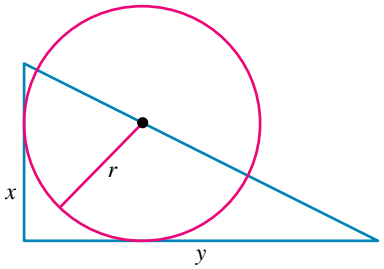
a. Prove the point theorem if P is inside the circle.

b. Prove the point theorem if P is outside the circle.

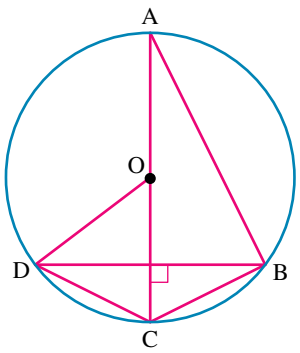
16. If the four sides of a parallelogram touch a circle, prove that the parallelogram is a rhombus.

Complex unfamiliar

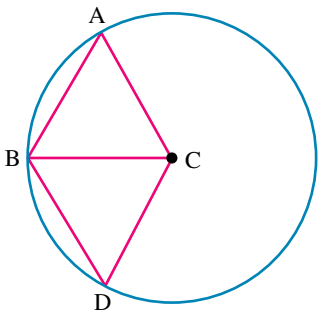
17. A circle of radius r is drawn so that its centre is on the hypotenuse of a right-angled triangle. If the circle touches the other sides of the triangle (lengths x and y), prove that $\frac{1}{r} = \frac{1}{x} + \frac{1}{y}$.



18. O is the centre of the circle shown, $\overline{OD} \cong \overline{CD}$ and $\overline{DB}$ is perpendicular to $\overline{AC}$. Find $\angle CAB$.



19. ABDC is a rhombus where A, B and D are points on a circle with centre C. If the area of the rhombus is $72\sqrt{3}$ units, what is the radius of the circle?



20. A tangent from a point P to a circle with centre O and diameter $\overline{AB}$ touches the circle at Q. If $\overline{QA} \parallel \overline{PO}$, demonstrate that $\overline{PB}$ is also a tangent.

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Units 1 & 2 Sit chapter test

Answers

Chapter 5 Circle geometry

Exercise 5.2 Review of congruent triangle tests

1–14. Sample responses can be found in the worked solutions in the online resources.

Exercise 5.3 Circle properties 1 — angles in a circle and chords

1. a. A, B, C, D, E, F
b. $\overline{OA}$, $\overline{OD}$, $\overline{OE}$
c. Major arcs include $\overset{\frown}{ABE}$, $\overset{\frown}{ABF}$, $\overset{\frown}{CDA}$.
d. $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{CE}$, $\overline{ED}$
e. $\angle AED$
f. $\angle AOE$, $\angle EOD$
g. $\angle ACE$, $\angle ADE$

2. a. $a = 90°$, $b = 45°$ **b.** $a = 32°$, $b = 32°$

3. a. 7 units **b.** 4 units

4–14. Sample responses can be found in the worked solutions in the online resources.

15. 2.55 cm

16. $\frac{r_1}{r_2} = \frac{1}{\cos 36°}$; the ratio remains the same.

Exercise 5.4 Circle properties 2 — tangents, secants and segments

1. a. $\overline{AB}$ **b.** $\overline{AE}$
c. Region CBE **d.** $\angle BEC$

2. 35

3–10. Sample responses can be found in the worked solutions in the online resources.

11. a. 8
b. $2\sqrt{527} = 45.9$ to 1 decimal place

12. A sample response can be found in the worked solutions in the online resources.

13. a. Right-angled triangle
b. $\overline{OA}$
c. They are radii. OB > OD, OB > OA
d. A sample response can be found in the worked solutions in the online resources.

14–18. Sample responses can be found in the worked solutions in the online resources.

Exercise 5.5 Circle properties 3 — cyclic quadrilaterals

1. $a = 81°$, $b = 99°$

2. $a = 90°$, $b = 105°$

3–14. Sample responses can be found in the worked solutions in the online resources.

15. $\angle A + \angle C = \angle B$
A sample response for the proof can be found in the worked solutions in the online resources.

16. Rhombus. A sample response can be found in the worked solutions in the online resources.

Exercise 5.6 Geometric proofs using vectors

1. a. $\underset{\sim}{b} - \underset{\sim}{a}$ **b.** $\underset{\sim}{a} - \underset{\sim}{b}$
c. $\underset{\sim}{c} - \underset{\sim}{b}$ **d.** $\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$

2. a. $\overrightarrow{OU} = \underset{\sim}{p} + \underset{\sim}{z}$
b. $\overrightarrow{OV} = \underset{\sim}{p} + \underset{\sim}{z} + \underset{\sim}{s}$
c. $\overrightarrow{TP} = \underset{\sim}{a} - \underset{\sim}{z}$
d. $\overrightarrow{RV} = \underset{\sim}{p} + \underset{\sim}{z}$
e. $\overrightarrow{PS} = \underset{\sim}{s} + \underset{\sim}{z} - \underset{\sim}{p}$

3, 4. Sample responses can be found in the worked solutions in the online resources.

5. a. i. $-\underset{\sim}{b}$ **ii.** $\underset{\sim}{a} - \underset{\sim}{b}$
iii. $\underset{\sim}{a} + \underset{\sim}{b}$ **iv.** $\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}$
v. $\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}$
b. AC = AB

6–16. Sample responses can be found in the worked solutions in the online resources.

17. a, b. Sample responses can be found in the worked solutions in the online resources.
b. 1 : 7
c. $(n-1)^2 : n^2 + n + 1$

5.7 Review: exam practice

1. a. 64° **b.** 32° **c.** 116° **d.** 90°

2. a. 90° **b.** 18 units

3. a. 60° **b.** 60°
c. Cyclic quadrilateral **d.** 30°

4. A sample response can be found in the worked solutions in the online resources.

5. Yes; a sample response can be found in the worked solutions in the online resources.

6. $y = 90° - x$

7–10. Sample responses can be found in the worked solutions in the online resources.

11. $\frac{50\pi}{4} - 25 = 14.3\,\text{cm}^2$ (to 1 decimal place)

12. A sample response can be found in the worked solutions in the online resources.

13. 30°

14–17. Sample responses can be found in the worked solutions in the online resources.

18. 30°

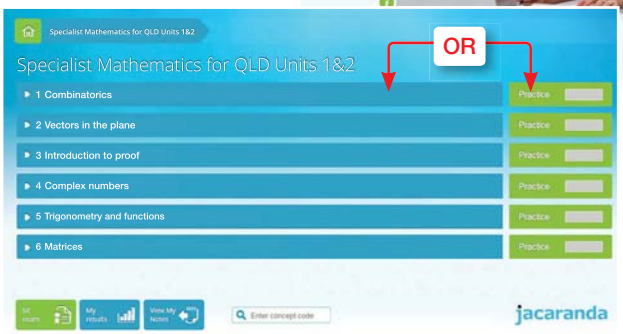
19. 12 units

20. A sample response can be found in the worked solutions in the online resources.

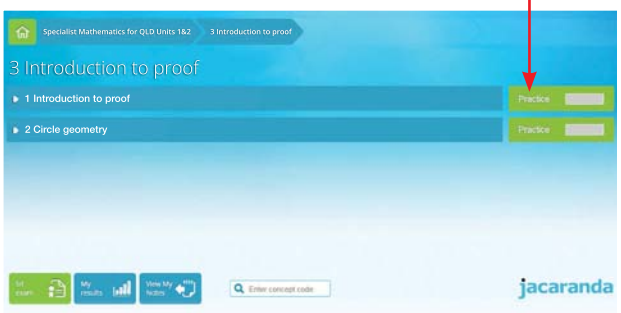
TOPIC 3 Introduction to proof

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

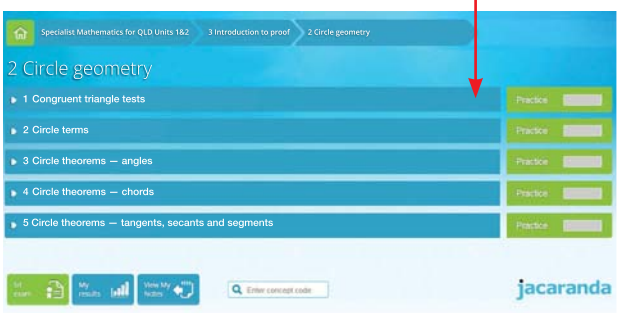
- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



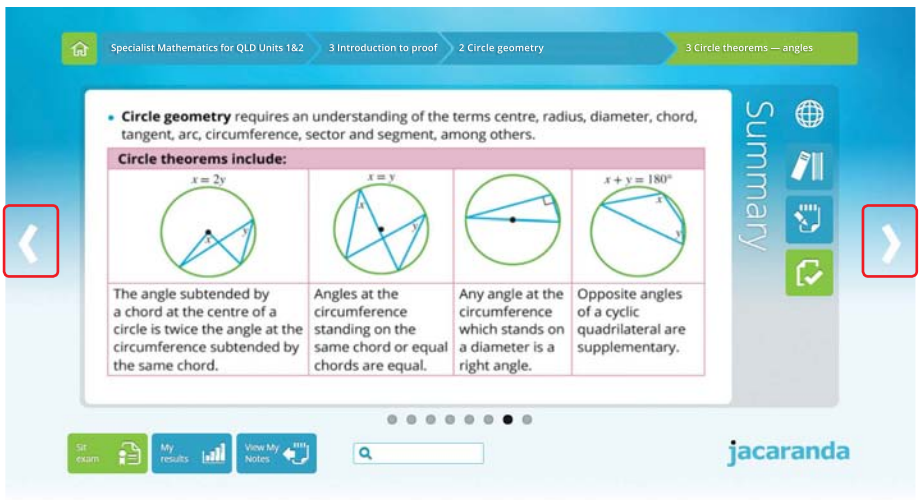
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 2

Specialist Mathematics: Unit 1 examination

Unit

Unit 1: Combinatorics, vectors and proof

Topic

Topic 1: Combinatorics
Topic 2: Vectors in the plane
Topic 3: Introduction to proof

Conditions

Technique	Response Type	Duration	Reading
Technology active	Short response	120 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet • Notes not permitted • Non-CAS graphics calculator permitted		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	40	

* Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Part A: Simple familiar — total marks: 24

Question 1 (6.5 marks)

a. Calculate $6!$

b. Calculate ${}^5P_2$.

c. Calculate ${}^6C_2$.

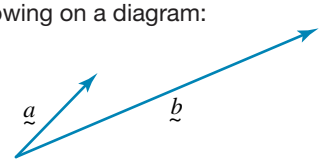
d. Calculate how many different 3-digit numbers can be made with the digits $3, 5, 7, 9$ if the digits cannot be repeated.

e. Calculate the number of teams consisting of 3 people that can be formed from a group of 10 people.

f. Calculate the number of arrangements that are possible with the letters in the word COFFEE.

Question 2 (1 mark)

Use vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to represent the following on a diagram:



a. $\underset{\sim}{b} - \underset{\sim}{a}$

b. $2\underset{\sim}{a}$

Question 3 (4 marks)

A $(-2, -4)$, B $(3, 1)$ and C $(-3, -2)$ are three points on a plane.

a. Determine the displacement vector $\overrightarrow{AB}$ in component form.

b. Determine the displacement vector $\overrightarrow{AB}$ in polar form.

c. Calculate $-2\overrightarrow{BC}$ in component form.

d. Calculate $2\overrightarrow{AB} + 3\overrightarrow{BC}$ in component form.

Question 4 (3 marks)

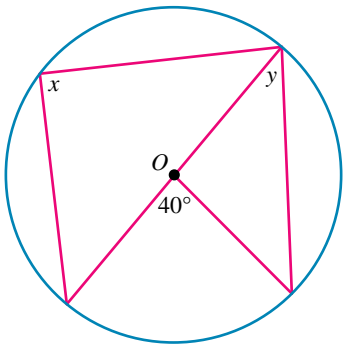
a. Express $\frac{1}{15}$ as a decimal.

b. Express $0.\dot{3}\dot{5}$ as a rational number.

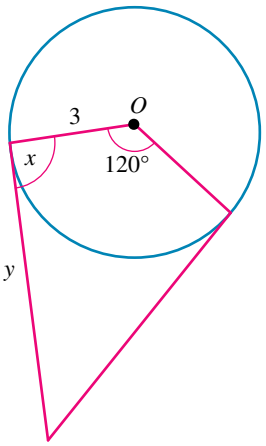
Question 5 (5 marks)

Calculate the values of the following unknowns. Identify the rules and definitions used.

a.



b.



Question 6 (4.5 marks)

Use proof by contradiction to prove that $\sqrt{11}$ is irrational.

Part B: Complex familiar — total marks: 8

Question 7 (3 marks)

Tienna has a bookshelf with places for seven books. She wants to display books from her two favourite topics, horses and Paris. She owns six books on each topic but wants to display more books about Paris.

If she wants to fill the bookshelf, determine the number of ways Tienna can select her books.

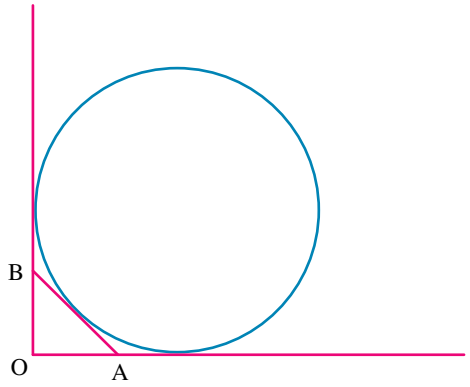
Question 8 (5 marks)

Prove that the sum of the squares of the lengths of a parallelogram's diagonals is equal to the sum of the squares of the lengths of the sides.

Part C: Complex unfamiliar — total marks: 8

Question 9 (5 marks)

A circle is drawn so that it touches the extension of two perpendicular line segments OA and OB as shown. If the circle also touches AB, determine the relationship between the perimeter of ΔAOB and the radius of the circle.



Question 10 (3 marks)

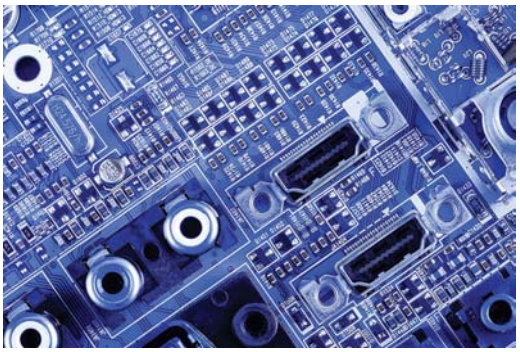
Integers are selected at random from the numbers $1 - 50$ until there is a pair of numbers with a difference that is a multiple of 5. William believes that after 6 numbers have been selected, a suitable pair is guaranteed.

Investigate the validity of William's claim.

CHAPTER 6
Complex numbers

6.1 Overview

6.1.1 Introduction



In 1545, the Italian mathematician Girolamo Cardano proposed what was then a startling mathematical expression:

$$40 = \left(5 + \sqrt{-15}\right)\left(5 - \sqrt{-15}\right)$$

This was a valid expression, yet it included the square root of a negative number, which seemed 'impossible'.

The definition of real numbers included whole numbers, fractions, decimals, irrational and rational numbers as subsets of the real number set. Whenever the square root of a negative number was encountered, it did not fit into any of these subsets and hence could not be classified as a real number. However, the solutions to quadratic equations sometimes force us to consider the square root of a negative number. Consider the two equations $x^2 + 2x + 26 = 0$ and $x^2 - 4x + 29 = 0$. How do the solutions of these equations relate to properties of the associated parabolas?

Up to this point, you have encountered solutions of quadratic equations that were classified as having rational, irrational or no real solutions. The study of complex numbers provides solutions to these previously unsolvable equations.

Why did the square roots of negative numbers become central to the study of a new set of numbers called complex numbers? It was partly curiosity and partly because people such as the Ancient Greek mathematician Diophantus and the 17th-century German mathematician Leibniz had found that real numbers could not solve all equations. Eventually it was shown that complex numbers could solve previously unsolvable problems. They are now used extensively in the fields of physics and engineering in areas such as electrical circuits and electromagnetic waves. Combined with calculus, complex numbers form an important part of the field of mathematics known as complex analysis.

LEARNING SEQUENCE

6.1 Overview
6.2 Introduction to complex numbers
6.3 Basic operations using complex numbers
6.4 Complex conjugates and division of complex numbers
6.5 The complex plane (the Argand plane)
6.6 Complex numbers in polar form
6.7 Basic operations on complex numbers in polar form
6.8 Roots of equations
6.9 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.2 Introduction to complex numbers

6.2.1 Square root of a negative number

The quadratic equation $x^2 + 1 = 0$ has no solutions for x in the Real Number System R because the equation yields $x = \pm\sqrt{-1}$ and there is no real number which, when squared, gives -1 as the result.

If, however, we define an **imaginary number** denoted by i such that:

$$i^2 = -1$$

Then the answer to the quadratic $x^2 + 1 = 0$, $x = \pm\sqrt{-1}$ becomes $x = \pm\sqrt{i^2} = \pm i$

For the general case $x^2 + a^2 = 0$, with $a \in R$, we can write:

$$\begin{aligned} x &= \pm\sqrt{-a^2} \\ &= \pm\sqrt{-1 \times a^2} \\ &= \pm\sqrt{i^2 \times a^2} \\ &= \pm ai \end{aligned}$$

Powers of i will produce $\pm i$ or ± 1. As $i^2 = -1$, if follows:

$$\begin{aligned} i^3 &= i^2 \times i \\ &= -1 \times i \\ &= -i \end{aligned} \qquad \begin{aligned} i^4 &= i^2 \times i^2 \\ &= -1 \times -1 \\ &= 1 \end{aligned} \qquad \begin{aligned} i^5 &= i^2 \times i^3 \\ &= -1 \times -i \\ &= i \end{aligned} \qquad \begin{aligned} i^6 &= (i^2)^3 \\ &= (-1)^3 \\ &= -1 \end{aligned}$$

The pattern is, quite obviously, even powers of i result in 1 or -1 and odd powers of i result in i or $-i$.

6.2.2 Definition of a complex number

A **complex number** (generally denoted by the letter z) is defined as a quantity consisting of a real number added to a multiple of the imaginary unit i. For real numbers a and b, $a + bi$ is a complex number. This is referred to as the standard or **Cartesian form**.

$C = \{z : z = a + bi$ where $a, b \in R\}$ defines the set of complex numbers.
The real part of z is a and is written as Re(z). That is, Re(z) $= a$.
The imaginary part of z is b and is written as Im (z). That is, Im (z) $= b$.

***Note*: Every real number x can be written as $a + 0i$ and so the set of real numbers is a subset of the set of complex numbers. That is, $R \subset C$.**

WORKED EXAMPLE 1

Using the imaginary number i, write a simplified expression for:

a. $\sqrt{-16}$ **b.** $\sqrt{-5}$.

THINK	WRITE
a. 1. Express the square root of -16 as the product of the square root of 16 and the square root of -1.	**a.** $\sqrt{-16} = \sqrt{16} \times \sqrt{-1}$

2. Substitute i^2 for -1. $= \sqrt{16} \times \sqrt{i^2}$
3. Take the square root of 16 and i^2. $= 4i$

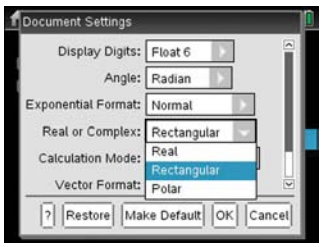
b. 1. Express the square root of -5 as the product of the square root of 5 and the square root of -1. **b.** $\sqrt{-5} = \sqrt{5} \times \sqrt{-1}$

2. Substitute i^2 for -1. $= \sqrt{5} \times \sqrt{i^2}$
3. Simplify. $= i\sqrt{5}$

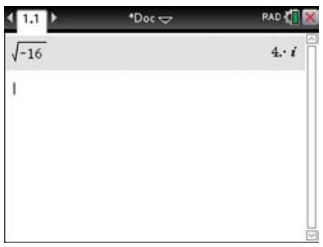
TI | THINK

a.1. In the Document Settings menu, change the Calculation Mode to Rectangular, then select OK.

WRITE



2. On a Calculator page, complete the entry line as: $\sqrt{-16}$ then press ENTER.

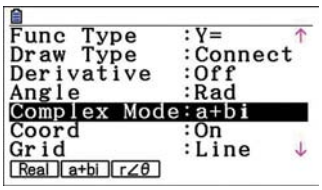


3. The answer appears on the screen. $\sqrt{-16} = 4i$

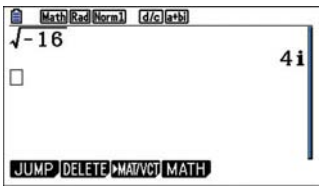
CASIO | THINK

a.1. In the SETCP menu, change the Complex Mode to $a + bi$, then press EXIT.

WRITE



2. On the Run-Matrix screen, complete the entry line as: $\sqrt{-16}$ then press EXE.



3. The answer appears on the screen. $\sqrt{-16} = 4i$

WORKED EXAMPLE 2

Write down the real and imaginary parts of the following complex numbers, z.

a. $z = -3 + 2i$ **b.** $z = -\frac{1}{2}i$

THINK

WRITE

a. 1. The real part is the 'non-i' term. **a.** $\text{Re}(z) = -3$

2. The imaginary part is the coefficient of the i term. $\text{Im}(z) = 2$

b. 1. The real part is the 'non-i' term. **b.** $\text{Re}(z) = 0$

2. The imaginary part is the coefficient of the i term. $\text{Im}(z) = -\frac{1}{2}$

WORKED EXAMPLE 3

Write $i^8 + i^5$ in the form $a + bi$ where a and b are real numbers.

THINK	WRITE
1. Simplify both i^8 and i^5 using the lowest possible power of i, recalling the index laws.	$i^8 = \left(i^2\right)^4 = (-1)^4 = 1$ $i^5 = i^4 \times i = \left(i^2\right)^2 \times i = (-1)^2 \times i = 1 \times i = i$
2. Add the two answers.	$i^8 + i^5 = 1 + i$

WORKED EXAMPLE 4

Simplify $z = i^4 - 2i^2 + 1$ and $w = i^6 - 3i^4 + 3i^2 - 1$ and show that $z + w = -4$.

THINK	WRITE
1. Replace terms with the lowest possible powers of i (remember $i^2 = -1$).	$z = i^4 - 2i^2 + 1$ $= \left(i^2\right)^2 - 2 \times -1 + 1$ $= (-1)^2 + 2 + 1$ $= 4$ $w = i^6 - 3i^4 + 3i^2 - 1$ $= \left(i^2\right)^3 - 3\left(i^2\right)^2 + 3 \times -1 - 1$ $= (-1)^3 - 3(-1)^2 - 3 - 1$ $= -1 - 3 - 3 - 1$ $= -8$
2. Add the two answers.	$z + w = i^4 - 2i^2 + 1 + i^6 - 3i^4 + 3i^2 - 1$ $= 4 - 8$ $= -4$

WORKED EXAMPLE 5

Evaluate each of the following.

a. **Re $(7 + 6i)$**

b. **Im (10)**

c. **Re $\left(2 + i - 3i^3\right)$**

d. **Im $\left(\dfrac{1 - 3i - i^2 - i^3}{2}\right)$**

THINK	WRITE
a. Recall the real part of the complex number $a + bi$ is a, so the real part of $7 + 6i$ is 7.	a. $\text{Re}\,(7 + 6i) = 7$
b. The number 10 can be expressed in complex form as $10 + 0i$ and so the imaginary part is 0.	b. $\text{Im}\,(10) = \text{Im}\,(10 + 0i)$ $= 0$

c. 1. Simplify $2 + i - 3i^3$ to the lowest possible power of i (remember $i^2 = -1$).

2. The real part is 2.

d. 1. Simplify the numerator of $\left(\frac{1 - 3i - i^2 - i^3}{2}\right)$ to the lowest possible value of i.

2. Simplify by dividing the numerator by 2.

3. The imaginary part is -1.

c. $\text{Re}\,(2 + i - 3i^3) = \text{Re}\,(2 + i - 3i \times i^2)$
$= \text{Re}\,(2 + i + 3i)$
$= \text{Re}\,(2 + 4i)$
$= 2$

d. $\text{Im}\left(\frac{1 - 3i - i^2 - i^3}{2}\right) = \text{Im}\left(\frac{1 - 3i + 1 + i}{2}\right)$
$= \text{Im}\left(\frac{2 - 2i}{2}\right)$
$= \text{Im}\frac{2(1 - i)}{2}$
$= \text{Im}\,(1 - i)$
$= -1$

TI | THINK

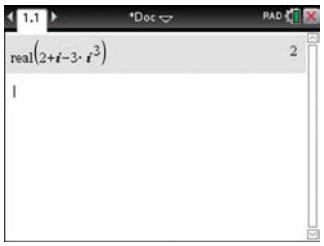
c.1. On a Calculator page, press MENU, then select:
2: Number
9: Complex Number Tools
2: Real Part.
Complete the entry line as:
real $(2 + i - 3i^3)$
then press ENTER.
Note: The symbol i can be found by pressing the π button.

2. The answer appears on the screen.

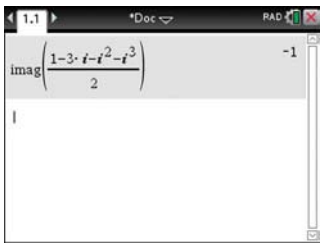
d.1. On a Calculator page, press MENU, then select:
2: Number
9: Complex Number Tools
3: Imaginary Part.
Complete the entry line as:
imag $\left(\frac{1 - 3i - i^2 - i^3}{2}\right)$
then press ENTER.

2. The answer appears on the screen.

WRITE



$\text{Re}\,(2 + i - 3i^3) = 2$



$\text{Im}\left(\frac{1 - 3i - i^2 - i^3}{2}\right) = -1$

CASIO | THINK

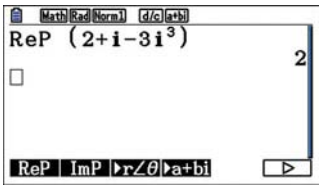
c.1. On a Run-Matrix screen, press OPTN, then select COMPLEX by pressing F3. Press F6 to scroll across to more menu options, then select ReP by pressing F1.
Complete the entry line as:
ReP $(2 + i - 3i^3)$
then press EXE.
Note: The symbol i can be found by pressing SHIFT 0.

2. The answer appears on the screen.

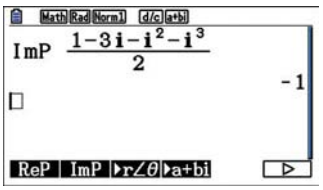
d.1. On a Run-Matrix screen, press OPTN, then select COMPLEX by pressing F3. Press F6 to scroll across to more menu options, then select ImP by pressing F2.
Complete the entry line as:
ImP $\frac{1 - 3i - i^2 - i^3}{2}$
then press EXE.

2. The answer appears on the screen.

WRITE



$\text{Re}\,(2 + i - 3i^3) = 2$



$\text{Im}\left(\frac{1 - 3i - i^2 - i^3}{2}\right) = -1$

Exercise 6.2 Introduction to complex numbers

Technology free

1. **WE1** Using the imaginary number i, write down expressions for:
   a. $\sqrt{-9}$ b. $\sqrt{-25}$ c. $\sqrt{-49}$ d. $\sqrt{-3}$
   e. $\sqrt{-11}$ f. $\sqrt{-7}$ g. $\sqrt{-\frac{4}{9}}$ h. $\sqrt{-\frac{36}{25}}$
2. **WE2** Write down the real and imaginary parts, respectively, of the following complex numbers, z.
   a. $9+5i$ b. $5-4i$ c. $-3-8i$ d. $11i-6$
   e. 27 f. $2i$ g. $-5+i$ h. $-17i$
3. **WE3** Write each of the following in the form $a+bi$, where a and b are real numbers.
   a. i^9+i^{10} b. i^9-i^{10} c. $i^{12}+i^{15}$ d. i^7-i^{11}
   e. $i^5+i^6-i^7$ f. $i\left(i^{13}+i^{16}\right)$ g. $2i-i^2+2i^3$ h. $3i+i^4-5i^5$
4. **WE4** Simplify $z=i^6+3i^7-2i^{10}-3$ and $w=4i^8-3i^{11}+3$ and show that $z+w=5$.
5. **WE5** Evaluate each of the following.
   a. $\text{Re}(-5+4i)$
   b. $\text{Re}(15-8i)$
   c. $\text{Re}(12i)$
   d. $\text{Im}(1-6i)$
   e. $\text{Im}(3+2i)$
   f. $\text{Im}(8)$
   g. $\text{Re}(i^5-3i^4+6i^6)$
   h. $\text{Im}\left(\dfrac{4i^9-5i^{14}-2i^7}{3}\right)$

6. Write $3-\dfrac{i^3-i+2}{i^2-i^4}$ in the form $a+bi$, where a and b are real numbers.
7. **MC** a. The value of $\text{Re}\left(i+i^3+i^5\right)$ is:
   A. -1 **B.** 3 **C.** 1 **D.** 0
   b. The value of $\text{Im}\left[i\left(2i^4-3i^2+5i\right)\right]$ is:
   A. 0 **B.** -5 **C.** 5 **D.** 10
   c. The expression $i+i^2-i^3+i^4-i^5+i^6$ simplifies to:
   A. i **B.** 0 **C.** $i-1$ **D.** $i+2$
   d. If $f(i)=\dfrac{1+i+i^2+\ldots+i^{11}}{4}$ which one of the statements below is true?
   A. $f(i)=2+i$ **B.** $\text{Re}\left[f(i)\right]=5$
   C. $\text{Im}\left[f(i)\right]=-\frac{1}{4}$ **D.** $f(i)=0$
8. If n is an even natural number, show that $(-1)^{\frac{n}{2}}=i^n$.
9. Simplify each of the following.
   a. $\dfrac{6i^3}{\sqrt{-9}}$ b. $\dfrac{20i^4}{\sqrt{-100}}$ c. $\dfrac{10i^5}{\sqrt{-50}}$ d. $\dfrac{8i^6}{\sqrt{-16}}$
10. Evaluate the following.
    a. $\text{Re}\left(3(4-6i)+i^8\right)$ b. $\text{Im}\left(2(2-5i)+i^7\right)$
    c. $\text{Re}\left(3(2-5i)-4i^{10}\right)$ d. $\text{Im}\left(4(2+3i)-5i^9\right)$

11. If $f(n) = 1 + i + i^2 + i^3 + i^4 + ... + i^n$, evaluate the following.

a. $f(6)$ **b.** $f(7)$

c. $f(2014)$ **d.** $f(2015)$

12. If $g(n) = 1 - i + i^2 - i^3 + i^4 + ... + (-i)^n$, evaluate the following.

a. $g(6)$ **b.** $g(7)$

c. $g(2014)$ **d.** $g(2015)$

6.3 Basic operations using complex numbers

6.3.1 Complex number arithmetic

Complex numbers can be added, subtracted, multiplied and divided. In general, the solutions obtained when performing these operations are presented in the standard form $z = a + bi$.

Addition of complex numbers

Addition is performed by adding the real and imaginary parts separately.

If $z = m + ni$ and $w = p + qi$, then $z + w = (m + p) + (n + q)i$
where $\text{Re}(z + w) = \text{Re}(z) + \text{Re}(w)$ and $\text{Im}(z + w) = \text{Im}(z) + \text{Im}(w)$.

Subtraction of complex numbers

If we write $z - w$ as $z + -w$, we can use the rule for addition of complex numbers to obtain:

$$\begin{aligned} z + -w &= (m + ni) + -(p + qi) \\ &= m + ni - p - qi \\ &= (m - p) + (n - q)i \end{aligned}$$

If $z = m + ni$ and $w = p + qi$, then $z - w = (m - p) + (n - q)i$.

WORKED EXAMPLE 6

For $z = 8 + 7i$, $w = -12 + 5i$ and $u = 1 + 2i$, calculate:
a. $z + w$ **b.** $w - z$ **c.** $u - w + z$.

THINK	WRITE
a. Use the addition rule for complex numbers.	**a.** $z + w = (8 + 7i) + (-12 + 5i)$ $= (8 - 12) + (7 + 5)i$ $= -4 + 12i$
b. Use the subtraction rule for complex numbers.	**b.** $w - z = (-12 + 5i) - (8 + 7i)$ $= (-12 - 8) + (5 - 7)i$ $= -20 - 2i$
c. Use both the addition rule and the subtraction rule.	**c.** $u - w + z = (1 + 2i) - (-12 + 5i) + (8 + 7i)$ $= (1 + 12 + 8) + (2 - 5 + 7)i$ $= 21 + 4i$

Multiplication by a constant (or scalar)

Consider part **a** in Worked example 6, where $z + w = -4 + 12i$. The real and imaginary parts share a common factor of 4. Hence, the equation could be rewritten as:

$$\begin{aligned} z + w &= -4 + 12i \\ &= 4(-1 + 3i) \end{aligned}$$

This is the equivalent of the complex number $-1 + 3i$ multiplied by a constant (or scalar), 4.
If we consider the general form $z = a + bi$ and $k \in R$, then:

$$\begin{aligned} kz &= k(a + bi) \\ &= ka + kbi \end{aligned}$$

WORKED EXAMPLE 7

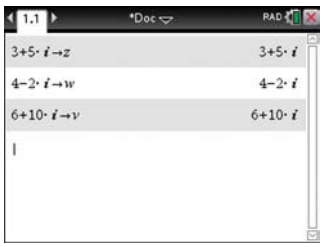
If $z = 3 + 5i$, $w = 4 - 2i$ and $v = 6 + 10i$, evaluate:
a. $3z + w$ **b.** $2z - v$ **c.** $4z - 3w + 2v$.

THINK	WRITE
a. 1. Calculate $3z + w$ by substituting values for z and w.	**a.** $3z + w = 3(3 + 5i) + (4 - 2i)$
2. Use the rule for adding complex numbers.	$= (9 + 15i) + (4 - 2i)$ $= (9 + 4) + (15 - 2)i$ $= 13 + 13i$ (or $13(1 + i)$)
b. 1. Calculate $2z - v$ by substituting values for z and v.	**b.** $2z - v = 2(3 + 5i) - (6 + 10i)$
2. Use the rule for subtraction of complex numbers.	$= 6 + 10i - 6 - 10i$ $= 0 + 0i$ $= 0$
c. 1. Calculate $4z - 3w + 2v$ by substituting values for z, w and v.	**c.** $4z - 3w + 2v = 4(3 + 5i) - 3(4 - 2i) + 2(6 + 10i)$
2. Use the addition rule and the subtraction rule to simplify.	$= 12 + 20i - 12 + 6i + 12 + 20i$ $= 12 + 46i$

TI | THINK

c.1. On a Calculator page, complete the entry line as:
$3 + 5i$
Press CTRL, then press VAR, then type 'z' and press ENTER to store $z = 3 + 5i$.
Repeat this step to store the complex numbers w and v.
Note: The symbol i can be found by pressing the π button.

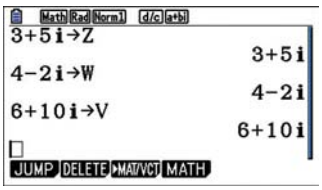
WRITE



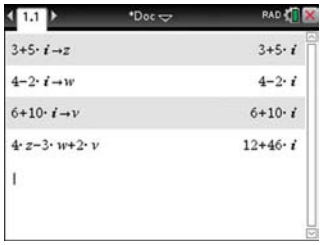
CASIO | THINK

c.1. On a Run-Matrix screen, complete the entry line as:
$3 + 5i$
Press the store button, then type 'Z' and press EXE to store $z = 3 + 5i$.
Repeat this step to store the complex numbers w and v.
Note: The symbol i can be found by pressing SHIFT 0.

WRITE



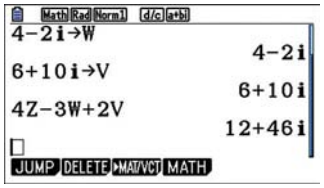
2. Complete the next entry line as:
$4z - 3w + 2v$
then press ENTER.



3. The answer appears on the screen.

$4z - 3w + 2v = 12 + 46i$

2. Complete the next entry line as:
$4Z - 3W + 2V$
then press EXE.



3. The answer appears on the screen.

$4z - 3w + 2v = 12 + 46i$

Multiplication of two complex numbers

The multiplication of two complex numbers also results in a complex number.

If $z = m + ni$ and $w = p + qi$,
then $z \times w = (m + ni)(p + qi)$
$= mp + mqi + npi + nqi^2$
$= (mp - nq) + (mq + np)i$ (since $i^2 = -1$).

If $z = m + ni$ and $w = p + qi$, then $z \times w = (mp - nq) + (mq + np)i$.

WORKED EXAMPLE 8

If $z = 6 - 2i$ and $w = 3 + 4i$, express zw in standard form.

THINK

1. Expand the brackets.
2. Express in the form $a + bi$ by substituting -1 for i^2 and simplifying the expression using the addition and subtraction rules.

WRITE

$zw = (6 - 2i)(3 + 4i)$
$= 18 + 24i - 6i - 8i^2$
$= 18 + 24i - 6i + 8$
$= 26 + 18i$

WORKED EXAMPLE 9

Simplify $(2 - 3i)(2 + 3i)$.

THINK

1. Expand the brackets.
2. Substitute -1 for i^2 and simplify the expression.

WRITE

$(2 - 3i)(2 + 3i) = 4 + 6i - 6i - 9i^2$
$= 4 - 9 \times -1$
$= 13$

WORKED EXAMPLE 10

Determine $\text{Re}(z^2w) + \text{Im}(zw^2)$ for $z = 4 + i$ and $w = 3 - i$.

THINK	WRITE
1. Express z^2w in the form $a + bi$.	$z^2w = (4 + i)^2(3 - i)$ $= (16 + 8i + i^2)(3 - i)$ $= (16 + 8i - 1)(3 - i)$ $= (15 + 8i)(3 - i)$ $= 45 - 15i + 24i - 8i^2$ $= 53 + 9i$
2. The real part, $\text{Re}\left(z^2w\right)$, is 53.	$\text{Re}\left(z^2w\right) = 53$
3. Express zw^2 in the form $a + bi$.	$zw^2 = (4 + i)(3 - i)^2$ $= (4 + i)(9 - 6i + i^2)$ $= (4 + i)(8 - 6i)$ $= 32 - 24i + 8i - 6i^2$ $= 38 - 16i$
4. The imaginary part, $\text{Im}\left(zw^2\right)$, is -16.	$\text{Im}\left(zw^2\right) = -16$
5. Calculate the value of $\text{Re}\left(z^2w\right) + \text{Im}\left(zw^2\right)$.	$\text{Re}\left(z^2w\right) + \text{Im}\left(zw^2\right) = 53 - 16$ $= 37$

6.3.2 Equality of two complex numbers

If $z = m + ni$ and $w = p + qi$, then $z = w$ if and only if $m = p$ and $n = q$.

The condition 'if and only if' (sometimes written in short form as **iff**) means that both of the following situations must apply.

- If $z = w$, then $m = p$ and $n = q$.
- If $m = p$, and $n = q$ then $z = w$.

WORKED EXAMPLE 11

Find the values of x and y that satisfy $(3 + 4i)(x + yi) = 29 + 22i$.

THINK	WRITE
1. Write the left-hand side of the equation.	LHS $= (3 + 4i)(x + yi)$
2. Expand the left-hand side of the equation.	$= 3x + 3yi + 4xi + 4yi^2$
3. Express the left-hand side in the form $a + bi$.	$= (3x - 4y) + (4x + 3y)i$
4. Equate the real parts and imaginary parts of both sides of the equation to create a pair of simultaneous equations.	$3x - 4y = 29$ [1] $4x + 3y = 22$ [2]

5. Simultaneously solve [1] and [2] for x and y. Multiply equation [1] by 3 and equation [2] by 4 so that y can be eliminated.	$9x - 12y = 87$ [3] $16x + 12y = 88$ [4]
6. Add the two new equations and solve for x.	Adding equations [3] and [4]: $25x = 175$ $x = 7$
7. Substitute $x = 7$ into equation [1] and solve for y.	Substituting $x = 7$ into equation [1]: $3(7) - 4y = 29$ $21 - 4y = 29$ $-4y = 8$ $y = -2$
8. State the solution.	Therefore, $x = 7$ and $y = -2$.
9. Check the solution by substituting these values into equation [2].	Check: $4 \times 7 + 3 \times -2 = 22$.

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Units 1 & 2 > Area 4 > Sequence 1 > Concepts 1 & 3

Complex numbers Summary screen and practice questions
Operations using complex numbers Summary screen and practice questions

Exercise 6.3 Basic operations using complex numbers

Technology free

1. WE6 For $z = 5 + 3i$, $w = -1 - 4i$, $u = 6 - 11i$ and $v = 2i - 3$, calculate:
a. $z + w$ **b.** $u - z$ **c.** $w + v$ **d.** $u - v$
e. $w - z - u$ **f.** $v + w - z$

2. WE7 If $z = -3 + 2i$, $w = -4 + i$ and $u = -8 - 5i$, evaluate:
a. $3w$ **b.** $2u + z$ **c.** $4z - 3u$ **d.** $3z + u + 2w$
e. $2z - 7w + 9u$ **f.** $3(z + 2u) - 4w$

3. WE8 Using z, w, u and v from question **2**, express each of the following in the standard form $a + bi$.
a. zw **b.** uv **c.** wu **d.** zu
e. u^2 **f.** $u(wv)$

4. WE9 Simplify the following.
a. $(10 + 7i)(9 - 3i)$ **b.** $(3 - 4i)(5 + 4i)$
c. $(8 - 2i)(4 - 5i)$ **d.** $(5 + 6i)(5 - 6i)$
e. $(2i - 7)(2i + 7)$ **f.** $(9 - 7i)^2$

5. For $z = -1 - 3i$ and $w = 2 - 5i$, calculate z^2w.

6. WE10 Determine $\text{Re}\left(z^2\right) - \text{Im}(zw)$ for $z = 1 + i$ and $w = 4 - i$.

Technology active

7. For $z = 3 + 5i$, $w = 2 - 3i$ and $u = 1 - 4i$, determine:
   a. $\text{Im}\left(u^2\right)$
   b. $\text{Re}\left(w^2\right)$
   c. $\text{Re}(uw) + \text{Im}(zw)$
   d. $\text{Re}(zu) - \text{Im}\left(w^2\right)$
   e. $\text{Re}\left(z^2\right) - \text{Re}(zw) - \text{Im}(uz)$
   f. $\text{Re}\left(u^2w\right) + \text{Im}\left(zw^2\right)$
8. **WE11** Find the values of a and b that satisfy each of the following.
   a. $(2 + 3i)(a + bi) = 16 + 11i$
   b. $(5 - 4i)(a + bi) = 1 - 4i$
   c. $(3i - 8)(a + bi) = -23 - 37i$
   d. $(7 + 6i)(a + bi) = 4 - 33i$
9. **MC** If $z = 8 - 7i$ and $w = 3 + 4i$, then:
   a. $\text{Re}(zw)$ is equal to:
      A. -4 **B.** 5 **C.** 11 **D.** 52
   b. $\text{Im}\left(w^2\right) + \text{Re}\left(z^2\right)$ is equal to:
      A. 76 **B.** 39 **C.** 105 **D.** 56
   c. $3z - 2w$ is equal to:
      A. $30 - 13i$ **B.** $30 - 29i$ **C.** $18 - 29i$ **D.** 18
10. Let $z_1 = 2 + 5i$, $z_2 = 3 - 2i$ and $z_3 = -18 - 7i$.
    a. Verify the associative law, that is, $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.
    b. Evaluate $5z_1 - 2z_2 + 3z_3$.
    c. Evaluate $\text{Re}(2z_1 - 3z_2 + 4z_3) + \text{Im}(2z_1 - 3z_2 + 4z_3)$.
    d. Evaluate the real numbers x and y if $x z_1 + y z_2 = z_3$.
11. Evaluate the values of the real numbers x and y if:
    a. $x(3 - 2i) + y(4 + 5i) = 23$
    b. $x(3 - 2i) - y(4 - 7i) = 13i$
    c. $x(4 - 3i) + y(3 - 4i) = 14 - 7i$
    d. $x(3 - 5i) + y(4 + 7i) + 7 + 43i = 0$

6.4 Complex conjugates and division of complex numbers

6.4.1 The conjugate of a complex number

The **conjugate of a complex number** is obtained by changing the sign of the imaginary component.

If $z = a + bi$, the conjugate $\bar{z}$ of z is defined as $\bar{z} = a - bi$.

Conjugates are useful because the multiplication (or addition) of a complex number and its conjugate results in a real number.

Multiplication of a complex number and its conjugate:

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 + b^2 \end{aligned}$$

where $a,\ b \in R$, and $a - bi$ and $a + bi$ are conjugates. You will use this result when dividing complex numbers.

Note: Compare this expression with the formula for the difference of two squares:

$$(m - n)(m + n) = m^2 - n^2$$

Addition of a complex number and its conjugate:

$$z + \overline{z} = a + bi + a - bi$$
$$= 2a$$

WORKED EXAMPLE 12

Write the conjugate of each of the following complex numbers.

a. $8 + 5i$ **b.** $-2 - 3i$ **c.** $4 + i\sqrt{5}$

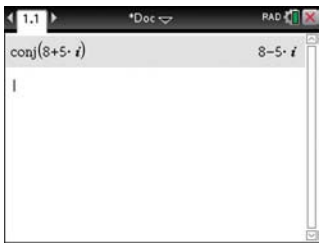
THINK	WRITE
a. Change the sign of the imaginary component.	**a.** $8 - 5i$
b. Change the sign of the imaginary component.	**b.** $-2 + 3i$
c. Change the sign of the imaginary component.	**c.** $4 - i\sqrt{5}$

TI | THINK

a.1. On a Calculator page, press MENU, then select:
2: Number
9: Complex Number Tools
1: Complex Conjugate.
Complete the entry line as:
conj $(8 + 5i)$
then press ENTER.
Note: The symbol i can be found by pressing the π button.

2. The answer appears on the screen.

WRITE



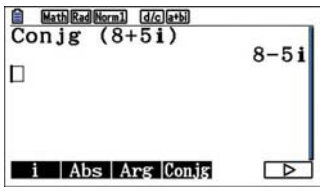
$8 - 5i$

CASIO | THINK

a.1. On a Run-Matrix screen, press OPTN, then select COMPLEX by pressing F3. Select Conjg by pressing F4.
Complete the entry line as:
Conjg $(8 + 5i)$
then press EXE.
Note: The symbol i can be found by pressing SHIFT 0.

2. The answer appears on the screen.

WRITE



$8 - 5i$

WORKED EXAMPLE 13

If $z = 5 - 2i$ and $w = 7 - i$, show that $\overline{z + w} = \overline{z} + \overline{w}$.

THINK	WRITE
1. Add the conjugates $\overline{z}$ and $\overline{w}$.	$\overline{z} + \overline{w} = (5 + 2i) + (7 + i) = 12 + 3i$
2. Add z to w.	$z + w = (5 - 2i) + (7 - i) = 12 - 3i$
3. Write down the conjugate of $z + w$.	$\overline{z + w} = 12 + 3i$
4. The conjugate of $z + w$ equals the sum of the individual conjugates.	$\overline{z + w} = \overline{z} + \overline{w}$

6.4.2 Division of complex numbers

The application of conjugates to division of complex numbers will now be investigated. Consider the complex numbers $z = a + bi$ and $w = c + di$. To find $\frac{z}{w}$ in the form $x + yi$ we must multiply both the numerator and denominator by the conjugate of w to make the denominator a real number only.

$$\frac{z}{w} = \frac{a+bi}{c+di}$$

$$= \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$ Multiply by the conjugate of $c + di$.

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$ Simplify the expressions in the numerator and in the denominator.

$$= \frac{ac+bd}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$ Express in the form $x + yi$.

Thus we can state:

If z and w are complex numbers in the form $a + bi$, then $\frac{z}{w}$ can also be expressed in the form $a + bi$ by simplifying:

$$\frac{z}{w} \times \frac{\overline{w}}{\overline{w}}$$

WORKED EXAMPLE 14

Express $\frac{2+i}{2-i}$ in the form $a + bi$.

THINK

1. Multiply both the numerator and denominator by the conjugate of $2 - i$ to make the denominator real.
2. Expand the expressions obtained in the numerator and denominator.
3. Substitute -1 for i^2 and simplify the expression.

WRITE

$$\frac{2+i}{2-i} = \frac{(2+i)}{(2-i)} \times \frac{(2+i)}{(2+i)}$$

$$= \frac{4+4i+i^2}{4-i^2}$$

$$= \frac{4+4i-1}{4+1}$$

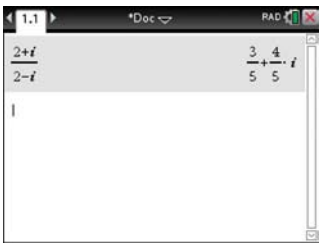
$$= \frac{3+4i}{5}$$

$$= \frac{3}{5} + \frac{4i}{5}$$

TI | THINK

1. On a Calculator page, complete the entry line as:
   $\frac{2+i}{2-i}$
   then press ENTER.
   Note: The symbol i can be found by pressing the π button.
2. The answer appears on the screen.

WRITE

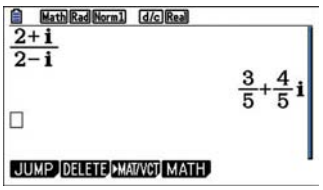


$$\frac{2+i}{2-i} = \frac{3}{5} + \frac{4}{5}i$$

CASIO | THINK

1. On a Run-Matrix screen, complete the entry line as:
   $\frac{2+i}{2-i}$
   then press EXE.
   Note: The symbol i can be found by pressing SHIFT 0.
2. The answer appears on the screen.

WRITE



$$\frac{2+i}{2-i} = \frac{3}{5} + \frac{4}{5}i$$

6.4.3 Multiplicative inverse of a complex number

Given a non-zero complex number z, there exists a complex number w such that $zw = 1$, with w being the **multiplicative inverse** of z denoted by $w = z^{-1} = \frac{1}{z}$. Hence, $zz^{-1} = 1$.

WORKED EXAMPLE 15

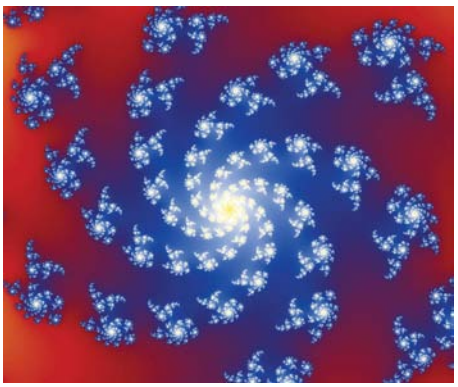
If $z = 3 + 4i$, determine z^{-1}.

THINK	WRITE
1. Write z^{-1} as a rational expression: $z^{-1} = \frac{1}{z}$.	$z^{-1} = \frac{1}{z} = \frac{1}{3+4i}$
2. Multiply both the numerator and denominator by the conjugate of $3 + 4i$.	$= \frac{1}{(3+4i)} \times \frac{(3-4i)}{(3-4i)}$ $= \frac{3-4i}{25}$
3. Write the expression in the form $a + bi$.	$= \frac{3}{25} - \frac{4i}{25}$

In general terms, the multiplicative inverse of a complex number can be written in standard (Cartesian) form, $a + bi$.

$$\textbf{If } z = a + bi, \textbf{ then } z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Complex numbers can be used to generate fractal patterns such as the Julia set shown.



WORKED EXAMPLE 16

If $z = 3 + i$ and $w = \frac{2}{4 - i}$, determine Im $(4z - w)$.

THINK	WRITE
1. Substitute for z and w in $4z - w$.	$4z - w = 4(3 + i) - \frac{2}{4 - i}$

2. Express $4z - w$ with a common denominator.

$$= \frac{4(3+i)(4-i)-2}{4-i}$$
$$= \frac{4(13+i)-2}{4-i}$$
$$= \frac{50+4i}{4-i}$$

3. Remove i from the denominator by multiplying the numerator and denominator by the conjugate of $4 - i$.

$$= \frac{(50+4i)}{(4-i)} \times \frac{(4+i)}{(4+i)}$$
$$= \frac{196+66i}{17}$$

4. Simplify the expression so that it is in the form $a + bi$.

$$= \frac{196}{17} + \frac{66i}{17}$$

5. State the imaginary component of $4z - w$.

$$\text{Im}(4z - w) = \frac{66}{17}$$

WORKED EXAMPLE 17

Prove that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

THINK	WRITE
1. When asked to 'prove', you should not use actual values for the pronumerals. Use general values of z_1, z_2, $\bar{z}_1$ and $\bar{z}_2$.	Let $z_1 = a + bi$. $\bar{z}_1 = a - bi$ Let $z_2 = c + di$. $\bar{z}_2 = c - di$
2. Generally, in a proof, do not work both sides of the equation at once. Calculate the LHS first.	LHS $= \overline{(a+bi) \times (c+di)}$ $= \overline{ac + adi + bci + bdi^2}$ $= \overline{(ac - bd) + (ad + bc)i}$ $= (ac - bd) - (ad + bc)i$
3. Calculate the RHS and show that it equals the LHS.	RHS $= (a - bi)(c - di)$ $= ac - adi - bci + bdi^2$ $= (ac - bd) - (ad + bc)i$ $=$ LHS Hence, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

on Resources

Digital document: SkillSHEET Complex numbers and their conjugates (doc-26834)

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Units 1 & 2 > Area 4 > Sequence 1 > Concept 4

Complex conjugates and division of complex numbers Summary screen and practice questions

Exercise 6.4 Complex conjugates and division of complex numbers

Technology free

1. **WE12** Write down the conjugate of each of the following complex numbers.
   a. $7 + 10i$ b. $5 - 9i$ c. $3 + 12i$
   d. $\sqrt{7} - 3i$ e. $2i + 5$ f. $-6 - \sqrt{11}\,i$
2. **WE13** If $z = 6 + 3i$ and $w = 3 - 4i$, show that $\overline{z - w} = \overline{z} - \overline{w}$.
3. **WE14** Express $\dfrac{2 + i}{3 - i}$ in the form $a + bi$.
4. Express each of the following in the form $a + bi$.
   a. $\dfrac{1 - i}{1 + i}$ b. $\dfrac{3 - 2i}{2 + 3i}$ c. $\dfrac{2 + 5i}{4 - 3i}$
   d. $\dfrac{4 - 3i}{5 + 2i}$ e. $\dfrac{4 - 5i}{2 - 7i}$ f. $\dfrac{2 + \sqrt{3}\,i}{\sqrt{5} - \sqrt{2}\,i}$
5. **WE15** Determine z^{-1} if z is equal to:
   a. $2 - i$ b. $3 + i$ c. $4 - 3i$
   d. $5 + 4i$ e. $2i - 3$ f. $\sqrt{3} - i\sqrt{2}$
6. If $676z = 10 - 24i$, express z^{-1} in the form $a + bi$.
7. **WE16** If $z = 2 - i$ and $w = \dfrac{1}{3 + i}$ determine each of the following:
   a. $\text{Re}(z + w)$ b. $\text{Im}(w - z)$ c. $\text{Re}(z^{-1} + w^{-1})$ d. $\text{Im}(3z + 2w)$
   e. $\text{Re}(4w - 2z)$

Technology active

8. Write $\dfrac{2 + i}{1 + i} + \dfrac{9 - 2i}{2 - i} + \dfrac{7 + i}{1 - i}$ in the form $a + bi$.
9. Simplify $\dfrac{(2 + 5i)^2(5i - 2)}{3(4 + 7i) - 2(5 + 8i)}$.
10. Determine the conjugate of $(5 - 6i)(3 - 8i)$.
11. **MC** If $z = 5 - 12i$, $w = -9 - i$ and $u = 15 - 6i$, then:
    a. $\text{Re}(z^{-1})$ is:
       A. 5 **B.** 12
       C. $\frac{12}{169}$ **D.** $\frac{5}{169}$
    b. $\text{Im}(z\overline{w})$ is equal to:
       A. -33 **B.** 103
       C. 113 **D.** 70
    c. The expression $\overline{2z - w + 3u}$ is equal to:
       A. $26 - 7i$ **B.** $64 + 41i$
       C. $46 - 29i$ **D.** $34 - 41i$

12. If $z = 6 + 8i$ and $w = 10 - 3i$:
    a. show that $\overline{zw} = \overline{z} \times \overline{w}$
    b. **WE17** prove the result by letting $z = a + bi$ and $w = c + di$.
13. Use the result $\overline{zw} = \overline{z} \times \overline{w}$ to prove that $\overline{z}^n = (\overline{z})^n$.

14. If $z = 4 + i$ and $w = 1 + 3i$:

a. show that $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

b. prove the result by letting $z = a + bi$ and $w = c + di$.

15. If $z = -5 - 4i$ and $w = 2i$, calculate Re $\left(z\overline{w} + \overline{z}w\right)$.

16. If $z_1 = 2 + 3i$, $z_2 = -4 - i$ and $z_3 = 5 - i$, calculate:

a. $2z_1 - z_2 - 4z_3$ **b.** $z_1\overline{z}_2 + z_2\overline{z}_3$ **c.** $\overline{z_1 z_2 z_3} - \overline{z_1}\,\overline{z_2}\,\overline{z_3}$

17. If $z_1 = a + bi$ and $z_2 = c + di$, show that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$.

18. **a.** If $z = 1 + i$, find z^4, z^8 and z^{12}.

b. Deduce from your results in **a** that $z^{4n} = (2i)^{2n}$, $n \in N$.

19. If $z = a + bi$, find the values of a and b such that $\frac{z-1}{z+1} = z + 2$.

20. Determine values for a and b so that $z = a + bi$ satisfies $\frac{z+i}{z+2} = i$.

21. If $z = a + bi$, determine the values of a and b such that $z = \sqrt{3 + 4i}$.

22. If $z = 2 - 3i$ and $w = 1 - 2i$:

a. calculate:

i. $z\overline{z}$ **ii.** $w\overline{w}$

b. show that:

i. $\overline{z + w} = \overline{z} + \overline{w}$ **ii.** $\overline{zw} = \overline{z} \times \overline{w}$ **iii.** $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

c. calculate:

i. $\overline{\left(\frac{1}{z}\right)}$ **ii.** $\overline{\left(\frac{1}{w}\right)}$

d. calculate $\overline{z^2 + w^2}$

e. calculate $\overline{z + zw}$

f. calculate $z^{-1}w^{-1}$.

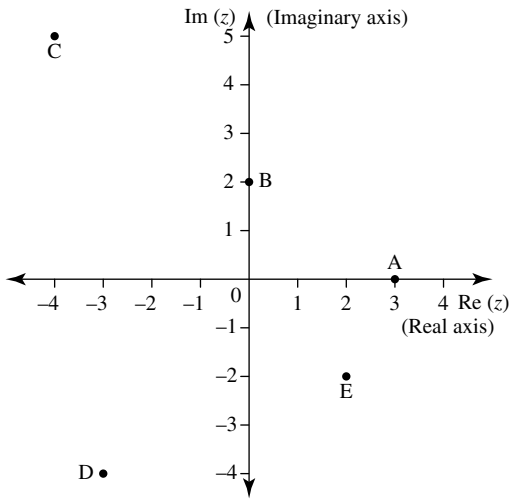
6.5 The complex plane (the Argand plane)

6.5.1 Plotting numbers in the complex plane

Thus far we have represented complex numbers in the standard form $z = a + bi$. This is also referred to as the Cartesian form. We know from previous studies that an ordered pair of real numbers (x, y) can be represented on the Cartesian plane. Similarly, we can consider the complex number $z = a + bi$ as $z = x + yi$, consisting of the ordered pair of real numbers (x, y), which can be plotted as a point (x, y) on the complex number plane.

This is also referred to as the **Argand plane** or an **Argand diagram** in recognition of the work done in this area by the Swiss mathematician Jean-Robert Argand.

The horizontal axis is referred to as the real axis and the vertical axis is referred to as the imaginary axis.



For example, the complex numbers $3 + 0i$ (A), $0 + 2i$ (B), $-4 + 5i$ (C), $-3 - 4i$ (D) and $2 - 2i$ (E) are shown on the Argand diagram below.

Note: The imaginary axis is labelled 1, 2, 3 … etc. (not i, $2i$, $3i$ …).

6.5.2 Geometrically multiplying a complex number by a scalar

WORKED EXAMPLE 18

Given the complex number $z = 1 + 2i$, represent the complex numbers z and $2z$ on the same Argand diagram. Comment on their relative positions.

THINK

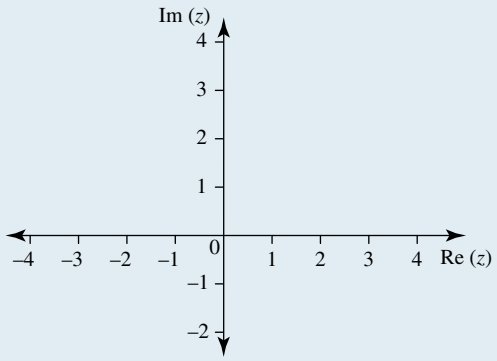
1. Draw an Argand plane.

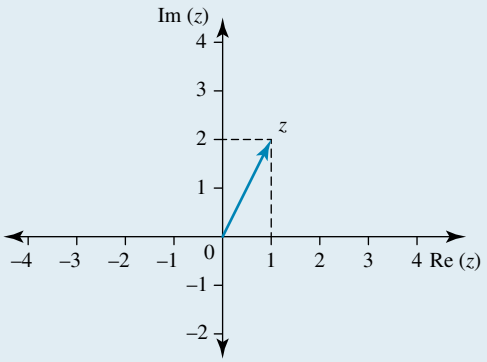
2. The complex number $z = 1 + 2i$ is of the form $z = x + yi$.
   It can be plotted on the Argand plane at $\text{Re}(z) = 1$ and $\text{Im}(z) = 2$

3. The complex number z has been multiplied by a scalar of 2. Calculate the complex number $2z$ from $z = 1 + 2i$.

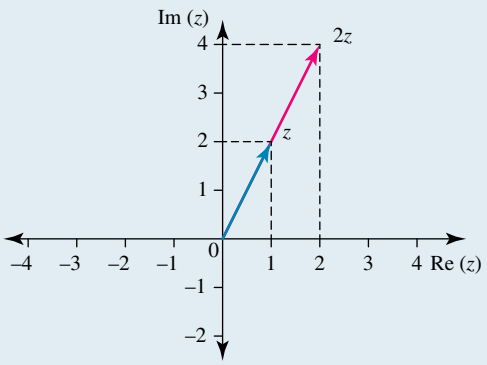
4. The complex number $2z = 2 + 4i$ is of the form $z = x + yi$.
   It can be plotted on the Argand plane at $\text{Re}(z) = 2$ and $\text{Im}(z) = 4$.

WRITE



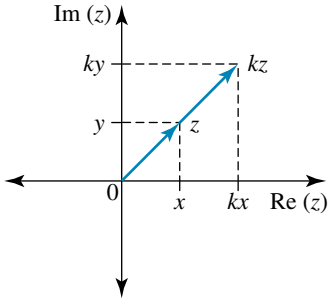


$z = 1 + 2i$
$2z = 2 + 4i$



We can see that $2z$ is a complex number with twice the length of z.

If $z = x + yi$, then $kz = kx + kyi$, where $k \in R$. The following diagram shows the situation for $x > 0$, $y > 0$ and $k > 1$.



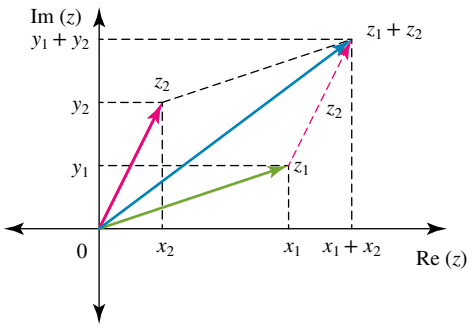
When a complex number is multiplied by a constant (or scalar), k, this produces a line segment in the same direction (or at 180° if $k < 0$) that is longer if $k < -1$ or $k > 1$, or shorter if $-1 < k < 0$ or $0 < k < 1$. This is called a **dilation**, which is a form of **linear transformation** using multiplication. When the complex number z is multiplied by k, it is said to be dilated by a factor of k.

6.5.3 Addition of complex numbers in the complex plane

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

If we now draw line segments from the origin (the point $0 + 0i$) to the points z_1 and z_2, then the addition of two complex numbers can be achieved using the same procedure as adding two vectors.

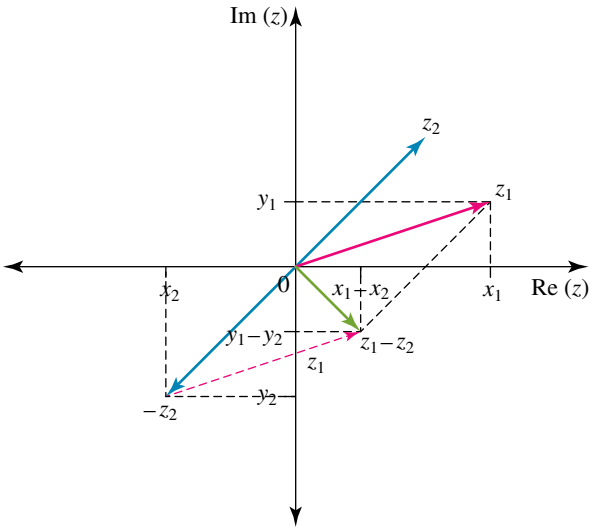


6.5.4 Subtraction of complex numbers in the complex plane

If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, then:

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (x_1 + y_1 i) - (x_2 + y_2 i) \\ z_1 - z_2 &= (x_1 - x_2) + (y_1 - y_2)\, i \end{aligned}$$

The subtraction of two complex numbers can be achieved using the same procedure as subtracting two vectors.



WORKED EXAMPLE 19

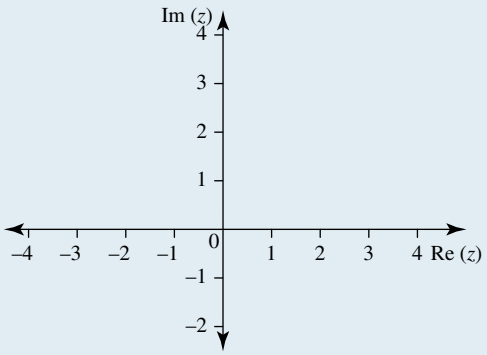
Given the complex numbers $u = 1 + i$ and $v = 2 - 3i$, represent the following complex numbers on separate Argand diagrams.

a. $u + v$ **b.** $u - v$

THINK

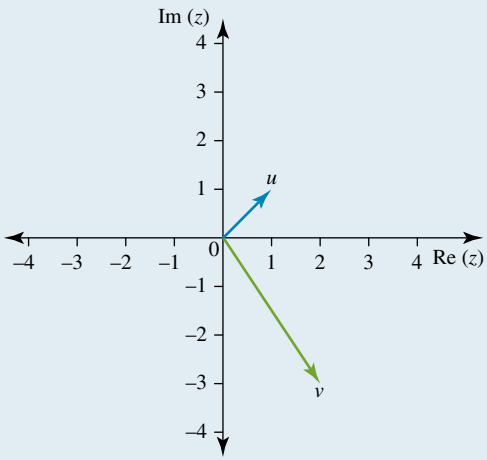
a. 1. Draw an Argand diagram.

WRITE



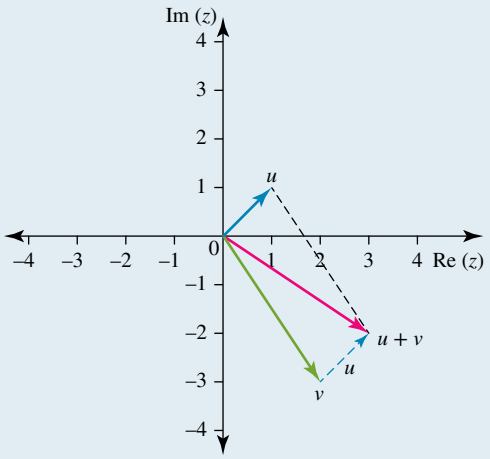
2. The complex numbers $u = 1 + i$ and $v = 2 - 3i$ are of the form $z = x + yi$. They can be plotted on the Argand plane at:
u: Re$(z) = 1$ and Im $(z) = 1$
v: Re$(z) = 2$ and Im $(z) = -3$

$u = 1 + i$
$v = 2 - 3i$

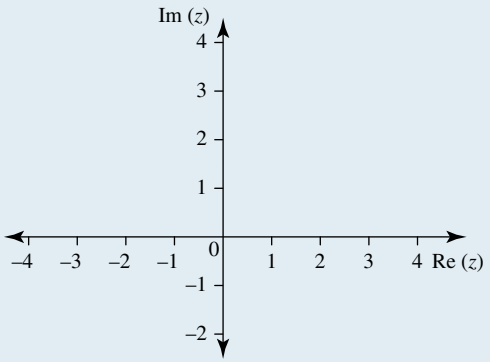


3. Evaluate the complex number $u + v$ and plot this complex number on the same Argand diagram.
$u + v$: $\text{Re}(z) = -1$ and $\text{Im}(z) = -2$

$u + v = 1 + i + 2 - 3i$
$= 3 - 2i$

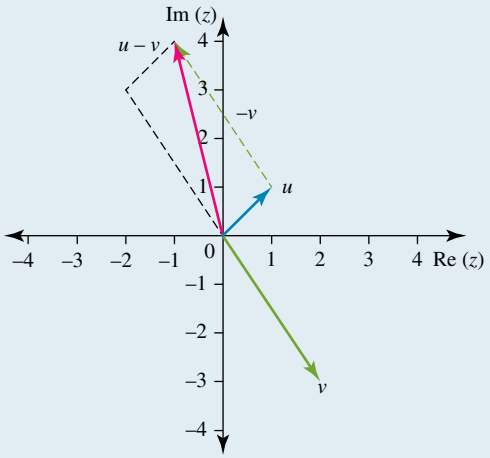


b. 1. Draw an Argand diagram.



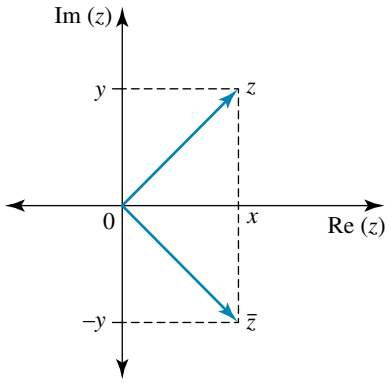
2. Plot the complex numbers u and v on an Argand plane.
Evaluate the complex number $u - v$ and plot this complex number on the same Argand diagram.
$u - v$: $\text{Re}(z) = -1$ and $\text{Im}(z) = 4$

$u = 1 + i$
$v = 2 - 3i$
$u - v = -1 + 4i$



6.5.5 Geometrical representation of a conjugate of a complex number

If $z = x + yi$, then the conjugate of z is given by $\bar{z} = x - yi$.

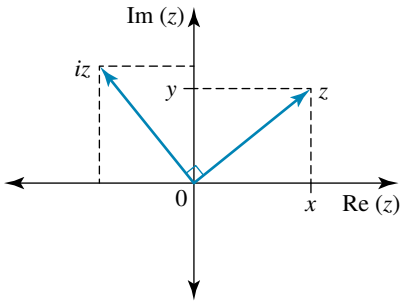


It can be seen that $\bar{z}$ is the **reflection** of the complex number in the real axis. Reflections are another form of linear transformation.

6.5.6 Multiplication by *i*

If $z = x + yi$, then iz is given by:

$$\begin{aligned} iz &= i(x + yi) \\ &= ix + i^2y \\ &= -y + xi \end{aligned}$$



It is clear from the diagram that iz is a **rotation** of 90° anticlockwise from z. Rotation is another form of transformation, using multiplication. The complex number z is said to be rotated 90° anticlockwise from z.

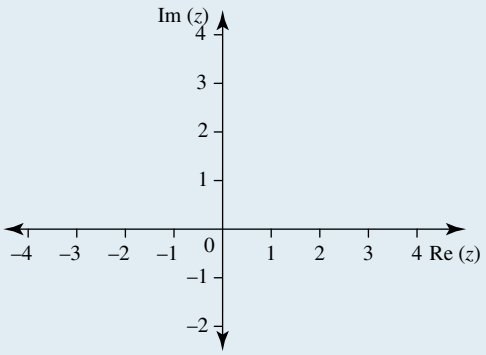
WORKED EXAMPLE 20

Given the complex number $u = 1 + 2i$, represent the complex numbers $\bar{u}$ and iu on the same Argand diagram and comment on their relative positions.

THINK

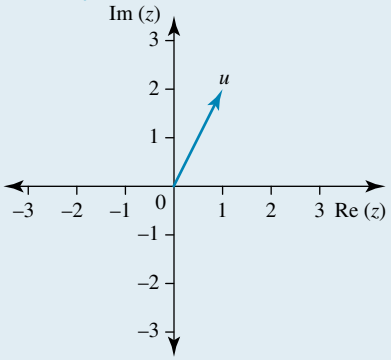
1. Draw an Argand diagram.

WRITE



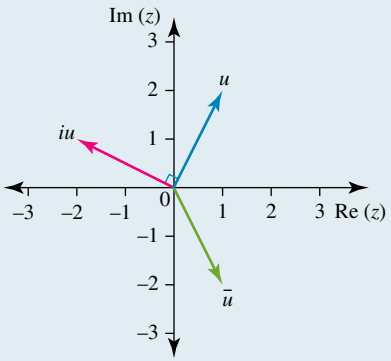
2. Plot the complex number u on an Argand plane.
u: $\text{Re}(z) = 1$ and $\text{Im}(z) = 2$

$u = 1 + 2i$



3. Evaluate the complex numbers $\overline{u}$ and iu, and plot these complex number on the same Argand diagram.
$\overline{u}$: $\text{Re}(z) = 1$ and $\text{Im}(z) = -2$
iu: $\text{Re}(z) = -2$ and $\text{Im}(z) = 1$

$\overline{u} = 1 - 2i$
$iu = i(1 + 2i)$
$= i + 2i^2$
$= -2 + i$



4. Comment on their relative positions.

$\overline{u}$ is the reflection of the complex number in the real axis. iu is a rotation of 90° anticlockwise from u.

on Resources

Digital document: SkillSHEET Operations with complex numbers (doc-26835)

studyon

Units 1 & 2 > Area 4 > Sequence 1 > Concept 2

The Argand plane Summary screen and practice questions

Exercise 6.5 The complex plane (the Argand plane)

Technology free

1. **WE18** Given the complex number $z = 2 + i$, represent the complex numbers z and $3z$ on the same Argand diagram. Comment on their relative positions.
2. Given the complex number $z = 4 - 2i$, represent the complex numbers z and $-\frac{z}{2}$ on the same Argand diagram. Comment on their relative positions.

3. **WE19** Given the complex numbers $u = 3 - 2i$ and $v = 1 + 2i$, represent the following complex numbers on separate Argand diagrams.
   a. $u + v$ b. $u - v$
4. Given the two vectors $\underset{\sim}{u} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j}$, evaluate and draw the vectors $\underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$.
5. **WE20** Given the complex number $u = -3 - 2i$, represent the complex numbers $\overline{u}$ and iu on the same Argand diagram and comment on their relative positions.
6. Given the complex number $v = -2 + 3i$, represent the complex numbers $\overline{v}$ and i^2v on the same Argand diagram and comment on their relative positions.

7. a. Given $z = 2 - 3i$, evaluate and plot each of the following on the same Argand diagram. Comment on their relative positions.
      i. $2z$ ii. $\overline{z}$ iii. iz
   b. Given $z = -4 - 4i$, evaluate and plot each of the following on the same Argand diagram. Comment on their relative positions.
      i. $-\frac{z}{2}$ ii. $\overline{z}$ iii. iz
8. a. Given the complex numbers $u = -1 - 2i$ and $v = 2 + 3i$, evaluate and plot each of the following on separate Argand diagrams.
      i. $u + v$ ii. $u - v$
   b. Given the complex numbers $u = 2 - 3i$ and $v = 1 + 4i$, evaluate and plot each of the following on separate Argand diagrams.
      i. $u + v$ ii. $u - v$
9. a. Given $z = 1 - i$, plot each of the following on one Argand diagram.
      i. z ii. i^2z iii. i^3z
   b. Given $z = -2 + 3i$, plot each of the following on the same Argand diagram.
      i. z ii. i^2z iii. i^3z
10. a. If $u = 1 + i$, evaluate and plot each of the following on one Argand diagram.
      i. u ii. $\frac{1}{u}$
   b. If $u = -\sqrt{3} - i$, evaluate and plot each of the following on one Argand diagram.
      i. u ii. $\frac{1}{u}$
11. a. Given that $u = 2 + 2i$ and $v = -1 - i$, evaluate and plot each of the following on one Argand diagram.
      i. uv ii. $\frac{u}{v}$
   b. Given that $u = 2\,i$ and $v = 1 - i$, evaluate and plot each of the following on one Argand diagram.
      i. uv ii. $\frac{u}{v}$
12. a. Given $z = 1 - i$, plot each of the following on one Argand diagram.
      i. z ii. z^2 iii. z^3
   b. Given $z = -1 + i$, plot each of the following on one Argand diagram.
      i. z ii. z^2 iii. z^3
13. a. If $\underset{\sim}{u} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{v} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$, show that the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.
   b. Evaluate the image of the point $(3, -4)$ when it is rotated 90° anticlockwise.
   c. Consider the vectors $\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j}$ and $\underset{\sim}{v} = -b\underset{\sim}{i} + a\underset{\sim}{j}$ where $a, b \in R$.
      i. Show that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.
      ii. Explain how this relates to the complex numbers $z = a + bi$ and iz.
      iii. Complex numbers and vectors are said to be isomorphic. What does this mean?

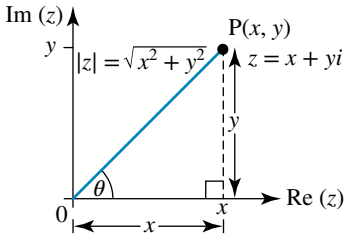
6.6 Complex numbers in polar form

6.6.1 The modulus of z

The **magnitude** (or **modulus** or **absolute value**) of the complex number $z = x + yi$ is the length of the line segment joining the origin to the point z. It is denoted by:

$|z|$, $|x + yi|$ or mod z.

The modulus of z is calculated using Pythagoras' theorem.



$|z| = \sqrt{x^2 + y^2}$ so that we have $z\bar{z} = |z|^2$.

WORKED EXAMPLE 21

Find the modulus of the complex number $z = 8 - 6i$.

THINK

Calculate the modulus by rule.
Because it forms the hypotenuse of a right-angled triangle, the modulus is always greater than or equal to Re (z) or Im (z).

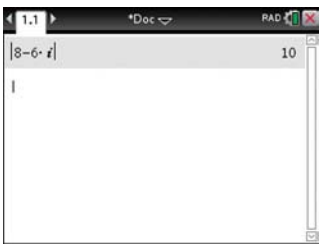
WRITE

$|z| = \sqrt{8^2 + (-6)^2}$
$= \sqrt{100}$
$= 10$

TI | THINK

1. On a Calculator page, press MENU, then select:
   2: Number
   9: Complex Number Tools
   5: Magnitude.
   Complete the entry line as:
   $|8 - 6i|$
   then press ENTER.
   Note: The symbol i can be found by pressing the π button.
2. The answer appears on the screen.

WRITE

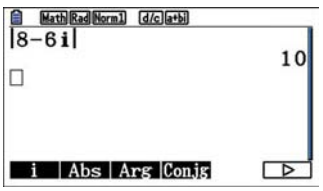


$|z| = 10$

CASIO | THINK

1. On a Run-Matrix screen, press OPTN, then select COMPLEX by pressing F3. Select Abs by pressing F2.
   Complete the entry line as:
   $|8 - 6i|$
   then press EXE.
   Note: The symbol i can be found by pressing SHIFT 0.
2. The answer appears on the screen.

WRITE



$|z| = 10$

WORKED EXAMPLE 22

If $z = 4 + 2i$ and $w = 7 + 6i$, represent the position of $w - z$ on an Argand diagram and calculate $|w - z|$.

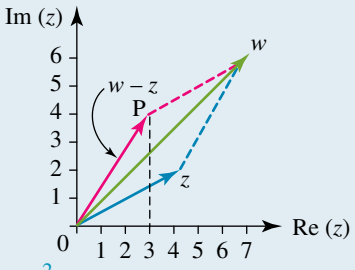
THINK

1. Calculate $w - z$.

WRITE

$w - z = 7 + 6i - (4 + 2i)$
$= 3 + 4i$

2. Represent it on an Argand diagram as a directed line segment OP.



3. Use Pythagoras' theorem to determine the length of OP.

$\overline{OP}^2 = 3^2 + 4^2 = 25$

$\overline{OP} = 5$

So $|w - z| = 5$.

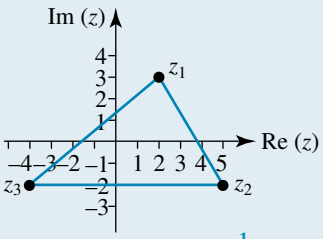
WORKED EXAMPLE 23

Represent $z_1 = 2 + 3i$, $z_2 = 5 - 2i$ and $z_3 = -4 - 2i$ on the complex number plane and calculate the area of the shape formed when the three points are connected by straight line segments.

THINK

1. Show the connected points on the complex number plane.

WRITE



2. Calculate the area of the triangle obtained. The length of the base and height can be found by inspection (base = 9, height = 5).

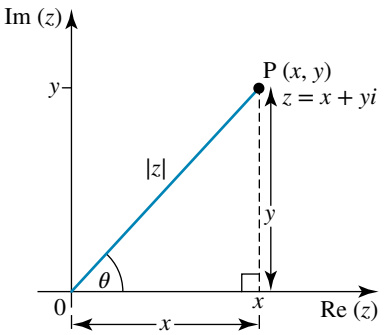
$$\text{Area of triangle} = \frac{1}{2} \times 9 \times 5$$
$$= 22.5 \text{ square units.}$$

6.6.2 The argument of z

The **argument of z**, arg (z), is the angle measurement anticlockwise of the positive Real axis.

In the figure, arg $(z) = \theta$, where

$$\sin\theta = \frac{y}{|z|} \text{ and } \cos\theta = \frac{x}{|z|} \text{ or } \tan\theta = \frac{y}{x}.$$



For non-zero z, an infinite number of arguments of z exist since, for a given z $\{\theta: \theta = \pm 2n\pi, n \in N\}$ (represented by the position of point P in the figure), because a clockwise or anticlockwise rotation consisting of multiples of 2π radians (or 360°) merely moves P to its original position.

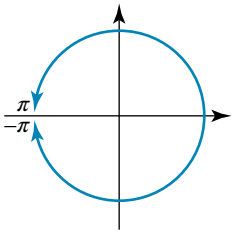
$\arg(z) = \theta$ where $\tan\theta = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\frac{y}{x}$

and $z \times i^n$, $n \in N$ produces an anti-clockwise rotation of $90n$ degrees.

To ensure that there is only one value of θ corresponding to z, we refer to the *principal value* or principal argument of θ and denote it by Arg (z). Note the capital A.

Arg (z) is the angle θ in the range $-\pi < \theta \leq \pi$ or $(-\pi, \pi]$.



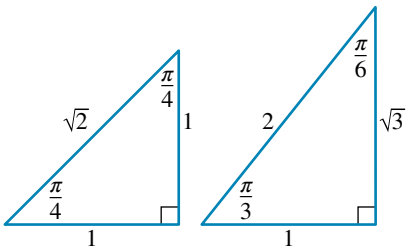
Exact values

Exact values are explored in greater detail in Chapter 8.

Angle	sin θ	cos θ	tan θ
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Some useful triangles

It will be easier if you remember these 2 triangles only — not the ratios shown above. Draw a quick sketch and work out each trigonometric ratio when you need to.



WORKED EXAMPLE 24

Find the Argument of z for each of the following in the interval $(-\pi, \pi]$.

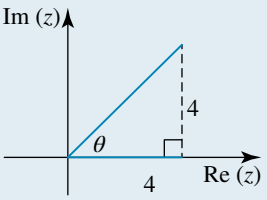
a. $z = 4 + 4i$ **b.** $z = 1 - \sqrt{3}i$

THINK

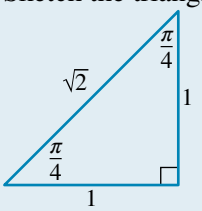
a. 1. Plot z.

WRITE

a.



2. Sketch the triangle that has sides in this 1:1 ratio.



From the diagram

$\theta = \frac{\pi}{4}$

$\therefore \text{Arg}(z) = \frac{\pi}{4}$

3. This result can be verified using an inverse trigonometric ratio, $\theta = \tan^{-1}\frac{y}{x}$.

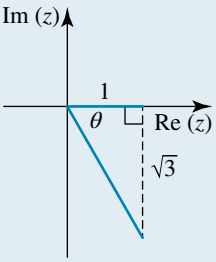
Check:

$\theta = \tan^{-1}\frac{4}{4}$

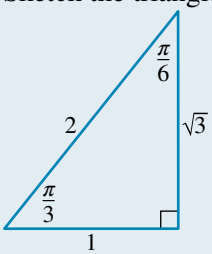
$= \frac{\pi}{4}$

b. 1. Plot z.

b.



2. Sketch the triangle that has sides in this ratio.



From the diagram

$\theta = \frac{\pi}{3}$

$\therefore \text{Arg}(z) = -\frac{\pi}{3}$

Remember: $-\pi < \theta \leq \pi$.

3. This result can be verified using an inverse trigonometric ratio, $\theta = \tan^{-1}\frac{y}{x}$.

Check:

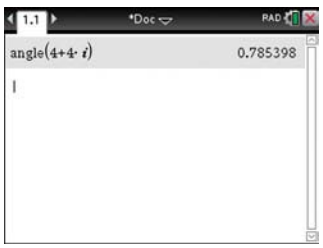
$\theta = \tan^{-1}\frac{-\sqrt{3}}{1}$

$= -\frac{\pi}{3}$

TI | THINK

a.1. Put the calculator in Radian mode. On a Calculator page, press MENU, then select:
2: Number
9: Complex Number Tools
4: Polar Angle.
Complete the entry line as:
angle $(4 + 4i)$
then press ENTER.
Note: The symbol i can be found by pressing the π button.

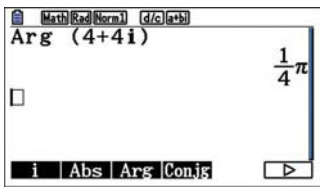
WRITE



CASIO | THINK

a.1. Put the calculator in Radian mode. On a Run-Matrix screen, press OPTN, then select COMPLEX by pressing F3. Select Arg by pressing F3. Complete the entry line as:
Arg $(4 + 4i)$
then press EXE.
Note: The symbol i can be found by pressing SHIFT 0.

WRITE



2. The answer appears on the screen.
Note: The calculator will give a decimal answer, not an exact answer.

$\text{Arg}(z) = 0.785$ radians (to 3 decimal places)

2. The answer appears on the screen.

$\text{Arg}(z) = \frac{\pi}{4}$

WORKED EXAMPLE 25

Convert each of the following into Arguments.

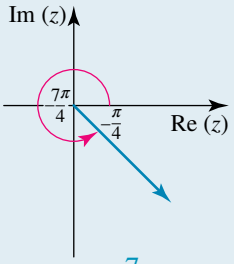
a. $\frac{7\pi}{4}$ **b.** $-\frac{5\pi}{2}$

THINK

a. 1. Sketch the angle.

WRITE

a.

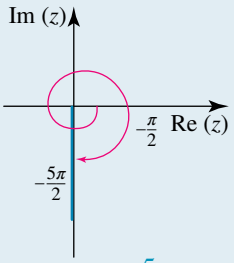


2. Since the given angle is positive, subtract multiples of 2π until it lies in the range $(-\pi, \pi]$.

$$\text{Arg}(z) = \frac{7\pi}{4} - 2\pi$$
$$= -\frac{\pi}{4}$$

b. 1. Sketch the angle.

b.



2. Since the given angle is negative, add multiples of 2π until it lies in the range $(-\pi, \pi]$.

$$\text{Arg}(z) = -\frac{5\pi}{2} + 2\pi$$
$$= -\frac{\pi}{2}$$

WORKED EXAMPLE 26

Find the modulus and Argument for each of the following complex numbers.

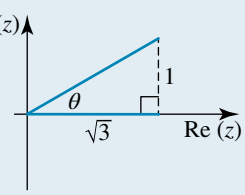
a. $\sqrt{3} + i$ **b.** $-1 - \sqrt{2}i$

THINK

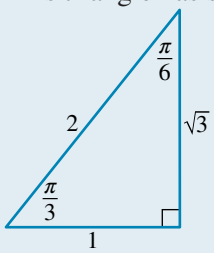
a. 1. Plot z.

WRITE

a.



2. This triangle has sides in the same ratio as



$|z| = 2$

$\text{Arg}\,(z) = \dfrac{\pi}{6}$

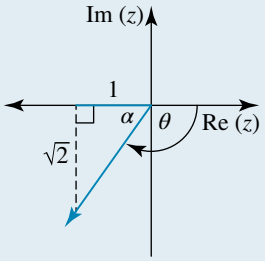
3. These results can be verified by calculating the modulus and principal argument by rule.

Check:

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

b. 1. Plot z.

b.



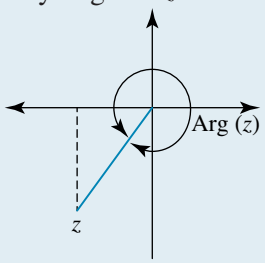
2. Find the modulus.

$$|z| = \sqrt{(-1)^2 + \left(-\sqrt{2}\right)^2}$$
$$= \sqrt{1+2}$$
$$= \sqrt{3}$$

3. The triangle in the third quadrant will be used to find α but the answer will be finally expressed as θ and Arg (z).

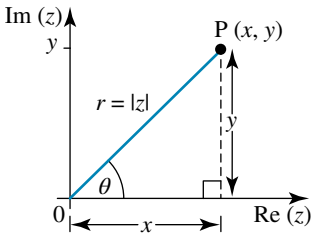
$$\alpha = \tan^{-1}\frac{y}{x}$$
$$= \tan^{-1}\frac{-\sqrt{2}}{-1}$$
$$= 0.955$$
$$\theta = -\pi + 0.955$$
$$\text{Arg}\,(z) = -2.187$$

4. Remember Arg (z) can be thought of as the quickest way to get to z.



6.6.3 Expressing complex numbers in polar form

Suppose $z = x + yi$ is represented by the point P (x, y) on the complex plane using Cartesian coordinates.



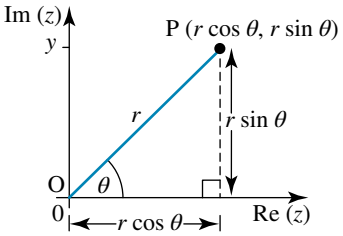
Using the trigonometric properties of a right-angled triangle, z can also be expressed in polar coordinates as follows. We have:

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\theta$$

where $|z| = r = \sqrt{x^2 + y^2}$ and $\theta = \text{Arg}\ (z)$.

Substituting these values, the point P (x, y) becomes the **polar form** P $(r\cos\theta,\ r\sin\theta)$ as shown.



Now $z = x + yi$ in Cartesian form becomes

$z = r\cos\theta + r\sin\theta i$ (after substitution of $x = r\cos\theta$, $y = r\sin\theta$)

$= r(\cos\theta + i\sin\theta)$

$= r\,\text{cis}\,\theta$, where cis θ is the abbreviated form of $\cos\theta + i\sin\theta$.

(*Note:* The acronym 'cis' is pronounced 'sis'.)

Cartesian form: $z = x + yi$
Polar form: $z = r\cos\theta + r\sin\theta\, i$
$= r\,\text{cis}\,\theta$

WORKED EXAMPLE 27

Express each of the following in polar form, $r\,cis\,\theta$, where $\theta = \text{Arg}\ (z)$.

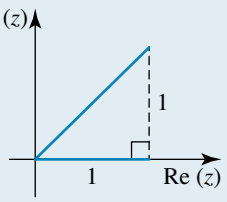
a. $z = 1 + i$ **b.** $z = 1 - \sqrt{3}i$

THINK

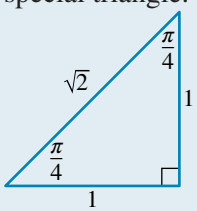
a. 1. Plot z.

WRITE

a.



2. The ratio of the sides of this triangle matches the following special triangle.



From the diagram:

$|z| = \sqrt{2}$

$\theta = \frac{\pi}{4}$

3. These results can be verified by rule:

(a) Find the value of r using $r = |z| = \sqrt{x^2 + y^2}$.

(b) Determine $\tan\theta$ from $\tan\theta = \frac{y}{x}$ and hence find θ.

Check:

$r = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\tan\theta = \frac{1}{1}$

$= 1$

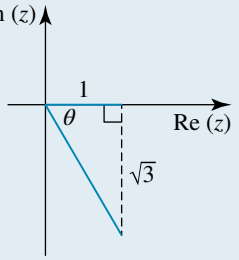
4. The angle θ is in the range $(-\pi, \pi]$, which is required.

5. Substitute the values of r and θ in $z = r\cos\theta + r\sin\theta i = r\,\text{cis}\,\theta$.

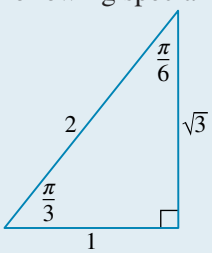
$$\theta = \frac{\pi}{4}$$
$$z = \sqrt{2}\cos\frac{\pi}{4} + \sqrt{2}\sin\frac{\pi}{4}i$$
$$= \sqrt{2}\,\text{cis}\frac{\pi}{4}$$

b. 1. Sketch z.

b.



2. The ratio of the sides of this triangle is the same as that in the following special triangle:



From the diagram
$r = 2$
$\theta = \frac{\pi}{3}$

$\text{Arg}\,(z) = -\frac{\pi}{3}$

3. These results can be verified by rule:
(a) Calculate the value of r.
(b) Determine the appropriate value of θ.

Check:
$$r = \sqrt{1 + \left(\sqrt{3}\right)^2}$$
$$= 2$$
$$\tan\theta = -\frac{\sqrt{3}}{1}$$
$$= -\sqrt{3}$$
$$\theta = -\frac{\pi}{3}$$

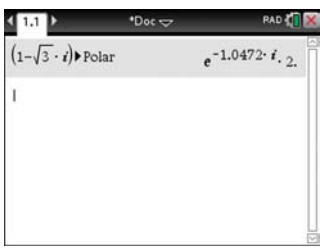
4. Substitute for r and θ in $z = r\cos\theta + r\sin\theta i$ and write in the form $r\,\text{cis}\,\theta$.

$$z = 2\cos\left(-\frac{\pi}{3}\right) + 2\sin\left(-\frac{\pi}{3}\right)i$$
$$= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$$

TI | THINK

b.1. Put the calculator in Radian mode. On a Calculator page, complete the entry line as: $1 - \sqrt{3}\,i$.
Press MENU, then select:
2: Number
9: Complex Number Tools
6: Convert to Polar
then press ENTER.
Note: The symbol i can be found by pressing the π button.

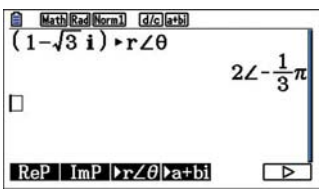
WRITE



CASIO | THINK

b.1. Put the calculator in Radian mode.
On a Run-Matrix screen, complete the entry line as: $1 - \sqrt{3}\,i$.
Press OPTN, then select COMPLEX by pressing F3. Press F6 to scroll across to more menu options, then select $r\angle\theta$ by pressing F3. Press EXE.
Note:
The symbol i can be found by pressing SHIFT 0.

WRITE



2. The answer appears on the screen. Note: The calculator will give a decimal answer, not an exact answer.	The answer is given in the form $e^{i\theta} \cdot \lvert z \rvert$, where $\theta = -1.047$ and $\lvert z \rvert = 2$. Rewriting this in the form $\lvert z \rvert \operatorname{cis}\theta$ gives $z = 2\operatorname{cis}(-1.047)$.
2. The answer appears on the screen.	$z = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$

6.6.4 Converting from polar form to Cartesian form

WORKED EXAMPLE 28

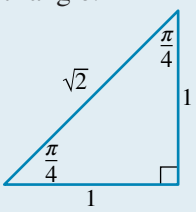
Express $3\operatorname{cis}\frac{\pi}{4}$ in Cartesian (or standard $a + bi$) form.

THINK

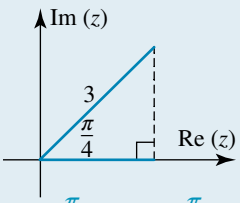
1. Sketch z.

2. Express $3\operatorname{cis}\frac{\pi}{4}$ in Cartesian form.

3. Simplify using exact values from the following triangle:



WRITE



$$3\operatorname{cis}\frac{\pi}{4} = 3\cos\frac{\pi}{4} + 3\sin\frac{\pi}{4}i$$

$$= 3 \times \frac{1}{\sqrt{2}} + 3 \times \frac{1}{\sqrt{2}}i$$

$$= \frac{3}{\sqrt{2}} + \left(\frac{3}{\sqrt{2}}\right)i$$

TI | THINK

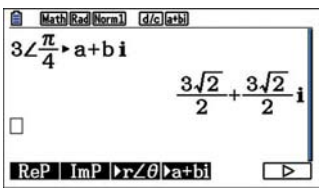
1. Put the calculator in Radian mode. On a Calculator page, complete the entry line as:
$3\angle\frac{\pi}{4}$
Press MENU, then select:
2: Number
9: Complex Number Tools
7: Convert to Rectangular
then press ENTER.
Note: The $\angle$ symbol can be found by pressing CTRL, then the Catalogue button.

WRITE

CASIO | THINK

1. Put the calculator in Radian mode. On a Run-Matrix screen, complete the entry line as:
$3\angle\frac{\pi}{4}$
Press OPTN, then select COMPLEX by pressing F3. Press F6 to scroll across to more menu options, then select $a + bi$ by pressing F4. Press EXE.
Note: The $\angle$ symbol can be found by pressing SHIFT, then the Variables button.

WRITE



2. The answer appears on the screen. *Note:* The calculator will give a decimal answer, not an exact answer.	$3 \text{ cis } \frac{\pi}{4} = 2.121 + 2.121i$ (to 3 decimal places)
2. The answer appears on the screen.	$3\text{cis}\frac{\pi}{4} = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i$

studyon

Units 1 & 2 > Area 4 > Sequence 1 > Concept 5

Complex numbers in polar form Summary screen and practice questions

Exercise 6.6 Complex numbers in polar form

Technology free

In the following questions give arg (z) or Arg (z) correct to 3 decimal places where the angle cannot be easily expressed as a common multiple of π.

1. a. Represent $z = 4 + 8i$ on an Argand diagram.

b. Calculate the *exact* distance of z from the origin. (Do not use your calculator.)

2. WE21 Find the modulus of each of the following.

a. $z = 5 + 12i$ **b.** $z = \sqrt{5} - 2i$ **c.** $z = -4 + 7i$

d. $z = -3 - 6i$ **e.** $z = \sqrt{3} + \sqrt{2}\,i$ **f.** $z = (2 + i)^2$

3. WE22 If $z = 3 + i$, $w = 4 - 3i$ and $u = -2 + 5i$ then:

i. represent each of the following on an Argand diagram

ii. calculate the magnitude in each case.

a. $z - w$ **b.** $u + z$ **c.** $w - u$ **d.** $w + z$ **e.** $z + w - u$ **f.** z^2

4. a. WE23 Show the points $z_1 = -3 + 0i$, $z_2 = 2 + 5i$, $z_3 = 7 + 5i$ and $z_4 = 9 + 0i$ on the complex number plane.

b. Calculate the area of the shape formed when the four points are connected by straight line segments in the order z_1 to z_2 to z_3 to z_4.

5. a. Show the points $z = -1 + 3i$, $u = 3$ and $w = 3 + 12i$ on the complex number plane.

b. Calculate the area of the triangle produced by joining the three points with straight line segments.

6. a. If the complex numbers $u = 3 - 4i$, $\overline{u}$, v and $\overline{v}$ form a square with an area of 64 square units, evaluate the complex number v.

b. If the complex numbers $u = -2 + 5i$, $\overline{u}$, v and $\overline{v}$ form a rectangle with an area of 60 square units, evaluate the complex number v.

c. If $a, b, c \in R^+$, evaluate the area of the rectangle formed by the complex numbers $u = a + bi$, $\overline{u}$, $v = -c - bi$ and $\overline{v}$.

7. a. Evaluate the area of the triangle that is formed by the complex numbers $u = 4 + 3i$ and iu and the origin O.

b. Evaluate the area of the triangle that is formed by the complex numbers $u = 12 + 5i$ and iu and the origin O.

c. If $a, b \in R^+$, evaluate the area of the triangle that is formed by the complex numbers $u = a + bi$, iu and the origin O.

8. **a.** Evaluate the area of the square that is formed by the complex numbers $u = 4 - 3i$, iu $u + iu$ and the origin O.
   b. If the area of the square formed by the complex numbers $u = 6 + bi$, iu $u + iu$ and the origin O is equal to 50 square units, evaluate the value of b.
   c. If $a, b \in R^+$, evaluate the area of the square that is formed by the complex numbers $u = a + bi$, iu $u + iu$ and the origin O.
9. **a.** Evaluate the equation and area of the circle that passes through the complex numbers $u = 3 - 4i$, iu and the origin O.
   b. Evaluate the equation and area of the circle that passes through the complex numbers $u = 5 + 12i$, iu and the origin O.
10. If $a, b \in R^+$, evaluate the equation and area of the circle that passes through the complex numbers $u = a + bi$, iu and the origin O.
11. **WE24** Find the Argument of z for each of the following in the interval $(-\pi, \pi]$. (Give exact answers where possible or to 3 decimal places.)
    a. $z = 3 + 2i$
    b. $z = \sqrt{3} + i$
    c. $z = 5 - 5i$
    d. $z = -4 + 8i$
    e. $z = -2 - 2\sqrt{3}\,i$
    f. $z = 6 - 10i$
    g. $z = 3i$
    h. $z = -\sqrt{7}$
    i. $z = -6i$
    j. $z = 55$

12. **WE25** Convert each of the following into principal arguments.
    a. $\frac{3\pi}{2}$ **b.** $-\frac{11\pi}{6}$ **c.** $\frac{15\pi}{8}$ **d.** $-\frac{5\pi}{4}$
    e. $\frac{19\pi}{6}$ **f.** $\frac{20\pi}{7}$ **g.** $-\frac{18\pi}{5}$ **h.** $-\frac{13\pi}{12}$
13. **WE26** Find the modulus and principal argument of each of the following complex numbers.
    a. $3 - 3i$ **b.** $-5 + 5i$ **c.** $-1 - \sqrt{3}\,i$ **d.** $4\sqrt{3} + 4i$
    e. $-7 - 10i$ **f.** $6i - 2$ **g.** $\left(\sqrt{3} + i\right)^2$
14. **WE27** Express each of the following in the polar form, $z = r\,\text{cis}\,\theta$, where $\theta = \text{Arg}\,(z)$.
    a. $z = -1 + i$ **b.** $z = \sqrt{6} + \sqrt{2}\,i$ **c.** $z = -\sqrt{5} - \sqrt{5}\,i$ **d.** $z = \sqrt{5} - \sqrt{15}\,i$
    e. $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ **f.** $z = -\frac{1}{4} + \frac{1}{4}i$
15. **WE28** Express each of the following complex numbers in Cartesian (or standard $a + bi$) form.
    a. $2\ \text{cis}\frac{2\pi}{3}$ **b.** $3\ \text{cis}\frac{\pi}{4}$ **c.** $\sqrt{5}\ \text{cis}\ \frac{5\pi}{6}$ **d.** $4\ \text{cis}\left(-\frac{\pi}{3}\right)$
    e. $\sqrt{7}\ \text{cis}\left(-\frac{7\pi}{4}\right)$ **f.** $8\ \text{cis}\ \frac{\pi}{2}$ **g.** $\sqrt{3}\ \text{cis}\,\pi$
16. **MC** If $z = 3 - 50i$ and $w = 5 + 65i$, the value of $|z + w|$ is:
    A. 64 **B.** 15 **C.** 17 **D.** 225
17. **MC** The perimeter of the triangle formed by the line segments connecting the points $2 - 4i$, $14 - 4i$ and $2 + i$ is:
    A. 13 **B.** 30 **C.** 10 **D.** 17

18. MC The principal argument of $4\sqrt{3} - 4i$ is:

A. $\frac{\pi}{6}$ **B.** $\frac{\pi}{3}$ **C.** $\frac{5\pi}{6}$ **D.** $-\frac{\pi}{6}$

19. MC In polar form, $5i$ is:

A. $\text{cis}\, 5\pi$ **B.** $\text{cis}\, \frac{5\pi}{2}$ **C.** $5\,\text{cis}\, 5\pi$ **D.** $5\,\text{cis}\, \frac{\pi}{2}$

20. MC The Cartesian form of $\sqrt{3}\,\text{cis}\left(-\frac{7\pi}{6}\right)$ is:

A. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **B.** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **C.** $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$ **D.** $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$

6.7 Basic operations on complex numbers in polar form

6.7.1 Addition and subtraction in polar form

In general there is no simple way to add or subtract complex numbers given in the polar form $r\,\text{cis}\,\theta$. For addition or subtraction, the complex numbers need to be expressed in Cartesian form first.

6.7.2 Multiplication, division and powers in polar form

De Moivre's theorem is very useful for multiplying and dividing complex numbers. One version of de Moivre's theorem is:

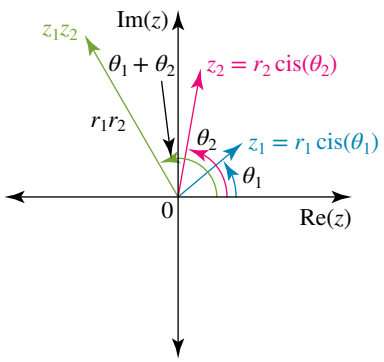
If $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$, then

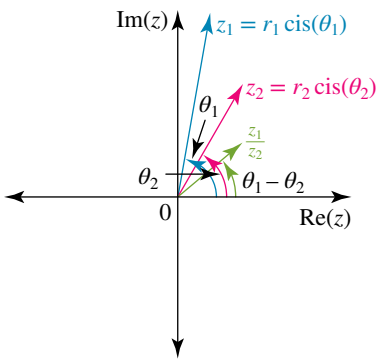
$$z_1 \times z_2 = r_1 \times r_2\,\text{cis}\,(\theta_1 + \theta_2)$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{cis}\,(\theta_1 - \theta_2).$$

The diagrams below illustrate this geometrically.





WORKED EXAMPLE 29

Express $5\,\text{cis}\,\frac{\pi}{4} \times 2\,\text{cis}\,\frac{5\pi}{6}$ in the form $r\,\text{cis}\,\theta$ where $\theta \in (-\pi, \pi]$.

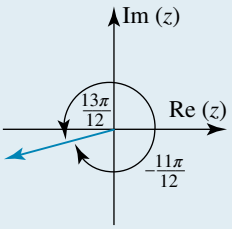
THINK

1. Simplify by recalling the multiplication version of de Moivre's theorem: $z_1 \times z_2 = r_1 \times r_2\,\text{cis}\,(\theta_1 + \theta_2)$.

WRITE

$$5\,\text{cis}\,\frac{\pi}{4} \times 2\,\text{cis}\,\frac{5\pi}{6} = (5 \times 2)\,\text{cis}\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)$$
$$= 10\,\text{cis}\left(\frac{13\pi}{12}\right)$$

2. Sketch this number.



3. Subtract 2π from θ to express the answer in the required form where $\theta \in (-\pi, \pi]$.

$$10\,\text{cis}\left(\frac{13\pi}{12}\right) = 10\,\text{cis}\left(-\frac{11\pi}{12}\right)$$

WORKED EXAMPLE 30

Express $z_1 z_2$ in Cartesian form if $z_1 = \sqrt{2}\,\text{cis}\,\frac{5\pi}{6}$ and $z_2 = \sqrt{6}\,\text{cis}\left(-\frac{\pi}{3}\right)$.

THINK

1. Recall the formula $z_1 \times z_2 = r_1 \times r_2\,\text{cis}\,(\theta_1 + \theta_2)$.

WRITE

$$z_1 \times z_2 = \sqrt{2}\,\text{cis}\,\frac{5\pi}{6} \times \sqrt{6}\,\text{cis}\left(-\frac{\pi}{3}\right)$$
$$= \left(\sqrt{2} \times \sqrt{6}\right)\text{cis}\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)$$
$$= 2\sqrt{3}\,\text{cis}\,\frac{\pi}{2}$$

2. Simplify and write the result in Cartesian form.

$$= 2\sqrt{3}\cos\frac{\pi}{2} + 2\sqrt{3}\sin\frac{\pi}{2}i$$
$$= 2\sqrt{3} \times 0 + 2\sqrt{3} \times 1i$$
$$= 2\sqrt{3}\,i$$

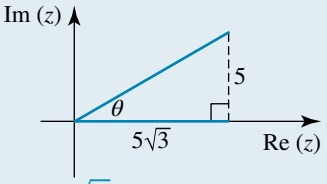
WORKED EXAMPLE 31

If $z = 5\sqrt{3} + 5i$ and $w = 3 + 3\sqrt{3}i$, express the product zw in polar form.

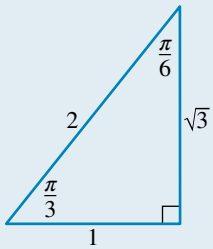
THINK

WRITE

1. Sketch z.



2. Factorise z to recognise the special triangle:



$z = 5\sqrt{3} + 1$

$= 5(\sqrt{3} + 1)$

Let $z = r_1 \operatorname{cis} \theta_1$.

$r_1 = 5 \times 2 = 10$

$\theta_1 = \dfrac{\pi}{6}$

The ratio of sides in z is 5 times that of the sides in this triangle.

3. Verify this by rule if you wish.

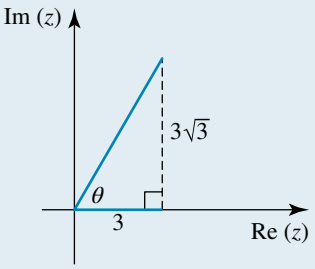
Alternatively:

$r_1 = \sqrt{\left(5\sqrt{3}\right)^2 + 5^2} = 10$

$\theta = \tan^{-1}\dfrac{\cancel{5}}{\cancel{5}\sqrt{3}}$, so $\theta_1 = \dfrac{\pi}{6}$

Therefore, $z = 10 \operatorname{cis} \dfrac{\pi}{6}$.

4. Sketch w. The ratio of sides in w is 3 times that of the sides in the special triangle shown in step **2**.



5. Write w in polar form.

Let $w = r_2 \operatorname{cis} \theta_2$.

$r_2 = 3 \times 2 = 6$

$\theta_2 = \dfrac{\pi}{3}$

6. Verify this by rule if you wish.

Alternatively:

$r_2 = \sqrt{3^2 + \left(3\sqrt{3}\right)^2} = 6$

$\tan\theta_2 = \dfrac{3\sqrt{3}}{3} = \sqrt{3}$, so $\theta_2 = \dfrac{\pi}{3}$

Therefore, $w = 6 \operatorname{cis} \dfrac{\pi}{3}$.

7. Determine zw using $z_1 \times z_2 = r_1 \times r_2 \operatorname{cis}(\theta_1 + \theta_2)$.

$$zw = 10 \operatorname{cis} \frac{\pi}{6} \times 6 \operatorname{cis} \frac{\pi}{3}$$
$$= 60 \operatorname{cis} \left(\frac{\pi}{6} + \frac{\pi}{3} \right)$$
$$= 60 \operatorname{cis} \frac{\pi}{2}$$

WORKED EXAMPLE 32

Express $10 \operatorname{cis} \left(-\frac{\pi}{3} \right) \div 5 \operatorname{cis} \frac{5\pi}{6}$ in the form $r \operatorname{cis} \theta$ where $\theta \in (-\pi, \pi]$.

THINK

1. Simplify by recalling the division version of de Moivre's theorem: $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$

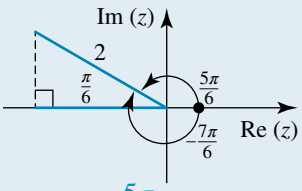
2. Sketch this number.

3. State θ, the principal argument.

4. State the result in polar form.

WRITE

$$10 \operatorname{cis} \left(-\frac{\pi}{3} \right) \div 5 \operatorname{cis} \frac{5\pi}{6} = 2 \operatorname{cis} \left(-\frac{\pi}{3} - \frac{5\pi}{6} \right)$$
$$= 2 \operatorname{cis} \left(-\frac{7\pi}{6} \right)$$



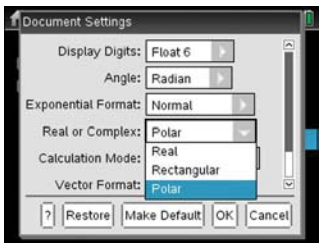
$$\text{Arg}(z) = \frac{5\pi}{6}$$

$$z = 2 \operatorname{cis} \frac{5\pi}{6}$$

TI | THINK

1. In the Document Settings menu, change the Angle to Radian and the Calculation Mode to Polar, then select OK.

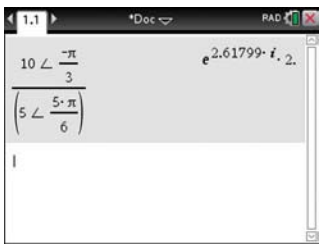
WRITE



2. On a Calculator page, complete the entry line as:

$$\frac{10 \angle \frac{-\pi}{3}}{5 \angle \frac{5\pi}{6}}$$

then press ENTER.
Note: The $\angle$ symbol can be found by pressing CTRL, then pressing the Catalogue button.



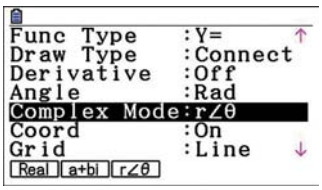
3. The answer appears on the screen.
Note: The calculator will give a decimal answer, not an exact answer.

The answer is given in the form $e^{i\theta} \cdot |z|$, where $\theta = 2.618$ and $|z| = 2$. Rewriting this in the form $|z| \operatorname{cis}\theta$ gives $z = 2 \operatorname{cis}(2.618)$.

CASIO | THINK

1. In the SETUP menu, change the Angle to Radian and the Complex Mode to $r\angle\theta$, then press EXIT.

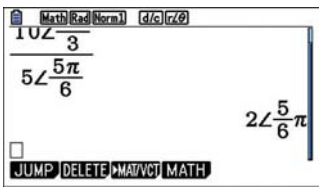
WRITE



2. On a Run-Matrix screen, complete the entry line as:

$$\frac{10 \angle \frac{-\pi}{3}}{5 \angle \frac{5\pi}{6}}$$

then press EXE.
Note: The $\angle$ symbol can be found by pressing SHIFT, then the Variables button.



3. The answer appears on the screen.

$$z = 2 \operatorname{cis} \left(\frac{5\pi}{6} \right)$$

Index powers of z

A logical consequence of the multiplication version of de Moivre's theorem involves the calculation of a power of z.

$z^n = z \times z \times \ldots \times z$ (n times)

> **If $z = r\,\text{cis}\,\theta$, then $z^n = r^n\,\text{cis}\,n\theta$.**
> **In other words,**
> **$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,n\theta$, where $\text{cis}\,\theta = \cos\theta + i\sin\theta$.**

The proofs required to establish these rules are outside the scope of this course and are not included here.

WORKED EXAMPLE 33

If $z = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$, calculate z^3.

THINK	WRITE
1. Recall de Moivre's theorem for index powers.	$z^n = r^n\,\text{cis}\,(n\theta)$
2. Substitute the values into de Moivre's theorem and simplify.	$z^3 = 2^3\,\text{cis}\left(3 \times \frac{5\pi}{6}\right)$ $= 8\,\text{cis}\left(\frac{5\pi}{2}\right)$
3. Remember $-\pi < \text{Arg}(z) \leq \pi$.	$\text{Arg}\left(\frac{5\pi}{2}\right) = \frac{\pi}{2}$
4. Write the answer.	$z^3 = 8\,\text{cis}\left(\frac{\pi}{2}\right)$

Negative powers of z

Your earlier studies have shown that $z^{-1} = \frac{1}{z}$. Similarly, $z^{-3} = \frac{1}{z^3}$.

WORKED EXAMPLE 34

Evaluate $(1 - i)^{-4}$.

THINK	WRITE
1. Convert $(1 - i)$ to polar form.	$\lvert 1 - i\rvert = \sqrt{1^2 + (-1)^2}$ $= \sqrt{2}$ $\text{Arg}\,(1 - i) = \tan^{-1}(-1)$ $= -\frac{\pi}{4}$ $1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

2. Write De Moivre's theorem and substitute in the values.

$$z^n = r^n \operatorname{cis}(n\theta)$$
$$(1-i)^{-4} = \left(\sqrt{2}\right)^{-4} \operatorname{cis}\left(4 \times \frac{\pi}{4}\right)$$

3. Simplify the expression, recalling the exact values of cos π and sin π.

$$= \frac{1}{4} \operatorname{cis} \pi$$
$$= \frac{1}{4}(\cos \pi + i \sin \pi)$$

4. Write the final expression.

$$(1-i)^{-4} = -\frac{1}{4}$$

6.7.3 Powers of complex numbers

Whole powers of z

As with real numbers, powers of complex numbers can be written as:

$$z^n = z \times z \times z \times z \times \ldots \times z \text{ to } n \text{ factors.}$$

Since $z = a + bi$ is a binomial (containing two terms), we can express z^n using Pascal's triangle to generate the coefficients of each term.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ \text{5th row} & \rightarrow & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} \quad \text{and so on.}$$

$(a + bi)^5$ can therefore be expanded using the elements of the fifth row of Pascal's triangle:

$$\begin{aligned} (a+bi)^5 &= 1a^5 + 5a^4(bi)^1 + 10a^3(bi)^2 + 10a^2(bi)^3 + 5a(bi)^4 + (bi)^5 \\ &= 1a^5 + 5a^4bi + 10a^3b^2i^2 + 10a^2b^3i^3 + 5ab^4i^4 + b^5i^5 \\ &= 1a^5 + 5a^4bi - 10a^3b^2 - 10a^2b^3i + 5ab^4 + b^5i \\ &= 1a^5 - 10a^3b^2 + 5ab^4 + 5a^4bi - 10a^2b^3i + b^5i \\ &= 1a^5 - 10a^3b^2 + 5ab^4 + \left(5a^4b - 10a^2b^3 + b^5\right)i \quad \text{grouped into standard form.} \end{aligned}$$

$$\operatorname{Re}\left[(a+bi)^5\right] = 1a^5 - 10a^3b^2 + 5ab^4$$
$$\operatorname{Im}\left[(a+bi)^5\right] = 5a^4b - 10a^2b^3 + b^5$$

WORKED EXAMPLE 35

Use Pascal's triangle to expand $(2 - 3i)^3$.

THINK

1. Use the third row of Pascal's triangle to expand (1 3 3 1). Use brackets to keep the negative sign of the second term.

WRITE

$$(2-3i)^3$$
$$= 1\left(2^3\right) + 3(2)^2(-3i) + 3(2)(-3i)^2 + (-3i)^3$$

2. Simplify the expression.

$= 8 - 36i + 54i^2 - 27i^3$
$= 8 - 36i - 54 + 27i$
$= -46 - 9i$

6.7.4 Trigonometric proofs with complex numbers

In Chapter 9 you will be exploring trigonometric identities and the different ways to prove if statements that are written as functions of one or more angles are true. It is possible to combine de Moivre's theorem and binomial expansion of complex numbers to prove some of these multi-angle identities.

If $z = r\,\text{cis}\,\theta$, then de Moivre's theorem tells us that:

$$\begin{aligned} z^n &= r^n \text{cis}\, n\theta \\ &= r^n(\cos n\theta + i \sin n\theta) \end{aligned} \qquad [1]$$

We can also write:

$$z^n = r^n(\cos\theta + i\sin\theta)^n \qquad [2]$$

Equating [1] and [2] gives us:

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

Equating the real and imaginary components:

$$\cos n\theta = \text{Re}\,(\cos\theta + i\sin\theta)^n$$
$$\sin n\theta = \text{Im}\,(\cos\theta + i\sin\theta)^n$$

It may be necessary to use the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ as part of your proof. The Pythagorean identities will be examined in greater detail in Chapter 9.

WORKED EXAMPLE 36

Use de Moivre's theorem to prove:

a. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

b. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

THINK

1. Both proofs involve 3θ, so it will be necessary to determine $(\cos\theta + i\sin\theta)^3$.
2. Determine the expansion using the third row (1 3 3 1) of Pascal's triangle.

WRITE

$(\cos\theta + i\sin\theta)^3$

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= 1(\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) \\ &\quad + 3(\cos\theta)(i\sin\theta)^2 + 1(i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta \\ &\quad + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta\end{aligned}$$

3. Simplify the expression, remembering that $i^2 = -1$ and $i^3 = -i$.
4. Group the real and imaginary terms together.

$$\begin{aligned}
&= \cos^3\theta + 3i\cos^2\theta\sin\theta \\
&\quad - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&\quad + 3i\cos^2\theta\sin\theta - i\sin^3\theta \\
(\cos\theta + i\sin\theta)^3 &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&\quad + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

a. 1. State the rule for $\cos n\theta$.

$$\cos n\theta = \text{Re}\,(\cos\theta + i\sin\theta)^n$$

2. Use the expansion of $(\cos\theta + i\sin\theta)^3$ to find $\cos 3\theta$.

$$\begin{aligned}
\cos 3\theta &= \text{Re}\,(\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta
\end{aligned}$$

3. We want to prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, so use the Pythagorean identity $\sin^2\theta = 1 - \cos^2\theta$ to replace $\sin^2\theta$.

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\
\therefore \cos 3\theta &= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

b. 1. State the rule for $\sin n\theta$.

$$\sin n\theta = \text{Im}\,(\cos\theta + i\sin\theta)^n$$

2. Use the expansion of $(\cos\theta + i\sin\theta)^3$ to find $\sin 3\theta$.

$$\begin{aligned}
\sin 3\theta &= \text{Im}\,(\cos\theta + i\sin\theta)^3 \\
&= 3\cos^2\theta\sin\theta - \sin^3\theta
\end{aligned}$$

3. We want to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, so use the Pythagorean identity $\cos^2\theta = 1 - \sin^2\theta$ to replace $\sin^2\theta$.

$$\begin{aligned}
\sin 3\theta &= 3\left(1 - \sin^2\theta\right)\sin\theta - \sin^3\theta \\
&= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\
\therefore \sin 3\theta &= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

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Units 1 & 2 > Area 4 > Sequence 1 > Concept 6

Operations on complex numbers in polar form Summary screen and practice questions

Exercise 6.7 Basic operations on complex numbers in polar form

Technology free

1. **WE29** Express each of the following in the form $r\,\text{cis}\,\theta$ where $\theta \in (-\pi, \pi]$.
   a. $2\ \text{cis}\,\frac{\pi}{4} \times 3\,\text{cis}\,\frac{\pi}{2}$
   b. $5\ \text{cis}\,\frac{2\pi}{3} \times 4\,\text{cis}\left(-\frac{\pi}{3}\right)$
   c. $6\ \text{cis}\,\frac{3\pi}{4} \times \sqrt{5}\,\text{cis}\,\pi$
   d. $\sqrt{3}\,\text{cis}\left(-\frac{5\pi}{6}\right) \times \sqrt{2}\,\text{cis}\left(-\frac{\pi}{2}\right)$
   e. $\sqrt{7}\,\text{cis}\left(-\frac{7\pi}{12}\right) \times 2\,\text{cis}\,\frac{5\pi}{12}$
2. **WE30** Express the resultant complex numbers in question 1 in Cartesian form.
3. **WE31** Express the following products in polar form.
   a. $(2 + 2i)\left(\sqrt{3} + i\right)$
   b. $\left(\sqrt{3} - 3i\right)\left(2\sqrt{3} - 2i\right)$
   c. $\left(-4 + 4\sqrt{3}\,i\right)(-1 - i)$

4. **WE32** Express each of the following in the form $r\,\text{cis}\,\theta$ where $\theta \in (-\pi, \pi]$.

a. $12\ \text{cis}\,\frac{5\pi}{6} \div 4\,\text{cis}\,\frac{\pi}{3}$

b. $36\ \text{cis}\,\frac{3\pi}{4} \div 9\,\text{cis}\left(-\frac{\pi}{6}\right)$

c. $\sqrt{20}\ \text{cis}\left(-\frac{\pi}{2}\right) \div \sqrt{5}\ \text{cis}\left(-\frac{\pi}{5}\right)$

d. $4\sqrt{3}\ \text{cis}\,\frac{4\pi}{7} \div \sqrt{6}\ \text{cis}\,\frac{11\pi}{14}$

e. $3\sqrt{5}\ \text{cis}\left(-\frac{7\pi}{12}\right) \div 2\sqrt{10}\ \text{cis}\,\frac{5\pi}{6}$

5. **WE33** If $z = \sqrt{3}\ \text{cis}\,\frac{3\pi}{4}$ and $w = 2\,\text{cis}\left(-\frac{\pi}{4}\right)$, express each of the following in polar form.

a. z^3 **b.** w^4 **c.** z^4 **d.** w^5

6. **WE34** If $z = 1 - i$ and $w = -\sqrt{3} + i$, write the following in standard form.

a. z^{-4} **b.** w^{-3} **c.** z^{-3} **d.** w^{-5} **e.** $\frac{z^3}{w^4}$ **f.** z^2w^3

7. Determine $(2 + 2i)^2(1 - \sqrt{3}\,i)^4$ in standard form.

8. Write $\dfrac{\left(\sqrt{3} - i\right)^6}{\left(2 - 2\sqrt{3}\,i\right)^3}$ in the form $x + yi$.

9. **WE35** Use Pascal's triangle to expand $(4 + 5i)^4$.

10. **MC**

a. $\sqrt{5}\ \text{cis}\left(-\frac{\pi}{3}\right) \times \sqrt{8}\ \text{cis}\left(-\frac{\pi}{6}\right)$ is equal to:

A. $6\sqrt{2}\,i$ **B.** $-2\sqrt{10}\,i$ **C.** $-6\sqrt{3}$ **D.** $-6i$

b. If $z = \left(\sqrt{6} + \sqrt{2}\right) + \left(\sqrt{6} - \sqrt{2}\right)i$, then $\sqrt{2}\,z^{-3}$ is:

A. $1 + i$ **B.** $\sqrt{2}\,i$ **C.** $\frac{1}{64}(1 - i)$ **D.** $\sqrt{2} + i$

c. If $z = -1 - \sqrt{3}\,i$ and $w = 2 + 2i$, then $\frac{w^4}{z^3}$ is equal to:

A. $-4 + 4i$ **B.** $2\sqrt{3}$ **C.** $-4i$ **D.** -8

11. If $z = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$ and $w = \sqrt{3}\,\text{cis}\,\frac{\pi}{6}$, find the modulus and the argument of $\frac{z^6}{w^4}$.

12. If $z = 4 + i$ and $w = -3 - 2i$, determine $(z + w)^9$.

13. Find $z^6 + w^4$, if $z = \sqrt{2} - \sqrt{2}\,i$ and $w = 2 - 2i$.

14. If $z_1 = \sqrt{5}\ \text{cis}\left(-\frac{2\pi}{5}\right)$, $z_2 = 2\,\text{cis}\,\frac{3\pi}{8}$ and $z_3 = \sqrt{10}\ \text{cis}\,\frac{\pi}{12}$, find the modulus and the argument of $\dfrac{z_1^2 \times z_2^3}{z_3^4}$.

15. **WE36** Use de Moivre's theorem to prove:

a. $\sin 2\theta = 2\sin\theta\cos\theta$

b. $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$

16. Use de Moivre's theorem to prove:

a. $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

b. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

6.8 Roots of equations

6.8.1 Linear factors of real quadratic polynomials

The **roots of an equation** are also known as the solutions of an equation. Geometrically, the roots of a function are the x-intercepts of the function.

Up to this point, linear factors of real quadratic polynomials have been limited to those over the set of real numbers. Now, linear factors can be determined over the complex number field.

WORKED EXAMPLE 37

Determine the linear factors over C for $z^2 - 4z + 29$.

THINK	WRITE
1. Complete the square. *Note:* The sum of two squares has no real factors.	$z^2 - 4z + 29$ $= z^2 - 4z + 4 + (29 - 4)$ $= (z - 2)^2 + 25$
2. Write the expression as the difference of two squares.	$= (z - 2)^2 - 25i^2$ (using $i^2 = -1$)
3. Write the expression as a product of linear factors.	$= (z - 2 + 5i)(z - 2 - 5i)$
4. Write the answer.	The linear factors of $z^2 - 4z + 29$ are $(z - 2 + 5i)(z - 2 - 5i)$.

6.8.2 The general solution of real quadratic equations

Consider the quadratic equation $az^2 + bz + c = 0$, where the coefficients a, b and c are real. Recall that the roots depend upon the discriminant, $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the equation has two distinct real roots.
- If $\Delta = 0$, the equation has one real repeated root.
- If $\Delta < 0$, the equation has no real roots.

With the introduction of complex numbers, it can now be stated that if $\Delta < 0$, then the equation has one pair of **complex conjugate roots** of the form $a \pm bi$.

WORKED EXAMPLE 38

Solve for z, given $z^2 - 6z + 25 = 0$.

THINK	WRITE
Method 1: Using completing the square	$z^2 - 6z + 25 = 0$
1. Complete the square.	$z^2 - 6z + = -25$ $z^2 - 6z + 9 = -25 + 9$ $(z - 3)^2 = -16$
2. Substitute i^2 with -1.	$(z - 3)^2 = 16i^2$
3. Solve for z.	$z - 3 = \pm 4i$
4. State the two solutions for z.	$z = 3 \pm 4i$

Method 2: Using the general quadratic formula

1. Write the general quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. State the values for a, b and c.

$$z^2 - 6z + 25 = 0$$
$$a = 1,\ b = -6,\ c = 25$$

3. Substitute the values into the formula and simplify.

$$z = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(25)}}{2}$$
$$= \frac{6 \pm \sqrt{-64}}{2}$$
$$= \frac{6 \pm \sqrt{64i^2}}{2}$$
$$= \frac{6 \pm 8i}{2}$$
$$= 3 \pm 4i$$

4. State the two solutions.

$$z = 3 \pm 4i$$

6.8.3 The relationship between roots and coefficients

For a quadratic with real coefficients, we have seen that if the discriminant is negative, then the roots occur in complex conjugate pairs. Here we find a relationship between the roots and the coefficients.

Given a quadratic $az^2 + bz + c = 0$ where $a \neq 0$, then:

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$

Let the roots of the quadratic equation be α and β, so the factors are $(z - \alpha)(z - \beta)$. Expanding the brackets, the expression becomes:

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0 \text{ or}$$

$$z^2 - (\text{sum of roots})\, z + \text{product of roots} = 0$$

$$\text{so that } \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

This gives us a relationship between the roots of a quadratic equation and its coefficients. Rather than formulating a problem as solving a quadratic equation, we now consider the reverse problem. That is forming a quadratic equation with real coefficients, given one of the roots.

WORKED EXAMPLE 39

Determine the equation of the quadratic $P(z)$, with real coefficients given that $P(4 - 3i) = 0$.

THINK	WRITE
1. State the value of the second root, recalling that complex conjugate roots are of the form $a \pm bi$.	$P(4 - 3i) = 0$ $\Rightarrow 4 - 3i$ is a root of $P(z)$. $\Rightarrow 4 + 3i$ is also a root of $P(z)$. Let $\alpha = 4 - 3i$ and let $\beta = 4 + 3i$.
2. Find the sum of the roots.	$\alpha + \beta = 8$
3. Find the product of the roots.	$\alpha\beta = 16 - 9i^2 = 25$
4. State the quadratic equation.	$P(z) = (z - 4 + 3i)(z - 4 - 3i) = z^2 - 8z + 25$

6.8.4 Complex equations reducible to quadratics

Although solving general quartic equations is beyond the scope of this course, there are special cases of polynomials of degree 4 involving only z^4 and z^2 terms. These equations can be solved by using a suitable substitution and reducing the quartic equation to a quadratic. For example, by using the substitution $u = z^2$, the equation $az^4 + bz^2 + c = 0$ becomes $au^2 + bu + c = 0$.

WORKED EXAMPLE 40

Solve for z if $z^4 + 3z^2 - 28 = 0$.

THINK	WRITE
1. Use a suitable substitution to reduce the quartic equation to a quadratic equation.	Let $u = z^2$, then $u^2 = z^4$. $z^4 + 3z^2 - 28 = 0$ $u^2 + 3u - 28 = 0$
2. Factorise the expression.	$(u + 7)(u - 4) = 0$
3. Substitute u with z^2.	$(z^2 + 7)(z^2 - 4) = 0$
4. Factorise the expression, using the difference of two squares.	$\left(z^2 - \left(\sqrt{7}i\right)^2\right)(z^2 - 2^2) = 0$ $\left(z + \sqrt{7}i\right)\left(z - \sqrt{7}i\right)(z + 2)(z - 2) = 0$
5. State all four roots.	$z = \pm\sqrt{7}i$ and $z = \pm 2$

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Units 1 & 2 > Area 4 > Sequence 1 > Concept 7

Factorising quadratic expressions over the complex number field Summary screen and practice questions

Exercise 6.8 Roots of equations

Technology free

1. **WE37** Determine the linear factors over C for $z^2 - 6z + 25$.
2. Determine the linear factors over C for $z^2 + 4z + 7$.
3. Determine the linear factors of each of the following
   a. $z^4 + z^2 - 20$
   b. $z^4 + 6z^2 - 27$
   c. $z^4 - 13z^2 - 48 = 0$
   d. $z^4 - 23z^2 - 50$
4. **WE38** Solve for z, if $z^2 - 4z + 29 = 0$.
5. Solve for z, if $z^2 + 2z + 26 = 0$.
6. Calculate the roots of each of the following.
   a. $z^2 + 2z + 17 = 0$
   b. $z^2 - 4z + 20 = 0$
   c. $z^2 - 6z + 13 = 0$
   d. $z^2 + 10z + 41 = 0$

7. **WE39** Determine the quadratic $P(z)$ with real coefficients given that $P(-5-2i)=0$.
8. Determine the quadratic $P(z)$ with real coefficients given that $P(5i)=0$.
9. Form a quadratic with integer coefficients for each of the following cases.
   a. -2 and $\frac{1}{3}$ are the roots
   b. $2-6i$ is a root.
   c. $-2+3i$ is a root.
   d. $-4-5i$ is a root.
10. Form a quadratic with integer coefficients for each of the following cases.
    a. $\frac{1}{2}$ and $-\frac{1}{5}$ are the roots.
    b. $4-\sqrt{3}\,i$ is a root.
    c. $-5-\sqrt{7}\,i$ is a root.
    d. $-3-\sqrt{8}\,i$ is a root.

11. a. Solve for z if $z^2+(2-i)z+(3-i)=0$.
    b. Determine the two numbers such that their sum is 6 and their product is 10.
    c. i. Show that $(3-2i)^2=5-12i$.
    ii. Solve for z if $z^2+(4-6i)z=10$.
12. a. Solve for z if $z^2+(4-i)z+(5-2i)=0$.
    b. Determine the two numbers such that their sum is 8 and their product is 25.
    c. i. Show that $(4-3i)^2=7-24i$.
    ii. Solve for z if $z^2+\sqrt{11}\,z+(1+6i)=0$.
13. **WE40** Solve for z if $z^4+z^2-30=0$.
14. Solve for z if $2z^4-z^2-21=0$.

Technology active

15. Solve for z in each of the following.
    a. $z(4-z)=8$
    b. $z(6-z)=10$
    c. $z(2-z)=26$
    d. $z(8-z)=41$
16. Solve for z in each of the following.
    a. $(z+2)(z-6)+25=0$
    b. $(z-2)(z+8)+30=0$
    c. $(z+2)(z-4)+12=0$
    d. $(z+6)(z+4)+7=0$
17. Find the roots of each of the following.
    a. $4z^2+12z+10=0$
    b. $4z^2+20z+29=0$
    c. $4z^2-12z+13=0$
    d. $9z^2-42z+53=0$
18. Solve each of the following.
    a. $z^4+3z^2-28=0$
    b. $z^4+11z^2+24=0$
    c. $z^4-21z^2-100=0$
    d. $z^4-2z^2-15=0$
19. Solve each of the following.
    a. $z^8-25z^4+144=0$
    b. $z^8-29z^4+100=0$
20. Solve each of the following.
    a. $z^6+7z^3-8=0$
    b. $z^6+19z^3-216=0$

6.9 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

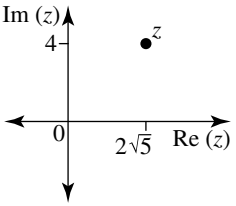
Simple familiar

Questions **1** and **2** refer to the complex number $z = 2\sqrt{5} - 4i$.

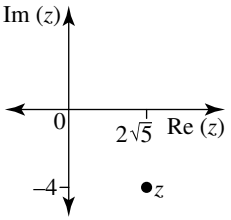
1. Identify the real and imaginary parts of z.

2. **MC** The Argand diagram which correctly represents z is:

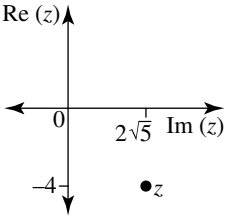
A.



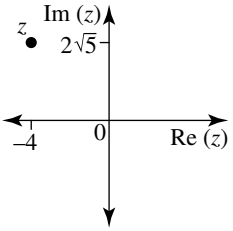
B.



C.



D.



3. Simplify $i^6 - i^3(i^2 - 1)$.

Questions **4** and **5** refer to the complex numbers $u = 5 - 1$ and $v = 4 + 3i$. You may choose to use a technology of your choice.

4. Evaluate the expression $2u - v$.

5. Evaluate the expression uv.

6. If $z = 3 - 8i$, then find; using a technology of your choice:

a. $\text{Im}\left(z^2\right)$

b. a and b if $z^3 = a + bi$.

7. If $z = 2 - 5i$, $u = -3 + i$ and $w = 1 + 2i$, evaluate:

a. $z - 2u + 3w$ **b.** $|z|$ **c.** $|u|\,|z + w|$ **d.** $\text{Im}\left(\dfrac{1 + 2i}{1 - i}\right)$

8. Simplify the expression $\dfrac{2i}{1 + i} - \dfrac{3}{2 - i}$ using a technology of your choice.

9. If $z = 6 - 2i$ and $w = 5 + 3i$, express $\dfrac{z}{w}$ in the form $a + bi$, $a, b \in R$. You may use technology.

10. Evaluate $\text{Arg}\,(2 - 2i)$.

11. If $z = -7 - 7i$, express z in polar form.

12. Determine the roots of the equation $z^2 + 2z - 5 = 0$ using a technology of your choice.

Complex familiar

13. How many degrees apart are two consecutive roots of $z^8 = 1$ on the unit circle?

14. If $z_1 = 10\,\text{cis}\dfrac{\pi}{4}$ and $z_2 = 5\,\text{cis}\left(-\dfrac{\pi}{6}\right)$, evaluate $z_1 z_2$ in polar form.

15. Evaluate $12\sqrt{2}\,\text{cis}\dfrac{3\pi}{4} \div 3\,\text{cis}\left(-\dfrac{\pi}{2}\right)$ in standard form.

16. Let $z = 2\,\text{cis}\frac{\pi}{3}$ and $w = 2\,\text{cis}\frac{\pi}{4}$.

a. Express $\frac{z}{w}$ in the form $r\cos\theta + r\sin\theta i$.

b. Express z and w in Cartesian form.

c. Express $\frac{z}{w}$ in Cartesian form.

d. Using the results of parts **a** and **c**, find the exact values for:

i. $\cos\frac{\pi}{12}$ **ii.** $\sin\frac{\pi}{12}$ **iii.** $\tan\frac{\pi}{12}$.

e. By letting $z = 2\,\text{cis}\frac{\pi}{4}$ and $w = 2\,\text{cis}\frac{\pi}{6}$ and following parts **a** to **c** for zw instead of $\frac{z}{w}$, deduce that $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$.

Complex unfamiliar

17. Let $u = 1 - i$.

a. **i.** Find $u\overline{u}$.

ii. Find $\text{Arg}\,u + \text{Arg}\left(2\overline{u}\right)$.

b. Let $z = x + yi$, $x, y \in R$, and $|z - \overline{u}| = |z - 2u|$.
Find the value of x when $y = 0$.

18. Consider the complex number z such that $z = 3 + 2i$.

a. Find the value for iz, i^2z, i^3z and i^4z. (Give answers in standard $x + yi$ form.)

b. Comment on the value of i^4z.

c. Plot each number from part **a** on the same Argand diagram.

d. Use a pair of compasses to draw a circle whose centre is at the origin and which passes through each point on the diagram.

e. Find the radius of the circle, giving your answer in exact (surd) form.

f. Carefully study the five points on your diagram. What transformation is required to transform:

i. point z into point iz?

ii. point iz into point i^2z?

iii. point i^2z into point i^3z?

g. On the Argand diagram, what transformation takes place when a complex number is multiplied by i?

h. For a complex number z such that $z = x + yi$, describe the curve that all points representing numbers of the form zi^n (that is, z, zi, zi^2, zi^3, and so on) would lie on an Argand diagram.

19. **a.** For any complex number $z = x + yi$ where both x and y are real, describe the transformation required to obtain $\frac{1}{z}$.

b. For any complex number $z = x + yi$ where both x and y are real, describe the transformation required to obtain iz.

20. **a.** Given $u = 2i$, evaluate u^4.

b. Hence, find all the values of z if $z^4 + 16 = 0$.

c. Plot all of the solutions of $z^4 + 16 = 0$ on an Argand diagram. Comment on their relative positions.

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Units 1 & 2 Sit chapter test

Answers

Chapter 6 Complex numbers

Exercise 6.2 Introduction to complex numbers

1. **a.** $3i$ **b.** $5i$ **c.** $7i$
d. $\sqrt{3}\,i$ **e.** $\sqrt{11}\,i$ **f.** $\sqrt{7}\,i$
g. $\frac{2}{3}i$ **h.** $\frac{6}{5}i$

2. **a.** 9, 5 **b.** 5, −4 **c.** −3, −8
d. −6, 11 **e.** 27, 0 **f.** 0, 2
g. −5, 1 **h.** 0, −17

3. **a.** $-1+i$ **b.** $1+i$ **c.** $1-i$
d. $0+0i$ **e.** $-1+2i$ **f.** $-1+i$
g. $1+0i$ **h.** $1-2i$

4. $z=-2-3i$, $w=7+3i$; sample responses can be found in the worked solutions in the online resources.

5. **a.** −5 **b.** 15 **c.** 0
d. −6 **e.** 2 **f.** 0
g. −9 **h.** 2

6. 4 – i

7. **a.** D **b.** C **c.** C **d.** D

8. Sample responses can be found in the worked solutions in the online resources.

9. **a.** −2 **b.** $-2i$ **c.** $\sqrt{2}$ **d.** $2i$

10. **a.** 13 **b.** −11 **c.** 10 **d.** 7

11. **a.** i **b.** 0 **c.** i **d.** 0

12. **a.** $-i$ **b.** 0 **c.** $-i$ **d.** 0

Exercise 6.3 Basic operations using complex numbers

1. **a.** $4-i$ **b.** $1-14i$ **c.** $-4-2i$
d. $9-13i$ **e.** $-12+4i$ **f.** $-9-5i$

2. **a.** $-12+3i$ **b.** $-19-8i$ **c.** $12+23i$
d. $-25+3i$ **e.** $-50-48i$ **f.** $-41-28i$

3. **a.** $7-23i$ **b.** $4+45i$ **c.** $-50-13i$
d. $63-37i$ **e.** $-85-132i$ **f.** $176-61i$

4. **a.** $111+33i$ **b.** $31-8i$ **c.** $22-48i$
d. 61 **e.** −53 **f.** $32-126i$

5. $14+52i$

6. −3

7. **a.** −8 **b.** −5 **c.** −9
d. 35 **e.** −30 **f.** −115

8. **a.** $a=5, b=-2$ **b.** $a=\frac{21}{41}, b=-\frac{16}{41}$
c. $a=1, b=5$ **d.** $a=-2, b=-3$

9. **a.** D **b.** B **c.** C

10. **a.** $-13-4i$ **b.** $-50+8i$
c. −89 **d.** $x=-3, y=-4$

11. **a.** $x=5, y=2$ **b.** $x=4, y=3$
c. $x=5, y=-2$ **d.** $x=3, y=-4$

Exercise 6.4 Complex conjugates and division of complex numbers

1. **a.** $7-10i$ **b.** $5+9i$ **c.** $3-12i$
d. $\sqrt{7}+3i$ **e.** $5-2i$ **f.** $-6+\sqrt{11}\,i$

2. Sample responses can be found in the worked solutions in the online resources.

3. $\frac{1}{2}+\frac{1}{2}i$

4. **a.** $0-i$
b. $0-i$
c. $-\frac{7}{25}+\frac{26}{25}i$
d. $\frac{14}{29}-\frac{23}{29}i$
e. $\frac{43}{53}+\frac{18}{53}i$
f. $\frac{2\sqrt{5}-\sqrt{6}}{7}+\frac{2\sqrt{2}+\sqrt{15}}{7}i$

5. **a.** $\frac{2}{5}+\frac{1}{5}i$ **b.** $\frac{3}{10}-\frac{1}{10}i$ **c.** $\frac{4}{25}+\frac{3}{25}i$
d. $\frac{5}{41}-\frac{4}{41}i$ **e.** $-\frac{3}{13}-\frac{2}{13}i$ **f.** $\frac{\sqrt{3}}{5}+\frac{\sqrt{2}}{5}i$

6. $10+24i$

7. **a.** $\frac{23}{10}$ **b.** $\frac{9}{10}$ **c.** $\frac{17}{5}$
d. $-\frac{16}{5}$ **e.** $-\frac{14}{5}$

8. $\frac{17}{2}+\frac{9}{2}i$

9. −29

10. $-33+58i$

11. **a.** D **b.** C **c.** B

12–14. Sample responses can be found in the worked solutions in the online resources.

15. −16

16. **a.** $-12+11i$ **b.** $-30-19i$ **c.** 0

17. Sample responses can be found in the worked solutions in the online resources.

18. **a.** −4, 16, −64
b. Sample responses can be found in the worked solutions in the online resources.

19. $a=-1, b=\pm\sqrt{2}$

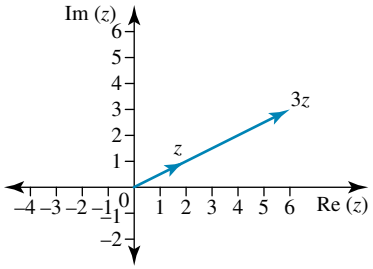
20. $a=-\frac{1}{2}, b=\frac{1}{2}$

21. $a=2, b=1$; $a=-2, b=-1$

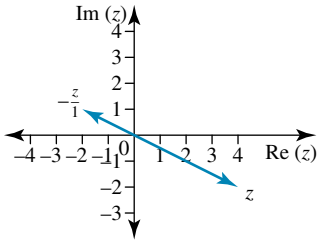
22. **a.** **i.** 13 **ii.** 5
b. **i. ii. iii.** Sample responses can be found in the worked solutions in the online resources.
c. **i.** $\frac{2}{13}-\frac{3}{13}i$
ii. $\frac{1}{5}-\frac{2}{5}i$
d. $-8+16i$
e. $-2+10i$
f. $-\frac{4}{65}+\frac{7}{65}i$

Exercise 6.5 The complex plane (the Argand plane)

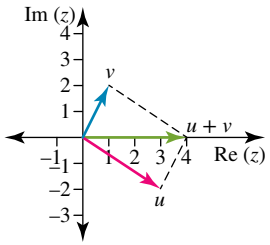
1. $3z = 3 + 6i$, $3z$ has a length of 3 times that of z.



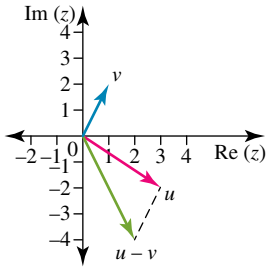
2. $-\frac{z}{2} = -2 + i$; $-\frac{z}{2}$ is in the opposite direction to z and half its length.



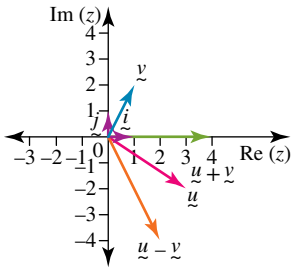
3. a. $u + v = 4$



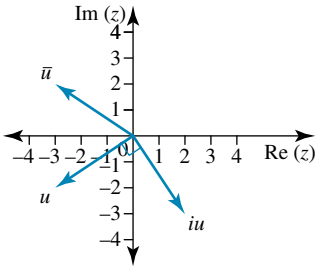
b. $u - v = 2 - 4i$



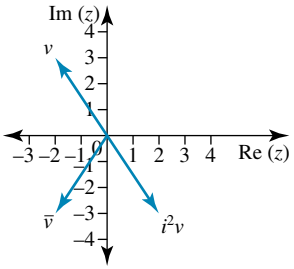
4.



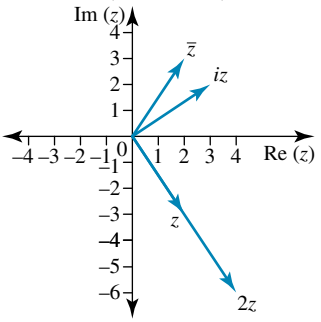
5. $\overline{u} = -3 + 2i$, , $\overline{u}$ is the reflection of the complex number in the real axis and iu is a rotation of 90° anticlockwise from u.



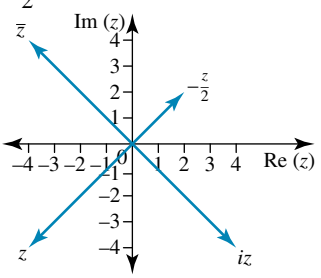
6. $\overline{v} = -2 - 3i$, $i^2v = 2 - 3i$, $\overline{v}$ is the reflection of the complex number in the real axis and i^2v is a rotation of 90° anti-clockwise from v.



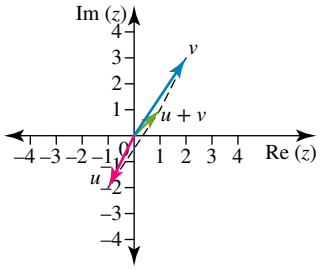
7. a. $2z = 4 - 6i$, $\overline{z} = 2 + 3i$, $iz = 3 + 2i$



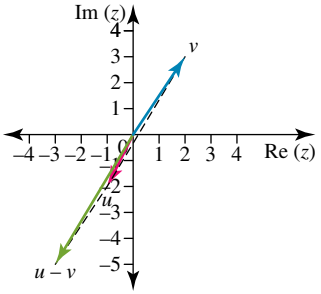
b. $-\frac{z}{2} = 2 + 2i$, $\overline{z} = -4 + 4i$, $iz = 4 - 4i$



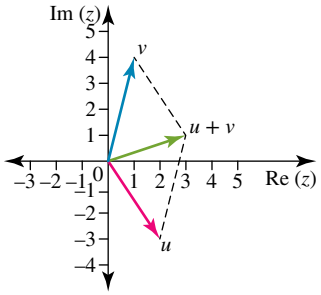
8. a. i. $u + v = 1 + i$



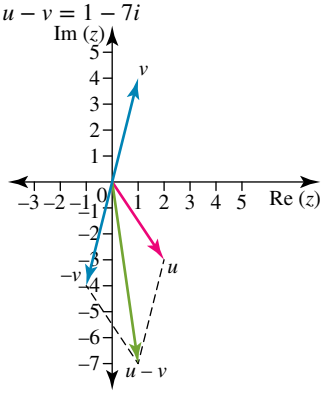
ii. $u - v = -3 - 5i$



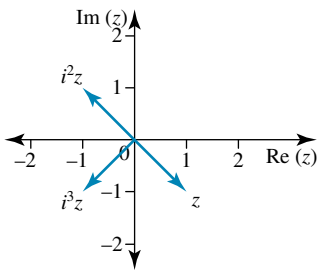
b. i. $u + v = 3 + i$



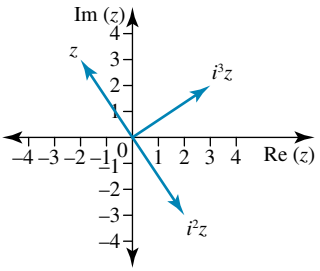
ii. $u - v = 1 - 7i$



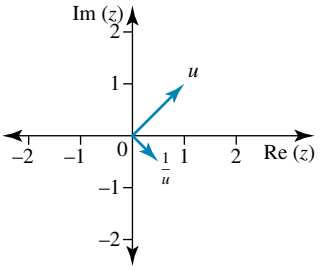
9. a. $z = 1 - i, i^2z = -1 + i, i^3z = -1 - i$



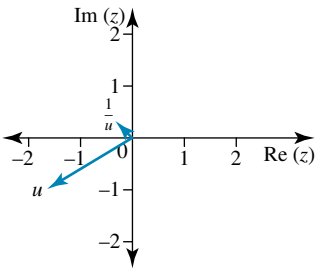
b. $z = -2 + 3i, i^2z = 2 - 3i, i^3z = 3 + 2i$



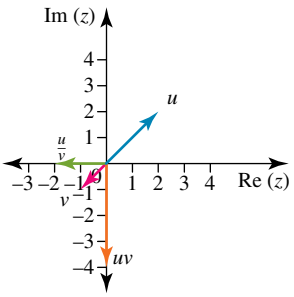
10. a. $u = 1 + i, \frac{1}{u} = \frac{1}{2} - \frac{1}{2}i$



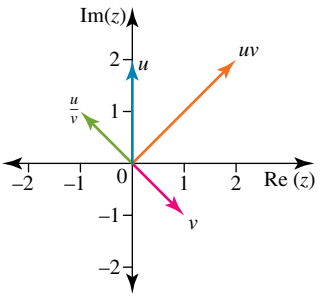
b. $u = -\sqrt{3} - i, \frac{1}{u} = -\frac{\sqrt{3}}{4} + \frac{1}{4}i$



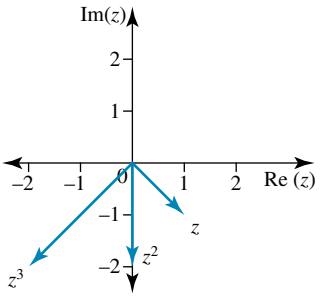
11. a. $uv = -4i, \frac{u}{v} = -2$



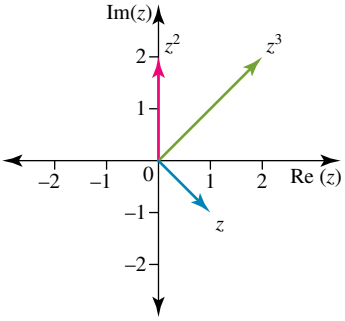
b. $uv = 2 + 2i, \frac{u}{v} = -1 + i$



12. a. $z^2 = -2i$, $z^3 = -2 - 2i$



b. $z^2 = 2i$, $z^3 = 2 + 2i$



13. a. Sample responses can be found in the worked solutions in the online resources.

b. (4, 3)

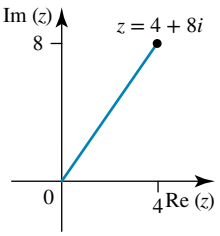
c. i. Sample responses can be found in the worked solutions in the online resources.

ii. Vector $\underset{\sim}{v}$ is a rotation of vector $\underset{\sim}{u}$ through an angle of 90°, which is similar to the complex numbers z and iz.

iii. Vectors and complex numbers have a similar structure. The term 'isomorphic' means having the same form and is used in many branches of mathematics to identify mathematical objects that have the same structural properties.

Exercise 6.6 Complex numbers in polar form

1. a.

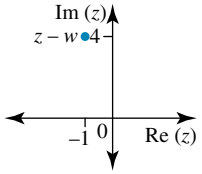


b. $|z| = 4\sqrt{5}$

2. a. 13 **b.** 3 **c.** $\sqrt{65}$

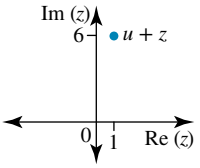
d. $3\sqrt{5}$ **e.** $\sqrt{5}$ **f.** 5

3. a. i.



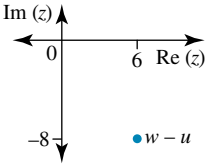
ii. $\sqrt{17}$

b. i.



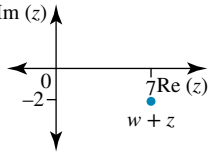
ii. $\sqrt{37}$

c. i.



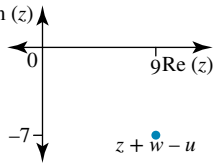
ii. 10

d. i.



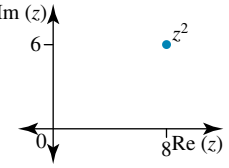
ii. $\sqrt{53}$

e. i.



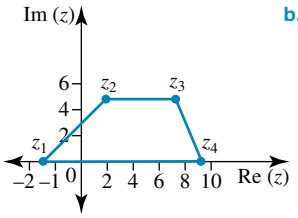
ii. $\sqrt{130}$

f. i.



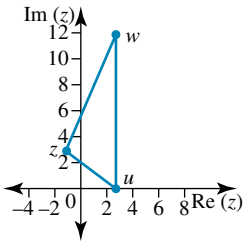
ii. 10

4. a.



b. 42.5 square units

5. a.



b. 24 square units

6. a. $v = -5 + 4i$

b. $v = 4 + 5i$

c. $2b(a + c)$

7. a. $\frac{25}{2}$ **b.** $\frac{169}{2}$ **c.** $\frac{1}{2}(a^2 + b^2)$

8. a. 25

b. $b = \pm\sqrt{14}$

c. $a^2 + b^2$

9. a. $x^2 - 7x + y^2 + y = 0$, $\frac{25\pi}{2}$

b. $x^2 + 7x + y^2 - 17y = 0$, $\frac{169\pi}{2}$

10. $x^2 - (a - b)x + y^2 - (a + b)y = 0$, $\frac{\pi}{2}(a^2 + b^2)$

11. **a.** 0.588 **b.** $\frac{\pi}{6}$ **c.** $-\frac{\pi}{4}$
d. 2.034 **e.** $-\frac{2\pi}{3}$ **f.** -1.030
g. $\frac{\pi}{2}$ **h.** π **i.** $-\frac{\pi}{2}$
j. 0

12. **a.** $-\frac{\pi}{2}$ **b.** $\frac{\pi}{6}$ **c.** $-\frac{\pi}{8}$
d. $\frac{3\pi}{4}$ **e.** $-\frac{5\pi}{6}$ **f.** $\frac{6\pi}{7}$
g. $\frac{2\pi}{5}$ **h.** $\frac{11\pi}{12}$

13. **a.** $3\sqrt{2}, -\frac{\pi}{4}$ **b.** $5\sqrt{2}, \frac{3\pi}{4}$ **c.** $2, -\frac{2\pi}{3}$
d. $8, \frac{\pi}{6}$ **e.** $\sqrt{149}, -2.182$ **f.** $2\sqrt{10}, 1.893$
g. $4, \frac{\pi}{3}$

14. **a.** $\sqrt{2}\,\text{cis}\frac{3\pi}{4}$ **b.** $2\sqrt{2}\,\text{cis}\frac{\pi}{6}$
c. $\sqrt{10}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ **d.** $2\sqrt{5}\,\text{cis}\left(-\frac{\pi}{3}\right)$
e. $\text{cis}\left(-\frac{2\pi}{3}\right)$ **f.** $\frac{\sqrt{2}}{4}\,\text{cis}\frac{3\pi}{4}$

15. **a.** $-1+\sqrt{3}\,i$ **b.** $\frac{3\sqrt{2}}{2}+\frac{3\sqrt{2}}{2}i$
c. $-\frac{\sqrt{15}}{2}-\frac{\sqrt{5}}{2}i$ **d.** $\frac{\sqrt{14}}{2}+\frac{\sqrt{14}}{2}i$
e. $\frac{\sqrt{14}}{2}+\frac{\sqrt{14}}{2}i$ **f.** $0+8i$
g. $-\sqrt{3}$

16. C **17.** B **18.** D **19.** D **20.** D

Exercise 6.7 Basic operations on complex numbers in polar form

1. **a.** $6\,\text{cis}\frac{3\pi}{4}$ **b.** $20\,\text{cis}\frac{\pi}{3}$
c. $6\sqrt{5}\,\text{cis}\left(-\frac{\pi}{4}\right)$ **d.** $\sqrt{6}\,\text{cis}\frac{2\pi}{3}$
e. $2\sqrt{7}\,\text{cis}\left(-\frac{\pi}{6}\right)$

2. **a.** $-3\sqrt{2}+3\sqrt{2}\,i$ **b.** $10+10\sqrt{3}\,i$
c. $3\sqrt{10}-3\sqrt{10}\,i$ **d.** $-\frac{\sqrt{6}}{2}+\frac{3\sqrt{2}}{2}i$
e. $\sqrt{21}-\sqrt{7}\,i$

3. **a.** $4\sqrt{2}\,\text{cis}\frac{5\pi}{12}$
b. $8\sqrt{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$
c. $8\sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$

4. **a.** $3\,\text{cis}\frac{\pi}{2}$ **b.** $4\,\text{cis}\frac{11\pi}{12}$
c. $2\,\text{cis}\left(-\frac{3\pi}{10}\right)$ **d.** $2\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{14}\right)$
e. $\frac{3\sqrt{2}}{4}\,\text{cis}\frac{7\pi}{12}$

5. **a.** $3\sqrt{3}\,\text{cis}\frac{\pi}{4}$ **b.** $16\,\text{cis}\,\pi$
c. $9\,\text{cis}\,\pi$ **d.** $32\,\text{cis}\frac{3\pi}{4}$

6. **a.** $-\frac{1}{4}+0i$
b. $0-\frac{1}{8}i$
c. $-\frac{1}{4}+\frac{1}{4}i$
d. $\frac{\sqrt{3}}{64}-\frac{1}{64}i$
e. $\frac{\sqrt{2}}{8}\cos\frac{\pi}{12}-\frac{\sqrt{2}}{8}\sin\frac{\pi}{12}i$
f. $16+0i$

7. $-64\sqrt{3}-64i$

8. $1+0i$

9. $-1519-720i$

10. **a.** B **b.** C **c.** D

11. $\frac{8}{9}, -\frac{\pi}{6}$

12. $16-16i$

13. $-64+64i$

14. $\frac{2}{5}, -\frac{\pi}{120}$

15, 16. Sample proofs can be found in the worked solutions in the online resources.

Exercise 6.8 Roots of equations

1. $(z-3+4i)(z-3-4i)$

2. $\left(z+2+\sqrt{3}\,i\right)\left(z+2-\sqrt{3}\,i\right)$

3. **a.** $(z+2)(z-2)\left(z+\sqrt{5}\,i\right)\left(z-\sqrt{5}\,i\right)$
b. $\left(z+\sqrt{3}\right)\left(z-\sqrt{3}\right)(z+3\,i)(z-3\,i)$
c. $(z+4)(z-4)\left(z+\sqrt{3}\,i\right)\left(z-\sqrt{3}\,i\right)$
d. $(z+5)(z-5)\left(z+\sqrt{2}\,i\right)\left(z-\sqrt{2}\,i\right)$

4. $2\pm5i$

5. $-1\pm5i$

6. **a.** $-1\pm4i$ **b.** $2\pm4i$
c. $3\pm2i$ **d.** $-5\pm4i$

7. $z^2+10z+29$

8. z^2+25

9. **a.** $3z^2+5z-2$ **b.** $z^2-4z+40$
c. $z^2+4z+13$ **d.** $z^2+8z+41$

10. **a.** $10z^2-32z-1$ **b.** $z^2-8z+19$
c. $z^2+10z+32$ **d.** $z^2+6z+17$

11. **a.** $-1+2i\,,\ -1-i$
b. $3\pm i$
c. **i.** Sample responses can be found in the worked solutions in the online resources.
ii. $-5+5i,\ 1+i$

12. **a.** $-2+\left(\frac{\sqrt{5}+1}{2}\right)i, -2+\left(\frac{1-\sqrt{5}}{2}\right)i$
b. $4\pm3i$

c. **i.** Sample responses can be found in the worked solutions in the online resources.

ii. $-\left(\frac{\sqrt{11}+4}{2}\right)+\frac{3}{2}i, -\left(\frac{\sqrt{11}-4}{2}\right)-\frac{3}{2}i$

13. $\pm\sqrt{5}, \pm\sqrt{6}i$

14. $\pm\sqrt{\frac{7}{2}}i, \pm\sqrt{3}$

15. **a.** $2 \pm 2i$ **b.** $3 \pm i$
c. $1 \pm 5i$ **d.** $4 \pm 5i$

16. **a.** $2 \pm 3i$ **b.** $-3 \pm \sqrt{5}\,i$
c. $1 \pm \sqrt{3}\,i$ **d.** $-5 \pm \sqrt{6}\,i$

17. **a.** $-\frac{3}{2} \pm \frac{1}{2}i$ **b.** $-\frac{5}{2} \pm i$
c. $\frac{3}{2} \pm i$ **d.** $\frac{7}{3} \pm \frac{2}{3}i$

18. **a.** $z = \pm 2, \pm\sqrt{7}i$ **b.** $z = \pm\sqrt{3}i, \pm 2\sqrt{2}i$
c. $z = \pm 5, \pm 2i$ **d.** $z = \pm\sqrt{5}, \pm\sqrt{3}i$

19. **a.** $\pm 2, \pm\sqrt{3}, \pm 2i, \pm\sqrt{3}i$
b. $\pm\sqrt{5}, \pm\sqrt{2}, \pm\sqrt{2}i \pm\sqrt{5}i$

20. **a.** $1, -2, 1 \pm \sqrt{3}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

b. $2, -3, -1 \pm \sqrt{3}i, \frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$

6.9 Review: exam practice

1. Real: $2\sqrt{5}$, imaginary: -4

2. B

3. $-1 - 2i$

4. $6 - 5i$

5. $23 + 11i$

6. **a.** -48 **b.** $a = -549, b = 296$

7. **a.** $11 - i$ **b.** $\sqrt{29}$
c. $6\sqrt{5}$ **d.** $-\frac{1}{2}$

8. $-\frac{1}{5} + \frac{2}{5}i$

9. $\frac{12}{17} - \frac{14}{17}i$

10. $-\frac{\pi}{4}$

11. $7\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

12. $-1 \pm 2i$

13. $45°$

14. $50\,\text{cis}\,\frac{\pi}{12}$

15. $-4 - 4i$

16. **a.** $\cos\frac{\pi}{2} + \sin\frac{\pi}{12}i$

b. $z = 1 + \sqrt{3}i,\ w = \sqrt{2} + \sqrt{2}i$

c. $\frac{\sqrt{2}+\sqrt{6}+\left(\sqrt{6}-\sqrt{2}\right)i}{4}$

d. **i.** $\frac{\sqrt{6}+\sqrt{2}}{4}$ **ii.** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **iii.** $2 - \sqrt{3}$

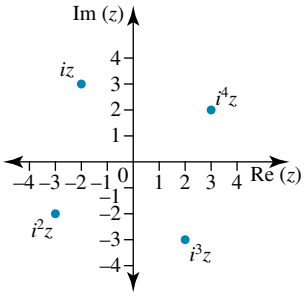
e. Sample responses can be found in the worked solutions in the online resources.

17. **a.** **i.** 2 **ii.** 0 **b.** $x = 3$

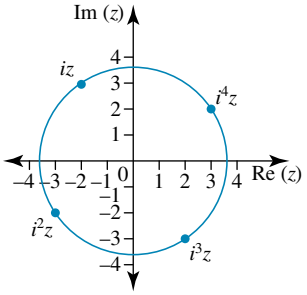
18. **a.** $iz = -2 + 3i, i^2z = -3 - 2i, i^3z = 2 - 3i, i^4z = 3 + 2i$

b. $i^4z = z$

c.



d.



e. $\sqrt{13}$

f. **i**, **ii**, **iii** Rotation of $\frac{\pi}{4}$ (90°) anticlockwise about the origin

g. Rotation of $\frac{\pi}{4}$ (90°) anticlockwise about the origin.

h. All points will lie on a circle with the origin as the centre and radius $\sqrt{x^2 + y^2}$.

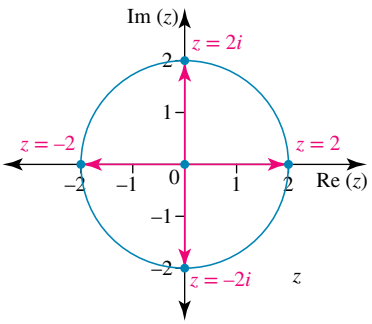
19. **a.** Reflecting z in the Real axis

b. Rotating z through $\frac{\pi}{4}$ (90°) in an anticlockwise direction

20. **a.** 16

b. $\pm 2i, \pm 2$

c.

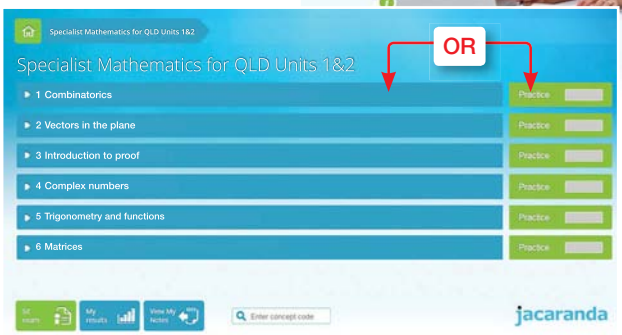


There are one pair of real roots and one pair of conjugate roots. The roots are equally spaced by $\frac{\pi}{2}$ and all lie on a circle of radius 2.

TOPIC 1 Complex numbers 1

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

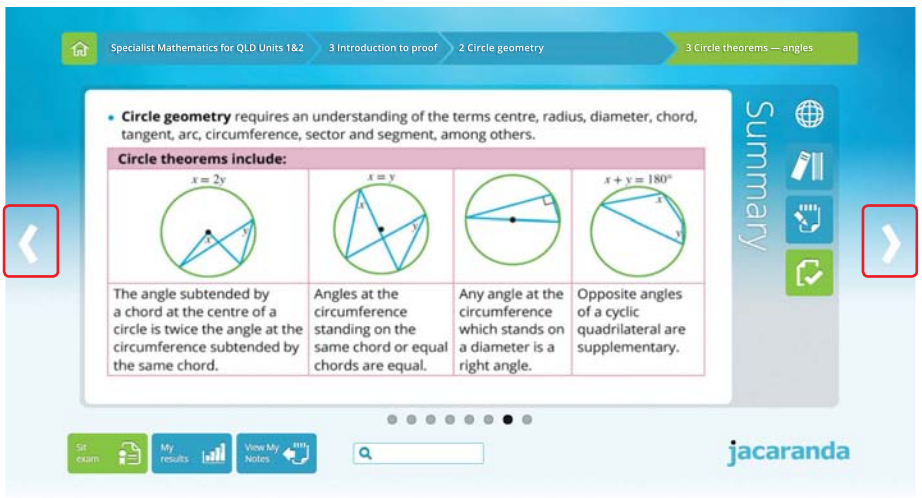
- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



- Select **Practice** at the sequence level to access all questions in the sequence.

- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



CHAPTER 7
Sketching graphs

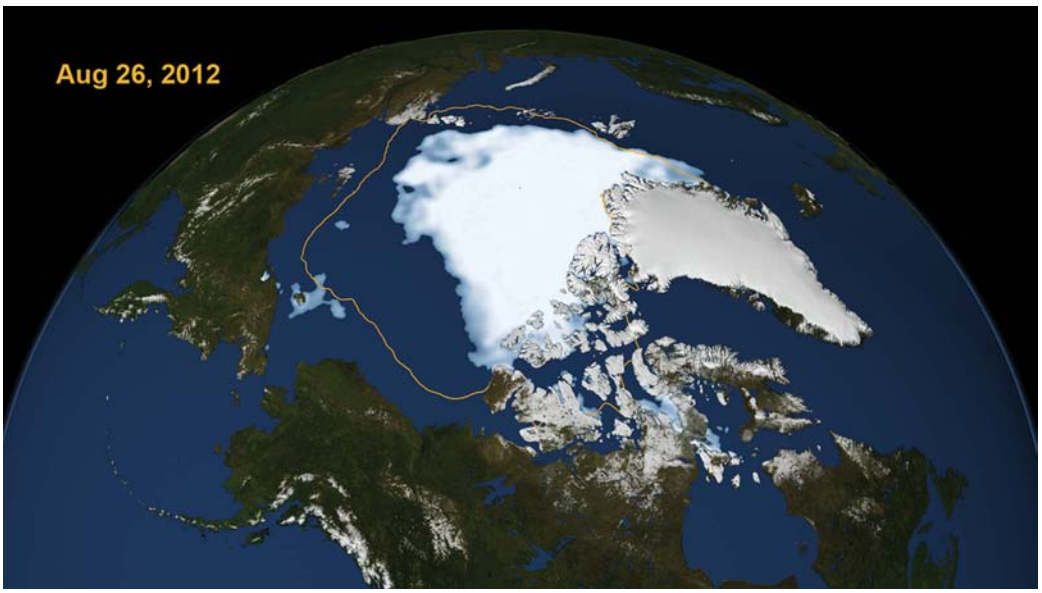
7.1 Overview

7.1.1 Introduction

Functions define special relationships between two variables — specifically, relationships where for every input value (x) there is one output value (y). Functions were first defined in the late 17th century by Leibniz as any quantity varying along a curve that is described by an equation. In 1734, Euler introduced the familiar $f(x)$ notation.

Today functions are used to model relationships between variables across a variety of disciplines, from forensic science and computer programming to climate science. For example, climate scientists use data about sea level, temperature, carbon dioxide and Arctic ice extent to define functions that allow them to create the complex models required to predict the extent of global climate change.

If we understand the relationship between two variables, it is possible to model their relationship. Knowing what a function looks like helps us understand why functions behave the way that they do. In this section you will explore the graphing of modulus functions, reciprocal functions and rational functions



LEARNING SEQUENCE

7.1 Overview
7.2 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$
7.3 Sketching graphs of reciprocal functions
7.4 Sketching graphs of rational functions
7.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.2 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

7.2.1 Review of relations and function

A mathematical **relation** is a set of ordered pairs. The pairs may be listed, for example $A = \{(-2, 4), (1, 5), (3, 4)\}$, or described by a rule such as $B = \{(x, y) : y = 2x\}$ or $C = \{(x, y) : y \leq 2x\}$. They may also be presented as a graph on coordinate axes.

For a set of ordered pairs (x, y), the **domain** is the set of all the x-values of the ordered pairs and the **range** is the set of all the y-values of the ordered pairs.

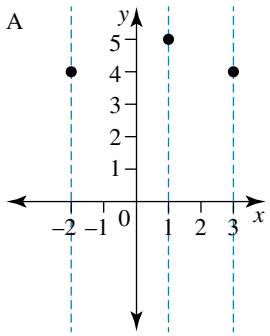
For $A = \{(-2, 4), (1, 5), (3, 4)\}$, the domain is $\{-2, 1, 3\}$ and the range is $\{4, 5\}$. For both $B = \{(x, y) : y = 2x\}$ and $C = \{(x, y) : y \leq 2x\}$, the domain is R and the range is R where R is the set of real numbers.

The graph of any polynomial relation normally has a domain of R. However, restrictions can be placed on the values of the variables. If there are restrictions, the relation is defined on a **restricted domain**.

A **function** is a particular type of relation: in the set of ordered pairs, every x-value is paired to a unique y-value.

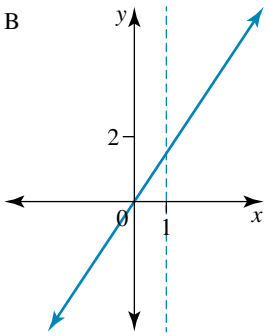
Functions can be recognised from their graphs by the **vertical line test**, as demonstrated in the graphs shown. If a vertical line can be drawn that cuts the graph at more than one place, the graph is not that of a function.

A



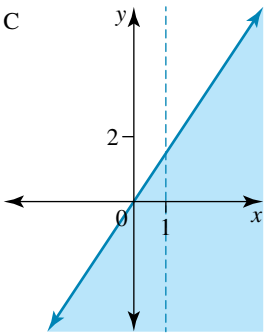
A vertical line cuts once.
$A = \{(-2, 4), (1, 5), (3, 4)\}$ is a function.

B



A vertical line cuts once.
$B = \{(x, y) : y = 2x\}$ is a function.

C



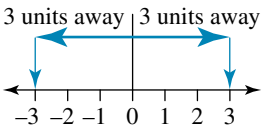
A vertical line cuts many times.
$C = \{(x, y) : y \leq 2x\}$ is a relation.

on Resources

Interactivity: Vertical and horizontal line test (int-2570)

7.2.2 An introduction to the modulus function

Sometimes only the **magnitude** of a number (that is, the size of a number) is required. This is referred to as the **absolute value** or **modulus** of a number.



Consider the numbers -3 and 3. They are both 3 units from the origin, as demonstrated in the line graph.

If x is a number, the absolute value of x is expressed as $|x|$. More formally, $|x| = \sqrt{x^2}$.

Consider 3 and -3 again. If $x = 3$, then $x^2 = 9$ and $\sqrt{9} = 3$. Additionally, if $x = -3$, then $x^2 = 9$ and $\sqrt{9} = 3$.

WORKED EXAMPLE 1

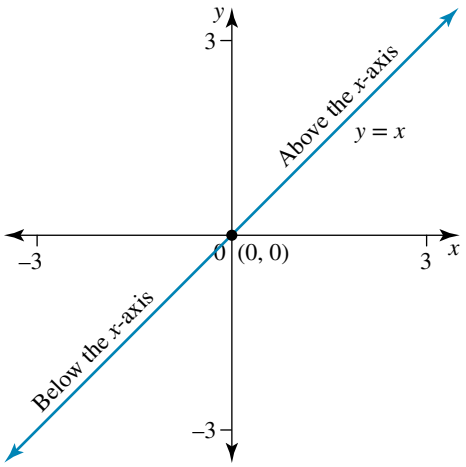
Solve $|4x| = 16$.

THINK	WRITE/DRAW
1. $\|16\| = 16$ and $\|-16\| = 16$, so there are 2 possible values for $4x$: 16 and -16.	$4x = 16$ or $4x = -16$
2. Solve the equations.	$x = 4$ or $x = -4$

7.2.3 Sketching $y = |x|$

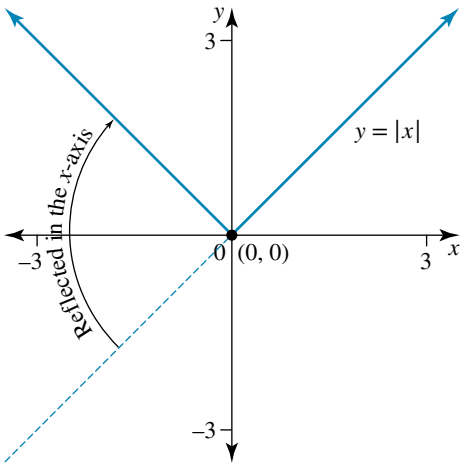
Consider the graph of $y = x$. When $x \geq 0$, $y \geq 0$, and when $x < 0, y < 0$.

x	-2	-1	0	1	2
y	-2	-1	0	1	2



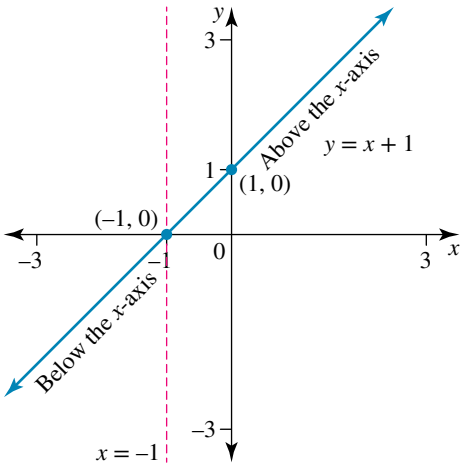
Now consider the graph $y = |x|$.
When $x \geq 0$, $y = x$ and $y = |x|$ are the same, but for $x < 0$, $y = |x|$ and $y = -x$ are the same.

x	-2	-1	0	1	2
$y = \|x\|$	2	1	0	1	2



7.2.4 Graphing $y = |f(x)|$ from $y = f(x)$

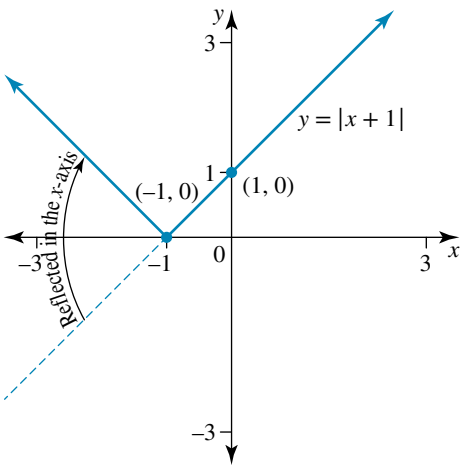
Consider the graph $y = x + 1$.
The graph crosses the x-axis when $y = 0$, that is, when $x = -1$.
For $x \geq -1$, $y \geq 0$; for $x < -1$, $y < 0$.



Now consider the graph $y = |x + 1|$.
For $x \geq -1$, $|x + 1| = x + 1$,
but for $x < -1$, $|x + 1| = -(x + 1)$.

This results in a reflection in the x-axis at $x = -1$.

When graphing $y = |f(x)|$, it is useful to consider the original graph of $y = f(x)$ to determine significant features.



WORKED EXAMPLE 2

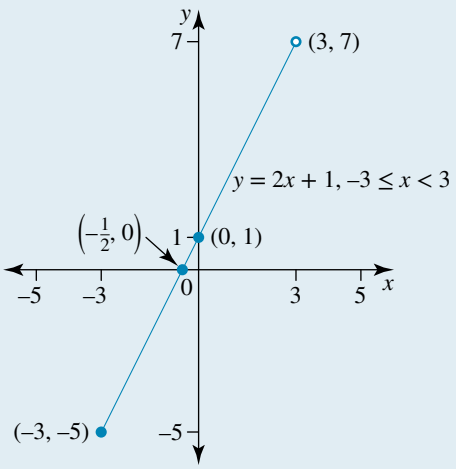
Use the graph of $y = 2x + 1$ to do the following.
a. Sketch $y = |2x + 1|$ for the domain $-3 \leq x < 3$.
b. State the range of the function.

THINK

a. 1. Sketch $y = 2x + 1$ for $-3 \leq x < 3$.
The gradient is 2 and the y-intercept is 1.

WRITE

a.

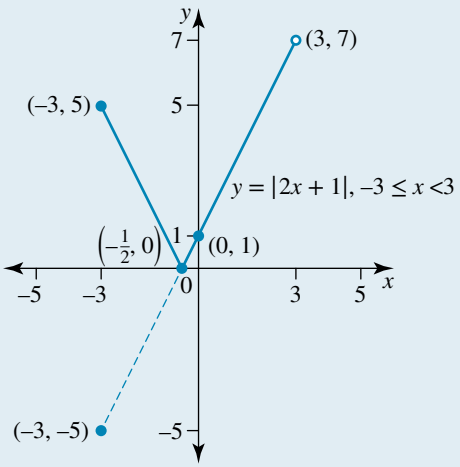


2. Identify where $2x + 1 \geq 0$.

$2x + 1 \geq 0$

$x \geq -\frac{1}{2}$

3. Sketch $y = |2x + 1|$ for $-3 \leq x < 3$.
Given the domain is restricted to $-3 \leq x < 3$, the graph has end points — a solid point at $x = -3$, as the restricted domain states $x \geq -3$, and a hollow point at $x = 3$, as the restricted domain states $x < 3$. The graph is reflected in the x-axis at the x-intercept.



b. To determine the range, consider the end points of the restricted domain. The minimum value is $y = 0$. The larger value between 5 and 7 is 7; however, as it is a hollow point, it cannot be included. Hence, the range is $0 \leq y < 7$ or $[0, 7)$.

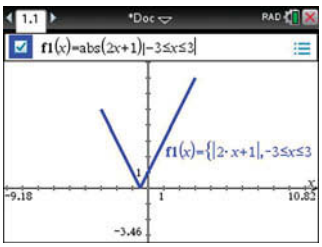
b. When $x = -3$,
$y = |2x + 1|$
$y = |2 \times -3 + 1|$
$y = 5$
When $x = 3$,
$y = |2x + 1|$
$y = |2 \times 3 + 1|$
$y = 7$
Hence, the range is $[0, 7)$.

TI | THINK

a.1. On a Graphs page, complete the entry line for function1 as:
$f1(x) = \text{abs}(2x + 1)|-3 \leq x \leq 3$
then press ENTER.

Note: The restricted domain $-3 \leq x < 3$ doesn't include the point at $x = 3$. However, on the calculator, the point at $x = 3$ needs to be included so that the calculator can find the coordinates of this point.

WRITE

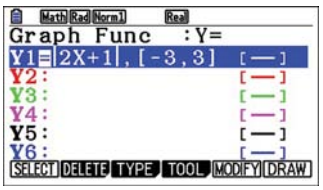


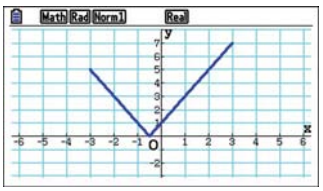
CASIO | THINK

a.1. On a Graph screen, press SHIFT 4, then use the down arrow to scroll down to Abs, then press EXE. Complete the entry line for Y1 as:
$Y1 = |2X + 1|, [-3, 3]$
then press EXE.

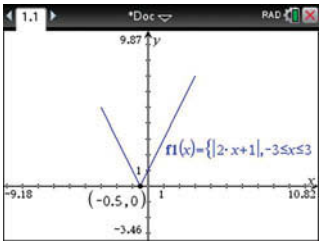
Note: The restricted domain $-3 \leq x < 3$ doesn't include the point at $x = 3$. However, on the calculator, the point at $x = 3$ needs to be included so that the calculator can find the coordinates of this point.

WRITE

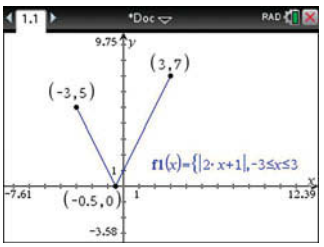




2. To find the coordinates of the vertex, press MENU, then select:
6: Analyze Graph
1: Zero.
Move the cursor to the left of the vertex when prompted for the lower bound, then press ENTER.
Move the cursor to the right of the vertex when prompted for the upper bound, then press ENTER.



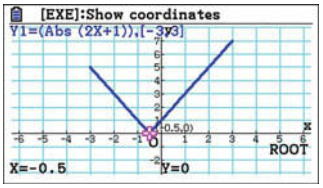
3. To label the end points, press MENU, then select:
5: Trace.
1: Graph Trace.
Type '3', then press ENTER twice. Type '3', then press ENTER twice.



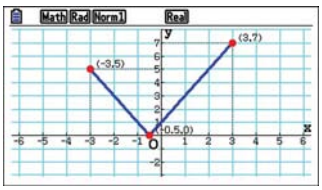
4. Sketch the graph.

Copy the graph from the screen, remembering that the end point at $x = -3$ is included, so it should be drawn with a solid point, and the end point at $x = 3$ is not included, so it should be drawn with a hollow point (as in **a3** above).

2. To find the coordinates of the vertex, select G-Solv by pressing F5. Select ROOT by pressing F1, then press EXE.



3. To label the end points, select Trace by pressing F1, then type '−3' and press EXE twice. Type '3', then press EXE twice.



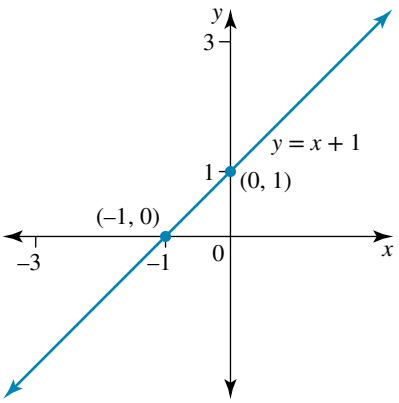
4. Sketch the graph.

Copy the graph from the screen, remembering that the end point at $x = -3$ is included, so it should be drawn with a solid point, and the end point at $x = 3$ is not included, so it should be drawn with a hollow point (as in **a3** above).

7.2.5 Graphing $y = f(|x|)$ from $y = f(x)$

Consider again the graph of $y = x + 1$.

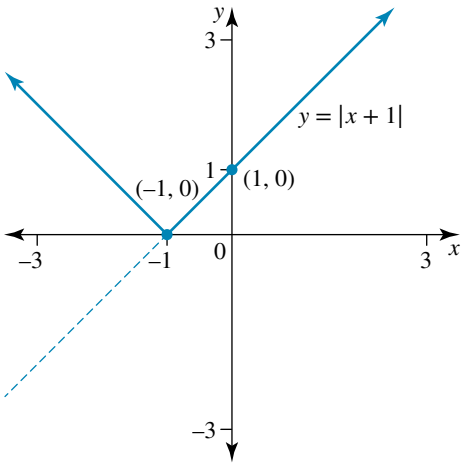
x	-2	-1	0	1	2
$y = x + 1$	-1	0	1	2	3



How would this change if we sketched $y = |x| + 1$?

For $x \geq 0$, $|x| = x$. Therefore, the graph would be unchanged for this interval, just the same as in the graph of $y = |x + 1|$.

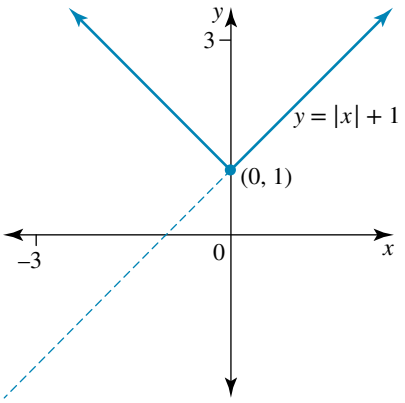
x	-2	-1	0	1	2
$y = \|x + 1\|$	-1	0	1	2	3



For $x \leq 0$, the graph $y = |x| + 1$ is reflected in the y-axis, as can be seen in the following table of values.

Note: This graph is a vertical shift of 1 unit from the graph of $y = |x|$.

x	-2	-1	0	1	2
$\|x\|$	2	1	0	1	2
$y = \|x\| + 1$	3	2	1	2	3



WORKED EXAMPLE 3

Use the graph of $y = 2x + 1$ to do the following.

a. Sketch $y = 2|x| + 1$ for the domain $-4 < x \leq 5$.

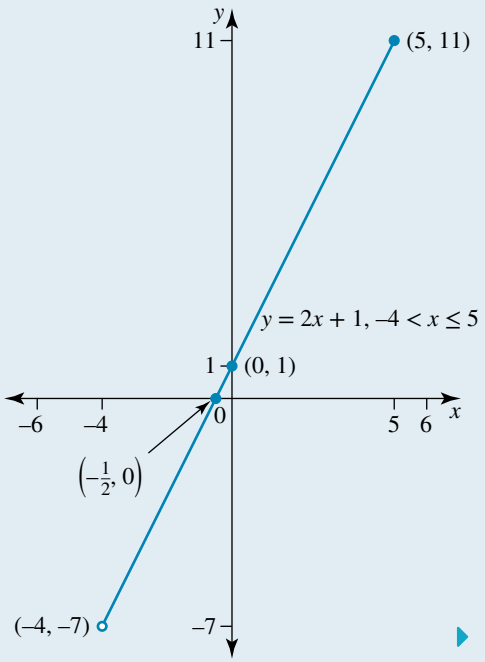
b. State the range of the function.

THINK

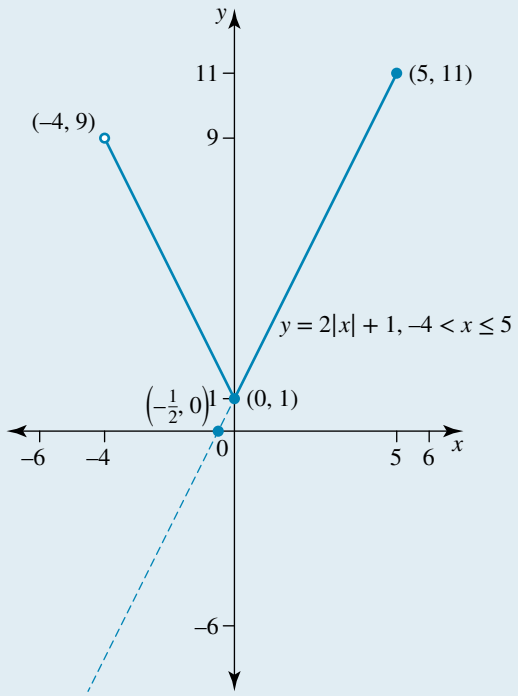
a. 1. Sketch $y = 2x + 1$ for $-4 < x \leq 5$.
The gradient is 2 and the y-intercept is 1.

WRITE

a.



2. $|x| = x$ for $x \geq 0$; therefore, the graph is unchanged for $x \geq 0$.
The graph is symmetrical about the y-axis.
Given the domain is restricted to $-4 < x \leq 5$, the graph has end points — a hollow point at $x = -4$, as the restricted domain states $x > 4$, and a solid point at $x = 5$, as the restricted domain states $x \leq 5$.



b. To determine the range, consider the end points of the restricted domain and the y-intercept.

b. When $x = -4$,
$y = 2|x| + 1$
$y = 2|-4| + 1$
$y = 9$
When $x = 5$,
$y = 2|x| + 1$
$y = 2|5| + 1$
$y = 11$
When $x = 0$, $y = 1$.
Hence, the range is $[1, 11]$.

7.2.6 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

The patterns observed above are true for any graph involving the modulus function.

When graphing $y = |f(x)|$,

$$y = \begin{cases} f(x), f(x) \geq 0 \\ -f(x), f(x) < 0 \end{cases}$$

This results in a reflection in the x-axis at the x-intercept for $y < 0$.

When graphing $y = f(|x|)$,

$$y = \begin{cases} f(x), x \geq 0 \\ f(-x), x < 0 \end{cases}.$$

This results in a reflection in the y-axis at the y-intercept for $x < 0$.

WORKED EXAMPLE 4

Use the graph of $y = (x - 4)(x - 2)$ to sketch the following.

a. $y = |(x - 4)(x - 2)|$ **b.** $y = (|x| - 4)(|x| - 2)$

THINK

a. 1. Sketch $y = (x - 4)(x - 2)$. The x-intercepts are $x = 4$ and $x = 2$. The x-value for the turning point is the midpoint of the intercepts. This gives the turning point as $(3, -1)$. The y-intercept is $y = 8$.

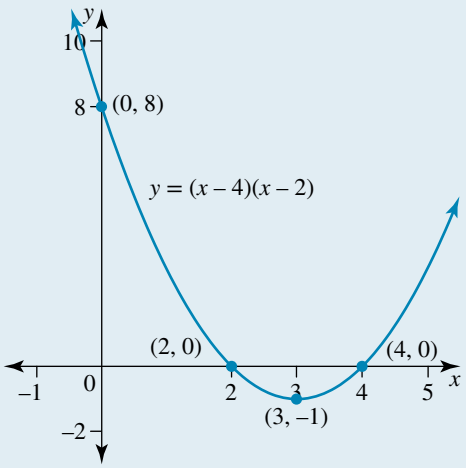
2. Identify where the graph is above and below the x-axis.

3. Sketch $y = |(x - 4)(x - 2)|$ by reflecting the graph for the interval $(2, 4)$ in the x-axis.

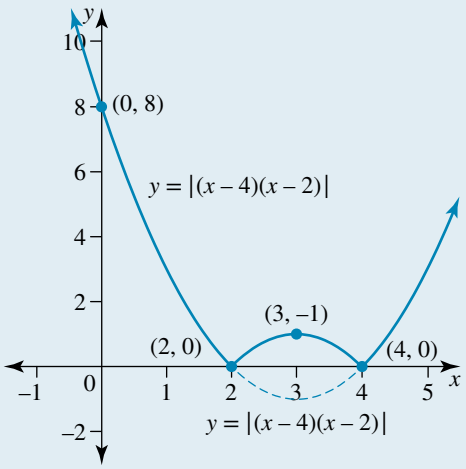
b. 1. The graph for $y = (|x| - 4)(|x| - 2)$ is the same as the graph of $y = (x - 4)(x - 2)$ for $x \geq 0$. For $x < 0$, reflect the graph in the y-axis.

WRITE

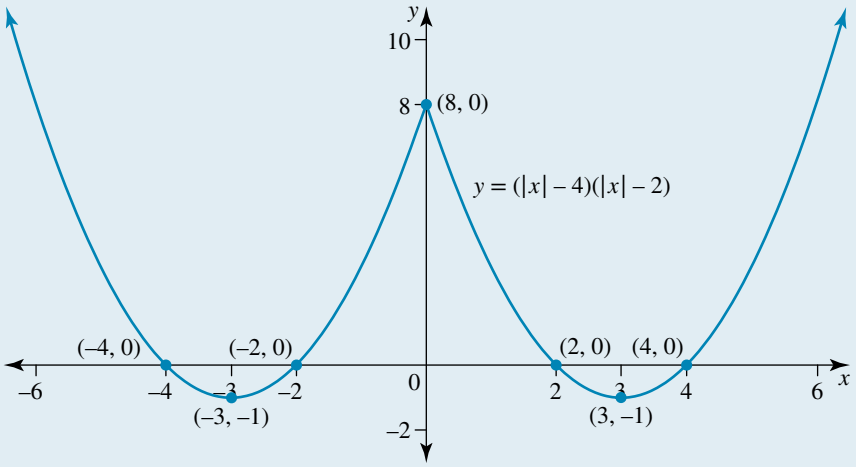
a.



The graph is at or above the x-axis for $x \leq 2$ and $x \geq 4$.
The graph is below the x-axis for $2 < x < 4$.



b.



studyon

Units 1 & 2 > Area 5 > Sequence 1 > Concepts 1 & 2

The modulus function Summary screen and practice questions
Graphs of the modulus function Summary screen and practice questions

Exercise 7.2 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

Technology free

1. Identify the true statements among the following.
   a. $|-5| = 5$ b. $|6| = -6$ c. $\sqrt{9} = \pm 3$ d. $|7| = 7$
2. WE1 Solve $|3x| = 12$.
3. Solve $|x + 1| = 3$.
4. Use the graph of $y = 2x - 3$ to sketch the following, and state the range of each.
   a. WE2 $y = |2x - 3|$ for the domain $-4 \leq x \leq 2$
   b. WE3 $y = 2|x| - 3$ for the domain $-2 < x \leq 4$
5. Use the graph of $y = 1 - 2x$ to sketch the following.
   a. $y = |1 - 2x|$ b. $y = 1 - 2|x|$
6. Use the graph of $y = 3x + 2$ to sketch the following.
   a. $y = |3x + 2|$ b. $y = 3|x| + 2$
7. Use the graph of $y = \frac{x}{4} + 1$ to sketch the following.
   a. $y = \left|\frac{x}{4} + 1\right|$ b. $y = \frac{|x|}{4} + 1$
8. WE4 Use the graph of $y = (x - 1)(x + 3)$ to sketch the following.
   a. $y = |(x - 1)(x + 3)|$ b. $y = (|x| - 1)(|x| + 3)$
9. Use the graph of $y = (1 - x)(x + 5)$ to sketch the following.
   a. $y = |(1 - x)(x + 5)|$ b. $y = (1 - |x|)(|x| + 5)$
10. Use the graph of $y = (x + 1)(x - 3)$ to sketch the following.
    a. $y = |(x + 1)(x - 3)|$ b. $y = (|x| + 1)(|x| - 3)$
11. Use the graph of $y = (3 - x)(5 - x)$ to sketch the following.
    a. $y = |(3 - x)(5 - x)|$ b. $y = (3 - |x|)(5 - |x|)$

Technology active

12. Use the graph of $y = (x - 1)(x + 1)(x - 2)$ to sketch the following.
    Note: It is not necessary to identify the turning points in your sketches.
    a. $y = |(x - 1)(x + 1)(x - 2)|$ b. $y = (|x| - 1)(|x| + 1)(|x| - 2)$
13. Use the sketch of $y = (x + 1)^2(x - 2)$ to sketch the following.
    Note: It is not necessary to identify the turning points in your sketches.
    a. $y = |(x + 1)^2(x - 2)|$ b. $y = (|x| + 1)^2(|x| - 2)$
14. Use the graph of $y = (x - 2)^2$ to sketch the following.
    a. $y = |(x - 2)^2|$ b. $y = (|x| - 2)^2$
    What do you notice about your answer for part a?
15. Use sketches of $y = |2x + 1|$ and $y = |1 - 2x|$ to determine what the graph of $y = |2x + 1| + |1 - 2x|$ looks like. You must justify your prediction.

7.3 Sketching graphs of reciprocal functions

7.3.1 Graphing reciprocal functions of linear equations

The **reciprocal** of any number or function is the **multiplicative inverse** of the original. When the original is multiplied by its reciprocal, the result will always be 1. The reciprocal of 3 is $\frac{1}{3}$, that is, $3 \times \frac{1}{3} = 1$. Similarly, for a function $f(x)$, the reciprocal is $\frac{1}{f(x)}$.

The reciprocal of a function is undefined if $f(x) = 0$. On a graph, this is shown as a vertical **asymptote**. As $\frac{1}{f(x)} \neq 0$, $y = 0$ is the horizontal asymptote of the reciprocal function.

The graph of $y = f(x)$ can be used to sketch $y = \frac{1}{f(x)}$ as shown in the following worked example.

The y-intercept of $y = f(x)$ is $f(0)$. Therefore, the y-intercept of $\frac{1}{f(x)}$ is $\frac{1}{f(0)}$.

WORKED EXAMPLE 5

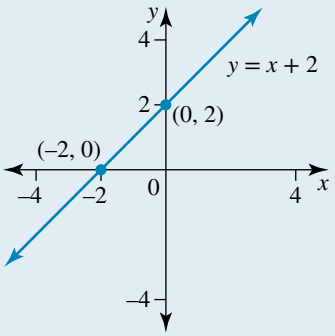
Use the graph of $y = x + 2$ to sketch $y = \frac{1}{x+2}$.

THINK

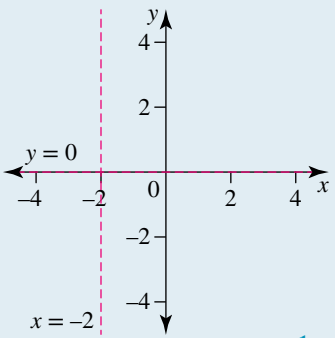
1. Sketch $y = x + 2$ and identify any intercepts.

WRITE

When $x = 0$, $y = 2$.
When $y = 0$, $x = -2$.



2. The x-intercept occurs at $x = -2$; this will be the vertical asymptote. The horizontal asymptote is $y = 0$. Sketch both asymptotes with a dashed line.



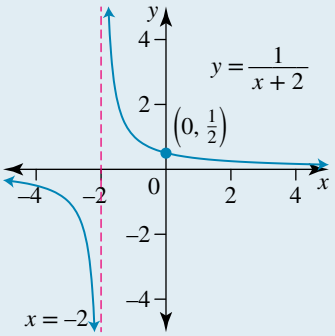
3. The y-intercept of $y = x + 2$ is $y = 2$. Therefore, the y-intercept of $y = \frac{1}{x+2}$ is $y = \frac{1}{2}$.

The y-intercept of $y = \frac{1}{x+2}$ is $y = \frac{1}{2}$.

4. Sketch the reciprocal functions, considering the following.

$y = x + 2$ is below the x-axis for $x < -2$. Therefore, $y = \dfrac{1}{x+2}$ is also below the x-axis for $x < -2$.

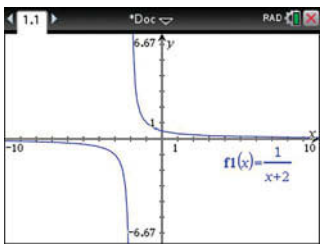
$y = x + 2$ is above the x-axis for $x > -2$. Therefore, $y = \dfrac{1}{x+2}$ is also above the x-axis for $x > -2$.



TI | THINK

1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = \dfrac{1}{x+2}$
then press ENTER.

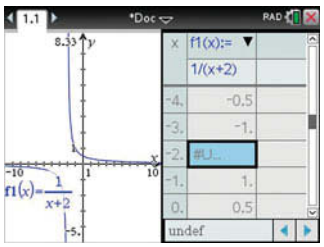
WRITE



2. The position of the horizontal asymptote can be identified from the graph.

The equation of the horizontal asymptote is $y = 0$.

3. To find the position of the vertical asymptote, press MENU, then select:
7: Table
1: Split-screen Table.
Use the up/down arrows to scroll until a cell containing #undef is found.



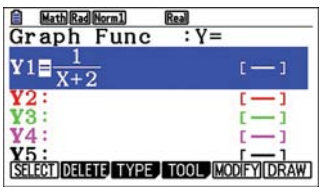
4. The position of the vertical asymptote can be read from the table.

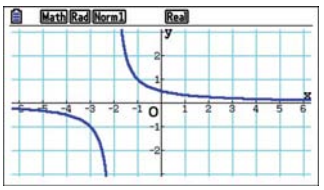
The equation of the vertical asymptote is $x = -2$.

CASIO | THINK

1. On a Graph screen, complete the entry line for Y1 as:
Y1 = 1 ÷ (X + 2)
then press EXE.
Select DRAW by pressing F6.

WRITE

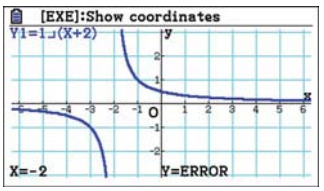


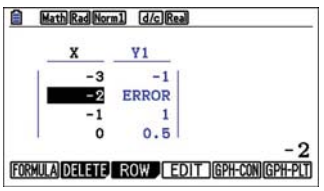


2. The position of the horizontal asymptote can be identified from the graph.

The equation of the horizontal asymptote is $y = 0$.

3. To find the position of the vertical asymptote, select Trace by pressing F1, then use the left/right arrows to trace along the graph until the screen reads Y = ERROR. Alternatively, on a Table screen, highlight Y1, then select TABLE by pressing F6. Use the up/down arrows to scroll until a cell containing ERROR is found.

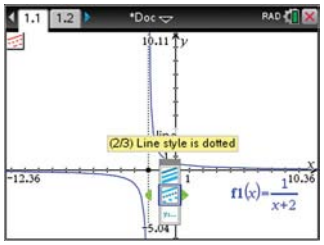


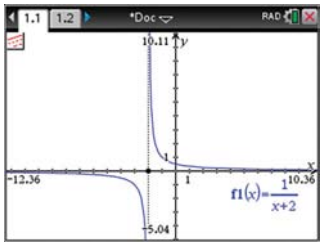


4. The position of the vertical asymptote can be read from the screen.

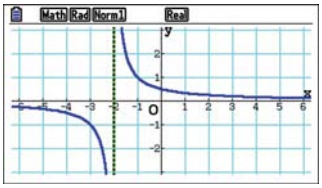
The equation of the vertical asymptote is $x = -2$.

5. To draw the vertical asymptote on the screen, press MENU, then select:
8: Geometry
4: Construction
1: Perpendicular.
Click on the x-axis, then click on the point on the x-axis where $x = 2$.
Press MENU, then select:
1: Actions
4: Attributes.
Click on the vertical line, then press the down arrow, then the right arrow to change the line style to dotted. Press ENTER.

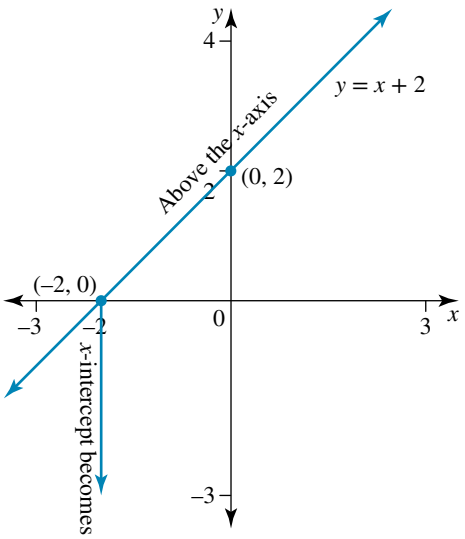




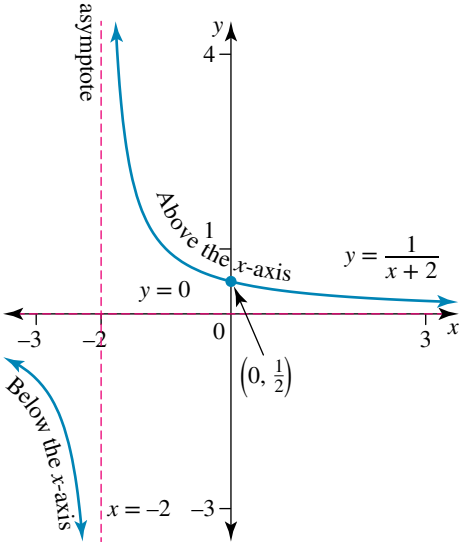
5. To draw the vertical asymptote on the screen, select DRAW by pressing F4. Press F6 to scroll across to more options, then select Vertical by pressing F4. Use the left arrow to move the vertical line to $x = -2$. Press SHIFT 5 to format the line, then select Broken for the Line Style. Press EXIT.



Compare the graphs of $y = x + 2$ and $y = \dfrac{1}{x+2}$.
For the graph of $y = x + 2$, the x-intercept occurs at $(-2, 0)$, that is, when $y = 0$.



The reciprocal of $y = x + 2$, $y = \dfrac{1}{x+2}$, is undefined at $y = \dfrac{1}{0}$, that is at $x = -2$. Hence, the x-intercept of the original $y = x + 2$ becomes the asymptote of $y = \dfrac{1}{x+2}$.



7.3.2 Graphing reciprocal functions for quadratic and cubic functions

The same process as shown above can be used to graph the reciprocal functions of quadratic and cubic functions.

WORKED EXAMPLE 6

Use the graph of $y = (x + 2)(x - 4)$ to sketch $y = \dfrac{1}{(x + 2)(x - 4)}$.

THINK

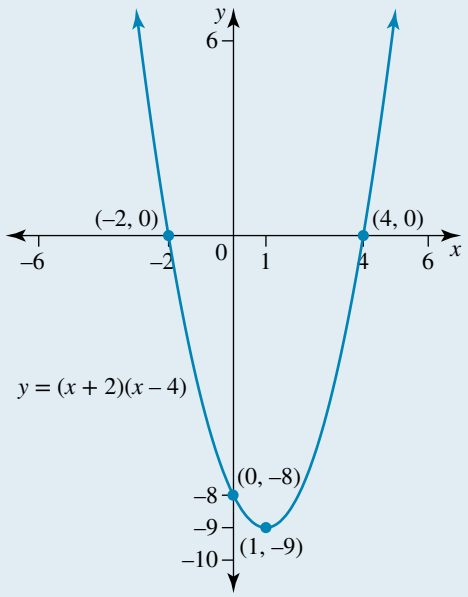
1. Identify the key points of $y = (x + 2)(x - 4)$.

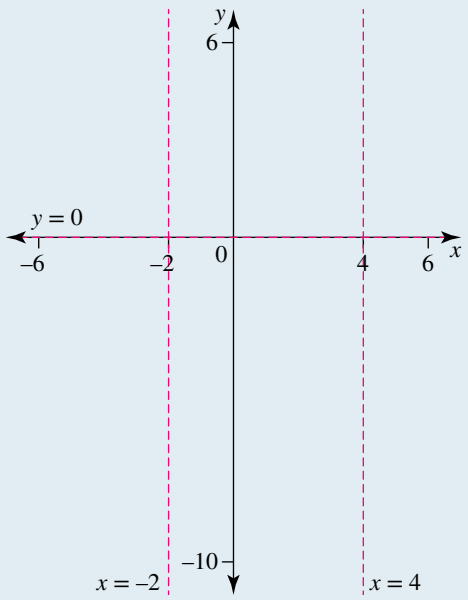
2. Sketch $y = (x + 2)(x - 4)$.

3. The x-intercepts occur at $x = -2$ and $x = 4$, so these become the vertical asymptotes. The horizontal asymptote is $y = 0$. Include all asymptotes by sketching with a dashed line.

WRITE

y-intercept: $x = 0,\ y = -8$
x-intercepts: $y = 0,\ x = -2$ or $x = 4$
Turning point: $x = 1,\ y = -9$





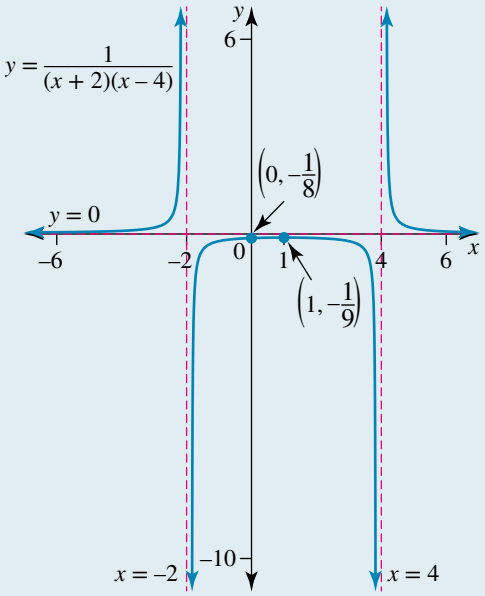
4. The y-intercept of $y = (x + 2)(x - 4)$ is -8. Therefore, the y-intercept of $y = \dfrac{1}{(x + 2)(x - 4)}$ is $\dfrac{-1}{8}$.

The y-intercept is $\left(0, \dfrac{-1}{8}\right)$.

5. The turning point of $y = (x + 2)(x - 4)$ is $(1, -9)$. Therefore, the turning point of $y = \dfrac{1}{(x + 2)(x - 4)}$ is $\left(1, \dfrac{-1}{9}\right)$.

The turning point is $\left(1, \dfrac{-1}{9}\right)$.

6. The graph of $y = (x + 2)(x - 4)$ is above the x-axis for $x < -2$ and $x > 4$. Therefore, $y = \dfrac{1}{(x + 2)(x - 4)}$ is above the x-axis in those regions. Similarly, the graph is below the x-axis for $-2 < x < 4$.
Sketch the reciprocal function.



If $y = f(x)$ does not have any x-intercepts, then $y = \dfrac{1}{f(x)}$ does not have any vertical asymptotes. The horizontal asymptote is still $y = 0$.

Notice that in the previous examples, at the extreme values of x ($x \to \infty$ and $x \to -\infty$) the graph approaches the horizontal asymptote. This is true for all graphs with a horizontal asymptote.

WORKED EXAMPLE 7

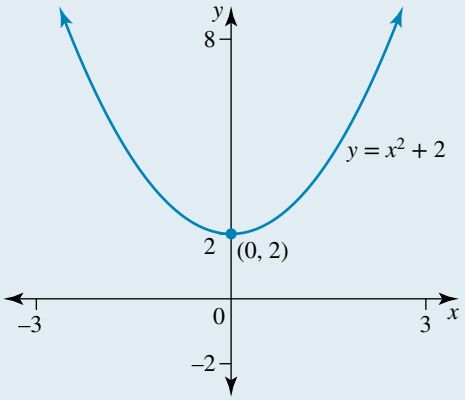
Use the graph of $y = x^2 + 2$ to sketch $y = \dfrac{1}{x^2 + 2}$.

THINK

1. Sketch $y = x^2 + 2$, identifying key points.

WRITE

y-intercept: $x = 0$, $y = 2$
x-intercepts: There are no x-intercepts.
Turning point: $x = 0$, $y = 2$



2. There are no x-intercepts and therefore no vertical asymptotes in the reciprocal function.

There are no vertical asymptotes.

The horizontal asymptote is still $y = 0$.

The horizontal asymptote is $y = 0$.

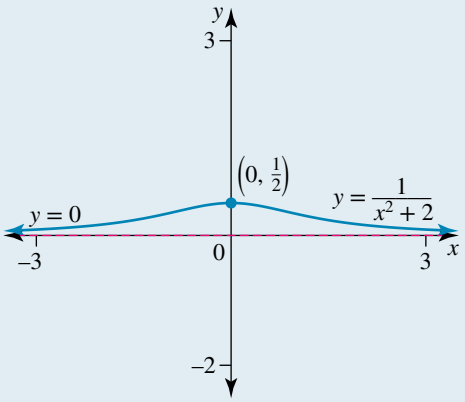
The turning point (and y-intercept) $(0, 2)$ will become $\left(0, \frac{1}{2}\right)$ in the reciprocal function.

The turning point (and y-intercept) is $\left(0, \frac{1}{2}\right)$.

3. $y = x^2 + 2$ is always above the x-axis, so $y = \frac{1}{x^2 + 2}$ will always be above the x-axis.

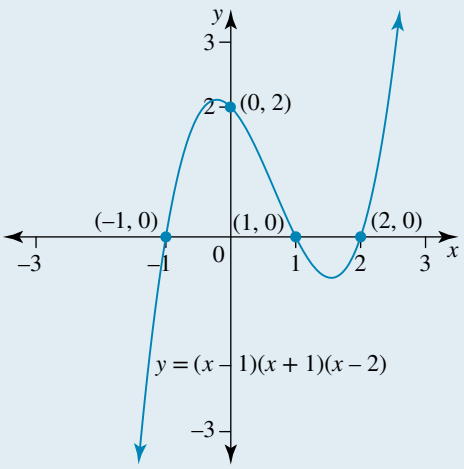
The graph will approach the horizontal asymptote of $y = 0$ as $x \to \infty$ and $x \to -\infty$.

Sketch the reciprocal function.



WORKED EXAMPLE 8

Use the given sketch of $y = (x - 1)(x + 1)(x - 2)$ to sketch $y = \frac{1}{(x - 1)(x + 1)(x - 2)}$.

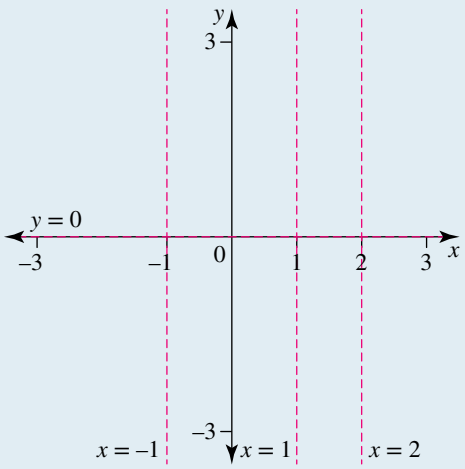


THINK

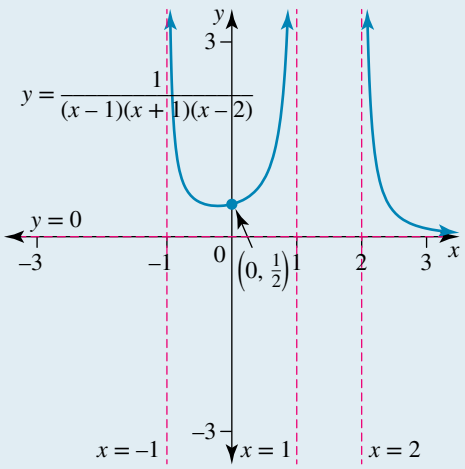
WRITE

1. The x-intercepts are at $x = -1$, $x = 1$ and $x = 2$. These become the vertical asymptotes.
The horizontal asymptote is $y = 0$. Sketch the asymptotes with dashed lines.

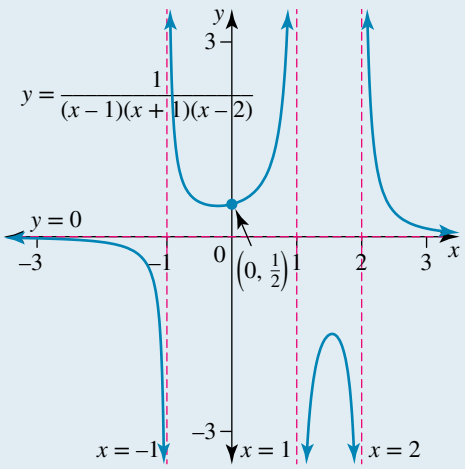
The vertical asymptotes are $x = -1$, $x = 1$ and $x = 2$.
The horizontal asymptote is $y = 0$.



2. The graph of $y = (x - 1)(x + 1)(x - 2)$ is above the x-axis for $-1 < x < 1$ and $x > 2$. The graph of $y = \dfrac{1}{(x - 1)(x + 1)(x - 2)}$ will also be above the axis for these intervals.
The y-intercept $(0, 2)$ will become $\left(0, \dfrac{1}{2}\right)$.



3. The graph of $y = (x - 1)(x + 1)(x - 2)$ is below the x-axis for $x < -1$ and $1 < x < 2$. The graph of $y = \dfrac{1}{(x - 1)(x + 1)(x - 2)}$ will also be below the axis for these intervals. The turning point (x, y) on the graph $y = (x - 1)(x + 1)(x - 2)$ between $1 < x < 2$ can be used to determine the turning point on the reciprocal, $\left(x, \dfrac{1}{y}\right)$.
Sketch the reciprocal function.



studyon

Units 1 & 2 | Area 5 | Sequence 1 | Concepts 3 & 4

Sketching graphs of reciprocal functions Summary screen and practice questions
Sketching graphs of quadratic reciprocal functions Summary screen and practice questions

Exercise 7.3 Sketching graphs of reciprocal functions

Technology free

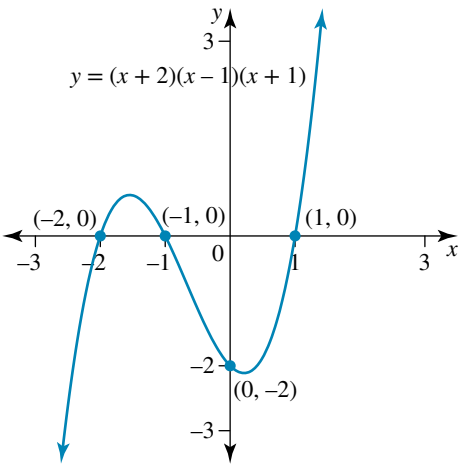
1. **a.** Sketch $y = x - 2$.
   b. Use your sketch of $y = x - 2$ to identify the asymptotes for $y = \frac{1}{x-2}$.
   c. Identify the y-intercept of $y = x - 2$ and use this to identify the y-intercept of $y = \frac{1}{x-2}$.
   d. Use your sketch of $y = x - 2$ to sketch $y = \frac{1}{x-2}$.
2. **a.** Sketch $y = x + 1$.
   b. Use your sketch of $y = x + 1$ to identify the asymptotes for $y = \frac{1}{x+1}$.
   c. Identify the y-intercept of $y = x + 1$ and use this to identify the y-intercept of $y = \frac{1}{x+1}$.
   d. Use your sketch of $y = x + 1$ to sketch $y = \frac{1}{x+1}$.
3. **WE5** Use the graph of $y = 3 - x$ to sketch $y = \frac{1}{3-x}$.
4. Use the graph of $y = 2x - 1$ to sketch $y = \frac{1}{2x-1}$.
5. **a.** Sketch $y = (x + 1)(x - 3)$.
   b. Use your sketch of $y = (x + 1)(x - 3)$ to identify the asymptotes for $y = \frac{1}{(x+1)(x-3)}$.
   c. Identify the y-intercept and turning point of $y = (x + 1)(x - 3)$ and use these to identify the y-intercept and turning point of $y = \frac{1}{(x+1)(x-3)}$.
   d. Use your sketch of $y = (x + 1)(x - 3)$ to sketch $y = \frac{1}{(x+1)(x-3)}$.
6. **WE6** Use the graph of $y = (x - 4)(x - 3)$ to sketch $y = \frac{1}{(x-4)(x-3)}$.
7. Use the graph of $y = (x - 1)(x + 3)$ to sketch $y = \frac{1}{(x-1)(x+3)}$.
8. Use the graph of $y = (2x - 1)(x + 3)$ to sketch $y = \frac{1}{(2x-1)(x+3)}$.
9. Use the graph of $y = (x - 2)^2$ to sketch $y = \frac{1}{(x-2)^2}$.
10. **WE7** Use the graph of $y = x^2 + 4$ to sketch $y = \frac{1}{x^2+4}$.

11. Use the graph of $y = -x^2 - 1$ to sketch $y = \dfrac{1}{-x^2 - 1}$.

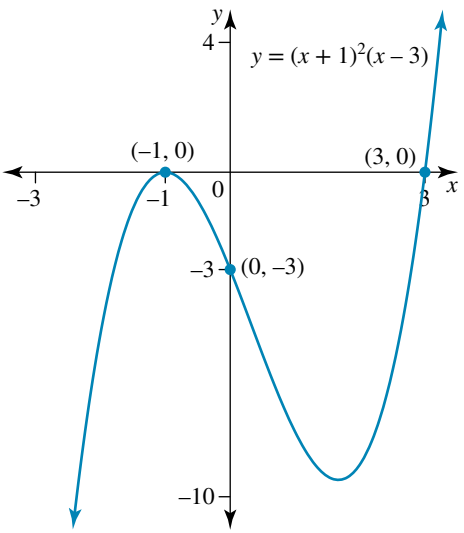
12. Use the graph of $y = x^2 + 3$ to sketch $y = \dfrac{1}{x^2 + 3}$.

13. Use the graph of $y = -x^2 - 2$ to sketch $y = \dfrac{-1}{x^2 + 2}$.

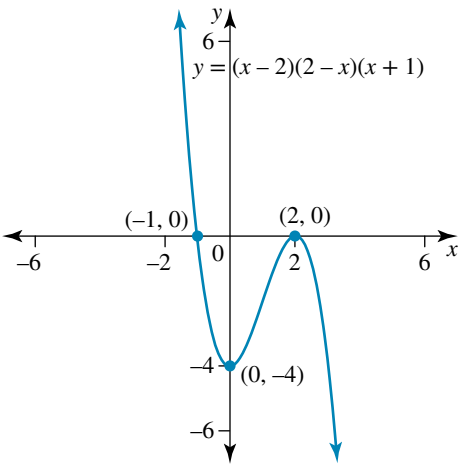
14. **WE8** Use the given sketch of $y = (x + 2)(x - 1)(x + 1)$ to sketch $y = \dfrac{1}{(x + 2)(x - 1)(x + 1)}$.



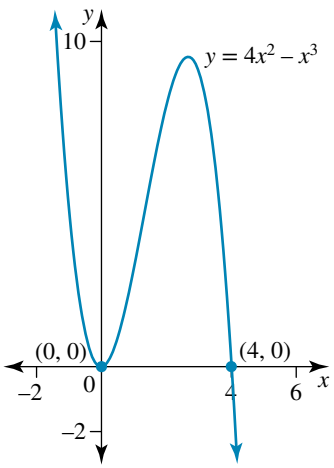
15. Use the given sketch of $y = (x + 1)^2(x - 3)$ to sketch $y = \dfrac{1}{(x + 1)^2(x - 3)}$.



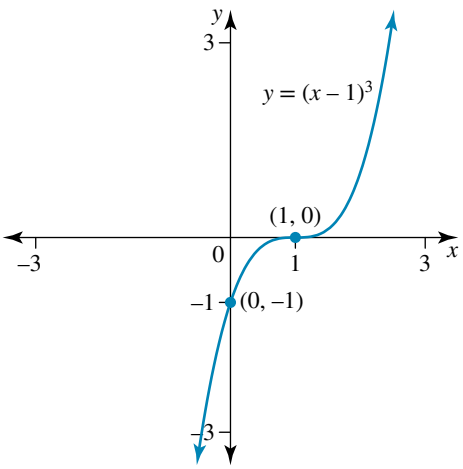
16. Use the graph of $y = (x - 2)(2 - x)(x + 1)$ to sketch $y = \dfrac{1}{(x - 2)(2 - x)(x + 1)}$.



17. Use the graph of $y = 4x^2 - x^3$ to sketch $y = \dfrac{1}{4x^2 - x^3}$.



18. Use the graph of $y = (x - 1)^3$ to sketch $y = \dfrac{1}{(x - 1)^3}$.



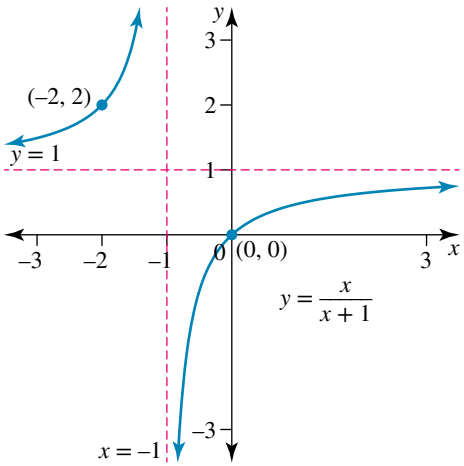
Technology active

19. Use the graph of $y = \dfrac{1}{x}$ to sketch:

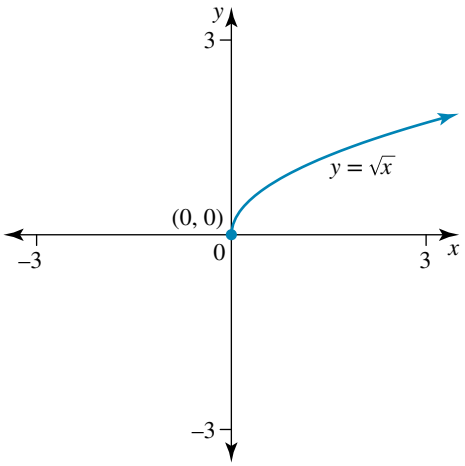
a. $y = \left|\dfrac{1}{x}\right|$ **b.** $y = \dfrac{1}{|x|}$

You may use technology to confirm your answers.
What do you notice about your answers for parts **a** and **b**?

20. The graph of $y = \dfrac{x}{x + 1}$ is shown. Use your understanding of reciprocal functions to sketch $y = \dfrac{x + 1}{x}$. You must justify the decisions you have made. You may use technology to confirm your decisions.



21. The graph of $y = \sqrt{x}$ is shown. Use your understanding of reciprocal functions to sketch $y = \dfrac{1}{\sqrt{x}}$. You must justify the decisions you have made. You may use technology to confirm your decisions.



7.4 Sketching graphs of rational functions

7.4.1 Rational functions

Recall that a rational number can be written in the form $\dfrac{a}{b}$ where a and b are integers, the only common factor between a and b is 1, and $b \neq 0$. Similarly, a **rational function** is a function that can be expressed as the quotient of two polynomials, such that $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ have no common factors of degree greater than or equal to 1, and $h(x) \neq 0$ (that is $R \setminus \{x: h(x) = 0\}$).

Consider the equation $y = \dfrac{x^2 + 3x + 2}{x + 1}$. It is **undefined** for $x = -1$. However, it can be simplified by factorising the numerator.

$$\begin{aligned} y &= \frac{x^2 + 3x + 2}{x + 1}, \; x \neq -1 \\ &= \frac{\cancel{(x+1)}\,(x + 2)}{\cancel{x+1}} \\ &= x + 2 \end{aligned}$$

This means that the sketch of $y = \dfrac{x^2 + 3x + 2}{x + 1}$ looks like the sketch of $y = x + 2$, except that there is an undefined point at $x = -1$. Hence, at $x = -1$, the graph is discontinuous. This is shown in the following worked example.

WORKED EXAMPLE 9

Sketch $y = \dfrac{x^2 + 3x + 2}{x + 1}$.

THINK

1. This function is undefined at $x = -1$.
2. Factorise the numerator.

WRITE

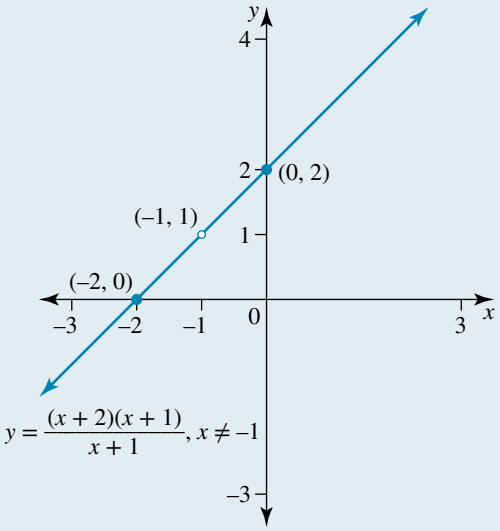
$x \neq -1$

$y = \dfrac{x^2 + 3x + 2}{x + 1}$

$y = \dfrac{\cancel{(x+1)}(x + 2)}{\cancel{x+1}}$

$y = x + 2$

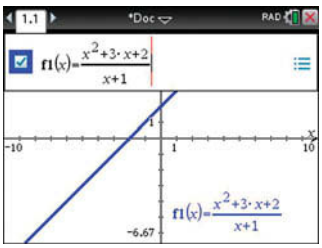
3. Sketch the line, demonstrating the discontinuity at $x = -1$. This is shown by a hollow circle.



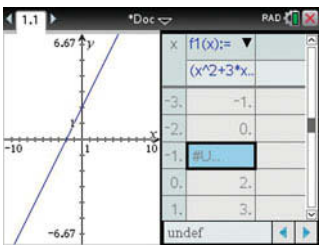
TI | THINK

1. On a Graphs page, complete the entry line for function 1 as:
$f1(x) = \frac{x^2 + 3x + 2}{x + 1}$
then press ENTER.

WRITE



2. To find any points of discontinuity, press MENU, then select:
7: Table
1: Split-screen Table.
Use the up and down arrows to find a cell containing #undef.



There is a point of discontinuity at $x = -1$.

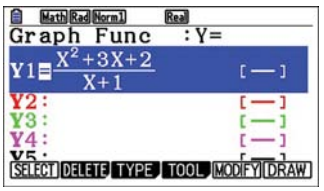
3. Sketch the graph.

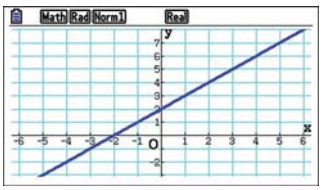
Copy the graph from the screen, making sure to add a hollow point at $x = -1$ to represent the point of discontinuity.

CASIO | THINK

1. On a Graph screen, complete the entry line for Y1 as:
Y1 = (X² + 3X + 2) ÷ (X + 1) then press EXE.
Select DRAW by pressing F6.

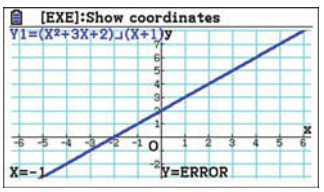
WRITE

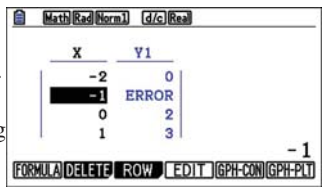




2. To find any points of discontinuity, select Trace by pressing F1, then use the left and right arrows to trace along the graph until the screen reads Y = ERROR.

Alternatively, on a Table screen, highlight Y1, then select TABLE by pressing F6. Use the up and down arrows to scroll until a cell containing ERROR is found.





There is a point of discontinuity at $x = -1$.

3. Sketch the graph.

Copy the graph from the screen, making sure to add a hollow point at $x = -1$ to represent the point of discontinuity.

7.4.2 Graphing improper fractions

A **proper fraction** occurs when the degree of the numerator is lower than the degree of the denominator, creating a rational expression. Conversely, in an **improper fraction**, when the degree of the numerator is higher than that of the denominator, it is necessary to rewrite the expression as a proper fraction. This is discussed in section 7.4.5.

Improper fractions, where the degree of the numerator and denominator are equal, can be rewritten to assist in sketching. For example, $\frac{x+1}{x} = 1 + \frac{1}{x}$ and $\frac{2x+5}{x+1} = 2 + \frac{3}{x+1}$.

When graphing a rational expression that is also an improper fraction, it is not necessary to complete the division, but only to determine the initial term as this becomes the horizontal asymptote.

WORKED EXAMPLE 10

Sketch the following.

a. $y = \frac{x+2}{x}$

b. $y = \frac{2x}{x+2}$

THINK

a. 1. The function is undefined at $x = 0$, so this is the vertical asymptote.

2. To find the horizontal asymptote, begin to simplify the expression.

3. Determine the x- and y-intercepts.

4. Sketch the asymptotes and intercepts.

WRITE

a. $x \neq 0$

Therefore, the vertical asymptote is $x = 0$.

Horizontal asymptote: $y = \frac{x}{x}$

$= 1$

Therefore, the horizontal asymptote is $y = 1$.

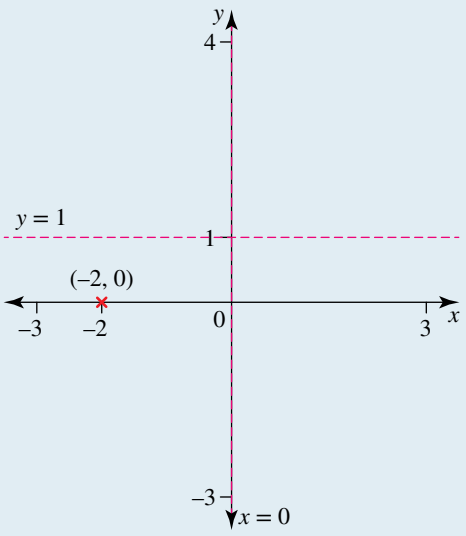
$x \neq 0$

Therefore, there is no y-intercept.

$y = 0$: $0 = \frac{x+2}{x}$

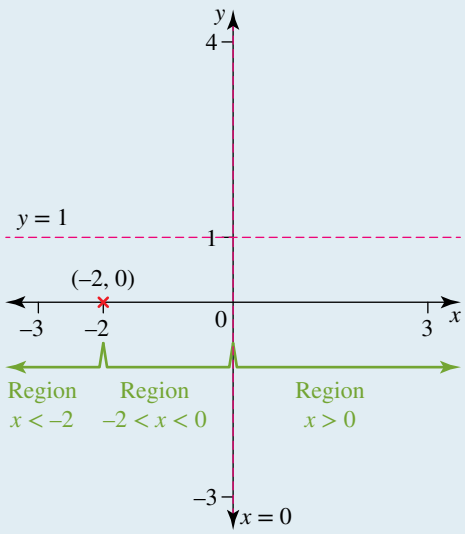
$x = -2$

Therefore, the x-intercept is $(-2, 0)$.

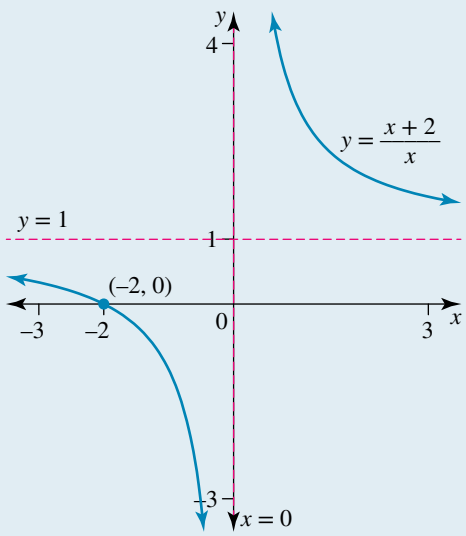


5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

x	$y = \dfrac{x+2}{x}$	
-3	$\dfrac{(-)}{(-)} = (+)$	Above the x-axis
-1	$\dfrac{(+)}{(-)} = (-)$	Below the x-axis
1	$\dfrac{(+)}{(+)} = (+)$	Above the x-axis



6. Sketch the function.



b. 1. The function $y = \dfrac{2x}{x+2}$ is undefined at $x = -2$, so this is the vertical asymptote.

2. To find the horizontal asymptote, begin to simplify the expression.

3. Determine the x- and y-intercepts.

b. $x \neq -2$

Therefore, the vertical asymptote is $x = -2$.

Horizontal asymptote: $y = \dfrac{2x}{x}$

$= 2$

Therefore, the horizontal asymptote is $y = 2$.

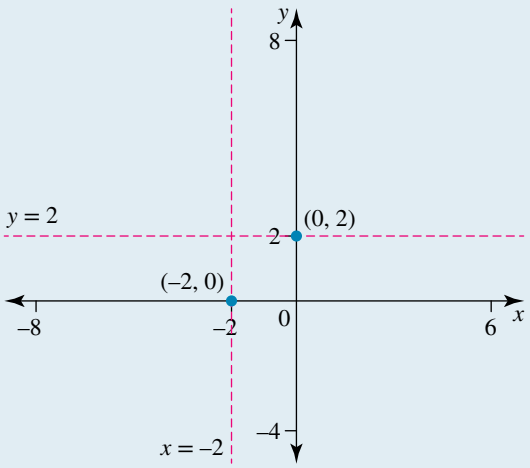
$x = 0$: $y = 0$

$y = 0$: $0 = \dfrac{2x}{x+2}$

$x = 0$

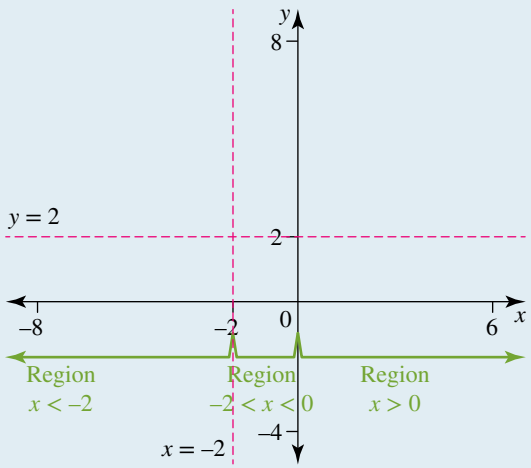
Therefore, the intercept (for both x and y) is $(0, 0)$.

4. Sketch the asymptotes and intercepts.

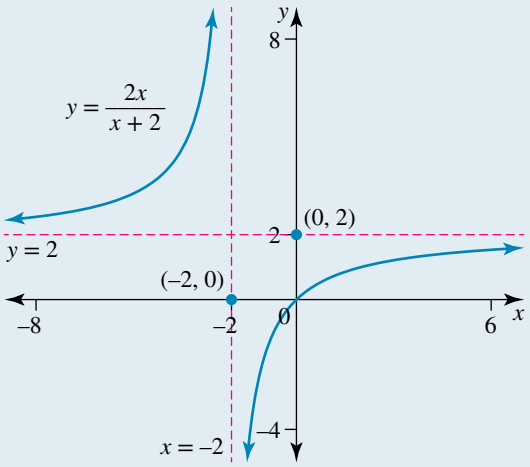


5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

x	$y = \dfrac{2x}{x+2}$	
-3	$\dfrac{(-)}{(-)} = (+)$	Above the x-axis
-1	$\dfrac{(-)}{(+)} = (-)$	Below the x-axis
2	$\dfrac{(+)}{(+)} = (+)$	Above the x-axis



6. Sketch the function.



WORKED EXAMPLE 11

Sketch the following.

a. $y = \dfrac{x^2 - x - 6}{x^2 - 1}$

b. $y = \dfrac{2x^2 + 5}{x^2 - 25}$

THINK

a. 1. Determine vertical asymptotes by solving $x^2 - 1 = 0$.

2. To find the horizontal asymptote, begin to simplify the expression.

3. Determine the x- and y-intercepts.

4. Sketch the asymptotes and intercepts.

WRITE

$x^2 - 1 = 0$

$(x + 1)(x - 1) = 0$

$x = \pm 1$

Therefore, the vertical asymptotes are $x = 1$ and $x = -1$.

Horizontal asymptote: $y = \dfrac{x^2}{x^2}$

$= 1$

Therefore, the horizontal asymptote is $y = 1$.

$x = 0$: $y = \dfrac{-6}{-1}$

$= 6$

Therefore, the y-intercept is $(0, 6)$.

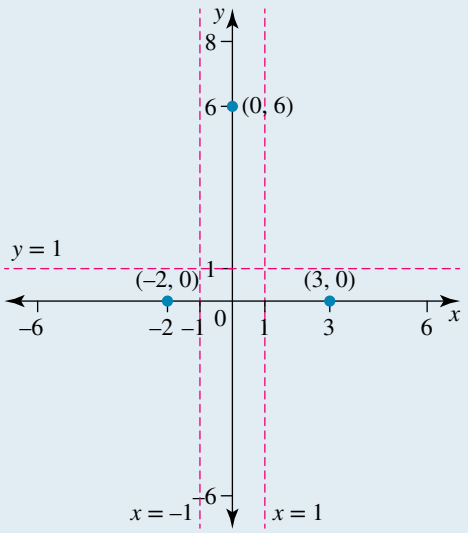
$y = 0$: $0 = \dfrac{x^2 - x - 6}{x^2 - 1}$

$0 = x^2 - x - 6$

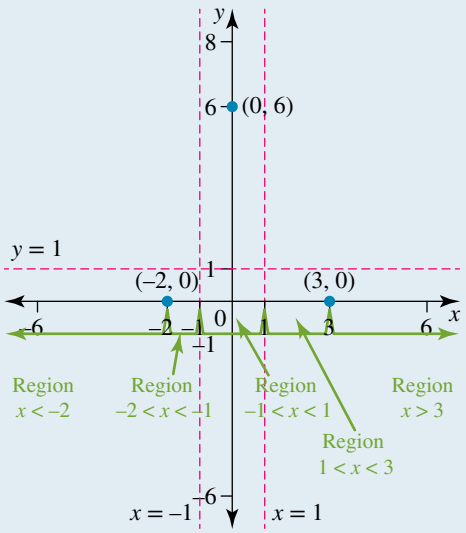
$= (x - 3)(x + 2)$

$x = 3, \ x = -2$

Therefore, the x-intercepts are $(3, 0)$ and $(-2, 0)$.

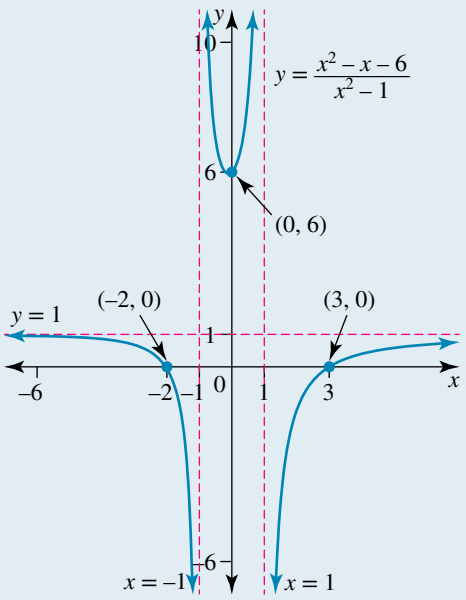


5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.



x	$y = \dfrac{x^2 - x - 6}{x^2 - 1} = \dfrac{(x-3)(x+2)}{(x-1)(x+1)}$	
-3	$\dfrac{(-)(-)}{(-)(-)} = (+)$	Above the x-axis
-1.5	$\dfrac{(-)(+)}{(-)(-)} = (-)$	Below the x-axis
0	$\dfrac{(-)(+)}{(-)(+)} = (+)$	Above the x-axis
2	$\dfrac{(-)(+)}{(+)(+)} = (-)$	Below the x-axis
4	$\dfrac{(+)(+)}{(+)(+)} = (+)$	Above the x-axis

6. Sketch the function.



b. 1. Determine vertical asymptotes by solving $x^2 - 25 = 0$.

$$x^2 - 25 = 0$$
$$(x + 5)(x - 5) = 0$$
$$x = \pm 5$$

Therefore, the vertical asymptotes are $x = 5$ and $x = -5$.

2. To find the horizontal asymptote, begin to simplify the expression.

$$\text{Horizontal asymptote: } y = \frac{2x}{x}$$
$$= 2$$

Therefore, the horizontal asymptote is $y = 2$.

3. Determine the x- and y-intercepts.

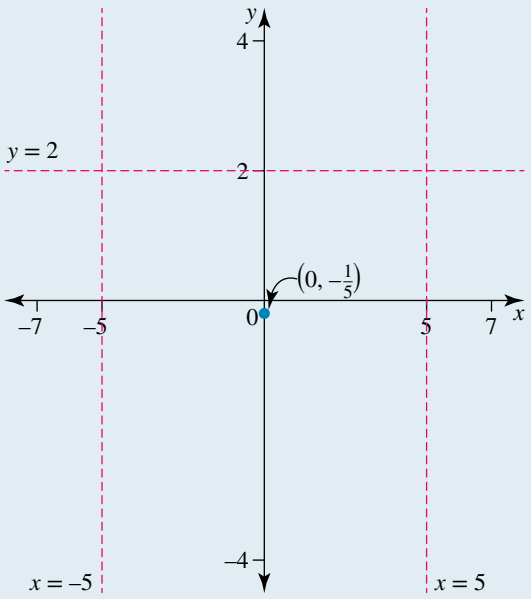
$$x = 0\text{: } y = \frac{5}{-25}$$
$$= -\frac{1}{5}$$

Therefore, the y-intercept is $\left(0, -\frac{1}{5}\right)$.

$$y = 0\text{: } 0 = \frac{2x^2 + 5}{x^2 - 25}$$
$$0 = 2x^2 + 5$$

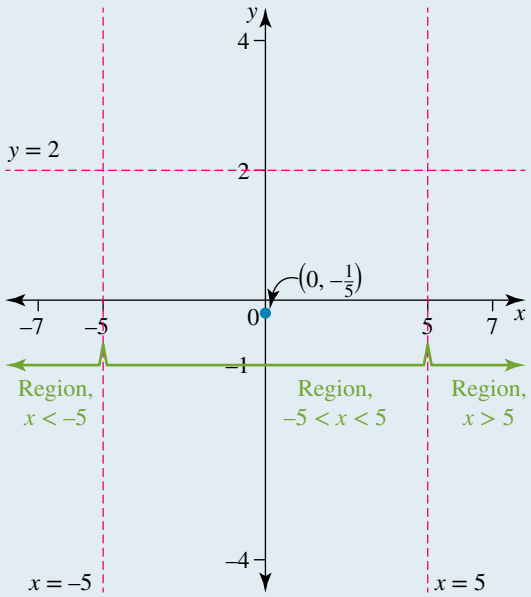
Therefore, there is no x-intercept.

4. Sketch the asymptotes and intercepts.

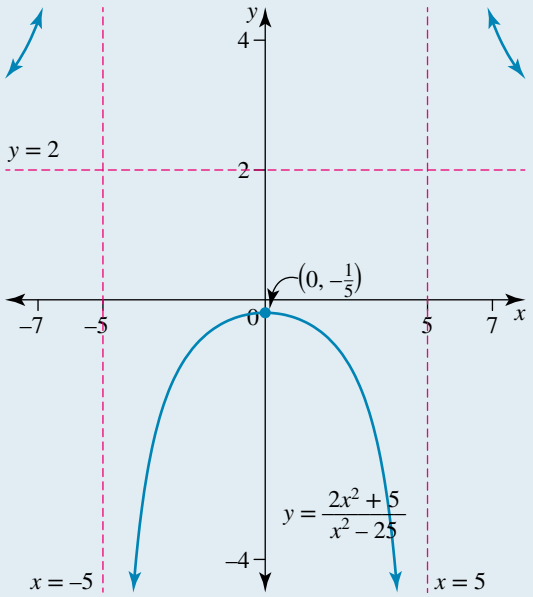


5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.
In this instance, the numerator is always positive.

x	$y = \dfrac{2x^2+5}{x^2-25} = \dfrac{2x^2+5}{(x+5)(x-5)}$	
-6	$\dfrac{(+)}{(-)(-)} = (+)$	Above the x-axis
0	$\dfrac{(+)}{(+)(-)} = (-)$	Below the x-axis
6	$\dfrac{(+)}{(+)(+)} = (+)$	Above the x-axis



6. Sketch the function.



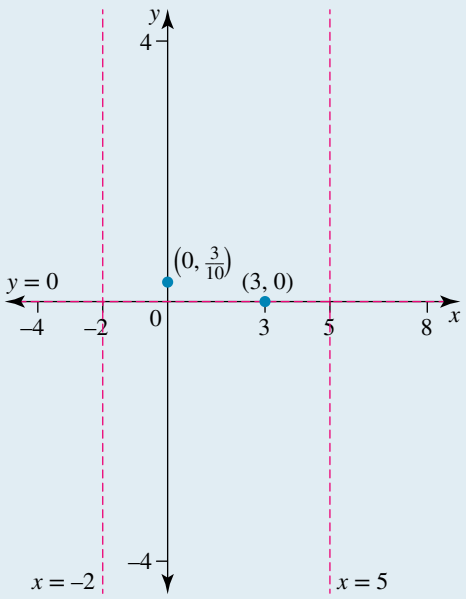
7.4.3 Graphs that cross the horizontal asymptote

It is possible for a graph to cross the horizontal asymptote, but not the vertical asymptote. The horizontal asymptote is approached at the extreme ends of the graph ($x \to \infty$ and $x \to -\infty$) but the graph may cross it elsewhere.

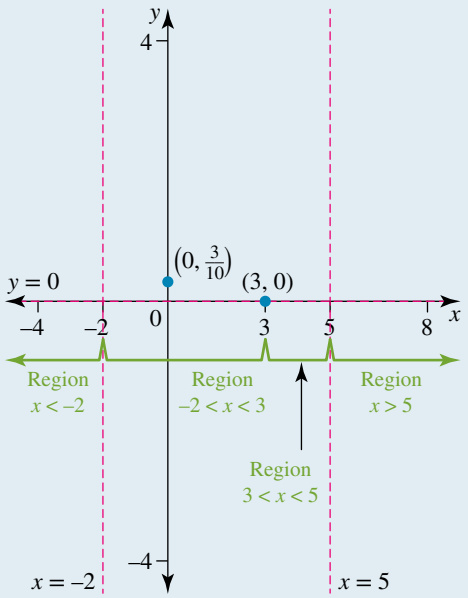
In Worked example 12 below, because $y = \dfrac{x-3}{x^2-3x-10}$ is a proper fraction, $y = 0$ is the horizontal asymptote. However, when $x = 3$, $y = 0$, so there is a value for x that lies on the horizontal asymptote. The graph approaches $y = 0$ as $x \to \pm\infty$.

WORKED EXAMPLE 12

Sketch $y = \dfrac{x-3}{x^2-3x-10}$.

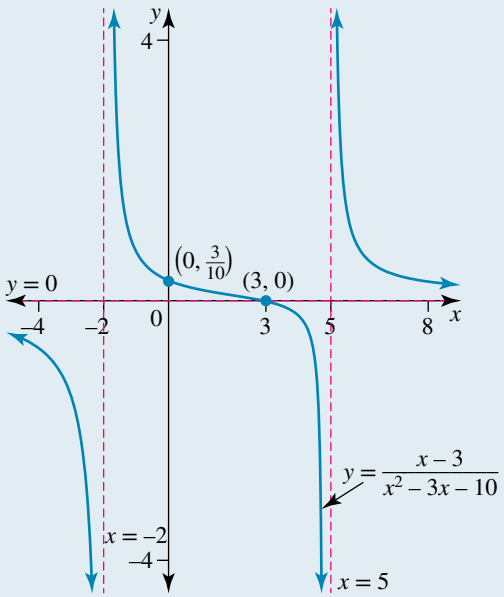
THINK	WRITE
1. Determine vertical asymptotes by solving $x^2 - 3x - 10 = 0$.	$x^2 - 3x - 10 = 0$ $(x-5)(x+2) = 0$ Therefore, the vertical asymptotes are $x = 5$ and $x = -2$.
2. The expression is a proper fraction, so $y = 0$ is the horizontal asymptote.	The horizontal asymptote is $y = 0$.
3. Determine the x- and y-intercepts.	$x = 0$: $y = \dfrac{-3}{-10}$ $= \dfrac{3}{10}$ Therefore, the y-intercept is $\left(0, \dfrac{3}{10}\right)$. $y = 0$: $x - 3 = 0$ $x = 3$ Therefore, the x-intercept is $(3, 0)$.
4. Sketch the asymptotes and intercepts.	

5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.



x	$y = \dfrac{x-3}{x^2-3x-10}$	
-3	$\dfrac{(-)}{(-)(-)} = (-)$	Below the x-axis
-1	$\dfrac{(-)}{(-)(+)} = (+)$	Above the x-axis
4	$\dfrac{(+)}{(-)(+)} = (-)$	Below the x-axis
6	$\dfrac{(+)}{(+)(+)} = (+)$	Above the x-axis

6. Sketch the function.



7.4.4 Graphs without vertical asymptotes

Consider a rational function in the form $\dfrac{f(x)}{g(x)}$. If there is no solution for $g(x) = 0$, then there will be no vertical asymptotes. The horizontal asymptote is identified in the usual manner.

WORKED EXAMPLE 13

Sketch the following.

a. $y = \dfrac{x+4}{x^2+2}$ **b.** $y = \dfrac{x^2-x-6}{x^2+1}$

THINK

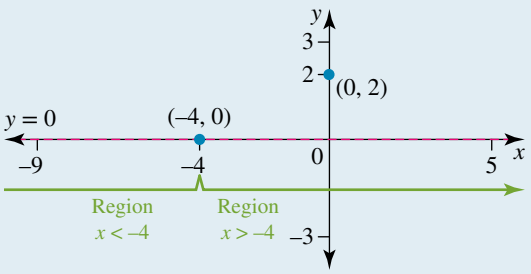
a. 1. There are no solutions to $x^2 + 2 = 0$, so there are no vertical asymptotes.

2. The expression is a proper fraction, so $y = 0$ is the horizontal asymptote.

3. Determine the x- and y-intercepts.

4. Sketch the asymptotes and intercepts.

5. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.
In this instance, the denominator is always positive.

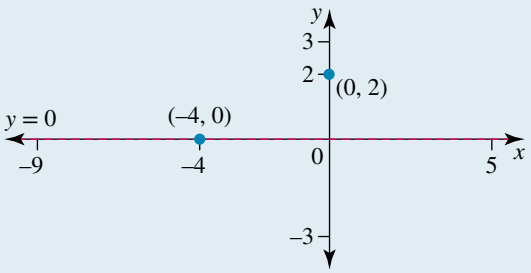


6. Sketch the function.
The function crosses the x-axis when $x = -4$ but must approach the horizontal asymptote as $x \to -\infty$. This means that there is a local minimum in the interval $(-\infty, -4)$.

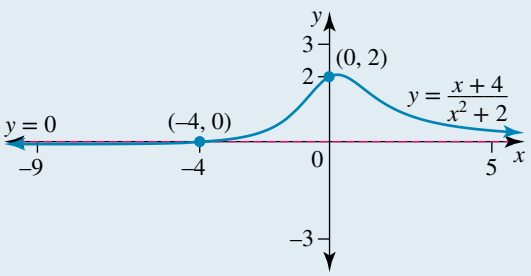
WRITE

a. $x^2 + 2 \neq 0$
Therefore, there are no vertical asymptotes.
The horizontal asymptote is $y = 0$.

$x = 0$: $y = \dfrac{4}{2}$
$= 2$
Therefore, the y-intercept is $(0, 2)$.
$y = 0$: $x + 4 = 0$
$x = -4$
Therefore, the x-intercept is $(-4, 0)$.



x	$y = \dfrac{x+4}{x^2+2}$	
5	$\dfrac{(-)}{(+)} = (-)$	Below the x-axis
0	$\dfrac{(+)}{(+)} = (+)$	Above the x-axis



Similarly, the function passes through the points $(-4, 0)$ and $(0, 2)$ but approaches the horizontal asymptote as $x \to \infty$. If $x = -1$, $y = 1$. If $x = 1$, $y = \frac{5}{3}$. Both of these y-values are smaller than the y-intercept of 2. Therefore, there is a local maximum in the interval $(-1, 1)$.

b. 1. There are no solutions to $x^2 + 1 = 0$, so there are no vertical asymptotes.

2. The expression is not a proper fraction. To find the horizontal asymptote, begin to simplify the expression.

3. Determine the x- and y-intercepts.

4. Does the graph cross the horizontal asymptote, $y = 1$?

5. Sketch the asymptotes and intercepts.

b. $x^2 + 1 \neq 0$

Therefore, there are no vertical asymptotes.

Horizontal asymptote: $y = \frac{x^2}{x^2}$

$= 1$

Therefore, the horizontal asymptote is $y = 1$.

$x = 0$: $y = \frac{-6}{1}$

$= -6$

Therefore, the x-intercept is $(0, -6)$.

$y = 0$: $x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3, x = -2$

Therefore, the y-intercepts are $(3, 0)$ and $(-2, 0)$.

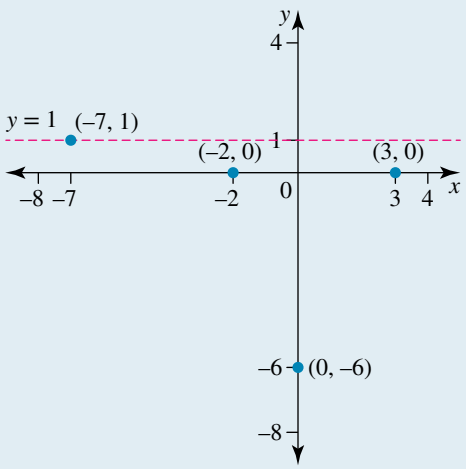
$\frac{x^2 - x - 6}{x^2 + 1} = 1$

$x^2 - x - 6 = x^2 + 1$

$0 = x + 7$

$x = -7$

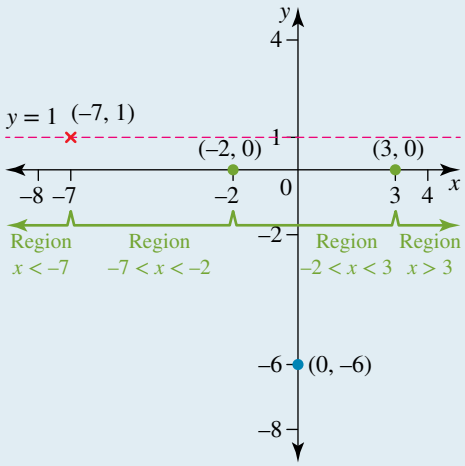
The graph crosses the horizontal asymptote at $(-7, 1)$.



6. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign. If the function is above the x-axis, it may also be useful to determine if it is above or below the horizontal asymptote.

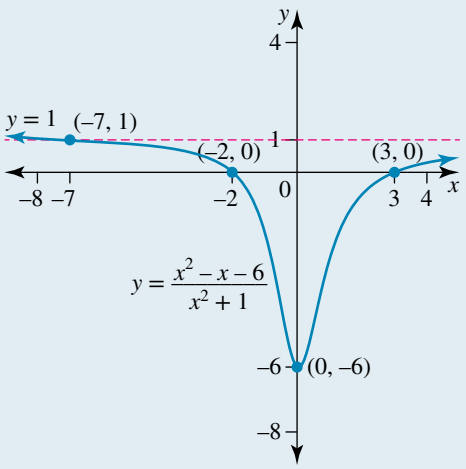
In this instance, the denominator is always positive.

x	$y = \dfrac{x^2 - x - 6}{x^2 + 1} = \dfrac{(x-3)(x+2)}{x^2+1}$	
-8	$\dfrac{(-)(-)}{(+)} = (+)$ $\dfrac{(-11)(-6)}{65} > 1$	Above the x-axis; above the horizontal asymptote
-3	$\dfrac{(-)(+)}{(+)} = (-)$	Below the x-axis.
0	$\dfrac{(-)(+)}{(+)} = (-)$	Below the x-axis.
4	$\dfrac{(+)(+)}{(+)} = (+)$ $\dfrac{(1)(6)}{17} < 1$	Above the x-axis; below the horizontal asymptote



7. Sketch the function.

The function crosses the horizontal asymptote when $x = -7$ but must approach the horizontal asymptote as $x \to -\infty$. This means that there is a local maximum in the interval $(-\infty, -7)$.



7.4.5 Oblique asymptotes

Sometimes, instead of a horizontal asymptote, the function may have an **oblique** asymptote (a line in the form $y = mx + c$). As with horizontal asymptotes, the function may cross an oblique asymptote, but as $x \to \pm\infty$, the function will approach the oblique asymptote.

The function $y = \dfrac{x+1}{x}$ can be rewritten as $y = 1 + \dfrac{1}{x}$. In this instance, the horizontal asymptote is $y = 1$.

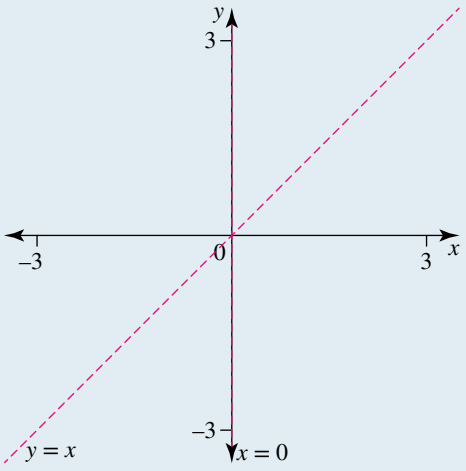
The function $y = \dfrac{x^2+1}{x}$ can be rewritten as $y = x + \dfrac{1}{x}$. In this instance, the oblique asymptote is $y = x$.

In the previous examples involving improper fractions, the degrees of the numerator and denominator were equal, so it was only necessary to begin the simplification to identify the horizontal asymptote. If the degree of the numerator is greater than that of the denominator, it is necessary to rewrite the expression so it includes a proper fraction. Technology may assist with this process.

WORKED EXAMPLE 14

Sketch the following.

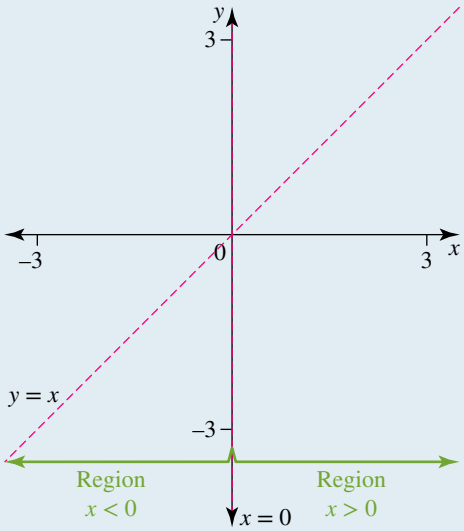
a. $y = \dfrac{x^2+1}{x}$ **b.** $y = \dfrac{x^3-8}{x^2+5x+6}$

THINK	WRITE
a. 1. Determine the vertical asymptotes.	The vertical asymptote is $x = 0$.
2. The function is an improper fraction, so rewrite it (using technology if required).	$y = \dfrac{x^2+1}{x}$ $= x + \dfrac{1}{x}$
3. Identify the oblique asymptote.	The oblique asymptote is $y = x$.
4. Determine the x- and y-intercepts.	$x = 0$: undefined. Therefore, there are no x-intercepts. $y = 0$: $x^2 + 1 = 0$. No solution. Therefore, there are no y-intercepts.
5. Sketch the asymptotes.	

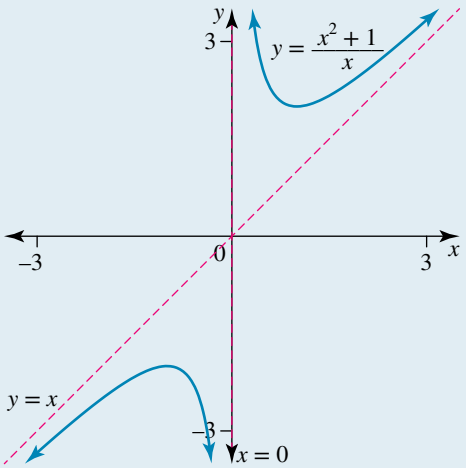
6. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

The numerator will always be positive.

x	$y = \dfrac{x^2 + 1}{x}$	
-1	$\dfrac{(+)}{(-)} = (-)$	Below the x-axis
1	$\dfrac{(+)}{(+)} = (+)$	Above the x-axis



7. Sketch the function.



b. 1. Determine the vertical asymptotes by solving $x^2 + 5x + 6 = 0$.

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, x = -2$

Therefore, the vertical asymptotes are $x = -3$ and $x = -2$.

2. The function is an improper fraction, so it needs to be rewritten as a proper fraction.

$$y = \frac{x^3 - 8}{x^2 + 5x + 6}$$

3. Rewrite the numerator as a product of a linear factor and the quadratic factor in the denominator, adding extra terms so that the numerator is still equivalent to $x^3 - 8$.

$$y = \frac{(x-5)(x^2+5x+6)+19x+22}{x^2+5x+6}$$

4. Split the fraction into two and simplify.
CAS (Computer Algebra Software) technology may be used to rewrite the expression as a proper fraction.

$$y = \frac{(x-5)\cancel{(x^2+5x+6)}}{\cancel{x^2+5x+6}} + \frac{19x+22}{x^2+5x+6}$$
$$= x - 5 + \frac{19x+22}{x^2+5x+6}$$

5. Identify the oblique asymptote.

The oblique asymptote is $y = x - 5$.

6. Find where the graph crosses the oblique asymptote by solving $\frac{19x+22}{x^2+5x+6} = 0$.

Use the asymptote $y = x - 5$ to identify the y-value.

The graph will cross the oblique asymptote when
$19x + 22 = 0$
$x = -1.16$
$y = -1.16 - 5$
$= -6.16$
Therefore, the graph crosses the oblique asymptote at $(-1.16, 6.16)$.

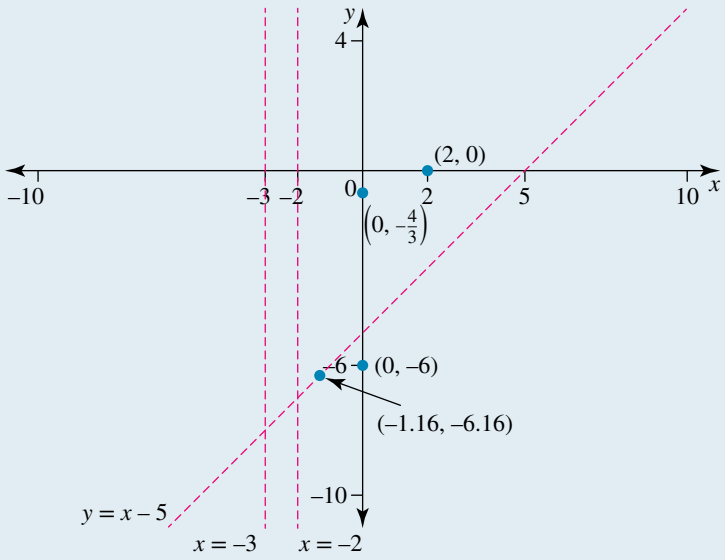
7. Determine the x- and y-intercepts.

$x = 0$: $y = \frac{-8}{6}$
$= \frac{-4}{3}$

Therefore, the y-intercept is $\left(0, -\frac{4}{3}\right)$.

$y = 0$: $x^2 - 8 = 0$
$x = 2$
Therefore, the x-intercept is $(2, 0)$.

8. Sketch the asymptotes and intercept

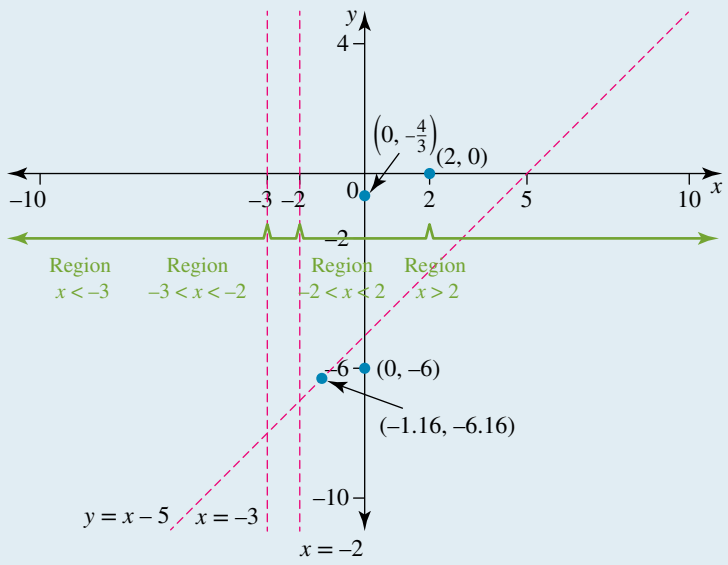


9. Choose values for x from important intervals to determine the sign of the function. It is not necessary to calculate the actual y-values, merely to determine the sign.

Although the numerator can be factorised as $(x-2)\left(x^2+2x+4\right)$, it is probably easier to use the form x^3-8.

If the oblique asymptote and the graph are on the same side of the x-axis, evaluate the sign of $\dfrac{19x+22}{(x+2)(x+3)}$ to determine if the function is above or below the asymptote.

x	$y=\dfrac{x^3-8}{x^2+5x+6}=x-5+\dfrac{19x+22}{x^2+5x+6}$	
-4	$\dfrac{(-)}{(-)(-)}=(-)$ $\left(\dfrac{19x+22}{(x+2)(x+3)}\right)=\dfrac{(-)}{(-)(-)}$ $=(-)$	Below the x-axis; below the oblique asymptote
-2.5	$\dfrac{(-)}{(+)(-)}=(+)$	Above the x-axis
-1.5	$\dfrac{(-)}{(+)(+)}=(-)$ $\left(\dfrac{19x+22}{(x+2)(x+3)}\right)=\dfrac{(-)}{(+)(+)}$ $=(-)$	Below the x-axis; below the oblique asymptote
0	$\dfrac{(-)}{(+)(+)}=(-)$ $\left(\dfrac{19x+22}{(x+2)(x+3)}\right)=\dfrac{(+)}{(+)(+)}$ $=(+)$	Below the x-axis; above the oblique asymptote
3	$\dfrac{(+)}{(+)(+)}=(+)$	Above the x-axis



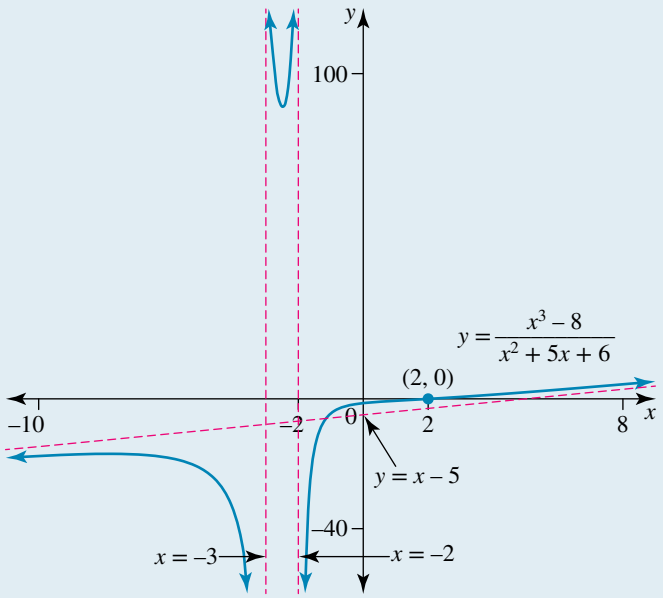
10. Sketch the function.

In the interval $(-\infty, -3)$, the function is below the oblique asymptote.

In the interval $(-3, -2)$, the function is above the x-axis. The point $(-2.5, 94.5)$ lies on the function in this interval.

In the interval $(-2, \infty)$, the function is initially below the oblique asymptote. It crosses over the asymptote at the point $(-1.16, -6.16)$ and then remains above the oblique asymptote.

The function will approach the oblique asymptotes as $x \to \pm\infty$



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Units 1 & 2 | Area 5 | Sequence 1 | Concepts 5 & 6

Sketching rational functions expressed as proper fractions Summary screen and practice questions
Sketching rational functions expressed as improper fractions Summary screen and practice questions

Exercise 7.4 Sketching graphs of rational functions

Technology free

1. **a.** Identify where the function $y = \dfrac{x^2 - x - 6}{x - 3}$ is undefined.

b. Simplify $y = \dfrac{x^2 - x - 6}{x - 3}$.

2. Consider the function $y = \dfrac{x + 3}{x}$.

a. Identify where the function is undefined.

b. Identify the horizontal and vertical asymptotes.

3. Consider the function $y = \dfrac{x^2 - 1}{x^2 - x - 6}$.

a. Identify where the function is undefined.

b. Identify the horizontal and vertical asymptotes.

4. Use your answer to question **1** to sketch $y = \dfrac{x^2 - x - 6}{x - 3}$.

5. **WE9** Sketch the following.

a. $y = \dfrac{x^2 - 4x + 4}{x - 2}$ **b.** $y = \dfrac{x^2 - x - 2}{x - 2}$ **c.** $y = \dfrac{x^2 - 1}{x + 1}$ **d.** $y = \dfrac{x^2 - 5x + 6}{2 - x}$

6. Use your answer to question **2** to sketch $y = \dfrac{x + 3}{x}$.

7. **WE10** Sketch the following.

a. $y = \dfrac{2x}{x + 3}$ **b.** $y = \dfrac{x - 2}{x}$ **c.** $y = \dfrac{3x}{x - 2}$ **d.** $y = \dfrac{2x + 1}{x + 3}$

8. Use your answer to question **3** to sketch $y = \dfrac{x^2 - 1}{x^2 - x - 6}$.

9. **WE11** Sketch the following.

a. $y = \dfrac{2x^2}{x^2 - 1}$ **b.** $y = \dfrac{x^2 - 2x - 3}{x^2 - 2x - 15}$ **c.** $y = \dfrac{x^2}{x^2 - 4}$ **d.** $y = \dfrac{4x^2 - 1}{x^2 - 3x - 4}$

10. **WE12** Sketch the following.

a. $y = \dfrac{x + 1}{x^2 - x - 6}$ **b.** $y = \dfrac{3x}{x^2 + 2x - 8}$ **c.** $y = \dfrac{x - 1}{x^2 - 2x + 1}$ **d.** $y = \dfrac{2 - x}{x^2 - 3x - 4}$

11. **WE13** Sketch the following.

a. $y = \dfrac{x + 2}{x^2 + 1}$ **b.** $y = \dfrac{x^2 - 1}{x^2 + 2}$ **c.** $y = \dfrac{x}{x^2 + 3}$ **d.** $y = \dfrac{x^2 - 4}{x^2 + 2}$

e. $y = \dfrac{1 - x}{x^2 + 3}$ **f.** $y = \dfrac{x + 2}{x^2 + 2}$

12. **WE14** Sketch the following.

a. $y = \dfrac{x^2 - 1}{x}$ **b.** $y = \dfrac{x^3}{x^2 - 2x - 3}$

13. Sketch the following.

a. $y = \dfrac{1 - x^2}{x}$ **b.** $y = \dfrac{x + 5}{x + 2}$ **c.** $y = \dfrac{x^2 - 2x - 3}{x^2 - 2x + 1}$ **d.** $y = \dfrac{x - 4}{4 - x^2}$

14. Sketch the following.

a. $y = \dfrac{1 - x^2}{x^2 + 1}$ **b.** $y = \dfrac{2x^2}{x^2 + 2}$

Technology active

15. Sketch the following.

a. $y = \dfrac{x^2 - 2}{x}$ **b.** $y = \dfrac{x^2}{x - 1}$

16. Sketch the following.

a. $y = \dfrac{x^2 - 2x + 1}{x^2 - x - 2}$ **b.** $y = \dfrac{x^2 + x - 6}{x^2 - 9}$

17. Sketch the following.

a. $y = \dfrac{2x^2 - x}{x - 1}$ **b.** $y = \dfrac{x^2 - 5x + 7}{x - 2}$

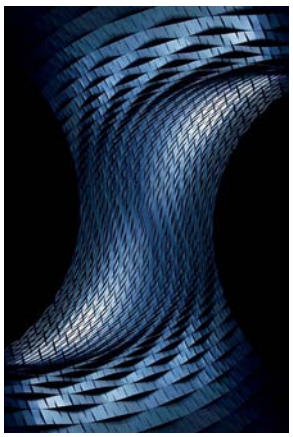
18. Sketch the following.

a. $y = \dfrac{x^3 + x - 1}{x^2 - 1}$ **b.** $y = \dfrac{2x^3 - 4x^2 - x + 5}{x^2 - x - 2}$

7.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar



1. **a.** Sketch $y = 1 - 5x$ and use your sketch to answer parts **b–d**.

b. Sketch $y = |1 - 5x|$.

c. Sketch $y = 1 - 5|x|$.

d. Sketch $y = \dfrac{1}{1 - 5x}$.

2. **a.** Sketch $y = 2x + 5$ and use your sketch to answer parts **b–d**.

b. Sketch $y = |2x + 5|$.

c. Sketch $y = 2|x| + 5$.

d. Sketch $y = \dfrac{1}{2x + 5}$.

3. **a.** Sketch $y = (1 - x)(x + 5)$ and use your sketch to answer parts **b–d**.

b. Sketch $y = |(1 - x)(x + 5)|$.

c. Sketch $y = (1 - |x|)(|x| + 5)$.

d. Sketch $y = \dfrac{1}{(1 - x)(x + 5)}$.

4. **a.** Sketch $y = 25 - x^2$ and use your sketch to answer parts **b–d**.

b. Sketch $y = |25 - x^2|$.

c. Sketch $y = 25 - |x|^2$.

d. Sketch $y = \dfrac{1}{25 - x^2}$.

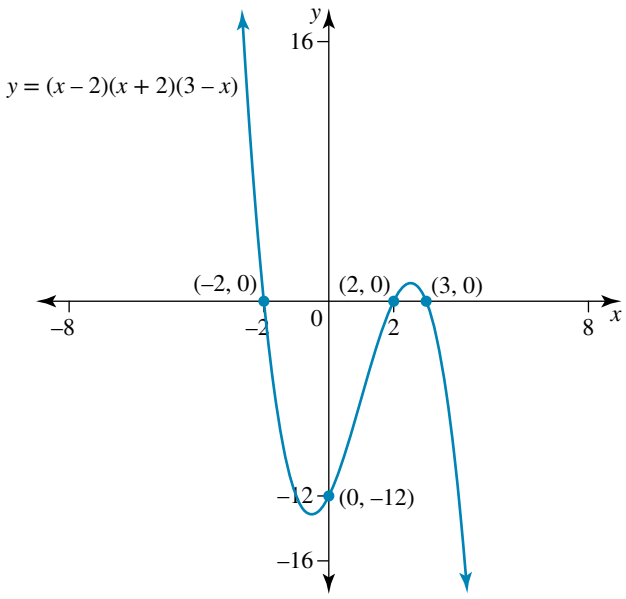
5. **a.** Sketch $y = -(x^2 + 2)$ and use your sketch to answer parts **b–d**.

b. Sketch $y = |-(x^2 + 2)|$.

c. Sketch $y = -(|x|^2 + 2)$.

d. Sketch $y = \dfrac{-1}{x^2 + 2}$.

6. Use the graph of $y = (x - 2)(x + 2)(3 - x)$ shown to sketch the following.



a. $y = |(x - 2)(x + 2)(3 - x)|$

b. $y = \dfrac{1}{(x - 2)(x + 2)(3 - x)}$

7. Sketch $y = \dfrac{x^2 - 1}{x - 1}$.

8. Sketch $y = \dfrac{x^2 + x - 6}{2 - x}$.

9. Sketch $y = \dfrac{x + 1}{x - 2}$.

10. Sketch $y = \dfrac{2x - 4}{x + 2}$.

11. Sketch $y = \dfrac{x^2 + 1}{x^2 - 9}$.

12. Sketch $y = \dfrac{x^2 + 1}{x^2}$.

Complex familiar

13. Sketch $y = \dfrac{3x^2}{1 - x^2}$.

14. Sketch $y = \dfrac{x^3 + 1}{x^2}$.

15. Sketch $y = \dfrac{x - 1}{x^2 - 3x - 10}$.

16. Sketch $y = \dfrac{x^3 + 1}{x^2 - 4}$.

Complex unfamiliar

Technology may be used to answer questions **17–20**.

17. a. Sketch $y = \dfrac{x + 1}{x + 2}$ and use your sketch to answer parts **b–d**.

b. Sketch $y = \left|\dfrac{x + 1}{x + 2}\right|$.

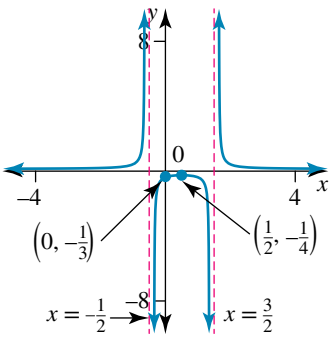
c. Sketch $y = \dfrac{|x| + 1}{|x| + 2}$.

d. Sketch $y = \dfrac{x + 2}{x + 1}$.

18. Sketch $y = \dfrac{(x + 3)(x - 7)(1 - x)}{x^2 + x - 2}$.

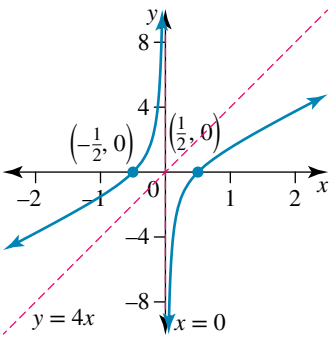
19. Use the graph of $f(x)$ shown to sketch the following.

a. $|f(x)|$ **b.** $\dfrac{1}{f(x)}$



20. Use the graph of $f(x)$ shown to sketch the following.

a. $f(|x|)$ **b.** $\dfrac{1}{f(x)}$



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Units 1 & 2 Sit chapter test

Answers

Chapter 7 Sketching graphs

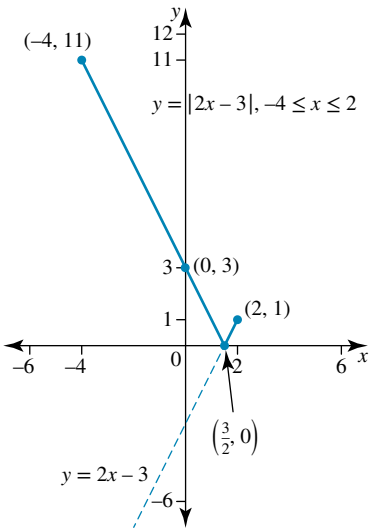
Exercise 7.2 Sketching graphs of $y = |f(x)|$ and $y = f(|x|)$ from $y = f(x)$

1. **a.** True **b.** False **c.** False **d.** True

2. $x = \pm 4$

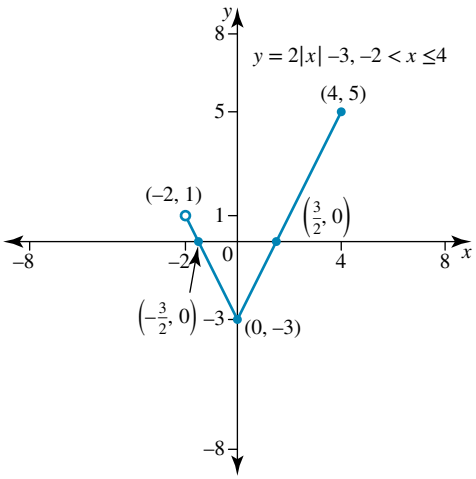
3. $x = 2, x = -4$

4. **a.**



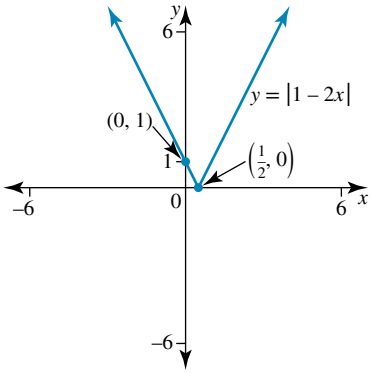
Range $\{-11, 1\}$

b.

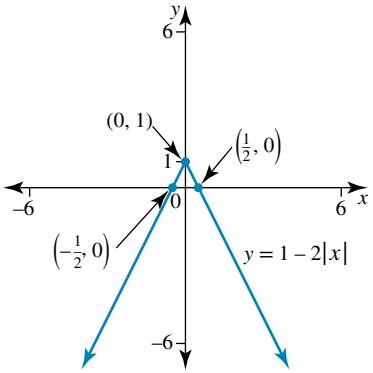


Range $\{1, 5\}$

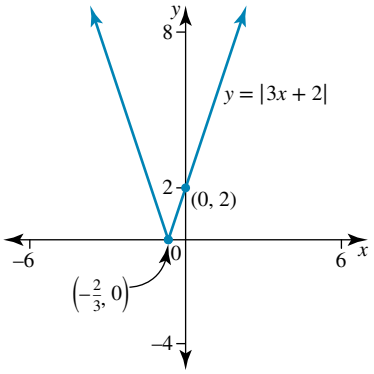
5. **a.**



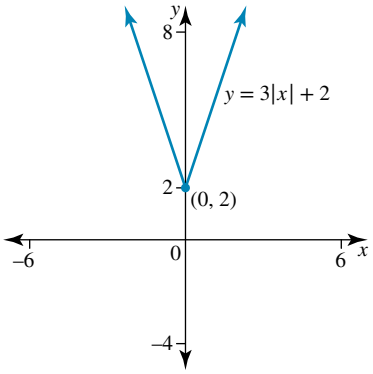
b.



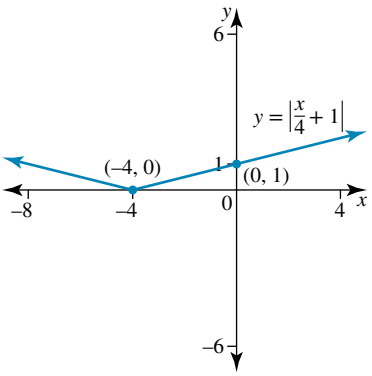
6. **a.**



b.



7. a.



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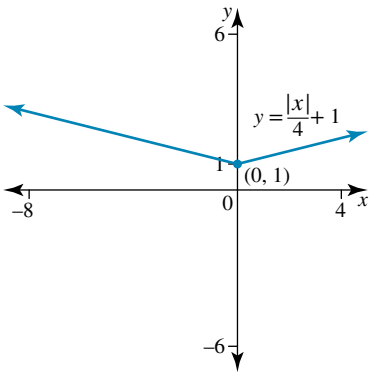

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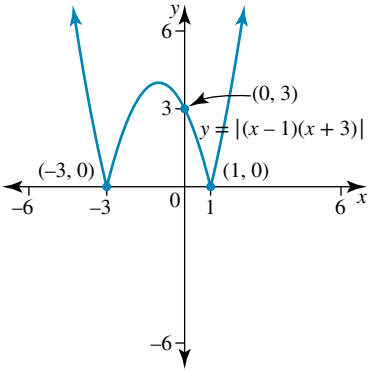

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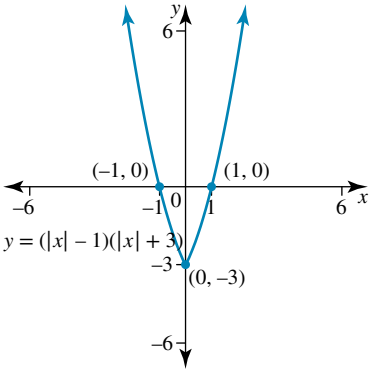

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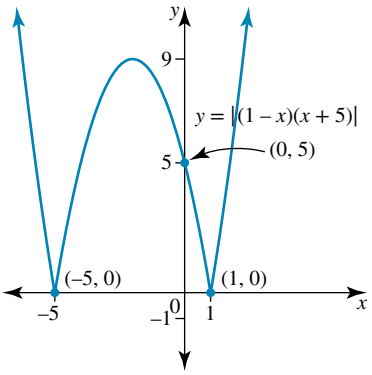

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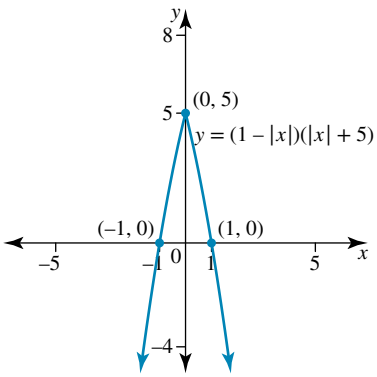

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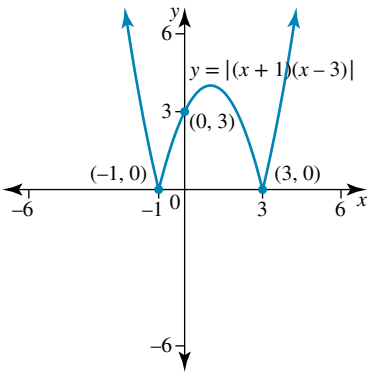

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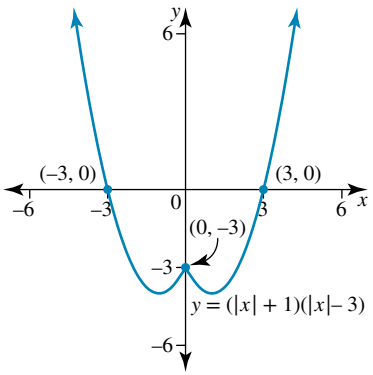
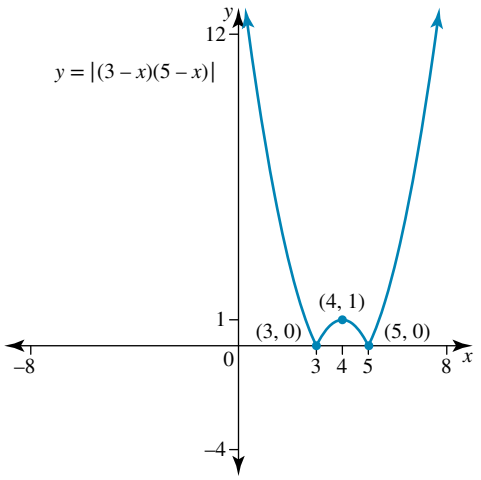
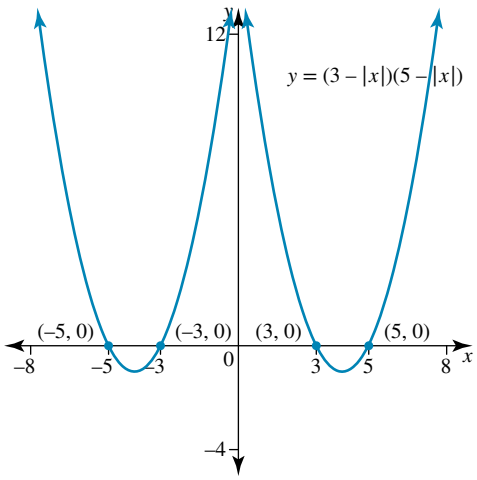


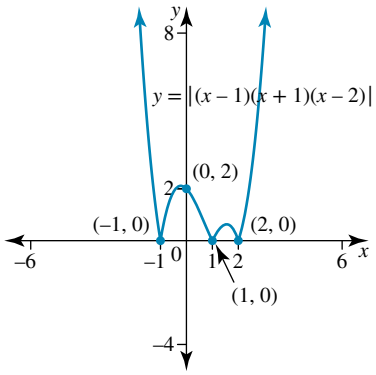
11. a.



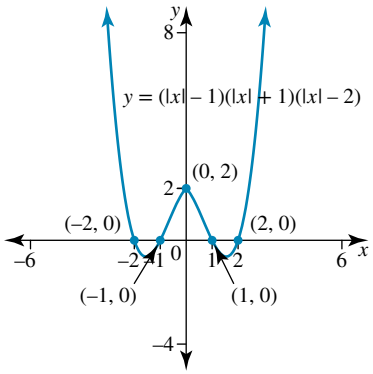
b.



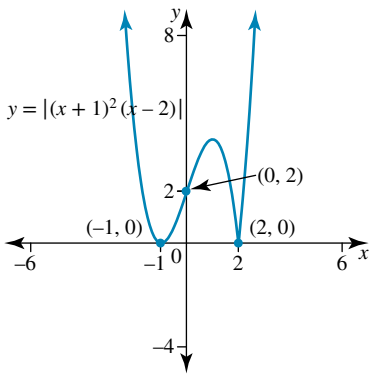
12. a.



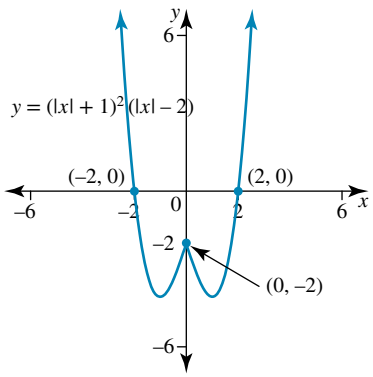
b.



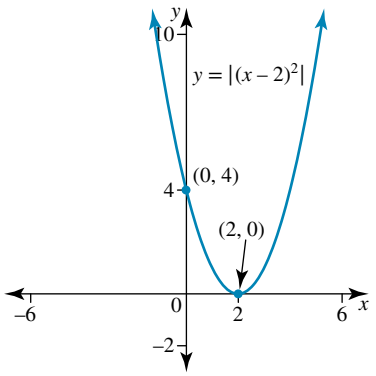
13. a.



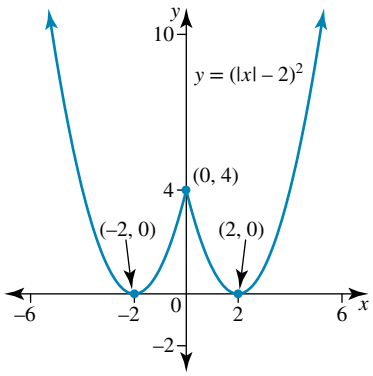
b.



14. a.

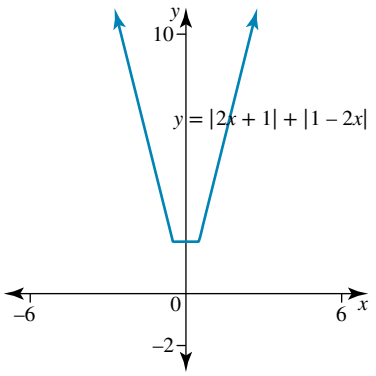


b.



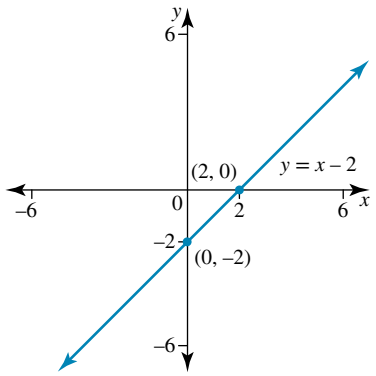
The graph for **a** is identical to the original graph.

15.



Exercise 7.3 Sketching graphs of reciprocal functions

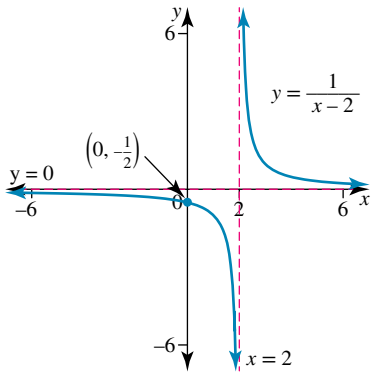
1. a.



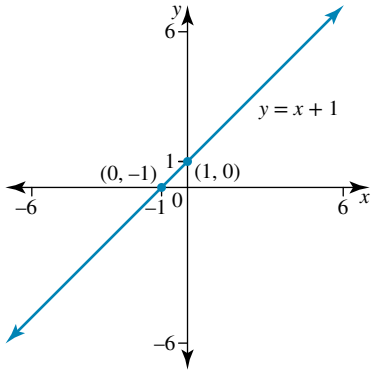
b. $x = 2, y = 0$

c. $\left(0, -\frac{1}{2}\right)$

d.



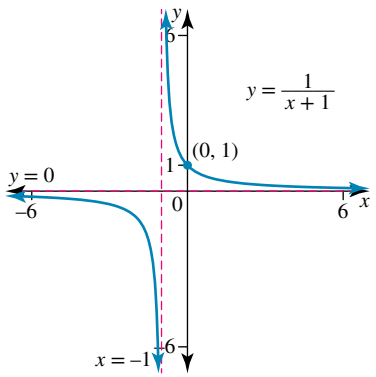
2. a.



b. $x = -1, y = 0$

c. $(0, 1)$

d.



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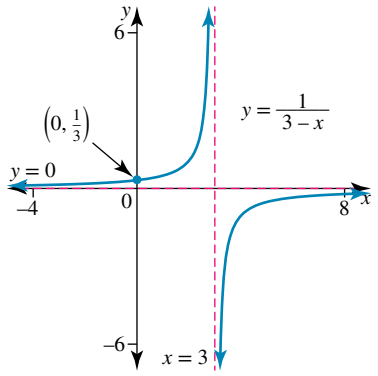

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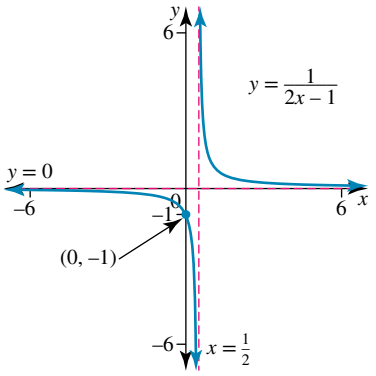
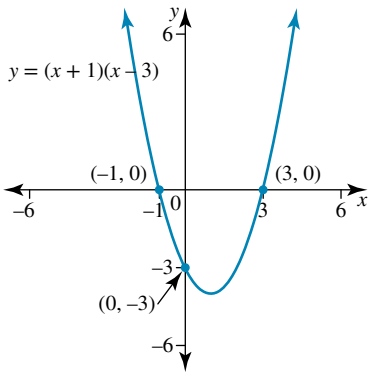


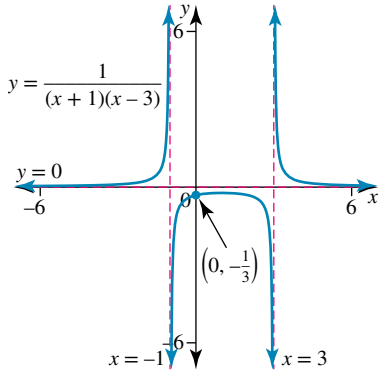
5. a.



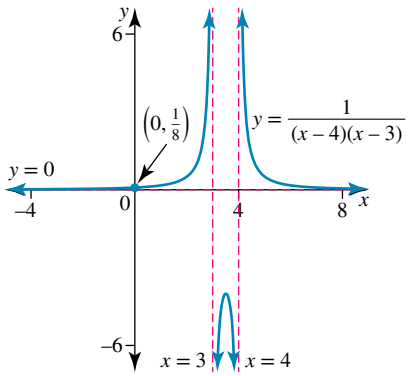
b. $x = -1, x = 3, y = 0$

c. y-intercept $\left(0, -\frac{1}{3}\right)$, turning point $\left(1, -\frac{1}{4}\right)$

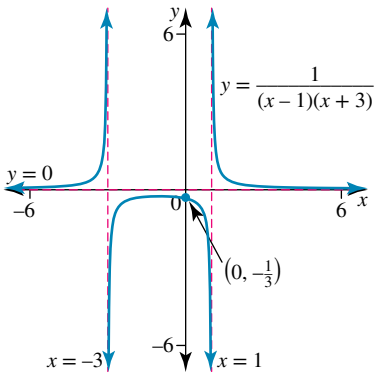
d.



6.



7.



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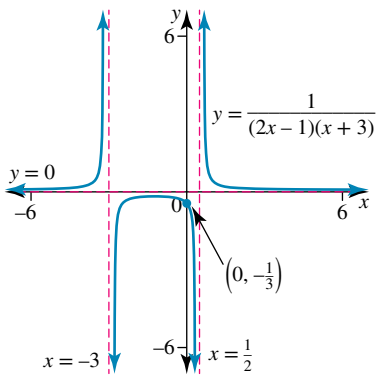

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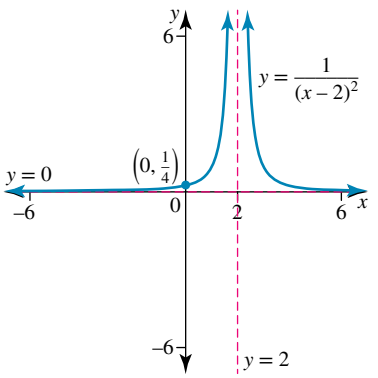

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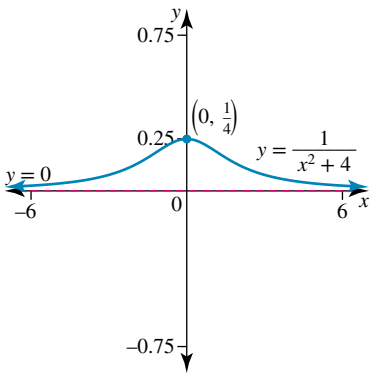
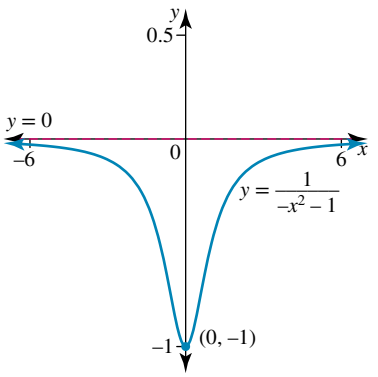
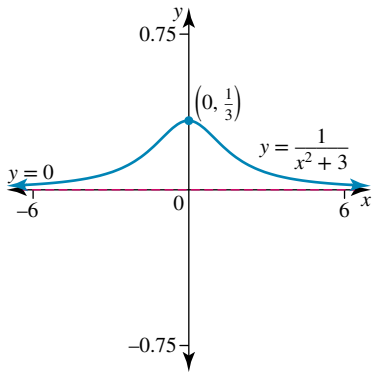


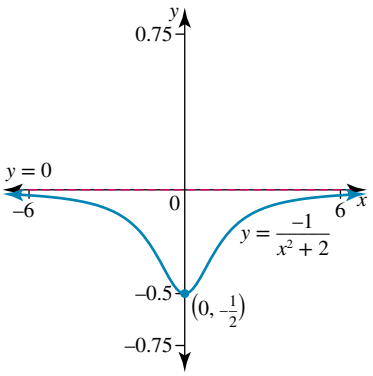
11.



12.



13.



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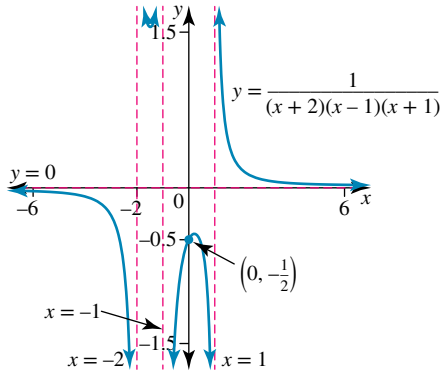

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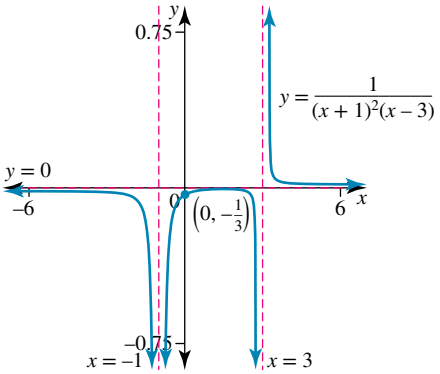

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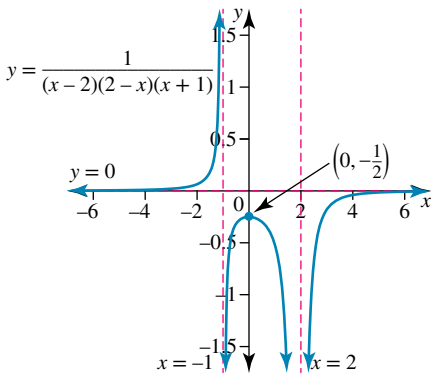

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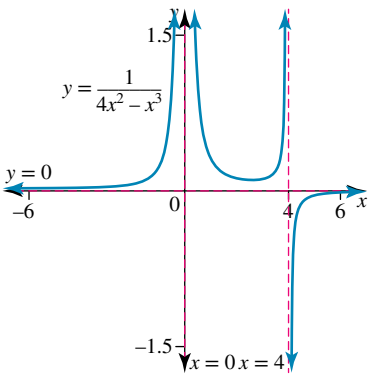

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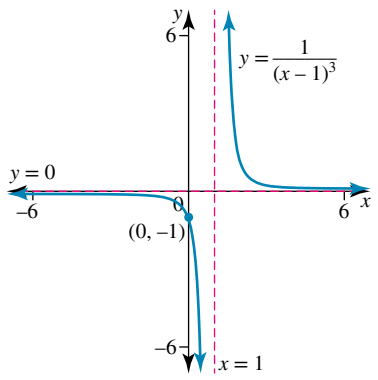
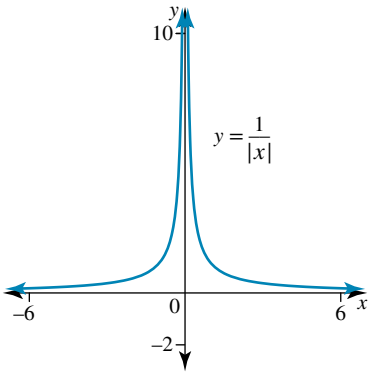


19. **a, b** The graphs for **a** and **b** are identical.



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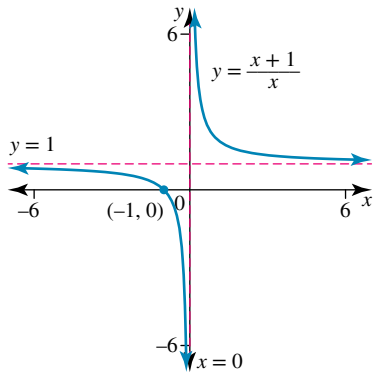

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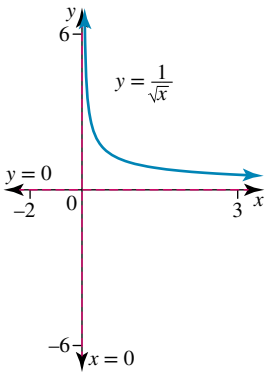


Exercise 7.4 Sketching graphs of rational functions

1. a. $x = 3$

b. $y = x + 2, x \neq 3$

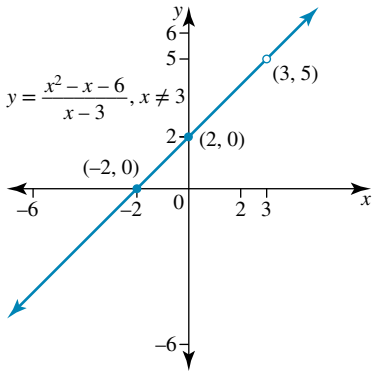
2. a. $x = 0$

b. $x = 0, y = 1$

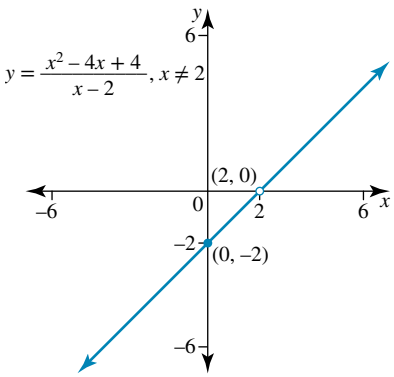
3. a. $x = -2, x = 3$

b. $x = -2, x = 3, y = 1$

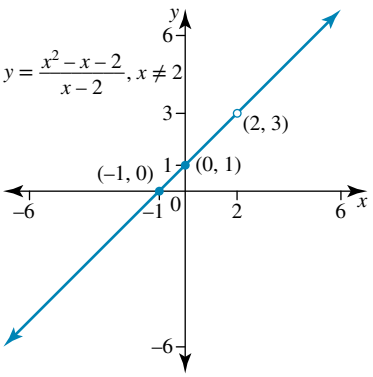
4.



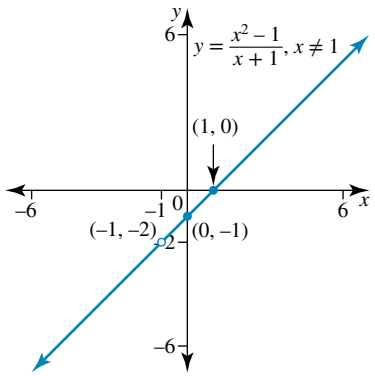
5. a.



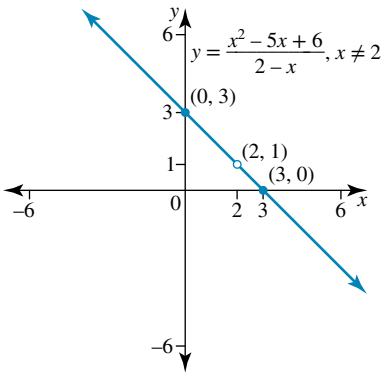
b.



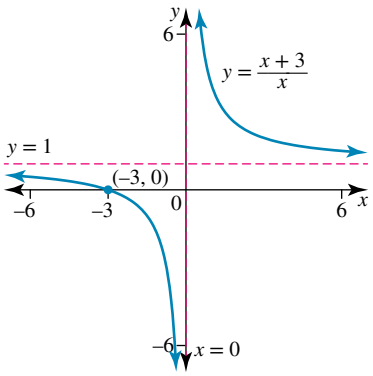
c.



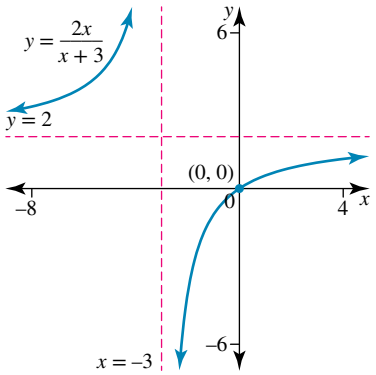
d.



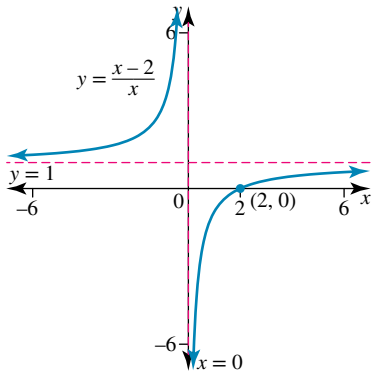
6.


y
6
$y = \frac{x+3}{x}$
$y = 1$
(–3, 0)
–6
–3
0
6
x
–6
$x = 0$

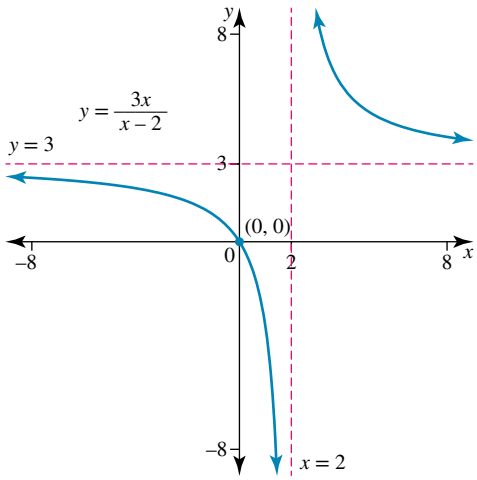
7. a.


y
6
$y = \frac{2x}{x+3}$
$y = 2$
(0, 0)
–8
0
4
x
–6
$x = -3$

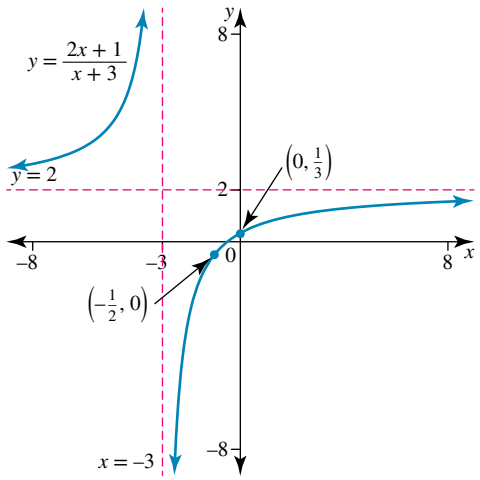
b.


y
6
$y = \frac{x-2}{x}$
$y = 1$
–6
0
2
(2, 0)
6
x
–6
$x = 0$

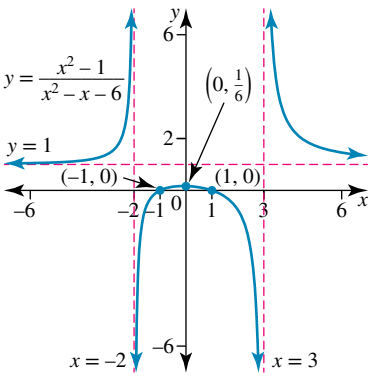
c.


y
8
$y = \frac{3x}{x-2}$
$y = 3$
3
(0, 0)
–8
0
2
8
x
–8
$x = 2$

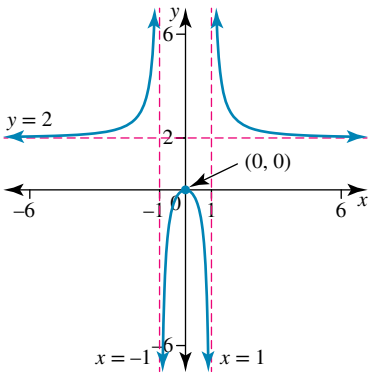
d.


y
8
$y = \frac{2x+1}{x+3}$
$y = 2$
2
$\left(0, \frac{1}{3}\right)$
–8
–3
0
8
x
$\left(-\frac{1}{2}, 0\right)$
$x = -3$
–8

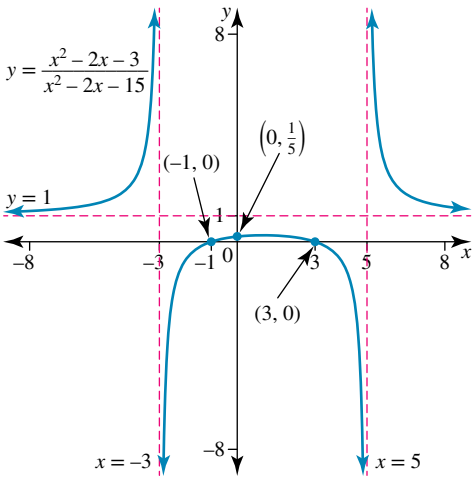
8.


y
6
$y = \frac{x^2-1}{x^2-x-6}$
$\left(0, \frac{1}{6}\right)$
2
$y = 1$
(–1, 0)
(1, 0)
–6
–2
–1
0
1
3
6
x
–6
$x = -2$
$x = 3$

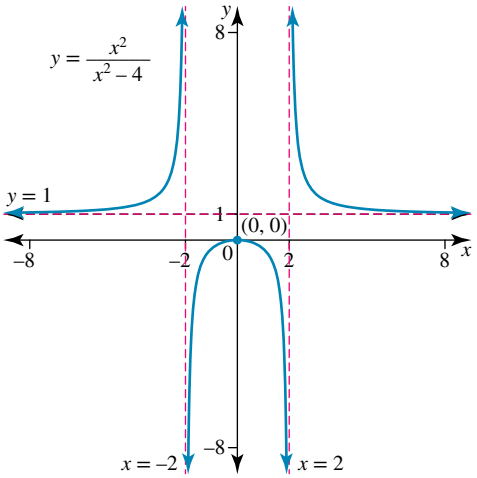
9. a.


y
6
y = 2
2
(0, 0)
–6
–1
0
1
6
x
–6
x = –1
x = 1

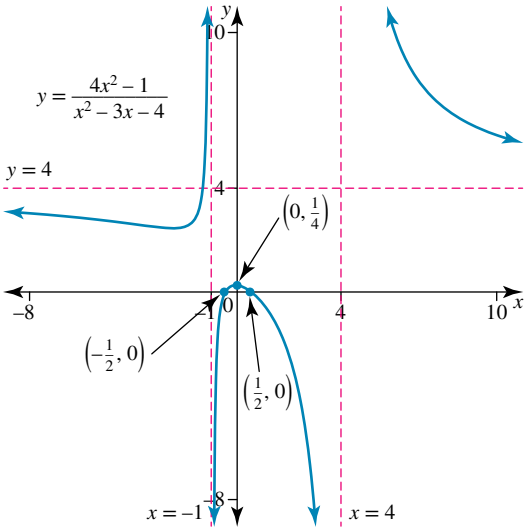
b.


y
8
y = x² – 2x – 3 / x² – 2x – 15
(0, 1/5)
(–1, 0)
y = 1
1
–8
–3
–1
0
3
5
8
x
(3, 0)
–8
x = –3
x = 5

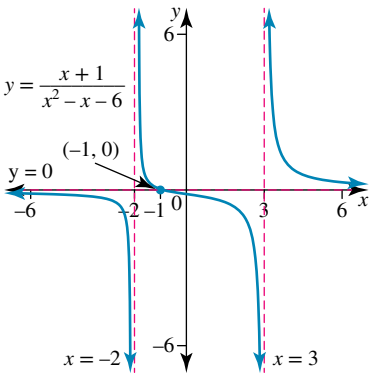
c.


y
8
y = x² / x² – 4
y = 1
1
(0, 0)
–8
–2
0
2
8
x
–8
x = –2
x = 2

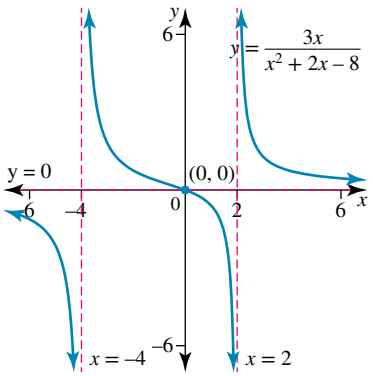
d.


y
10
y = 4x² – 1 / x² – 3x – 4
y = 4
4
(0, 1/4)
–8
–1
0
4
10
x
(–1/2, 0)
(1/2, 0)
–8
x = –1
x = 4

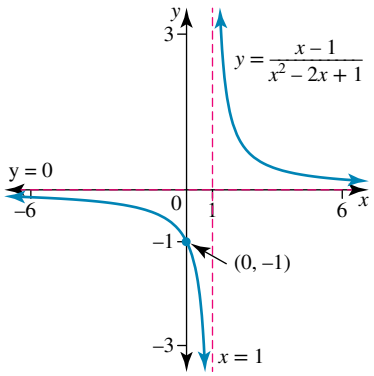
10. a.


y
6
y = x + 1 / x² – x – 6
(–1, 0)
y = 0
–6
–2
–1
0
3
6
x
–6
x = –2
x = 3

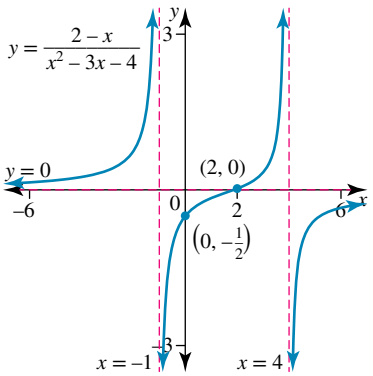
b.


y
6
y = 3x / x² + 2x – 8
y = 0
(0, 0)
–6
–4
0
2
6
x
–6
x = –4
x = 2

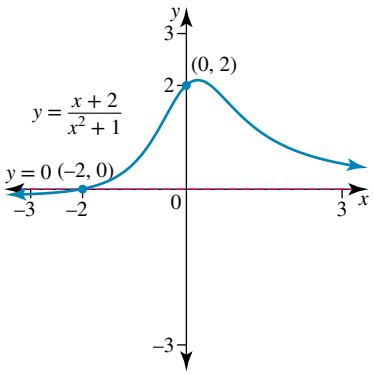
c.


y
3
$y = \frac{x-1}{x^2-2x+1}$
y = 0
–6
0
1
6
x
–1
(0, –1)
–3
x = 1

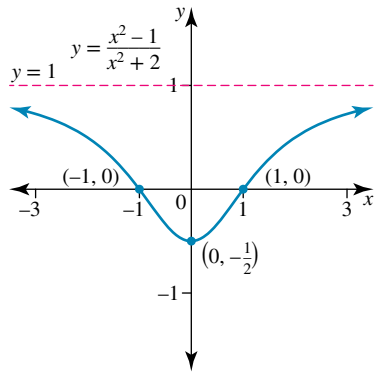
d.


y
3
$y = \frac{2-x}{x^2-3x-4}$
y = 0
(2, 0)
–6
0
2
6
x
$(0, -\frac{1}{2})$
–3
x = –1
x = 4

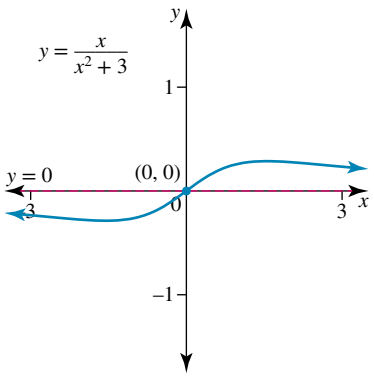
11. a.


y
3
(0, 2)
2
$y = \frac{x+2}{x^2+1}$
y = 0 (–2, 0)
–3
–2
0
3
x
–3

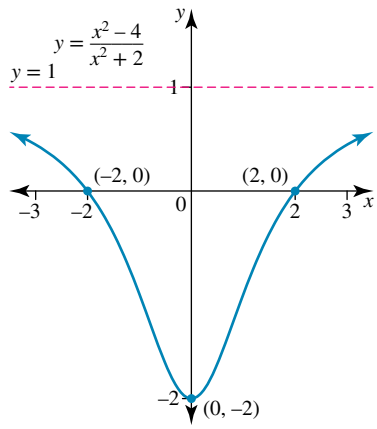
b.


y
$y = \frac{x^2-1}{x^2+2}$
y = 1
1
(–1, 0)
(1, 0)
–3
–1
0
1
3
x
$(0, -\frac{1}{2})$
–1

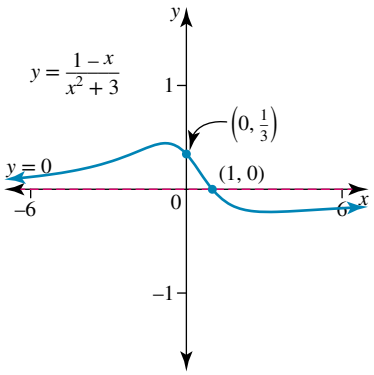
c.


y
$y = \frac{x}{x^2+3}$
1
y = 0
(0, 0)
–3
0
3
x
–1

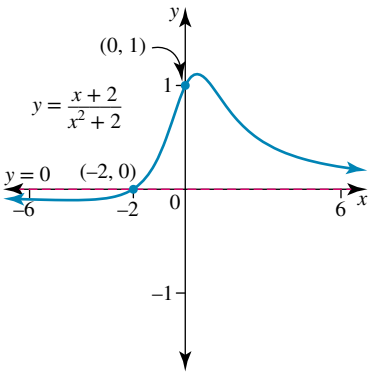
d.


y
$y = \frac{x^2-4}{x^2+2}$
y = 1
1
(–2, 0)
(2, 0)
–3
–2
0
2
3
x
–2
(0, –2)

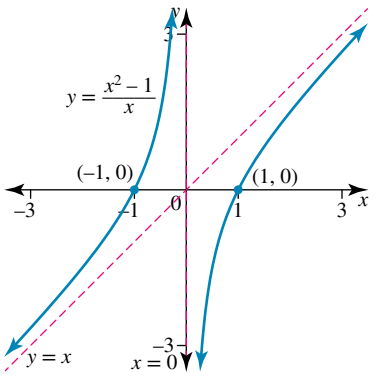
e.


y
$y = \frac{1-x}{x^2+3}$
1
$(0, \frac{1}{3})$
y = 0
(1, 0)
–6
0
6
x
–1

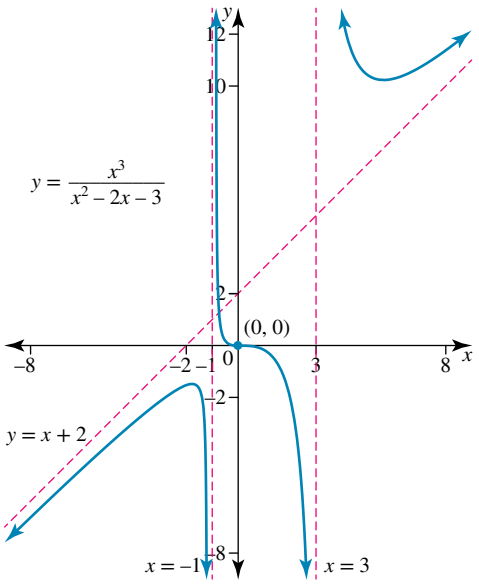
f.


y
(0, 1)
1
$y = \frac{x+2}{x^2+2}$
y = 0
(–2, 0)
–6
–2
0
6
x
–1

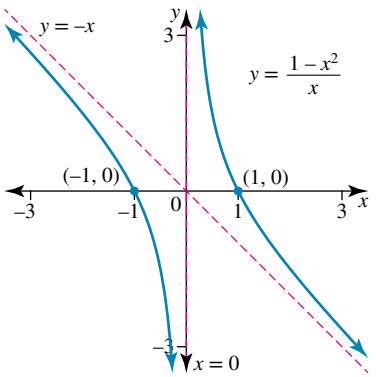
12. a.



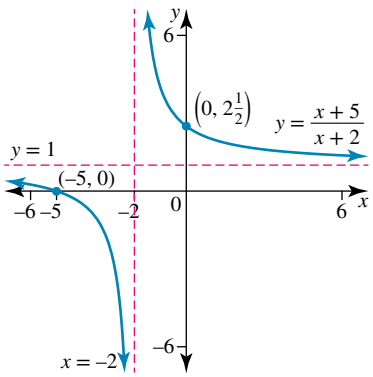
b.



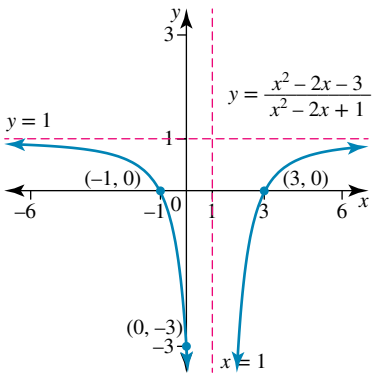
13. a.



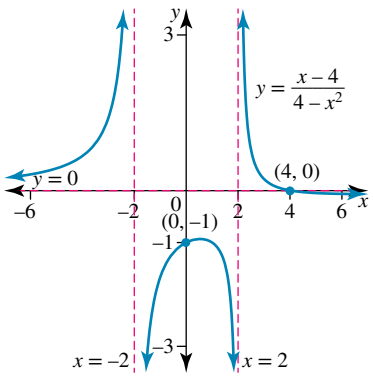
b.



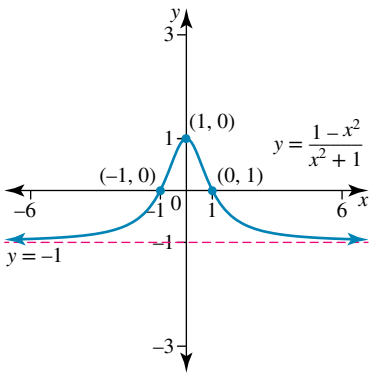
c.



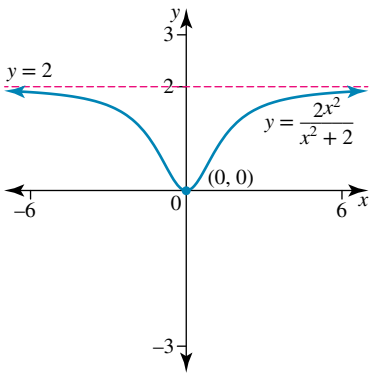
d.



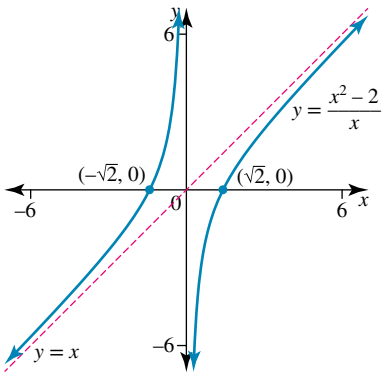
14. a.



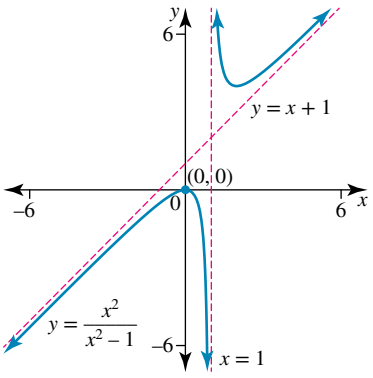
b.



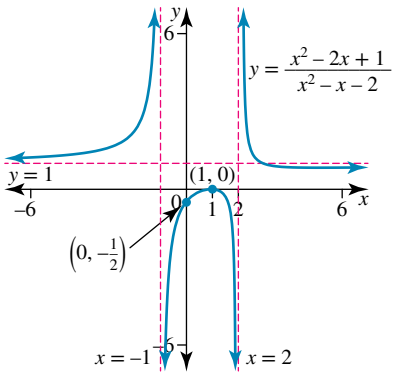
15. a.



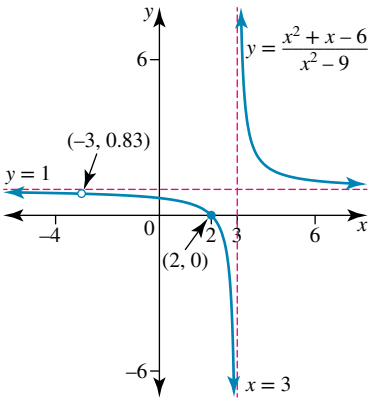
b.



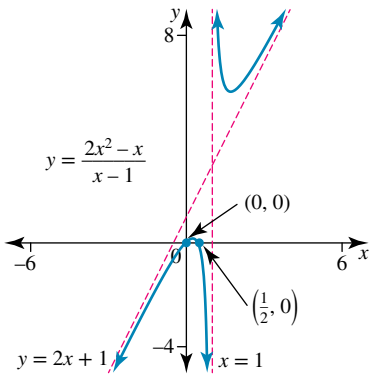
16. a.



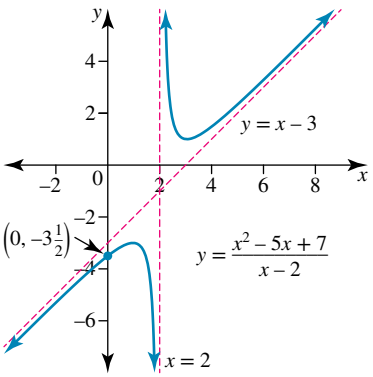
b.



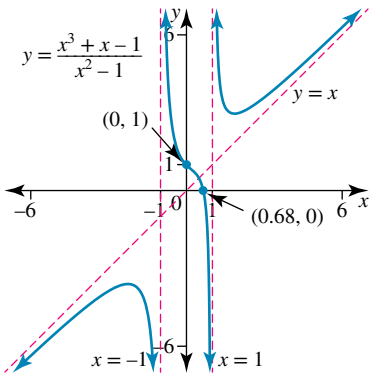
17. a.



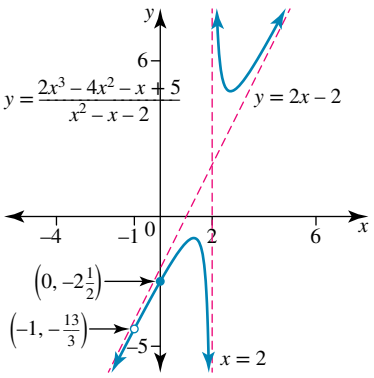
b.



18. a.

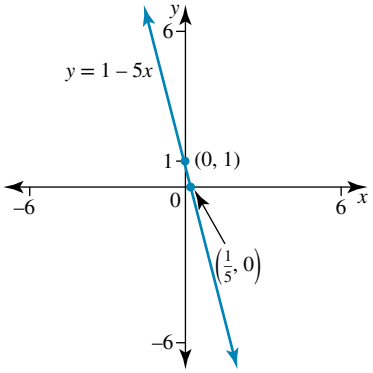


b.

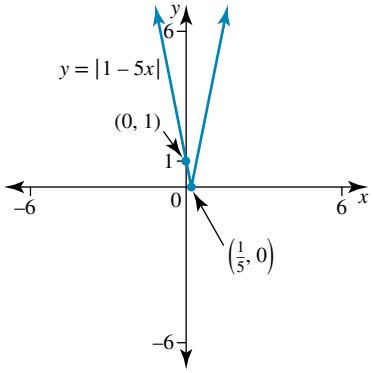


7.5 Review: exam practice

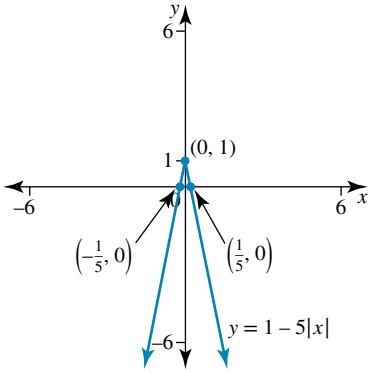
1. a.



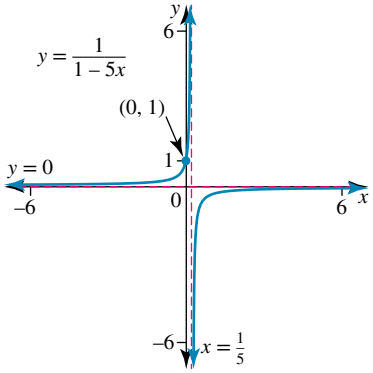
b.



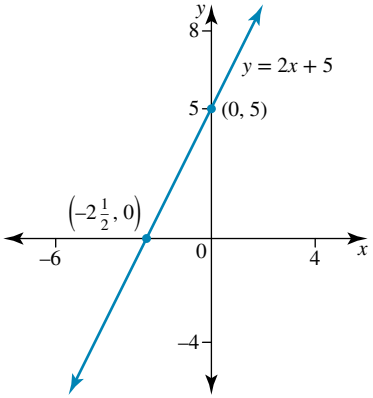
c.



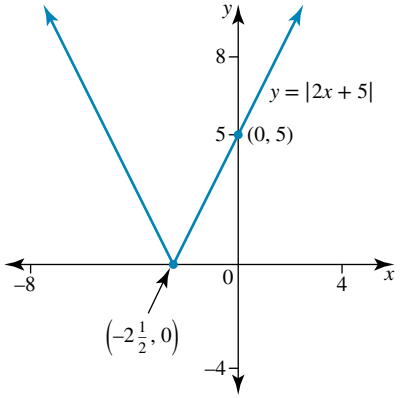
d.



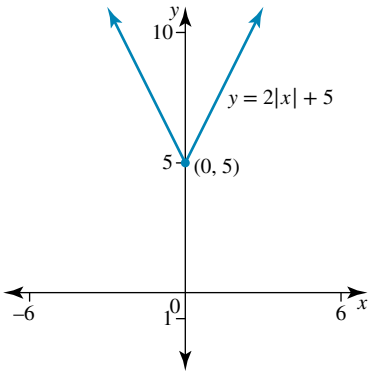
2. a.



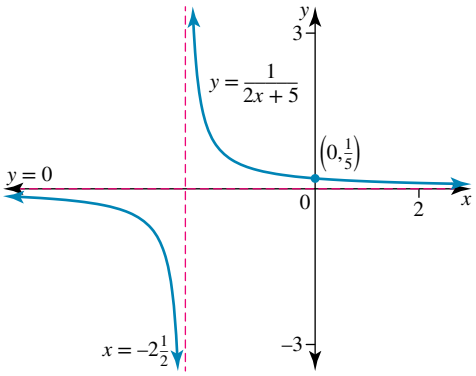
b.



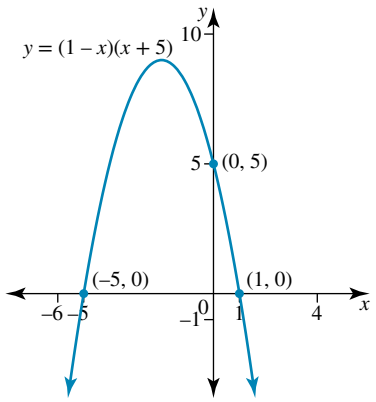
c.



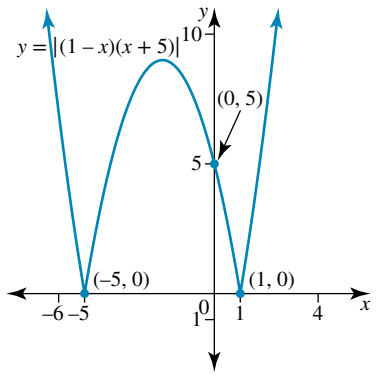
d.



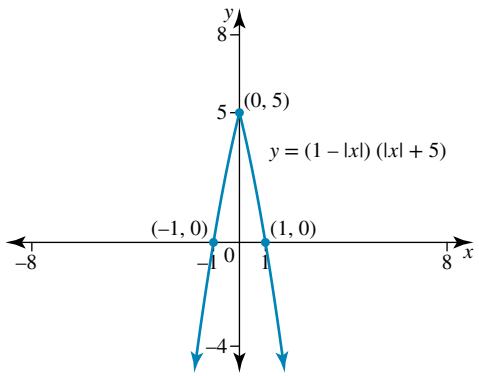
3. a.


y = (1 – x)(x + 5)
(0, 5)
(–5, 0)
(1, 0)

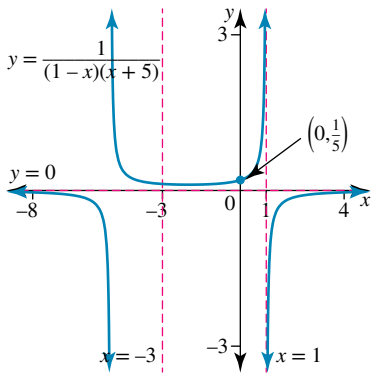
b.


y = |(1 – x)(x + 5)|
(0, 5)
(–5, 0)
(1, 0)

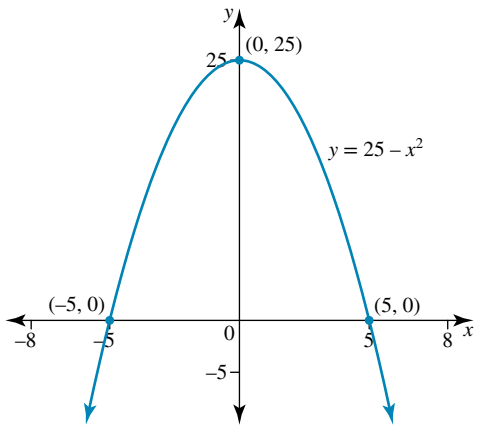
c.


y = (1 – |x|) (|x| + 5)
(0, 5)
(–1, 0)
(1, 0)

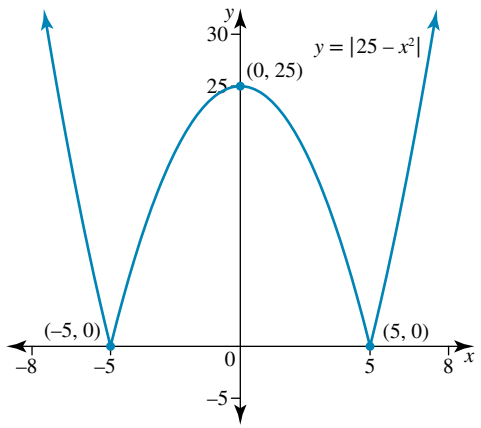
d.


y = 1/((1 – x)(x + 5))
(0, 1/5)
y = 0
x = –3
x = 1

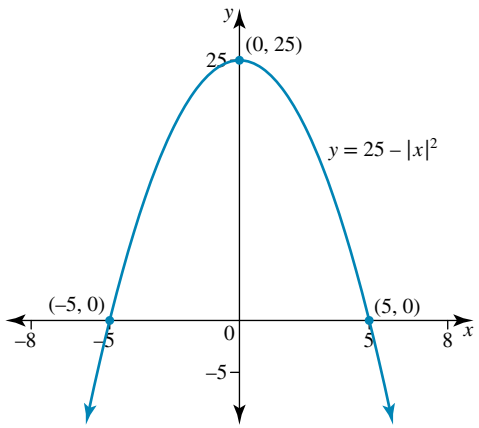
4. a.


y = 25 – x²
(0, 25)
(–5, 0)
(5, 0)

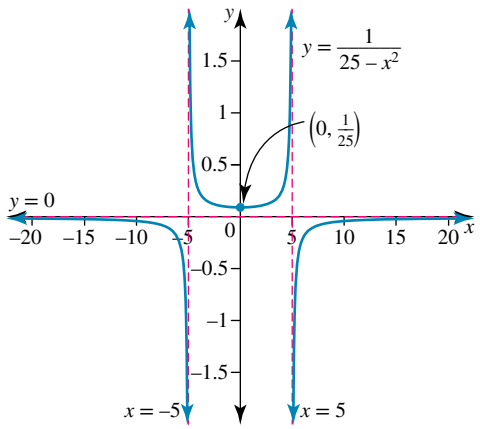
b.


y = |25 – x²|
(0, 25)
(–5, 0)
(5, 0)

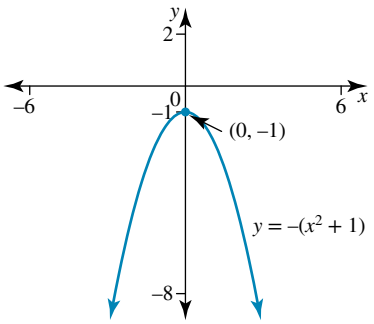
c.


y = 25 – |x|²
(0, 25)
(–5, 0)
(5, 0)

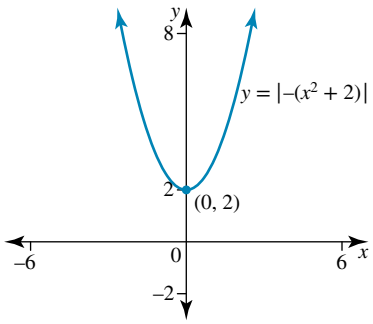
d.


y = 1/(25 – x²)
(0, 1/25)
y = 0
x = –5
x = 5

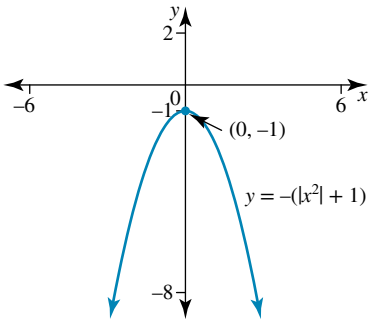
5. a.


y
2
−6
0
6
x
−1
(0, −1)
y = −(x² + 1)
−8

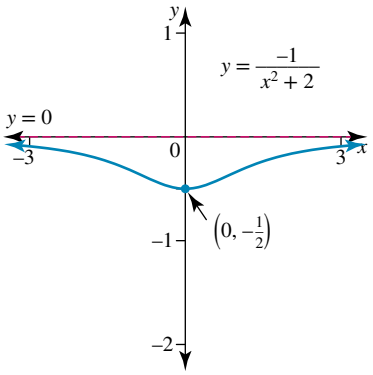
b.


y
8
y = |−(x² + 2)|
2
(0, 2)
−6
0
6
x
−2

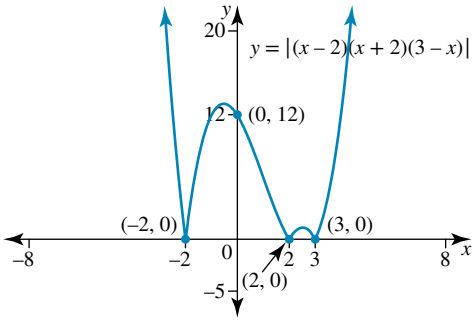
c.


y
2
−6
0
6
x
−1
(0, −1)
y = −(|x²| + 1)
−8

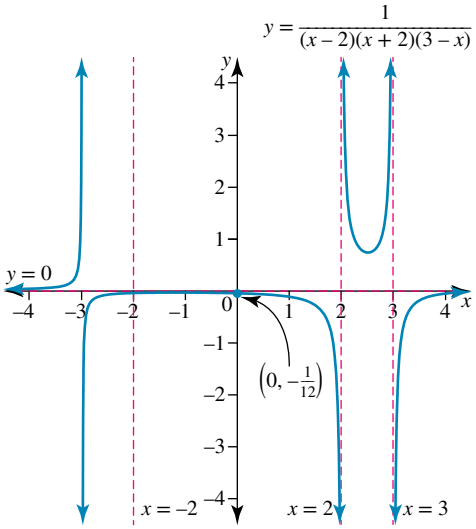
d.


y
1
y = −1/(x² + 2)
y = 0
−3
0
3
x
(0, −1/2)
−1
−2

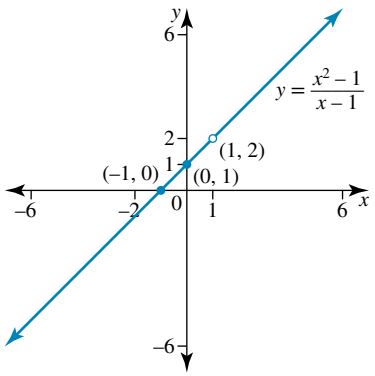
6. a.


y
20
y = |(x − 2)(x + 2)(3 − x)|
12
(0, 12)
(−2, 0)
(3, 0)
−8
−2
0
2
3
8
x
(2, 0)
−5

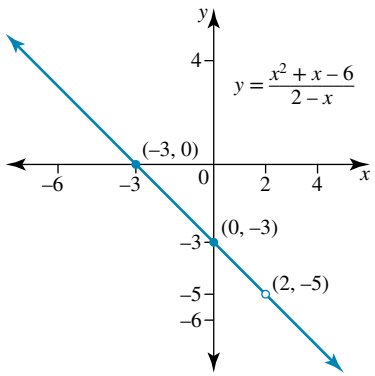
b.


y = 1/((x − 2)(x + 2)(3 − x))
y
4
3
2
1
y = 0
−4
−3
−2
−1
0
1
2
3
4
x
−1
−2
−3
−4
(0, −1/12)
x = −2
x = 2
x = 3

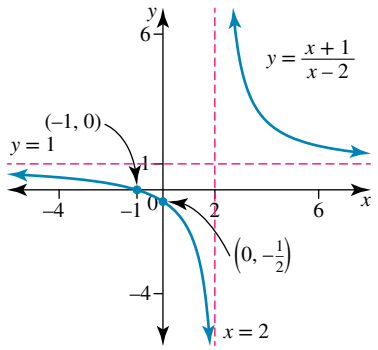
7.


y
6
y = (x² − 1)/(x − 1)
2
1
(1, 2)
(−1, 0)
(0, 1)
−6
−2
0
1
6
x
−6

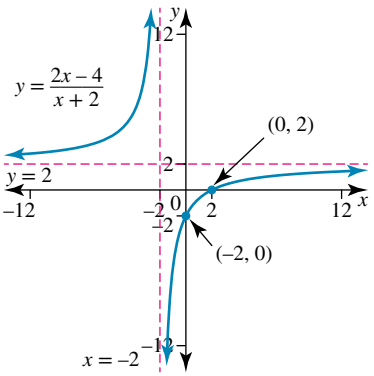
8.


y
4
y = (x² + x − 6)/(2 − x)
(−3, 0)
−6
−3
0
2
4
x
(0, −3)
−3
(2, −5)
−5
−6

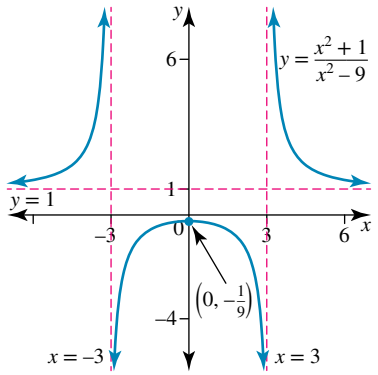
9.



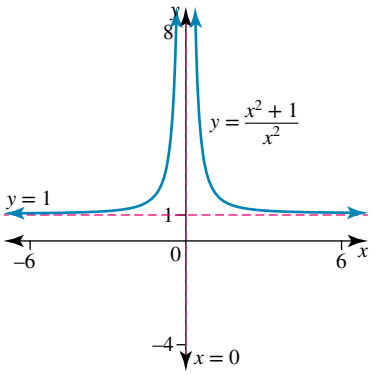
10.



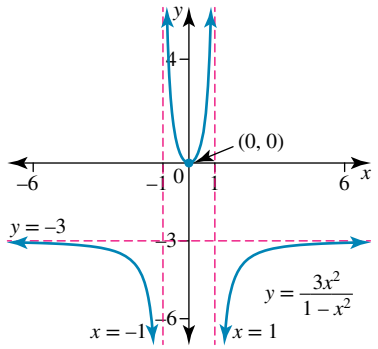
11.



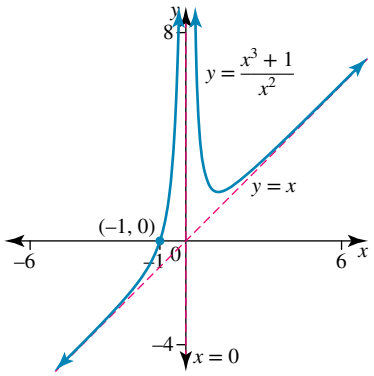
12.



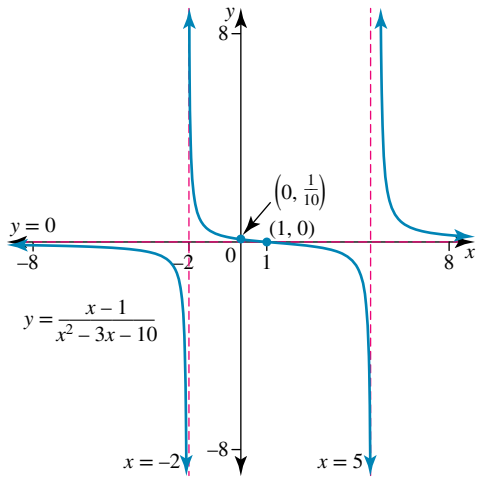
13.



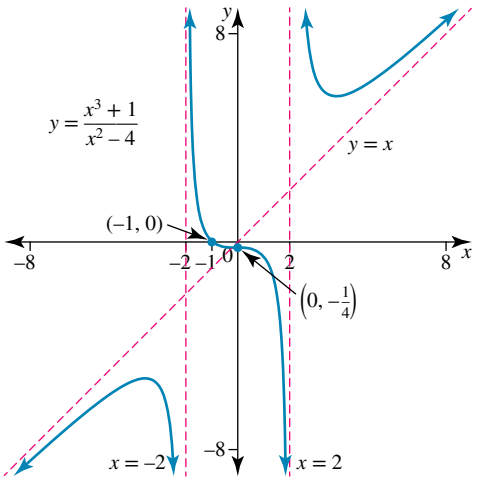
14.



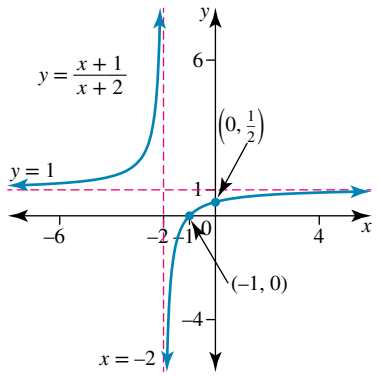
15.



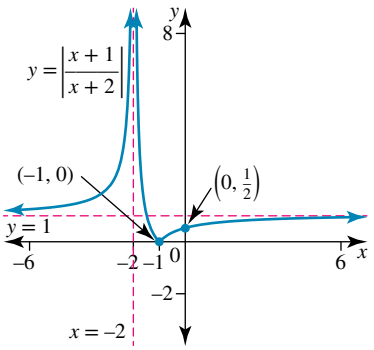
16.



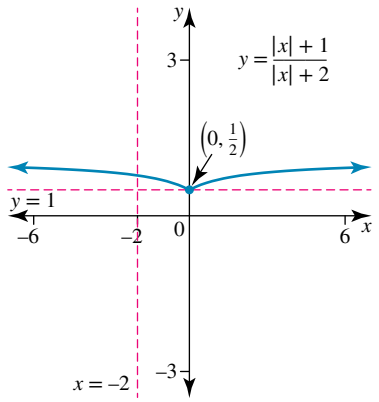
17. a.


y
x
y = (x + 1)/(x + 2)
6
1
−4
−6
−2
−1
0
4
y = 1
x = −2
(0, 1/2)
(−1, 0)

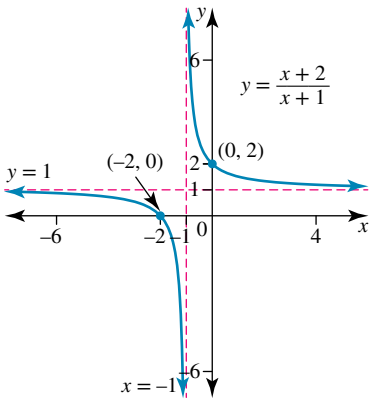
b.


y
x
y = |(x + 1)/(x + 2)|
8
−2
6
−6
−2
−1
0
y = 1
x = −2
(−1, 0)
(0, 1/2)

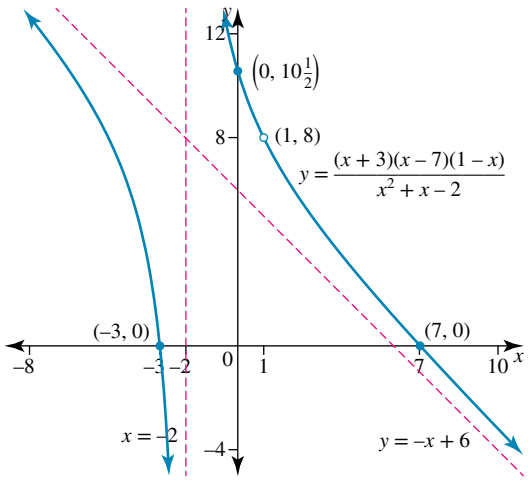
c.


y
x
y = (|x| + 1)/(|x| + 2)
3
−3
−6
−2
0
6
y = 1
x = −2
(0, 1/2)

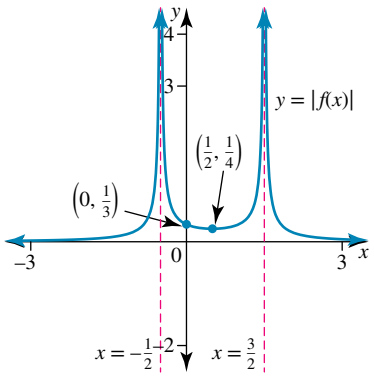
d.


y
x
y = (x + 2)/(x + 1)
6
2
1
−6
−2
−1
0
4
y = 1
x = −1
(−2, 0)
(0, 2)

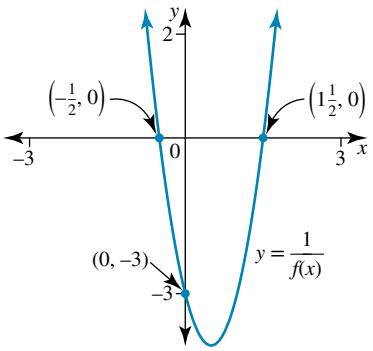
18.


y
x
12
8
−4
−8
−3
−2
0
1
7
10
(0, 10 1/2)
(1, 8)
y = (x + 3)(x − 7)(1 − x)/(x² + x − 2)
(−3, 0)
(7, 0)
x = −2
y = −x + 6

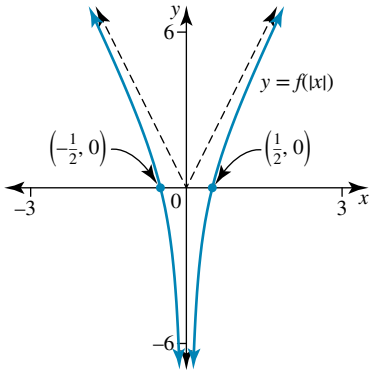
19. a.


y
4
3
2
y = |f(x)|
(1/2, 1/4)
(0, 1/3)
–3
0
3
x
x = –1/2
x = 3/2

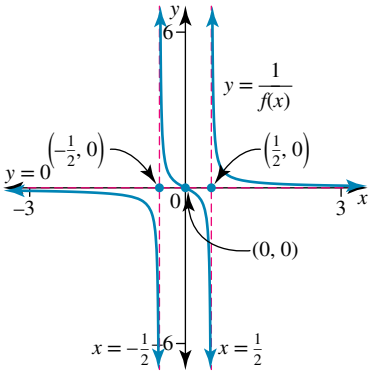
b.


y
2
(–1/2, 0)
(1 1/2, 0)
–3
0
3
x
(0, –3)
–3
y = 1/f(x)

20. a.


y
6
y = f(|x|)
(–1/2, 0)
(1/2, 0)
–3
0
3
x
–6

b.


y
6
y = 1/f(x)
(–1/2, 0)
(1/2, 0)
y = 0
–3
0
3
x
(0, 0)
x = –1/2
–6
x = 1/2

CHAPTER 8
Trigonometric functions

8.1 Overview

8.1.1 Introduction

Trigonometry is the branch of mathematics that deals with triangles and the relationship between the angles and sides of a triangle. Derived from *trigonon*, the ancient Greek word for triangle, trigonometry was originally devised around 1700 BCE as a tool for astronomers.

Hipparchus, the Greek astronomer and mathematician, is known as the father of trigonometry for compiling the first trigonometric tables around 150 BCE. From the 4th century onwards Indian mathematicians made significant contributions to trigonometry, with Bhaskara the First and Brahmagupta developing formulas for determining sine values. Islamic mathematicians built on the work of the Greeks and Indians, and by the 10th century were using all six trigonometric functions, with mathematician Abū al-Wafā' al-Būzjānī developing sine tables to 8 decimal places. Islamic mathematicians were the first to use triangulation to determine distances between points. Today, the sine and cosine functions are used in many fields, for example in medical techniques such as CAT and MRI scans.

AN EGYPTIAN ASTRONOMER; OR, HIPPARCHUS AT ALEXANDRIA.

LEARNING SEQUENCE

8.1 Overview
8.2 Review of trigonometry
8.3 Solving trigonometric equations
8.4 The tangent function
8.5 The reciprocal functions
8.6 Modelling periodic functions
8.7 Review: exam practice

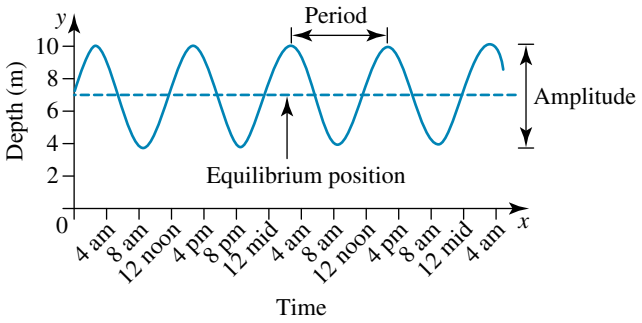
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Review of trigonometry

8.2.1 Periodic functions

The trigonometric functions are examples of **periodic functions** — functions that repeat themselves over a period of time.

In the graph shown, the depth of water in an inlet has been measured over time. The resulting graph is an example of a periodic function.



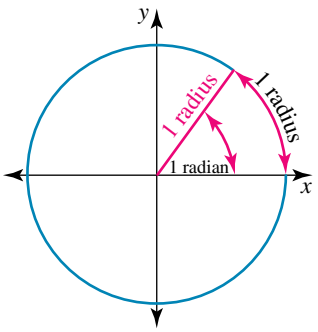
The length of time that it takes for a periodic function to begin repeating itself is called the **period**, T, of the function. In the function shown, the graph repeats itself every 12 hours, so the period is 12 hours.

In many periodic functions, the graph will oscillate between a maximum and minimum value. The centre of the oscillation is known as the **equilibrium position** or mean position. In the graph shown above, the maximum position is 10 m and the minimum position is 4 m, so the equilibrium position is 7 metres.

The maximum variation of the function from its equilibrium position is called the **amplitude**. The maximum of 10 and the minimum of 4 are both 3 m away from the equilibrium position of 7. The amplitude is 3 m.

8.2.2 Radian measure

You should be familiar with measuring angles in degrees (°), and will recall that there are 360° in a full circle. An alternative unit for angle measurement is the **radian**.



Consider the circle shown, which shows a sector that cuts off an arc on the circle equal to the radius of the circle.

This angle is defined to be equal to 1 radian, which can be written 1^c. In many examples no unit is given, in which case radians are assumed.

The circumference of a circle $= 2\pi r$ units.

The angle swept in one revolution is equivalent to 2π radians. In the unit circle, when $r = 1$, the circumference of the circle is $2\pi \times 1$; that is, $360° = 2\pi$.

Therefore, for all circles 2π radians $= 360°$, so:

$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$1° = \frac{\pi^c}{180°}$$

These relationships are used to convert degrees to radians and vice versa. In most cases an angle measured in radians will be given in terms of π. This is an exact radian measure.

WORKED EXAMPLE 1

Convert the following angles into exact radians.

a. $120°$ **b.** $70°$

THINK

a. 1. Multiply 120 by $\frac{\pi}{180°}$.

2. Simplify by dividing through by the highest common factor, 60.

b. 1. Multiply 70 by $\frac{\pi}{180°}$.

2. Simplify by dividing through by the highest common factor, 10.

WRITE

a. $120° = 120° \times \frac{\pi}{180°}$

$= \frac{2\pi}{3}$

b. $70° = 70° \times \frac{\pi}{180°}$

$= \frac{7\pi}{18}$

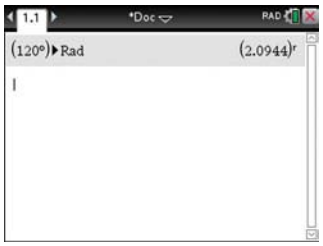
TI | THINK

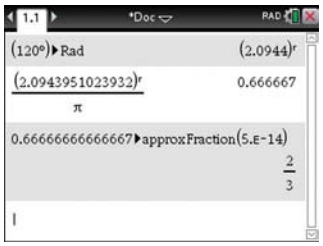
a.1. On a Calculator page, complete the entry line as:
120°
Press the Catalogue button and select ▷ Rad, then press ENTER.
Note: The degree symbol can be found by pressing the π button.

2. To express the answer as a multiple of π, complete the next entry line as:
$\frac{ans}{\pi}$
then press ENTER.
Press Menu, then select:
2: Number
2: Approximate to Fraction
then press ENTER.

3. The answer appears on the screen.

WRITE





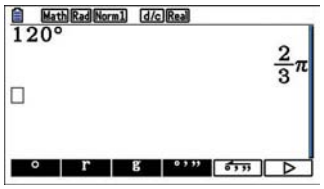
$120° = \frac{2\pi^c}{3}$

CASIO | THINK

a.1. Put the Calculator in Radian mode.
On the Run-Matrix screen, complete the entry line as:
120°
then press EXE.
Note: The degree symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.

2. The answer appears on the screen.

WRITE



$120° = \frac{2\pi^c}{3}$

WORKED EXAMPLE 2

Convert the following angles to degrees without the use of a calculator.

a. $\frac{5\pi}{6}$ **b.** 0.8π

THINK

a. 1. Multiply $\frac{5\pi}{6}$ by $\frac{180°}{\pi}$.

WRITE

a. $\frac{5\pi}{6} = \frac{5\cancel{\pi}}{6} \times \frac{180°}{\cancel{\pi}}$

2. Cancel out the π. $= \dfrac{5 \times 180^\circ}{6}$

3. Simplify. $= 150^\circ$

b. 1. Multiply 0.8π by $\dfrac{180^\circ}{\pi}$. **b.** $0.8\pi = 0.8\cancel{\pi} \times \dfrac{180^\circ}{\cancel{\pi}}$

2. Cancel out the π. $= 0.8 \times 180^\circ$

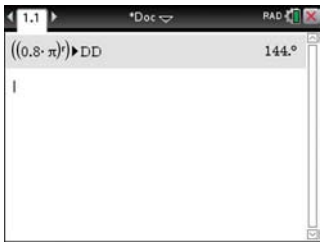
3. Simplify. $= 144^\circ$

TI | THINK

b.1. On a Calculator page, complete the entry line as:
$(0.8\pi)^r$
Press the Catalogue button and select ▷ DD, then press ENTER.
Note: The calculator uses ^{r} as the radian symbol. This symbol can be found by pressing the π button.

The answer appears on the screen.

WRITE

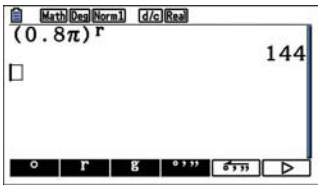


$(0.8\pi)^c = 144^\circ$

CASIO | THINK

b.1. Put the Calculator in Degree mode.
On the Run-Matrix screen, complete the entry line as:
$(0.8\pi)^r$
then press EXE.
Note: The calculator uses ^{r} as the radian symbol. This symbol can be found by pressing OPTN, pressing F6 to scroll across to more menu options, then selecting ANGLE by pressing F5.
The answer appears on the screen.

WRITE



$(0.8\pi)^c = 144^\circ$

8.2.3 Exact values and angles of any magnitude

Calculated trigonometric ratios (sin, cos and tan) of most angles are rational approximations, correct to several decimal places.

However, for a few particular angles, exact trigonometric values can be determined. Consider the right-angled isosceles triangle shown.

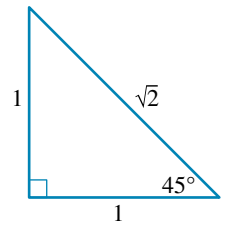
This right-angled triangle with equal sides of 1 unit has a hypotenuse of $\sqrt{2}$ units and angles of 90° and 45°. We can therefore calculate the exact trigonometric values for 45°.

Using: SOH CAH TOA

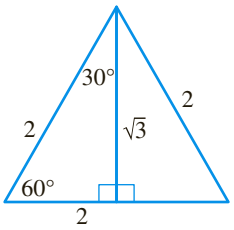
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

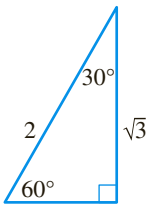
$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{1}{1} = 1$$



To find the exact trigonometric values for 30° and 60°, consider the equilateral triangle with each side length 2 units. The triangle has been bisected, giving a perpendicular height of $\sqrt{3}$ units.



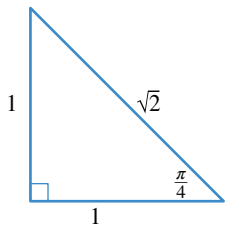


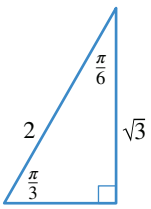
Using: SOH CAH TOA

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

These triangles should be memorised for both degrees and radians. The triangles using radian measures are shown in the figures below.





The results are summarised in this table.

Angle (degrees)	Angle (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

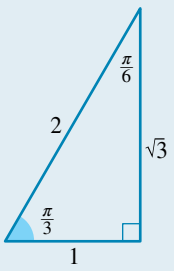
WORKED EXAMPLE 3

Without the use of a calculator, determine the exact value of:

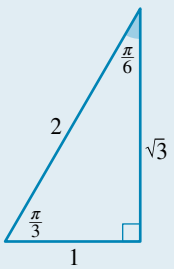
a. $\sin\frac{\pi}{3}$ **b.** $\tan\frac{\pi}{6}$.

THINK

a. Read the exact value from the table or use a special triangle,
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$.



b. Read the exact value from the table or use a special triangle,
$\tan\theta = \frac{\text{opp}}{\text{adj}}$.



WRITE

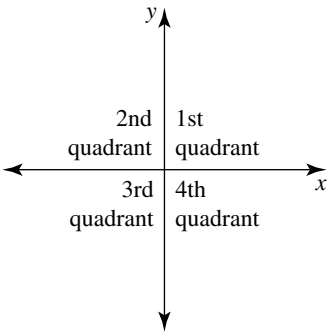
a. $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

b. $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

8.2.4 The unit circle

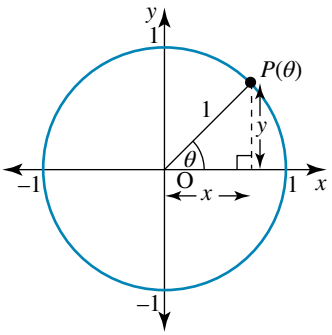
Trigonometric ratios can be considered using a **unit circle**.

The coordinate axes divide the unit circle into four quadrants as shown in the diagram.



Quadrant 1 is where $0° < \theta < 90°$.
Quadrant 2 is where $90° < \theta < 180°$.
Quadrant 3 is where $180° < \theta < 270°$.
Quadrant 4 is where $270° < \theta < 360°$.

The position of point P can be described using the coordinates x and y and the angle, θ. The point, P, which is on the circle, is an angle of θ from the positive x-axis and can be expressed as $P(\theta)$.



Recall that $\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$$\cos\theta = \frac{x}{1}$$

$\therefore$ The x-coordinate of P is $x = \cos\theta$.

Recall that $\sin\theta = \frac{\text{opp}}{\text{hyp}}$

$$\sin\theta = \frac{y}{1}$$

$\therefore$ The y-coordinate of P is $y = \sin\theta$.

The point, P, can also be denoted as $P(\cos\theta, \sin\theta)$.

Note: $-1 \leq \sin\theta \leq 1$, for all θ and $-1 \leq \cos\theta \leq 1$, for all θ.

$\tan\theta = \frac{y}{x}$ or $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

The unit circle can be divided into symmetrical sections, as shown in the diagram.

Relationships between the circular functions — sine, cosine and tangent — can be established based on these symmetrical properties.

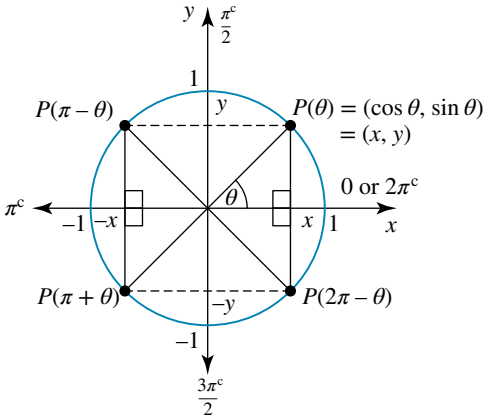
For simplicity, assume θ is an acute angle, although the following properties hold for any θ.

Quadrant 1

As already seen: $\sin\theta = y$

$\cos\theta = x$

$\tan\theta = \frac{y}{x}$



Quadrant 2

By symmetry: $\sin(\pi - \theta) = y = \sin\theta$

$\cos(\pi - \theta) = -x = -\cos\theta$

$\tan(\pi - \theta) = \frac{y}{-x} = -\tan\theta$

Quadrant 3

By symmetry: $\sin(\pi + \theta) = -y = -\sin\theta$

$\cos(\pi + \theta) = -x = -\cos\theta$

$\tan(\pi + \theta) = \frac{-y}{x} = \tan\theta$

Quadrant 4

By symmetry: $\sin(2\pi - \theta) = -y = -\sin\theta$

$\cos(2\pi - \theta) = x = \cos\theta$

$\tan(2\pi - \theta) = \frac{-y}{x} = -\tan\theta$

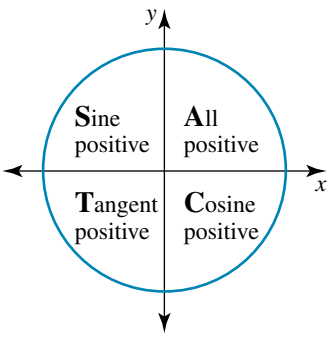
Notes

- These relationships also apply if degrees are used in place of radians — that is, if π is replaced by 180° or 2π is replaced by 360°.
- An angle measurement is assumed to be in radians unless the degree symbol is given.

Since $y = \sin\theta$, sine is positive in quadrants 1 and 2 and sine is negative in quadrants 3 and 4.

Since $y = \cos\theta$, cosine is positive in quadrants 1 and 4 and cosine is negative in quadrants 2 and 3.

Since $\tan = \frac{y}{x}$, tangent is positive in quadrants 1 and 3 and tangent is negative in quadrants 2 and 4.



The diagram summarises which of the trigonometric functions is positive in each quadrant.

The word 'CAST' will assist in recalling this.

You may also use a short phrase, such as 'All Stations To Central'.

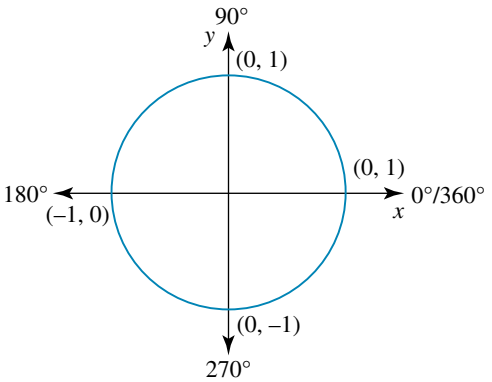
Note that for angles which are greater than 360° we can find the quadrant by making more than one revolution. The equivalent angle in the first revolution can be found by repeatedly subtracting 360° until an angle between 0 and 360° is found. For example, an angle of 560° is in quadrant 3 (560° − 360° = 200°).

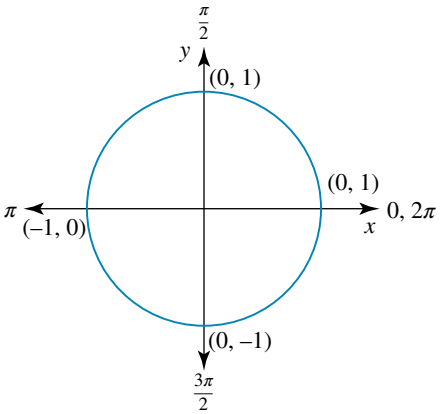
For angles that are negative, we move in the opposite (clockwise) direction. To find the equivalent positive angle, add 360° to the negative angle. For example, an angle of −210° will lie in quadrant 2 (−210° + 360° = 150°).

In addition to considering angles within quadrants 2, 3 and 4 we need to be able to calculate the trigonometric ratios for the angles on the boundary of each quadrant. Exact values of the boundary angles can be found using the following figure and the fact that:

$$\sin\theta = y,\ \cos\theta = x \text{ and } \tan\theta = \frac{y}{x}.$$

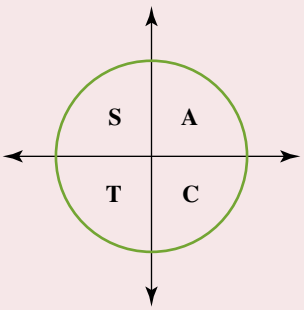
As with the exact values studied earlier, we should know these results in both degrees and radians. This figure can also be drawn in radians.





Angle (degrees)	Angle (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
90°	$\frac{\pi}{2}$	1	0	Undefined
180°	π	0	−1	0
270°	$\frac{3\pi}{2}$	−1	0	Undefined
360°	2π	0	1	0

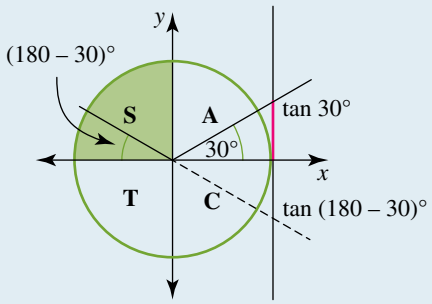
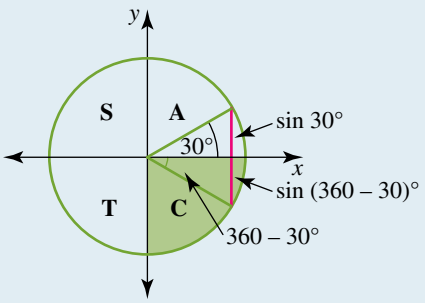
Provided an angle is expressed as $\pi \pm \theta$ or $2\pi \pm \theta$, the trigonometric function (sin or cos) remains the same, only the sign (+ or −) may change. Use this diagram to determine the sign.



WORKED EXAMPLE 4

Find, without using a calculator, the exact value of the following.

a. $\tan\ 150°$ **b.** $\sin\ 330°$

THINK	WRITE
a. 1. Express $\tan 150°$ as $\tan(180 - 30)°$.	**a.** $\tan\ 150° = \tan(180 - 30)°$
2. Sketch the angle on a unit circle and relate it to the first quadrant.	
3. By symmetry $\tan(180 - 30)° = -\tan 30°$.	$\tan 150° = -\tan 30°$
4. Replace $\tan 30°$ with its exact value, $\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$.	$= -\frac{1}{\sqrt{3}}$ or $-\frac{\sqrt{3}}{3}$
b. 1. Express $\sin\ 330°$ as $\sin(360 - 30)°$.	**b.** $\sin\ 330° = \sin(360 - 30)°$
2. Sketch the angle on a unit circle and relate it to the first quadrant.	
3. By symmetry $\sin(360 - 30)° = -\sin\ 30°$.	$\sin 330° = -\sin 30°$
4. Replace $\sin 30°$ with its exact value, $\frac{1}{2}$.	$= -\frac{1}{2}$

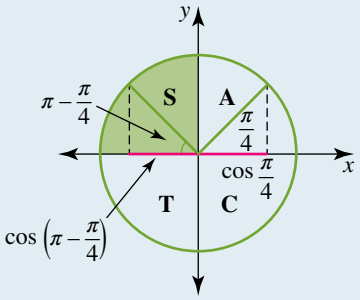
WORKED EXAMPLE 5

Determine, without using a calculator, the exact value of the following.

a. $\cos\frac{3\pi}{4}$ **b.** $\tan\frac{7\pi}{6}$

THINK	WRITE
a. 1. Express $\cos\frac{3\pi}{4}$ as $\cos\left(\pi - \frac{\pi}{4}\right)$.	**a.** $\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$

2. Sketch the angle on a unit circle and relate it to the first quadrant.



3. By symmetry $\cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4}$.

4. Replace $\cos\frac{\pi}{4}$ with its exact value of $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$.

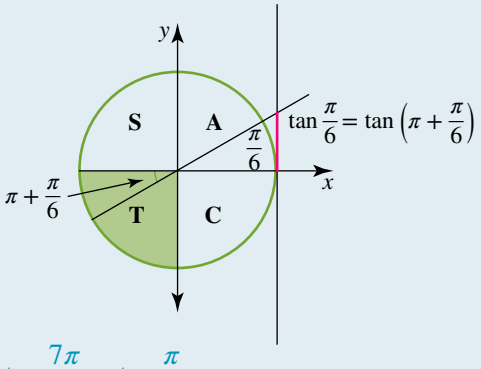
$\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}$

b. 1. Express $\tan\frac{7\pi}{6}$ as $\tan\left(\pi + \frac{\pi}{6}\right)$.

b. $\tan\frac{7\pi}{6} = \tan\left(\pi + \frac{\pi}{6}\right)$

2. Sketch the angle on a unit circle and relate it to the first quadrant.



3. By symmetry $\tan\left(\pi + \frac{\pi}{6}\right) = \tan\frac{\pi}{6}$.

4. Replace $\tan\frac{\pi}{6}$ with its exact value $\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$.

$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6}$

$= \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$

Resources

- **Interactivity:** The 'All Sin Tan Cos' rule (int-2583)
- **Interactivity:** The unit circle (int-2582)
- **Interactivity:** Symmetry points and quadrants (int-2584)

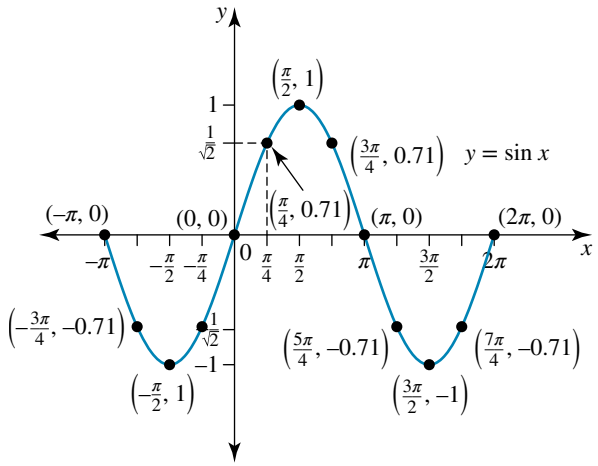
8.2.5 Graphs of the sine and cosine function

The graph of $y = \sin x$

To get an idea of what the graph of $y = \sin x$ looks like, we may first construct a table of values.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \sin x$	0	−0.71	−1	−0.71	0	0.71	1	0.71	0	−0.71	−1	−0.71	0

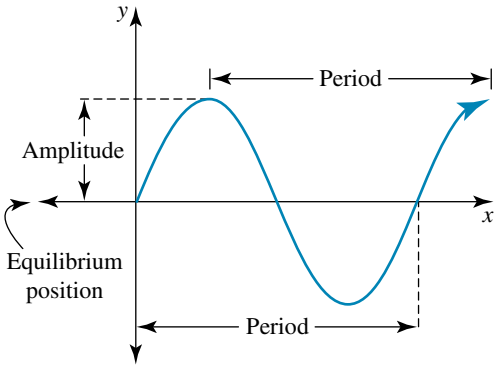
Next, these values are plotted on a set of coordinate axes and a smooth curve is drawn to join the points. The shape may be verified by using a graphics calculator. The **range** (the set of possible y-values for the functions) of $y = \sin x$ is $\{y: -1 \leq y \leq 1\}$.



As the curve repeats itself after every 2π units, the period is 2π (or 360° if the angles are measured in degrees).

The equilibrium position of the curve (or mean position) is $y = 0$. The maximum and minimums are 1 and -1 respectively. This means that the range of $\sin x$ is $[-1, 1]$. As the maximum distance from the equilibrium value is 1, the amplitude is 1.

Note: When dealing with graphs of periodic functions such as $\sin x$, we assume that the units for x are radians unless otherwise stated.

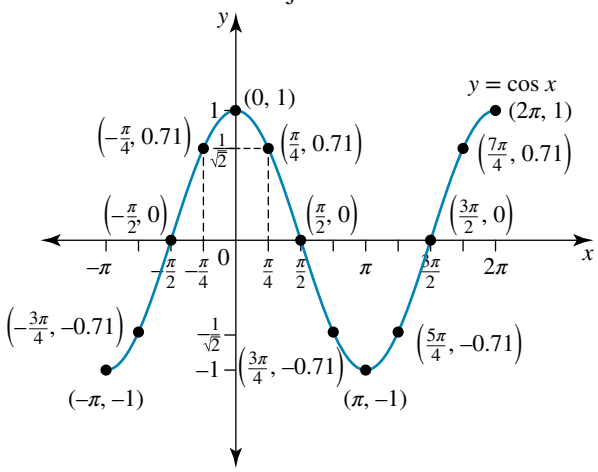


The graph of $y = \cos x$

As for the graph of $y = \sin x$, the graph of $y = \cos x$ can be established by first completing a table of values.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \cos x$	−1	−0.71	0	0.71	1	0.71	0	−0.71	−1	−0.71	0	0.71	0

When these points are plotted on a set of axes and joined with a smooth curve, the graph looks like this:



Verify the shape of this curve using a graphics calculator. It can be seen that the basic shape of the curve $y = \cos x$ is the same as that of $y = \sin x$. It is also a periodic function with period $= 2\pi$ and amplitude $= 1$.

Note: The graph of $y = \cos x$ is exactly the same as that of $y = \sin x$ translated $\frac{\pi}{2}$ radians or 90° to the left.

Properties of the graphs

The graphs of $y = \sin(x)$ and $y = \cos(x)$ both have the same shape. This shape is known as a **sinusoidal curve**.

Both curves have a range of $\{y: -1 \leq y \leq 1\} = [-1, 1]$, meaning that the **amplitude** of the graphs is 1.

Both graphs repeat themselves every 2π, meaning that the graphs are **periodic** with a period of 2π.

Their **domain** is R, but as they repeat themselves, we often will just draw the graphs over one **cycle**, that is showing one period, or draw the graphs over $[0, 2\pi]$ or $[-\pi, \pi]$.

Graphs of $y = a\sin(bx)$ and $y = a\cos(bx)$ where $a \in R$, $b \in Q^+$

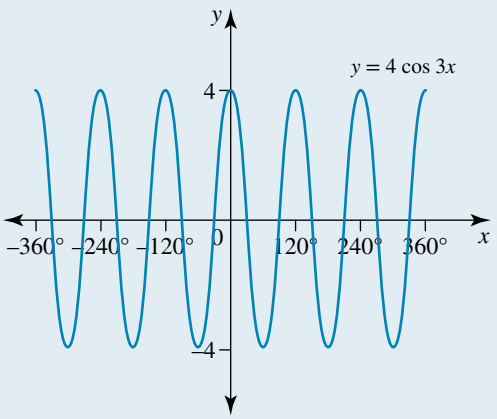
The graphs of $y = a\sin(bx)$ and $y = a\cos(bx)$ both have an amplitude of $a = |a|$ and a range of $[-a, a]$.

As $\sin(bx + 2\pi) = \sin(bx)$ and $\cos(bx + 2\pi) = \cos(bx)$, the graphs are periodic with b cycles in 2π. Hence, the period of one cycle, T, is $T = \frac{2\pi}{b}$.

WORKED EXAMPLE 6

Sketch the graph of the function $y = 4\cos 3x$ over $-360° \leq x \leq 360°$ and state the period and amplitude.

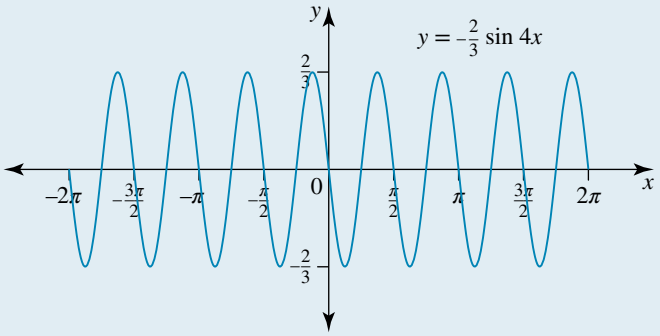
THINK	WRITE
1. Recall the general form for graphs of the cosine function, then state the function being considered.	$y = a\cos bx$ $y = 4\cos 3x$ $\therefore a = 4,\ b = 3$
2. Recall the formula for the period.	Period $= \frac{360°}{b}$
3. Substitute $b = 3$.	$= \frac{360°}{3}$
4. Simplify the value of the period.	$= 120°$
5. The amplitude is the value in front of cos, written as a positive value, $\|a\|$.	Amplitude $= 4$
6. Draw a set of axes.	
7. Since the amplitude is 4, mark or imagine horizontal guidelines at $y = -4$ and 4.	
8. Sketch one cycle of the graph every period (every 120°) along the x-axis, for $-360° \leq x \leq 360°$, showing key x-values.	



WORKED EXAMPLE 7

Sketch the graph of the function $y = -\frac{2}{3}\sin 4x$ over $-2\pi \leq x \leq 2\pi$ and state the period and amplitude.

THINK	WRITE
1. Recall the general form for graphs of the sine function, then state the function being considered.	$y = a\sin bx$ $y = -\frac{2}{3}\sin 4x$ $\therefore a = -\frac{2}{3},\ b = 4$
2. Recall the formula for the period.	Period $= \frac{2\pi}{b}$
3. Substitute $b = 4$.	$= \frac{2\pi}{4}$
4. Simplify the value of the period.	$= \frac{\pi}{2}$
5. The amplitude is the value in front of sin, written as a positive value, $\|a\|$.	Amplitude $= \frac{2}{3}$
6. Draw a set of axes.	
7. Since the amplitude is $\frac{2}{3}$, mark or imagine horizontal guidelines at $y = -\frac{2}{3}$ and $\frac{2}{3}$.	
8. Determine the x-intercepts and mark these on the graph.	
9. Sketch one cycle of the graph every period (every $\frac{\pi}{2}$) along the x-axis, for $-2\pi \leq x \leq 2\pi$.	



Vertical translations

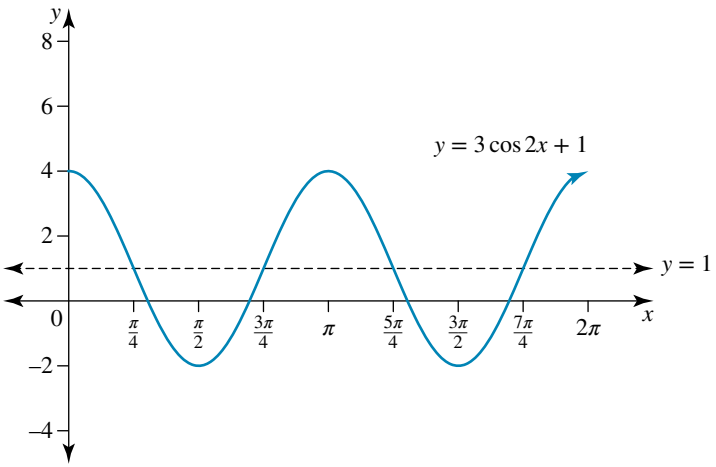
If we add a constant to the function, the graph is moved up or down and is said to be **translation** parallel to the y-axis.

The graph of $f(x) = 3\cos 2x + 1$ or $f(x) = 1 + 3\cos 2x$ is shown below. The amplitude is 3, the period is π, the domain is R and the range is $-2 \leq y \leq 4$.

The x-intercepts are found by solving $3\cos 2x + 1 = 0$. So $\cos 2x = \frac{1}{3}$ (cosine is negative in quadrants 2 and 3). The basic angle is inverse cosine of $\frac{1}{3} = 1.231^{c}$.

$$2x = \pi - 1.231,\ \pi + 1.231,\ 3\pi - 1.231,\ 3\pi + 1.231$$
$$2x = 1.911,\ 4.373,\ 8.194,\ 10.656$$
$$x = 0.955,\ 2.186,\ 4.097,\ 5.328$$

The equilibrium position is $y = 1$.



The general function for vertically translated graphs is:

$$f(x) = a\cos bx + d$$
$$f(x) = a\sin bx + d$$

where $a, d \in R$, $b \in Q^+$, the amplitude $= |a|$, the graphs are periodic with period $T = \dfrac{2\pi}{b}$ and the vertical translation is d.

WORKED EXAMPLE 8

Sketch the graph of $f(x) = 3 + 6\cos(3x)$ over the domain $[0, 2\pi]$, stating the amplitude, period and range of the function. Determine the coordinates of all axial intercepts, the maximum and minimum points, and the end points.

THINK	WRITE
1. Recall the general formula and state the the function being considered.	$f(x) = a\cos bx + d$ $f(x) = 6\cos 3x + 3$ $\therefore a = 6, b = 3, d = 3.$
2. Write down the period, and amplitude.	Period: $\dfrac{2\pi}{b} = \dfrac{2\pi}{3}$ Amplitude: 6
3. The end points are when $x = 0$ and $x = 2\pi$.	$f(0) = 3 + 6\cos 0$ $= 9$ $f(2\pi) = 3 + 6\cos(2\pi)$ $= 9$
4. To find the coordinates of the x-intercepts, let $y = 0$ and solve for x.	$3 + 6\cos(3x) = 0$ $6\cos(3x) = -3$ $\cos(3x) = -\dfrac{1}{2}$ $3x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}, \dfrac{10\pi}{3}, \dfrac{14\pi}{3}, \dfrac{16\pi}{3}$ $x = \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{8\pi}{9}, \dfrac{10\pi}{9}, \dfrac{14\pi}{9}, \dfrac{16\pi}{9}$
5. Write these x-intercepts as coordinates.	$\left(\dfrac{2\pi}{9}, 0\right), \left(\dfrac{4\pi}{9}, 0\right), \left(\dfrac{8\pi}{9}, 0\right), \left(\dfrac{10\pi}{9}, 0\right),$ $\left(\dfrac{14\pi}{9}, 0\right), \left(\dfrac{16\pi}{9}, 0\right)$

6. Since $\cos(3x)$ has a maximum value of 1, the maximum value is $3 + 6 = 9$.

The maximum occurs when $\cos(3x) = 1$.
The maximum is $f(x) = 3 + 6$
$= 9$

$\cos(3x) = 1$
$3x = 0, 2\pi, 4\pi, 6\pi$
$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

7. Write these maximum turning points as coordinates.

$(0, 9), \left(\frac{2\pi}{3}, 9\right), \left(\frac{4\pi}{3}, 9\right), (2\pi, 9)$

8. Since $\cos(2x)$ has a minimum value of -1, the minimum value is $3 - 6 = -3$.

The minimum occurs when $\cos(3x) = -1$.
The minimum is $f(x) = 3 - 6$
$= -3$

$\cos(3x) = 1$
$3x = \pi, 3\pi, 5\pi$
$x = \frac{\pi}{3}, \pi, \frac{5\pi}{5}$

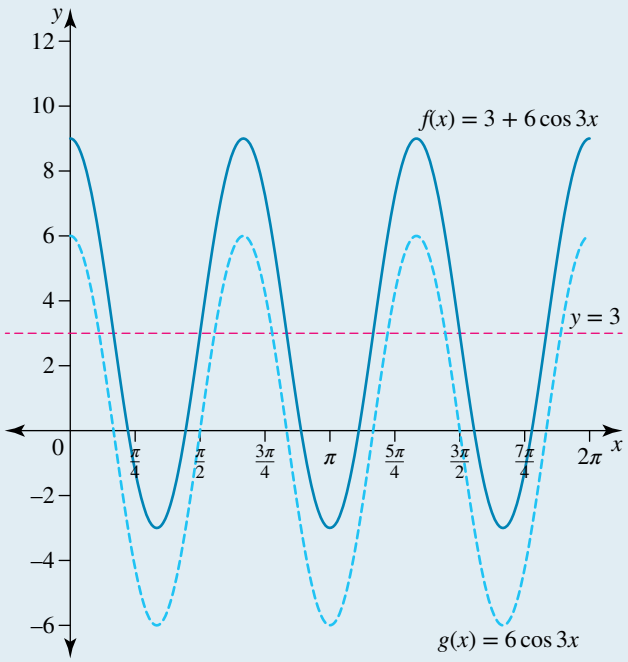
9. Write these minimum turning points as coordinates.

$\left(\frac{\pi}{3}, -3\right), (\pi, -3), \left(\frac{5\pi}{5}, -3\right)$

10. State the range, considering the maximum and minimum turning points.

Range: $[-3, 9]$

11. Sketch $g(x) = 6\cos 3x$ in pencil and use this to sketch $f(x) = 3 + 6\cos 3x$ by shifting the graph up 3 units.

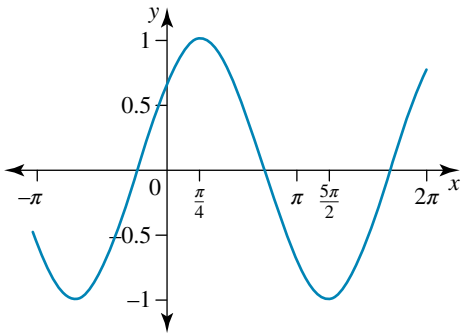


Horizontal translations

The graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ is the graph of $f(x) = \sin x$ translated $\frac{\pi}{4}$ units to the right.

The graph of $f(x) = \sin\left(x + \frac{\pi}{4}\right)$ is the graph of $f(x) = \sin x$ translated $\frac{\pi}{4}$ units to the left.

The graph of $f(x) = \sin\left(x + \frac{\pi}{4}\right)$ is shown. It is drawn between $[-\pi, 2\pi]$, the amplitude is 1, the period is 2π, the domain is R and the range is $-1 \leq y \leq 1$. The y-intercept occurs when $x = 0$.



$$f(x) = \sin\left(x + \frac{\pi}{4}\right), \ -\pi \leq x \leq 2\pi$$

Confirm these graphs with your graphics calculator.

The x-intercepts occur when $\sin\left(x + \frac{\pi}{4}\right) = 0$.

So $$x + \frac{\pi}{4} = 0, \ \pi, \ 2\pi$$

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

This graph is a translation of the basic graph of the function $f(x) = \sin x$, $\frac{\pi}{4}$ units parallel to the x-axis in a positive direction.

The general function for horizontally translated graphs is:

$$f(x) = a \sin b\,(x \pm c)$$
$$f(x) = a \cos b\,(x \pm c)$$

where $(x + c)$ is a horizontal translation of c units to the left and $(x - c)$ is a horizontal translation of c units to the right.

WORKED EXAMPLE 9

Sketch the graph $f(x) = 4 \sin\left(2x - \frac{\pi}{3}\right)$ over the domain $[-\pi, \ \pi]$. State the period, amplitude, and range of the function. Determine the coordinates of all axial intercepts, the maximum and minimum points, and the end points.

THINK	WRITE
1. Recall the general formula in the form $f(x) = a \sin(b(x + c))$ and rewrite $f(x) = 4 \sin\left(2x - \frac{\pi}{3}\right)$.	$f(x) = 4 \sin\left(2x - \frac{\pi}{3}\right)$ $= 4 \sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ $\therefore a = 4, \ b = 2, \ c = -\frac{\pi}{6}$
2. Write down the period and amplitude.	Period $= \frac{2\pi}{b} = \pi$ Amplitude: 4
3. The end points are when $x = -\pi$ and $x = \pi$.	$f(-\pi) = 4 \sin\left(-2\pi - \frac{\pi}{3}\right)$ $= 4 \sin\left(-\frac{\pi}{3}\right)$ $= -4 \sin\left(\frac{\pi}{3}\right)$ $= -2\sqrt{3}$

$$f(\pi) = 4\sin\left(-2\pi - \frac{\pi}{3}\right)$$
$$= 4\sin\left(-\frac{\pi}{3}\right)$$
$$= -2\sqrt{3}$$

4. To find the coordinates of the x-intercepts, let $y = 0$ and solve for x.

$$4\sin\left(2x - \frac{\pi}{3}\right) = 0$$
$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 0$$
$$2\left(x - \frac{\pi}{6}\right) = -2\pi, -\pi, 0, \pi$$
$$x - \frac{\pi}{6} = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}$$
$$x = -\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$$

5. Write these x-intercepts as coordinates.

$$\left(-\frac{5\pi}{6}, 0\right), \left(-\frac{\pi}{3}, 0\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{2\pi}{3}, 0\right)$$

6. The maximum value is 4 and occurs when $\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 1$.

The maximum occurs when
$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 1.$$
The maximum is 4.
$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = 1$$
$$2\left(x - \frac{\pi}{6}\right) = -\frac{3\pi}{2}, \frac{\pi}{2}$$
$$x - \frac{\pi}{6} = -\frac{3\pi}{4}, \frac{\pi}{4}$$
$$x = -\frac{7\pi}{12}, \frac{5\pi}{12}$$

7. Write these maximum turning points as coordinates.

$$\left(-\frac{7\pi}{12}, 4\right), \left(\frac{5\pi}{12}, 4\right)$$

8. The minimum value is -4 and occurs when $\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = -1$.

The minimum occurs when
$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = -1.$$
The minimum is -4.
$$\sin\left(2\left(x - \frac{\pi}{6}\right)\right) = -1$$
$$2\left(x - \frac{\pi}{6}\right) = -\frac{\pi}{2}, \frac{3\pi}{2}$$
$$x - \frac{\pi}{6} = -\frac{\pi}{4}, \frac{3\pi}{4}$$
$$x = -\frac{\pi}{12}, \frac{11\pi}{12}$$

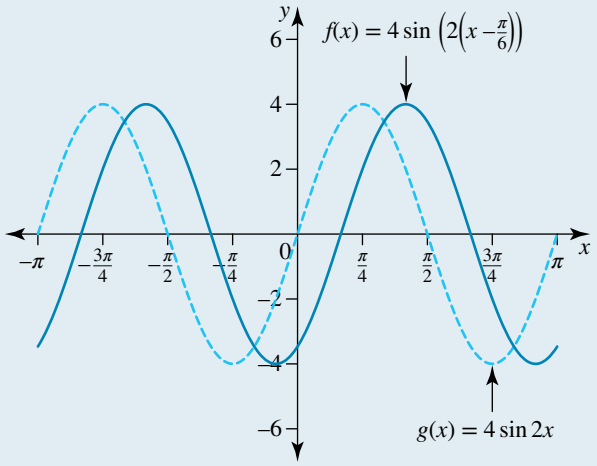
9. Write these minimum turning points as coordinates.

$$\left(-\frac{\pi}{12}, -4\right), \left(\frac{11\pi}{12}, -4\right)$$

10. State the range, considering the maximum and minimum turning points.

Range: [–4, 4]

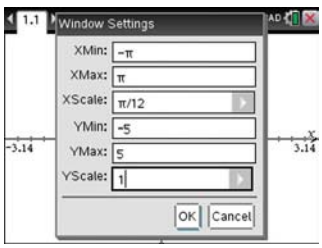
11. Sketch $g(x) = 4\sin 2x$ in pencil and use this to sketch $f(x) = 4\sin\left(2\left(x - \frac{\pi}{6}\right)\right)$ by shifting the graph $\frac{\pi}{6}$ units to the right.



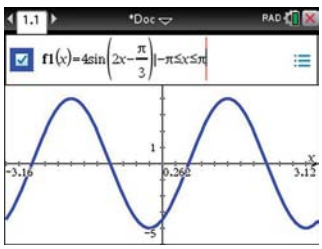
TI | THINK

1. On a Graphs page, set the Graphing Angle to Radian. Press MENU, then select:
4: Window/Zoom
1: Window Settings …
Complete the fields as:
XMin: $-\pi$
XMax: π
XScale: $\pi/12$
YMin: –5
YMax: 5
YScale: 1
then select OK.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

WRITE



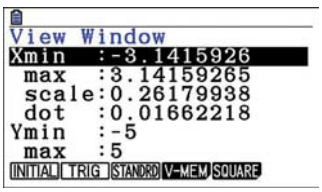
2. Complete the entry line for function 1 as:
$f1(x) = 4\sin(2x - \frac{\pi}{3})| -\pi \le x \le \pi$
then press ENTER.



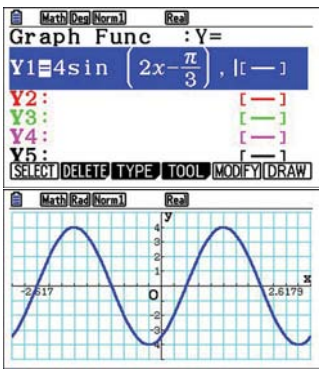
CASIO| THINK

1. Put the Calculator in Radian mode.
On a Graph screen, press SHIFT then F3 to open the V_WIN. Complete the fields as:
Xmin: $-\pi$
max: π
scale: $\pi/12$
Ymin: –5
max: 5
scale: 1
then press EXIT.
Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

WRITE



2. Complete the entry line for y1 as:
$y1 = 4\sin(2x - \frac{\pi}{3})$,
$[-\pi, \pi]$ then press EXE.
Select DRAW by pressing F6.



3. Identify the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has x-intercepts at $\left(-\frac{5\pi}{6}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{\pi}{6}, 0\right)$ and $\left(\frac{2\pi}{3}, 0\right)$. Be sure to mark these points when sketching the graph.

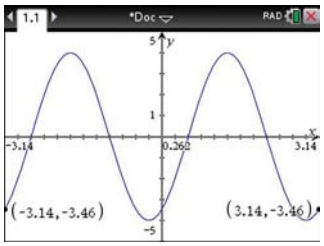
4. Identify the coordinates of the maximums from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has maximums at $\left(-\frac{7\pi}{12}, 4\right)$ and $\left(\frac{5\pi}{12}, 4\right)$. Be sure to mark these points when sketching the graph.

5. Identify the coordinates of the minimums from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{12}, -4\right)$ and $\left(\frac{11\pi}{12}, -4\right)$. Be sure to mark these points when sketching the graph.

6. To find the coordinates of the endpoints, press MENU, then select:
5: Trace
1: Graph Trace.
Type '$-\pi$' then press ENTER twice. Type 'π' then press ENTER twice.
Note: The calculator will display decimal approximations, not exact values.



The endpoints are at approximately (–3.14, –3.46) and (3.14, –3.46).

3. Identify the coordinates of the x-intercepts from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has x-intercepts at $\left(-\frac{5\pi}{6}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{\pi}{6}, 0\right)$ and $\left(\frac{2\pi}{3}, 0\right)$. Be sure to mark these points when sketching the graph.

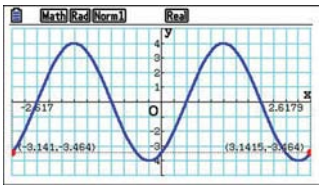
4. Identify the coordinates of the maximums from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has maximums at $\left(-\frac{7\pi}{12}, 4\right)$ and $\left(\frac{5\pi}{12}, 4\right)$. Be sure to mark these points when sketching the graph.

5. Identify the coordinates of the minimums from the graph.

The x-axis scale is $\frac{\pi}{12}$, so it can be seen that the graph has minimums at $\left(-\frac{\pi}{12}, -4\right)$ and $\left(\frac{11\pi}{12}, -4\right)$. Be sure to mark these points when sketching the graph.

6. To find the coordinates of the endpoints, press SHIFT, then F1 to select Trace.
Type '$-\pi$', then press EXE twice. Type 'π', then press EXE twice.
Note: The calculator will display decimal approximations, not exact values.



The endpoints are at approximately (–3.14, –3.46) and (3.14, –3.46).

Combining the transformations

If we take a general trigonometric function $f(x) = a\sin(b(x + c)) + d$ or $f(x) = a\cos(b(x + c)) + d$, it has an amplitude of a, a period of $\frac{2\pi}{b}$, a horizontal translation of c units to the left (or to the right if c is negative) and a vertical translation of d units up (or down if d is negative).

WORKED EXAMPLE 10

Sketch the graph of $y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$ for $0 \leq x \leq 2\pi$, and state the period and amplitude.

THINK

1. Recall the general formula for combined transformations and rewrite the equation in this form.

WRITE/DRAW

$y = a\cos(b(x + c)) + d$

$y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$

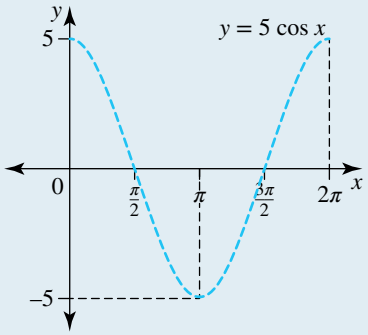
$\therefore a = 5,\ b = 1,\ c = \frac{\pi}{4},\ d = 5$

2. Determine the period and amplitude. (The coefficient of x is effectively equal to 1 as $x = 1x$, hence $b = 1$.)

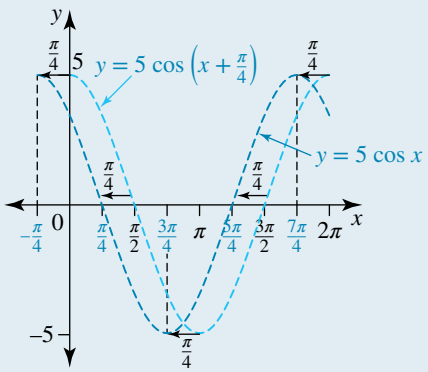
$$\text{Period} = \frac{2\pi}{b} = 2\pi$$

$$\text{Amplitude} = 5$$

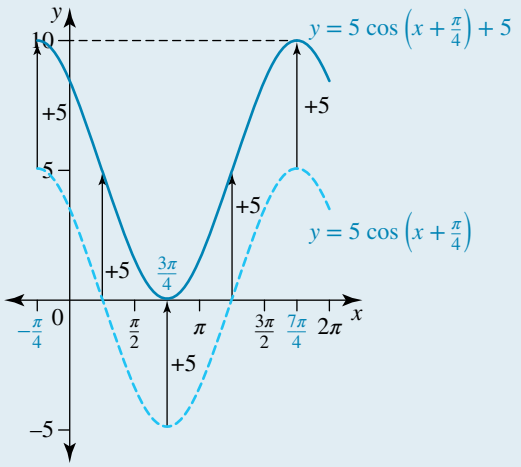
3. Using a pencil, sketch a basic cosine shape with the period and amplitude above.



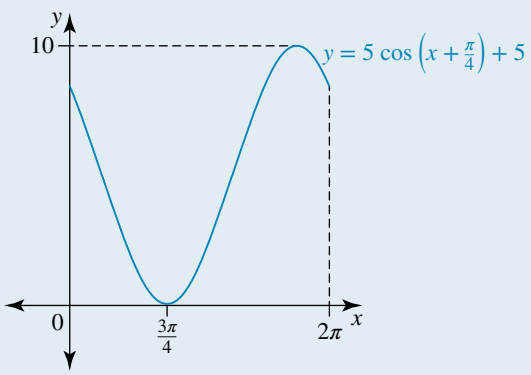
4. $c = \frac{\pi}{4}$, so apply the x-translation of $\frac{\pi}{4}$ (the + sign means to the left). Again, use pencil.



5. $d = 5$, so apply the y-translation of +5 (+ means up).



6. Erase the pencilled stages.

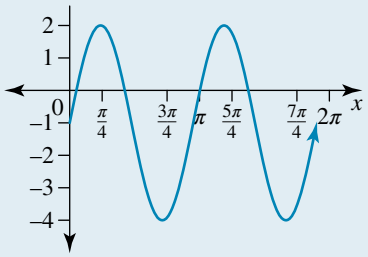


8.2.6 Finding the equation of a given trigonometric graph

Sometimes it is necessary to be able to determine the equation of a trigonometric function from a graph.

WORKED EXAMPLE 11

The graph shown is a trigonometric function of the form $y = a \sin bx + d$. Determine the values of a, b and d. Hence, state the equation of the function.



THINK

1. The amplitude (a) is half the distance between the maximum and minimum values. Remember to consider the absolute value to determine the distance.
2. The period is the interval from one point on the graph to the next point where the graph begins to repeat itself. The period is $\frac{2\pi}{b}$.
3. The equilibrium position is $y = -1$, so the graph has been translated down 1 unit.
4. Substitute values of a, b, d into the given form $y = a\sin(bx) + d$.

WRITE

$y = a \sin bx + d$

Amplitude, $a = \frac{1}{2}(2 + 4)$

$= 3$

The period is $\frac{2\pi}{b} = \pi$.

$b\pi = 2\pi$

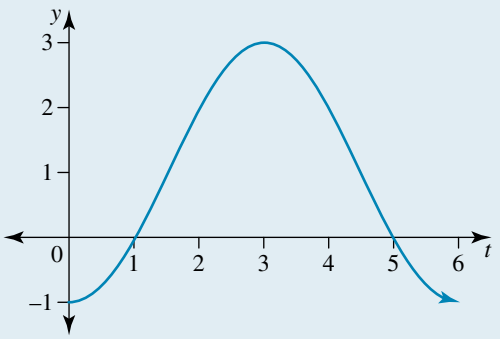
$b = 2$

$d = -1$

The equation of the function is $y = 3 \sin 2x - 1$.

WORKED EXAMPLE 12

This graph is a trigonometric function of the form $y = a \cos bt + d$. Determine the equation of the function.



THINK	WRITE
1. The amplitude $\|a\|$ is half the distance between the maximum and minimum values.	$y = a\cos bt + d$ Amplitude $a = \frac{1}{2}(3 + 1)$ $= 2$
2. The period is the amount of time taken to complete the pattern once.	Period $= \frac{2\pi}{b}$ $= 6,$ so $b = \frac{\pi}{3}$
3. The line through the centre of the graph, the equilibrium position, is $y = 1$, so the graph has been translated up one unit.	$d = 1$
4. We know that the graph is a cosine graph so it must be inverted; that is, A is negative.	$a = -2$
5. The equation of this trigonometric function is in the form $y = a\cos bt + d$. Replace a, b and d to obtain the equation of the function.	$y = -2\cos\frac{\pi}{3}t + 1$

on Resources

- **Interactivity:** Sine and cosine graphs (int-2976)
- **Interactivity:** Oscillation (int-2977)

studyon

Units 1 & 2 > Topic 2 > Chapter 5 > Concepts 1–5

The radian measure Summary screen and practice questions
Exact values Summary screen and practice questions
The unit circle Summary screen and practice questions
Symmetry properties Summary screen and practice questions

Exercise 8.2 Review of trigonometry

Technology free

1. **WE 1** Convert the following angles into exact radians.
   a. 30° b. 45° c. 60° d. 20° e. 50° f. 90°
2. Convert the following angles into exact radians.
   a. 270° b. 360° c. 150° d. −225° e. −420° f. 300°

3. **WE 2** Convert the following angles to degrees without the use of a calculator.

   a. $\frac{\pi}{5}$ **b.** $\frac{2\pi}{3}$ **c.** $\frac{2\pi}{9}$ **d.** $\frac{11\pi}{9}$ **e.** 3.6π **f.** $-\frac{\pi}{6}$

4. **WE 3** Determine the exact value of each of the following.

   a. $\sin\frac{\pi}{4}$ **b.** $\cos\frac{\pi}{3}$ **c.** $\tan\frac{\pi}{3}$ **d.** $\sin\frac{\pi}{6}$ **e.** $\tan\frac{\pi}{4}$ **f.** $\cos\frac{\pi}{6}$

5. **WE 4** Determine the exact value of each of the following.

   a. $\sin 150°$ **b.** $\cos 135°$ **c.** $\tan 240°$ **d.** $\tan 330°$ **e.** $\sin 240°$ **f.** $\cos 210°$

6. **WE 5** Determine the exact value of each of the following.

   a. $\cos\frac{5\pi}{4}$ **b.** $\sin\frac{5\pi}{6}$ **c.** $\tan\frac{7\pi}{4}$ **d.** $\sin\frac{2\pi}{3}$ **e.** $\cos\frac{7\pi}{3}$ **f.** $\tan\left(-\frac{\pi}{6}\right)$

Technology active

7. **WE 6** Sketch the graphs of the following functions for $-360° \leq x \leq 360°$ and state the period and amplitude of each.

   a. $y = 2\cos 4x$

   b. $y = 5\sin\frac{x}{3}$

   c. $y = -4\cos\frac{x}{2}$

   d. $y = \frac{3}{2}\sin\frac{2x}{3}$

   e. $y = 1.8\cos\frac{x}{3}$

   f. $y = -3\cos\frac{3x}{4}$

8. **WE 7** Sketch the graphs of the following functions for $-2\pi \leq x \leq 2\pi$ and state the period and amplitude of each.

   a. $y = \sin 2x$ **b.** $y = 2\cos x$

   c. $y = 3\sin\frac{x}{2}$ **d.** $y = 4\cos 2x$

   e. $y = \frac{1}{2}\sin 3x$ **f.** $y = \frac{2}{3}\cos\frac{x}{2}$

9. **WE 8** Sketch the graphs of the following over the domain $[0, 2\pi]$. State the amplitude, period and range of each function. Determine the coordinates of all axial intercepts, the maximum and minimum points, and the end points.

   a. $f(x) = \sin x + 1$ **b.** $f(x) = 2\cos x - 2$

   c. $f(x) = \sin 3x - 1$ **d.** $f(x) = 3\sin\frac{1}{2}x + 4$

10. **WE 9** Sketch the graphs of the following over the domain $[-\pi, \pi]$. State the period, amplitude, and range of each function. Determine the coordinates of all axial intercepts, the maximum and minimum points, and the end points.

    a. $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ **b.** $f(x) = 3\cos\left(x - \frac{\pi}{3}\right)$

    c. $f(x) = 2\sin(2x + \pi)$ **d.** $f(x) = -3\cos(3x + \pi)$

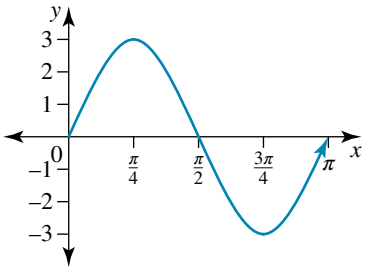
11. **WE 10** Sketch the graphs of the following for $0 \leq x \leq 2\pi$, and state the period and amplitude.

    a. $y = \cos\left(3x - \frac{\pi}{2}\right) + 1$ **b.** $y = 2\sin(2x - \pi) - 2$

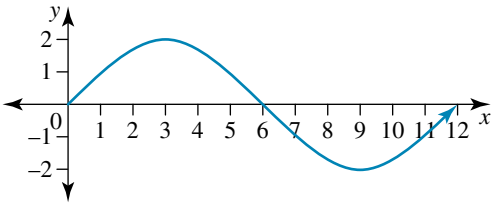
    c. $y = 2\sin\left(x - \frac{\pi}{4}\right) - 1$ **d.** $y = \cos\frac{1}{2}(x - \pi) + 1$

12. **WE 11** The equations of the following graphs are of the form $y = a \sin bx$. Determine the values of a and b. Hence, state the equation of each function.

a.

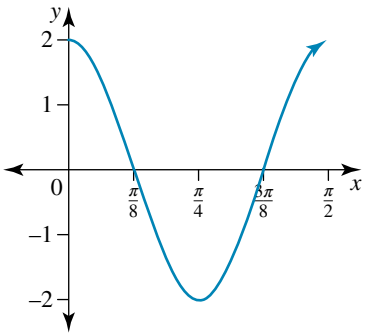


b.

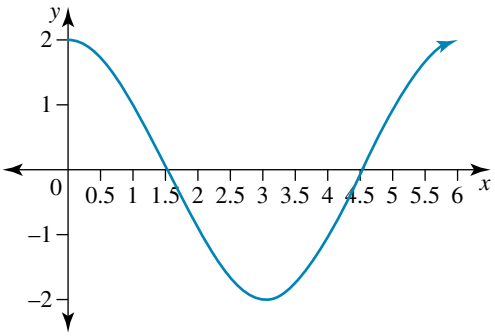


13. **WE 12** The equations of the following graphs are of the form $y = a \cos bx$. Determine the values of a and b. Hence, state the equation of each function.

a.

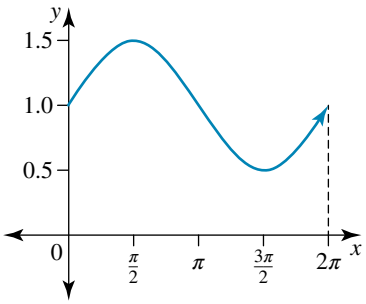


b.

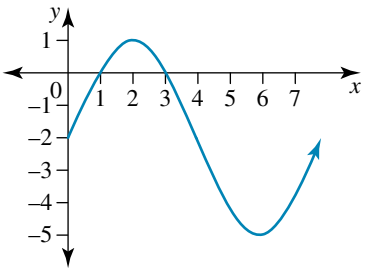


14. The equations of the following graphs are of the form $y = a \sin bx + d$. Determine the values of a, b, and d. Hence state the equation of the function.

a.

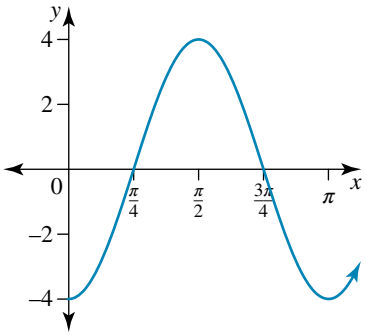


b.

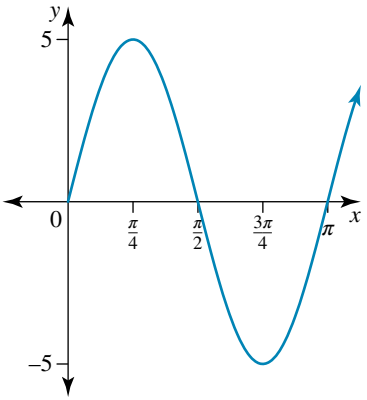


15. The equations of the following graphs are of the form $y = a \cos(b(x + c))$. Determine the values of a, b, and c. Hence, state the equation of the function.

a.

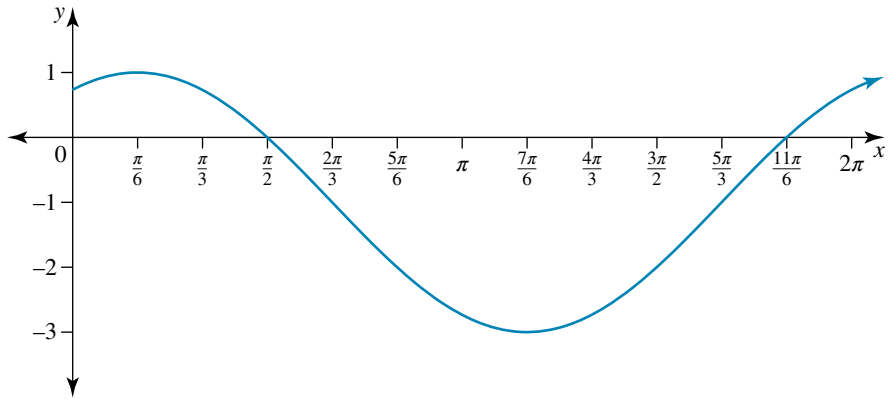


b.

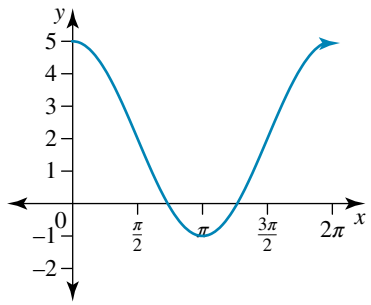


16. The equations of the following graphs are of the form $y = a\sin(b(x + c)) + d$. Determine the values of a, b, c and d. Hence state the equation of the function.

a.



b.

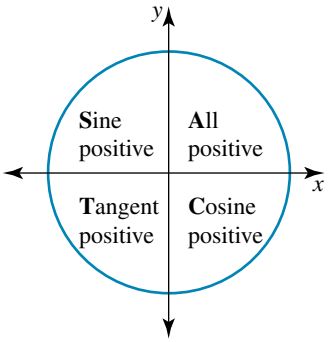


8.3 Solving trigonometric equations

8.3.1 Simple trigonometric equations

As $\sin\theta$ and $\cos\theta$ are periodic functions, there will be an infinite number of solutions to a trigonometric equation such as $\cos\theta = \frac{9}{13}$. In this case, the cosine ratio is positive in the first and fourth quadrants. The first quadrant solution is $\cos^{-1}\frac{9}{13} \approx 46°$ and the fourth quadrant solution is $360° - 46° \approx 314°$.

For a negative trigonometric ratio, we solve the corresponding positive equation to find a first quadrant angle to use, then find the corresponding angles in the negative quadrants.

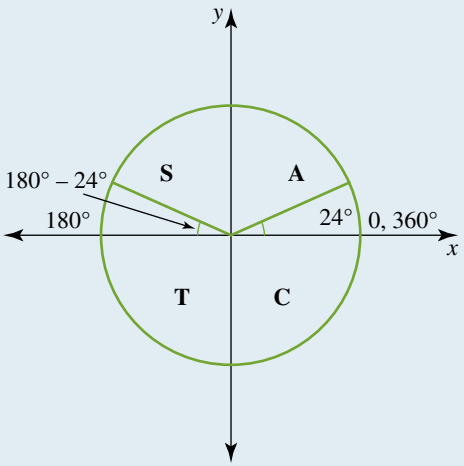


WORKED EXAMPLE 13

Solve $\sin\theta = 0.407$ over the domain $0° \leq \theta \leq 360°$, correct to the nearest degree.

THINK	WRITE
a. 1. Write the equation.	**a.** $\sin\theta = 0.407$
2. Use your calculator to find the first quadrant angle.	First quadrant angle $= 24°$

3. The sine ratio is positive in the first and second quadrants.



4. Find the second quadrant angle by subtracting 24° from 180°.

$180° - 24° = 156°$

5. Write the answer.

$\theta = 24°$ or $156°$

When solving trigonometric equations, we assume that the units for x are radians unless otherwise stated.

WORKED EXAMPLE 14

Solve the equation $\sin\theta = -\frac{\sqrt{3}}{2}$ over the domain $0 \leq \theta \leq 2\pi$.

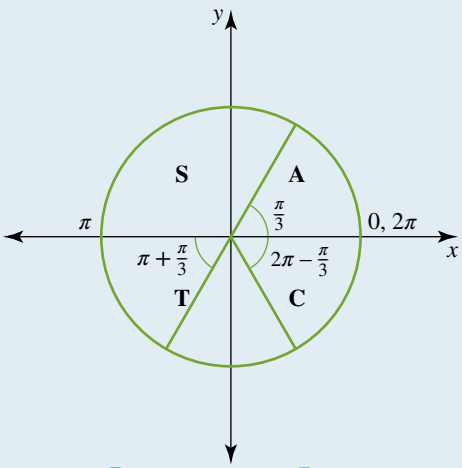
THINK

1. Write the equation.
2. Use the special triangles to find the first quadrant angle.
3. The sine ratio is negative in the third and fourth quadrants.

WRITE

$\sin\theta = -\frac{\sqrt{3}}{2}$

First quadrant angle $= \frac{\pi}{3}$



4. Find the third quadrant angle by adding $\frac{\pi}{3}$ to π and the fourth quadrant angle by subtracting $\frac{\pi}{3}$ from 2π.

$\theta = \pi + \frac{\pi}{3}$ or $\theta = 2\pi - \frac{\pi}{3}$

$= \frac{4\pi}{3}$ $= \frac{5\pi}{3}$

5. Write the answer.

$\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

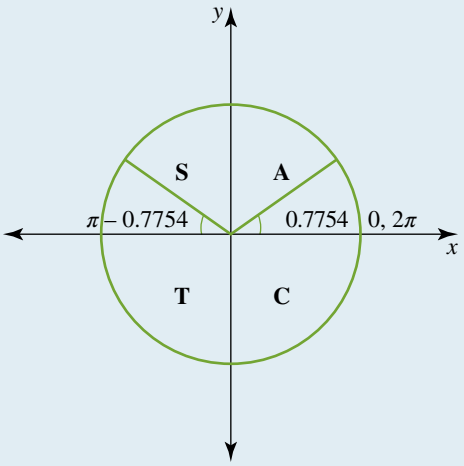
8.3.2 Changing the domain

In many cases the equation that we have to solve may not be in the domain $0 \leq x \leq 2\pi$. We may be asked to solve the equation in the domain $0 \leq x \leq 4\pi$ (2 revolutions) or $-2\pi \leq x \leq 2\pi$ (also 2 revolutions, but one in the negative sense).

To find the solutions to a trigonometric equation beyond the first revolution we simply add or subtract 2π to the first revolution solutions.

WORKED EXAMPLE 15

Solve $\sin \alpha = 0.7$ over the domain $0 \leq \alpha \leq 4\pi$, to 4 decimal places.

THINK	WRITE
1. Write the equation.	$\sin \alpha = 0.7$
2. Use your calculator to find the first quadrant angle.	First quadrant angle ≈ 0.7754
3. The sine ratio is positive in the first and second quadrants.	
4. Find the second quadrant angle by subtracting 0.7754 from π.	$\alpha = \pi - 0.7754$ $= 2.3662$
5. Find the solutions between 2π and 4π adding 2π to each of the first revolution solutions.	$\alpha = 0.7754 + 2\pi$ $\quad$ $\alpha = 2.3662 + 2\pi$ $= 7.0586$ $\quad$ $= 8.6494$
6. Write the answer.	$\alpha = 0.7754, 2.3662, 7.0586, 8.6494$

In many equations, you will need to make the trigonometric ratio the subject of the equation.

WORKED EXAMPLE 16

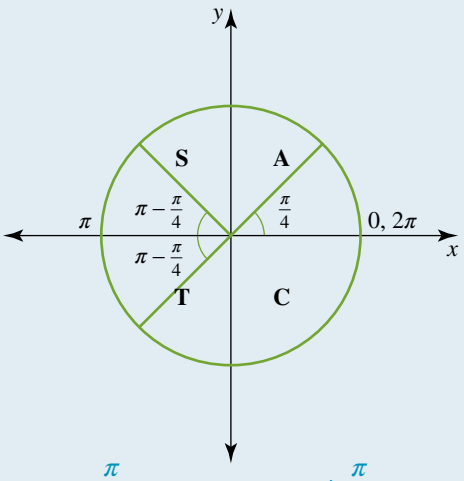
Determine exact solutions to $\sqrt{2}\cos x + 1 = 0$ over the domain $-2\pi \leq x \leq 2\pi$.

THINK	WRITE
1. Write the equation.	$\sqrt{2}\cos x + 1 = 0$
2. Make $\cos x$ the subject of the equation.	$\sqrt{2}\cos x = -1$ $\cos x = \frac{-1}{\sqrt{2}}$

3. Use the special triangles to find the first quadrant angle.

$$\text{First quadrant angle} = \frac{\pi}{4}$$

4. The cosine ratio is negative in the second and third quadrants.



5. Find the second quadrant angle by subtracting $\frac{\pi}{4}$ from π. Find the third quadrant angle by adding $\frac{\pi}{4}$ to π.

$$x = \pi - \frac{\pi}{4} \qquad x = \pi + \frac{\pi}{4}$$
$$= \frac{3\pi}{4} \qquad = \frac{5\pi}{4}$$

6. To find the solutions between -2π and 0, subtract 2π from each of the first revolution solutions.

$$x = \frac{3\pi}{4} - 2\pi \qquad x = \frac{5\pi}{4} - 2\pi$$
$$= -\frac{5\pi}{4} \qquad = -\frac{3\pi}{4}$$

7. Write the answer.

$$x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$$

8.3.3 Further trigonometric equations

When solving trigonometric equations of the form $a\sin(x \pm c) + d = 0$ or $a\cos(x \pm c) + d = 0$ for $x \in [0,\ 2\pi]$, we must be careful to generate all solutions. This may mean adding 2π to the base angle (quadrant 1 solution) to determine all the solutions within the required interval.

WORKED EXAMPLE 17

Calculate the exact values of x for $x \in [0,\ 2\pi]$ if $2\sin\left(x + \frac{\pi}{3}\right) - \sqrt{3} = 0$.

THINK

1. Rearrange the equation to make $\sin\left(x + \frac{\pi}{3}\right)$ the subject.

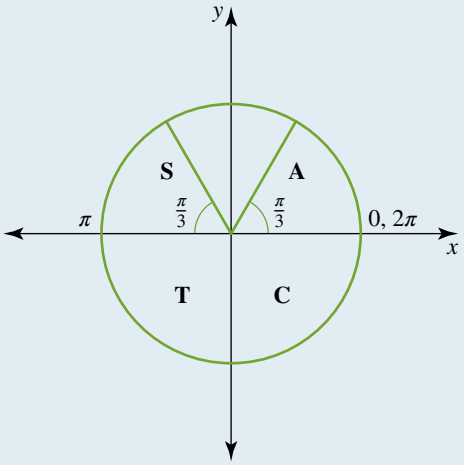
WRITE

$$2\sin\left(x + \frac{\pi}{3}\right) - \sqrt{3} = 0$$
$$2\sin\left(x + \frac{\pi}{3}\right) = \sqrt{3}$$
$$\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

2. Use special triangles to find the first quadrant angle.

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

3. The sine ratio is positive in the first and second quadrants.



4. Solve for $x + \frac{\pi}{3}$. We want to find all of the values of $x \in [0, 2\pi]$. It is possible that there may be some values of $x + \frac{\pi}{3}$ that are outside of $[0, 2\pi]$. To allow for this, add 2π to the quadrant 1 solution to obtain an additional solution.

$$x + \frac{\pi}{3} = \frac{\pi}{3}, x + \frac{\pi}{3} = \pi - \frac{\pi}{3}, x + \frac{\pi}{3} = 2\pi + \frac{\pi}{3}$$

5. Solve for x. Check that all solutions are between 0 and 2π.

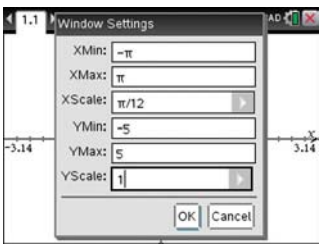
$$x = 0, x = \frac{\pi}{3} \text{ or } x = 2\pi$$

The solutions are $x = 0, \frac{\pi}{3}, 2\pi$.

TI | THINK

1. On a Graphs page, set the Graphing Angle to Radian. Press MENU, then select:
   4: Window/Zoom
   1: Window Settings …
   Complete the fields as:
   XMin: 0
   XMax: 2π
   XScale: $\pi/3$
   YMin: –5
   YMax: 5
   YScale: 1
   then select OK.
   Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

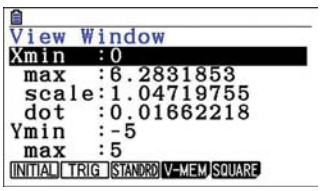
WRITE



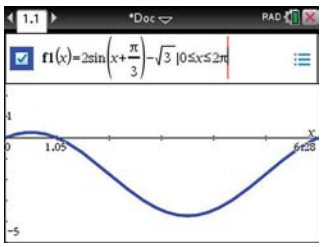
CASIO | THINK

1. Put the Calculator in Radian mode.
   On a Graph screen, press SHIFT, then F3 to open the V_WIN. Complete the fields as:
   Xmin: 0
   max: 2π
   scale: $\pi/3$
   Ymin: –5
   max: 5
   scale: 1
   then press EXIT.
   Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be easily read from the graph.

WRITE



2. Complete the entry line for function 1 as: $f1(x) = 2\sin(x + \frac{\pi}{3}) - \sqrt{3} \,|\, 0 \le x \le 2\pi$ then press ENTER.



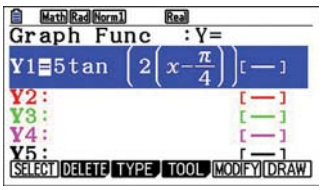
3. Identify the coordinates of the *x*-intercepts from the graph.

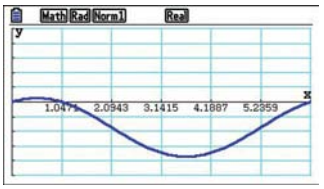
The *x*-axis scale is $\frac{\pi}{3}$, so it can be seen that the graph has *x*-intercepts at $(0, 0)$, $\left(\frac{\pi}{3}, 0\right)$ and $(2\pi, 0)$. Be sure to mark these points when sketching the graph.

4. The answer appears on the screen.

The *x*-intercepts of the graph represent the solutions to the equation $2\sin(x + \frac{\pi}{3}) - \sqrt{3} = 0$ over the domain $[0, 2\pi]$.
$\therefore\ x = 0, \frac{\pi}{3}$ and 2π.

2. Complete the entry line for y1 as: $y1 = 2\sin(x + \frac{\pi}{3}) - \sqrt{3}, [0, 2\pi]$ then press EXE. Select DRAW by pressing F6.





3. Identify the coordinates of the *x*-intercepts from the graph.

The *x*-axis scale is $\frac{\pi}{3}$, so it can be seen that the graph has *x*-intercepts at $(0, 0)$, $\left(\frac{\pi}{3}, 0\right)$ and $(2\pi, 0)$. Be sure to mark these points when sketching the graph.

4. The answer appears on the screen.

The *x*-intercepts of the graph represent the solutions to the equation $2\sin(x + \frac{\pi}{3}) - \sqrt{3} = 0$ over the domain $[0, 2\pi]$.
$\therefore\ x = 0, \frac{\pi}{3}$ and 2π.

When solving trigonometric equations of the form $a\sin(bx) + d = 0$ or $a\cos(bx) + d = 0$, if $x \in [0, 2\pi]$, then $nx \in [0, 2\pi n]$, so we must add 2π to the base angle n times. Equations such as these can have up to $2n$ solutions.

WORKED EXAMPLE 18

Calculate the exact values of x for $x \in [0, 2\pi]$ if $2\cos(2x) - 1 = 0$.

THINK

1. Rearrange the equation to make $\cos 2x$ the subject.

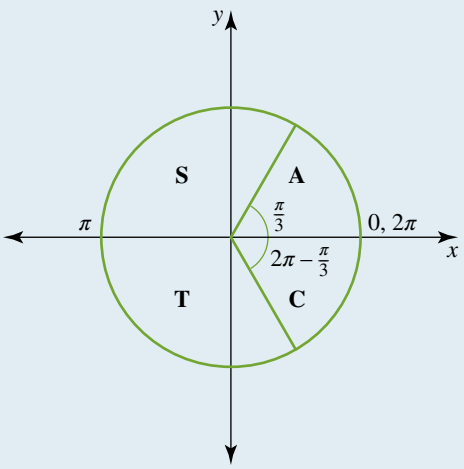
WRITE

$2\cos(2x) - 1 = 0$
$2\cos(2x) = 1$
$\cos(2x) = \frac{1}{2}$

2. Use special triangles to find the first quadrant angle.

$\cos\frac{\pi}{3} = \frac{1}{2}$

3. The cosine ratio is positive in the first and fourth quadrants.



4. Solve for $2x$. It is possible that there may be some values of $2x$ that are outside of $[0, 2\pi]$. To allow for this, add 2π to the first and fourth quadrant solutions to obtain the additional solutions.

First quadrant solution: $2x = \frac{\pi}{3}$

Fourth quadrant solution: $2x = 2\pi - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

$2x = \frac{\pi}{3} + 2\pi$ or $2x = \frac{5\pi}{3} + 2\pi$

$= \frac{7\pi}{3}$ $= \frac{11\pi}{6}$

5. Solve for x. Check that all solutions are between 0 and 2π.

$x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x = \frac{7\pi}{6}, x = \frac{11\pi}{6}$

The solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

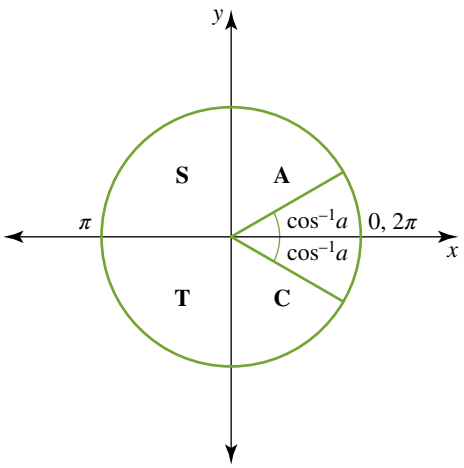
8.3.4 General solutions of the cosine and sin functions

General solutions of $y = \cos x$

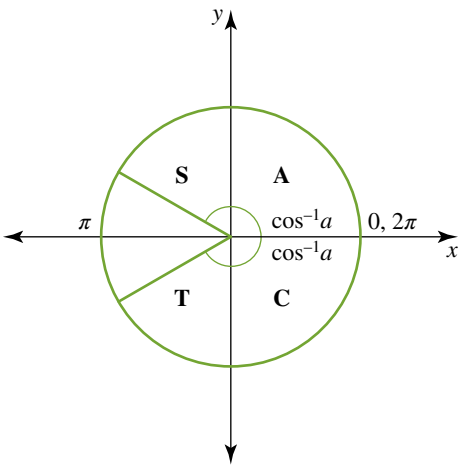
Because trigonometric functions have an infinite number of solutions, sometimes we want to develop a general formula that describes the solution in terms of n, where n is an integer.

Consider the solutions to $\cos x = a$, where $-1 \leq a \leq 1$. If a is positive, then the solutions are in quadrants 1 and 4. If a is negative, then the solutions are in quadrants 2 and 3.

$\cos x = a,\ 0 \leq a \leq 1$



$\cos x = a,\ -1 \leq a \leq 0$



Combining these solutions and extrapolating for an infinite number of solutions, the **general solution** is given by $x = 2n\pi \pm \cos^{-1} a,\ n \in Z$.

WORKED EXAMPLE 19

Determine the general solution to $\cos x = \frac{1}{2}$.

THINK

1. Rearrange to the form $\cos^{-1} a$.

WRITE

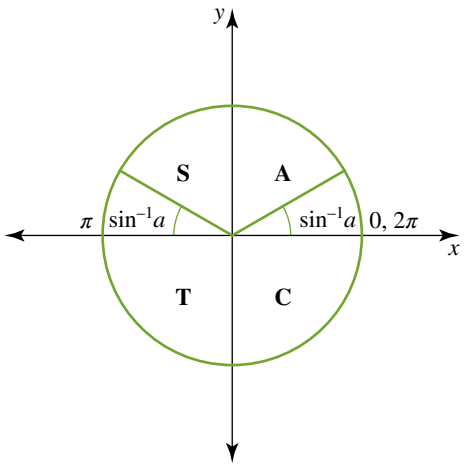
$x = \cos^{-1} \frac{1}{2}$

2. Recall the general solution formula and substitute $\cos^{-1}\frac{1}{2}$.	$x = 2n\pi \pm \cos^{-1} a,\ n \in Z$ $= 2n\pi \pm \cos^{-1}\frac{1}{2}$
3. Use special triangles to recognise $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.	$x = 2n\pi \pm \frac{\pi}{3}$
4. Simplify by factorising $\frac{\pi}{3}$.	$x = \frac{\pi}{3}(6n \pm 1)$
5. State the general solution.	$x = \frac{\pi}{3}(6n \pm 1),\ n \in Z$

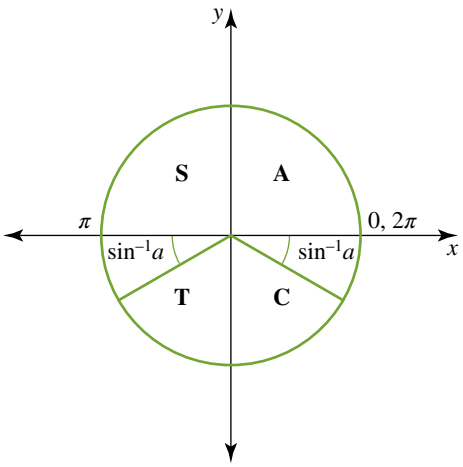
The general solution to $y = \sin x$

Consider the solutions to $\sin x = a$, where $-1 \leq a \leq 1$. If a is positive, then the solutions are in quadrants 1 and 2. If a is negative, then the solutions are in quadrants 3 and 4.

$\sin x = a,\ 0 \leq a \leq 1$



$\sin x = a,\ -1 \leq a \leq 0$



Note: $\sin^{-1} a$ will be negative.

The solutions can be described by $x = \sin^{-1} a$ or $x = \pi - \sin^{-1} a$. This means that the general solutions are $x = 2n\pi + \sin^{-1} a,\ (2n+1)\pi - \sin^{-1} a, n \in Z$.

WORKED EXAMPLE 20

Determine the general solution to $\sin x = -\frac{\sqrt{3}}{2}$.

THINK	WRITE
1. Rearrange to the form $\sin^{-1} a$.	$x = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
2. Recognise $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. As $\sin x$ is negative, $-\frac{\pi}{2} < x < 0$.	$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

3. Recall the formula for the general solution and substitute for $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

$x = 2n\pi + \sin^{-1} a,\ n \in Z \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1} a,\ n \in Z$

$= 2n\pi - \frac{\pi}{3} \qquad = (2n+1)\pi + \frac{\pi}{3}$

4 Simplify by factorising $\frac{\pi}{3}$.

$x = \frac{\pi}{3}(6n-1) \quad \text{or} \quad x = \frac{\pi}{3}[3(2n+1)+1]$

$= \frac{\pi}{3}(6n+4)$

4. State the general solutions.

$x = \frac{\pi}{3}(6n-1), \frac{\pi}{3}(6n+4),\ n \in Z$

General solutions of transformed sine and cosine functions

When solving equations of the form $a\sin(nx+b)+c=0$ or $a\cos(nx+b)+c=0$, it is necessary to rearrange the equation so that $\sin(nx+b)$ or $\cos(nx+b)$ is the subject.

WORKED EXAMPLE 21

Determine the general solution to $\sqrt{2}\ \sin\left(3x+\frac{\pi}{4}\right)+1=0$.

THINK

1. Rearrange the equation to make $\sin\left(3x+\frac{\pi}{4}\right)$ the subject.

2. Use exact values to find $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

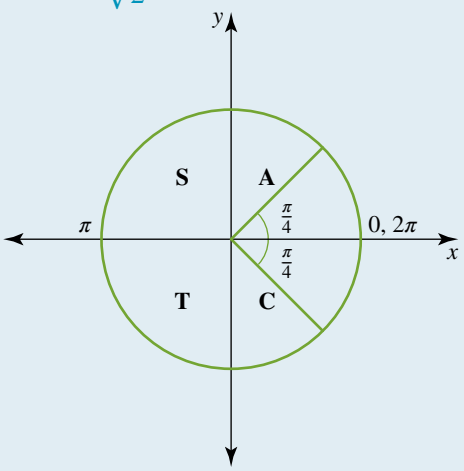
WRITE

$\sqrt{2}\sin\left(3x+\frac{\pi}{4}\right)+1=0$

$\sqrt{2}\sin\left(3x+\frac{\pi}{4}\right)=-1$

$\sin\left(3x+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$



3. Recall the general solution to $\sin x = a$ is $x = 2n\pi + \sin^{-1} a$, $(2n + 1)\pi - \sin^{-1} a$, $n \in Z$.

   Use this formula to find equations for $3x + \frac{\pi}{4}$.

$$\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$3x + \frac{\pi}{4} = 2n\pi - \frac{\pi}{4} \quad \text{or} \quad 3x + \frac{\pi}{4} = 2(n+1)\pi + \frac{\pi}{4},\ n \in Z$$

4. Solve for x and simplify your answer.

$$3x = 2n\pi - \frac{\pi}{2} \quad \text{or} \quad 3x = (2n+1)\pi$$

$$x = \frac{2n\pi}{3} - \frac{\pi}{6} \qquad x = \frac{(2n+1)\pi}{3}$$

$$= \frac{4\pi n}{6} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}(4n - 1)$$

studyon

Units 1 & 2 > Topic 5 > Chapter 2 > Concepts 5–7

The sine function Summary screen and practice questions
The cosine function Summary screen and practice questions
Solving trigonometric equations Summary screen and practice questions

Exercise 8.3 Solving trigonometric equations

Technology active

1. **WE 13** Solve each of the following trigonometric equations over the domain $0° \leq \theta \leq 360°$, correct to the nearest degree.
   a. $\sin\theta = 0.6$ b. $\cos\theta = -0.25$ c. $\sin\theta = -0.85$
   d. $\cos\theta = 0.195$ e. $\sin\theta = -0.333$ f. $\cos\theta = 0.757$
2. Determine exact solutions to each of the following trigonometric functions over the domain $0° \leq \theta \leq 360°$.
   a. $\sin\theta = \frac{\sqrt{3}}{2}$ b. $\cos\theta = \frac{\sqrt{2}}{2}$
   c. $\sin\theta = -\frac{1}{2}$ d. $\cos\theta = -\frac{1}{2}$
   e. $\sin\theta = -\frac{\sqrt{2}}{2}$ f. $\cos\theta = \frac{\sqrt{3}}{2}$

3. **WE 14** Solve each of the following equations over the domain $0 \leq x \leq 2\pi$, to 2 decimal places.
   a. $\sin x = 0.8$ b. $\cos x = -0.5$ c. $\sin x = -0.327$ d. $\cos x = 0.707$
4. Solve each of the following equations over the domain $0 \leq x \leq 2\pi$, giving your answers as exact values.
   a. $\sin x = \frac{\sqrt{3}}{2}$ b. $\cos x = -\frac{1}{2}$ c. $\cos x = \frac{\sqrt{2}}{2}$ d. $\sin x = -\frac{1}{2}$
5. **WE 15** Solve each of the following equations over the domain $0 \leq x \leq 4\pi$, to 4 decimal places.
   a. $\cos x = -0.6591$ b. $\sin x = 0.9104$ c. $\cos x = 0.48$ d. $\sin x = -0.371$

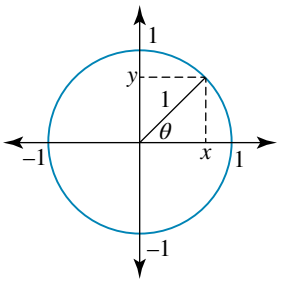
6. **WE 15** Solve each of the following equations over the domain $-2\pi \leq x \leq 2\pi$, to 2 decimal places.
   a. $\sin x = 0.2686$ **b.** $\cos x = -0.7421$ **c.** $\sin x = -0.5432$ **d.** $\cos x = 0.1937$
7. Solve each of the following equations over the domain $0° \leq \theta \leq 360°$.
   a. $\sin\theta = 1$ **b.** $\cos\theta = 0$ **c.** $\sin\theta = 0$ **d.** $\cos\theta = -1$
8. Solve each of the following equations over the domain $0 \leq x \leq 2\pi$.
   a. $\sin x = 0$ **b.** $\cos x = 0$ **c.** $\cos x = 1$ **d.** $\sin x = -1$
9. **WE 16** Determine exact solutions to each of the following equations over the domain $0 \leq x \leq 2\pi$.
   a. $2\sin x = 1$ **b.** $2\cos x = \sqrt{3}$ **c.** $2\sin x + \sqrt{3} = 0$ **d.** $2\cos x + \sqrt{2} = 0$
10. Solve each of the following equations over the domain $0 \leq x \leq 2\pi$, to 2 decimal places.
    a. $4\sin x = 1$ **b.** $3\cos x = -2$ **c.** $4 + \sin x = 3$ **d.** $1 + 2\cos x = \sqrt{2}$
11. **WE 17** Calculate all the solutions as exact values to the following equations over the domain $-2\pi \leq x \leq 2\pi$.
    a. $2\sin x - 1 = 0$ **b.** $3\cos x = 0$ **c.** $2\sin x + \sqrt{3} = 0$ **d.** $\sqrt{2}\cos x = 1$
12. **WE 18** Calculate all the solutions to the following equations over the domain $-2\pi \leq x \leq 2\pi$. Give your solutions correct to 2 decimal places.
    a. $4\sin x + 2 = 6$ **b.** $3\cos x - 3 = 0$ **c.** $\frac{1}{2}\cos x + 4 = 4.21$ **d.** $2\sin x - 5 = -4$
13. **WE 19** Determine the general solutions of the following.
    a. $\cos x = \frac{\sqrt{2}}{2}$ **b.** $2\cos x - \sqrt{3} = 0$
    c. $\sqrt{2}\cos x + 1 = 0$ **d.** $2\cos x + 1 = 0$

14. **WE 20** Determine the general solutions of the following equations.
    a. $\sin x = \frac{1}{2}$ **b.** $\sqrt{2}\sin x = 1$
    c. $2\sin x + \sqrt{3} = 0$ **d.** $2\sin x + 1 = 0$
15. **WE 21** Determine the general solutions of the following equations.
    a. $2\sin 2x + \sqrt{3} = 0$ **b.** $2\cos 2x + \sqrt{3} = 0$
    c. $\sqrt{2}\sin\left(3x - \frac{\pi}{4}\right) - 1 = 0$ **d.** $2\cos\left(3\left(x - \frac{\pi}{12}\right)\right) - 1 = 0$
16. Determine the solutions to each of the following equations.
    a. $\cos x = \cos 2x$, $x \in [0, 2\pi]$ **b.** $\sin 2x = \sin x$, $x \in [-\pi, \pi]$

8.4 The tangent function

8.4.1 Features of the tangent function

The graph of $y = \tan x$



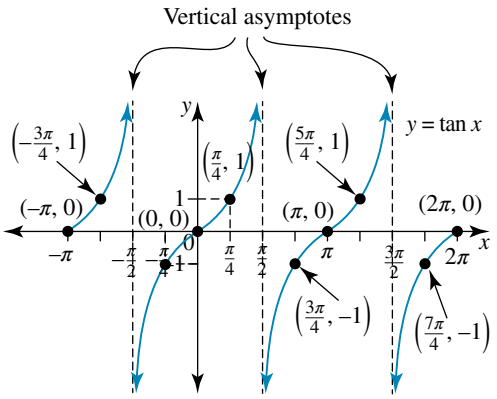
Consider the unit circle with an angle of θ at the centre.

From this circle we can see that $\sin\theta = y$, $\cos\theta = x$ and $\tan\theta = \frac{y}{x}$. From this we are able to conclude that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. We can use this information to generate the table of values below, or we can use our knowledge of special angles.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$y = \tan x$	0	1	Undefined	−1	0	1	Undefined	−1	0	1	Undefined	−1	0

Note the presence of some undefined y-values. This is because $\tan x = \dfrac{\sin x}{\cos x}$ and $\cos x = 0$ at these values.
These undefined values are shown as vertical asymptotes through the given value of x for which they occur.
The graph of $y = \tan x$ is shown in the following figure.
Verify this graph using a graphics calculator as shown with $y = \tan x$.
The features of the graph of $y = \tan x$ are:

1. It has vertical asymptotes at $x = \ldots -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$
2. It has no amplitude.
3. It has a period of π.
4. It has a range of R (the set of all Real numbers).



Notes:

1. As $\tan x = \dfrac{\sin x}{\cos x}$, the x-intercepts occur when $\sin x = 0$ and the asymptotes occur when $\cos x = 0$.
2. Multiples of $\dfrac{\pi}{8}$ could be used to give more points to plot and a clearer indication of the shape of the graph.

8.4.2 Sketching the tangent function

To sketch a function for $y = a\tan(b(x + c)) + d$, some key points can be identified by rewriting the equation as $y = a\dfrac{\sin(b(x + c))}{\cos(b(x + c))} + d$.

As the period of $\tan x$ is π, the period of $y = a\tan(b(x + c)) + d$ will be $\dfrac{\pi}{b}$.

Vertical asymptotes will occur when $\cos(b(x + c)) = 0$.

$$b(x + c) = \frac{\pi}{2} + n\pi,\ n \in Z$$
$$= \frac{\pi}{2}(1 + 2n)$$
$$x = \frac{\pi(2n + 1)}{2b} - c$$

The vertical shift will be d units up if d is positive and d units down if d is negative.
The horizontal shift will be c units to the left if c is positive and c units right if c is negative.
Although the graph does not have an amplitude, knowing that $\tan\dfrac{\pi}{4} = 1$ can be used give an indication of the scale of the graph. This means that if $b(x + c) = \dfrac{\pi}{4}$, then $y = a + d$.

WORKED EXAMPLE 22

Sketch the graph of $y = 5\tan\left(2\left(x - \dfrac{\pi}{4}\right)\right) + 1$ over the domain $-2\pi \leq x \leq 2\pi$.

THINK	WRITE
1. Recall the general formula for the target function and identify a, b, c and d.	$y = a\tan b(x + c) + d$ $a = 5,\ b = 2,\ c = -\dfrac{\pi}{4},\ d = 1$

2. Evaluate the period and the vertical shift.

$$\text{Period} = \frac{\pi}{b}$$
$$= \frac{\pi}{2}$$

Vertical shift: up 1

3. Rewrite the equation using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

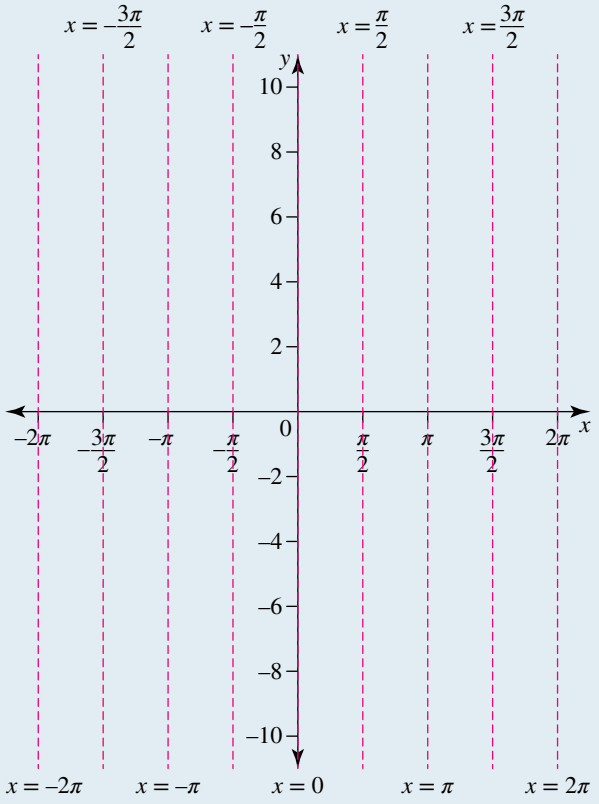
$$y = 5\tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$$
$$= 5\frac{\sin\left(2\left(x - \frac{\pi}{4}\right)\right)}{\cos\left(2\left(x - \frac{\pi}{4}\right)\right)} + 1$$

4. Identify the vertical asymptotes by solving when the denominator is equal to 0. Find one solution and use the period to identify the others. Mark the asymptotes on the graph.

$$\cos\left(2\left(x - \frac{\pi}{4}\right)\right) = 0$$
$$\left(2\left(x - \frac{\pi}{4}\right)\right) = \frac{\pi}{2}$$
$$x - \frac{\pi}{4} = \frac{\pi}{4}$$
$$x = \frac{\pi}{2}$$

The asymptotes will be π units apart.
The asymptotes will be

$$x = \frac{\pi}{2} - \frac{5\pi}{2}, \frac{\pi}{2} - \frac{4\pi}{2}, \frac{\pi}{2} - \frac{3\pi}{2}, \frac{\pi}{2} - \frac{2\pi}{2}, \frac{\pi}{2} - \frac{\pi}{2},$$
$$\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2} + \frac{2\pi}{2}, \frac{\pi}{2} + \frac{3\pi}{2}$$
$$= -2\pi, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$$



5. As the vertical shift up is 1, find one point when $y = 1$ and use the period to identify the other points. Mark these on the graph.

$$5\frac{\sin\left(2\left(x-\frac{\pi}{4}\right)\right)}{\cos\left(2\left(x-\frac{\pi}{4}\right)\right)}+1=1$$

$$5\frac{\sin\left(2\left(x-\frac{\pi}{4}\right)\right)}{\cos\left(2\left(x-\frac{\pi}{4}\right)\right)}=0$$

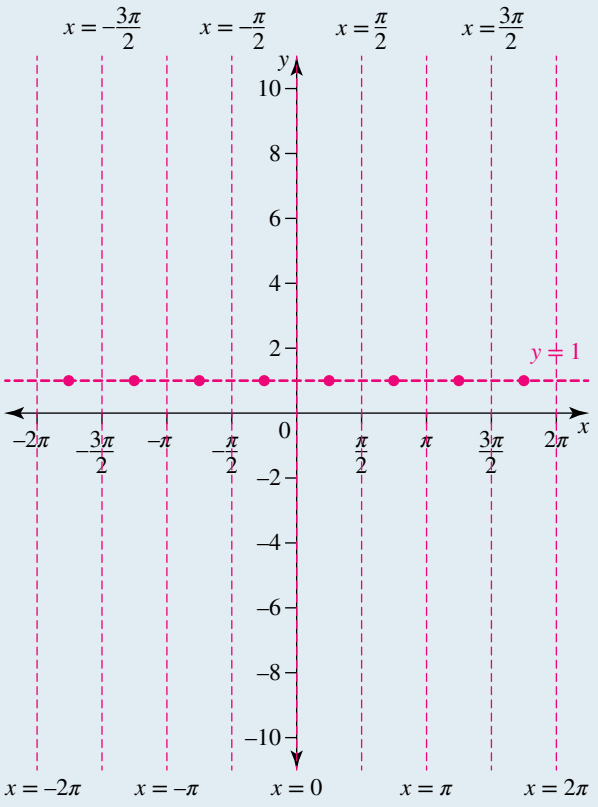
$$\sin\left(2\left(x-\frac{\pi}{4}\right)\right)=0$$

$$2\left(x-\frac{\pi}{4}\right)=0$$

$$x-\frac{\pi}{4}=0$$

$$x=\frac{\pi}{4}$$

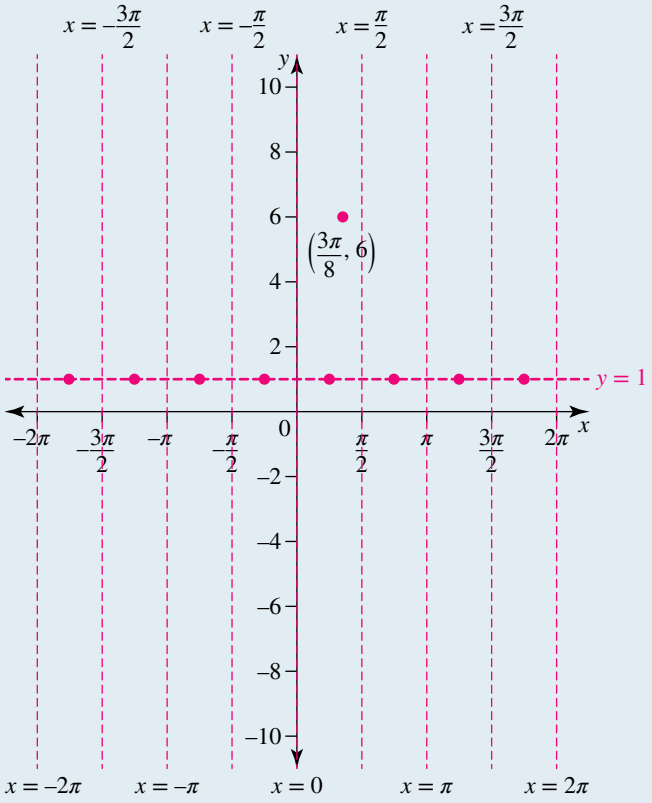
$$x=\frac{\pi}{4}-\frac{4\pi}{2},\frac{\pi}{4}-\frac{3\pi}{2},\frac{\pi}{4}-\frac{2\pi}{2},\frac{\pi}{4}-\frac{\pi}{2},\frac{\pi}{4},$$
$$\frac{\pi}{4}+\frac{\pi}{2},\frac{\pi}{4}+\frac{2\pi}{2},\frac{\pi}{4}+\frac{3\pi}{2}$$
$$=\frac{-7\pi}{4},-\frac{5\pi}{4},-\frac{3\pi}{4},-\frac{\pi}{4},\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$$



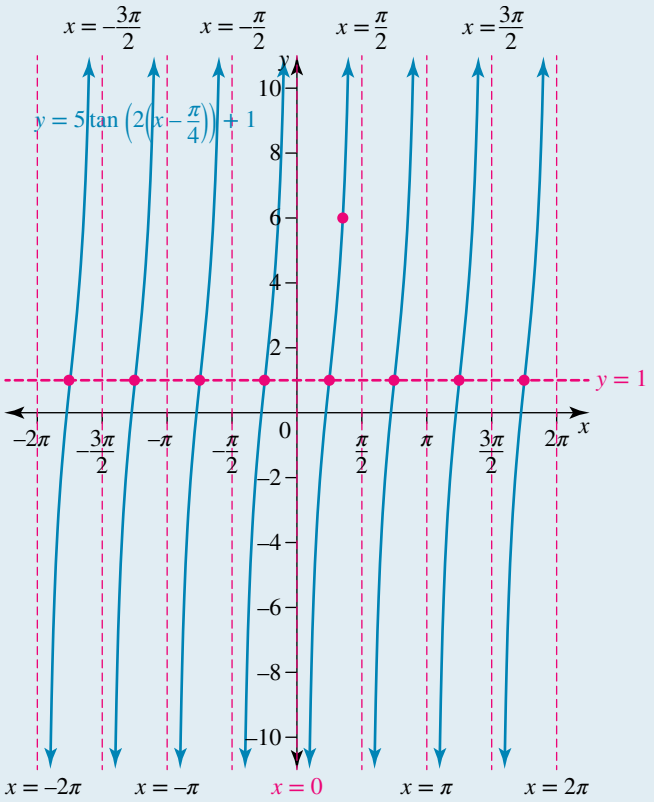
6. Use $\tan\frac{\pi}{4} = 1$ to find a point on the graph.

$$\tan\frac{\pi}{4} = 1$$
$$2\left(x - \frac{\pi}{4}\right) = \frac{\pi}{4}$$
$$x - \frac{\pi}{4} = \frac{\pi}{8}$$
$$x = \frac{3\pi}{8}$$

Therefore, when $x = \frac{3\pi}{8}$, $y = 5 \times 1 + 1$
$= 6$



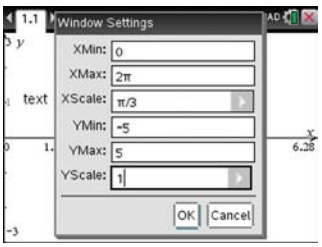
7. Sketch the graph.



TI | THINK

1. On a Graphs page, set the Graphing Angle to Radian.
   Press MENU, then select:
   4: Window/Zoom
   1: Window Settings …
   Complete the fields as:
   XMin: -2π
   XMax: π
   XScale: $\pi/4$
   YMin: –10
   YMax: 10
   YScale: 1
   then select OK.
   Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

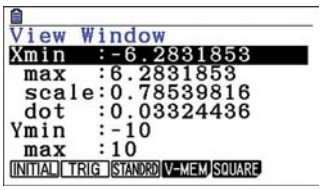
WRITE



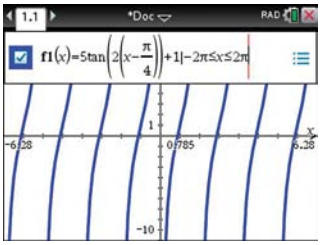
CASIO | THINK

1. Put the Calculator in Radian mode.
   On a Graph screen, press SHIFT, then F3 to open the V_WIN.
   Complete the fields as:
   Xmin: -2π
   max: π
   scale: $\pi/4$
   Ymin: –10
   max: 10
   scale: 1
   then press EXIT.
   Note: The calculator will only give decimal approximations for intercepts, minimums and maximums, so it is important to have the x-axis scale as a multiple of π so that important points can be read easily from the graph.

WRITE



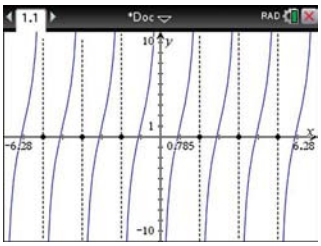
2. Complete the entry line for function 1 as: $f1(x) = 5\tan(2(x - \frac{\pi}{4})) + 1|-2\pi \le x \le 2\pi$ then press ENTER.



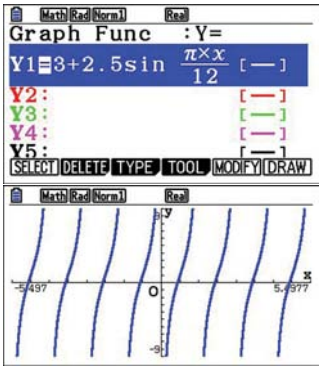
3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{4}$, so it can be seen that the graph has vertical asymptotes at $x = -2\pi$, $-\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

4. To draw the vertical asymptotes, press MENU, then select
8: Geometry
4: Construction
1: Perpendicular.
Click on the x-axis, then click on the point $\left(-\frac{\pi}{2}, 0\right)$.
Repeat this step to draw the other vertical asymptotes.
Note: To change the style of the vertical lines, place the cursor on the line and press CTRL, then press MENU, then select
3: Attributes.



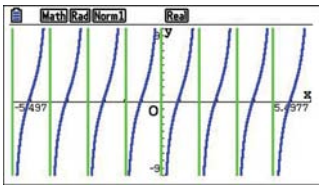
2. Complete the entry line for y1 as: $y1 = 5\tan\left(2\left(x - \frac{\pi}{4}\right)\right) + 1, [-2\pi, 2\pi]$ then press EXE. Select DRAW by pressing F6.



3. Identify the positions of the vertical asymptotes.

The x-axis scale is $\frac{\pi}{4}$, so it can be seen that the graph has vertical asymptotes at $x = -2\pi$, $-\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

4. To draw the vertical asymptotes, select Sketch by pressing SHIFT, then F4. Press F6 to scroll across to more menu options, then select Vertical by pressing F4. Use the left/right arrows to position the vertical line, then press EXE. Repeat this step to draw the other vertical asymptotes.



Resources

Interactivity: The tangent function (int-2978)

8.4.3 Solving the tangent function

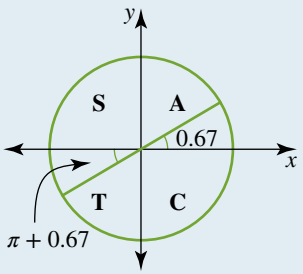
Equations involving the tangent function can be solved using a similar method to solving sine and cosine functions.

WORKED EXAMPLE 23

Solve the equation $\tan\theta = 0.8$ over the domain $0 \le \theta \le 2\pi$. Give the answer correct to 2 decimal places.

THINK	WRITE
1. Write the equation.	$\tan\theta = 0.8$

2. Use your calculator to find the first quadrant angle.
   First quadrant angle ≈ 0.67
3. The tangent ratio is positive in the first and third quadrants.



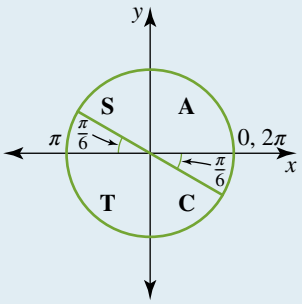
4. Find the third quadrant angle by adding 0.67 to π.
   $\pi + 0.67 = 3.81$
5. Write the answer.
   $\theta = 0.67$ or 3.81

WORKED EXAMPLE 24

Determine exact solutions to $\sqrt{3}\tan 2x + 1 = 0$ over the domain $0 \leq x \leq 2\pi$.

THINK | **WRITE**

1. Write the equation.
   $\sqrt{3}\tan 2x + 1 = 0$
2. Rearrange the equation to make $\tan 2x$ the subject.
   $\sqrt{3}\tan 2x = -1$
   $\tan 2x = -\frac{1}{\sqrt{3}}$
3. Use the special triangles to find the first quadrant angle.
   $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$
4. The tangent ratio is negative in quadrants 2 and 4.



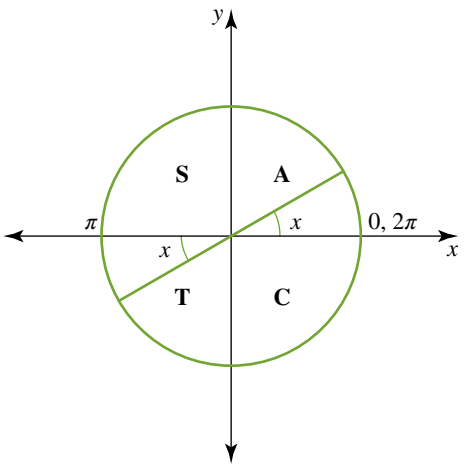
$2x = \pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}$
$= \frac{5\pi}{6}, \frac{11\pi}{6}$

5. As we wish to find all $x \in [0, 2\pi]$, it is necessary to find all $2x \in [0, 4\pi]$. Add 2π to each solution.
   $2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{5\pi}{6} + 2\pi, \frac{11\pi}{6} + 2\pi$
   $= \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$
6. Solve for x.
   $x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

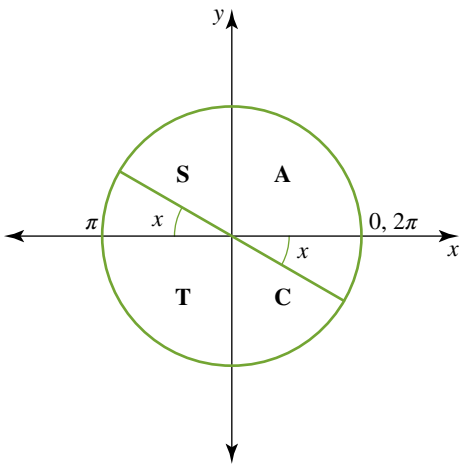
8.4.4 General solutions of the tangent function

Consider the solutions to $\tan x = a$. If a is positive, the solution is in quadrants 1 and 3. If a is negative, the solution is in quadrants 2 and 4. In both cases, the solutions are separated by π radians (or 180°).

$\tan x = a, a > 0.$



$\tan x = a, a < 0$



The general solution is given by $x = n\pi + \tan^{-1} a, n \in Z$.

WORKED EXAMPLE 25

Determine the general solution to $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$.

THINK

1. Use special triangles to find the first quadrant angle, $\tan^{-1}(\sqrt{3})$.
2. Recall the general solution to $\tan x = a$ is $x = n\pi + \tan^{-1} a, n \in Z$. Use this formula to find equations for $x - \frac{\pi}{4}$.
3. Simplify by factorising $\frac{\pi}{12}$.

WRITE

$\tan\frac{\pi}{3} = \sqrt{3}$

$x - \frac{\pi}{4} = n\pi + \frac{\pi}{3}, n \in Z$

$x = n\pi + \frac{\pi}{3} + \frac{\pi}{4}$

$= n\pi + \frac{7\pi}{12}$

$x = \frac{\pi}{12}(12n + 7), n \in Z$

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Units 1 & 2 › Topic 5 › Chapter 2 › Concept 8

The tangent function Summary screen and practice questions

Exercise 8.4 The tangent function

Technology free

1. Sketch the following graphs for $0° \leq x \leq 360°$, stating the period and asymptotes.
   - **a.** $y = 2\tan x$
   - **b.** $y = -\tan x$
   - **c.** $y = \tan x + 1$
   - **d.** $y = \tan(x + 30°)$

2. **WE 22** Sketch the following graphs over the domain $-2\pi \leq x \leq 2\pi$, stating the period and asymptotes.
   - **a.** $y = 3\tan x + 1$
   - **b.** $y = 4\tan\dfrac{x}{2}$
   - **c.** $y = -4\tan\dfrac{x}{2} + 3$
   - **d.** $y = 2\tan\left(x + \dfrac{\pi}{6}\right) + 1$
   - **e.** $y = 2\tan\left(\dfrac{x + \pi}{4}\right) - 3$
   - **f.** $y = 0.5\tan\left(2x + \dfrac{\pi}{3}\right) - 1$

Technology active

3. Solve the following over the domain $0° \leq x \leq 360°$, correct to the nearest degree.
   - **a.** $\tan\theta = 5.72$
   - **b.** $\tan\theta = -0.837$
   - **c.** $\tan\theta = 3.421$
   - **d.** $\tan\theta = -1.871$
4. Determine exact solutions to each of the following trigonometric equations over the domain $0° \leq \theta \leq 360°$.
   - **a.** $\tan\theta = \sqrt{3}$
   - **b.** $\tan\theta = -\dfrac{\sqrt{3}}{3}$
   - **c.** $\tan\theta = 1$
   - **d.** $\tan\theta = \dfrac{1}{\sqrt{3}}$
5. **WE 23** Solve each of the following over the domain $0 \leq x \leq 2\pi$. Give your answers correct to 2 decimal places.
   - **a.** $\tan x = 1.5$
   - **b.** $\tan x = -0.39$
   - **c.** $\tan x = 6.32$
   - **d.** $\tan x = -1.4$
6. Solve each of the following over the domain $0 \leq x \leq 2\pi$.
   - **a.** $\tan x = 1$
   - **b.** $\tan x = -\sqrt{3}$
   - **c.** $\tan x = \dfrac{\sqrt{3}}{3}$
   - **d.** $\tan x = -1$
7. Solve each of the following equations over the domain $0 \leq x \leq 4\pi$ to 2 decimal places.
   - **a.** $\tan x = 0.58$
   - **b.** $\tan x = -2.1$
   - **c.** $\tan x = 1.57$
   - **d.** $\tan x = -3.87$
8. Solve each of the following equations over the domain $-2\pi \leq x \leq 2\pi$ to 2 decimal places.
   - **a.** $\tan x = -0.4776$
   - **b.** $\tan x = 3$
   - **c.** $\tan x = 5.421$
   - **d.** $\tan x = -2.67$
9. Solve $\tan\theta = 0$ over the domain $0° \leq \theta \leq 360°$.
10. Solve $\tan x = 0$ over the domain $-2\pi \leq x \leq 2\pi$

11. Determine the exact solutions to each of the following equations over the domain $0 \leq x \leq 2\pi$.

a. $2\tan x = 2$ **b.** $\sqrt{3}\tan x + 3 = 0$

c. $3\tan x + 3 = 0$ **d.** $4\tan x = 0$

12. Solve each of the following equations over the domain $0 \leq x \leq 2\pi$, to 2 decimal places.

a. $2\tan x - 7 = 0$ **b.** $3\tan x + 9 = 0$

c. $3\tan x + 1 = 0$ **d.** $\tan x + 3 = 1$

13. Determine the exact solutions to the following trigonometric equations over the domain $-2\pi \leq x \leq 2\pi$.

a. $\tan x + \sqrt{3} = 0$ **b.** $\sqrt{3}\tan x - 1 = 0$

c. $2\tan x - 2 = 0$ **d.** $\dfrac{1}{\tan x} = \sqrt{3}$

14. **WE 24** Determine exact solutions to the following equations over the domain $0 \leq x \leq 2\pi$.

a. $\sqrt{3}\tan 2x - 1 = 0$ **b.** $\tan\dfrac{x}{2} = 1$

c. $\tan 3x = \sqrt{3}$ **d.** $\sqrt{3}\tan\dfrac{2x}{3} + 1 = 0$

15. Determine x over the domain $0 \leq x \leq 2\pi$ for each of the following equations.

a. $\sqrt{3}\tan\left(2x + \dfrac{\pi}{4}\right) = 1$ **b.** $4\tan\left(3x + \dfrac{\pi}{2}\right) + 4 = 0$

c. $3 - \sqrt{3}\tan\left(2x + \dfrac{\pi}{6}\right) = 0$ **d.** $\sqrt{2} = \sqrt{2}\tan \pi x$

16. **WE 25** Determine the general solutions to the following equations.

a. $\tan x = \sqrt{3}$ **b.** $\tan x = -1$

c. $3\tan x = \sqrt{3}$ **d.** $1 - \tan x = 0$

17. Determine the general solutions to the following equations.

a. $\tan 2x = -\dfrac{\sqrt{3}}{3}$ **b.** $2\tan\left(x - \dfrac{\pi}{6}\right) = 2$

c. $\sqrt{3}\tan\left(2x + \dfrac{\pi}{4}\right) = 3$ **d.** $3\tan\left(3x - \dfrac{2\pi}{3}\right) + 3 = 0$

18. Calculate all solutions between 0 and 2π for the following equations. Give exact values where appropriate; otherwise, give answers correct to 4 decimal places.

a. $\sin x = \cos x$ **b.** $\sin 2x = \cos 2x$

c. $\sin 2x = \sqrt{3}\cos 2x$ **d.** $\sqrt{3}\sin 3x = \cos 3x$

e. $\sin 3x + 2\cos 3x = 0$ **f.** $\sin x + 3\cos x = 0$

8.5 The reciprocal functions

8.5.1 Naming the reciprocal functions

The reciprocal of the sine function is called the **cosecant function**; it may be abbreviated to cosec or csc. It is defined as $\text{cosec}\,(x) = \dfrac{1}{\sin(x)}$, provided that $\sin(x) \neq 0$.

The reciprocal of the cosine function is called the **secant function**, often abbreviated to sec. It is defined as $\sec(x) = \dfrac{1}{\cos(x)}$, provided that $\cos(x) \neq 0$.

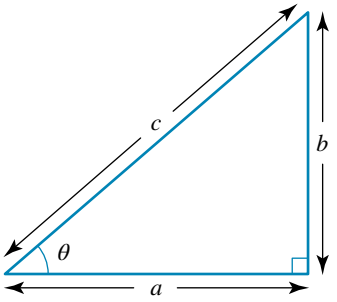
The reciprocal of the tangent function is called the **cotangent function**, often abbreviated to cot. It is defined as $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$, provided that $\sin(x) \neq 0$.

Note that the reciprocal functions are not the inverse trigonometric functions.
These functions can also be defined in terms of right-angled triangles.

$$\text{cosec}(x) = \frac{1}{\sin(x)} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{c}{b}$$

$$\sec(x) = \frac{1}{\sin(x)} = \frac{1}{\frac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{c}{a}$$

$$\cot(x) = \frac{1}{\tan(x)} = \frac{1}{\frac{\text{opposite}}{\text{adjacent}}} = \frac{\text{adjacent}}{\text{opposite}} = \frac{a}{b}$$



8.5.2 Exact values

The exact values for the reciprocal trigonometric functions for angles that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (or 30° and 45°) can be found from the corresponding trigonometric values by finding the reciprocal. Note that we may need to simplify the resulting expression or rationalise the denominator.

WORKED EXAMPLE 26

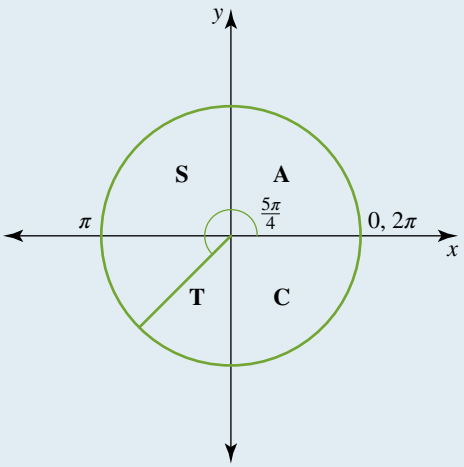
Determine the exact value of $\text{cosec}\left(\frac{5\pi}{4}\right)$.

THINK

1. Rewrite $\text{cosec}\left(\frac{5\pi}{4}\right)$ by recalling $\text{cosec}(\theta) = \frac{1}{\sin(\theta)}$.
2. $\frac{5\pi}{4}$ is in quadrant 3. Sine is negative in that quadrant.

WRITE

$$\text{cosec}\left(\frac{5\pi}{4}\right) = \frac{1}{\sin\left(\frac{5\pi}{4}\right)}$$



$$\text{cosec}\left(\frac{5\pi}{4}\right) = \frac{1}{\sin\left(\pi + \frac{\pi}{4}\right)}$$

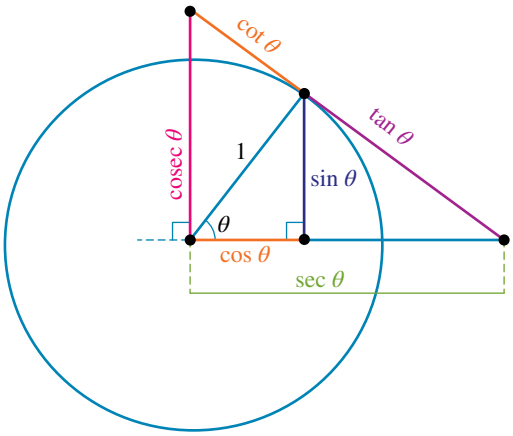
$$= \frac{1}{-\sin\left(\frac{\pi}{4}\right)}$$

3. Use $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and simplify the fraction.

$$\begin{aligned}\operatorname{cosec}\left(\frac{5\pi}{4}\right) &= \frac{1}{-\frac{1}{\sqrt{2}}} \\ &= 1 \times \frac{-\sqrt{2}}{1} \\ &= -\sqrt{2}\end{aligned}$$

In a circle of radius 1, for an acute angle θ, it is possible to locate lengths that are $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$.

Through the use of right-angled triangles and the unit circle, if one trigonometric ratio for an angle is known, it is possible to calculate all of the ratios.



WORKED EXAMPLE 27

If $\operatorname{cosec}\theta = \frac{7}{4}$ and $\frac{\pi}{2} \leq \theta \leq \pi$, determine the exact value of $\cot\theta$.

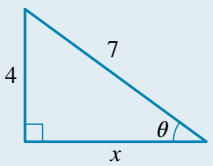
THINK

1. Recall the definition $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ and solve for $\sin\theta$.

WRITE

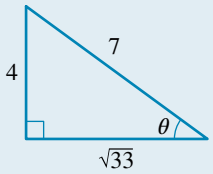
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\frac{7}{4} = \frac{1}{\sin\theta}$$
$$\sin\theta = \frac{4}{7}$$

2. Recall that $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.
   Draw a right-angled triangle and mark an angle with an opposite side length of 4 and a hypotenuse of length 7. Label the unknown side length as x.

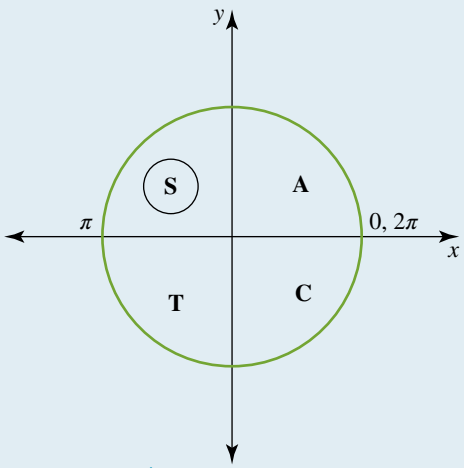


3. Use Pythagoras' theorem to find the unknown side length.

$$x = \sqrt{7^2 - 4^2}$$
$$= \sqrt{49 - 16}$$
$$= \sqrt{33}$$



4. Given that $\frac{\pi}{2} < \theta < \pi$, θ is in the second quadrant, and although $\sin\theta$ is positive, $\tan\theta$ is negative.
Determine $\tan\theta$ using the value of x from step 3.



$$\tan\theta = -\frac{4}{\sqrt{33}}$$

5. Use $\tan\theta$ to calculate $\cot\theta$.

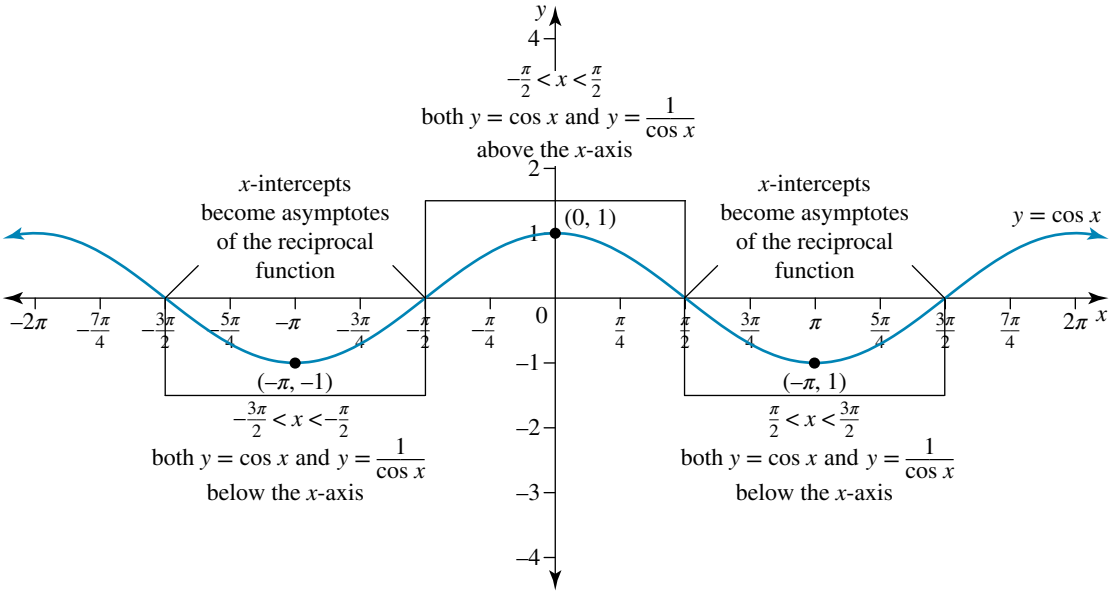
$$\cot\theta = \frac{1}{\tan\theta}$$
$$= \frac{1}{-\frac{4}{\sqrt{33}}}$$
$$= -\frac{\sqrt{33}}{4}$$
$$\therefore \cot\theta = -\frac{\sqrt{33}}{4}$$

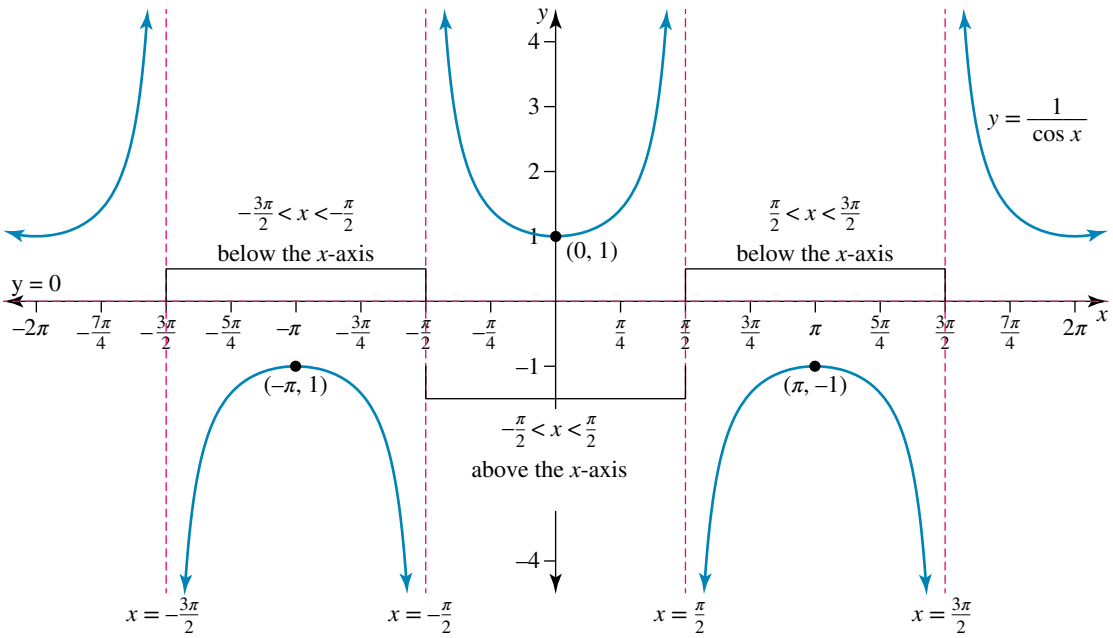
8.5.3 Sketching the reciprocal functions

In Chapter 7, we looked at how the graph of $f(x)$ can be used to find the graph of $\frac{1}{f(x)}$. This method will be used to graph $\sec x = \frac{1}{\cos x}$, $\operatorname{cosec} x = \frac{1}{\sin x}$ and $\cot x = \frac{1}{\tan x}$.

The graph of $y = \sec x$

Consider the graph of $y = \cos x$.

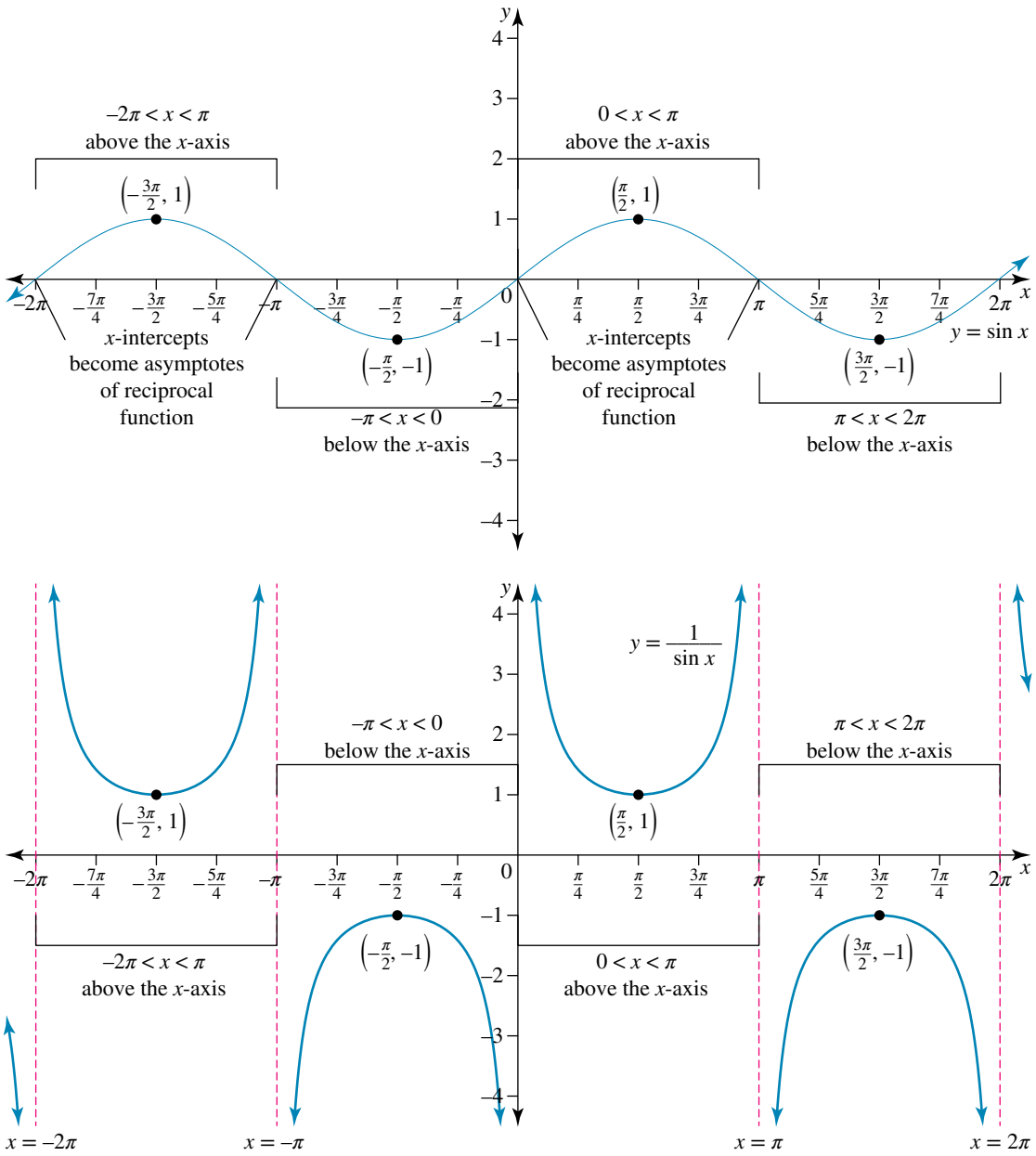




The graph of $y = \operatorname{cosec} x$

In a similar fashion, the graph of $y = \sin x$ can be used to determine the graph of $y = \dfrac{1}{\sin x}$ (also known as $y = \operatorname{cosec} x$).

Consider the graph of $y = \sin x$.



WORKED EXAMPLE 28

Use the graph of $y = 2\cos x$ to sketch $y = \frac{1}{2}\sec x$ over the domain $-2\pi \leq x \leq 2\pi$.

THINK

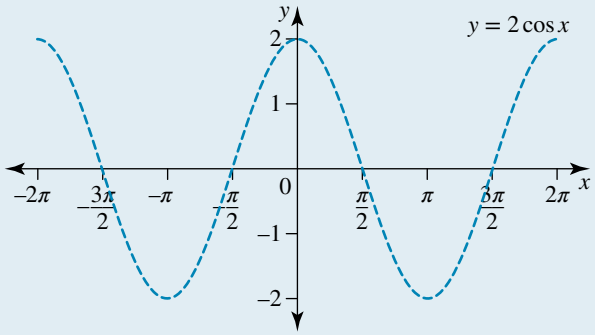
1. Rewrite $y = \frac{1}{2}\sec x$ in terms of $\cos x$.

WRITE

$$y = \frac{1}{2}\sec x$$
$$= \frac{1}{2} \times \frac{1}{\cos x}$$
$$= \frac{1}{2\cos x}$$

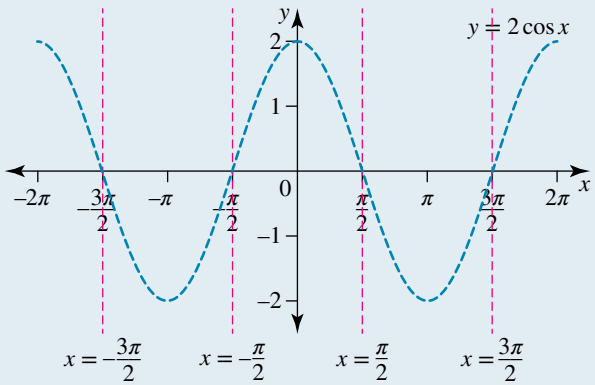
2. Sketch $y = 2\cos x$ by identifying the amplitude, period, horizontal shift and vertical shift. Identify the end points at $x = -2\pi$ and $x = 2\pi$.

$y = 2\cos x$
Amplitude: 2 Period: 2π
Horizontal shift: 0 Vertical shift: 0
$2\cos(-2\pi) = 2\cos(2\pi) = 2$

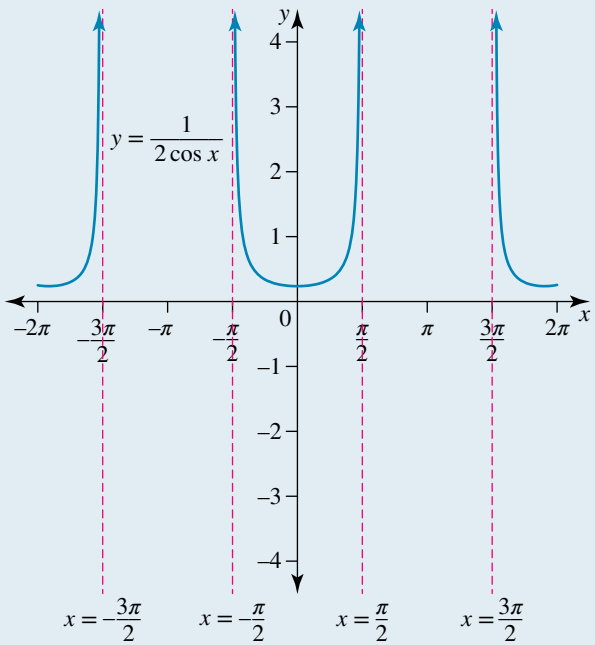


3. Find the x-intercepts and hence the vertical asymptotes for the reciprocal graph.

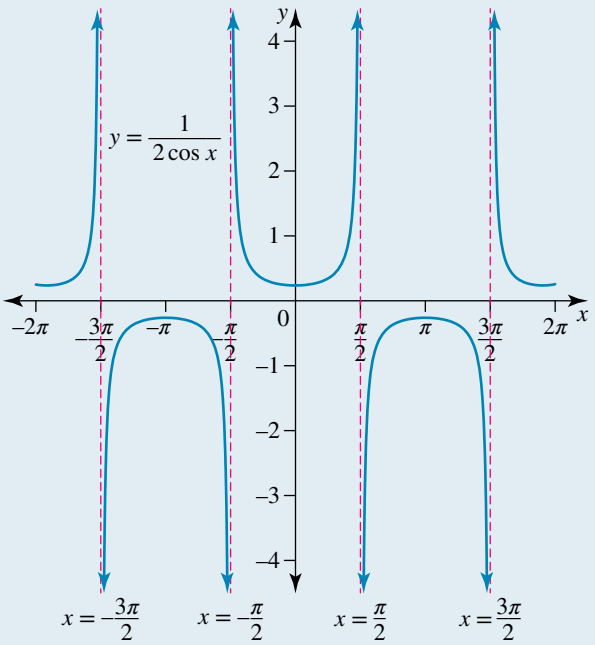
The x-intercepts occur at
$x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
These will be the vertical asymptotes for the reciprocal function.



4. The graph of $y = 2\cos x$ is above the x-axis in the regions $-2\pi \leq x < -\frac{3\pi}{2}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x \leq 2\pi$. The graph of $y = \frac{1}{2\cos x}$ will also be above the x-axis in these regions.
A maximum value of $y = 2$ is reached in the original graph, meaning that a minimum of $y = \frac{1}{2}$ will be reached in the reciprocal function.



5. The graph of $y = 2\cos x$ is below the x-axis in the regions $-\frac{3\pi}{3} < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$.
Therefore, $y = \frac{1}{2\cos x}$ is also below the x-axis in these regions. The minimum of $y = -2$ will become a maximum of $y = -\frac{1}{2}$.



WORKED EXAMPLE 29

Use the graph of $y = \sin\left(\frac{x}{2}\right)$ to sketch $y = \text{cosec}\left(\frac{x}{2}\right)$ over the domain $-2\pi \leq x \leq 2\pi$.

THINK

1. Rewrite $y = \text{cosec}\left(\frac{x}{2}\right)$ in terms of $\sin\left(\frac{x}{2}\right)$.

2. Sketch $y = \sin\left(\frac{x}{2}\right)$ by identifying the amplitude, period, horizontal shift and vertical shift.
Identify the end points at $x = -2\pi$ and $x = 2\pi$.

WRITE

$y = \text{cosec}\left(\frac{x}{2}\right)$

$= \frac{1}{\sin\left(\frac{x}{2}\right)}$

$y = \sin\left(\frac{x}{2}\right)$

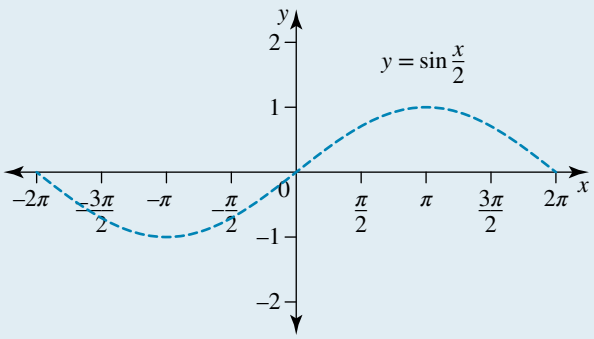
Amplitude: 1

Period: $\frac{2\pi}{\frac{1}{2}} = 4\pi$

Horizontal shift: 0

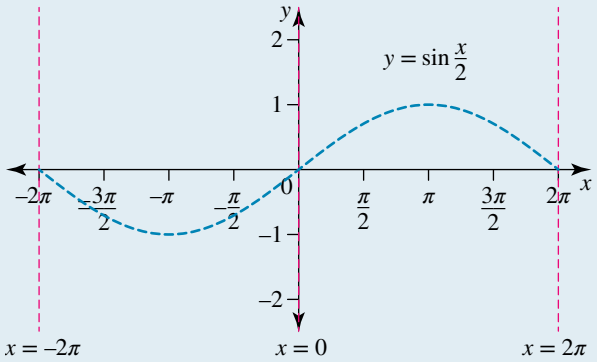
Vertical shift: 0

$2\sin(-2\pi) = 2\sin(2\pi) = 0$

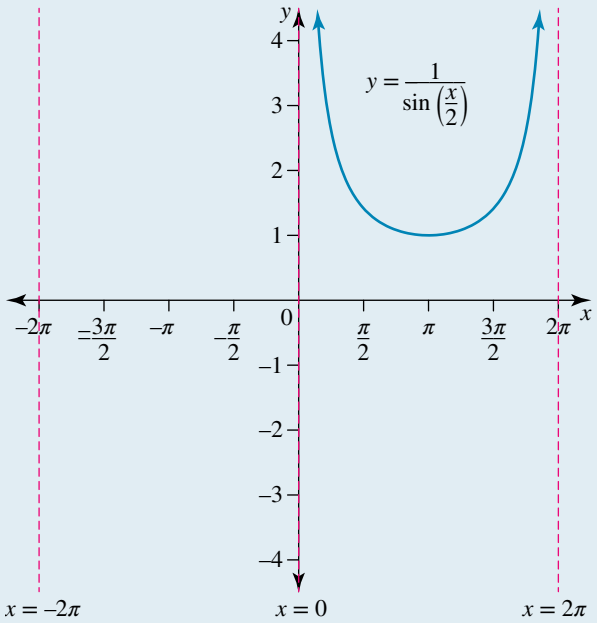


3. Find the x-intercepts and hence the vertical asymptotes for the reciprocal graph.

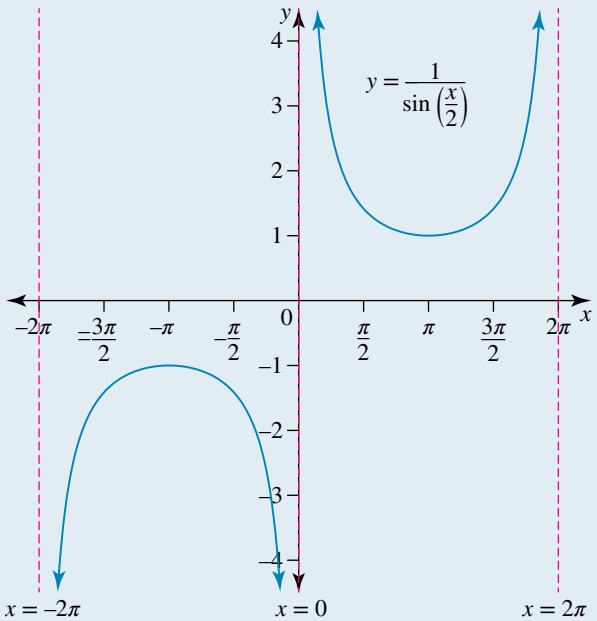
The x-intercepts occur at $x = -2\pi$, $x = 0$ and $x = 2\pi$. These will be the vertical asymptotes for the reciprocal function.



4. The graph of $y = \sin\left(\frac{x}{2}\right)$ is above the x-axis in the region $0 < x \leq 2\pi$. The graph of $y = \dfrac{1}{\sin\left(\frac{x}{2}\right)}$ will also be above the x-axis in these regions.
A maximum value of $y = 1$ is reached in the original graph, meaning that a minimum of $y = 1$ will be reached in the reciprocal function.

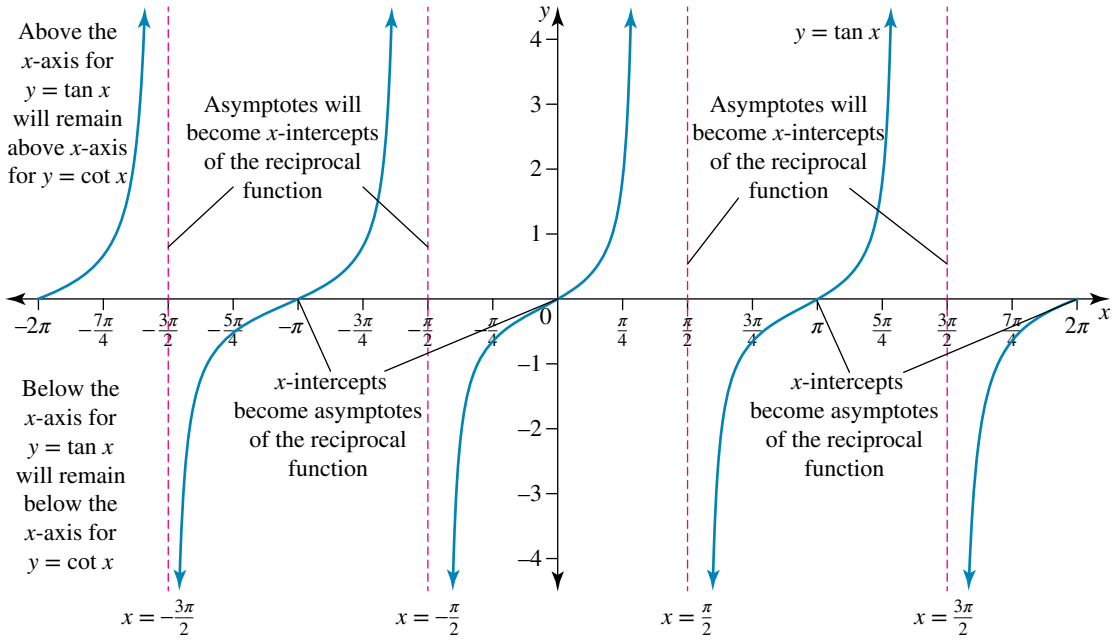


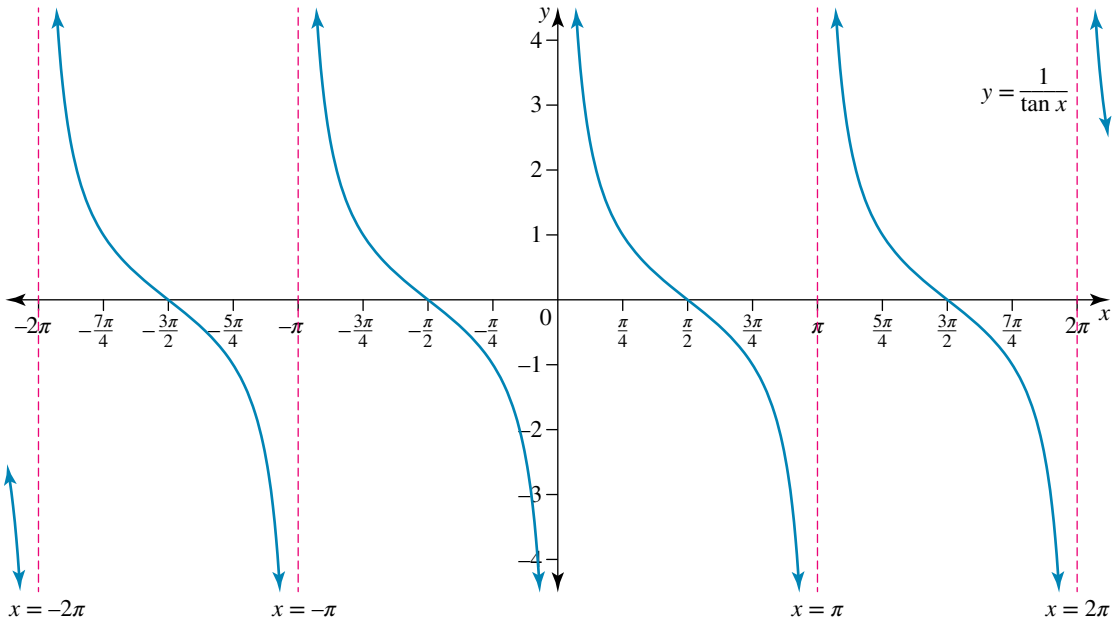
5. The graph of $y = \sin\left(\frac{x}{2}\right)$ is below the x-axis in the region $-2\pi \leq x < 0$.
Therefore, $y = \dfrac{1}{\sin\left(\frac{x}{2}\right)}$ is also below the x-axis in these regions. The minimum of $y = -1$ will become a maximum of $y = -1$.



The graph of $y = \cot x$

The graph of $y = \tan x$ can also be used to find the graph of $y = \dfrac{1}{\tan x}$ (also known as $y = \cot x$).





WORKED EXAMPLE 30

Use the graph of $y = \tan\left(\dfrac{x}{2}\right)$ to sketch $y = \cot\left(\dfrac{x}{2}\right)$ over the domain $-2\pi \leq x \leq 2\pi$.

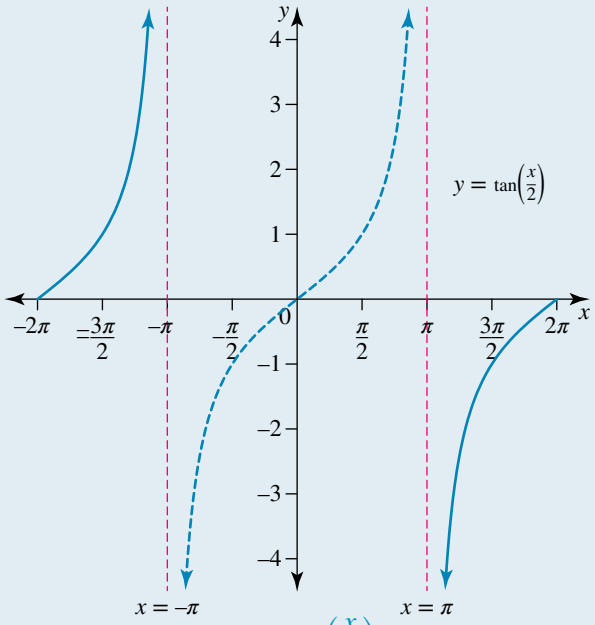
THINK	WRITE
1. Rewrite $y = \cot\left(\dfrac{x}{2}\right)$ in terms of $\tan\left(\dfrac{x}{2}\right)$.	$y = \cot\left(\dfrac{x}{2}\right)$ $= \dfrac{1}{\tan\left(\frac{x}{2}\right)}$

2. Sketch $y = \tan\left(\frac{x}{2}\right)$ by identifying the period, horizontal shift and vertical shift. Identify the end points at $x = -2\pi$ and $x = 2\pi$.

$y = \tan\left(\frac{x}{2}\right)$ Period: $\frac{\pi}{\frac{1}{2}} = 2\pi$

Horizontal shift: 0 Vertical shift: 0

$\tan(-2\pi) = \tan 2\pi = 0$

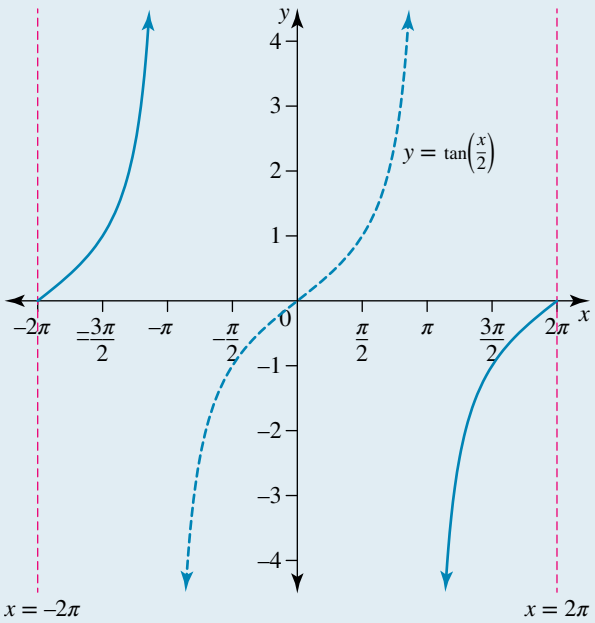


3. The asymptotes of $y = \tan\left(\frac{x}{2}\right)$ will become the x-intercepts of $y = \cot\left(\frac{x}{2}\right)$.

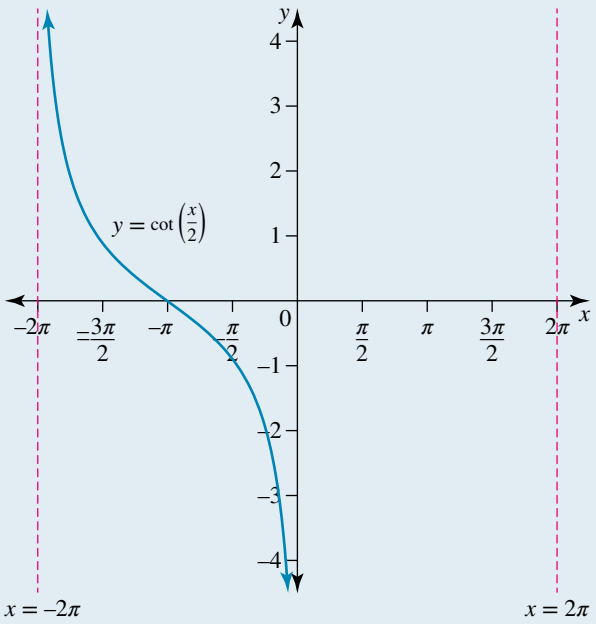
The x-intercepts of $y = \cot\left(\frac{x}{2}\right)$ will be $x = \pm\pi$.

4. Find the x-intercepts and hence the vertical asymptotes for the reciprocal graph.

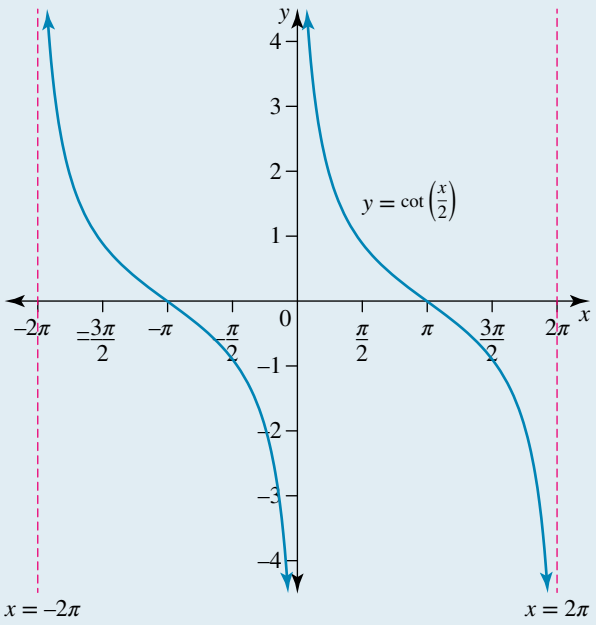
The x-intercepts of $y = \tan\left(\frac{x}{2}\right)$ are $x = -2\pi$, $x = 0$ and $x = 2\pi$. These will become the vertical asymptotes of $y = \cot\left(\frac{x}{2}\right)$.



5. If we consider the region $-2\pi \leq x < 0$, the graph of $y = \tan\left(\frac{x}{2}\right)$ is initially above the x-axis between $x = -2\pi$ and $x = -\pi$ and is then below the x-axis. This will also be true for the reciprocal function.



6. In a similar fashion, the graph for $x = 0$ to $x = 2\pi$ can be obtained.



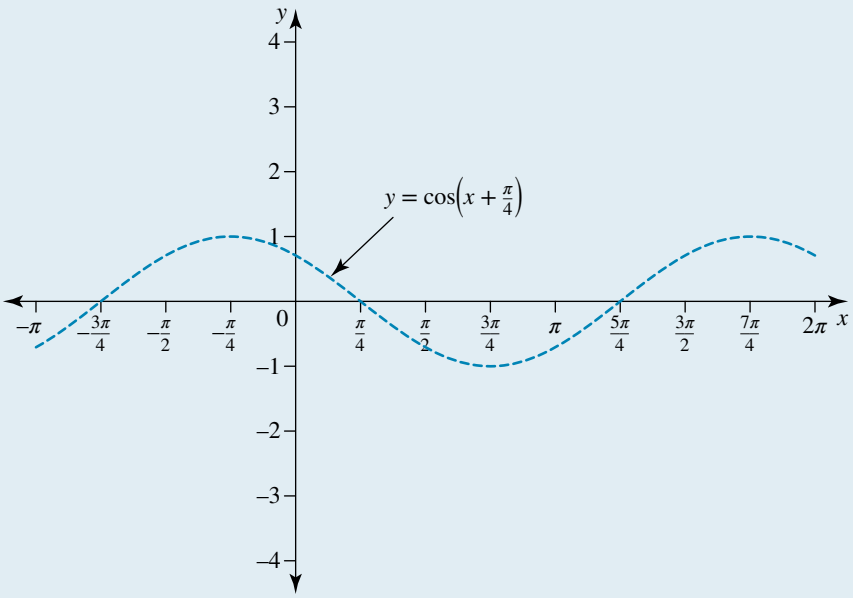
Transformations of the reciprocal functions

Transformations can also be applied to the reciprocal functions.

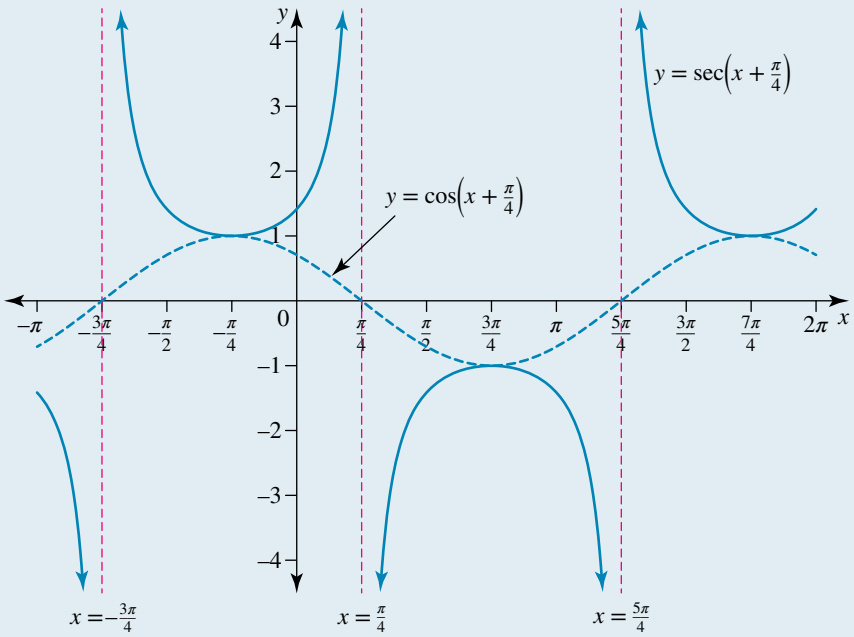
WORKED EXAMPLE 31

Sketch the graph of $y = \sec\left(x + \frac{\pi}{4}\right) + 1$ over the domain $[-\pi, 2\pi]$.

THINK	WRITE
1. Sketch the graph of $y = \cos\left(x + \frac{\pi}{4}\right)$ to find the graph of $y = \sec\left(x + \frac{\pi}{4}\right)$.	$y = \cos\left(x + \frac{\pi}{4}\right)$ Amplitude: 1 Period: 2π Horizontal shift: $\frac{\pi}{4}$ left Vertical shift: 0 End points: $\cos\left(-\pi + \frac{\pi}{4}\right) = \cos\left(-\frac{3\pi}{4}\right)$ $= -\frac{\sqrt{2}}{2}$ $\cos\left(2\pi + \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ $= \frac{\sqrt{2}}{2}$



2. Now consider $y = \sec\left(x + \frac{\pi}{4}\right)$. The x-intercepts of $y = \cos\left(x + \frac{\pi}{4}\right)$ will become the asymptotes. Sections of the graph above the x-axis will remain above the x-axis, and sections below the x-axis will remain below the x-axis.	$y = \sec\left(x + \frac{\pi}{4}\right)$ The asymptotes will be $x = -\frac{\pi}{4}$, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$. The graph of $y = \cos\left(x + \frac{\pi}{4}\right)$ and $y = \sec\left(x + \frac{\pi}{4}\right)$:

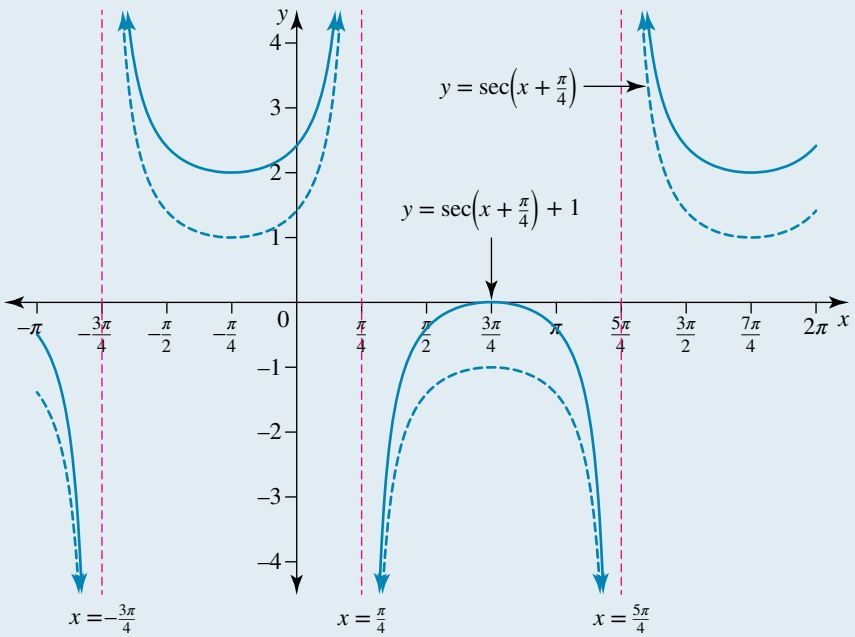


3. To graph

$y = \sec\left(x + \frac{\pi}{4}\right) + 1$,

move $y = \sec\left(x + \frac{\pi}{4}\right)$ up 1.

The graph of $y = \sec\left(x + \frac{\pi}{4}\right) + 1$:



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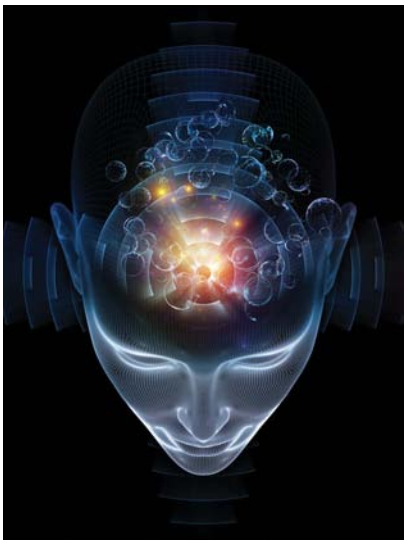
Units 1 & 2 > Topic 5 > Chapter 2 > Concepts 9 and 10

The reciprocal functions Summary screen and practice questions
Graphs of reciprocal functions Summary screen and practice questions

Exercise 8.5 The reciprocal functions

Technology free



1. **WE 26** Determine the exact value of each of the following.
   a. $\operatorname{cosec}\left(\frac{2\pi}{3}\right)$
   b. $\operatorname{cosec}\left(\frac{\pi}{3}\right)$
   c. $\operatorname{cosec}\left(\frac{5\pi}{6}\right)$
   d. $\operatorname{cosec}\left(\frac{7\pi}{4}\right)$
2. Determine the exact value of each of the following.
   a. $\sec\left(\frac{\pi}{6}\right)$
   b. $\sec\left(-\frac{7\pi}{6}\right)$
   c. $\sec\left(\frac{4\pi}{3}\right)$
   d. $\sec\left(-\frac{7\pi}{4}\right)$
3. Determine the exact value of each of the following.
   a. $\cot\left(\frac{\pi}{6}\right)$
   b. $\cot\left(\frac{2\pi}{3}\right)$
   c. $\cot\left(\frac{7\pi}{4}\right)$
   d. $\cot\left(\frac{11\pi}{6}\right)$
4. a. **WE 27** If $\operatorname{cosec}\theta = \frac{5}{2}$ and $\frac{\pi}{2} < \theta < \pi$, determine the exact value of $\cot\theta$.
   b. If $\cot\theta = 4$ and $\pi < \theta < \frac{3\pi}{2}$, determine the exact value of $\sec\theta$.
   c. If $\sin(\theta) = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$, determine the exact value of $\sec\theta$.
   d. If $\operatorname{cosec}\theta = 4$ and $\frac{\pi}{2} < \theta < \pi$, determine the exact value of $\cot\theta$.
   e. If $\cos\theta = -\frac{3}{7}$ and $\pi < \theta < \frac{3\pi}{2}$, determine the exact value of $\cot\theta$.
   f. If $\sec\theta = -\frac{5}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, determine the exact value of $\operatorname{cosec}\theta$.
5. a. If $\cos(x) = \frac{3}{7}$ and $\frac{3\pi}{2} < x < 2\pi$, determine the exact value of $\operatorname{cosec}(x)$.
   b. If $\sec(x) = \frac{8}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, determine the exact value of $\cot(x)$.
   c. If $\operatorname{cosec}(x) = 4$ and $\frac{\pi}{2} < x < \pi$, determine the exact value of $\tan(x)$.
   d. If $\cot(x) = -\frac{5}{6}$ and $\frac{\pi}{2} < x < \pi$, determine the exact value of $\operatorname{cosec}(x)$.
   e. If $\sec(x) = -7$ and $\pi < x < \frac{3\pi}{2}$, determine the exact value of $\cot(x)$.
   f. If $\cot(x) = 4$ and $\pi < x < \frac{3\pi}{2}$, determine the exact value of $\operatorname{cosec}(x)$.
6. a. **WE 28** Use the graph of $y = 4\cos x$ to sketch $y = \frac{1}{4}\sec x$ over the domain $-2\pi \leq x \leq 2\pi$.
   b. Use the graph of $y = 2\sin x$ to sketch $y = \frac{1}{2}\operatorname{cosec} x$ over the domain $-2\pi \leq x \leq 2\pi$.
7. Sketch $y = \frac{1}{4}\operatorname{cosec} x$ over the domain $[-\pi, \pi]$.
8. a. **WE 29** Use the graph of $y = \sin 2x$ to sketch $y = \operatorname{cosec} 2x$ over the domain $-2\pi \leq x \leq 2\pi$.
   b. Use the graph of $y = \cos 2x$ to sketch $y = \sec 2x$ over the domain $-2\pi \leq x \leq 2\pi$.

9. Sketch $y = \sec\dfrac{x}{2}$ over the domain $[-\pi, \pi]$.

10. a. **WE 30** Use the graph of $y = \tan 2x$ to sketch $y = \cot 2x$ over the domain $-2\pi \leq x \leq 2\pi$.

    b. Use the graph of $y = \tan 3x$ to sketch $y = \cot 3x$ over the domain $-2\pi \leq x \leq 2\pi$.

11. Sketch $y = \cot\left(\dfrac{x}{3}\right)$ over the domain $[-3\pi, 3\pi]$.

12. a. **WE 31** Sketch the graph of $y = \frac{1}{2}\sec\left(x + \dfrac{\pi}{4}\right) - 1$ over the domain $[-\pi, 2\pi]$.

    b. Sketch the graph of $y = \cot\left(x + \dfrac{\pi}{4}\right) + 1$ over the domain $[-\pi, 2\pi]$.

13. Sketch the following over the domain of $[-\pi, \pi]$.

    a. $y = 2\sec x - 1$

    b. $y = \dfrac{2}{\sin\left(x + \frac{\pi}{4}\right)}$

    c. $y = 0.25\,\text{cosec}\left(x - \dfrac{\pi}{4}\right)$

    d. $y = 3\sec\left(2x + \dfrac{\pi}{2}\right) - 2$

14. a. If $\text{cosec}\,(x) = \dfrac{p}{q}$ where $p, q \in R^+$ and $\dfrac{\pi}{2} < x < \pi$, determine $\sec(x) - \cot(x)$.

    b. If $\sec x = \dfrac{a}{b}$ where $a, b \in R^+$ and $\dfrac{3\pi}{2} < x < 2\pi$, determine $\cot(x) - \text{cosec}\,(x)$.

Technology active

15. Use the graph of $y = \sin x + 2$ to sketch $y = \dfrac{1}{\sin x + 2}$ over the domain $\left[-\dfrac{5\pi}{2}, \dfrac{5\pi}{2}\right]$.

16. a. Use the graph of $y = \cos^2 x$ to sketch $y = \dfrac{1}{\cos^2 x}$ over the domain $\left[-\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right]$. Sketch both graphs on the same set of axes.

    b. Hence, determine the graph of $y = \tan^2 x$ for the same domain.

8.6 Modelling periodic functions

8.6.1 Periodic phenomena in the real-world

Many situations arise in science and nature where relationships between two variables exhibit periodic behaviour. Tide heights, sound waves, biorhythms and ovulation cycles are examples.

In these situations trigonometric functions can be used to model the behaviour of the variables. The independent variable, x, is often a measurement such as time. When modelling with trigonometric functions you should work in radians unless otherwise instructed.

WORKED EXAMPLE 32

***E. coli* is a type of bacterium. Its concentration, P parts per million (ppm), at a particular beach over a 12-hour period t hours after 6 am, is described by the function:**

$$P = 0.05\sin\frac{\pi t}{12} + 0.1.$$

a. Determine the i maximum and ii minimum *E. coli* levels at this beach.

b. What is the level at 3 pm?

THINK	WRITE
a. Write the function.	**a.** $P = 0.05 \sin \frac{\pi t}{12} + 0.1$
i. 1. The maximum value of the sine function is 1.	**i.** The maximum P occurs when $\sin \frac{\pi t}{12} = 1$.
2. Substitute $\sin \frac{\pi t}{12} = 1$ into the equation for P and evaluate.	Maximum $P = 0.05(1) + 0.1$ $= 0.05$
3. State the solution.	The maximum *E. coli* level is 0.15 ppm.
ii. 1. The minimum value of the sine function is -1.	**ii.** The minimum P occurs when $\sin \frac{\pi t}{12} = -1$.
2. Substitute $\sin \frac{\pi t}{12} = -1$ into the equation for P and evaluate.	Minimum $P = 0.05(-1) + 0.1$ $= 0.05$
3. State the solution.	The minimum *E. coli* level is 0.05 ppm.
b. 1. At 3 pm it is 9 hours since 6 am.	**b.** At 3 pm, $t = 9$.
2. Substitute $t = 9$ into the equation for P, and evaluate.	When $t = 9$, $P = 0.05 \sin \frac{9\pi}{12} + 0.1$ $= 0.05 \sin \frac{3\pi}{12} + 0.1$ $= 0.05 \frac{1}{\sqrt{2}} + 0.1$ $= 0.035 + 0.1$ $= 0.135$
3. State the solution.	The *E. coli* level at 3 pm is approximately 0.135 ppm.

WORKED EXAMPLE 33

The depth of the tide, d metres, is given by $d(t) = 3 + 2.5 \sin\left(\frac{\pi t}{12}\right)$, where t is the time in hours and $t \geq 0$.

a. Determine the depth after 2 hours.

b. Determine the maximum depth and the first two times this occurs.

c. Determine the minimum depth and the first two times this occurs.

d. Determine the period, amplitude and range of the function $d(t)$.

e. Determine the first four times when the depth is 1.75 metres.

f. Sketch the graph of d versus t, showing 2 cycles.

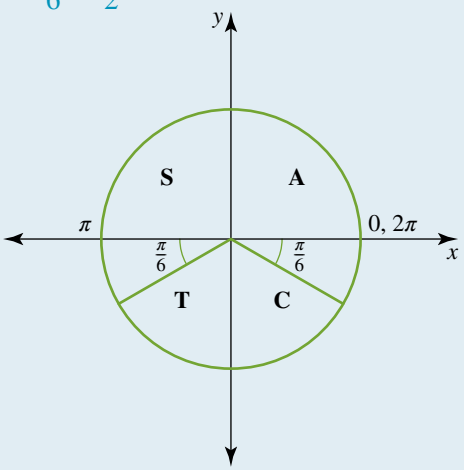
THINK	WRITE
a. Two hours is at $t = 2$. Substitute $t = 2$ and find $d(2)$.	$d(2) = 3 + 2.5 \sin\left(\frac{\pi}{6}\right) = 4.25$
b. 1. The maximum occurs when $\sin\left(\frac{\pi t}{12}\right) = 1$. Use this to determine the maximum depth.	The maximum occurs when $\sin\left(\frac{\pi t}{12}\right) = 1$. $d = 3 + 2.5 \times 1$ $= 5.5$ The maximum height will be 5.5 m.
2. To find when the maximum occurs, determine when $\sin\left(\frac{\pi t}{12}\right) = 1$.	$\sin\left(\frac{\pi t}{12}\right) = 1$. Use $\sin\frac{\pi}{2} = 1$ $\frac{\pi t}{12} = \frac{\pi}{2}$ or $\frac{\pi t}{12} = 2\pi + \frac{\pi}{2}$ $t = \frac{\pi}{2} \times \frac{12}{\pi}$ $= \frac{5\pi}{2}$ $= 6$ $t = \frac{5\pi}{2} \times \frac{12}{\pi}$ $= 30$ The maximums will occur after 6 and 30 hours.
c. 1. The minimum occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$. Use this to determine the minimum depth.	The minimum occurs when $\sin\left(\frac{\pi t}{12}\right) = -1$ $d = 3 + 2.5 \times -1$ $= 0.5$ The minimum height will be 0.5 m.
2. To find when the minimum occurs, solve $\sin\left(\frac{\pi t}{12}\right) = -1$.	$\sin\left(\frac{\pi t}{12}\right) = -1$. Use $\sin\frac{3\pi}{2} = -1$ $\frac{\pi t}{12} = \frac{3\pi}{2}$ or $\frac{\pi t}{12} = 2\pi + \frac{3\pi}{2}$ $t = \frac{3\pi}{2} \times \frac{12}{\pi}$ $= \frac{7\pi}{2}$ $= 18$ $t = \frac{7\pi}{2} \times \frac{12}{\pi}$ $= 42$ The minimums will occur after 18 and 42 hours.
d. The function is in the form $d(t) = a\sin(b(t + c)) + d$, where A is the amplitude and the period is $T = \frac{2\pi}{b}$. The range of the function is the minimum to the maximum.	$a = 2.5$ The amplitude is 2.5 m. $b = \frac{\pi}{12}$ The period is $T = \frac{2\pi}{b}$ $= 2\pi \times \frac{12}{\pi}$ $= 24$ The period is 24 hours. The range is [0.5, 5.5].

e. Solve $d(t) = 3 + 2.5\sin\left(\frac{\pi t}{12}\right) = 1.75$.

$$3 + 2.5\sin\left(\frac{\pi t}{12}\right) = 1.75$$
$$2.5\sin\left(\frac{\pi t}{12}\right) = 1.75 - 3$$
$$= -1.25$$
$$\sin\left(\frac{\pi t}{12}\right) = \frac{-1.25}{2.5}$$
$$= -\frac{1}{2}$$

f. Use $\sin\frac{\pi}{6} = \frac{1}{2}$ and draw a diagram to identify when $\sin\left(\frac{\pi t}{12}\right) = -\frac{1}{2}$. The solutions for $\frac{\pi t}{12}$ will be in quadrants 3 and 4. Adding 2π to each of the solutions will result in the four required solutions.

$$\sin\frac{\pi}{6} = \frac{1}{2}$$



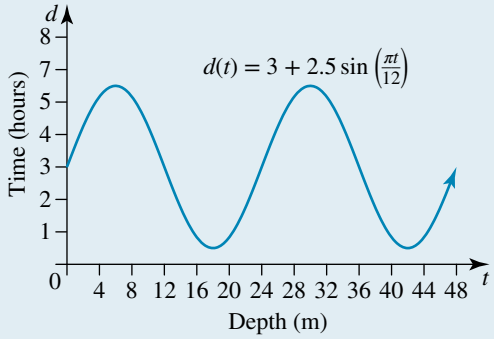
$$\frac{\pi t}{12} = \pi + \frac{\pi}{6} \quad \text{or} \quad \frac{\pi t}{12} = 2\pi - \frac{\pi}{6}$$
$$= \frac{7\pi}{6} \qquad = \frac{11\pi}{6}$$
$$t = \frac{7\pi}{6} \times \frac{12}{\pi} \qquad t = \frac{11\pi}{6} \times \frac{12}{\pi}$$
$$= 14 \qquad = 22$$

$$\frac{\pi t}{12} = \frac{7\pi}{6} + 2\pi \quad \text{or} \quad \frac{\pi t}{12} = \frac{11\pi}{6} + 2\pi$$
$$= \frac{19\pi}{6} \qquad = \frac{23\pi}{6}$$
$$t = \frac{19\pi}{6} \times \frac{12}{\pi} \qquad t = \frac{23\pi}{6} \times \frac{12}{\pi}$$
$$= 38 \qquad = 46$$

The height will be 1.75 m at 14, 22, 38 and 46 hours.

g. Use the period, range, maximum and minimum to sketch the function for 2 cycles.

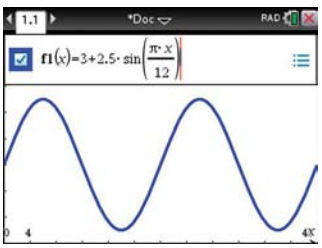
The period is 24 hours, so 2 cycles is 48 hours. The range is [0.5, 5.5].



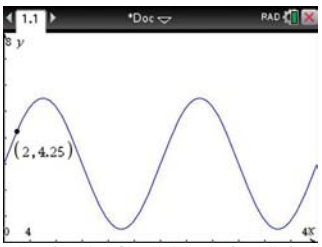
TI | THINK

f.1. On a Graphs page, complete the entry line for function 1 as: $f1(x) = 3+2.5 \sin\left(\frac{\pi x}{12}\right)$ then press ENTER.

WRITE



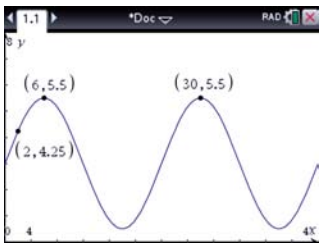
a.1. Press MENU, then select:
5: Trace
1: Graph Trace.
Type '2', then press ENTER twice.



2. The answer appears on the screen.

The depth after 2 hours is 4.25 m.

b.1. To find the maximums, press MENU, then select: 6: Analyze Graph 3: Maximum. Move the cursor to the left of the maximum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the maximum when prompted for the upper bound, then press ENTER. Repeat this step to find the other maximum.



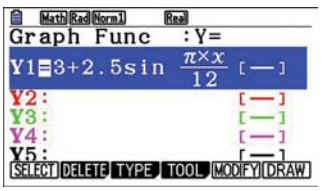
2. The answer appears on the screen.

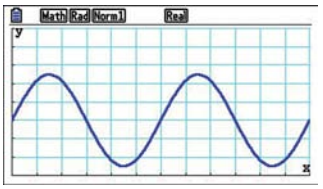
The maximum depth is 5.5 m, and this occurs after 6 and 30 hours.

CASIO | THINK

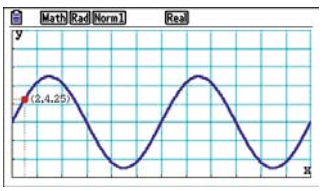
f.1. On a Graph screen, complete the entry line for y1 as: $y1 = 3 + 2.5 \sin\left(\frac{\pi x}{12}\right)$ then press EXE. Select DRAW by pressing F6.

WRITE





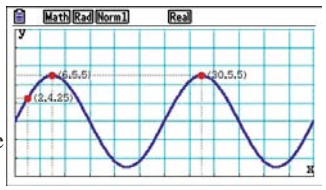
a.1. Select TRACE by pressing SHIFT, then F1. Type '2', then press EXE twice.



2. The answer appears on the screen.

The depth after 2 hours is 4.25 m.

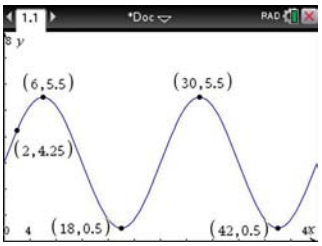
b.1. To find the maximums, select G-Solv by pressing SHIFT, then F5, then select MAX by pressing F2. With the cursor on the first maximum, press EXE, then use the left/right arrows to move to the next maximum and press EXE.



2. The answer appears on the screen.

The maximum depth is 5.5 m, and this occurs after 6 and 30 hours.

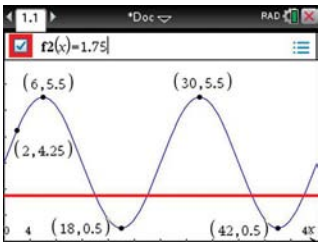
c.1. To find the minimums, press MENU, then select:
6: Analyze Graph
2: Minimum.
Move the cursor to the left of the minimum when prompted for the lower bound, then press ENTER. Move the cursor to the right of the minimum when prompted for the upper bound, then press ENTER.
Repeat this step to find the other minimum.



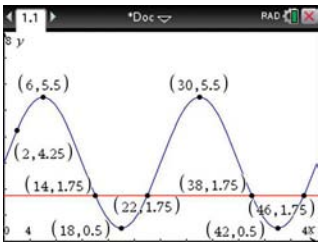
2. The answer appears on the screen.

The minimum depth is 0.5 m, and this occurs after 18 and 42 hours.

e.1. Press TAB to bring up the function entry line and complete the entry line for function 2 as:
$f2(x) = 1.75$
then press EXE.



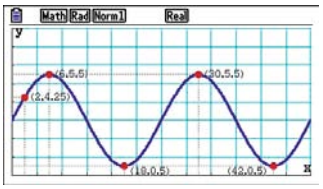
2. To find the points of intersection, press MENU, then select:
6: Analyze Graph
4: Intersection.
Move the cursor to the left of the point of intersection when prompted for the lower bound, then press ENTER. Move the cursor to the right of the point of intersection when prompted for the upper bound, then press ENTER.
Repeat this step to find the other points of intersection.



3. The answer appears on the screen.

The height will be 1.75 m at 14, 22, 38 and 46 hours.

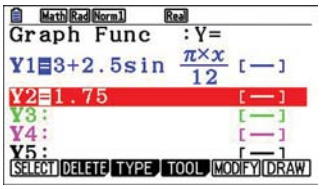
c.1. To find the minimums, select G-Solv by pressing SHIFT, then F5, then select MIN by pressing F3. With the cursor on the first minimum, press EXE, then use the left/right arrows to move to the next minimum and press EXE.

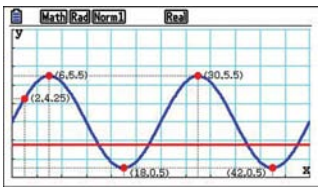


2. The answer appears on the screen.

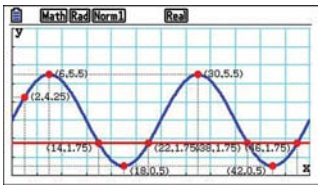
The minimum depth is 0.5 m, and this occurs after 18 and 42 hours.

e.1. Return to the function entry screen and complete the entry line for y2 as:
$y2 = 1.75$
then press EXE.
Select DRAW by pressing F6.





2. To find the points of intersection, select G-Solv by pressing SHIFT, then F5, then select INTSECT by pressing F5. With the cursor on the first point of intersection, press EXE. Use the left/right arrows to move to the next point of intersection and mark it on the graph by pressing EXE.
Repeat this step to find the other points of intersection.



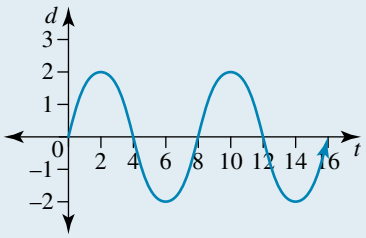
3. The answer appears on the screen.

The height will be 1.75 m at 14, 22, 38 and 46 hours.

Sometimes it is necessary to determine the equation of the trigonometric function that models the situation.

WORKED EXAMPLE 34

A river flows through Tony's property. Tony measures the water level by measuring the distance above or below a marker in the water. Tony lets d be the level of water above or below the marker and t be the number of hours since midnight. Tony's results are shown in the graph.
Express d as a function of t.



THINK	WRITE
1. The graph is in the basic shape of a sine graph.	The graph is of the form $y = a\sin bx$.
2. The amplitude of the graph is 2.	Amplitude $= 2$: $y = 2\sin bx$
3. The period of the function is 4. Put this equal to the period and solve for b.	$\frac{2\pi}{b} = 4$ $2\pi = 4b$ $b = \frac{2\pi}{4}$ $= \frac{\pi}{2}$
4. Substitute into the general form $y = a\sin bx$ and write the equation.	$y = 2\sin\frac{\pi}{2}x$

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Units 1 & 2 > Topic 5 > Chapter 2 > Concept 11

Modelling periodic functions Summary screen and practice questions

Exercise 8.6 Modelling periodic functions

Technology free

1. WE 32 The height above the ground, h metres, of a child on a swing at any time t seconds after being released is:

$$h = 1 + 0.6\cos\frac{\pi t}{2}.$$

Determine:

a. the maximum height of the swing

b. the height after:

i. 3 seconds **ii.** $\frac{4}{3}$ seconds.

Technology active

2. The temperature, T °C, inside a building on a given day is given by the function

$$T = 8 \sin \frac{\pi t}{12} + 18$$

where t is the number of hours after 8 am.

a. What is the maximum temperature in the building and the time at which it first occurs?

b. Determine the temperature at 8 pm.

c. Determine the temperature at 6 pm.

d. Determine the temperature at 12 midnight.

3. The displacement, x mm, of a harp string t seconds after it is initially plucked is modelled by the function

$$x(t) = 12 \sin 20\pi t.$$

a. State the amplitude of the function.

b. State the period of the function.

c. How many vibrations (that is, cycles) will the string complete in 1 second?

d. Determine the displacement after 0.08 seconds.

4. **WE 33** The temperature, T °C, in an office room is given by

$$T(t) = 15 + 3.5 \cos\left(\frac{\pi t}{12}\right)$$

where t is the time in hours and $t \geq 0$.

a. Determine the temperature after 2 hours.

b. Determine the maximum temperature and the first two times this occurs.

c. Determine the minimum temperature and the first two times this occurs.

d. Determine the period, amplitude and range of the function $T(t)$.

e. Determine the first four times when the temperature is 16.75 °C.

f. Sketch the graph of T versus t, showing two cycles.

5. A rod is vibrating. The displacement, y cm, of the end of the rod satisfies $y = 1.8 \cos(10\pi t)$ where t is the time in seconds and $t \geq 0$.

a. Sketch the graph of y versus t, showing two cycles.

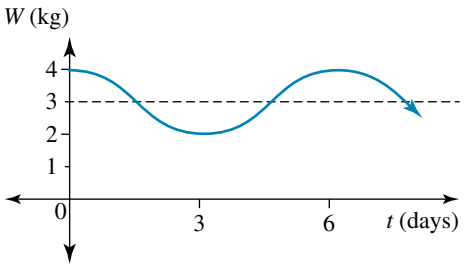
b. Determine the first four times when $y = 0.9$ cm.

6. The motion of a piston satisfies $s = 16 \sin\left(\frac{5\pi t}{4}\right)$, where s is the displacement in cm and t is the time in seconds, $t \geq 0$.

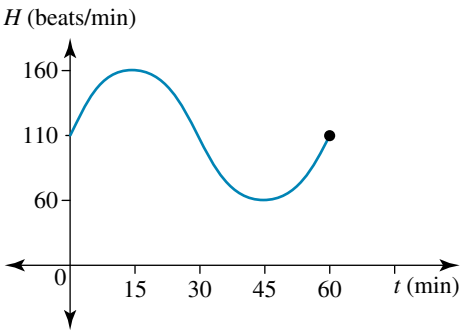
a. Sketch the graph of s versus t, showing two cycles.

b. Determine the first four times when $s = 8$ cm.

7. The voltage, v volts, in a circuit satisfies $v = 120\sqrt{2}\cos(100\pi t)$, where t is the time in seconds and $t \geq 0$.
   a. Sketch the graph of v versus t, showing two cycles.
   b. Determine the first four times when $v = 120$ volts.
8. **WE 34** The mass of a rabbit over a period of time is modelled by the graph.



   a. State i the amplitude and ii the period.
   b. Express W as a function of t.
9. The diagram shows the heart rate of an athlete during a particular hour of a workout.



   a. Determine the initial heart rate.
   b. State the amplitude.
   c. State the period.
   d. Express H as a function of t.
10. The temperature, T °C, in a house on a summer's day is given by $T(t) = c + b\cos(nt)$, where t is the time in hours. If the maximum and minimum temperatures are 24 °C and 18 °C and this cycle repeats for 24 hours, determine the values of c, b and n.
11. A model for the height above the ground, h metres, of a capsule on the London Eye is given by $h(t) = 68 - 67\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes and $t \geq 0$.
   a. State the greatest height of the capsule above the ground and the first time this occurs.
   b. Determine the period and amplitude of the motion of the function $h(t)$.
   c. During one cycle, for how long are passengers more than 80 metres above the ground?
   d. Sketch the graph of h against t, showing one cycle.

12. A swinging door has an angular displacement of θ radians at a time t seconds, given by $\theta = 4^{-\frac{t}{5}} \cos(2\pi t)$, for $t \geq 0$.

a. Determine the initial angular displacement of the door.

b. Determine the angular displacement of the door after 5 seconds.

c. Determine the angular displacement of the door after 10 seconds.

d. After a long time, what is the angular displacement of the door?

e. Sketch the graph of θ versus t over the first 10 seconds.

f. How many times does the door swing wider that 0.3 radians?

13. A cyclist rides one lap of a circular track at a constant speed so that her distance, d metres, from her starting point at any time, t seconds, after starting is:

$$d = 50 - 50 \cos \frac{\pi t}{30}$$

Calculate:

a. the time taken to complete one lap

b. the radius of the track

c. the maximum distance from the start

d. the length of the track

e. her distance from her starting point after 15 seconds

f. her distance from her starting point after 40 seconds.

14. The depth of water, d metres, at a port entrance is given by the function $d(t) = 4.5 + 1.5 \sin \frac{\pi t}{12}$ where t is in hours.

a. Determine **i** the maximum and **ii** the minimum depth at the port entrance.

b. A certain ship needs the depth at the port entrance to be more than 5 metres. The ship can be loaded and unloaded, and in and out of the port, in 9 hours. Assuming that the ship enters the port just as the depth at the entrance passes 5 metres, will the ship be able to exit 9 hours later? How long will it have to spare, or by how many minutes will it miss out?

15. The temperature in an office is controlled by a thermostat. The preferred temperature P can be set to values between 18 and 25 degrees Celsius.

The temperature (T °C) in the office at time t hours after 9 am is given by the rule $T = P + 2.4 \sin(\pi t)$.

If the preferred temperature on the thermostat has been set to 23 °C:

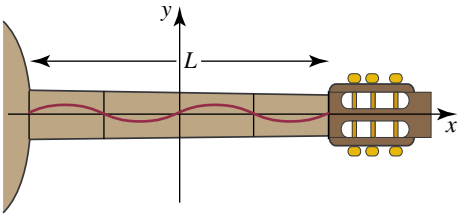
a. State the maximum and minimum temperatures

b. Determine the temperature at **i** noon and **ii** 3.30 pm

c. Sketch the graph of the function between 9 am and 5 pm.

d. Freddy feels thirsty if the temperature is above 24.2 degrees. Determine the length of time between 9 am and 5 pm that Freddy feels thirsty.

16. A 'standing wave' on a guitar string may be approximated by the function $y = 0.3 \sin \frac{\pi}{20} x$ where x cm and y cm are defined on the diagram shown.



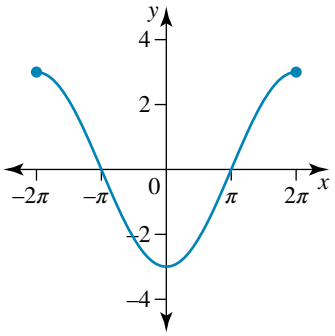
   a. Determine the period of the standing wave.
   b. If the frets coincide with the mean positions of the wave, determine the value of L.
   c. If the frets were to be spaced 16 cm apart, what would the value of L be?
   d. Determine the equation of the standing wave that will have an amplitude of 0.3 cm with frets 16 cm apart if the frets are at each mean position.

8.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Without using a calculator, state the exact value of each of the following.
   a. $\sin 30°$ b. $\cos 60°$ c. $\tan 45°$ d. $\sin \frac{\pi}{4}$ e. $\cos \frac{\pi}{6}$ f. $\tan \frac{\pi}{3}$
2. Consider the function $f(x) = -2 \sin 3x$. Without using technology:
   a. state the amplitude
   b. state the range
   c. state the period
   d. sketch the function over the domain $[0, 2\pi]$.
3. State the rule for this graph.



4. Sketch the graphs of the following functions without using technology.
   a. $y = -4 \sin \frac{x}{2}$ for the domain $-2\pi \leq x \leq 4\pi$
   b. $y = 1.5 \cos 2x$ for the domain $-\pi \leq x \leq \pi$
5. Solve the following trigonometric equations over the domain $0° \leq \theta \leq 360°$, correct to the nearest degree where appropriate.
   a. $\sin \theta = 0.9$ b. $\cos \theta = -0.4$ c. $\tan \theta = 1.6$
   d. $\sin \theta = -\frac{\sqrt{3}}{2}$ e. $\cos \theta = \frac{1}{2}$ f. $\tan \theta = 1$
6. Determine the exact solutions to each of the following over the domain $0 \leq x \leq 2\pi$ without the aid of technology.
   a. $\cos x = -\frac{\sqrt{3}}{2}$ b. $\tan x = -\sqrt{3}$ c. $\sin x = -\frac{\sqrt{2}}{2}$
   d. $2 \sin x = \sqrt{3}$ e. $2 \cos x = -1$ f. $4 \tan x = -4$

For questions 7 to 9, sketch the graphs without the use of technology.

7. Sketch $y = 2 \sec 2(x + 45°)$ over the domain $-180° \leq x \leq 180°$.
8. Sketch $y = 1 - \text{cosec}\left(\frac{1}{2}x + \pi\right)$ over the domain $-2\pi \leq x \leq 2\pi$.
9. Sketch $y = \tan 3x - 3$ over the domain $x \in [0, \pi]$.
10. Determine the general solution to $2 \tan\left(2x - \frac{\pi}{3}\right) - 2 = 0$.

You may choose to use technology to answer questions **11** and **12**.

11. **a.** Determine the general solution for $\cos(3x) + \sqrt{3}\sin(3x) = 0$.

b. Solve $\cos(3x) + \sqrt{3}\sin(3x) = 0$ for $x \in [0, 2\pi]$.

12. **a.** Determine the general solution for $\cot(2x) = \cos(2x)$.

b. Solve $\cot(2x) = \cos(2x)$ for $x \in [0, 2\pi]$.

Complex familiar

You may choose to use technology to answer questions **13** to **16**.

13. The number of rabbits in a national park is observed for one year. At any time t months after observation begins, the number of rabbits is modelled by the function $P = 2 - 0.8\sin\dfrac{\pi t}{6}$ where P is in thousands.

a. Determine the minimum, maximum and equilibrium number of rabbits.

b. Determine the period and amplitude of the function.

c. Sketch a graph of the function.

d. Determine the population after 5 months.

14. The sound level of a siren follows the rule $L(t) = 6\sin \pi t + 80$, where L is the sound level measured in decibels (dB) and t is the time in seconds.

a. Determine the amplitude and period of $L(t)$?

b. Determine the maximum and how long does it take for the sound to reach its first maximum?

c. Determine the first time the sound reaches 83 dB.

d. Sketch the graph of $L(t)$ for $0 \leq t \leq 2$.

15. The height (in centimetres) that a clock's pendulum swings above its base can be approximated by the function $H = 14 + 5.9\cos\left(\dfrac{7\pi t}{4}\right)$ at any time t seconds after it is released. Give answers to the following correct to 1 decimal place.

a. Identify the maximum and minimum heights that the pendulum reaches.

b. Identify the height after 1 minute.

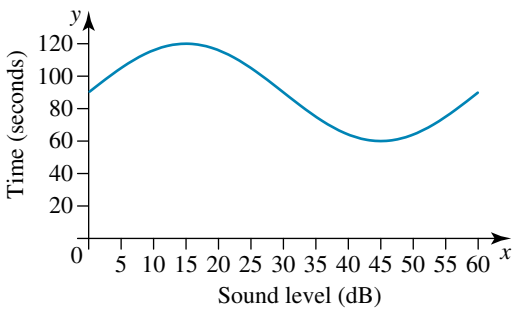
c. Sketch the graph of the function for the first 2 seconds

d. Calculate the number of times the pendulum swings in 1 minute.

e. Calculate the length of time that the pendulum is below 14 cm as it travels from one side to the other.

f. The pendulum is found to be losing time and needs its swing adjusted to 75 swings per minute. Determine the new function, $H(t)$, that approximates the height of the pendulum.

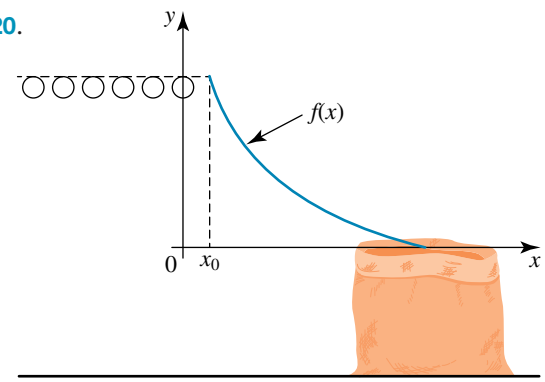
16. The level of sound in a recording studio is being monitored. The sound engineers know that the level can be modelled by a function of the form $y = a + b\sin nx$. If in 60 seconds (the period) of a recording session the range of sound intensity is 60 to 120 decibels, determine the values of a, b and n.



Complex unfamiliar

You may choose to use technology to answer questions **17** to **20**.

17. At the end of a conveyor belt that transports completed toys from Santa's factory is a chute that allows the toys to fall into a sack. The chute has the shape of a cot function of the form $f(x) = a\cot bx$, $x_0 \leq x \leq 3.5$, where x_0, a and b are real numbers. The end of the chute is 3.0 m horizontally away from the end of the conveyor belt, which is 2.0 m above the top of the sack. That is, $f(3.5) = 0$ and $f(0.50) = 2.0$.



a. Demonstrate that $x_0 = 0.50$.

b. Determine the value of b and hence a.

c. When the toys are a horizontal distance of 1.0 m from the sack, calculate the value of x and how high above the sack they are.

d. A new chute is to be modelled on the cosec function, so that $y = a\text{cosec}(bx)$ for $0.5 \leq x \leq 3.5$. The new chute will have the same starting height and position but will be horizontal when it reaches the sack. Determine the equation of the model for the new chute.

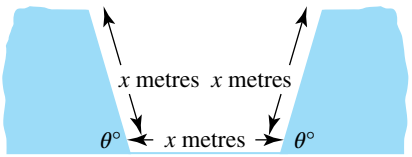
18. Using appropriate technology, graph each of the following functions. Discuss the features of both waveforms and determine the period and amplitude for each.

a. $y = \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \ldots$ **b.** $y = \sin x + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots$

19. Mara is a civil engineer who is working on the construction of an irrigation channel like the one illustrated. This channel has a trapezoidal cross-section in which the bottom and sides are x metres in length. Mara requires $\theta°$ to be acute and for the sides to be equally inclined to the horizontal.

a. Derive an expression for the area of the cross-section of the channels as a function of θ.

b. Using appropriate technology, assist Mara to determine the value of $\theta°$ for which the cross-sectional area is a maximum.



20. At a jetty in north Queensland, Ariel measures the depth of water at various times on 4 February. Her observations yield the following results:

Time	6 am	7 am	8 am	9 am	10 am	11am	12 noon	1 pm	2 pm
Depth (metres)	1.4	1.8	2.3	2.6	2.5	2.2	1.9	1.3	0.8
Time	3 pm	4 pm	5 pm	6 pm	7 pm	8 pm	9 pm	10 pm	11 pm
Depth (metres)	0.4	0.7	1.1	1.4	1.8	2.3	2.4	2.5	2.4

Ariel has information that suggests that boats which safely moor at this jetty can do so only when the depth of the water is above 1.8 metres.

a. To predict future tidal behaviour, Ariel decides to model her observed data using appropriate functions of a graphics calculator. Assist Ariel to create a mathematical model that best reflects the given date. Use 3 decimal places in your answer.

b. Use the model to determine if boats may safely moor at the end of the jetty on 7 February after 6 pm.

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Units 1 & 2 Sit chapter test

Answers

Chapter 8 Trigonometric functions

Exercise 8.2 Review of trigonometry

1. **a.** $\frac{\pi}{6}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{3}$ **d.** $\frac{\pi}{9}$ **e.** $\frac{5\pi}{18}$ **f.** $\frac{\pi}{2}$

2. **a.** $\frac{3\pi}{2}$ **b.** 2π **c.** $\frac{5\pi}{6}$ **d.** $-\frac{5\pi}{4}$ **e.** $-\frac{7\pi}{3}$ **f.** $\frac{5\pi}{3}$

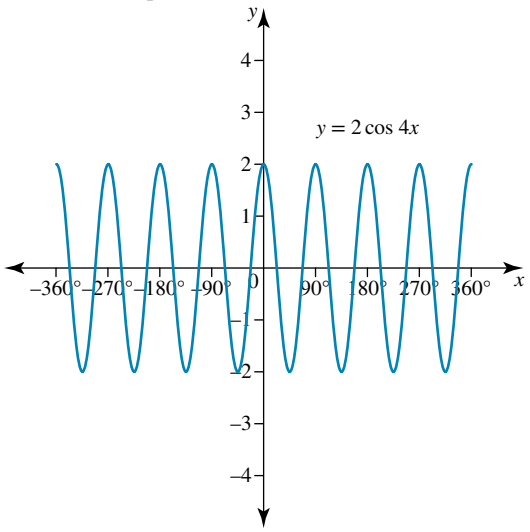
3. **a.** $36°$ **b.** $120°$ **c.** $40°$ **d.** $220°$ **e.** $648°$ **f.** $-30°$

4. **a.** $\frac{\sqrt{2}}{2}$ **b.** $\frac{1}{2}$ **c.** $\sqrt{3}$ **d.** $\frac{1}{2}$ **e.** 1 **f.** $\frac{\sqrt{3}}{2}$

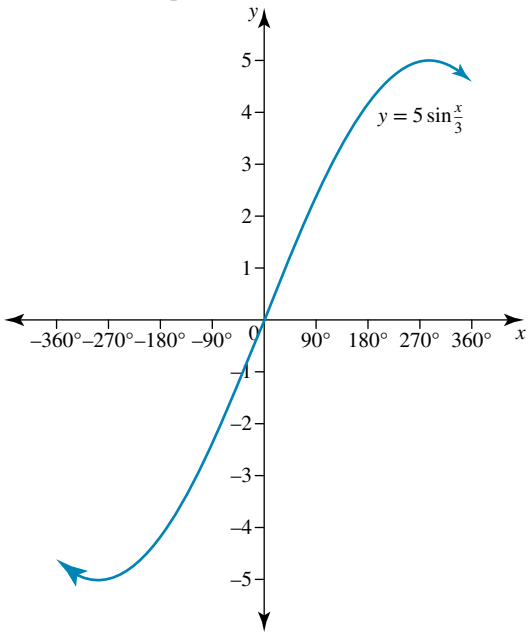
5. **a.** $\frac{1}{2}$ **b.** $-\frac{\sqrt{2}}{2}$ **c.** $\sqrt{3}$ **d.** $-\frac{\sqrt{3}}{3}$ **e.** $-\frac{\sqrt{3}}{2}$ **f.** $-\frac{\sqrt{3}}{2}$

6. **a.** $-\frac{\sqrt{2}}{2}$ **b.** $\frac{1}{2}$ **c.** -1 **d.** $\frac{\sqrt{3}}{2}$ **e.** $\frac{1}{2}$ **f.** $-\frac{\sqrt{3}}{3}$

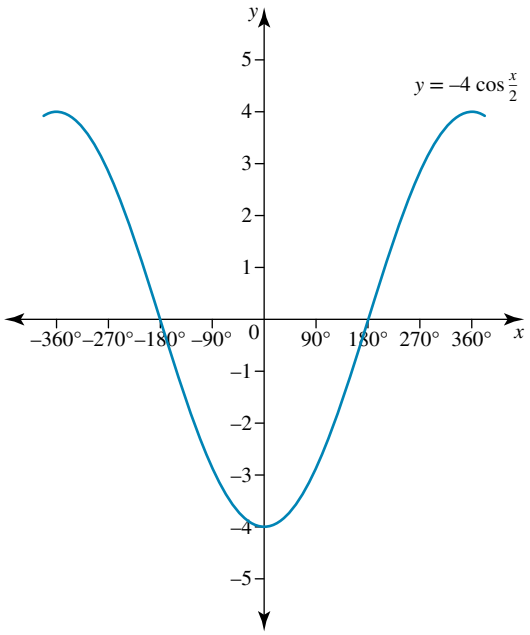
7. **a.** Period 90°, amplitude 2



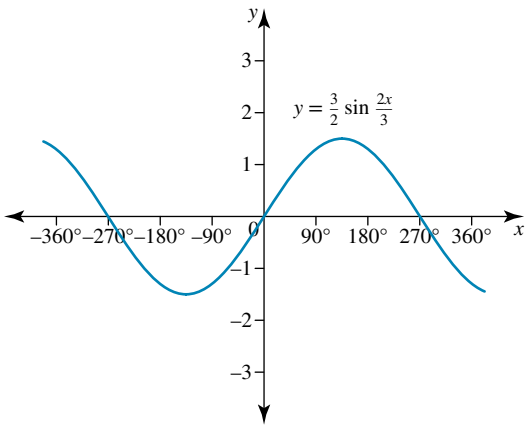
b. Period 1080°, amplitude 5



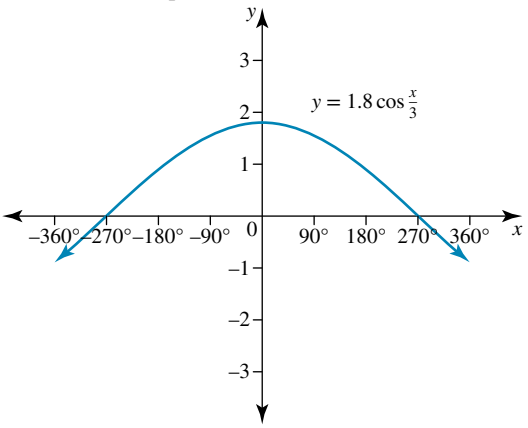
c. Period 720°, amplitude 4



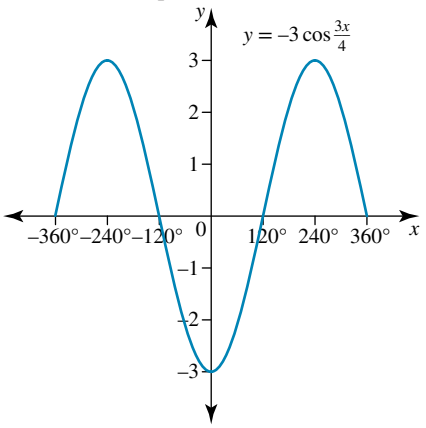
d. Period 540°, amplitude $\frac{3}{2}$



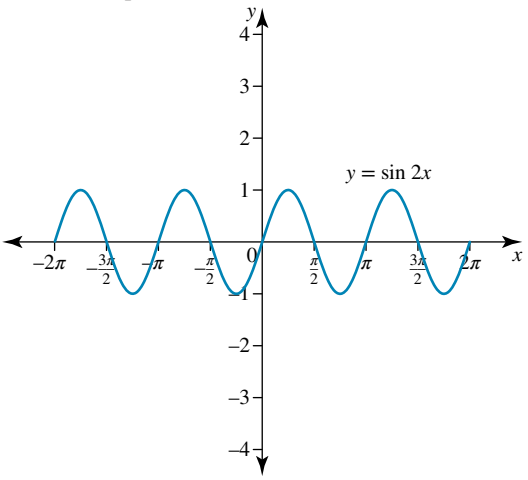
e. Period 1080°, amplitude 1.8



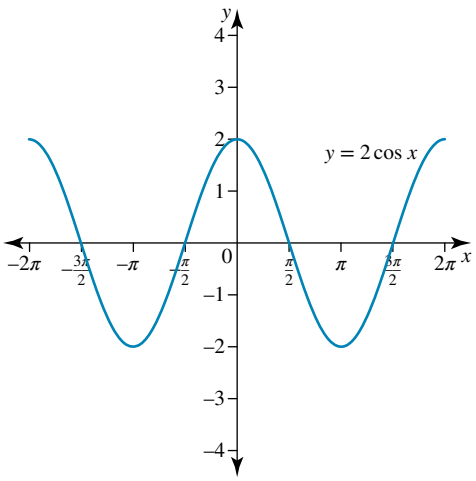
f. Period 480°, amplitude 3



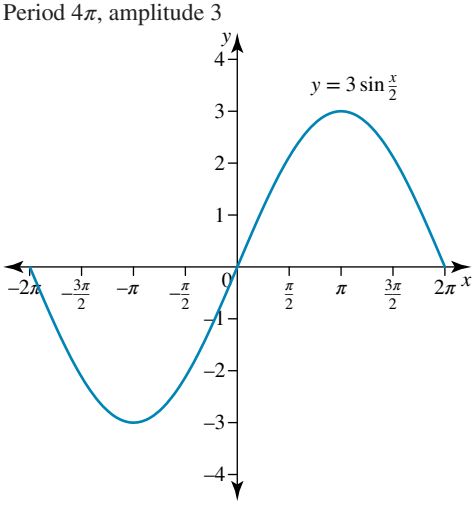
8. a. Period π, amplitude 1



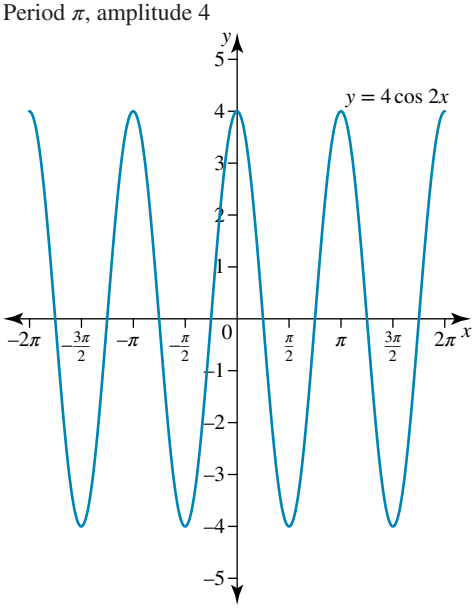
b. Period 2π, amplitude 2



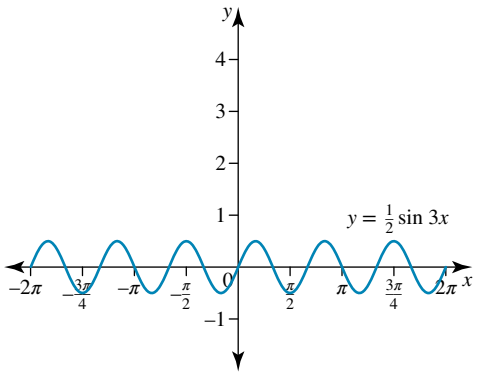
c. Period 4π, amplitude 3



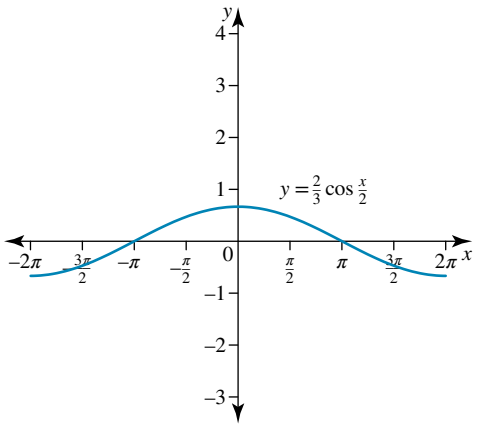
d. Period π, amplitude 4



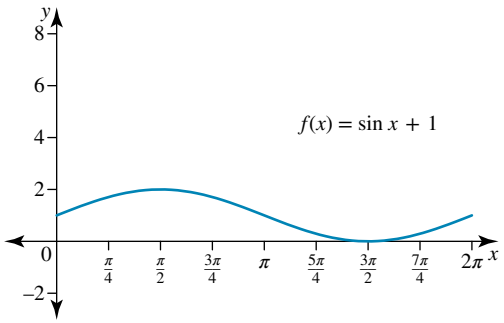
e. Period $\frac{2\pi}{3}$, amplitude $\frac{1}{2}$



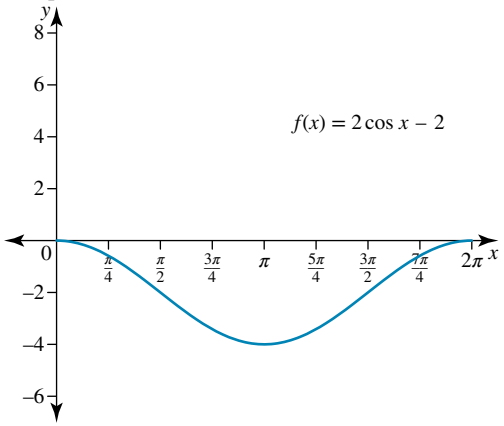
f. Period 4π, amplitude $\frac{2}{3}$



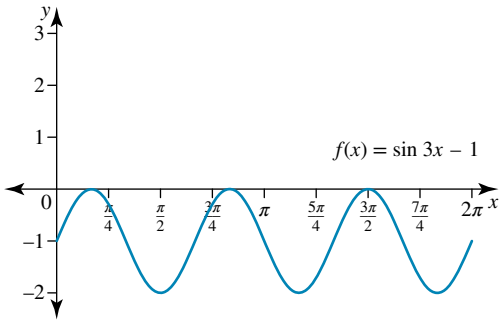
9. a. Amplitude 1
Period 2π
Range $[0, 2]$
y-intercept $(0, 1)$
x-intercepts $\left(\frac{3\pi}{2}, 0\right)$
Maximum $\left(\frac{\pi}{2}, 2\right)$
Minimum $\left(\frac{3\pi}{2}, 0\right)$
End points $(0, 1)$, $(2\pi, 1)$



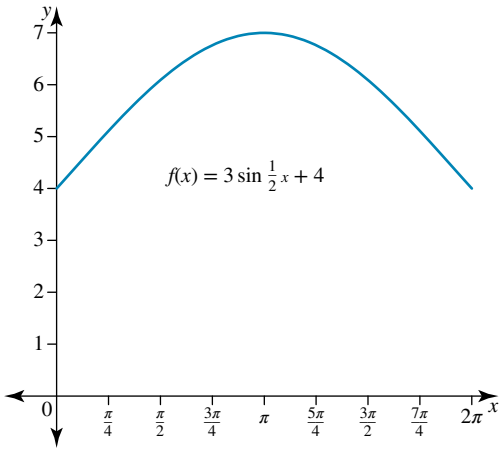
b. Amplitude 1
Period 2π
Range $[-4, 0]$
y-intercept $(0, 0)$
x-intercepts $(0, 0)$, $(2\pi, 0)$
Maximum $(0, 0)$, $(2\pi, 0)$
Minimum $(\pi, -4)$
End points $(0, 0)$, $(2\pi, 0)$



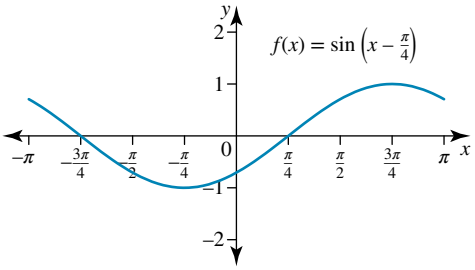
c. Amplitude 1
Period $\frac{2\pi}{3}$
Range $[-2, 0]$
y-intercept $(0, -1)$
x-intercepts $\left(\frac{\pi}{6}, 0\right)$, $\left(\frac{5\pi}{6}, 0\right)$, $\left(\frac{9\pi}{6}, 0\right)$
Maximum $\left(\frac{\pi}{6}, 0\right)$, $\left(\frac{5\pi}{6}, 0\right)$, $\left(\frac{9\pi}{6}, 0\right)$
Minimum $\left(\frac{\pi}{2}, -2\right)$, $\left(\frac{7\pi}{6}, -2\right)$, $\left(\frac{11\pi}{6}, -2\right)$
End points $(0, -1)$, $(2\pi, -1)$



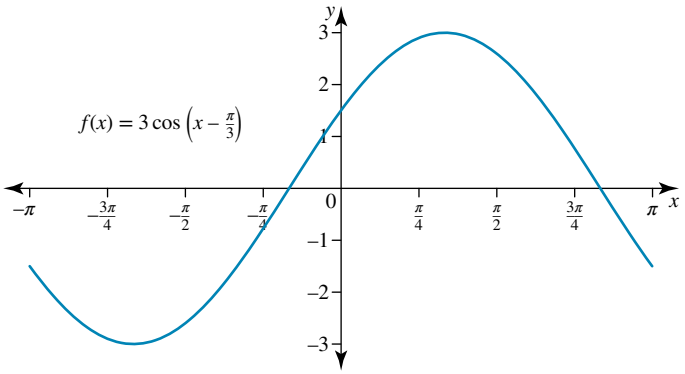
d. Amplitude 3
Period 2π
Range $[4, 7]$
y-intercept $(0, 4)$
x-intercepts: none
Maximum $(\pi, 7)$
Minimum: not in this domain
End points $(0, 4)$, $(2\pi, 4)$



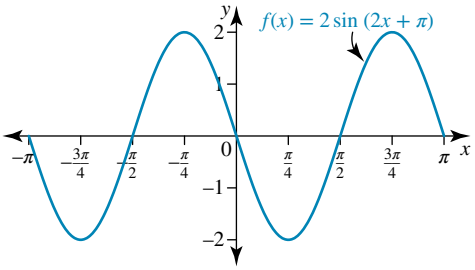
10. a. Amplitude 1
Period 2π
Range $[-1, 1]$
y-intercept $\left(0, -\frac{\sqrt{2}}{2}\right)$
x-intercepts $\left(-\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right)$
Maximum $\left(\frac{3\pi}{4}, 1\right)$
Minimum $\left(-\frac{\pi}{4}, -1\right)$
End points $\left(\pm\pi, \frac{\sqrt{2}}{2}\right)$



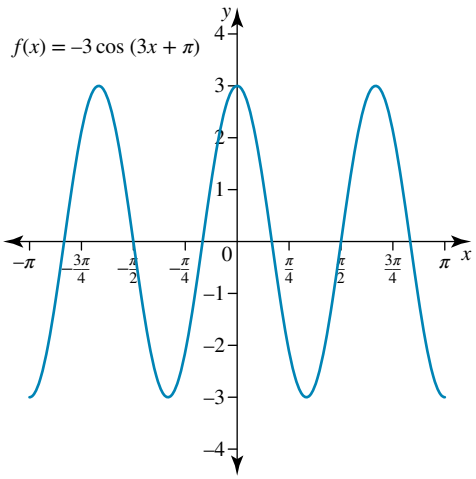
b. Amplitude 3
Period 2π
Range $[-3, 3]$
y-intercept $\left(0, \frac{3}{2}\right)$
x-intercepts $\left(-\frac{\pi}{6}, 0\right), \left(\frac{5\pi}{6}, 0\right)$
Maximum $\left(\frac{\pi}{3}, 3\right)$
Minimum $\left(\frac{4\pi}{3}, -3\right)$
End points $\left(\pm\pi, -\frac{3}{2}\right)$



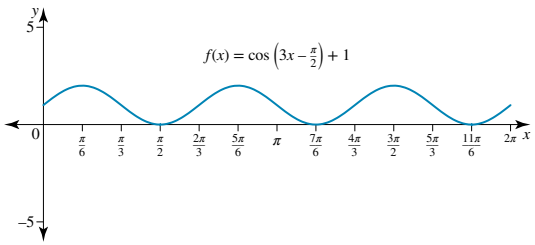
c. Amplitude 2
Period π
Range $[-2, 2]$
y-intercept $(0, 0)$
x-intercepts $\left(\pm\frac{3\pi}{2}, 0\right), (\pm\pi, 0), (0, 0)$
Maximum $\left(-\frac{\pi}{4}, 2\right), \left(\frac{3\pi}{4}, 2\right)$
Minimum $\left(-\frac{3\pi}{4}, -2\right), \left(\frac{\pi}{4}, -2\right)$
End points $(\pm\pi, 0)$



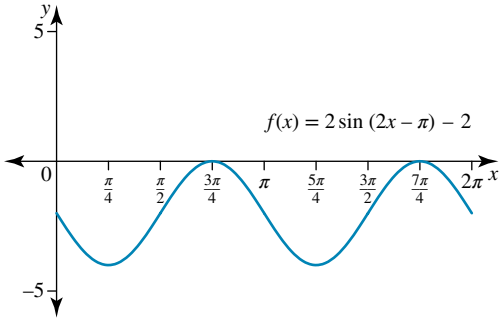
d. Amplitude 3
Period $\frac{2\pi}{3}$
Range $[-3, 3]$
y-intercept $(0, 3)$
x-intercepts $\left(\pm\frac{5\pi}{6}, 0\right), \left(\pm\frac{\pi}{2}, 0\right), \left(\pm\frac{\pi}{6}, 0\right)$
Maximum $\left(\pm\frac{2\pi}{3}, 3\right), (0, 3)$
Minimum $\left(\pm\frac{\pi}{3}, -3\right), (\pm\pi, 3)$
End points $(\pm\pi, -3)$



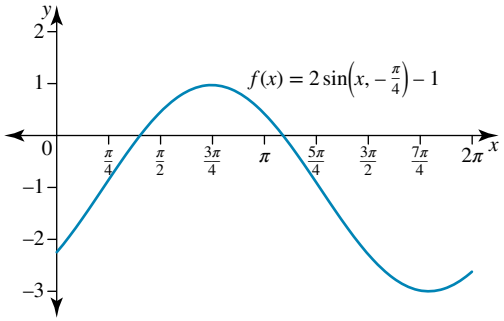
11. a. Period $\frac{2\pi}{3}$, amplitude 1



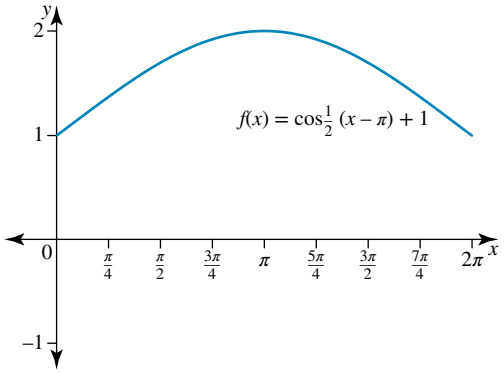
b. Period π, amplitude 2



c. Period 2π, amplitude 2



d. Period 4π, Amplitude 1



12. a. $a = 3, b = 2, y = 3\sin 2x$ **b.** $a = 2, b = \frac{\pi}{6}, y = 2\sin\frac{\pi}{6}x$

13. a. $a = 2, b = 4, y = 2\cos 4x$ **b.** $a = 2, b = \frac{\pi}{3}, y = 2\cos\frac{\pi}{3}x$

14. a. $a = \frac{1}{2}, b = 1, d = 1\ y = \frac{1}{2}\sin x + 1$ **b.** $a = 3, b = \frac{\pi}{4}, d = -2\ y = 3\sin\frac{\pi x}{4} - 2$

15. a. $a = 4, b = 2, c = \frac{\pi}{2}\ y = 4\cos\left(2\left(x \pm \frac{\pi}{2}\right)\right)$ **b.** $a = 5, b = 2, c = -\frac{\pi}{4}\ y = 5\cos\left(2\left(x - \frac{\pi}{4}\right)\right)$

16. a. $a = 2, b = 1, c = \frac{\pi}{3}, d = -1, y = 2\sin\left(x + \frac{\pi}{3}\right) - 1$ **b.** $a = 3, b = 1, c = \frac{\pi}{2}, d = 2, y = 3\sin\left(x + \frac{\pi}{2}\right) + 2$

Exercise 8.3 Solving trigonometric equations

1. a. 37°, 143° **b.** 104°, 256° **c.** 238°, 302° **d.** 79°, 281°
e. 199°, 341° **f.** 41°, 319°

2. a. 60°, 120° **b.** 45°, 315° **c.** 210°, 330° **d.** 120°, 240°
e. 225°, 315° **f.** 30°, 330°

3. a. 0.93c, 2.21c **b.** 2.09c, 4.19c **c.** 5.95c, 3.47c **d.** 0.79c, 5.50c

4. a. $\frac{\pi}{3}, \frac{2\pi}{3}$ **b.** $\frac{2\pi}{3}, \frac{4\pi}{3}$ **c.** $\frac{\pi}{4}, \frac{7\pi}{4}$ **d.** $\frac{7\pi}{6}, \frac{11\pi}{6}$

5. a. 2.2904, 3.9928, 8.5736, 10.2760 **b.** 1.1442, 1.9973, 7.4274, 8.2805
c. 1.0701, 5.2130, 7.3533, 11.4962 **d.** 3.5217, 5.9031, 9.8049, 12.1863

6. a. −6.01, −3.41, 0.27, 2.87 **b.** −3.88, −2.41, 2.41, 3.88
c. −2.57, −0.57, 3.71, 5.71 **d.** −4.90, −1.38, 1.38, 4.90

7. a. 90° **b.** 90°, 270° **c.** 0°, 180°, 360° **d.** 180°

8. a. 0, π, 2π **b.** $\frac{\pi}{2}, \frac{3\pi}{2}$ **c.** 0, 2π **d.** $\frac{3\pi}{2}$

9. a. $\frac{\pi}{6}, \frac{5\pi}{6}$ **b.** $\frac{\pi}{6}, \frac{11\pi}{6}$ **c.** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **d.** $\frac{3\pi}{4}, \frac{5\pi}{4}$

10. a. 0.25, 2.89 **b.** 2.30, 3.98

c. $\frac{3\pi}{2}$ (or 4.71) **d.** 1.36, 4.92

11. a. $-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$ **b.** $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

c. $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d.** $-\frac{7\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}$

12. a. −4.71, 1.57 **b.** −6.28, 0, 6.28

c. −5.15, −1.14, 1.14, 5.15 **d.** −5.76, −3.67, 0.52, 2.62

13. a. $x = \frac{\pi}{4}(8n \pm 1), n \in Z$ **b.** $x = \frac{\pi}{6}(12n \pm 1), n \in Z$

c. $x = \frac{\pi}{4}(8n \pm 3), n \in Z$ **d.** $x = \frac{2\pi}{3}(3n \pm 1), n \in Z$

14. a. $\frac{\pi}{6}(12n + 1), \frac{\pi}{6}(12n + 5), n \in Z$ **b.** $\frac{\pi}{4}(8n + 1), \frac{\pi}{4}(8n + 3), n \in Z$

c. $\frac{2\pi}{3}(3n + 2), \frac{\pi}{3}(6n - 1), n \in Z$ **d.** $\frac{\pi}{6}(12n - 1), \frac{\pi}{6}(12n + 7), n \in Z$

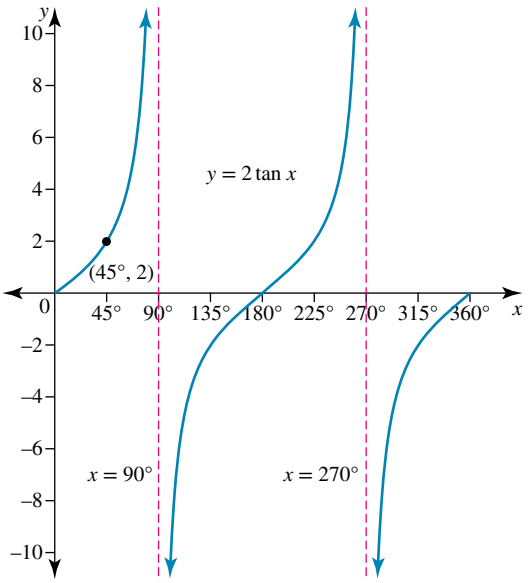
15. a. $\frac{\pi}{3}(3n + 1), \frac{\pi}{6}(6n + 1), n \in Z$ **b.** $\frac{\pi}{12}(12n \pm 5), n \in Z$

c. $\frac{\pi}{3}(2n + 1), \frac{\pi}{6}(4n + 1), n \in Z$ **d.** $\frac{\pi}{36}(24n - 1), \frac{\pi}{36}(24n + 7), n \in Z$

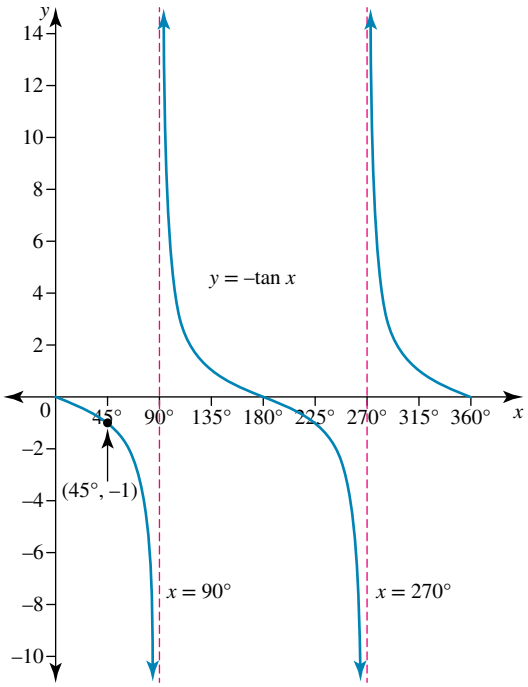
16. a. $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **b.** $-\pi, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \pi$

Exercise 8.4 The tangent function

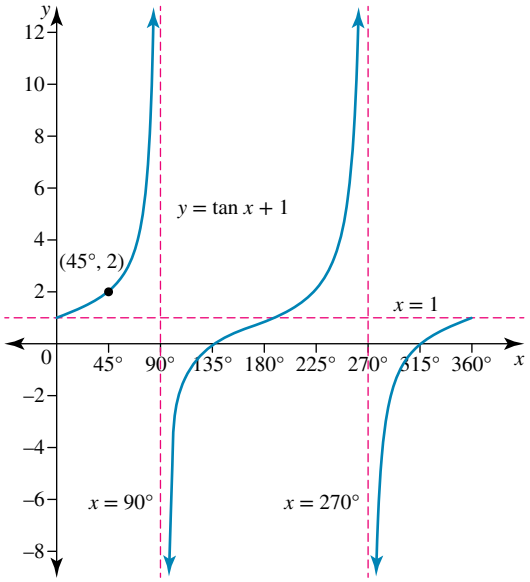
1. a. Period 180°, asymptotes $x = 90°$, $x = 270°$



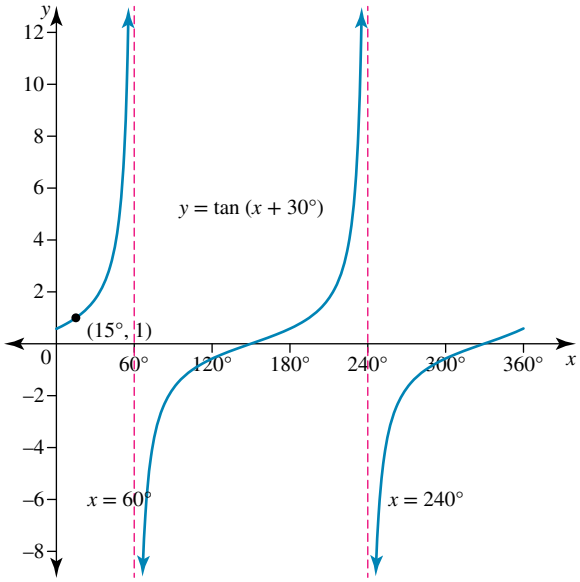
b. Period 180°, asymptotes $x = 90°$, $x = 270°$



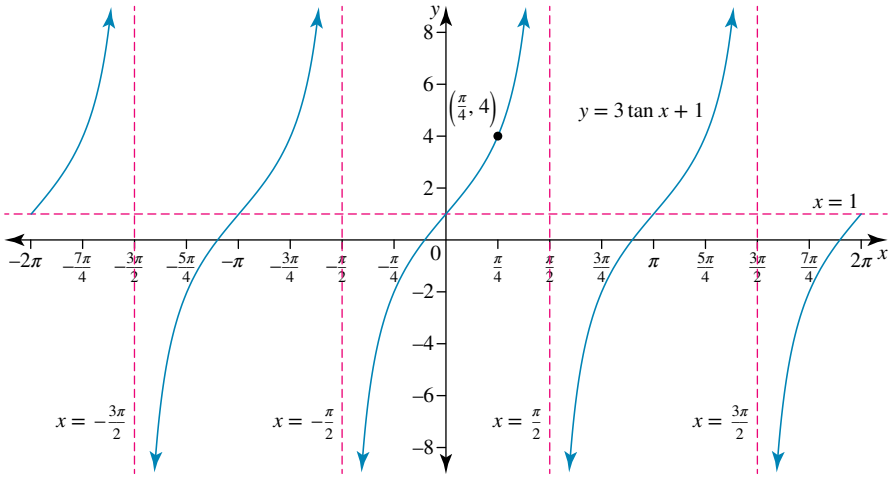
c. Period 180°, asymptotes $x = 90°$, $x = 270°$



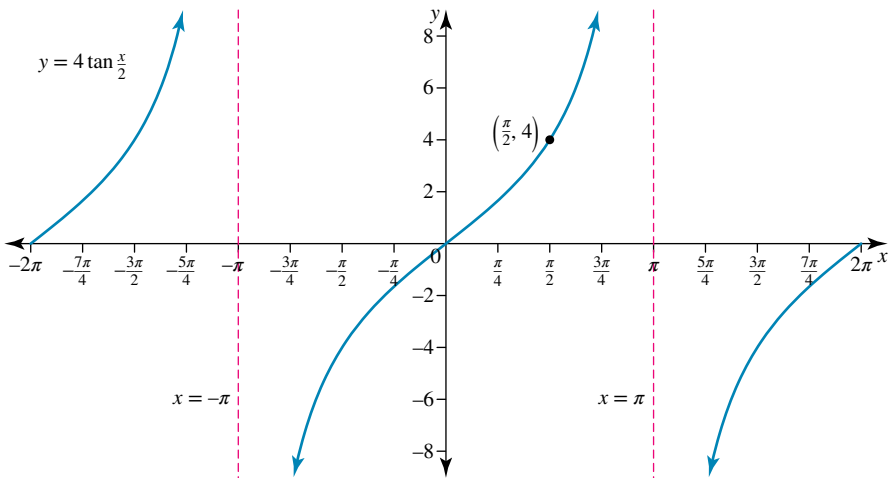
d. Period 180°, asymptotes $x = 60°$, $x = 240°$



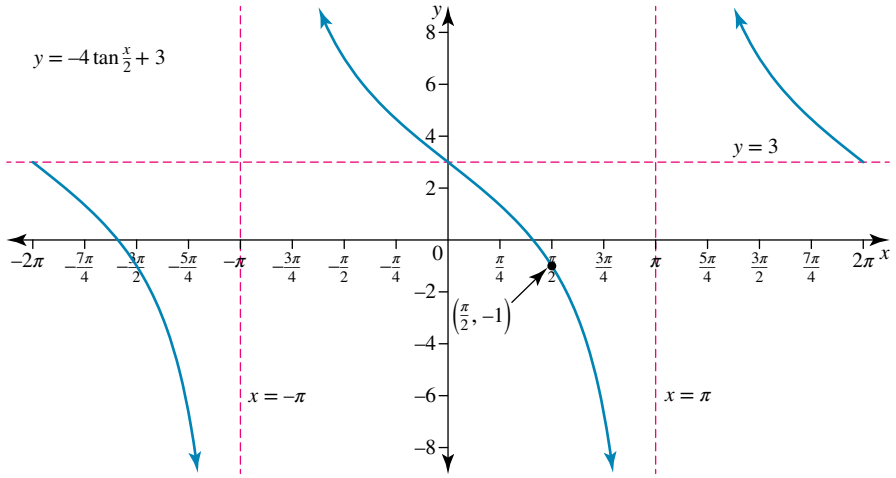
2. a. Period π, asymptotes $x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$



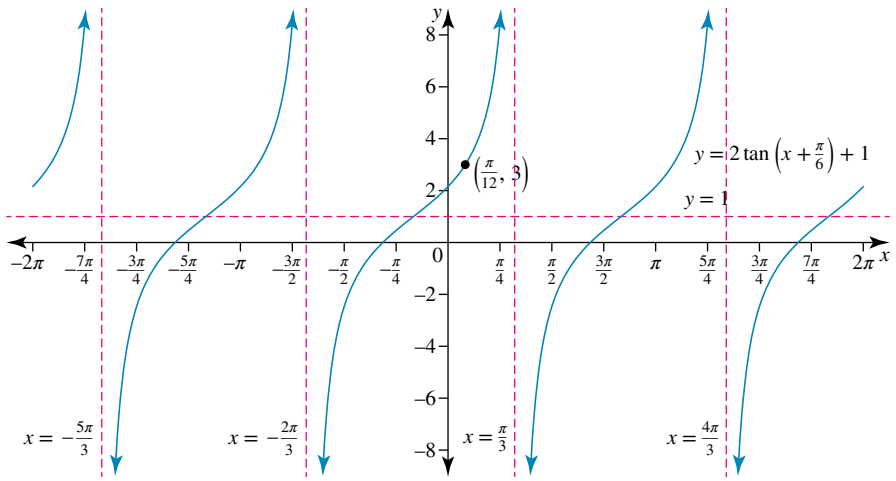
b. Period 2π, asymptotes $x = \pm\pi$



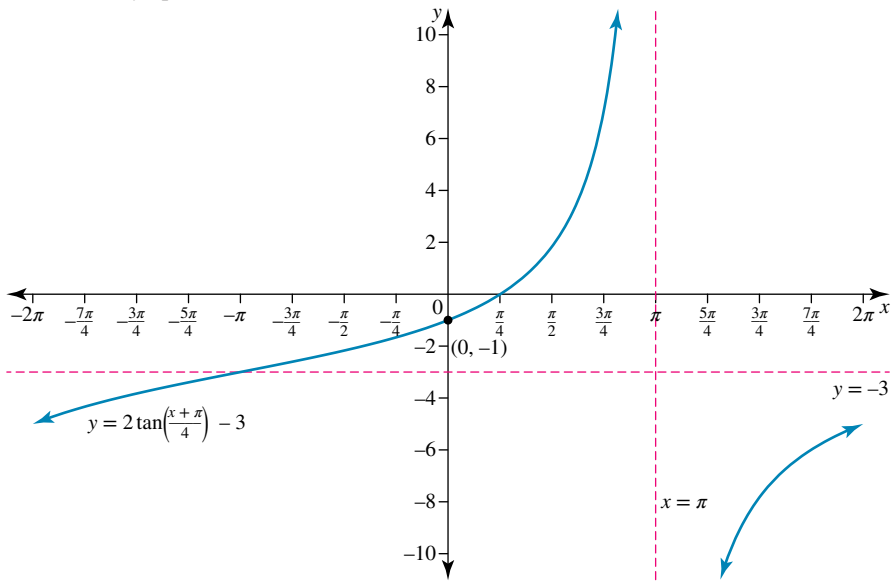
c. Period 2π, asymptotes $x = \pm\pi$



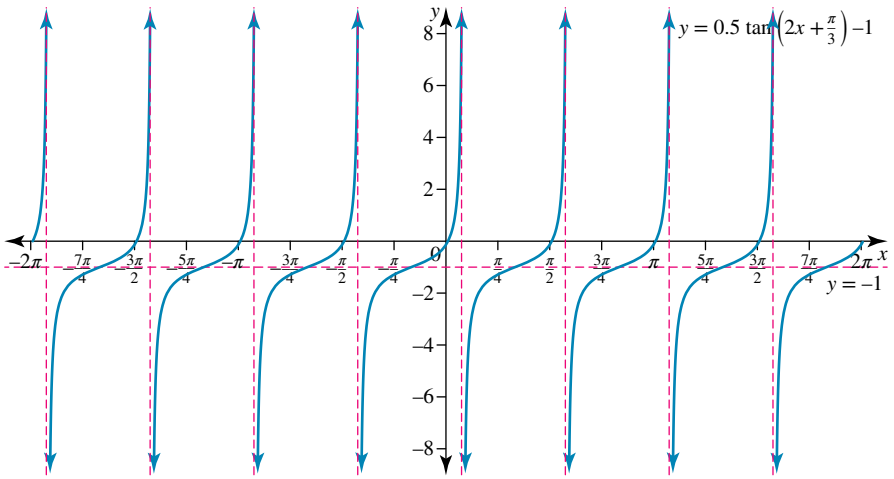
d. Period π, asymptotes $x = -\frac{5\pi}{3}, x = -\frac{2\pi}{3}, x = \frac{\pi}{3}, x = \frac{4\pi}{3}$



e. Period 4π, asymptotes $x = \pi$



f. Period $\frac{\pi}{2}$, asymptotes $x = -\frac{23\pi}{12}, -\frac{17\pi}{12}, -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$



3. a. 80°, 260° **b.** 140°, 320° **c.** 74°, 254° **d.** 118°, 298°

4. a. 60°, 240° **b.** 150°, 330° **c.** 45°, 225° **d.** 30°, 210°

5. a. 0.98^c, 4.12^c **b.** 2.77^c, 5.91^c **c.** 1.41^c, 4.56^c **d.** 2.19^c, 5.33^c

6. a. $\frac{\pi}{4}, \frac{5\pi}{4}$ **b.** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **c.** $\frac{\pi}{6}, \frac{7\pi}{6}$ **d.** $\frac{3\pi}{4}, \frac{7\pi}{4}$

7. a. 0.53, 3.67, 6.81, 9.95 **b.** 2.02, 5.16, 8.30, 11.44
c. 1.00, 4.15, 7.29, 10.43 **d.** 1.82, 4.97, 8.11, 11.25

8. a. −3.59, −0.45, 2.69, 5.83 **b.** −5.03, −1.89, 1.25, 4.39
c. −4.89, −1.75, 1.39, 4.53 **d.** −4.35, −1.21, 1.93, 5.07

9. 0°, 180°, 360°

10. $-2\pi, -\pi, 0, \pi, 2\pi$

11. a. $\frac{\pi}{4}, \frac{5\pi}{4}$ **b.** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **c.** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **d.** $0, \pi, 2\pi$

12. a. 1.29, 4.43 **b.** 1.90, 5.04 **c.** 2.82, 5.96 **d.** 2.03, 5.18

13. a. $-\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$ **b.** $-\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}$ **c.** $-\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4},$ **d.** $-\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}, \frac{7\pi}{6}$

14. a. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ **b.** $\frac{\pi}{2}$
c. $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$ **d.** $\frac{5\pi}{4}$

15. a. $\frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$ **b.** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$
c. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ **d.** $\frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}, \frac{17}{4}, \frac{21}{4}, \frac{25}{4}$

16. a. $\frac{\pi}{3}(3n+1), n \in Z$ **b.** $\frac{\pi}{4}(4n-1), n \in Z$ **c.** $\frac{\pi}{6}(6n+1), n \in Z$ **d.** $x = \frac{\pi}{4}(4n+1), n \in Z$

17. a. $\frac{\pi}{12}(6n-1), n \in Z$ **b.** $\frac{\pi}{12}(12n+5), n \in Z$ **c.** $\frac{\pi}{24}(12n+1), n \in Z$ **d.** $\frac{\pi}{36}(12n+5), n \in Z$

18. a. $\frac{\pi}{4}, \frac{5\pi}{4}$ **b.** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
c. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$ **d.** $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$
e. 0.6871, 1.7253, 2.7725, 3.8197, 4.8669, 5.9141 **f.** 1.8925, 5.0341

Exercise 8.5 The reciprocal functions

1. a. $\frac{2\sqrt{3}}{3}$ **b.** $\frac{2\sqrt{3}}{3}$ **c.** 2 **d.** $-\sqrt{2}$

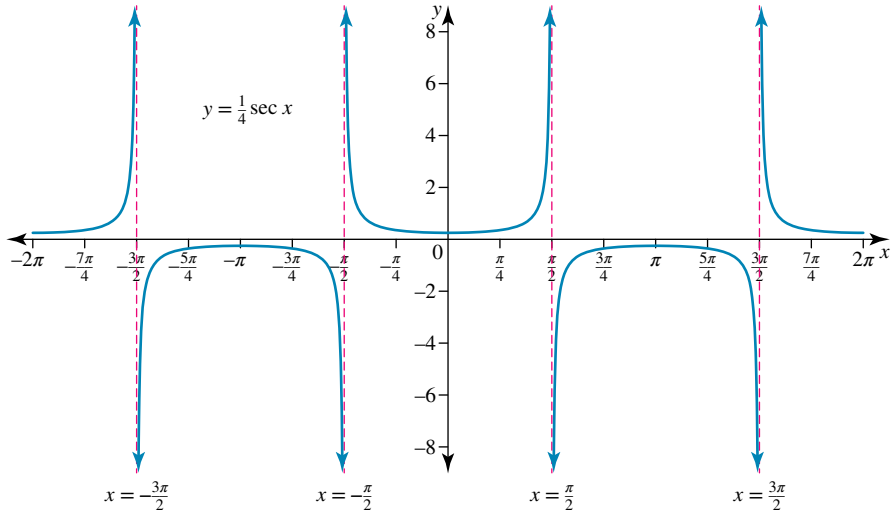
2. a. $\frac{2\sqrt{3}}{3}$ **b.** $-\frac{2\sqrt{3}}{3}$ **c.** −2 **d.** $\sqrt{2}$

3. **a.** $\sqrt{3}$ **b.** $-\frac{\sqrt{3}}{3}$ **c.** -1 **d.** $-\sqrt{3}$

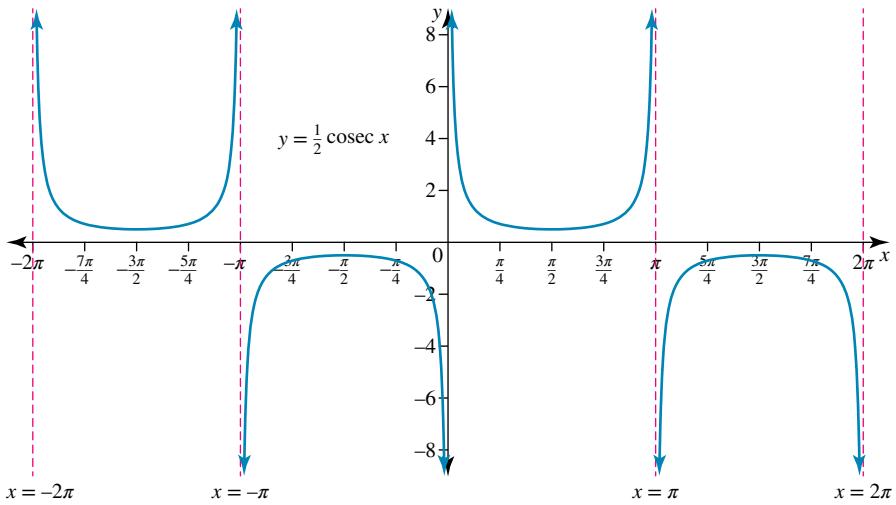
4. **a.** $-\frac{\sqrt{21}}{2}$ **b.** $-\frac{\sqrt{17}}{4}$ **c.** $\frac{3\sqrt{8}}{8}$ **d.** $-\sqrt{15}$ **e.** $\frac{3\sqrt{40}}{40}$ **f.** $\frac{5\sqrt{21}}{21}$

5. **a.** $-\frac{7\sqrt{40}}{40}$ **b.** $-\frac{5\sqrt{39}}{39}$ **c.** $-\frac{\sqrt{15}}{15}$ **d.** $\frac{\sqrt{61}}{6}$ **e.** $\frac{\sqrt{3}}{12}$ **f.** $-\sqrt{17}$

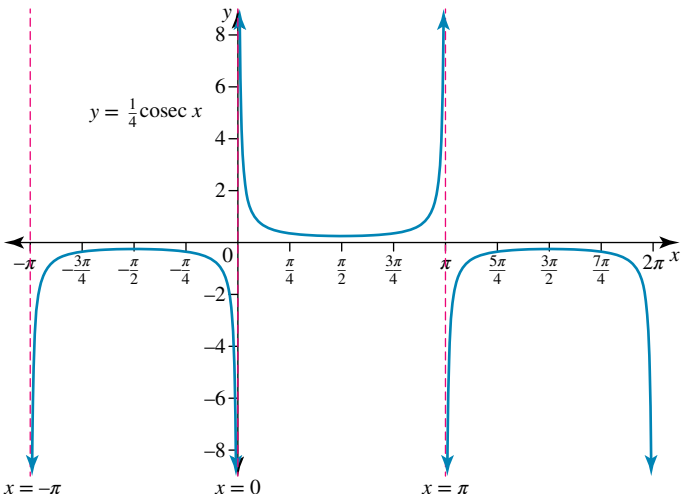
6. **a.**



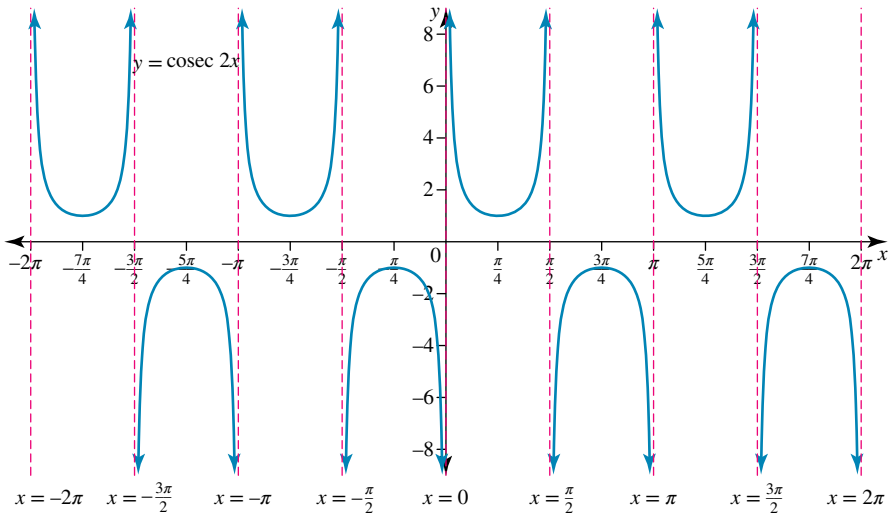
b.



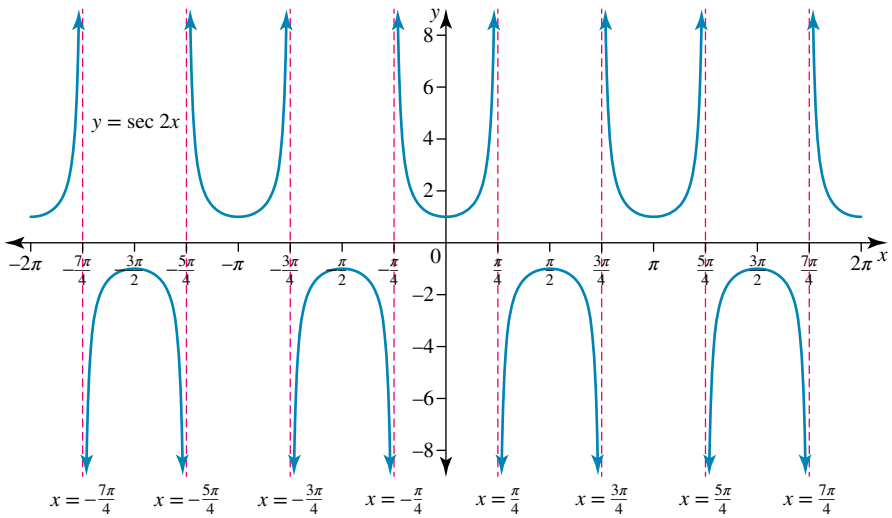
7.



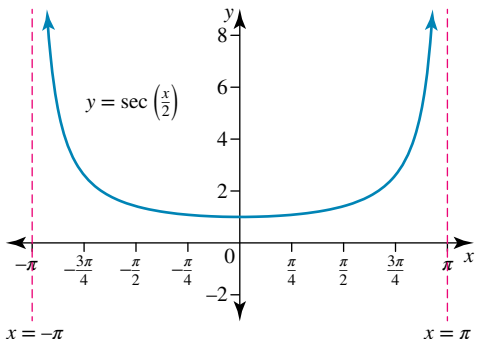
8. a.


y = cosec 2x
x = −2π
x = −3π/2
x = −π
x = −π/2
x = 0
x = π/2
x = π
x = 3π/2
x = 2π

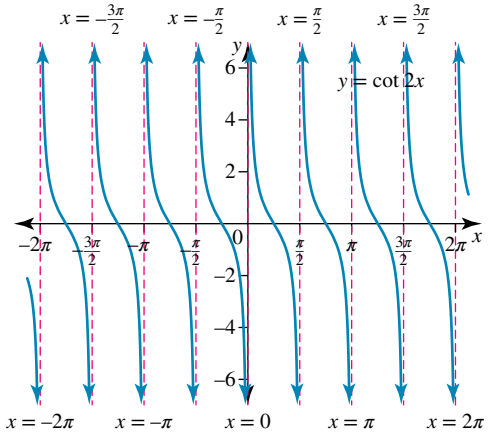
b.


y = sec 2x
x = −7π/4
x = −5π/4
x = −3π/4
x = −π/4
x = π/4
x = 3π/4
x = 5π/4
x = 7π/4

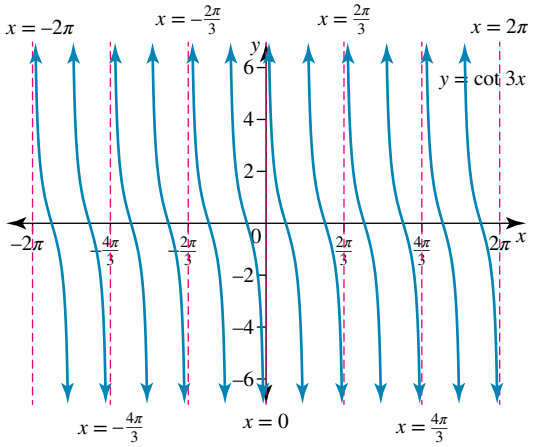
9.


y = sec (x/2)
x = −π
x = π

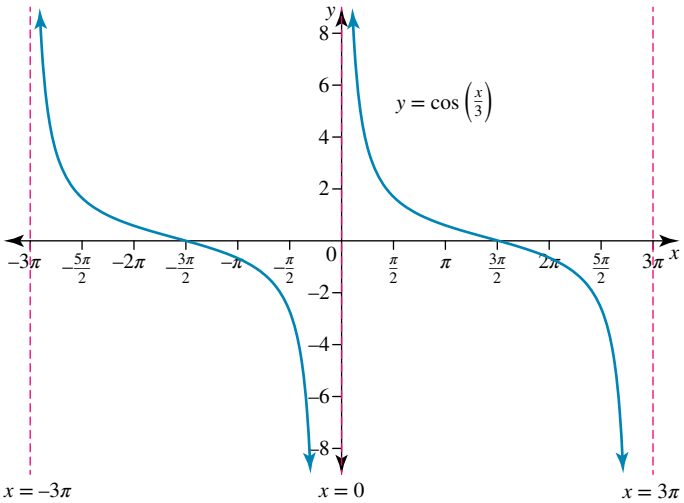
10. a.


$x = -\frac{3\pi}{2}$
$x = -\frac{\pi}{2}$
$x = \frac{\pi}{2}$
$x = \frac{3\pi}{2}$
$y = \cot 2x$
$x = -2\pi$
$x = -\pi$
$x = 0$
$x = \pi$
$x = 2\pi$

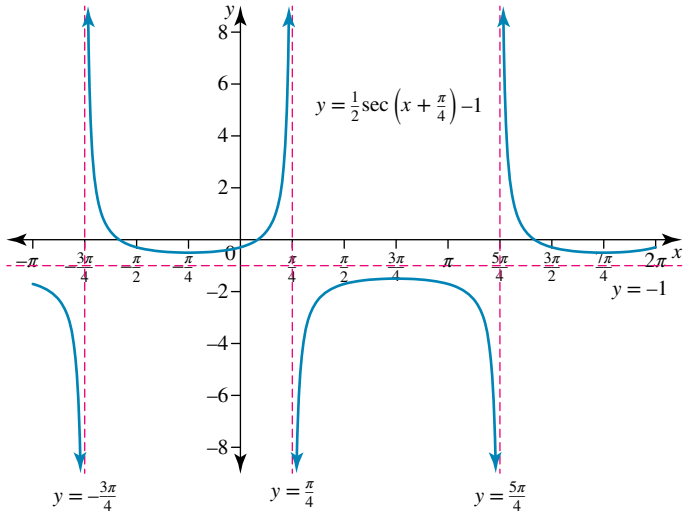
b.


$x = -2\pi$
$x = -\frac{2\pi}{3}$
$x = \frac{2\pi}{3}$
$x = 2\pi$
$y = \cot 3x$
$x = -\frac{4\pi}{3}$
$x = 0$
$x = \frac{4\pi}{3}$

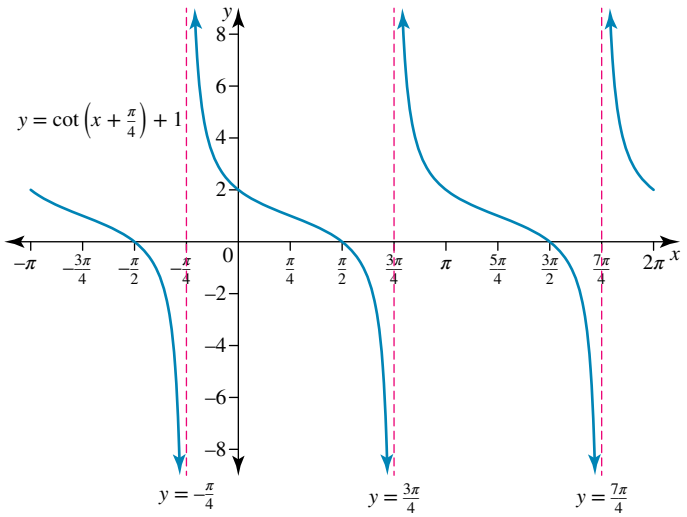
11.


$y = \cos\left(\frac{x}{3}\right)$
$x = -3\pi$
$x = 0$
$x = 3\pi$

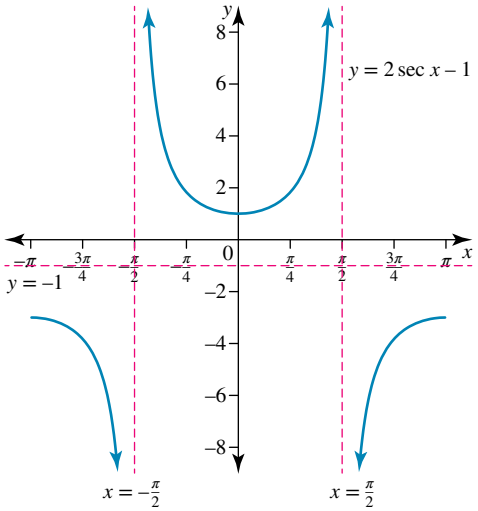
12. a.


$y = \frac{1}{2}\sec\left(x + \frac{\pi}{4}\right) - 1$
$y = -1$
$y = -\frac{3\pi}{4}$
$y = \frac{\pi}{4}$
$y = \frac{5\pi}{4}$

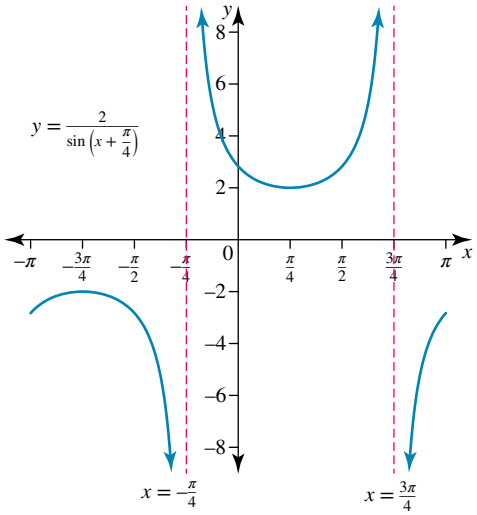
b.



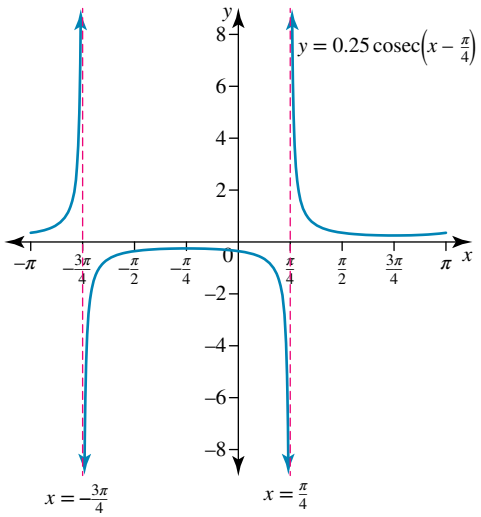
13. a.



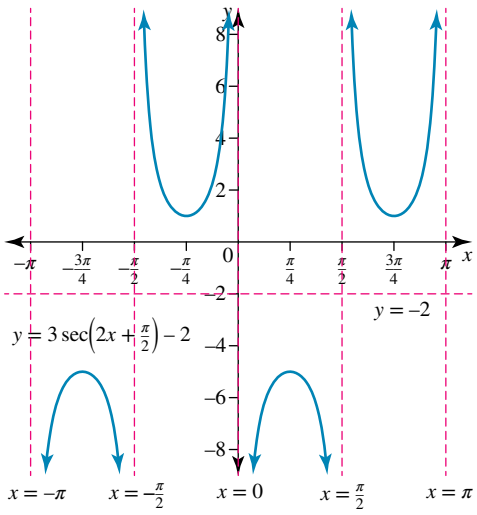
b.



c.

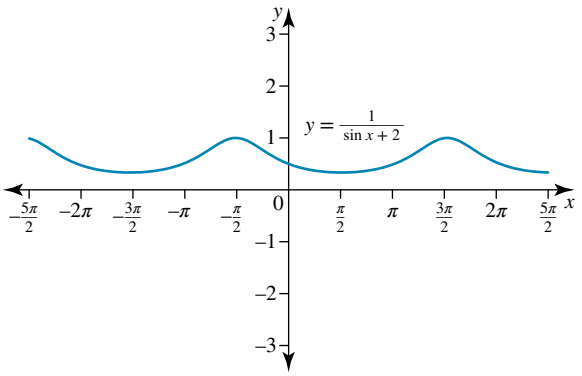


d.

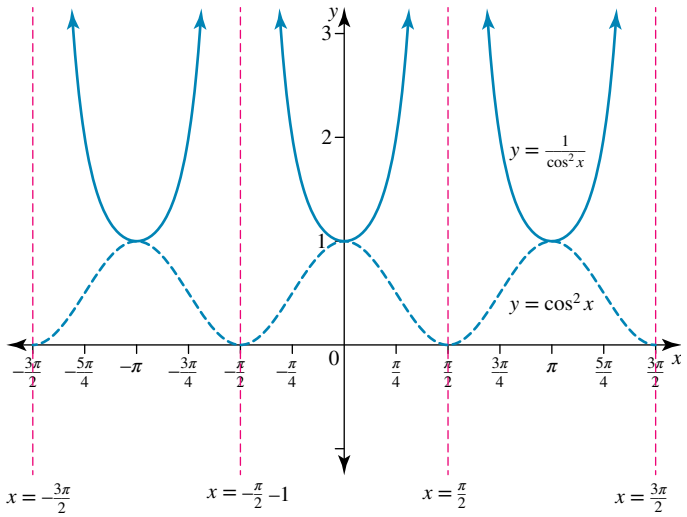


14. a. $\dfrac{p^2 - q^2 - pq}{q\sqrt{p^2 - q^2}}$ **b.** $\dfrac{\sqrt{a^2 - b^2}}{a + b}$

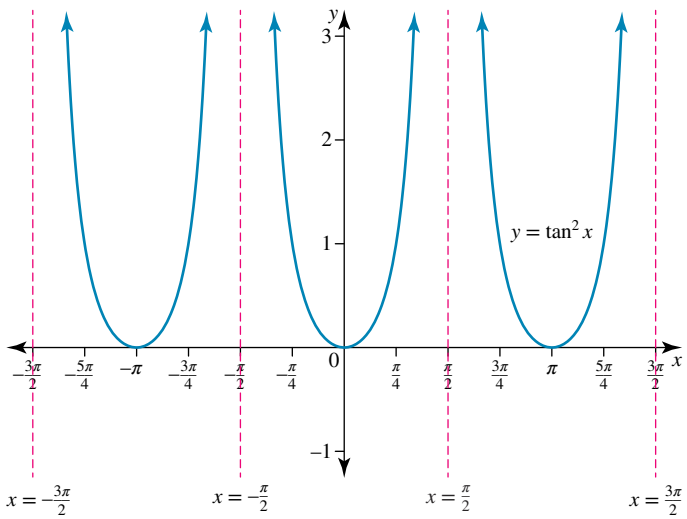
15.



16. a.



b.



Exercise 8.6 Modelling periodic functions

1. a. 1.6 m

b. i. 1 m **ii.** 0.7 m

2. a. 26 °C at 2 pm **b.** 18 °C **c.** 22 °C **d.** 11.07 °C

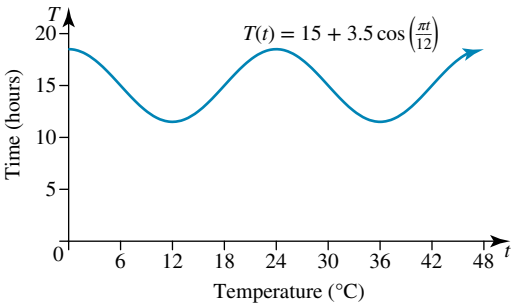
3. a. 12 mm

b. $\frac{1}{10}$ s

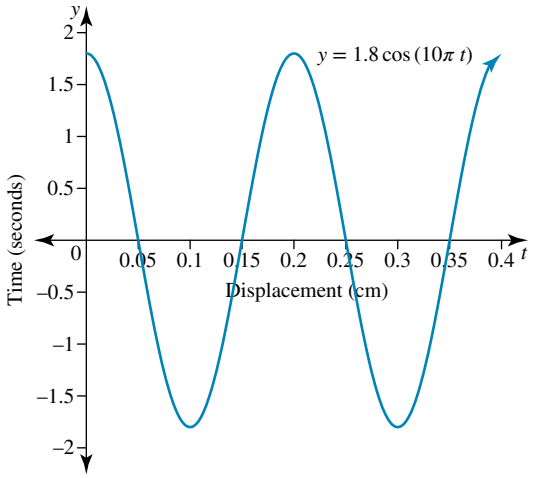
c. 10 vibrations

d. If displacement is positive to the right, the string is 11.41 mm to the left, or vice versa.

4. a. 18 °C
b. 18.5 °C; 0, 24 h
c. 11.5 °C; 12, 36 h
d. Period 24 hours, amplitude 3.5 °C, range [11.5, 18.5]
e. 4, 20, 28 and 44 h
f.

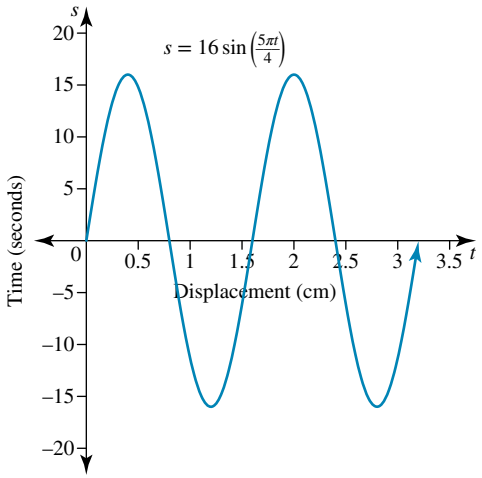


5. a.



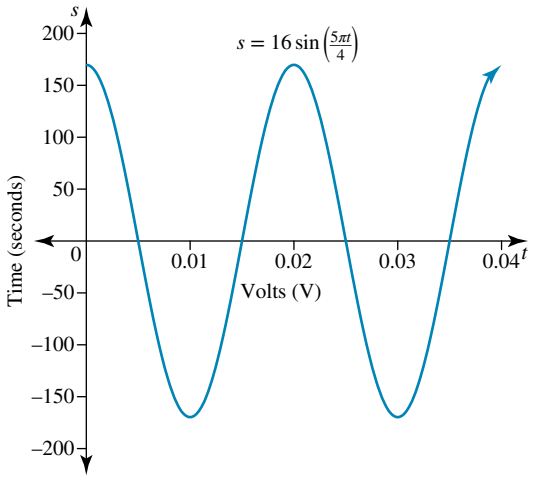
b. $\frac{1}{30}$, $\frac{1}{6}$, $\frac{7}{30}$, $\frac{11}{30}$ s

6. a.



b. $\frac{2}{15}$, $\frac{2}{3}$, $\frac{26}{15}$, $\frac{34}{15}$ s

7. a.



b. $\frac{1}{400}$ (0.0025), $\frac{7}{400}$ (0.0175), $\frac{9}{400}$ (0.0225), $\frac{3}{80}$ (0.0375) s

8. a. i. 1 kg **ii.** 6 days

b. $W = \cos\left(\frac{\pi t}{3}\right) + 3$

9. a. 110 beats/minute

b. 50 beats/minute

c. 60 minutes

d. $H = 50\sin\left(\frac{\pi t}{30}\right) + 110$

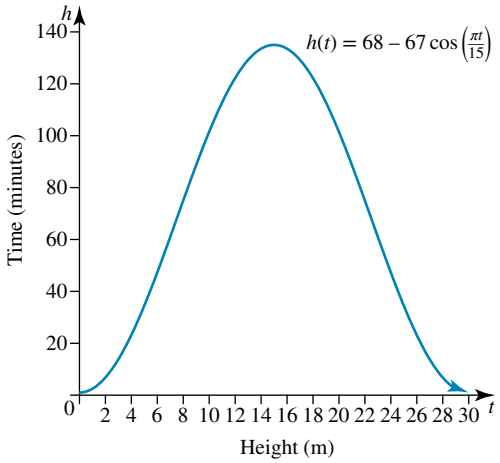
10. $c = 21, b = 3, n = \frac{\pi}{12}$

11. a. 135 m, 15 minutes

b. Amplitude 67 m, period 30 minutes

c. 13.28 minutes

d.



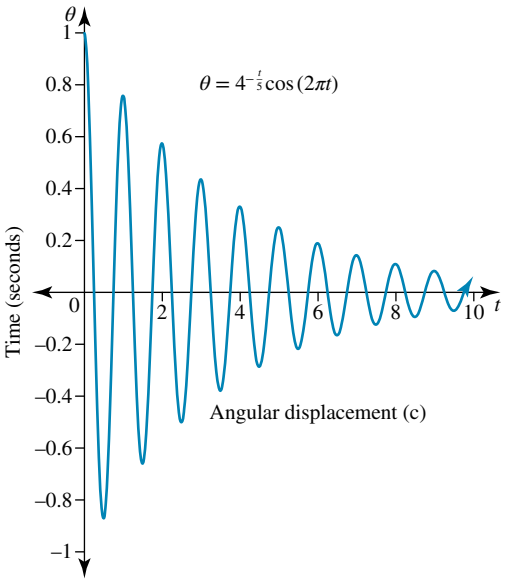
12. a. 1 radian

b. 0.25 radians

c. 0.0625 radians

d. 0 radians

e.



f. 9 times

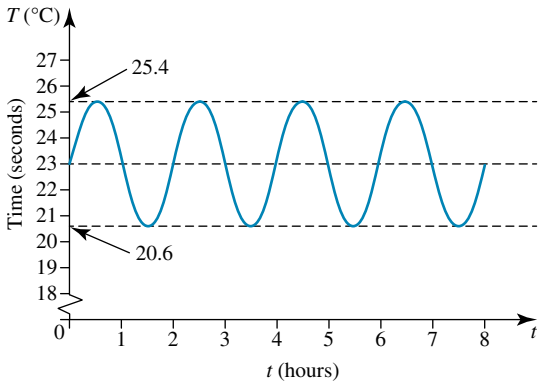
13. a. 60 seconds **b.** 50 m **c.** 100 m **d.** 314.16 m **e.** 50 m **f.** 75 m

14. a. Maximum 6 m, minimum 3 m

b. Yes. It will have 24 minutes to spare.

15. a. Maximum 25.4 °C, minimum 20.6 °C

b. i. 23.0 °C **ii.** 25.4 °C

c. 

d. 2 hours 40 minutes

16. a. 40 cm **b.** 80 cm **c.** 64 cm **d.** $y = 0.3 \sin\left(\frac{\pi x}{16}\right)$

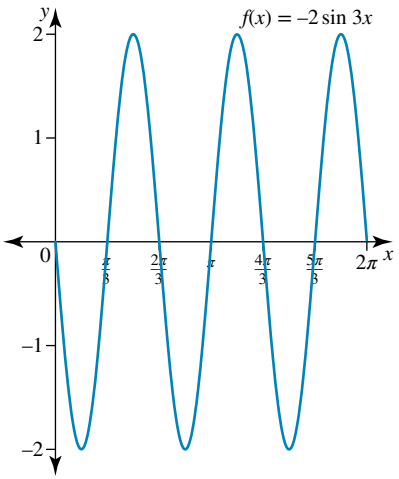
8.7 Review: exam practice

1. a. $\frac{1}{2}$ **b.** $\frac{1}{2}$ **c.** 1 **d.** $\frac{\sqrt{2}}{2}$ **e.** $\frac{\sqrt{3}}{2}$ **f.** $\sqrt{3}$

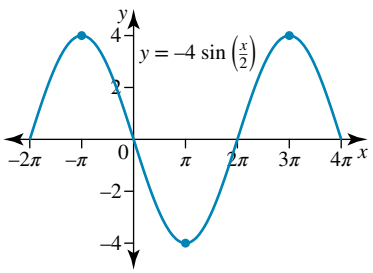
2. a. 2

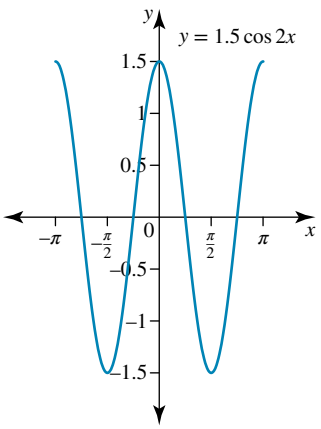
b. $[-2, 2]$

c. $\frac{2\pi}{3}$

d. 

3. $y = 3 \cos \frac{x}{2}$

4. a. 

b. 

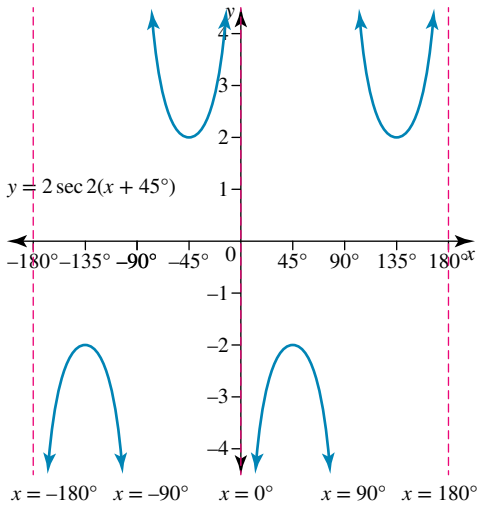
5. a. 64°, 116° **b.** 114°, 246° **c.** 58°, 238° **d.** 240°, 300°

e. 60°, 300° **f.** 45°, 225°

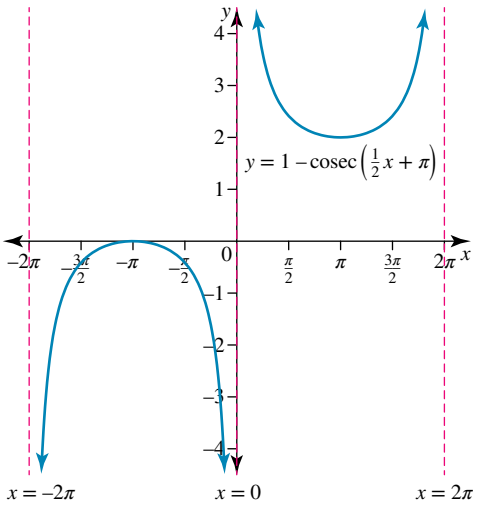
6. a. $\frac{5\pi}{6}, \frac{7\pi}{6}$ **b.** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **c.** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **d.** $\frac{\pi}{3}, \frac{2\pi}{3}$

e. $\frac{2\pi}{3}, \frac{4\pi}{3}$ **f.** $\frac{3\pi}{4}, \frac{7\pi}{4}$

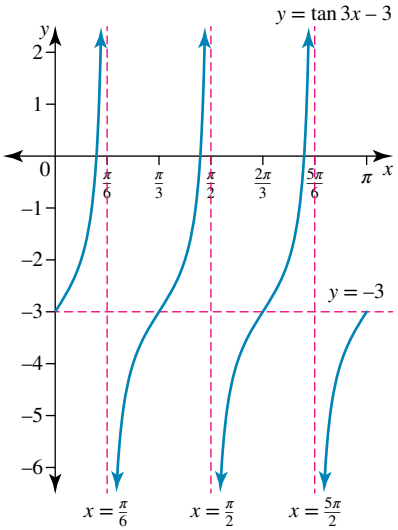
7.



8.



9.



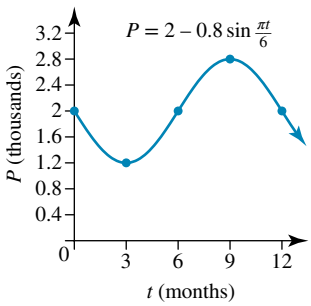
10. $x = \frac{\pi}{24}(12n - 5)$

11. a. $\frac{\pi}{18}(6n - 1)$ **b.** $\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}$

12. a. $\frac{\pi}{4}(4n \pm 1)$ **b.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

13. a. Maximum 2800, minimum 1200, equilibrium 2000 **b.** Period 12 months, amplitude 800

c.



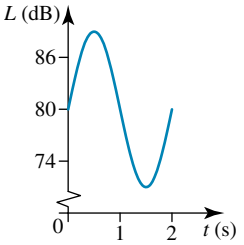
d. 1600

14. a. Amplitude 6 dB, period 2 s

b. Maximum 86 dB, first time $\frac{1}{2}$ s

c. $\frac{1}{6}$ s

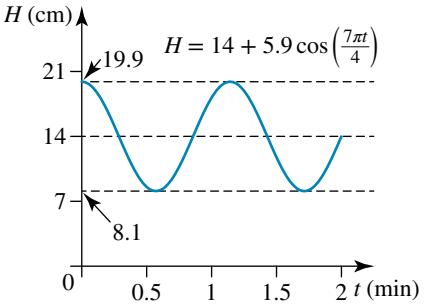
d. $L(t) = 6\sin \pi t + 80$



15. a. Maximum 19.9 cm, minimum 8.1 cm

b. 8.1 cm

c.



d. 52.5

e. $\frac{4}{7}$ s

f. $H(t) = 14 + 5.9\cos\left(\frac{5\pi t}{4}\right)$

16. $y = 90 + 30\sin\left(\frac{\pi x}{30}\right)$

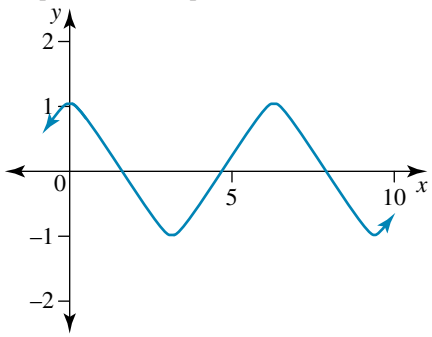
17. a. Sample responses can be found in the worked solutions in the online resources.

b. $b = \frac{\pi}{7}$, $a = 2\tan\left(\frac{\pi}{14}\right)$

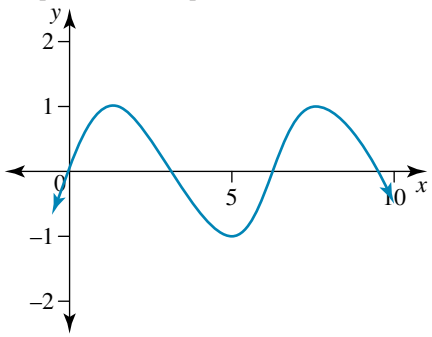
c. $x = 2.5$, 0.22 m above the sack

d. $y = 2\sin\frac{\pi}{14}\operatorname{cosec}\left(\frac{\pi x}{7}\right)$, $0 \leq x \leq 3.5$

18. a. Amplitude 1.1715, period 6.283



b. Amplitude 0.9887, period 6.283



19. a. Area $= x^2 \sin\theta\,(1 + \cos\theta)$

b. $\theta_{\max} = 60°$

20. a. $d = 0.981\sin(0.521t - 0.202) + 1.580$

b. Between 7:15 pm and 12:23 am the next day

CHAPTER 9
Trigonometric identities

9.1 Overview

9.1.1 Introduction

An identity is a relationship that hold true for all possible values of the variable or variables. This chapter explores trigonometric identities — identities that use functions of one or more angles. You have already used the trigonometric identity $\tan(A) = \frac{\sin(A)}{\cos(A)}$. Identities are useful for simplifying expressions involving trigonometric functions, which becomes important in the study of calculus. These trigonometric identities, and calculus more broadly, form the building blocks of architecture and are used across all fields of engineering.

The use of trigonometric identities in architecture is both aesthetic and practical. Many modern buildings, as demonstrated by the images in this chapter, make use of trigonometry, geometry and calculus to create complex, beautiful facades. But this use of mathematics in architecture is more than superficial. Architects must ensure the buildings they design can withhold the loads and forces which act upon these structures. Although these loads and forces are typically defined by vectors, the vector components can be resolved through the use of trigonometric functions relative to the angle the force makes with an axis. Architects also use trigonometry for passive solar design to create comfortable and energy-efficient buildings.

The Russian mathematician Pafnuty Chebyshev (1821–1894) is better known for his work in the fields of probability, statistics, number theory and differential equations, but he also devised recurrence relations for trigonometric multiple angles that we explore later in this chapter.

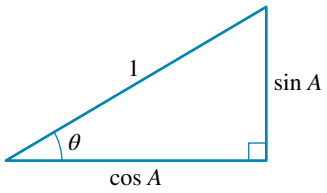
LEARNING SEQUENCE

9.1 Overview
9.2 Pythagorean identities
9.3 Compound angle formulas
9.4 Multiple angle formulas
9.5 Product–sum identities
9.6 Convert $a\cos(x) + b\sin(x)$ to $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$
9.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.2 Pythagorean identities

9.2.1 Using the Pythagorean identity to solve simple trigonometric unknowns



Consider the right-angled triangle with a hypotenuse of 1 as shown. Applying Pythagoras' theorem to this triangle results in the **identity** $\sin^2(A) + \cos^2(A) = 1$. This is known as the **Pythagorean identity**.

WORKED EXAMPLE 1

If $\sin(A) = 0.4$ and $0° < A < 90°$, find $\cos(A)$ correct to 3 decimal places.

THINK

a. 1. Use the identity $\sin^2(A) + \cos^2(A) = 1$.

2. Substitute 0.4 for $\sin(A)$.

3. Solve the equation for $\cos(A)$ correct to 3 decimal places.

4. Retain the positive answer only as cosine is positive in the first quadrant.

WRITE

a. $\sin^2(A) + \cos^2(A) = 1$

$(0.4)^2 + \cos^2(A) = 1$

$\cos^2(A) = 1 - 0.16$

$= 0.84$

$\cos(A) = \pm\sqrt{0.84}$

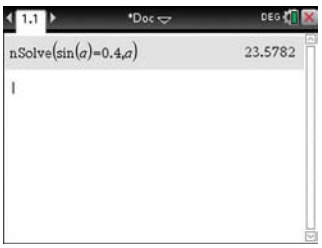
$= 0.917 \text{ or } -0.917$

For $0° < A < 90°$, cos is positive so $\cos(A) = 0.917$.

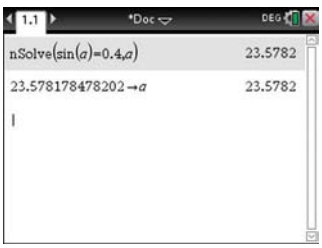
TI | THINK

1. Put the calculator in DEGREE mode.
On a Calculator page, select MENU, then select:
3: Algebra
1: Numerical Solve
Complete the entry line as:
nSolve(sin(*a*) = 0.4, *a*)
then press ENTER.

WRITE



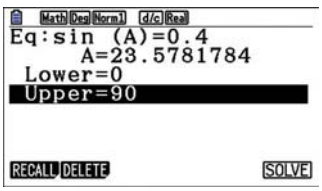
2. To store this answer as the value for *a*, press ctrl then var, then type "*a*". Press ENTER.



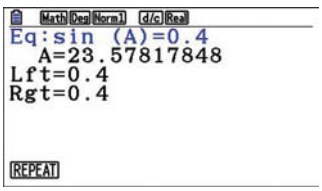
CASIO | THINK

1. Put the calculator in DEGREE mode.
On an Equation screen, select Solver by pressing F3.
Complete the entry line for the equation as:
sin(*A*) = 0.4
then press EXE.
Set the lower bound to 0 and the upper bound to 90.

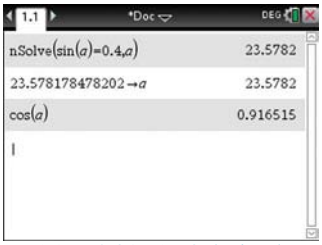
WRITE



2. Select SOLVE by pressing F6.



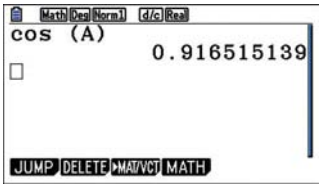
3. Complete the next entry line as:
cos(a)
then press ENTER.



4. The answer appears on the screen.

cos(A) = 0.917 (to 3 decimal places).

3. On a Run-Matrix screen, complete the entry line as:
cos(A)
then press EXE.



4. The answer appears on the screen.

cos(A) = 0.917 (to 3 decimal places).

WORKED EXAMPLE 2

Find all possible values of sin(A) over the domain $0 \leq A \leq 2\pi$ if cos(A) = 0.75.

THINK

1. Use the identity $\sin^2(A) + \cos^2(A) = 1$.
2. Substitute 0.75 for cos(A).
3. Solve the equation for sin(A) correct to 3 decimal places.
4. Retain both the positive and negative solutions, since the angle could be in either the first or fourth quadrants.

WRITE

$\sin^2(A) + \cos^2(A) = 1$
$\sin^2(A) + (0.75)^2 = 1$
$\sin^2(A) = 1 - 0.5625$
$= 0.4375$
$\sin(A) = \pm\sqrt{0.4375}$
$= 0.661 \text{ or } -0.661$

TI | THINK

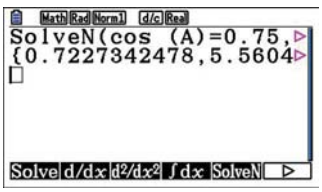
1. Put the calculator in RADIAN mode.
On a Calculator page, select MENU, then select:
3: Algebra
1: Numerical Solve
Complete the entry line as:
nSolve(cos(a) = 0.75, a)
then press ENTER.
This gives the solution found in the 1st quadrant.

WRITE

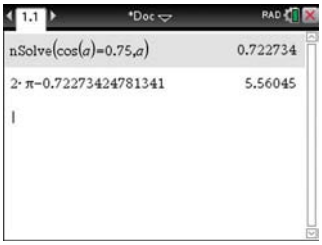
CASIO | THINK

1. Put the calculator in RADIAN mode.
On a Run-Matrix screen, press OPTN, then select CALC by pressing F4.
Select SolveN by pressing F5.
Complete the entry line as:
SolveN(cos(A) = 0.75, A, 0, 2π)
Press b, then press OPTN.
Select LIST by pressing F1, then select List by pressing F1 again. Type '1', then press EXE.
This will store all solutions in List1.

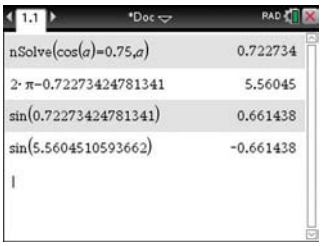
WRITE



2. To find the solution found in the 4th quadrant, complete the next entry line as:
$2\pi - ans$
then press ENTER.



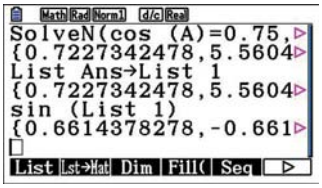
3. Complete the next entry line as:
sin(0.722734)
then press ENTER.
Complete the next entry line as:
sin(5.56045)
then press ENTER.
Note: Use the up arrow to highlight a previous answer and press ENTER to paste it onto the current entry line.



4. The answers appear on the screen.

$\sin(A) = 0.661, -0.661$

2. Complete the next entry line as:
sin(List1)
then press EXE.
This calculates sine of all values in List 1.



3. The answers appear on the screen.

$\sin(A) = 0.661, -0.661$

9.2.2 Other Pythagorean identities

The Pythagorean identity can be manipulated to demonstrate other relationships, which can then be used to simplify more complex trigonometric expressions.

Remember that $\tan(A) = \dfrac{\sin(A)}{\cos(A)}$, $\cot A = \dfrac{\cos(A)}{\sin(A)}$, $\sec(A) = \dfrac{1}{\cos(A)}$ and $\text{cosec}(A) = \dfrac{1}{\sin(A)}$.

Divide each term in the original Pythagorean identity, $\sin^2(A) + \cos^2(A) = 1$, by $\cos^2(A)$:

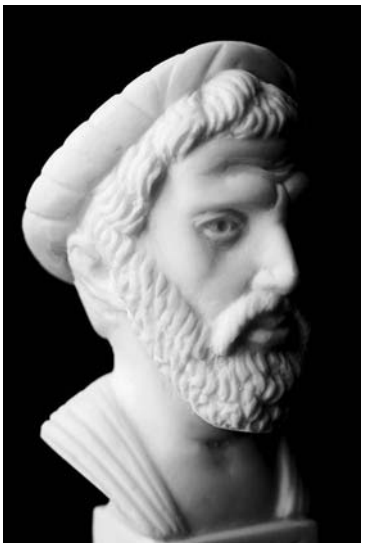
$$\frac{\sin^2(A)}{\cos^2(A)} + \frac{\cos^2(A)}{\cos^2(A)} = \frac{1}{\cos^2(A)}$$

$$\tan^2(A) + 1 = \sec^2(A)$$

Divide each term in the original Pythagorean identity, $\sin^2(A) + \cos^2(A) = 1$, by $\sin^2(A)$:

$$\frac{\sin^2(A)}{\sin^2(A)} + \frac{\cos^2(A)}{\sin^2(A)} = \frac{1}{\sin^2(A)}$$

$$1 + \cot^2(A) = \text{cosec}^2(A)$$



Pythagorean identities:

$$\mathbf{\sin^2(A) + \cos^2(A) = 1}$$

$$\mathbf{\tan^2(A) + 1 = \sec^2(A)}$$

$$\mathbf{\cot^2(A) + 1 = \text{cosec}^2(A)}$$

WORKED EXAMPLE 3

Use a Pythagorean identity to simplify $\cos^2(A)\tan^2(A)$.

THINK	WRITE
1. Use the identity $\tan^2(A) + 1 = \sec^2(A)$.	Using $\tan^2(A) = \sec^2(A) - 1$, $\cos^2(A)\tan^2(A) = \cos^2(A)\left(\sec^2(A) - 1\right)$
	$= \cos^2(A)\sec^2(A) - \cos^2(A)$
2. As $\sec(A) = \dfrac{1}{\cos(A)}$, $\cos^2(A)\sec^2(A) = 1$.	$= 1 - \cos^2(A)$
3. $\sin^2(A) + \cos^2(A) = 1$ so $\sin^2(A) = 1 - \cos^2(A)$.	$= \sin^2(A)$

Sometimes it is easier to simplify expressions involving $\tan(A)$, $\cot(A)$, $\sec(A)$ and $\operatorname{cosec}(A)$ by rewriting them in terms of $\sin(A)$ and $\cos(A)$.

WORKED EXAMPLE 4

Simplify $\dfrac{\tan^2(A) - 1}{\tan^2(A) + 1}$.

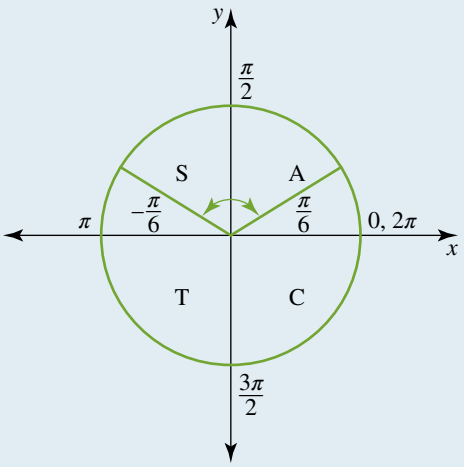
THINK	WRITE
1. Substitute $\tan(A) = \dfrac{\sin(A)}{\cos(A)}$.	$\dfrac{\tan^2(A) - 1}{\tan^2(A) + 1} = \dfrac{\frac{\sin^2(A)}{\cos^2(A)} - 1}{\frac{\sin^2(A)}{\cos^2(A)} + 1}$
2. To assist with simplifying the fraction, multiply by $\dfrac{\cos^2(A)}{\cos^2(A)}$.	$= \dfrac{\frac{\sin^2(A)}{\cos^2(A)} - 1}{\frac{\sin^2(A)}{\cos^2(A)} + 1} \times \dfrac{\cos^2(A)}{\cos^2(A)}$
	$= \dfrac{\frac{\sin^2(A)\,\cancel{\cos^2(A)}}{\cancel{\cos^2(A)}} - \cos^2(A)}{\frac{\sin^2(A)\,\cancel{\cos^2(A)}}{\cancel{\cos^2(A)}} + \cos^2(A)}$
	$= \dfrac{\sin^2(A) - \cos^2(A)}{\sin^2(A) + \cos^2(A)}$
3. Use $\sin^2(A) + \cos^2(A) = 1$ to simplify the denominator.	$= \dfrac{\sin^2(A) - \cos^2(A)}{1}$
	$= \sin^2(A) - \cos^2(A)$

9.2.3 Quadratic trigonometric equations

In your studies so far, you have factorised quadratic expressions and solved quadratic equations. These skills can also be used to factorise and solve equations when the quadratic is written in terms of a **trigonometric function**.

WORKED EXAMPLE 5

Solve the equation $2\sin^2(A) = \sin(A)$ for A over the domain $0 \leq A \leq 2\pi$.

THINK	WRITE
1. Write the equation.	$2\sin^2(A) = \sin(A)$
2. Move $\sin(A)$ to the left of the equation to make the equation equal to zero.	$2\sin^2(A) - \sin(A) = 0$
3. This is a quadratic in terms of $\sin(A)$. It can be easier to identify the factors if you rewrite the quadratic in terms of a variable.	Let $a = \sin(A)$. $2a^2 - a = 0$ $a(2a - 1) = 0$
4. Rewrite the equation in terms of $\sin(A)$.	$\sin(A)(2\sin(A) - 1) = 0$
5. Using the Null Factor Law, solve each factor equal to 0.	$\sin(A) = 0$ or $2\sin(A) - 1 = 0$
6. Solve $\sin(A) = 0$.	$\sin(A) = 0 \quad \sin(A) = \frac{1}{2}$ $A = 0, \pi, 2\pi$
7. Solve $\sin(A) = \frac{1}{2}$, remembering that $\sin(A)$ will be positive in quadrants 1 and 2. Sin $\left(\frac{\pi}{6}\right) = \frac{1}{2}$ is a known trigonometric ratio.	 $\sin(A) = \frac{1}{2}$ The quadrant 1 solution becomes: $A = \frac{\pi}{6}$ The quadrant 2 solution becomes: $A = \frac{5\pi}{6}$
8. Combine all 5 solutions to the equation.	$A = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, 2\pi$

Sometimes it is necessary to use the Pythagorean identities to rewrite the expression as a quadratic in terms of one trigonometric function.

WORKED EXAMPLE 6

Use a Pythagorean identity to factorise $\text{cosec}^2(A) - \cot(A) - 3$.

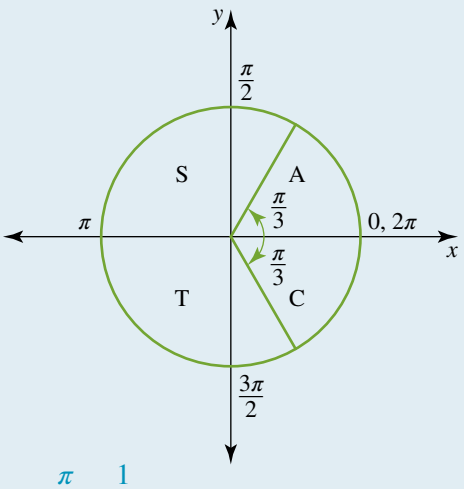
THINK	WRITE
1. Use the identity $1 + \cot^2(A) = \text{cosec}^2(A)$ and then collect like terms.	$\text{cosec}^2(A) - \cot(A) - 3 = 1 + \cot^2(A) - \cot(A) - 3$ $= \cot^2(A) - \cot(A) - 2$
2. This is a quadratic in terms of $\cot(A)$. It can be easier to identify the factors if you rewrite the quadratic in terms of a variable.	Let $a = \cot(A)$. $\cot^2(A) - \cot(A) - 2 = a^2 - a - 2$ $= (a - 2)(a + 1)$
3. Rewrite the factors in terms of $\cot(A)$.	$\text{cosec}^2(A) - \cot(A) - 3 = (\cot(A) - 2)(\cot(A) + 1)$

WORKED EXAMPLE 7

Solve the equation $2\sin^2(A) = \cos(A) + 1$ for A over the domain $0 \leq \theta \leq 2\pi$.

THINK	WRITE
1. Write the equation.	$2\sin^2(A) = \cos(A) + 1$
2. Make the substitution $\sin^2(A) = 1 - \cos^2(A)$.	$2\left(1 - \cos^2(A)\right) = \cos(A) + 1$
3. Form a quadratic equation by expanding the brackets and then bringing all of the terms to one side. Before we factorise a quadratic, we normally rearrange to write the square term with a positive coefficient. This is a quadratic equation in terms of $\cos(A)$.	$2 - 2\cos^2(A) = \cos(A) + 1$ $0 = \cos(A) + 1 + 2\cos^2(A) - 2$ $0 = 2\cos^2(A) + \cos(A) - 1$
4. Factors of quadratic trigonometric equations can be more easily identified if the quadratic is rewritten in terms of a variable.	Let $a = \cos(A)$. $2a^2 + a - 1 = 0$
5. Factorise the equation.	$(2a - 1)(a + 1) = 0$
6. Rewrite the factors in terms of $\cos(A)$.	$(2\cos(A) - 1)(\cos(A) + 1) = 0$
7. Using the Null Factor Law, solve each factor equal to 0.	$2\cos(A) - 1 = 0$ or $\cos(A) + 1 = 0$ $\cos(A) = \frac{1}{2}$ $\quad$ $\cos(A) = -1$

8. Solve $\cos(A) = \frac{1}{2}$, remembering that $\cos(A)$ will be positive in quadrants 1 and 4. Cos $\left(\frac{\pi}{3}\right) = \frac{1}{2}$ is a known trigonometric ratio.



$\cos\frac{\pi}{3} = \frac{1}{2}$

The quadrant 1 solution becomes:

$A = \frac{\pi}{3}$

The quadrant 4 solution becomes:

$A = 2\pi - \frac{\pi}{3}$

$= \frac{5\pi}{3}$

9. Solve $\cos(A) = -1$.
   $\cos(\pi) = -1$ is a known trigonometric ratio.

$\cos(A) = -1$

$A = \pi$

10. Combine all solutions.

$A = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

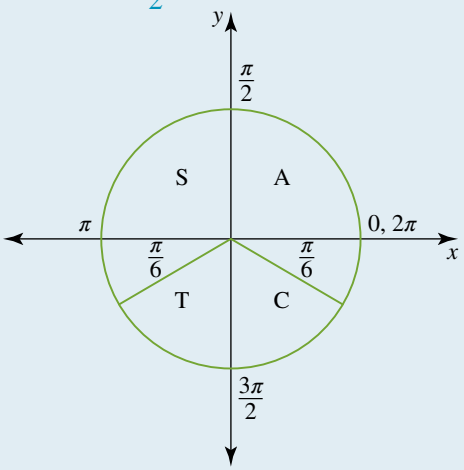
WORKED EXAMPLE 8

Find the general solution to $2\sin^2(2A) - \sin(2A) - 1 = 0$.

THINK	WRITE
1. This is a quadratic in terms of $\sin(2A)$. Factors of quadratic trigonometric equations can be more easily identified if the quadratic is rewritten in terms of a variable.	Let $a = \sin(2A)$. $2a^2 - a - 1 = 0$
2. Factorise the equation.	$(2a + 1)(a - 1) = 0$
3. Rewrite the factors in terms of $\sin(2A)$.	$(2\sin(2A) + 1)(\sin(2A) - 1) = 0$
4. Using the Null Factor Law, solve each factor equal to 0.	$2\sin(2A) + 1 = 0$ or $\sin(2A) - 1 = 0$ $\sin(2A) = -\frac{1}{2}$ $\quad$ $\sin(2A) = 1$

5. Solve $\sin(2A) = -\frac{1}{2}$, remembering that $\sin(2A)$ will be negative in quadrants 3 and 4, and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ is a known trigonometric ratio.

$\sin(2A) = -\frac{1}{2}$



As a general solution is needed, each solution for $2A$ will require $2n\pi$, $n \in Z$.

The quadrant 3 solution becomes:

$$2A = -\pi + \frac{\pi}{6} + 2n\pi$$
$$= -\frac{5\pi}{6} + 2n\pi$$
$$A = -\frac{5\pi}{12} + n\pi$$
$$= \pi\left(-\frac{5}{12} + n\right)$$
$$= \pi\left(\frac{-5 + 12n}{12}\right)$$
$$= \frac{\pi}{12}(12n - 5)$$

The quadrant 4 solution becomes:

$$2A = -\frac{\pi}{6} + 2n\pi$$
$$A = -\frac{\pi}{12} + n\pi$$
$$= \pi\left(-\frac{1}{12} + n\right)$$
$$= \pi\left(\frac{12n - 1}{12}\right)$$
$$= \frac{\pi}{12}(12n - 1)$$

6. Solve $\sin(2A) = 1$ using $\sin\left(\frac{\pi}{2}\right) = 1$.

$\sin(2A) = 1$

The general solution becomes:

$$2A = \frac{\pi}{2} + 2n\pi$$

$$A = \frac{\pi}{4} + n\pi$$
$$= \pi\left(\frac{1}{4} + n\right)$$
$$= \pi\left(\frac{1+4n}{4}\right)$$
$$= \frac{\pi}{4}(1+4n)$$

7. Combine all solutions. $A = \frac{\pi}{12}(12n-5), \frac{\pi}{12}(12n-1), \frac{\pi}{4}(4n+1), n \in Z$

studyon

Units 1 & 2 > Area 5 > Sequence 3 > Concept 1

Pythagorean identities Summary screen and practice questions

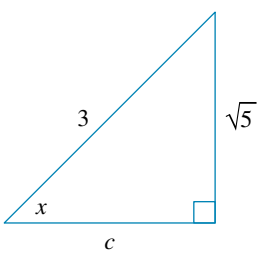
Exercise 9.2 Pythagorean identities

Technology free

1. Copy and complete the table, correct to 3 decimal places:

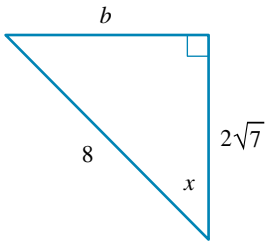
A	30°	81°	129°	193°	260°	350°	−47°
$\sin^2(A)$							
$\cos^2(A)$							
$\sin^2(A) + \cos^2(A)$							

2. **WE1** If $\sin(A) = 0.8$ and $0° < A < 90°$, find, correct to 3 decimal places:
   a. $\cos(A)$ b. $\tan(A)$
3. If $\cos(A) = 0.3$ and $0° < A < 90°$, find, correct to 3 decimal places:
   a. $\sin(A)$ b. $\tan(A)$
4. **WE2** Find all possible values of the following, correct to 3 decimal places.
   a. $\cos(x)$ if $\sin(x) = 0.4$ b. $\cos(x)$ if $\sin(x) = -0.7$
   c. $\sin(x)$ if $\cos(x) = 0.24$ d. $\sin(x)$ if $\cos(x) = -0.9$
5. Use the diagram to find the exact value of:
   a. $\sin(x)$ b. $\cos(x)$



6. Use the diagram to find the exact value of:

a. $\cos(x)$ **b.** $\tan(x)$



7. Given that $\sin(A) = \frac{\sqrt{5}}{4}$ and $\frac{3\pi}{2} < A < 2\pi$, find:

a. the exact value of $\cos(A)$

b. the exact value of $\tan(A)$.

8. **WE3** Use the Pythagorean identity to simplify $\tan^2(A) - \sin^2(A)\tan^2(A)$.

9. **WE4** Simplify $\frac{\text{cosec}(A)\cos^2(A)}{1 + \text{cosec}(A)}$.

10. **WE5** Solve each of the following equations over the domain $0 \leq x \leq 2\pi$.

a. $\sin^2(x) - \sin(x) = 0$ **b.** $\cos^2(x) + \cos(x) = 0$

c. $2\sin^2(x) + \sqrt{3\sin(x)} = 0$ **d.** $2\cos^2(x) + \cos(x) - 1 = 0$

e. $\sin^2(x) + 3\sin(x) - 4 = 0$ **f.** $\sin^2(x) - \sin(x) - 1 = 0$

11. **WE6** Use a Pythagorean identity to factorise each of the following.

a. $1 + \sin(A) - 2\cos^2(A)$ **b.** $\sec^2(x) - \tan(x) - 3$

c. $2\cot^2(\alpha) - \text{cosec}(\alpha) + 1$ **d.** $2\cos^2(3x) - 2\sin^2(3x) + 1$

e. $\text{cosec}^2(3A) - 2\text{cosec}(3A) + 2\cot^2(3A) + 1$ **f.** $\tan^2(2\beta) - \sec(2\beta) - 1$

12. **WE7** Solve each of the following equations over the domain $0 \leq A \leq 2\pi$.

a. $2\cos^2(A) = 1 + \sin(A)$ **b.** $2\sin^2(A) + \sin(A) - 1 = 0$

c. $2\sin^2(A) - 1 = 0$ **d.** $1 + \cos(A) = 2\sin^2(A)$

e. $\sin^2(A) = 1 + \cos(A)$ **f.** $2\cos^2(A) = 5 + 5\sin(A)$

13. **WE8** Find the general solution to each of the following equations.

a. $2\sin^2(2A) - 3\sin(2A) + 1 = 0$

b. $2\cos^2(2A) + \cos(2A) - 1 = 0$

c. $2\sin^2(4x) + \sin(4x) = 0$

d. $\cos^2(4x) - \cos(4x) = 0$

e. $2\cos^2(3x) + \sqrt{3}\cos(3x) = 0$

f. $2\sin^2(3x) - \sqrt{3}\sin(3x) = 0$

14. Find the general solution to each of the following equations.

a. $\tan^2(x) + \left(\sqrt{3} + 1\right)\tan(x) + \sqrt{3} = 0$

b. $\tan^2(x) + \left(\sqrt{3} - 1\right)\tan(x) - \sqrt{3} = 0$

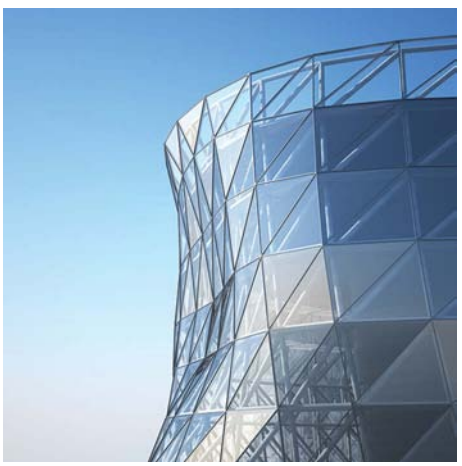
15. Find the general solution to each of the following equations.

a. $2\sin^3(x) + \sin^2(x) - 2\sin(x) - 1 = 0$

b. $2\cos^3(x) - \cos^2(x) - 2\cos(x) + 1 = 0$

c. $\tan^3(x) - \tan^2(x) - \tan(x) + 1 = 0$

d. $\tan^4(x) - 4\tan^2(x) + 3 = 0$



16. Solve the following equation for A.

$$\frac{1}{\sin^2(A)} - \frac{1}{\cos^2(A)} - \frac{1}{\tan^2(A)} - \frac{1}{\cot^2(A)} - \frac{1}{\sec^2(A)} - \frac{1}{\operatorname{cosec}^2(A)} = -3,\ A \in [0,\ 2\pi]$$

17. If $\sin^{16} A = \frac{1}{9}$, solve the following equation for A.

$$\frac{1}{\cos^2(A)} + \frac{1}{1+\sin^2(A)} + \frac{2}{1+\sin^4(A)} + \frac{4}{1+\sin^8(A)}.$$

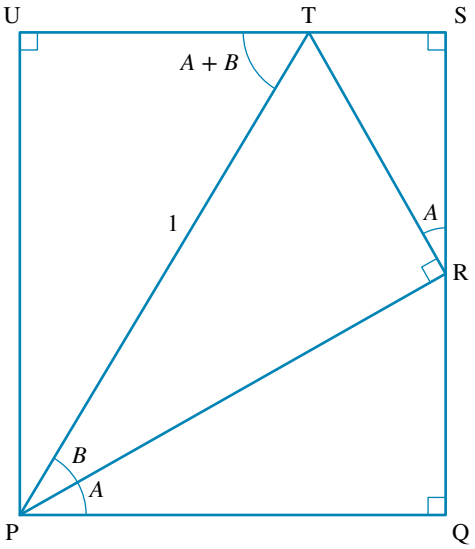
Technology active

18. What is the range of $\sin^4(x) + \cos^2(x)$?

9.3 Compound angle formulas

9.3.1 Angle sum and angle difference formulas

Consider a rectangle PQSU with right-angled triangles PQR and PRT as shown. The length of PT is 1.



Consider ΔPRT.

$$\cos(B) = \frac{PR}{PT} \qquad \sin(B) = \frac{RT}{PT}$$
$$= \frac{PR}{1} \qquad = \frac{RT}{1}$$
$$PR = \cos(B) \qquad RT = \sin(B)$$

Consider ΔPQR.

$$\cos(A) = \frac{PQ}{PR} \qquad \sin(A) = \frac{RQ}{PR}$$
$$= \frac{PQ}{\cos(B)} \qquad = \frac{RQ}{\cos(B)}$$
$$PQ = \cos(A)\cos(B) \qquad RQ = \sin(A)\cos(B)$$

Consider ΔRST.

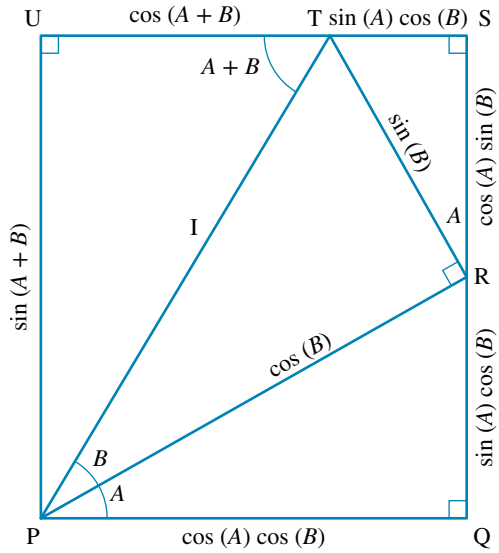
$$\sin(A) = \frac{ST}{RT} \qquad \cos(A) = \frac{RS}{RT}$$
$$= \frac{ST}{\sin(B)} \qquad = \frac{RS}{\sin(B)}$$
$$ST = \sin(A)\sin(B) \qquad RS = \cos(A)\sin(B)$$

Consider ΔPTU.

As UT ∥ PQ, $\angle UTP = \angle TPQ$ (Z)
$= A + B$

$$\cos(A+B) = \frac{UT}{PT} \qquad \sin(A+B) = \frac{PU}{PT}$$
$$= \frac{UT}{1} \qquad = \frac{PU}{1}$$
$$UT = \cos(A+B) \qquad PU = \sin(A+B)$$

Adding these lengths gives us the following diagram.



In the rectangle, PU = QS. Therefore, $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.
As PQ = US, $\cos(A)\ \cos(B) = \cos(A+B) + \sin(A)\sin(B)$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

If we consider $A+(-B)$ and remember that $\cos(-B) = \cos(B)$ and $\sin(-B) = -\sin(B)$, then the identities become:

$$\sin(A + (-B)) = \sin(A)\cos(-B) + \cos(A)\sin(-B)$$
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

$$\cos(A + (-B)) = \cos(A)\cos(-B) - \sin(A)\sin(-B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

These are known as the **angle sum** and **angle difference formulas**.

The angle sum formulas:

$$\mathbf{\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)}$$
$$\mathbf{\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)}$$

The angle difference formulas:

$$\mathbf{\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)}$$
$$\mathbf{\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)}$$

Each of these rules can be used in both ways, depending on the question.

WORKED EXAMPLE 9

Evaluate $\sin 22° \cos 38° + \cos 22° \sin 38°$, giving your answer as an exact value.

THINK	WRITE
1. The formula $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ can be used.	Let $A = 22°$ and $B = 38°$. $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ $\sin 22° \cos 38° + \cos 22° \sin 38° = \sin(22° + 38°)$
2. Simplify the expression	$= \sin 60°$
3. If the exact value is known, then simplify further.	$= \frac{\sqrt{3}}{2}$

The compound angle formulas can be used to simply expressions, particularly when the sum or difference involves an angle with a known trigonometric ratio.

WORKED EXAMPLE 10

Expand $2\cos\left(\theta + \frac{\pi}{3}\right)$.

THINK	WRITE
1. The formula $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ can be used.	Let $A = \theta$ and $B = \frac{\pi}{3}$. $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $2\cos\left(\theta + \frac{\pi}{3}\right) = 2\left(\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right)$
2. Exact values $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ are known, so use these to simplify.	$= 2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right)$
3. Simplify.	$= \cos\theta - \sqrt{3}\sin\theta$
4. State the solution.	$2\cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta - \sqrt{3}\sin\theta$

WORKED EXAMPLE 11

Use one of the compound angle formulas to simplify $\cos\left(\frac{3\pi}{2} - \theta\right)$.

THINK	WRITE
1. The formula $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ can be used.	Let $A = \frac{3\pi}{2}$ and $B = \theta$. $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ $\cos\left(\frac{3\pi}{2} - \theta\right) = \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta$

2. Exact values $\cos\left(\frac{3\pi}{2}\right) = 0$ and $\sin\left(\frac{3\pi}{2}\right) = -1$ are known, so use these to simplify.
3. Simplify.
4. State the solution.

$= 0 \times \cos\theta - 1 \times \sin\theta$

$= -\sin\theta$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

9.3.2 Finding exact values

In previous work, we have used the exact values for 30° and 45° $\left(\text{or } \frac{\pi}{6} \text{ and } \frac{\pi}{4}\right)$ and multiples of these values.

As $45° - 30° = 15° \left(\text{or } \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}\right)$, it is possible to use the compound angle formulas to evaluate exact values for multiples of 15° $\left(\frac{\pi}{12}\right)$.

WORKED EXAMPLE 12

Find the exact value of $\sin\left(\frac{13\pi}{12}\right)$.

THINK

1. Express $\frac{13\pi}{12}$ in terms of multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$.
2. The compound angle formulas $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ can be used.
3. As $\frac{5\pi}{6}$ is in quadrant 2, $\sin\frac{5\pi}{6}$ is positive and $\cos\frac{5\pi}{6}$ is negative.

WRITE

$\frac{13\pi}{12} = \frac{10\pi}{12} + \frac{3\pi}{12}$

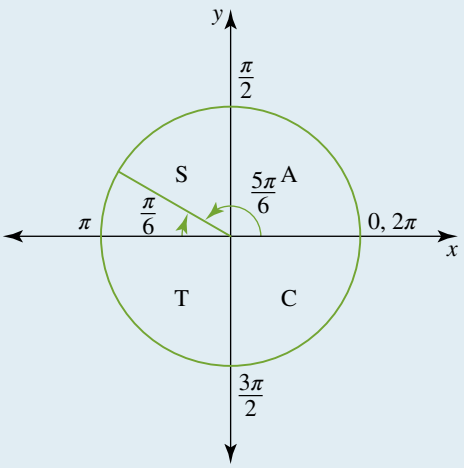
$= \frac{5\pi}{6} + \frac{\pi}{4}$

Let $A = \frac{5\pi}{6}$ and $B = \frac{\pi}{4}$.

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$\sin\frac{13\pi}{12} = \sin\left(\frac{5\pi}{6} + \frac{\pi}{4}\right)$

$= \sin\frac{5\pi}{6}\cos\frac{\pi}{4} + \cos\frac{5\pi}{6}\sin\frac{\pi}{4}$



$$\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} \qquad \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}$$
$$= \frac{1}{2} \qquad = -\frac{\sqrt{3}}{2}$$

4. $\frac{\pi}{4}$ is in quadrant 1, so both $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$ will be positive.

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

5. Substitute values for $\sin\frac{5\pi}{6}$, $\cos\frac{5\pi}{6}$, $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$ and simplify.

$$\sin\frac{13\pi}{12} = \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

9.3.3 Using compound angle formulas and Pythagorean identities

Sometimes it is necessary to use the Pythagorean identities to find the unknown ratios before it is possible to use the compound angle formulas.

WORKED EXAMPLE 13

If $\cos(A) = \frac{12}{13}$ and $\sin(B) = \frac{7}{25}$, where $0 < A < \frac{\pi}{2}$ and $\frac{\pi}{2} < B < \pi$, find the exact value of $\sin(A - B)$.

THINK

1. To calculate $\sin(A - B)$ we will need to calculate $\sin(A)$ and $\cos(B)$. The Pythagorean identity can be used.

WRITE

$$\sin^2(A) + \cos^2(A) = 1$$
$$\sin^2(A) + \left(\frac{12}{13}\right)^2 = 1$$
$$\sin^2(A) = 1 - \frac{144}{169}$$
$$= \frac{25}{169}$$
$$\sin(A) = \pm\frac{5}{13}$$

As $0 < A < \frac{\pi}{2}$, $\sin(A) > 0$,

$\sin(A) = \frac{5}{13}$

$\sin^2(B) + \cos^2(B) = 1$

$\left(\frac{7}{25}\right)^2 + \cos^2(B) = 1$

$\cos^2(B) = 1 - \frac{49}{625}$

$= \frac{576}{625}$

$\cos(B) = \pm\frac{24}{25}$

As $\frac{\pi}{2} < B < \pi$, $\cos(B) < 0$,

$\cos(B) = -\frac{24}{25}$.

2. Use the compound angle formula $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ to simplify and solve.

$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

$= \frac{5}{13} \times -\frac{24}{25} - \frac{12}{13} \times \frac{7}{25}$

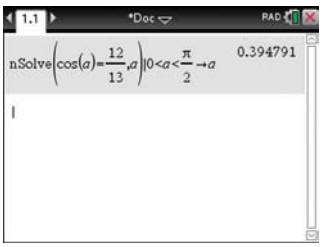
$= -\frac{120}{325} - \frac{84}{325}$

$= -\frac{204}{325}$

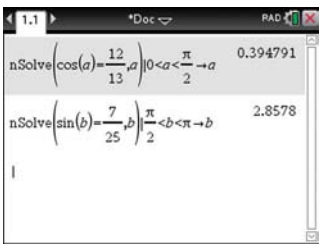
TI | THINK

1. Put the calculator in RADIAN mode.
On a Calculator page, select MENU, then select:
3: Algebra
1: Numerical Solve.
Complete the entry line as: nSolve$(\cos(a) = \frac{12}{13}, a) | 0 < a < \frac{\pi}{2}$
Press ctrl, var and A to store the result as a. Press ENTER.

WRITE



2. Select MENU, then select:
3: Algebra
1: Numerical Solve.
Complete the entry line as:
nSolve$(\sin(b) = \frac{7}{25}, b) | \frac{\pi}{2} < b < \pi$
Press ctrl, var and B to store the result as b. Press ENTER.



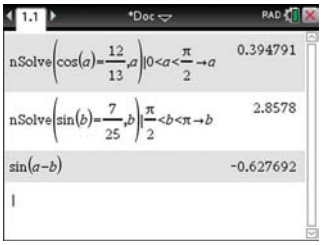
CASIO | THINK

WRITE

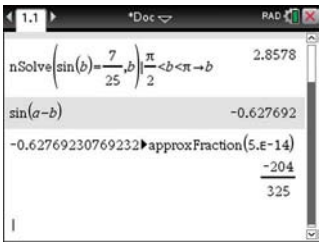
1. Form an expression for A. For $0 < A < \frac{\pi}{2}$, if $\cos A = \frac{12}{13}$, then $A = \cos^{-1}\left(\frac{12}{13}\right)$.[1]

2. Form an expression for B . $\sin B = \frac{7}{25}$ For $\frac{\pi}{2} < B < \pi$, $\sin(\pi - B) = \frac{7}{25}$ and $B = \pi - \sin^{-1}\left(\frac{7}{25}\right)$ [2]

3. Complete the next entry line as:
$\sin(a - b)$
Then press ENTER.



4. Press MENU, then select:
2: Number
2: Approximate to Fraction.
Then press ENTER.



5. The answer appears on the screen.

$$\sin(A - B) = \frac{-204}{325}$$

3. Form an expression for $\sin(A - B)$.

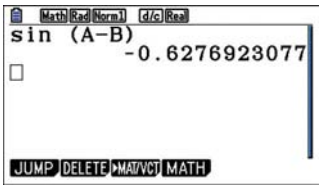
Substituting [1] and [2] into $\sin(A - B)$ gives:

$$\sin(A - B) = \sin\left(\cos^{-1}\left(\frac{12}{13}\right) - \left(\pi - \sin^{-1}\left(\frac{7}{25}\right)\right)\right)$$

4. Put the calculator in RADIAN mode.
On a Run-Matrix screen, complete the entry line as:

$$\sin\left(\cos^{-1}\left(\frac{12}{13}\right) - \left(\pi - \sin^{-1}\left(\frac{7}{25}\right)\right)\right)$$

Then press EXE.



5. The answer appears on the screen.

$$\sin(A - B) = \frac{-204}{325}$$

9.3.4 Proofs using the compound angle formulas

The compound angle formulas can be used to prove other identities.

WORKED EXAMPLE 14

Prove that $\sin(A + B)\sin(A - B) = \sin^2(A) - \sin^2(B)$.

THINK	WRITE
1. When trying to decide where to begin, it is often best to begin with the more complicated side (the side with more terms or with the compound angles). In this case, it is the left-hand side.	$\text{LHS} = \sin(A + B)\sin(A - B)$
2. Substitute the compound angle formulas.	$= (\sin(A)\cos(B) + \cos(A)\sin(B))$ $\times (\sin(A)\cos(B) - \cos(A)\sin(B))$
3. Expand the brackets. Notice in this equation the difference of two squares pattern can be used.	$= \sin^2(A)\cos^2(B) - \cos^2(A)\sin^2(B)$
4. The right-hand side of the identity is written in terms of sine. Use the Pythagorean identity to replace the cosines.	$= \sin^2(A)\left(1 - \sin^2(B)\right) - \left(1 - \sin^2(A)\right)\sin^2(B)$
5. Simplify. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is proven.	$= \sin^2(A) - \sin^2(A)\sin^2(B) - \sin^2(B) + \sin^2(A)\sin^2(B)$ $= \sin^2(A) - \sin^2(B)$ $= \text{RHS}$ $\therefore \sin(A + B)\ \sin(A - B) = \sin^2(A) - \sin^2(B)$

Exercise 9.3 Compound angle formulas

Technology free

1. **WE9** Evaluate the following, giving your answers as exact values.
   - **a.** $\sin 27° \cos 33° + \cos 27° \sin 33°$
   - **b.** $\cos 47° \cos 43° - \sin 47° \sin 43°$
   - **c.** $\cos 76° \cos 16° + \sin 76° \sin 16°$
   - **d.** $\cos 63° \sin 18° - \sin 63° \cos 18°$
2. **WE10** Expand the following.
   - **a.** $\sqrt{2} \sin\left(\theta - \frac{\pi}{4}\right)$
   - **b.** $2 \sin\left(\theta + \frac{\pi}{3}\right)$
   - **c.** $2 \cos\left(\theta - \frac{\pi}{6}\right)$
   - **d.** $\sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right)$
3. **WE11** Use one of the compound angle formulas to simplify each of the following.
   - **a.** $\sin\left(\frac{\pi}{2} - \theta\right)$
   - **b.** $\cos\left(\frac{\pi}{2} - \theta\right)$
   - **c.** $\sin(\pi + \theta)$
   - **d.** $\cos(\pi - \theta)$
4. Use one of the compound angle formulas to simplify each of the following.
   - **a.** $\sin\left(\frac{3\pi}{2} - \theta\right)$
   - **b.** $\cos\left(\frac{3\pi}{2} + \theta\right)$
   - **c.** $\tan(\pi - \theta)$
   - **d.** $\tan(\pi + \theta)$
5. Simplify each of the following.
   - **a.** $\sin\left(x + \frac{\pi}{3}\right) - \sin\left(x - \frac{\pi}{3}\right)$
   - **b.** $\cos\left(\frac{\pi}{3} + x\right) - \cos\left(\frac{\pi}{3} - x\right)$
   - **c.** $\cos\left(\frac{\pi}{6} - x\right) - \cos\left(\frac{\pi}{6} + x\right)$
   - **d.** $\tan\left(x + \frac{\pi}{4}\right) \tan\left(x - \frac{\pi}{4}\right)$
6. **WE12** Find the exact values of the following.
   - **a.** $\cos\left(\frac{7\pi}{12}\right)$
   - **b.** $\sin\left(\frac{11\pi}{12}\right)$
   - **c.** $\tan\left(\frac{\pi}{12}\right)$
   - **d.** $\tan\left(\frac{5\pi}{12}\right)$
7. **WE13** Given that $\cos(A) = \frac{4}{5}$ and $\sin(B) = \frac{12}{13}$ and that A and B are both acute angles, find the exact values of:
   - **a.** $\cos(A - B)$
   - **b.** $\tan(A + B)$
8. Given that $\sin(A) = \frac{5}{13}$ and $\tan(B) = \frac{24}{7}$ and A is obtuse and B is acute, find the exact values of:
   - **a.** $\sin(A + B)$
   - **b.** $\cos(A + B)$
9. Given that $\sec(A) = \frac{7}{2}$ and $\operatorname{cosec}(B) = \frac{3}{2}$ and A is acute but B is obtuse, find the value exact values of:
   - **a.** $\cos(A + B)$
   - **b.** $\sin(A - B)$
10. Given that $\operatorname{cosec}(A) = \frac{1}{a}$ and $\sec(B) = \frac{1}{b}$ and A and B are both acute, evaluate $\tan(A + B)$.

11. Given that $\sin(A) = \frac{a}{a+1}$ and $\cos(B) = \frac{a}{a+2}$ and A and B are both acute, evaluate $\tan(A - B)$.

12. **WE14** Prove that $\cos(A + B)\cos(A - B) = \cos^2(A) - \sin^2(B)$.

13. If ABCD is a cyclic quadrilateral, show that $\cos(A) + \cos(B) + \cos(C) + \cos(D) = 0$.

14. If ABC is a triangle, prove that $\tan\left(\frac{A}{2}\right) = \cot\left(\frac{B+C}{2}\right)$.

15. If $\tan(x) - \tan(y) = m$ and $\cot(y) - \cot(x) = n$, prove that $\frac{1}{m} + \frac{1}{n} = \cot(x - y)$.

16. If $\tan(B) = \frac{\sin(A)\cos(A)}{2 + \cos^2(A)}$, prove that $3\tan(A - B) = 2\tan(A)$.

9.4 Multiple angle formulas

9.4.1 Double angle formulas

If the trigonometric ratio of a particular angle is known, it is possible to find the trigonometric ratio of multiples of that angle.

Using the compound angle formulas, it is possible to determine formulas for double angles.

Using $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$, we can find an expression for $\sin(2A)$.

$$\begin{aligned}\sin(2A) &= \sin(A + A)\\ &= \sin(A)\cos(A) + \cos(A)\sin(A)\\ &= 2\sin(A)\cos(A)\end{aligned}$$

Similarly, using $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$:

$$\begin{aligned}\cos(2A) &= \cos(A + A)\\ &= \cos(A)\cos(A) - \sin(A)\sin(A)\\ &= \cos^2(A) - \sin^2(A)\end{aligned}$$

Using $\cos^2(A) = 1 - \sin^2(A)$: $\cos 2A = \left(1 - \sin^2(A)\right) - \sin^2(A)$

$$= 1 - 2\sin^2(A)$$

Using $\sin^2(A) = 1 - \cos^2(A)$: $\cos 2A = \cos^2(A) - \left(1 - \cos^2(A)\right)$

$$= 2\cos^2(A) - 1$$

These are known as the **double angle formulas**.

Double angle formulas:

$$\begin{aligned}\mathbf{\sin(2A)} &= \mathbf{2\sin(A)\cos(A)}\\ \mathbf{\cos(2A)} &= \mathbf{\cos^2(A) - \sin^2(A)}\\ &= \mathbf{2\cos^2(A) - 1}\\ &= \mathbf{1 - 2\sin^2(A)}\end{aligned}$$

Using the double angle formulas to find exact values

The double angle formulas can be used to find exact values of expressions. The formulas may be used in either direction.

WORKED EXAMPLE 15

Find the exact value of $\sin\left(\frac{7\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right)$.

THINK

1. As the expression is a multiple of sine and cosine of the same angle, the formula $\sin(2A) = 2\sin(A)\cos(A)$ can be used.
2. Substitute $A = \frac{7\pi}{12}$ and simplify.
3. $\frac{7\pi}{6}$ is in quadrant 3; $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ is a known trigonometric ratio and is negative in quadrant 3.

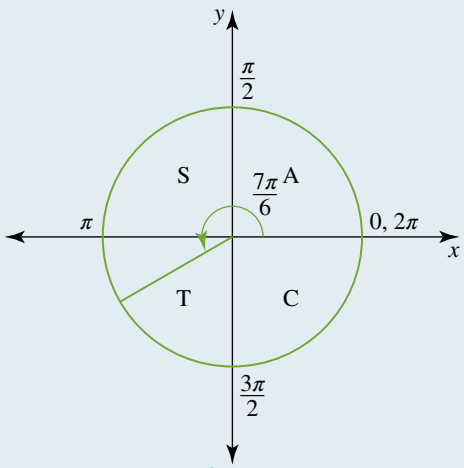
WRITE

$2\sin(A)\cos(A) = \sin(2A)$

$\sin(A)\cos(A) = \frac{1}{2}\sin(2A)$

$\sin\left(\frac{7\pi}{12}\right)\cos\left(\frac{7\pi}{12}\right) = \frac{1}{2}\sin\left(2 \times \frac{7\pi}{12}\right)$

$= \frac{1}{2}\sin\left(\frac{7\pi}{6}\right)$



$= \frac{1}{2}\sin\left(\pi + \frac{\pi}{6}\right)$

$= -\frac{1}{2}\sin\left(\frac{\pi}{6}\right)$

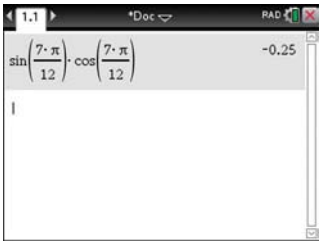
$= -\frac{1}{2} \times \frac{1}{2}$

$= -\frac{1}{4}$

TI | THINK

1. Put the calculator in RADIAN mode.
   On a Calculator page, complete the entry line as: $\sin\left(\frac{7\pi}{12}\right) \times \cos\left(\frac{7\pi}{12}\right)$
   Then press ENTER.

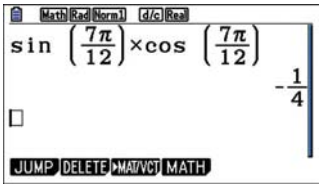
WRITE



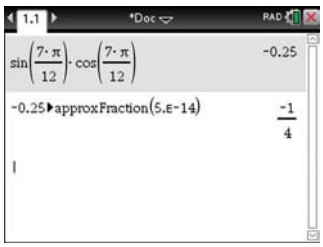
CASIO | THINK

1. Put the calculator in RADIAN mode.
   On a Run-Matrix screen, complete the entry line as: $\sin\left(\frac{7\pi}{12}\right) \times \cos\left(\frac{7\pi}{12}\right)$
   Then press EXE.

WRITE



2. Type '–0.25', then press MENU and select:
2: Number
2: Approximate to Fraction.
Then press ENTER.



3. The answer appears on the screen. $\sin\left(\frac{7\pi}{12}\right) \times \cos\left(\frac{7\pi}{12}\right) = -\frac{1}{4}$

2. The answer appears on the screen. $\sin\left(\frac{7\pi}{12}\right) \times \cos\left(\frac{7\pi}{12}\right) = -\frac{1}{4}$

WORKED EXAMPLE 16

If $0 \leq A \leq \frac{\pi}{2}$ and $\cos(A) = \frac{1}{4}$, find the exact values of:

a. $\cos(2A)$ **b.** $\sin(2A)$ **c.** $\tan(2A)$

THINK

WRITE

a. 1. As we know the value for $\cos(A)$, use $\cos(2A) = 2\cos^2(A) - 1$.

$\cos(2A) = 2\cos^2(A) - 1$

2. Substitute for $\cos(A)$.

$$= 2 \times \left(\frac{1}{4}\right)^2 - 1$$
$$= \frac{2}{16} - 1$$
$$= -\frac{14}{16}$$
$$= -\frac{7}{8}$$

b. 1. As $\sin(2A) = 2\sin(A)\cos(A)$, it is necessary to calculate $\sin(A)$. The Pythagorean identity can be used.

$\sin^2(A) + \cos^2(A) = 1$

$$\sin^2(A) + \left(\frac{1}{4}\right)^2 = 1$$
$$\sin^2(A) + \frac{1}{16} = 1$$
$$\sin^2(A) = \frac{15}{16}$$
$$\sin(A) = \pm\frac{\sqrt{15}}{4}$$

2. $0 \leq A \leq \frac{\pi}{2}$ so $\sin(A) > 0$.

As $0 \leq A \leq \frac{\pi}{2}$, $\sin(A) = \frac{\sqrt{15}}{4}$.

3. Use the formula to find $\sin(2A)$.

$$\begin{aligned}\sin(2A) &= 2\sin(A)\cos(A)\\ &= 2 \times \frac{\sqrt{15}}{4} \times \frac{1}{4}\\ &= \frac{\sqrt{15}}{8}\end{aligned}$$

c. Use the identity $\tan(2A) = \dfrac{\sin(2A)}{\cos(2A)}$.

$$\begin{aligned}\tan(2A) &= \frac{\sin(2A)}{\cos(2A)}\\ &= \frac{\frac{\sqrt{15}}{8}}{-\frac{7}{8}}\\ &= \frac{\sqrt{15}}{8} \times -\frac{8}{7}\\ &= -\frac{\sqrt{15}}{7}\end{aligned}$$

9.4.2 Using the double angle formulas to solve trigonometric equations

Previously, the Pythagorean identity was used to rewrite expressions using a single trigonometric ratio. In a similar fashion, the double angle formulas can be used to rewrite the expressions in terms of the same angle.

WORKED EXAMPLE 17

If $\sin 2x + \sqrt{3}\cos x = 0$, solve for $x \in [0, 2\pi]$.

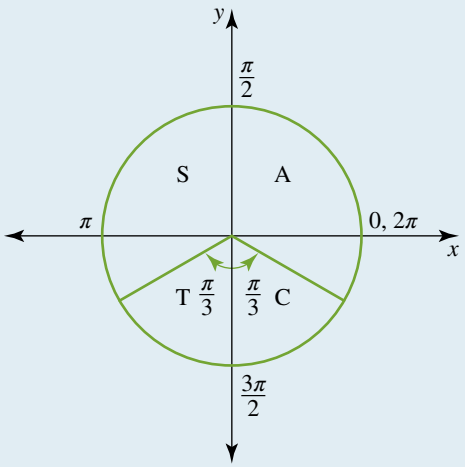
THINK

1. The equation involves trigonometric ratios of both x and $2x$. Use the double angle formula to rewrite $\sin 2x$ in terms of $\sin x$ and $\cos x$.
2. Factorise by identifying the common factor of $\cos x$.
3. Solve each factor equal to 0.
4. Solve $\cos x = 0$.

WRITE

$$\sin 2x + \sqrt{3}\cos x = 0$$
$$2\sin x \cos x + \sqrt{3}\cos x = 0$$
$$\cos x\left(2\sin x + \sqrt{3}\right) = 0$$
$$\cos x = 0 \text{ or } 2\sin x + \sqrt{3} = 0$$
$$\sin x = -\frac{\sqrt{3}}{2}$$
$$\cos x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

5. Solve $\sin x = -\frac{\sqrt{3}}{2}$, remembering that $\sin x$ is negative in quadrants 3 and 4. Sin $\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ is a known trigonometric ratio.



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$x = \pi + \frac{\pi}{3}$ or $x = 2\pi - \frac{\pi}{3}$

$= \frac{4\pi}{3}$ $= \frac{5\pi}{3}$

6. Combine the solutions.

$x = \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

9.4.3 Proofs using the double angle formulas

The double angle formulas can be combined with the other formulas covered in this chapter and the Pythagorean identities to prove trigonometric identities. When deciding where to start, it is often best to begin with the more complex side and work to simplify it.

WORKED EXAMPLE 18

Prove the identity $\frac{\cos(2\alpha)\cos(\alpha) + \sin(2\alpha)\sin(\alpha)}{\sin(3\alpha)\cos(\alpha) - \cos(3\alpha)\sin(\alpha)} = \frac{1}{2}\text{cosec}(\alpha)$.

THINK

1. The left-hand side is the more complex side, so begin with that side.
2. Use the angle difference formulas. The numerator can be simplified using $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$, and the denominator can be simplified using $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$.

WRITE

$\text{LHS} = \frac{\cos(2\alpha)\cos(\alpha) + \sin(2\alpha)\sin(\alpha)}{\sin(3\alpha)\cos(\alpha) - \cos(3\alpha)\sin(\alpha)}$

$= \frac{\cos(2\alpha - \alpha)}{\sin(3\alpha - \alpha)}$

3. Simplify.

4. Substitute $\sin 2\alpha = 2\sin\alpha\cos\alpha$ for the denominator.

5. Simplify the fraction and use $\text{cosec}(A) = \frac{1}{\sin(A)}$.

$$= \frac{\cos(\alpha)}{\sin(2\alpha)}$$
$$= \frac{\cos(\alpha)}{2\sin(\alpha)\cos(\alpha)}$$
$$= \frac{1}{2\sin(\alpha)}$$
$$= \frac{1}{2}\text{cosec}(\alpha)$$
$$= \text{RHS}$$

6. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is proven.

$$\therefore \frac{\cos(2\alpha)\cos(\alpha) + \sin(2\alpha)\sin(\alpha)}{\sin(3\alpha)\cos(\alpha) - \cos(3\alpha)\sin(\alpha)} = \frac{1}{2}\text{cosec}(\alpha)$$

9.4.4 Half-angle formulas

The double angle formulas express $\sin(2A)$ and $\cos(2A)$ in terms of $\sin(A)$ and $\cos(A)$. Using these formulas, it is also possible to express $\sin(A)$ and $\cos(A)$ in terms of $\sin\left(\frac{A}{2}\right)$ and $\cos\left(\frac{A}{2}\right)$, which are the **half-angle formulas**.

Half-angle formulas:

$$\sin(A) = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$$
$$\cos(A) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$
$$= 2\cos^2\left(\frac{A}{2}\right) - 1$$
$$= 1 - 2\sin^2\left(\frac{A}{2}\right)$$

WORKED EXAMPLE 19

Prove the identity $\text{cosec}(A) - \cot(A) = \tan\left(\frac{A}{2}\right)$.

THINK

1. The left-hand side is the more complex side, so begin with that side.
2. Remember that $\text{cosec}(A) = \frac{1}{\sin(A)}$ and $\cot A = \frac{\cos(A)}{\sin(A)}$ and make the substitutions.

WRITE

$$\text{LHS} = \text{cosec}(A) - \cot(A)$$
$$= \frac{1}{\sin(A)} - \frac{\cos(A)}{\sin(A)}$$

3. As the fractions have a common denominator, simplify the expression by adding the fractions.

$$= \frac{1 - \cos(A)}{\sin(A)}$$

4. On the right-hand side, the angle is $\frac{A}{2}$, so use the half angle formulas for $\cos(A)$ and $\sin(A)$. There are three options to use as a substitution for $\cos(A)$. In this instance, as $\tan\left(\frac{A}{2}\right) = \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$, writing $\cos(A)$ in terms of $\sin\left(\frac{A}{2}\right)$ would be the first option to try.

$$= \frac{1 - \left(1 - 2\sin^2\left(\frac{A}{2}\right)\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

5. Simplify the fraction.

$$= \frac{1 - 1 + 2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}$$

$$= \tan\left(\frac{A}{2}\right)$$

$$= \text{RHS}$$

6. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is proven.

$$\therefore \text{cosec}(A) - \cot(A) = \tan\left(\frac{A}{2}\right)$$

9.4.5 Multiple angle formulas

When the multiple of the angle is greater than 2, the formulas are referred to as **multiple angle formulas**.

These formulas were proved using complex numbers and de Moivre's formula in Chapter 6, in Worked example 35 and the accompanying exercise.

Proof of these multiple angle formulas may also be demonstrated using the compound angle formulas and the Pythagorean identities. In the worked example and the exercise that follow, these multiple angle proofs will be demonstrated using both de Moivre's theorem and the compound angle formulas.

Multiple angle formulas:

$$\sin(3A) = 3\sin(A) - 4\sin^3(A)$$

$$\cos(3A) = 4\cos^3(A) - 3\cos(A)$$

$$\sin(4A) = \cos(A)\left(4\sin(A) - 8\sin^3(A)\right)$$

$$\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$$

WORKED EXAMPLE 20

Prove the multiple angle formula $\cos(3A) = 4\cos^3(A) - 3\cos(A)$:
a. using the compound angle formulas
b. using de Moivre's formula.

THINK	WRITE
a 1. Although the right-hand side has more terms, by using the compound angle formulas and then the double angle formulas, it is possible to rewrite cos(3A) in terms of sin(A) and cos(A). So begin with the left-hand side.	$\text{LHS} = \cos(3A)$
2. Use the multiple angle formula $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ by using $3A = 2A + A$.	$= \cos(2A + A)$ $= \cos(2A)\cos(A) - \sin(2A)\sin(A)$
3. Use the double angle formulas to rewrite cos(2A) and sin(2A). As the right-hand side is written in terms of cos(A), use that option for cos(2A).	$= \left(2\cos^2(A) - 1\right)\cos(A) - (2\sin(A)\cos(A))\sin(A)$ $= 2\cos^3(A) - \cos(A) - 2\sin^2(A)\cos(A)$
4. Substitute $\sin^2(A) = 1 - \cos^2(A)$.	$= 2\cos^3(A) - \cos(A) - 2\left(1 - \cos^2(A)\right)\cos(A)$ $= 2\cos^3(A) - \cos(A) - 2\cos(A) + 2\cos^3(A)$ $= 4\cos^3(A) - 3\cos(A)$
5. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is proven.	$= \text{RHS}$ $\therefore \cos 3(A) = 4\cos^3(A) - 3\cos(A)$
b. 1. The proof involves (3A) so it will be necessary to determine $(\cos(A) + i\sin(A))^3$.	$(\cos(A) + i\sin(A))^3$
2. Determine the expansion using the third row (1 3 3 1) of Pascal's triangle.	$(\cos(A) + i\sin(A))^3 = 1(\cos(A))^3 + 3(\cos(A))^2(i\sin(A))$ $+ 3(\cos(A))(i\sin(A))^2 + 1(i\sin(A))^3$ $= \cos^3(A) + 3i\cos^2(A)\sin(A)$ $+ 3i^2\cos(A)\sin^2(A) + i^3\sin^3(A)$
3. Simplify the expression, remembering that $i^2 = -1$ and $i^3 = -i$.	$= \cos^3(A) + 3i\cos^2(A)\sin(A)$ $- 3\cos(A)\sin^2(A) - i\sin^3(A)$
4. Group the real and imaginary terms together.	$= \cos^3(A) - 3\cos(A)\sin^2(A)$ $+ 3i\cos^2(A)\sin(A) - i\sin^3(A)$ $(\cos(A) + i\sin(A))^3 = \cos^3(A) - 3\cos(A)\sin^2(A)$ $+ i\left(3\cos^2(A)\sin(A) - \sin^3(A)\right)$

5. State the rule for $\cos n\theta$.
6. Use the expansion of $(\cos\theta + i\sin\theta)^3$ to find the left-hand side term, $\cos 3\theta$.
7. We want to prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ so use the Pythagorean identity $\sin^2\theta = 1 - \cos^2\theta$ to replace $\sin^2\theta$.

$$\cos(nA) = \text{Re}\,(\cos(A) + i\sin(A))^n$$
$$\begin{aligned}\text{LHS} &= \cos(3A)\\ &= \text{Re}\,(\cos(A) + i\sin(A))^3\\ &= \cos^3(A) - 3\cos(A)\sin^2(A)\\ \cos(3A) &= \cos^3(A) - 3\cos(A)\left(1 - \cos^2(A)\right)\\ &= \cos^3(A) - 3\cos(A) + 3\cos^3(A)\\ &= 4\cos^3(A) - 3\cos(A)\\ &= \text{RHS}\end{aligned}$$
$$\therefore \cos(3A) = 4\cos^3(A) - 3\cos(A)$$

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Units 1 & 2 > Area 5 > Sequence 3 > Concept 2

Compound and double angle formulas Summary screen and practice questions

Exercise 9.4 Multiple angle formulas

Technology free

1. **WE15** Find the exact values of the following.
   a. $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$
   b. $\cos^2(112°30') - \sin^2(112°30')$
   c. $2\sin^2(22°30') - 1$
   d. $\cos^2\left(\frac{\pi}{12}\right)$

2. **WE16** Given that $\sec(A) = \frac{8}{3}$ and $0 \le A \le \frac{\pi}{2}$, find the exact values of the following.
   a. $\cos(2A)$
   b. $\sin(2A)$
   c. $\tan(2A)$
   d. $\cot(2A)$
3. **WE17** Solve each of the following equations for $x \in [0, 2\pi]$.
   a. $\sin(2x) - \cos(x) = 0$
   b. $\sin(x) - \sin(2x) = 0$
   c. $\cos(2x) - \cos(x) = 0$
   d. $\sin(x) - \cos(2x) = 0$
4. Solve each of the following equations for $x \in [0, 2\pi]$.
   a. $\tan(x) = \sin(2x)$
   b. $\sin(2x) = \sqrt{3}\cos(x)$
   c. $\sin(4x) = \sin(2x)$
   d. $\cos(2x) = \sin(4x)$
5. **WE18** Prove the following identities.
   a. $\dfrac{\sin(2A)\cos(A) - \cos(2A)\sin(A)}{\cos(2A)\cos(A) + \sin(2A)\sin(A)} = \tan(A)$
   b. $\dfrac{\cos(2A)\cos(A) + \sin(2A)\sin(A)}{\sin(2A)\cos(A) - \cos(2A)\sin(A)} = \cot(A)$
   c. $\dfrac{\sin(3A)}{\sin(A)} - \dfrac{\cos(3A)}{\cos(A)} = 2$
   d. $\dfrac{\cos(3A)}{\sin(A)} + \dfrac{\sin(3A)}{\cos(A)} = 2\cot(2A)$

6. Prove the following identities.

a. $\dfrac{\tan(3A)+\tan(A)}{\tan(3A)-\tan(A)} = 2\cos(2A)$

b. $\dfrac{\tan(A)-\tan(B)}{\tan(A)+\tan(B)} = \dfrac{\sin(A-B)}{\sin(A+B)}$

c. $\dfrac{\sin(A)-\cos(A)}{\sin(A)+\cos(A)} - \dfrac{\sin(A)+\cos(A)}{\sin(A)-\cos(A)} = 2\tan(2A)$

d. $\dfrac{\cos(A)+\sin(A)}{\cos(A)-\sin(A)} + \dfrac{\cos(A)-\sin(A)}{\cos(A)+\sin(A)} = 2\sec(2A)$

7. **WE19** Prove the following identities.

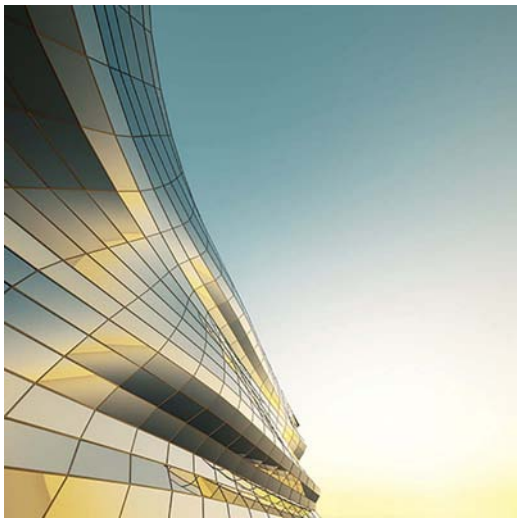
a. $\dfrac{\sin(A)}{1-\cos(A)} = \cot\left(\dfrac{A}{2}\right)$

b. $\dfrac{\sin(A)}{1+\cos(A)} = \tan\left(\dfrac{A}{2}\right)$

c. $\dfrac{1-\cos(A)+\sin(A)}{1+\cos(A)+\sin(A)} = \tan\left(\dfrac{A}{2}\right)$

d. $\dfrac{\sin\left(\frac{A}{2}\right)+\sin(A)}{1+\cos(A)+\cos\left(\frac{A}{2}\right)} = \tan\left(\dfrac{A}{2}\right)$

8. Prove the following identities.



a. $\sin(A+B)\sin(A-B) = \sin^2(A)-\sin^2(B)$

b. $\tan(A+B)\tan(A-B) = \dfrac{\tan^2(A)-\tan^2(B)}{1-\tan^2(A)\tan^2(B)}$

c. $\cot(A+B) = \dfrac{\cot(A)\cot(B)-1}{\cot(A)+\cot(B)}$

d. $\cot(A-B) = \dfrac{\cot(A)\cot(B)+1}{\cot(B)-\cot(B)}$

9. Prove the following identities.

a. $\sin(2A) = \dfrac{2\tan(A)}{1+\tan^2(A)}$

b. $\cos(2A) = \dfrac{1-\tan^2(A)}{1+\tan^2(A)}$

c. $\dfrac{\cos^3(A)-\sin^3(A)}{\cos(A)-\sin(A)} = 1+\dfrac{1}{2}\sin(2,A)$

d. $\dfrac{\cos^3 A+\sin^3 A}{\cos(A)+\sin(A)} = 1-\dfrac{1}{2}\sin(2,A)$

10. In a triangle ABC with side lengths a, b and c, where C is a right angle, c is the hypotenuse, and a is opposite A, show that:

a. $\sin(2A) = \dfrac{2ab}{c^2}$

b. $\cos(2A) = \dfrac{b^2-a^2}{c^2}$

c. $\tan(2A) = \dfrac{2ab}{b^2-a^2}$

d. $\sin\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c-b}{2c}}$

e. $\cos\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c+b}{2c}}$

f. $\tan\left(\dfrac{A}{2}\right) = \sqrt{\dfrac{c-b}{c+b}}$

11. **WE20** Prove the multiple angle formula $\sin(3A) = 3\sin(A) - 4\sin^3(A)$:
    a. using the compound angle formulas
    b. using de Moivre's formula.
12. Prove each of the following multiple angle formulas.
    i. using the compound angle formulas
    ii. using de Moivre's formula.
    a. $\sin(4A) = \cos(A)\left(4\sin(A) - 8\sin^3(A)\right)$
    b. $\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$
13. Chebyshev, the Russian mathematician mentioned in the overview, developed a number of recurrence relationships for trigonometric multiple angles.
    a. Prove the relationship $\cos(nx) = 2\cos(x)\cos[(n-1)x] - \cos[(n-2)x]$.
    b. Use this relationship and the formula for $\cos 2x$ to demonstrate that $\cos(5x) = 16\cos^5(x) - 20\cos^3(x) + 5\cos(x)$.
14. Chebyshev's recurrence formula for multiple angles of the sine function is given by $\sin nx = 2\cos(x)\sin[(n-1)x] - \sin[(n-2)x]$.
    a. Prove Chebyshev's recurrence formula for multiple angles of the sine function.
    b. Use this formula and the formula for $\sin 2x$ to demonstrate that $\sin(6x) = \cos(x)\left(32\sin^5(x) - 32\sin^3(x) + 6\sin(x)\right)$.

9.5 Product–sum identities

9.5.1 Expressing sums as products and products as sums

The angle sum and angle difference formulas can be written as products of the functions, allowing a sum to be expressed as a product and vice versa.

Starting with the angle sum and angle difference formulas:

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \quad [1]$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B) \quad [2]$$

[1] + [2]: $\sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

[1] – [2]: $\sin(A+B) - \sin(A-B) = 2\cos(A)\sin(B)$

$$\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

In a similar fashion,

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \quad [3]$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B) \quad [4]$$

[3] + [4]: $\cos(A+B) + \cos(A-B) = 2\cos(A)\cos(B)$

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

[3] – [4]: $\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B)$

$$\sin(A)\sin(B) = -\frac{1}{2}[\cos(A+B) - \cos(A-B)] \quad \text{or}$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

For the trigonometric functions sine and cosine, to write products of the functions as sums of the functions, the following formulas can be used.

Product–sum identities

$$\boldsymbol{\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]}$$

$$\boldsymbol{\cos(A)\sin(B) = \frac{1}{2}[\sin(A+B) - \sin(A-B)]}$$

$$\boldsymbol{\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]}$$

$$\boldsymbol{\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]}$$

For the purposes of the proof of the sum identities, let $\alpha = A + B$ and $\beta = A - B$. Then:

$$\alpha + \beta = 2A \qquad \text{and} \qquad \alpha - \beta = 2B.$$

$$A = \frac{\alpha+\beta}{2} \qquad\qquad B = \frac{\alpha-\beta}{2}$$

These substitutions can be used to express the sums of trigonometric functions as products of trigonometric functions.

$$\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}[\sin\alpha + \sin\beta]$$

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \sin\alpha + \sin\beta$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}[\sin\alpha - \sin\beta]$$

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = \sin\alpha - \sin\beta$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}[\cos\alpha + \cos\beta]$$

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \cos\alpha + \cos\beta$$

$$\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = -\frac{1}{2}[\cos\alpha - \cos\beta]$$
$$-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = \cos\alpha - \cos\beta$$
$$2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = \cos\beta - \cos\alpha$$

For consistency with the other formulas, these identities are expressed in terms of A and B as follows.

Sum–product identities

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos(B) - \cos(A) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

9.5.2 Using exact values

By rewriting products as sums or sums as products, it may be possible to rewrite expressions in terms of angles that the trigonometric ratios are known for.

WORKED EXAMPLE 21

Evaluate $\dfrac{\sin 85^\circ + \sin 35^\circ}{\cos 85^\circ + \cos 35^\circ}$, giving your answer as an exact value.

THINK

1. Simplify the numerator using $\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ and the denominator using $\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.
2. Simplify the terms and cancel any common factors.

WRITE

Let $A = 85^\circ$ and $B = 35^\circ$.

$$\frac{\sin 85^\circ + \sin 35^\circ}{\cos 85^\circ + \cos 35^\circ} = \frac{2\sin\left(\frac{85^\circ+35^\circ}{2}\right)\cos\left(\frac{85^\circ-35^\circ}{2}\right)}{2\cos\left(\frac{85^\circ+35^\circ}{2}\right)\cos\left(\frac{85^\circ-35^\circ}{2}\right)}$$
$$= \frac{2\sin 60^\circ \cos 25^\circ}{2\cos 60^\circ \cos 25^\circ}$$
$$= \frac{\sin 60^\circ}{\cos 60^\circ}$$
$$= \tan 60^\circ$$
$$= \sqrt{3}$$

WORKED EXAMPLE 22

Demonstrate that $\cos 70° - \sqrt{2}\cos 25° + \cos 20° = 0$.

THINK	WRITE
1. Using the $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, sum-product identity, it is possible to rewrite $\cos 70° + \cos 20°$.	$\text{LHS} = \cos 70° - \sqrt{2}\cos 25° + \cos 20°$ $= (\cos 70° + \cos 20°) - \sqrt{2}\cos 25°$ $= \left(2\cos\left(\frac{70° + 20°}{2}\right)\cos\left(\frac{70° - 20°}{2}\right)\right) - \sqrt{2}\cos 25°$ $= 2\cos 45° \cos 25° - \sqrt{2}\cos 25°$
2. Substitute $\cos 45° = \frac{\sqrt{2}}{2}$.	$= 2 \times \frac{\sqrt{2}}{2}\cos 25° - \sqrt{2}\cos 25°$ $= \sqrt{2}\cos 25° - \sqrt{2}\cos 25°$ $= 0$ $= \text{RHS}$
3. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is demonstrated.	$\therefore \cos 70° - \sqrt{2}\cos 25° + \cos 20° = 0$

9.5.3 Trigonometric proofs

Like the formulas introduced earlier in the chapter, the product as sum formulas and the sum as product formulas can be used to prove other identities.

WORKED EXAMPLE 23

Prove the identity $\frac{\sin(5\alpha) - \sin(3\alpha)}{\cos(5\alpha) + \cos(3\alpha)} = \tan(\alpha)$.

THINK	WRITE
1. The left-hand side is the more complex side, so begin with that side.	$\text{LHS} = \frac{\sin(5\alpha) - \sin(3\alpha)}{\cos(5\alpha) + \cos(3\alpha)}$
2. The numerator can be rewritten using $\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$ and the denominator can be rewritten using $\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.	Let $A = 5\alpha$ and $B = 3\alpha$. $\text{LHS} = \frac{2\cos\left(\frac{5\alpha+3\alpha}{2}\right)\sin\left(\frac{5\alpha-3\alpha}{2}\right)}{2\cos\left(\frac{5\alpha+3\alpha}{2}\right)\cos\left(\frac{5\alpha-3\alpha}{2}\right)}$

3. Simplify the terms, noting that
$\frac{5A + 3A}{2} = \frac{8A}{2}, \frac{5A - 3A}{2} = \frac{2A}{2}$
$= 4A \qquad = A$
and $\tan(A) = \frac{\sin A}{\cos A}$.

4. As it has been demonstrated that the left-hand side is equal to the right-hand side, the identity is demonstrated.

$= \frac{2\cos 4\alpha \sin \alpha}{2\cos 4\alpha \cos \alpha}$
$= \frac{\sin \alpha}{\cos \alpha}$
$= \tan \alpha$
$= \text{RHS}$

$\therefore \frac{\sin 5\alpha - \sin 3\alpha}{\cos 5\alpha + \cos 3\alpha} = \tan \alpha$

9.5.4 Solving trigonometric equations

By rewriting sums of trigonometric ratios as products, it is possible to use the Null Factor Law to solve equations.

WORKED EXAMPLE 24

Find the general solution for x of $\cos(4x) + \cos(2x) = 0$.

THINK

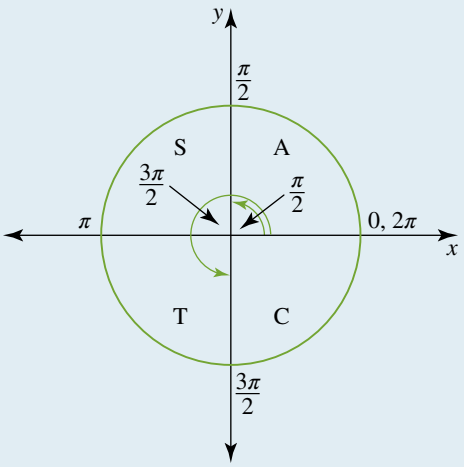
1. Rewriting the left-hand side as a product allows the Null Factor Law to be used.

WRITE

Use
$\cos(A) + \cos(B) = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$
where $A = 4x$ and $B = 2x$.
$\cos(4x) + \cos(2x) = 0$
$2\cos\left(\frac{4x + 2x}{2}\right)\cos\left(\frac{4x - 2x}{2}\right) = 0$
$2\cos 3x \cos x = 0$

2. Set each factor to zero and solve.
3. Solve $\cos x = 0$. Note that $\cos\frac{\pi}{2} = 0$ and $\cos\frac{3\pi}{2} = 0$. These solutions are π apart, so the general solution is $\frac{\pi}{2} + n\pi, n \in Z$.

$\cos 3x = 0$ or $\cos x = 0$



$$\cos x = 0$$
$$x = \frac{\pi}{2} + n\pi,\ n \in Z$$
$$= \pi\left(\frac{1}{2} + n\right)$$
$$= \pi\left(\frac{1+2n}{2}\right)$$
$$= \frac{\pi(2n+1)}{2}$$

4. Solve $\cos 3x = 0$.

$\cos 3x = 0$
Using the previous solution:
$$3x = \frac{\pi(2n+1)}{2},\ n \in Z$$
$$x = \frac{\pi(2n+1)}{6}$$

5. Combine the solutions.

$$x = \frac{\pi(2n+1)}{2}, \frac{\pi(2n+1)}{6},\ n \in Z$$

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Units 1 & 2 > Area 5 > Sequence 3 > Concept 3

Other trigonometric identities Summary screen and practice questions

Exercise 9.5 Product–sum identities

Technology free

1. WE21 Evaluate the following, giving your answers as exact values.

a. $\dfrac{\sin 95° + \sin 25°}{\cos 95° + \cos 25°}$

b. $\dfrac{\sin 95° - \sin 25°}{\cos 95° - \cos 25°}$

2. Evaluate the following. Give your answers as exact values.

a. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

b. $\dfrac{\cos 75° + \cos 15°}{\sin 75° + \sin 15°}$

c. $\dfrac{\cos 75° - \cos 15°}{\sin 75° + \sin 15°}$

d. $\dfrac{\sin 75° - \sin 15°}{\cos 75° - \cos 15°}$

3. Evaluate the following. Give your answers as exact values.
   a. $\sin 105° + \sin 15°$
   b. $\sin 105° - \sin 15°$
   c. $\cos 105° + \cos 15°$
   d. $\cos 105° - \cos 15°$
4. **WE22** Demonstrate the following.
   a. $\cos 80° - \sqrt{3}\cos 50° + \cos 20° = 0$
   b. $\sin 80° - \sqrt{3}\sin 50° + \sin 20° = 0$
5. Demonstrate the following.
   a. $\cos 10° + \cos 110° + \cos 130° = 0$
   b. $\cos 20° + \cos 100° + \cos 220° = 0$
   c. $\cos 40° - \sqrt{2}\cos 5° + \cos 50° = 0$
   d. $\sin 40° - \sqrt{2}\cos 5° + \sin 50° = 0$
6. **WE23** Prove the identity $\dfrac{\sin(5A) - \sin(3A)}{\cos(5A) - \cos(3A)} = -\cot(4A)$.
7. Prove the identity $\dfrac{\cos(4A) + \cos(2A)}{\sin(4A) - \sin(2A)} = \cot A$.
8. Prove the following identities.
   a. $\dfrac{\sin(2A) + \sin(4A)}{\cos(2A) + \cos(4A)} = \tan(3A)$
   b. $\dfrac{\sin(2A) + \sin(2B)}{\cos(2A) + \cos(2B)} = \tan(A + B)$
9. Prove the following identities.
   a. $\dfrac{\sin(2A) + \sin(2B)}{\sin(2A) - \sin(2B)} = \dfrac{\tan(A + B)}{\tan(A - B)}$
   b. $\dfrac{\cos(2A) + \cos(2B)}{\cos(2A) - \cos(2B)} = -\cot(A + B)\cot(A - B)$
10. Demonstrate that the following are true.
    a. $\sin(2A) + \sin(6A) + \sin(8A) = 4\cos(A)\cos(3A)\sin(4A)$
    b. $\sin(2A) + \sin(4A) + \sin(6A) = 4\cos(A)\cos(2A)\sin(3A)$
11. Demonstrate that the following are true.
    a. $\dfrac{\sin(A) + 2\sin(3A) + \sin(5A)}{\sin(3A) + 2\sin(5A) + \sin(7A)} = \dfrac{\sin(3A)}{\sin(5A)}$
    b. $\dfrac{\cos(2A) - \cos(6A)}{\sin(2A) + 2\sin(4A) + \sin(6A)} = \tan(A)$

12. **WE24** Find the general solutions of the following.
    a. $\cos(2x) + \cos(3x) = 0$
    b. $\sin(2x) + \sin(4x) = 0$
13. Find the general solutions to the following.
    a. $\sin(x) + \sin(3x) = 0$
    b. $\cos(x) + \cos(3x) = 0$
14. Find the general solutions to the following.
    a. $\sin(x) + \sin(2x) + \sin(3x) = 0$
    b. $\cos(x) + \cos(2x) + \cos(3x) = 0$
    c. $\sin(x) + \cos(x) + \sin(3x) = 0$
    d. $\cos(x) + \sin(2x) - \cos(3x) = 0$
15. Prove the following.
    a. $\dfrac{\sin(A) + \sin(B) - \sin(A + B)}{\sin(A) + \sin(B) + \sin(A + B)} = \tan\left(\dfrac{A}{2}\right)\tan\left(\dfrac{B}{2}\right)$
    b. $\sin(A) + \sin(B) + \sin(C) - \sin(A + B + C) = 4\sin\left(\dfrac{A + B}{2}\right)\sin\left(\dfrac{B + C}{2}\right)\sin\left(\dfrac{A + C}{2}\right)$

16. a. Prove that $\dfrac{\sin(3A)+\sin(5A)}{\cos(3A)+\cos(5A)} = \tan(4A)$.

b. Prove that $\dfrac{\sin(3A)+\sin(5A)+\sin(7A)}{\cos(3A)+\cos(5A)+\cos(7A)} = \tan(5A)$.

c. Prove that $\dfrac{\sin(3A)+\sin(5A)+\sin(7A)+\sin(9A)}{\cos(3A)+\cos(5A)+\cos(7A)+\cos(9A)} = \tan(6A)$.

d. Deduce the simplification of

$$\frac{\sin(3A)+\sin(5A)+\sin(7A)+\sin(9A)+\sin(11A)+\sin(13A)}{\cos(3A)+\cos(5A)+\cos(7A)+\cos(9A)+\cos(11A)+\cos(13A)}.$$

9.6 Convert $a\cos(x)+b\sin(x)$ to $R\cos(x\pm\alpha)$ or $R\sin(x\pm\alpha)$

9.6.1 Express $a\cos(x)+b\sin(x)$ in the form $R\cos(x\pm\alpha)$ or $R\sin(x\pm\alpha)$

In Chapter 8, you learned how to graph trigonometric functions. Using the compound angle formulas, it is possible to rewrite expressions of the form $a\cos(x)+b\sin(x)$ as either $R\cos(x\pm\alpha)$ or $R\sin(x\pm\alpha)$. This makes it easier to visualise what the graph might look like and to identify the maximum and minimum values.

WORKED EXAMPLE 25

Express $\sqrt{3}\cos x+\sin x$ in the form $R\cos(x-\alpha)$.

THINK

1. Let $\sqrt{3}\cos x+\sin x$ equal $R\cos(x-\alpha)$ and use the compound angle formulas to expand $R\cos(x-\alpha)$.

2. Equate the coefficients of $\cos x$ and $\sin x$.

3. As R is a factor of both equations, it can be eliminated by finding $\dfrac{[2]}{[1]}$. This will also allow us to use $\tan(\alpha)=\dfrac{\sin(\alpha)}{\cos(\alpha)}$.

WRITE

Let $\sqrt{3}\cos x+\sin x = R\cos(x-\alpha)$.

$$\begin{aligned}\sqrt{3}\cos x+\sin x &= R\cos(x-\alpha)\\ &= R(\cos x\cos(\alpha)+\sin x\sin(\alpha))\\ &= R\cos x\cos(\alpha)+R\sin x\sin(\alpha)\end{aligned}$$

Equating the coefficients of $\cos x$ and $\sin x$:

$$\sqrt{3} = R\cos(\alpha) \quad [1]$$
$$1 = R\sin(\alpha) \quad [2]$$

$$\begin{aligned}\frac{[2]}{[1]}:\frac{1}{\sqrt{3}} &= \frac{R\sin(\alpha)}{R\cos(\alpha)}\\ &= \frac{\sin(\alpha)}{\cos(\alpha)}\\ &= \tan(\alpha)\end{aligned}$$

4. From equations [1] and [2], both $\sin(\alpha)$ and $\cos(\alpha)$ are positive, so α is in quadrant 1. Remember $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$. Solve for α.

$$\alpha = \frac{\pi}{6}$$

5. Use either equation [1] or [2] to solve for R.

$$\begin{aligned}[1]: \sqrt{3} &= R\cos\frac{\pi}{6}\\ &= R \times \frac{\sqrt{3}}{2}\\ R &= \sqrt{3} \times \frac{2}{\sqrt{3}}\\ &= 2\end{aligned}$$

6. State the final result.

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$$

9.6.2 General transformations and applications

To make it easier to visualise the graph and to interpret the function, R is normally a positive number and α is generally in the first quadrant. Note that $-R\cos x = R\cos(x + \pi)$ and $-R\sin x = R\sin(x + \pi)$. Additionally, $R\cos x = R\sin\left(x + \frac{\pi}{2}\right)$ and $R\sin x = R\cos\left(x - \frac{\pi}{2}\right)$. This means that it is possible to write $a\cos x + b\sin x$ as either a sine or a cosine function, and the magnitude of α will vary depending on the function chosen.

For R to be positive and α to be in quadrant 1, the following transformations are used:

$a\sin x \pm b\cos x$ in the form $R\sin(x \pm \alpha)$
$a\cos x \pm b\sin x$ in the form $R\cos(x \mp \alpha)$

WORKED EXAMPLE 26

Given the function $f(x) = 2\sin x - 2\sqrt{3}\cos x$, $x \in [0, 2\pi]$:
a. express $f(x)$ in the form $R\sin(x - \alpha)$
b. determine the coordinates of the maximum and minimum of $f(x)$
c. sketch $f(x)$
d. solve $f(x) = 2$.

THINK

a. 1. Let $2\sin(x) - 2\sqrt{3}\cos(x)$ equal $R\sin(x - \alpha)$ and use the compound angle formulas to expand $R\sin(x - \alpha)$.

2. Equate the coefficients of $\cos x$ and $\sin x$.

WRITE

a. Let $2\sin(x) - 2\sqrt{3}\cos(x) = R\sin(x - \alpha)$.

$$\begin{aligned}2\sin(x) - 2\sqrt{3}\cos(x) &= R\sin(x - \alpha)\\ &= R(\sin(x)\cos(\alpha) - \cos(x)\sin(\alpha))\\ &= R\sin(x)\cos(\alpha) - R\cos(x)\sin(\alpha)\end{aligned}$$

Equating the coefficients of $\cos(x)$ and $\sin(x)$:

$$2 = R\cos(\alpha) \quad [1]$$
$$2\sqrt{3} = R\sin(\alpha) \quad [2]$$

3. As R is a factor of both equations, it can be eliminated by finding $\frac{[2]}{[1]}$. This will also allow us to use $\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$.

$$\frac{[2]}{[1]}: \frac{2\sqrt{3}}{2} = \frac{R\sin(\alpha)}{R\cos(\alpha)}$$

4. From equations [1] and [2], both $\sin(\alpha)$ and $\cos(\alpha)$ are positive, so α is in quadrant 1. Remember $\tan\frac{\pi}{3} = \sqrt{3}$. Solve for α.

$$\sqrt{3} = \frac{\sin(\alpha)}{\cos(\alpha)}$$
$$= \tan(\alpha)$$
$$\alpha = \frac{\pi}{3}$$

5. Use either equation [1] or [2] to solve for R.

$$[1]: 2 = R\cos\frac{\pi}{6}$$
$$= R \times \frac{1}{2}$$
$$R = 2 \times 2$$
$$= 4$$

6. State the final result.

$$2\sin x - 2\sqrt{3}\cos x = 4\sin\left(x - \frac{\pi}{3}\right)$$

b. 1. R will be the amplitude of the graph. There is no vertical shift.

$R = 4$; therefore, the amplitude of the graph is 4. This means that the maximum is 4 and the minimum is -4.

2. Identify where the maximum occurs. (The largest possible value of a sine function is 1 and $\sin\left(\frac{\pi}{2}\right) = 1$). Note we are looking for $x \in [0, 2\pi]$.

The maximum will occur when

$$\sin\left(x - \frac{\pi}{3}\right) = 1$$
$$x - \frac{\pi}{3} = \frac{\pi}{2}$$
$$x = \frac{\pi}{2} + \frac{\pi}{3}$$
$$= \frac{3\pi}{6} + \frac{2\pi}{6}$$
$$= \frac{5\pi}{6}$$

The maximum point is $\left(\frac{5\pi}{6}, 4\right)$.

3. Identify where the minimum occurs. (The smallest possible value of a sine function is -1 and $\sin\left(\frac{3\pi}{2}\right) = 1$). Note we are looking for $x \in [0, 2\pi]$.

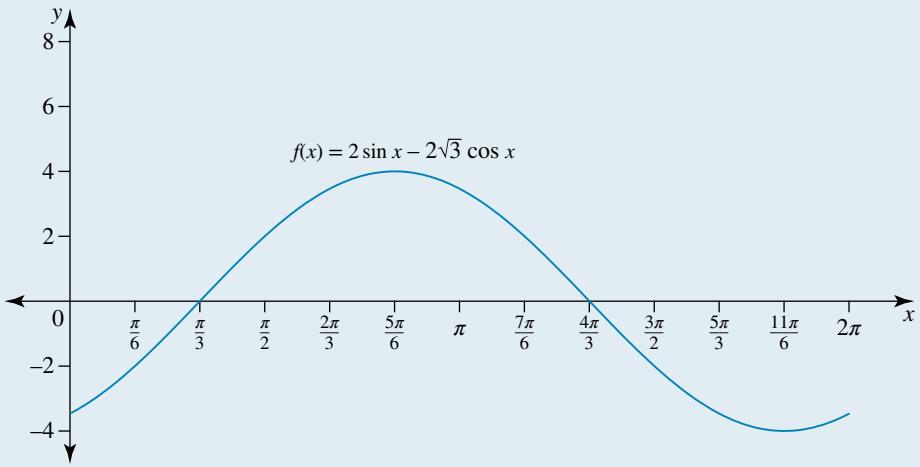
The minimum will occur when

$$\sin\left(x - \frac{\pi}{3}\right) = -1$$
$$x - \frac{\pi}{3} = \frac{3\pi}{2}$$
$$x = \frac{3\pi}{2} + \frac{\pi}{3}$$
$$= \frac{9\pi}{6} + \frac{2\pi}{6}$$
$$= \frac{11\pi}{6}$$

The minimum point is $\left(\frac{11\pi}{6}, -4\right)$.

c. Use the maximum and minimum points to sketch the function over the required domain.

c. $f(x) = 2\sin x - 2\sqrt{3}\cos x,\ x \in [0, 2\pi]$



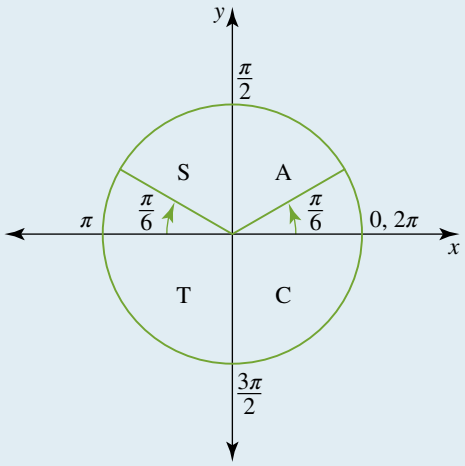
d. 1. To solve $f(x)$, use the form $R\sin(x - \alpha)$.

d. $2\sin(x) - 2\sqrt{3}\cos(x) = 2$

$$4\sin\left(x - \frac{\pi}{3}\right) = 2$$

2. Solve for x. Remember that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and sine is positive in quadrants 1 and 2.

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$$



$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Therefore: $x - \frac{\pi}{3} = \frac{\pi}{6}$ or $x - \frac{\pi}{3} = \pi - \frac{\pi}{6}$

$$x = \frac{\pi}{6} + \frac{\pi}{3} \qquad x = \pi - \frac{\pi}{6} + \frac{\pi}{3}$$
$$= \frac{\pi}{6} + \frac{2\pi}{6} \qquad = \frac{6\pi}{6} - \frac{\pi}{6} + \frac{2\pi}{6}$$
$$= \frac{3\pi}{6} \qquad = \frac{7\pi}{6}$$
$$= \frac{\pi}{2}$$

3. Combine the solutions.

$$x = \frac{\pi}{2} \text{ or } x = \frac{7\pi}{6}$$

Exercise 9.6 Convert $a\cos(x) + b\sin(x)$ to $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$

Technology free

1. **WE25** Express $\cos(x) + \sin(x)$ in the form $R\cos(x - \alpha)$.
2. Express $\sqrt{3}\cos(x) - \sin(x)$ in the form $R\cos(x + \alpha)$.
3. Express the following in the form $R\sin(x + \alpha)$.
   a. $\sin(x) + \cos(x)$
   b. $\sqrt{3}\sin(x) + \cos(x)$
4. Express the following in the form $R\sin(x - \alpha)$.
   a. $\sin(x) - \cos(x)$
   b. $\sqrt{3}\sin(x) - \cos(x)$
5. Express the following in the form $R\cos(x - \alpha)$.
   a. $4\sqrt{2}\cos(x) + 4\sqrt{2}\sin(x)$
   b. $7\cos(x) + 24\sin(x)$
6. Express the following in the form $R\cos(x + \alpha)$.
   a. $\cos(x) - \sin(x)$
   b. $\cos(x) - \sqrt{3}\sin(x)$
7. **WE26** Given the function $f(x) = \sqrt{2}\sin(x) - \sqrt{2}\cos(x)$, $x \in [0, 2\pi]$:
   a. express $f(x)$ in the form $R\sin(x - \alpha)$
   b. determine the coordinates of the maximum and minimum of $f(x)$
   c. sketch $f(x)$
   d. solve $f(x) = 1$.
8. Consider the function $f(x) = 5\sin(x) - 12\cos(x)$, $x \in [0, 2\pi]$. In your answer,use 3 decimal places where necessary.
   a. Express $f(x)$ in the form $R\cos(x + \alpha)$.
   b. Determine the coordinates of the maximum and minimum of $f(x)$.
   c. Sketch $f(x)$.
   d. Solve $f(x) = 6.5$.
9. For the function $f(x) = 3\sin(x) - 4\cos(x)$, $x \in [0°, 360°]$. In your answer, use 2 decimal places where necessary.
   a. Express $f(x)$ in the form $R\sin(x - \alpha)$.
   b. Determine the coordinates of the maximum and minimum of $f(x)$.
   c. Sketch $f(x)$.
   d. Solve $f(x) = 2.5$.
10. For the function $f(x) = 12\cos(x) + 5\sin(x)$, $x \in [0, 2\pi]$. In your answer, use 2 decimal places where necessary.
    a. Express $f(x)$ in the form $R\cos(x - \alpha)$.
    b. Determine the coordinates of the maximum and minimum of $f(x)$.
    c. Sketch $f(x)$.
    d. Solve $f(x) = 6.5$ and hence find the general solution of $12\cos(x) + 5\sin(x) = 6.5$.

Technology active

11. a. Express $24\cos(y) - 7\sin(y)$ in the form $R\cos(y + \alpha)$, giving your answer to 3 decimal places as necessary.

b. A particle moves in a straight line. Its displacement, x, at time t seconds is given by $x = 30 + 24\cos(3t) - 7\sin(3t)$ for $t \geq 0$. Calculate the maximum and minimum displacement from the origin, O, and the first time these occur, to the nearest hundredth of a second.

12. a. Express $2\sin(x) + 2\sqrt{3}\cos(x)$ in the form $R\sin(x + \alpha)$.

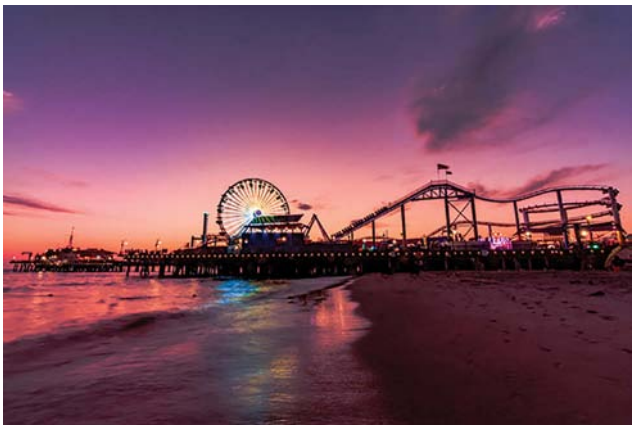
b. The depth of water near a pier changes with the tides according to the rule

$$h(t) = 5 + 2\sin\left(\frac{\pi t}{12}\right) + 2\sqrt{3}\cos\left(\frac{\pi t}{12}\right)$$

where t hours is the time, for $t \geq 0$, and h is depth in metres. Determine the maximum and minimum depth and the first times these occur, to the nearest hour.

c. Identify the first two times when the depth of the water is 7 metres.

d. Sketch the graph of the depth of the water, showing one cycle.



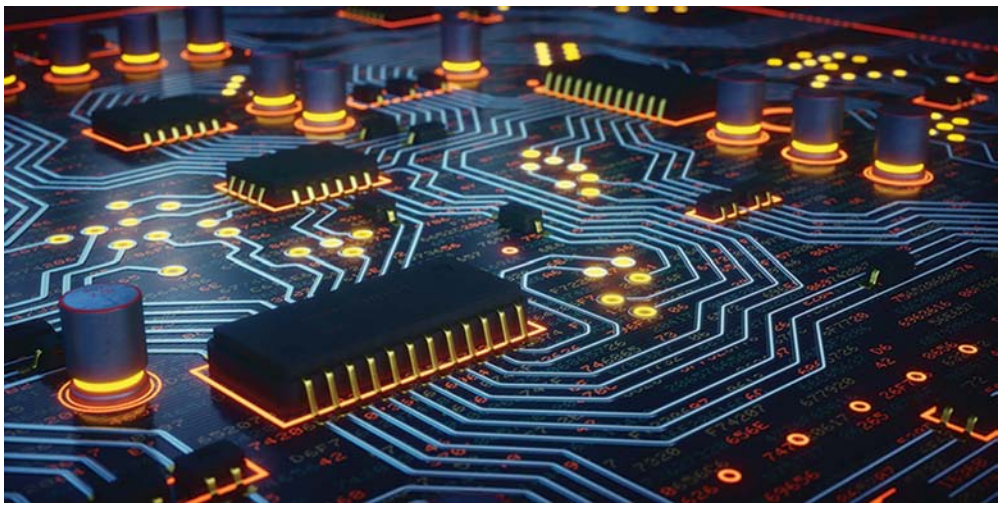
13. Calculate the maximum and minimum values of each of the following.

a. $\dfrac{4}{5 + 2.4\cos(x) - 3.2\sin(x)}$

b. $\dfrac{\sqrt{3}}{2 - 4\cos(x) - 4\sqrt{3}\sin(x)}$

14. a. At time t, the current I in a circuit is given by $I = 40\cos(300t) + 9\sin(300t)$. Express the current in the form $R\cos(nt - \alpha)$ where α is in decimal degrees, correct to 2 decimal places.

b. At a time t, the current I in a circuit is given by $I = 11\cos(800t) - 60\sin(800t)$. Express the current in the form $R\cos(nt + \alpha)$ where α is in decimal degrees, correct to 2 decimal places.

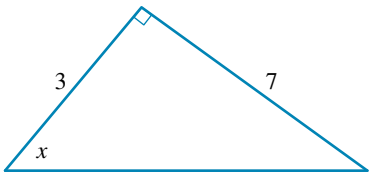


9.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Use the diagram to find the exact values of:
   a. $\cos(x)$ b. $\sin(x)$



2. Simplify $\left(2\tan\frac{x}{2}\right)^2 - \left(2\sec\frac{x}{2}\right)^2$.
3. If $\sec(x) = a$, $a > 1$, $0 < x < \frac{\pi}{2}$, find expressions for:
   a. $\cot(x)$ b. $\text{cosec}(x)$
4. Determine the values of $\cos(x)$ and $\sin(x)$ if $\cos(2x) = \frac{7}{25}$.
5. If $\sin(A) = \frac{1}{9}$, $\frac{\pi}{2} < A < \pi$, calculate the exact value of $\sin(2A)$.
6. If $\cos(A) = \frac{1}{3}$ and $\cos(B) = \frac{1}{4}$, where A and B are both acute angles, calculate $\cos(A + B)$ as exact values.
7. Use multiple angle formulas to simplify $\frac{\sin(3A)}{\sin(A)} + \frac{\cos(3A)}{\cos(A)}$.
8. Determine the exact value of $\tan\left(\frac{13\pi}{12}\right)$.

9. Find the general solution for x of the equation $\sin(2x) = \sin(x)$.
10. Use appropriate formulas to simplify $\frac{\sin(A) + \sin(B)}{\cos(A) + \cos(B)}$.
11. Prove the following.
    a. $\cos^3(\theta) = \frac{1}{4}(\cos(3\theta) + 3\cos(\theta))$ b. $\cos^4(\theta) = \frac{1}{8}(\cos(4\theta) + 4\cos(2\theta) + 3)$
12. Consider $\cos(x) - \cos(3x) + \cos(5x) = 0$.
    a. Determine the general solution for x.
    b. Hence, calculate x if $x \in [0, 2\pi]$.

Complex familiar

13. a. Prove that $\tan(\theta) = \cot(\theta) - 2\cot(2\theta)$.
    b. Hence, demonstrate that $\tan(\theta) + 2\tan(2\theta) = \cot(\theta) - 4\cot(4\theta)$.
    c. Use appropriate formulas to simplify $\tan(\theta) + 2\tan(2\theta) + 4\tan(4\theta)$.
14. At a time t, the current in a circuit, I_1, is given by $I_1 = 40\cos(800t) + 9\sin(800t)$ and the current in another circuit, I_2, is given by $I_2 = 9\cos(800t) - 40(\sin 800t)$. You may use technology to answer the following.
    a. Express I_1 in the form $R\cos(nt - \alpha)$ and I_2 in the form $R\cos(nt + \alpha)$, where α is in degrees correct to the nearest minute.
    b. Express the total current, $I_T = I_1 + I_2$, as a single cosine function.

15. If $\tan(\alpha) = \frac{1}{5}$ and $\tan(\beta) = \frac{1}{239}$:

a. demonstrate that $\tan(4\alpha) = \frac{120}{119}$

b. hence, determine that $\tan(4\alpha - \beta) = 1$ and $4\alpha - \beta = \frac{\pi}{4}$ is a solution.

16. Prove the following.

a. $\tan(3A) = \frac{3\tan(A) - \tan^3(A)}{1 - 3\tan^2(A)}$

b. $\tan(4A) = \frac{4\tan(A)\left(1 - \tan^2(A)\right)}{1 - 6\tan^2(A) + \tan^4(A)}$

Complex unfamiliar

17. If $\alpha + \beta = \theta$ and $\tan(\alpha) : \tan(\beta) = a : b$, prove that $\sin(\alpha - \beta) : \sin(\theta) = (a - b) : (a + b)$.

18. If $\sin^4(\theta) + \cos^4(\theta) = 1$, prove that θ is a multiple of $\frac{\pi}{2}$.

19. If $\alpha + \beta + \gamma = \pi$, prove that:

a. $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 4\sin(\alpha)\sin(\beta)\sin(\gamma)$

b. $\cos(\alpha) + \cos(\beta) + \cos(\gamma) = 1 + 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

20. Prove that if $\sin(\alpha - \beta) = \sin^2(\alpha) - \sin^2(\beta)$, then either $\alpha - \beta = n\pi$ or $\alpha + \beta = 2n\pi + \frac{\pi}{2}$, $n \in Z$.

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Units 1 & 2 Sit chapter test

Answers

Chapter 9 Trigonometric identities

Exercise 9.2 Pythagorean identities

1.

$A°$	30°	81°	129°	193°	260°	350°	−47°
$\sin^2 A$	0.25	0.976	0.604	0.051	0.970	0.030	0.535
$\cos^2 A$	0.75	0.024	0.396	0.949	0.030	0.970	0.465
$\sin^2 A + \cos^2 A$	1.00	1.000	1.000	1.000	1.000	1.000	1.000

2. **a.** 0.600 **b.** 1.333

3. **a.** 0.954 **b.** 3.180

4. **a.** ± 0.917 **b.** ± 0.714 **c.** ± 0.971 **d.** ± 0.436

5. **a.** $\frac{\sqrt{5}}{3}$ **b.** $\frac{2}{3}$

6. **a.** $\frac{\sqrt{7}}{4}$ **b.** $\frac{3}{\sqrt{7}}$

7. **a.** $\frac{\sqrt{11}}{4}$ **b.** $-\sqrt{\frac{5}{11}}$

8. $\sin^2 A$

9. $1 - \sin A$

10. **a.** $0, \frac{\pi}{2}, \pi, 2\pi$ **b.** $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ **c.** $0, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

d. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **e.** $\frac{\pi}{2}$ **f.** $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

11. **a.** $(2\sin(A) - 1)(\sin(A) + 1)$

b. $(\tan x - 2)(\tan x + 1)$

c. $(2\,\text{cosec}(\alpha) + 1)(\text{cosec}(\alpha) - 1)$

d. $(1 - 2\sin(3x))(1 + 2\sin(3x))$ or $(2\cos(3x) + 1)(2\cos 3x - 1)$

e. $(\text{cosec}\,(3A) - 1)(3\,\text{cosec}\,(3A) + 1)$

f. $(\sec(2\beta) + 1)(\sec(2\beta) - 2)$

12. **a.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **b.** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **c.** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

d. $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **e.** $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ **f.** $\frac{3\pi}{2}$

13. **a.** $\frac{\pi}{12}(12n + 1), \frac{\pi}{12}(12n + 5), \frac{\pi}{4}(4n + 1), n \in Z$

b. $\frac{\pi}{3}(6n \pm 1), \frac{\pi}{2}(2n + 1), n \in Z$

c. $\frac{\pi}{24}(12n - 5), \frac{\pi}{24}(12n - 1), \frac{n\pi}{4}, n \in Z$

d. $\frac{n\pi}{2}, \frac{\pi}{8}(2n + 1), n \in Z$

e. $\frac{\pi}{18}(12n + 5), \frac{\pi}{18}(12n - 5), \frac{\pi}{6}(2n + 1), n \in Z$

f. $\frac{\pi}{9}(6n + 1), \frac{2\pi}{9}(3n + 1), \frac{n\pi}{3}, n \in Z$

14. **a.** $\frac{\pi}{3}(3n - 1), \frac{\pi}{4}(4n - 1), n \in Z$

b. $\frac{\pi}{3}(3n - 1), \frac{\pi}{4}(4n + 1), n \in Z$

15. **a.** $\frac{\pi}{6}(12n + 1), \frac{\pi}{6}(12n + 5), \frac{\pi}{2}(2n + 1), n \in Z$

b. $\frac{\pi}{3}(6n \pm 1), n\pi, n \in Z$

c. $\frac{\pi}{4}(2n + 1), n \in Z$

d. $\frac{\pi}{4}(2n + 1), \frac{\pi}{3}(3n \pm 1), n \in Z$

16. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

17. 9

18. $\left[\frac{3}{4}, 1\right]$

Exercise 9.3 Compound angle formulas

1. a. $\frac{\sqrt{3}}{2}$ **b.** 0 **c.** $\frac{1}{2}$ **d.** $-\frac{\sqrt{2}}{2}$

2. a. $\sin(\theta) - \cos(\theta)$ **b.** $\sin(\theta) + \sqrt{3}\cos(\theta)$
c. $\sqrt{3}\cos(\theta) + \sin(\theta)$ **d.** $\cos(\theta) - \sin(\theta)$

3. a. $\cos(\theta)$ **b.** $\sin(\theta)$ **c.** $-\sin(\theta)$ **d.** $-\cos(\theta)$

4. a. $-\cos(\theta)$ **b.** $\sin(\theta)$ **c.** $-\tan(\theta)$ **d.** $\tan(\theta)$

5. a. $\sqrt{3}\cos(x)$ **b.** $-\sqrt{3}\sin x$
c. $\sin(x)$ **d.** -1

6. a. $\frac{\sqrt{2}-\sqrt{6}}{4}$ **b.** $\frac{\sqrt{6}-\sqrt{2}}{4}$
c. $2-\sqrt{3}$ **d.** $2+\sqrt{3}$

7. a. $\frac{56}{65}$ **b.** $-\frac{63}{16}$

8. a. $-\frac{253}{325}$ **b.** $-\frac{204}{325}$

9. a. $-\frac{8\sqrt{5}}{21}$ **b.** $-\frac{19}{21}$

10. $\dfrac{ab + \sqrt{1-a^2}\sqrt{1-b^2}}{b\sqrt{1-a^2} - a\sqrt{1-b^2}}$

11. $\dfrac{a^2 - 2\sqrt{a+1}\sqrt{2a+1}}{a\left(\sqrt{2a+1} + 2\sqrt{a+1}\right)}$

12–16. Sample responses can be found in the worked solutions in the online resources.

Exercise 9.4 Multiple angle formulas

1. a. $\frac{\sqrt{2}}{4}$ **b.** $-\frac{\sqrt{2}}{2}$ **c.** $-\frac{\sqrt{2}}{2}$ **d.** $\frac{\sqrt{3}+2}{4}$

2. a. $-\frac{23}{32}$ **b.** $\frac{3\sqrt{55}}{32}$ **c.** $-\frac{3\sqrt{55}}{23}$ **d.** $-\frac{23\sqrt{55}}{165}$

3. a. $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$
b. $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$
c. $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$
d. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

4. a. $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$
b. $\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$
c. $0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}, 2\pi$
d. $\frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{7\pi}{4}$

5–14. Sample responses can be found in the worked solutions in the online resources.

Exercise 9.5 Product–sum identities

1. a. $\sqrt{3}$ **b.** $-\frac{\sqrt{3}}{3}$

2. a. $\frac{\sqrt{3}}{3}$ **b.** 1 **c.** $-\frac{\sqrt{3}}{3}$ **d.** -1

3. a. $\frac{\sqrt{6}}{2}$ **b.** $\frac{\sqrt{2}}{2}$ **c.** $\frac{\sqrt{2}}{2}$ **d.** $-\frac{\sqrt{6}}{2}$

4–11. Sample responses can be found in the worked solutions in the online resources.

12. a. $\frac{\pi}{5}(2n+1), \pi(2n+1), n \in Z$
b. $\frac{n\pi}{3}, \frac{\pi}{2}(2n+1), n \in Z$

13. a. $\frac{n\pi}{2}, \frac{\pi}{2}(2n+1), n \in Z$
b. $\frac{\pi}{4}(2n+1), \frac{\pi}{2}(2n+1), n \in Z$

14. a. $\frac{n\pi}{2}, \frac{\pi}{3}(6n \pm 2), n \in Z$
b. $\frac{\pi}{4}(2n+1), \frac{\pi}{3}(6n \pm 2), n \in Z$
c. $\frac{\pi}{2}(2n+1), \frac{\pi}{12}(12n-1), \frac{\pi}{12}(12n-5), n \in Z$
d. $\frac{n\pi}{2}, \frac{\pi}{6}(12n-1), \frac{\pi}{6}(12n-5), n \in Z$

15. Sample responses can be found in the worked solutions in the online resources.

16. a–c. Sample responses can be found in the worked solutions in the online resources.
d. $\tan(8A)$

Exercise 9.6 Convert $a\cos(x) + b\sin(x)$ to $R\cos(x \pm \alpha)$ or $R\sin(x \pm \alpha)$

1. $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

2. $2\cos\left(x + \frac{\pi}{6}\right)$

3. a. $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$
b. $2\sin\left(x + \frac{\pi}{6}\right)$

4. a. $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$
b. $2\sin\left(x - \frac{\pi}{6}\right)$

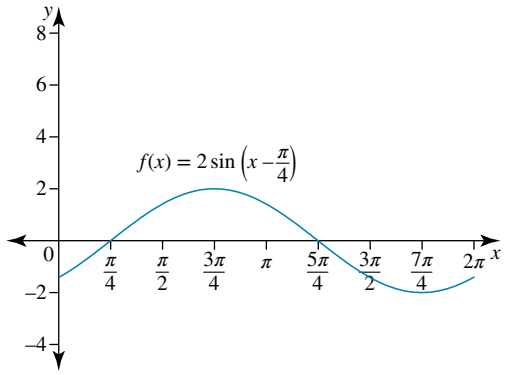
5. a. $8\cos\left(x - \frac{\pi}{4}\right)$
b. $25\cos(x - 1.287)$

6. a. $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$
b. $2\cos\left(x + \frac{\pi}{3}\right)$

7. a. $f(x) = 2\sin\left(x - \frac{\pi}{4}\right)$

b. Maximum $\left(\frac{3\pi}{4}, 2\right)$, minimum $\left(\frac{7\pi}{4}, -2\right)$

c.

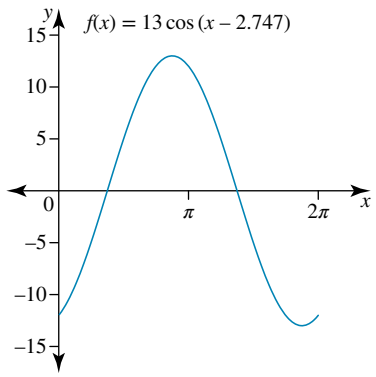


d. $\frac{5\pi}{12}, \frac{13\pi}{12}$

8. a. $f(x) = 13\cos(x - 2.747)$

b. Maximum $(2.747, 13)$, minimum $(5.888, -13)$

c.

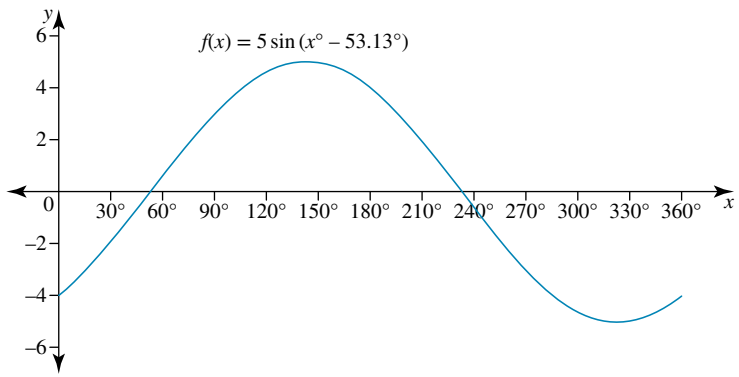


d. 1.7, 3.794

9. a. $f(x) = 5\sin(x° - 53.13°)$

b. Maximum $(143.13°, 5)$, minimum $(323.13°, -5)$

c.

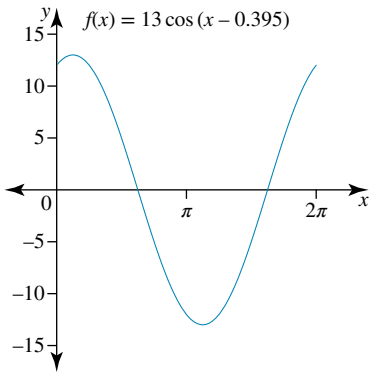


d. 83.13°, 203.13°

10. a. $f(x) = 13\cos(x - 0.395)$

b. Maximum $(0.395, 13)$, minimum $(3.537, -13)$

c.



d. $x = 1.442,\ 5.631$; general solution: $x = 1.442 + 2n\pi,\ 5.631 + 2n\pi,\ n \in Z$

11. a. $f(x) = 25\cos(x + 0.284)$

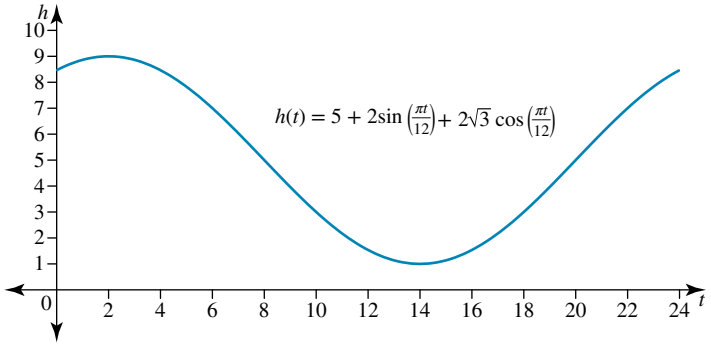
b. Maximum of 55 after 2.00 seconds, minimum of 5 after 0.95 seconds

12. a. $2\sin(x) + 2\sqrt{3}\cos(x) = 4\sin\left(x + \frac{\pi}{3}\right)$

b. Maximum of 9 after 2 hours, minimum of 1 after 14 hours

c. 6, 10 hours

d.



13. a. Maximum 4, minimum $\frac{4}{9}$

b. Maximum $\frac{\sqrt{3}}{10}$, minimum $-\frac{\sqrt{3}}{6}$

14. a. $I = 41\cos(300t - 12.68°)$

b. $I = 61\cos(800t - 79.61°)$

9.7 Review: exam practice

1. a. $\frac{3\sqrt{58}}{58}$ **b.** $\frac{7\sqrt{58}}{58}$

2. -4

3. a. $\frac{\sqrt{a^2 - 1}}{a^2 - 1}$ **b.** $\frac{a\sqrt{a^2 - 1}}{a^2 - 1}$

4. $\cos x = \pm\frac{4}{5}$, $\sin x = \pm\frac{3}{5}$

5. $-\frac{8\sqrt{5}}{81}$

6. $\frac{1-2\sqrt{30}}{12}$

7. $4\cos 2A$

8. $2 - \sqrt{3}$

9. $\frac{\pi}{3}(6n \pm 1),\ n\pi,\ n \in Z$

10. $\tan\left(\frac{A + B}{2}\right)$

11. a, b. Sample responses can be found in the worked solutions in the online resources.

12. a. $x = \frac{\pi}{6}(6n \pm 1), \frac{\pi}{6}(2n + 1), n \in Z$

b. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

13. a, b. Sample responses can be found in the worked solutions in the online resources.

c. $\cot(\theta) - 8\cot(8\theta)$

14. a. $I_1 = 41\cos(800t - 12°41'), I_2 = 41\cos(800t + 77°19')$

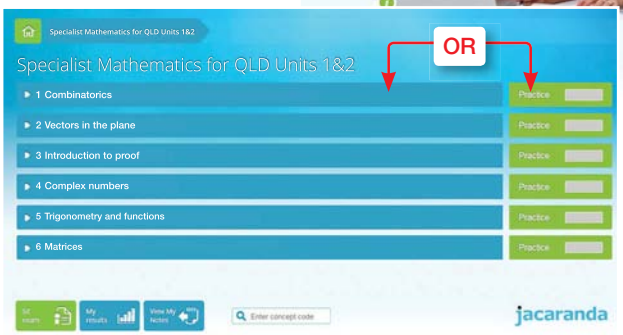
b. $I_T = 41\sqrt{2}\cos(800t + 32°19')$

15–20. Sample responses can be found in the worked solutions in the online resources.

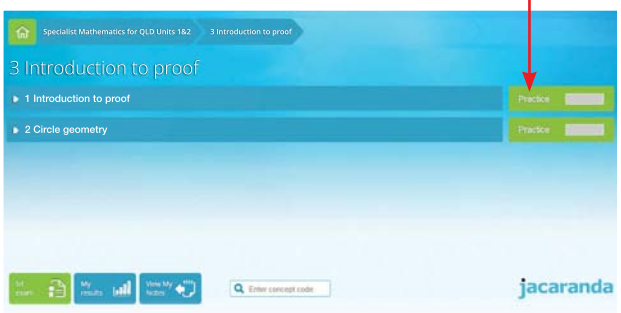
TOPIC 2 Trigonometry and functions

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.

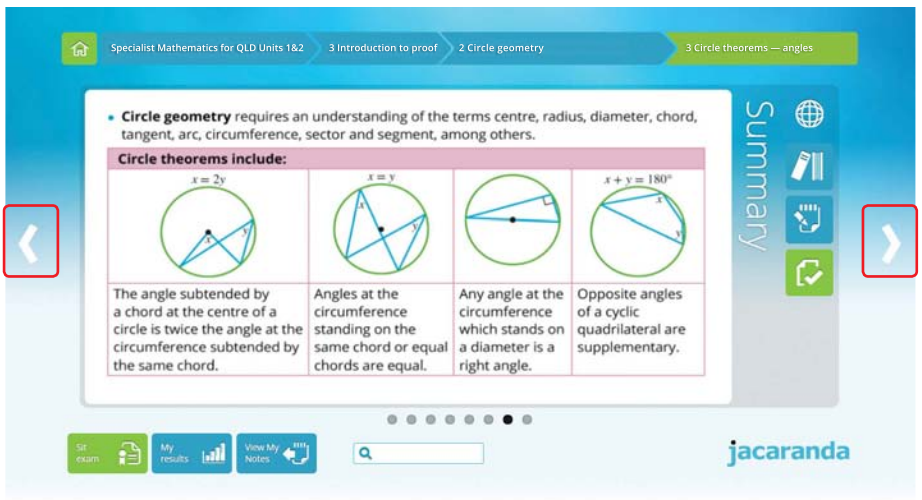


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



CHAPTER 10
Matrix arithmetic

10.1 Overview

10.1.1 Introduction

Matrices were first used to solve systems of linear equations. Today they are used in many fields, including engineering, physics, economics and statistics. They are used primarily in the encoding and decoding of information. You have probably used matrix technology without realising it in encrypted messaging and internet banking.

The British mathematician Alan Turing led a team of mathematicians who were responsible for breaking the Germans' Enigma code during World War II. Their work helped the Allies to win the war in Europe and also led to the creation of the first computer. Turing built a machine to help with the decoding process. This was the start of the computer age, and now every computer-generated image is the result of matrix mathematics.

Turing and his team's code-breaking work at Bletchley Park was the subject of the 2014 movie *The Imitation Game* starring Benedict Cumberbatch as Dr Turing.

LEARNING SEQUENCE

10.1 Overview
10.2 Addition, subtraction and scalar multiplication of matrices
10.3 Matrix multiplication
10.4 Determinants and inverses
10.5 Matrix equations and solving 2×2 linear equations
10.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Addition, subtraction and scalar multiplication of matrices

10.2.1 Introduction to matrices

The table shows the final medal tally for the top four countries at the 2016 Rio de Janero Olympic Games.

Country	Gold	Silver	Bronze
United States of America	46	37	38
Great Britain	27	23	17
People's Republic of China	26	18	26
Russian Federation	19	18	19

This information can be presented in a matrix, without the country names, and without the headings for gold, silver and bronze:

$$\begin{bmatrix} 46 & 37 & 38 \\ 27 & 23 & 17 \\ 26 & 18 & 26 \\ 19 & 18 & 19 \end{bmatrix}$$

The data is presented in a rectangular array arranged in rows and columns, and is called a **matrix**. It conveys information such as that the second country won 27 gold, 23 silver and 17 bronze medals. This matrix has four rows and three columns. The numbers in the matrix, in this case representing the number of medals won, are called **elements of the matrix**.

Matrices

In general we enclose a matrix in square brackets and usually use capital letters to denote it. The size or **order of a matrix** is important, and is determined by the number of rows and the number of columns, strictly in that order.

Consider the following matrix A:

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$$

A is a 2×2 matrix: it has two rows and two columns. When the number of rows and columns in a matrix are equal, it is called a **square matrix**.

$$B = \begin{bmatrix} 2 & -5 & -3 \\ -4 & 2 & -5 \\ 1 & 3 & 4 \end{bmatrix}$$

B is a 3×3 matrix: it has three rows and three columns.

$$C = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

C is a 3×1 matrix as it has three rows and one column. If a matrix has only one column, it is also called a **column** or **vector matrix**.

$$D = \begin{bmatrix} 3 & -2 \end{bmatrix}$$

D is a 1×2 matrix as it has one row and two columns. If a matrix has only one row it is also called a **row matrix**.

$$E = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$$

E is a 3×2 matrix as it has three rows and two columns.

$$F = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$$

F is a 2×3 matrix as it has two rows and three columns.

Each element in a matrix can also be identified by its position in the matrix. For example, in matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, the element 3 is in the first row and first column, so $a_{11} = 3$. The element 5 is in the first row and second column, so $a_{12} = 5$. The element 4 is in the second row and first column, so $a_{21} = 4$. Finally, the element 7 is in the second row and second column, so $a_{22} = 7$.

In general, we can write a 2×2 matrix as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A general matrix of order $m \times n$ can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix}$$

Here, a_{14} denotes the element in the first row and fourth column, a_{43} denotes the element in the fourth row and third column, and a_{ij} denotes the element in the ith row and jth column.

10.2.2 Operations on matrices

Operations include addition, subtraction and multiplication of two matrices. Division of matrices is not possible.

Equality of matrices

Two matrices are equal, if and only if, they have the same order and each of the corresponding elements are equal.

For example, if $\underset{3 \times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}$, then $x = 1, y = -2$ and $z = 3$; if $\underset{2 \times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}}$, then $a = 3,\ b = 5,$ $c = 4$ and $d = 7$.

Addition and subtraction of matrices

Only two matrices of the same order can be added or subtracted. To add or subtract two matrices, we add or subtract the elements in the corresponding positions. For example, if $P = \underset{2\times 2}{\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}}$ and $Q = \underset{2\times 2}{\begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix}}$, then:

$$P + Q = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2+4 & 3-2 \\ -1+6 & 5+3 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 5 & 8 \end{bmatrix}$$

$$P - Q = P + (-Q) = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 2-4 & 3+2 \\ -1-6 & 5-3 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -7 & 2 \end{bmatrix}$$

In general, for 2×2 matrices, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then:

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

Note that none of the matrices defined by

$$A = \underset{2\times 2}{\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}}, \quad B = \underset{3\times 3}{\begin{bmatrix} 2 & -5 & -3 \\ -4 & 2 & -5 \\ 1 & 3 & 4 \end{bmatrix}}, \quad C = \underset{3\times 1}{\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}, \quad D = \underset{1\times 2}{\begin{bmatrix} 3 & -2 \end{bmatrix}}, \quad E = \underset{3\times 2}{\begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}}, \quad F = \underset{2\times 3}{\begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}}$$

can be added or subtracted from one another, as they are all of different orders.

WORKED EXAMPLE 1

At a football match one food outlet sold 280 pies, 210 hotdogs and 310 boxes of chips. Another food outlet sold 300 pies, 220 hotdogs and 290 boxes of chips. Represent each data set as a 1×3 matrix, and determine the total number of pies, hotdogs and chips sold by these two outlets.

THINK	WRITE
1. Use a 1×3 matrix to represent the number of pies, hotdogs and chips sold.	$\begin{bmatrix} \text{pies} & \text{hotdogs} & \text{chips} \end{bmatrix}$
2. Write the matrix for the sales from the first outlet.	$S_1 = \begin{bmatrix} 280 & 210 & 310 \end{bmatrix}$
3. Write the matrix for the sales from the second outlet.	$S_2 = \begin{bmatrix} 300 & 220 & 290 \end{bmatrix}$
4. Use the rules of addition of matrices to detrmine the sum of these two matrices.	$S_1 + S_2 = \begin{bmatrix} 280 + 300 & 210 + 220 & 310 + 290 \end{bmatrix}$ $= \begin{bmatrix} 580 & 430 & 600 \end{bmatrix}$

Scalar multiplication of matrices

A **scalar**, as introduced in Chapter 2, is an entity with magnitude only; that is, it is a real number. To multiply any matrix by a scalar, we multiply every element in the matrix by the scalar.

$$\text{If } P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

then:

$$2P = 2\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 2 & 2 \times 3 \\ 2 \times -1 & 2 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ -2 & 10 \end{bmatrix}$$

WORKED EXAMPLE 2

Given the matrices $A = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $B = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ y \end{bmatrix}$ determine the values of x and y if $xA + 2B = C$.

THINK	WRITE
1. Substitute for the given matrices.	$xA + 2B = C$ $x\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 2\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$
2. Apply the rules for scalar multiplication.	$\begin{bmatrix} 3x \\ -5x \end{bmatrix} + \begin{bmatrix} -10 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$
3. Apply the rules for addition of matrices.	$\begin{bmatrix} 3x - 10 \\ -5x + 8 \end{bmatrix} = \begin{bmatrix} 2 \\ y \end{bmatrix}$

4. Apply the rules for equality of matrices.	$3x - 10 = 2$ $-5x + 8 = y$
5. Solve the first equation for x.	$3x = 12$ $x = 4$
6. Substitute for x into the second equation and solve this equation for y.	$-5x + 8 = y$ $y = 8 - 20$ $y = -12$
7. Answer the question.	$\therefore\ x = 4,\ \ y = 12$

10.2.3 Special matrices

The zero matrix

The 2×2 **null matrix** or **zero matrix** O, with all elements equal to zero, is given by $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

When matrices $A + B = O$, matrix B is the **additive inverse** of A. Hence, $B = O - A$.

If $A = \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -4 \\ -5 & 3 \end{bmatrix}$,

$$A + B = \begin{bmatrix} 2 & 4 \\ 5 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 + (-2) & 4 + (-4) \\ 5 + (-5) & -3 + 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Hence, B is the additive inverse of A.

The identity matrix

The 2×2 **identity matrix** I is defined by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This square matrix has ones on the leading diagonal and zeros on the other diagonal. An example of an identity matrix is shown in Worked example 3c.

WORKED EXAMPLE 3

Given the matrices $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$, determine the matrix X if:

a. $X = 2A - 3B$ **b.** $A + X = O$ **c.** $X = B + 2A - 3I$

THINK	WRITE
a. 1. Substitute for the given matrices.	**a.** $X = 2A - 3B$ $= 2\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} - 3\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$
2. Apply the rules for scalar multiplication.	$= \begin{bmatrix} 6 & 10 \\ 8 & 14 \end{bmatrix} - \begin{bmatrix} -15 & -9 \\ 9 & 12 \end{bmatrix}$
3. Apply the rules for subtraction of matrices.	$= \begin{bmatrix} 21 & 19 \\ -1 & 2 \end{bmatrix}$

b. 1. Transpose the equation to make X the subject.

2. State the final answer.

b. $A + X = O$

$X = O - A$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} -3 & -5 \\ -4 & -7 \end{bmatrix}$

c. 1. Substitute for the given matrices.

2. Apply the rules for scalar multiplication.

3. Apply the rules for addition and subtraction of matrices.

4. State the final answer.

c. $X = B + 2A - 3I$

$= \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 10 \\ 8 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

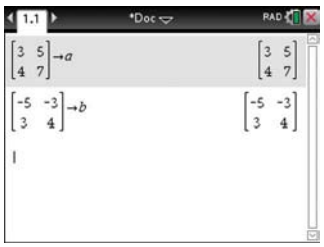
$= \begin{bmatrix} -5+6-3 & -3+10+0 \\ 3+8-0 & 4+14-3 \end{bmatrix}$

$= \begin{bmatrix} -2 & 7 \\ 11 & 15 \end{bmatrix}$

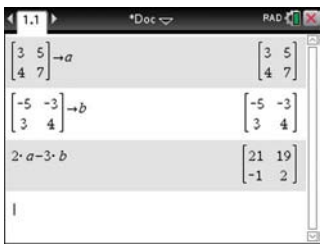
TI | THINK

a.1. On a Calculator page, press the template button and complete the entry line as:
$\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$
Press CTRL, then press VAR, then type 'a' and press ENTER to store matrix A.
Repeat this step to store matrix B.

WRITE



2. Complete the next entry line as: $2a - 3b$ then press ENTER.



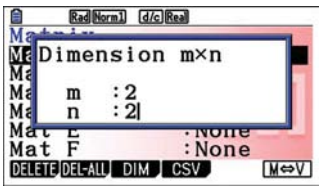
3. The answer appears on the screen.

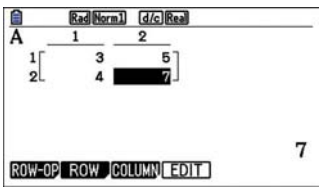
$X = \begin{bmatrix} 21 & 19 \\ -1 & 2 \end{bmatrix}$

CASIO | THINK

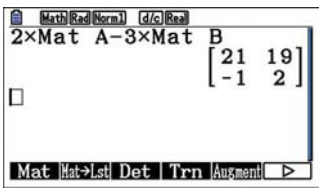
a.1. On a Run-Matrix screen, select MAT/VCT by pressing F3.
Select Matrix A by pressing EXE and change its dimensions to 2 by 2.
Press EXE.
Enter the elements of Matrix A, then press EXIT.
Repeat this step to store Matrix B, then press EXIT to return to the Run-Matrix screen.

WRITE





2. Complete the entry line as:
$2 \times$ Mat A $- 3 \times$ Mat B
then press EXE.
Note: To find 'Mat', press OPTN, select MAT/VCT by pressing F2, then select Mat by pressing F1.



3. The answer appears on the screen.

$X = \begin{bmatrix} 21 & 19 \\ -1 & 2 \end{bmatrix}$

studyon

Units 1 & 2 > Area 6 > Sequence 1 > Concept 1

Addition, subtraction and scalar multiplication Summary screen and practice questions

Exercise 10.2 Addition, subtraction and scalar multiplication of matrices

Technology free

1. **WE 1** At football matches, commentators often quote player statistics. In one particular game, the top ranked player on the ground had 25 kicks, 8 marks and 10 handballs. The second ranked player on the same team on the ground had 20 kicks, 6 marks and 8 handballs, while the third ranked player on the same team on the ground had 18 kicks, 5 marks and 7 handballs. Represent this data as 1×3 matrices and determine the total number of kicks, marks and handballs by these three players.

2. At the end of a doubles tennis match, one player had 2 aces, 3 double faults, 25 forehand winners and 10 backhand winners, while his partner had 4 aces, 5 double faults, 28 forehand winners and 7 backhand winners. Represent this data as 2×2 matrices and determine the total number of aces, double faults, forehand and backhand winners for these players.

3. **WE 2** Given the matrices $A = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 \\ y \end{bmatrix}$, determine the values of x and y if $xA + 2B = C$.

4. Given the matrices $A = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ and $C = \begin{bmatrix} 6 \\ y \\ z \end{bmatrix}$, determine the values of x, y and z if $xA - 2B = C$.

5. **WE 3** If $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$, determine the matrix X given the following.

   a. $X = 3A - 2B$

   b. $2A + X = O$

6. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$, determine the values of a, b, c and d given the following.

   a. $A + 2I - 2B = O$

   b. $3I + 4B - 2A = O$

7. If $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, determine the matrix C given the following.

   a. $C = A + B$

   b. $3A + 2C = 4B$

8. Consider these matrices: $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$

   a. Determine the following matrices.

      i. $B + C$

      ii. $A + B$

   b. Verify the Associative Law for matrix addition: $A + (B + C) = (A + B) + C$.

9. If $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, determine the matrix C given the following.

a. $3A = C - 2B$ **b.** $C + 3A - 2B = O$

10. Given the matrices $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine the matrix C if the following apply.

a. $3A + C - 2B + 4I = O$ **b.** $4A - C + 3B - 2I = O$

11. If $A = \begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}$ and $B = \begin{bmatrix} 2 & y \\ y & -3 \end{bmatrix}$, determine the values of x and y given the following.

a. $A + B = \begin{bmatrix} 7 & 4 \\ 9 & 2 \end{bmatrix}$ **b.** $B - A = \begin{bmatrix} -1 & 1 \\ -4 & -6 \end{bmatrix}$

12. If $D = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & -2 \end{bmatrix}$ and $E = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 4 & -3 \end{bmatrix}$, determine the matrix C given the following.

a. $C = D + E$ **b.** $3D + 2C = 4E$

13. a. Given $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ write down the values of a_{11}, a_{12}, a_{21} and a_{22}.

b. Find the 2×2 matrix B if $b_{11} = 3$, $b_{12} = -2$, $b_{21} = -3$ and $b_{22} = 5$.

14. a. Find the 2×2 matrix A whose elements are $a_{ij} = 2i - j$ for $j \neq i$ and $a_{ij} = ij$ for $j = i$.

b. Find the 2×2 matrix A whose elements are $a_{ij} = i + j$ for $i < j$, $a_{ij} = i - j + 1$ for $i > j$ and $a_{ij} = i + j + 1$ for $i = j$.

Technology active

15. The **trace of a matrix** A denoted by $\text{tr}(A)$ is equal to the sum of leading diagonal elements. For 2×2 matrices, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\text{tr}(A) = a_{11} + a_{22}$. Consider the following matrices: $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$

a. Calculate the following.

i. $\text{tr}(A)$ **ii.** $\text{tr}(B)$ **iii.** $\text{tr}(C)$

b. Is $\text{tr}(A + B + C) = \text{tr}(A) + \text{tr}(B) + \text{tr}(C)$?

c. Is $\text{tr}(2A + 3B - 4C) = 2\text{tr}(A) + 3\text{tr}(B) - 4\text{tr}(C)$?

16. If $A = \begin{bmatrix} 12 & 10 & 4 \\ 8 & 6 & 8 \\ 14 & 12 & 10 \end{bmatrix}$ and $B = \begin{bmatrix} 15 \\ 2 \\ 4 \end{bmatrix}$, use your calculator to calculate $A \times B$.

What is the order of AB?

10.3 Matrix multiplication

10.3.1 Multiplication of matrices

At the end of the first quarter of an AFL football match between Sydney and Melbourne the scores were as shown.

This information is represented in a matrix as:

$$\begin{matrix} & \text{Goals} & \text{Behinds} \\ \text{Sydney} \\ \text{Melbourne} \end{matrix} \begin{bmatrix} 12 & 15 \\ 9 & 10 \end{bmatrix}$$

One goal in AFL football is worth 6 points and one behind is worth 1 point.

This information is represented in a matrix as:

$$\begin{matrix} \text{Goals} \\ \text{Behinds} \end{matrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

To get the total points scored by both teams the matrices are multiplied.

$$\begin{bmatrix} 12 & 15 \\ 9 & 10 \end{bmatrix} \times \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \times 6 + 15 \times 1 \\ 9 \times 6 + 10 \times 1 \end{bmatrix} = \begin{bmatrix} 87 \\ 64 \end{bmatrix}$$

Multiplying matrices in general

Two matrices A and B may be multiplied together to form the product AB if the number of columns in the first matrix A, is equal to the number of rows in the second matrix B. If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then the product AB will be of order $m \times p$. The number of columns in the first matrix must be equal to the number of rows in the second matrix otherwise matrix multiplication is not defined.

Order of matrix A
m rows and n columns
$m \times n$

Order of matrix B
n rows and p columns
$n \times p$

Columns in matrix A must equal number of rows in matrix B.

Therefore, matrix multiplication is defined and the order of the product AB will be $m \times p$ (the outside numbers).

The product is obtained by multiplying each element in each row of the first matrix by the corresponding elements of each column in the second matrix.

In general, if $A = \underset{2 \times 2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$ **and** $B = \underset{2 \times 1}{\begin{bmatrix} d \\ e \end{bmatrix}}$, **then** $AB = \underset{2 \times 1}{\begin{bmatrix} ad + be \\ cd + de \end{bmatrix}}$.

For 2×2 **matrices, if** $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ **and** $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, **then:**

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

In general, $AB \neq BA$.

WORKED EXAMPLE 4

Given the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, C = \begin{bmatrix} 3 & 5 \\ -1 & 6 \end{bmatrix}, D = \begin{bmatrix} 1 & 7 & 4 \end{bmatrix}$$

i. determine if matrix multiplication is defined for each of the products given below
ii. state the order of the product if matrix multiplication is defined.

a. AB **b.** AC **c.** DB **d.** BD

THINK	WRITE
a. 1. State the order of each of the matrices.	**a.** A: 3×2 matrix B: 3×1 matrix C: 2×2 matrix D: 1×3 matrix
2. Determine if the product is defined by checking that the number of columns of the first matrix is the same as the number of rows of the second matrix. *Note:* Check that the inside numbers are the same.	**i.** AB: A B 3×2 and 3×1 No, AB is not defined. **ii.** AC: A C 3×2 and 2×2 Yes, AC is defined. **iii.** DB: D B 1×3 and 3×1 Yes, DB is defined. **iv.** BD: B D 3×1 and 1×3 Yes, BD is defined.
b. If matrix multiplication is defined, the outside numbers give the order of the product.	**b. i.** Not defined **ii.** 3×2 **iii.** 1×1 **iv.** 3×3

WORKED EXAMPLE 5

Given the matrices $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$, determine the following matrices.

a. AB **b.** BA **c.** B^2

THINK	WRITE
a. 1. Substitute for the given matrices.	**a.** $AB = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$

2. Check that matrix multiplication is defined. Since A and B are both 2×2 matrices, the product AB will also be a 2×2 matrix.

3. Recall the rules for matrix multiplication and apply.

$$= \begin{bmatrix} 3 \times -5 + 5 \times 3 & 3 \times -3 + 5 \times 4 \\ 4 \times -5 + 7 \times 3 & 4 \times -3 + 7 \times 4 \end{bmatrix}$$

4. Simplify and give the final result.

$$AB = \begin{bmatrix} 0 & 11 \\ 1 & 16 \end{bmatrix}$$

b. 1. Substitute for the given matrices.

b. $$BA = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$$

2. Check that matrix multiplication is defined. Since both A and B are 2×2 matrices, the product BA will also be a 2×2 matrix.

3. Recall the rules for matrix multiplication and apply.

$$= \begin{bmatrix} -5 \times 3 + -3 \times 4 & -5 \times 5 + -3 \times 7 \\ 3 \times 3 + 4 \times 4 & 3 \times 5 + 4 \times 7 \end{bmatrix}$$

4. Simplify and give the final result.

$$BA = \begin{bmatrix} -27 & -46 \\ 25 & 43 \end{bmatrix}$$

c. 1. $B^2 = B \times B$. Substitute for the given matrix.

c. $$B^2 = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$$

2. Since B is a 2×2 matrix, B^2 will also be a 2×2 matrix. Apply the rules for matrix multiplication.

$$= \begin{bmatrix} -5 \times -5 + -3 \times 3 & -5 \times -3 + -3 \times 4 \\ 3 \times -5 + 4 \times 3 & 3 \times -3 + 4 \times 4 \end{bmatrix}$$

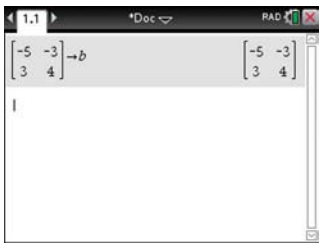
3. Simplify and give the final result.

$$B^2 = \begin{bmatrix} 16 & 3 \\ -3 & 7 \end{bmatrix}$$

TI | THINK

c.1. On a Calculator page, press the Template button and complete the entry line as:
$\begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$
Press CTRL, then press VAR, then type '*b*' and press ENTER to store matrix B.

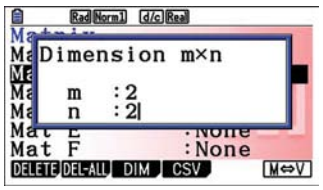
WRITE

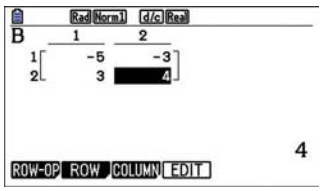


CASIO | THINK

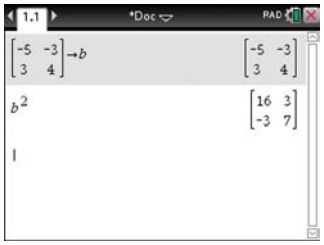
c.1. On a Run-Matrix screen, select MAT/VCT by pressing F3.
Select Matrix B by pressing
EXE and change its dimensions to 2 by 2.
Press EXE.
Enter the elements of Matrix B, then press EXIT.
Press EXIT again to return to the Run-Matrix screen.

WRITE



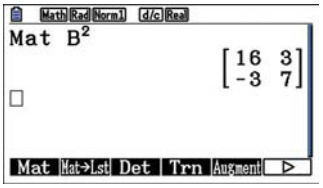


2. Complete the next entry line as: b^2 then press ENTER.



3. The answer appears on the screen. $B^2 = \begin{bmatrix} 16 & 3 \\ -3 & 7 \end{bmatrix}$

2. Complete the entry line as: Mat B^2 then press EXE. *Note:* To find 'Mat', press OPTN, select MAT/VCT by pressing F2, then select Mat by pressing F1.



3. The answer appears on the screen. $B^2 = \begin{bmatrix} 16 & 3 \\ -3 & 7 \end{bmatrix}$

Worked example 5 shows that matrix multiplication in general is not commutative: $AB \neq BA$, although there are exceptions. It is also possible that one product is defined and the other is not defined, and that the products may have different orders. Note that squaring a matrix (when defined) is not the square of each individual element.

WORKED EXAMPLE 6

Given the matrices $E = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$, determine the following matrices.

a. EF **b.** FE

THINK	WRITE
a. 1. Substitute for the given matrices.	**a.** $EF = \begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}$
2. Check that matrix multiplication is defined. Since E is a 3×2 matrix and F is a 2×3 matrix, the product EF will be a 3×3 matrix.	
3. Recall the rules for matrix multiplication and apply.	$= \begin{bmatrix} 3\times2+5\times4 & 3\times3+5\times-5 & 3\times4+5\times-2 \\ -4\times2+2\times4 & -4\times3+2\times-5 & -4\times4+2\times-2 \\ -1\times2+3\times4 & -1\times3+3\times-5 & -1\times4+3\times-2 \end{bmatrix}$
4. Simplify and give the final result.	$EF = \begin{bmatrix} 26 & -16 & 2 \\ 0 & -22 & -20 \\ 10 & -18 & -10 \end{bmatrix}$
b. 1. Substitute for the given matrices.	**b.** $FE = \begin{bmatrix} 2 & 3 & 4 \\ 4 & -5 & -2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -4 & 2 \\ -1 & 3 \end{bmatrix}$
2. Check that matrix multiplication is defined. Since F is a 2×3 matrix and E is a 3×2 matrix, the product EF will be a 2×2 matrix.	

3. Recall the rules for matrix multiplication and apply.

$$= \begin{bmatrix} 2\times3+3\times-4+4\times-1 & 2\times5+3\times2+4\times3 \\ 4\times3+-5\times-4+2\times-1 & 4\times5+-5\times2+-2\times3 \end{bmatrix}$$

4. Simplify and give the final result.

$$FE = \begin{bmatrix} -10 & 28 \\ 34 & 4 \end{bmatrix}$$

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Units 1 & 2 > Area 6 > Sequence 1 > Concept 2

Matrix multiplication Summary screen and practice questions

Exercise 10.3 Matrix multiplication

Technology free

1. State the order of each of the following matrices.

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 7 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ -1 & 6 \end{bmatrix}, D = \begin{bmatrix} 8 & -1 & 1 \end{bmatrix}$$

2. **WE 4** Given the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 7 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ -1 & 6 \end{bmatrix}, D = \begin{bmatrix} 8 & -1 & 1 \end{bmatrix}$$

   i. determine if matrix multiplication is defined for each of the products, given below
   ii. state the order of the product, if matrix multiplication is defined.

   a. AB b. BD c. DA d. CD e. DB f. AC

3. **WE 5** Given the matrices $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$, determine the following matrices.

   a. AB b. BA
   c. A^2 d. B^2

4. Given the matrices $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine the following matrices.

   a. AO b. OA c. AI d. IA

   What observations can you make from completing parts a to d?

5. **a.** Given the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, verify the following.

i. $AI = IA = A$ **ii.** $AO = OA = O$

b. Given the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, verify the following.

i. $AI = IA = A$ **ii.** $AO = OA = O$

6. **a.** Given the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, verify that $(I - A)(I + A) = I - A^2$.

b. If $A = \begin{bmatrix} 3 & 0 \\ -4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}$, verify that $AB = O$ where $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Does $BA = O$?

c. If $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}$, verify that $AB = O$ where $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Does $BA = O$?

7. Given the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$, determine the following matrices.

a. AX **b.** XA

8. WE 6 Given the matrices $D = \begin{bmatrix} 2 & -1 \\ -3 & 5 \\ -1 & -4 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 5 \end{bmatrix}$, determine the following matrices.

a. DE **b.** ED

9. Given the matrices $C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & -2 \end{bmatrix}$, determine the following matrices.

a. CD **b.** DC

10. If $A = \begin{bmatrix} x & -3 \\ 2 & x \end{bmatrix}$ and $B = \begin{bmatrix} 2 & x \\ x & -3 \end{bmatrix}$, calculate the value of x given the following.

a. $AB = \begin{bmatrix} 3 & 18 \\ 13 & 3 \end{bmatrix}$ **b.** $BA = \begin{bmatrix} -16 & 10 \\ 10 & 24 \end{bmatrix}$ **c.** $A^2 = \begin{bmatrix} -2 & 12 \\ -8 & -2 \end{bmatrix}$ **d.** $B^2 = \begin{bmatrix} 8 & -2 \\ -2 & 13 \end{bmatrix}$

Use the matrices $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$ to answer, if possible, questions **11** and **12**.

11. Calculate the following.

a. $A + B$ **b.** $A + C$ **c.** $B + C$ **d.** AB **e.** BA **f.** AC

12. Calculate the following.

a. CA **b.** BC **c.** CB **d.** ABC **e.** CBA **f.** CAB

Technology active

13. Given $D = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -4 \end{bmatrix}$ and $E = \begin{bmatrix} 6 & 2 \\ 3 & -1 \\ 3 & -3 \end{bmatrix}$, determine the following matrices.

a. DE **b.** ED **c.** $E + D$ **d.** D^2

14. Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$.

a. Verify the Distributive Law: $A(B + C) = AB + AC$.
b. Verify the Associative Law for Multiplication: $A(BC) = (AB)C$.
c. Is $(A + B)^2 = A^2 + 2AB + B^2$? Explain.
d. Show that $(A + B)^2 = A^2 + AB + BA + B^2$.

15. a. If $P = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$, determine the matrices

P^2, P^3 and P^4, and deduce the matrix P^n.

b. If $Q = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, determine the matrices

Q^2, Q^3 and Q^4, and deduce the matrix Q^n.

c. If $R = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$, determine the matrices

R^2, R^3 and R^4, and deduce the matrix R^n.

d. If $S = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$, determine the matrices S^2, S^3 and S^4, and deduce the matrices S^8 and S^9.

16. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, evaluate the matrix $A^2 - 6A + 11I$.

17. a. If $B = \begin{bmatrix} 4 & 5 \\ -2 & -3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, evaluate the matrix $B^2 - B - 22I$.

b. If $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, evaluate the matrix $C^2 - 5C + 14I$.

18. If $D = \begin{bmatrix} d & -4 \\ -2 & 8 \end{bmatrix}$, evaluate the matrix $D^2 - 9D$.

19. The trace of a matrix A, denoted by $\text{tr}(A)$, is equal to the sum of leading diagonal elements. For 2×2 matrices, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\text{tr}(A) = a_{11} + a_{22}$.

Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$.

a. Calculate the following.
i. $\text{tr}(AB)$ **ii.** $\text{tr}(BA)$ **iii.** $\text{tr}(A)\,\text{tr}(B)$
b. Does $\text{tr}(ABC)$ equal $\text{tr}(A)\,\text{tr}(B)\,\text{tr}(C)$?

10.4 Determinants and inverses

10.4.1 Determinant of a 2×2 matrix

Associated with a square matrix is a single number called the **determinant of a matrix**. For a matrix A the determinant of the matrix A is denoted by $\det(A)$ or the symbol Δ. The determinant is represented not by the square brackets that we use for matrices, but by straight lines; that is,

$$\text{if } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

To evaluate the determinant, multiply the elements in the leading diagonal and subtract the product of the elements in the other diagonal.

$$\textbf{det } (A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

WORKED EXAMPLE 7

Calculate the determinant of the matrix $F = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$.

THINK

1. Apply the definition $\det(F) = ad - bc$ by multiplying the elements in the leading diagonal. Subtract the product of the elements in the other diagonal.
2. State the value of the determinant.

WRITE

$$F = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$$

$$\det(F) = 3 \times 7 - 5 \times 4$$
$$= 21 - 20$$
$$\det(F) = 1$$

10.4.2 Determinant of a 3×3 matrix

The determinant of a 3×3 matrix involves evaluating the determinants of three 2×2 matrices.

$$\text{If } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

The 3 sub-determinants are referred to as **minors**.

The coefficients of each sub-determinant are the elements of row 1 of the 3×3 matrix, that is, elements a, b and c.

The second coefficient, b, is given a negative sign

Minors are formed by removing the row and column of each of these coefficients and using the remaining elements to give three 2×2 matrices.

For example,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ becomes } a\begin{bmatrix} e & f \\ h & i \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ becomes } -b\begin{bmatrix} d & f \\ g & i \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ becomes } c\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

$$\textbf{If } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \ \det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Note: Although you may not be required to calculate the determinant of matrices of higher order than 2×2 without technology, the formula is included here to enhance your understanding. Worked example 9 demonstrates the calculation of higher-order determinants using technology.

WORKED EXAMPLE 8

Evaluate $\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -1 & 2 & 0 \end{vmatrix}$.

THINK

1. Use elements of row 1 as the coefficients of the minors.
2. Evaluate the minors.

WRITE

$$\begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= 2(-1 \times 0 - 2 \times 2) - 1(1 \times 0 - 2 \times -1) + 3(1 \times 2 - -1 \times -1)$$

$$= 2(0 - 4) - 1(0 + 2) + 3(2 - 1)$$

$$= -8 - 2 + 3$$

$$= -7$$

The determinant of a transpose matrix

A matrix, A, can be transposed to form a new matrix, A^T, by swapping the rows with the columns, as follows:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

For a square matrix, A, its transpose, A^T, is formed by flipping the other elements over the main diagonal. The determinants of these two matrices are equal.

For square matrices:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

$$\det(A) = \det\left(A^T\right)$$

WORKED EXAMPLE 9

If $A = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & 3 \\ 0 & 2 & 5 \end{bmatrix}$, use technology to calculate the determinant of A and the determinant of the transpose of matrix A, A^T. What conclusion can you make?

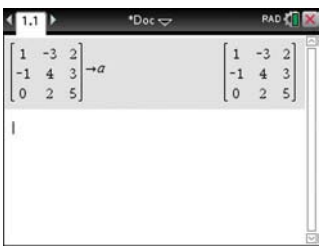
TI | THINK

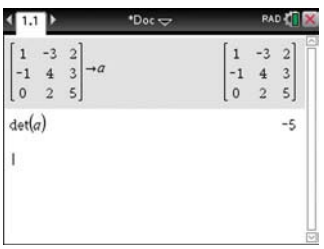
1. On a Calculator page, press the template button and complete the entry line as:
$\begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & 3 \\ 0 & 2 & 5 \end{bmatrix}$
Press CTRL, then press VAR, then type 'a' and press ENTER to store matrix A.

2. Press MENU then select
7: Matrix & Vector
3: Determinant
Complete the entry line as:
det (a)
then press ENTER.

3. The answer appears on the screen.

WRITE





$\det(A) = -5$

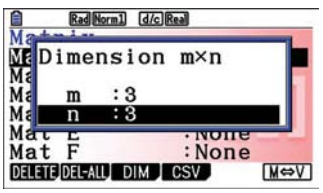
CASIO | THINK

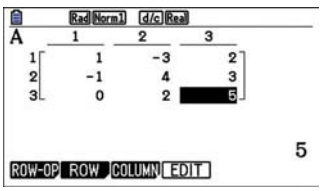
1. On a Run-Matrix screen, select MAT/VCT by pressing F3.
Select Matrix A by pressing
EXE and change its dimensions to 3 by 3.
Press EXE.
Enter the elements of Matrix A,
then press EXIT. Press EXIT again to return to the Run-Matrix screen.

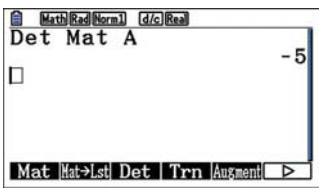
2. Complete the entry line as:
Det Mat A
then press EXE.
Note: 'Det' and 'Mat' can be found by pressing OPTN, then selecting MAT/VCT by pressing F2.

3. The answer appears on the screen.

WRITE

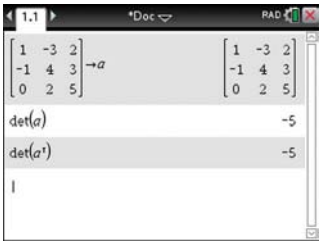






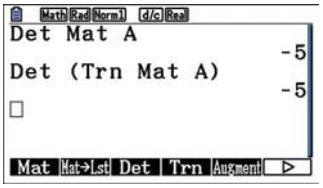
$\det(A) = -5$

4. Press MENU, then select:
   7: Matrix & Vector
   3: Determinant.
   Type 'a'.
   Press MENU, then select:
   7: Matrix & Vector
   2: Transpose
   and press ENTER.



5. The answer appears on the screen. $\det(A^{\mathrm{T}}) = -5$
6. State your conclusion. $\therefore\ \det(A) = \det(A^{\mathrm{T}})$

4. Complete the next entry line as:
   Det (Trn Mat A)
   then press EXE.
   Note: 'Det', 'Trn' and 'Mat' can be found by pressing OPTN, then selecting MAT/VCT by pressing F2.



5. The answer appears on the screen. $\det(A^{\mathrm{T}}) = -5$
6. State your conclusion. $\therefore\ \det(A) = \det(A^{\mathrm{T}})$

10.4.3 Inverse of a 2×2 matrix

The identity matrix I, defined by $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, has the property that for a 2×2 non-zero matrix A,

$AI = IA = A$.

When any square matrix is multiplied by its **multiplicative inverse**, the identity matrix I is obtained.

This is the same as multiplying 3 by its multiplicative inverse, $\frac{1}{3}$; the result is 1.

That is, $3 \times \frac{1}{3} = 1$ and $\frac{1}{3} \times 3 = 1$.

Thus, 3 is the multiplicative inverse of $\frac{1}{3}$ and $\frac{1}{3}$ is the multiplicative inverse of 3.

For a matrix, A, the multiplicative inverse is called the inverse matrix and is denoted by A^{-1}, and $AA^{-1} = A^{-1}A = I$. Note that $A^{-1} \neq \frac{1}{A}$ as division of matrices is not defined.

Consider the products of $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 \times 7 + 5 \times -4 & 3 \times -5 + 5 \times 3 \\ 4 \times 7 + 7 \times -4 & 4 \times -5 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 7 \times 3 + -5 \times 4 & 7 \times 5 + -5 \times 7 \\ -4 \times 3 + 3 \times 4 & -4 \times 5 + 3 \times 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now for the matrix $A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, the determinant $\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix} = 3 \times 7 - 5 \times 4 = 1$.

$A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$ is obtained from the matrix A by swapping the elements on the leading diagonal, and placing a negative sign on the other two elements.

Use the following rules to find the inverse matrix, A^{-1}, of a 2×2 matrix A:

1. Calculate the determinant of A.
$$\det(A) = ad - bc$$
2. Divide 1 by the determinant.
$$\frac{1}{\det(A)} = \frac{1}{ad - bc}$$

3. Swap the elements in the leading diagonal, a and d, and change the signs of the elements on the other diagonal, $-b$ and $-c$.

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4. The inverse matrix, A^{-1}, is given by:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

WORKED EXAMPLE 10

Determine the inverse of the matrix $P = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and verify that $PP^{-1} = P^{-1}P = I$.

THINK

1. Calculate the determinant.
   If $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|P| = ad - bc$.

WRITE

$$\begin{aligned} |P| &= \begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} \\ &= 2 \times 5 - 3 \times -1 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

2. To find the inverse of matrix P, recall and apply the rule $P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$P^{-1} = \frac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$$

3. Substitute and evaluate PP^{-1}.

$$PP^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} \times \frac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$$

4. Apply the rules for scalar multiplication and multiplication of matrices.

$$= \frac{1}{13}\begin{bmatrix} 2 \times 5 + 3 \times 1 & 2 \times -3 + 3 \times 2 \\ -1 \times 5 + 5 \times 1 & -1 \times -3 + 5 \times 2 \end{bmatrix}$$

5. Simplify the matrix product to show that $PP^{-1} = I$.

$$= \frac{1}{13}\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6. Substitute and evaluate $P^{-1}P$.

$$P^{-1}P = \frac{1}{13}\begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

7. Use the rules for scalar multiplication and multiplication of matrices.

$$= \frac{1}{13}\begin{bmatrix} 5 \times 2 + -3 \times -1 & 5 \times 3 + -3 \times 5 \\ 1 \times 2 + 2 \times -1 & 1 \times 3 + 2 \times 5 \end{bmatrix}$$

8. Simplify the matrix product to show that $P^{-1}P = I$.

$$= \frac{1}{13}\begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

9. Answer the question.

$$PP^{-1} = P^{-1}P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Singular matrices

If a matrix has a zero determinant then the inverse matrix does not exist, and the original matrix is termed a **singular matrix**.

$\frac{1}{\det(A)} = \frac{1}{0}$ is not defined as we cannot divide by zero.

Hence, if $\det(A) = 0$, then A^{-1} does not exist and matrix A is singular.

WORKED EXAMPLE 11

Demonstrate that the matrix $\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$ is singular.

THINK

1. Evaluate the determinant.
2. Since the determinant is zero, the matrix $\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$ is singular.

WRITE

$$\begin{vmatrix} -3 & 2 \\ 6 & -4 \end{vmatrix} = (-3 \times -4) - (2 \times 6)$$
$$= 12 - 12$$
$$= 0$$
$$\therefore \begin{vmatrix} -3 & 2 \\ 6 & -4 \end{vmatrix} = 0$$

WORKED EXAMPLE 12

If $A = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$, stating the values of p, q and r. Hence evaluate the matrix $pA^2 + qA + rI$.

THINK

1. Substitute to find the matrix $A - kI$. Recall and apply the rules for scalar multiplication and subtraction of matrices.
2. Evaluate the determinant of the matrix $A - kI$.
3. Simplify the determinant of the matrix $A - kI$.
4. Equate the matrices and state the values of p, q and r.
5. Determine the matrix A^2.

WRITE

$$A - kI = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} - k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2-k & 4 \\ 3 & 5-k \end{bmatrix}$$

$$\det(A - kI) = \begin{vmatrix} -2-k & 4 \\ 3 & 5-k \end{vmatrix}$$
$$= (-2-k)(5-k) - 4 \times 3$$
$$= -(2+k)(5-k) - 1$$
$$= -\left(10 + 3k - k^2\right) - 12$$
$$= k^2 - 3k - 22$$

$$pk^2 + qk + r = k^2 - 3k - 22$$
$$\therefore p = 1; q = 3; r = -22$$

$$A^2 = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix}$$

6. Substitute for p, q and r and evaluate the matrix $A^2 - 3A - 22I$.

$$A^2 - 3A - 22I = \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix} - 3\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix} - 22\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Simplify by applying the rules for scalar multiplication of matrices.

$$= \begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix} - \begin{bmatrix} -6 & 12 \\ 9 & 15 \end{bmatrix} - \begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix}$$

8. Simplify and apply the rules for addition and subtraction of matrices.

$$\therefore A^2 - 3A - 22I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

10.4.4 Inverse of a 3 x 3 matrix

Calculators can be used to find the inverse of a 3 × 3 matrix.

WORKED EXAMPLE 13

Using technology, determine the inverse of the following matrix.

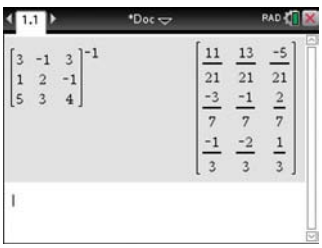
$$A = \begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{5} & \mathbf{3} & \mathbf{4} \end{bmatrix}$$

TI | THINK

1. On a Calculator page, press the template button and complete the entry line as:
$\begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & -1 \\ 5 & 3 & 4 \end{bmatrix}^{-1}$
then press ENTER.

2. The answer appears on the screen.

WRITE



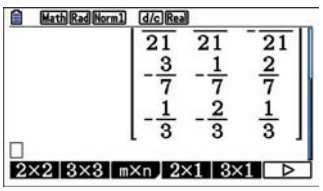
$$A^{-1} = \begin{bmatrix} \frac{11}{21} & \frac{13}{21} & \frac{-5}{21} \\ \frac{-3}{7} & \frac{-1}{7} & \frac{2}{7} \\ \frac{-1}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

CASIO | THINK

1. On a Run-Matrix screen, select MATH by pressing F4, then select MAT/VCT by pressing F1, then select 3 × 3 by pressing F2. Complete the entry line as:
$\begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & -1 \\ 5 & 3 & 4 \end{bmatrix}^{-1}$
then press EXE.

2. The answer appears on the screen.

WRITE



$$A^{-1} = \begin{bmatrix} \frac{11}{21} & \frac{13}{21} & \frac{-5}{21} \\ \frac{-3}{7} & \frac{-1}{7} & \frac{2}{7} \\ \frac{-1}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$$

studyon

Units 1 & 2 > Area 6 > Sequence 1 > Concept 3

Determinants and inverses Summary screen and practice questions.

Exercise 10.4 Determinants and inverses

Technology free

1. **WE 7** Calculate the determinant of the matrix $G = \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$.

2. The matrix $\begin{bmatrix} x & 5 \\ 3 & x+2 \end{bmatrix}$ has a determinant equal to 9. Calculate the values of x.

3. **WE 8** Evaluate the following. You may choose to use technology.

   a. $\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 5 \\ -1 & 3 & 6 \end{vmatrix}$

   b. $\begin{vmatrix} -1 & -1 & 0 \\ -3 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix}$

4. **WE 10** Determine the inverse of the matrix $A = \begin{bmatrix} 4 & -2 \\ 5 & 6 \end{bmatrix}$ and verify that $AA^{-1} = A^{-1}A = I$.

5. Determine the inverse matrix of each of the following matrices.

   a. $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 0 & -3 \\ 2 & -1 \end{bmatrix}$

6. The inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} p & 3 \\ 3 & q \end{bmatrix}$. Calculate the values of p and q.

7. **WE 11** Demonstrate that the matrix $\begin{bmatrix} 1 & -2 \\ -5 & 10 \end{bmatrix}$ is singular.

8. Calculate the value of x if the matrix $\begin{bmatrix} x & 4 \\ 3 & x+4 \end{bmatrix}$ is singular.

9. Consider the matrix $P = \begin{bmatrix} 6 & -2 \\ 4 & 2 \end{bmatrix}$.

   a. Calculate the following.

      i. $\det(P)$ ii. P^{-1}

   b. Verify that $PP^{-1} = P^{-1}P = I$.

   c. Calculate the following.

      i. $\det\left(P^{-1}\right)$ ii. $\det(P)\det\left(P^{-1}\right)$

10. Calculate the value of x for each of the following.

    a. $\begin{vmatrix} x & -3 \\ 4 & 2 \end{vmatrix} = 6$

    b. $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix} = 4$

11. Calculate the values of x if each of the following are singular matrices.

    a. $\begin{bmatrix} x & -3 \\ 4 & 2 \end{bmatrix}$

    b. $\begin{vmatrix} x & 3 \\ 4 & x \end{vmatrix}$

12. Given $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -3 & 5 \end{bmatrix}$, determine, if possible, the following matrices.

    a. $(AB)^{-1}$ b. A^{-1} c. C^{-1} d. $(ABC)^{-1}$

13. **WE 12** If $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$, stating the values of p, q and r. Hence, evaluate the matrix $pA^2 + qA + rI$.

14. If $A = \begin{bmatrix} 4 & -8 \\ -3 & 2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine the value of k for which the determinant of the matrix $A - kI$ is equal to zero.

15. If $A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, express the determinant of the matrix $A - kI$, $k \in R$ in the form $pk^2 + qk + r$, stating the values of p, q and r. Hence, evaluate the matrix $pA^2 + qA + rI$.

Technology active

Questions **16** and **17** refer to the following matrices.

$$A = \begin{bmatrix} 2 & -3 \\ -1 & -4 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

16. a. Calculate $\det(A)$, $\det(B)$ and $\det(C)$.

b. Is $\det(AB) = \det(A)\det(B)$?

c. Verify that $\det(ABC) = \det(A)\det(B)\det(C)$.

17. a. Calculate the matrices A^{-1}, B^{-1}, C^{-1}.

b. Is $(AB)^{-1} = A^{-1}B^{-1}$?

c. Is $(AB)^{-1} = B^{-1}A^{-1}$?

d. Is $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$?

18. WE 9 If $A = \begin{bmatrix} 4 & 6 & 8 \\ -2 & 3 & 6 \\ -3 & 2 & -1 \end{bmatrix}$, use technology to calculate the determinants of A and the determinant of the transpose of matrix A, A^T. What conclusion can you make?

19. WE 13 Use technology to determine the inverse of each of the following matrices.

a. $A = \begin{bmatrix} 5 & -1 & 2 \\ -2 & 3 & -1 \\ 6 & 4 & -4 \end{bmatrix}$

b. $B = \begin{bmatrix} 12 & 8 & 0 & 4 \\ 8 & 4 & 8 & 8 \\ 4 & 12 & 8 & 12 \\ 8 & 4 & 4 & 8 \end{bmatrix}$

20. Consider the matrices $B = \begin{bmatrix} -3 & 5 \\ -2 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a. Calculate the values of k for which the determinant of the matrix $B - kI = 0$.

b. Determine the matrix $Q^{-1}BQ$.

21. Let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

a. Determine the following matrices.

i. $R\left(\frac{\pi}{2}\right)$ **ii.** $R\left(\frac{\pi}{6}\right)$ **iii.** $R\left(\frac{\pi}{3}\right)$ **iv.** R^2 **v.** R^{-1}

b. Verify that $R\left(\frac{\pi}{6}\right)R\left(\frac{\pi}{3}\right) = R\left(\frac{\pi}{2}\right)$.

c. Verify that $R(\alpha)R(\beta) = R(\alpha + \beta)$.

10.5 Matrix equations and solving 2 × 2 linear equations

10.5.1 Matrix equations

Inverse matrices are used to solve matrix equations as division of matrices is not possible.

Consider the matrix equations $AX = B$ and $XA = B$, where A, B and X are matrices, and X needs to be found.

If $AX = B$, pre-multiply both sides by A^{-1}, the inverse of matrix A. Remember order of multiplication is important when multiplying matrices.

$\Rightarrow A^{-1}AX = A^{-1}B$

Since $A^{-1}A = I$,

$IX = A^{-1}B$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore X = A^{-1}B.$

If $XA = B$, multiply both sides of the equation by A^{-1}, remembering order of multiplication is important.

$\Rightarrow XAA^{-1} = BA^{-1}$

Since $AA^{-1} = I$,

$XI = BA^{-1}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore X = BA^{-1}.$

In general, if $AX = B$, then $X = A^{-1}B$, and if $XA = B$, then $X = BA^{-1}$.

WORKED EXAMPLE 14

Given the matrices $A = \begin{bmatrix} 3 & -4 \\ 5 & -6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & -2 \end{bmatrix}$, determine matrix X if:

a. $AX = C$ **b.** $XA = D$.

THINK

a. 1. If $AX = C$, multiply both sides by the inverse matrix, A^{-1} and solve for X.

2. Recall the determinant rule and apply to evaluate the determinant of the matrix A.
$\det(A) = ad - bc$

3. Recall the inverse matrix formula to determine the inverse matrix A^{-1}.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

WRITE

a. $AX = C$

$A^{-1}AX = A^{-1}C$

$IX = X = A^{-1}C$

$\therefore X = A^{-1}C$

$\det(A) = \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} = 3 \times -6 - (-4 \times 5) = 2$

$A^{-1} = \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix}$

4. Substitute for the given matrices.

$$X = \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

5. X is a 2×1 column matrix. Apply the rules to multiply the matrices.

$$= \frac{1}{2}\begin{bmatrix} -6 \times -1 + 4 \times 2 \\ -5 \times -1 + 3 \times 2 \end{bmatrix}$$

6. State the answer.

$$= \frac{1}{2}\begin{bmatrix} 14 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ \frac{11}{2} \end{bmatrix}$$

b. 1. If $XA = D$, multiply both sides by the inverse matrix A^{-1}, and solve for X

2. Substitute for the given matrices.

b. $XA = D$

$XAA^{-1} = DA^{-1}$

$XI = X = DA^{-1}$

$\therefore X = DA^{-1}$

$$X = \begin{bmatrix} 3 & -2 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -6 & 4 \\ -5 & 3 \end{bmatrix}$$

3. X is a 1×2 matrix. Apply the rules to multiply the matrices.

$$X = \frac{1}{2}\begin{bmatrix} 3 \times -6 + -2 \times -5 & 3 \times 4 + -2 \times 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -8 & 6 \end{bmatrix}$$

4. State the answer.

$$= \begin{bmatrix} -4 & 3 \end{bmatrix}$$

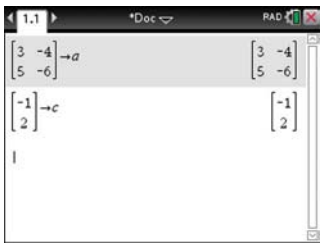
TI | THINK

WRITE

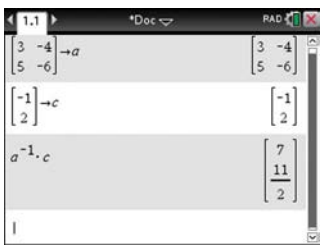
a.1. On a Calculator page, press the template button and complete the entry line as:

$\begin{bmatrix} 3 & -4 \\ 5 & -6 \end{bmatrix}$

Press CTRL, then press VAR, then type 'a'and press ENTER to store matrix A.
Repeat this step to store Matrix C.



2. Complete the next entry line as:
$a^{-1} \times c$
then press ENTER.



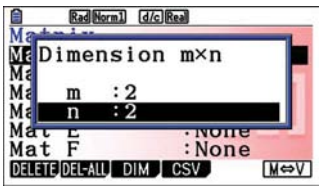
3. The answer appears on the screen.

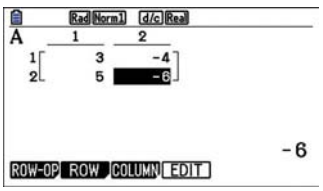
$$X = \begin{bmatrix} 7 \\ \frac{11}{2} \end{bmatrix}$$

CASIO | THINK

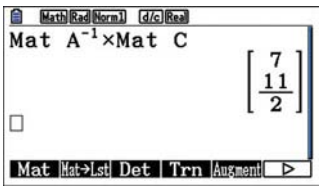
WRITE

a.1. On a Run-Matrix screen, select MAT/VCT by pressing F3.
Select Matrix A by pressing EXE and change its dimensions to 2 by 2.
Press EXE.
Enter the elements of Matrix A, then press EXIT.
Repeat this step to store Matrix C, then press EXIT to return to the Run-Matrix screen.





2. Complete the next entry line as:
Mat A^{-1} × Mat C
then press ENTER.
Note: To find 'Mat', press OPTN, select MAT/VCT by pressing F2, then select Mat by pressing F1.



3. The answer appears on the screen.

$$X = \begin{bmatrix} 7 \\ \frac{11}{2} \end{bmatrix}$$

10.5.2 Solving 2×2 linear equations

Consider the two linear equations $ax + by = e$ and $cx + dy = f$.

These equations can be written in matrix form as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$, this equation is of the form $AX = B$, which can be solved for X, as $X = A^{-1}B$.

WORKED EXAMPLE 15

Solve for x and y using inverse matrices.

$$\mathbf{4}x + \mathbf{5}y = \mathbf{6}$$
$$\mathbf{3}x + \mathbf{2}y = \mathbf{8}$$

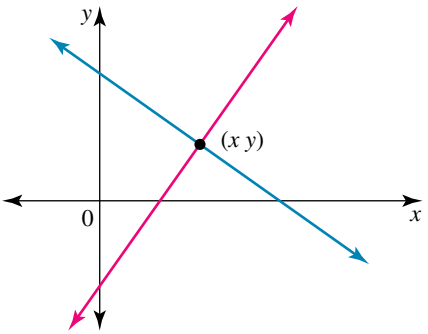
THINK	WRITE
1. First rewrite the two equations as a matrix equation.	$\begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$
2. Write down the matrices A, X and B.	$A = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$
3. Write as an equation and solve for X.	$AX = B$ $\therefore X = A^{-1}B$
4. Recall the determinant rule and calculate for matrix A.	$\Delta = \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = 4 \times 2 - 3 \times 5 = -7$
5. Recall the inverse matrix A^{-1} rule, and apply with rules for scalar multiplication to simplify this inverse.	$A^{-1} = \frac{1}{-7} \begin{bmatrix} 2 & -5 \\ -3 & 4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$
6. The unknown matrix X satisfies the equation $X = A^{-1}B$. Write the equation in matrix form.	$X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix}$
7. Apply the rules for matrix multiplication. The product is a 2×1 matrix.	$= \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -2 \times 6 + 5 \times 8 \\ 3 \times 6 + -4 \times 8 \end{bmatrix}$
8. Apply the rules for scalar multiplication, and the rules for equality of matrices.	$X = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 28 \\ -14 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$
9. State the final answer.	$x = 4$ and $y = -2$

10.5.3 Geometrical interpretation of solutions

There are 3 possible cases for the solutions of systems of linear equations as follows:

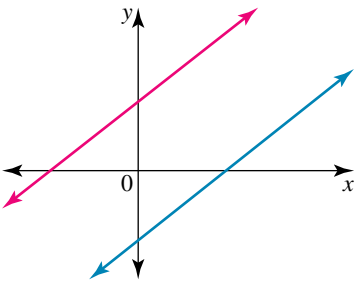
- a unique solution
- no solution
- infinitely many solutions.

If the determinant is non-zero ($\Delta \neq 0$), then these two equations are consistent. Graphically, the two lines have different gradients and therefore they intersect at a unique point resulting in a *unique solution*.



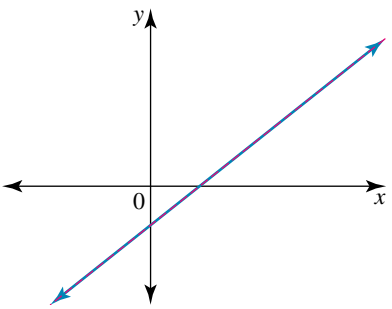
If the determinant is zero ($\Delta = 0$), then there are two possibilities.

The lines are parallel, which indicates that there is *no solution*. Graphically the two lines have the same gradient but different *y*-intercepts.



OR

The lines are multiples of one another, that is, they have the same gradient and the same *y*-intercept (they overlap). This indicates that there is an *infinite number of solutions*.



WORKED EXAMPLE 16

Solve the following simultaneous linear equations using matrices and interpret the solution geometrically.

$$3x - 2y = 6$$
$$-6x + 4y = -10$$

THINK	WRITE
1. First write the two equations as a matrix equation.	$\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$
2. Write down the matrices A, X and B.	$A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ -10 \end{bmatrix}$
3. Write as an equation and solve for X.	$AX = B$ $\therefore X = A^{-1}B$

4. Recall the determinant rule and calculate for matrix A.

$$\Delta = \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = (3 \times 4) - (-2 \times -6) = 0$$

5. The inverse matrix A^{-1} does not exist. This method cannot be used to solve the simultaneous equations.

The matrix A is singular, as the determinant equals zero, which means the lines may be parallel or the exact same line.

6. Rearrange both equations into the form $y = mx + c$.

$$3x - 2y = 6 \rightarrow y = \frac{3x - 6}{2} \rightarrow y = \frac{3}{2}x - 3$$
$$-6x + 4y = -10 \rightarrow y = \frac{6x - 10}{4} \rightarrow y = \frac{3}{2}x - \frac{5}{2}$$

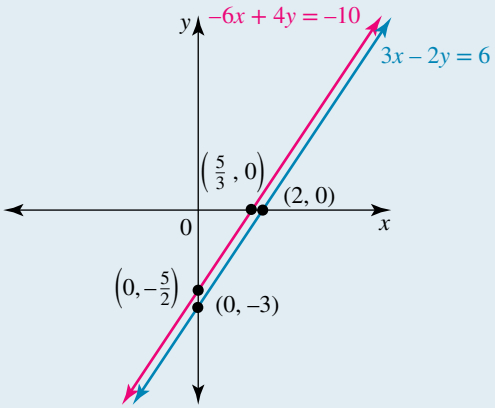
7. As the gradients are equal, the lines are parallel (y-intercepts are different).
Alternatively:

The gradients of both lines are $\frac{3}{2}$, and the y-intercepts are -3 and $-\frac{5}{2}$. The lines are parallel.

8. Apply another method to solving simultaneous equations: the graphical method. Since both equations represent straight lines, determine the x- and y-intercepts.

Line [1]: $3x - 2y = 6$ crosses the x-axis at $(2, 0)$ and the y-axis at $(0, -3)$.
Line [2]: $-6x + 4y = -10$ crosses the x-axis at $\left(\frac{5}{3}, 0\right)$ and the y-axis at $\left(0, -\frac{5}{2}\right)$.

9. Sketch the graphs. Note that the two lines are parallel and therefore have no points of intersection.



10. State the final answer.

There is no solution.

WORKED EXAMPLE 17

Solve the following linear simultaneous equations for x and y, using matrices and interpret the solution geometrically.

$$3x - 2y = 6$$
$$-6x + 4y = -12$$

THINK

1. First write the two equations as a matrix equation.

WRITE

$$\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$$

2. Write down the matrices A, X and B.

$$A = \begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$$

3. Write as an equation and solve for X.

$$AX = B$$
$$\therefore X = A^{-1} B$$

4. Recall the determinant rule and calculate for matrix A.

$$\Delta = \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = (3 \times 4) - (-2 \times -6) = 0$$

5. The inverse matrix A^{-1} does not exist. This method cannot be used to solve the simultaneous equations.

The matrix A is singular, as the determinant equals zero, which means the lines may be parallel or the exact same line.

6. Rearrange both equations into the form $y = mx + c$.

$$3x - 2y = 6 \rightarrow y = \frac{3x - 6}{2} \rightarrow y = \frac{3}{2}x - 3$$
$$-6x + 4y = -12 \rightarrow y = \frac{6x - 12}{4} \rightarrow y = \frac{3}{2}x - 3$$

7. The lines have the same gradients and y-intercept. They are the same line.

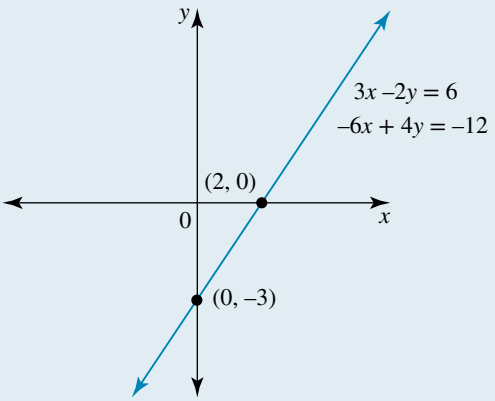
$$\text{Gradients} = \frac{3}{2}$$
y-intercept $= -3$

Alternatively:

8. Apply another method of solving simultaneous equations: the graphical method. Determine the x- and y-intercepts.

Line [1]: $3x - 2y = 6$ crosses the x-axis at $(2, 0)$ and the y-axis at $(0, -3)$.
Line [2]: $-6x + 4y = -12$ is actually the same line, since [2] $= -2 \times$ [1].

9. Sketch the graphs. Note that since the lines overlap, there is an infinite number of points of intersection.



Since $3x - 2y = 6$, $x = \frac{6 + 2y}{3}$.
If $y = 0$, $x = 2$: $(2, 0)$
If $y = 1$, $x = \frac{8}{3}$: $\left(\frac{8}{3}, 1\right)$
If $y = 2$, $x = \frac{10}{3}$: $\left(\frac{10}{3}, 2\right)$
If $y = 3$, $x = 4$: $(4, 3)$
In general, let $y = t$ so that $x = \frac{6 + 3t}{3}$.
As a coordinate: $\left(\frac{6 + 2t}{3}, t\right)$

10. State the final answer.

There is an infinite number of solutions of the form $\left(2 + \frac{2t}{3}, t\right)$ where $t \in R$.

WORKED EXAMPLE 18

Determine the values of k for which the equations $kx - 3y = k - 1$ and $10x - (k + 1)y = 8$ have:
a. a unique solution
b. no solution
c. an infinite number of solutions.
(You are not required to find the solution set.)

THINK	WRITE
1. First write the two equations as matrix equations.	a. $\begin{bmatrix} k & -3 \\ 10 & -(k+1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k-1 \\ 8 \end{bmatrix}$
2. Write out the determinant, as it is the key to answering this question.	$\Delta = \begin{vmatrix} k & -3 \\ 10 & -(k+1) \end{vmatrix}$
3. Evaluate the determinant in terms of k.	$\Delta = -k(k+1) + 30$ $= -k^2 - k + 30$ $= -\left(k^2 + k - 30\right)$ $\Delta = -(k+6)(k-5)$
4. Let the determinant equal zero and solve for k.	$k = -6, k = 5$
a. If $\Delta \neq 0$, the solution is unique; that is, there is a unique solution when $k \neq -6$ and $k \neq 5$.	There is a unique solution when $\Delta \neq 0$, that is, when $k \neq -6$ and $k \neq 5$, or $k \in R \backslash \{-6, 5\}$.
b, c. There is either no solution or an infinite number of solutions when $\Delta = 0$.	
b. 1. Substitute $k = -6$ into the two equations.	b. $-6x - 3y = -7 \Rightarrow 2x + y = \frac{7}{3}$ $\Rightarrow y = -2x + \frac{7}{3}$ $10x + 5y = 8 \Rightarrow 2x + y = \frac{8}{5}$ $\Rightarrow y = -2x + \frac{8}{5}$
2. The gradients are the same. The two equations represent parallel lines with different y-intercepts. Interpret the answer.	When $k = -6$ there is no solution, as the lines are parallel.
c. 1. Substitute $k = 5$ into the two equations.	c. $5x - 3y = 4 \Rightarrow y = \frac{5x - 4}{3} \Rightarrow y = \frac{5}{3}x - \frac{4}{3}$ $10x - 6y = 8 \Rightarrow y = \frac{10x - 8}{6} \Rightarrow y = \frac{5}{3}x - \frac{4}{3}$
2. The lines have the same gradient and y-intercept. They are the same line.	Gradient$= \frac{5}{3}$ y-intercept$= \frac{-4}{3}$
3. The two equations are multiples of one another. Interpret the answer.	When $k = 5$ there are an infinite number of solutions.

Exercise 10.5 Matrix equations and solving 2 × 2 linear equations

Technology free

1. **WE 14** If $A = \begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$, $C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & -5 \end{bmatrix}$, determine matrix X given the following.

a. $AX = C$ **b.** $XA = D$

2. If $B = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 3 \end{bmatrix}$, determine matrix X given the following:

a. $BX = C$ **b.** $XB = D$

3. **WE 15** Solve for x and y using inverse matrices.

$$3x - 4y = 23$$
$$5x + 2y = 21$$

4. Solve for x and y using inverse matrices.

$$2x + 5y = -7$$
$$3x - 2y = 18$$

5. Solve each of the following simultaneous linear equations using inverse matrices.

a. $2x + 3y = 4$, $-x + 4y = 9$ **b.** $4x + 5y = -6$, $2x - 3y = 8$ **c.** $x - 2y = 8$, $5x + 4y = -2$ **d.** $-2x + 7y + 3 = 0$, $3x + y + 7 = 0$

6. Consider the matrices $A = \begin{bmatrix} 1 & -2 \\ 5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -7 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -5 \\ -19 \end{bmatrix}$ and $D = \begin{bmatrix} 7 & 14 \end{bmatrix}$.

Calculate the matrix X in each of the following cases.

a. $AX = C$ **b.** $XA = B$ **c.** $AX = B$ **d.** $XA = D$

7. If $P = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -1 \\ -3 & 6 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, calculate the matrix X given the following.

a. $XP - Q = O$ **b.** $PX - Q = O$

8. **WE 16** Solve the following simultaneous linear equations using matrices and interpret the solution geometrically.

$$4x - 3y = 12$$
$$-8x + 6y = -18$$

9. **WE 17** Solve the following simultaneous linear equations using matrices and interpret the solution geometrically.

$$4x - 3y = 12$$
$$-8x + 6y = -24$$

10. **WE 18** Determine the values of k for which the equations $(k + 1)x - 2y = 2k$ and $-6x + 2ky = -8$ have:

a. a unique solution
b. no solution
c. an infinite number of solutions.

(You are not required to find the solution set.)

11. Determine the value of k if the following simultaneous linear equations have no solution.

$$5x - 4y = 20$$
$$kx + 2y = -8$$

12. **a.** The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points (12, 6) and (8, 3).

i. Write down two simultaneous equations that can be used to solve for a and b.

ii. Using inverse matrices, determine the values of a and b.

b. The line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the points (4, 5) and (−4, −15).

i. Write down two simultaneous equations that can be used to solve for a and b.

ii. Using inverse matrices, determine the values of a and b.

13. Calculate the value of k if the following simultaneous equations for x and y have an infinite number of solutions.

$$5x - 4y = 20$$
$$kx + 2y = -10$$

14. Determine the values of k for which the following simultaneous linear equations have:

i. no solution

ii. an infinite number of solutions.

a. $x - 3y = k$
$-2x + 6y = 6$

b. $3x - 5y = k$
$-6x + 10y = 10$

15. Demonstrate that each of the following does not have a unique solution. Describe the solution set and solve if possible.

a. $x - 2y = 3$
$-2x + 4y = -6$

b. $2x - y = 4$
$-4x + 2y = -7$

16. Determine the values of k for which the following systems of equations have:

i. a unique solution

ii. no solution

iii. an infinite number of solutions

(You are not required to find the solution set.)

a. $(k-2)x - 2y = k - 1$
$-4x + ky = -6$

b. $(k+1)x + 5y = 4$
$6x + 5ky = k + 6$

c. $(k-1)x - 3y = k + 2$
$-4x + 2ky = -10$

d. $2x - (k-2)y = 6$
$(k-5)x - 2y = k - 3$

17. Determine the values of p and q for which the following systems of equations has:

i. a unique solution

ii. no solution

iii. an infinite number of solutions.

(You are not required to find the solution set.)

a. $-2x + 3y = p$
$qx - 6y = 7$

b. $4x - 2y = q$
$3x + py = 10$

c. $3x - py = 6$
$7x - 2y = q$

d. $px - y = 3$
$-3x + 2y = q$

Technology active

18. Consider matrices $A = \begin{bmatrix} -2 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 19 \\ 12 & -7 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 3 \end{bmatrix}$. Find the matrix X in each of the following cases.

a. $AX = C$ **b.** $XA = B$ **c.** $AX = B$ **d.** $XA = D$

19. a, b, c and d are all non-zero real numbers.

a. If $P = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, determine P^{-1} and verify that $PP^{-1} = P^{-1}P = I$.

b. If $Q = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, determine Q^{-1} and verify that $QQ^{-1} = Q^{-1}Q = I$.

20. a, b, c and d are all non-zero real numbers.

a. If $R = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, determine R^{-1} and verify that $RR^{-1} = R^{-1}R = I$.

b. If $S = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$, determine S^{-1} and verify that $SS^{-1} = S^{-1}S = I$.

c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, determine A^{-1} and verify that $AA^{-1} = A^{-1}A = I$.

10.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Find the values of x if each of the following is a singular matrix.

a. $\begin{bmatrix} x & \frac{1}{x} \\ 8 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} x+1 & -3 \\ -2 & x \end{bmatrix}$

2. Find the value of x in each of the following.

a. $\begin{vmatrix} x & x \\ 8 & 2 \end{vmatrix} = 12$ **b.** $\begin{vmatrix} \frac{1}{x} & x \\ -2 & 3 \end{vmatrix} = 7$

3. Consider the following matrices.

$$A = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -4 & 1 \\ 3 & -5 & 2 \\ 7 & -4 & 8 \end{bmatrix}, C = \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ -7 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & 1 & -5 \\ 3 & -4 & 7 \end{bmatrix}, E = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

a. Can any of the matrices be added to or subtracted from one another?

b. Determine all possible products.

c. Determine the order of the products which were possible in part **b**.

4. Solve each of the following simultaneous linear equations using inverse matrices.

a. $3x + 4y = 6$, $2x + 3y = 5$ **b.** $x + 4y = 5$, $3x - y = -11$ **c.** $-4x + 3y = 13$, $2x - y = 5$ **d.** $-2x + 5y = 15$, $3x - 2y = 16$

5. If $A = \begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 3 & 5 \end{bmatrix}$, determine matrix X given the following.

a. $AX = B$ **b.** $XA = B$

6. Demonstrate that each of the following does not have a unique solution. Describe the solution set and solve if possible.

a. $2x - 3y = 5$
$-4x + 6y = -11$

b. $3x - 4y = 5$
$-6x + 8y = -10$

7. Solve each of the following using inverse matrices.

a. $x + y = 6a$
$4x - 3y = 3a + 14b$

b. $3bx - 2ay = 0$
$bx + ay = 5ab$

c. $x - y = 6b$
$3x - 4y = 17b - a$

d. $\frac{x}{a} + \frac{y}{b} = 2a + b$
$\frac{2x}{b} + \frac{3y}{a} = 2a + 6b$

8. Determine the values of k for which the following simultaneous linear equations have:

i. no solution

ii. an infinite number of solutions.

a. $2x - 5y = 5$
$-4x + 10y = k$

b. $4x - 6y = 8$
$-2x + 3y = k$

9. Determine the values of k for which the following systems of equations have:

i. a unique solution

ii. no solution

iii. an infinite number of solutions.

a. $2x + (k + 1)y = 4$
$kx + 6y = k - 4$

b. $3x - ky = 3k$
$(k + 1)x - 10y = 6k$

10. Determine the values of p and q for which the following systems of equations have:

i. a unique solution

ii. no solution

iii. an infinite number of solutions.

a. $px + 2y = 5$
$-3x - 5y = q$

b. $4x + 3y = q$
$5x + py = 7$

11. Determine the values of k for which the system of equations $2x + (k - 1)y = 4$ and $kx + 6y = k + 4$ have.

a. a unique solution

b. no solution

c. an infinite number of solutions.

12. Let $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} c & d \end{bmatrix}$.

a. Demonstrate that AB exists but $(AB)^{-1}$ does not.

b. Find BA and $(BA)^{-1}$.

Complex familiar

You may wish to use a technology of your choice to answer questions **13** to **16**.

13. Calculate the values of x if each of the following is a singular matrix.

a. $\begin{bmatrix} x - 2 & -2 \\ 12 & 6 \end{bmatrix}$ **b.** $\begin{bmatrix} x + 1 & x - 1 \\ 4 & 3 \end{bmatrix}$ **c.** $\begin{bmatrix} x + 2 & 3 \\ 5 & x \end{bmatrix}$ **d.** $\begin{bmatrix} x + 3 & 5 \\ 4 & x + 2 \end{bmatrix}$

14. Consider the following matrices.

$A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 11 \\ -5 & -8 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 25 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -4 \end{bmatrix}$

Determine the matrix X in each of the following cases.

a. $AX = C$ **b.** $XA = B$ **c.** $AX = B$ **d.** $XA = D$

15. If $B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ express the determinant of the matrix $B - kI,\ k \in R$ in the form $pk^2 + qk + r$ starting the values of p, q and r. Hence, evaluate the matrix $pB^2 + qB + rI$.

16. a. If $A = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, express the determinant of the matrix $A - kI$ in the form $pk^2 + qk + r$starting the values of p, q and r. Hence show that $pA^2 + qA + rI = O$.

b. If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, express the determinant of the matrix $B - kI$ in the form $pk^2 + qk + r$ stating the values of p, q and r. Hence, show that $pB^2 + qB + rI = O$.

Complex unfamiliar

You may wish to use a technology of your choice to answer questions **17** to **20**.

17. Consider the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a. Calculate the values of k for which the determinant of the matrix $A - kI = 0$.

b. Determine the matrix $P^{-1}AP$.

18. Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$ and $D = [1\ \ -2]$.

Find the matrix X in the each of the following cases.

a. $AX = C$ **b.** $XA = B$ **c.** $AX = B$ **d.** $XA = D$

19. a. If the product of two matrices $AB = I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine $A^{-1}$and B^{-1}.

b. If the product of two matrices $AB = 2I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine A^{-1} and B^{-1}.

c. If $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, calculate A^2 and hence show that $A = A^{-1}$.

d. If $M = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$, calculate M^3 and hence show $M^{-1} = M^2$.

20. Consider the following system of simultaneous equations.

$$\begin{aligned} kx - y + z &= 8 \\ 3x + ky + 2z &= 2 \\ x + 3y + z &= -6 \end{aligned}$$

Determine for what values of k there is:

a. a unique solution **b.** no solution **c.** an infinite number of solutions.

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Units 1 & 2 Sit chapter test

Answers

Chapter 10 Matrix arithmetic

Exercise 10.2 Addition, subtraction and scalar multiplication of matrices

1. $\begin{bmatrix} 63 & 19 & 25 \end{bmatrix}$; 63 kicks, 19 marks, 25 handballs
2. $\begin{bmatrix} 6 & 8 \\ 53 & 17 \end{bmatrix}$; 6 aces, 8 double faults, 53 forehand winners, 17 backhand winners
3. $x = 2; y = 18$
4. $x = 3; y = -16; z = 11$
5. a. $\begin{bmatrix} -10 & 4 \\ 11 & 21 \end{bmatrix}$ b. $\begin{bmatrix} 4 & -8 \\ -6 & -10 \end{bmatrix}$
6. a. $a = 2; b = 8; c = -2; d = -8$
   b. $a = \frac{11}{2}; b = 8; c = -2; d = -\frac{9}{2}$
7. a. $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ b. $\frac{1}{2}\begin{bmatrix} 15 \\ 14 \end{bmatrix}$
8. a. i. $\begin{bmatrix} 5 & 3 \\ 7 & 1 \end{bmatrix}$ ii. $\begin{bmatrix} 6 & 8 \\ 1 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 7 & 6 \\ 6 & 5 \end{bmatrix}$
   Sample response can be found in the worked solutions in the online resources.
9. a. $\begin{bmatrix} 11 & 8 \\ -3 & 16 \end{bmatrix}$ b. $\begin{bmatrix} 5 & -16 \\ 15 & 4 \end{bmatrix}$
10. a. $\begin{bmatrix} 1 & -16 \\ 15 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 14 & 10 \\ -3 & 21 \end{bmatrix}$
11. a. $x = 5; y = 7$ b. $x = 3; y = -2$
12. a. $\begin{bmatrix} 3 & 2 & 9 \\ -2 & 6 & -5 \end{bmatrix}$ b. $\frac{1}{2}\begin{bmatrix} 5 & -20 & 1 \\ 13 & 10 & -6 \end{bmatrix}$
13. a. $a_{11} = 2; a_{12} = 3; a_{21} = -1; a_{22} = 4$
    b. $\begin{bmatrix} 3 & -2 \\ -3 & 5 \end{bmatrix}$
14. a. $\begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 3 \\ 2 & 5 \end{bmatrix}$
15. a. i. 6 ii. 9 iii. 5
    b. Yes
    c. Yes
16. 3×1 matrix $\begin{bmatrix} 216 \\ 164 \\ 274 \end{bmatrix}$

Exercise 10.3 Matrix multiplication

1. $A = 3 \times 2$
   $B = 3 \times 1$
   $C = 2 \times 2$
   $D = 1 \times 3$
2. a. i. No ii. –
   b. i. Yes ii. 3×3
   c. i. Yes ii. 1×2
   d. i. No ii. –
   e. i. Yes ii. 1×1
   f. i. Yes ii. 3×2
3. a. $\begin{bmatrix} -8 & -20 \\ 1 & -3 \end{bmatrix}$ b. $\begin{bmatrix} 8 & 28 \\ -7 & -19 \end{bmatrix}$
   c. $\begin{bmatrix} 16 & 12 \\ 9 & 37 \end{bmatrix}$ d. $\begin{bmatrix} 0 & -4 \\ 1 & 5 \end{bmatrix}$
4. a. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
   c. $\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$ d. $\begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$
5. a. i. $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ ii. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
   b. i. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ii. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
6. a. $\begin{bmatrix} 0 & -18 \\ 6 & -12 \end{bmatrix}$ b. No; $\begin{bmatrix} 0 & 0 \\ 10 & 0 \end{bmatrix}$
   c. No; $\begin{bmatrix} 0 & 0 \\ ax + by & 0 \end{bmatrix}$
7. a. $\begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ b. Does not exist
8. a. $\begin{bmatrix} 0 & 8 & -11 \\ 7 & -26 & 34 \\ -9 & 14 & -17 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 21 \\ 11 & -42 \end{bmatrix}$
9. a. $\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$ b. [7]
10. a. $x = -3$ b. $x = -4$
    c. $x = -2$ d. $x = 2$
11. a. Does not exist b. Does not exist
    c. Does not exist d. $\begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}$
    e. $[-13]$ f. Does not exist
12. a. $\begin{bmatrix} 6 \\ 13 \end{bmatrix}$ b. $\begin{bmatrix} 21 & -13 \end{bmatrix}$
    c. Does not exist d. $\begin{bmatrix} -21 & 13 \\ 42 & -26 \end{bmatrix}$
    e. Does not exist f. $\begin{bmatrix} 18 & -30 \\ 39 & -65 \end{bmatrix}$
13. a. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 2 & -8 & 4 \\ 5 & -20 & 10 \\ 9 & -36 & 18 \end{bmatrix}$
    c. Does not exist d. Does not exist
14. a. $\begin{bmatrix} 31 & 9 \\ 23 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 19 & -24 \\ -81 & -76 \end{bmatrix}$
    c. LHS $= \begin{bmatrix} 44 & 56 \\ 7 & 9 \end{bmatrix}$
    RHS $= \begin{bmatrix} 55 & 25 \\ 4 & -2 \end{bmatrix}$
    d. $\begin{bmatrix} 44 & 56 \\ 7 & 9 \end{bmatrix}$
    No, since $BA \neq AB$.

15. a. $P^2 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}$; $P^3 = \begin{bmatrix} -1 & 0 \\ 0 & 64 \end{bmatrix}$;
$P^4 = \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}$; $P^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 4^n \end{bmatrix}$

b. $Q^2 = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$; $Q^3 = \begin{bmatrix} 8 & 0 \\ 0 & -27 \end{bmatrix}$; $Q^4 = \begin{bmatrix} 16 & 0 \\ 0 & 81 \end{bmatrix}$;
$Q^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-3)^n \end{bmatrix}$

c. $R^2 = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$; $R^3 = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}$; $R^4 = \begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$;
$R^n = \begin{bmatrix} 1 & 0 \\ 3n & 1 \end{bmatrix}$

d. $S^2 = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$; $S^3 = \begin{bmatrix} 0 & 18 \\ 12 & 0 \end{bmatrix}$; $S^4 = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$;
$S^8 = \begin{bmatrix} 1296 & 0 \\ 0 & 1296 \end{bmatrix}$; $S^9 = \begin{bmatrix} 0 & 3888 \\ 2592 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

17. a. $\begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} d^2 - 9d + 8 & 4 - 4d \\ 2 - 2d & 0 \end{bmatrix}$

19. a. **i.** 39 **ii.** 39 **iii.** 54
b. No

Exercise 10.4 Determinants and inverses

1. -22

2. $x = 4, -6$

3. a. -53 **b.** -33

4. $\frac{1}{34}\begin{bmatrix} 6 & 2 \\ -5 & 4 \end{bmatrix}$

5. a. $\frac{1}{4}\begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$ **b.** $\frac{1}{6}\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}$
c. $\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ **d.** $\frac{1}{6}\begin{bmatrix} -1 & 3 \\ -2 & 0 \end{bmatrix}$

6. $p = -4$; $q = -2$

7. $\Delta = 0$

8. $x = -6, 2$

9. a. **i.** 20 **ii.** $\frac{1}{10}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$
b. Sample responses can be found in the worked solutions in the online resources.
c. **i.** $\frac{1}{20}$ **ii.** 1

10. a. $x = -3$ **b.** $x = \pm 4$

11. a. $x = -6$ **b.** $x = \pm 2\sqrt{3}$

12. a. Does not exist **b.** Does not exist
c. $\frac{1}{22}\begin{bmatrix} 5 & -4 \\ 3 & 2 \end{bmatrix}$ **d.** Does not exist

13. $p = 1$, $q = -7$, $r = 13$; $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

14. $k = -2, 8$

15. $p = 1$, $q = 2$, $r = -11$; $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

16. a. -11; 2; 10
b. Yes
c. Sample responses can be found in the worked solutions in the online resources.

17. a. $A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & 3 \\ 1 & 2 \end{bmatrix}$
$B^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$
$C^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$
b. No
c. Yes
d. Yes

18. $\det(A) = -140$
$\det\left(A^{\mathrm{T}}\right) = -140$
$\det(A) = \det\left(A^{\mathrm{T}}\right)$

19. a. $\begin{bmatrix} \frac{4}{39} & -\frac{2}{39} & \frac{5}{78} \\ \frac{7}{39} & \frac{16}{39} & -\frac{1}{78} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$ **b.** $\begin{bmatrix} \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 \\ \frac{1}{12} & 0 & \frac{1}{12} & -\frac{1}{6} \\ 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{5}{48} & -\frac{3}{16} & \frac{1}{48} & \frac{1}{3} \end{bmatrix}$

20. a. $k = -1, 2$ **b.** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

21. a. **i.** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **ii.** $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$
iii. $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ **iv.** $\begin{bmatrix} \cos 2\theta & -\sin(2\theta) \\ \sin 2\theta & \cos(2\theta) \end{bmatrix}$
v. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
b, c Sample responses can be found in the worked solutions in the online resources.

Exercise 10.5 Matrix equations and solving 2×2 linear equations

1. a. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **b.** $\frac{1}{2}\begin{bmatrix} -5 & -2 \end{bmatrix}$

2. a. $\frac{1}{11}\begin{bmatrix} -2 \\ 7 \end{bmatrix}$ **b.** $\frac{1}{11}\begin{bmatrix} -7 & 3 \end{bmatrix}$

3. $x = 5$; $y = -2$

4. $x = 4$; $y = -3$

5. a. $x = -1$, $y = 2$ **b.** $x = 1$, $y = -2$
c. $x = 2$, $y = -3$ **d.** $x = -2$, $y = -1$

6. a. $\frac{1}{7}\begin{bmatrix} -29 \\ 3 \end{bmatrix}$ **b.** $\frac{1}{14}\begin{bmatrix} 7 & 7 \\ -38 & -12 \end{bmatrix}$

c. $\frac{1}{14}\begin{bmatrix} -2 & 8 \\ -22 & -3 \end{bmatrix}$ **d.** $\begin{bmatrix} -3 & 2 \end{bmatrix}$

7. a. $\frac{1}{10}\begin{bmatrix} 11 & 3 \\ -30 & 0 \end{bmatrix}$ **b.** $\frac{1}{10}\begin{bmatrix} 2 & 8 \\ -9 & 9 \end{bmatrix}$

8. No solution

9. $\left(3 + \frac{3t}{4}, t\right)\ t \in R$

10. a. $k \in R \setminus \{-3, 2\}$ **b.** $k = -3$

c. $k = 2$

11. $k = -\frac{5}{2}$

12. a. i. $\frac{12}{a} + \frac{6}{b} = 1, \quad \frac{8}{a} + \frac{3}{b} = 1$

ii. $a = 4;\ b = -3$

b. i. $\frac{4}{a} + \frac{5}{b} = 1, \quad -\frac{4}{a} - \frac{15}{b} = 1$

ii. $a = 2;\ b = -5$

13. $k = -\frac{5}{2}$

14. a. i. $k \neq -3$ **ii.** $k = -3$

b. i. $k \neq -5$ **ii.** $k = -5$

15. a. $(2t + 3, t),\ t \in R$ **b.** No solution

16. a. i. $k \in R \setminus \{-2, 4\}$

ii. $k = -2$

iii. $k = 4$

b. i. $k \in R \setminus \{-3, 2\}$

ii. $k = -3$

iii. $k = 2$

c. i. $k \in R \setminus \{-2, 3\}$

ii. $k = -2$

iii. $k = 3$

d. i. $k \in R \setminus \{1, 6\}$ **ii.** $k = 1$

iii. $k = 6$

17. a. i. $q \neq 4,\ p \in R$ **ii.** $q = 4,\ p \neq -\frac{7}{2}$

iii. $q = 4,\ p = -\frac{7}{2}$

b. i. $p \neq -\frac{3}{2},\ q \in R$ **ii.** $p = -\frac{3}{2},\ q \neq \frac{40}{3}$

iii. $p = -\frac{3}{2},\ q = \frac{40}{3}$

c. i. $p \neq \frac{6}{7},\ q \in R$ **ii.** $p = \frac{6}{7},\ q \neq 14$

iii. $p = \frac{6}{7},\ q = 14$

d. i. $p \neq \frac{3}{2},\ q \in R$ **ii.** $p = \frac{3}{2},\ q \neq -6$

iii. $p = \frac{3}{2},\ q = -6$

18. a. $\frac{1}{11}\begin{bmatrix} -6 \\ 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 2 \\ -4 & 1 \end{bmatrix}$

c. $\frac{1}{11}\begin{bmatrix} 13 & -58 \\ 16 & 31 \end{bmatrix}$ **d.** $\frac{1}{22}\begin{bmatrix} 17 & 3 \end{bmatrix}$

19. a. $P^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$ **b.** $Q^{-1} = \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$

20. a. $R^{-1} = \begin{bmatrix} 0 & \frac{1}{c} \\ \frac{1}{b} & -\frac{a}{bc} \end{bmatrix}$

b. $S^{-1} = \begin{bmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{bmatrix}$

c. $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

10.6 Review: exam practice

1. a. $x = \pm 2$ **b.** $x = -3, 2$

2. a. $x = -2$ **b.** $x = \frac{1}{2}, 3$

3. a. No, they all have different orders.

b. AB does not exist.

AC does not exist.

$AD = \begin{bmatrix} 4 & 13 & -39 \\ 18 & -13 & 13 \end{bmatrix}$

$AE = \begin{bmatrix} 29 \\ 7 \end{bmatrix}$

BA does not exist.

$BC = \begin{bmatrix} -5 & -26 \\ -10 & -33 \\ -39 & -32 \end{bmatrix}$

BD does not exist.

BE does not exist.

$CA = \begin{bmatrix} 3 & -22 \\ 20 & 18 \\ -29 & 22 \end{bmatrix}$

CB does not exist.

$CD = \begin{bmatrix} -6 & 19 & -43 \\ 17 & -19 & 30 \\ -8 & -15 & 49 \end{bmatrix} CE = \begin{bmatrix} 23 \\ -5 \\ -39 \end{bmatrix}$

DA does not exist.

$DB = \begin{bmatrix} -28 & 7 & -36 \\ 43 & -20 & 51 \end{bmatrix} DC = \begin{bmatrix} 42 & -13 \\ -44 & -18 \end{bmatrix}$

DE does not exist.

EA does not exist.

EB does not exist.

EC does not exist.

ED does not exist.

c. D: 2×3, AE: 2×3, BC: 3×2, CA: 3×2, CD: 3×3, CE: 3×1, DB: 2×3, DC: 2×2

4. a. $x = -2, y = 3$ **b.** $x = -3, y = 2$

c. $x = 14, y = 23$ **d.** $x = 10, y = 7$

5. a. $\frac{1}{5}\begin{bmatrix} -13 & 1 \\ -7 & -6 \end{bmatrix}$ **b.** $\frac{1}{5}\begin{bmatrix} 0 & -5 \\ 17 & -19 \end{bmatrix}$

6. a. No solution. Lines are parallel.

b. Infinite solutions, lines are identical. $\left(\frac{4t + 5}{3}, t\right), t \in R$

7. a. $x = 3a + 2b,\ y = 3a - 2b$

b. $x = 2a,\ y = 3b$

c. $x = a + 7b,\ y = a + b$

d. $x = ab,\ y = 2ab$

8. a. i. No solution when $k \neq -8$

ii. Infinite number of solutions when $k = -8$

b. i. No solution when $k \neq -4$

ii. Infinite number of solutions when $k = -4$

9. a. i. $k \in R\backslash\{-4, 3\}$ **ii.** $k = 3$

iii. $k = -4$

b. i. $k \in R\backslash\{-6, 5\}$ **ii.** $k = -6$

iii. $k = 5$

10. a. i. $p \neq \frac{6}{5},\ q \in R$ **ii.** $p = \frac{6}{5},\ q \neq -\frac{25}{2}$

iii. $p = \frac{6}{5},\ q = -\frac{25}{2}$

b. i. $p \neq \frac{15}{4},\ q \in R$ **ii.** $p = \frac{15}{4},\ q \neq \frac{28}{5}$

iii. $p = \frac{15}{4},\ q = \frac{28}{5}$

11. a. $k \in R\backslash\{-3, 4\}$

b. No solution when $k = -3$

c. An infinite number of solutions when $k = 4$

12. a. $AB = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$

b. $BA = [ac + bd]$

$(BA)^{-1} = \left[\frac{1}{ac + bd}\right]$

13. a. $x = -2$ **b.** $x = 7$

c. $x = -5,\ 3$ **d.** $x = -7,\ 2$

14. a. $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$

c. $\frac{1}{11}\begin{bmatrix} -5 & 36 \\ 10 & 49 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 1 \end{bmatrix}$

15. $k^2 - 7k + 2;\ p = 1,\ q = -7,\ r = 2;\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

16. a. $p = 1, q = 2, r = -11,\ A^2 + 2A - 11$

Sample responses can be found in the worked solutions in the online resources.

b. $p = 1, q = -(a + d), r = ad - bc; k^2 - (a + d)k + ad - bc$

Sample responses can be found in the worked solutions in the online resources.

17. a. $k = -1,\ 4$ **b.** $\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$

18. a. $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **b.** $\frac{1}{11}\begin{bmatrix} 21 & -2 \\ 5 & -12 \end{bmatrix}$

c. $\frac{1}{11}\begin{bmatrix} 10 & 29 \\ 8 & -1 \end{bmatrix}$ **d.** $\frac{1}{11}\begin{bmatrix} 2 & -7 \end{bmatrix}$

19. a. $A^{-1} = B, B^{-1} = A$

b. $A^{-1} = \frac{1}{2}B, B^{-1} = \frac{1}{2}A$

c. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Sample responses can be found in the worked solutions in the online resources.

d. $M^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Sample responses can be found in the worked solutions in the online resources.

20. a. There is a unique solution when $k \neq 5$ or $k \neq 2$.

b. There is no solution when $k = 5$.

c. There is an infinite number of solutions when $k = 2$.

CHAPTER 11
Matrix transformations

11.1 Overview

11.1.1 Introduction

Following on from the introduction to matrices in Chapter 10, we will now investigate their transformations. A transformation is a movement or change in a geometric shape through translation, reflection, rotation and dilation. A translation can be thought of as a 'slide', a reflection as a 'mirror image', a rotation as a 'spin' about a point and a dilation as an enlargement or reduction of a shape or figure.

One of the most visually arresting examples of the manipulation of images through transformations is in the fascinating art of Maurits Cornelis Escher. Although this Dutch artist was not formally mathematically trained, he drew great inspiration from the mathematical ideas he read about.

Today the transformation of points, and hence images, through matrix transformations is the basis of computer-based animation. Early animations, such as cartoons, were originally drawn by hand — a very time-consuming process. In much the same way, you have previously manipulated single points and simple images individually through algebra and trigonometry. In this chapter you will see how matrix transformations of reflection, dilation and rotation can be combined into a single new matrix. This combination of the movements of several individual matrices into one operation allows movement of every vertex in three-dimensional space. When matrix transformations are combined with the imagination of animators, designers and storytellers, images are created that have the power to shock, awe and inspire.

LEARNING SEQUENCE

11.1 Overview
11.2 Translations
11.3 Reflections and rotations
11.4 Dilations
11.5 Combinations of transformations
11.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Translations

11.2.1 Matrix transformations

A **transformation** is a function which maps the points of a set X, called the **pre-image** or original point onto a set of points Y, called the **image**.

A transformation is a change of position of points, lines, curves or shapes in a plane, or a change in shape due to an enlargement or reduction by a scale factor.

Each point of the plane is transformed or mapped onto another point.

The transformation, T, is written as:

$$T: \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix}$$

The matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is the column matrix representing the coordinates of the point $\mathrm{P}(x, y)$, the pre-image or original point. The matrix $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represents the coordinates of the point $\mathrm{P}'(x', y')$, the image of $\mathrm{P}(x, y)$ after a transformation.

Any transformation that can be represented by a 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is called a **linear transformation**. The origin never moves under a linear transformation.

An **invariant point** or **fixed point** is a point of the domain of the function which is mapped onto itself after a transformation, that is, the pre-image point is the same as the image point and is unchanged by the transformation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \Rightarrow x' = x \text{ and } y' = y$$

For example, a reflection in the line $y = x$ leaves every point on the line $y = x$ unchanged.

The transformations which will be studied in this topic are:

- translations
- reflections
- rotations
- dilations.

11.2.2 Translations

A **translation** is a transformation where each point in the plane is moved a given distance in a horizontal and/or vertical direction.

Consider a marching band marching in perfect formation. As the leader of the marching band moves from a position $\mathrm{P}(x, y)$ a steps across and b steps up to a new position $\mathrm{P}'(x', y')$, all members of the band will also move to a new position $\mathrm{P}'(x + a, y + b)$. Their new position could be defined as:

$\mathrm{P}'(x', y') = \mathrm{P}'(x + a, y + b)$ where a represents the horizontal translation and b represents the vertical translation and P' is the image of P.

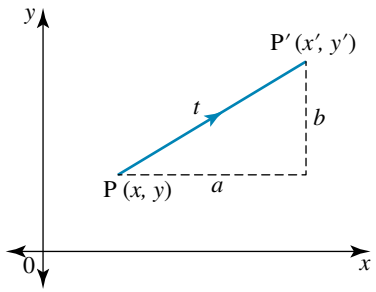
The matrix equation for a translation can be given as:

$$\begin{matrix}\mathbf{P'} & & \mathbf{P} & & \mathbf{T} \\ \begin{bmatrix} x' \\ y' \end{bmatrix} & = & \begin{bmatrix} x \\ y \end{bmatrix} & + & \begin{bmatrix} a \\ b \end{bmatrix}\end{matrix}$$

where a represents the horizontal translation and b the vertical translation.

The matrix $\begin{bmatrix} a \\ b \end{bmatrix}$ is called the **translation matrix** or column vector and is denoted by T. It represents the horizontal and vertical displacement, as shown in the diagram:

$$x' = x + a$$
$$y' = y + b$$



Each x-coordinate is moved a units parallel to the x-axis and each y-coordinate in moved b units parallel to the y-axis

Note t (lower case) denotes the translation itself and T (upper case) denotes the matrix of the translation.

WORKED EXAMPLE 1

A cyclist in a race needs to move from the front position at (0, 0) across 2 positions, to the left, so that the other cyclists can pass. Write the translation matrix and determine the cyclist's new position.

THINK

1. Write down the translation matrix, T, using the information given.

WRITE

The cyclist moves across to the left by 2 units. Translating 2 units to the left means each x -coordinate decreases by 2.

$$T = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

2. Recall and apply the matrix transformation for a translation equation.

$$\begin{matrix}\mathbf{P'} & & \mathbf{P} & & \mathbf{T} & & \\ \begin{bmatrix} x' \\ y' \end{bmatrix} & = & \begin{bmatrix} x \\ y \end{bmatrix} & + & \begin{bmatrix} -2 \\ 0 \end{bmatrix} & = & \begin{bmatrix} x-2 \\ y+0 \end{bmatrix}\end{matrix}$$

3. Substitute the pre-image point into the matrix equation.

The pre-image point is (0, 0).

$$\begin{matrix}\mathbf{P'} & & \\ \begin{bmatrix} x' \\ y' \end{bmatrix} & = & \begin{bmatrix} 0-2 \\ 0+0 \end{bmatrix}\end{matrix}$$

4. State the cyclist's new position by calculating the coordinates of the image point from the matrix equation.
Note: The *y*-coordinate has not changed; that is, the cyclist is still in front, but to the left.

$$\begin{matrix} P' & & P & & T \\ \begin{bmatrix} x' \\ y' \end{bmatrix} & = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & + & \begin{bmatrix} -2 \\ 0 \end{bmatrix} \end{matrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

The new position is $(-2, 0)$.

11.2.3 Translations of an object

Matrix addition can be used to find the Cartesian coordinates of a translated object when an object is moved or translated from one location to another on the coordinate plane without changing its size or orientation.

Consider the triangle ABC with coordinates A $(-1, 3)$, B $(0, 2)$ and C $(-2, 1)$. It is to be moved 3 units to the right and 1 unit down. To find the coordinates of the vertices of the translated $\Delta A'B'C'$, we can use matrix addition.

First, the coordinates of the triangle ΔABC can be written as a **coordinate matrix**. The coordinates of the vertices of a figure are arranged as columns in the matrix.

$$\Delta ABC = \begin{matrix} A & B & C \end{matrix} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Secondly, translating the triangle 3 units to the right means each *x*-coordinate increases by 3.

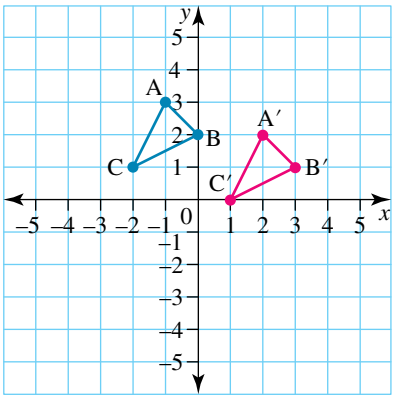
Translating the triangle 1 unit down means that each *y*-coordinate decreases by 1.

The translation matrix that will do this is $\begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix}$.

Finally, to find the coordinates of the vertices of the translated triangle $\Delta A'B'C'$ add the translation matrix to the coordinate matrix.

$$\begin{matrix} A & B & C \end{matrix} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ -1 & -1 & -1 \end{bmatrix} = \begin{matrix} A' & B' & C' \end{matrix} \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

The coordinates of the vertices of the translated triangle $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ are A′ $(2, 2)$, B′ $(3, 1)$ and C′ $(1, 0)$.



WORKED EXAMPLE 2

Determine the translation matrix if ΔABC with coordinates A $(-1, 3)$, B $(0, 2)$ and C $(-2, 1)$ is translated to $\Delta A'B'C'$ with coordinates A′ $(2, 4)$, B′ $(3, 3)$ and C′ $(1, 2)$.

THINK

1. Write the coordinates of ΔABC as a coordinate matrix.

WRITE

The coordinates of the vertices of a figure are arranged as columns in the matrix.

2. Write the coordinates of the vertices of the translated triangle $\Delta A'B'C'$ as a coordinate matrix.

$$\Delta ABC = \begin{matrix} A & B & C \end{matrix} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\Delta A'B'C' = \begin{matrix} A' & B' & C' \end{matrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix}$$

3. Calculate the translation matrix by recalling the matrix equation: $P' = P + T$. Rearrange the equation to make T the subject and apply the rules of matrix subtraction.

$$\begin{matrix} P' \end{matrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix} = \begin{matrix} P \end{matrix} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix} + T$$

$$T = \begin{matrix} P' \end{matrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 2 \end{bmatrix} - \begin{matrix} P \end{matrix} \begin{bmatrix} -1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

4. Translating the triangle 3 units to the right means that each x -coordinate increases by 3. Translating the triangle 1 unit up means that each y-coordinate increases by 1.

The translation matrix is:

$$T = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

on Resources

Interactivity: Translations matrix (int-6294)

11.2.4 Translations of a curve

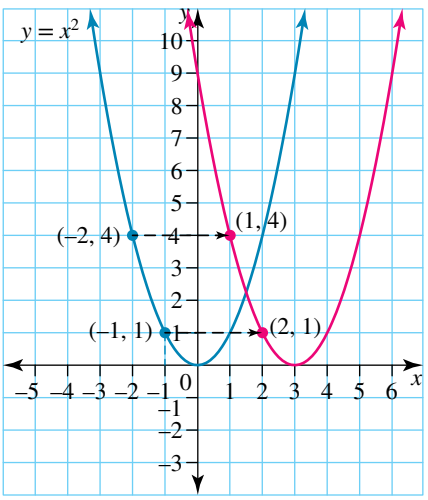
A translation of a curve maps every original point (x, y) of the curve onto a new unique and distinct image point (x', y').

Consider the parabola with the equation $y = x^2$.

If the parabola is translated 3 units in the positive direction of the x-axis, what is the image equation and what happens to the coordinates?

As seen from the table of values, each coordinate (x, y) has a new coordinate pair or image point $(x + 3, y)$.

x	y	(x, y)	$x' = x + 3$	$y' = y$	(x', y')
−3	9	(−3, 9)	−3 + 3	9	(0, 9)
−2	4	(−2, 4)	−2 + 3	4	(1, 4)
−1	1	(−1, 1)	−1 + 3	1	(2, 1)
0	0	(0, 0)	0 + 3	0	(3, 0)
1	1	(1, 1)	1 + 3	1	(4, 1)
2	4	(2, 4)	2 + 3	4	(5, 4)
3	9	(3, 9)	3 + 3	9	(6, 9)



The matrix equation for the translation of any point on the curve $y = x^2$ can be written as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The image equations for the two coordinates are $x' = x + 3$ and $y' = y$.
Rearranging the image equations to make the pre-image coordinates the subject, we get

$$x' = x + 3 \leftrightarrow x = x' - 3 \text{ and } y = y'.$$

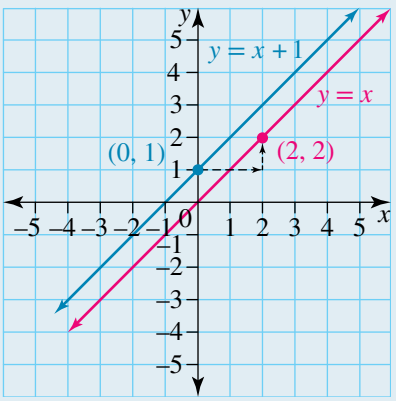
To find the image equation, substitute the image expressions into the pre-image equation.

$$y = x^2$$
$$y = y' \qquad x = x' - 3$$
$$\therefore y' = (x' - 3)^2$$

The equation of the parabola, $y = x^2$, after a translation of 3 units in the positive direction of the x-axis is $y = (x - 3)^2$.

WORKED EXAMPLE 3

Determine the equation of the image of the line with equation $y = x + 1$ after it is transformed by the translation matrix $T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

THINK	WRITE
1. Recall the matrix equation for the transformation and apply it to the given transformation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
2. State the image equations for the two coordinates.	$x' = x + 2$ and $y' = y + 1$
3. Rearrange the equations to make the pre-image coordinates x and y the subjects.	$x = x' - 2$ and $y = y' - 1$
4. Substitute the image equations into the pre-image equation to find the image equation.	$y = x + 1$ $y' - 1 = (x' - 2) + 1$ $y' = x'$ The image equation is $y = x$.
5. Graph the image and pre-image equation to verify the translation.	

WORKED EXAMPLE 4

Determine the equation of the image of the parabola with equation $y = x^2$ after it is transformed by the translation matrix $T = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

THINK

1. Recall the matrix equation for the transformation and apply it to the given translation.
2. State the image equations for the two coordinates.
3. Rearrange the equations to make the pre-image coordinates x and y the subjects.
4. Substitute the image expressions into the pre-image equation to find the image equation.
5. Graph the image and pre-image equation to verify the translation.

WRITE

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$x' = x - 3$ and $y' = y + 1$

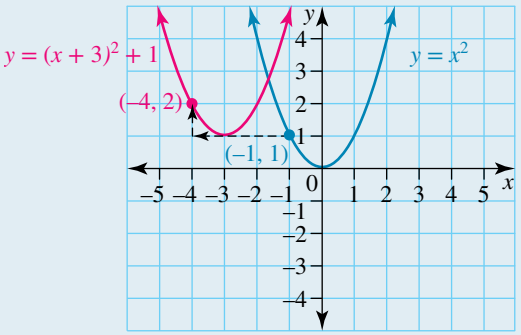
$x = x' + 3$ and $y = y' - 1$

$y = x^2$

$y' - 1 = (x' + 3)^2$

$y' = (x' + 3)^2 + 1$

The image equation is $y = (x+3)^2 + 1$.



studyon

Units 1 & 2 > Topic 6 > Chapter 2 > Concept 1 **Translations** Summary screen and practice questions

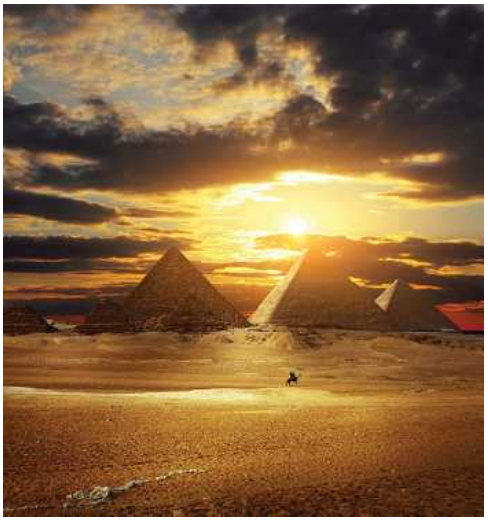
Exercise 11.2 Translations

Technology free

1. **WE 1** A chess player moves his knight 1 square to the right and 2 squares up from position (2, 5). Write the translation matrix and determine the new position of the knight.
2. Find the image of the point $(-1, 0)$ using the matrix equation for the translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

3. Find the image of the point (1, 2) using the matrix equation for translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

4. Find the image of the point (3, −4) using the matrix equation for translation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



5. **WE 2** Determine the translation matrix if ΔABC with coordinates A (0, 0), B (2, 3) and C(−3, 4) is translated to ΔA′B′C′ with coordinates A′(1, −2), B′ (3, 1) and C′(−2, 2).
6. Determine the translation matrix if ΔABC with coordinates A (3, 0), B (2, 4) and C(−2, −5) is translated to ΔA′B′C′ with coordinates A′ (4, 2), B′ (3, 6) and C′(−1, −3).
7. The image equations are given by $x' = x + 2$ and $y' = y + 1$. Express the translation in matrix equation form.
8. **a.** On a Cartesian plane, draw $\Delta ABC = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ and $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}$.

   b. Calculate the translation matrix if $\Delta ABC = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 3 & 1 \end{bmatrix}$ is translated to $\Delta A'B'C' = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 0 \end{bmatrix}$.
9. **WE 3** Determine the equation of the image of the the line with equation $y = x - 3$ after it is transformed by the translation matrix $T = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.
10. Determine the equation of the image of the line with equation $y = x - 1$ after it is transformed by the translation matrix $T = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
11. Determine the equation of the image of the line with equation $y = x + 3$ after it is transformed by the translation matrix $T = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
12. **WE 4** Determine the equation of the image of the parabola with equation $y = x^2$ after it is transformed by the translation matrix $T = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
13. Determine the equation of the image of the parabola with equation $y = x^2 + 1$ after it is transformed by the translation matrix $T = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$.
14. Determine the equation of the image of the parabola with equation $y = x^2 - 2$ after it is transformed by the translation matrix $T = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$.
15. Determine a translation matrix that maps the line with equation $y = x$ onto the line with equation $y = x + 2$.
16. Determine a translation matrix that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = (x - 7)^2 + 3$.

17. Determine the translation equation that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = x^2 - 4x + 10$.
18. Determine the translation equation that maps the circle with equation $x^2 + y^2 = 9$ onto the circle with equation $(x - 1)^2 + y^2 = 9$.
19. Determine the translation equation that maps the parabola with equation $y = x^2$ onto the parabola with equation $y = (x - a)^2 + b$.
20. Determine the translation equation that maps the circle with equation $x^2 + y^2 = r^2$ onto the circle with equation $(x - a)^2 + y^2 = r^2$.

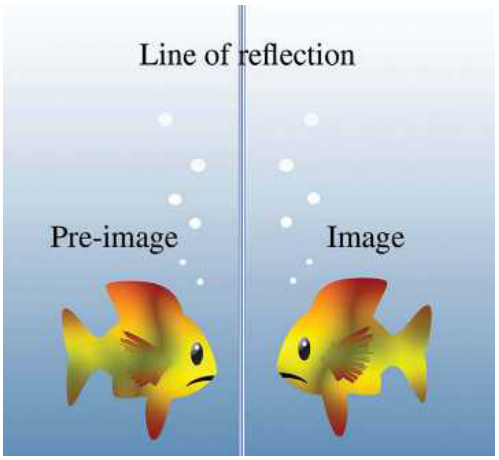
11.3 Reflections and rotations

11.3.1 Reflections

A **reflection** is a transformation in a line. This line is called a line of reflection, and the image point is a mirror image of the pre-image point. The line of reflection is called the **mediator**.

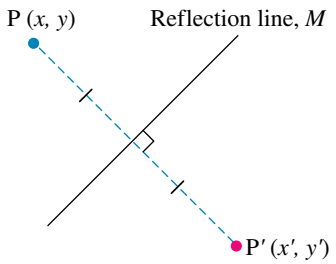
In a reflection:

- the image, P′, and the pre-image, P, are equidistant from the mediator
- the line of reflection is perpendicular to the line joining the image, P′, and the pre-image, P.



The following reflections will be considered:

- reflection in the x-axis ($y = 0$)
- reflection in the y-axis ($x = 0$)
- reflections in lines that pass through the origin.
- reflection in the line $y = x \tan\theta$.



11.3.2 Reflection in the x-axis ($y = 0$)

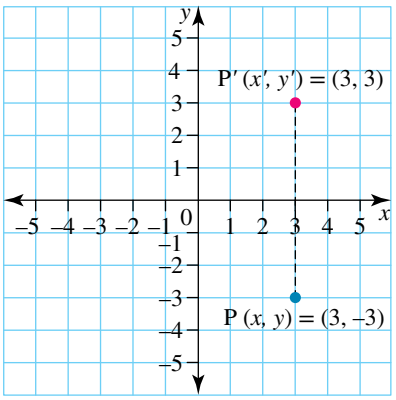
The reflection in the x-axis maps the point P (x, y) onto the point P′ (x', y'), giving the image point $(x', y') = (x, -y)$.

The matrix for a reflection in the x-axis is:

$$M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

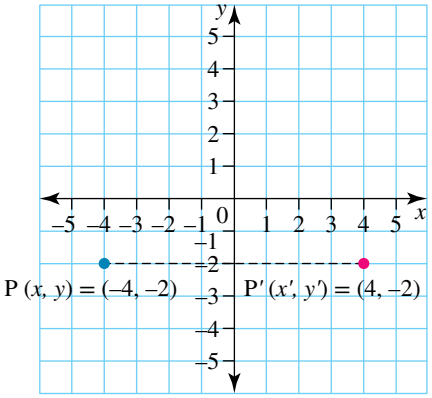
In matrix form, the reflection for any point in the x-axis is:

$$\begin{matrix} \text{P}' \\ \begin{bmatrix} x' \\ y' \end{bmatrix} \end{matrix} = \begin{matrix} \text{T} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{matrix} \begin{matrix} \text{P} \\ \begin{bmatrix} x \\ y \end{bmatrix} \end{matrix}$$



11.3.3 Reflection in the y-axis ($x = 0$)

The reflection in the y-axis maps the point P(x, y) onto the point P$'(x', y')$, giving the image point $(x', y') = (-x, y)$.



The matrix for a reflection in the y-axis is:

$$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

In matrix form, the reflection for any point in the y-axis is:

$$\begin{matrix} \text{P}' \\ \begin{bmatrix} x' \\ y' \end{bmatrix} \end{matrix} = \begin{matrix} \text{T} \\ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix} \begin{matrix} \text{P} \\ \begin{bmatrix} x \\ y \end{bmatrix} \end{matrix}$$

on Resources

Interactivity: Reflection matrix (int-6295)

WORKED EXAMPLE 5

Determine the image of the point $(-2, 3)$:
a. after a reflection in the x-axis
b. after it is transformed by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Comment on your answer.

THINK	WRITE
a. 1. Recall the reflection matrix for reflection in the x-axis.	a. $M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2. Apply the matrix equation for reflection in the x-axis matrix, P$'$ = TP.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Substitute the pre-image point into the matrix equation.	The pre-image point is $(-2, 3)$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
4. Calculate the coordinates of the image point.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$ The image point is $(-2, -3)$.
b. 1. Recall the matrix equation, P$'$ = TP.	b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
2. Substitute the pre-image point into the matrix equation.	The pre-image point is $(-2, 3)$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

3. Calculate the coordinates of the image point.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The image point is (2, 3). It is a reflection in the y-axis from the pre-image point.

TI | THINK

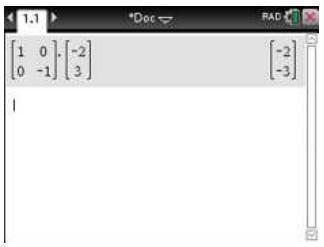
a.1. On a Calculator page, complete the entry line as:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

then press ENTER.
Note: Matrix templates can be found by pressing the templates button.

2. The answer appears on the screen.

WRITE



The image point is (−2, −3).

CASIO | THINK

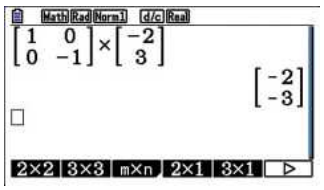
a.1. On a Run-Matrix screen, complete the entry line as:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

then press EXE.
Note: Matrix templates can be found by selecting MATH by pressing F4, then selecting MAT/VCT by pressing F1.

2. The answer appears on the screen.

WRITE



The image point is (−2, −3) .

WORKED EXAMPLE 6

Determine the equation of the image of the graph of $y = (x + 1)^2$ after it is reflected in the y-axis.

THINK

1. Recall the reflection matrix for reflection in the y-axis and apply it to the matrix equation.

2. Determine the image coordinates.

3. Rearrange the equations to make the pre-image coordinates x and y the subjects.

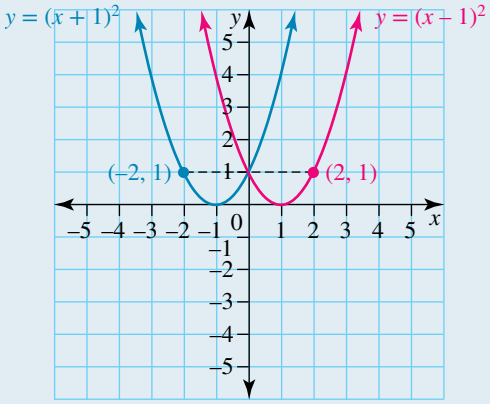
4. Substitute the image expressions into the pre-image equation $y = (x + 1)^2$ to find the equation of the image.

WRITE

$$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = -x$
$y' = y$
$x = -x'$
$y = y'$
$y = (x + 1)^2$
$y' = (-x' + 1)^2$
$y' = (x' - 1)^2$
The equation of the image is $y = (x - 1)^2$.

5. Graph the image and the pre-image equation to verify the reflection.



Alternatively:

6. Recall the reflection matrix for reflection in the y-axis and apply it to the matrix equation, $P' = TP$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

7. Determine the pre-image coordinates by using the inverse of the transformational matrix, $P = T^{-1}P'$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

8. Multiply and simplify the matrix equation.

$$x = -x'$$
$$y = y'$$

9. Substitute the image expressions into the pre-image equation $y = (x + 1)^2$ to find the image equation.

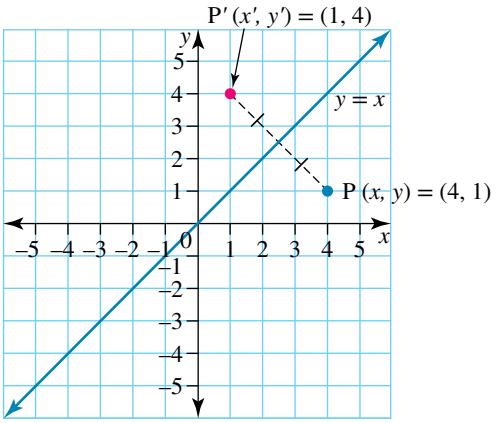
$$y = (x + 1)^2$$
$$y' = (-x' + 1)^2$$
$$y' = (x' - 1)^2$$

The equation of the image is $y = (x - 1)^2$.

11.3.4 Reflection in a line that passes through the origin (0,0)

Reflection in the line with equation $y = x$

A reflection in the line $y = x$ maps the point $P(x, y)$ onto the point $P'(x', y')$, giving the image point $(x', y') = (y, x)$.



The matrix for a reflection in the line $y = x$ is

$$M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In matrix form, a reflection in the line $y = x$ is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Similarly, in matrix form, a reflection in the line $y = -x$ is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

WORKED EXAMPLE 7

Determine the coordinates of the image of the point (3, 1) after a reflection in the line with equation $y = x$.

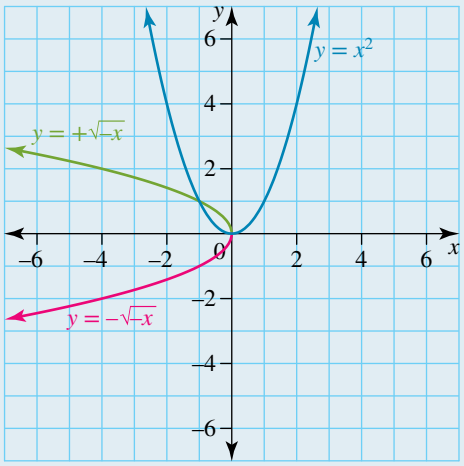
THINK	WRITE
1. Recall the reflection matrix for reflection in the line $y = x$.	$M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2. Use the matrix equation for a reflection about the line with equation $y = x$: $\text{P}' = \text{TP}$.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Substitute the pre-image point into the matrix equation.	The pre-image point is (3, 1). $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
4. Calculate the coordinates of the image point.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ The image point is (1, 3).

WORKED EXAMPLE 8

Determine the equation of the image of $y = x^2$ after a reflection in the line $y = -x$.

THINK	WRITE
1. Recall the matrix for a reflection in the line $y = -x$.	$M_{y=-x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
2. Write the matrix equation for a reflection in the line $y = -x$; $\text{P}' = \text{TP}$.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Multiply the matrices.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$
4. Find the image equations.	$x' = -y$ and $y' = -x$
5. Rearrange the equations to make x and y the subjects.	$y = -x'$ and $x = -y'$
6. Substitute the image expressions into the pre-image equation $y = x^2$ and make y the subject.	$y = x^2$ $-x' = (-y')^2$ $-x' = y'^2$ $\therefore y' = \pm\sqrt{-x'}$
7. Answer the question.	The equation of the image is $y = \pm\sqrt{-x}$.

8. Graph the image and the pre-image to verify the reflection.



11.3.5 Reflection in the line $y = x \tan\theta$

The line $y = x \tan\theta$ might be more easily recognised as $y = mx$, where m is the gradient of the line which passes through the origin.

Remember that the gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and tangent ratio $= \dfrac{\text{rise}}{\text{run}}$.

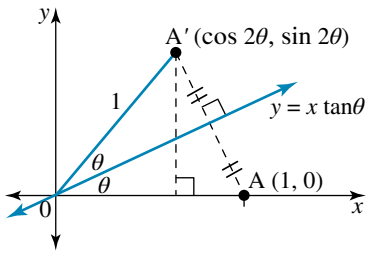
Therefore the tangent and gradient ratios provide the same information: $\dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Carefully examine these diagrams that illustrate reflection of the points (1, 0) and (0, 1) in the line $y = x \tan\theta$.

Note the following from these diagrams.

For the point A (1, 0):

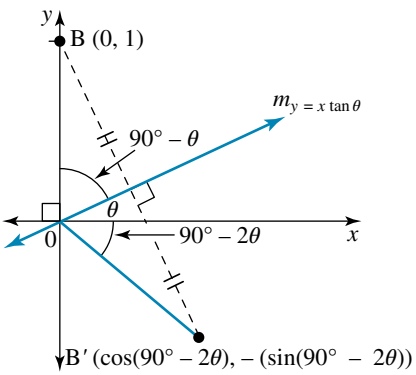
1. Point A is reflected to a point equidistant from and perpendicular to, the line $y = x \tan\theta$.
2. The angle from the x-axis to A′ is 2θ.
3. The x-coordinate of the right-angled triangle is $\cos 2\theta$.
4. The y-coordinate of this triangle is $\sin 2\theta$.
5. Hence, point (1, 0) → (cos 2θ, sin 2θ).



For the point B,(0 :1)

1. Point B is reflected to a point equidistant from, and perpendicular to, the line $y = x \tan\theta$.
2. $\angle MOB = 90° - \theta$; therefore, $\angle MOB' = 90° - \theta$.
3. Therefore, $\angle XOB' = (90° - \theta) - \theta = 90° - 2\theta$.
4. The x-coordinate = cos (90° − 2θ).
5. The y-coordinate = − sin (90° − 2θ) because the angle is in the fourth quadrant.
6. Hence, point (0, 1) → [cos (90° − 2θ), − sin (90° − 2θ)].
7. Using trigonometric ratios, this simplifies to yield (sin 2θ, − cos 2θ). (Remember that sin 30° = cos 60°, etc.)

Hence, the general reflection matrix in the line $y = x \tan\theta$ is:



$$M_{y = x \tan\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

WORKED EXAMPLE 9

Determine the equation of the image of the line $y = -x - 1$ after a reflection in the line $y = \sqrt{3}x$.

THINK	WRITE
1. Recall the general reflection matrix for a reflection in the line $y = x\tan\theta$.	$M_{y=x\tan\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
2. Determine θ by equating the general form of the equation with the given equation and recall the exact value.	$y = \sqrt{3}x$ $y = x\tan\theta$ $\therefore \tan\theta = \sqrt{3}$ $\theta = \tan^{-1}(\sqrt{3})$ $\therefore \theta = \frac{\pi}{3}$
3. Substitute $\frac{\pi}{3}$ for θ into the general reflection matrix for a reflection in the line $y = x\tan\theta$.	$M_{y=x\tan\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ $M_{y=\sqrt{3}x} = \begin{bmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{bmatrix}$
4. Simplify the reflection matrix.	$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$
5. State the matrix equation for a reflection in the line $y = \sqrt{3}x$.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = M_{y=\sqrt{3}x} \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
6. Rearrange the matrix equation to make the pre-image point (x, y) the subject	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$

7. Find the inverse of $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ and simplify the matrix equation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-\frac{1}{4}-\frac{3}{4}} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

8. Multiply the matrices to find the pre-image expressions.

$$x = -\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$$

$$y = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$$

9. Substitute the image expressions into the pre-image equation $y = -x - 1$ to find the equation of the image

$y = -x - 1$ becomes

$$\frac{\sqrt{3}}{2}x' + \frac{1}{2}y' = \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' - 1$$

$$\frac{1}{2}y' + \frac{\sqrt{3}}{2}y' = \frac{1}{2}x' - \frac{\sqrt{3}}{2}x' - 1$$

10. Make y' the subject and simplify the equations by rationalising the denominator.

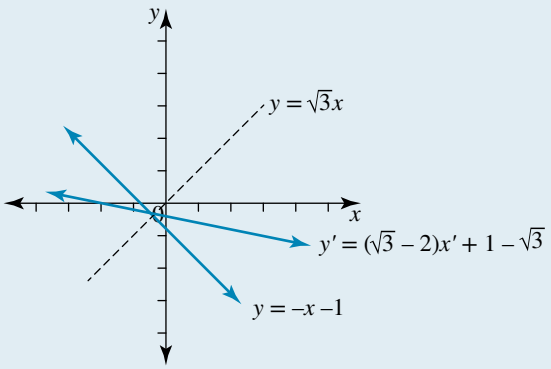
$$\frac{1+\sqrt{3}}{2}y' = \frac{1-\sqrt{3}}{2}x' - 1$$

$$y' = \left(\sqrt{3} - 2\right)x' + 1 - \sqrt{3}$$

11. Answer the question.

The equation of the image of the line $y = -x - 1$ is $y = (\sqrt{3} - 2)x + 1 - \sqrt{3}$.

12. Sketch the image and the pre-image graphs to verify the reflection.

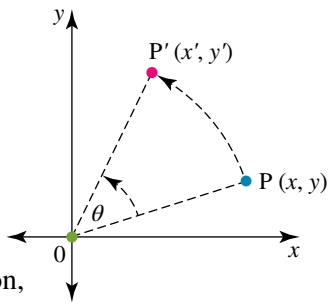


A summary of the matrices for reflections are shown in the following table.

Reflection in	Matrix	Matrix equation
x-axis	$M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
y-axis	$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
line $y = x$	$M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
line $y = -x$	$M_{y=-x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
line $y = x \tan\theta$	$M_{y=x\tan\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

11.3.6 Rotations

A **rotation** is a linear transformation in which the point P (x, y) is rotated about a fixed point called the centre or **point of rotation**. This point is usually taken as the origin. The pre-image point, P (x, y), can be rotated through an angle, θ, in an anti-clockwise or clockwise direction to get the image point, P′ (x', y').



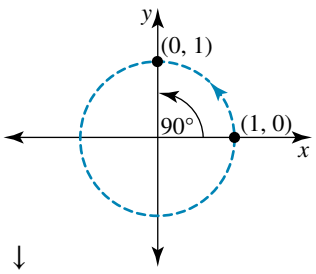
- In a rotation, each point rotates through the same angle of rotation, θ.
- The pre-image, P, and the image, P′, are equidistant from the origin. That is, OP = OP′.
- Rotation in an anti-clockwise direction is considered to be a positive rotation, and rotation in a clockwise direction is considered to be a negative rotation
- R_θ is the symbol used for the matrix rotation in an anti-clockwise direction; $R_{-\theta}$ is used for clockwise direction.

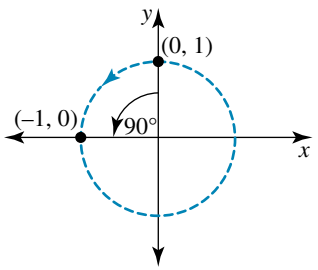
11.3.7 Special rotations in an anticlockwise direction

In this section we will discuss transformations involving rotations of 90°, 180°, 270° and 360°, as well as general rotations.

Rotation of 90°

Consider the figure shown.





As the plane rotates through $\theta = 90°$ about the origin, point (1, 0) will map to point (0, 1) and point (0, 1) will map to point (−1, 0).

Hence, the identity matrix, I, is altered to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (↓ ↓) to achieve a rotation of 90° about the origin.

It is most important that you recognise the pattern that is displayed by the columns in the matrix and the coordinates of the image points. This concept forms the basis of the next section of work and totally eliminates 'remembering' formulas so that you will be able to understand what is happening to the points.

Hence, $R_{90^\circ} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and is the matrix of rotation.

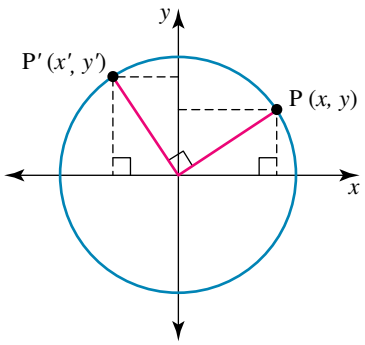
In general terms

$$(x, y) \rightarrow (-y, x)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = -y$$

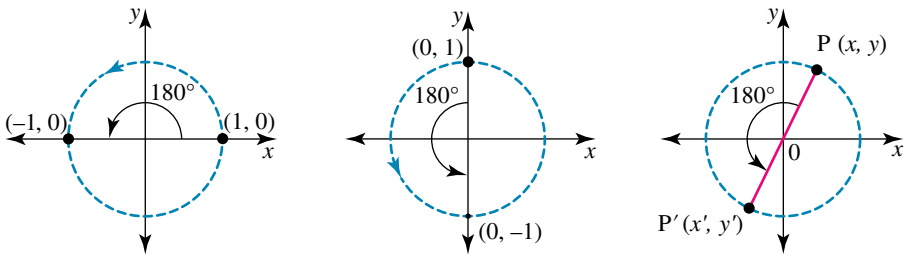
$$y' = x$$



As mentioned earlier, these rotation matrices should not be learned. They are quite similar and can be too readily confused. Sketch the original (1, 0) and (0, 1) points and then use their images to build the rotation matrices.

Rotation of 180°

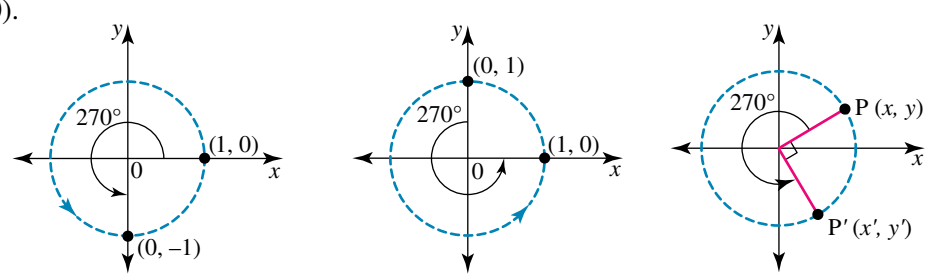
In the diagrams below, notice that point (1, 0) is mapped onto point (−1, 0) and point (0, 1) is mapped onto (0, −1).



Therefore, $R_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ where $(x, y) \rightarrow (-x, -y)$.

Rotation of 270°

In the diagrams below, notice that point (1, 0) is mapped onto point (0, −1) and point (0, 1) is mapped onto point (1, 0).



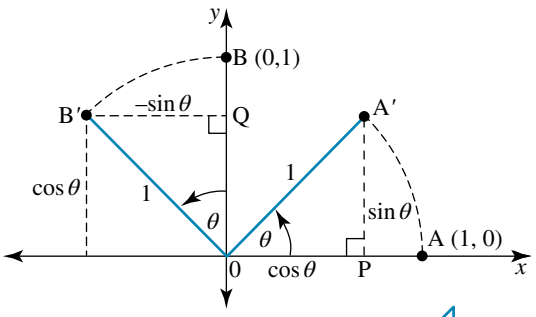
Therefore, $R_{270^\circ} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ where $(x, y) \rightarrow (y, -x)$.

Rotation of 360°

$R_{360^\circ} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ because R_{360° essentially leaves the original unchanged (or mapped onto itself).

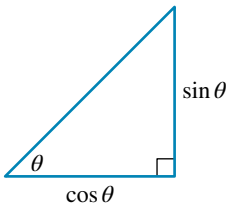
General rotation of θ

Consider the points A (1, 0) and B (0, 1) that are rotated through angle θ about the origin in an anti-clockwise direction.



Careful examination of the diagram shows that point A (1, 0) is mapped onto point A′ $(\cos\theta, \sin\theta)$ and point B (0, 1) is mapped onto point B′ $(-\sin\theta, \cos\theta)$, where

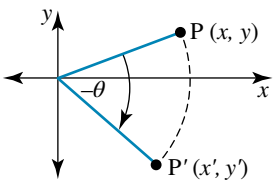
$$\cos\theta = x \quad \text{(horizontal)}$$
$$\text{and } \sin\theta = y \quad \text{(vertical)}$$



R_θ is the matrix rotation for an anticlockwise rotation through θ about the origin.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$R_{-\theta}$ is the matrix rotation when θ is taken in a clockwise, negative rotation about the origin as shown.



$$R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Both R_θ and $R_{-\theta}$ can be used to confirm the specific cases of R_{90}, R_{180}, and R_{270}.

In summary:

- **The matrix rotation for an anti-clockwise rotation through θ about the origin is:**

$$\boldsymbol{R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

- **The matrix rotation for a clockwise rotation through θ about the origin is:**

$$\boldsymbol{R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}$$

- **The matrix equation for an anti-clockwise rotation of any point is:**

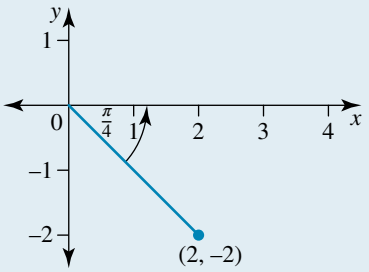
$$\boldsymbol{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

and the matrix equation for a clockwise rotation is:

$$\boldsymbol{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

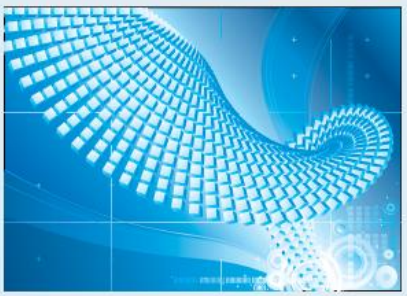
WORKED EXAMPLE 10

The point $(2, -2)$ is rotated $\frac{\pi}{4}$ about the origin in an anti-clockwise direction. Find the coordinates of the image of the point after this transformation.

THINK	WRITE
1. Sketch the point $(2, -2)$ and the angle $\frac{\pi}{4}$.	
2. Recall the rotation matrix equation for an anti-clockwise rotation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Substitute the pre-image point, $(2, -2)$, and $\frac{\pi}{4}$ for θ and evaluate.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $= \begin{bmatrix} \frac{2}{\sqrt{2}} & +\frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & -\frac{2}{\sqrt{2}} \end{bmatrix}$ $= \begin{bmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{bmatrix}$
4. Rationalise the denominator and simplify.	$= \begin{bmatrix} \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ 0 \end{bmatrix}$ $= \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}$
5. State the coordinates of the image point.	The image point is $(2\sqrt{2}, 0)$.

WORKED EXAMPLE 11

Determine the equation of the image of the line $y = -x + 4$ under a rotation of 30° about the origin in an anti-clockwise direction.



THINK	WRITE
1. Recall the rotation matrix equation for an anti-clockwise rotation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
2. Substitute 30° for θ and evaluate.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Rearrange the matrix equation to make the pre-image point (x, y) the subject.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$
4. Find the inverse of $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and simplify the matrix equation.	$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\left(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} \times \frac{1}{2}\right)} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\frac{3}{4} + \frac{1}{4}} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$
5. Multiply the matrices to find the pre-image expressions.	$x = \frac{\sqrt{3}}{2}x' + \frac{1}{2}y'$ $y = -\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$

6. Substitute the image expressions into the pre-image equation $y = -x + 4$ to find the equation of the image.

$$y = -x + 4$$
$$-\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' = -\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right) + 4$$
$$-\frac{1}{2}x' + \frac{\sqrt{3}}{2}y' = -\frac{\sqrt{3}}{2}x' - \frac{1}{2}y' + 4$$

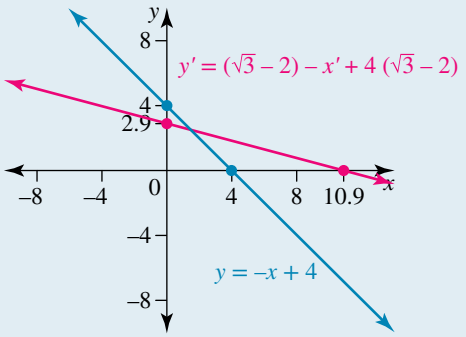
7. Make y' the subject and simplify the equations by rationalising the denominator.

$$y' = \frac{\left(1 - \sqrt{3}\right)x'}{\sqrt{3} + 1} + \frac{8}{\sqrt{3} + 1}$$
$$= \left(\sqrt{3} - 2\right)x' + 4\left(\sqrt{3} - 1\right)$$

8. Answer the question.

The equation of the image of the line $y = -x + 4$ is $y' = \left(\sqrt{3} - 2\right)x' + 4\left(\sqrt{3} - 1\right)$.

9. Use technology to sketch the image and the pre-image graphs to verify the rotation.



studyon

Units 1 & 2 > Topic 6 > Chapter 2 > Concepts 2 & 3

Reflections Summary screen and practice questions
Rotations Summary screen and practice questions

Exercise 11.3 Reflections and rotations

Technology free

1. **WE 5a** Determine the image of the point $(-3, -1)$ after a reflection in the x-axis.
2. **WE 5b** Determine the image of the point $(5, -2)$ after it is transformed by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
3. **WE 6** Determine the equation of the image of the graph of $y = (x - 2)^2$ after it is reflected in the y-axis.
4. Determine the equation of the image of the graph of $y = x^2 + 1$ after it is reflected in the x -axis.
5. **WE 7** Determine the image of the point $(-2, 5)$ after a reflection in the line with equation $y = x$.

6. Determine the image of the point $(9, -6)$ after a reflection in the line with equation $y = -x$.
7. **WE 8** Determine the equation of the image of $y = x^2$ after a reflection in the line $y = x$.
8. Determine the equation of the image of $y = 2x^2 + 1$ after a reflection in the line $y = -x$.
9. The line with equation $y = -x + 3$ is transformed according to the matrix equations given. Determine the equation of the image of the line after this transformation.
   a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
10. The parabola with equation $y = x^2 + 2x + 1$ is transformed according to the matrix equations given. Determine the equation of the image of the parabola after this transformation.
    a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Technology active

11. **WE 9** Determine the equation of the image of the line $y = x + 1$ after a reflection in the line $y = \frac{\sqrt{3}}{3}x$.
12. **WE 10** The point $(5, 4)$ is rotated $\frac{\pi}{3}$ about the origin in a clockwise direction. Determine the coordinates of the image of the point after this transformation.
13. Determine the matrices for the following rotations about the origin.
    a. 90° clockwise b. 180° clockwise c. 45° anti-clockwise d. $\frac{\pi}{6}$ anti-clockwise
14. a. Determine the coordinates of the image of the point A $(7, -6)$ rotated 270° about the origin in a clockwise direction.
    b. Determine the coordinates of the image of the point A $(7, -6)$ rotated 90° about the origin in an anti-clockwise direction.
    c. Show that a clockwise rotation of 270° is the same as an anti-clockwise rotation of 90° about the origin.
15. **WE 11** Determine the equation of the image of the line $y = -3x + 1$ under a rotation of 45° about the origin in an anti-clockwise direction.
16. Determine the equation of the image of the line $y = 2x + 1$ under a rotation of 90° about the origin in a clockwise direction.

11.4 Dilations

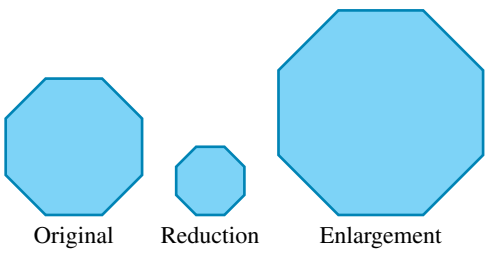
11.4.1 Dilations from the *x*- and *y*-axes

A **dilation** is a linear transformation that changes the size of a figure. The figure is enlarged or reduced parallel to either axis or both axes. A dilation requires a centre point and a scale factor.

A dilation is defined by a scale factor denoted by λ.

If $\lambda > 1$, the figure is enlarged.

If $0 < \lambda < 1$, the figure is reduced.

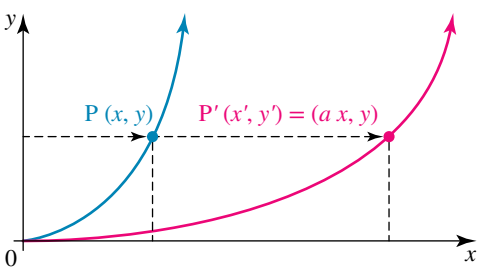


Dilation from the *y*-axis or parallel to the *x*-axis

A dilation from the *y*-axis or parallel to the *x*-axis is represented by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a\,x \\ y \end{bmatrix}$$

where a is the dilation factor.



The points (x, y) are transformed onto points with the same y-coordinate.

The point moves away from the y-axis in the direction of the x-axis by a factor of a. This determines the horizontal enlargement of the figure if $a > 1$ or the horizontal compression if $0 < a < 1$.

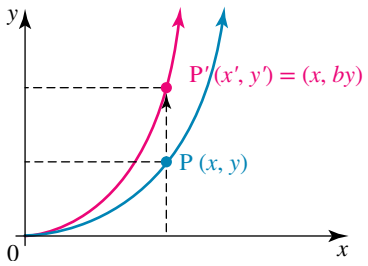
Dilations from the x-axis or parallel to the y-axis

A dilation from the x-axis or parallel to the y-axis is represented by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ by \end{bmatrix}$$

where b is the dilation factor.

The point moves away from the x-axis in the direction of the y-axis by a factor of b. This determines the vertical enlargement of the figure if $b > 1$ or if $0 < b < 1$, the vertical compression.



Resources

Interactivity: Dilation and enlargement matrices (int-6297)

WORKED EXAMPLE 12

Determine the coordinates of the image of the point $(3, -4)$ under a dilation of factor $\frac{1}{2}$ from the x-axis.

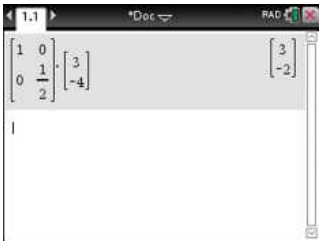
THINK	WRITE
1. Recall the dilation matrix for dilations from the x-axis.	$\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$
2. Identify the dilation factor and use the matrix equation for dilation by substituting the value of λ.	The dilation factor is $b = \frac{1}{2}$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
3. Substitute the pre-image point into the matrix equation.	The pre-image point is $(3, -4)$. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix}$
4. Calculate the coordinates of the image point.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ The image point is $(3, -2)$.

TI | THINK

1. On a Calculator page, complete the entry line as:
$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$
then press ENTER.
Note: Matrix templates can be found by pressing the templates button.

2. The answer appears on the screen.

WRITE



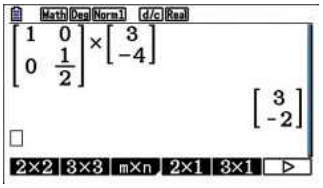
The image point is $(3, -2)$.

CASIO | THINK

1. On a Run-Matrix screen, complete the entry line as:
$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$
then press EXE.
Note: Matrix templates can be found by selecting MATH by pressing F4, then selecting MAT/VCT by pressing F1.

2. The answer appears on the screen.

WRITE



The image point is $(3, -2)$.

WORKED EXAMPLE 13

Determine the equation of the image of the parabola with equation $y = x^2$ after it is dilated by a factor of 2 from the y-axis.

THINK

1. Recall the matrix equation for dilation.

2. Find the expressions of the image coordinates in terms of the pre-image coordinates.

3. Rearrange the equations to make the pre-image coordinates x and y the subjects.

4. Substitute the image values into the pre-image equation to find the image equation.

5. Graph the image and the pre-image equations to verify the translation.

WRITE

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \end{bmatrix}$$

$x' = 2x$ and $y' = y$

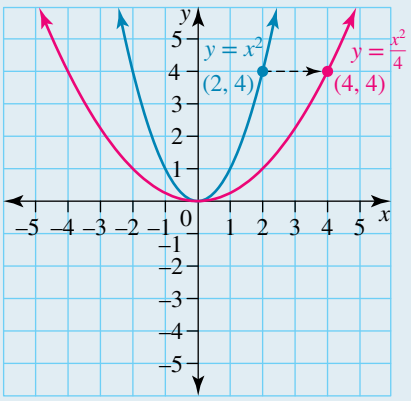
$x = \frac{x'}{2}$ and $y = y'$

$y = x^2$

$y' = \left(\frac{x'}{2}\right)^2$

$= \frac{(x)^2}{4}$

The equation of the image is $y = \frac{x^2}{4}$.



11.4.2 Dilation from both *x*- and *y*-axes

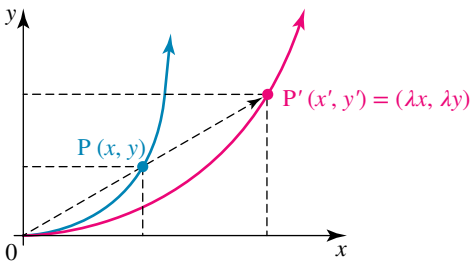
A dilation parallel to both the *x*-axis and *y*-axis can be represented by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

where a and b are the dilation factors in the *x*-axis and *y*-axis directions respectively.

- When $a \neq b$ the object is skewed.
- When $a = b = \lambda$, the size of the object is enlarged or reduced by the same factor, and the matrix equation is:

$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \lambda I \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \text{ where } \lambda \text{ is the dilation factor.} \end{aligned}$$



WORKED EXAMPLE 14

Jo has fenced a rectangular vegetable patch with fence posts at A (0, 0), B (3, 0), C (3, 4) and D (0, 4).

a. She wants to increase the size of the vegetable patch by a dilation factor of 3 in the *x*-direction and a dilation factor of 1.5 in the *y*-direction. Where should Jo relocate the fence posts?

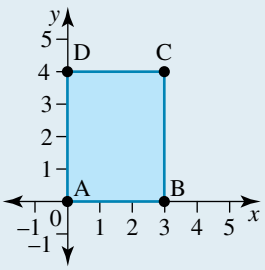
b. Jo has noticed that the vegetable patch in part a is too long and can only increase the vegetable patch size by a dilation factor of 2 in both the *x*-direction and the *y*-direction. Where should she relocate the fence posts? Will this give her more area to plant vegetables? Explain.

THINK

a. 1. Draw a diagram to represent this situation.

WRITE

a.



2. State the coordinates of the vegetable patch as a coordinate matrix.

The coordinates of the vegetable patch ABCD can be written as a coordinate matrix.

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

3. State the dilation matrix.

$$\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix}$$

4. Multiply the dilation matrix by the coordinate matrix to calculate the new fence post coordinates.

$$\begin{bmatrix} 3 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 9 & 9 & 0 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

The new fence posts are located at
A′ (0, 0), B′ (9, 0), C′(9, 6) and D′ (0, 6).

b. 1. State the coordinates of the vegetable patch as a coordinate matrix.

b. The coordinates of the vegetable patch ABCD can be written as a coordinate matrix:

$$\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

2. State the dilation matrix.

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

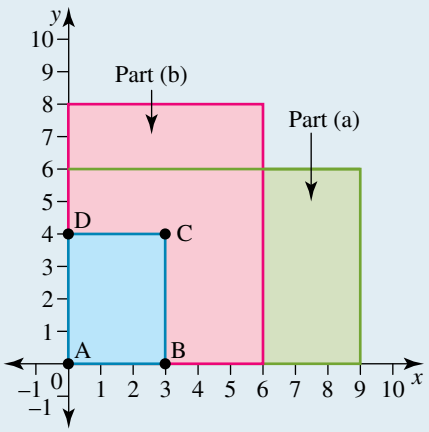
3. Calculate the new fence post coordinates A′B′C′D′ by multiplying the dilation matrix by the coordinate matrix.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 6 & 0 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

The new fence posts are located at
A′ (0, 0), B′ (6, 0), C′ (6, 8) and D′ (0, 8).

4. Draw a diagram of the original vegetable patch, and the two transformed vegetable patches on the same Cartesian plane.



5. Determine the area for each vegetable patch.

The vegetable patch size when dilated by a factor of 3 in the x-direction and a dilation factor of 1.5 in the y-direction gives an area of 54 units^2.
When dilated by a factor of 2 in both the x-direction and the y-direction, the vegetable patch has an area of 48 units^2.The farmer will have less area to plant vegetables in the second option.

studyon

Units 1 & 2 > Area 6 > Sequence 2 > Concept 4 **Dilations** Summary screen and practice questions

on Resources

Digital document: SkillSHEET Building a transformation matrix (doc-26836)
Digital document: SkillSHEET Finding the image of a curve after transformation (doc-26837)

Exercise 11.4 Dilations

Technology free

1. **WE12** Determine the coordinates of the image of the point $(2, -1)$ after a dilation of factor 3 from the x-axis.
2. Determine the coordinates of the image of $(-1, 4)$ after a dilation factor of 2 parallel to the y-axis.
3. A man standing in front of a carnival mirror looks like he has been dilated 3 times wider. Write a matrix equation for this situation.
4. Find the coordinates of the image of the point $(-1, 4)$ after a dilation by $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ from the y-axis.
5. Determine the coordinates of the image of $(2, -5)$ after a dilation of 3 parallel to the x-axis.
6. **WE13** Determine the equation of the image of the parabola with equation $y = x^2$ after it is dilated by a factor of 3 from the y-axis.
7. Determine the equation of the image of the parabola with equation $y = x^2$ after it is dilated by a factor of $\frac{1}{2}$ from the x-axis.
8. **WE14** A farmer has fenced a vegetable patch with fence posts at A (0, 0), B (3, 0), C (3, 4) and D (0, 4). She wants to increase the vegetable patch size by a dilation factor of 1.5 in the x-direction and a dilation factor of 3 in the y-direction. Where should she relocate the fence posts?

9. Jack wants to plant flowers on a flower patch with corners at A (2, 1), B (4, 1), C (3, 2) and D (1, 2). He wants to increase the flower patch size by a dilation factor of 2 in both the x-direction and the y-direction. Where should he relocate the new corners of the flower patch?
10. A transformation T is given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
    a. Determine the coordinates of image of the point A $(-1, 3)$.
    b. Describe the transformation represented by T.
11. Determine the equation of the image of the line with equation $2y + x = 3$ after it is dilated by $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
12. Determine the equation of the image of the parabola with equation $y = x^2 - 1$ after it is dilated by $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.
13. Determine the equation of the image of the hyperbola with equation $y = \dfrac{1}{x+1}$ after it is dilated by a factor of 2 from the y-axis.

14. The equation $y = 2\sqrt{x}$ is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

a. Determine the transformations represented by this matrix equation.

b. What is the equation of the image of $y = 2\sqrt{x}$ after this transformation?

15. Determine the equation of the image of the circle with equation $x^2 + y^2 = 4$ after it is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

16. The coordinates of ΔABC can be written as a coordinate matrix $\begin{bmatrix} -2 & -1 & -3 \\ 0 & 3 & 2 \end{bmatrix}$. It has undergone a transformation T given by $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

a. Determine the dilation factor, λ, if the image coordinate point A′ is $(-3, 0)$.

b. Calculate the coordinates of the vertices of $\Delta A'B'C'$.

Technology active

17. Calculate the dilation factor from the y-axis when the graph of $y = \frac{1}{x^2}$ maps on to the graph of $y = \frac{1}{3x^2}$.

18. **a.** Determine the equation of the image of $x+2y = 2$ under a dilation by a factor of 3 parallel to the x-axis.

b. Is there an invariant point?

11.5 Combinations of transformations

11.5.1 Double transformation matrices

A **combined transformation** is made up of two or more transformations.

If a linear transformation T_1 of a plane is followed by a second linear transformation T_2, then the results may be represented by a single transformation matrix T.

When transformation T_1 is applied to the point $P(x, y)$ it results in $P'(x', y')$.

When transformation T_2 is then applied to $P'(x', y')$ it results in $P''(x'', y'')$.

Summarising in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = T_1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2 \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Substituting $T_1 \begin{bmatrix} x \\ y \end{bmatrix}$ for $\begin{bmatrix} x' \\ y' \end{bmatrix}$ into $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2 \begin{bmatrix} x' \\ y' \end{bmatrix}$ results in $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = T_2 T_1 \begin{bmatrix} x \\ y \end{bmatrix}$.

To form the single transformation matrix T, the first transformation matrix T_1 must be *pre-multiplied* by the second transformation matrix T_2. The order of multiplication is important.

This is written as:

$$\boldsymbol{T = T_2 T_1}$$

Common transformation matrices used for combinations of transformations	
$M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x-axis
$M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y-axis
$M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = x$
$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Anti-clockwise (positive) rotation about origin
$D_{\lambda_1, 1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 1 \end{bmatrix}$	Dilation in one direction parallel to the x-axis or from the y-axis
$D_{1, \lambda_2} = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	Dilation in one direction parallel to the y-axis or from the x-axis
$D_{\lambda_1, \lambda_2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	Dilation parallel to both the x- and y-axes (λ_1 and λ_2 are the dilation factors)

Note: Translations are not linear transformations. The combined effect of two translations $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is found by addition, $\begin{bmatrix} a+c \\ b+d \end{bmatrix}$.

WORKED EXAMPLE 15

Determine the single transformation matrix T that describes a reflection in the x-axis followed by a dilation of factor 3 from the y-axis.

THINK

1. Determine the transformation matrices being used.

WRITE

T_1 = reflection in the x-axis

$T_1 : M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

T_2 = dilation of factor 3 from the y-axis

$T_2 : D_{3,1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

2. State the combination of transformations matrix and simplify.

$T = T_2 T_1$

$T = D_{3,1} M_x$

$$T = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

3. State the single transformation matrix.

The single transformation matrix is:

$$T = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

WORKED EXAMPLE 16

Calculate the coordinates of the image of the point P (2, 3) under a reflection in the *y*-axis followed by a rotation of 90° about the origin in an anti-clockwise direction.

THINK

WRITE

1. State the transformation matrices, T_1 and T_2.

T_1 = reflection in the *y*-axis

T_2 = rotation of 90° anti-clockwise

$$T_1 = M_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T_2 = R_{90°} = \begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix}$$

2. Determine the single transformation matrix and simplify.

$T = T_2 T_1$

$T = R_{90°} M_y$

$$T = \begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

The single transformation matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

3. State the single transformation matrix equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

4. Substitute the pre-image (2, 3) into the matrix equation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

5. Calculate the coordinates of the image point.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

6. State the answer.

The coordinates of the image point are $(-3, -2)$.

TI | THINK

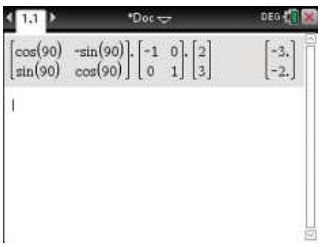
1. Put the calculator in Degree mode.On a Calculator page, complete the entry line as:

$$\begin{bmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then press ENTER.
Note: Matrix templates can be found by pressing the templates button.

2. The answer appears on the screen.

WRITE



The image point is $(-3, -2)$.

CASIO | THINK

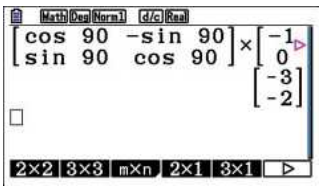
1. On a Run-Matrix screen, complete the entry line as:

$$\begin{bmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

then press EXE.
Note: Matrix templates can be found by selecting MATH by pressing F4, then selecting MAT/VCT by pressing F1.

2. The answer appears on the screen.

WRITE



The image point is $(-3, -2)$.

11.5.2 Inverse transformation matrices

The inverse of a transformation matrix will transform the image of a point or shape back to its original position.

Note: $\begin{bmatrix} x' \\ y' \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$

By determining the inverse of a matrix, we get:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

WORKED EXAMPLE 17

Determine the coordinates of the pre-image point, P (x, y), under a reflection in the x-axis followed by a rotation of 90° about the origin in a clockwise direction of the image point P′ $(-3, -2)$.

THINK	WRITE
1. State the transformation matrices, T_1 and T_2.	$T_1 = \text{reflection in the } x\text{-axis}$ $T_2 = \text{rotation of } 90° \text{ clockwise}$ $T_1 = M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $T_2 = R_{-90°} = \begin{bmatrix} \cos(90°) & \sin(90°) \\ -\sin(90°) & \cos(90°) \end{bmatrix}$
2. Determine the single transformation matrix and simplify.	$T = T_2 T_1$ $T = R_{-90°} M_x$ $T = \begin{bmatrix} \cos(90°) & \sin(90°) \\ -\sin(90°) & \cos(90°) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ The single transformation matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.
3. State the single transformation matrix equation.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
4. Substitute the image point $P'(-3, -2)$ into the matrix equation.	$\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
5. Using $\begin{bmatrix} x \\ y \end{bmatrix} = T^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$, rearrange the equation to make the image point (x, y) the subject.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ -2 \end{bmatrix}$
6. Find the inverse matrix for the single transformation matrix.	$T^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
7. Simplify the matrix equation.	$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ $= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
8. State the answer.	The coordinates of the pre-image point are $(2, 3)$.

WORKED EXAMPLE 18

A triangle ABC is transformed under the transformation matrix $T = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ to give vertices at A′ (−1, −7), B′ (4, 18) and C′ (11, 7). Determine the vertices A, B and C.

THINK	WRITE
1. Write the image vertices, A′ (−1 , −7) , B′ (4, 18) and C′ (11, 7), of ΔABC as a coordinate matrix.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$
2. Write the pre-image vertices, A, B and C, of ΔABC as a coordinate matrix.	Let A = (a, b) , B = (c, d) , C = (e, f). $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$
3. Recall the matrix equation and substitute known values.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$
4. Rearrange the equation to make the image points the subject.	$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$
5. Determine the inverse of the transformation matrix.	$T^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$
6. Simplify the matrix equation.	$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$ $\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$ $= \frac{1}{10} \begin{bmatrix} 10 & -20 & 30 \\ -10 & 30 & 40 \end{bmatrix}$ $= \begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \end{bmatrix}$
7. Answer the question.	The vertices of triangle ABC are A (1, −1), B (−2, 3) and C (3, 4).

TI | THINK

	WRITE
1. Write the image vertices, A′ (−1, −7), B′ (4, 18) and C′ (11, 7) , of ΔABC as a coordinate matrix.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$
2. Write the pre-image vertices of ΔABC as a coordinate matrix. The pre-image vertices are A, B and C.	$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$
3. State the matrix equation.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

CASIO | THINK

	WRITE
1. Write the image vertices, A′ (−1, −7),B′ (4, 18) and C′ (11, 7), of ΔABC as a coordinate matrix.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$
2. Write the pre-image vertices of ΔABC as a coordinate matrix. The pre-image vertices are A, B and C.	$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$
3. State the matrix equation.	$\begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$

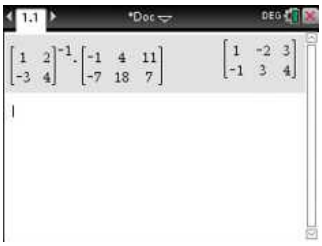
4. Rearrange the equation to make the image points the subject.

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$$

5. On a Calculator page, complete the entry line as:

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$$

then press ENTER.
Note: Matrix templates can be found by pressing the templates button.



6. The answer appears on the screen.

The vertices of triangle ABC are A $(1, -1)$, B $(-2, 3)$ and C $(3, 4)$.

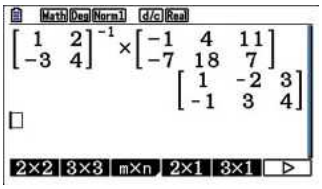
4. Rearrange the equation to make the image points the subject.

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$$

5. On a Run-Matrix screen, complete the entry line as:

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{bmatrix}$$

then press EXE.
Note: Matrix templates can be found by selecting MATH by pressing F4, then selecting MAT/VCT by pressing F1.



6. The answer appears on the screen.

The vertices of triangle ABC are A $(1, -1)$, B $(-2, 3)$ and C $(3, 4)$.

11.5.3 Interpreting the determinant of the transformation matrix

A single transformation matrix is represented by T. When a shape is transformed by this transformation matrix, the magnitude of the determinant of matrix T gives the ratio of the image area to the original area.

Area of image = $|\det(T)| \times$ area of object

where $|\det(T)|$ represents the area scale factor for the transformation.
If $|\det(T)|$ is negative, then the transformation will have involved some reflection.

Recall that the determinant of a 2×2 matrix, $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is $\det(T) = ad - bc$.

WORKED EXAMPLE 19

The triangle ABC is mapped by the transformation represented by $T = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ onto the triangle A′B′C′. Given that the area of ΔABC is 8 units^2, calculate the area of A′B′C′.

THINK

1 Recall the determinant formula and calculate for the given matrix.

WRITE

$$T = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

$$\det(T) = 1 + 3 = 4$$

2 Analyse the determinant.	The determinant is 4, which means that the area of the image is 4 times the area of the original object.
3 Calculate the area of the image, $\Delta A'B'C'$.	Area of image $= \lvert\det(T)\rvert \times$ area of object Area of $\Delta A'B'C' = 4 \times 8$ $= 32$
4 Answer the question.	The area of $\Delta A'B'C'$ is 32 units^2.

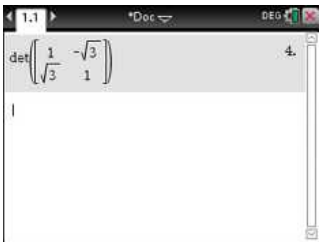
TI | THINK

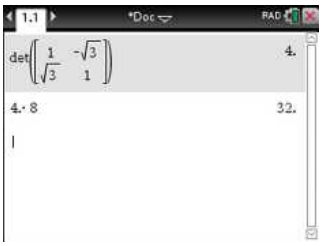
1. On a Calculator page, press MENU, then select:
   7: Matrix & Vector
   3: Determinant.
   Complete the entry line as:

$$\det\left(\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)$$

   then press ENTER.
   Note: Matrix templates can be found by pressing the templates button.
2. Complete the next entry line as: ans × 8 then press ENTER.
2. The answer appears on the screen.

WRITE





The area is 32 units^2.

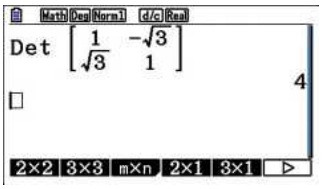
CASIO | THINK

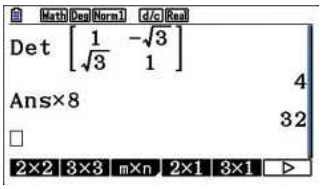
1. On a Run-Matrix screen, press OPTN and select MAT/VCT by pressing F2, then select Det by pressing F3. Complete the entry line as:

$$\det\left(\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)$$

   then press EXE.
   Note: Matrix templates can be found by selecting MATH by pressing F4, then selecting MAT/VCT by pressing F1.
2. The answer appears on the screen.

WRITE





The area is 32 units^2.

studyon

Units 1 & 2 > Topic 6 > Chapter 2 > Concepts 5 **Combinations** Summary screen and practice questions

Exercise 11.5 Combinations of transformations

Technology free

1. **WE 15** Determine the single transformation matrix T that describes a reflection in the y-axis followed by a dilation factor of 3 from the x-axis.
2. Determine the single transformation matrix T that describes a reflection in the line $y = x$ followed by a dilation of factor 2 from both the x- and y-axes.

3. **WE 16** Calculate the coordinates of the image of the point P$(1, -3)$ under a reflection in the x-axis followed by a rotation of 180° about the origin in an anti-clockwise direction.
4. Calculate the coordinates of the image of the point P$(-2, 2)$ under a reflection in the line $y = x$ followed by a rotation of 45° about the origin in an anti-clockwise direction.
5. Describe fully a sequence of two geometrical transformations represented by $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.
6. Determine the equation of the image of $y = x^2$ under a double transformation: a reflection in the x-axis followed by a dilation factor of 2 parallel to both the x- and y-axes.
7. Determine the equation of the image of $y = \sqrt{x}$ under a double transformation: a reflection in the y-axis followed by a dilation of 3 parallel to the x-axis.
8. **a.** State the transformations that have undergone $T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
   b. Determine the image of the curve with equation $2x - 3y = 12$.
9. **a.** State the transformations that have undergone $T\left(\begin{bmatrix} x' \\ y' \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
   b. Determine the image of the curve with equation $y = 2x^2 - 1$.
10. State the image of P (x, y) for a translation of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the x-axis.

Technology active

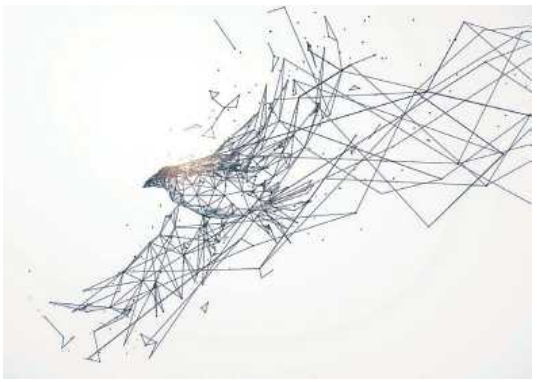
11. **WE 17** Determine the coordinates of the pre-image point, P(x, y), under a reflection in the x-axis followed by a rotation of 90° about the origin in an anti-clockwise direction of the image point P′$(1, -2)$.
12. **a.** Calculate the coordinates of the image of point P′(x', y') when the point P (x, y) undergoes a double transformation: a reflection in the y-axis followed by a translation of 4 units in the positive direction of the x-axis.
    b. Reverse the order of the pair of transformations in part **a**. Is the image different?
13. Find the coordinates of the pre-image point, P(x, y), under a reflection in the y-axis followed by a dilation of factor 3 from the y-axis of the image point P′$(-3, 6)$.
14. **WE 18** A triangle ABC is transformed under the transformation matrix $T = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$ to give vertices at A′$(0, 0)$, B′$(4, 18)$ and C′$(9, 15)$. Determine the vertices A, B and C.
15. **WE 19** The triangle ABC is mapped by the transformation represented by $T = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ onto the triangle A′B′C′. Given that the area of ABC is 4 units^2, calculate the area of A′B′C′.
16. A rectangle ABCD is transformed under the transformation matrix $T = \begin{bmatrix} 3 & 2 \\ 5 & 8 \end{bmatrix}$, to give vertices at A′$(0, 0)$, B′$(3, 0)$, C′$(3, 2)$ and D′$(0, 2)$.
    a. Determine the vertices of the square ABCD.
    b. Calculate the area of the transformed figure ABCD.
17. The triangle ABC is mapped by the transformation represented by $T = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ onto the triangle A′B′C′. Given that the area of ABC is 10 units^2, calculate the area of A′B′C′.
18. A rectangle ABCD with vertices at A $(0, 0)$, B $(2, 0)$, C $(2, 3)$ and D $(0, 3)$ is transformed under the transformation matrix $T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Calculate the new area of the transformed rectangle.

19. If D_λ denotes a dilation factor of λ parallel to both axes, what single dilation would be equivalent to D_λ^2?
20. Check whether the transformation 'a reflection in the y-axis followed by a reflection in the line $y = x$' is the same as 'a reflection in the line $y = x$ followed by a reflection in the y-axis'.

11.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar



1. Determine the single transformation matrix for a dilation of factor 7 from the y-axis followed by a reflection in the line $y = x$.
2. The point $(-3, 2)$ is translated by the matrix $\begin{bmatrix} -1 \\ -5 \end{bmatrix}$. Determine the new coordinates of the point.
3. Determine the transformation matrix for a dilation of factor λ from the x-axis followed by a reflection in the line $y = x$, then a reflection in the x-axis.
4. Calculate the coordinates of the image of the points A $(0, 1)$ and C $(3, 2)$ under the transformation defined by $\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$.
5. State the coordinates of the image of the point P (x, y) under a translation defined by $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the x axis.
6. Determine the equation of the image of $y = 2x^2$ under the translation defined by $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
7. Determine the matrix equation that represents a dilation of factor 3 from the y-axis followed by a translation of 2 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis.
8. Determine the coordinates of the pre-image point D (a, b) under a transformation defined by $T = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ of the image point D′ $(10, 3)$.
9. Determine the equation of the image of the graph of $y = \sqrt{x}$ after a dilation of factor 3 from the y -axis followed by a translation of 2 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis.
10. Determine the final image point when point P $(2, -1)$ undergoes two reflections. It is firstly reflected in the x-axis and then reflected in the line $y = x$.
11. Calculate the coordinates of the vertices of the image of pentagon ABCDE with A $(2, 5)$, B $(4, 4)$, C $(4, 1)$, D $(2, 0)$ and E $(0, 3)$ after a reflection in the y-axis.
12. A triangle ABC with vertices A $(2, -1)$, B $(-4, 0)$ and C $(5, 2)$ is rotated 45° anticlockwise.
    a. Calculate the coordinates of vertices A′, B′ and C′ of the rotated triangle.
    b. Compare the area of the triangle A′B′C′ to that of triangle ABC.

Complex familiar

You may wish to use a technology of your choice to answer questions 13 to 16.

13. Under a certain transformation, the circle $(x - 2)^2 + (y - 2)^2 = 4$ becomes $(x + 2)^2 + (y - 2)^2 = 4$. It is claimed that three different transformations could have achieved this outcome. Investigate this claim, giving details of the possible transformations and their matrices of transformation.

14. The line $y = 2x - 1$ undergoes a succession of translations defined by $T_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ and $T_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Verify that the order in which these translations occur has no effect on the result.

15. Find the equation of the image of the line $y = -2x + 2$ under a rotation of $\frac{\pi}{2}$ about the origin in an anticlockwise direction. Sketch the original and the image.

16. Determine the image of the point P $(1, 2)$ after a reflection in the line $y = \sqrt{3}x$.

Complex unfamiliar

You may wish to use a technology of your choice to answer questions **17** to **20**.

17. A farmer has fenced a vegetable patch with fence posts at A $(0, 0)$, B $(3, 0)$, C $(3, 4)$ and D $(0, 4)$. She wants to increase the size of the vegetable patch by a dilation factor of 1.5 in the x-direction and a dilation factor of 3 in the y-direction. Where should she relocate the fence posts?

18. Avril has plotted all of the locations for a cartoon on the Cartesian plane. The vertices of the outline of a character's house have coordinates $(-3, 5)$, $(-3, 7)$, $(5, 7)$ and $(5, 5)$. She wants to move the points so that they are reflected in the y-axis.
To add more animation to her cartoon, she wants not only to reflect the outline of the house in the y-axis, but also to combine it with a dilation factor of 2 in the x-axis and then a translation of 2 units across in the negative x-direction and 5 units down in the negative y-direction.

a. Plot the points on a set of axes.
b. State the transformation matrix for a reflection in the y-axis.
c. State the transformation matrix equation.
d. What are the new coordinates of the house?
e. Plot the new coordinates of the house on the same set of axes as the original coordinates.
f. Write the transformation matrix for each of the transformations.
g. State the single transformation matrix required to animate the house.
h. State the transformation matrix equation.
i. Determine the new coordinates of the house after the combinations of transformations.

19. A computer designer wants to animate a marshmallow being squashed. The marshmallow is modelled by the unit circle equation $x^2 + y^2 = 1$. To make it look squashed, it has to undergo the following transformations for the animation:

- a dilation factor of 2 from the y-axis and factor of 3 from the x-axis
- a translation of 1 unit across the x-axis and 2 units up the y-axis.

a. State the transformation matrix for each of the transformations.
b. State the transformation matrix equation required to squash the marshmallow.
c. Determine the equation of the image of the squashed marshmallow. What is the shape of the squashed marshmallow?
d. On the same set of axes, draw both the pre-image and image equations representing the marshmallow.
e. Determine the area of the squashed marshmallow using the determinant.

20. Mark wants to build a deck for entertaining outside. To plan out his deck, he uses a coordinate grid where each square represents 1 metre. The coordinates for the vertices of his deck are $(2, 2)$, $(8, 2)$, $(2, 6)$ and $(8, 6)$.

- **a.** Calculate the area of the deck.
- **b.** Mark decides to increase the deck by dilating it by a factor of 1.5 from the x-axis. State the transformation matrix required to transform the deck.
- **c.** Determine the new coordinates of the deck.
- **d.** Calculate the new area of the deck.
- **e.** Mark ordered enough decking wood to cover an area of $40\,\text{m}^2$. Does he have enough to build his new deck?
- **f.** On the blueprint of Mark's house, the coordinates of the corners of the garage are $(1, 5)$, $(1, 25)$, $(31, 5)$ and $(31, 25)$. On the blueprint, 1 unit represents 1 cm.
  - **i.** If the scale of the blueprint is $\frac{1}{30}$ of the actual structure, find the coordinates of the garage when it is built.
  - **ii.** Calculate the area of Mark's garage.
  - **iii.** Mark has two cars. Each car needs a parking area of $15\,\text{m}^2$ in the garage. Does he have enough space to park both cars in his garage?

- **g.** Mark decides to transform a garden bed in the form of a triangle with vertices at A $(-2, -2)$, B $(0, 2)$ and C $(2, -2)$ using the transformation equations $x' = x + y$ and $y' = x - y$.
  - **i.** Determine the general transformation matrix.
  - **ii.** Determine the new coordinates of Mark's garden bed.
  - **iii.** Calculate the scale factor by which the area has been increased.

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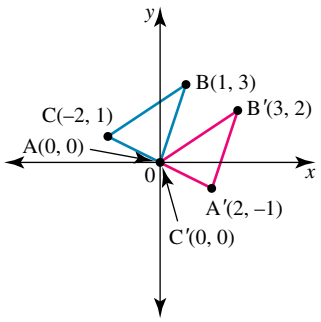
Units 1 & 2 Sit chapter test

Answers

Chapter 11 Matrix transformations

Exercise 11.2 Translations

1. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ Image point is $(3,\ 7)$
2. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ Image point is $(-6,\ 2)$
3. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ Image point is $(4,\ 0)$
4. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ Image point is $(2,\ -2)$
5. $T = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$
6. $T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$
7. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
8. a.



   b. $T = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$
9. $y = x + 1$
10. $y = x - 6$
11. $y = x + 6$
12. $y = (x - 2)^2 - 1$
13. $y = (x + 3)^2 + 1$
14. $y = (x - 7)^2 - 6$
15. $T = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$
16. $T = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$
17. $T = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$
18. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
19. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$
20. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}$

Exercise 11.3 Reflections and rotations

1. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ Image point is $(-3,\ 1)$
2. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$ Image point is $(-5,\ -2)$
3. $y = (x + 2)^2$
4. $y = -x^2 - 1$
5. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}, (5,\ -2)$
6. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}, (6,\ -9)$
7. $y = \pm\sqrt{x}$
8. $y = \pm\dfrac{\sqrt{-2\,(x + 1)}}{2}$
9. a. $y = x - 3$ b. $y = x + 3$
10. a. $y = -(x^2 + 2x + 1)$ b. $y = x^2 - 2x + 1$
11. $y = -\left(\sqrt{3} - 2\right)x + 1 - \sqrt{3}$
12. $\left(\dfrac{5}{2} + 2\sqrt{3}, -\dfrac{5\sqrt{3}}{2} + 2\right)$
13. a. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
    c. $\dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ d. $\dfrac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$
14. a. $(6,\ 7)$ b. $(6,\ 7)$
    c. The points are the same.
15. $y = -\dfrac{x}{2} + \dfrac{\sqrt{2}}{4}$
16. $y = \dfrac{-x + 1}{2}$

Exercise 11.4 Dilations

1. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, (2,\ -3)$
2. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}, (-1,\ 8)$
3. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
4. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, (-2,\ 4)$
5. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, (6,\ -5)$
6. $y = \left(\dfrac{x}{3}\right)^2 = \dfrac{x^2}{9}$
7. $y = \dfrac{x^2}{2}$
8. $(0,\ 0),\ (4.5,\ 0),\ (4.5,\ 12),\ (0,\ 12)$
9. $(4,\ 2),\ (8,\ 2),\ (6,\ 4),\ (2,\ 4)$

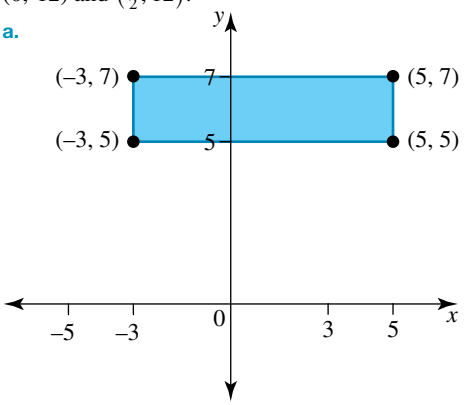
10. a. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, $(-3, 6)$
    b. A dilation of 3 parallel to the x-axis and a dilation of 2 parallel to the y-axis
11. $x + y = 3$
12. $y = \left(\frac{x}{3}\right)^2 - 1 = \frac{x^2}{9} - 1$
13. $y = \frac{2}{x+2}$
14. a. A dilation of 2 from the y-axis and dilation of 3 from the x-axis
    b. $y = 3\sqrt{2x}$
15. $\frac{x^2}{4} + y^2 = 4$
16. a. $\lambda = \frac{3}{2}$
    b. $\begin{bmatrix} -3 & -\frac{3}{2} & -\frac{9}{2} \\ 0 & 6 & 4 \end{bmatrix}$;
    A′ $(-3, 0)$, B′ $\left(-\frac{3}{2}, 6\right)$, C′ $\left(-\frac{9}{2}, 4\right)$
17. $\lambda = \pm\frac{1}{\sqrt{3}}$ or $\pm\frac{\sqrt{3}}{3}$
18. a. $y = -\frac{x}{6} + 1$
    b. Invariant point is $(0, 1)$.

Exercise 11.5 Combinations of transformations

1. $T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$
2. $T = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$
3. $(-1, -3)$
4. $(2\sqrt{2}, 0)$
5. Dilation of factor 2 to both axes followed by an anticlockwise rotation of 90° about the origin
6. $y = -\frac{x^2}{2}$
7. $y = \sqrt{-\frac{x}{3}}$
8. a. Reflection in line $y = x$ followed by a reflection in x-axis
   b. $3x + 2y = -12$ or $y = -\frac{3}{2}x - 6$
9. a. Dilation of factor 2 from the y-axis followed by a reflection in the x-axis and translation of 1 unit in the positive x-direction and 2 units in the positive y-direction.
   b. $y = -\frac{1}{2}(x-1)^2 + 3$
10. $(x + 2, -y + 1)$
11. $(-2, 1)$
12. a. $(-x + 4, y)$ b. Yes: $(-x - 4, y)$
13. $(1, 6)$
14. A$(0, 0)$, B$\left(\frac{-2}{17}, \frac{17}{7}\right)$, C$(3, 0)$
15. 20 units^2
16. a. A$(0, 0)$, B$\left(\frac{12}{7}, \frac{-15}{14}\right)$, C$\left(\frac{10}{7}, \frac{-9}{14}\right)$, D$\left(\frac{-2}{7}, \frac{3}{7}\right)$
    b. $\frac{3}{7}$ units^2
17. 70 units^2
18. 30 units^2
19. D_λ^2 gives a dilation factor of λ^2 parallel to both axes.
20. Not the same

11.6 Review: exam practice

1. $\begin{bmatrix} 0 & 1 \\ 7 & 0 \end{bmatrix}$
2. $(-4, -3)$
3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}$
4. A′$(3, 0)$, C′$(12, -3)$
5. $(x + 2, -y + 1)$ or $(x + 2, -(y - 1))$
6. $y = 2x^2 - 8x + 7$
7. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 5 \end{bmatrix}$
8. $(-3, 5)$
9. $y = \frac{\sqrt{3(x+2)}}{3} + 5$
10. $(1, 2)$
11. A′$(-2, 5)$, B′$(-4, 5)$, C′$(-4, 1)$, D′$(-2, 0)$, E′$(0, 3)$
12. a. A′$\left(\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$, B′$\left(-2\sqrt{2}, 2\sqrt{2}\right)$, C′$\left(\frac{7}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$
    b. The areas are the same.
13. The circle has not changed size. Possible transformations include:
    - a translation of 4 units to the left, $T = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$
    - a reflection in the y-axis, $M_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
    - an anticlockwise rotation of 90°, $R_{90°} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
14. Sample responses can be found in the worked solutions in the online resources.
15. $y = \frac{x}{2} + 1$
16. P′$\left(\frac{2\sqrt{3}-1}{2}, \frac{\sqrt{3}+2}{2}\right)$
17. She should relocate the fence to points $(0, 0)$, $\left(\frac{9}{2}, 0\right)$, $(0, 12)$ and $\left(\frac{9}{2}, 12\right)$.
18. a.



b. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

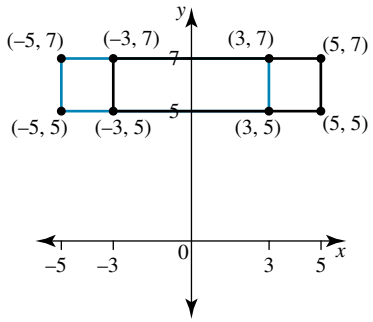
c. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & 5 & 5 \\ 5 & 7 & 7 & 5 \end{bmatrix}$

$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 3 & -5 & -5 \\ 5 & 7 & 7 & 5 \end{bmatrix}$

(3, 5), (3, 7), (−5, 7), (−5, 5)

e.



f. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

g. $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

h. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

i. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -3 & 5 & 5 \\ 5 & 7 & 7 & 5 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 & -12 & -12 \\ 0 & 2 & 2 & 0 \end{bmatrix}$

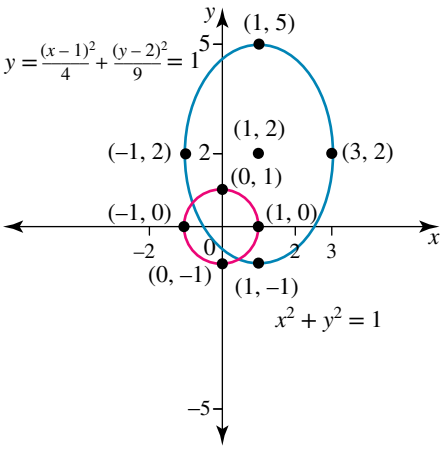
(4, 0), (4, 2), (−12, 2), (−12, 0)

19. a. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

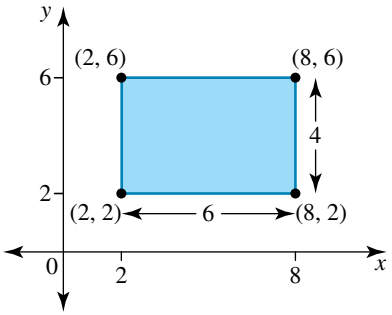
c. $\frac{(x-1)^2}{4} + \frac{(y'-2)}{9} = 1$; oval shape

d.



e. 6π units^2

20. a.



Area $= 6 \times 4 = 24$ m^2

b. $\begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix}$

c. (2, 3), (8, 3), (2, 9), (8, 9)

d. Area $= 6 \times 6 = 36$ m^2

e. Yes, he only needs 36 m^2 and has 4 m^2 to spare.

f. i. (30, 150), (30, 750), (930, 150), (930, 750)

ii. 54 m^2

iii. He needs 30 m^2 for the car; he has enough space.

g. i. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

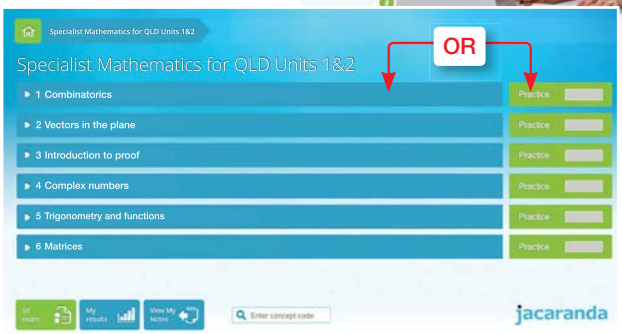
ii. A′ (−4, 0), B′ (2, −2), C′ (0, 4)

iii. Area factor = 2

TOPIC 3 Matrices

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

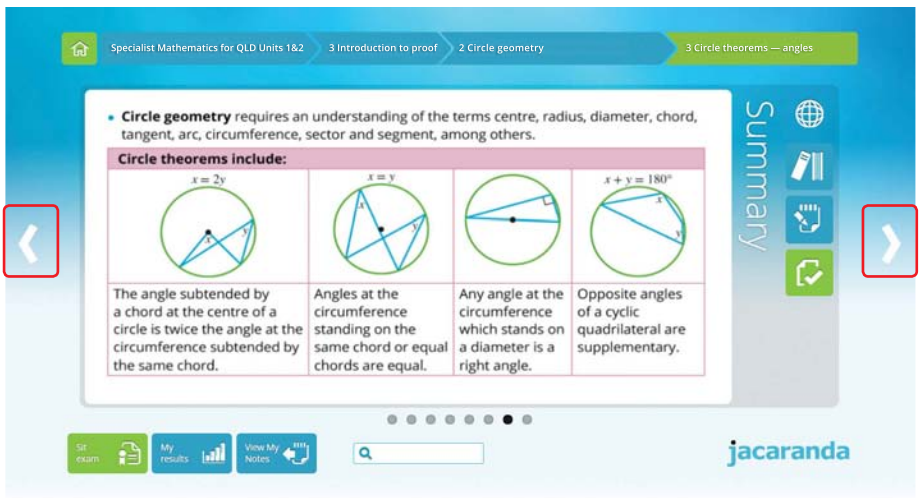
- Select your **course** *Specialist Mathematics for Queensland Units 1&2* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level ***OR*** select **Practice** to answer all practice questions available for each area.



- Select **Practice** at the sequence level to access all questions in the sequence.

- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 3

Specialist Mathematics: Unit 2 examination

Unit

Unit 2: Complex numbers, trigonometry, functions and matrices

Topic

Topic 1: Complex numbers
Topic 2: Trigonometry and functions
Topic 3: Matrices

Conditions

Technique	Response Type	Duration	Reading
Technology active	Short response	120 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet: • Notes not permitted • Non-CAS graphics calculator		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5, and 6	43	

* Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Part A: Simple familiar — total marks: 25.5

Question 1 (6.5 marks)

If $\sin(A) = \frac{3}{5}$ and $\cos(B) = \frac{5}{13}$ where A and B are acute angles:

a. calculate $\tan(A)$

b. calculate $\sin(A + B)$.

Question 2 (7.5 marks)

If $u = 5 - i$ and $v = 4 + 3i$:

a. state $\overline{u}$

b. calculate $|v|$

c. use an Argand diagram to represent $u - v$

d. calculate $2u - v$

e. calculate uv

f. calculate $\frac{u}{v}$.

Question 3 (4 marks)

If $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$:

a. calculate AB algebraically

b. calculate $A - 2B$ algebraically.

Question 4 (3.5 marks)

Solve $\begin{bmatrix} 2 & -1 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$.

Justify the reasonableness of your solution.

Question 5 (4 marks)

Use the graph of $f(x) = 2x - 6$ to sketch:

a. $\frac{1}{f(x)}$

b. $|f(x)|$

Part B: Complex familiar — total marks: 8.5

Question 6 (3.5 marks)

Prove that $\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$.

Question 7 (5 marks)

Determine the factors of $P(z) = z^3 + 6z^2 + 21z + 26$ over C.
Verify the reasonableness of your solution.

Part C: Complex unfamiliar — total marks: 9

Question 8 (3.5 marks)

Triangle T has vertices $(0, 2)$, $(0, 7)$ and $(k, 1)$. Matrix $\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$ is used to transform T to T'. If the area of T' is 165, determine the value of k.

Question 9 (3 marks)

Evaluate $\cos\left(2\sec^{-1}\left(\frac{x-1}{4}\right)\right)$ in terms of x.

Question 10 (2.5 marks)

Sine functions can be used to model the behaviour of musical notes. The function $y = \sin 2\pi fx$ can be used to model a note with frequency f where f is the frequency in Hertz and x is time in seconds. When notes are played together, the functions can be added together to model the resulting sound.
If notes with frequencies 10 Hz and 15 Hz are played together, determine the period of the resulting sound wave.

PRACTICE ASSESSMENT 4

Specialist Mathematics: Units 1 & 2 examination

Unit

Unit 1: Combinatorics, vectors and proof
Unit 2: Complex numbers, trigonometry, functions and matrices

Topic

Unit 1: Topic 1: Combinatorics
Topic 2: Vectors in the plane
Topic 3: Introduction to proof
Unit 2: Topic 1: Complex numbers
Topic 2: Trigonometry and functions
Topic 3: Matrices

Conditions

Technique	Response Type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 90 minutes Paper 2: 90 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet • Notes not permitted • Non-CAS graphics calculator permitted for Paper 2 ONLY		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem-solving *Assessment objective: 1, 2, 3, 4, 5 and 6	40	35	

* Queensland Curriculum & Assessment Authority, *Specialist Mathematics General Senior Syllabus 2019 v1.1*, Brisbane, 2018.
For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

A detailed breakdown of the examination marks summary can be found in the PDF version of this assessment instrument in your eBookPLUS.

Paper 1 - Technology free

Part A: Simple familiar — total marks: 30

Question 1 (6 marks)

Consider the digits 1, 2, 5 and 7. If the digits cannot be repeated:

a. calculate the number of three-digit numbers that can be created

b. calculate the number of even three-digit numbers

c. calculate the number of three-digit numbers that are greater than 500

d. calculate the number of even three-digit numbers that are less than 500.

Question 2 (4 marks)

Consider the points A $(1, 2)$, B $(-3, 5)$ and C $(-5, -1)$.

a. Calculate $\overrightarrow{AB}$.

b. Calculate a unit vector parallel to $\overrightarrow{AB}$.

c. Calculate $\overrightarrow{AC}$.

d. Calculate $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

Question 3 (4 marks)

Consider the statement 'if x is even, then x^2 is even'.

a. Identify the converse statement.

b. Explain if the original statement is implication ($\Rightarrow$) or equivalence ($\Leftrightarrow$).

c. Identify the contrapositive statement.

d. Demonstrate that the original statement is true.

Question 4 (4 marks)

If $u = 4\text{cis}\left(\frac{\pi}{3}\right)$ and $v = 12\text{cis}\left(\frac{\pi}{6}\right)$:

a. calculate uv and express the answer in the form $a + bi$

b. calculate $\frac{u}{v}$ and express the answer in the form $a + bi$.

Question 5 (5 marks)

Sketch the graph of $y = \sec(2x) + 1$ over the domain $0 \leq x \leq 2\pi$.
Identify key features.

Question 6 (7 marks)

a. Identify the matrix T that will rotate points $90°$ anticlockwise about the origin.

b. Use T to transform the point $(-4, 7)$.

c. Identify a single matrix Q that will reflect a point in the line $y = \sqrt{3}x$ and then perform a dilation that would map a point in the form (x, y) to $(-x, 3y)$.

d. Use an appropriate calculation to verify that matrix Q will result in the same transformation as performing the individual transformations.

Part B: Complex familiar — total marks: 6

Question 7 (6 marks)

Solve the equation $4\sin^2(x) + 12\cos(x) - 9 = 0$ over the domain $0° \leq x \leq 360°$.
Justify the reasonableness of your solution.

Part C: Complex unfamiliar — total marks: 4

Question 8 (4 marks)

A circle is inscribed inside a right-angled triangle with sides 3, 4 and 5 cm. Determine the radius of the circle.

Paper 2 - Technology active

Part A: Simple familiar — total marks: 15

Question 1 (5 marks)

$A = \begin{bmatrix} -2 & 5 \\ 7 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 0 \\ 5 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 59 \\ 32 \end{bmatrix}$

Use appropriate technology to evaluate the following.

a. A^3

b. A^{-1}

c. AB

d. X if $AX = C$

Question 2 (5 marks)

$w = 2 + 11i$, $z = -4 + 3i$

Use appropriate technology to evaluate the following.

a. wz

b. $\frac{w}{z}$

c. $w^2 + \overline{z}$

d. z in polar form

Question 3 (5 marks)

Consider a class of 20 students.

a. If a team of 5 students is to be selected from the class, calculate the number of different teams that could be chosen.

b. If a particular student in the class, Teagan, must be in the team, calculate the number of different 5-person teams that could be selected.

c. If students William and Olivia cannot be selected, calculate the number of different 5-person teams that could be selected.

d. If the class needs to elect a class captain and a class vice-captain, calculate the number of possible pairs.

Part B: Complex familiar — total marks: 9

Question 4 (5 marks)

By writing $\sqrt{3}\cos(x) - 2\sin(x)$ in the form $A\cos(x + \alpha)$, determine the maximum and minimum values and the first time within the domain $0 \leq x \leq 2\pi$ that these occur.

Question 5 (4 marks)

Consider the vectors $\underset{\sim}{u} = 3\hat{\imath} + \hat{\jmath}$ and $\underset{\sim}{v} = 2\hat{\jmath}$.

a. Determine the angle between vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$.

b. Determine the vector resolute of $\underset{\sim}{v}$, parallel to $\underset{\sim}{u}$.

Part C: Complex unfamiliar — total marks: 11

Question 6 (5 marks)

Data about tide times on Monday was collected.

Time	6 am	7 am	8 am	9 am	10 am	11 am	12 pm	1 pm	2 pm
Depth (m)	2.4	2.8	3.3	3.6	3.5	3.2	2.9	2.3	1.8
Time	3 pm	4 pm	5 pm	6 pm	7 pm	8 pm	9 pm	10 pm	11 pm
Depth (m)	1.4	1.7	2.1	2.4	2.8	3.3	3.4	3.5	3.4

Boats can come into the harbour when the water is more than 2.8 m deep.

a. Develop a suitable model to determine if a boat can come into harbour at 2 pm on Thursday.

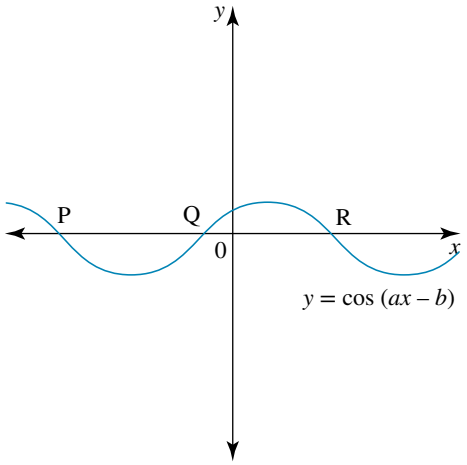
b. If the boat can enter the harbour, determine how long it will be able to do so. If the boat cannot enter the harbour, determine how long it will need to wait until it is able to enter.

Question 7 (3 marks)

A group of people need to select a team consisting of at least 2 members of the group. One of the group members suggested that they list all of the possible team compositions, but another member calculated that there would be approximately 250 different teams that could be formed from the members of the groups.
Determine the number of group members.

Question 8 (3 marks)

Below is a graph of a function in the form $y = \cos(ax - b)$, $a > 0$, $0 < b < \pi$. If the points marked are P $\left(-\frac{13\pi}{5}, 0\right)$, Q $\left(-\frac{3\pi}{5}, 0\right)$ and R $\left(\frac{7\pi}{5}, 0\right)$, determine the values of a and b.



Evaluate the reasonableness of your solution.

GLOSSARY

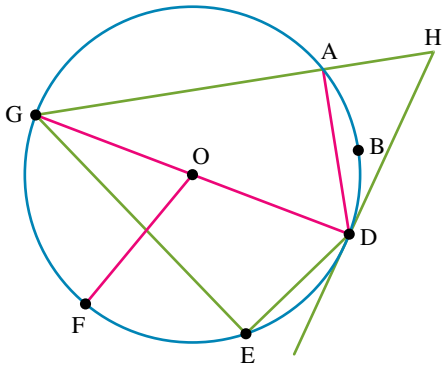
absolute value: $|x|$, the magnitude of a number

addition principle: If there are n ways of performing operation A and m ways of performing operation B, then there are $m + n$ ways of performing A *or* B.

additive inverse: When two matrices $A + B = O$, matrix B is an additive inverse of A.

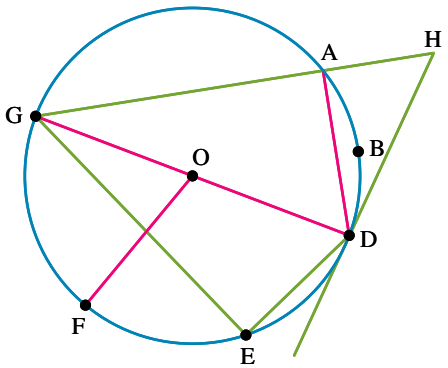
alternate segment: the chord $\overline{AD}$ divides the circle into two segments. The segment closest to the tangent $\overline{HD}$ contains $\angle ADH$. The alternate segment contains $\angle AGD$ and this angle is known as the angle in the alternate segment.

$\angle ADH = \angle AGD$

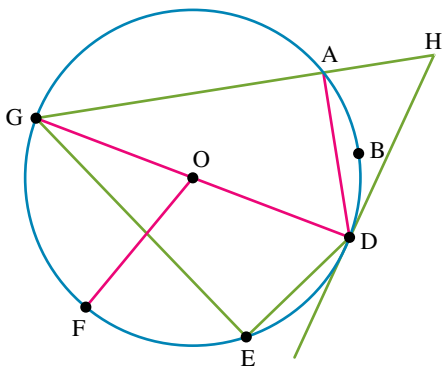


amplitude: the maximum variation of a function from its equilibrium (middle) position

angle at the centre: the angle made by two radii intersecting the centre of a circle. As O is the centre of the circle, $\angle FOD$ is called an angle at the centre subtended by the arc $\overset{\frown}{FED}$.

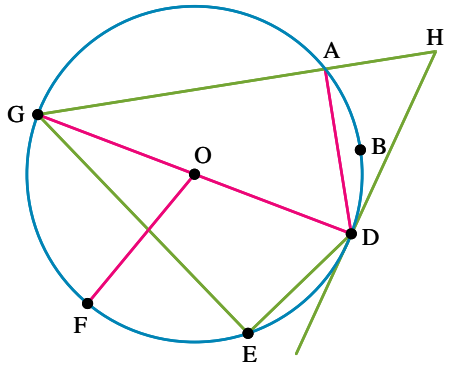


angle at the circumference: the angle at the circumference of a circle. Consider the angle $\angle AGD$. As G is on the circumference of the circle, then $\angle AGD$ is called an angle at the circumference subtended by the arc $\overset{\frown}{ABD}$. The same angle can also be described as standing on chord $\overline{AD}$.



angle difference formulas: $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$,
$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

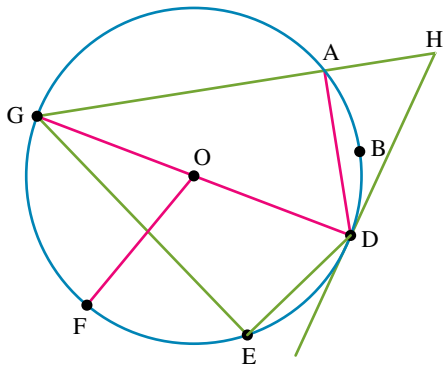
angle in a semicircle: the angle at the circumference of a semi-circle subtended by the arc of the semicircle. As $\overline{GD}$ is a diameter, $\angle GAD$ is called an angle in a semicircle.



angle sum formulas: $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$,
$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

applied forces: forces acting on objects in physical contact with each other; include tensile forces, contact forces, compressive forces

arc: Three points on a circle describe an arc. $\overset{\frown}{ABD}$ begins at point A and then passes through B to end at D. A minor arc is smaller than a semicircle, and a major arc is larger than a semicircle.



Argand diagram: plotting a complex number $x + yi$ as a point (x, y) on the complex plane with real and imaginary axes (the Argand plane)

Argand plane: geometric representation of complex numbers $x + yi$ with real and imaginary axes. The axes are denoted by 'Re (z)' and 'Im (z)' respectively.

argument of z: $\text{Arg}(z)$; the angle measurement anticlockwise of the positive real axis;
$\arg(z) = \theta \quad \sin\theta = \frac{y}{|z|}, \cos\theta = \frac{x}{|z|}, \tan\theta = \frac{y}{x}$

arrangement: a way of choosing things where order is important; see **permutation**

asymptote: the graphical representation of an undefined function $f(x) = 0$. Graphically it is a line that a graph approaches but never reaches. A horizontal asymptote shows the long-term behaviour as $x \to \infty$; a vertical asymptote may occur where a function is undefined, such as at $x = 0$ for the hyperbola $y = \frac{1}{x}$

axiom: a proposition that is assumed to be true; from Greek, meaning 'agreed starting point'. Every area of mathematics has its own basic axioms.

balanced forces: An object at rest will stay at rest or an object in motion will maintain its motion under balanced forces. Objects under balanced forces are in equilibrium.

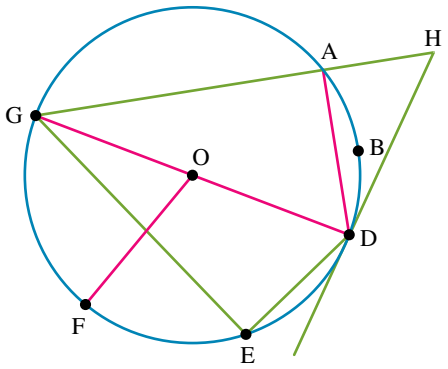
binomial theorem: a rule for expanding expressions of the form $(x + y)^n$:
$(x + y)^n = x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots {}^nC_rx^{n-r}y^r + \ldots y^n$

Cartesian form of a complex number: $x + yi$; consists of an ordered pair of numbers (x, y) which can be plotted on the complex plane or Argand plane

Cartesian form of a vector: From the origin to point (x, y), the Cartesian form of vector $\underset{\sim}{u}$ is $u = x\hat{\imath} + y\hat{\jmath}$.

centre of the circle: O is the centre of the circle. One of Euclid's postulates states that any circle can be defined by its centre and its radius.

chord: an interval jointing two points on a circle. $\overline{AG}$, $\overline{AD}$ and $\overline{GD}$ are all chords.



circumference: the distance around a circle (that is, the length of the circle if it was opened up and straightened into a line segment). It may also be used to refer to the edge of the circle. Points on the circumference are really points on the circle.

column matrix: a matrix with only one column; a vector matrix

combination: the number of ways of choosing r things from n distinct things where order is not important; ${}^{n}\mathrm{C}_{r} = \dfrac{n!}{r!\,(n-r)!}$, also written as $\binom{n}{r}$

combined transformation: a combination of two or more transformations

complex conjugate roots: the complex solutions for quadratic equations with $\Delta < 0$ (i.e. those that have no real solutions); expressed in the form $a \pm bi$

complex number: $z = a + bi$; a real number added to a multiple of the imaginary unit i, when a and b are real numbers; $\mathrm{Re}\,(z) = a$, $\mathrm{Im}\,(z) = b$

component (of a vector): The components of a vector $\underset{\sim}{u}$ are the two horizontal and vertical vectors (parallel to the unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$) that sum to $\underset{\sim}{u}$.

conjugate of a complex number: The conjugate, $\bar{z}$, of a complex number z has the opposite sign of the imaginary component. Multiplication or addition of a complex number and its conjugate results in a real number.

contradiction: In proof by contradiction, the opposite of what you are trying to prove is assumed to be true. Eventually, the proof will reach a statement that cannot be true, meaning that the initial assumption must be false.

contrapositive: a statement that contains the negation of both terms of the original statement but in the reverse order. For example, the contrapositive of 'if P, then Q' is 'if not Q, then not P'. The contrapositive of a true statement is also true.

converse: The converse of a statement has the same propositions but in reverse order. The converse may be false, even if the original statement is true. For example, the converse of 'if P, then Q' is 'if Q, then P'; symbolically, the converse of $\mathrm{P} \Rightarrow \mathrm{Q}$ is $\mathrm{Q} \Rightarrow \mathrm{P}$ or $\mathrm{P} \Leftarrow \mathrm{Q}$.

coordinate matrix: a matrix that represents the coordinates of points as columns in the matrix

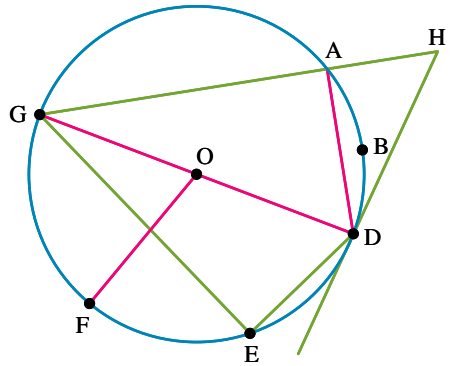
cosecant function: the reciprocal of the sine function; $cosec\,(x) = \dfrac{1}{\sin(x)}$, $\sin(x) \neq 0$

cotangent function: the reciprocal of the tangent function; $\cot(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$, $\sin(x) \neq 0$

counter example: an example which shows a statement is not true

cycle: one complete iteration of a periodic function over a single period

cyclic quadrilateral: a quadrilateral whose vertices all lie on a circle. ADEG is a cyclic quadrilateral.



De Moivre's theorem: $z_1 \times z_2 = r_1 \times r_2 \text{cis}(\theta_1 + \theta_2)$ and $(r \text{ cis } \theta)^n = r^n \text{cis } n\theta$, where $\text{cis}\theta = \cos\theta + i \sin\theta$; useful for multiplying and dividing complex numbers

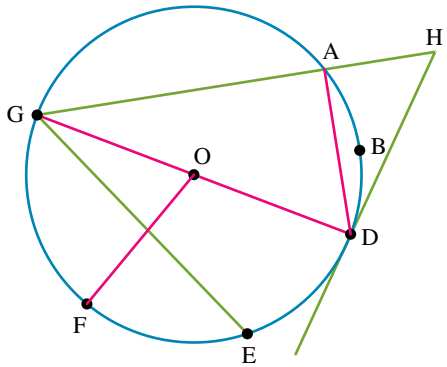
definition: the agreed meaning of a term

derived unit: a unit defined in terms of the standard units

determinant of a matrix: a value associated with a square matrix, evaluated by multiplying the elements in the leading diagonal and subtracting the product of the elements in the other diagonal; for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

diameter: a chord that passes through the centre of a circle. $\overline{GD}$ is a chord and a diameter.



dilation: a linear transformation that enlarges or reduces the size of a figure by a scale factor k parallel to either axis or both

domain: the set of possible x-values for a function; the set of all x-values of the ordered pairs (x, y) that make up a relation

dot product: or scalar product; multiplication of two vectors which results in a scalar

double angle formulas: $\sin(2A) = 2\sin(A)\cos(A)$,

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= 2\cos^2(A) - 1 \\ &= 1 - 2\sin^2(A)\end{aligned}$$

elements: the members of a set; $a \in A$ means a is an element of, or belongs to, the set A. If a is not an element of the set A, this is written as $a \notin A$.

elements of a matrix: the numbers in the matrix

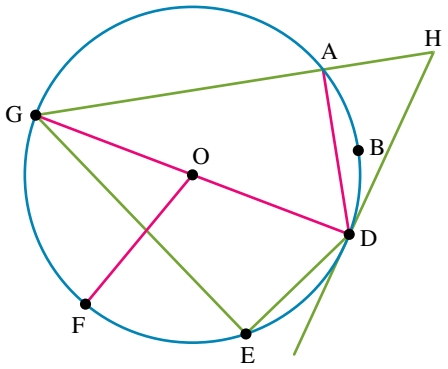
equilibrium: a condition when all forces acting on an object are balanced; that is, the net or resultant force is zero

equilibrium position: the centre of oscillation between the maximum and minimum values of a periodic function

equivalent statements: Statements P and Q are equivalent if the converse of P ⇒ Q is true; that is, P ⇒ Qand Q ⇒ P; symbol ⇔.

existential quantifier: there exists; symbolically, ∃. For a propositional function, consider if there exists a value for the variable which will make the statement true.

exterior angle to a cyclic quadrilateral: When one side of a cyclic quadrilateral is extended to an exterior point, the exterior angle is created between this extended segment and the cyclic quadrilateral. ∠DAH is an exterior angle.



external (to a circle): Points outside the circle are external to the circle.

factorial: $n!$, multiplying each of the integers from n down to 1; $0! = 1$

field force: force acting on objects without physical contact, for example gravity

fixed point: a point of the domain of a function which is mapped onto itself after a transformation; an invariant point

force: a push or pull on an object; includes field force and applied force (e.g. contact, tensile)

function: a relation in which the set of ordered pairs $(x,\ y)$ have each x-coordinate paired to a unique y-coordinate; typically expressed as $y = f(x)$

***g*:** gravitational field strength; a measure of the force of gravity acting on a unit mass; equals 9.8 N/kg at the surface of the Earth

general solution: a formula that describes the solution of a function in terms of n, where n is an integer.
The general solution of $\cos x = a, -1 \leq a \leq 1$ is $x = 2n\pi \pm \cos^{-1} a, n \in Z$.
The general solution of $\sin x = a, -1 \leq a \leq 1$ is $x = 2n\pi + \sin^{-1} a, (2n+1)\pi - \sin^{-1} a, n \in Z$.
The general solution of $\tan x = a$ is $x = n\pi + \tan^{-1} a, n \in Z$.

half-angle formulas: $\sin(A) = 2\sin\dfrac{(A)}{2}\cos\dfrac{(A)}{2}$,

$$\cos(A) = \cos^2\frac{(A)}{2} - \sin^2\frac{(A)}{2}$$
$$= 2\cos^2\frac{(A)}{2} - 1$$
$$= 1 - 2\sin^2\frac{(A)}{2}$$

identical vector: a vector with the same magnitude and direction as the original

identity: a relationship that is true for all possible values of the variable or variables

identity matrix: I; a square matrix that has 1s down the leading diagonal and 0s on the other diagonal; for example, the 2×2 identity matrix is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iff: the condition 'if and only if'; when all stated conditions must apply

image: a point or figure after a transformation

imaginary number: i; the root of the equation $x^2 = -1$; that is, $x = \pm\sqrt{i^2}$

implication: if P, then Q; symbolically, P ⇒ Q

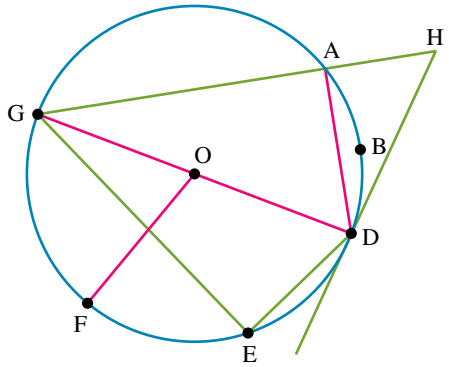
improper fraction: a fraction in which the degree of the numerator is a greater number or a higher degree than the denominator

inclusion–exclusion principle: For two sets S and T: $n(S \cup T) = n(S) + n(T) - n(S \cap T)$
For three sets S, T and R: $n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T) - n(T \cap R) - n(S \cap R) + n(S \cap T \cap R)$

inertial mass: resistance to acceleration

integers: All of the positive and negative numbers and zero: $\ldots\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\ldots$

interior opposite angle in a cyclic quadrilateral: the angle inside the quadrilateral that does not share any rays with the exterior angle. If the exterior angle is ∠DAH, then the interior opposite angle in GADE is ∠GED.



intersection: the set containing the elements common to both A and B; denoted as $A \cap B$

invariant point: a point of the domain of a function which is mapped onto itself after a transformation; a fixed point

irrational numbers: real numbers not including the rational numbers; that is, real numbers that cannot be expressed in the form $\frac{a}{b}$, where $b \neq 0$

kinematics: the study of the motion of objects

linear transformation: a transformation of a vector in which the origin does not move. A linear transformation can be represented by a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Linear transformations include rotations around the origin, reflection in lines passing through the origin and dilations. Translations are not linear transformations.

magnitude: $|x|$; the absolute value of a number

magnitude (of a vector): $|z|$ or r; the length of any directed line segment representing the vector z

major arc: an arc that is larger than a semicircle

mathematical statement: a mathematical sentence that is either true or false; a proposition. For example, '5 is a prime number' is a true mathematical statement.

matrix: a rectangular array used to store and display data

mediator: a line of reflection

minor arc: an arc that is smaller than a semicircle

minors: The determinants of a smaller square matrix formed by deleting one row and one column from a larger square matrix

modulus: the absolute value of a number

modulus of a complex number: $|z|$; for a complex number, $z = x + iy$, the length of the line segment from the origin to point z; $|z| = \sqrt{x^2 + y^2}$

multiple angle formulas: $\sin(3A) = 3\sin(A) - 4\sin^3(A)$,
$\cos(3A) = 4\cos^3(A) - 3\cos(A)$,
$\sin(4A) = \cos(A)\big(4\sin(A) - 8\sin^3(A)\big)$,
$\cos(4A) = 8\cos^4(A) - 8\cos^2(A) + 1$

multiplication principle: If there are n ways of performing operation A and m ways of performing operation B, then there are $m \times n$ ways of performing A *and* B.

multiplicative inverse: A number multiplied by its multiplicative inverse equals 1. That is, $z^{-1} = \frac{1}{z}$, $zz^{-1} = 1$.

multiplicative inverse (of a matrix): A^{-1}; the inverse of a square matrix A such that $AA^{-1} = A^{-1}A = 1$.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \det A \neq 0$

natural numbers: the counting numbers 1, 2, 3, …

negation: the opposite of a mathematical statement. If the statement is P, the negation is 'not P'; symbolically, $\neg$ P .

net force: the vector sum of all real forces acting on an object; resultant force

newton (N): the SI unit of weight, the amount of force required to accelerate 1 kg of mass 1 m/s^2

null matrix: a square matrix that consists entirely of '0' elements; a zero matrix

oblique: slanting or inclined; not parallel or perpendicular to a line or surface

order of a matrix: the size of a matrix

particle: an object taken as a point; used to model an object in Newtonian dynamics as a representation of the object's motion. The original object's shape, change in shape and/or internal movements are irrelevant.

Pascal's identity: a theorem dealing with binomial coefficients: ${}^{n}\mathrm{C}_{r} = {}^{n-1}\mathrm{C}_{r-1} + {}^{n-1}\mathrm{C}_{r}$ for $0 < r < n$

Pascal's triangle: a triangle formed by rows of the binomial coefficients of the terms in the expansion of $(a + b)^n$ with n as the row number.

period: the length of time it takes a periodic function to repeat itself

periodic functions: functions that repeat over time

permutation: the number of ways of choosing r things from n distinct things when order is important; ${}^{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+1)$

pigeon-hole principle: If there are $(nk + 1)$ pigeons to be placed in n pigeon-holes, then there is at least one pigeon-hole with $(k + 1)$ pigeons in it.

point of rotation: the fixed point around which a rotation occurs

points on a circle: any point that lies on a circle

polar form (of a complex number): the complex number z expressed in terms of θ; $z = r\,\mathrm{cis}\theta$

polar form (of a vector): a vector expressed in the form $[r, \theta]$ where r is the magnitude of the vector and $\theta = \tan^{-1}\frac{y}{x}$, which is the direction of the vector (the angle the vector makes to the positive direction of the x-axis)

position vector: a vector that defines a point by magnitude and direction relative to the origin

postulate: In geometry, a postulate is a statement that is assumed to be true without proof. (This is usually called an axiom in other areas of mathematics.)

pre-image: the original figure before a transformation

probability: the long-term proportion or relative frequency of the occurrence of an event

projection: A vector $\underset{\sim}{v}$ can be defined by a projection acting in the direction of vector $\underset{\sim}{u}$ and a projection acting perpendicular to $\underset{\sim}{u}$.

proof: demonstration that a statement is always true using definitions, postulates and previously proven statements in a formal sequence of steps

proper fraction: a fraction in which the degree of the numerator is lower than the degree of the denominator, creating a rational expression

proposition: a sentence that is either true or false; a mathematical statement. For example, '5 is a prime number' is a true proposition.

propositional function: a proposition that includes variables

Pythagorean identities: For any angle A, $\sin^2(A) + \cos^2(A) = 1$, $\tan^2(A) + 1 = \sec^2(A)$ and $\cot^2(A) + 1 = \operatorname{cosec}^2(A)$

quantifiers: used with a propositional function to give information about the scope of the function's variables. There are 2 main quantifiers:

- the universal quantifier (for all/for each): $\forall$
  (consider if the function is true for all possible values of the variable)
- the existential quantifier (there exists): $\exists$
  (consider if there is a value for the variable that will make the statement true).

radian: the angle created by an arc on a circle equal in length to the circle's radius; symbol ^{c}. Radians are typically measured in π, as $\pi^c = 180°$.

radius: any line segment joining a point on the circle to the centre

range: the set of possible y-values for a function; the set of all y-values of the ordered pairs (x, y) that make up a relation

rational function: a function expressed as the quotient of two polynomials $f(x) = \dfrac{g(x)}{h(x)}$, where $h(x) \neq 0$

rational numbers: real numbers that can be written in the form $\dfrac{a}{b}$ where a and b are integers, the only common factor between a and b is 1, and $b \neq 0$.

real numbers: numbers that can be represented on a number line

reciprocal: the multiplicative inverse of the original; 1 divided by the original. The reciprocal of $f(x)$ is $\dfrac{1}{f(x)}$.

reflection: a transformation of a point, line or figure defined by a line of reflection, where the image point is a mirror image of the pre-image point

relation: any set of ordered pairs (x, y)

resistive forces: forces that occur when two objects move or attempt to move relative to one another, for example air drag and friction

restricted domain: a subset of a function's maximal domain, often due to practical limitations on the independent variable in modelling situations

resultant force: the vector sum of all real forces acting on an object; net force

roots of an equation: the solutions of an equation

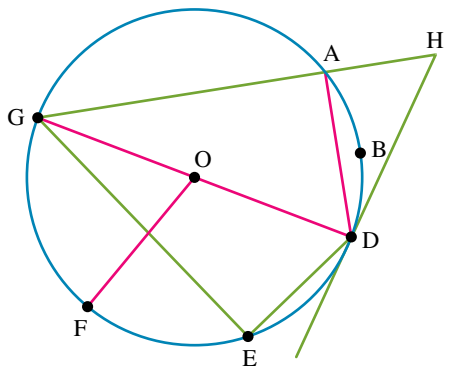
rotation: a transformation of a point, line or figure where each point is moved a constant amount around a fixed point

row matrix: a matrix with only one row

scalar: a quantity that has only magnitude, no direction

scalar resolute: $\hat{\underset{\sim}{u}} \cdot \underset{\sim}{v}$; the magnitude of vector $\underset{\sim}{v}$ acting in the direction of vector $\underset{\sim}{u}$

secant: a line that touches a circle at two distinct points. $\overline{GH}$ is a secant.



secant function: the reciprocal of the cosine function; $\sec(x) = \dfrac{1}{\cos(x)}$, $\cos(x) \neq 0$

segment: A chord divides a circle into two regions; the larger region is the major segment and the small region is the minor segment.

selection: the number of ways of choosing things when order is not important; see **combination**

set: a collection of elements

SI: Système International d'Unités; defines seven base units of measure from which all other SI units can be defined

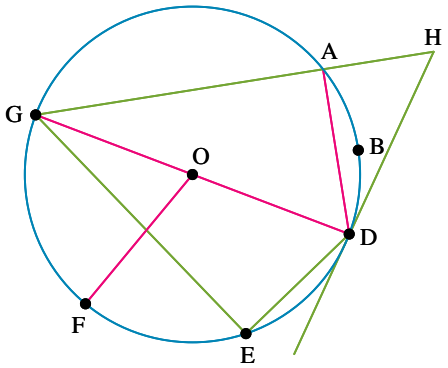
singular matrix: a matrix that has a zero determinant and hence has no multiplicative inverse

sinusoidal curve: the regular shape of graphs based on $y = \sin(x)$ and $y = \cos(x)$

square matrix: a matrix with an equal number of rows and columns

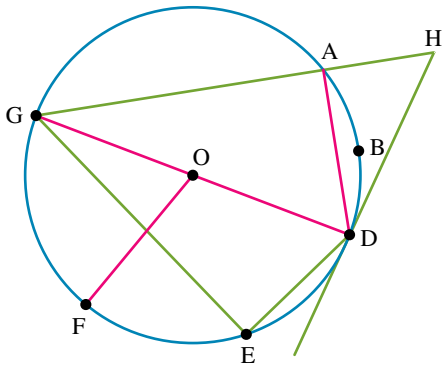
standard form (of a complex number) the expression of a complex number as a real number added to a multiple of the imaginary unit i, when a and b are real numbers; $z = a + bi$

standing: $\angle$AGD is called an angle at the circumference standing on chord $\overline{\text{AD}}$ (or subtended by the arc $\overset{\frown}{\text{ABD}}$).



statics: a situation when the net or resultant force acting on an object is zero; the object is in equilibrium

subtend: An arc on a circle subtends the angle at the circumference opposite it. The arc $\overset{\frown}{\text{ABD}}$ is said to subtend the angle $\angle$AGD, where A, B, D and G are points on the circle.



surd: a root of a number that does not have an exact answer. Surds are irrational numbers. Surds themselves are exact numbers, for example $\sqrt{6}$; their decimal approximations are not.

tangent: a line that touches a circle at one point

theorem: a mathematical statement that can be shown to be true using a proof.

thrust: the force required to move an object forward

trace of a matrix: tr(A) for matrix A; the sum of the leading diagonal elements

transformation: a geometric operation that may change the shape and/or the position of a point or set of points

translation: a transformation of a point, line or figure where each point in the plane is moved a given distance in a horizontal or vertical direction

translation matrix: the representation of a translation by a matrix

trigonometric function: sine (sin), cosine (cos) or tangent (tan). On a unit circle, cos θ is the x-coordinate of the trigonometric point [θ]; sin θ is the y-coordinate of the trigonometric point [θ]; and tan θ is the

length of the intercept that the line through the origin and the trigonometric point $[\theta]$ cuts off on the tangent drawn to the unit circle at (1, 0).

unbalanced forces: forces acting on an object to cause a change in motion

undefined: a mathematical expression that does not have a meaning, such as division by zero. Division of a number by zero cannot be undone by multiplication by zero, $\frac{x}{0} \times 0 \neq x$; hence, division by zero is undefined.

union: the set of all elements in any one set and in a combination of sets; $A \cup B$ contains the elements in A or in B or in both A and B

unit circle: a circle of radius 1 about the origin, from which trigonometric ratios can be considered

unit vectors: the vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ with magnitudes of 1 unit, which allow a vector to be resolved into its components; $u = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$

universal quantifier: for all; symbolically $\forall$. For a propositional function, consider if the function is true for all possible values of the variable.

universal set: the complete set of objects being considered; usually represented by ξ

vector: a quantity that has both magnitude and direction

vector matrix: a matrix with only one column

vector resolute: the direction of vector $\underset{\sim}{v}$ acting parallel or perpendicular to the direction of vector $\underset{\sim}{u}$; $\underset{\sim}{v}_{\parallel} = \underset{\sim}{u} - (\underset{\sim}{u} \cdot \hat{\underset{\sim}{v}})\,\hat{\underset{\sim}{v}}$; $\underset{\sim}{v}_{\perp} = (\underset{\sim}{u}.\hat{\underset{\sim}{v}})\,\hat{\underset{\sim}{v}}$

vertical line test: a test that determines whether a graph is that of a function; any vertical line that cuts the graph of a function does so exactly once

weight: a vector quantity equal to the mass of an object multiplied by the acceleration due to gravity; $\underset{\sim}{W} = m\underset{\sim}{g}$; weight of mass m in gravitational field $\underset{\sim}{g}$

zero matrix: a square matrix that consists entirely of '0' elements

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Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
1	6.5			1
2	1			2
3	4			2
4	3			3
5	5			3
6	4.5			3
7		3		1
8		5		3
9			5	3
10			3	1, 3
Totals	**24**	**8**	**8**	
Percentage	**60%**	**20%**	**20%**	

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
1	6.5			2
2	7.5			1
3	4			3
4	3.5			3
5	4			2
6		3.5		2
7		5		1
8			3.5	3
9			3	2
10			2.5	2
Totals	**25.5**	**8.5**	**9**	
Percentage	**59%**	**20%**	**21%**	

Examination marks summary

Question number	Simple familiar (SF)	Complex familiar (CF)	Complex unfamiliar (CU)	Topics
Paper 1				
1	6			Unit 1, Topic 1
2	4			Unit 1, Topic 2
3	4			Unit 1, Topic 3
4	4			Unit 2, Topic 1
5	5			Unit 2, Topic 2
6	7			Unit 2, Topic 3
7		6		Unit 2, Topic 2
8			4	Unit 1, Topic 3
Paper 2				
1	5			Unit 2, Topic 3
2	5			Unit 2, Topic 1
3	5			Unit 1, Topic 1
4		5		Unit 2, Topic 2
5		4		Unit 1, Topic 2
6			5	Unit 2, Topic 2
7			3	Unit 1, Topic 1
8			3	Unit 2, Topic 2
Totals	**45**	**15**	**15**	
Percentage	**60%**	**20%**	**20%**	