MATHS Quest

YEAR 12

MATHS B

FOR QUEENSLAND

SECOND EDITION

jacaranda Maths Quest

eBook plus
www.jacplus.com.au

Nick **Simpson** Robert **Rowland**

MATHS Quest

YEAR 12

MATHS B

FOR QUEENSLAND

SECOND EDITION

jacaranda Maths Quest

MATHS Quest

YEAR 12

MATHS B

FOR QUEENSLAND

SECOND EDITION

Nick **Simpson**
Robert **Rowland**

jacaranda plus

Second edition published 2010 by
John Wiley & Sons Australia, Ltd
42 McDougall Street, Milton, Qld 4064

First edition published 2003

Typeset in 10.5/12.5 pt Times

National Library of Australia
Cataloguing-in-Publication data

Author: Simpson, N. P. (Nicholas Patrick), 1957–
Title: Maths quest year 12 maths B for Queensland / Nick Simpson and Robert Rowland.
Edition: 2nd ed.
ISBN: 978 1 7421 6037 5 (pbk.)
978 1 7421 6174 7 (web.)
Series: Maths Quest series
Notes: Includes index.
Target audience: For secondary school age.
Subjects: Mathematics — Textbooks.
Mathematics — Problems, exercises, etc.
Other authors/contributors: Rowland, Robert, 1963–

Dewey Number: 510

Cover photograph and internal design images: © Digital Vision

Typeset in India by Aptara

Printed in Singapore by
C.O.S. Printers Pte Ltd

10 9 8

Contents

CHAPTER 10

▶ The normal distribution 373

Introduction

Maths Quest Maths B Year 12 for Queensland 2nd edition is one of the exciting *Maths Quest* resources specifically designed for the Queensland senior mathematics syllabuses beginning in 2009. It has been written and compiled by practising Queensland Maths B teachers. It breaks new ground in mathematics textbook publishing.

This resource contains:
- a student textbook with accompanying student website (eBookPLUS)
- a teacher edition with accompanying teacher website (eGuidePLUS)
- a solutions manual containing fully worked solutions to all questions contained in the student textbook.

Student textbook

Full colour is used throughout to produce clearer graphs and headings, to provide bright, stimulating photos and to make navigation through the text easier.

Clear, concise *theory sections* contain *worked examples, graphics calculator tips* and *highlighted important text* and *remember boxes*.

Worked examples in a Think/Write format provide clear explanation of key steps and suggest how solutions can be presented.

Exercises contain many carefully graded skills and application problems, including multiple-choice questions. Cross-references to relevant worked examples appear beside the first 'matching' question throughout the exercises.

Investigations, often suggesting the use of technology, provide further discovery learning opportunities.

Each chapter concludes with a *summary* and *chapter review* exercise containing questions that help consolidate students' learning of new concepts.

As part of the chapter review, there is also a *Modelling and problem solving* section. This provides students with further opportunities to practise their skills.

Technology is fully integrated within the resource. To support the use of graphics calculators, instructions for two models of calculator are presented in worked examples and graphics calculator tips throughout the text. The two models of graphics calculator featured are the Casio *fx*-9860G AU and the TI-Nspire CAS. (Note that the screen shots shown in this text for the TI-Nspire CAS calculator were produced using OS 1.7. Screen displays may vary depending on the operating system in use.)

For those students using the TI-89 Titanium model of graphics calculator, an appendix containing matching instructions has been included at the back of the book.

Student website — eBookPLUS

The accompanying student website contains an electronic version of the entire student textbook plus the additional learning resources listed below.

WorkSHEETs — editable Word 97 documents that may be completed on screen, or printed and completed later.

SkillSHEETs — printable pages that contain additional examples and problems designed to help students revise required concepts.

Test Yourself activities — multiple-choice quizzes for students to test their skills after completing each chapter.

Teacher edition

The teacher edition textbook contains everything in the student textbook and more. To support teachers assisting students in the class, answers appear in red next to most questions in the exercises and investigations. Each chapter is annotated with relevant syllabus information.

Teacher website — eGuidePLUS

The accompanying teacher website contains everything in the student website plus the following resources:

- two tests per chapter (with fully worked solutions)
- fully worked solutions to WorkSHEETs
- a syllabus planning document
- assessment tasks (and answers)
- fully worked solutions to all questions in the student textbook.

Solutions manual

Maths Quest Maths B Year 12 for Queensland Solutions Manual contains the fully worked solutions to every question and investigation in the *Maths Quest Maths B Year 12 for Queensland 2nd edition* student textbook.

Fully worked solutions are available for all titles in the *Maths Quest for Queensland* senior series.

Maths Quest is a rich collection of teaching and learning resources within one package.

About eBookPLUS

Next generation teaching and learning

This book features eBookPLUS: an electronic version of the entire textbook and supporting multimedia resources. It is available for you online at the JacarandaPLUS website (**www.jacplus.com.au**).

Using the JacarandaPLUS website

To access your eBookPLUS resources, simply log on to **www.jacplus.com.au** using your existing JacarandaPLUS login and enter the registration code. If you are new to JacarandaPLUS, follow the three easy steps below.

Step 1. Create a user account

The first time you use the JacarandaPLUS system, you will need to create a user account. Go to the JacarandaPLUS home page (**www.jacplus.com.au**), click on the button to create a new account and follow the instructions on screen. You can then use your nominated email address and password to log in to the JacarandaPLUS system.

Step 2. Enter your registration code

Once you have logged in, enter your unique registration code for this book, which is printed on the inside front cover of your textbook. The title of your textbook will appear in your bookshelf. Click on the link to open your eBookPLUS.

Step 3. View or download eBookPLUS resources

Your eBookPLUS and supporting resources are provided in a chapter-by-chapter format. Simply select the desired chapter from the drop-down list. Your eBookPLUS contains the entire textbook's content in easy-to-use HTML. The student resources panel contains supporting multimedia resources for each chapter.

Once you have created your account, you can use the same email address and password in the future to register any JacarandaPLUS titles you own.

Using eBookPLUS references

eBookPLUS logos are used throughout the printed books to inform you that a multimedia resource is available for the content you are studying.

Searchlight IDs (e.g. **INT-0001**) give you instant access to multimedia resources. Once you are logged in, simply enter the searchlight ID for that resource and it will open immediately.

Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com/customer/portal/articles/463717**.

Troubleshooting

- Go to the JacarandaPLUS help page at **www.jacplus.com.au/jsp/help.jsp**.
- Contact John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
  Phone: 1800 JAC PLUS (1800 522 7587)

Acknowledgements

The authors and publisher would like to thank the following copyright holders, organisations and individuals for their assistance and for permission to reproduce copyright material in this book.

Illustrative material

• © Corbis Australia, pp. **45**, **59**, **96**, **307**; • © Coo-ee Picture Library, pp. **84**(slide rule), **309**; • © Digital Stock/Corbis Corporation, pp. **78**(iceberg), **159**, **162**, **410**; • © Digital Vision, p. **42**; • © Fairfax Photo Library, p. **131**/Gary Medlicott; • © Getty Images, pp. **158**/Andy Whale, **205**/Marcus Lyon, **315**/Photographer's Choice/Craig van der Lende, **349**/Photographer's Choice/Craig van der Lende, **373**/The Image Bank/John & Lisa Merrill, **374**/UpperCut Images/Keith Brofsky, **406**(schoolgirl)/First Light/Huy Lam; • © Image Disk Photography, p. **370**; • © John Wiley & Sons Australia, pp. **84**(calculator)/Renee Bryon, **84**(table), **89**(car)/Kari-Ann Tapp, **336**; • © Photodisc, pp. **43**, **70**(comet), **70**(runner), **72**, **78**(bee), **87**, **94**, **106**, **117**, **126**, **130**, **131**(scuba diver), **133**(hands on face), **134**(box), **134**(radioactive), **149**(bacteria), **149**(coffee), **151**, **152**, **154**(hands on face), **161**, **176**, **191**, **200**, **201**, **208**, **219**, **245**, **259**, **323**, **325**, **357**, **362**, **386**, **390**, **400**(nails), **400**(child), **406**(die), **411**(tyres), **411**(lamp), **412**, **415**; • © Photolibrary, pp. **62**(workers)/Index Stock Imagery/Ed Lallo, **62**(family)/F1 Online/Paul Paul, **81**/Peter J. Robinson, **84**(computer)/Flirt Collection/Chris Collins, **89**(petri dishes)/Phillip Hayson, **103**/SPL/Dr Gary Settles, **114**/Peter J. Robinson, **196**/Chris Cheadle, **199**/Japan Travel Bureau, **207**/Dennis Olson, **232**/Stephanie Adams, **235**/Glow Images, Inc., **246**/Glow Images, Inc., **273**/Simon Fraser, **275**/Superstock/ Martin Child, **306**/Visions LLC, **308**/Superstock/Martin Child, **316**/Commercial Eye/Iconica, **348**/Flirt Collection /Lucidio Studio Inc., **356**/Tips Italia /Wesley Hitt, **391**(meat)/Hot Stock; • © PureStock, p. **12**; • © Stockbyte, pp. **133**(shocked), **133**(excited), **133**(worried), **133**(happy), **154**(shocked), **154**(excited), **154**(worried), **154**(happy), **391**(fishing), **401**; • © Tony Intihar, p. **125**(children); • © Viewfinder Australia Photo Library, p. **272**.

Screenshots

• Screenshots from TI-89 Titanium and TI-Nspire CAS reproduced with permission from Texas Instruments • Screenshots from Casio *fx*-980G AU reproduced with permission from CASIO.

Software

Microsoft® Excel
Screenshots reproduced by permission of Microsoft Corporation.
Note: Microsoft Software has been used only in screenshots.
Screenshots appear on pages 140, 248.
Microsoft Excel is a registered trademark of the Microsoft Corporation in the United States and/or other countries.

Every effort has been made to trace the ownership of copyright material. Information that will enable the publisher to rectify any error or omission in subsequent reprints will be welcome. In such cases, please contact the Permissions Section of John Wiley & Sons Australia.

1

Modelling change and rates of change

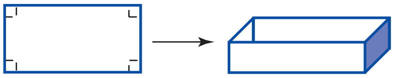
syllabus reference

Introduction to functions
Rates of change

In this chapter

Introduction

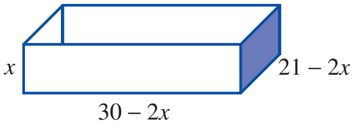
Aaron's box



Aaron was seated at his desk about to begin an essay that was due to be handed in on the following day. He had writer's block. He stared at the A4 paper in front of him and waited for inspiration. Aaron took the A4 paper and began to cut squares from each corner. He folded the paper and made an open box.

He began to wonder — as any good mathematician would — if the size of the squares cut from the original piece of A4 paper were changed, would the volume change as well? How does the volume relate to the size of the square removed?

This situation can be modelled by letting the square removed from each corner have side length x. As A4 paper measures 21 cm by 30 cm, the box has the following dimensions.



The volume of the box, V, is given by

$$V = \text{Area of base} \cdot \text{height}$$
$$= (30 - 2x)(21 - 2x)x$$

This relationship between volume and x can be thoroughly investigated using this equation.

Before continuing, however, we shall review and in some cases extend some of the techniques for dealing with equations such as this.

Using functions to model change

In this section we shall look at functions and function notation, and examine how functions can be used to model change.

Recall that a *function* is a relation that links a number, x, with another number, y. We write $y = f(x)$ and say that y is a function of x.

In the example of Aaron's box given above, the volume, V, of the box is a function of the side length, x, of the square. This is written:

$$V(x) = (30 - 2x)(21 - 2x)x.$$

The function $V(x)$ is called a *polynomial function* because it involves only positive powers of x.

The functions $G(t) = \sin(t)$ or $f(d) = \dfrac{1}{d^2}$ are not polynomial functions.

Functions can be combined. Consider the following worked example.

WORKED Example 1

If $f(x) = \frac{2}{x}$ and $g(x) = (1 + x)^2$:

a **calculate $f[g(2)]$**

b **write an expression for $f[g(x)]$**

c **calculate $g[f(2)]$**

d **write an expression for $g[f(x)]$.**

THINK	WRITE
a 1 Begin by evaluating $g(2)$. 2 Then calculate $f(9)$.	**a** $g(2) = (1 + 2)^2 = 9$ $f(9) = \frac{2}{9}$ Therefore $f[g(2)] = \frac{2}{9}$.
b Repeat the process used in **a** with x instead of 2.	**b** $g(x) = (1 + x)^2$ $f[(1 + x)^2] = \frac{2}{(1+x)^2}$ Therefore $f[g(x)] = \frac{2}{(1+x)^2}$.
c 1 Begin by evaluating $f(2)$. 2 Then calculate $g(1)$.	**c** $f(2) = \frac{2}{2} = 1$ $g(1) = (1 + 1)^2 = 4$ Therefore $g[f(2)] = 4$.
d Repeat the process used in **c** with x instead of 2.	**d** $f(x) = \frac{2}{x}$ $g\left(\frac{2}{x}\right) = \left(1 + \frac{2}{x}\right)^2$ Therefore $g[f(x)] = \left(1 + \frac{2}{x}\right)^2$.

Another concept associated with a function, $f(x)$, is the *inverse*, which is written $f^{-1}(x)$.

$$\text{If } y = f(x) \text{ then } x = f^{-1}(y).$$

It should be noted that to be precise in the use of functions and their inverses we must be careful in the use of the domain and range of the function. For example, if $f(x) = x^2$, then clearly $f(3) = 9$ and $f(-3) = 9$.

What is the value of $f^{-1}(9)$? Should $f^{-1}(9) = 3$ or -3?

In this chapter, we consider only general questions involving the inverse function and not the intricacies of restricted range and domain.

WORKED Example 2

If $f(x) = 3x - 2$,
a calculate $f^{-1}(4)$ **b give an expression for $f^{-1}(y)$.**

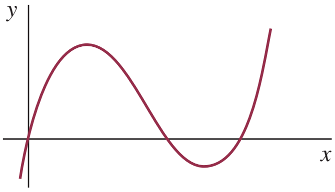
THINK	WRITE
a What value of x will give $f(x) = 4$?	**a** $f(x) = 4$ $3x - 2 = 4$ $x = 2$ Therefore $f^{-1}(4) = 2$.
b ① What value of x will give $f(x) = y$?	**b** $f(x) = y$ $3x - 2 = y$
② Make x the subject of the equation, then $x = f^{-1}(y)$.	$3x = y + 2$ $x = \dfrac{y+2}{3}$ Therefore $f^{-1}(y) = \dfrac{y+2}{3}$.

The graph of a function $y = f(x)$ provides a powerful representation for understanding the relationship between x and y. The ideal tool for drawing graphs is the graphics calculator.

Again consider the function $V(x) = (30 - 2x)(21 - 2x)x$. We can sketch the graph of the polynomial $V(x)$ and use this to understand the relationship between V and x. There are numerous questions we may choose to ask:

What is the volume, V, when $x = 4$ cm?
What is the value of x when $V = 100$ cm^3?
What is the maximum value of the volume, V?

A graphics calculator provides the easiest means of answering these questions.



WORKED Example 3

If $V(x) = (30 - 2x)(21 - 2x)x$, calculate the value of V when $x =$
a 5 cm **b 6 cm** **c 6.5 cm.**

THINK **WRITE/DISPLAY**

For the Casio *fx*-9860G AU

① To calculate the given values of V, press:
- MENU
- 7 (TABLE).

Complete the entry line as:
$Y1 = (30 - 2x)(21 - 2x)x$
then press EXE.

```
Table Func   :Y=
Y1=(30-2X)×(21-2[—]
Y2:            [—]
Y3:            [—]
Y4:            [—]
Y5:            [—]
Y6:            [—]
```

② To set the table range, press:
- F5 (SET).

Enter the values as shown, then press EXE.

```
Table Setting
X

 Start:5
 End  :50
 Step :0.5
```

THINK | **WRITE/DISPLAY**

③ To display the table, press:
- (F6) (TABLE).

X	Y1	Y'1
5	1100	-90
5.5	1045	-129
6	972	-162
6.5	884	-189

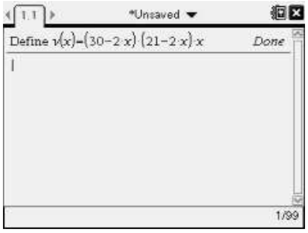
④ Write the answers.

a $V(5) = 1100\text{ cm}^3$
b $V(6) = 972\text{ cm}^3$
c $V(6.5) = 884\text{ cm}^3$.

For the TI-Nspire CAS

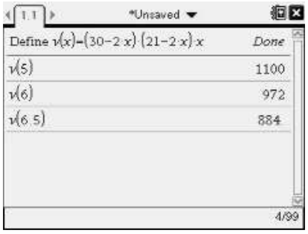
① To calculate the given values of V, open a Calculator page.
Press:
- MENU (menu)
- 1: Actions (1)
- 1: Define (1).

Complete the entry line as:
Define $V(x) = (30 - 2x) \cdot (21 - 2x) \cdot x$
then press ENTER (enter).



Define v(x)=(30−2·x)·(21−2·x)·x Done

② Complete the entry lines as:
$V(5)$
$V(6)$
$V(6.5)$
pressing ENTER (enter) after each line.



Define v(x)=(30−2·x)·(21−2·x)·x Done
v(5) 1100
v(6) 972
v(6.5) 884

③ Write the answers.

a $V(5) = 1100\text{ cm}^3$
b $V(6) = 972\text{ cm}^3$
c $V(6.5) = 884\text{ cm}^3$.

WORKED Example 4

If $V(x) = (30 - 2x)(21 - 2x)x$, calculate the value(s) of x when $V = 900\text{ cm}^3$ for x lying between 0 and 15.

THINK | **WRITE/DISPLAY**

For the Casio *fx*-9860G AU

① To solve $V(x) = 900$ for $0 \le x \le 15$, press:
- (MENU)
- 5 (GRAPH).

Complete the entry line as:
$Y1 = (30 - 2x)(21 - 2x)x$
then press (EXE).

```
Graph Func  :Y=
Y1■(30-2X)×(21-2[—]
Y2:            [—]
Y3:            [—]
Y4:            [—]
Y5:            [—]
Y6:            [—]
```

Continued over page

THINK

WRITE/DISPLAY

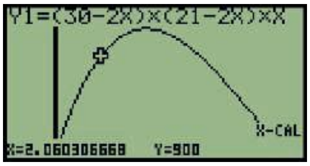
② To set the V-Window for an appropriate domain and range, press:
- SHIFT
- F3 (V-Window).

Enter the values as shown and then press EXE.

```
View Window
Xmin  :-2
 max  :11
 scale:1
 dot  :0.1031746
Ymin  :-10
 max  :1300
```

③ To sketch the graph and find the values for x, press:
- F6 (DRAW)
- SHIFT
- F5 (G-Solv)
- F6
- F2 (X-CAL).

Enter the Y Value as 900 and then press EXE. Use the right arrow to find the other value of x.



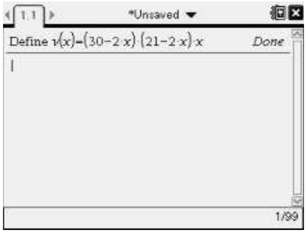
```
Y1=(30-2X)×(21-2X)×X
X=2.060306668     Y=900
```

④ Write the answer.

Solving $V(x) = 900$, for $0 \le x \le 15$
$x = 2.06031$ or $x = 6.4144$

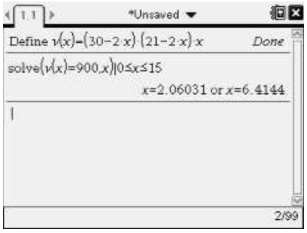
For the TI-Nspire CAS

① To solve $V(x) = 900$ for $0 \le x \le 15$, on a Calculator page, complete the entry line as:
Define $V(x) = (30 - 2x) \cdot (21 - 2x) \cdot x$
then press ENTER.



```
Define v(x)=(30-2·x)·(21-2·x)·x          Done
```

② Complete the entry line as:
solve $(V(x) = 900, x)|0 \le x \le 15$.



```
Define v(x)=(30-2·x)·(21-2·x)·x          Done
solve(v(x)=900,x)|0≤x≤15
                     x=2.06031 or x=6.4144
```

③ Write the answer.

Solving $V(x) = 900$, for $0 \le x \le 15$
$x = 2.06031$ or $x = 6.4144$

WORKED Example 5

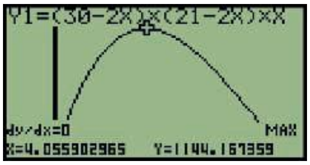
If $V(x) = (30 - 2x)(21 - 2x)x$, calculate the value of x between 0 and 10.5 which maximises the value of the volume.

THINK

WRITE/DISPLAY

For the Casio *fx*-9860G AU

① To find the value of x for which $V(x)$ is a maximum, where $0 \le x \le 10.5$, sketch the graph with an appropriate window and then press:
- SHIFT
- F5 (G-Solv)
- F2 (MAX).



```
Y1=(30-2X)×(21-2X)×X
dy/dx=0
X=4.055902965     Y=1144.167359
```

THINK

WRITE/DISPLAY

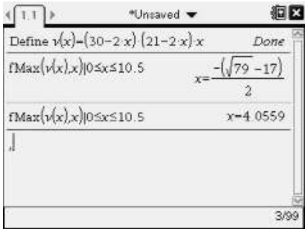
② Write the answer.

The maximum value of $V(x)$ for $0 \le x \le 10.5$ is 1144.17 cm^3 when $x = 4.0559$ cm.

For the TI-Nspire CAS

① To find the value of x for which $V(x)$ is a maximum, where $0 \le x \le 10.5$, on a Calculator page, complete the entry line as:
Define $V(x) = (30 - 2x) \cdot\ (21 - 2x) \cdot\ x$
then press ENTER (enter).
Complete the entry line as:
fMax($v(x)$, x)|$0 \le x \le 10.5$
then press ENTER (enter).
To obtain an approximate value for the exact value given, press:

- Ctrl (ctrl)
- ENTER (enter).

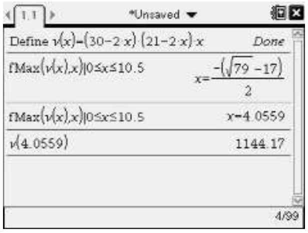


Define $v(x)=(30-2\cdot x)\cdot(21-2\cdot x)\cdot x$ *Done*

fMax($v(x)$,x)|$0\le x\le 10.5$ $x=\frac{-(\sqrt{79}-17)}{2}$

fMax($v(x)$,x)|$0\le x\le 10.5$ $x=4.0559$

② To find the maximum value of $V(x)$ when $x = 4.0559$, complete the entry line as:
$V(4.0559)$
then press ENTER (enter).



Define $v(x)=(30-2\cdot x)\cdot(21-2\cdot x)\cdot x$ *Done*

fMax($v(x)$,x)|$0\le x\le 10.5$ $x=\frac{-(\sqrt{79}-17)}{2}$

fMax($v(x)$,x)|$0\le x\le 10.5$ $x=4.0559$

$v(4.0559)$ 1144.17

③ Write the answer.

The maximum value of $V(x)$ for $0 \le x \le 10.5$ is 1144.17 cm^3 when $x = 4.0559$ cm.

WORKED Example 6

Find the intersection of the curves $y = x^3 - 3x^2 + 8$ and $y = 2x + 7$.

THINK

WRITE/DISPLAY

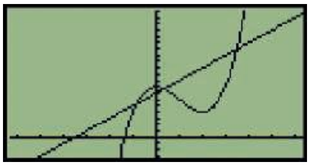
For the Casio *fx*-9860G AU

① To find the intersection of the curves, press:

- (MENU)
- 5 (GRAPH).

Complete the entry line as:
$Y1 = x^3 - 3x^2 + 8$
$Y2 = 2x + 7$
pressing (EXE) after each entry.
Adjust the V-Window if necessary and press:

- (F6) (DRAW).

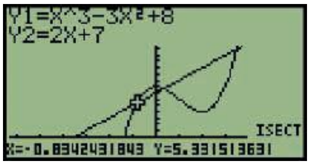


Continued over page ➔

THINK | **WRITE/DISPLAY**

② To identify the points of intersection, press:
- SHIFT
- G-Solv
- F5 (ISCT).

To find further points of intersection, press the right arrow.

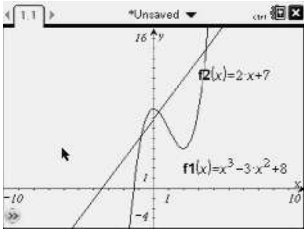


③ Write the answer.

The intersections of the curves $y = x^3 - 3x^2 + 8$ and $y = 2x + 7$ occur at (−0.83, 5.33), (0.34, 7.69) and (3.49, 13.98).

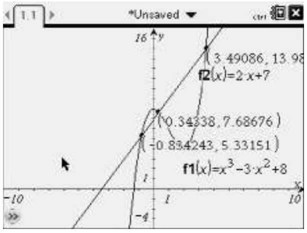
For the TI-Nspire CAS

① To sketch the curves, on a Graphs page, complete the entry lines as:
$f1(x) = x^3 - 3x^2 + 8$
$f2(x) = 2x + 7$
pressing ENTER enter after each line.



② To find the intersection of the curves, press:
- MENU menu
- 7: Points and Lines 7
- 3: Intersection Point(s) 3.

Use the NavPad to move the pointer and press Click once on each graph.



③ Write the answer.
Note: The TI-Nspire CAS calculator can find these points of intersection algebraically. On a Calculator page, complete the entry line as:
solve($x^3 - 3x^2 + 8 = 2x + 7, x$)
then press ENTER enter.

The intersections of the curves $y = x^3 - 3x^2 + 8$ and $y = 2x + 7$ occur at (−0.83, 5.33), (0.34, 7.69) and (3.49, 13.98).

remember

1. If $y = f(x)$, then the inverse of $f(x)$ is written as $f^{-1}(y)$ and $x = f^{-1}(y)$.
2. In analysing functions, the graphics calculator can:
   - (a) graph a function
   - (b) generate a table of values
   - (c) find values of x, given y
   - (d) find the maximum value on an interval
   - (e) find the intersection between curves.

EXERCISE 1A Using functions to model change

WORKED Example 1

1 If $f(x) = x^2 + 1$ and $g(x) = 3x - 5$:

a calculate $f[g(2)]$

b write an expression for $f[g(x)]$

c calculate $g[f(2)]$

d write an expression for $g[f(x)]$.

2 If $r(t) = (1 + t)^2$ and $s(t) = t^2 + 1$:

a calculate $r[s(2)]$

b calculate $s[r(2)]$

c find $s[r(t)]$

d find $r[s(t)]$.

WORKED Example 2

3 If $f(x) = 4 - 2x$,

a calculate $f^{-1}(-6)$

b give an expression for $f^{-1}(y)$.

4 If $g(x) = \frac{2}{x}$,

a calculate $g(3)$

b calculate $g^{-1}(3)$

c find an expression for $g^{-1}(y)$.

WORKED Example 3

5 For the polynomial $P(x) = x^3 - 3x^2 + 8$, calculate the values of:

a $P(4)$

b $P(4.5)$

c $P(5)$.

WORKED Example 4

6 For the polynomial $P(x) = x^3 - 3x^2 + 8$, calculate the values of x for which:

a $P(x) = 15$

b $P(x) = -4$.

WORKED Example 5

7 For the polynomial $P(x) = x^3 - 3x^2 + 8$, calculate the values of x for which:

a $P(x)$ is a maximum between $x = -1$ and $x = 1$

b $P(x)$ is a minimum between $x = 0$ and $x = 4$.

WORKED Example 6

8 Find the coordinates of the intersection of $y = x^2 - 3x$ and $y = -x^2 + x + 4$.

9 The height, $H(t)$, of a ball thrown vertically upwards is given by: $H(t) = 40t - 5t^2$, where t is the time in seconds.

a Complete the table at right, which gives the height of the ball for various times.

b At what times is the ball 10 m above the ground?

t	$H(t)$
4	
5	
6	
7	

10 As it leaves a bus stop, a bus accelerates so that its distance from the stop is:

$$d(t) = 0.22t^2 \text{ for } t < 10 \text{ seconds.}$$

Larry is trying to catch the bus, but is 40 m from the bus stop as the bus leaves. He chases the bus, running at 7.5 m/s.

a Write an expression for Larry's distance from the bus stop at time t.

b Graph both distances using the graphics calculator.

c Does Larry ever catch the bus?

11 The depth of water over a rock shoal in a river is given by the formula:

$$d(t) = 2 \sin\left(\frac{\pi}{12}t + 1\right) + 3.1$$

where t is in radians and gives time after midnight.

A boat will hit the rocks if the depth of water over the shoal is less than 2.5 m.

a Draw the graph of d for a period of 24 hours.

b For what times, in this 24 hours, is the water over the shoal too shallow for the boat to travel over it safely?

Graphing polynomial functions

eBook plus

Interactivity:
Simultaneous quadratic and linear equations
int-0261

A graphics calculator is certainly useful when it comes to drawing a graph and getting information from the graph. However, as anyone who has tried to graph a function using a graphics calculator will tell you, it is not always easy to have the view window set appropriately and have the relevant features of the graph on screen.

It is useful to have a general idea of what the graph ought to look like. This will help you to find a suitable view window to look at the graph, and you will be able to identify and correct any errors that you make during the process. Further, a number of modelling contexts require you to have an understanding of the features and trends in a graph — not simply the specific information given by one particular graph. For these reasons we shall now examine in some detail the graphs of polynomial functions.

Quadratic graphs

Revision of quadratic functions

1. The general form of the quadratic function is $y = ax^2 + bx + c, x \in \text{R}$.
2. The graph of a quadratic function is called a parabola and:
   (a) for $a > 0$, the graph has a minimum value
   (b) for $a < 0$, the graph has a maximum value
   (c) the y-intercept is c
   (d) the turning point has an x-coordinate of $x = \dfrac{-b}{2a}$
   (e) the x-intercepts are found by solving the equation $ax^2 + bx + c = 0$.
3. The equation $ax^2 + bx + c = 0$ can be solved by either:
   (a) factorising and using the Null Factor Law
   or
   (b) using the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

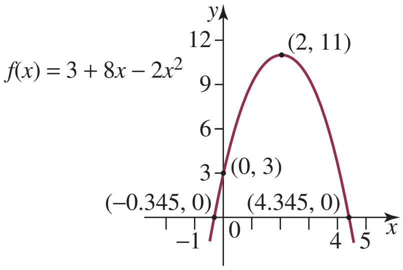
eBook plus

Digital doc:
SkillSHEET 1.1
Solving quadratic equations using the quadratic formula

WORKED Example 7

Sketch the graph of $y = 3 + 8x - 2x^2$, showing the turning point and all intercepts.

THINK	WRITE
1 To find the y-intercept, find y when $x = 0$.	When $x = 0$, $y = 3$
2 State the y-intercept.	The y-intercept is 3.
3 To find the x-intercepts, let the quadratic equal zero.	When $y = 0$, $3 + 8x - 2x^2 = 0$
4 Solve for x using the quadratic formula.	$x = \dfrac{-8 \pm \sqrt{8^2 - 4(-2)(3)}}{2(-2)}$ $= \dfrac{-8 \pm \sqrt{88}}{-4}$ $= \dfrac{-8 \pm 2\sqrt{22}}{-4}$ $= \dfrac{-4 \pm \sqrt{22}}{-2}$ $= 2 - \dfrac{\sqrt{22}}{2}$ or $2 + \dfrac{\sqrt{22}}{2}$
5 Give approximate answers for the surds.	$x \approx -0.345$ or 4.345

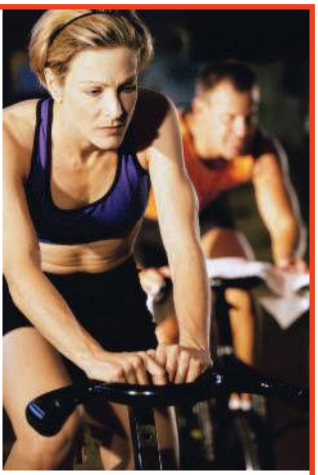
THINK	**WRITE**
6 State the x-intercepts.	The x-intercepts are -0.345 and 4.345.
7 Write the original rule using decreasing powers of x.	$y = -2x^2 + 8x + 3$
8 Find the x-coordinate of the turning point.	$x = \frac{-b}{2a}$ $= \frac{-8}{-4}$ $= 2$
9 Find the y-coordinate of the turning point.	$y = 3 + 8 \cdot 2 - 2 \cdot 2^2$ $= 11$
10 State the turning point.	The turning point is $(2, 11)$.
11 Draw a set of axes and mark the coordinates of the turning point and the points where the graph crosses the axes. 12 Sketch a parabola through these points.	

WORKED Example 8

If a graph has an equation of the form $y = x^2 + bx + c$ with a turning point at $(-2, -1)$, give the rule.

THINK	**WRITE**
1 (a) The general equation is $y = ax^2 + bx + c$.	The coefficient of x^2 is 1, so $a = 1$.
(b) The turning point is $(-2, -1)$.	x-coordinate of turning point $= -2$
(c) The x-coordinate of the turning point is $\frac{-b}{2a}$.	Therefore, $\frac{-b}{2a} = -2$ $\frac{-b}{2} = -2$ $-b = -4$ $b = 4$
2 Find c by substituting $(-2, -1)$ into the equation.	$y = x^2 + 4x + c$ $-1 = (-2)^2 + 4 \cdot -2 + c$ $-1 = 4 - 8 + c$ $-1 = -4 + c$ $c = 3$
3 Write the answer in a sentence.	Therefore the equation is $y = x^2 + 4x + 3$.

WORKED Example 9

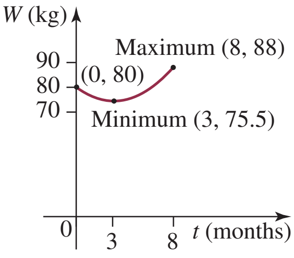


The weight (W) of a person t months after a gymnasium program is started is given by the function:

$$W(t) = \frac{t^2}{2} - 3t + 80, \text{ where } t \in [0, 8] \text{ and } W \text{ is in kilograms.}$$

Find:

a **the minimum weight of the person**
b **the maximum weight of the person.**

THINK	WRITE
1 Find the turning point.	$x = \frac{-b}{2a}$ $= \frac{3}{1}$ $= 3$ $y = \frac{3^2}{2} - 3 \cdot 3 + 80$ $= 75.5$
2 State the coordinates of the turning point.	The turning point is (3, 75.5).
3 Find the end point value for W when $t = 0$.	When $t = 0$ $W = \frac{0^2}{2} - 3(0) + 80$ $= 80$
4 State its coordinates.	One end point is (0, 80).
5 Find the end point value of W when $t = 8$.	When $t = 8$ $W = \frac{8^2}{2} - 3(8) + 80$ $= 88$
6 State its coordinates.	The other end point is (8, 88).
7 Draw a set of axes and mark the points on it. 8 Sketch a parabola between the end points. 9 Locate the maximum and minimum values of W on the graph.	
a State the minimum weight from the graph.	**a** The minimum weight is 75.5 kg.
b State the maximum weight from the graph.	**b** The maximum weight is 88 kg.

Cubic graphs

Cubic functions are polynomials of degree 3. In this section, we will look at how graphs of cubic functions may be sketched by finding intercepts and recognising basic shapes.

Forms of cubic functions

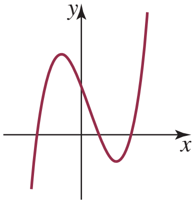
Cubic functions may take several forms. The three main forms are described below.

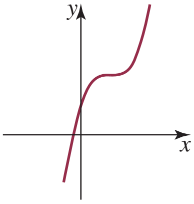
General form

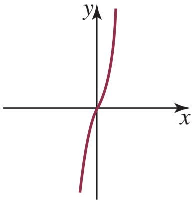
The general form of a cubic function is

$$y = ax^3 + bx^2 + cx + d$$

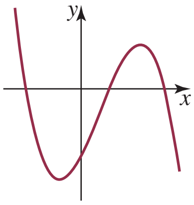
If a is positive (that is, $a > 0$), the function is called a *positive cubic*. Several positive cubics appear below.

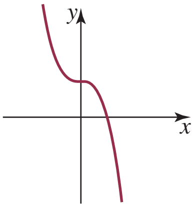




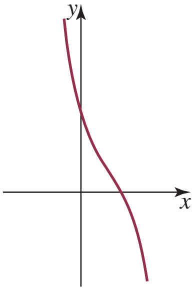


If a is negative (that is, $a < 0$), the function is called a *negative cubic*. Several negative cubics appear below.





You may wish to investigate in more detail the type of equation required to produce each of the above graphs.

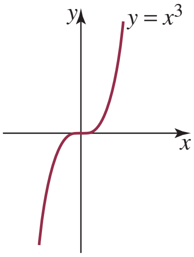


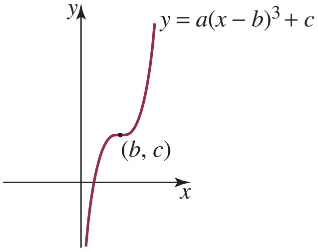
Basic form

Some (but certainly not all) cubic functions are transformations of the basic function $y = x^3$, and may be expressed in the form

$$y = a(x - b)^3 + c$$

For example, $y = 2(x - 3)^3 + 5$ is the graph of $y = x^3$ translated +3 in the x direction, +5 in the y direction, and dilated by a factor of 2 in the y direction.





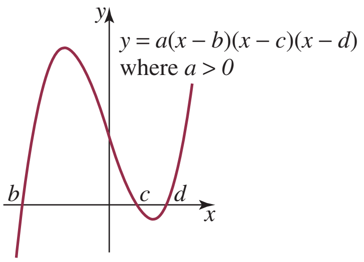
The point (b, c) is called a *point of inflection*.

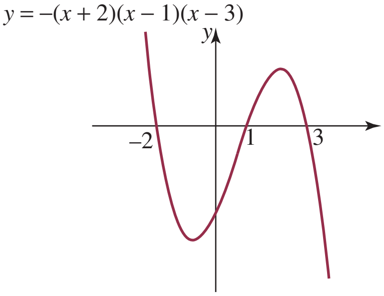
Factor form

Cubic functions of the type

$$y = a(x - b)(x - c)(x - d)$$

are said to be in factor form, where b, c and d are the x-intercepts. Often a cubic function in general form may be factorised to express it in factor form.

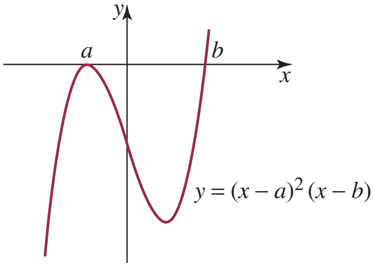




Repeated factors

A twice-only repeated factor in a factorised cubic function indicates a turning point that just touches the x-axis.

Verify this for several cases using a graphics calculator.



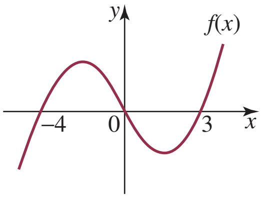
WORKED Example 10

For each of the following graphs, find the rule and express it in factorised form. Assume that $a = \pm 1$.

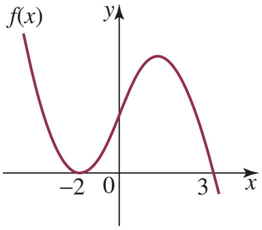
eBook*plus*

Tutorial:
Worked example 10
int-0519

a



b



THINK	WRITE
a ① Find a by deciding whether the graph is a positive or negative cubic.	**a** Positive cubic, so $a = 1$.
② Use the x-intercepts –4, 0 and 3 to find the factors.	The factors are $(x + 4)$, x and $(x - 3)$.
③ Express $f(x)$ as a product of a and its factors.	$f(x) = 1(x + 4)x(x - 3)$
④ Simplify.	$f(x) = x(x + 4)(x - 3)$

THINK		**WRITE**
b	1 Find a by deciding whether the graph is a positive or negative cubic.	**b** Negative cubic, so $a = -1$.
	2 Use the x-intercept, -2, which is also a turning point, to find the repeated factor.	$(x + 2)^2$ is a factor.
	3 Use the other x-intercept, 3, to find the other factor.	$(x - 3)$ is also a factor.
	4 Express $f(x)$ as a product of a and its factors.	$f(x) = -1(x + 2)^2(x - 3)$
	5 Simplify.	$f(x) = (3 - x)(x + 2)^2$

Quartic graphs

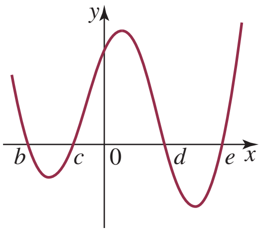
To draw the graph of a quartic function such as

$$y = x^4 - 3x^3 + x^2 - 4x + 3$$

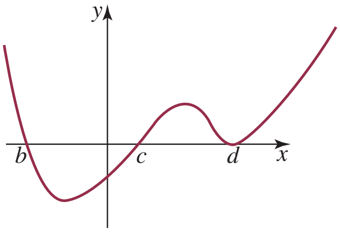
is very difficult and involves algebra beyond the scope of the syllabus. To understand the behaviour of a quartic function we will consider it in the form

$$y = a(x - b)(x - c)(x - d)(x - e).$$

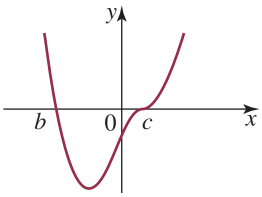
The following graphs are cases where a is positive.



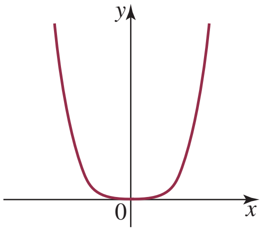
A factor may occur twice, for example $y = a(x - b)(x - c)(x - d)^2$.



A factor may occur three times, for example $y = a(x - b)(x - c)^3$.



A factor may occur four times, for example $y = a(x - b)^4$.

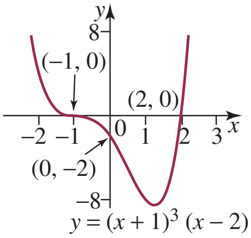


If a is negative, each of these graphs is reflected in the x-axis.

WORKED Example 11

Sketch the graph of $y = (x + 1)^3(x - 2)$.

THINK	WRITE
① State the function.	$y = (x + 1)^3(x - 2)$
② Find the y-intercept.	When $x = 0$, $y = (1)^3(-2)$ $= -2$
③ State the y-intercept.	The y-intercept is -2.
④ Find the x-intercept.	When $y = 0$, $(x + 1)^3(x - 2) = 0$
⑤ Solve for x.	$x = -1$ or 2
⑥ State the x-intercepts.	The x-intercepts are -1 and 2.
⑦ Sketch the graph. (The cubed factor indicates a point of inflection.)	There is a point of inflection at $(-1, 0)$.



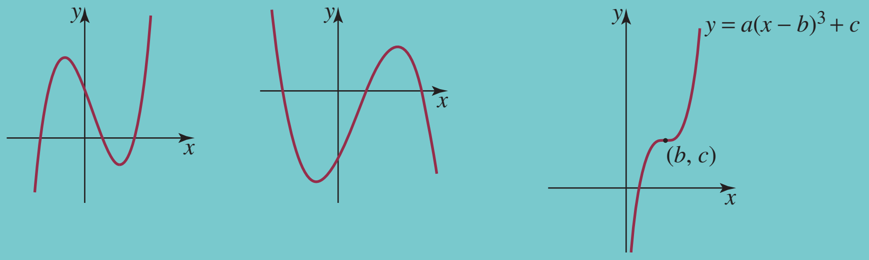
remember

Quadratic graphs

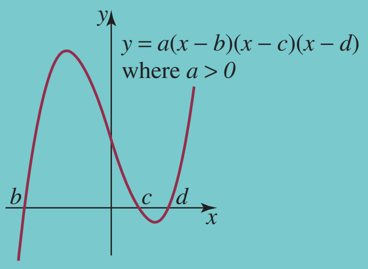
1. General equation is $y = ax^2 + bx + c$.
2. The quadratic formula is given by the equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
3. The x-coordinate of the turning point is $x = \dfrac{-b}{2a}$.

Cubic graphs

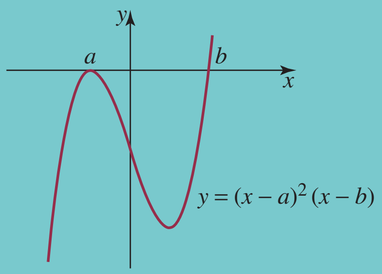
4. The general equation is $y = ax^3 + bx^2 + cx + d$.
5. Basic shapes of cubic graphs:


(a) Positive cubic (b) Negative cubic (c) Basic form

(d) Factor form



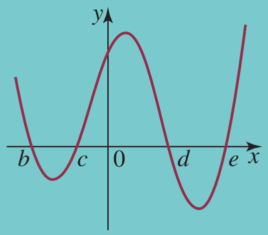
(e) Repeated factor



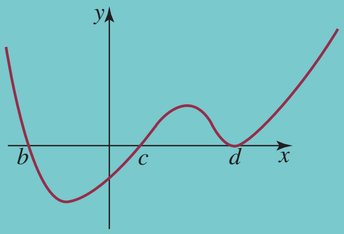
If $a < 0$, then the reflections through the x-axis of the types of graph in the above figures are obtained.

Quartic graphs

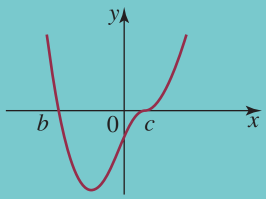
6. Basic shapes of quartic graphs:
   (a) Positive quartic in factor form
   $y = a(x - b)(x - c)(x - d)(x - e)$
   where $a > 0$.



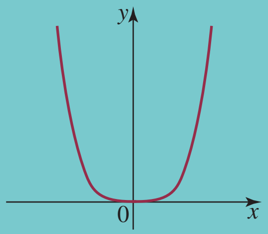
   (b) A factor may occur twice, for example
   $y = a(x - b)(x - c)(x - d)^2$.



   (c) A factor may occur three times, for example
   $y = a(x - b)(x - c)^3$.



   (d) A factor may occur four times, for example
   $y = a(x - b)^4$.



If a is negative, each of these graphs is reflected in the x-axis.

EXERCISE 1B Graphing polynomial functions

WORKED Example 7

1 Sketch the graphs of each of the following functions, showing all intercepts.

a $f(x) = x^2 - 6x + 8$ **b** $f(x) = x^2 + 6x + 8$
c $f(x) = x^2 - 5x + 4$ **d** $f(x) = 6 - x - x^2$
e $f(x) = 10 + 3x - x^2$ **f** $f(x) = 2x^2 + 5x - 3$
g $f(x) = 6x^2 - x - 12$ **h** $f(x) = 15 + x - 6x^2$

2 Find the turning point for each of the functions in question **1**.

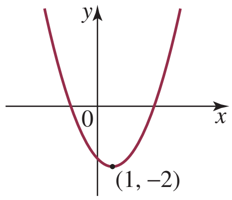
WORKED Example 8

eBook plus

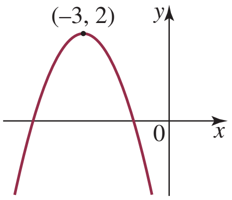
Digital docs:
SkillSHEET 1.2
Identifying domain and range for quadratic graphs
Spreadsheet
050 Quadratic graphs

3 Each of the functions graphed below is of the form $y = x^2 + bx + c$. For each function, give the rule.

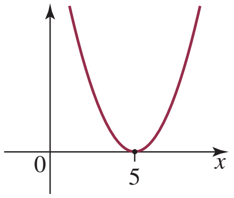
a



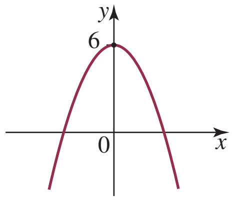
b



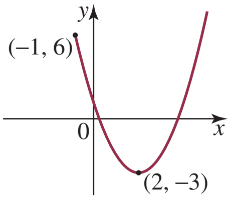
c



d



e



4 **multiple choice**

Consider the function with the rule $y = x^2 - 2x - 3$.

a It has x-intercepts:

A (1, 0) and (3, 0) **B** (–1, 0) and (3, 0) **C** (1, 0) and (–3, 0)
D (2, 0) and (–1, 0) **E** (0, –1) and (0, 3)

b It has a turning point with coordinates:

A (–1, 0) **B** (2, –3) **C** (1, –4) **D** (–1, –4) **E** (1, 0)

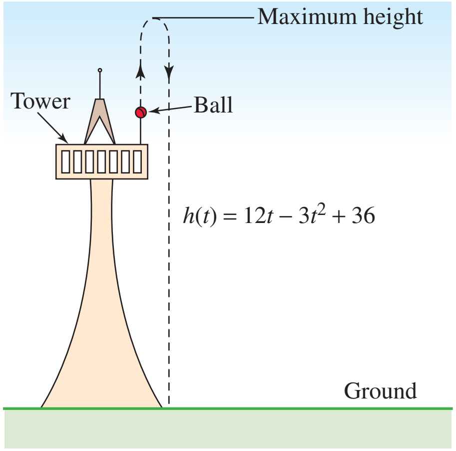
WORKED Example 9

5 The volume of water in a tank, V m^3, over a 10-month period is given by the function $V(t) = 2t^2 - 16t + 40$, where t is in months and $t \in [0, 10]$. Find:

a the minimum volume of water in the tank
b the maximum volume of water in the tank.

6 A ball thrown upwards from a tower attains a height above the ground given by the function $h(t) = 12t - 3t^2 + 36$, where t is the time in seconds and h is in metres. Find:

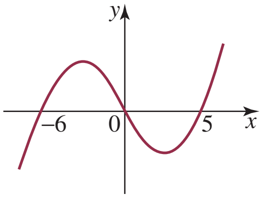
a the maximum height above the ground that the ball reaches
b the time taken for the ball to reach the ground.



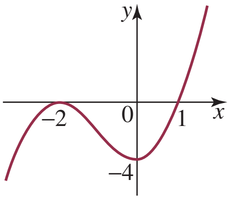
WORKED Example 10

7 For each of the following graphs, find the rule and express it in factorised form. Assume that $a = \pm 1$.

a



b

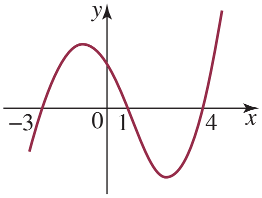


eBook plus

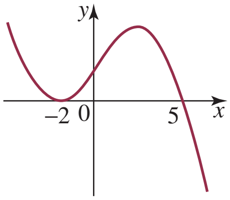
Digital doc:
Spreadsheet
007 Cubic graphs – factor form

8 Match each of the following graphs to the most appropriate rule below.

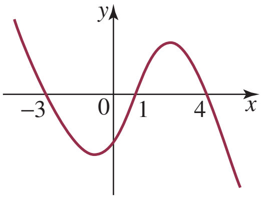
a



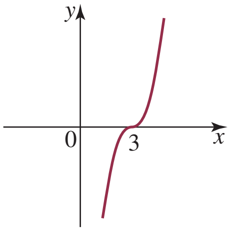
b



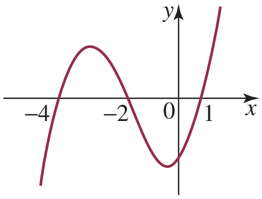
c



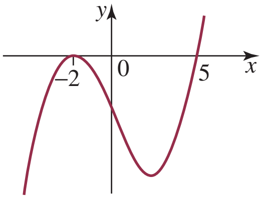
d



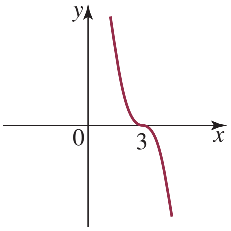
e



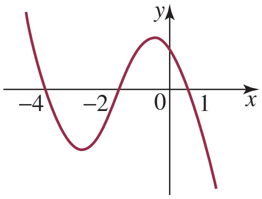
f



g



h



i $y = (x - 3)^3$

ii $y = (x + 3)(1 - x)(x - 4)$

iii $y = (x + 4)(x + 2)(1 - x)$

iv $y = (x + 2)^2(5 - x)$

v $y = (x + 3)(x - 1)(x - 4)$

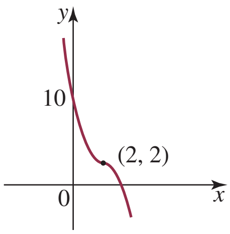
vi $y = (x + 4)(x + 2)(x - 1)$

vii $y = (3 - x)^3$

viii $y = (x + 2)^2(x - 5)$

9 **multiple choice**

The function graphed in the figure could have the following rule:



A $y = (x - 2)^3 + 2$

B $y = (x + 2)^3 + 2$

C $y = (2 - x)^3 + 2$

D $y = (x + 2)^3 - 2$

E $y = (x - 2)^3$

eBook*plus*

Digital docs:

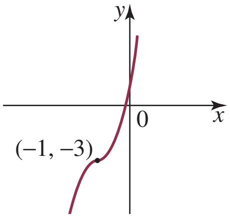
Spreadsheets
009 Cubic graphs – $y = a(x - b)^3 + c$ form
073 Quartic graphs – factor form

WorkSHEET 1.1

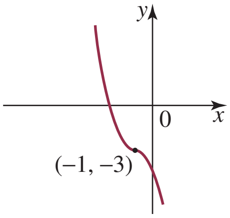
10 **multiple choice**

The graph of $f(x) = 5(x + 1)^3 - 3$ is best represented by:

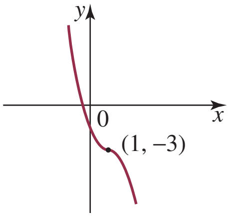
A



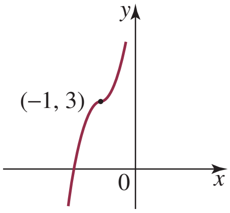
B



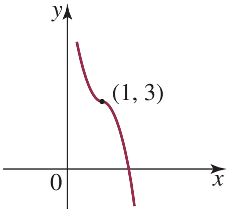
C



D



E



11 The function $f(x) = x^3 + ax^2 + bx - 64$ has x-intercepts (–2, 0) and (4, 0). Find the values of a and b.

12 The distance, d km, of a group of hikers from their starting point t hours after setting off on a hike can be modelled by the function with the rule:

$$d(t) = at^2(b - t)$$

The hikers are 3 km from the start after 2 hours and return to the starting point after 5 hours.

a Find the values of a and b.

b Hence, give the rule for $d(t)$.

c Sketch the graph of $d(t)$.

d Find to the nearest 100 metres the maximum distance of the hikers from their starting point and the time, to the nearest minute, that it occurs.

WORKED Example 11

13 Sketch the graph of each of the following functions.

a $y = x(x - 1)^3$

b $y = (2 - x)(x^2 - 4)(x + 3)$

c $y = x^4 - x^2$

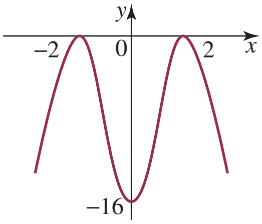
d $y = -(x - 2)^2(x + 1)^2$

e $y = (x + 2)^3(x - 3)$

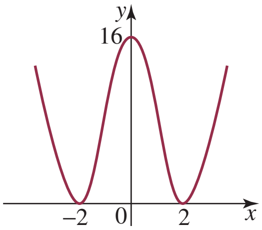
14 **multiple choice**

Consider the function $f(x) = (x - 2)^2(x + 2)^2$. The graph of $f(x)$ is best represented by:

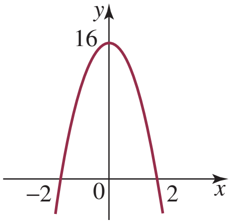
A



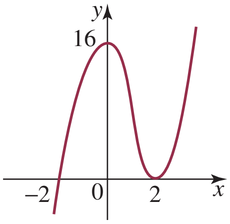
B



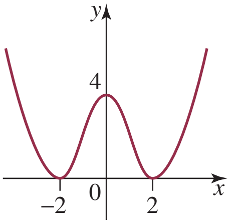
C



D



E



15 The function $f(x) = x^4 + ax^3 - 4x^2 + bx + 6$ has x-intercepts (2, 0) and (–3, 0). Find the values of a and b.

Quartics and beyond

Using your graphics calculator or other suitable technology, investigate the following.

1 Investigate graphs of functions of the form $f(x) = x^n$ for values of n from 4 to 9.

2 What do graphs of functions for which n is even have in common?

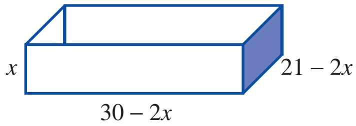
3 What does an odd value of n do to the graph?

4 Investigate graphs of functions of the form $y = (x - a)^n(x - b)^m(x - c)^p$ for various values of the pronumerals in the equation, for m, n and $p \leq 4$. Write a report on your findings.

Review of differentiation

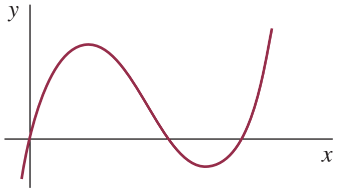
Aaron's box revisited

Let us return to our consideration of the volume of the open box created from a sheet of A4 paper with four corners removed. The side length of the square removed is x cm.



The volume of the box, V, is given by

$$\begin{aligned} V &= \text{area of base} \cdot \text{height} \\ &= (30 - 2x)(21 - 2x)x \end{aligned}$$



The graph of V versus x we have seen earlier.

One question of interest is to find a legitimate value of x that maximises the volume, V. We solved this earlier, in worked example 5, using a graphics calculator. We found that the maximum value of V is 1144.2 cm^3 when x = 4.1 cm^3.

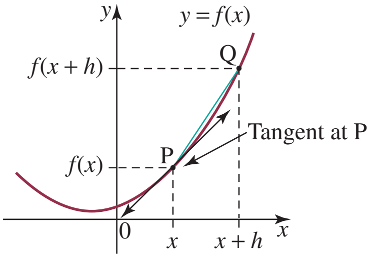
To solve this problem without using the power of a graphics calculator, we need to use a mathematical concept called *differentiation*. Differentiation, which you met in year 11, is part of a larger subject called *calculus*. Although the graphics calculator can provide a solution to a specific problem such as this, differentiation provides a more powerful way of investigating a wide range of problems in such diverse areas as population biology and space travel.

We begin with a review of differentiation from year 11.

The gradient function or derivative of *f*(*x*)

To find the gradient of the graph of $y = f(x)$ at the point P, we begin by considering the gradient of the straight line joining P to Q, called the *secant*.

$$\begin{aligned} \text{Gradient of secant} &= \frac{\text{rise}}{\text{run}} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



If the point Q moves closer and closer to P, we find the gradient of the tangent to the curve $y = f(x)$ at P is:

$$\text{Gradient at P} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

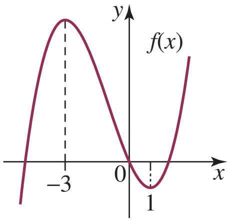
The gradient of $y = f(x)$ at the point x is given by a function called the *derived function* or the *derivative*, and is written as $f'(x)$ or $\frac{dy}{dx}$.

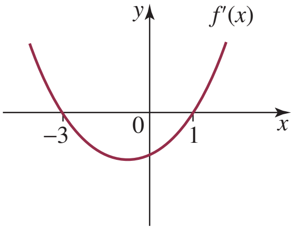
WORKED Example 12

eBookplus

Tutorial:
Worked example 12
int–0552

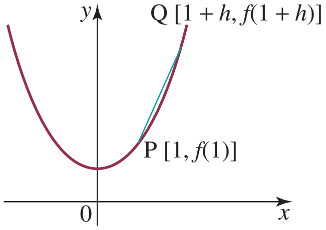
Sketch the gradient function of the following function and state its domain.



THINK	WRITE
1 Find when $f'(x) = 0$.	$f'(x) = 0$ if $x = -3$ and $x = 1$.
2 Find when $f'(x) > 0$.	$f'(x) > 0$ if $x < -3$ and $x > 1$.
3 Find when $f'(x) < 0$.	$f'(x) < 0$ if $-3 < x < 1$.
4 Sketch the graph of the gradient function.	
5 Find the domain by determining where $f(x)$ is smooth and continuous.	Domain is R.

WORKED Example 13

Find the gradient of the chord PQ drawn to the curve $f(x) = x^2 + 2$ in the diagram:
a by hand **b using the TI-Nspire CAS calculator.**



THINK	WRITE/DISPLAY
a Find the gradient and simplify.	**a**
$\text{Gradient} = \dfrac{\text{rise}}{\text{run}}$.	$\text{Gradient} = \dfrac{f(x+h) - f(x)}{h},\ h \neq 0$

$$= \frac{f(1+h) - f(1)}{h}$$

$$= \frac{(1+h)^2 + 2 - (1^2 + 2)}{h}$$

$$= \frac{3 + 2h + h^2 - 3}{h}$$

$$= \frac{2h + h^2}{h}$$

$$= \frac{h(2+h)}{h}$$

$$= 2 + h$$

The gradient is $2 + h$.

Continued over page

THINK

b Using the TI-Nspire CAS calculator

① On a Calculator page, press:
- MENU menu
- 1: Actions 1
- 1: Define 1.

Complete the entry line as:
Define $f(x) = x^2 + 2$
then press ENTER enter.

② To find the gradient of the chord PQ, complete the entry line as:

$$\frac{f(1+h)-f(1)}{h}$$

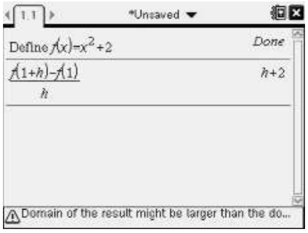
then press ENTER enter.

③ Write the answer.

WRITE/DISPLAY

b

```
Define f(x)=x^2+2                         Done
1/99
```



```
Define f(x)=x^2+2                         Done
(f(1+h)-f(1))/h                           h+2
Domain of the result might be larger than the do...
```

The gradient of the chord PQ is $2 + h$.

The definition of the derivative of $y = f(x)$ is given by:

If $y = f(x)$ then

$$\frac{dy}{dx} \text{ or } f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Using this definition to find the derivative of a function is called *differentiation from first principles* and, as the following example shows, it can be an involved process.

WORKED Example 14

Use first principles to differentiate $g(x) = x^2 - x$.

THINK

① Find $g(x + h)$ and simplify.

② Find $g'(x)$ using first principles.

WRITE

$$g(x+h) = (x+h)^2 - (x+h)$$
$$= x^2 + 2xh + h^2 - x - h$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}, h \neq 0$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h - (x^2 - x)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - h}{h}$$
$$= \lim_{h \to 0} \frac{h(2x + h - 1)}{h}$$
$$= \lim_{h \to 0} (2x + h - 1)$$
$$= 2x - 1$$

There is an easier method of calculating the derivatives of polynomial functions. It is called *differentiation by the rule*.

$$\text{If } y = ax^n \text{ then}$$
$$\frac{dy}{dx} = nax^{n-1}.$$

WORKED Example 15

Differentiate $y = x^4 - \frac{3}{2}x^2 + 7$.

THINK	WRITE
1 Write the equation.	$y = x^4 - \frac{3}{2}x^2 + 7$
2 Differentiate each of the three terms separately.	$\frac{dy}{dx} = 4x^{4-1} - \frac{3}{2}(2)x^{2-1} + 0$ $= 4x^3 - 3x$

WORKED Example 16

Find the derivative of:

i $f(x) = \frac{1}{x} + \frac{1}{\sqrt{x}}$ **ii** $f(x) = \frac{x + \sqrt{x}}{x^2}$:

a **by hand** **b** **using the TI-Nspire CAS calculator.**

THINK	WRITE/DISPLAY
a i 1 Write the equation.	**a i** $f(x) = \frac{1}{x} + \frac{1}{\sqrt{x}}$
2 Rewrite $\frac{1}{x}$ and $\frac{1}{\sqrt{x}}$ using negative indices.	$f(x) = x^{-1} + x^{-\frac{1}{2}}$
3 Differentiate each term.	$f'(x) = -x^{-1-1} - \frac{1}{2}x^{-\frac{1}{2}-1}$ $= -x^{-2} - \frac{x^{-\frac{3}{2}}}{2}$
4 Write the function in the form originally given.	$= -\frac{1}{x^2} - \frac{1}{2\sqrt{x^3}}$
ii 1 Write the equation.	**ii** $f(x) = \frac{x + \sqrt{x}}{x^2}$
2 Rewrite $\sqrt{x}$ using indices.	$= \frac{x + x^{\frac{1}{2}}}{x^2}$

Continued over page

THINK | **WRITE/DISPLAY**

③ Separate the function into two terms expressed in index form.

$$= \frac{x}{x^2} + \frac{x^{\frac{1}{2}}}{x^2}$$

④ Simplify each term.

$$= x^{-1} + x^{-\frac{3}{2}}$$

⑤ Differentiate each term.

$$f'(x) = -1x^{-1-1} - \frac{3}{2}x^{-\frac{3}{2}-1}$$

$$= -x^{-2} - \frac{3}{2}x^{-\frac{5}{2}}$$

⑥ Simplify $f'(x)$.

$$= -\frac{1}{x^2} - \frac{3}{2x^{\frac{5}{2}}}$$

$$= -\frac{1}{x^2} - \frac{3}{2\sqrt{x^5}}$$

b For the TI-Nspire CAS

① On a Calculator page, press:

- MENU (menu)
- 4: Calculus (4)
- 1: Derivative (1).

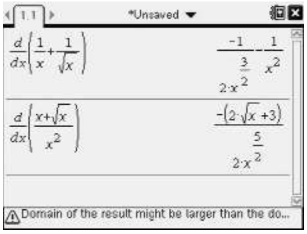
Complete the entry lines as:

$$\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{\sqrt{x}}\right)$$

$$\frac{d}{dx}\left(\frac{x + \sqrt{x}}{x^2}\right)$$

pressing ENTER (enter) after each line.

b



$\frac{d}{dx}\left(\frac{1}{x}+\frac{1}{\sqrt{x}}\right)$ $\quad \frac{-1}{2\cdot x^{\frac{3}{2}}}-\frac{1}{x^2}$

$\frac{d}{dx}\left(\frac{x+\sqrt{x}}{x^2}\right)$ $\quad \frac{-(2\cdot\sqrt{x}+3)}{2\cdot x^{\frac{5}{2}}}$

Domain of the result might be larger than the do...

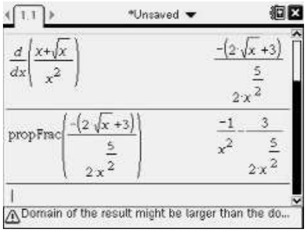
② To separate the answer to part **ii** into two separate fractions, press:

- MENU (menu)
- 2: Number (2)
- 7: Fraction Tools (7)
- 1: Proper Fraction (1).

Complete the entry line as:

$$\text{propfrac}\left(\frac{^{-}(2\sqrt{x}+3)}{2x^{\frac{5}{2}}}\right)$$

then press ENTER (enter).



$\frac{d}{dx}\left(\frac{x+\sqrt{x}}{x^2}\right)$ $\quad \frac{-(2\cdot\sqrt{x}+3)}{2\cdot x^{\frac{5}{2}}}$

$\text{propFrac}\left(\frac{-(2\cdot\sqrt{x}+3)}{2\cdot x^{\frac{5}{2}}}\right)$ $\quad \frac{-1}{x^2}-\frac{3}{2\cdot x^{\frac{5}{2}}}$

Domain of the result might be larger than the do...

③ Write the answer.

i $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{\sqrt{x}}\right) = \frac{^{-}1}{2x^{\frac{3}{2}}} - \frac{1}{x^2}$

ii $\frac{d}{dx}\left(\frac{x + \sqrt{x}}{x^2}\right) = \frac{^{-}1}{x^2} - \frac{3}{2x^{\frac{5}{2}}}$

WORKED Example 17

If $f(x) = x^3 - 2x^2 + \frac{8}{x}$, find:

a $f'(x)$

b $f'(2)$.

THINK	WRITE
a ① Write the equation.	**a** $f(x) = x^3 - 2x^2 + \frac{8}{x}$
② Express $f(x)$ so each term is in index form.	$= x^3 - 2x^2 + 8x^{-1}$
③ Differentiate $f(x)$ to obtain the gradient function $f'(x)$.	$f'(x) = 3x^2 - 4x - 8x^{-2}$
④ Simplify $f'(x)$.	$= 3x^2 - 4x - \frac{8}{x^2}$
b Evaluate $f'(2)$.	**b** $f'(2) = 3(2)^2 - 4(2) - \frac{8}{2^2}$ $= 12 - 8 - 2$ $= 2$ Therefore the gradient of $f(x)$ is 2 when $x = 2$.

remember

1. If $y = f(x)$, the derivative, $\frac{dy}{dx}$ or $f'(x)$, gives the gradient of the curve at x.
2. The derivative, $f'(x)$, can be found from first principles using
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
3. Rules for differentiating ax^n
   (a) If $f(x) = ax^n$ then $f'(x) = nax^{n-1}$, where a and n are constants.
   (b) If $f(x) = c$ then $f'(x) = 0$, where c is a constant.
   (c) If $f(x) = ag(x)$, where a is a constant then $f'(x) = ag'(x)$.
4. If $f(x) = g(x) + h(x)$ then $f'(x) = g'(x) + h'(x)$.
   Differentiate each term of a function separately.
5. To evaluate $f(a)$ where a is a constant, replace every x in the $f(x)$ equation with the value of a. For example, if $f(x) = 2x^2 - 3x + 1$, $f(2) = 2 \cdot 2^2 - 3 \cdot 2 + 1 = 3$.

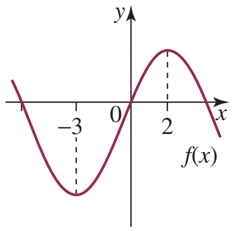
EXERCISE 1C Review of differentiation

If possible, use the TI-Nspire CAS calculator to complete questions **2** and **9** and to check your answers to question **10**.

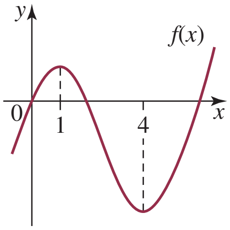
WORKED Example 12

1 For each cubic function $f(x)$ graphed below, sketch the gradient function.

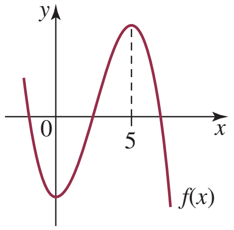
a



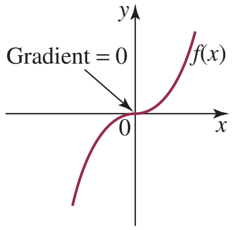
b



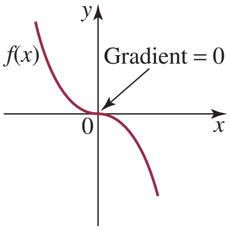
c



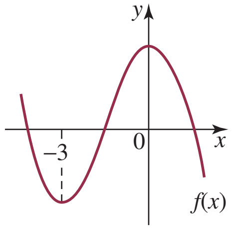
d



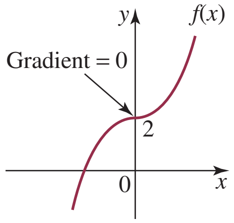
e



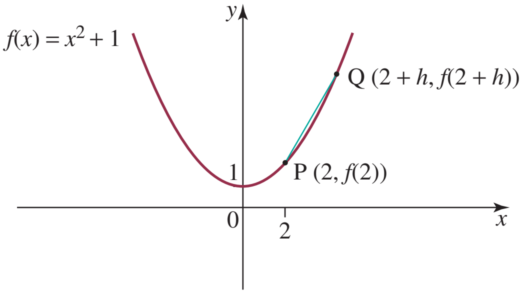
f



g



Questions **2** and **3** refer to the following diagram. Consider the chord $\overline{PQ}$ drawn to the curve $f(x)$ as shown.



WORKED Example 13

2 Find the gradient of the chord PQ drawn to the curve $f(x) = x^2 + 1$ in the diagram at right.

3 **multiple choice**

The gradient of the tangent at P in the diagram is:

A h **B** $4 + h$ **C** $4h$
D $4 + h^2$ **E** 4

4 **a** Find the gradient of the chord PQ to the function $f(x) = x(x + 2)$ if the x-coordinates of P and Q are -1 and $-1 + h$ respectively.

b Hence, find the gradient of the function at P.

5 **multiple choice**

The gradient of a function $f(x)$ at the point where $x = 3$ is:

A $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ **B** $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(x)}{h}$ **C** $\lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

D $\dfrac{f(3+h) - f(3)}{h}$ **E** $\dfrac{f(x+h) - f(x)}{h}$

WORKED Example 14

6 Use first principles to differentiate $f(x)$ if:

a $f(x) = 3x + 5$ **b** $f(x) = x^2 - 3$ **c** $f(x) = x^2 + 6x$

d $f(x) = (x - 4)(x + 2)$ **e** $f(x) = 8 - 3x^2$ **f** $f(x) = x^3 + 2.$

7 Use first principles to find $\frac{dy}{dx}$ if:

a $y = 9 - 4x$ **b** $y = x^2 + 3x$ **c** $y = x^2 - 2x + 7$

d $y = 3x^2 + 8x - 5$ **e** $y = x^3 - 4x$ **f** $y = 5x - 2x^3$

g $y = \frac{x^2}{3}$ **h** $y = -x^2 - 2x.$

8 Find the derivative of each of the following.

a $y = x^6$ **b** $y = 3x^2$ **c** $y = 5x^4$ **d** $y = x^{20}$

e $y = -4x^3$ **f** $y = -5x$ **g** $y = \frac{1}{2}x^3$ **h** $y = \frac{x^4}{3}$

i $y = 10$ **j** $y = 8x^5$

WORKED Example 15

9 Differentiate each of the following.

a $f(x) = 4x^3 + 5x$ **b** $g(x) = -5x^2 + 6x + 1$

c $h(x) = 9 + \frac{x^3}{5}$ **d** $h(x) = 4 - 3x + 6x^2 + x^3$

e $g(x) = 7x^{11} + 6x^5 - 8$ **f** $f(x) = \frac{2x^5}{5} + \frac{x^3}{3} + 10$

g $f(x) = -6x + 3x^2 - 4x^3$ **h** $g(x) = 7x^2 - 4x + \frac{2}{3}$

i $h(x) = (x + 4)(x - 1)$ **j** $f(x) = (x^2 + 2x)(3x - 6)$

WORKED Example 16

eBookplus

Digital doc:
SkillSHEET 1.3
Index laws

10 Find the derivative of each of the following.

a $\frac{2}{x^3}$ **b** $3\sqrt{x}$ **c** $x^{\frac{1}{3}}$

d $4x^{\frac{5}{4}}$ **e** $\sqrt{x} - 2x^2$ **f** $\frac{1}{x} + x^2$

g $x^{-\frac{1}{2}} + x^{\frac{2}{3}}$ **h** $\frac{x + 3}{x}$ **i** $\frac{x^2 + x^3}{x}$

j $\frac{3}{4x}$ **k** $\frac{2}{5x^2}$ **l** $\frac{2}{\sqrt{x}} + 3x^{-2}$

m $\frac{1}{3}x^3 - 4x + x^{-3}$ **n** $\frac{x + x^{\frac{1}{4}}}{\sqrt{x}}$

WORKED Example 17

11 If $f(x) = 2x^5 - 10x + 5$ find:

a $f'(x)$ **b** $f'(2).$

12 **multiple choice**

If $f(x) = x^2 - 6x$ then $f'(4)$ is equal to:

A 8 **B** −12 **C** 12 **D** 2 **E** −16

13 **multiple choice**

The value of $f'(9)$ if $f(x) = x^2 + x^{\frac{3}{2}} - 10x$ is:

A $\frac{1}{2}$ **B** 18 **C** $12\frac{1}{2}$ **D** 8 **E** 0

14 Find $g'(-2)$ if $g(x) = \dfrac{1}{x^2} + 3x - 8$.

15 Find the gradient of the curve $y = \dfrac{5}{x^4}$ at the point where **a** $x = 2$ and **b** $x = 0$.

16 Find the gradient of $f(x) = 2x^3 - x^2 + \dfrac{1}{\sqrt{x}}$ at the point where x equals:

a 1 **b** 4 **c** 9.

17 If $g(x) = \sqrt[3]{x} + 4x$, find: **a** $g'(x)$ **b** $g'(1)$ **c** $g'(8)$ **d** $g'(-8)$.

18 Show that the derivative of $y = k$, where k is a constant, is zero.

19 For each of the following:

- **i** expand the brackets
- **ii** differentiate the expanded expression
- **iii** factorise.

a $(x + 1)^2$ **b** $(x + 1)^3$ **c** $(2x + 1)^2$ **d** $(2x + 1)^3$ **e** $(3x + 1)^2$ **f** $(3x + 1)^3$

20 Using the results of question **12** give the derivative of $(ax + b)^n$ in factorised form. (a, b, n are constants.)

Rules for differentiation

Clearly, it is much easier to differentiate a function $f(x) = 3x^3$ by the rule as opposed to using first principles; but what happens when we try to differentiate a more complex expression such as $g(x) = (x^3 + 2)^4$? In situations such as this we can use one of three rules for differentiating functions: the chain rule, the product rule or the quotient rule.

The chain rule is used when the expression to be differentiated is a function of a function. For example $f(x) = (x^3 + 2)^4$, with one function, $x^3 + 2$, being acted on by a second function: 'raise to the fourth power'.

The product rule is used when the expression to be differentiated is a product of two functions. For example, $f(x) = x^3(x + 2)^2$ is the product of x^3 and $(x + 2)^2$.

The quotient rule is used when the expression to be differentiated is a quotient of two functions. For example,

$$f(x) = \frac{x^3}{(x+2)^2}$$

is the quotient of x^3 and $(x + 2)^2$.

We now look at these rules in more detail.

Chain rule

A function which can be expressed as a composition of two simpler functions is called a composite function. For example, $y = (x + 3)^2$ can be expressed as $y = u^2$ where $u = x + 3$.

That is, to obtain y from x, the first function to be performed is to add 3 to $x (u = x + 3)$, then this function has to be 'squared' ($y = u^2$).

Or if $x = 1$, to obtain y first calculate $1 + 3$ $(= 4)$, then secondly 'square' the result, 4^2, giving $y = 16$.

Composite functions can be differentiated using the chain rule. For example, using the previous function, $y = (x + 3)^2$:

Let $u = x + 3$, so $y = u^2$.

Then $\dfrac{du}{dx} = 1$ and $\dfrac{dy}{du} = 2u$.

But we require $\dfrac{dy}{dx}$ and $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. This is known as the chain rule. It is known as the chain rule because u provides the 'link' between y and x.

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 2u \cdot 1 \\ &= 2(x + 3) \cdot 1 \quad \text{(replacing } u \text{ with } x + 3) \\ &= 2(x + 3) \end{aligned}$$

The chain rule is used when it is necessary to differentiate a 'function of a function' as above.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

WORKED Example 18

If $y = (3x - 2)^3$ is expressed as $y = u^n$, find:

a $\dfrac{dy}{du}$ **b** $\dfrac{du}{dx}$ **and hence** **c** $\dfrac{dy}{dx}$.

eBook*plus*

Tutorial:
Worked example 18
int-0554

THINK	WRITE
a 1 Write the equation.	**a** $y = (3x - 2)^3$
2 Express y as a function of u.	Let $y = u^3$ where $u = 3x - 2$.
3 Differentiate y with respect to u.	$\dfrac{dy}{du} = 3u^2$
b 1 Express u as a function of x.	**b** $u = 3x - 2$
2 Differentiate u with respect to x.	$\dfrac{du}{dx} = 3$
c 1 Find $\dfrac{dy}{dx}$ using the chain rule.	**c** $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ $= 3u^2 \cdot 3$ $= 9u^2$
2 Replace u as a function of x.	$= 9(3x - 2)^2$

WORKED Example 19

If $f(x) = \dfrac{1}{\sqrt{2x^2 - 3x}}$ find $f'(x)$.

THINK	WRITE
① Write the equation.	$f(x) = \dfrac{1}{\sqrt{2x^2 - 3x}}$
② Express $f(x)$ in index form, that is, as $y = [g(x)]^n$.	$y = (2x^2 - 3x)^{-\frac{1}{2}}$
③ Express y as a function of u.	Let $y = u^{-\frac{1}{2}}$ where $u = 2x^2 - 3x$.
④ Differentiate y with respect to u.	$\dfrac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$
⑤ Express u as a function of x.	$u = 2x^2 - 3x$
⑥ Differentiate u with respect to x.	$\dfrac{du}{dx} = 4x - 3$
⑦ Find $f'(x)$ using the chain rule.	$\dfrac{dy}{dx} = f'(x) = -\frac{1}{2}u^{-\frac{3}{2}} \cdot \ (4x - 3)$
⑧ Replace u as a function of x and simplify.	$= -\frac{1}{2}(2x^2 - 3x)^{-\frac{3}{2}}(4x - 3)$ $= \dfrac{-(4x-3)}{2(2x^2 - 3x)^{\frac{3}{2}}}$ $= \dfrac{3 - 4x}{2\sqrt{(2x^2 - 3x)}^3}$

A quicker way to apply the chain rule when a function can be expressed in index form is as follows.

If $f(x) = [g(x)]^n$ then $f'(x) = n[g(x)]^{n-1} \cdot \ g'(x)$. That is, differentiate the bracket and then what is inside the bracket; ‘outside then inside’.

WORKED Example 20

Find the derivative of $f(x) = (x^2 - 2x)^3$.

THINK	WRITE
① Write the equation.	$f(x) = (x^2 - 2x)^3$
② Let $g(x)$ equal what is inside the bracket.	$g(x) = x^2 - 2x$
③ Find $g'(x)$.	$g'(x) = 2x - 2$
④ Use the rule $f'(x) = n[g(x)]^{n-1} \cdot \ g'(x)$ to differentiate $f(x)$.	$f'(x) = 3(x^2 - 2x)^{3-1} \cdot \ (2x - 2)$ $= 3(x^2 - 2x)^2 \cdot \ [2(x - 1)]$ $= 6(x - 1)(x^2 - 2x)^2$
⑤ Simplify $f'(x)$ as far as possible.	$= 6(x - 1)[x(x - 2)x(x - 2)]$ $= 6x^2(x - 1)(x - 2)^2$

Product rule

Any function that is a product of two simpler functions, for example,

$$f(x) = (x + 2)(x - 5)$$

can be differentiated using the product rule of differentiation.

The product rule is stated as follows.

$$\text{If } y = uv \text{ then } \frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

Or if $f(x) = u(x) \cdot v(x)$ then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.

WORKED Example 21

If $y = (3x - 1)(x^2 + 4x + 3)$ is expressed as $y = uv$, find:

a u and v **b** $\frac{du}{dx}$ and $\frac{dv}{dx}$ **c** $\frac{dy}{dx}$ using $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$.

THINK	WRITE
a 1 Write the equation.	**a** $y = (3x - 1)(x^2 + 4x + 3)$
2 Identify u and v, two functions of x which are multiplied together.	Let $u = 3x - 1$ and $v = x^2 + 4x + 3$.
b 1 Differentiate u with respect to x.	**b** $\frac{du}{dx} = 3$
2 Differentiate v with respect to x.	$\frac{dv}{dx} = 2x + 4$
c 1 Apply the product rule to find $\frac{dy}{dx}$.	**c** $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$ $= (3x - 1)(2x + 4) + (x^2 + 4x + 3) \cdot 3$
2 Expand and simplify where possible.	$= 6x^2 + 10x - 4 + 3x^2 + 12x + 9$ $= 9x^2 + 22x + 5$

Quotient rule

The quotient rule is used to differentiate functions which are rational expressions (that is, one function divided by another). For example,

$$f(x) = \frac{x^2 - 6x + 3}{5x + 2}$$

The quotient rule is stated as follows.

$$\text{If } y = \frac{u}{v} \text{ then } \frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

Or if $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$.

WORKED Example 22

If $y = \dfrac{3 - x}{x^2 + 4x}$ is expressed as $y = \dfrac{u}{v}$, find:

a u **and** v **b** $\dfrac{du}{dx}$ **and** $\dfrac{dv}{dx}$ **c** $\dfrac{dy}{dx}$.

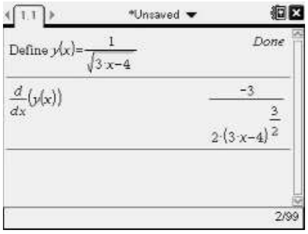
THINK	WRITE
a 1 Write the equation.	**a** $y = \dfrac{3 - x}{x^2 + 4x}$
2 Identify u and v.	Let $u = 3 - x$ and $v = x^2 + 4x$.
b 1 Differentiate u with respect to x.	**b** $\dfrac{du}{dx} = -1$
2 Differentiate v with respect to x.	$\dfrac{dv}{dx} = 2x + 4$
c 1 Apply the quotient rule to obtain $\dfrac{dy}{dx}$.	**c** $\dfrac{dy}{dx} = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$ $= \dfrac{(x^2 + 4x) \cdot -1 - (3 - x)(2x + 4)}{(x^2 + 4x)^2}$
2 Simplify $\dfrac{dy}{dx}$ where possible, factorising the final answer where appropriate.	$= \dfrac{-x^2 - 4x - (12 + 2x - 2x^2)}{(x^2 + 4x)^2}$ $= \dfrac{-x^2 - 4x - 12 - 2x + 2x^2}{(x^2 + 4x)^2}$ $= \dfrac{x^2 - 6x - 12}{(x^2 + 4x)^2}$ $= \dfrac{x^2 - 6x - 12}{x^2(x + 4)^2}$

Graphics Calculator tip! Using the TI-Nspire CAS calculator

The **TI-Nspire CAS** can be used to do almost all the calculations in this section on calculus. This does not necessarily mean the by-hand skills associated with derivatives are redundant. Rather, your understanding can be reinforced with appropriate calculator use.

The following examples showcase some of the calculus that can be done with the **TI-Nspire CAS** calculator.

1. To calculate the derivative of $y(x) = \dfrac{1}{\sqrt{3x - 4}}$, on a Calculator page, complete the entry line as:
   Define $y(x) = \dfrac{1}{\sqrt{3x - 4}}$
   then press:
   - ENTER enter
   - MENU menu



Define $y(x)=\dfrac{1}{\sqrt{3\cdot x-4}}$ Done

$\dfrac{d}{dx}(y(x))$ $\dfrac{-3}{2\cdot(3\cdot x-4)^{\frac{3}{2}}}$

- 4: Calculus 4
- 1: Derivative 1.

Complete the entry line as:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y(x))$$

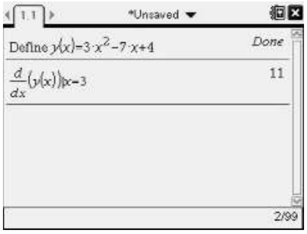
then press ENTER enter.

2. To calculate the derivative of $y = 3x^2 - 7x + 4$ at the point $x = 3$, on a Calculator page, complete the entry line as:
Define $y(x) = 3x^2 - 7x + 4$
then press:
   - ENTER enter
   - MENU menu
   - 4: Calculus 4
   - 1: Derivative 1.

   Complete the entry line as:

$$\frac{\mathrm{d}}{\mathrm{d}x}(y(x))|x = 3$$

then press ENTER enter.



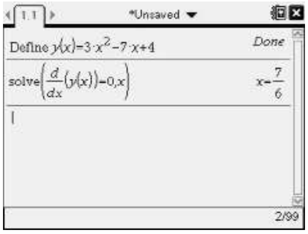
Define $y(x)=3\cdot x^2-7\cdot x+4$ Done

$\frac{d}{dx}(y(x))|x=3$ 11

3. To find the point where the derivative of $y = 3x^2 - 7x + 4$ is equal to 0, on a Calculator page, complete the entry line as:
Define $y(x) = 3x^2 - 7x + 4$.
Complete the entry line as:

$$\text{solve}\left(\frac{\mathrm{d}}{\mathrm{d}x}(y(x)) = 0,\ x\right)$$

then press ENTER enter.



Define $y(x)=3\cdot x^2-7\cdot x+4$ Done

$\text{solve}\left(\frac{d}{dx}(y(x))=0,x\right)$ $x=\frac{7}{6}$

remember

Chain rule for composite functions

1. A composite function is a function composed of two (or more) functions.
2. Composite functions can be differentiated using the chain rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.
3. A short way of applying the chain rule is:
If $f(x) = [g(x)]^n$ then $f'(x) = n[g(x)]^{n-1} \cdot g'(x)$.

Product rule

4. If $y = u \cdot v$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = u \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + v \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.
5. If $f(x) = u(x) \cdot v(x)$ then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.

Quotient rule

6. If $y = \frac{u}{v}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v \cdot \frac{\mathrm{d}u}{\mathrm{d}x} - u \cdot \frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.
7. If $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$.

EXERCISE 1D Rules for differentiation

In this exercise, your teacher will indicate the level of use of your calculator.

1 If each of the following functions is expressed in the form $y = u^n$, state **i** u and **ii** n.

a $y = (5x - 4)^3$ **b** $y = \sqrt{3x + 1}$ **c** $y = \dfrac{1}{(2x + 3)^4}$

d $y = \dfrac{1}{7 - 4x}$ **e** $y = (5x + 3)^{-6}$ **f** $y = (4 - 3x)^{\frac{4}{3}}$

2 **multiple choice**

If $y = (x + 3)^5$ is expressed as $y = u^5$ then:

A $u = (x + 3)^5$ **B** $u = x + 3$ **C** $u = x$

D $u = 3$ **E** $u = x^5$

WORKED Example 18

3 If each of the following composite functions is expressed as $y = u^n$, find:

i $\dfrac{dy}{du}$ **ii** $\dfrac{du}{dx}$ and hence **iii** $\dfrac{dy}{dx}$.

a $y = (3x + 2)^2$ **b** $y = (7 - x)^3$ **c** $y = \dfrac{1}{2x - 5}$

d $y = \dfrac{1}{(4 - 2x)^4}$ **e** $y = \sqrt{5x + 2}$ **f** $y = \dfrac{3}{\sqrt{3x - 2}}$

g $y = 3(2x^2 + 5x)^5$ **h** $y = (4x - 3x^2)^{-2}$ **i** $y = \left(x + \dfrac{1}{x}\right)^6$

j $y = 4(5 - 6x)^{-4}$

For questions **4**, **5** and **6**, $y = \sqrt{x^2 - 3x + 2}$ is expressed as $y = u^n$.

4 **multiple choice**

$\dfrac{dy}{du} =$

A $\frac{1}{2}u$ **B** $u^{-\frac{1}{2}}$ **C** $\dfrac{1}{2\sqrt{u}}$ **D** $\frac{1}{2}u^{\frac{3}{2}}$ **E** $\frac{1}{2}u^{\frac{1}{2}}$

5 **multiple choice**

$\dfrac{du}{dx} =$

A $2x - 3$ **B** $x^2 - 3x + 2$ **C** $x^2 - 3x$

D $\frac{1}{2}x^2 - \frac{3}{2}x + 1$ **E** $x - 3$

6 **multiple choice**

Using the chain rule, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is equal to:

A $\frac{2x-3}{u}$
B $\frac{2x-3}{2\sqrt{x^2-3x+2}}$
C $\frac{x^2-3x+2}{2\sqrt{u}}$
D $\frac{1}{2}(2x-3)$
E $\frac{1}{2}(2x-3)(x^2-3x+2)^{\frac{1}{2}}$

7 Use the chain rule to find the derivative of the following.

a $y = (8x + 3)^4$
b $y = (2x - 5)^3$
c $f(x) = (4 - 3x)^5$
d $y = \sqrt{3x^2 - 4}$
e $f(x) = (x^2 - 4x)^{\frac{1}{3}}$
f $g(x) = (2x^3 + x)^{-2}$
g $g(x) = \left(x - \frac{1}{x}\right)^6$
h $y = (x^2 - 3x)^{-1}$

WORKED Example 19

8 If $f(x) = \frac{1}{\sqrt{4x+7}}$, find $f'(x)$.

9 Use the chain rule to find the derivative of the following. (*Hint:* Simplify first using index notation and the laws of indices.)

a $y = \frac{\sqrt{6x-5}}{6x-5}$
b $f(x) = \frac{(x^2+2)^2}{\sqrt{x^2+2}}$

WORKED Example 20

10 Find the derivative of:

a $f(x) = (x^2 + 5x)^8$
b $y = (x^3 - 2x)^2$
c $f(x) = (x^3 + 2x^2 - 7)^{\frac{1}{5}}$
d $y = (2x^4 - 3x^2 + 1)^{\frac{3}{2}}$.

11 If $f(x) = (2x - 1)^6$, find $f'(3)$.

12 If $g(x) = (x^2 - 3x)^{-2}$, find $g'(-2)$.

13 If $f(x) = \sqrt{x^2 - 2x + 1}$, find:

a $f(3)$
b $f'(x)$
c $f'(3)$
d $f'(x)$ when $x = 2$.

14 Find the gradient of the function $h(x) = \sqrt{3x^2 + 2x}$ at the point where $x = 2$.

15 Find the value of $f'(-1)$ if $f(x) = \frac{3}{\sqrt{5-4x}}$.

WORKED Example 21

16 If $y = (x + 3)(2x^2 - 5x)$ is expressed as $y = u \cdot v$, find:

a u and v
b $\frac{\mathrm{d}u}{\mathrm{d}x}$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$
c $\frac{\mathrm{d}y}{\mathrm{d}x}$ using the product rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = u \cdot \frac{\mathrm{d}v}{\mathrm{d}x} + v \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$.

17 Find the derivative of:

a $x^2(x+1)^3$ **b** $x^3(x+1)^2$

c $\sqrt{x}\,(x+1)^5$ **d** $x^{\frac{3}{2}}(x-2)^3$

e $x(x-1)^{-2}$ **f** $x\sqrt{x+1}$.

WORKED Example 22

18 If $y = \dfrac{x+3}{x+7}$ is expressed as $y = \dfrac{u}{v}$, find:

a u and v

b $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$

c $\dfrac{dy}{dx}$ using the quotient rule, $\dfrac{dy}{dx} = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$.

19 If $f(x) = \dfrac{x^2+2x}{5-x}$ is expressed as $f(x) = \dfrac{u(x)}{v(x)}$, find:

a $u(x)$ and $v(x)$

b $u'(x)$ and $v'(x)$

c $f'(x)$ using the quotient rule.

20 Find the derivative of each of the following.

a $\dfrac{2x}{x^2-4x}$ **b** $\dfrac{x^2+7x+6}{3x+2}$ **c** $\dfrac{4x-7}{10-x}$ **d** $\dfrac{5-x^2}{x^{\frac{3}{2}}}$

21 **multiple choice**

If $h(x) = \dfrac{8-3x^2}{x}$ then $h'(x)$ equals:

A $\dfrac{9x^2-8}{x^2}$ **B** $\dfrac{8-9x^2}{x^2}$ **C** $\dfrac{-3x^2+8}{x^2}$

D $\dfrac{-3x^2-8}{x^2}$ **E** $\dfrac{-3x^2+8}{x}$

eBook plus

Digital doc:
WorkSHEET 1.2

22 Find the derivative of each of the following.

a $x(x^2+1)^3$ **b** $\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$ **c** $\dfrac{(x^2+1)^3}{x}$

d $\dfrac{1}{(x^2-3)^5}$ **e** $\dfrac{\sqrt{x}(x+1)^3}{x-1}$

23 Use the TI-Nspire CAS calculator to check your answers to question **22**.

summary

Modelling change

- If $y = f(x)$ then the inverse of $f(x)$ is written as $f^{-1}(y)$ and $x = f^{-1}(y)$.
- In analysing functions the graphics calculator can:
  1. graph a function
  2. generate a table of values
  3. find values of x, given y
  4. find the maximum value on an interval
  5. find the intersection between curves.

Graphing polynomials

- Quadratic graphs
  1. General equation is $y = ax^2 + bx + c$.
  2. The quadratic formula is given by the equation $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
  3. The x-coordinate of the turning point is $x = \dfrac{-b}{2a}$.
- Cubic graphs
  The general equation is $y = ax^3 + bx^2 + cx + d$.
- Quartic graphs
  1. Positive quartics in factor form are $y = a(x - b)(x - c)(x - d)(x - e)$.
  2. A factor may occur twice, for example $y = a(x - b)(x - c)(x - d)^2$.
  3. A factor may occur three times, for example $y = a(x - b)(x - c)^3$.
  4. A factor may occur four times, for example $y = a(x - b)^4$.

  If a is negative, each of these graphs is reflected in the x-axis.

Review of differentiation

- The gradient of $y = f(x)$ at the point x is given by a function called the *derived function* or the *derivative* and is written as $f'(x)$ or $\dfrac{dy}{dx}$.
- Differentiation from first principles:

  If $y = f(x)$ then $\dfrac{dy}{dx}$ or $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.
- Differentiation by the rule: If $y = ax^n$ then $\dfrac{dy}{dx} = nax^{n-1}$.

The rules for differentiation

- Chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- Product rule If $y = uv$ then

  $$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$
- Quotient rule If $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \dfrac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$

CHAPTER review

1A

1 If $f(x) = x^3 - 4$ and $g(x) = 4x + 2$:
- **a** calculate $f[g(2)]$
- **b** write an expression for $f[g(x)]$
- **c** calculate $g[f(2)]$
- **d** write an expression for $g[f(x)]$.

1A

2 **multiple choice**

If $p(t) = (1 - t)^2$ and $q(t) = t^2 - 1$, then an expression for $q[p(t)]$ is:
- **A** $(1 - t)^2 \cdot (t^2 - 1)$
- **B** $(t^2 - 1) \cdot (1 - t)^2$
- **C** $(1 - t)^4 - 1$
- **D** $(1 - t^2)^2 - 1$
- **E** $1 - (1 - t^2)^2$

1A

3 If $f(x) = 4x - 2$:
- **a** calculate $f^{-1}(6)$
- **b** give an expression for $f^{-1}(y)$.

1A

4 **multiple choice**

For the polynomial $P(x) = x^3 - 5x^2 + 6$, use your graphics calculator to find the positive value of x for which $P(x)$ is a minimum. The value of x is:
- **A** 3.33
- **B** 12.5
- **C** 0.5
- **D** 6.5
- **E** 2.45

1B

5 Sketch the graph of $y = 8 - 2x - x^2$, by labelling the turning point and all intercepts.

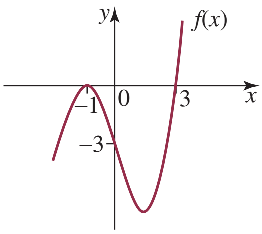
1B

6 Sketch the graph of $y = 3x^2 + 8x - 3$, $x \in [-3, 0)$.

1B

7 **multiple choice**

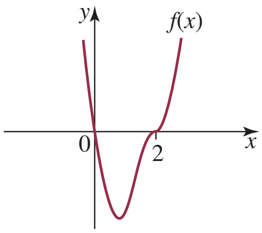
The rule for the graph shown at right could be:
- **A** $f(x) = (x - 1)^2(x + 3)$
- **B** $f(x) = (x + 1)(x - 3)^2$
- **C** $f(x) = (x + 1)^2(3 - x)$
- **D** $f(x) = (x^2 - 1)(x + 3)$
- **E** $f(x) = (x - 3)(x + 1)^2$



1B

8 **multiple choice**

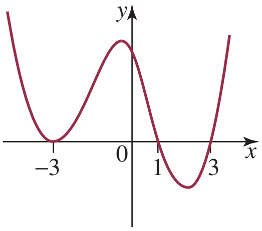
The rule for the graph shown at right could be:
- **A** $f(x) = x(x + 2)^3$
- **B** $f(x) = -x(x - 2)^2$
- **C** $f(x) = x^2(x - 2)^2$
- **D** $f(x) = x(x - 2)^3$
- **E** $f(x) = x(2 - x)^2$



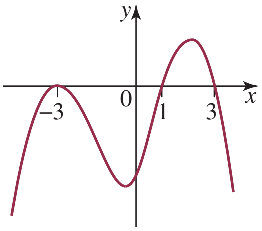
9 **multiple choice** 1B

The graph of $y = (x + 3)^2(x - 1)(x - 3)$ is best represented by:

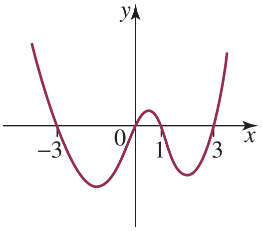
A



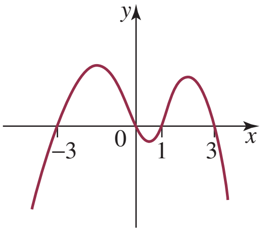
B



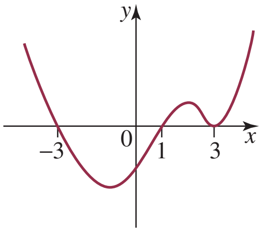
C



D



E



10 **a** Find the derivative of $f(x) = x^3 + 2x$ using first principles. 1C

b Hence find the gradient at the point where $x = 1$.

11 **a** Find the gradient function if $g(x) = \frac{x^3}{3} - 4x$. 1C

b Find the gradient of $g(x)$ when $x = 3$.

12 If $h(x) = \frac{3x^4}{2} + \frac{x^3}{4} - 3x$, find: 1C

a $h'(x)$

b **i** $h'(-1)$ **ii** $h'(2)$.

13 **multiple choice** 1C

The derivative of $f(x) = 4x^3 - x^2 + 3x$ is:

A $12x^2 - 2x + 3$ **B** $4x^2 - 2x + 3$ **C** $12x^2 - 2x$

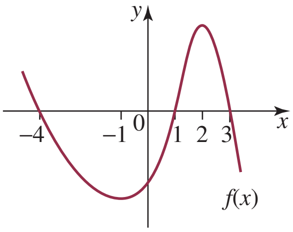
D $12x^2 - x + 3$ **E** $4x^2 - 2x$

14 **multiple choice** 1C

The derivative of $g(x) = \frac{1}{x^2} - 2\sqrt{x}$ is:

A $-\frac{1}{x} - \frac{2}{\sqrt{x}}$ **B** $-\frac{2}{x^3} - \frac{2}{\sqrt{x}}$ **C** $-\frac{2}{x^3} - \frac{1}{\sqrt{x}}$ **D** $-\frac{2}{x} - \frac{2}{\sqrt{x}}$ **E** $-\frac{2}{x^3} - \frac{1}{x^{\frac{3}{2}}}$

15 The graph of a cubic function is shown at right. Sketch the graph of its gradient function. 1C



16 Calculate the derivative of: 1D

a $(4 - x^2)^3$ **b** $x^2(x + 3)^4$ **c** $\frac{x^3}{x^2 + 1}$.

1D

17 **multiple choice**

The derivative of $(2x + 5)^6$ is:

A $6(2x + 5)^5$

B $12x(2x + 5)^5$

C $6x(2x + 5)^5$

D $12(2x + 5)^5$

E $12(2x + 5)^4$

1D

18 **multiple choice**

The derivative of $\frac{1}{\sqrt{4x-9}}$ is:

A $2\sqrt{4x-9}$

B $\frac{-2}{\sqrt{(4x-9)^3}}$

C $\frac{4}{(4x-9)^{\frac{3}{2}}}$

D $4\sqrt{4x-9}$

E $\frac{2}{4x-9}$

1D

19 **multiple choice**

The derivative of $\frac{2x+1}{x-2}$ is:

A $\frac{4x-5}{(x-2)^2}$

B $\frac{-3}{(x-2)^2}$

C $\frac{4x-3}{(x-2)^2}$

D $4x - 5$

E $\frac{-5}{(x-2)^2}$

1D

20 **multiple choice**

If $g(x) = (x^2 + 3x - 7)^5$ then $g'(x)$ is equal to:

A $5(x^2 + 3x - 7)^4$

B $(2x + 3)(x^2 + 3x - 7)^4$

C $5(2x + 3)^4$

D $5(2x + 3)(x^2 + 3x - 7)^4$

E $(x^2 + 3x - 7)^4$

Modelling & problem solving

1 A 'rogue satellite' has its distance from Earth, d thousand kilometres, modelled by a cubic function of time, t days after launch. After 1 day it reaches a maximum distance from Earth of 4000 kilometres, then after 2 days it is 2000 kilometres away. It effectively returns to Earth after 3 days, then moves further and further away.

a What is the satellite's initial distance from Earth?

b Sketch the graph of d versus t for the first 6 days of travel.

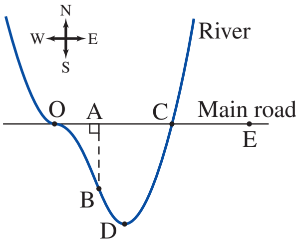
c Express d as a function of t. The moon is approximately 240 000 kilometres from Earth.

d Which is closer to Earth after 8 days, the satellite or the moon? By how far? The satellite is programmed to self-destruct. This happens when it is 490 000 kilometres from Earth.

e What is the 'life span' of the satellite?

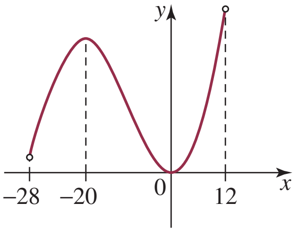
f State the domain and range of $d(t)$.

2 The diagram shows a main road passing through O, A, C and E. The road crosses a river at point O and 3 kilometres further along the road at point C. Between O and C, the furthest the river is from the road is 8.54 kilometres, at a point D, 2.25 kilometres east of a north–south line through O. Point A is 1 km east of point O. If point O is taken as the origin and the road as the x-axis, then the path of the river can be modelled by a quartic function, as shown in blue.



a Give the coordinates of C and D.
b Find the rule for the quartic function, $f(x)$.
c How far is the river from the main road along the track AB?
d A canoeing race, of at least 17 kilometres in length, along the river is being organised. It is suggested that the race could start at O and finish at C. Is this course satisfactory? Why?

3 A section of a roller-coaster ride follows part of the curve with the equation $y = \frac{1}{200}(x^3 + 30x^2)$ as shown on the right.



a For what values of x (domain) is the gradient:
- **i** zero?
- **ii** positive?
- **iii** negative?

b Sketch the gradient function.
c Use the graph of the gradient function to find the value of x where the gradient is steepest over the domain [–25, 10].
d Find $\frac{dy}{dx}$.
e Find the gradient where x equals:
- **i** –25
- **ii** –10
- **iii** 10.

f Does this verify your answer to part **c**? Briefly explain.
g What is the highest point reached by the roller-coaster? (Give your answer in metres.)

eBook*plus*

Digital doc:
Test Yourself
Chapter 1

eBook*plus* ACTIVITIES

1B Graphing polynomial functions

Digital docs

- SkillSHEET 1.1: Practise solving quadratic equations using the quadratic formula (*page 10*)
- SkillSHEET 1.2: Practise identifying domain and range for quadratic graphs (*page 18*)
- Spreadsheet 050: Investigate quadratic graphs (*page 18*)
- Spreadsheet 007: Investigate cubic graphs — factor form (*page 19*)
- Spreadsheet 009: Investigate cubic graphs of the form $y = a(x - b)^3 + c$ (*page 20*)
- Spreadsheet 073: Investigate quartic graphs — factor form (*page 20*)
- WorkSHEET 1.1: Determine equations for quadratic graphs, use the discriminant, substitute values into quadratic expressions and use quadratic graphs to model paths of objects (*page 20*)

Tutorial

- **WE10** Int-0519: Watch a tutorial on using the discriminant (*page 14*)

Interactivity

- Simultaneous quadratic and linear equations int-0261: Consolidate your understanding of simultaneous quadratic and linear graphs (*page 10*)

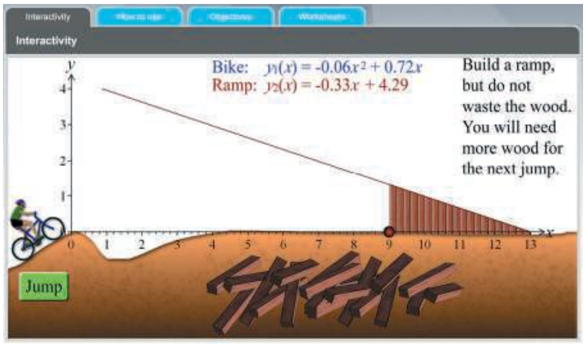
1C Review of differentiation

Digital doc

- SkillSHEET 1.3: Practise index laws (*page 29*)

Tutorial

- **WE12** Int-0552: Watch a tutorial on sketching the gradient function (*page 23*)



Bike: $y_1(x) = -0.06x^2 + 0.72x$
Ramp: $y_2(x) = -0.33x + 4.29$

Build a ramp, but do not waste the wood. You will need more wood for the next jump.

1D Rules for differentiation

Digital doc

- WorkSHEET 1.2: Sketch gradient functions given graphs, identify when gradients are less than, greater than or equal to zero, calculate gradients at any given point and use first principles and differentiation rules to determine the derivative (*page 38*)

Tutorial

- **WE18** Int-0554: Watch a tutorial on using the chain rule (*page 31*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 43*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

2

Applications of differentiation

syllabus reference

Rates of change
Optimisation

In this chapter

Introduction

In the previous chapter, the concept of a function was used to model relationships between variables. Whether the relationship was between volume and side length, weight loss and time, or height and time, the function and its graph proved to be a powerful tool for analysis.

In that chapter, the concept of the derivative, the rate of change of one variable with another, was revisited. Now we shall use the derivative to further analyse functions and show how such analysis can be applied to solving practical problems. In particular the derivative is used to assist in sketching curves, find equations of tangents and normals, optimise functions and analyse rates of change.

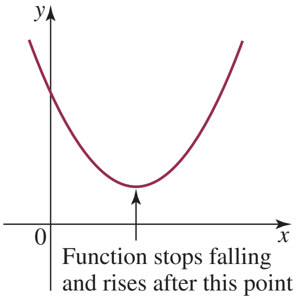
Sketching curves

When the graphs of polynomial functions are being sketched, four main characteristics should be featured:

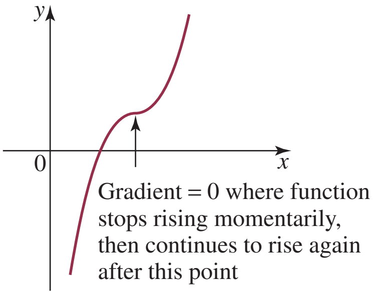
1. the basic shape (whenever possible)
2. the y-intercept
3. the x-intercept
4. the stationary points.

Stationary points

A *stationary point* is a point on a graph where the function momentarily stops rising or falling; that is, it is a point where the gradient is zero.



or



The *stationary point* (or turning point) of a quadratic function can be found by completing a perfect square in the form $y = (x + h)^2 + k$ to obtain $(-h, k)$, but for cubics, quartics or higher-degree polynomials there is no similar procedure. Differentiation enables stationary points to be found for any polynomial function where the rule is known.

The gradient of a function $f(x)$ is $f'(x)$.

Stationary points occur wherever the gradient is zero.

$f(x)$ has stationary points when $f'(x) = 0$

or

y has stationary points when $\dfrac{dy}{dx} = 0$.

The solution of $f'(x) = 0$ gives the x-value or values where stationary points occur.

If $f'(a) = 0$, a stationary point occurs when $x = a$ and $y = f(a)$. So the coordinate of the stationary point is $(a, f(a))$.

Types of stationary points

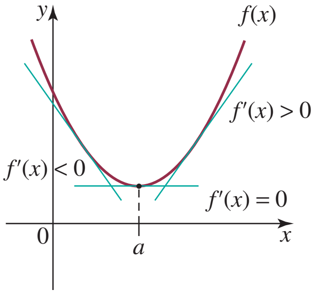
There are four types of stationary point.

1. A local minimum turning point at $x = a$.

If $x < a$, then $f'(x) < 0$ (immediately to the left of $x = a$, the gradient is negative).

If $x = a$, then $f'(x) = 0$ (at $x = a$ the gradient is zero).

If $x > a$, then $f'(x) > 0$ (immediately to the right of $x = a$, the gradient is positive).



2. A local maximum turning point at $x = a$.

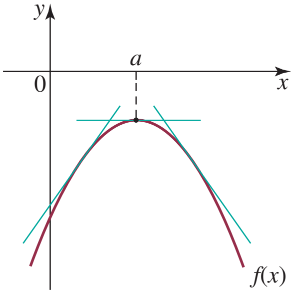
If $x < a$, then $f'(x) > 0$.

If $x = a$, then $f'(x) = 0$.

If $x > a$, then $f'(x) < 0$.

The two cases (1 and 2) can be called ‘turning points’ because the gradients each side of the stationary point are opposite in sign (that is, the graph turns).

The term ‘local turning point at $x = a$’ implies ‘in the vicinity of $x = a$’, as polynomials can have more than one stationary point.



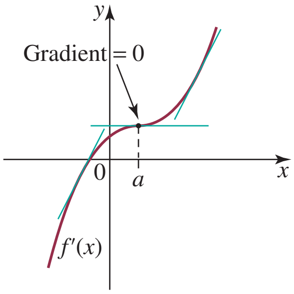
3. A positive stationary point of horizontal inflection at $x = a$.

If $x < a$, then $f'(x) > 0$.

If $x = a$, then $f'(x) = 0$.

If $x > a$, then $f'(x) > 0$.

That is, the gradient is positive either side of the stationary point.



4. A negative stationary point of horizontal inflection at $x = a$.

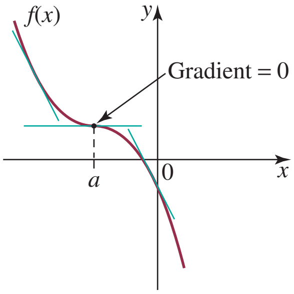
If $x < a$, then $f'(x) < 0$.

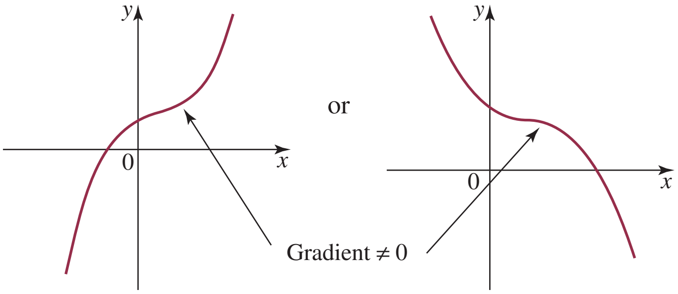
If $x = a$, then $f'(x) = 0$.

If $x > a$, then $f'(x) < 0$.

In cases 3 and 4 above the word ‘stationary’ implies that the gradient is zero.

Not all points of inflection are stationary points.





When determining the nature of stationary points it is helpful to complete a ‘gradient table’, which shows the sign of the gradient either side of any stationary points. This is known as the *first derivative* test.

Gradient tables are demonstrated in the examples that follow.

WORKED Example 1

a **Find the stationary points for $y = x^3 + 6x^2 - 15x + 2$.**
b **Determine the nature (or type) of each stationary point.**
c **Sketch the graph.**

THINK

a ① Write the equation.

② Find $\frac{dy}{dx}$.

③ Solve for x if $\frac{dy}{dx} = 0$ to locate stationary points.

④ Substitute the x solutions into the equation and evaluate to find the corresponding y-values.

⑤ State the stationary points.

b ① For each stationary point find $\frac{dy}{dx}$ immediately to the left and right to determine the nature of the stationary points. We will use $x = -6$, $x = 0$ and $x = 2$.

② Complete a gradient table and state the type of each stationary point from the definitions on the previous page.

WRITE

a $y = x^3 + 6x^2 - 15x + 2$

$$\frac{dy}{dx} = 3x^2 + 12x - 15$$

For stationary points,

$$\frac{dy}{dx} = 0$$

$$3x^2 + 12x - 15 = 0$$
$$3(x^2 + 4x - 5) = 0$$
$$3(x + 5)(x - 1) = 0$$
$$(x + 5)(x - 1) = 0$$
$$x = -5 \quad \text{or} \quad x = 1$$

When $x = -5$, $y = (-5)^3 + 6(-5)^2 - 15(-5) + 2$
$= 102$

When $x = 1$, $y = (1)^3 + 6(1)^2 - 15(1) + 2$
$= -6$

The stationary points are (–5, 102) and (1, –6)

b Nature: if $x = -6$, $\frac{dy}{dx} = 3(-6)^2 + 12(-6) - 15$
$= 21$ (that is, positive).

If $x = 0$, $\frac{dy}{dx} = 3(0)^2 + 12(0) - 15$
$= -15$ (that is, negative).

If $x = 2$, $\frac{dy}{dx} = 3(2)^2 + 12(2) - 15$
$= 21$ (that is, positive).

Gradient table:

x	**–6**	**–5**	**0**	**1**	**2**
$\frac{dy}{dx}$	+	0	–	0	+
Slope	/	‾	\	_	/

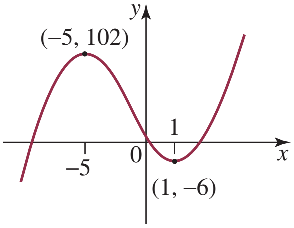
Therefore (–5, 102) is a local maximum turning point and (1, –6) is a local minimum turning point.

THINK

c Sketch the graph using the stationary points.

WRITE

c



WORKED Example 2

Sketch the graph of $g(x) = x^2(4 - x^2)$, clearly indicating all stationary points and axis intercepts:

a **by hand** **b** **using a graphics calculator.**

THINK

a 1 Write the rule for $g(x)$.

2 Expand $g(x)$ to make it easier to differentiate.

3 Differentiate $g(x)$.

4 Solve $g'(x) = 0$.

5 Find $g(x)$ for each value of x where $g'(x) = 0$.

6 Complete a gradient table to determine the types of stationary points.

7 State the stationary points and their types.

WRITE/DISPLAY

a $g(x) = x^2(4 - x^2)$

$g(x) = 4x^2 - x^4$

$g'(x) = 8x - 4x^3$

For stationary points,

$$g'(x) = 0$$
$$8x - 4x^3 = 0$$
$$4x(2 - x^2) = 0$$
$$x = 0 \quad \text{or} \quad x^2 = 2$$
$$x = 0 \text{ or } x = -\sqrt{2} \text{ or } \sqrt{2}$$

When $x = 0$, $g(0) = 0$

When $x = -\sqrt{2}$, $g(-\sqrt{2}) = 4(-\sqrt{2})^2 - (-\sqrt{2})^4$
$= 4$

When $x = \sqrt{2}$, $g(\sqrt{2}) = 4(\sqrt{2})^2 - (\sqrt{2})^4$
$= 4$

Therefore the stationary points are $(-\sqrt{2}, 4)$, $(0, 0)$ and $(\sqrt{2}, 4)$.

Gradient table:

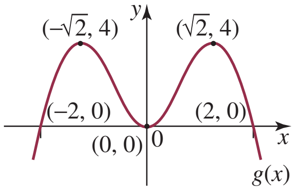
x	-2	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	2
$g'(x)$	+	0	–	0	+	0	–
Slope	/	—	\	_	/	—	\

Therefore $(-\sqrt{2}, 4)$ is a local maximum stationary point.

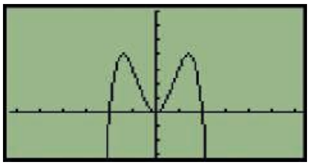
$(0, 0)$ is a local minimum stationary point.

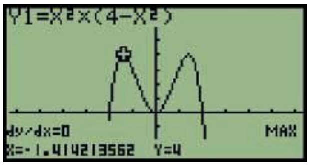
$(\sqrt{2}, 4)$ is a local maximum stationary point.

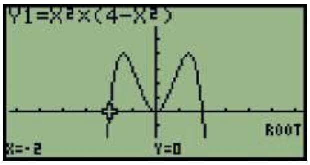
Continued over page →

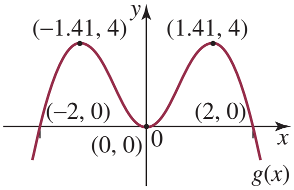
THINK	**WRITE/DISPLAY**
8 Solve $g(x) = 0$ to determine the x-intercepts.	x-intercepts: When $g(x) = 0$, $x^2(4 - x^2) = 0$ $x^2 = 0$ or $x^2 = 4$ $x = 0$ or $x = -2$ or 2 The x-intercepts are $(-2, 0)$, $(0, 0)$ and $(2, 0)$.
9 Find $g(0)$ to determine the y-intercept.	y-intercept: When $x = 0$, $g(0) = 0^2(4 - 0^2)$ $= 0$ The y-intercept is $(0, 0)$.
10 Sketch the graph of $g(x)$.	

For the Casio *fx*-9860G AU

b 1 To sketch the graph of $g(x) = x^2(4 - x^2)$, press: • MENU • 5 (Graph). Complete the entry line as: $Y1 = x^2(4 - x^2)$ then press EXE. Choose a suitable V-Window and press F6 (DRAW).	**b** 

2 To identify stationary points and axis intercepts, press:
• SHIFT
• F5 (G-Solv)
• F2 (MAX).
	
	`Y1=X²×(4-X²)`
	`dy/dx=0`
	`MAX`
	`X=-1.414213562 Y=4`

3 Repeat the above steps to find approximate values for maximum and minimum stationary points and intercepts.	
	`Y1=X²×(4-X²)`
	`ROOT`
	`X=-2 Y=0`

4 Sketch the graph, showing all important features. *Note:* This calculator will give only approximate values for the turning points.	

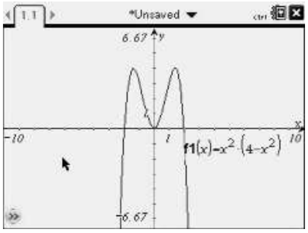
THINK

WRITE/DISPLAY

For the TI-Nspire CAS

b 1 To sketch the graph of $g(x) = x^2(4 - x^2)$, open a Graphs page.
Complete the entry line as:
$f1(x) = x^2(4 - x^2)$
then press ENTER enter.

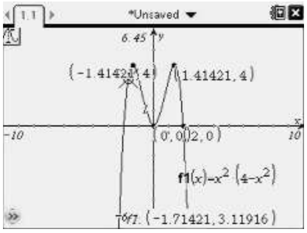
b



2 To identify important features, press:

- MENU menu
- 5: Trace 5
- 1: Graph Trace 1.

Use the arrow keys to trace along the graph; the maximum, minimum and zeros (x-intercepts) will be displayed as the cursor moves along the curve.



3 To find the turning points in exact form, on a Calculator page, press:

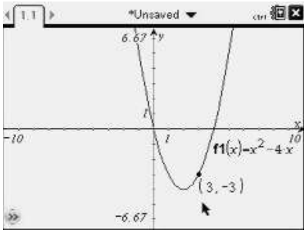
- MENU menu
- 3: Algebra 3
- 1: Solve 1
- MENU menu
- 4: Calculus 4
- 1: Derivative 1.

Complete the entry line as:

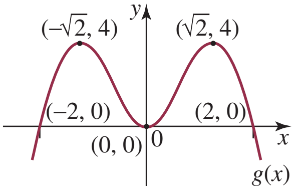
$$\text{solve}\left(\frac{d}{dx}(f1(x)) = 0,\ x\right)$$

then press ENTER enter.

The y-coordinates of the turning points can be found as shown.



4 Sketch the graph, showing all important features.

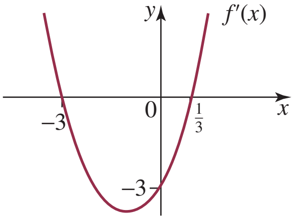


WORKED Example 3

If $f(x) = x^3 + 4x^2 - 3x - 7$

a **sketch the graph of $f'(x)$**

b **state the values of x where $f(x)$ is i increasing and ii decreasing.**

THINK	WRITE
a ① Write the rule for $f(x)$.	**a** $f(x) = x^3 + 4x^2 - 3x - 7$
② Differentiate $f(x)$ to find $f'(x)$.	$f'(x) = 3x^2 + 8x - 3$
③ Solve $f'(x) = 0$ to find the x-intercepts of $f'(x)$.	x-intercepts: When $f'(x) = 0$, $3x^2 + 8x - 3 = 0$ $(3x - 1)(x + 3) = 0$ $x = \frac{1}{3}$ or -3 The x-intercepts of $f'(x)$ are $(\frac{1}{3}, 0)$ and $(-3, 0)$.
④ Evaluate $f'(0)$ to find the y-intercept of $f'(x)$.	y-intercept: When $x = 0$, $f'(0) = -3$ so the y-intercept of $f'(x)$ is $(0, -3)$.
⑤ Sketch the graph of $f'(x)$ (an upright parabola).	
b **i** By inspecting the graph of $f'(x)$ deduce where $f'(x)$ is positive (that is, above the x-axis).	**b** **i** $f'(x) > 0$ where $x < -3$ and $x > \frac{1}{3}$ so $f(x)$ is increasing where $x < -3$ and $x > \frac{1}{3}$.
ii By inspecting the graph of $f'(x)$, deduce where $f'(x)$ is negative (that is, below the x-axis).	**ii** $f'(x) < 0$ where $-3 < x < \frac{1}{3}$ so $f(x)$ is decreasing where $-3 < x < \frac{1}{3}$.

remember

1. A stationary point (SP) occurs when $f'(x) = 0$.
2. There are four types of stationary points:
   - (a) *Local maximum* where the gradient is positive on the left of the SP and negative on the right
   - (b) *Local minimum* where the gradient is negative on the left of the SP and positive on the right
   - (c) *Positive point of horizontal inflection* where the gradient is positive on both sides of the SP
   - (d) *Negative point of horizontal inflection* where the gradient is negative on both sides of the SP.

EXERCISE 2A Sketching curves

In this exercise, your teacher will indicate the level of use of your calculator.

WORKED Example 1a, b

1 Find the stationary points and their nature for each of the following functions.

a $y = 8 - x^2$
b $f(x) = x^3 - 3x$
c $g(x) = 2x^2 - 8x$
d $f(x) = 4x - 2x^2 - x^3$
e $g(x) = 4x^3 - 3x^4$
f $y = x^2(x + 3)$
g $y = 5 - 6x + x^2$
h $f(x) = x^3 + 8$
i $y = -x^2 - x + 6$
j $y = 3x^4 - 8x^3 + 6x^2 + 5$
k $g(x) = x(x^2 - 27)$
l $y = x^3 + 4x^2 - 3x - 2$
m $h(x) = 12 - x^3$
n $g(x) = x^3(x - 4)$

WORKED Example 1c

2 Sketch the graph of each function in question **1**, clearly indicating all stationary points.

3 **a** Find the stationary points of the function $f(x) = x^3 - 2x^2 - 7x - 4$ and state their nature.
b Sketch the graph of $f(x)$.

4 **a** Find the stationary points, and their nature, for the curve $y = x^3 - x^2 - 16x + 16$.
b Sketch the graph.

5 **a** Find the stationary points of the function $g(x) = x^4 - 4x^2$ and state their nature.
b Sketch the graph of $g(x)$.

6 **a** If $y = x^4 - 6x^2 + 8x - 3$ then find each stationary point and its nature.
b Show that the point (1, 0) lies on the curve.
c Sketch the graph.

7 **a** If $y = x^4 + x^3 - 5x^2 - 6$ then find each stationary point and its nature.
b Find the y-intercept.
c Sketch the graph.

WORKED Example 2

8 Sketch the graph of each of the following functions, clearly indicating all stationary points and y-intercepts.

a $f(x) = x^4 - x^2$
b $f(x) = x^3 - 3x^2$
c $g(x) = x^3 + 3x^4$
d $g(x) = x^3 - 4x^2 + 4x$
e $h(x) = x^3 - 4x^2 - 11x + 30$
f $h(x) = x(x + 3)(x - 5)$
g $f(x) = x^4 - 2x^2 + 1$
h $f(x) = x(x^2 + 1)$
i $g(x) = x^3 + 9x^2 + 24x + 20$
j $h(x) = (x^2 - 1)^3$

9 **multiple choice**

If $f'(x) < 0$ where $x > 2$ and $f'(x) > 0$ where $x < 2$, then at $x = 2$, $f(x)$ has a:

A local minimum
B local maximum
C point of inflection
D discontinuous point
E gradient of 2

10 **multiple choice**

The function $f(x) = x^3 + x^2 - 8x - 3$ has stationary points when x is equal to:

A -2 and $\frac{4}{3}$
B 3 only
C -2 only
D 3 and $\frac{4}{3}$
E 0 and 2

11 **multiple choice**

The graph of $y = x^4 + x^3$ has:

A a local maximum where $x = 0$

B a local minimum where $x = 0$

C a local minimum where $x = -\frac{3}{4}$

D a local maximum where $x = -\frac{3}{4}$

E a local maximum where $x = -\frac{4}{3}$

12 **multiple choice**

A quadratic function has a turning point (2, 1) and a y-intercept of (0, 9). The equation must be:

A $y = (x - 2)^2 + 9$

B $y = (x - 1)^2 + 8$

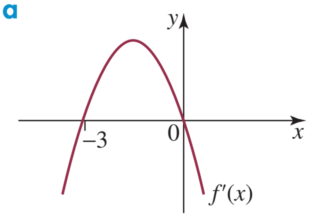
C $y = (x - 2)^2 + 1$

D $y = 2(x - 2)^2 + 9$

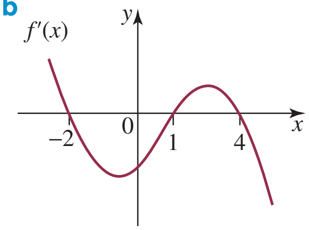
E $y = 2(x - 2)^2 + 1$

13 The graphs of $f'(x)$ are shown below. Find all values of x for which $f(x)$ has stationary points and state their nature.

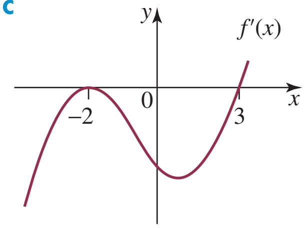
a



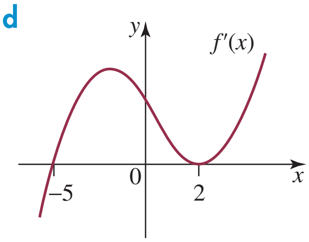
b



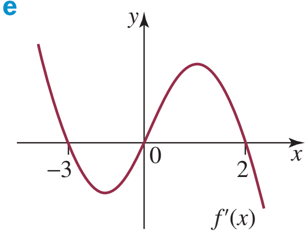
c



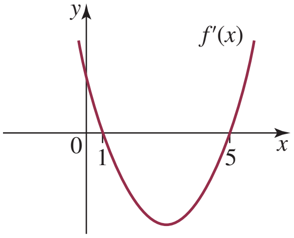
d



e



f



14 Show that $f(x) = x^2 - 4x + 3$ is decreasing for $x < 2$ and increasing for $x > 2$.

WORKED Example 3

15 For each of the following functions **i** sketch $f'(x)$ and, hence, state the values of x where $f(x)$ is **ii** increasing and **iii** decreasing.

a $f(x) = \frac{1}{3}x^3 + 2x^2 + 2$ **b** $g(x) = x^3 + 2x^2 - 7x - 5$ **c** $h(x) = x^4 + 4x^3 + 4x^2$

16 If $y = f(x)$ has the following properties then sketch its graph.

$f'(x) = 0$ if $x = -2$ and $x = 3$ $f'(x) < 0$ if $-2 < x < 3$ $f'(x) > 0$ for all other x

17 If $f(x) = x^3 + ax^2 + bx$ has stationary points at $x = 2$ and $x = -\frac{4}{3}$, find:

a the value of a and b

b the nature of the stationary points.

18 If $f(x) = x^4 + ax^2 + b$ has a stationary point (1, 4), find:

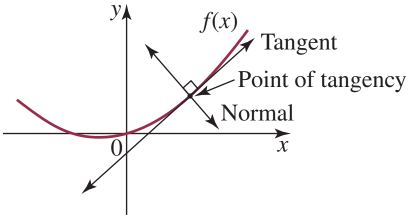
a a and b

b the other stationary points

c the nature of each stationary point.

Equations of tangents and normals

As we have seen, a tangent to a curve is a straight line which touches the curve at a given point and represents the gradient of the curve at that point. A normal to a curve is a straight line passing through the point where the tangent touches the curve and is perpendicular (at right angles) to the tangent at that point.



If the gradient of the tangent to a curve is *m*, then the gradient of the normal is $-\frac{1}{m}$ (as the product of the gradients of 2 perpendicular lines equals −1).

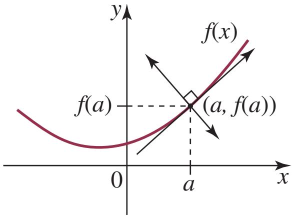
The equation of a straight line passing through the point (x_1, y_1) and with a gradient of *m* is:

$$y - y_1 = m(x - x_1)$$

The gradient of the tangent at $x = a$ is $f'(a)$.

Therefore the gradient of the normal is $\frac{-1}{f'(a)}$.

The equation of the tangent at $x = a$ is $y - f(a) = f'(a)\,(x - a)$ and the equation of the normal is $y - f(a) = \left(\frac{-1}{f'(a)}\right)(x - a)$.



eBook plus

Digital docs:

SkillSHEET 2.1
Finding equations of normals

SkillSHEET 2.2
Finding the equation of a straight line

WORKED Example 4

Find the equation of the tangent to $y = x^3 - 2x + 3$ at the point (1, 2).

THINK	WRITE
1 Write the equation.	$y = x^3 - 2x + 3$
2 Find $\frac{dy}{dx}$.	$\frac{dy}{dx} = 3x^2 - 2$
3 Evaluate $\frac{dy}{dx}$ where $x = 1$ to find the gradient of the tangent where $x = 1$.	At $x = 1$, $\frac{dy}{dx} = 3 - 2$ $= 1$ So gradient of tangent is 1.
4 Substitute (1, 2) for (x_1, y_1) and $m = 1$ into the rule for the equation of a straight line.	Equation of tangent at the point (1, 2) is $y - y_1 = m(x - x_1)$ $y - 2 = 1(x - 1)$ $y - 2 = x - 1$
5 Rearrange the rule to a simple form.	$y = x + 1$

WORKED Example 5

Find the equation of the normal to the curve with the equation $y = 3x^2 - 1$ at $x = 1$.

THINK	WRITE
1 Write the equation.	$y = 3x^2 - 1$
2 Evaluate y when $x = 1$.	At $x = 1$, $y = 3 - 1 = 2$
3 Find $\frac{dy}{dx}$.	$\frac{dy}{dx} = 6x$
4 Find the gradient at $x = 1$.	When $x = 1$, $\frac{dy}{dx} = 6$. So the gradient is 6.
5 Evaluate the gradient of the normal which is $\frac{-1}{\left(\frac{dy}{dx}\right)}$.	Gradient of normal is $-\frac{1}{6}$.
6 Determine the equation of the normal at (1, 2).	Equation of normal is $y - y_1 = m(x - x_1)$ $y - 2 = -\frac{1}{6}(x - 1)$ $-6y + 12 = x - 1$ $x + 6y - 13 = 0$ or $x + 6y = 13$

WORKED Example 6

Find the tangent to the curve $y = x^2 - 4x$ at the points $x = 0$, $x = 3$ and $x = 5$.

THINK

1 To find the tangent to the curve $y = x^2 - 4x$ at the points $x = 0$, $x = 3$ and $x = 5$, on a Graphs page, complete the entry line as:

$f1(x) = x^2 - 4x$

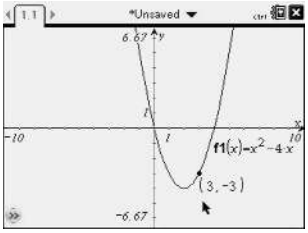
then press ENTER enter.

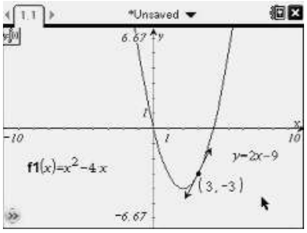
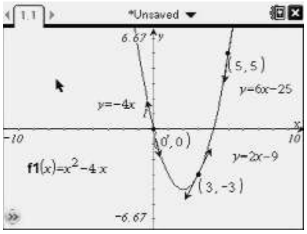
To place a point on the curve, press:

- MENU menu
- 7: Points and Lines 7
- 1: Point 1.

Use the arrow keys to move the pointer to any point on the graph. When the coordinates appear, press ENTER enter. Edit the x-coordinate of this point to display $(3, {}^{-}3)$.

WRITE/DISPLAY



THINK	WRITE/DISPLAY
2 To draw a tangent at a point, press: • MENU menu • 7: Points and Lines 7 • 7: Tangent 7. Use the arrow keys to move the pointer to the point on the curve (3, $^-3$). Press ENTER enter and the tangent will appear. Then press: • MENU menu • 1: Actions 1 • 7: Coordinates & Equations 7. Use the arrow keys to move the pointer to the tangent and the equation will appear.	
3 Repeat the steps above for $x = 0$ and $x = 5$.	
4 Write the answer.	For the curve $y = x^2 - 4x$ the equation of the tangent at the point (0, 0) is $y = -4x$, the equation of the tangent at the point ($^-3$, 3) is $y = 2x - 9$ and the equation of the tangent at the point (5, 5) is $y = 6x - 25$.

remember

1. If m is the gradient of the tangent, then $-\frac{1}{m}$ is the gradient of the normal.
2. The equation of a straight line passing through the point (x_1, y_1) with gradient m is
$$y - y_1 = m(x - x_1)$$
3. Parallel lines have the same gradient.

EXERCISE 2B Equations of tangents and normals

WORKED Example 4

1 Find the equation of the tangent to the curve $y = x^2 + x$ at the point (2, 6).

2 Find the equations of the tangent to the curve $y = x^2 + 5x - 6$ at the points where it crosses the x-axis.

WORKED Example 5

3 Find the equation of the normal to the curve $y = 3x^2 - 5x + 4$ at the point where $x = 1$.

4 Find the equation of the normal to the curve $y = \frac{1}{2}x^2 + 3x - 7$ at the point where it crosses the y-axis.

eBook plus

Digital docs:
Spreadsheet
023 Tangent & normal
WorkSHEET 2.1

5 For each of the following functions, find the equation of:
- **i** the tangent
- **ii** the normal at the given value of x.

a $y = x^2 + 1,\ x = 1$
b $y = x^3 - 6x,\ x = -2$
c $y = \dfrac{1}{x},\ x = 2$
d $y = (x - 1)(x^2 + 2),\ x = -1$
e $y = \sqrt{x},\ x = 4$
f $y = \sqrt{2x + 3},\ x = 3$
g $y = x(x + 2)(x - 1),\ x = -1$
h $y = x^3 - 3x^2 + 4x,\ x = 0$
i $y = 2x^3 + x^2 - 6x + 2,\ x = 1$

6 **multiple choice**

If $y = (2x + 3)^4$ then at the point $(-1, 1)$

a the equation of the normal is:

A $y + 4x - 3 = 0$
B $8y + x - 7 = 0$
C $y - 4x + 5 = 0$
D $y + 2x + 8 = 0$
E $2y - x = 0$

b the value of x where the gradient of the tangent is parallel to the x-axis is:

A $-\frac{2}{3}$
B $\frac{1}{3}$
C $-\frac{3}{2}$
D $-\frac{1}{3}$
E $\frac{2}{3}$

7 Find the equation of the tangent to $f(x) = x^2 + 4x + 1$ which is parallel to the line $y = 2x + 3$.

8 Find the equation of the tangent to $y = \dfrac{x^2 + 1}{x^2 - 1}$ at $x = 0$.

WORKED Example 6

9 Using appropriate technology, find the equation of the tangent to the curve $y = 3x^2 + x - 2$ when $x = -\dfrac{1}{2}$.

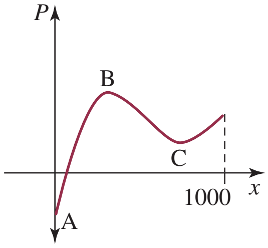
Maximum and minimum problems when the function is known

In many practical situations we need to find the maximum or minimum value of a function. For example, it is important for manufacturers to minimise the cost of running their business; it is also important that they maximise profits.

Suppose that the graph shown below gives the profit, P, for manufacturing x components in a week. The number of components that can be manufactured in a week varies from 0 to 1000.

The point B is called a *local maximum* and clearly $\dfrac{dP}{dx} = 0$ at B.

The point C is called a *local minimum* and clearly $\dfrac{dP}{dx} = 0$ at C.



Looking at the graph we can see that the greatest value of P occurs at B. The least value occurs, not at C, but at A when $x = 0$. Note that the point C is a local minimum but it is not the least value of $f(x)$.

In general, to find the greatest or least value of a function, $f(x)$, on an interval $a \leq x \leq b$:

1. Find $f'(x)$.
2. Find the values of x for which $f'(x) = 0$.
3. Compute the values of $f(x)$ using x-values obtained in step 2. Compute $f(a)$ and $f(b)$.
4. The greatest (least) value of $f(x)$ is the largest (smallest) of the values obtained in step 3.

WORKED Example 7

The population of a colony of birds at time t months, where $0 \leq t \leq 3$, after observation began can be modelled by the function:

$P(t) = 20t(9 - t^2) + 300$,

where P is the number of birds.

Find:

- **i the initial population**
- **ii the greatest number of birds**

a by hand

b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a i 1 Write the rule for $P(t)$.	**a i** $P(t) = 20t(9 - t^2) + 300$
2 The initial value occurs when $t = 0$ and is $P(0)$.	$P(0) = 0 + 300$ $= 300$ So the initial population of birds is 300.
ii 1 Find $P'(t)$ using the product rule.	**ii** $P'(t) = (20t)(-2t) + (9 - t^2)(20) + 0$ $= -40t^2 + 180 - 20t^2$ $= 180 - 60t^2$
2 Solve $P'(t) = 0$.	For maximum and/or minimum, $P'(t) = 0$ $-60t^2 + 180 = 0$ $t^2 = 3$ $t = \pm\sqrt{3}$ Reject $t = -\sqrt{3}$ so $t \approx 1.73$
3 Compute $P(t)$ for points where $\frac{dP}{dt} = 0$ and for the end points.	$P(1.73) = 507.8$ $P(0) = 300$ $P(3) = 300$
4 Write the answer in a sentence.	The greatest number of birds is 507.

Continued over page

THINK

b ① To find the initial population when $P(t) = 20t(9 - t^2) + 300$, on a Calculator page, press:

- MENU (menu)
- 1: Actions (1)
- 1: Define (1).

Complete the entry lines as:

Define $p(t) = 20t \cdot \ (9 - t^2) + 300$

$p(0)$

pressing ENTER (enter) after each line.

② To find the time when the number of birds is a maximum, complete the entry line as:

$\text{fMax}(P(t),t)\,|\,0 \le t \le 3$

then press ENTER (enter).

To find the maximum number of birds, complete the entry line as:

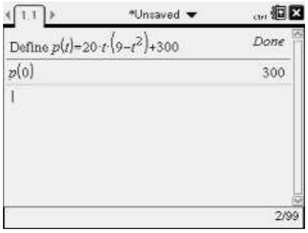
$p(0)$

then press ENTER (enter).

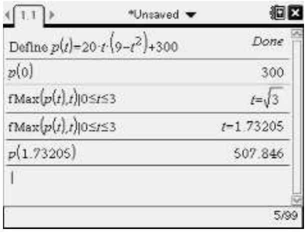
③ Write the answer. (Remember the number of birds must be an integer.)

WRITE/DISPLAY

b



```
Define p(t)=20·t·(9-t²)+300        Done
p(0)                               300
|
                                  2/99
```



```
Define p(t)=20·t·(9-t²)+300        Done
p(0)                               300
fMax(p(t),t)|0≤t≤3               t=√3
fMax(p(t),t)|0≤t≤3          t≈1.73205
p(1.73205)                     507.846
|
                                  5/99
```

i For $P(t) = 20t(9 - t^2) + 300$, the initial population of birds is $P(0) = 300$.

ii Solve $P'(t) = 0$ for $0 \le t \le 3$.
The maximum number of birds is 507 after 1.73 months.

WORKED Example 8

eBook*plus*

Tutorial:
Worked example 8
int-0560

The displacement of a particle moving in a straight line from the origin at any time, t, is $x(t) = \frac{1}{3}t^3 - 4t^2 + 12t + 1$, $0 \le t \le 7$.

Find the maximum and minimum displacement.

THINK

① Write the rule.

② Find $x'(t)$.

③ Solve $x'(t) = 0$.

WRITE

$$x(t) = \tfrac{1}{3}t^3 - 4t^2 + 12t + 1$$

$$x'(t) = t^2 - 8t + 12$$

For maximum and/or minimum,

$$t^2 - 8t + 12 = 0$$
$$(t - 2)(t - 6) = 0$$
$$t = 2 \text{ or } t = 6$$

THINK **WRITE**

4 Test for maximum and minimum using the first derivative test.

Gradient table:

t	0	2	4	6	8
$x'(t)$	+	0	−	0	+
Slope	/	−	\	_	/

When $t = 2$ a local maximum occurs and when $t = 6$ a local minimum occurs.

5 Evaluate $x(2)$ to find the local maximum.

$$\begin{aligned} x(2) &= \tfrac{1}{3}(2)^3 - 4(2)^2 + 12(2) + 1 \\ &= \tfrac{8}{3} - 16 + 24 + 1 \\ &= 11\tfrac{2}{3} \end{aligned}$$

so a local maximum occurs at $(2, 11\frac{2}{3})$.

6 Evaluate $x(6)$ to find the local minimum.

$$\begin{aligned} x(6) &= \tfrac{1}{3}(6)^3 - 4(6)^2 + 12(6) + 1 \\ &= 72 - 144 + 72 + 1 \\ &= 1 \end{aligned}$$

so a local minimum occurs at (6, 1).

7 Evaluate $x(0)$ and $x(7)$ to see if the end points give a smaller or larger value.

$x(0) = 1$ which is equal to the local minimum above

$$\begin{aligned} x(7) &= \tfrac{1}{3}(7)^3 - 4(7)^2 + 12(7) + 1 \\ &= 114\tfrac{1}{3} - 196 + 84 + 1 \\ &= 3\tfrac{1}{3} \end{aligned}$$

which is between the local maximum and minimum above.

Therefore the maximum displacement is $11\frac{2}{3}$ units from the origin and the minimum displacement is 1 unit from the origin.

remember

1. To find a maximum and/or minimum, let $f'(x) = 0$.
2. If the function has a limited domain, $f'(x)$ may not be useful for finding maximum and/or minimum values.
3. Always prove your maximum and/or minimum value using a slope or gradient diagram.

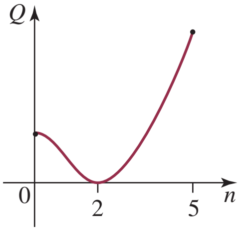
EXERCISE 2C Maximum and minimum problems when the function is known

1 For each of the following, write the expression for the derivative indicated.

a $C = 5x - 2x^2 + 10, \dfrac{\mathrm{d}C}{\mathrm{d}x}$ **b** $P = n^3 + 2n^2 - 5\sqrt{n}, \dfrac{\mathrm{d}P}{\mathrm{d}n}$

c $V = \frac{1}{4}h^4 - \frac{2}{3}h^3, \dfrac{\mathrm{d}V}{\mathrm{d}h}$

For questions **2** and **3**, the function $Q = n^3 - 3n^2 + 5$, $0 \le n \le 5$, is graphed at right.



2 **multiple choice**

The minimum value occurs where n equals:

A -2 **B** 0 **C** 1
D 2 **E** 5

3 **multiple choice**

The maximum value occurs where n is equal to:

A 0 **B** 2 **C** 5 **D** 1 **E** -1

WORKED Example 7

4 The profit, \$$P$ per week, of a small manufacturing company is related to the number of workers, n, by:

$$P = -\tfrac{1}{2}n^3 + 96n + 600$$

Find:

a the number of workers needed for maximum profit per week

b the maximum profit per week.

5 The cost, \$$C$, of producing x-metre lengths of a certain tow rope is:

$C = \frac{1}{5}x^2 - 8x + 100$ per rope, $x > 0$

What is the length of the cheapest tow rope that can be produced?

WORKED Example 8

6 The number of people, P, visiting a certain beach on a particular day in January depends on the number of hours, x, that the temperature is below 30 °C according to the rule $P = x^3 - 12x^2 + 21x + 105$ where $x \ge 0$.

Find the value of x for the maximum and minimum number of people who visit the beach.

7 The number of rabbits, N, which feed on a particular farmland on any night can be modelled as $N = -\frac{1}{3}x^3 + 5x^2 + 75x + 500$ where x is the average overnight temperature in °C and $x \geq -5$.
Find:

a the temperature for the minimum and maximum number of rabbits
b the minimum and maximum number of rabbits that will be found on the farmland.

8 The length of a snake, L cm, at time t weeks after it is born is modelled as:

$$L = 12 + 6t + 0.01(20 - t)^2, \; 0 \leq t \leq 20$$

Find:

a the length at **i** birth and **ii** 20 weeks
b R, the rate of growth, at any time, t **c** the maximum and minimum growth rate.

9 The profit, \$$P$, per item that a store makes by selling n items of a certain type each day is $P = 40\sqrt{n+25} - 200 - 2n$.

a Find the number of items that need to be sold to maximise the profit on each item.
b What is the maximum profit per item?
c Hence, find the total profit per day by selling this number of items.

Maximum and minimum problems when the function is unknown

When solving maximum and/or minimum problems when the function is not given directly, a rule for the function must be obtained from the given information. This rule should have the quantity being maximised or minimised in terms of one variable only. Sometimes a diagram will assist in establishing the rule. Then solve the problem using differentiation.

These steps should be followed.

1. Draw a diagram if appropriate.
2. Identify the quantity to be maximised or minimised.
3. Express this quantity in terms of one variable only.
4. Solve $f'(x) = 0$.
5. Verify it is a maximum or minimum using the first derivative test.
6. Answer the question which has been asked.

WORKED Example 9

The sum of two positive numbers is 10. Find the numbers if the sum of their squares is a minimum.

THINK	WRITE
1 Let x = one number and y = the other number and form an equation.	Let x = 1st number and y = 2nd number. $x + y = 10$
2 Express y in terms of x.	So $y = 10 - x$
3 Write an expression for $S(x)$, the sum of the squares of x and y, in terms of x only.	$S(x) = x^2 + y^2$ $= x^2 + (10 - x)^2$

Continued over page →

THINK	**WRITE**
4 Simplify the equation.	$= x^2 + 100 - 20x + x^2$ $= 2x^2 - 20x + 100$
5 Find $S'(x)$.	$S'(x) = 4x - 20$
6 Find x for minimum S by solving $S'(x) = 0$.	Function has a stationary point when $S'(x) = 0$ $4x - 20 = 0$ $4x = 20$ $x = 5$
7 Verify that it is a minimum by the first derivative test.	Gradient table:

x	4	5	6
$S'(x)$	–	0	+
Slope	\	_	/

So $x = 5$ gives a minimum for S.

8 Find y.	When $x = 5$, $y = 10 - 5$ $= 5$
9 State the two numbers.	Therefore, the two numbers which give a minimum of their squares are both 5.

Note: The actual sum was not asked for in this example.

WORKED Example 10

eBook*plus*

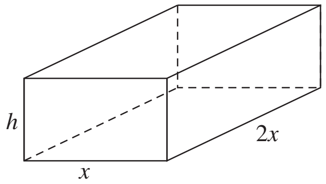
Tutorial:
Worked example 10
int-0561

A cuboid container with a base length twice its width is to be made with 48 m^2 of metal.

a **Show that the height $h = \frac{8}{x} - \frac{2x}{3}$, where x is the width of the base.**

b **Express the volume, V, in terms of x.**

c **Find the maximum volume.**

THINK	**WRITE**
a 1 Draw a diagram of a cuboid.	**a** 
2 Let x = width of base and hence express length in terms of x.	Let x = width and h = height so length = $2x$

THINK | **WRITE**

③ Calculate the total surface area (T.S.A) of the cuboid in terms of x and h only.

$$\begin{aligned}\text{T.S.A.} &= 2[2x(x) + 2x(h) + x(h)]\\ &= 2(2x^2 + 3xh)\\ &= 4x^2 + 6xh\end{aligned}$$

④ Express h in terms of x.

As T.S.A. = 48 m^2

$$\begin{aligned}4x^2 + 6xh &= 48\\ 6xh &= 48 - 4x^2\\ h &= \frac{48 - 4x^2}{6x}\\ &= \frac{48}{6x} - \frac{4x^2}{6x}\\ h &= \frac{8}{x} - \frac{2x}{3}\end{aligned}$$

b ① Find the volume, V, in terms of x and h.

b $V = x(2x)h$

② Express the volume in terms of x by substituting for h.

$$\begin{aligned}V(x) &= 2x^2\left(\frac{8}{x} - \frac{2x}{3}\right)\\ &= 16x - \frac{4x^3}{3}\end{aligned}$$

c ① Solve $V'(x) = 0$.

c $V'(x) = 16 - 4x^2 = 0$ for maximum or minimum

$4x^2 = 16$

$x^2 = 4$

$x = 2$ or -2 (reject -2 as width cannot be negative)

② Verify that $x = 2$ gives a maximum.

Gradient table:

x	**1**	**2**	**3**
$V'(x)$	+	0	–
Slope	/	—	\

The maximum volume is achieved when $x = 2$ m.

③ Substitute $x = 2$ into the rule for $V(x)$ to obtain the maximum volume.

$$\begin{aligned}V(2) &= 16(2) - \frac{4(2)^3}{3}\\ &= 32 - \tfrac{32}{3}\\ &= 32 - 10\tfrac{2}{3}\\ &= 21\tfrac{1}{3}\end{aligned}$$

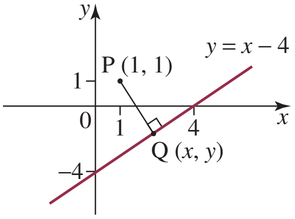
Therefore the maximum volume is $21\tfrac{1}{3}$ m^3.

WORKED Example 11

Find the minimum distance from the straight line with equation $y = x - 4$ to the point (1, 1):

a by hand

b using the TI-Nspire CAS calculator.



THINK | **WRITE/DISPLAY**

a 1 The minimum distance between a straight line and a point is a perpendicular line from a point on the straight line, Q (x, y), to the point P (1, 1).

a Let Q be the point (x, y).

2 From the given rule for the straight line, find y in terms of x.

As Q is on the line $y = x - 4$ then Q is $(x, x - 4)$.

3 Find the distance $d(x)$ between P and Q in terms of x only using the formula for the distance between 2 points.

$$\begin{aligned} d(x) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x - 1)^2 + (x - 4 - 1)^2} \\ &= \sqrt{(x - 1)^2 + (x - 5)^2} \\ &= \sqrt{x^2 - 2x + 1 + x^2 - 10x + 25} \\ &= \sqrt{2x^2 - 12x + 26} \end{aligned}$$

4 Differentiate $d(x)$ using the chain rule (or differentiate the expression inside the square root only, as the smallest value of this expression will give the minimum distance when the square root is taken).

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 - 12x + 26)^{\frac{1}{2}}$$

$$= \frac{1}{2} \cdot (4x - 12) \cdot \frac{1}{(2x^2 - 12x + 26)^{\frac{1}{2}}}$$

$$= \frac{4x - 12}{2\sqrt{2x^2 - 12x + 26}}$$

OR

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 - 12x + 26)$$

$$= 4x - 12$$

5 Solve for x where the derivative equals zero.

For maximum or minimum,

$$\frac{4x - 12}{2\sqrt{2x^2 - 12x + 26}} = 0$$

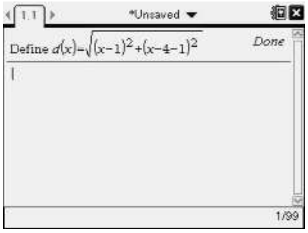
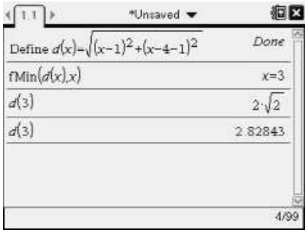
$$\begin{aligned} 4x - 12 &= 0 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

6 Verify that $x = 3$ gives a minimum.

Gradient table:

x	2	3	4
Derivative $(4x - 12)$	–	0	+
Slope	\	_	/

So $x = 3$ gives the minimum distance.

THINK	**WRITE/DISPLAY**
7 Evaluate $d(3)$ to obtain the minimum distance. The exact answer is $\sqrt{8}$ and 2.828 is an approximate answer.	$d(3) = \sqrt{2(3)^2 - 12(3) + 26}$ $= \sqrt{18 - 36 + 26}$ $= \sqrt{8}$ or ≈ 2.828 Therefore the minimum distance is $\sqrt{8}$ units or approximately 2.828 units.
For the TI-Nspire CAS	
b 1 The minimum distance between a straight line and a point is a perpendicular line from a point on the straight line, Q (x, y), to the point P (1, 1).	**b** Let Q be the point (x, y).
2 The y-coordinate of Q can be expressed in terms of x, as Q is on the line $y = x - 4$.	P is (1, 1). Q is $(x, x - 4)$.
3 The distance between P and Q can be found using the formula for distance between two points.	$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $d(x) = \sqrt{(x - 1)^2 + (x - 4 - 1)^2}$
4 On a Calculator page, complete the entry line as: Define $d(x) = \sqrt{(x - 1)^2 + (x - 4 - 1)^2}$ then press ENTER ⏎.	 Define $d(x)=\sqrt{(x-1)^2+(x-4-1)^2}$ Done
5 To find the value of x for which $d(x)$ is a minimum, complete the entry lines as: fMin($d(x)$, x) $d(3)$ pressing ENTER ⏎ after each line.	 Define $d(x)=\sqrt{(x-1)^2+(x-4-1)^2}$ Done fMin($d(x)$,x) $x=3$ $d(3)$ $2\cdot\sqrt{2}$ $d(3)$ 2.82843
6 Write the answer.	Using a CAS calculator, $d(x)$ is a minimum when $x = 3$. Therefore the minimum distance is approximately 2.83 units.

remember

1. Express the quantity being maximised or minimised in terms of one variable only.
2. Find the derivative and let it equal zero to find the maximum or minimum.
3. Test for maximum and/or minimum using the first derivative test (gradient table).
4. Use exact answers where appropriate. If using an approximate answer show by using the ≈ symbol.
5. Ensure you have answered the question asked.

EXERCISE 2D Maximum and minimum problems when the function is unknown

In this exercise, your teacher will indicate the level of use of your calculator.

WORKED Example 9

1 The sum of two positive numbers is 10. Find the numbers if their product is a maximum.

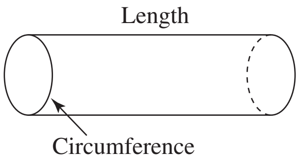
2 The sum of two positive numbers is 8. Find the numbers if the sum of the cube of one and the square of the other is a minimum.

3 A rectangular frame is to be made from a piece of wire 120 cm long.

- **a** If the width is x cm, show that the length is $60 - x$.
- **b** Find an expression for the area of the rectangle in terms of x.
- **c** Hence, find the dimensions for maximum area.
- **d** Find the maximum area.

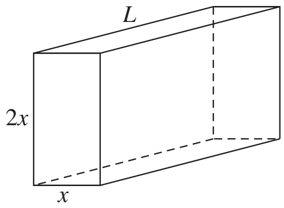
4 Find the area of the largest rectangular paddock that can be achieved with 400 metres of fencing.

5 Cylindrical, cardboard postal tubes are made with the restriction that the sum of the length and the circular circumference are 120 cm. What should the dimensions be for maximum volume?



6 The frame of a container in the shape of a cuboid is shown at right.
If it is to be made with a total length of 18 metres of steel edging, find:

- **a** the value of L in terms of x
- **b** the expression for the volume in terms of x only
- **c** the length of each edge for maximum volume
- **d** the maximum volume.

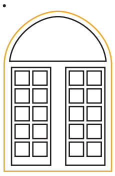


WORKED Example 10

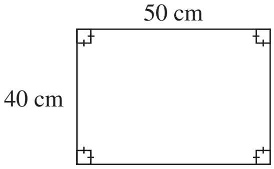
7 A cuboid with a square base is to be made with 200 cm^2 of material.

- **a** Show that the height, $h = \frac{50}{x} - \frac{x}{2}$, where x is the side length of the base.
- **b** Express the volume, V, in terms of x.
- **c** Find the maximum volume (to the nearest unit).

8 A window frame is in the shape of a semicircle joined to a rectangle. Find the maximum area of a window using 300 cm of framework.

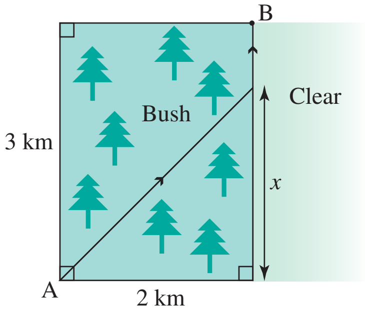


9 A rectangular sheet of cardboard, 50 cm by 40 cm, is to have square corners cut out so it can be folded into a rectangular tray.
Find the maximum volume possible for such a tray.



10 A bushwalker can walk at 5 km/h through clear land and 3 km/h through bushland. If she has to get from point A to point B following a route indicated at right, find the value of x so that the route is covered in a minimum time.

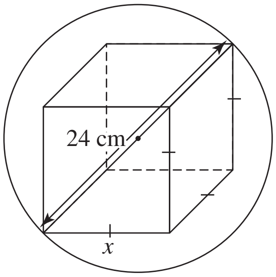
$\left(\textit{Note:}\ \text{time} = \dfrac{\text{distance}}{\text{speed}}\right)$



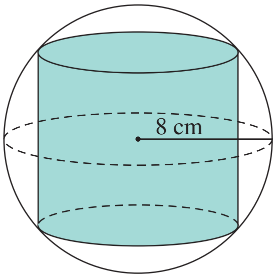
11 The cost of running a train at a constant speed of v km/h is $C = 50 + \dfrac{v^2}{1000}$ dollars/hour.

a Find the time taken for an 800-km journey in terms of v.
b Hence, find an expression for the cost of an 800-km journey.
c Find the most economical speed for this journey.

12 Find the side length of the largest cube which can fit inside a sphere of diameter 24 cm.

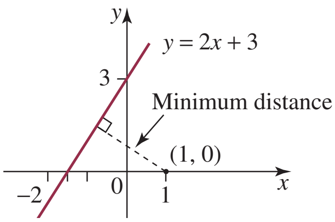


13 A cylinder of cheese is to be removed from a spherical piece of cheese of radius 8 cm. What is the maximum volume of the cylinder of cheese? (Express the answer to the nearest unit.)

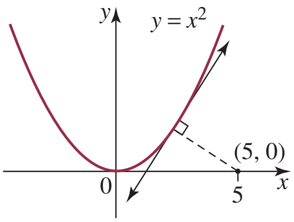


WORKED Example 11

14 Find the minimum distance from the line $y = 2x + 3$ to the point (1, 0).



15 Find the minimum distance from the parabola $y = x^2$ to the point (5, 0). (Express the answer to the nearest hundredth.)



eBook*plus*

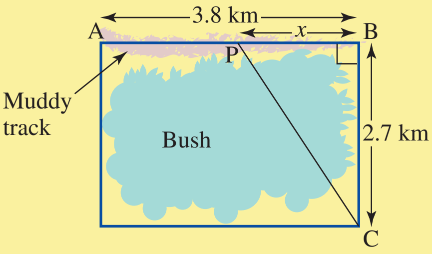
Digital doc:
WorkSHEET 2.2

16 A comet is following a path in the same plane as Mars according to the rule $y = \frac{1}{2}x^2 - 4$ where Mars is the origin and x and y are in millions of kilometres.

What is the closest distance the comet gets to Mars?

Cross-country run

Joanne is training for a cross-country run. Part of her course requires Joanne to run along a muddy track and then through dense bush. Joanne needs to work out how far along the muddy track she should run before she should go through the bush. She knows that she can average 5 m/s along the muddy track and 2 m/s in the bush. The muddy track is 3.8 km long and the bush is 2.7 km deep.



Find:

1. The distance from P to C.
2. Write an expression for the time taken to travel from P to C.
3. Write an expression for the total distance from A to C via P.
4. Write an expression for the total time taken for Joanne to travel from A to C.
5. Find the value of x, which will give Joanne the minimum time to run the course.
6. Find the time it takes Joanne to run the course.
7. Joanne's friend Peter, who is helping Joanne train, does not believe that her value for x is correct. He believes that Joanne should run 1.5 km along the muddy track before heading through the bush. What is the time difference for Joanne to run the course following Peter's advice?

Rates of change

eBook plus

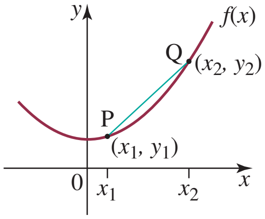
Interactivity: **Rates of change** int-0253

The derivative of a function is used to find the gradient of a function or the rate of change of a function.

The derivative of y with respect to x is $\frac{dy}{dx}$, which gives the instantaneous rate of change of y with respect to x.

If $\frac{dy}{dx} > 0$ then y is increasing as x increases (gradient is positive) and if $\frac{dy}{dx} < 0$ then y is decreasing as x increases (gradient is negative).

The average rate of change of a function is like the gradient of a straight line between two points.



The average rate of change from $x = x_1$ to $x = x_2$ is:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Note: Rates of change are often calculated with respect to time, but not always. If you are required to find the rate of change with respect to some quantity other than time then the quantity must be stated. If this quantity is not stated then the rate of change is taken as being with respect to time.

WORKED Example 12

a **Find the rate of change of the surface area of a melting ice cube with respect to its side length.**

b **What is the rate of change when $x = 2$ cm? (Assume that the ice cube remains in the shape of a cube.)**

THINK	WRITE
a 1 Express the surface area, S, of the cube in terms of its side length, x.	**a** $S = 6x^2$ (total surface area of a cube)
2 Find $\frac{dS}{dx}$.	$\frac{dS}{dx} = 12x$
3 Place a negative sign in front of the rate as the ice cube is melting (that is, the rate is decreasing).	But $\frac{dS}{dx} = -12x$ because the surface area is decreasing.
b 1 Substitute $x = 2$ into $\frac{dS}{dx}$.	**b** When $x = 2$ $\frac{dS}{dx} = -12(2)$ $= -24$
2 State the solution.	Therefore, the rate of change of the surface area when $x = 2$ cm is -24 cm^2/cm (it is decreasing at a rate of 24 cm^3/cm).

WORKED Example 13

The number of mosquitoes, N, around a dam on a certain night can be modelled by the equation

$$N = \frac{400}{2t+1} + 100t + 1000$$

where t equals hours after sunset. Find:

- **a** **the initial number of mosquitoes**
- **b** **the average rate of change in the first 4 hours**
- **c** **the rate of change at any time, t**
- **d** **the rate of change when $t = 4$ hours.**

THINK	WRITE
a ① Write the rule.	**a** $N = \frac{400}{2t+1} + 100t + 1000$
② Find N when $t = 0$.	When $t = 0$, $N = \frac{400}{1} + 0 + 1000$ $= 1400$ The initial number of mosquitoes is 1400.
b ① Find N when $t = 4$.	**b** When $t = 4$, $N = \frac{400}{2 \cdot 4 + 1} + 100 \cdot 4 + 1000$ $= 44.4 + 400 + 1000$ $= 1444.4$ that is, 1444 mosquitoes (as the number has not reached 1445).
② Calculate the average rate between $t = 0$ and $t = 4$.	The average rate of change in the first 4 hours $= \frac{1444 - 1400}{4}$ $= \frac{44}{4}$ $= 11$ mosquitoes per hour.
c Differentiate N with respect to t, to find $\frac{dN}{dt}$.	**c** $\frac{dN}{dt} = \frac{-800}{(2t+1)^2} + 100$
d Find $\frac{dN}{dt}$ when $t = 4$.	**d** When $t = 4$ the rate of change is: $\frac{dN}{dt} = -\frac{800}{9^2} + 100$ $= 90.1$ mosquitoes per hour.

remember

1. The average rate of change is like the gradient of a straight line between two points.
2. The average rate of change $= \dfrac{y_2 - y_1}{x_2 - x_1}$.
3. The instantaneous rate of change of $y = f(x)$ at $x = x_1$ is given by $f'(x_1)$.
4. The quantities acceleration (a), velocity (v) and displacement (x) are related. $a = \dfrac{dv}{dt}$ and $v = \dfrac{dx}{dt}$.

EXERCISE 2E Rates of change

eBook plus

Digital doc: **SkillSHEET 2.3** Subtracting function values

In the following exercise, use a graphics calculator to assist with any graphing.

1 Express the following in simplest mathematical notation.
- **a** The rate of change of volume, V, with respect to radius, r.
- **b** The rate of change of surface area, S, with respect to height, h.
- **c** The rate of change of area, A, with respect to time, t.
- **d** The rate of change of cost, C, with respect to distance, x.
- **e** The rate of change of intensity, I, with respect to pressure p.
- **f** The rate of change of velocity.

WORKED Example 12

2 **a** Find the rate of change of the area, $A = \pi r^2$, of an increasing circular oil spill with respect to the radius, r.
- **b** What is the rate of change when $r = 10$ metres?

3 **a** Find the rate of change of the volume, $V = \frac{4}{3}\pi r^3$, of a deflating spherical balloon with respect to the radius, r.
- **b** Hence, find the rate when $r = 5$ cm.

4 A sugar cube dissolves in a cup of tea. Find the rate of decrease of its surface area, S, when the side length, x, is 0.5 cm.

WORKED Example 13

5 The height of a projectile t seconds after being fired is $h = 10 + 20t - 5t^2$ metres. Find:
- **a** the initial height of the projectile
- **b** the average rate of change in the first 3 seconds
- **c** the rate of change at any time, t
- **d** the rate of change of height after **i** 1 second and **ii** 3 seconds. Explain the difference between these two answers.

6 The volume of water (in litres) which has flowed through a swimming pool filter t minutes after starting it is $V = \frac{1}{100}\left(30t^3 - \dfrac{t^4}{4}\right)$ where $0 \le t \le 90$.
- **a** At what rate is water flowing through the filter at any time, t?
- **b** Sketch a graph of $\dfrac{dV}{dt}$ over the domain [0, 90].
- **c** When is the rate of flow greatest?

7 A particle moves in a straight line so that its displacement from a point, O, at any time, t, is $x = \sqrt{3t^2 + 4}$. Find:

a the velocity as a function of time

b the acceleration as a function of time

c the velocity and acceleration when $t = 2$.

8 If a particle is moving in a straight line so that its displacement from the origin at any time, t, is $x = t^3 - 12t^2 + 36t$, find:

a the velocity

b the time and displacement when the velocity is zero

c the acceleration when the velocity is zero.

9 **multiple choice**

If $V = t^3 - 2t^2 + 8\sqrt{t}$, where V is in m^3 and t is in hours

a the rate of change of V with respect to t when $t = 4$ is:

A 48 m^3/h **B** 32 m^3/h **C** 28 m^3/h **D** 34 m^3/h **E** –20 m^3/h

b the average rate of change from $t = 1$ to $t = 4$ is:

A $13\frac{2}{3}$ m^3/h **B** $10\frac{1}{3}$ m^3/h **C** 3 m^3/h **D** $-10\frac{1}{3}$ m^3/h **E** 0 m^3/h

10 The amount of chlorine in a jug of water t hours after it was filled from a tap is $C = \frac{20}{t+1}$, where C is in millilitres. Find the rate of decrease of chlorine 9 hours after being poured.

summary

Stationary points

- A function $f(x)$ has stationary points wherever its derivative $f'(x) = 0$.
- If $f'(a) = 0$, then $f(x)$ has a stationary point $(a, f(a))$ which is:
  1. A local minimum turning point if:
     for $x < a, f'(x) < 0$
     and $x > a, f'(x) > 0$.
  2. A local maximum turning point if:
     for $x < a, f'(x) > 0$
     and $x > a, f'(x) < 0$.
  3. A positive stationary point of inflection if:
     for $x < a, f'(x) > 0$
     and $x > a, f'(x) > 0$.
  4. A negative stationary point of inflection if:
     for $x < a, f'(x) < 0$
     and $x > a, f'(x) < 0$.

Equations of tangents and normals

- The gradient of the tangent at $x = a$ to the curve $y = f(x)$ is $f'(a)$.
- The gradient of the normal at $x = a$ to the curve $y = f(x)$ is $\dfrac{-1}{f'(a)}$.
- The equation of a straight line with gradient m and passing through the point (x_1, y_1) is:

$$y - y_1 = m(x - x_1)$$

Maximum and minimum problems

- When solving maximum or minimum problems follow these steps:
  1. draw a diagram if appropriate
  2. identify the quantity to be maximised or minimised (say, $f(x)$)
  3. express the quantity in terms of one variable only (say x)
  4. solve $f'(x) = 0$
  5. verify it is a maximum or minimum using the first derivative test
  6. sketch a graph to confirm the maximum or minimum found
  7. answer the question.

Rates of change

- The instantaneous rate of change of $y = f(x)$ at $x = x_1$ is given by $f'(x_1)$.
- The average rate of change of $y = f(x)$ between $x = x_1$ and $x = x_2$ is:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

CHAPTER review

2A **1** **multiple choice**

The graph of $y = (x + 2)^3$ has:

A 1 turning point
B 2 turning points
C 1 point of inflection
D 2 points of inflection
E 0 stationary points

2A **2** **multiple choice**

The graph of $x^3 + 2x^2 + x - 2$ has:

A 2 points of inflection
B 1 point of inflection
C 1 turning point and 1 point of inflection
D 3 turning points
E 2 turning points

2A **3** **multiple choice**

The graph of $\frac{1}{3}x^3 - 4x^2 - 9x + 5$ has a local maximum turning point at:

A $(-1, 9\frac{2}{3})$
B $(9, 20)$
C $(9, 18)$
D $(1, -10\frac{2}{3})$
E $(-1, 7\frac{1}{3})$

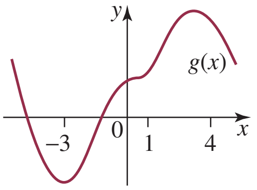
2A **4** **multiple choice**

If the graph of $g(x)$ has the following properties:

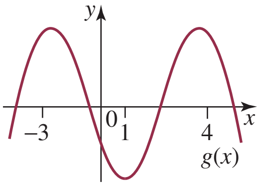
- **i** $g'(x) = 0$ if $x = -3$, 1 and 4
- **ii** $g'(x) < 0$ if $x < -3$ and $1 < x < 4$
- **iii** $g'(x) > 0$ for all other x

then the graph of $g(x)$ could be:

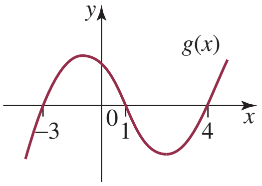
A



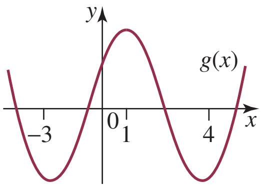
B



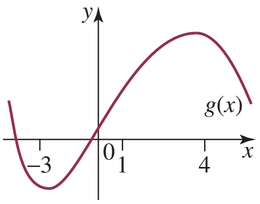
C



D



E



2A **5** Determine the stationary points and their nature for the function $f(x) = 2x^3 + 3x^2 - 36x + 5$.

2A **6** Sketch the graph of $y = x^2(16 - x^2)$, clearly indicating all stationary points and intercepts.

2B **7** **multiple choice**

The equation of a line with a gradient of 2 and passing through the point $(1, -3)$ is:

A $y = 2x + 5$
B $y = 2x$
C $y = 2x - 1$
D $y = 2x - 5$
E $y = x + 1$

8 **multiple choice** 2B

The equation of the tangent to the curve $y = x(x + 2)(x - 1)$ at the point where $x = -1$ is:

A $y - x + 1 = 0$ **B** $y - x + 3 = 0$ **C** $y + x + 3 = 0$

D $y + x - 1 = 0$ **E** $y - x - 1 = 0$

9 Find the equation of the tangent to the curve $y = 6 - 3x + 2x^2 - x^3$ at the point where $x = 1$. 2B

10 Find the equation of the tangent to the curve $y = \frac{x^2}{4} - 5$ which is parallel to the line $y - 3x - 7 = 0$. 2B

11 Find the equation of the normals to the curve $y = x + \frac{1}{x}$ at the point where the normals are parallel to the line $3y + 4x - 10 = 0$. 2B

12 **multiple choice** 2C

The graph of $f'(x)$ shown at right indicates that the graph of $f(x)$ has:

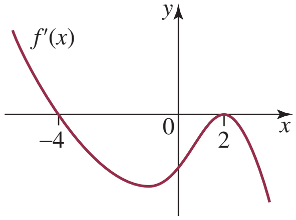
A a turning point at $x = 2$ and $x = -4$

B a turning point at $x = 2$ and point of inflection at $x = -4$

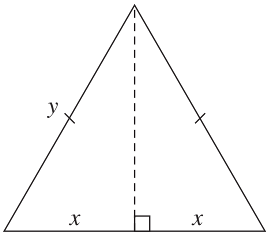
C a turning point at $x = -4$ and point of inflection at $x = 2$

D 3 stationary points

E 2 points of inflection at $x = -4$ and $x = 2$



Questions **13** to **17** follow from the isosceles triangle below which has a perimeter of 40 cm.



13 **multiple choice** 2D

The value of y in terms of x is:

A $40 - 2x$ **B** $20 - x$ **C** $40 - x$ **D** $20 - 2x$ **E** x^2

14 **multiple choice** 2D

The height of the triangle in terms of x is:

A $\sqrt{400 - 40x}$ **B** $20 - \sqrt{40x}$ **C** $\sqrt{400 - 40x + 2x^2}$

D $\sqrt{400 - 40x + x^2}$ **E** $\sqrt{40x}$

15 **multiple choice** 2D

The area in terms of x is:

A $(20 - x)\sqrt{400 - 40x}$ **B** $x\sqrt{400 - 40x + x^2}$ **C** $2x\sqrt{400 - 40x + x^2}$

D $x\sqrt{400 - 40x}$ **E** $2x\sqrt{400 - 40x}$

16 **multiple choice** 2D

The maximum area of the triangle is obtained if x equals:

A $6\frac{2}{3}$ cm **B** 10 cm **C** 20 cm **D** 5 cm **E** $10\frac{2}{5}$ cm

2D

17 **multiple choice**

Therefore, the maximum area possible is:

A $50\sqrt{2}$ cm^2 **B** 64 cm^2 **C** $32\sqrt{5}$ cm^2 **D** $\frac{40\sqrt{3}}{9}$ **E** $\frac{400\sqrt{3}}{9}$

2E

18 **multiple choice**

The average rate of change of the function $f(x) = x^4 - 3x^3 + 5x$ between $x = 1$ and $x = 3$ is:

A 15 **B** 6 **C** 12 **D** $4\frac{1}{2}$ **E** 16

2E

19 An iceberg in the shape of a cube is slowly melting. The rate of change of the surface area of the iceberg in m^3/m when the side length is 40 metres is:

A 480 **B** –240 **C** 720 **D** –480 **E** 240

Modelling & problem solving

1 The number of bees, N, in a hive can be modelled by the function $N = 2t(50 - t) + 180$ where t is the number of days the hive has existed. What is the maximum number of bees in the hive?

2 The area of a certain triangular shape is:

$$A = \frac{2x^2}{3(x-6)}, x > 6$$

where x is the length of the base of the triangle.

a Find the value of x for minimum area.

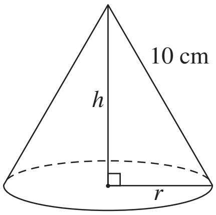
b Hence, find the minimum area.

3 Consider the cone with a slant side of 10 cm shown at right.

a Show that the height, h, is $\sqrt{100 - r^2}$.

b Show that the volume of the cone is $V = \frac{1}{3}\pi r^2\sqrt{100 - r^2}$.

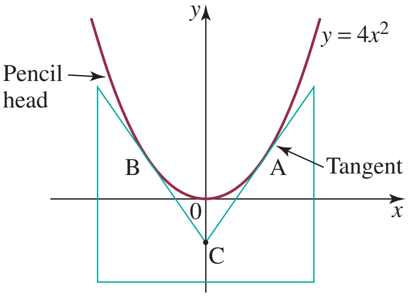
c Hence, find the maximum volume.



4 A manufacturing company is required to produce cylindrical cans (for tuna) of volume 50 cm^3. Tin used to produce the cans costs 40 cents per 100 cm^2.

a Find the area of tin required, A, in terms of the radius, r.
b Find the radius of the can (to the nearest tenth) for minimum area.
c Hence, find the minimum area (to the nearest tenth).
d How much is the cost of tin to produce 10 000 such cans?

5 The cross-section of a pencil head when it is first placed in a sharpening device is shown below.



The gradient of the tangent at point A is 4.

The equation of the pencil head is $y = 4x^2$.
If the x- and y-axes are as indicated and all distances are in centimetres, find:

a the coordinates of points A and B
b the equation of the tangent to the curve $y = 4x^2$ at point A
c the coordinate of point C
d the minimum distance from the pencil head to point C
e the length of the pencil head if it starts at the point where the normal at point A meets the y-axis.

eBook*plus*

Digital doc:
Test Yourself
Chapter 2

eBook*plus* ACTIVITIES

2B Equations of tangents and normals

Digital docs

- SkillSHEET 2.1: Practise finding equations of normals (*page 55*)
- SkillSHEET 2.2: Practise finding the equation of a straight line (*page 55*)
- Spreadsheet 023: Investigate tangents and normals (*page 58*)
- WorkSHEET 2.1: Find stationary points of functions and sketch their graphs, find when the rate of change is increasing or decreasing and find equations of tangents and normals at given points on a curve (*page 58*)

2C Maximum and minimum problems when the function is known

Tutorial

- **WE8** Int-0560: Watch a tutorial on applications of maximum and minimum problems (*page 60*)

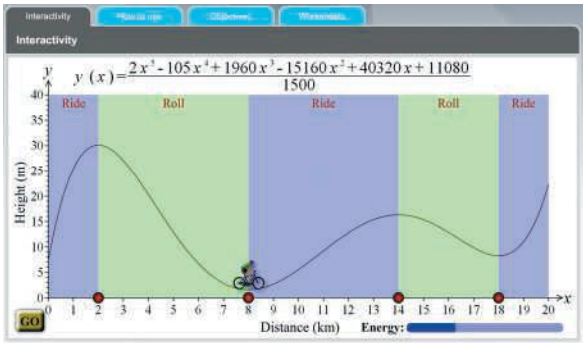
2D Maximum and minimum problems when the function is unknown

Digital doc

- WorkSHEET 2.2: Apply differentiation skills to real-life problems (*page 70*)

Tutorial

- **WE10** Int-0561: Watch a worked example on maximising volume (*page 64*)



2E Rates of change

Digital doc

- SkillSHEET 2.3: Practise subtracting function values (*page 73*)

Interactivity

- Rates of change int-0253: Consolidate your understanding of rates of change (*page 71*)

Chapter review

Digital doc

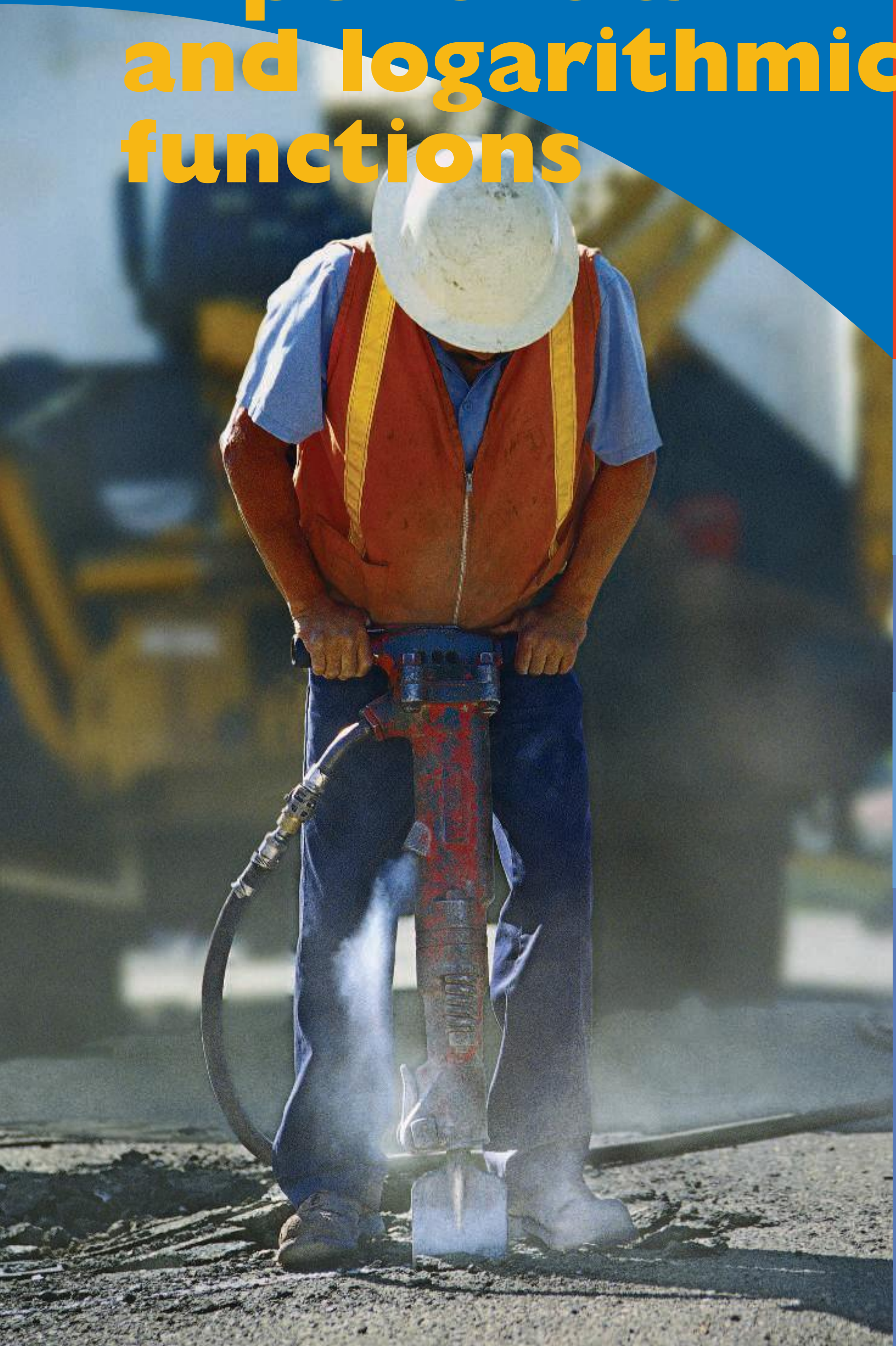
- Test Yourself: Take the end-of-chapter test to test your progress (*page 79*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

3

Exponential and logarithmic functions



syllabus reference

Exponential and logarithmic functions and applications

In this chapter

Introduction

Cell growth — seeking a model

A researcher is observing the growth of a particular type of cell in a laboratory experiment. He obtains the following data.

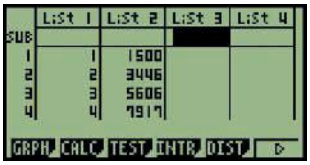
Day (D)	1	2	3	4
Number of cells (N)	1500	3446	5606	7917

Using a graphics calculator, we can try to find the model — that is, a formula relating the number of cells (N) to the number of days (D) — that most closely fits these data.

For the Casio *fx*-9860G AU

1. Press:
   - MENU
   - 2 (STAT).

   Enter the data in the two lists.



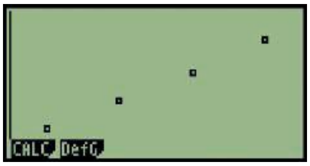
	List 1	List 2	List 3	List 4
SUB				
1	1	1500		
2	2	3446		
3	3	5606		
4	4	7917		

2. To graph this data on a scatterplot, press:
   - F1 (GRPH)
   - F6 (SET).

   Set the fields as shown.

```
StatGraph1
Graph Type   :Scatter
XList        :List1
YList        :List2
Frequency    :1
Mark Type    :▫
```

3. Press:
   - EXIT
   - F1 (GRH1).



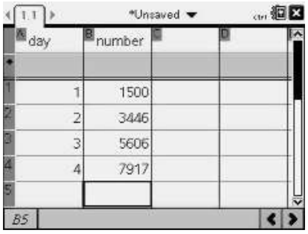
4. This scatterplot could fit an exponential curve. A number of different models can be tested. The model that best fits this data is the power model: $y = a \cdot x^b$. To find the values of a and b, press:
   - F1 (CALC)
   - F6
   - F4 (Pwr).

```
PowerReg
   a =1499.98976
   b =1.20000985
   r =0.99999999
  r²=0.99999999
 MSe=1.0506E-09
y=a·x^b
```

For the TI-Nspire CAS

1. Open a Lists & Spreadsheet page.

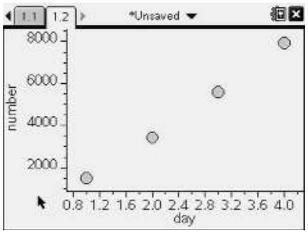
   Label Column A ‘Day’ and Column B ‘Number’, then enter the data in the appropriate columns.



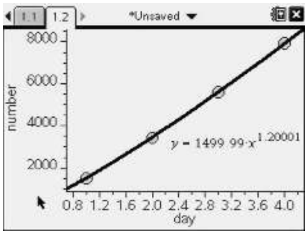
day	number
1	1500
2	3446
3	5606
4	7917

2. To graph this data on a scatterplot, open a Data & Statistics page.

   Tab to the horizontal axis and select Day and then tab to the vertical axis and select Number.



3. This scatterplot could fit an exponential curve. A number of different models can be tested. The model that best fits this data is the power model: $y = a \cdot\ x^b$. To find the values of a and b, press:
   - MENU (menu)
   - 4: Analyze ⟨4⟩
   - 6: Regression ⟨6⟩
   - 7: Show Power ⟨7⟩.



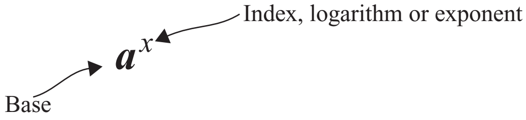
These procedures yield the relationship:

$$N = 1500(D)^{1.2}$$

Is there a method of producing this relationship whereby one can see what is happening rather than relying on the calculator? In the section that follows we shall see that in data sets such as this, logarithms can be used for detecting power relationships.

The index laws

A number in index form has two parts, the base and the index, logarithm or exponent and is written in the form:



The index laws are:

1. When numbers with the same base are multiplied, the indices are added.

$$a^x \cdot a^y = a^{x+y}$$

2. When numbers with the same base are divided, the indices are subtracted.

$$a^x \mid a^y = a^{x-y}$$

$$\text{or } \frac{a^x}{a^y} = a^{x-y}$$

3. When numbers with an index or exponent are raised to another index or exponent, the indices are multiplied.

$$(a^x)^y = a^{xy}$$

4. When numbers have an index of zero the answer is one.

$$a^0 = 1$$

5. When a number has a negative index, it becomes a fraction with a positive index.

$$a^{-x} = \frac{1}{a^x} \quad \text{and} \quad \frac{1}{a^{-x}} = a^x$$

6. When a number has a fractional index, the denominator of the fraction becomes the root.

$$a^{\frac{1}{y}} = \sqrt[y]{a} \quad \text{and} \quad a^{\frac{x}{y}} = \sqrt[y]{a^x} \text{ or } a^{\frac{x}{y}} = (\sqrt[y]{a})^x$$

WORKED Example 1

Simplify: $\dfrac{(2x^2y^3)^3 \cdot 3(xy^4)^2}{6x^4 \cdot 2xy^4}$.

THINK	WRITE
1 Remove the brackets by multiplying the indices.	$\dfrac{(2x^2y^3)^3 \cdot 3(xy^4)^2}{6x^4 \cdot 2xy^4} = \dfrac{2^3x^6y^9 \cdot 3x^2y^8}{12x^5y^4}$
2 Add the indices of x and add the indices of y. Simplify 2^3 to 8 and multiply the whole numbers.	$= \dfrac{24x^8y^{17}}{12x^5y^4}$
3 Subtract the indices of x and y. Divide 24 by 12.	$= 2x^3y^{13}$

For negative indices and fractional or decimal indices, the same rules apply.

WORKED Example 2

Write in simplest form:

a $64^{\frac{2}{3}}$ **b** $32^{-0.4}$.

THINK	WRITE
a ① Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.	**a** $64^{\frac{2}{3}} = \sqrt[3]{64^2}$
② Rewrite using $\sqrt[y]{a^x} = (\sqrt[y]{a})^x$.	$=(\sqrt[3]{64})^2$
③ Simplify by taking the cube root of 64.	$= 4^2$
④ Square 4.	$= 16$
b ① Write as a fraction with a positive index.	**b** $32^{-0.4} = \dfrac{1}{32^{0.4}}$
② Change 0.4 to $\frac{4}{10}$.	$= \dfrac{1}{32^{\frac{4}{10}}}$
③ Simplify the fractional index.	$= \dfrac{1}{32^{\frac{2}{5}}}$
④ Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.	$= \dfrac{1}{(\sqrt[5]{32})^2}$
⑤ Simplify by taking the 5th root of 32.	$= \dfrac{1}{2^2}$
⑥ Square 2.	$= \frac{1}{4}$

WORKED Example 3

Simplify, leaving your answer with positive indices.

a $a^{-2}b^4 \cdot (a^3b^{-4})^{-1}$ **b** $\left(\dfrac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^2}\right)^{-1} | \left(\dfrac{3a^{-\frac{3}{2}}b^2}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^2$

THINK	WRITE
a ① Remove the brackets by multiplying the indices.	**a** $a^{-2}b^4 \cdot (a^3b^{-4})^{-1} = a^{-2}b^4 \cdot a^{-3}b^4$
② Add the indices of a and of b.	$= a^{-5}b^8$
③ Place a^5 in the denominator with a positive index.	$= \dfrac{b^8}{a^5}$

Continued over page →

THINK	WRITE
b ① Remove the brackets by multiplying the indices.	**b** $\left(\dfrac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^{2}}\right)^{-1} \div \left(\dfrac{3a^{-\frac{3}{2}}b^{2}}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^{2}$ $= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \div \dfrac{3^{2}a^{-3}b^{4}}{a^{\frac{3}{2}}b}$
② When dividing by a fraction, invert and multiply.	$= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \cdot \dfrac{a^{\frac{3}{2}}b}{9a^{-3}b^{4}}$
③ Add the indices of *a* and of *b* in the numerator and add the indices of *b* in the denominator. Multiply the numbers.	$= \dfrac{ab^{2}}{27a^{-3}b^{2}}$
④ Subtract the indices of *a* and *b*.	$= \dfrac{a^{4}}{27}$

If the expression contains different numbers which do not have the same base, write each number as a product of prime factors.

WORKED Example 4

Simplify: $\dfrac{3^{n} \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2n} \cdot 8^{n}}$.

THINK	WRITE
① Write each number as the product of prime factors.	$\dfrac{3^{n} \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2n} \cdot 8^{n}}$ $= \dfrac{3^{n} \cdot (3 \cdot 2)^{n+1} \cdot (2^{2} \cdot 3)^{n-1}}{3^{2n} \cdot 2^{3n}}$
② Remove the brackets.	$= \dfrac{3^{n} \cdot 3^{n+1} \cdot 2^{n+1} \cdot 2^{2n-2} \cdot 3^{n-1}}{3^{2n} \cdot 2^{3n}}$
③ Add the indices of numbers with base 3 in the numerator and add indices of numbers with base 2 in the numerator.	$= \dfrac{3^{3n} \cdot 2^{3n-1}}{3^{2n} \cdot 2^{3n}}$
④ Subtract the indices.	$= 3^{n} \cdot 2^{-1}$
⑤ Write the term with a negative index in the denominator with a positive index.	$= 3^{n} \cdot \dfrac{1}{2}$
⑥ Simplify.	$= \dfrac{3^{n}}{2}$

When numbers with the same base are added or subtracted, the normal rules for addition and subtraction apply. If the indices are negative, the terms are rewritten as fractions with positive indices.

WORKED Example 5

An antique chair worth \$15 000 is increasing in value by 10% each year.

a **Write an equation for the value of the chair, \$*v*, in terms of the time, *t*, in years.**

b **Hence, find the value of the chair after 10 years. Give your answer correct to the nearest hundred dollars.**

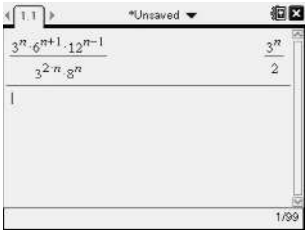
THINK	WRITE
a 1 Find by what percentage the chair appreciates each year.	**a** Appreciates by (100 + 10)% or 110%.
2 Write this as a decimal.	$110\% = \frac{110}{100}$ $= 1.1$
3 Find the value after 1 year.	After 1 year it is worth \$(15 000 · 1.1) = \$16 500
4 Find the value after 2 years.	After 2 years it is worth \$(15 000 · 1.1) · (1.1) $= \$(15\,000 \cdot 1.1^2)$ = \$18 150
5 Find the value after 3 years.	After 3 years it is worth $\$(15\,000 \cdot 1.1^3) = \$19\,965$
6 Hence find the formula. *Note:* A formula does not include the dollar sign.	$v = 15\,000 \cdot 1.1^t$
b 1 Substitute $t = 10$ in the equation.	**b** $v = 15\,000 \cdot 1.1^{10}$
2 Evaluate 1.1^{10}.	$= 15\,000 \cdot 2.593\,742\,5$
3 Calculate *v* and express your answer correct to the nearest hundred dollars.	$= 38\,906.138$ $\approx 38\,900$ (to nearest 100)
4 Write your answer in a sentence.	The value of the chair after 10 years is about \$38 900.

Graphics Calculator tip! Simplifying expressions involving indices

The **TI-Nspire CAS** can be used to simplify expressions of the form $\frac{3^n \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2n} \cdot 8^n}$. On a Calculator page, complete the entry line as:

$$\frac{3^n \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2n} \cdot 8^n}$$

then press ENTER (enter).



$\frac{3^n \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2 \cdot n} \cdot 8^n}$ $\frac{3^n}{2}$

remember

1. $a^x\ a^y = a^{x+y}$
2. $a^x \mid a^y = a^{x-y}$ or $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $a^0 = 1$
5. $a^{-x} = \dfrac{1}{a^x}$ and $\dfrac{1}{a^{-x}} = a^x$
6. $a^{\frac{1}{y}} = \sqrt[y]{a}$ and $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$

EXERCISE 3A The index laws

WORKED Example 1

1 Simplify:

a $x^3 \cdot x^4$ **b** $x^7 \mid x^2$ **c** $(x^2)^5$

d $(x^{-3})^2$ **e** $\dfrac{x^4 \cdot x^5}{x^3}$ **f** $\dfrac{(x^2)^3 \cdot x^5}{(x^5)^2}$

g $\dfrac{5x^2y^4 \cdot 4x^5y}{2^2x^3y^2}$ **h** $\dfrac{3x^3y^5 \cdot 10xy^4}{5x^2y^6}$ **i** $\dfrac{(2xy^2)^3 \cdot 5(x^4y)^2}{4x^5y^3 \cdot 3x^2y^3}$

j $\dfrac{(3^2x^3y)^2 \cdot 2(xy^3)^5}{4x^4y^2 \cdot 3x^5y}$

WORKED Example 2

2 Simplify without using a calculator.

a $27^{\frac{2}{3}}$ **b** $16^{\frac{3}{4}}$ **c** $25^{-\frac{3}{2}}$ **d** $100\,000^{-\frac{3}{5}}$

e $81^{0.25}$ **f** $36^{1.5}$ **g** $\left(\frac{9}{49}\right)^{\frac{1}{2}}$ **h** $\left(\frac{27}{64}\right)^{\frac{2}{3}}$

i $\left(\frac{243}{32}\right)^{-\frac{3}{5}}$ **j** $\left(\frac{256}{81}\right)^{-\frac{3}{4}}$

WORKED Example 3

3 Simplify, leaving your answer with positive indices.

a $3x^{-3}y^2 \cdot (x^2y)^{-4}$ **b** $x^4y^{-1} \cdot (x^{-2}y^3)^{-1}$

c $2x^{\frac{1}{2}}y^{\frac{2}{3}} \cdot \left(9x^{\frac{3}{2}}y^2\right)^{\frac{1}{2}}$ **d** $5x^{-\frac{1}{3}}y^{\frac{3}{4}} \cdot \left(8^{\frac{1}{3}}x^{\frac{2}{3}}y^{-\frac{1}{2}}\right)^2$

e $\left(x^{-2}y^{\frac{1}{2}}\right)^{-\frac{3}{2}} \cdot \left(9x^{-\frac{1}{5}}y^{-\frac{1}{2}}\right)^{\frac{5}{2}}$ **f** $16^{\frac{1}{2}}\left(x^{\frac{2}{5}}y^{-\frac{1}{4}}\right)^{-\frac{1}{2}} \cdot \left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$

g $\left(\dfrac{a^{\frac{3}{2}}b^{-2}c}{3a^{-\frac{1}{2}}bc^{-2}}\right)^{-2} \mid 3\left(\dfrac{a^{\frac{2}{3}}b^3}{a^{-1}c^2}\right)^3$ **h** $\left(\dfrac{a^{-\frac{3}{2}}b^{\frac{3}{4}}}{ab^2}\right)^{-2} \mid \left(\dfrac{9a^{-3}b}{4a^2b^3}\right)^{\frac{1}{2}}$

WORKED Example 4

4 Simplify:

a $2^n \cdot 4^{n+1} \cdot 8^{n-1}$ **b** $3^n \cdot 9^{n-1} \cdot 27^{n+1}$ **c** $2^{n-1} \cdot 3^n \cdot 6^{n+1}$

d $2^n \cdot 3^{n+1} \cdot 9^n$ **e** $\dfrac{3^2 \cdot 2^{-3}}{9^{\frac{3}{2}}} \cdot 16$ **f** $\dfrac{5^2 \cdot 3^{-1}}{125 \cdot 9^{-2}} \mid \dfrac{27}{5}$

5 Simplify, writing your answer as a single fraction with positive indices.

a $x^{-1} + \frac{1}{x^{-1}}$

b $(x^{-1} + x^{-2})^2$

c $\frac{1}{x^{-1} + 1} + \frac{1}{x^{-1} - 1}$

d $2x(x^2 - y^2)^{-1} - (x - y)^{-1}$

6 If $a = 2^3$, $b = 2^{-3}$, $c = 6^2$ and $d = 3^{-1}$, find:

a $\frac{a^2 b}{c^{\frac{1}{2}}}$

b $\frac{a^{\frac{1}{3}} b^{-1} d}{c^2}$.

7 **multiple choice**

$3^{-x} + 3^x$ is equal to:

A 1

B $\frac{1 + 3^{2x}}{3^x}$

C 3^{-x^2}

D 6

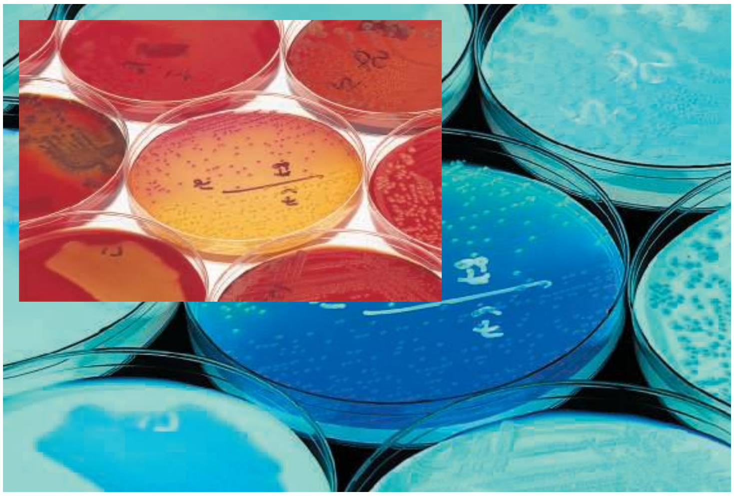
E $\frac{1 + 3^x}{3^x}$

WORKED Example 5

8 A population of organisms is growing so that the number of organisms, N, after t days is given by the formula $N = 500 \cdot 2^{0.1t}$.

a Find the number of organisms after 10 days.

b Find the size of the population after 15 days. Give your answer to the nearest whole number.



9 **multiple choice**

A car worth \$10 000 is depreciating at 20% per annum, so that each year the car is worth 80% of its value the previous year. A model for the value of the car, \$$V$, in terms of the time, t, in years is:

A $V = 10\,000 \cdot 20^t$

B $V = 10\,000 \cdot (0.2)^t$

C $V = 0.8 \cdot 10\,000^t$

D $V = 10\,000(80)^t$

E $V = 10\,000(0.8)^t$

10 A ball is dropped from a window h m above the ground. When it lands on the ground it rebounds to 80% of its height. The equation showing the height of the ball, h metres, after r rebounds is: $h = 10 \cdot (0.8)^r$.

a From how far above the ground was the ball dropped?

b How far above the ground does the ball reach on the fourth rebound? Give your answer to the nearest centimetre.

c How far has the ball travelled when it hits the ground for the fourth time?

Logarithms and laws of logarithms

The number 81 can be written as:

$$81 = 3^4.$$

That is, given the base 3 and the exponent 4 we can find the number 81 by calculating $3 \cdot 3 \cdot 3 \cdot 3$.

Note, however, that we need a calculator to compute $3^{4.5}$:

$$3 \wedge 4.5 \text{ gives } 140.296$$
$$140.296 = 3^{4.5}.$$

What do we do if we are given the number and the base, but need to find the power?

$$100 = 10^x$$

In this case, an easy calculation shows that $100 = 10^2$.

However, how do we find x in an equation such as the following?

$$200 = 10^x.$$

This is where logarithms are useful.

If $200 = 10^x$

then $x = \log_{10} 200$

$= 2.301$ (from calculator)

In general, **if $N = a^x$**

then $x = \log_a N$.

Notes

1. $a > 0$
2. $\log_a N$ is defined only if $N > 0$

These statements can be used to form three simple and useful laws of logarithms.

1. It is known that $a^0 = 1$. Therefore, $\mathbf{\log_a 1 = 0}$.
2. It is known that $a^1 = a$. Therefore, $\mathbf{\log_a a = 1}$.
3. It is known that there are no values of x for which $10^x = 0$ and $a^x = 0$. Therefore, $\log_{10} 0$ is undefined and **$\log_a 0$ is undefined**.

WORKED Example 6

Evaluate:

a $\log_2 1$ **b** $\log_5 5$.

THINK	WRITE
a Log of 1 to any base is equal to zero: $\log_a 1 = 0$.	**a** $\log_2 1 = 0$
b If the number and base are equal the answer is 1: $\log_a a = 1$.	**b** $\log_5 5 = 1$

WORKED Example 7

Write in index form:

a $\log_2 8 = 3$ **b** $\log_x 81 = 4$.

THINK	WRITE
a Use $a^x = y \Rightarrow \log_a y = x$.	**a** $\log_2 8 = 3 \Leftrightarrow 2^3 = 8$
b Use $a^x = y \Rightarrow \log_a y = x$.	**b** $\log_x 81 = 4 \Leftrightarrow x^4 = 81$

When written in index form an expression can be rewritten in logarithmic form.

$$3^{-1} = \tfrac{1}{3} \Leftrightarrow \log_3\left(\tfrac{1}{3}\right) = -1$$

This is using the equivalent statements $a^x = y \Rightarrow \log_a y = x$.

Other laws can be derived from the index laws.

4. Let $m = a^x$. Therefore, $\log_a m = x$.
   Let $n = a^y$. Therefore, $\log_a n = y$.
   $mn = a^x \cdot a^y$
   $mn = a^{x+y}$ Therefore, $\log_a mn = x + y$
   $= \log_a m + \log_a n$.
   $$\boldsymbol{\log_a mn = \log_a m + \log_a n}$$

5. $\dfrac{m}{n} = \dfrac{a^x}{a^y} = a^{x-y}$

   Therefore, $\log_a \dfrac{m}{n} = x - y$

   and $\boldsymbol{\log_a \dfrac{m}{n} = \log_a m - \log_a n}$.

6. $m = a^x \Rightarrow \log_a m = x$ known from the basic statement $a^x = y \Rightarrow \log_a y = x$.
   $m^p = (a^x)^p = a^{xp}$ Write each side of the equation with an index p and simplify.
   $m^p = a^{xp}$
   $\log_a m^p = xp$ Rewrite using an equivalent log equation, where a is the base and xp is the index.
   $\boldsymbol{\log_a m^p = p \log_a m}$ Replace x with its log equivalent, $x = \log_a m$, from step 1.

7. Change-of-base rule Suppose $b = a^x$, then $x = \log_a b$.
   Consider $N = b^y$, then $y = \log_b N$.
   But $N = b^y = (a^x)^y = a^{xy}$.
   Therefore, $\log_a N = xy = \log_a b \log_b N$.
   Thus,
   $$\boldsymbol{\log_b N = \frac{\log_a N}{\log_a b}}$$
   This is called the *change-of-base rule*.

WORKED Example 8

Simplify without a calculator:

a $\log_{10} 5 + \log_{10} 2$ **b** $\log_4 20 - \log_4 5$ **c** $\log_2 16$ **d** $\log_5 \sqrt[5]{x}$.

THINK	WRITE
a ① Rewrite using $\log_a mn = \log_a m + \log_a n$.	**a** $\log_{10} 5 + \log_{10} 2 = \log_{10}(5 \cdot 2)$
② Simplify.	$= \log_{10} 10$
③ Simplify using $\log_a a = 1$.	$= 1$
b ① Rewrite using $\log_a \dfrac{m}{n} = \log_a m - \log_a n$.	**b** $\log_4 20 - \log_4 5 = \log_4\left(\dfrac{20}{5}\right)$
② Simplify.	$= \log_4 4$
③ Simplify using $\log_a a = 1$.	$= 1$

Continued over page →

THINK	WRITE
c ① Rewrite 16 as a number with base 2.	**c** $\log_2 16 = \log_2 2^4$
② Rewrite using $\log_a m^p = p \log_a m$.	$= 4 \log_2 2$
③ Simplify using $\log_a a = 1$.	$= 4 \cdot 1$
	$= 4$
d ① Rewrite using $\sqrt[y]{a} = a^{\frac{1}{y}}$.	**d** $\log_5 \sqrt[5]{x} = \log_5 x^{\frac{1}{5}}$
② Rewrite using $\log_a m^p = p \log_a m$.	$= \frac{1}{5} \log_5 x$

WORKED Example 9

Simplify: $\log_3 27 + \log_3 9 - \log_3 81$.

THINK	WRITE
① Write 27, 9 and 81 as numbers to base 3.	$\log_3 27 + \log_3 9 - \log_3 81$ $= \log_3 3^3 + \log_3 3^2 - \log_3 3^4$
② Simplify using $\log_a m^p = p \log_a m$.	$= 3 \log_3 3 + 2 \log_3 3 - 4 \log_3 3$
③ Rewrite using $\log_3 3 = 1$.	$= 3 + 2 - 4$
④ Simplify.	$= 1$

WORKED Example 10

eBook plus

Tutorial:
Worked example 10
int-0529

Simplify: a $2 + \log_{10} 3$ b $3 \log_3 6 - 3 \log_3 18$ c $\dfrac{\log_3 9}{\log_3 27}$.

THINK	WRITE
a ① Write 2 as $2 \log_{10} 10$ because $\log_{10} 10 = 1$.	**a** $2 + \log_{10} 3 = 2 \log_{10} 10 + \log_{10} 3$
② Rewrite using $\log_a m^p = p \log_a m$.	$= \log_{10} 10^2 + \log_{10} 3$
③ Rewrite using $\log_a mn = \log_a m + \log_a n$.	$= \log_{10} (10^2 \cdot 3)$
④ Write 10^2 as 100.	$= \log_{10} (100 \cdot 3)$
⑤ Multiply the numbers in the brackets.	$= \log_{10} 300$
b ① Rewrite using $\log_a m^p = p \log_a m$.	**b** $3 \log_3 6 - 3 \log_3 18 = \log_3 6^3 - \log_3 18^3$
② Rewrite using $\log_a \frac{m}{n} = \log_a m - \log_a n$.	$= \log_3 \frac{6^3}{18^3}$
③ Write 6^3 as $6 \cdot 6 \cdot 6$ and 18^3 as $18 \cdot 18 \cdot 18$.	$= \log_3 \left(\frac{6 \cdot 6 \cdot 6}{18 \cdot 18 \cdot 18} \right)$
④ Simplify.	$= \log_3 \left(\frac{1}{3^3} \right)$
⑤ Write the number with base 3.	$= \log_3 3^{-3}$
⑥ Rewrite using $\log_a m^p = p \log_a m$.	$= -3 \log_3 3$
⑦ Simplify using $\log_a a = 1$.	$= -3 \cdot 1$
	$= -3$

THINK

c 1 Write the numbers with the same base. It is not possible to cancel the 9 and the 27 because they cannot be separated from the log.

2 Rewrite using $\log_a m^p = p\log_a m$.

3 Cancel the logs because they are the same.

WRITE

c $\dfrac{\log_3 9}{\log_3 27} = \dfrac{\log_3 3^2}{\log_3 3^3}$

$= \dfrac{2\log_3 3}{3\log_3 3}$

$= \dfrac{2}{3}$

WORKED Example 11

Find the value of $\log_2 18$.

THINK

1 The number 18 is not a simple power of 2. However, a calculator cannot compute logs to base 2 directly; therefore use the change-of-base rule.

2 Choose a value of 10 for a. Use a calculator to compute the logs to base 10.

WRITE

$\log_b N = \dfrac{\log_a N}{\log_a b}$

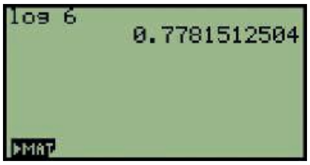
$\log_2 18 = \dfrac{\log_{10} 18}{\log_{10} 2}$

$= \dfrac{1.255}{0.301}$

$= 4.17$

Graphics Calculator tip! Evaluating logarithms to base 10

The **Casio *fx*-9860G AU** can be used to evaluate logarithms to base 10.



```
log 6
                  0.7781512504
```

To find the approximate value of $\log_{10} 6$, press:

- MENU
- 1 (RUN).

Complete the entry line as:

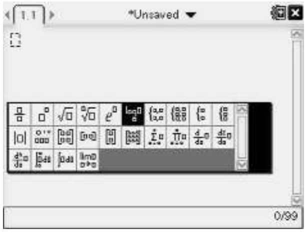
log 6

then press EXE.

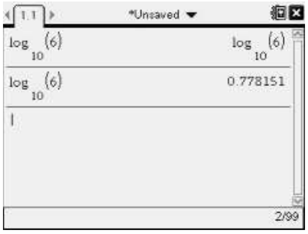
The **TI-Nspire CAS** can be used to evaluate logarithms to any base.

1. To find the approximate value of $\log_{10} 6$, on a Calculator page, press:
   - Ctrl ctrl
   - · .

   Use the NavPad to highlight the logarithm template and then press ENTER enter.



2. Complete the fields as shown.
   This template can be used for any base.



```
log_10(6)        log_10(6)
log_10(6)        0.778151
```

WORKED Example 12

The apparent brightness of a star can be found using the formula $B = 6 - 2.5\log_{10} A$, where A is the actual brightness of that star. Find the apparent brightness of a star with actual brightness of 3.16.

THINK	WRITE
① Write the formula.	$B = 6 - 2.5\log_{10} A$
② Substitute 3.16 for A.	When $A = 3.16$, $B = 6 - 2.5\log_{10}(3.16)$
③ Evaluate $\log_{10}(3.16)$ using a graphics calculator.	$= 6 - 2.5 \cdot 0.5$
④ Simplify.	$= 4.75$
⑤ Write your answer in a sentence.	The apparent brightness of the star is 4.75.

remember

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a 0$ is undefined.
4. $\log_a x$ does not exist when $x < 0$.
5. $\log_a mn = \log_a m + \log_a n$
6. $\log_a \dfrac{m}{n} = \log_a m - \log_a n$
7. $\log_a m^p = p\log_a m$
8. $\log_b N = \dfrac{\log_a N}{\log_a b}$

EXERCISE 3B Logarithms and laws of logarithms

WORKED Example 6

1 Evaluate the following.

a $\log_3 1$ **b** $\log_5 1$ **c** $\log_2 2$ **d** $\log_6 6$ **e** $\log_4 0$ **f** $\log_2 0$

WORKED Example 7

2 Write the following in index form.

a $\log_2 16 = 4$ **b** $\log_3 27 = 3$ **c** $\log_x 25 = 2$

d $\log_x 64 = 6$ **e** $\log_5 125 = x$ **f** $\log_{10} 10\,000 = x$

g $\log_3 x = 5$ **h** $\log_2 x = 7$ **i** $\log_5 \left(\frac{1}{5}\right) = -1$

j $\log_7 \left(\frac{1}{49}\right) = -2$

3 Write the following in logarithmic form.

a $2^3 = 8$ **b** $2^5 = 32$ **c** $3^4 = 81$ **d** $3^2 = 9$

e $4^3 = x$ **f** $4^5 = x$ **g** $5^x = 125$ **h** $5^x = 25$

i $2^{-1} = \frac{1}{2}$ **j** $3^{-2} = \frac{1}{9}$ **k** $x^3 = 27$ **l** $p^4 = 16$

4 Evaluate using your calculator. Give your answer correct to 3 decimal places.

a $\log_{10} 3$ **b** $\log_{10} 4$ **c** $\log_{10} 0.5$ **d** $\log_{10} 0.8$

e $\log_{10} 20$ **f** $\log_{10} 60$ **g** $\log_{10} 300$ **h** $\log_{10} 500$

WORKED Example 8

5 Simplify without using a calculator.

a $\log_6 3 + \log_6 2$ **b** $\log_{10} 5 + \log_{10} 2$ **c** $\log_3 6 - \log_3 2$

d $\log_2 10 - \log_2 5$ **e** $\log_2 32$ **f** $\log_3 81$

g $\log_5 \left(\frac{1}{5}\right)$ **h** $\log_3 \left(\frac{1}{27}\right)$

6 Simplify:

a $\log_2 \sqrt{x}$ **b** $\log_3 \sqrt[3]{x}$ **c** $3 \log_3 \sqrt[3]{x}$

d $4 \log_4 \sqrt[4]{x}$ **e** $\log_2 \sqrt{\frac{x^4}{y^2}}$ **f** $\log_3 \sqrt[5]{\frac{x^5}{y^{10}}}$

WORKED Example 9

7 Simplify without using a calculator.

a $\log_4 10 + \log_4 2 - \log_4 5$ **b** $\log_5 10 + \log_5 2 - \log_5 4$

c $\log_5 25 + \log_5 125 - \log_5 625$ **d** $\log_5 4 + \log_5 \left(\frac{1}{4}\right)$

e $\frac{1}{2} \log_{10} 16 + \log_{10} 5^2$ **f** $\frac{1}{2} \log_{10} 25 + \log_{10} 2$

g $\log_3 2 - \log_3 10 + \log_3 15$ **h** $\log_5 4 - \log_5 8 + \log_5 10$

i $\log_2 16 + \log_2 8 + \log_2 4$ **j** $\log_3 6 + \log_3 x^{\frac{1}{2}} = 1$

WORKED Example 10

8 Simplify without using a calculator.

a $4 \log_2 12 - 4 \log_2 6$ **b** $3 \log_2 3 - 3 \log_2 6$

c $2 + \log_5 10 - \log_5 2$ **d** $2 + \log_5 2 - \log_5 10$

e $1 + \log_2 5$ **f** $3 + \log_3 2$

g $\dfrac{\log_2 64}{\log_2 8}$ **h** $\dfrac{\log_5 125}{\log_5 25}$

i $\dfrac{\log_a \sqrt{x}}{\log_a x}$ **j** $\dfrac{\log_a x^2}{\log_a x^3}$

9 Simplify without using a calculator.

- **a** $5\log_3 x + \log_3 x^2 - \log_3 x^7$
- **b** $3\log_2 x + \log_2 x^3 - \log_2 x^6$
- **c** $3\log_4 x - 5\log_4 x + 2\log_4 x$
- **d** $4\log_6 x - 5\log_6 x + \log_6 x$
- **e** $\log_{10} x^2 + 3\log_{10} x - 2\log_{10} x$
- **f** $4\log_{10} x - \log_{10} x + \log_{10} x^2$
- **g** $\log_5 (x+1) + \log_5 (x+1)^2$
- **h** $\log_4 (x-2)^3 - 2\log_4 (x-2)$

WORKED Example 11

10 Calculate, to 3 decimal places, the value of:

- **a** $\log_2 40$
- **b** $\log_6 100$
- **c** $\log_{100} 50$

11 **multiple choice**

$2\log_{10} 5 - \log_{10} 20 + \log_{10} 8$ is equal to:

- **A** $\log_{10} 2$
- **B** $-\log_{10} 2$
- **C** 1
- **D** -1
- **E** $\log_{10} 4$

12 **multiple choice**

If $\log_a b = 2$, then b is equal to:

- **A** 0
- **B** 1
- **C** 2
- **D** a
- **E** a^2

13 If $y = a\log_{10} x$, find x when $a = 2$ and $y = 3$. Give your answer correct to 3 decimal places.

14 Find the logarithm of 5 to base 10. Give your answer correct to 3 decimal places.

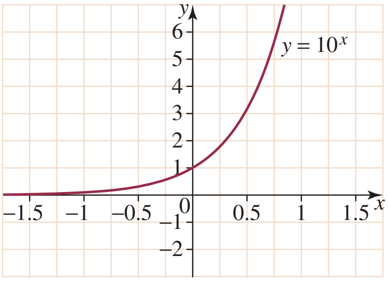
WORKED Example 12

15 Earthquake intensity is often reported on the Richter scale. The magnitude of R is given by: $R = \log_{10}\left(\frac{a}{T}\right) + B$ where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with the increasing distance from the epicentre of the earthquake.

Find the magnitude of the earthquake if the amplitude of the ground motion is 10 microns, the period is 1 second and the empirical factor is 6.8.

Graphs of exponential and logarithmic functions

Consider the graph of $y = 10^x$ as shown at right.



Three features of this graph are worthy of note:

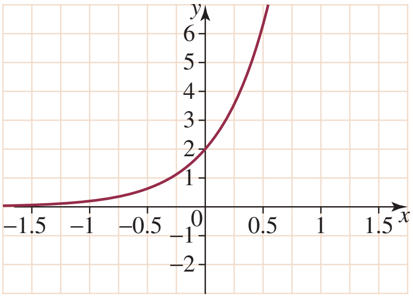
- The y-intercept is (0, 1).
- As $x \to -\infty$ the graph has an asymptote, $y = 0$.
- As $x \to \infty$, $y \to \infty$.

WORKED Example 13

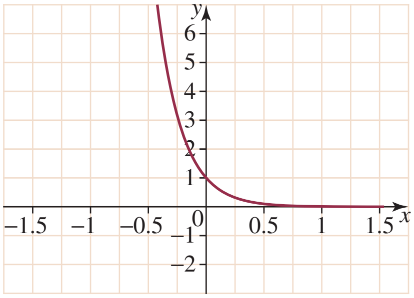
Using previous knowledge of the transformation of the graphs of functions, match each equation (a–d) with a suitable graph (P–S).

a $y = -2 \cdot 10^{2x}$ **b** $y = 10^{2x}$ **c** $y = 2 \cdot 10^{2x}$ **d** $y = 10^{-2x}$

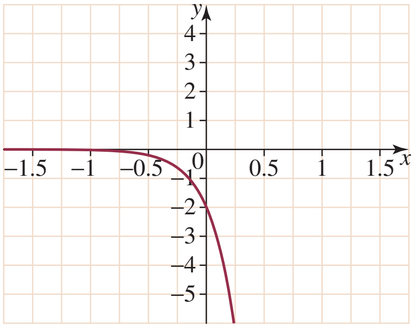
P



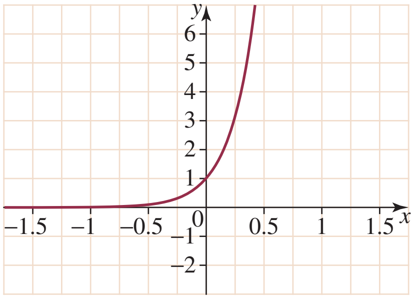
Q



R



S



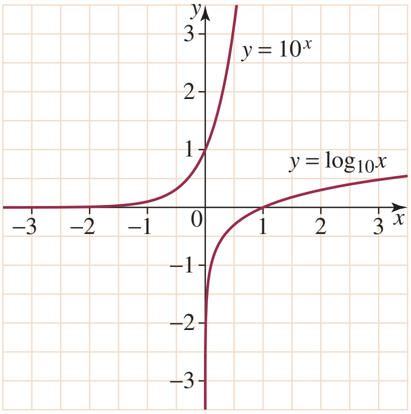
THINK	WRITE
a ① Compare $y = -2 \cdot 10^{2x}$ to the graph of $y = 10^x$. The graph of $y = -10^x$ is obtained by reflecting $y = 10^x$ through the x-axis.	**a** A reflection in the x-axis
② The graph of $y = 2 \cdot 10^x$ is obtained by dilating the graph of $y = 10^x$ by a factor of 2 in the y direction.	A dilation of factor 2 in the y direction
③ The graph of $y = 10^{2x}$ is obtained by dilating the graph of $y = 10^x$ by a factor of $\frac{1}{2}$ in the x direction.	A dilation of $\frac{1}{2}$ in the x direction
④ Write the answer.	The graph R represents the graph of $y = -2 \cdot 10^{2x}$.

Continued over page

THINK	**WRITE**
b Compare $y = 10^{2x}$ to the graph of $y = 10^x$. The graph of $y = 10^{2x}$ is obtained by dilating the graph of $y = 10^x$ by a factor of $\frac{1}{2}$ in the x direction.	**b** A dilation of $\frac{1}{2}$ in the x direction The graph S represents the graph of $y = 10^{2x}$.
c Compare the graph of $y = 2 \cdot 10^x$ to the graph of $y = 10^x$. The graph of $y = 2 \cdot 10^x$ is obtained by dilating the graph of $y = 10^x$ by a factor of 2 in the y direction.	**c** A dilation of factor 2 in the y direction The graph P represents the graph of $y = 2 \cdot 10^x$.
d Compare the graph of $y = 10^{-2x}$ to the graph of $y = 10^x$. The graph of $y = 10^{-2x}$ is obtained by reflecting the graph of $y = 10^x$ in the x axis and dilating it by a factor of $\frac{1}{2}$ in the x direction.	**d** A reflection in the x axis followed by a dilation of factor of $\frac{1}{2}$ in the x direction. The graph Q represents the graph of $y = 10^{-2x}$.

The graphs of $y = 10^x$ and $y = \log_{10} x$

From the equation $y = \log_{10} x$ we can deduce that $x = 10^y$. You will recall the relationship between the graphs of inverse functions: the graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by a reflection in the line $y = x$. From the graphs of these functions it is clear that they are inverses.



Indicial equations

The equation $a^x = y$ is an example of a general indicial equation and $2^x = 32$ is an example of a more specific indicial equation.

To solve one of these equations it is necessary to write both sides of the equation with the same base if the unknown is an index or with the same index if the unknown is the base.

WORKED Example 14

Solve for x in each of the following.

a $2^x = 32$ **b** $5^{-x} = 125$ **c** $3^x = \frac{1}{27}$ **d** $2 \cdot 3^x = 162$ **e** $2^{x-1} = 16$

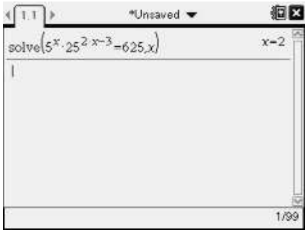
THINK	**WRITE**
a 1 Write 32 with base 2, the same as the left-hand side.	**a** $2^x = 32$ $= 2^5$
2 The indices are equal because the base is 2 on each side of the equation.	$x = 5$

THINK	**WRITE**
b 1 Write 125 with base 5.	**b** $5^{-x} = 125$ $= 5^3$
2 Equate the indices.	$-x = 3$
3 Multiply both sides by −1.	$x = -3$
c 1 Write 27 with base 3.	**c** $3^x = \frac{1}{27}$ $= \frac{1}{3^3}$
2 Write $\frac{1}{3^3}$ as a number with base 3.	$3^x = 3^{-3}$
3 Equate the indices.	$x = -3$
d 1 Divide both sides by 2 to leave 3^x on the left-hand side.	**d** $2 \cdot 3^x = 162$ $3^x = 81$
2 Write 81 as a number with base 3.	$= 3^4$
3 Equate the indices.	$x = 4$
e 1 Write 16 with base 2.	**e** $2^{x-1} = 16$ $= 2^4$
2 Equate the indices.	$x - 1 = 4$
3 Solve for x.	$x = 5$

In more complex equations it is necessary to write more than two terms with the same base. The bases 2, 3 and 5 are the most commonly employed in order to simplify problems.

WORKED Example 15

Solve for x in $5^x \cdot 25^{2x-3} = 625$: a by hand b using the TI-Nspire CAS calculator.

THINK	**WRITE/DISPLAY**
a 1 Write all numbers with the same base.	**a** $5^x \cdot 25^{2x-3} = 625$ $5^x \cdot (5^2)^{2x-3} = 5^4$
2 Simplify.	$5^x \cdot 5^{2(2x-3)} = 5^4$
3 Remove the brackets in the index.	$5^x \cdot 5^{4x-6} = 5^4$
4 Add the indices on the left-hand side.	$5^{5x-6} = 5^4$
5 Equate the indices.	$5x - 6 = 4$
6 Solve the equation.	$5x = 10$ $x = 2$
b 1 To solve $5^x \cdot 25^{2x-3} = 625$ for x, on a Calculator page, complete the entry line as: solve($5^x \cdot 25^{2x-3} = 625$, x) then press ENTER enter.	**b**  solve($5^x \cdot 25^{2x-3}=625$,x) x=2
2 Write the answer.	Solve $5^x \cdot 25^{2x-3} = 625$ for x. $x = 2$.

Sometimes it is possible to use the methods for solving quadratic equations to help solve indicial equations. Remember that $2^{2x} = (2^x)^2$.

WORKED Example 16

eBookplus

Tutorial:
Worked example 16
int-0530

Solve for x in the following.

a $(2^x - 16)(2^x + 4) = 0$ **b** $3^{2x} - 12 \cdot 3^x + 27 = 0$ **c** $4^x - 2^{x+3} + 16 = 0$

THINK	WRITE
a 1 Use the Null Factor Law to solve by making each bracket equal to zero.	**a** $(2^x - 16)(2^x + 4) = 0$ $(2^x - 16) = 0$ or $(2^x + 4) = 0$
2 Solve each equation.	$2^x = 16$ or $2^x = -4$
3 Write 16 as a number with base 2 but –4 can not be written with base 2.	$2^x = 2^4$ or no real solution
4 Solve by equating the indices.	$x = 4$
b 1 Write 3^{2x} as $(3^x)^2$.	**b** $3^{2x} - 12 \cdot 3^x + 27 = 0$ $(3^x)^2 - 12 \cdot 3^x + 27 = 0$
2 Let $3^x = a$ to make a simpler quadratic equation to solve.	$a^2 - 12a + 27 = 0$ where $a = 3^x$
3 Factorise.	$(a - 3)(a - 9) = 0$
4 Use the Null Factor Law by making each bracket equal to zero.	$a - 3 = 0,\ a - 9 = 0$
5 Solve for a.	$a = 3,\ a = 9$
6 Substitute back $a = 3^x$.	$3^x = 3,\ 3^x = 9$
7 Write numbers with base 3.	$3^x = 3^1,\ 3^x = 3^2$
8 Equate the indices.	$x = 1,\ x = 2$
c 1 Rewrite 4^x as $(2^x)^2$ and 2^{x+3} as $2^x \cdot 2^3$.	**c** $4^x - 2^{x+3} + 16 = 0$ $(2^x)^2 - 2^x \cdot 2^3 + 16 = 0$
2 Rewrite 2^3 as 8.	$(2^x)^2 - 2^x \cdot 8 + 16 = 0$
3 Let $2^x = a$ to make a simpler quadratic equation to solve.	$a^2 - a \cdot 8 + 16 = 0$ where $a = 2^x$
4 Replace $a \cdot 8$ with $8a$ because the coefficient precedes the pronumeral.	$a^2 - 8a + 16 = 0$
5 Factorise.	$(a - 4)(a - 4) = 0$
6 Use the Null Factor Law.	$a - 4 = 0,\ a - 4 = 0$
7 Solve for a.	$a = 4$ and $a = 4$
8 The two factors are equal because $a^2 - 8x + 16 = 0$ is a perfect square.	
9 Substitute back $a = 2^x$.	$2^x = 4$
10 Write 4 as a number with base 2.	$2^x = 2^2$
11 Solve by equating the indices.	$x = 2$

Remember to always make the right-hand side equal to zero when solving quadratic equations.

It is a good idea to substitute your answer back into the original equation to check the accuracy of your work.

WORKED Example 17

Solve for x in the following.

a $x^3 = 8$ **b** $x^{\frac{1}{2}} = 3$ **c** $(x - 2)^4 = 625$

THINK	WRITE
a ① Write 8 as a number cubed so that both sides of the equation have the same index.	**a** $x^3 = 8$ $x^3 = 2^3$
② Equate the bases.	$x = 2$
b ① Square both sides because x^2 is the inverse of $x^{\frac{1}{2}}$ and this will give x a power of 1.	**b** $x^{\frac{1}{2}} = 3$ $\left(x^{\frac{1}{2}}\right)^2 = 3^2$
② Simplify.	$x = 9$
c ① Write 625 as a number with an index of 4.	**c** $(x - 2)^4 = 625$ $= 5^4$
② Equate the bases because the indices are the same.	$x - 2 = 5$
③ Solve for x.	$x = 7$

If the base is not the same and the numbers cannot be written with the same base then logarithms can be used. It is possible to take the logarithm of both sides of an equation provided the same base is used.

WORKED Example 18

Solve for x in the following. Give your answers correct to 3 decimal places.

a $5^x = 10$ **b** $2^{(x + 1)} = 12$

THINK	WRITE
a ① Take the logarithm of both sides to base 10 so that a calculator can be used.	**a** $5^x = 10$ $\log_{10} 5^x = \log_{10} 10$
② Use $\log_a m^p = p \log_a m$ and $\log_a a = 1$.	$x \log_{10} 5 = 1$
③ Divide both sides by $\log_{10} 5$.	$x = \dfrac{1}{\log_{10} 5}$
④ Use a calculator to simplify.	$x \approx 1.431$ (3 decimal places)
b ① Take the logarithm of both sides to base 10.	**b** $2^{(x + 1)} = 12$ $\log_{10} 2^{(x + 1)} = \log_{10} 12$
② Use $\log_a m^p = p \log_a m$ to simplify.	$(x + 1) \log_{10} 2 = \log_{10} 12$
③ Divide both sides by $\log_{10} 2$.	$(x + 1) = \dfrac{\log_{10} 12}{\log_{10} 2}$
④ Use a calculator to simplify the right-hand side. Keep all digits in the calculator until the final answer, then round off as required.	$x + 1 \approx 3.585$
⑤ Solve for x.	$x \approx 3.585 - 1$ $x \approx 2.585$ (3 decimal places)

Inequalities are worked in exactly the same way except that there is a change of sign when dividing or multiplying both sides of the inequality by a negative number.

WORKED Example 19

Solve for *x*, giving answers to 3 decimal places:

i $2^x > 5$

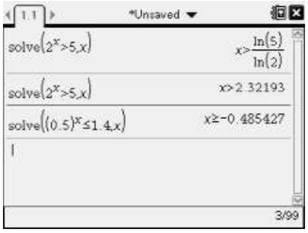
ii $0.5^x \leq 1.4$

a **by hand**

b **using the TI-Nspire CAS calculator.**

THINK	WRITE/DISPLAY
a **i** ① Take the logarithm of both sides to base 10.	**a** **i** $2^x > 5$ $\log_{10} 2^x > \log_{10} 5$
② Use $\log_a m^p = p \log_a m$ to simplify.	$x \log_{10} 2 > \log_{10} 5$
③ Divide both sides by $\log_{10} 2$ which is positive because 2 is a number greater than 1.	$x > \dfrac{\log_{10} 5}{\log_{10} 2}$
④ Use a calculator to simplify.	$x > 2.322$ (3 decimal places)
ii ① Take the logarithm of both sides to base 10.	**ii** $(0.5)^x \leq 1.4$ $\log_{10}(0.5)^x \leq \log_{10} 1.4$
② Use $\log_a m^p = p \log_a m$ to simplify.	$x \log_{10}(0.5) \leq \log_{10} 1.4$
③ Divide both sides by $\log_{10}(0.5)$. $\log_{10}(0.5) < 0$ because $0 < 0.5 < 1$, so change the sign.	$x \geq \dfrac{\log_{10} 1.4}{\log_{10} 0.5}$
④ Use your calculator to simplify.	$x \geq -0.485$ (3 decimal places)
b ① On a Calculator page, complete the entry lines as: solve$(2^x > 5, x)$ solve$((0.5)^x \leq 1.4, x)$ pressing ENTER after each line.	**b** (screenshot below)
② Write the answer.	**i** Solve $2^x > 5$ for x $x > 2.322$ (3 decimal places). **ii** Solve $0.5^x \leq 1.4$ for x $x \geq {}^{-}0.485$ (3 decimal places).

b

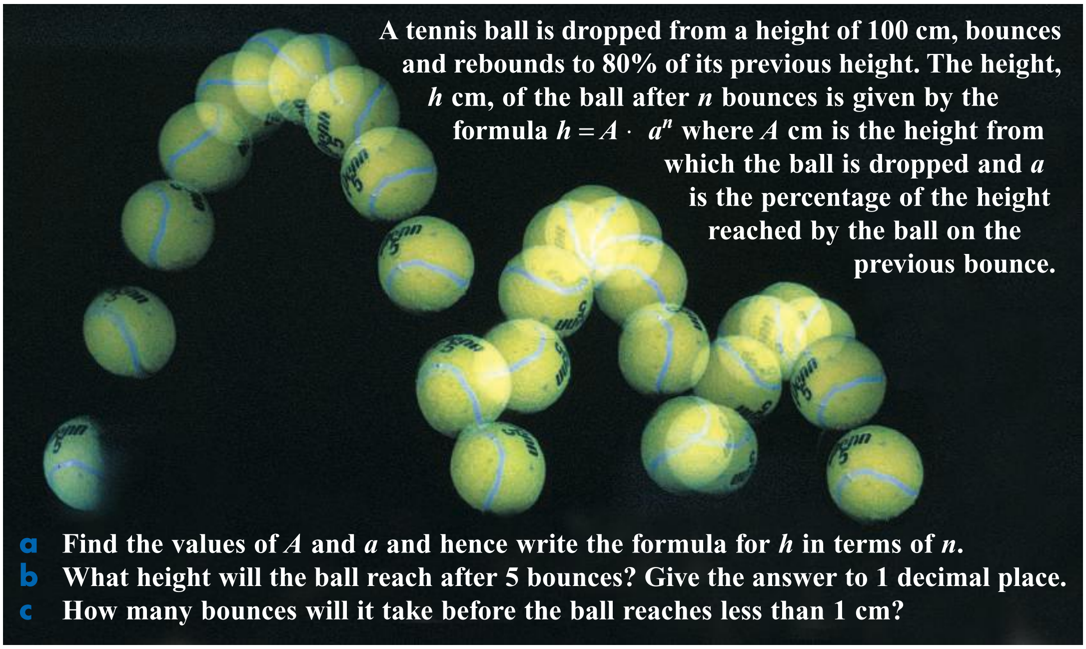


$\text{solve}(2^x>5,x)$ $\quad x>\dfrac{\ln(5)}{\ln(2)}$

$\text{solve}(2^x>5,x)$ $\quad x>2.32193$

$\text{solve}((0.5)^x\leq 1.4,x)$ $\quad x\geq -0.485427$

WORKED Example 20



A tennis ball is dropped from a height of 100 cm, bounces and rebounds to 80% of its previous height. The height, h cm, of the ball after n bounces is given by the formula $h = A \cdot a^n$ where A cm is the height from which the ball is dropped and a is the percentage of the height reached by the ball on the previous bounce.

a **Find the values of A and a and hence write the formula for h in terms of n.**
b **What height will the ball reach after 5 bounces? Give the answer to 1 decimal place.**
c **How many bounces will it take before the ball reaches less than 1 cm?**

THINK	**WRITE**
a ① The ball is dropped from a height of A cm.	**a** $A = 100$
② a is the percentage of the height reached by the ball on the previous bounce.	$a = 80\% = \frac{80}{100} = 0.8$
③ Substitute the values for A and a into the formula $h = A \cdot a^n$.	$h = A \cdot a^n$ $h = 100 \cdot (0.8)^n$
b ① Substitute 5 for n.	**b** $h = 100 \cdot (0.8)^5$
② Evaluate using a calculator.	$= 32.768$
③ Write your answer in a sentence.	The ball bounces to 32.8 cm after 5 bounces.
c ① Substitute 1 for h.	**c** $1 = 100 \cdot (0.8)^n$
② Divide both sides by 100.	$(0.8)^n = 0.01$
③ Take log of both sides to base 10.	$\log_{10}(0.8)^n = \log_{10}(0.01)$
④ Use $\log_a m^p = p \log_a m$ to simplify.	$n \log_{10}(0.8) = \log_{10}(0.01)$
⑤ Divide both sides by $\log_{10}(0.8)$.	$n = \frac{\log_{10} 0.01}{\log_{10} 0.8}$
⑥ Evaluate.	$= 20.64$
⑦ Bounces must be in whole numbers.	≈ 21
⑧ After 20 bounces the ball reaches more than 1 cm but after 21 bounces the ball reaches less than 1 cm because it bounces to a smaller and smaller height. Write the answer in a sentence.	The ball reaches less than one centimetre after 21 bounces.

remember

1. The equations $a^x = y$ and $2^x = 32$ are indicial equations.
2. Write numbers with the same base to help simplify problems. The most common ones to use are 2, 3 and 5.
3. If the base is the same equate the indices.
4. If the indices are the same equate the bases.
5. Use the Null Factor Law to solve quadratic equations.
6. A negative number cannot be expressed in index form, for example, -4 cannot be expressed with base 2.
7. $a^{2x} = (a^x)^2$
8. Take the logarithm of both sides of an equation or inequation using the same base.
9. Change the sign of an inequality when multiplying or dividing by a negative number.
10. $\log_a x > 0$ if $x > 1$
11. $\log_a x < 0$ if $0 < x < 1$

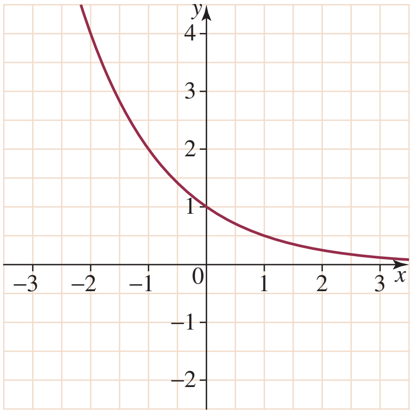
EXERCISE 3C Indicial equations

WORKED Example 13

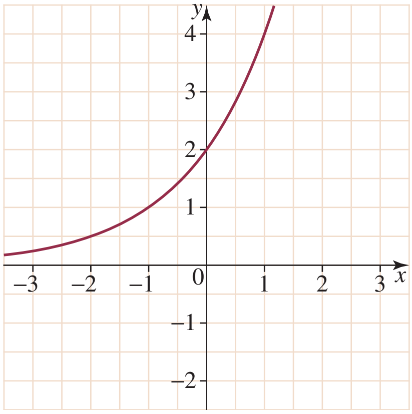
1 Using previous knowledge of the transformation of the graphs of functions, match each equation (**a**–**d**) with a suitable graph (P–S).

a $y = 2^{-x}$ **b** $y = -2^x$ **c** $y = 2^{2x}$ **d** $y = 2 \cdot 2^x$

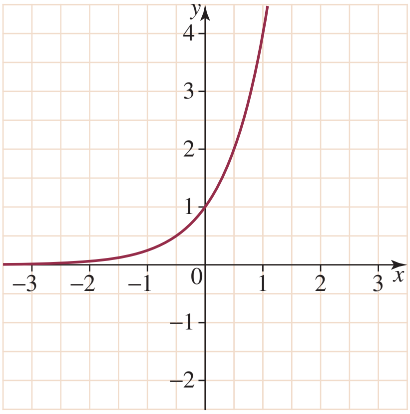
P



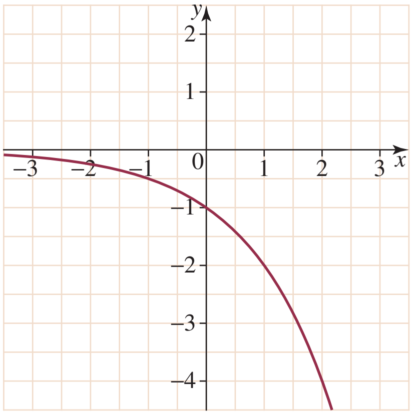
Q



R



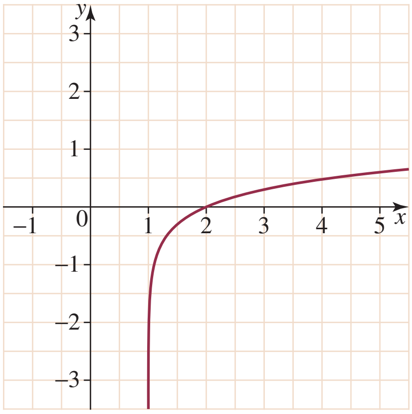
S



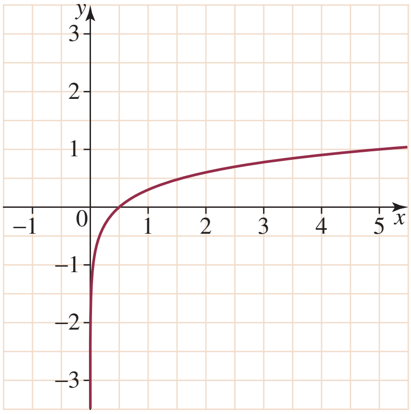
2 Using previous knowledge of the transformation of the graphs of functions, match each equation (**a**–**d**) with a suitable graph (P–S).

a $y = -\log_{10} x$ **b** $y = 2\log_{10} x$ **c** $y = \log_{10} 2x$ **d** $y = \log_{10}(x - 1)$

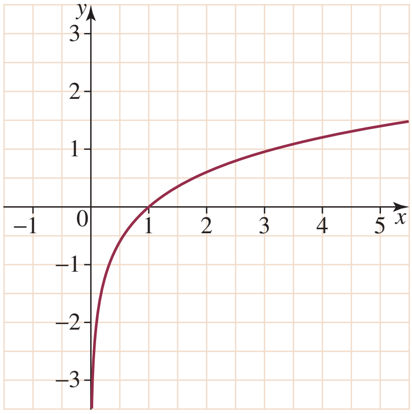
P



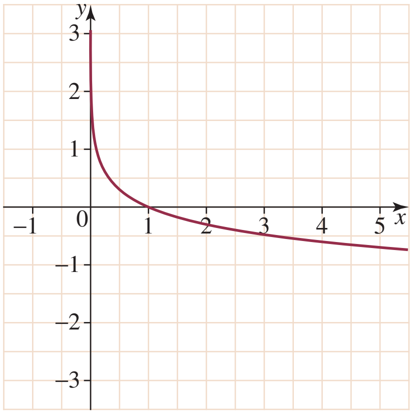
Q



R



S



WORKED Example 14

3 Solve for x in each of the following.

a $3^x = 81$ **b** $2^x = 64$ **c** $5^{-x} = 25$ **d** $10^{-x} = 1000$

e $\frac{1}{2^x} = 32$ **f** $\frac{1}{3^x} = 27$ **g** $7^x = \frac{1}{49}$ **h** $5^x = \frac{1}{125}$

i $243^x = 3$ **j** $10\,000^x = 10$

eBook*plus*

Digital docs:

SkillSHEET 3.1
Index form

SkillSHEET 3.2
Solving equations

SkillSHEET 3.3
Solving indicial equations by equating the bases

4 Solve for x in each of the following.

a $3 \cdot 2^x = 48$ **b** $2 \cdot 5^x = 250$ **c** $3^{x+1} = 27$ **d** $6^{x-2} = 216$

e $5^{2x-1} = \frac{1}{125}$ **f** $3^{3x-2} = \frac{1}{243}$ **g** $4^{x-2} = 1$ **h** $2^{2x-6} = 1$

WORKED Example 15

5 Solve for x in each of the following.

a $3^x \cdot 3^{x-1} = 243$ **b** $5^x \cdot 5^{2x+1} = 625$ **c** $2^x \cdot 4^{x-1} = 16$ **d** $4^x \cdot 16^{x+2} = 64$

e $\frac{3^{3x+1}}{9^{x-2}} = 81$ **f** $\frac{5^{2x-1}}{25^{4x-3}} = 625$ **g** $125^x = \frac{5^{\frac{1}{2}} \cdot 625^{-\frac{3}{4}}}{25}$ **h** $128^x = \frac{2^{\frac{3}{4}} \cdot 8^{\frac{1}{4}}}{16^{\frac{1}{8}}}$

WORKED Example 16

6 Solve for x in the following.

a $(3^x - 9)(3^x - 1) = 0$ **b** $(5^x - 125)(5^x - 5) = 0$ **c** $2^{2x} - 6 \cdot 2^x + 8 = 0$

d $6^{2x} - 7 \cdot 6^x + 6 = 0$ **e** $25^x + 4 \cdot 5^x - 5 = 0$ **f** $4^x - 6 \cdot 2^x - 16 = 0$

g $4^{2x} - 20 \cdot 4^x = -64$ **h** $3^{2x} - 36 \cdot 3^x = -243$ **i** $4^x = 33 \cdot 2^x - 32$

j $9^x = 2 \cdot 3^x + 3$

WORKED Example 17

7 Solve for x in the following.

a $x^5 = 243$
b $x^3 = 125$
c $x^{\frac{1}{2}} = 4$
d $x^{\frac{1}{3}} = 3$
e $(x - 1)^3 = 8$
f $(2x + 1)^4 = 81$
g $5(x + 2)^2 = 80$
h $7(x - 3)^5 = 224$

WORKED Example 18

8 Solve for x in the following. Give your answer correct to 3 decimal places.

a $2^x = 5$
b $3^x = 12$
c $(0.3)^{x - 1} = 10$
d $(1.2)^{2x} = 7$
e $(1.4)^{2 - x} = 6$
f $(0.6)^{1 - 2x} = 8$
g $3 \cdot 5^x = 27$
h $2 \cdot 4^x = 22$
i $5 \cdot 7^x = 1$
j $4 \cdot 6^x = 1$
k $2^x \cdot 3^{x + 1} = 10$
l $5^x \cdot 2^{x - 1} = 100$

WORKED Example 19

9 Solve for x in the following. Give your answer correct to 3 decimal places.

a $3^x > 5$
b $5^x < 2$
c $2^{2x} \leq 7$
d $4^{3x} \geq 6$
e $(0.2)^x > 3$
f $(0.3)^x < 4$
g $7^x \geq 0.5$
h $6^x \leq 0.6$
i $(0.4)^x > 0.2$
j $(0.5)^x < 0.9$

eBook plus

Digital docs:
SkillSHEET 3.4
Solving linear equations
WorkSHEET 3.1

10 **multiple choice**

The value of x for which $5 \cdot 2^x = 1255$, correct to 3 decimal places is:

A 7.971
B 897.750
C 897.749
D 7.972
E 2.059

11 **multiple choice**

The solution to the equation $10^{2x} = 3 \cdot 10^x + 4$ is:

A $\log_{10}(-1), \log_{10}4$
B $-1, 4$
C $10^x + 1, 10^x - 1$
D 0, 0.602
E $\log_{10}4$

WORKED Example 20

12 If \$10 000 is invested at 6% p.a. compound interest, the amount after n years is given by the formula $A = 10\,000(1.06)^n$. Find:

a the amount after 5 years, giving your answer to the nearest dollar
b the number of years it takes for the investment to be worth \$16 000, giving your answer to the nearest year
c how long it would take for the investment to double in value.

13 A new car costing \$25 000 is depreciating at 15% per year.

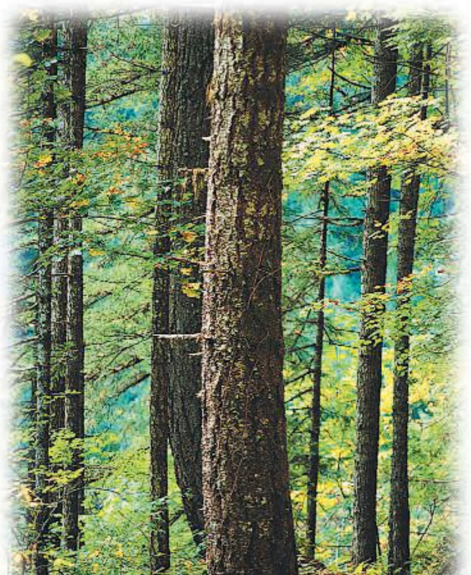
a What is the value of the car after one year?
b What percentage is that of the original cost?

The value of the car at the end of any year can be found by using the formula $V = C\left(1 - \frac{r}{100}\right)^n$, where \$$V$ is the value of the car, C is the original cost of the car, r is the rate of depreciation, and n is the number of years after purchase.

c Write the formula for V in terms of n, substituting values for C and r.
d What will the car be worth after 6 years?
e How long will it take before the car is worth \$500?
f Will the car ever be worth \$0? Give reasons for your answer.

14 The diameter of a tree trunk increases according to the formula: $D = A10^{0.04t}$, where D cm is the diameter of the trunk t years after it is first measured and A cm is the diameter of the trunk when it is first measured.

a Write an equation for D in terms of t if the trunk had a diameter of 20 cm when it was first measured.
b When will the diameter be 25 cm?
c After how many years will the diameter be greater than 30 cm?



Logarithmic equations using any base

The equation $\log_a y = x$ is an example of a general logarithmic equation. Laws of logarithms and indices are used to solve these equations.

WORKED Example 21

Solve for x in the following equations.

a $\log_2 x = 3$ **b** $\log_3 x^4 = -16$ **c** $\log_5 (x - 1) = 2$

THINK	WRITE
a 1 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**a** $\log_2 x = 3$ $2^3 = x$
2 Rearrange and simplify.	$x = 8$
b 1 Rewrite using $\log_a m^p = p \log_a m$.	**b** $\log_3 x^4 = -16$ $4 \log_3 x = -16$
2 Divide both sides by 4.	$\log_3 x = -4$
3 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$3^{-4} = x$
4 Rearrange and simplify.	$x = \frac{1}{3^4}$ $= \frac{1}{81}$
c 1 Rewrite using $a^x = y \Leftrightarrow \log_a y = {}^x$.	**c** $\log_5 (x - 1) = 2$ $5^2 = x - 1$
2 Solve for x.	$x - 1 = 25$ $x = 26$

The base of a logarithmic function and the base of an exponential function must be a positive real number other than 1.

WORKED Example 22

Solve for x in each of the following.

a $\log_x 4 = 2$ **b** $\log_x 81 = \frac{4}{3}$ **c** $\log_x \left(\frac{1}{125}\right) = -3$

THINK	WRITE
a 1 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**a** $\log_x 4 = 2$ $x^2 = 4$
2 Solve the quadratic equation.	$x^2 - 4 = 0$ $(x - 2)(x + 2) = 0$ $x - 2 = 0$ or $x + 2 = 0$ $x = \pm 2$
3 Check to see if solutions are valid.	$x = 2$ is the only solution.

This is the only solution. The solution $x = -2$ is not valid because the base of a logarithmic function must be a positive real number other than 1.

Continued over page →

THINK	WRITE
b ① Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**b** $\log_x 81 = \frac{4}{3}$ $x^{\frac{4}{3}} = 81$
② Cube both sides.	$x^4 = 81^3$
③ Take the 4th root of both sides.	$\sqrt[4]{x^4} = \sqrt[4]{81^3}$
④ Rewrite.	$= (\sqrt[4]{81})^3$
⑤ Take the 4th root of x^4 and of 81.	$x = 3^3$
⑥ Simplify.	$= 27$
c ① Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**c** $\log_x \left(\frac{1}{125}\right) = -3$ $x^{-3} = \frac{1}{125}$
② Write 125 as a number with a similar index.	$= \frac{1}{5^3}$
③ Rewrite with the same indices.	$= 5^{-3}$
④ Equate the bases.	$x = 5$

WORKED Example 23

Solve for x in the following.

a $\log_2 16 = x$ **b** $\log_3 \frac{1}{3} = x$ **c** $\log_9 3 = x$

THINK	WRITE
a ① Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**a** $\log_2 16 = x$ $2^x = 16$
② Write 16 with base 2.	$= 2^4$
③ Equate the indices.	$x = 4$
b ① Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**b** $\log_3 \frac{1}{3} = x$ $3^x = \frac{1}{3}$ $= \frac{1}{3^1}$
② Write $\frac{1}{3}$ with base 3.	$3^x = 3^{-1}$
③ Equate the indices.	$x = -1$

THINK	**WRITE**
c ① Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	**c** $\log_9 3 = x$ $9^x = 3$
② Write 9 with base 3.	$(3^2)^x = 3$
③ Remove the brackets.	$3^{2x} = 3^1$
④ Equate the indices.	$2x = 1$
⑤ Solve.	$x = \frac{1}{2}$

WORKED Example 24

Solve for x in the following.

a $\log_2 4 + \log_2 x - \log_2 8 = 3$ **b** $\log_{10} x + \log_{10} (x - 3) = \log_{10} 4$

THINK	**WRITE**
a ① Simplify the left-hand side. Use $\log_a mn = \log_a m + \log_a n$ and $\log_a \frac{m}{n} = \log_a m - \log_a n$.	**a** $\log_2 4 + \log_2 x - \log_2 8 = 3$ $\log_2 \left(\frac{4 \cdot x}{8}\right) = 3$
② Simplify LHS.	$\log_2 \frac{x}{2} = 3$
③ Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = \frac{x}{2}$
④ Solve.	$x = 2 \cdot 2^3$ $= 2 \cdot 8$ $= 16$
b ① Simplify the left-hand side by using $\log_a mn = \log_a m + \log_a n$.	**b** $\log_{10} x + \log_{10} (x - 3) = \log_{10} 4$ $\log_{10} x(x - 3) = \log_{10} 4$
② Equate the logs.	$x(x - 3) = 4$
③ Expand.	$x^2 - 3x = 4$
④ Solve the quadratic equation.	$x^2 - 3x - 4 = 0$ $(x - 4)(x + 1) = 0$ $x = 4$ or $x = -1$ $x = 4$ is the only solution.
⑤ $x > 0$, $x - 3 > 0$ because it is not possible to take the logarithm of a negative number, $x > 3$.	

Cell growth — a model revealed

At the beginning of this chapter we looked at a researcher who was growing cells in a laboratory, and the problem he faced in finding a model, or mathematical formula, relating cell number and days of growth. In other words, given a set of data, how could he determine a power relationship ($y = ax^n$) between the variables x and y?

If a power relationship, $y = ax^n$, is suspected, begin by taking the logarithms of both sides of the equation.

$$\begin{aligned} y &= ax^n \\ \log_{10} y &= \log_{10} ax^n \\ &= \log_{10} a + \log_{10} x^n \\ &= \log_{10} a + n \log_{10} x \end{aligned}$$

That is, $\log_{10} y = \log_{10} a + n \log_{10} x$.

If we graph $\log_{10} y$ (Y) against $\log_{10} x$ (X) a straight line will result. The equation of the straight line is:

$$Y = \log_{10} a + nX$$

The gradient of the straight line is n, and the Y-intercept is $\log_{10} a$.

Before graphics calculators and computers became easily accessible, people used log-log graph paper to investigate relationships suspected to be of the form $y = ax^n$.

WORKED Example 25

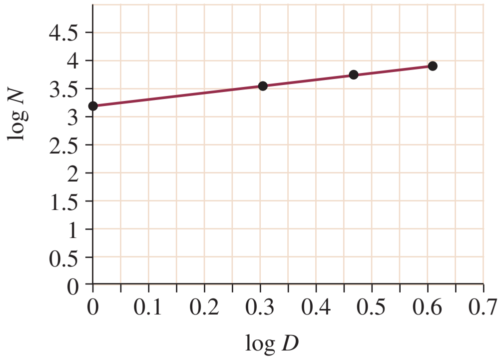
A researcher is observing the growth of a particular type of cell in a laboratory experiment. The following data are obtained.

Day (D)	1	2	3	4
Number of cells (N)	1500	3446	5606	7917

If a relationship of the form $N = AD^p$ is suspected, use a graphical method to calculate values of A and p, and hence state the relationship.

THINK	WRITE
1 State the form of the relationship.	$N = AD^p$
2 Take the log of both sides to base 10. *Note*: by convention we often write log a instead of $\log_{10} a$.	$\log N = \log AD^p$
3 Rewrite using $\log_a mn = \log_a m + \log_a n$.	$\log N = \log A + \log D^p$
4 Rewrite using $\log_a m^p = p \log_a m$.	$\log N = \log A + p \log D$
5 Change order of terms.	$\log N = p \log D + \log A$
6 Relate to the equation of a straight line.	This is of the form of a straight line $y = mx + c$ where $y = \log N$ and $x = \log D$.
7 Complete a table of values to obtain points to plot. We want to graph $\log N$ against $\log D$.	

D	1	2	3	4
$\log D$	0	0.301	0.477	0.602
N	1500	3446	5606	7917
$\log N$	3.176	3.537	3.749	3.899

THINK	**WRITE**
8 Use the coordinate points to draw a graph of log N against log D.	
9 Read the value of the vertical intercept from the graph. This gives us the value of log A.	$\log A = 3.176$
10 State the value of A. (Use the table to obtain this.)	$A = 1500$
11 Calculate the gradient of the line. This gives us the value of p.	$\text{Gradient} = \dfrac{3.899 - 3.176}{0.602 - 0}$ $= 1.20$ so $p = 1.2$
12 Substitute the values of A and p to state the relationship.	$N = AD^p$ $N = 1500\ D^{1.2}$

remember

1. In a logarithmic equation the unknown can be:
   - (a) the number, $\log_2 x = 5$
   - (b) the base, $\log_x 8 = 3$
   - (c) the logarithm, $\log_2 4 = x$.
2. In the expression $\log_a x$, a is a positive real number other than 1.
3. Use laws of logarithms and indices to solve the equations.
4. Check that all solutions are valid.
5. To investigate a power relationship between y and x, $y = Ax^n$, graph $\log_{10} y$ against $\log_{10} x$. If the graph is a straight line, then:
   y-intercept gives $\log_{10} A$
   gradient gives n.

EXERCISE 3D Logarithmic equations using any base

WORKED Example 21

1 Solve for x in the following.

a $\log_5 x = 2$

b $\log_3 x = 4$

c $\log_2 x = -3$

d $\log_4 x = -2$

e $\log_{10} x^2 = 4$

f $\log_2 x^3 = 12$

g $\log_3 (x + 1) = 3$

h $\log_5 (x - 2) = 3$

i $\log_4 (2x - 3) = 0$

j $\log_{10} (2x + 1) = 0$

k $\log_2 (-x) = -5$

l $\log_3 (-x) = -2$

m $\log_5 (1 - x) = 4$

n $\log_{10} (5 - 2x) = 1$

WORKED Example 22

2 Solve for x in the following.

a $\log_x 9 = 2$

b $\log_x 16 = 4$

c $\log_x 25 = \frac{2}{3}$

d $\log_x 125 = \frac{3}{4}$

e $\log_x \left(\frac{1}{8}\right) = -3$

f $\log_x \left(\frac{1}{64}\right) = -2$

g $\log_x 6^2 = 2$

h $\log_x 4^3 = 3$

WORKED Example 23

3 Solve for x in the following.

a $\log_2 8 = x$

b $\log_3 9 = x$

c $\log_5 \left(\frac{1}{5}\right) = x$

d $\log_4 \left(\frac{1}{16}\right) = x$

e $\log_4 2 = x$

f $\log_8 2 = x$

g $\log_6 1 = x$

h $\log_8 1 = x$

i $\log_{\frac{1}{2}} 2 = x$

j $\log_{\frac{1}{3}} 9 = x$

WORKED Example 24a

4 Solve for x in the following.

a $\log_2 x + \log_2 4 = \log_2 20$

b $\log_5 3 + \log_5 x = \log_5 18$

c $\log_3 x - \log_3 2 = \log_3 5$

d $\log_{10} x - \log_{10} 4 = \log_{10} 2$

e $\log_4 8 - \log_4 x = \log_4 2$

f $\log_3 10 - \log_3 x = \log_3 5$

g $\log_6 4 + \log_6 x = 2$

h $\log_2 x + \log_2 5 = 1$

i $3 - \log_{10} x = \log_{10} 2$

j $5 - \log_4 8 = \log_4 x$

WORKED Example 24

5 Solve for x in the following.

a $\log_2 x + \log_2 6 - \log_2 3 = \log_2 10$

b $\log_2 x + \log_2 5 - \log_2 10 = \log_2 3$

c $\log_3 5 - \log_3 x + \log_3 2 = \log_3 10$

d $\log_5 4 - \log_5 x + \log_5 3 = \log_5 6$

e $\log_5 x + \log_5 (x - 2) = \log_5 3$

f $\log_3 x + \log_3 (x + 2) = \log_3 8$

g $\log_4 x + \log_4 (x - 6) = 2$

h $\log_5 x + \log_5 (x + 20) = 3$

i $\log_5 (x + 1) + \log_5 (x - 3) = 1$

j $\log_6 (x - 2) + \log_6 (x + 3) = 1$

6 **multiple choice**

If $\log_a x = 0.7$, then $\log_a x^2$ is equal to:

A 0.49 **B** 1.4 **C** 0.35 **D** 0.837 **E** 0

7 **multiple choice**

If $\log_{10} x = a$, then $(\log_{10} x)^2 + \log_{10} x - 6$ becomes:

A $(\log_{10} a)^2 + \log_{10} a - 6$ **B** $a^2 + a + 6$ **C** $\log_{10} x^3 - 6$

D $(a - 2)(a + 3)$ **E** $\log_{10} 10^6 x^3$

8 Solve for x in the following.

a $(\log_{10} x)^2 + \log_{10} x - 2 = 0$ (*Hint*: Let $a = \log_{10} x$)

b $(\log_{10} x)^2 - 2\log_{10} x - 3 = 0$

c $(\log_2 x)^2 - 2\log_2 x = 8$

d $(\log_2 x)^2 + 3\log_2 x = 4$

e $(\log_3 x)^2 - \log_3 x^4 + 3 = 0$

f $(\log_5 x)^2 - \log_5 x^3 + 2 = 0$

g $\log_2 x^4 = (\log_2 x)^2$

h $\log_3 x^3 = (\log_3 x)^2$

i $\log_{10}(x^2 + 2x - 5) = 1$

j $\log_3(x^2 - 3x - 7) = 1$

9 If $\dfrac{\log_{10} x}{\log_{10} 2} = 4$, find x.

WORKED Example 25

10 Use graphical methods to determine a relationship of the form $N = At^n$ in the following set of data:

a

t	1	2	3	4
N	200	566	1039	1600

b

t	1	2	3	4
N	840	1188	1454	1680

c

t	1	2	3	4
N	283	655	1044	1470

11 An object falls from a high tower. The distance it falls in a certain time is recorded in the table below.

***t* (s)**	1	2	3	4	5
***d* (m)**	4.7	18.8	42.3	75.2	117.5

a If a relationship of the form $d = At^n$ exists, find values for A and n.

b Use this relationship to predict the value of d after 7 seconds.

12 A sidewalk restaurant has installed a heater to warm diners on cool winter evenings. The amount of heat, H W/m^2, reaching a particular point decreases as the distance, d m, from the heater increases. Measurements were taken of the relationship between H and d and are recorded in the table below.

***d* m**	1.8	2.2	2.9	3.3
***H* W/m^2**	1242	919	607	500

a It is suspected that the relationship between H and d is of the form $H = Ad^n$. Calculate the values of A and n.

b Use this model to predict the value of d for which the heat, H, reaching diners falls below 300.

13 The decibel (dB) scale for measuring loudness, d, is given by the formula $d = 10\log_{10}(I \cdot\ 10^{12})$, where I is the intensity of sound in watts per square metre.

a Find the number of decibels of sound if the intensity is 1.

b Find the number of decibels of sound produced by a jet engine at a distance of 50 metres if the intensity is 10 watts per square metre.

c Find the intensity of sound if the sound level of a pneumatic drill 10 metres away is 90 decibels.

d Find how the value of d changes if the intensity is doubled. Give your answer to the nearest decibel.

e Find how the value of d changes if the intensity is 10 times as great.

f By what factor does the intensity of sound have to be multiplied in order to add 20 decibels to the sound level?

Exponential equations (base *e*)

Euler's number, e, named after an 18th century Swiss mathematician, is a very important number used in problems involving natural growth and natural decay. Like π, it is irrational and has to be approximated: $e = 2.718\,281\,828\,459\ \ldots$ The number e can be used to find the value of an investment after a period of time, or the temperature of a liquid after it has been cooling.

To find the value of e, take the expression $\left(1 + \frac{1}{n}\right)^n$ and evaluate it for increasing values of n.

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

$n = 1$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1}\right)^1 = 2$

$n = 2$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{2}\right)^2 = 2.25$

$n = 3$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{3}\right)^3 = 2.370\,37$

$n = 5$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{5}\right)^5 = 2.48832$

$n = 10$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10}\right)^{10} = 2.59374$

$n = 100$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{100}\right)^{100} = 2.70481$

$n = 1000$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1000}\right)^{1000} = 2.716\,92$

$n = 10\,000$ $\quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10\,000}\right)^{10\,000} = 2.71815$

As n increases, $\left(1 + \frac{1}{n}\right)^n$ becomes closer and closer to 2.718 281 or e, or

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$$

Scientific calculators and graphics calculators have an e^x function which is treated in the same way as any other number.

An answer given in terms of e is an exact answer whereas the calculator gives an approximation.

The laws of indices apply in the same way if e is the base.

1. $e^x \cdot e^y = e^{x+y}$
2. $e^x \mid e^y = e^{x-y}$
3. $(e^x)^y = e^{xy}$
4. $e^0 = 1$
5. $e^{-x} = \dfrac{1}{e^x}$
6. $e^{\frac{x}{y}} = \sqrt[y]{e^x}$

WORKED Example 26

Solve for x in $e^{3x} = e$.

THINK	WRITE
1 Write the equation.	$e^{3x} = e$
2 Write e with a power of 1.	$e^{3x} = e^1$
3 Equate the indices.	$3x = 1$
4 Solve for x.	$x = \frac{1}{3}$

WORKED Example 27

Solve for x to 3 decimal places: i $e^x = 3$ ii $e^x - 3e^{-x} = 2$
a by hand b using the TI-Nspire CAS calculator.

THINK	WRITE
a i 1 Write the equation.	**a i** $e^x = 3$
2 Using the definition of a logarithm, if $e^x = 3$ then $x = \log_e 3$.	$x = \log_e 3$
3 Solve for x, using LN on your calculator.	$x \approx 1.099$
ii 1 Write the equation.	**ii** $e^x - 3e^{-x} = 2$
2 Write e^{-x} as $\dfrac{1}{e^x}$.	$e^x - \dfrac{3}{e^x} = 2$
3 Multiply every term by e^x.	$(e^x)^2 - 3 = 2e^x$
4 Make the right-hand side equal to zero.	$(e^x)^2 - 2e^x - 3 = 0$
5 Let $e^x = a$.	$a^2 - 2a - 3 = 0$ where $a = e^x$
6 Factorise and solve for a.	$(a - 3)(a + 1) = 0$ $a - 3 = 0$ or $a + 1 = 0$ $a = 3$ or $a = -1$
7 Substitute e^x for a.	$e^x = 3$ or $e^x = -1$
8 Solve for x by taking the log of both sides to base e.	$\log_e e^x = \log_e 3$
9 $e^x = 3$ is the only solution because $e^x = -1$ has no real solution.	$x \approx 1.099$ (3 decimal places)

Continued over page →

THINK

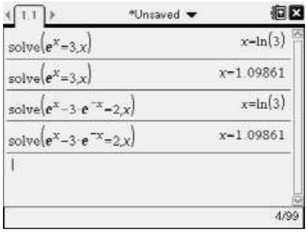
For the TI-Nspire CAS

b ① On a Calculator page, complete the entry lines as:
solve($e^x = 3, x$)
solve($e^x - 3e^{-x} = 2, x$)
pressing ENTER after each line.

② Write the answer.

WRITE/DISPLAY

b



```
*Unsaved
solve(e^x=3,x)            x=ln(3)
solve(e^x=3,x)            x=1.09861
solve(e^x-3·e^-x=2,x)     x=ln(3)
solve(e^x-3·e^-x=2,x)     x=1.09861
                              4/99
```

i Solve $e^x = 3$ for x
$x = 1.099$ (3 decimal places).

ii Solve $e^x - 3e^{-x} = 2$ for x
$x = 1.099$ (3 decimal places).

remember

1. Euler's number e is an irrational number which is approximated to 2.718 (3 decimal places).
2. Evaluate e by using the 'e^x' button on the calculator.
3. The number e is an exact answer, whereas the calculator gives an approximation.
4. The laws of indices apply in the same way if e is the base.
5. Use the LN button to take the log of a number to base e. The LOG button means $\log_{10}$.
6. $\log_e x = \ln x$.
7. $e^x > 0$, that is, $e^x = -1$ has no real solution.

EXERCISE 3E Exponential equations (base *e*)

1 Evaluate the following, giving your answer correct to 3 decimal places.

a e^2 **b** e^4 **c** $e^{\frac{1}{2}}$ **d** $e^{\frac{1}{3}}$ **e** $\sqrt[4]{e}$

f $\sqrt[5]{e}$ **g** $\ln 4$ **h** $\ln 5$ **i** $\log_e 1.5$ **j** $\log_e 3.6$

WORKED Example 26

2 Solve for x in each of the following.

a $e^x = e$ **b** $e^x = e^2$ **c** $e^{x-2} = e^4$ **d** $e^{2x} = e^{-1}$

e $e^{-x+1} = \frac{1}{e}$ **f** $e^{x-2} = \frac{1}{e^2}$ **g** $e^{3x+6} = \sqrt{e}$ **h** $e^{2x-1} = \sqrt{e^3}$

WORKED Example 27i

3 Solve for x in each of the following, giving your answer correct to 3 decimal places.

a $e^x = 2$ **b** $e^x = 5$ **c** $e^x = \frac{1}{2}$ **d** $e^x = \frac{1}{4}$

e $e^x = 1.3$ **f** $e^x = 2.6$ **g** $2e^x = 6$ **h** $3e^x = 12$

i $5e^x = 12$ **j** $4e^x = 10$

WORKED Example 27ii

4 Solve for x in each of the following, giving your answer correct to 3 decimal places.

- **a** $(e^x - 1)(e^x + 2) = 0$
- **b** $(e^x + 3)(e^x - 1) = 0$
- **c** $(e^{-x} - 2)(e^{2x} - 3) = 0$
- **d** $(3e^{-x} - 2)(2e^x - 1) = 0$
- **e** $(2e^x + 1)(e^x - 4) = 0$
- **f** $(3e^x - 2)(e^x + 4) = 0$
- **g** $(e^x)^2 - e^x = 0$
- **h** $(e^x)^2 - e \cdot e^x = 0$
- **i** $(e^x)^2 - 7e^x + 10 = 0$
- **j** $2(e^x)^2 + e^x - 15 = 0$
- **k** $6 - 11e^x + 3e^{2x} = 0$
- **l** $18 - 23e^x + 7e^{2x} = 0$

5 Solve for x in each of the following, giving your answer correct to 3 decimal places.

- **a** $e^x - 4e^{-x} = 0$
- **b** $e^x - 25e^{-x} = 0$
- **c** $e^x - 15e^{-x} - 2 = 0$
- **d** $5e^x - 12e^{-x} - 11 = 0$
- **e** $3e^x + 6e^{-x} - 11 = 0$
- **f** $4e^x + 6e^{-x} - 11 = 0$
- **g** $e^x + 2e^{-x} = 3$
- **h** $2e^x - 5e^{-x} = 9$

6 Solve for x in each of the following, giving your answer correct to 3 decimal places.

- **a** $e^x > 1$
- **b** $e^x < e$
- **c** $e^x < 2$
- **d** $e^x > 5$
- **e** $e^{2x} \geq 4$
- **f** $e^{x+1} \leq 6$
- **g** $e^{1-x} \leq 10$
- **h** $e^{1-x} \geq 15$
- **i** $e^{-x} > 0.75$
- **j** $e^{-x} < 0.26$
- **k** $e^{2x} + 0.6 < 1$
- **l** $e^{x+1} - 7.2 > 2.3$

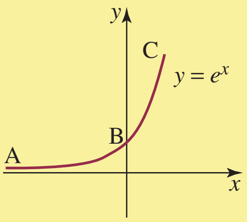
7 If $y = Ae^{-kt}$, and $y = 19.6$ when $t = 2$, and $y = 19.02$ when $t = 5$, find the value of the constants A and k. Give your answers correct to 2 decimal places.

8 For a body which has a higher temperature than its surroundings, Newton's Law of Cooling is given by the formula $\theta = \theta_0 e^{-kt}$, where θ is the difference between the temperature of the body and its surroundings after t minutes and θ_0 is the difference between the original temperature of the body and its surroundings. If the temperature of a freshly poured bowl of soup is 90°C in a room with a constant temperature of 18°C, and it cools to 65°C after 10 minutes, find the value of k. Give your answer correct to 2 decimal places.

The graph of $y = Ne^{kx}$

The diagram at right shows the graph of $y = e^x$ and uses the letters A, B and C to indicate key parts of the graph.

In this investigation you will use your graphics calculator to observe and report on the effect of changing N and k in the equation $y = Ne^{kx}$.



1 $N = 2$. On the same axes, graph the equations $y = 2e^x$ and $y = e^x$.
In your book, sketch the view window.
Write a sentence summarising the effect of changing N from 1 to 2.

2 $N = -1$. On the same axes, graph the equations $y = -1 \cdot e^x$ and $y = e^x$.
In your book, sketch the view window.
Write a sentence summarising the effect of changing N from 1 to -1.

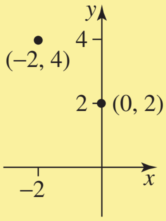
3 $k < 0$. On the same axes, graph the equations $y = e^x$ and $y = e^{-x}$.
In your book, sketch the view window.
Write a sentence summarising the effect of changing k from 1 to −1.

4 $0 < k < 1$. On the same axes, graph the equations $y = e^x$ and $y = e^{0.5x}$.
In your book, sketch the view window.
Write a sentence summarising the effect of changing k from 1 to 0.5.

5 $k > 1$. On the same axes, graph the equations $y = e^x$ and $y = e^{2x}$.
In your book, sketch the view window.
Write a sentence summarising the effect of changing k from 1 to 2.

Challenge

Use your calculator to obtain a guess-and-check solution to the following problem. Find the values of k and N such that the graph of $y = Ne^{kx}$ passes through (−2, 4) and (0, 2), the points shown.



Equations with natural (base *e*) logarithms

Graphics Calculator tip! Evaluating natural logarithms (base *e*)

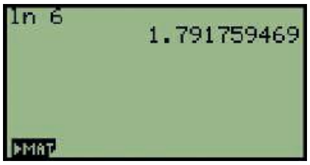
The **Casio *fx*-9860G AU** can be used to evaluate logarithms to base e.

To find the approximate value of $\log_e 6$, press:

- MENU
- 1 (RUN).

Complete the entry line as:
ln 6

then press EXE.

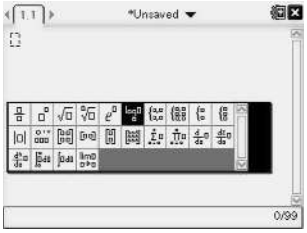


```
ln 6
                1.791759469
▶MAT
```

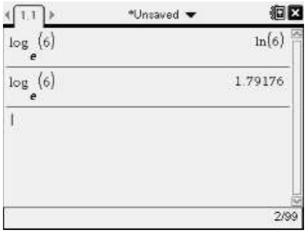
The **TI-Nspire CAS** can be used to evaluate logarithms to any base.

1. To find the approximate value of $\log_e 6$, on a Calculator page, press:
   - Ctrl ctrl
   - .

Use the NavPad to highlight the logarithm template and then press ENTER enter.
Alternatively the ln button on the keypad may be used.



2. Complete the fields as shown.
   This template can be used for any base.



$\log_e(6)$ ln(6)

$\log_e(6)$ 1.79176

2/99

The laws of logarithms apply in the same way for base e as they do for base 10.

$$e^x = y \Leftrightarrow \log_e y = x.$$

WORKED Example 28

Solve for x giving your answer correct to 3 decimal places, given $\log_e x = 3$.

THINK	WRITE
① Rewrite using $e^x = y \Leftrightarrow \log_e y = x$.	$\log_e x = 3$ $e^3 = x$
② Evaluate using e^x on your calculator.	$x \approx 20.086$

WORKED Example 29

Solve for x giving your answer correct to 3 decimal places where appropriate.

a $\log_e 3 = \log_e x$ **b** $\log_e x + \log_e 3 = \log_e 6$

THINK	WRITE
a Since the base is the same, equate the numbers.	**a** $\log_e 3 = \log_e x$ $x = 3$
b ① Rewrite using $\log_e mn = \log_e m + \log_e n$.	**b** $\log_e x + \log_e 3 = \log_e 6$ $\log_e 3x = \log_e 6$
② Equate the number parts.	$3x = 6$
③ Solve for x.	$x = 2$

remember

1. $e^x = y \Leftrightarrow \log_e y = x$
2. $\log_e 1 = \ln 1 = 0$
3. $\log_e e = \ln e = 1$
4. $\log_e 0 = \ln 0$ is undefined.
5. $\log_e mn = \log_e m + \log_e n$ or $\ln mn = \ln m + \ln n$
6. $\log_e \frac{m}{n} = \log_e m - \log_e n$ or $\ln \frac{m}{n} = \ln m - \ln n$
7. $\log_e m^p = p \log_e m$ or $\ln m^p = p \ln m$

EXERCISE 3F Equations with natural (base e) logarithms

WORKED Example 28

1 Solve for x in each of the following giving exact answers when appropriate, otherwise, correct to 3 decimal places.

a $\log_e x = 1$ **b** $\log_e x = 2$ **c** $\log_e x = -2$
d $\log_e x = -1$ **e** $\log_e x = 0.3$ **f** $\log_e x = 0.5$
g $\log_e x = -0.15$ **h** $\log_e x = -0.69$

2 Solve for x giving exact answers when appropriate, otherwise, correct to 3 decimal places.

a $\log_e 2x = 2$ **b** $\log_e 3x = 1$ **c** $\log_e x^3 = 3$
d $\log_e x^2 = 2$ **e** $\log_e x^2 = 0.4$ **f** $\log_e x^3 = 0.9$
g $\log_e (x - 1) = -1$ **h** $\log_e (2x + 1) = -2$ **i** $\log_e (-x) = 0.36$
j $\log_e (-x) = 0.72$ **k** $\log_e (1 - x) = -0.54$ **l** $\log_e (2 + x) = -0.83$

WORKED Example 29

3 Solve for x giving exact answers when appropriate, otherwise, correct to 3 decimal places.

a $\log_e x = \log_e 2$ **b** $\log_e x = \log_e 5$ **c** $\log_e x + \log_e 3 = \log_e 9$
d $\log_e x + \log_e 2 = \log_e 8$ **e** $\log_e x - \log_e 5 = \log_e 2$ **f** $\log_e x - \log_e 4 = \log_e 3$
g $1 + \log_e x = \log_e 6$ **h** $1 - \log_e x = \log_e 7$ **i** $\log_e 4 - \log_e x = \log_e 2$
j $\log_e 5 - \log_e x = \log_e 25$

4 Solve for x giving exact answers.

a $\log_e x + \log_e 5 - \log_e 10 = \log_e 3$ **b** $\log_e x + \log_e 3 - \log_e 9 = \log_e 4$
c $2\log_e 3 + \log_e x - \log_e 2 = \log_e 3$ **d** $3\log_e 2 + \log_e x - \log_e 4 = \log_e 5$
e $\log_e 6 + \log_e 2 - \log_e x = \log_e 4$ **f** $\log_e 4 + \log_e 3 - \log_e x = \log_e 2$
g $\log_e x + \log_e (x + 1) = \log_e 2$ **h** $\log_e x + \log_e (2x - 1) = \log_e 3$
i $\log_e (x - 1) + \log_e (x + 2) = \log_e 4$ **j** $\log_e (x + 1) + \log_e (2x - 1) = \log_e 5$

5 **multiple choice**

If $\ln y = \ln x + \ln a$, then an equation relating x and y which does not involve logarithms is:

A $y = x + a$ **B** $y = ax$ **C** $y = x - a$ **D** $y = \frac{x}{a}$ **E** $y = \frac{a}{x}$

6 **multiple choice**

In the equation $2\log_e x - \log_e 3x = a$, $x =$

A $3e^a$ **B** $-a$ **C** $3a$ **D** $\log_e 6a$ **E** no solution

7 Write the following equation without logarithms and with y as the subject.

$$2\log_e x + 1 = \log_e y$$

8 If $\log_e x = a$ and $y = e^a$, express y in terms of x.

9 Solve for x the equation $e^{\ln x} = 2$.

eBookplus

Digital doc:
WorkSHEET 3.2

10 Five grams of a radioactive substance is decaying so that the amount, A grams, that is left after t days, is given by the formula $A = 5e^{-kt}$.

- **a** Find the value of A when the number of grams of the radioactive substance has been halved.
- **b** Rewrite the equation with the new value of A.
- **c** Rearrange the equation so that t is the subject of the equation.
- **d** If $k = 0.005$, find how long it will take for the number of grams of the radioactive substance to be halved. Give your answer correct to the nearest day.

Graphing inverse functions

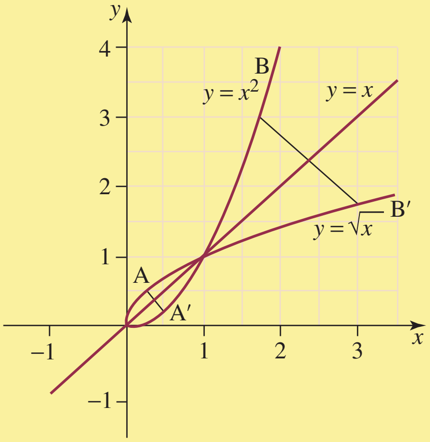
eBookplus

Interactivity:
Inverses
int-0248

Recall that if $y = f(x)$, the inverse of $f(x)$ is such that $x = f^{-1}(y)$.

We can use this relationship to sketch the graph of $f^{-1}(x)$ if we know the graph of $f(x)$.

Consider for example $f(x) = x^2$ for $x \geq 0$.



The graph of the inverse function is obtained by *reflecting* each point on the curve $y = f(x)$ in the line $y = x$. The graph of $y = f^{-1}(x)$ shown is the graph of $y = \sqrt{x}$.

1 Check, by using algebra, that $y = \sqrt{x}$ is the inverse of $y = x^2$.

An inverse relationship exists between the exponential function, a^x, and the logarithmic function, $\log_a x$.

2 On your graphics calculator sketch the graph of $y = e^x$. Copy this graph into your book.

3 On this copy add the line $y = x$.

4 By reflecting the graph of $y = e^x$ in the line $y = x$, obtain a sketch of the curve $y = \log_e x$.

5 Show that $e^{\log_e a} = \log_e e^a$.

Exponential and logarithmic modelling

eBookplus

eLesson:
Exponential modelling
eles-0091

Exponential and logarithmic functions can be used to model many real situations involving natural growth and decay.

Continuous growth and decay can be modelled by the equation $A = A_0e^{kt}$, where A_0 represents the initial value, t represents the time taken and k represents the rate constant.

For continuous growth, k is positive, but for continuous decay, k is negative. Logarithms to base 10, often called *common logarithms*, are used in scientific formulas for measuring the intensity of earthquakes, the acidity of solutions and the intensity of sound.

WORKED Example 30

eBook plus

Tutorial:
Worked example 30
int-0533

In the town of Ill Ness, the number of cases of a particular disease, D, can be modelled by the equation $D = D_0e^{kt}$, where t is the time in years. Using available medication the number of cases is being reduced by 20% each year. There are 10 000 people with the disease today.

a **How many people will have the disease after one year?**
b **Find the value of k correct to 3 decimal places.**
c **Write the equation substituting values for k and D_0.**
d **Find how long it would take for the number of people with the disease to be halved. Give your answer correct to the nearest year.**
e **How long would it take for the number of people with the disease to be reduced to 100? Give your answer correct to the nearest year.**

THINK	WRITE
a ① Find the percentage of people with the disease after one year.	**a** (100 − 20)% = 80%
② Find 80% of the original number.	80% of 10 000 = 8000
③ Write a sentence.	Therefore, 8000 people will have the disease after one year.
b ① Substitute $t = 0$ and $D = 10\,000$ into the given equation.	**b** $D = D_0e^{kt}$ When $t = 0$ and $D = 10\,000$, $10\,000 = D_0e^{k\cdot 0}$ $10\,000 = D_0 \cdot 1$ $10\,000 = D_0$ So $D = 10\,000e^{kt}$ [1]
② Substitute $t = 1$ and $D = 8000$ into [1], and solve for k.	When $t = 1$ and $D = 8000$, $8000 = 10\,000e^{k\cdot 1}$ $\frac{8000}{10\,000} = e^k$ $0.8 = e^k$ $\log_e 0.8 = \log_e e^k$ $-0.223 = k\log_e e$ $-0.223 = k \cdot 1$ $k = -0.223$
c Use the given equation $D = D_0e^{kt}$.	**c** $D = 10\,000e^{-0.223t}$
d ① Substitute 5000 for D.	**d** $D = 10\,000e^{-0.223t}$ When $D = 5000$, $5000 = 10\,000e^{-0.223t}$
② Simplify by dividing both sides by 10 000.	$0.5 = e^{-0.223t}$
③ Take $\log_e$ of both sides.	$\log_e 0.5 = -0.223t$
④ Solve for t.	$t = \frac{\log_e 0.5}{-0.223}$ ≈ 3.108 (3 decimal places)
⑤ Write a sentence.	It would take about 3 years.

THINK	WRITE
e ① Write the equation.	**e** $D = 10\,000e^{-0.223t}$
② Substitute 100 for D.	When $D = 100$. $100 = 10\,000e^{-0.223t}$
③ Simplify by dividing by 10 000.	$0.01 = e^{-0.223t}$
④ Take $\log_e$ of both sides.	$\log_e 0.01 = -0.223t$
⑤ Solve for t.	$t \approx 20.651$ (3 decimal places)
⑥ Write the answer in a sentence.	It would take about 21 years.

An earthquake formula

A number of earthquakes have been recorded in the Calamari Desert. The earthquakes' Richter magnitude, M, can be modelled by the equation $M = A\log_{10}E + B$ where E is the energy in joules.

1 If the Richter magnitude of an earthquake is 8.5 with an energy of 10^{17} joules, substitute the values into the equation $M = A\log_{10}E + B$.

2 If the Richter magnitude of an earthquake is 3.8 with an energy of 10^{10} joules, substitute the values into the equation $M = A\log_{10}E + B$.

3 Using the two equations, find the values of A and B. Give answers to 2 decimal places.

4 Substitute the values of A and B into the equation for finding the Richter magnitude of an earthquake.

5 If the magnitude of an earthquake is 7.8, find the energy, rounding the index to the nearest whole number.

6 If the energy of an earthquake is 10^{12} joules, find the magnitude of the earthquake. Give your answer correct to 2 decimal places.

remember

1. Read the question carefully.
2. Make a note of all the information that has been given.
3. Give your answers to the correct number of decimal places.
4. Continuous growth and decay is modelled by the equation $A = A_0e^{kt}$, A_0 represents the initial value (that is, when $t = 0$) and k represents the rate constant.

EXERCISE 3G Exponential and logarithmic modelling

WORKED Example 30

1 Changing δ-gluconolactone into gluconic acid can be modelled by the equation $y = y_0e^{-0.6t}$ where y is the number of grams of δ-gluconolactone present t hours after the process has begun. Suppose 200 grams of δ-gluconolactone is to be changed into gluconic acid.

- **a** Find the value of y_0.
- **b** Write the equation replacing y_0 with your answer.
- **c** How many grams of δ-gluconolactone will be present after 1 hour? Give your answer correct to the nearest gram.
- **d** How long will it take to reduce the amount of δ-gluconolactone to 50 grams? Give your answer correct to the nearest quarter of an hour.

2 The decay of radon-222 gas is given by the equation $y = y_0e^{-0.18t}$ where y is the amount of radon remaining after t days. When $t = 0$, $y = 10$ g. Give all answers to the nearest whole number.

- **a** Find the value of y_0.
- **b** Write the equation substituting your value of y_0.
- **c** What will be the mass after 1 day?
- **d** How many days will it take for the mass to reach 1 g?

3 The equation $y = A + B\log_e x$ relates two variables, x and y. The table at right shows values of x and y.

x	1	2	3
y	3	4.386	m

- **a** Find the value of A and B correct to the nearest whole number.
- **b** Write the equation relating x and y substituting values for A and B.
- **c** Using your new equation, find the value of m correct to 3 decimal places.
- **d** If $y = 7.6$, find x correct to the nearest whole number.

4 An amount of $1000 is invested in a building society where the 5% p.a. interest paid is compounded continuously. The amount in the account after t years can be modelled by the equation $A = A_0e^{rt}$, where r is the continuous interest rate.

- **a** Find the value of A_0 and r.
- **b** Write the equation substituting values of A_0 and r.
- **c** Find the amount in the bank after **i** 1 year **ii** 10 years. Give your answer correct to the nearest dollar.
- **d** How long will it take for the investment to double in value? Give your answer to the nearest year.

5 The number of people living in Boomerville at any time, t years, after the first settlers arrived can be modelled by the equation $P = P_0e^{kt}$. Suppose 500 people arrived on 1 January 1850, and by 1 January 1860 there were 675 people.

- **a** What is the value of P_0?
- **b** Find the value of k correct to 2 decimal places.
- **c** Write the equation substituting values for P_0 and k.
- **d** What will be the population on 1 January 1900? Give your answer to the nearest 10 people.
- **e** When will the population be 2000?

Career profile

TONY INTIHAR — Aircraft Maintenance Engineer

Qualifications:
Aviation — Electrical/Instrument/Radio
Employer: Pel-Air Aviation

I entered this field because of an interest in aviation technology after spending many years in the Navy. Whilst in the Navy I undertook additional study and gained further technical qualifications.

A typical day at work includes preparing aircraft for flying, and repairing and servicing aircraft. When aircraft return at the end of a day, I fix any reported faults and ensure they are prepared for the next day.

I find Mathematics a most valuable and essential tool in understanding the electronics theory associated with avionic systems. It helps to remove the guesswork when diagnosing system faults.

One example of a formula I use (which requires an understanding of logarithms) is:

$$\text{Power change} = 10 \log_{10} \frac{P_1}{P_2} \text{ decibels}$$

to determine the power gain in aircraft audio system amplifiers.

Questions

1. Suggest why Tony's job carries a great deal of responsibility.
2. Use the formula Tony quoted to calculate the power gain for an amplifier when $P_1 = 10\,000$ and $P_2 = 1000$.
3. Investigate Mathematics prerequisites for courses in aviation technology.

St Louis’ Gateway Arch

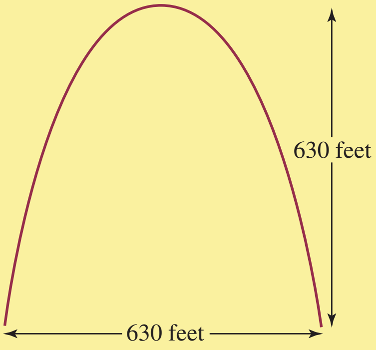
Historically, St Louis, Missouri, was an important starting point for pioneers on their way west. To commemorate its role as the gateway to the West, the city of St Louis built a 630-foot stainless steel structure called the *Gateway Arch*. The design and construction of this imposing arch was an amazing feat of engineering.

The Gateway Arch, designed by Finnish architect Eero Saarinen, takes the shape of a *catenary* curve.

A catenary curve can be described by the following function:

$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$ where a is a non-zero constant.

By selecting suitable axes, devise an equation that will produce the Gateway Arch.



summary

Index laws

- $a^x \cdot a^y = a^{x+y}$
- $a^x \mid a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^0 = 1$
- $a^{-x} = \frac{1}{a^x}$ and $\frac{1}{a^{-x}} = a^x$
- $a^{\frac{1}{y}} = \sqrt[y]{a}$ and $a^{\frac{x}{y}} = \sqrt[y]{a^x}$
- $a^x = y \Leftrightarrow \log_a y = x$

Logarithms and laws of logarithms

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a 0$ is undefined
- $\log_a mn = \log_a m + \log_a n$
- $\log_a \frac{m}{n} = \log_a m - \log_a n$
- $\log_a m^p = p \log_a m$
- $\log_b N = \frac{\log_a N}{\log_a b}$ (change-of-base rule)

Indicial equations

- To solve indicial equations:
  1. write all terms with the same base, write terms with the smallest possible base or take the logarithm of both sides of the equation
  2. then solve the equation.
- A negative number cannot be expressed in index form.
- If $0 < x < 1$, then $\log_a x < 0$ and if $x > 1$ then $\log_a x > 0$.
- It is not possible to take the logarithm of a negative number.

Exponential equations (base *e*)

- Euler's number $e = \lim_{h \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\,828\,459\ldots$
- The laws of indices and logarithms apply in the same way when using *e*.

Equations with natural logarithms (base *e*)

- To solve logarithmic equations use the laws of logarithms and indices.
- To investigate a power relationship between y and x, $y = Ax^n$
  1. Graph $\log_{10} y$ against $\log_{10} x$.
  2. If the graph is a straight line, then
     (a) y-intercept gives $\log_{10} A$
     (b) gradient gives n.

CHAPTER review

3A **1** **multiple choice**

If $a > 1$, the solution of x for the equation $x = a^2$ is:

A 1
B a negative number less than 1
C a positive number less than 1
D a negative number greater than 1
E a positive number greater than 1

3A **2** Simplify $4x^{\frac{3}{5}}y^{\frac{2}{3}} \cdot \left(2x^{-3}y^{\frac{4}{5}}\right)^{-2}$, leaving your answers with positive indices.

3B **3** **multiple choice**

$2\log_3 x + 4\log_3 x - \log_3 x^6$ is equal to:

A 0
B $\log_3\left(\frac{6x}{x^6}\right)$
C $\log_3(6x - x^6)$
D $\log_3\left(\frac{8x}{x^6}\right)$
E $6\log_3(x - x^6)$

3B **4** **multiple choice**

$3\log_2 6 + \log_2 3^2 - \log_2 108$ is equal to:

A $3\log_2 117$
B $3\log_2(0.5)$
C $\log_2 18$
D $3\log_2 2$
E 1

3B **5** If $\log_2 5 = 2.321$ and $\log_2 9 = 3.17$, find $\log_2 \frac{5}{9}$.

3B **6** Evaluate $\dfrac{\log_2 32}{\log_2 8}$.

3C **7** **multiple choice**

The solution(s) to the equation $(2^x - 1)(2^{2x} - 4) = 0$ is/are:

A 0, 1
B 0, 2
C 1, 2
D 1, 4
E 2, 4

3C **8** Find $\{x: 0.3^x > 4.6\}$.

3D **9** **multiple choice**

If $\log_e 2x = a$, then x is equal to:

A $2e^a$
B $2a^e$
C $\frac{e^a}{2}$
D $\frac{a^e}{2}$
E e^{2a}

3D **10** Solve for x in $\log_5 4 + \log_5 x = \log_5 24$.

11 **multiple choice** 3E

The solution(s) of the equation $e^x - 12e^{-x} = -4$ is/are:

A $\log_e 2, \log_e 6$ **B** $e^x - 2, e^x + 6$ **C** $2, -6$ **D** $\log_e 2$ **E** $\log_e 6$

12 **multiple choice** 3E

If $\log_e x = a$, then $e^{2a} + 3e^a - 2e^{-a}$ is equal to:

A $a^2 + 3a - \frac{2}{a}$ **B** $x^2 + 3x - \frac{2}{x}$ **C** $2\log_e a + 3\log_e a - \frac{2}{\log_e a}$

D $\log_e x^2 + 3\log_e x - \frac{2}{\log_e x}$ **E** $(e^a + 2)(e^a + 1)$

13 If $7e^{2x} = 1234$, find x, giving your answer correct to 2 decimal places. 3E

14 **multiple choice** 3E

If $e^{x+4} = e^{2x-1}$, then x is equal to:

A $-\frac{5}{3}$ **B** 5 **C** e^5 **D** -5 **E** $e^{-\frac{5}{3}}$

15 $A = A_0e^{-kx}$. If $A = 50$ when $x = 0$ and $A = 20$ when $x = 5$, find the values of A_0 and k. Give your answer correct to 2 decimal places. 3E

16 $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +$ 3E

a Write the next 3 terms.
b Substitute $x = 1$ in the equation using the first 7 terms.
c Show that $e \approx 2.7182$.

17 If $\log_e a = 0.6932$ and $\log_e\left(\frac{a}{3}\right) = x$, find the value of x, giving your answer correct to 2 decimal places. 3F

18 **multiple choice** 3F

The solution(s) to the equation $2\ln x = \ln(x + 4) + \ln 2$ is/are:

A $-2, 4$ **B** $2, -4$ **C** 1 **D** 2 **E** 4

19 Solve for x in 3F
$\log_e 5 + \log_e x - \log_e 2 = \log_e 10$.

20 **multiple choice** 3G

The air pressure P cm of mercury at h km above sea level can be modelled by the equation $P = 50e^{-0.2h}$. One kilometre above sea level the pressure has:

A increased by approximately 9 cm
B decreased by approximately 9 cm
C increased by approximately 41 cm
D decreased by approximately 41 cm
E neither increased nor decreased significantly.

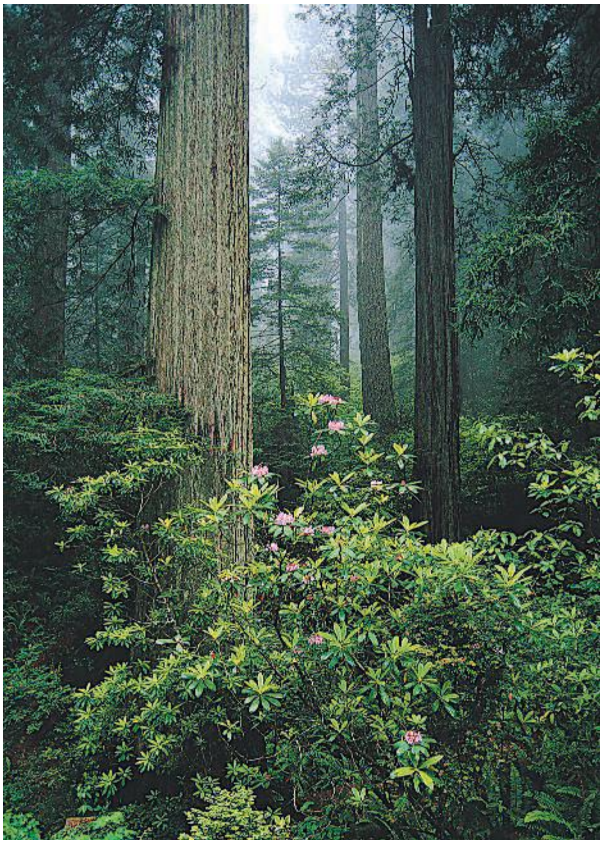
Modelling and problem solving

1 A cup of soup cools to the temperature of the surrounding air. Newton's Law of Cooling can be written as $T - T_S = (T_0 - T_S)e^{-kt}$ where T is the temperature of the object after t minutes, and T_S is the temperature of the surrounding air. The soup cooled from 90 °C to 70 °C after 6 minutes in a room with an air temperature of 15 °C.

- a Find the values of T_S, T_0 and k correct to 2 decimal places.
- b Write the equation substituting the values for T_S, T_0 and k.
- c Find the temperature of the soup after 10 minutes. Give your answer to the nearest degree.
- d How long would it take for the soup to be 40 °C? Give your answer to the nearest minute.
- e If the soup is placed in a refrigerator in which the temperature is 2 °C, how long will it take for the soup to reach 40 °C? Use the same value of k and give your answer to the nearest minute.

2 The diameter of a tree for a period of its growth can be modelled by the equation $D = D_0e^{kt}$ where t is the number of years after the beginning of the period. The diameter of the tree grew from 50 cm to 60 cm in the first 2 years that measurements were taken.

- a Find the values of D_0 and k.
- b Write the equation using these values.
- c How much will it have grown in the first 5 years? Round to the nearest centimetre.
- d How long will it take for the tree's diameter to double? Round to the nearest year.



3 The decay of a radioactive substance can be modelled by the equation $M = M_0e^{-kt}$, where M grams is the mass of the substance after t years. After 10 years the mass of the substance is 98 grams and after 20 years the mass is 96 grams.

- a What was the mass of the substance initially? Give your answer to the nearest gram.
- b Find the value of k. Give your answer to 3 decimal places.
- c Write the equation using these values.
- d Find the mass of the substance after 50 years.
- e How long would it take for the mass to be halved?

4 The number of bacteria present in a culture at any time, t hours, can be modelled by the equation $N = N_0e^{kt}$.

- a If the original number is doubled in 3 hours, find k correct to 2 decimal places.
- b Write the equation substituting the value of k.
- c Find the original number of bacteria if there were 2500 bacteria after 4 hours. Give the answer correct to the nearest thousand.
- d Write the equation substituting your value for the original population.
- e Find the number of bacteria present after 8 hours. Give your answer correct to the nearest thousand.

5 The intensity of light d metres below the surface of the sea can be modelled by the equation $I = I_0e^{-kd}$. Divers in the Sea of Loga have found that the intensity of light is halved when a diver is 5 metres below the surface of the water.

a Find the value of k correct to 4 decimal places.

b Write the equation substituting the value of k.

c Find the percentage of light available at a depth of 10 metres.

d If artificial light is necessary when the intensity of light is less than 0.1 of the intensity at the surface ($I < 0.1I_0$), find how deep a diver can go before artificial light is necessary.

6 A school in the suburb of Bienvenue opened with 30 students in February 2008. It has been found for the first years after opening that the number of students enrolled in the school t years after opening can be modelled by the equation $N = N_0e^{kt}$. There were 45 students enrolled in February 2009.

a Find the values of N_0 and k.

b Write the equation substituting the values for N_0 and k.

c How many students will there be 5 years after the opening?

d How many years will it take for the school to have 1000 pupils?

Another school in the suburb of Enbaisse has a declining student population. The number of students enrolled at any one time can be modelled by the equation $E = E_0e^{-rt}$. There are 1000 students enrolled in February 2008 and 900 in February 2009.

e Find the values of E_0 and r.

f Write the equation substituting the values for E_0 and r.

g How many students will be enrolled after 5 years?

h How many years will it take for the two schools to have approximately the same number of pupils?

i What will the population be then? Use the calculator value in the working and do not round off until the final answer.

eBook*plus*

Digital doc:
Test Yourself
Chapter 3

eBook*plus* ACTIVITIES

3B Logarithms and laws of logarithms

Tutorial

- **WE10** Int-0529: Watch a tutorial on simplifying logarithmic expressions (*page 92*)

3C Indicial equations

Digital docs

- SkillSHEET 3.1: Practise index form (*page 105*)
- SkillSHEET 3.2: Practise solving equations (*page 105*)
- SkillSHEET 3.3: Practise solving indicial equations by equating the bases (*page 105*)
- SkillSHEET 3.4: Practise solving linear equations (*page 106*)
- WorkSHEET 3.1: Simplify expressions involving indices and logarithms and solve indicial equations (*page 106*)

Tutorial

- **WE16** Int-0530: Watch a tutorial on solving exponential equations (*page 100*)

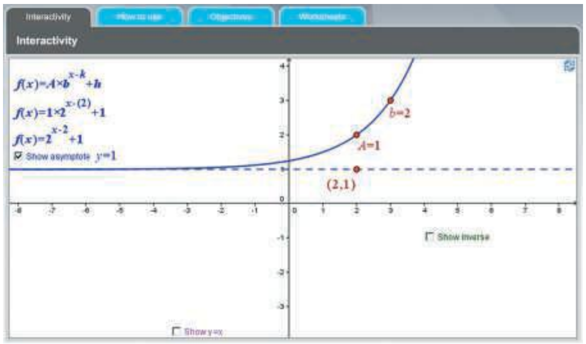
3F Equations with natural (base e) logarithms

Digital doc

- WorkSHEET 3.2: Solve logarithmic and indicial equations with and without a calculator (*page 121*)

Interactivity

- Inverses int-0248: Consolidate your understanding of inverses (*page 121*)



3G Exponential and logarithmic modelling

eLesson

- Exponential modelling eles-0091: Watch an eLesson on exponential modelling (*page 121*)

Tutorial

- **WE30** Int-0533: Watch a tutorial on how to apply exponential modelling techniques (*page 122*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 131*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

4

Derivatives of exponential and logarithmic functions

syllabus reference

Exponential and logarithmic functions and applications

Rates of change

In this chapter

Introduction

A problem in the post

A parcel is attracting nervous attention at Australia Post. The parcel is radioactive and inspectors are trying to identify the material. Suppose they make the following measurements.

There are 50 grams of the material and it is decaying at a rate of 0.25 grams per day.

One of the characteristics of radioactive material is its *half-life*. That is the time it takes for half the material to decay. Below is a table giving the half-lives of common radioactive elements.

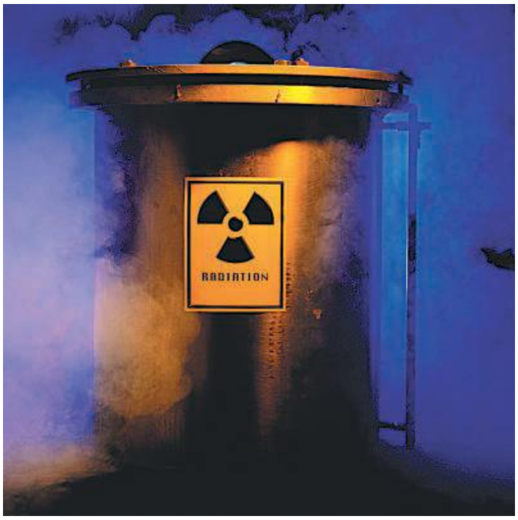
Isotope	**Half-life**
Carbon-14	5730 years
Phosphorus-32	14.2 days
Chlorine-36	301 000 years
Iodine-131	8 days
Polonium-210	138 days
Uranium-238	$4.5 \cdot 10^9$ years

Can we identify the element from this information?

From the ideas developed in the previous chapter it should be clear that an exponential equation would provide a good model for radioactive decay; that is, the amount of radioactive material, N, could be related to time, t, by the equation

$$N = N_0 e^{-kt}.$$

However, to find the value of the decay constant, k, in this particular problem, the derivative of N needs to be considered because the information relates to the *rate* of change in N.



Inverses

eBook plus

Interactivity:
Relations and their inverses
int-0250

Inverse operations are opposite operations. Addition and subtraction are inverse operations and multiplication and division are inverse operations.

Squaring and taking the square root are also inverse operations. The equation of the inverse of the function $y = e^x$ can be found by interchanging the x and y so that $y = e^x$ becomes $x = e^y$. Using $a^x = y \Leftrightarrow \log_a y = x$, $x = e^y$ becomes $\log_e x = y$ or $y = \log_e x$. Therefore $y = e^x$ and $y = \log_e x$ are the equations of inverse functions.

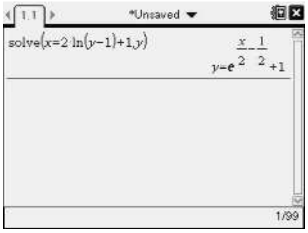
WORKED Example 1

Find the inverse of $y = 3e^{x+1}$.

THINK	WRITE
① Interchange x and y to write the inverse equation.	$y = 3e^{x+1}$ Inverse is $x = 3e^{y+1}$
② Divide both sides by 3.	$e^{y+1} = \frac{x}{3}$
③ Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$\log_e\left(\frac{x}{3}\right) = y + 1$
④ Make y the subject.	$y = \log_e\left(\frac{x}{3}\right) - 1$

WORKED Example 2

Find the inverse of $y = 2\log_e(x - 1) + 1$:
a by hand b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a ① Interchange x and y to write the inverse equation.	**a** $y = 2\log_e(x - 1) + 1$ Inverse is $x = 2\log_e(y - 1) + 1$
② Subtract 1 from both sides.	$2\log_e(y - 1) = x - 1$
③ Divide both sides by 2.	$\log_e(y - 1) = \frac{x-1}{2}$
④ Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$e^{\frac{x-1}{2}} = y - 1$
⑤ Add 1 to both sides.	$y = e^{\frac{x-1}{2}} + 1$
For the TI-Nspire CAS	
b ① Interchange x and y to write the inverse equation.	**b** $y = 2\log_e(x - 1) + 1$ Inverse is $x = 2\log_e(y - 1) + 1$
② To rearrange the equation of the inverse to make y the subject, on a Calculator page, complete the entry line as: solve($x = 2\ln(y - 1) + 1$, y) then press ENTER (enter).	 solve(x=2 ln(y−1)+1,y) $y=e^{\frac{x}{2}-\frac{1}{2}}+1$
③ Write the answer.	$y = e^{\frac{x-1}{2}} + 1$

remember

1. The equation $y = \log_e x$ is an inverse function of $y = e^x$.
2. To find the inverse of a function, interchange x and y and then make y the subject.

EXERCISE 4A Inverses

WORKED Example 1

eBook plus

Digital doc: **SkillSHEET 4.1** Finding the equation of the inverse

1 Find the equation of the inverse of the following.

a $y = 2e^x$
b $y = 3e^x$
c $y = e^{x+1}$
d $y = e^{x-1}$
e $y = e^{2x-1}$
f $y = e^{3x+1}$
g $y = e^{1-x}$
h $y = e^{2-x}$
i $y = e^{2-3x}$
j $y = e^{1-2x}$

2 Find the equation of the inverse of the following.

a $y = 1 + e^x$
b $y = 2 + e^x$
c $y = 2 - e^x$
d $y = 1 - e^x$
e $y = 1 - 2e^x$
f $y = 2 - 3e^x$
g $y = 2 + e^{x+1}$
h $y = 1 + e^{x-2}$
i $y = 3 - 2e^{x-2}$
j $y = 2 - 3e^{x+1}$

WORKED Example 2

3 Find the equation of the inverse of the following.

a $y = 2\log_e x$
b $y = 3\log_e x$
c $y = \log_e(x + 1)$
d $y = \log_e(x - 1)$
e $y = \log_e(2x - 1)$
f $y = \log_e(3x + 1)$
g $y = \log_e(1 - x)$
h $y = \log_e(2 - x)$
i $y = \log_e(2 - 3x)$
j $y = \log_e(1 - 2x)$

4 Find the equation of the inverse of the following.

a $y = 1 + \log_e x$
b $y = 2 + \log_e x$
c $y = 2 - \log_e x$
d $y = 1 - \log_e x$
e $y = 2 + 3\log_e x$
f $y = 3 + 5\log_e x$
g $y = 1 - \log_e(x + 2)$
h $y = 2 - \log_e(x - 1)$
i $y = 3 + 2\log_e(x - 1)$
j $y = 1 - 3\log_e(x + 2)$

5 **multiple choice**

If $y = 5\log_e(3x - 2) + 1$, the equation of the inverse is:

A $\dfrac{e^{\frac{x-1}{5}}}{3} + 2$
B $\dfrac{e^{\frac{x-1}{5}}}{3} - 2$
C $\dfrac{e^{\frac{x-1}{5}} + 2}{3}$
D $\dfrac{e^{\frac{x-1}{5}} - 2}{3}$
E $\dfrac{e^{\frac{x+2}{3}}}{5} - 1$

6 **multiple choice**

If $y = 5e^{2x+1} - 1$, the equation of the inverse is:

A $\frac{1}{2}\ln\left(\frac{x+1}{5}\right) + 1$
B $\frac{1}{2}\ln\left(\frac{x+1}{5}\right) - 1$
C $\dfrac{5\ln(x-1)+1}{2}$
D $\frac{1}{2}\ln\left(\frac{x+1}{5}\right) + \frac{1}{2}$
E $\frac{1}{2}\ln\left(\frac{x+1}{5}\right) - \frac{1}{2}$

The derivative of e^x

If $f(x) = e^x$ then using first principles

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h \to 0} \frac{e^h - 1}{h} \end{aligned}$$

Note that $\lim_{h \to 0} \frac{e^h - 1}{h}$ can be deduced by using a calculator and substituting values of h close to zero.

h	$\frac{e^h - 1}{h}$
0.01	1.0050
0.0001	1.000 05
0.000 001	1.000 000

That is, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

Therefore, $$\begin{aligned} f'(x) &= e^x \cdot 1 \\ &= e^x \end{aligned}$$

If $f(x) = e^x$ then $f'(x) = e^x$.

WORKED Example 3

Differentiate $y = e^{-5x}$.

THINK	WRITE
1 Write the equation.	$y = e^{-5x}$
2 Express u as a function of x and find $\frac{du}{dx}$.	Let $u = -5x$ So $\frac{du}{dx} = -5$
3 Express y as a function of u and find $\frac{dy}{du}$.	$y = e^u$ So $\frac{dy}{du} = e^u$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = -5e^u$
5 Replace u as a function of x.	$= -5e^{-5x}$

This example shows that if $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$.

WORKED Example 4

Find the derivative of $y = e^{2x+1}$.

THINK	WRITE
① Write the equation.	$y = e^{2x+1}$
② Express u as a function of x and find $\frac{du}{dx}$.	Let $u = 2x + 1$ So $\frac{du}{dx} = 2$
③ Express y as a function of u and find $\frac{dy}{du}$.	$y = e^u$ So $\frac{dy}{du} = e^u$
④ Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = e^u \cdot 2$ $= 2e^u$
⑤ Replace u as a function of x.	$= 2e^{2x+1}$

WORKED Example 5

Differentiate: **i** $f(x) = e^x(e^x - 2)$ **ii** $f(x) = \frac{e^{2x} - 2e^{-x}}{e^x}$:

a by hand **b** using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a i ① Write the equation.	**a i** $f(x) = e^x(e^x - 2)$
② Expand.	$= e^{2x} - 2e^x$
③ Differentiate.	$f'(x) = 2e^{2x} - 2e^x$
④ Factorise in order to leave the answer in the form it was given.	$= 2e^x(e^x - 1)$
ii ① Write the equation.	**ii** $f(x) = \frac{e^{2x} - 2e^{-x}}{e^x}$
② Write each term in the numerator over each term in the denominator.	$= \frac{e^{2x}}{e^x} - \frac{2e^{-x}}{e^x}$
③ Divide the numerator of each term by its denominator using the laws of indices.	$= e^{2x-x} - 2e^{-x-x}$ $= e^x - 2e^{-2x}$
④ Differentiate each term.	$f'(x) = e^x + 4e^{-2x}$
⑤ Write your answer in the form it was given.	$= e^x + \frac{4}{e^{2x}}$

For the TI-Nspire CAS

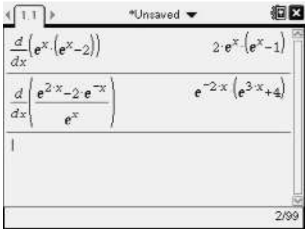
b ① To calculate the derivatives, on a Calculator page, complete the entry lines as:

$$\frac{d}{dx}(e^x(e^x - 2))$$

$$\frac{d}{dx}\left(\frac{e^{2x} - 2e^{-x}}{e^x}\right)$$

then press ENTER.

b



$\frac{d}{dx}(e^x\cdot(e^x-2))$ $2\cdot e^x\cdot(e^x-1)$

$\frac{d}{dx}\left(\frac{e^{2\cdot x}-2\cdot e^{-x}}{e^x}\right)$ $e^{-2\cdot x}\cdot(e^{3\cdot x}+4)$

THINK	**WRITE/DISPLAY**
② Write the answer.	**i** $\frac{d}{dx}(e^x(e^x - 2)) = 2e^x(e^x - 1)$ **ii** $\frac{d}{dx}\left(\frac{e^{2x} - 2e^{-x}}{e^x}\right) = e^{-2x}(e^{3x} + 4)$

WORKED Example 6

Find the derivative of $y = e^{x^3 - x}$.

THINK	**WRITE**
① Write the equation.	$y = e^{x^3 - x}$
② Express u as a function of x and find $\frac{du}{dx}$.	Let $u = x^3 - x$ So $\frac{du}{dx} = 3x^2 - 1$
③ Express y as a function of u and find $\frac{dy}{du}$.	$y = e^u$ So $\frac{dy}{du} = e^u$
④ Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = e^u \cdot (3x^2 - 1)$ $= (3x^2 - 1)e^u$
⑤ Replace u as a function of x.	$= (3x^2 - 1)e^{x^3 - x}$

This example shows that if $f(x) = e^{g(x)}$ then $f'(x) = g'(x)e^{g(x)}$.

WORKED Example 7

If $y = 3^x$, find the derivative, $\frac{dy}{dx}$.

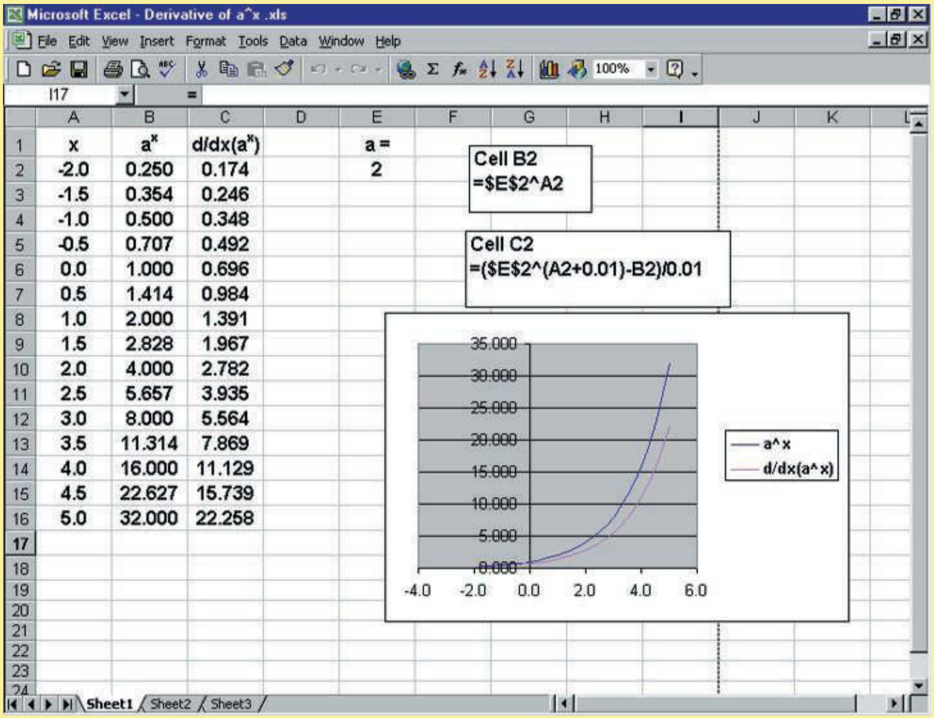
THINK	**WRITE**
① We can only differentiate exponential functions of the form e^{kx}. So express 3^x in this form.	If $3 = e^t$ then $t = \log_e 3$ Therefore $3 = e^{\log_e 3}$ and $y = 3^x$ $= (e^{\log_e 3})^x$ $= e^{\log_e 3 \cdot x}$
② If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$.	$\frac{dy}{dx} = \log_e 3 \cdot e^{\log_e 3 \cdot x}$ $= \log_e 3 \cdot 3^x$

The derivative of a^x

eBook plus

Digital doc:
Spreadsheet
220 Derivative of a^x

Use the following spreadsheet to investigate the derivative of a^x. You will note that the spreadsheet shows the graph of $y = a^x$ and $y = \frac{\mathrm{d}}{\mathrm{d}x}(a^x)$ for values of a that can be varied.



	x	a^x	d/dx(a^x)		a =
1	x	a^x	d/dx(a^x)		a =
2	-2.0	0.250	0.174		2
3	-1.5	0.354	0.246		
4	-1.0	0.500	0.348		
5	-0.5	0.707	0.492		
6	0.0	1.000	0.696		
7	0.5	1.414	0.984		
8	1.0	2.000	1.391		
9	1.5	2.828	1.967		
10	2.0	4.000	2.782		
11	2.5	5.657	3.935		
12	3.0	8.000	5.564		
13	3.5	11.314	7.869		
14	4.0	16.000	11.129		
15	4.5	22.627	15.739		
16	5.0	32.000	22.258		

Cell B2
=E2^A2

Cell C2
=(E2^(A2+0.01)-B2)/0.01

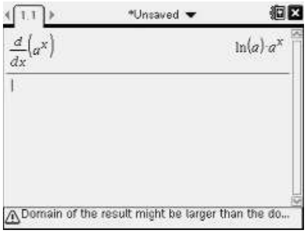
Set $a = 2$. How does the graph of $y = 2^x$ compare with the graph of the derivative of $y = 2^x$?
Set $a = 3$. How does the graph of $y = 3^x$ compare with the graph of the derivative of $y = 3^x$?
Adjust the value of a until the graphs overlap. That is $\frac{\mathrm{d}y}{\mathrm{d}x} = y$. What value of a achieves this?

Graphics Calculator tip! Finding the derivative of $y = a^x$

For the TI-Nspire CAS

To find the derivative of $y = a^x$, on a Calculator page, complete the entry line as:

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x).$$



$\frac{d}{dx}(a^x)$ $\ln(a)\cdot a^x$

Domain of the result might be larger than the do...

remember

1. If $f(x) = e^x$, $f'(x) = e^x$.
2. If $f(x) = e^{kx}$, $f'(x) = ke^{kx}$.
3. If $f(x) = ae^{kx+c}$, $f'(x) = ake^{kx+c}$.
4. If $f(x) = ae^{g(x)}$, $f'(x) = g'(x) \cdot ae^{g(x)}$.

EXERCISE 4B The derivative of e^x

WORKED Example 3

1 Differentiate each of the following using the chain rule.

a $y = e^{10x}$ **b** $y = e^{\frac{1}{3}x}$ **c** $y = e^{\frac{x}{4}}$

d $y = e^{-x}$ **e** $y = 2e^{3x}$ **f** $y = 4e^{-5x}$

g $y = -6e^{-2x}$ **h** $y = 5e^{0.2x}$ **i** $y = -2e^{-11x}$

WORKED Example 4

2 Find the derivative of each of the following.

a $y = e^{6x-2}$ **b** $y = e^{8-6x}$ **c** $y = 2e^{5x+3}$

d $y = 4e^{7-2x}$ **e** $y = -3e^{8x+1}$ **f** $y = -2e^{6-5x}$

g $y = 10e^{6-9x}$ **h** $y = -5e^{3x+4}$ **i** $y = 6e^{-7x}$

j $y = 2e^{\frac{x}{2}+1}$ **k** $y = 3e^{2-\frac{x}{3}}$ **l** $y = -4e^{\frac{x}{4}+5}$

3 **multiple choice**

The derivative of $y = e^{3x+2}$ is equal to:

A $3e^{3x+2}$ **B** $(3x+2)e^{3x+2}$ **C** $3e^{3x}$

D $3xe^{3x+2}$ **E** $3xe^{3x}$

WORKED Example 5

4 Differentiate each of the following.

a $f(x) = 2(e^x + 1)$ **b** $f(x) = 3e^{2x}(e^x + 1)$ **c** $f(x) = 5(e^{-4x} + 2x)$

d $f(x) = (e^x + 2)(e^{-x} + 3)$ **e** $f(x) = \dfrac{3e^{3x} + e^{-6x}}{e^x}$ **f** $f(x) = \dfrac{4e^{7x} - 2e^{-x}}{e^{-2x}}$

g $f(x) = e^x + e^2$ **h** $f(x) = 4e^{5x} + 2x^2 - e^{-3}$

WORKED Example 6

5 Find the derivative of each of the following.

a $y = e^{x^2+3x}$ **b** $y = e^{x^2-3x+1}$ **c** $y = e^{x^2-2x}$

d $f(x) = e^{2-5x}$ **e** $f(x) = e^{6-3x+x^2}$ **f** $g(x) = e^{x^3+3x-2}$

g $h(x) = 3e^{4x^2-7x}$ **h** $y = -5e^{1-2x-3x^2}$ **i** $y = e^{(2x+1)^3}$

j $f(x) = e^{(4-x)^4}$ **k** $g(x) = e^{(x+2)^{-2}}$ **l** $y = e^{\sqrt{3x+4}}$

m $f(x) = e^{(x+1)^{\frac{1}{3}}}$ **n** $h(x) = e^{(x^2+3x)^2}$

6 **multiple choice**

The derivative of $6e^{x^3-5x}$ is equal to:

A $3x^2 - 5$ **B** $6(3x^2-5)e^{x^3-5x}$ **C** $(3x^2-5)e^{x^3-5x}$

D $6(x^3-5x)e^{x^3-5x}$ **E** $6(3x^2-5)e^{3x^2-5}$

WORKED Example 7

7 Find the derivative, $\dfrac{dy}{dx}$, of each of the following.

a $y = 4^x$ **b** $y = 10^x$ **c** $y = \dfrac{1}{2^x}$

8 If $f(x) = 5e^{9-4x}$, find the exact value of $f'(2)$.

9 If $g(x) = 2e^{x^2 - 3x + 2}$, give the exact value of $g'(0)$.

10 Find the exact value of $h'(-1)$ if $h(x) = -5e^{x^3 + 2x}$.

The derivative of $\log_e x$

The inverse of the function $f(x) = e^x$ is $f^{-1}(x) = \log_e x$.
If $y = \log_e x$ then $e^y = x$ as shown earlier.

Let $x = e^y$

$$\frac{dx}{dy} = e^y$$

But $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ and $e^y = x$

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{e^y}$

$$= \frac{1}{x}$$

That is, if $f(x) = \log_e x$ then $f'(x) = \dfrac{1}{x}$.

WORKED Example 8

Differentiate $y = \log_e 7x$.

THINK	WRITE
1 Write the equation.	$y = \log_e 7x$
2 Express u as a function of x and find $\frac{du}{dx}$.	$u = 7x$, so $\frac{du}{dx} = 7$
3 Express y as a function of u and find $\frac{dy}{du}$.	$y = \log_e u$, so $\frac{dy}{du} = \frac{1}{u}$
4 Find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{1}{7x} \cdot 7$ $= \frac{1}{x}$

If $f(x) = \log_e kx$, where k is a constant, then $f'(x) = \dfrac{1}{x}$.

WORKED Example 9

Find the derivative of $y = 2\log_e(3x - 4)$.

THINK	WRITE
1 Write the equation.	$y = 2\log_e(3x - 4)$
2 Express u as a function of x.	Let $u = 3x - 4$.
3 Differentiate u with respect to x.	$\frac{du}{dx} = 3$
4 Express y as a function of u.	$y = 2\log_e u$

THINK	**WRITE**
⑤ Differentiate y with respect to u.	$\frac{dy}{du} = 2 \cdot \frac{1}{u}$ $= \frac{2}{u}$
⑥ Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{2}{u} \cdot 3$ $= \frac{6}{u}$
⑦ Replace u with $3x - 4$.	$= \frac{6}{3x - 4}$

WORKED Example 10

Differentiate $y = \log_e(x^2 + 4x - 1)$.

THINK	**WRITE**
① Write the equation.	$y = \log_e(x^2 + 4x - 1)$
② Let u equal the section in brackets.	Let $u = x^2 + 4x - 1$
③ Differentiate u with respect to x.	$\frac{du}{dx} = 2x + 4$
④ Express y as a function of u.	$y = \log_e u$
⑤ Differentiate y with respect to u.	$\frac{dy}{du} = \frac{1}{u}$
⑥ Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = \frac{1}{u} \cdot (2x + 4)$
⑦ Replace u with what is in the brackets.	$= \frac{2x + 4}{x^2 + 4x - 1}$

This example shows that if $f(x) = \log_e[g(x)]$ then $f'(x) = \frac{g'(x)}{g(x)}$.

WORKED Example 11

If $y = \log_{10} x$, find the derivative, $\frac{dy}{dx}$.

THINK	**WRITE**
We can only differentiate logs to base e. Use the change of base rule.	$y = \log_{10} x$ $= \frac{\log_e x}{\log_e 10}$
Note that $\frac{1}{\log_e 10}$ is a constant.	$= \frac{1}{\log_e 10} \cdot \log_e x$
	$\frac{dy}{dx} = \frac{1}{\log_e 10} \cdot \frac{1}{x}$

remember

1. If $f(x) = \log_e x$, then $f'(x) = \frac{1}{x}$.
2. If $f(x) = \log_e kx$, then $f'(x) = \frac{1}{x}$.
3. If $f(x) = \log_e (g(x))$, then $f'(x) = \frac{g'(x)}{g(x)}$.
4. If $f(x) = \log_a x$ then $f'(x) = \frac{1}{\log_e a} \cdot \frac{1}{x}$ or $\frac{1}{x \log_e a}$.

EXERCISE 4C The derivative of $\log_e x$

1 If $y = \log_e 4x$ is expressed as $y = \log_e u$, then find:

a u **b** $\frac{du}{dx}$ **c** $\frac{dy}{du}$ **d** $\frac{dy}{dx}$ using the chain rule.

WORKED Example 8

2 Differentiate each of the following.

a $y = \log_e 10x$ **b** $y = \log_e 5x$

c $y = \log_e (-x)$ **d** $y = \log_e (-6x)$

e $y = 3 \log_e 4x$ **f** $y = -6 \log_e 9x$

g $y = \log_e \left(\frac{x}{2}\right)$ **h** $y = \log_e \left(\frac{x}{3}\right)$

i $y = 4 \log_e \left(\frac{x}{5}\right)$ **j** $y = -5 \log_e \left(-\frac{2x}{3}\right)$

3 **multiple choice**

The derivative of $\log_e 8x$ is:

A 8 **B** $\frac{1}{8}x$ **C** $\frac{8}{x}$ **D** $\frac{1}{x}$ **E** $\log_e 8$

4 **multiple choice**

To differentiate $y = \log_e (3x + 7)$ using the chain rule:

a 'u' would be used to represent

A $3x + 7$ **B** $3x$ **C** $\log_e x$ **D** $\log_e 3x$ **E** x

b $\frac{dy}{du}$ and $\frac{du}{dx}$ are respectively

A $\frac{1}{u}$ and $3x + 7$ **B** $\frac{1}{u}$ and $3x$ **C** 3 and $\frac{1}{3x}$

D $\frac{1}{u}$ and 3 **E** 1 and 3

c Hence $\frac{dy}{dx}$ is equal to

A 3 **B** $\frac{1}{x}$ **C** $\frac{3}{3x+7}$

D $\frac{1}{3x+7}$ **E** $\frac{3}{x}$

WORKED Example 9

5 Find the derivative of each of the following.

a $y = \log_e(2x+5)$ **b** $y = \log_e(6x+1)$

c $y = \log_e(3x-4)$ **d** $y = \log_e(8x-1)$

e $y = \log_e(3-5x)$ **f** $y = \log_e(2-x)$

g $y = \log_e(4-7x)$ **h** $y = 6\log_e(5x+2)$

i $y = 8\log_e(4x-2)$ **j** $y = -4\log_e(12x+5)$

k $y = -7\log_e(8-9x)$

WORKED Example 10

6 Differentiate the following.

a $y = \log_e 3x^4$ **b** $y = \log_e(x^2+3)$

c $y = \log_e(x^2+4x)$ **d** $y = \log_e(x^2-3x+2)$

e $y = \log_e(x^3+2x^2-7x)$ **f** $y = \log_e(x^2-2x^3+x^4)$

g $y = \log_e\sqrt{2x+1}$ **h** $y = \log_e\sqrt{3-4x}$

i $y = \log_e\sqrt{x^2+2}$ **j** $y = \log_e(x+3)^{\frac{1}{4}}$

k $y = \log_e(5x+2)^{\frac{1}{3}}$ **l** $f(x) = \log_e(2-3x)^{\frac{1}{5}}$

m $f(x) = \log_e\left(\frac{1}{x+3}\right)$ **n** $f(x) = \log_e(3x-2)^4$

o $f(x) = \log_e(5x+8)^{-2}$ **p** $f(x) = \log_e\left(\frac{2}{4+3x}\right)$

7 **multiple choice**

Using the chain rule the derivative of $f(x) = \log_e(x^2-5x+2)$ would be:

A $\frac{1}{x^2-5x+2}$ **B** $\frac{-5}{x^2-5x+2}$ **C** $2x-5$

D $\frac{1}{x(2x-5)}$ **E** $\frac{2x-5}{x^2-5x+2}$

WORKED Example 11

8 Find the derivative, $\frac{dy}{dx}$, of:

a $y = \log_2 x$ **b** $y = \log_4 x$ **c** $y = \log_{10}(x^2)$

9 Find the gradient of the function $f(x) = 6\log_e(4-3x)$ when $x = -1$.

10 If $g(x) = 3\log_e(3x+5)$ find the value of $g'(0)$.

11 Find the exact value of $f'(2)$ if $f(x) = 3x^2 + 4\log_e(x^2+x)$.

12 If $y = e^{\log_e x}$, find:

a $\frac{dy}{dx}$

b the exact gradient when **i** $x = 1$ **ii** $x = 2$ **iii** $x = 4$ **iv** $x = 10$.
Can you explain this result?

eBook*plus*

Digital doc: WorkSHEET 4.1

13 If $f(x) = e^{\log_e x^2}$, find:

- **a** $f'(x)$
- **b** the exact value of **i** $f'(1)$ **ii** $f'(5)$ **iii** $f'(-2)$.

Derivatives of exponential and logarithmic functions

Recall the rules for differentiation:

The chain rule: $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The product rule: If $y = u(x) \cdot v(x)$

$$\text{then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The quotient rule: If $y = \frac{u(x)}{v(x)}$

$$\text{then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

WORKED Example 12

Find the derivative of: a $y = \log_e(e^x + 1)$ **b** $y = \dfrac{3}{e^x + 1}$**.**

THINK

a $\log_e(e^x + 1)$ is a function of a function. Use the chain rule.

b $\dfrac{3}{e^x + 1}$ is a function of a function. Use the chain rule.

WRITE

a
$$u = e^x + 1$$
$$y = \log_e u$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u} \cdot e^x$$
$$= \frac{e^x}{e^x + 1}$$

b
$$u = e^x + 1$$
$$y = \frac{3}{u}$$
$$= 3u^{-1}$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -3u^{-2} \cdot e^x$$
$$= \frac{-3e^x}{(e^x + 1)^2}$$

WORKED Example 13

Find the derivative of:

i $x \log_e x$

ii x^2e^{2x}

a **by hand**

b **using the TI-Nspire CAS calculator.**

THINK

a **i** $x \log_e x$ is a product. Use the product rule.

WRITE/DISPLAY

a **i** Let $y = x \log_e x$

$$u = x$$

$$v = \log_e x$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$$

$$= 1 \cdot \log_e x + x \cdot \frac{1}{x}$$

$$= \log_e x + 1$$

THINK

ii x^2e^{2x} is a product. Use the product rule.

WRITE/DISPLAY

ii Let $y = x^2e^{2x}$

$= u \cdot v$ where $u = x^2$, $v = e^{2x}$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \cdot 2e^{2x} + e^{2x} \cdot 2x$$

$$= 2x^2e^{2x} + 2xe^{2x}$$

For the TI-Nspire CAS

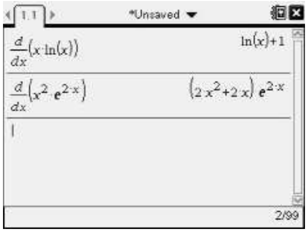
b ① To calculate the derivatives, on a Calculator page, complete the entry lines as:

$$\frac{d}{dx}(x \cdot \ln(x))$$

$$\frac{d}{dx}(x^2 \cdot e^{2x})$$

pressing ENTER enter after each line.

b



$\frac{d}{dx}(x \cdot \ln(x))$ $\ln(x)+1$

$\frac{d}{dx}(x^2 \cdot e^{2 \cdot x})$ $(2 \cdot x^2+2 \cdot x) \cdot e^{2 \cdot x}$

② Write the answer.

i $\frac{d}{dx}(x \cdot \log_e(x)) = \log_e x + 1$

ii $\frac{d}{dx}(x^2 \cdot e^{2x}) = e^{2x}(2x^2 + 2x)$

WORKED Example 14

Find the derivative of $\frac{\log_e x}{x}$.

THINK	WRITE
$\frac{\log_e x}{x}$ is a quotient. Use the quotient rule.	Let $y = \frac{\log_e x}{x}$ $= \frac{u}{v}$ where $u = \log_e x$, $v = x$ $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $= \frac{x \cdot \frac{1}{x} - \log_e x \cdot 1}{x^2}$ $= \frac{1 - \log_e x}{x^2}$

remember

Use the appropriate rule for differentiation by identifying whether the function can be expressed as:

1. a function of a function (Chain rule)
2. the product of two functions (Product rule)
3. the division of two functions (Quotient rule).

EXERCISE 4D

Derivatives of exponential and logarithmic functions

WORKED Example 12

1 Use the chain rule to find the derivatives of the following.

a $y = e^{x^2}$ **b** $y = \log_e (x^2 + 1)$ **c** $y = \sqrt{1 + e^x}$

d $y = \log_e (e^x + 1)$ **e** $y = e^{\log_e x + 1}$

WORKED Example 13

2 Use the product rule to find the derivatives of the following functions.

a x^2e^x **b** $x \log_e x$ **c** $e^x \log_e x$ **d** xe^{-x}

WORKED Example 14

3 Use the quotient rule to find the derivatives of the following functions.

a $\frac{1}{e^x + 1}$ **b** $\frac{x}{\log_e x}$ **c** $\frac{e^x}{e^x + 1}$ **d** $\frac{e^x}{x}$

4 **multiple choice**

Which of the following has the property that $\frac{dy}{dx} = xy$?

A $y = x^2$ **B** $y = \frac{1}{x}$ **C** $y = e^{x^2}$

D $y = e^{\frac{1}{2}x^2}$ **E** $y = \log_e (x^2)$

5 Calculate the derivatives of the following expressions.

a $\log_e (e^x + x)$ **b** $\log_e(\sqrt{x^2 + 1})$ **c** $x^2 e^{\frac{1}{x}}$

d $e^{2 \log_e x}$ **e** $\log_e [\log_e (x^2 + 1)]$

Applications of derivatives of exponential functions

We have seen in chapter 3 how functions involving e^x can be used to model population growth, radioactive decay, the growth of investments and the cooling of heated objects. However, in these earlier situations you were presented with a model and asked to interpret it. For example, the equation $y = y_0 e^{-0.18t}$ is used to describe the decay of radon-222 gas, where y is the amount of gas remaining after t days. In this section we shall use our knowledge of e^x and its derivative to develop equations like $y = y_0 e^{-0.18t}$.

In modelling that involves exponential growth and decay, two key phrases appear regularly, and one needs to understand how they are to be interpreted using mathematical symbols. These phrases are *rate of change of* and *is proportional to*.

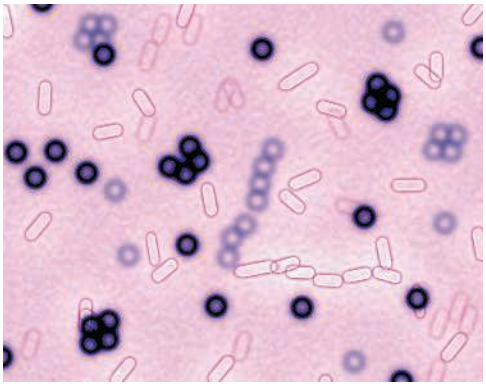
Rate of change of → $\frac{dy}{dt}$ (Usually, *rate* implies change with respect to time, although there are some settings where a different variable is specified.)

N is proportional to P → $N = kP$ where k is called the *constant of proportionality*.

Consider some possible settings where these phrases may be used.

The rate of growth of the number, N, of bacteria is proportional to the number of bacteria already present.

$$\frac{dN}{dt} = kN$$



The temperature, T, of a body cools at a rate proportional to the current temperature.

$$\frac{dT}{dt} = kT \text{ (or } \frac{dT}{dt} = -kT)$$

An equation of the form $\frac{dy}{dx} = ky$ is called a *differential equation* and its solution is not immediately obvious. Do not confuse an equation of this form with one that looks similar: $\frac{dy}{dx} = kx$. The latter equation has a solution that can be obtained easily by inspection, namely $y = \frac{1}{2}kx^2 + c$.

The equation $\frac{dy}{dx} = ky$ describes exponential growth and is so widely applicable that its solution is well known:

If $\frac{dy}{dx} = ky$ then $y = Ae^{kx}$ where A is the value of y when x is 0 — the initial value of y.

Proof

Assume $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$

$$\frac{dx}{dy} = \frac{1}{ky}$$

$$x = \frac{1}{k}\log_e y + c$$

Then $kx = \log_e y + c'$

$$kx + c'' = \log_e y$$

$$y = e^{xk + c''}$$

$$= e^{kx} \cdot e^{c''}$$

So $y = Ae^{kx}$ where $A = e^{c''}$

WORKED Example 15

Find an expression for y in terms of x if $\frac{dy}{dx} = ky$ under the following conditions:
$y = 20$ when $x = 0$ and
$y = 50$ when $x = 15$.

THINK	WRITE
① The equation $\frac{dy}{dx} = ky$ is a standard form whose solution is memorised.	$\frac{dy}{dx} = ky$ $\rightarrow y = Ae^{kx}$
② Use the information given to find A. (Remember, $e^0 = 1$.)	When $x = 0$, $y = 20$ $20 = Ae^0$ $A = 20$
③ Use the second piece of information to find k.	When $x = 15$, $y = 50$. $50 = 20e^{15k}$ $\frac{50}{20} = e^{15k}$ $2.5 = e^{15k}$
④ Take logs of both sides.	$\log_e 2.5 = 15k$ $k = \frac{\log_e 2.5}{15}$ $= 0.06$
⑤ Write the answer.	$y = 20e^{0.06x}$

WORKED Example 16

The rate of change in the number of bacteria, N, in a culture is proportional to the number present. Initially there were 500 present and after 3 hours this had grown to 2000.

a **Develop a model or equation for finding the number of bacteria at time t.**

b **After how long had the bacteria doubled?**

c **How many bacteria were present when $t = 5$ hours?**

THINK	WRITE
a 1 Rate of change of N is proportional to N.	**a** $\frac{dN}{dt} = kN$
2 If $\frac{dy}{dx} = ky$ then $y = Ae^{kx}$.	$N = Ae^{kt}$
3 The word 'initially' means 'when $t = 0$'.	When $t = 0, N = 500$ $500 = Ae^0$ $A = 500$ Thus $N = 500e^{kt}$
4 Use other information to find k.	When $t = 3, N = 2000$ $2000 = 500e^{3k}$ $4 = e^{3k}$ $3k = \log_e 4$ $k = \frac{\log_e 4}{3}$ $= 0.46$ Thus $N = 500e^{0.46t}$
b 1 Find t when $N = 1000$.	**b** $1000 = 500e^{0.46t}$ $2 = e^{0.46t}$ $\log_e 2 = 0.46t$ $t = \frac{\log_e 2}{0.46}$ $= 1.5$
2 Answer the question.	The bacteria take 1.5 hours to double in number.
c Find N when $t = 5$.	**c** The number of bacteria when $t = 5$ is $N = 500e^{0.46 \cdot 5}$ $= 4987$

A problem in the post — what's the substance?

The problem that began our consideration of the derivative of e^x was related to the radioactive decay of a material. In the following worked example we will find the half-life of the element and hence attempt to identify it from the table of known isotopes.

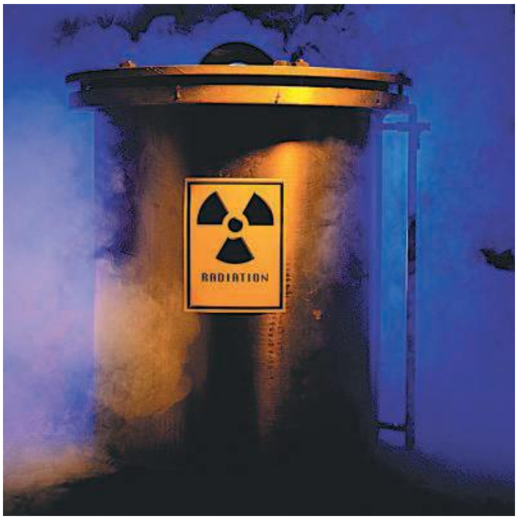
WORKED Example 17

The rate of decay of a radioactive element is known to be exponential. Initially, when there is 50 grams of the substance, the rate of decay is 0.25 grams per day.

a Find an expression for the number of grams, *N*, of the isotope.

b How long does it take for the amount of isotope to decrease to 25 grams? That is, what is the half-life of the isotope?

THINK	WRITE
a 1 The quantities that are related are the number of grams of the radioactive isotope, *N*, and time, *t*, in minutes.	**a** Let N = number of grams remaining, and t = time elapsed in days.
2 The rate of change is exponential.	$N = Ae^{kt}$
3 Initially, there is 50 grams of material. Use this information to find the value of *A*.	At $t = 0$, $N = 10$ $50 = Ae^0$ $A = 50$ So $N = 50e^{kt}$ and $\frac{dN}{dt} = 50ke^{kt}$
4 Initially, the rate of change is –0.25 g/d. The change is negative because it is decreasing. Use this information to find the value of *k*.	At $t = 0$, $\frac{dN}{dt} = -0.25$ $-0.25 = 50ke^0$ $k = -0.005$
5 Write the answer.	Therefore $N = 50e^{-0.005t}$.
b 1 Use the result from **a** and substitute $N = 25$.	**b** $N = 50e^{-0.005t}$ $25 = 50e^{-0.005t}$ $0.5 = e^{-0.005t}$ $\log_e 0.5 = -0.005t$ $t = 138.6$
2 Answer the question.	The half-life of the isotope is 138.6 days.



In worked example 17, we calculated the half-life of the isotope. By referring to a table of known half-lives we now may be able to identify the mysterious element.

Isotope	Half-life
Carbon-14	5730 years
Phosphorus-32	14.2 days
Chlorine-36	301 000 years
Iodine-131	8 days
Polonium-210	138 days
Uranium-238	$4.5 \cdot 10^9$ years

The substance is probably polonium-210, a material commonly used in film processing.

remember

1. Rate of change of → $\frac{dy}{dt}$
2. *N is proportional to P* → $N = kP$ where k is called the *constant of proportionality.*
3. If $\frac{dy}{dx} = ky$, then $y = Ae^{kx}$ where A is the value of y when x is 0.

EXERCISE 4E Applications of derivatives of exponential functions

WORKED Example 15

1 Find an expression for y in terms of x if $\frac{dy}{dx} = ky$ under the following conditions.

$y = 30$ when $x = 0$, and
$y = 80$ when $x = 5$.

2 Find an expression for y in terms of x if $\frac{dy}{dx} = ky$ under the following conditions.

$y = 20$ when $x = 0$, and
$y = 7$ when $x = 12$.
(Note that k will be a negative number because y is decreasing.)

3 Find an expression for N in terms of t if:
the rate of change of N is proportional to N,
the initial value of N is 100, and
when $t = 15$, $N = 320$.

4 **multiple choice**

A quantity G increases with S at a rate proportional to G. When $S = 0$, $G = R$. Which of the following formulas best describes the relationship between G and S?

A $\frac{dG}{dS} = RS$ **B** $\frac{dG}{dR} = RS$ **C** $\frac{dS}{dG} = kS$ **D** $\frac{dG}{dS} = kS$ **E** $\frac{dG}{dS} = kG$

WORKED Example 16

5 Over a number of years the value of an investment, V, increases at a rate proportional to V. Initially the investment is worth \$10 000, and after 4 years this has increased to \$18 000.

a Find an expression for V in terms of the number of years, t.
b What is the value of the investment after 10 years?
c When will the value of the investment reach \$50 000?

6 In its early stages, the spread of a rumour can be modelled using an exponential function; that is, the number of people, N, who hear a rumour grows in proportion to the number of people who have heard the rumour. This model is valid only in the early stages, as the number of 'new' people who may hear the rumour over time becomes small. Suppose that initially, only 5 people have heard the rumour, and 4 days later 150 people have heard it.

a Find an equation relating the number of people, N, who have heard the rumour, to the number of days, t.
b Predict the number of people who have heard the rumour 7 days after it started.
c How long would you predict it would take before 10 000 people have heard the rumour?

7 The half-life of radon-222 is 4 days; that is, in a sample of the substance half of the remaining material decays every 4 days. There was originally 50 g of radon-222.

a Develop an equation relating the amount of material remaining, N, in terms of the number of days, t.

b How much of the material is left after 12 days?

c When is there only 10 g of material remaining?

8 **multiple choice**

The half-life of carbon-14, an isotope of carbon, is 5700 years. In a fossil, a sample of carbon-14 revealed 3 grams of material that was undecayed carbon-14 and 9 grams of material that was decayed carbon-14. The best estimate of the age of the carbon-14 sample is:

A 3800 years
B 5700 years
C 8300 years
D 11 400 years
E 17 100 years

WORKED Example 17

9 The rate of decay of a radioactive element is known to be exponential. Initially, when there is 30 grams of the substance, the rate of decay is 12 milligrams per minute.

a Find an expression for the number of grams, N, of the element.

b How long does it take for the amount of the element to decrease to 5 grams?

10 The rate of increase in the population, P, of a country per year is 2% of the population. In 2002 the population was 6 million.

a Write a formula for $\frac{dP}{dt}$ where t is the time in years.

b Write an equation for P in terms of t.

c Predict the number of years it will take the population to reach 7 million.

eBook plus

Digital doc:
WorkSHEET 4.2

11 The rate of decay of a radioactive source falls from 3000 counts per minute to 2000 counts per minute 5 minutes later. From this information determine the half-life of the substance.

summary

- The equation $y = \log_e x$ is an inverse function of $y = e^x$.
- To find the inverse of a function, interchange x and y and then make y the subject of the equation.
-

y	$\frac{dy}{dx}$
e^x	e^x
e^{kx}	ke^{kx}
$e^{g(x)}$	$g'(x)e^{g(x)}$
$\log_e x$	$\frac{1}{x}$
$\log_e kx$	$\frac{1}{x}$
$\log_e[g(x)]$	$\frac{g'(x)}{g(x)}$
$\log_a x$	$\frac{1}{\log_e a} \cdot \frac{1}{x}$

- The chain rule for differentiation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
- The product rule for differentiation: If $y = u \cdot v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.
- The quotient rule for differentiation: If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.
- In word problems, *rate of change of* → $\frac{dy}{dt}$.

  N is proportional to P → $N = kP$ where k is called the *constant of proportionality.*
- If $\frac{dy}{dx} = ky$ then $y = Ae^{kx}$ where A is the value of y when x is 0.

CHAPTER review

1 **multiple choice**

4A If $2^a = \sqrt{\frac{x^2}{y}}$, then y is equal to:

A $\frac{x}{2^a}$ **B** $\left(\frac{x}{2^a}\right)^2$ **C** $\frac{x^2}{2^a}$ **D** $\frac{x}{2^{2a}}$ **E** $\frac{x^2}{4a}$

4A 2 **multiple choice**

The equation which is the inverse of $y = e^x - 1$ is:

A $y = \log_e x - 1$ **B** $y = \log_e (x - 1)$ **C** $y = \log_e x + 1$

D $y = \log_e (x + 1)$ **E** $y = e^{-x} - 1$

4A 3 Find the inverse of $y = 2e^{2x-1}$.

4A 4 Find the inverse of $y = \log_e (2x + 4)$.

4B 5 **multiple choice**

If $y = 5e^{-6x}$ then $\frac{dy}{dx}$ is equal to:

A $-30e^{-6x}$ **B** $-6e^{-6x}$ **C** $5e^{-6x-1}$ **D** $-30e^{-6x-1}$ **E** $5e^{-7x}$

4B 6 **multiple choice**

If $y = e^{4x+7}$ then $\frac{dy}{dx}$ is:

A $4e^{4x+6}$ **B** e^{4x+6} **C** e^{4x+7} **D** $4e^{4x+7}$ **E** $4e^{3x+7}$

4B 7 Calculate $\frac{dy}{dx}$ if $y =$

a e^{3x-1} **b** 3^{2x} **c** $\frac{1}{e^{2x}}$

4B 8 If $g(x) = 100e^{3-2x}$, calculate the value of $g'(3)$.

4C 9 **multiple choice**

The derivative of $\log_e(3x - 2)$ is:

A $\frac{1}{3x-2}$ **B** $\frac{1}{3x}$ **C** $\frac{1}{x}$ **D** $\frac{1}{3(3x-2)}$ **E** $\frac{3}{3x-2}$

10 **multiple choice** 4C

The derivative of $2\log_e(x^2+x)$ is:

A $\frac{2(2x+1)}{x^2+x}$ **B** $\frac{2(2x+1)}{x}$ **C** $\frac{2x+1}{x^2+x}$

D $\frac{2x}{x^2+x}$ **E** $\frac{4x}{x^2+x}$

11 Differentiate the following expressions. 4C

a $\log_e 2x$ **b** $\log_e(3-4x)$ **c** $\log_{10}(2x)$

12 The derivative of $\log_e x$ is exactly equal to the derivative of $\log_e 100x$. Explain why different functions have the same derivative. 4C

13 **multiple choice** 4D

If $f(x) = x^2e^{2x}$ then $f'(x)$ is equal to:

A $2xe^{2x} + 2x^2e^{2x}$ **B** $2xe^{2x}$ **C** $4xe^{2x}$

D $2xe^{2x} - 2x^2e^{2x}$ **E** $2xe^{2x} + x^2e^{2x}$

14 **multiple choice** 4D

If $g(x) = 2x\log_e 3x$ then $g'(x)$ must be:

A $2\log_e 3x + \frac{2}{3}$ **B** $2\log_e 3x + 2$ **C** $2\log_e 3x + 6x$

D $2\log_e 3x - \frac{2}{3}$ **E** $2\log_e 3x + 6x\log_e 3x$

15 **multiple choice** 4D

The derivative of $\frac{e^{4x}}{x^2}$ is:

A $\frac{(x-2)e^{4x}}{x^3}$ **B** $\frac{2(1-2x)}{x^3}$ **C** $\frac{2(2x-1)e^{4x}}{x^3}$

D $\frac{x^2e^{4x}-2e^{4x}}{x^4}$ **E** $\frac{2e^{4x}}{x^3}$

16 **multiple choice** 4E

If $\frac{dR}{dt} = kR$ then which of the following is *not* possible?

A $R = 10e^{kt}$ **B** $R = 20e^{kt}$ **C** $R = -20e^{kt}$

D $R = ke^{10t}$ **E** $R = 10ke^{kt}$

17 If $\frac{dy}{dx} = 3y$ and $y = 20$ when $x = 0$: 4E

a Find an equation for y in terms of x.

b Find the value of y when $x = 5$.

c Find the value of x when $y = 400$.

Modelling and problem solving

1 The value of an investment, V, increases at a rate proportional to V. Initially, the value of the investment is \$4500 and then 3 years later V has doubled.

a Find an equation for V in terms of t, the time in years.

b When is the value of the investment \$15 000?

2 The rate at which a liquid cools is proportional to the difference in temperature between the liquid and the temperature of the surrounding medium. This is known as Newton's Law of Cooling.

A liquid originally at 180 °C is allowed to cool in a room where the temperature is 25 °C. It is noted that after 10 minutes have elapsed, the temperature of the liquid has fallen to 100 °C.

Use the variable θ to represent the difference between the temperature and the surrounding medium.

a Write an equation for $\frac{d\theta}{dt}$.

b From the equation for $\frac{d\theta}{dt}$ write an equation for θ in terms of t.

c When will the liquid cool to 40 °C?

d What will the temperature of the liquid be after 35 minutes?

3 The population of cheetahs, P, in a national park in Africa since 1 January 2010 can be modelled as $N = 100te^{-\frac{t}{12}} + 500$ where t is the number of years.

a When does this model predict that the maximum population will be reached?

b What is the maximum population of cheetahs that will be reached?

c How many cheetahs will there be on 1 January in: **i** 2034 and **ii** 2094?

4 The weight (in kg) of a bodybuilder t months after starting a training program is

$$W = 5t - 20\log_e(t + 1) + 90,\ 0 \le t \le 15$$

a Find the weight of the bodybuilder at the start of the program.

b Find the minimum weight obtained and how many months it takes to reach it.

5 The number of people with the flu virus, N, in a particular town t days after a vaccine is introduced is $N = 3000 - 500\log_e(8t + 1)$.

a How many people are infected in the town before the vaccine is introduced?

b Find the average rate of change over the first 5 days.

c Find the rate of change of the number of people in the town infected with flu.

d Find the rate of change after 5 days.

6 The population of rabbits on a particular island t weeks after a virus is introduced is modelled by $P = 1200e^{-0.1t}$, where P is the number of rabbits.

Find:

a the time taken for the population to halve (to the nearest week)

b the rate of decrease of the population after: **i** 2 weeks and **ii** 10 weeks.

After 15 weeks the virus has become ineffective and the population of rabbits starts to increase again according to the model

$$P = P_0 + 10(t - 15)\log_e(2t - 29)$$

where t is the number of weeks since the virus was first introduced.

Find:

c the value of P_0

d the population after 30 weeks

e the rate of change of the population after **i** 20 weeks and **ii** 30 weeks

f how many weeks the population takes to get back to its original number.

eBookplus

Digital doc:
Test Yourself
Chapter 4

eBook*plus* ACTIVITIES

4A Inverses

Digital doc

- SkillSHEET 4.1: Practise finding the equation of the inverse (*page 136*)

Interactivity

- Relations and their inverses int-0250: Consolidate your understanding of relations and their inverses (*page 135*)

4B The derivative of e^x

Digital doc

- Spreadsheet 220: Investigate the derivative of a^x (*page 140*)

4C The derivative of $\log_e x$

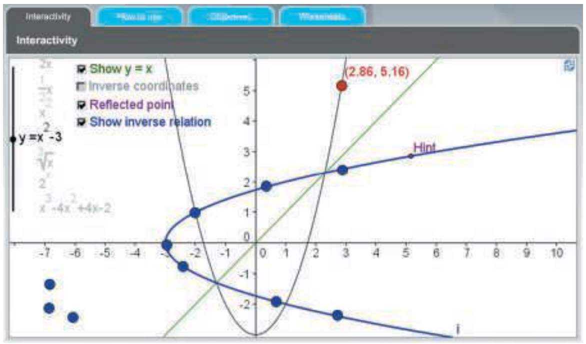
Digital doc

- WorkSHEET 4.1: Find derivatives of exponential and logarithmic functions (*page 146*)

4E Applications of derivatives of exponential functions

Digital doc

- WorkSHEET 4.2: Find derivatives of exponential and logarithmic functions using the chain, product and quotient rules where appropriate (*page 154*)



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 159)*.

To access eBookPLUS activities, log on to

www.jacplus.com.au

Periodic functions

5

syllabus reference

Periodic functions and applications

In this chapter

Introduction

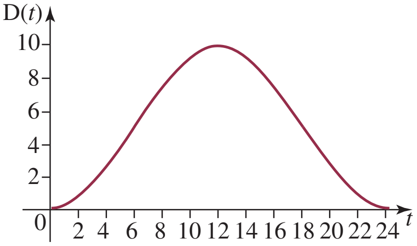
Midnight on Marus

At the South Pole in midsummer on the planet Marus, the red sun of its solar system does not set. It dips towards the horizon until, at 'midnight', its lower rim just touches it. The sun then rises again until at 'midday' its lower rim is at an angle of $D(t)°$ to the horizontal. The sun then sinks again, continuing in this pattern. The angle above the horizontal can be modelled by the relation:

$$D(t) = a - b \sin (nt + c)$$

where t is the time in hours after midnight and a, b, c and n are positive constants.

The graph of $D(t)$ for 24 hours is shown on the axes in the figure below.

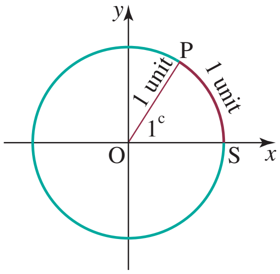


Can you find the values of a, b, c and n and hence write the rule for $D(t)$?

In this chapter, we shall revise the work covered in Year 11 on periodic functions, and extend the work previously covered on graphing this type of function.

Revision of radians and the unit circle

Revision of basic concepts



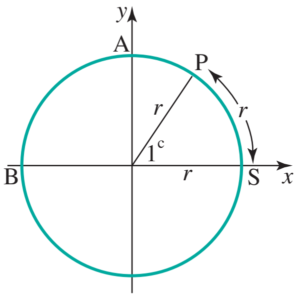
A unit circle

Angles are measured in degrees or radians.

To define a radian we can use a circle which has a radius of one unit. This circle is called the *unit circle*. If we take a piece of string that is the same length as the radius and place it along the circumference of the circle from S to P to form an arc, then the angle formed by joining S and P to O, the centre of the circle, measures one radian.

The radius of the circle can be any length and can still be regarded as a unit. As long as the arc is the same length as the radius, the angle will always measure one radian.

In general, therefore, a radian is the angle formed at the centre of any circle by radii meeting an arc which is the same length as the radius of the circle. Note the following.



A radian

1. One radian is written as 1^c (or 1 radian can be written as 1).
2. The circumference of a circle is $2\pi r$ units in length.
3. If the radius is one unit, as in the case of the unit circle, then the circumference is 2π units, and the angle at the centre of the circle is 2π radians.
4. 2π radians = 360°
5. The length of the semicircle from S through A to B is half the circumference and is π units.
6. π radians = 180°
7. An arc length of r units subtends an angle of 1 radian.
8. An arc length of $2\pi r$ units subtends an angle of 2π radians.
9. An arc length of a quarter of a circle is $\frac{2\pi r}{4}$ units $\left(\text{that is, } \frac{\pi r}{2}\right)$ and subtends an angle of $\frac{\pi}{2}$ radians.

Finding the number of degrees in one radian

Since $\pi^c = 180°$

we have $1^c = \frac{180°}{\pi} = 57.296°$ (correct to 3 decimal places)

Changing radians to degrees

Radians are converted to degrees using the following equation.

$$\mathbf{1^c = \frac{180°}{\pi}}$$

WORKED Example 1

Change the following to degrees, giving the answer correct to 2 decimal places.

a 2^c **b** 6.3^c **c** $\frac{9\pi}{10}$

THINK	WRITE
a ① Multiply the number of radians by $\frac{180}{\pi}$.	**a** $2^c = 2 \cdot \frac{180^\circ}{\pi}$
② Simplify where possible.	$= \frac{360^\circ}{\pi}$
③ Write the answer correct to 2 decimal places.	$\approx 114.59^\circ$
b ① Multiply the number of radians by $\frac{180}{\pi}$.	**b** $6.3^c = 6.3 \cdot \frac{180^\circ}{\pi}$
② Simplify. ③ Write to 2 decimal places.	$\approx 360.96^\circ$
c ① Multiply the number of radians by $\frac{180}{\pi}$.	**c** $\frac{9\pi}{10}^c = \frac{9\pi}{10} \cdot \frac{180^\circ}{\pi}$
② Simplify by cancelling π.	$= 162^\circ$

Changing degrees to radians

Degrees are converted to radians using the following equation.

$$180^\circ = \pi^c$$

$$1^\circ = \frac{\pi}{180}$$

WORKED Example 2

Change the following to radians.

a 2° **b** 36.35° **c** 150°

THINK	WRITE
a ① Multiply the number of degrees by $\frac{\pi}{180}$.	**a** $2^\circ = 2 \cdot \frac{\pi}{180}$
② Simplify, leaving your answer as an exact number of radians or round off to an appropriate number of decimal places.	$= \frac{\pi}{90}$ $\approx 0.035^c$
b ① Multiply the number of degrees by $\frac{\pi}{180}$.	**b** $36.35^\circ = 36.35 \cdot \frac{\pi}{180}$
② Simplify. *Note:* In this example it is not appropriate to leave your answer in exact form.	$\approx 0.634^c$
c ① Multiply the number of degrees by $\frac{\pi}{180}$.	**c** $150^\circ = 150 \cdot \frac{\pi}{180}$
② Simplify, leaving your answer in exact form.	$= \frac{150\pi}{180}$ $= \frac{5\pi}{6}$

Special angles

Note the following special cases with which you need to be familiar.

$180° = \pi$

$90° = \frac{\pi}{2}$ Divide both sides by 2.

$60° = \frac{\pi}{3}$ Divide both sides by 3.

$45° = \frac{\pi}{4}$ Divide both sides by 4.

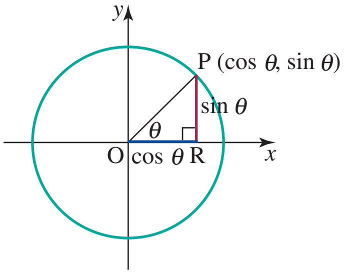
$30° = \frac{\pi}{6}$ Divide both sides by 6.

Basic definitions of sine, cosine and tangent

Sine and cosine

In the unit circle the vertical distance PR is defined as sine θ (or $\sin \theta$) and the horizontal distance OR is defined as cosine θ (or $\cos \theta$).

The coordinates of the point P are $(\cos \theta, \sin \theta)$ where θ can be in radians or degrees.
The x-coordinate of P is $\cos \theta$ and the y-coordinate of P is $\sin \theta$.



$\sin \theta$ and $\cos \theta$

Boundary angles

In Year 11 we noted the value of the trigonometric ratios for the boundary angles.

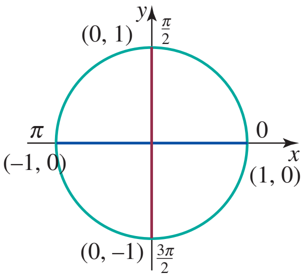
$\sin 0 = 0$ $\sin 0° = 0$

$\sin \frac{\pi}{2} = 1$ $\sin 90° = 1$

$\sin \pi = 0$ $\sin 180° = 0$

$\sin \frac{3\pi}{2} = -1$ $\sin 270° = -1$

$\sin 2\pi = 0$ $\sin 360° = 0$



$\cos 0 = 1$ $\cos 0° = 1$

$\cos \frac{\pi}{2} = 0$ $\cos 90° = 0$

$\cos \pi = -1$ $\cos 180° = -1$

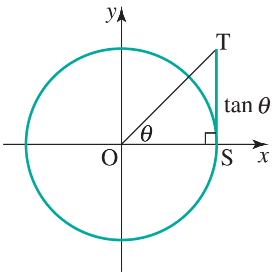
$\cos \frac{3\pi}{2} = 0$ $\cos 270° = 0$

$\cos 2\pi = 1$ $\cos 360° = 1$

Special cases

Tangent

Using the unit circle, the vertical distance TS is defined as $\tan \theta$.
TS is the tangent to the circle which intersects with the x-axis and $\angle TOS = \theta$.



$\tan \theta$

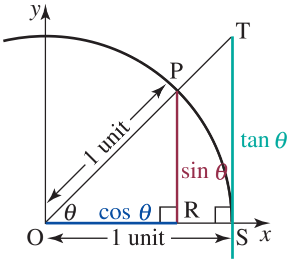
Using Pythagoras' theorem in triangle OPR (figure below),

$$PR^2 + OR^2 = OP^2.$$

So, $$\sin^2 \theta + \cos^2 \theta = 1$$

This is called the Pythagorean identity and is used in many trigonometric equations.

From the diagram below, $\triangle$OPR is similar to $\triangle$OTS (equiangular).



Identities

Therefore:
$$\frac{TS}{OS} = \frac{PR}{OR}$$
$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Boundary angle

Note the boundary angles for the tangent ratio.

$\tan 0 = 0$	$\tan \frac{\pi}{2}$ is undefined	$\tan \pi = 0$	$\tan \frac{3\pi}{2}$ is undefined	$\tan 2\pi = 0$
$\tan 0° = 0$	$\tan 90°$ is undefined	$\tan 180° = 0$	$\tan 270°$ is undefined	$\tan 360° = 0$

It can be seen that tan 90° is undefined because $\tan 90° = \frac{\sin 90°}{\cos 90°} = \frac{1}{0}$ which is undefined.

remember

1. Radians
   (a) 1^c = the size of the angle formed where the length of an arc is equal to the radius of the circle.
   (b) $\pi^c = 180°$
   (c) Angles are in radians unless a degree symbol is shown.
2. To change radians to degrees, multiply by $\frac{180}{\pi}$.
3. To change degrees to radians, multiply by $\frac{\pi}{180}$.
4. Identities that apply in both degrees and radians
   (a) $\sin^2 \theta + \cos^2 \theta = 1$ (b) $\tan\theta = \frac{\sin\theta}{\cos\theta}$
5. If an angle is expressed without units, we assume it is measured in radians.

EXERCISE 5A Revision of radians and the unit circle

WORKED Example 1

1 Change the following to degrees, giving answers correct to 2 decimal places.

a 3^c **b** 5^c **c** 4.8^c **d** 2.56^c **e** $\frac{7\pi}{20}^c$ **f** $\frac{3\pi}{10}^c$ **g** $\frac{5\pi}{6}^c$ **h** $\frac{5\pi}{4}^c$

WORKED Example 2

2 Change the following to radians. Give exact answers for **a**, **b**, **c** and **d**. Write other answers correct to 2 decimal places.

a 5° **b** 15° **c** 120° **d** 130° **e** 63.9°
f 78.82° **g** 235° **h** 260° **i** 310° **j** 350°

3 Evaluate using a calculator. Give answers correct to 3 decimal places.

a sin 0.4 **b** sin 0.8 **c** cos 1.4
d cos 1.7 **e** tan 2.9 **f** tan 2.4

eBook*plus*

Digital docs:

SkillSHEET 5.1
Changing degrees to radians

SkillSHEET 5.2
Calculating tangent ratios

Spreadsheet
077 The unit circle

4 Evaluate using a calculator, giving your answers to 3 decimal places.

a sin 75° **b** sin 68° **c** cos 160°
d cos 185° **e** tan 265° **f** tan 240°

5 Evaluate the following.

a $\sin 0$ **b** $\sin \pi$ **c** $\cos 2\pi$
d $\cos \pi$ **e** $\tan \frac{3\pi}{2}$ **f** $\tan \frac{\pi}{2}$

6 Evaluate the following.

a sin 90° **b** sin 360° **c** cos 180°
d cos 0° **e** tan 270° **f** tan 720°

7 Evaluate without using your calculator.

a $\sin^2 20° + \cos^2 20°$ **b** $\cos^2 50° + \sin^2 50°$ **c** $\sin^2 \pi + \cos^2 \pi$
d $\sin^2 2.5 + \cos^2 2.5$ **e** $\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2}$ **f** $\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2}$
g $2 \sin^2 \alpha + 2 \cos^2 \alpha$ **h** $5 \sin^2 \beta + 5 \cos^2 \beta$

8 Write the following in order from smallest to largest.

a sin 35°, sin 70°, sin 120°, sin 150°, sin 240°
b cos 0.2, cos 1.5, cos 3.34, cos 5.3, cos 6.3

9 If $\sin \theta = \frac{8}{17}$ and $\cos \theta = \frac{15}{17}$, find $\tan \theta$.

10 If $\sin A = 0.6$ and $\cos A = 0.8$, find $\tan A$. Draw a triangle marking in the position of angle A and possible lengths of the sides.

11 **multiple choice**

$\frac{\pi}{3}$ radians is equal to:

A 0° **B** 30° **C** 45° **D** 60° **E** 90°

12 **multiple choice**

The expression $1 - \sin^2 \alpha$ is equal to:

A 1 **B** $\cos^2 \alpha$ **C** $\cos \alpha$ **D** $\tan \alpha$ **E** $\tan^2 \alpha$

13 The temperature T°C inside a shop t hours after 2 am is given by

$$T = 15 - 3\cos\frac{\pi}{12}t$$

Calculate the exact temperature after 4 hours and the temperature to the nearest tenth of a degree at 9.00 am.

Symmetry and exact values

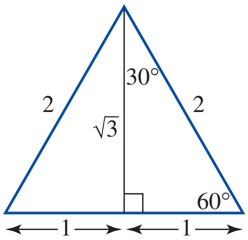
Exact values

Using the equilateral triangle (of side length 2 units) shown at right, the following exact values can be found.

$$\begin{aligned} \sin 30° &= \sin\frac{\pi}{6} \\ &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} \sin 60° &= \sin\frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} \cos 30° &= \cos\frac{\pi}{6} \\ &= \frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \cos 60° &= \cos\frac{\pi}{3} \\ &= \frac{1}{2} \end{aligned}$$

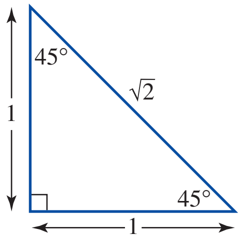
$$\begin{aligned} \tan 30° &= \tan\frac{\pi}{6} \\ &= \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \end{aligned} \qquad \begin{aligned} \tan 60° &= \tan\frac{\pi}{3} \\ &= \sqrt{3} \end{aligned}$$



Exact values of sine, cosine, tangent, 30°, 60°

Using the right isosceles triangle shown, the following exact values can be found.

$$\begin{aligned} \sin 45° &= \sin\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} \cos 45° &= \cos\frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} \tan 45° &= \tan\frac{\pi}{4} \\ &= 1 \end{aligned}$$



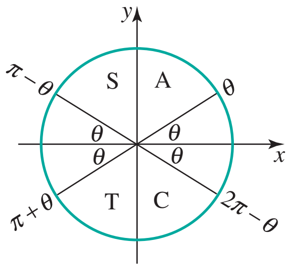
Exact values of sin, cos, tan 45°

The unit circle and symmetry properties

The unit circle is symmetrical so that the magnitude of sine, cosine and tangent at the angles shown are the same in each quadrant but the sign varies.

For a positive angle, θ, we rotate in an anticlockwise direction from the positive x-axis.

In the first quadrant sin, cos and tan are all positive.
In the second quadrant only sin is positive.
In the third quadrant only tan is positive.
In the fourth quadrant only cos is positive.
This can be remembered as all students to class.



Symmetrical properties

1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
sin θ	sin $(\pi - \theta) = \sin\theta$	sin $(\pi + \theta) = -\sin\theta$	sin $(2\pi - \theta) = -\sin\theta$
cos θ	cos $(\pi - \theta) = -\cos\theta$	cos $(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
tan θ	tan $(\pi - \theta) = -\tan\theta$	tan $(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

WORKED Example 3

Without using a calculator, find the exact value of:

a $\sin 150°$ **b** $\cos \frac{5\pi}{4}$.

eBookplus

Tutorial:
Worked example 3
int-0546

THINK	WRITE
a ① Find the equivalent first quadrant angle.	**a** $\sin 150° = \sin (180 - 30)°$
② As 150° is in the 2nd quadrant, sine is positive.	$= \sin 30°$
③ Write the exact value.	$= \frac{1}{2}$
b ① Find the equivalent first quadrant angle using $\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}$.	**b** $\cos \frac{5\pi}{4} = \cos \left(\pi + \frac{\pi}{4}\right)$
② Decide on the sign required. As it is in the 3rd quadrant, cosine is negative.	$= -\cos \frac{\pi}{4}$
③ Write the exact value.	$= -\frac{\sqrt{2}}{2}$

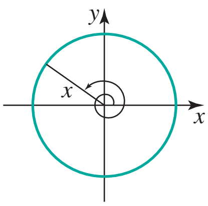
Angles are not restricted to values between 0 and 2π; that is, the domain is not restricted to $0 \le x \le 2\pi$. If an angle is greater than 2π radians, it is necessary to subtract multiples of 2π so that the angle is within one turn of the unit circle. Each 2π radians is a complete turn of the circle.

WORKED Example 4

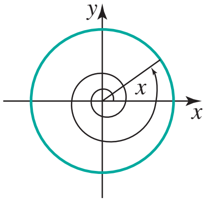
If $\sin x = 0.6$, $\cos x = 0.8$, and x is in the first quadrant, find:

a $\sin (3\pi - x)$ **b** $\cos (4\pi + x)$.

THINK	WRITE/DRAW
a ① Write 3π as $2\pi + \pi$ — that is, one complete cycle and then the angle $\pi - x$.	**a** $\sin (3\pi - x) = \sin (2\pi + \pi - x)$ $= \sin (\pi - x)$
② As it is in the 2nd quadrant, sine is positive.	$= \sin x$
③ Substitute the given value.	$= 0.6$



Continued over page

THINK	**WRITE/DRAW**
b ① $4\pi = 2 \cdot 2\pi$ — that is, two rotations plus angle x.	**b** $\cos(4\pi + x)$ 
② As it is in the 1st quadrant, cosine is positive.	$= \cos x$
③ Substitute the given value.	$= 0.8$

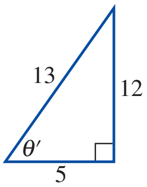
WORKED Example 5

eBookplus

Tutorial:
Worked example 5
int-0547

If $\sin\theta = \frac{12}{13}$ and $\frac{\pi}{2} < \theta < \pi$ find $\cos\theta$ and hence find $\tan\theta$.

Solution 1

THINK	**WRITE**
① In the given triangle we can find the length of the third side by using Pythagoras' theorem. Let θ' be the first quadrant angle corresponding to θ.	$13^2 = 5^2 + 12^2$ 
② $\cos\theta' = \dfrac{\text{adjacent}}{\text{hypotenuse}}$	$\cos\theta' = \frac{5}{13}$
③ θ is in the second quadrant, and therefore negative.	$\cos\theta = -\frac{5}{13}$
④ $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$	$\tan\theta' = \frac{12}{5}$
⑤ θ is in the second quadrant.	$\tan\theta = -\frac{12}{5}$

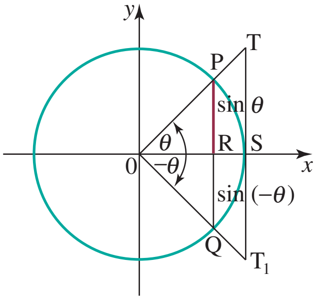
Solution 2

THINK	**WRITE**
① Use the identity $\sin^2\theta + \cos^2\theta = 1$.	$\sin^2\theta + \cos^2\theta = 1$
② Rearrange.	$\cos^2\theta = 1 - \sin^2\theta$
③ Substitute values.	$= 1 - \frac{144}{169}$
④ Evaluate.	$= \frac{25}{169}$ $\cos\theta = \pm\frac{5}{13}$
⑤ cos is negative in the 2nd quadrant.	Take $\cos\theta = -\frac{5}{13}$ as $\frac{\pi}{2} < \theta < \pi$
⑥ Use $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.	$\tan\theta = \dfrac{\frac{12}{13}}{-\frac{5}{13}}$ $\tan\theta = -\frac{12}{5}$

Negative angles

For negative angles, move in a clockwise direction.

In the diagram, $RQ = -PR$, and
$OR = OR$ so
$T_1S = -TS$



Negative angles

An alternative way to find tan (−θ) is:

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\ \theta} = -\tan\ \theta$$

$$\sin(-\theta) = -\sin\ \theta$$
$$\cos(-\theta) = \cos\ \theta$$
$$\tan(-\theta) = -\tan\ \theta$$

The diagram shows $0 < \theta < \frac{\pi}{2}$; however, these relationships are true for all θ.

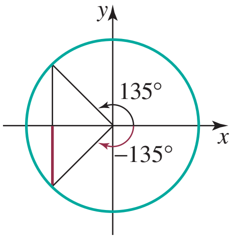
sin (−150°) = −sin 150° cos (−190°) = cos 190° tan (−280°) = −tan 280°

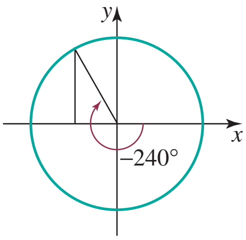
WORKED Example 6

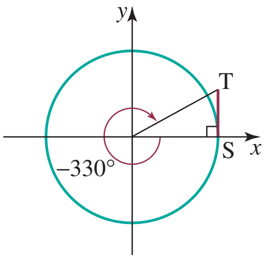
Find the exact value of:

a **sin (−135°)** **b** **cos (−240°)** **c** **tan (−330°).**

THINK	WRITE/DRAW
a 1 sin (−135°) = −sin 135°	**a** sin (−135°) = −sin 135°
2 135° is in the 2nd quadrant.	= −sin (180° − 45°)
3 Sine is positive in the 2nd quadrant.	= −sin 45°
4 Give the exact value.	$= -\frac{\sqrt{2}}{2}$
b 1 cos (−240°) = cos 240°	**b** cos (−240°) = cos 240°
	= cos (180° + 60°)
2 Cosine is negative in the 3rd quadrant.	= −cos 60°
3 Give the exact value.	$= -\frac{1}{2}$
c 1 tan (−330°) = tan 30°	**c** tan (−330°) = tan 30°
2 Give the exact value.	$= \frac{1}{\sqrt{3}}$
	$= \frac{\sqrt{3}}{3}$







WORKED Example 7

Find the exact value of the following: a by hand b using the TI-Nspire CAS calculator.

i $\sin\left(-\frac{4\pi}{3}\right)$ **ii** $\tan\left(-\frac{5\pi}{6}\right)$.

THINK **WRITE/DISPLAY**

a i ① $\sin(-\theta) = -\sin\theta$

a i $\sin\left(-\frac{4\pi}{3}\right) = -\sin\frac{4\pi}{3}$

② $\frac{4\pi}{3}$ is in the 3rd quadrant.

$= -\sin\left(\pi + \frac{\pi}{3}\right)$

③ Sine is negative in the 3rd quadrant.

$= -\left(-\sin\frac{\pi}{3}\right)$

$= \sin\frac{\pi}{3}$

④ Give the exact value.

$= \frac{\sqrt{3}}{2}$

ii ① $\tan(-\theta) = -\tan\theta$

ii $\tan\left(-\frac{5\pi}{6}\right) = -\tan\frac{5\pi}{6}$

② $\frac{5\pi}{6}$ is in the 2nd quadrant.

$= -\tan\left(\pi - \frac{\pi}{6}\right)$

③ Tangent is negative in the 2nd quadrant.

$= -\left(-\tan\frac{\pi}{6}\right)$

$= \tan\frac{\pi}{6}$

④ Give the exact value.

$= \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$

For the TI-Nspire CAS

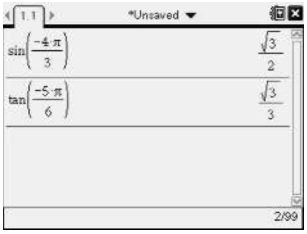
b ① On a Calculator page, complete the entry lines as:

$\sin\left(\frac{^-4\pi}{3}\right)$

$\tan\left(\frac{^-5\pi}{6}\right)$

pressing ENTER (enter) after each line.

b



$\sin\left(\frac{-4\cdot\pi}{3}\right)$ $\frac{\sqrt{3}}{2}$

$\tan\left(\frac{-5\cdot\pi}{6}\right)$ $\frac{\sqrt{3}}{3}$

② Write the exact value.

i $\frac{\sqrt{3}}{2}$

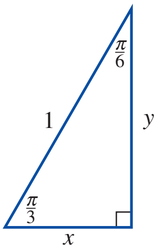
ii $\frac{\sqrt{3}}{3}$

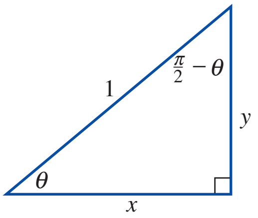
Complementary angles

Complementary angles add to 90° or $\frac{\pi}{2}$ radians.

Therefore, 30° and 60° are complementary angles.

In other words $\frac{\pi}{6}$ and $\frac{\pi}{3}$ are complementary angles and θ and $\left(\frac{\pi}{2}-\theta\right)$ are also complementary angles.





The sine of an angle is equal to the cosine of its complement. Therefore, sin 60° = cos 30°. We say that sine and cosine are *complementary* functions.

The complement of the tangent of an angle is the cotangent (or cot) — that is, tangent and cotangent are complementary functions (as well as reciprocal functions).

$$\begin{aligned}\cot\theta &= \frac{1}{\tan\theta}\\ &= \frac{1}{\frac{y}{x}}\\ &= \frac{x}{y}\\ &= \tan\left(\frac{\pi}{2}-\theta\right)\end{aligned}$$

1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$	$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta$	$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$	$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$
$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$	$\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$	$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$	$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$
$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$	$\tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$	$\tan\left(\frac{3\pi}{2}-\theta\right)=\cot\theta$	$\tan\left(\frac{3\pi}{2}+\theta\right)=-\cot\theta$

WORKED Example 8

eBook*plus*

Tutorial:
Worked example 8
int-0548

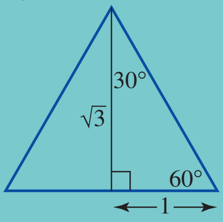
If sin $\theta = 0.4$, cos $x = 0.5$ and tan $\alpha = 0.6$, find:

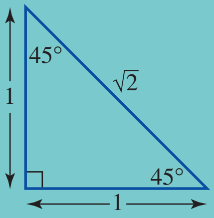
a cos $\left(\frac{\pi}{2}+\theta\right)$ **b** tan $\left(\frac{3\pi}{2}+\alpha\right)$.

THINK	WRITE
a ① Cosine and sine are complementary functions and cosine is negative in the second quadrant.	**a** $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$
② Substitute the given value for sin θ.	$=-0.4$
b ① Tangent and cotangent are complementary functions and tangent is negative in the 4th quadrant.	**b** $\tan\left(\frac{3\pi}{2}+\alpha\right)=-\cot\alpha$ $=-\frac{1}{\tan\alpha}$
② Substitute the given value for tan α.	$=-\frac{1}{0.6}$
③ Calculate.	$=-1.667$

remember

1. Exact values can be determined by using the equilateral triangle and the right isosceles triangle.





	0° 0	30° $\frac{\pi}{6}$	45° $\frac{\pi}{4}$	60° $\frac{\pi}{3}$	90° $\frac{\pi}{2}$	180° π	270° $\frac{3\pi}{2}$	360° 2π
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.	0

2. In the unit circle, sine, cosine and tangent are positive in the 1st quadrant, sine is positive in the 2nd quadrant, tangent is positive in the 3rd quadrant and cosine is positive in the 4th quadrant. Symmetry properties:

$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

3. Negative angles
   $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$
4. Complementary angles
   (a) Complementary angles add to 90° or $\frac{\pi}{2}$ radians.
   (b) The sine of an angle is equal to the cosine of its complement.

$\sin\left(\frac{\pi}{2} - \theta\right)$ $= \cos\theta$	$\sin\left(\frac{\pi}{2} + \theta\right)$ $= \cos\theta$	$\sin\left(\frac{3\pi}{2} - \theta\right)$ $= -\cos\theta$	$\sin\left(\frac{3\pi}{2} + \theta\right)$ $= -\cos\theta$
$\cos\left(\frac{\pi}{2} - \theta\right)$ $= \sin\theta$	$\cos\left(\frac{\pi}{2} + \theta\right)$ $= -\sin\theta$	$\cos\left(\frac{3\pi}{2} - \theta\right)$ $= -\sin\theta$	$\cos\left(\frac{3\pi}{2} + \theta\right)$ $= \sin\theta$
$\tan\left(\frac{\pi}{2} - \theta\right)$ $= \cot\theta$	$\tan\left(\frac{\pi}{2} + \theta\right)$ $= -\cot\theta$	$\tan\left(\frac{3\pi}{2} - \theta\right)$ $= \cot\theta$	$\tan\left(\frac{3\pi}{2} + \theta\right)$ $= -\cot\theta$

EXERCISE 5B Symmetry and exact values

eBookplus

Digital docs:
SkillSHEET 5.3
Rationalising the denominator
Spreadsheet
077 The unit circle

Note: Give answers as surds with rational denominators.

WORKED Example 3a

1 Without using a calculator, find the exact values of the following.

a $\sin 120°$ **b** $\cos 135°$ **c** $\tan 330°$ **d** $\cos 225°$
e $\sin 210°$ **f** $\tan 150°$ **g** $\sin 315°$ **h** $\cos 300°$
i $\tan 225°$ **j** $\cos 390°$ **k** $\sin 405°$ **l** $\tan 420°$

WORKED Example 3b

2 Find the exact values of the following.

a $\sin \frac{3\pi}{4}$ **b** $\cos \frac{5\pi}{6}$ **c** $\tan \frac{2\pi}{3}$ **d** $\cos \frac{4\pi}{3}$
e $\sin \frac{5\pi}{4}$ **f** $\tan \frac{7\pi}{6}$ **g** $\sin \frac{11\pi}{6}$ **h** $\cos \frac{5\pi}{3}$
i $\tan \frac{7\pi}{4}$ **j** $\cos \frac{9\pi}{4}$ **k** $\sin \frac{13\pi}{6}$ **l** $\tan \frac{7\pi}{6}$

WORKED Example 4

3 If $\sin x = 0.3$, $\cos a = 0.5$, $\tan b = 2.4$ and x, a and b are in the first quadrant, find the value of the following.

a $\sin(\pi - x)$ **b** $\cos(\pi - a)$ **c** $\tan(2\pi - b)$
d $\cos(\pi - x)$ **e** $\sin(\pi - a)$ **f** $\tan(\pi + b)$
g $\sin(2\pi - x)$ **h** $\cos(2\pi - a)$ **i** $\tan(\pi - b)$
j $\cos(2\pi + x)$ **k** $\sin(2\pi + a)$ **l** $\tan(2\pi + b)$
m $\sin(3\pi - x)$ **n** $\cos(3\pi + a)$ **o** $\tan(3\pi - b)$

4 If $\sin \alpha = \frac{7}{25}$ and $\cos \alpha = \frac{24}{25}$, find $\tan \alpha$ and show that $\sin^2 \alpha + \cos^2 \alpha = 1$.

WORKED Example 5

5 **a** If $\sin x = \frac{1}{2}$, and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and hence find $\tan x$.

b If $\cos x = -\frac{1}{\sqrt{2}}$, and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\tan x$.

c If $\sin x = -\frac{\sqrt{3}}{2}$, and $\frac{3\pi}{2} < x < 2\pi$, find $\cos x$ and $\tan x$.

d If $\tan x = \sqrt{3}$, and $\pi < x < \frac{3\pi}{2}$, find $\sin x$ and $\cos x$.

WORKED Example 6

6 Find the exact value of the following.

a $\sin(-30°)$ **b** $\cos(-45°)$ **c** $\tan(-60°)$
d $\cos(-150°)$ **e** $\sin(-120°)$ **f** $\tan(-135°)$
g $\sin(-225°)$ **h** $\cos(-210°)$ **i** $\tan(-240°)$
j $\cos(-330°)$ **k** $\sin(-315°)$ **l** $\tan(-300°)$
m $\sin(-420°)$ **n** $\cos(-390°)$ **o** $\tan(-405°)$

7 Evaluate $(\sin \alpha + \cos \alpha)^2 + (\sin \alpha - \cos \alpha)^2$.

WORKED Example 7

8 Find the exact values of:

a $\sin\left(-\frac{\pi}{3}\right)$ **b** $\cos\left(-\frac{\pi}{6}\right)$ **c** $\tan\left(-\frac{\pi}{4}\right)$ **d** $\cos\left(-\frac{3\pi}{4}\right)$

e $\sin\left(-\frac{2\pi}{3}\right)$ **f** $\tan\left(-\frac{5\pi}{6}\right)$ **g** $\sin\left(-\frac{7\pi}{6}\right)$ **h** $\cos\left(-\frac{5\pi}{4}\right)$

i $\tan\left(-\frac{4\pi}{3}\right)$ **j** $\cos\left(-\frac{5\pi}{3}\right)$ **k** $\sin\left(-\frac{13\pi}{6}\right)$ **l** $\tan\left(-\frac{9\pi}{4}\right)$

9 Show that $\cos^2\left(-\frac{\pi}{4}\right) + \sin^2\left(-\frac{\pi}{4}\right) = 1$.

WORKED Example 8

10 If $\sin\theta = 0.3$, $\cos x = 0.7$ and $\tan\alpha = 0.4$, find the value of the following.

a $\sin\left(\frac{\pi}{2} - x\right)$ **b** $\cos\left(\frac{3\pi}{2} + \theta\right)$ **c** $\tan\left(\frac{\pi}{2} - \alpha\right)$ **d** $\cos\left(\frac{\pi}{2} + \theta\right)$

e $\sin\left(\frac{3\pi}{2} - x\right)$ **f** $\tan\left(\frac{\pi}{2} + \alpha\right)$ **g** $\sin\left(\frac{\pi}{2} + x\right)$ **h** $\cos\left(\frac{3\pi}{2} - \theta\right)$

i $\tan\left(\frac{3\pi}{2} - \alpha\right)$ **j** $\cos\left(\frac{3\pi}{2} + \theta\right)$ **k** $\sin\left(\frac{3\pi}{2} - x\right)$ **l** $\tan\left(\frac{3\pi}{2} + \alpha\right)$

11 **multiple choice**

If $x = \frac{\pi}{12}$, $3\sin 2x$ is equal to:

A $\frac{3\sqrt{3}}{2}$ **B** $\frac{3\sqrt{2}}{2}$ **C** $\frac{3}{2}$ **D** $\frac{1}{2}$ **E** $\frac{\pi}{2}$

12 **multiple choice**

The expression $1 - \sin^2\left(\frac{\pi}{2} - \theta\right)$ is equal to:

A $\cos\left(\frac{\pi}{2} - \theta\right)$ **B** $\sin\left(\frac{\pi}{2} - \theta\right)$ **C** $\sin^2\left(\frac{\pi}{2} - \theta\right)$

D $\cos^2\left(\frac{\pi}{2} - \theta\right)$ **E** 0

eBook*plus*

Digital doc:
SkillSHEET 5.4
Problem solving using trigonometry

13 A weight on a spring moves so that its speed, v cm per second, is given by the formula

$$v = 10 + 2\sin\frac{\pi t}{6}$$

a Find the initial speed.
b Find the speed of the weight after 5 seconds.
c What is the greatest speed that the weight can reach?

14 The height, H, in metres, that sea water reaches up a particular tree trunk near a holiday resort is governed by the equation $H = 0.4\cos\frac{\pi t}{12} + 0.5$, where t is the number of hours past midnight.

Find the height of water up the trunk at:

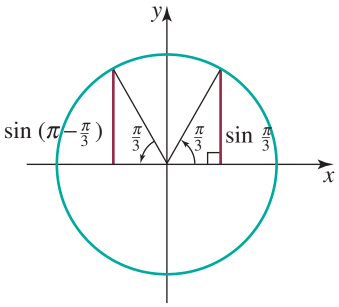
a midnight **b** 8 am
c 8 pm.

Further trigonometric equations

From the general equation sin $x = a$, we can find an infinite number of solutions. An example of this general equation is: $\sin x = \frac{\sqrt{3}}{2}$.

One of the solutions is $x = \frac{\pi}{3}$, because $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

However, we also know that $\sin\left(\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ because sine is positive in the second quadrant.



For this equation there are two solutions between 0 and 2π. They are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. (There are no solutions in the third and fourth quadrants because here, sine is negative.) $x = 0$ and $x = 2\pi$ are solutions for $\cos x = 1$ over the domain $0 \le x \le 2\pi$.

To find a greater number of solutions we can go around the unit circle as many times as we wish, finding new solutions each time. Since

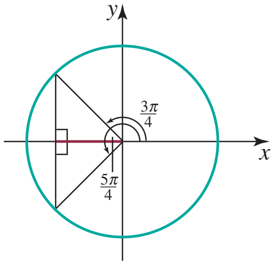
$$\sin\left(2\pi + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin\left(3\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

in the domain $0 \le x \le 4\pi$ there are 4 solutions: $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{7\pi}{3}$ and $\frac{8\pi}{3}$.

We can also go in a negative direction. In the domain $-2\pi \le x \le 0$ there are 2 solutions: $\frac{-4\pi}{3}$ and $\frac{-5\pi}{3}$.

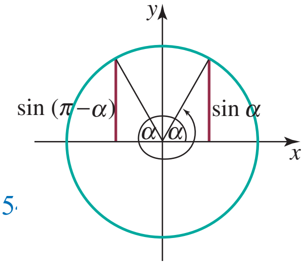
WORKED Example 9

Find all solutions to the equation $\cos x = -\frac{\sqrt{2}}{2}$ in the domain $0 \le x \le 2\pi$.

THINK	**WRITE**
1 Write the question.	$\cos x = -\frac{\sqrt{2}}{2}$
2 Find the equivalent angle in the first quadrant, ignoring the sign.	Basic angle is $\frac{\pi}{4}$.
3 In the 2nd and 3rd quadrants cos x is negative.	
4 Write the appropriate values of x in these quadrants.	2nd quadrant: $x = \pi - \frac{\pi}{4}$ 3rd quadrant: $x = \pi + \frac{\pi}{4}$ 
5 Simplify.	$x = \frac{3\pi}{4}, \frac{5\pi}{4}$

WORKED Example 10

Find all solutions to the equation sin $\alpha = 0.7$ in the domain $0 \le \alpha \le 4\pi$. Give your answers correct to 4 decimal places.



THINK	WRITE
① Find the equivalent angle in the 1st quadrant, ignoring the sign.	The basic angle is 0.7754.
② Find in which quadrants sin α is positive.	sin α is positive in the 1st and 2nd quadrants.
③ Look at angles beyond 2π because the domain is $[0, 4\pi]$ by adding the period to values in the first cycle (that is, finding values in the 5th and 6th quadrants). In $y = \sin \alpha$ the period is 2π.	From 0 to 2π: $\alpha = 0.7754, \pi - 0.775$ $= 0.7754, 2.3662$ From 0 to 4π: $\alpha = 0.7754, 2.3662,$ $0.7754 + 2\pi, 2.3662 + 2\pi$
④ Simplify giving answers correct to 4 decimal places.	$\alpha = 0.7754, 2.3662, 7.0586, 8.6494$

You will notice that the third solution can be found by adding 2π to the first solution and the fourth solution can be found by adding 2π to the second solution. This is because we have turned through an angle of 2π radians (1 revolution) beyond the original angle.

WORKED Example 11

Find all solutions to the equation $\sqrt{2}\cos x + 1 = 0$ in the domain $0 \le x \le 2\pi$.

THINK	WRITE
① Make cos x the subject of the equation.	$\sqrt{2}\cos x + 1 = 0$ $\sqrt{2}\cos x = -1$ $\cos x = -\frac{1}{\sqrt{2}}$
② Find the angle in the 1st quadrant.	Basic angle $= \frac{\pi}{4}$.
③ Decide in which quadrants cos x is negative.	$\cos x < 0$ in quadrants 2 and 3 $x = \pi - \frac{\pi}{4}, \pi + \frac{\pi}{4}$
④ Solve using only those solutions in the given domain.	$x = \frac{3\pi}{4}, \frac{5\pi}{4}$

WORKED Example 12

Find all solutions to the equation $2 \sin 2\theta = -\sqrt{3}$ in the domain $0° \le x \le 360°$.

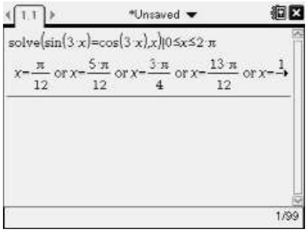
THINK	WRITE
① Simplify the trigonometric equation.	$2 \sin 2\theta = -\sqrt{3} \qquad 0° \le \theta \le 360°$
② Since θ is multiplied by 2 in the equation, multiply the domain by 2 also.	$\sin 2\theta = \frac{-\sqrt{3}}{2} \qquad 0° \le 2\theta \le 720°$
③ Find the 'first quadrant (basic) angle', ignoring the negative sign.	Basic angle = 60°
④ Now find values of 2θ between 0 and 720° which satisfy the equation. Note that $\sin \theta$ is negative in the 3rd and 4th quadrants.	$2\theta = 180° + 60°, 360° - 60°, 360° + 180° + 60°, 360° + 360° - 60°$ $2\theta = 240°, 300°, 480°, 660°$
⑤ Solve for θ.	$\theta = 120°, 150°, 240°, 330°$
⑥ Note that all values of θ are between 0 and 360°.	

WORKED Example 13

Find all solutions between 0 and 2π for the equation $\sin 3x = \cos 3x$:
a by hand **b** using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a ① Divide both sides by $\cos 3x$.	**a** $\sin 3x = \cos 3x \qquad 0 \le x \le 2\pi$ $\tan 3x = 1 \qquad 0 \le x \le 2\pi, \cos 3x \ne 0$
② Adjust the domain as shown.	$0 \le 3x \le 6\pi$
③ Find the basic angle.	Basic angle = $\frac{\pi}{4}$.
④ Solve for x between 0 and 6π. Note that tan is positive in the 1st and 3rd quadrants.	$3x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}$ $= \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$ $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}$
Cancel where possible.	$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$
Check that all answers are between 0 and $2\pi\left(2\pi = \frac{24\pi}{12} = \frac{8\pi}{4}\right)$. They are.	

Continued over page →

THINK	WRITE/DISPLAY
b ① On a Calculator page, press: • MENU (menu) • 3: Algebra (3) • 1: Solve (1). Complete the entry line as: solve(sin(3x)=cos(3x), x)| $0 \le x \le 2\pi$ then press ENTER (enter).	**b**  solve(sin(3·x)=cos(3·x),x)|0≤x≤2·π $x=\frac{\pi}{12}$ or $x=\frac{5\cdot\pi}{12}$ or $x=\frac{3\cdot\pi}{4}$ or $x=\frac{13\cdot\pi}{12}$ or $x=\frac{1}{}$ ▸
② Write the solutions.	$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$

remember

1. Given a general equation such as sin $x = a$, there can be an infinite number of solutions. The domain is usually restricted and it is important to find all values for x within this domain.
2. If the domain is given in radians, then the solution(s) to x should be in radians. If the domain is given in degrees, then the solution(s) to x should be in degrees.
3. Adjust the domain to match what has been done to the angle in the question.
4. Sine is positive in the 1st and 2nd quadrants, cosine is positive in the 1st and 4th quadrants and tangent is positive in the 1st and 3rd quadrants.
5. Find all the solutions within the specified domain.
6. If given $\cos^2 x = a$, then you will need to find values in all 4 quadrants (as x will have both positive and negative values).
7. If the equation is of the form sin $ax = k$ cos ax, divide both sides by cos ax to change the equation to tan ax, provided that cos $ax \neq 0$.

EXERCISE 5C Further trigonometric equations

WORKED Example 9

1 Find all solutions to the equations below in the domain $0 \le \theta \le 2\pi$.

a $\cos\theta = 0$ **b** $\sin\theta = -\frac{1}{\sqrt{2}}$ **c** $\cos\theta = \frac{1}{\sqrt{2}}$ **d** $\sin\theta = -1$ **e** $\cos\theta = -\frac{\sqrt{3}}{2}$

2 Find all the values of θ between 0° and 360° for which:

a $\sin\theta = 1$ **b** $\cos\theta = \frac{1}{2}$ **c** $\sin\theta = \frac{\sqrt{3}}{2}$ **d** $\cos\theta = -1$ **e** $\sin\theta = \frac{1}{\sqrt{2}}$

WORKED Example 10

3 For each equation below, find all the values of x between 0 and 4π. Give answers correct to 4 decimal places.

a $\cos x = -0.6591$ **b** $\sin x = 0.9104$ **c** $\cos x = 0.48$ **d** $\sin x = -0.371$

eBook plus

Digital doc:
Spreadsheet
075 Trig equations

4 Find all the values of x between 0° and 360° for which:

a $\sin x = 0.2686$ **b** $\cos x = -0.7421$ **c** $\sin x = -0.5432$ **d** $\cos x = 0.1937$

Give answers correct to 2 decimal places.

WORKED Example 11

5 Find all the solutions to the following equations in the domain $0 \le x \le 2\pi$.

a $2\sin x = 1$ **b** $3\cos x = 0$ **c** $2\sin x = -\sqrt{3}$ **d** $\sqrt{2}\cos x = 1$

WORKED Example 12

6 Find all the solutions to the following equations in the domain $0° \le x \le 360°$. Give exact answers where possible, otherwise give answers correct to 2 decimal places.

a $\cos 2x = 1$
b $2 \sin 2x = -1$
c $2 \cos 3x = -\sqrt{2}$
d $2 \sin 3x = \sqrt{3}$
e $\sin 3x = -0.1254$
f $3 \cos 2x = 0.5787$
g $4 \sin \frac{1}{2}x = 0.913$
h $\sqrt{2} \cos x = -0.2751$

7 Find all the solutions between 0 and 2π to the following equations. Give exact answers where possible, otherwise give answers correct to 4 decimal places.

a $4 \sin x + 2 = 6$
b $3 \cos x - 3 = 0$
c $\cos \frac{x}{2} + 4 = 4.21$
d $\sin \frac{x}{3} + 5 = 5.32$
e $2 \sin 3x - 5 = -4$
f $\sqrt{2} \cos 3x + 2 = 3$
g $2 \cos 2x + \sqrt{3} = 0$
h $\frac{1}{3} \sin \frac{1}{2}x - 1 = -0.8039$

WORKED Example 13

8 Find all values between 0 and 2π. Give exact answers for questions **a** to **d**. Otherwise give answers correct to 4 decimal places.

a $\sin x = \cos x$
b $\sin 2x = \cos 2x$
c $\sin 2x = \sqrt{3} \cos 2x$
d $\sqrt{3} \sin 3x = \cos 3x$
e $\sin 3x + 2 \cos 3x = 0$
f $\sin x + 3 \cos x = 0$

eBook plus

Digital doc:
WorkSHEET 5.1

9 A particle moves in a straight line so that its distance, x metres, from a point O is given by the equation $x = 3 + 4 \sin 2t$, where t is the time in seconds after the particle begins to move.

a Find the distance from O when the particle begins to move.
b Find the time when the particle first reaches O. Give your answer correct to 2 decimal places.

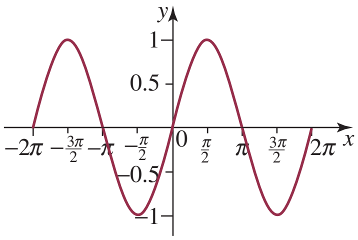
Further trigonometric graphs

Graphs of the sine and cosine functions

The graph of the function $f(x) = \sin x$ is drawn below. It is drawn over the domain $-2\pi \le x \le 2\pi$. The graph repeats itself every 2π radians. We say that it has a period of 2π. Half the distance between the maximum and minimum values is 1 so we say that the amplitude is 1.

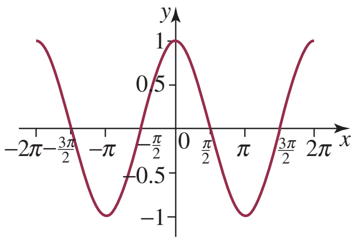
It is possible to take any value of x for the function $f(x) = \sin x$ so the domain of the whole function is R.

The range is $-1 \le y \le 1$. The graph is shown at right.



$f(x) = \sin x, -2\pi \le x \le 2\pi$

The graph of the function $f(x) = \cos x$ is the same shape but in a different position. The period is also 2π. The amplitude is 1, the domain is R and the range is $-1 \le y \le 1$. The graph is shown at right.



$f(x) = \cos x, -2\pi \le x \le 2\pi$

Complementary functions

In the previous section we looked at complementary angles. Use your graphics calculator to graph the following.

1 **a** $y = \sin x$ **b** $y = \cos\left(\frac{\pi}{2} - x\right)$

2 **a** $y = \cos x$ **b** $y = \sin\left(\frac{\pi}{2} - x\right)$

The resulting graphs will confirm these earlier results.

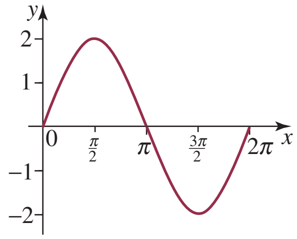
Dilation

y-dilation

If we change the amplitude, the distance between the maximum value and the minimum value also changes.

The graph of $f(x) = 2 \sin x$ is shown at right. It is drawn between 0 and 2π. The amplitude is 2. The period is still 2π. The domain is R and the range is $-2 \leq y \leq 2$.

This graph is a dilation of the basic graph of $f(x) = \sin x$ by factor 2 parallel to the y-axis. It has been stretched vertically.



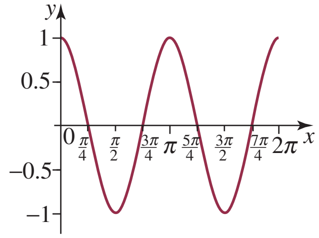
$f(x) = 2 \sin x, 0 \leq x \leq 2\pi$

x-dilation

If we change the coefficient of x the period changes and the width of the graph changes.

The graph of $f(x) = \cos 2x$ is shown at right. It is also drawn from 0 to 2π. The amplitude is 1 and the period is π. This can be found by dividing 2π by the coefficient of x. In this case $\frac{2\pi}{2} = \pi$. The domain is R. The range is $-1 \leq y \leq 1$.

The x-intercepts can be found by solving the equation $\cos 2x = 0$.



$f(x) = \cos 2x, 0 \leq x \leq 2\pi$

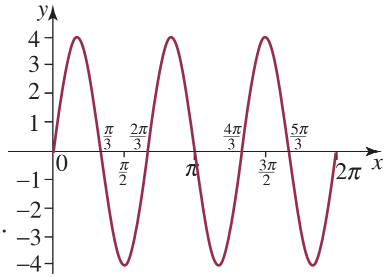
$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\text{so} \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

This graph is a dilation of the basic graph of $f(x) = \cos x$ of factor $\frac{1}{2}$ parallel to the x-axis. The period has been halved or the graph has been ‘squashed up’.

The decimal approximation for these solutions can be found using a graphics calculator. The graphics calculator can also be used to check the number and approximate value of the solutions when solving trigonometric equations.

x- and *y*-dilation



$f(x) = 4 \sin 3x,\ 0 \le x \le 2\pi$

The graph of $f(x) = 4 \sin 3x$ is shown at right. It is drawn from 0 to 2π. The amplitude is 4 and the period is $\frac{2\pi}{3}$. The domain is R. The range is $-4 \le y \le 4$.

The x-intercepts are found by solving $4 \sin 3x = 0$.

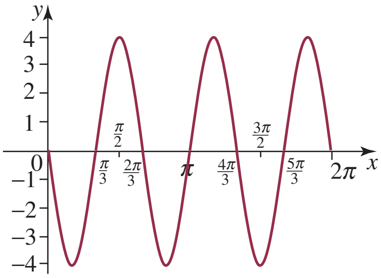
$$\begin{aligned} \text{So} \qquad \sin 3x &= 0 \\ 3x &= 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi \\ x &= 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi \end{aligned}$$

This graph has a dilation factor of 4 parallel to the y-axis, and of $\frac{1}{3}$ parallel to the x-axis.

Dilation

1. On your graphics calculator graph each of the following.
   $Y1 = \sin x$ $Y2 = 2 \sin x$ $Y3 = 3 \sin x$ $Y4 = \frac{1}{2} \sin x$
   From this you should be able to see the effect of a y-dilation. Repeat this for the corresponding cosine graphs.
2. On your graphics calculator graph each of the following.
   $Y1 = \sin x$ $Y2 = \sin 2x$ $Y3 = \sin 3x$ $Y4 = \sin \frac{1}{2}x$
   From this you should be able to see the effect of an x-dilation. Repeat this for the corresponding cosine graphs.
3. Sketch several graphs of the form $y = A \sin Bx$ and $y = A \cos Bx$, adjusting the value of A and B to have a look at the dilation effects.

Reflection



$f(x) = -4 \sin 3x,\ 0 \le x \le 2\pi$

If the coefficient of the function is negative, the graph is turned upside down or reflected in the x-axis. This does not alter the amplitude, which is always positive.

The graph of $f(x) = -4 \sin 3x$ is shown at right. You will notice that it is the graph of $f(x) = 4 \sin 3x$ turned upside down. (The graph of $f(x) = 4 \sin 3x$ is shown on the previous page.)

The amplitude is 4 and the period is $\frac{2\pi}{3}$. The domain is R. The range is $-4 \le y \le 4$.

The x-intercepts are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.

This graph has dilation factor of 4 parallel to the y-axis and of $\frac{1}{3}$ parallel to the x-axis.

If the function $f(x) = 4 \sin 3x$ is reflected in the x-axis, the result is $f(x) = -4 \sin 3x$. If we reflected the graph of $f(x) = 4 \sin 3x$ in the y-axis the result would still be $f(x) = -4 \sin 3x$.

If we reflect $f(x) = 2 \cos 3x$ in the x-axis the result is $f(x) = -2 \cos 3x$ but if we reflect it in the y-axis the graph does not change. This is because $f(x) = 2 \cos 3x$ is symmetrical about the y-axis.

Reflection

Check the reflections in both the x- and y-axes by graphing the following functions.

1 To confirm a reflection in the x-axis, graph the following function pairs.

  a $Y1 = 2\cos 3x$
  $Y2 = -2\cos 3x$ (or $Y1 = -Y1$)

  b $Y1 = 2\sin 3x$
  $Y2 = -2\sin 3x$ (or $Y2 = -Y1$)

2 To confirm a reflection in the y-axis, graph the following function pairs.

  a $Y1 = 2\cos 3x$
  $Y2 = 2\cos(-3x)$

  b $Y1 = 2\sin 3x$
  $Y2 = 2\sin(-3x)$

3 Explore the reflective effect of graphing $Y1 = 2\sin(3x)$ and $Y2 = -2\sin(-3x)$.

Translation

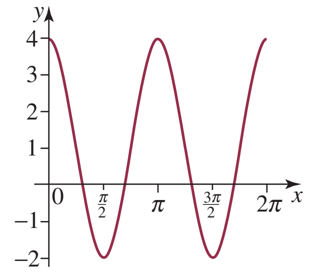
y-translation

If we add a constant to the function, the graph is moved up or down and is said to be a *translation* parallel to the y-axis.

The graph of $f(x) = 3\cos 2x + 1$ or $f(x) = 1 + 3\cos 2x$ is shown at right. The amplitude is 3, the period is π, the domain is R and the range is $-2 \le y \le 2$.

The x-intercepts are found by solving $3\cos 2x + 1 = 0$. So $\cos 2x = -\frac{1}{3}$ (cosine is negative in quadrants 2 and 3). The basic angle is inverse cosine of $\frac{1}{3} = 1.231^c$.

$2x = \pi - 1.231, \pi + 1.231, 3\pi - 1.231, 3\pi + 1.231$
$2x = 1.911, 4.373, 8.194, 10.656$
$x = 0.955, 2.186, 4.097, 5.328$



$f(x) = 3\cos 2x + 1, 0 \le x \le 2\pi$

x-translation

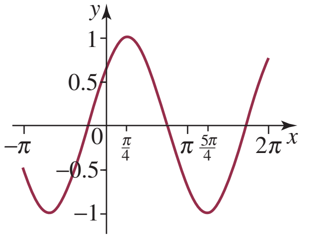
The graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$ is the graph of $f(x) = \sin x$ translated $\frac{\pi}{4}$ units to the right.

The graph of $f(x) = \sin\left(x + \frac{\pi}{4}\right)$ is the graph of $f(x) = \sin x$ translated $\frac{\pi}{4}$ units to the left.

The graph of $f(x) = \sin\left(x + \frac{\pi}{4}\right)$ is shown at right.

It is drawn between $[-\pi, 2\pi]$.
The amplitude is 1, the period is 2π, the domain is R and the range is $-1 \le y \le 1$. The y-intercept occurs when $x = 0$.

Confirm these graphs with your graphics calculator.



$f(x) = \sin\left(x + \frac{\pi}{4}\right), -\pi \le x \le 2\pi$

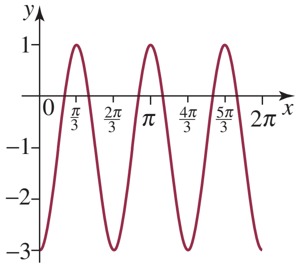
The x-intercepts occur when $\sin\left(x + \frac{\pi}{4}\right) = 0$.

So $$x + \frac{\pi}{4} = 0, \pi, 2\pi$$

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

This graph is a translation of the basic graph of the function $f(x) = \sin x$, $\frac{\pi}{4}$ units parallel to the x-axis in a positive direction.

If we take a general trigonometric function $f(x) = A \sin(Bx + C) + D$, it has an amplitude of A, a period of $\frac{2\pi}{B}$, a horizontal translation of $\frac{C}{B}$ units to the left and a vertical translation of D units up.



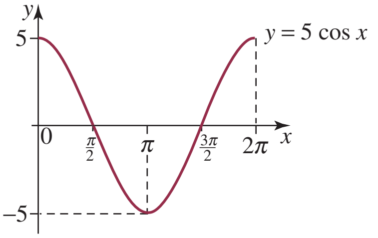
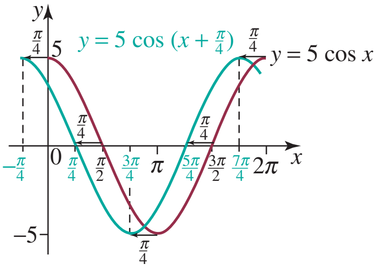
The graph of $f(x) = 2\cos(3x - \pi) - 1$ is shown at right for $0 \le x \le 2\pi$.

The amplitude is 2, the period is $\frac{2\pi}{3}$, the domain is R and the range is $-3 \le y \le 1$.

This graph is a dilation of the basic graph of $f(x) = \cos x$ of factor 2 parallel to the y-axis, and factor $\frac{1}{3}$ parallel to the x-axis. It is translated $\frac{\pi}{3}$ units to the left and 1 unit down.

WORKED Example 14

Sketch the graph of $y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$ for $0 \le x \le 2\pi$, and state the period and amplitude.

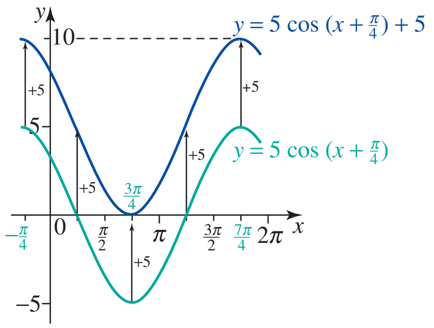
THINK	**WRITE/DRAW**
① Write the equation.	$y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$
② Find the period. (The coefficient of x is effectively equal to 1 as $x = 1x$.)	$\text{Period} = \frac{2\pi}{1}$ $= 2\pi$
③ State the amplitude.	Amplitude = 5
④ Using a pencil, sketch a basic cosine shape with the period and amplitude above.	
⑤ Apply the x-translation of $\frac{\pi}{4}$ (the + sign means to the left). Again, use pencil.	

Continued over page →

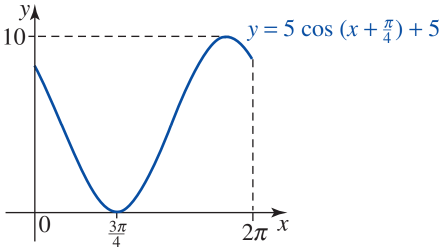
THINK

WRITE/DRAW

6 Apply the y-translation of +5 (+ means up).



7 Erase the pencilled stages.



Graphics Calculator tip! Graphing periodic functions

Worked example 14 can be done on a graphics calculator.

For the Casio *fx*-9860G AU

THINK

DISPLAY

1. To sketch the graph of $y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$ for $0 \le x \le 2\pi$, press:
   - MENU
   - 5 (GRAPH).

   Complete the entry line as:
   $Y1 = 5\cos(x + \pi \mid 4) + 5$
   then press EXE.

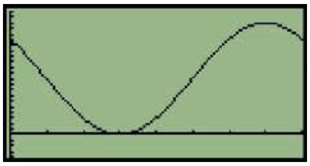
Graph Func :Y=
Y1■5cos (X+π÷4)+[—]
Y2: [—]
Y3: [—]
Y4: [—]
Y5: [—]
Y6: [—]

2. To set the domain and range, press:
   - SHIFT
   - F3 (V-Window).

   Enter values in Viewing Window as shown, then press EXE.

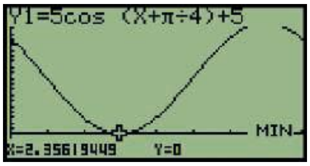
View Window
Xmin :0
max :6.2831853
scale:0.78539816
dot :0.04986655
Ymin :-2
max :11

3. To obtain the graph, press:
   - F6 (DRAW).



4. To find the minimum, press:
   - SHIFT
   - F5 (G-Solv)
   - F3 (MIN).

   Other significant points can be found in a similar manner.



Y1=5cos (X+π÷4)+5

X=2.35619449 Y=0

For the TI-Nspire CAS

THINK

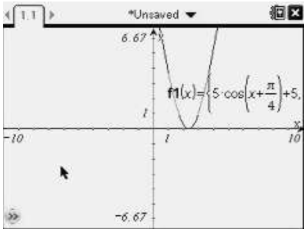
DISPLAY

1. To sketch the graph of $y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$ for $0 \le x \le 2\pi$, open a Graphs page.

   Complete the entry line as:

   $$f1(x) = 5\cos\left(x + \frac{\pi}{4}\right) + 5 \,|\, 0 \le x \le 2\pi$$

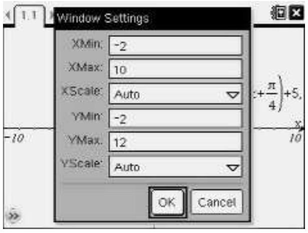
   then press ENTER enter.



2. To adjust the window, press:
   - MENU menu
   - 4: Window/Zoom 4
   - 1: Window Settings 1.

   Complete the fields as shown, then select OK.

   Note: Press Tab tab to move between fields.



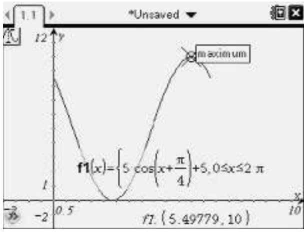
Window Settings

XMin: -2
XMax: 10
XScale: Auto
YMin: -2
YMax: 12
YScale: Auto

3. The graph is displayed with the specified domain.

   To find significant points on the graph, press:
   - MENU menu
   - 5: Trace 5
   - 1: Graph Trace 1.

   Use the NavPad to move the cursor along the curve.



4. To find significant points on the graph in exact form, open a Calculator page.

   Complete the entry lines as:

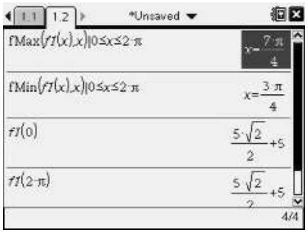
   fMax($f1(x), x$) | $0 \le x \le 2\pi$

   fMin($f1(x), x$) | $0 \le x \le 2\pi$

   $f1(0)$

   $f1(2\pi)$

   pressing ENTER enter after each entry.



fMax(f1(x),x)|0≤x≤2·π — $x=\frac{7\cdot\pi}{4}$

fMin(f1(x),x)|0≤x≤2·π — $x=\frac{3\cdot\pi}{4}$

f1(0) — $\frac{5\cdot\sqrt{2}}{2}+5$

f1(2·π) — $\frac{5\cdot\sqrt{2}}{2}+5$

remember

1. Graphs of the form $y = A \sin (Bx + C) + D$ and $y = A \cos (Bx + C) + D$ are transformations of $y = \sin x$ and $y = \cos x$.
2. A is the amplitude and is a dilation parallel to the y-axis. If A is negative, the amplitude is still positive but the graph is a reflection in the x-axis.
3. D is the vertical translation or a translation parallel to the y-axis. If D is positive, the graph is translated D units up, and if D is negative, the graph is translated D units down.
4. The range is $-A + D \leq y \leq A + D$.
5. The period is $\frac{2\pi}{B}$.
6. The factor B is the horizontal dilation which the graph has been dilated by a factor of $\frac{1}{B}$ parallel to the x-axis.
7. The value of $\frac{C}{B}$ is the horizontal translation or a translation parallel to the x-axis. If the ratio $\frac{C}{B}$ is positive, the graph is translated $\frac{C}{B}$ units to the left and if $\frac{C}{B}$ is negative, the graph is translated $\frac{C}{B}$ units to the right.

EXERCISE 5D Further trigonometric graphs

WORKED Example 14

1 State the period and amplitude of the graphs of each of the following.

a $y = \cos x$

b $y = \sin x$

c $y = 4 \sin x$

d $y = \frac{1}{3} \cos x$

e $y = 2 \cos 3x$

f $y = 3 \sin 2x$

g $y = 3 \sin \frac{1}{2}x$

h $y = 2 \cos \frac{1}{3}x$

i $y = -\frac{1}{3} \cos 2x$

j $y = -4 \sin 3x$

eBook*plus*

Digital docs:

SkillSHEET 5.5
Determining the period and amplitude of sine and cosine graphs

Spreadsheets
059 Sine graphs
006 Cosine graphs

2 Sketch the graphs of each of the following for one complete cycle stating the amplitude, the period and the range.

a $y = \frac{2}{3} \cos \theta$

b $y = 4 \sin \theta$

c $y = 3 \sin 2\theta$

d $y = 2 \cos 3\theta$

e $y = \frac{1}{2} \cos 3\theta$

f $y = \frac{1}{3} \sin 2\theta$

g $y = 4 \sin \frac{1}{2}\theta$

h $y = 3 \cos \frac{1}{3}\theta$

3 Sketch the graph of the function $f(\theta) = 4 \cos 3\theta$ for $0 \le \theta \le 2\pi$. State the amplitude, period and range.

4 Sketch the graph of the function $y = 2 \sin 2x$ for $-\pi \le x \le \pi$. State the amplitude, period and range.

5 From the basic graphs of $y = \sin x$ and $y = \cos x$, state the horizontal translation and the vertical translation for each of the following.

a $y = \sin\left(x + \dfrac{\pi}{3}\right) + 3$

b $y = \cos\left(x - \dfrac{\pi}{2}\right) + 1$

c $y = 3 \cos\left(x - \dfrac{\pi}{4}\right) - 2$

d $y = 2 \sin\left(x + \dfrac{\pi}{3}\right) - 1$

6 Sketch the graphs of the following for one complete cycle stating the amplitude, the period and the range.

a $y = \sin x + 1$

b $y = \cos x - 1$

c $y = 2 \cos x - 2$

d $y = 2 \sin x + 3$

e $y = \sin 3x - 1$

f $y = \cos 2x + 1$

g $y = 3 \cos 3x - 2$

h $y = \frac{1}{2} \sin 2x + 3$

i $y = 3 \sin \frac{1}{2}x + 4$

j $y = 2 \cos \frac{1}{3}x - 1$

7 Sketch the graphs of the following for $0 \le \theta \le 2\pi$. State the period and amplitude.

a $y = \sin\left(\theta - \dfrac{\pi}{4}\right)$

b $y = \cos\left(\theta + \dfrac{\pi}{2}\right)$

c $y = 3 \cos\left(\theta - \dfrac{\pi}{3}\right)$

d $y = 2 \sin\left(\theta - \dfrac{\pi}{4}\right)$

e $y = 2 \sin (2\theta + \pi)$

f $y = 3 \cos (3\theta + \pi)$

g $y = \cos\left(3\theta - \dfrac{\pi}{2}\right) + 1$

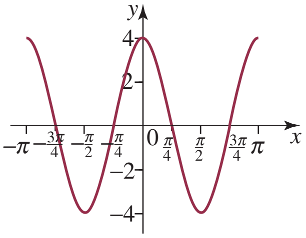
h $y = 2 \sin (2\theta - \pi) - 2$

i $y = 2 \sin\left(\theta - \dfrac{\pi}{4}\right) - 1$

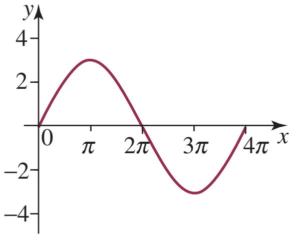
j $y = \cos \frac{1}{2}(\theta - \pi) + 1$

8 Write down the amplitude, period and range of the following graphs.

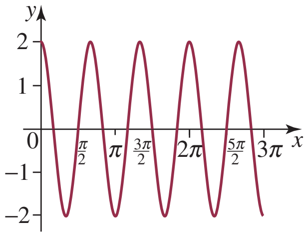
a



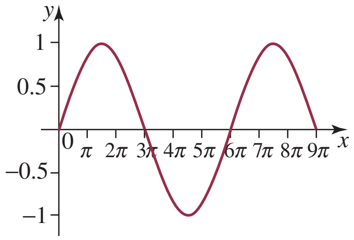
b



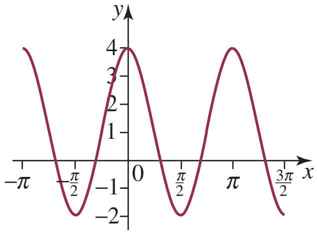
c



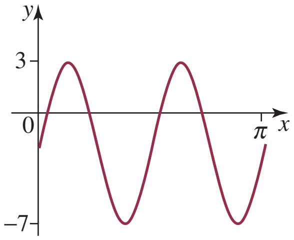
d



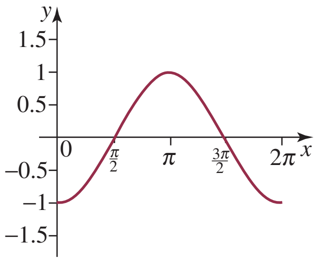
e



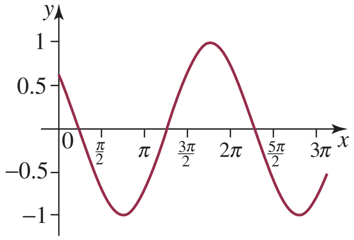
f



g



h



9 State the maximum and minimum values for each of the following.

a $y = \cos x$

b $y = \sin x$

c $y = 3 \sin x$

d $y = 2 \cos x$

e $y = 2 \cos 3x$

f $y = 3 \sin 2x$

g $y = 4 \sin 2x + 1$

h $y = \cos 4x - 2$

i $y = 2 \cos (x - \pi) - 3$

j $y = 4 \sin (3x + 3\pi) + 1$

k $y = \frac{1}{2} \sin (3x + 6\pi) + 2$

l $y = \frac{1}{3} \cos (2x - 4\pi) - 4$

10 Sketch the graphs of the following over the domain $0 \le x \le 2\pi$ and state the period, amplitude and range.

a $y = -\cos x$

b $y = -\sin x$

c $y = -2 \sin x$

d $y = -3 \cos x$

e $y = -3 \cos 2x$

f $y = -\sin 3x$

g $y = 1 - 4 \sin x$

h $y = -2 \cos 2x - 2$

i $y = -\frac{1}{2} \cos 3(x + \pi) + 1$

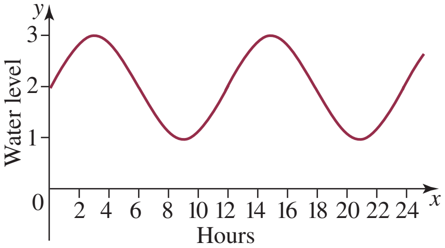
j $y = 2 - \sin (x - 2\pi)$

11 If the graph of $y = \sin x + 1$ is translated $\frac{\pi}{3}$ to the left, what is the new equation?

12 If the graph of $y = 2 \cos 3x - 2$ is translated $\frac{\pi}{4}$ to the right and 3 units up, what is the new equation?

13 If the graph of $y = 3 \sin (x - \pi) + 1$ is translated $\frac{\pi}{3}$ to the left and 3 units down, what is the new equation?

14 The level of the water in the Banksia River was measured at hourly intervals from midnight and the results recorded. The graph below shows the results.



Find:

- **a** the amplitude
- **b** the period
- **c** the maximum height of the river
- **d** the minimum height of the river
- **e** at what times the river has maximum height
- **f** at what times the river has a minimum height
- **g** the equation of the curve which is of the form $y = A \sin B(x) + D$.

eBook plus

Digital doc:
SkillSHEET 5.6
Using addition of ordinates

15 Using addition of ordinates, sketch the following graphs for the domain $0 \le x \le 2\pi$.

a $y = \sin x + \cos 2x$
b $y = \cos x + \sin 2x$
c $y = 2 \sin x + \cos x$
d $y = 2 \cos x + \sin x$
e $y = 2 \sin x + \cos 2x$
f $y = 2 \cos x + \sin 2x$
g $y = 2 \sin 2x + \cos x$
h $y = 2 \cos 2x + \sin x$

Graphs of the form $y = \tan Bx$

1 Use a graphics calculator or graphing software to produce graphs of the following between $x = -2\pi$ and $x = 2\pi$.

Sketch each graph into your workbook, showing x-intercepts in terms of π.

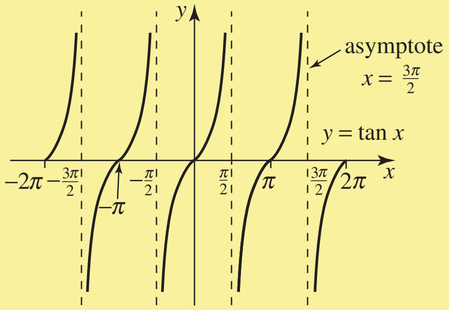
a $y = \tan x$
b $y = \tan 2x$
c $y = \tan 3x$
d $y = \tan 4x$
e $y = \tan \dfrac{x}{2}$
f $y = \tan \dfrac{x}{4}$

2 **a** Insert B and n into the following expression to make a true statement:

The graph of $y = \tan Bx$ has x-intercepts at $x = \pm\dfrac{\_\_\_\_\_\,x}{\_\_\_\_\_}$, where $n = 0, 1, 2, \ldots$

b Copy and complete: The graph of $y = \tan Bx$ has period _______, and range _______.

3 Consider your graph from question **1a**. The graph approaches (but never quite reaches) the vertical line $x = \dfrac{\pi}{2}$. Such a line is called an *asymptote*.
Write the equations of the asymptotes on your graphs from question **1**.



4 Copy and complete: The equations of asymptotes to the right of the origin for the graph of $y = \tan Bx$ are:

$$x = \frac{\pi}{2B},\ x = \frac{3\pi}{2B},\ \_\_\_\_\_,\ \_\_\_\_\_ \text{ etc.}$$

5 What is the general equation for asymptotes to the left of the origin?

6 Sketch the following between the x-values shown without using graphing technology. Show the position of all x-intercepts and asymptotes.

a $y = \tan 6x$ between $x = -\pi$ and π.
b $y = \tan \dfrac{x}{5}$ between $x = 0$ and 10π.
c $y = \tan \pi x$ between $x = 0$ and 3.
d $y = \tan \dfrac{\pi x}{2}$ between $x = -4$ and 4.

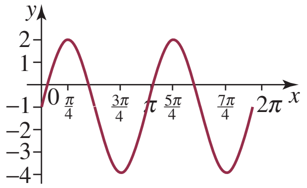
Finding equations of trigonometric graphs

Sometimes it is necessary to be able to find the equation of a trigonometric function from a graph.

eBook plus

Interactivity:
Sine and cosine graphs
int-0251

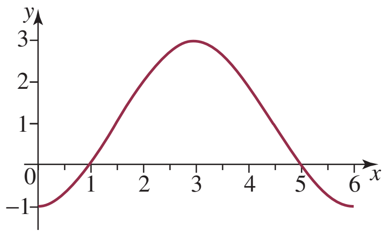
WORKED Example 15



The graph shown is a trigonometric function of the form $y = A \sin Bx + D$. Find the values of A, B and D. Hence find the equation of the function.

THINK	WRITE
① The amplitude (A) is half the distance between the maximum and minimum values.	$y = A \sin Bx + D$ Amplitude $A = \frac{1}{2}(2 + 4)$ $= 3$
② The period is the interval from one point on the graph to the next point where the graph begins to repeat itself. The period is $\frac{2\pi}{B}$.	The period is $\frac{2\pi}{B} = \pi$, so $B = 2$.
③ The line through the centre of the graph is $y = -1$, so the graph has been translated down 1 unit.	$D = -1$. So the equation is $y = 3 \sin 2x - 1$.

WORKED Example 16



This graph is a trigonometric function of the form $y = A \cos Bt + D$. Find the equation of the function.

THINK	WRITE
① The amplitude (a) is half the distance between the maximum and minimum values.	$y = A \cos Bt + D$ Amplitude $A = \frac{1}{2}(3 + 1)$ $= 2$
② The period is the amount of time taken to complete the pattern once.	Period $= \frac{2\pi}{B}$ $= 6$, so $B = \frac{\pi}{3}$
③ The graph has been translated up one unit.	$D = 1$
④ We know that the graph is a cosine graph so it must be inverted; that is, a is negative.	$A = -2$
⑤ The equation of this trigonometric function is in the form $y = A \cos Bt + D$. Replace A, B and D to obtain the equation of the function.	$y = -2 \cos \frac{\pi}{3} t + 1$

remember

1. Trigonometric functions can be expressed in the form of $y = A \sin (Bx + C) + D$ or $y = A \cos (Bx + C) + D$.
2. The amplitude A can be found by halving the distance between the maximum and minimum values.
3. The period is the interval from one point on the graph to the next point where the graph begins to repeat itself. The period is $\frac{2\pi}{B}$.
4. The vertical shift is D.
5. The horizontal shift is $\frac{C}{B}$.

EXERCISE 5E Finding equations of trigonometric graphs

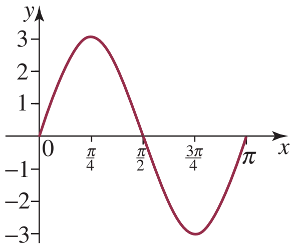
eBook plus

Digital docs:
Spreadsheets
059 Sine graphs
006 Cosine graphs

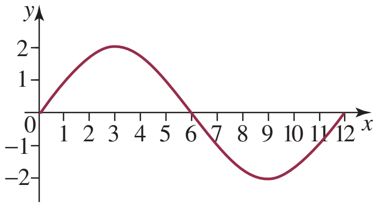
WORKED Example 15

1 The equations of the following graphs are of the form $y = A \sin Bx$. Find the values of A and B. Hence, find the equation of each function.

a



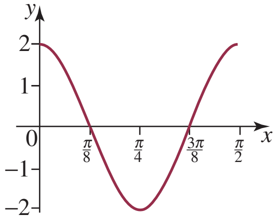
b



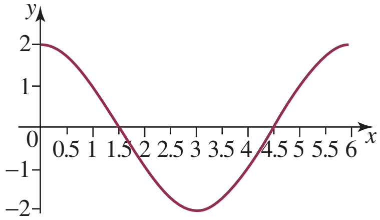
WORKED Example 16

2 The equations of the following graphs are of the form $y = A \cos Bx$. Find the values of A and B. Hence, find the equation of each function.

a

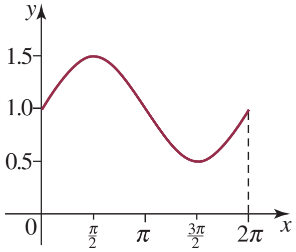


b

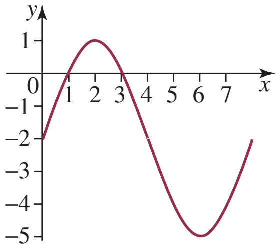


3 The equations of the following graphs are of the form $y = A \sin Bx + D$. Find the values of A, B, and D and hence write the equation of the function.

a

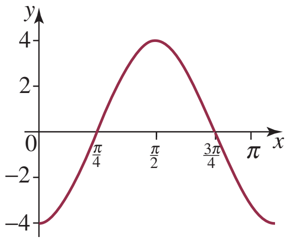


b

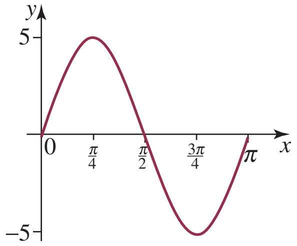


4 The equations of the following graphs are of the form $y = A\cos(Bx + C)$. Find the values of A, B and C. Hence, write the equation of the function.

a

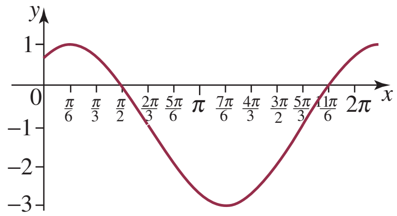


b

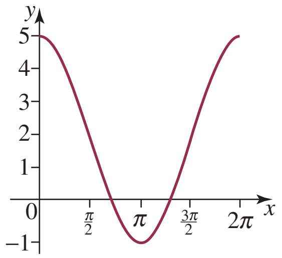


5 The equations of the following graphs are of the form $y = A\sin(Bx + C) + D$. Find the values of A, B, C and D. Hence, write the equation of the function.

a

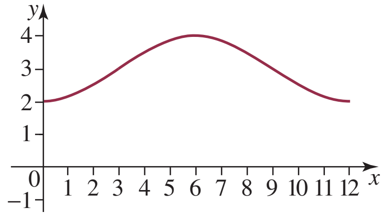


b

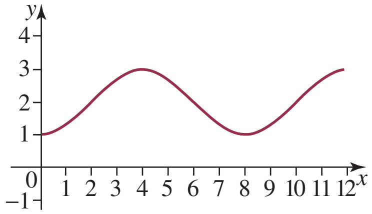


6 The equations of the following graphs are of the form $y = D + A\cos Bx$. Find the values of A, B and D. Hence, write the equation of the function.

a



b



7 **multiple choice**

If the amplitude is 2, the period is 6 and there is a vertical translation of −2, then the equation of the form $y = A\sin Bx + D$ is:

A $y = 2\sin 6x - 2$

B $y = 6\sin 2x - 2$

C $y = 2 - 2\sin\frac{\pi}{3}x$

D $y = 2\sin\frac{\pi}{3}x - 2$

E $y = 2\sin\frac{\pi}{6}x - 2$

eBook*plus*

Digital doc:
WorkSHEET 5.2

8 **multiple choice**

If the period is π, the range is $-2 \le x \le 4$, and the horizontal translation is $\frac{\pi}{4}$, the equation for the trigonometric function of the form $y = A\cos(Bx + C) + D$ is:

A $y = 3 - 2\cos\left(\pi x + \frac{\pi}{4}\right) + 1$

B $y = 3\cos\left(2x + \frac{\pi}{2}\right) - 1$

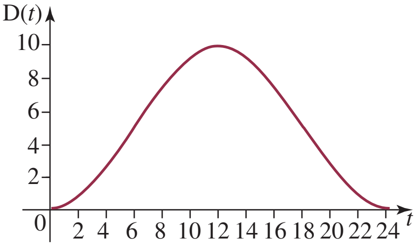
C $y = 2\cos\left(3x + \frac{3\pi}{4}\right) + 1$

D $y = 2\cos\left(3x - \frac{3\pi}{4}\right) - 1$

E $y = 3\cos\left(2x - \frac{\pi}{2}\right) + 1$

Midnight on Marus — following the sun

At the beginning of this chapter we considered the path of the red sun on the planet Marus. The graph was of the form $D(t) = a - b \sin (nt + c)$ where a, b, c and n were positive constants and t was the time in hours after midnight. The graph of $D(t)$ is shown on the next page.



1. State the values of a, b, c and n and hence write the rule for $D(t)$.
2. What would be the angle above the horizon at 6.00 am and 9.00 pm? Give an exact answer where possible, otherwise give your answer correct to 2 decimal places.
3. Use your graph to find at what times the angle to the horizontal is 8°. When does the sun reach this angle again?
4. By using an appropriate equation, check your answer and account for any difference in your two solutions.

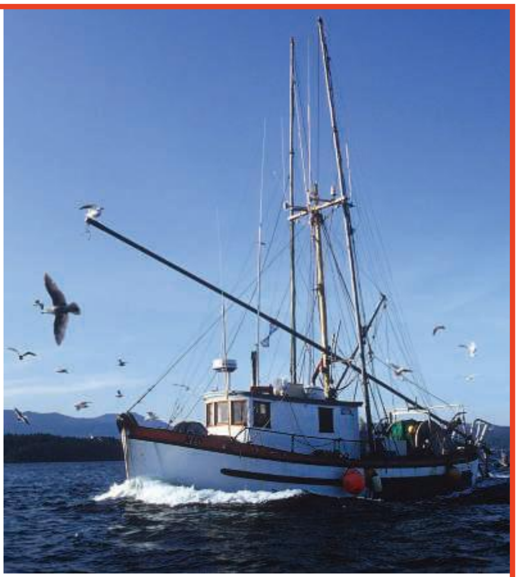
Trigonometric modelling and problem solving

In real life there are many examples of periodic behaviour. Sine and cosine functions such as

$$y = A \sin (Bx + C) + D \text{ and } y = A \cos (Bx + C) + D$$

are often used to model this behaviour.

WORKED Example 17



While out in his trawler John North, a fisherman, notes that the height of the tide in the harbour can be found by using the equation:

$$h = 5 + 2 \cos \frac{\pi}{6} t,$$

where h metres is the height of the tide and t is the number of hours after midnight.

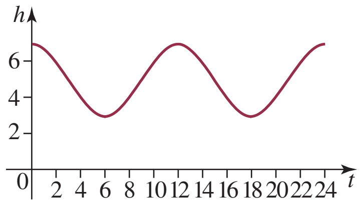
a **What is the height of the high tide and when does it occur in the first 24 hours?**

b **What is the difference in height between high and low tides?**

c **Sketch the graph of h for $0 \le t \le 24$.**

d **John North knows that his trawler needs a depth of 6 metres to enter the harbour. Between what hours is he able to bring his boat back into the harbour?**

THINK	WRITE
a ① Write the given equation.	**a** $h = 5 + 2 \cos \frac{\pi}{6} t$
② For high tide, find the maximum value of h.	For maximum h, $\cos \frac{\pi}{6} t = 1$ So $h = 5 + 2 \cdot 1$ $= 7$

THINK	**WRITE**
③ Find when high tide occurs.	$\frac{\pi}{6}t = 0, 2\pi, 4\pi, \ldots$ $t = 0, 12, 24, \ldots$ A high tide of height 7 m occurs at midnight, noon the next day, and midnight the next night.
b ① Find the minimum value of h.	**b** For minimum h, $\cos \frac{\pi}{6}t = -1$ So $h = 5 + 2 \cdot -1$ $= 3$
② Find the difference between high and low tides.	The difference between high and low tides is $7 - 3 = 4$ metres.
c Use the information from the previous page to sketch the graph.	**c** 
d ① Find t using the equation when $h = 6$.	**d** When $h = 6$, $5 + 2\cos \frac{\pi}{6}t = 6$ $2\cos \frac{\pi}{6}t = 1$ $\cos \frac{\pi}{6}t = \frac{1}{2}$ $\frac{\pi}{6}t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$ $t = 2, 10, 14, 22, \ldots$
② Write the answer in words.	From the graph we can see that John North can bring his boat back into harbour before 2 am, between 10 am and 2 pm and between 10 pm and 2 am the next morning.

remember

1. Read the question carefully to find out what is being asked.
2. Answer the questions in words where appropriate.
3. Ensure that graphs are sketched over the given domain.

Varying amplitude

Consider a trigonometric function where the amplitude itself is a function,

for example, $f(x) = x \sin x$ or $g(x) = e^{\frac{x}{5}} \sin x$.

1 Use a graphics calculator or graphing software to produce a graph of each of these functions over the domain $0 \le x \le 10$.
2 Investigate the effect of changing k on the graph of functions of the type $f(x) = e^{kx} \sin x$.
3 Suggest a format for an equation used to describe the diminishing level of a drug in the body after x hours. Sketch several examples of the graph of such a model.
4 Give another example of a situation that may be modelled by a graph with decreasing amplitude.
5 Investigate functions of the type $f(x) = a^{-x^2} \sin x$.

EXERCISE 5F Trigonometric modelling and problem solving

WORKED Example 17

eBook plus

Digital doc:
Spreadsheet
024 Function grapher

1 Competition is severe, so Fred Greenseas decides that he will catch more fish in an inlet several kilometres east of the place where John North fishes. There is a sandbar at the entrance to the inlet and the depth of water in metres on the sandbar is modelled by the function $d(t) = 6 + 2.5 \sin \frac{\pi t}{6}$ where t is the number of hours after 12 noon.

a What is the greatest depth of the water on the sandbar and when does it first occur?
b How many hours pass before there is once again the maximum depth of water on the sandbar?
c What is the least amount of water on the sandbar?
d Sketch the graph of d for $0 \le t \le 24$.
e Fred Greenseas needs a depth of 7.25 metres to cross the sandbar. Between what hours is he able to enter and leave the inlet?

2 A student wanting to catch fish to sell at a local market on Sunday has discovered that more fish are in the water at the end of the pier when the depth of water is greater than 8.5 metres. The depth of the water (in metres) is given by $d = 7 + 3 \sin \frac{\pi}{6}t$, where t hours is the number of hours after midnight on Friday.

a What is the maximum and minimum depth of the water at the end of the pier?
b Sketch a graph of d against t from midnight on Friday until midday on Sunday.
c When does the water first reach maximum depth?
d Between what hours should the student be on the pier in order to catch the most fish?
e If the student can fish for only two hours at a time, when should she fish in order to sell the freshest fish at the market from 10.00 am on Sunday morning?

3 The mean daily maximum temperature in Tarabon, an experimental town in a glass dome, is modelled by the function $T(m) = 18 + 7\cos\frac{\pi}{6}m$, where T is in degrees Celsius and m is the number of months after 1 January 1997.

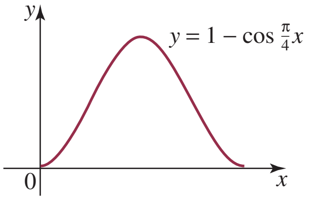
- **a** What was the mean daily maximum temperature in March 1997, and in August 1997?
- **b** What is the highest mean daily maximum temperature in Tarabon? In which months does it occur?
- **c** What would the mean daily maximum temperature be in February 1998?
- **d** If the pattern continued, how many months would pass before the mean daily maximum temperature would be the same again as it was in February 1998?

4 The height above the ground of the middle of a skipping rope as it is being turned in a child's game is found by using the equation $h = a \sin nt + c$, where t is the number of seconds after the rope has begun to turn. During the game, the maximum height the rope reaches is 1.8 metres and it takes 2 seconds for the rope to complete a full turn.

- **a** Find the values of a, n and c and hence write the equation of h in terms of t.
- **b** Sketch the graph of h against t for $0 \le t \le 5$.
- **c** After how much time will the rope be 25 cm above the ground? Give your answer correct to the nearest tenth of a second.

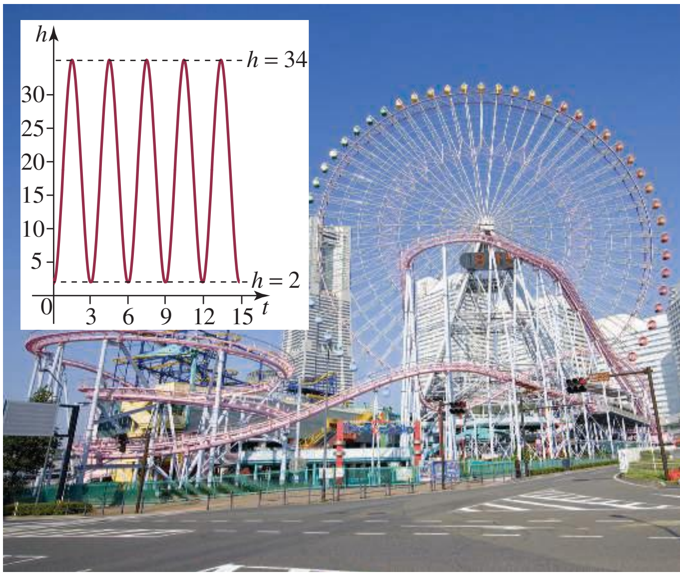
5 The graph at right shows the path of a small lizard as it runs over a hill.

- **a** Calculate the height of the hill, in metres, and hence find the amplitude of the trigonometric function.
- **b** If the ground was flat, how far would the lizard run to reach the same spot on the other side of the hill? Hence, find the period of the function.



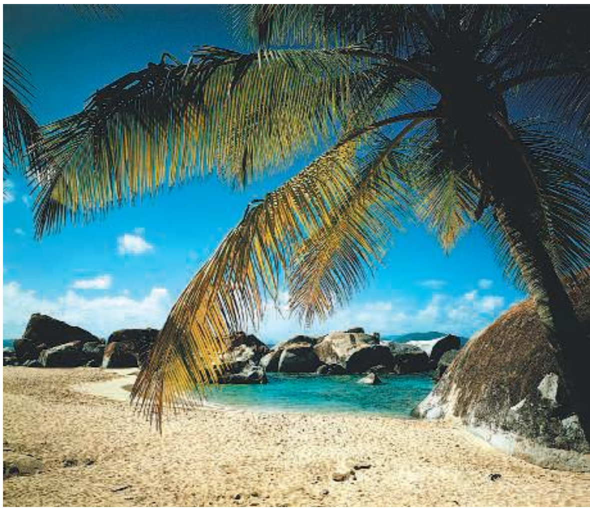
6 When Sloane and Michael were riding on a Ferris wheel, they realised that as the wheel turned clockwise, the height of their seat in metres after t minutes could be modelled by the function $h(t) = a - b\cos nt$.

The graph of h against t is shown for the whole ride.



- **a** State the values of a, b and n.
- **b** Write down the rule for $h(t)$.
- **c** How many times do Sloane and Michael reach the highest point during their ride and how far above the ground is it?
- **d** How high are they when the ride begins to move?
- **e** How many minutes pass before they are at this point again?
- **f** How far above the ground are they after 1 minute?
- **g** If the ride began when the boys were at the height found in part **f**, what would the function become?
- **h** Draw a graph of the new function for the first 6 minutes of the ride.

7 On a summer's day, the hourly temperature, which can be approximated to a cosine curve, was recorded. The maximum temperature was 30 °C and occurred at 3.00 pm. The minimum temperature was 10 °C and occurred at 3.00 am. The temperature was first recorded at 12 midnight, then every hour for 24 hours.



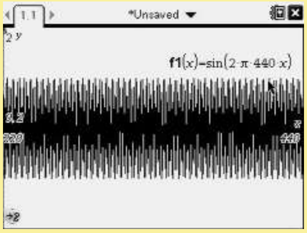
a What is the amplitude of the function?
b What is the period of the function?
c What is the middle value of the function?
d How far has this middle value been translated upwards from the *x*-axis?
e When do the maximum and minimum temperatures occur?
f Using the above, write an equation that will model this function.
g Check the accuracy of your work by using your equation to find the temperature after 3, 9, 15 and 21 hours.
h What is the temperature at midnight? Give your answer correct to the nearest degree. Check that your answer makes sense.

Beats

When a guitar string is struck it vibrates. These vibrations send out waves (sound waves) through the air. The pitch of a note produced by plucking a guitar string depends on the frequency of the vibrations of the string.

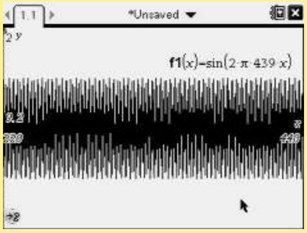
At times, when a guitar is being tuned, the note produced by plucking one string has almost, but not exactly, the same frequency as the neighbouring string and then an interesting phenomenon occurs. A slow throbbing sound is produced. The origin of this can be understood by looking at the result of adding two waves of almost the same frequency.

The graph at right represents the sound made by the first string:



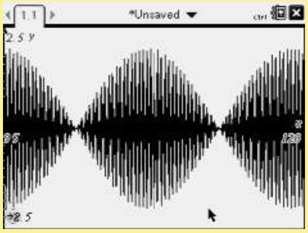
$f1(x)=\sin(2\cdot\pi\cdot 440\cdot x)$

The graph of the sound made by the second string is:

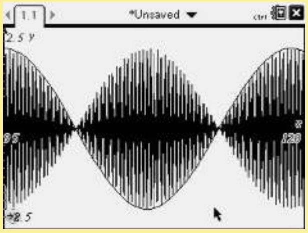


$f1(x)=\sin(2\cdot\pi\cdot 439\cdot x)$

The combined sound is represented by a graph that is the sum of the two individual graphs. The graph depicted at right has the equation $y = \sin(2\pi \cdot\ 440x) + \sin(2\pi \cdot\ 439x)$.



You can see that although the underlying oscillations are still rapid, there is an enveloping wave of much slower frequency. In this case the enveloping curve is $y = 2\cos(\pi x)$.



The slow, throbbing sound that is heard when these two notes, with almost identical frequency, are played is described by this enveloping curve. In this case the frequency of the throbbing sound is 1 cycle per second.

The mathematical description of this phenomenon relies on the following trigonometric identities.

$$\sin(A + B) = \sin A\cos B + \sin B\cos A$$
$$\sin(A - B) = \sin A\cos B - \sin B\cos A$$

In our case:

$\sin(2\pi \cdot\ 440x) + \sin(2\pi \cdot\ 439x)$

$= \sin(880\pi x) + \sin(878\pi x)$

$= \sin(879\pi x + \pi x) + \sin(879\pi x - \pi x)$

$= \sin(879\pi x)\cos(\pi x) + \sin(\pi x)\cos(879\pi x) +$
$\sin(879\pi x)\cos(\pi x) - \sin(\pi x)\cos(879\pi x)$

$= 2\sin(879\pi x)\cos(\pi x)$

The underlying vibration.

The throbbing, or beats effect.

For you to try:

Recreate these graphs on your calculator. (You may need to experiment with the Xmax value in your window settings to find a suitable visual display.)

Recreate this phenomenon on a guitar or sound generator.

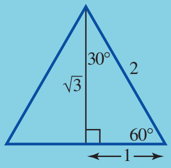
summary

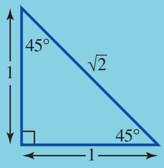
Revision of radians and the unit circle

- 1^c = the size of the angle formed where the length of an arc is equal to the radius of the circle.
- $\pi^c = 180°$
- Angles are in radians unless a degree symbol is shown.
- To change radians to degrees, multiply by $\frac{180}{\pi}$.
- To change degrees to radians, multiply by $\frac{\pi}{180}$.
- Identities
  1. $\sin^2\theta + \cos^2\theta = 1$ 2. $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Symmetry and exact values

- Exact values can be determined by using the equilateral triangle and the right isosceles triangle.





	0° (0)	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$	180° (π)	270° $\left(\frac{3\pi}{2}\right)$	360° (2π)
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.	0

- The unit circle is symmetrical so that the magnitude of sine, cosine and tangent are the same in each quadrant but the sign varies. All values of sine, cosine and tangent are positive in the 1st quadrant, sine is positive in the 2nd quadrant, tangent is positive in the 3rd quadrant and cosine is positive in the 4th quadrant.

$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

- Negative angles
  $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$
- Complementary angles add to 90° or $\frac{\pi}{2}$ radians.
- The sine of an angle is equal to the cosine of its complement.

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$	$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$	$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

Trigonometric equations

- Given a general equation such as sin $x = a$, there can be an infinite number of solutions. The domain is usually restricted and it is important to find all values for x within this domain.
- If the domain is given in radians, then the solution(s) to x should be in radians. If the domain is given in degrees, then the solution(s) to x should be in degrees.
- Adjust the domain to match what has been done to the angle in the question.
- Sine is positive in the 1st and 2nd quadrants, cosine is positive in the 1st and 4th quadrants and tangent is positive in the 1st and 3rd quadrants.
- Find all the solutions within the specified domain.
- If the equation is of the form $\sin ax = k \cos ax$, divide both sides by $\cos ax$ to change the equation to $\tan ax$.

Trigonometric graphs

Sine and cosine graphs

- Graphs of the form $y = A \sin (Bx + C) + D$ and $y = A \cos (Bx + C) + D$ are transformations of $y = \sin x$ and $y = \cos x$.
- The amplitude A is a dilation parallel to the y-axis. If A is negative, the amplitude is still positive but the graph is a reflection in the x-axis.
- The vertical translation D is a translation parallel to the y-axis.
- The range is $-A + D \le y \le A + D$.
- The period is $\frac{2\pi}{B}$.
- The factor B is the horizontal dilation where the graph has been dilated by a factor of $\frac{1}{B}$ parallel to the x-axis.
- The value of $\frac{C}{B}$ is the horizontal translation or a translation parallel to the x-axis.

Tangent graphs

The graph of $y = \tan x$ has the following properties.

- It has no amplitude.
- The period is π.
- There are x-intercepts at $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$
- There are vertical asymptotes at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- The range is R.

The graph of $y = \tan Bx$ has the following properties.

- It has no amplitude.
- The period is $\frac{\pi}{B}$.
- There are x-intercepts at $x = \pm\frac{n\pi}{B}$ where $n = 0, 1, 2, \ldots$
- There are vertical asymptotes at $x = \pm\frac{(2n + 1)\pi}{2B}$ where $n = 0, 1, 2, \ldots$
- The range is R.

Trigonometric modelling and problem solving

- Read the question carefully to find out what is being asked.
- Answer the questions in words where appropriate.
- Ensure that graphs are sketched over the given domain.

CHAPTER review

5A **1** What is 320° expressed in radians?

5A **2** What is $\frac{13\pi}{6}^{c}$ expressed in degrees?

5A **3** If $\sin\theta = -\frac{8}{15}$, and $\pi < x < \frac{3\pi}{2}$, find: **a** $\cos\theta$ **b** $\tan\theta$.

5B **4** What is the value of the expression $\sin\frac{5\pi}{3}$? Give your answer as a fraction.

5B **5** If $x = \frac{\pi}{12}$, find $5\cos 3x$.

5C **6** Find the solution of the equation $4\sin x = -2\sqrt{3}$ between π and $\frac{3\pi}{2}$.

5C **7** Find all solutions to the equation $3\sin 2\theta = 1.56$ over the domain $0 \le x \le \frac{\pi}{2}$. Give your answers correct to 3 decimal places.

5D **8** A trigonometric function is given by $f(x) = 3\cos(2x + \pi) - 1$. Find:
a the amplitude **b** the period **c** the range.

5D **9** Using addition of ordinates for the graph $y = 2\cos x - 3\sin x$, find the value of y at $x = \frac{\pi}{3}$.

5D **10** What is the period of the graph of $y = \tan\frac{x}{3}$?

5D **11** Sketch on the same set of axes:

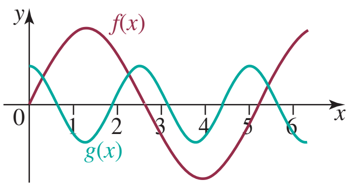
(a) $y = 2\sin\frac{x}{2}x$ from $x = 0$ to $x = 6$

(b) $y = 2\cos\frac{\pi}{2}x$ from $x = 0$ to $x = 6$.

5D **12** **a** State the transformations which are required to change the graph of $f(x) = \cos x$ to the graph of $f(x) = 2\cos\left(3x + \frac{\pi}{2}\right) - 1$.

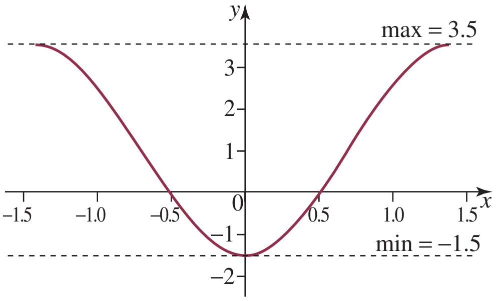
b If the graph of $f(x) = 2\cos\left(3x + \frac{\pi}{2}\right) - 1$ was reflected in the x-axis, what would be the equation of the resulting graph?

5D **13** From the figure, find $f(x) + g(x)$.



5E

14 For the graph shown at right, state the amplitude, period and range.



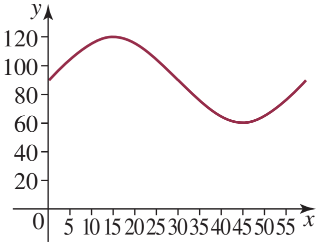
5E

15 If the graph of $y = \cos x$ is translated $\frac{\pi}{3}$ units to the right and reflected in the x-axis, what is the new equation?

Modelling and problem solving

5F

1 The level of sound in a recording studio is being monitored, and is known to model a function of the form $y = a + b \sin nx$. If in 60 seconds (period) of a recording session the range of sound intensity is 60 to 120 decibels, find the values of a, b and n.



2 The depth of water at an inlet can be modelled using the equation $D = 14 + 5 \sin \frac{4\pi t}{13}$, where t is the hours since high tide at midnight on January 1.

- **a** What is the maximum depth of water in the inlet?
- **b** What is the minimum depth of water in the inlet?
- **c** What is the period of the function?
- **d** What is the amplitude of the function?
- **e** Sketch a graph of D against t for two cycles.

eBook*plus*

Digital doc:
Test Yourself
Chapter 5

eBook*plus* ACTIVITIES

5A Revision of radians and the unit circle

Digital docs

- SkillSHEET 5.1: Practise changing degrees to radians (*page 167*)
- SkillSHEET 5.2: Practise calculating tangent ratios (*page 167*)
- Spreadsheet 077: Investigate properties of the unit circle (*page 167*)

5B Symmetry and exact values

Digital docs

- SkillSHEET 5.3: Practise rationalising the denominator (*page 175*)
- Spreadsheet 077: Investigate properties of the unit circle (*page 175*)
- SkillSHEET 5.4: Practise problem solving using trigonometry (*page 176*)

Tutorials

- **WE3** Int-0546: Watch a tutorial on exact values in degrees and radians (*page 169*)
- **WE5** Int-0547: Watch a tutorial on determining trigonometric ratios (*page 170*)
- **WE8** Int-0548: Watch a tutorial on complementary angle formulae (*page 173*)

5C Further trigonometric equations

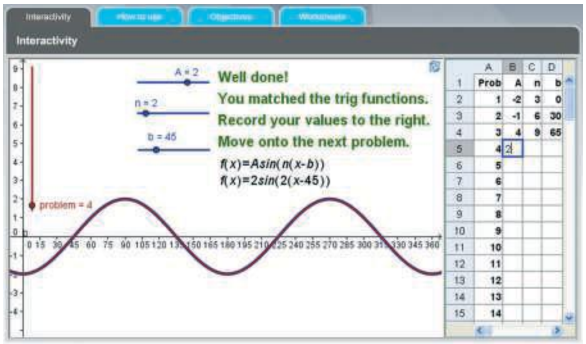
Digital docs

- Spreadsheet 075: Investigate trigonometric equations (*page 180*)
- WorkSHEET 5.1: Evaluate tangent ratios in exact values, solve trigonometric equations and answer application questions (*page 181*)

5D Further trigonometric graphs

Digital docs

- SkillSHEET 5.5: Practise determining the period and amplitude of sine and cosine graphs (*page 188*)
- Spreadsheet 059: Investigate sine graphs (*page 188*)
- Spreadsheet 006: Investigate cosine graphs (*page 188*)
- SkillSHEET 5.6: Practise using addition of ordinates (*page 192*)



Well done!
You matched the trig functions.
Record your values to the right.
Move onto the next problem.

$f(x)=A\sin(n(x-b))$
$f(x)=2\sin(2(x-45))$

	A	B	C	D
1	Prob	A	n	b
2	1	-2	3	0
3	2	-1	6	30
4	3	4	9	65
5	4	2		
6	5			
7	6			
8	7			
9	8			
10	9			
11	10			
12	11			
13	12			
14	13			
15	14			

5E Finding equations of trigonometric graphs

Digital docs

- Spreadsheet 059: Investigate sine graphs (*page 194*)
- Spreadsheet 006: Investigate cosine graphs (*page 194*)
- WorkSHEET 5.2: Sketch graphs of periodic functions, apply transformations and use addition of ordinates (*page 195*)

Interactivity

- Sine and cosine graphs int-0251: Consolidate your understanding of graphs of periodic functions (*page 193*)

5F Trigonometric modelling and problem solving

Digital doc

- Spreadsheet 024: Investigate graphs of functions (*page 198*)

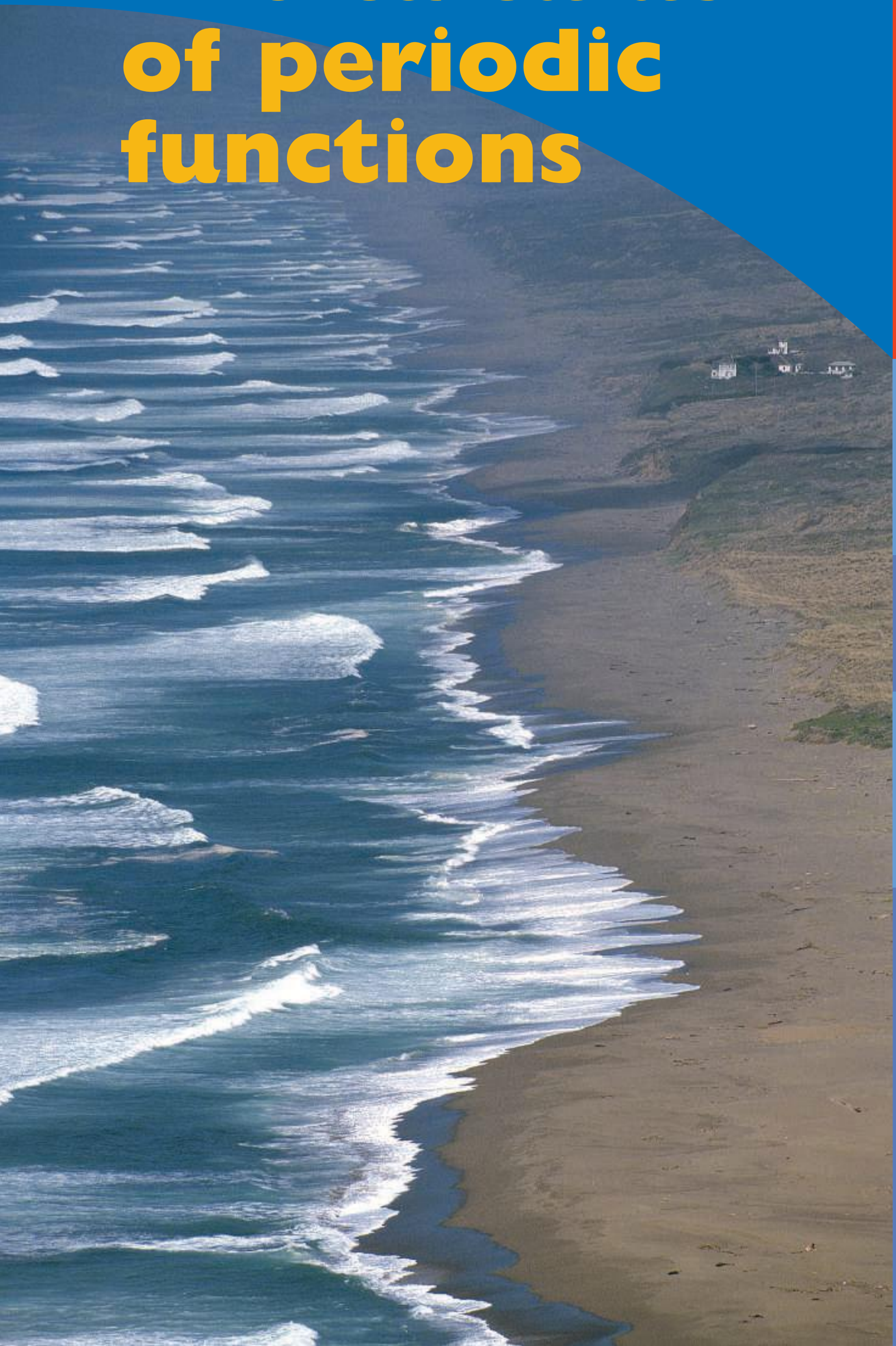
Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 205*).

To access eBookPLUS activities, log on to

www.jacplus.com.au



6

The calculus of periodic functions

syllabus reference

Periodic functions and applications

In this chapter

Introduction

The REAC warehouse — is it secure?

The Rastenburg Electrical Appliances Company installs a searchlight to assist the night guard to protect the front of its warehouse. The searchlight oscillates across the front of the warehouse, a distance of 100 metres. The searchlight completes one full oscillation every 2 minutes. To break into the warehouse, an intruder would need to scale a fence and run 50 metres to a window or door without being spotted by the night guard. Is the warehouse secure?

In this chapter we shall examine the mathematics of oscillations such as those made by the warehouse searchlight.

The derivatives of sin *x* and cos *x*

We shall now examine the derivatives of sin *x* and cos *x* using a graphics calculator. Examples are given for the Casio *fx*-9860G AU and the TI-Nspire CAS.

For the Casio *fx*-9860G AU

1. To draw the graph of sin *x*, press:
   - **MENU**
   - 5 (GRAPH).

   Complete the entry line as:

   $Y1 = \sin x$

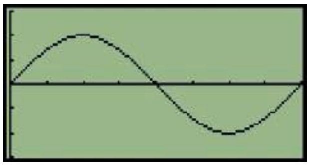
   then press **EXE**.

   To set the Viewing Window, press:
   - **SHIFT**
   - **F3** (V-Window)

   and enter the settings as shown.

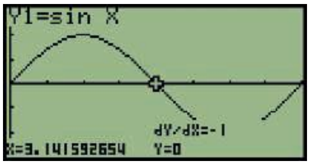
```
View Window
Xmin  :0
 max  :6.2831853
 scale:0.78539816
 dot  :0.04986655
Ymin  :-1.5
 max  :1.5
```

2. To draw the function press:
   - **F6** (DRAW).



3. To evaluate the gradient of the tangent at any point, press:
   - **SHIFT**
   - **MENU** (SET UP).

   Scroll down to Derivative and press:
   - **F1** (On)
   - **EXE**
   - **F6** (DRAW)
   - **SHIFT**
   - **F1** (Trace).



```
Y1=sin X
dY/dX=-1
X=3.141592654   Y=0
```

4. Using the left and right arrow keys to display the derivative at any point, complete the table at right.

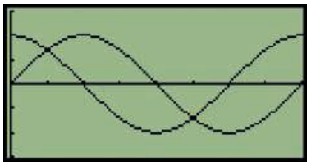
X	
0	
$\frac{\pi}{4}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	
π	
$\frac{5\pi}{4}$	
$\frac{3\pi}{2}$	
$\frac{7\pi}{4}$	

5. To graph the derivative of sin x, press:
   - **EXIT**
   - **OPTN**
   - **F2** (CALC)
   - **F1** (d/dx).

   Complete the entry line as:

   $Y2 = \text{d}/\text{d}x(\sin x)$

   then press:
   - **EXE**
   - **F6** (DRAW).

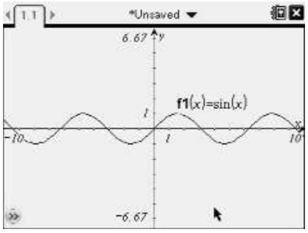


For the TI-Nspire CAS

1. To draw the graph of sin x, open a Graphs page.

   Complete the entry line as:

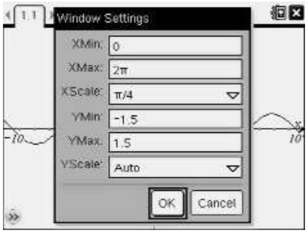
   $f1(x) = \sin(x)$.



2. To adjust the window settings, press:
   - MENU menu
   - 4: Window/Zoom 4
   - 1: Window settings 1.

   Enter the values in the fields as shown, and then select OK.

   Press Tab tab to move between fields.


Window Settings

XMin: 0

XMax: 2π

XScale: π/4

YMin: -1.5

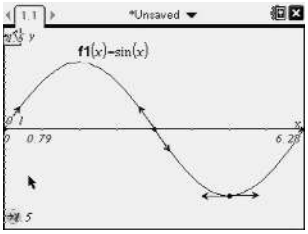
YMax: 1.5

YScale: Auto

3. To draw a tangent to the curve at $x = 0$, press:
   - MENU menu
   - 7: Points & Lines 7
   - 7: Tangent 7.

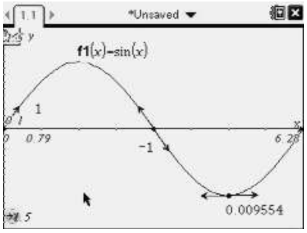
   Move the pointer until it hovers over the point (0, 0) and a tangent line appears; then press ENTER enter.

   Repeat this step to draw tangents at $x = \pi$ and $x = \frac{3\pi}{2}$.



4. To find the gradient of these tangents, press:
   - MENU menu
   - 8: Measurement 8
   - 3: Slope 3.

   Move the pointer over the tangent; the tangent line will flash and the value of the gradient for that tangent will appear. Press ENTER enter and then Esc esc. Repeat this for the other tangents.



5. Repeat steps 3 and 4 for differing values of x to complete the table at right.

   Note: To find the values of $\frac{dy}{dx}$ algebraically, open a Calculator page.

   Press:
   - Menu menu
   - 4: Calculus 4
   - 1: Derivative 1.

   Complete the entry line as:

   $\frac{d}{dx}(\sin(x)) \mid x = 0$

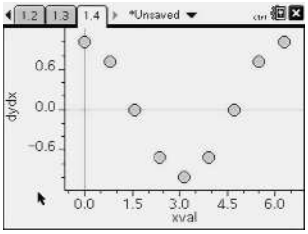
   then press ENTER enter. Repeat for all values of x.

X	$\frac{dy}{dx}$
0	
$\frac{\pi}{4}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	
π	
$\frac{5\pi}{4}$	
$\frac{3\pi}{2}$	
$\frac{7\pi}{4}$	
2π	

6. To draw a scatterplot of this data, open a Lists & Spreadsheet page.

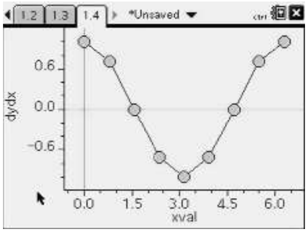
   Label Column A as xval and Column B as dydx and enter the values from the table.

   Open a Data & Statistics page.



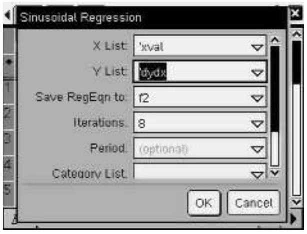
Tab to the x-axis and click to place xval as the independent variable. Tab to the y-axis to place dydx as the dependent variable.

7. To create an x–y line, press:
   - MENU menu
   - 1: Plot Type 1
   - 6: XY Line Plot 6.



8. To obtain an equation for this curve, return to the Lists & Spreadsheets page (1.2), then press:
   - MENU menu
   - 4: Statistics 4
   - 1: Stat Calculations 1
   - C: Sinusoidal Regression C.

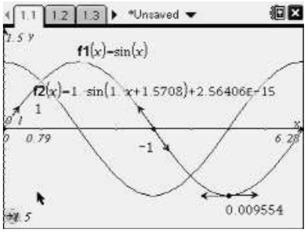
   Enter the values in the fields as shown and then select OK.
   This will paste the equation into $f2(x)$.


Sinusoidal Regression

X List: xval
Y List: dydx
Save RegEqn to: f2
Iterations: 8
Period: (optional)
Category List:

9. Press:
   - Ctrl ctrl
   - Back Arrow ◀
   - ENTER enter.

   The graph of the gradient function will appear.



Identifying the graph

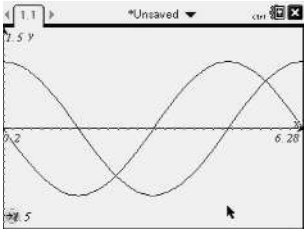
By comparing the graphics calculator displays and the screen dumps above, you should recognise the graph of $y = \frac{d}{dx}(\sin x)$ as being the same as $y = \cos x$.

$$\frac{d}{dx}(\sin x) = \cos x, \text{ when } x \text{ is in radians.}$$

Repeat the previous graphics calculator exercise to draw the graph of $y = \cos x$ and its derivative.

For the graphs of $y = \cos x$ and its derivative, you should have obtained the screen display at right.

The derivative of $y = \cos x$ is a reflection of $y = \sin x$ in the x-axis. We can therefore conclude that the derivative of $y = \cos x$ is $-\sin x$.



$$\frac{d}{dx}(\cos x) = -\sin x, \text{ when } x \text{ is in radians.}$$

Any function that involves a sin or a cos function can be differentiated using the above two rules together with the chain rule.

WORKED Example 1

Find the derivative of $y = \sin 5x$.

THINK	**WRITE**
1 Write the equation.	$y = \sin 5x$
2 Express u as a function of x and find $\frac{du}{dx}$.	Let $u = 5x$, so $\frac{du}{dx} = 5$
3 Express y as a function of u and find $\frac{dy}{du}$.	$y = \sin u$, so $\frac{dy}{du} = \cos u$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = 5\cos u$
5 Replace u with $5x$.	$= 5\cos 5x$

This example shows that if $f(x) = \sin ax$, then $f'(x) = a\cos ax$.
Similarly if $f(x) = \cos ax$, then $f'(x) = -a\sin ax$.

WORKED Example 2

eBook plus

Tutorial:
Worked example 2
int-0555

Differentiate $y = \cos(x^2 + 2x - 3)$:

a by hand **b using the TI-Nspire CAS calculator.**

THINK	**WRITE/DISPLAY**
a 1 Write the equation.	**a** $y = \cos(x^2 + 2x - 3)$
2 Express u as a function of x and find $\frac{du}{dx}$.	Let $u = x^2 + 2x - 3$, so $\frac{du}{dx} = 2x + 2$
3 Express y as a function of u and find $\frac{dy}{du}$.	$y = \cos u$, so $\frac{dy}{du} = -\sin u$
4 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = -\sin u \cdot (2x + 2)$
5 Replace u with the part in brackets in the rule and simplify.	$= -2(x + 1)\sin(x^2 + 2x - 3)$

For the TI-Nspire CAS

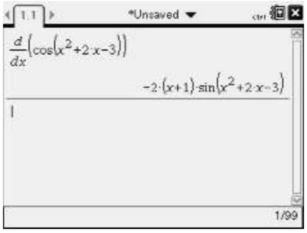
b 1 On a Calculator page, press:
- Menu menu
- 4: Calculus 4
- 1: Derivative 1.

Complete the entry line as:

$$\frac{d}{dx}(\cos(x^2 + 2x - 3))$$

then press ENTER enter.

b



$\frac{d}{dx}(\cos(x^2+2\cdot x-3))$

$-2\cdot(x+1)\cdot\sin(x^2+2\cdot x-3)$

2 Write the solution.

$$\frac{dy}{dx} = {}^{-}2(x + 1)\sin(x^2 + 2x - 3)$$

This example shows that the chain rule can be applied as follows.
If $f(x) = \sin[g(x)]$ then $f'(x) = g'(x)\cos[g(x)]$.
If $f(x) = \cos[g(x)]$ then $f'(x) = -g'(x)\sin[g(x)]$.

remember

1. If $f(x) = \sin x$ then $f'(x) = \cos x$
2. If $f(x) = \cos x$ then $f'(x) = -\sin x$
3. If $f(x) = \sin ax$ then $f'(x) = a \cos ax$
4. If $f(x) = \cos ax$ then $f'(x) = -a \sin ax$
5. If $f(x) = \sin [g(x)]$ then $f'(x) = g'(x) \cos [g(x)]$
6. If $f(x) = \cos [g(x)]$ then $f'(x) = -g'(x) \sin [g(x)]$

EXERCISE 6A The derivatives of sin x and cos x

WORKED Example 1

1 Find the derivative of each of the following:

a $y = \sin 8x$ **b** $y = \sin (-6x)$ **c** $y = \sin x$

d $y = \sin \frac{x}{3}$ **e** $y = \sin \left(-\frac{x}{2}\right)$ **f** $y = \sin \frac{2x}{3}$

2 Differentiate each of the following:

a $y = \cos 3x$ **b** $y = \cos (-2x)$ **c** $y = \cos \frac{x}{3}$

d $y = \cos 21x$ **e** $y = \cos (-7x)$ **f** $y = \cos \frac{\pi x}{4}$

g $y = \cos \left(-\frac{x}{8}\right)$ **h** $y = \cos \frac{2x}{5}$

3 **multiple choice**

a The derivative of $\sin 6x$ is:

A $6 \cos 6x$ **B** $6 \cos x$ **C** $6 \sin x$ **D** $-6 \cos 6x$ **E** $\frac{1}{6} \cos 6x$

b The derivative of $\cos 4x$ is:

A $4 \sin 4x$ **B** $4 \sin x$ **C** $-4 \sin x$ **D** $4 \cos 4x$ **E** $-4 \sin 4x$

c The derivative of $\sin (-4x)$ is:

A $4 \cos (-4x)$ **B** $4 \cos (-x)$ **C** $-4 \cos 4x$ **D** $-4 \cos (-4x)$ **E** $-4 \sin 4x$

d The derivative of $\cos (-8x)$ is:

A $8 \cos (-8x)$ **B** $8 \sin (-8x)$ **C** $-8 \sin (-8x)$ **D** $-8 \sin (-x)$ **E** $8 \sin (-x)$

e The derivative of $\sin \frac{x}{5}$ is:

A $5 \cos \frac{x}{5}$ **B** $-\frac{1}{5} \cos \frac{x}{5}$ **C** $\frac{1}{5} \cos \frac{x}{5}$ **D** $\frac{1}{5} \sin \frac{x}{5}$ **E** $\frac{1}{5} \cos x$

4 If $y = \sin (4x + 3)$ is expressed as $y = \sin u$, find:

a $\frac{dy}{du}$ **b** $\frac{du}{dx}$ **c** $\frac{dy}{dx}$ using the chain rule.

5 If $y = \cos(3x + 1)$ is expressed as $y = \cos u$, find:

a $\frac{dy}{du}$ **b** $\frac{du}{dx}$ **c** $\frac{dy}{dx}$ using the chain rule.

6 Differentiate each of the following:

a $y = \sin(2x + 3)$ **b** $y = \sin(6 - 7x)$ **c** $y = \sin(5x - 4)$

d $y = \sin\left(\frac{3x+2}{4}\right)$ **e** $y = \sin\left(\frac{8-7x}{3}\right)$ **f** $y = 5\pi \sin 2\pi x$

g $y = -4 \sin \frac{3x}{8}$

7 Differentiate each of the following:

a $y = \cos(3x - 2)$ **b** $y = \cos(4x + 7)$ **c** $y = \cos(8 - x)$

d $y = \cos(6 - 5x)$ **e** $y = \cos\left(\frac{2x+3}{3}\right)$ **f** $y = \cos\left(\frac{4x-1}{5}\right)$

g $y = 4\pi \cos 10\pi x$ **h** $y = -6 \cos(-2x)$

WORKED Example 2

8 Find the derivative of each of the following.

a $\cos(x^2 - 4x + 3)$ **b** $\sin(10 - 5x + x^2)$

c $\sin(e^x)$ **d** $\cos(x^2 + 7x)$

e $\cos(4x - x^2)$ **f** $\sin(x^2 + 3x)$

g $\cos(\log_e x)$ **h** $\sin(e^{4x})$

i $\sin(\log_e 3x)$ **j** $\cos\left(\frac{1}{x}\right)$

k $\sin[\log_e(2x - 1)]$ **l** $\sin(3e^{2x})$

m $\cos(2e^{3x})$ **n** $3\cos(\log_e 10x)$

o $4\sin(x^3 + 2x^2)$ **p** $-8\sin\left(-\frac{3x}{5}\right)$

q $-2\cos\left(\frac{x}{4}\right)$ **r** $\cos(x^2 + 2x) + \sin(3x - 9)$

s $\sin(x^2 - 4) - 3\cos(8 - 3x)$ **t** $-5\cos\left(\frac{3x+7}{10}\right) + 6\sin\left(\frac{5-4x}{3}\right) + 4x^3$

9 If $f(x) = 3\sin(x^2 + x)$ find $f'(1)$ (answer correct to 3 decimal places).

10 Find the gradient of the curve $g(x) = 2\cos(x^3 - 3x)$ at the point where $x = 0$.

11 For each of the following functions find:

i $f'(x)$ and

ii the exact value of $f'\left(\frac{\pi}{6}\right)$.

a $f(x) = e^{\sin x}$ **b** $f(x) = e^{\cos x}$ **c** $f(x) = \log_e(\sin x)$ **d** $f(x) = \log_e(\cos x)$

Further differentiation of trigonometric functions

eBook plus

Interactivity:
Mixed problems on differentiation
int-0252

We need to be able to use the product rule as applied to functions that have a periodic factor or factors.

WORKED Example 3

Find $\frac{d}{dx}(x \sin x)$.

THINK	WRITE
① Write the equation.	Let $y = x \sin x$ $= u \cdot v$
② Identify u and v, two functions of x that are multiplied together.	where $u = x$ and $v = \sin x$.
③ Find $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \cos x$
④ Find $\frac{dy}{dx}$ using the product rule.	$\frac{dy}{dx} = x \cos x + \sin x \cdot 1$
⑤ Simplify where possible.	$= x \cos x + \sin x$

WORKED Example 4

Find $\frac{d}{dx}\left(\frac{x}{\cos x}\right)$: a by hand b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a ① Rewrite the equation using a negative index.	**a** Let $y = x\,(\cos x)^{-1}$ $= u \cdot v$
② Identify u and v, two functions of x that are multiplied together.	where $u = x$ and $v = (\cos x)^{-1}$.
③ Find $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -1(\cos x)^{-2}.(-\sin x)$ $= \frac{\sin x}{\cos^2 x}$
④ Find $\frac{dy}{dx}$ using the product rule.	$\frac{dy}{dx} = x\,\frac{\sin x}{\cos^2 x} + (\cos x)^{-1}$
⑤ Simplify where possible.	$= \frac{x \sin x}{\cos^2 x} + \frac{1}{\cos x}$ $= \frac{x \sin x + \cos x}{\cos^2 x}$ So $\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{x \sin x + \cos x}{\cos^2 x}$

Continued over page →

THINK

For the TI-Nspire CAS

b ① On a Calculator page, press:

- MENU menu
- 4: Calculus 4
- 1: Derivative 1.

Complete the entry line as:

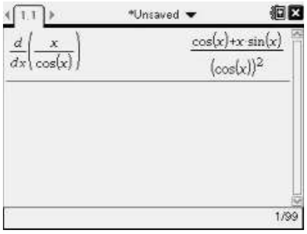
$$\frac{d}{dx}\left(\frac{x}{\cos(x)}\right)$$

then press ENTER enter.

② Write the solution.

WRITE/DISPLAY

b



$\frac{d}{dx}\left(\frac{x}{\cos(x)}\right)$ $\frac{\cos(x)+x\cdot\sin(x)}{(\cos(x))^2}$

$$\frac{dy}{dx}=\frac{\cos x+x\sin x}{\cos^2 x}$$

remember

1. Use the product rule when the function is the product of two simpler functions, which we call u and v.
2. When the function is written in fraction form, write the function as a product by raising the denominator to a power of -1.
3. The product rule is: if $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

EXERCISE 6B Further differentiation of trigonometric functions

WORKED Example 3

1 Find $\frac{d}{dx}(x\cos x)$.

2 Use the product rule to differentiate each of the following.

a $y = x^2 \sin x$ **b** $y = 3x \sin x$ **c** $y = x^5 \cos(3x + 1)$

d $y = \sin x \cos x$ **e** $y = 8 \sin 5x \log_e 5x$ **f** $y = 5 \cos 2x \sin x$

g $y = \sin\frac{4x}{3}\cos x$ **h** $y = 2x^{-3}\sin(2x + 3)$ **i** $y = 4e^{-5x}\sin(2 - x)$

j $y = \frac{1}{\sqrt{x}}\cos 6x$ **k** $y = \sin x \log_e x$ **l** $y = \pi x \cos 2\pi x$

3 **multiple choice**

The derivative of $f(x) = x^2 \sin 2x$ is:

A $f'(x) = 2x \cos 2x$

B $f'(x) = 4x \cos 2x$

C $f'(x) = 2x \sin 2x + x^2 \cos 2x$

D $f'(x) = 2x \sin 2x + 2x^2 \cos 2x$

E $f'(x) = 2x \sin x + 2x^2 \cos x$

4 Find $g'(0)$ if $g(x) = 5e^{2x} \cos 4x$.

5 Find the value of $f'(-2)$ if $f(x) = (x^2 + 2) \sin (4 - 3x)$ (answer correct to 3 decimal places).

WORKED Example 4

6 Find $\frac{d}{dx}\left(\frac{x}{\sin x}\right)$.

7 Find the derivative of each of the following.

a $y = \frac{\sin x}{x}$

b $y = \frac{\sin 4x}{\cos 2x}$

c $y = \frac{\cos x}{x}$

d $y = \frac{\cos x}{e^x}$

e $y = \frac{\sin \sqrt{x}}{x}$

f $y = \frac{2 \cos (3 - 2x)}{x^2}$

8 **multiple choice**

The derivative of $f(x) = \frac{\sin 4x}{4x + 1}$ is:

A $f'(x) = \frac{4(4x + 1) \cos x - 4 \sin 4x}{(4x + 1)^2}$

B $f'(x) = \frac{(4x - 1) \cos 4x - 4 \sin 4x}{(4x + 1)^2}$

C $f'(x) = \frac{4(4x - 1) \cos 4x - 4 \sin 4x}{4x + 1}$

D $f'(x) = \frac{4(4x - 1) \cos 4x - 4 \sin 4x}{(4x + 1)^2}$

E $f'(x) = \frac{4 \sin 4x - 4(4x + 1) \cos 4x}{(4x + 1)^2}$

9 By writing $\tan x$ as $\frac{\sin x}{\cos x}$ find $\frac{d}{dx}(\tan x)$.

10 Evaluate $f'(-\pi)$, $f'(0)$, $f'(\pi)$ for each of the following and explain the values of x for which the function exists.

a $f(x) = \ln (\sin x)$

b $f(x) = \sin (\ln x)$

c $f(x) = e^x \sin x$

d $f(x) = \ln x. \sin x$

(Remember that $\ln x = \log_e x$)

11 Consider $y = x \sin x \cos x$ in the domain $-2\pi \le x \le 2\pi$.

a Evaluate $\frac{dy}{dx}$.

b Show that at $x = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{1}{2}$.

c Investigate **a** and **b** using a graphing program or graphics calculator.

eBook plus

Digital doc:
WorkSHEET 6.1

Applications of differentiation

In previous work we have learned to use differentiation to find the equation of the tangent and normal to a curve, as well as to solve problems involving maximums and minimums. These skills can be applied to functions that involve the use of the periodic functions.

WORKED Example 5

Find the equation of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$:
a by hand **b using the TI-Nspire CAS calculator.**

THINK	WRITE/DISPLAY
a 1 Write the equation.	**a** $y = \sin x$
2 Differentiate the function.	$\frac{dy}{dx} = \cos x$
3 Substitute $x = \pi$ to find the gradient of the tangent.	At $x = \pi$ $\frac{dy}{dx} = \cos \pi$ $= -1$ The gradient of the tangent is -1.
4 Substitute $(\pi, 0)$ for (x_1, y_1) and $m = -1$ into the rule for the equation of a straight line, $y - y_1 = m(x - x_1)$.	The equation of the tangent at $(\pi, 0)$ is $y - 0 = -1(x - \pi)$.
5 Rearrange the rule to a simple form.	$y = -x + \pi$ $x + y - \pi = 0$

For the TI-Nspire CAS

b 1 On a Calculator page, press:
- MENU menu
- 4: Calculus 4
- 9: Tangent Line 9.

Complete the entry line as:

tangentLine(sin(x), x, π)

then press ENTER enter.

b

```
tangentLine(sin(x),x,π)                π−x
```

2 Write the solution. $y = \pi - x$

Periodic motions can be modelled by a trigonometric equation. By differentiating these functions we are then able to solve problems relating to maximums and minimums.

Remember that the following steps are used when solving a maximum/minimum problem.

1. Find $f'(x)$ to obtain the gradient function.
2. Solve for x where $f'(x) = 0$ (to find the values of x where the maximums or minimums occur).
3. Apply the first derivative test to see if the point is a maximum or minimum.
4. Substitute the appropriate value of x into $f(x)$ to obtain the maximum or minimum.

WORKED Example 6

The population of a colony of frogs rises and falls according to the breeding season. The population can be modelled by the equation $P(t) = 100 \sin \frac{\pi t}{2} + 500$, where t is the number of months since the beginning of the year.

Find:

a **the population at the beginning of the year**

b **the first time at which the population is greatest.**

THINK / **WRITE**

a ① Write the equation.

$$P(t) = 100 \sin \frac{\pi t}{2} + 500$$

② Substitute $t = 0$.

$$P(0) = 100 \sin \frac{\pi \cdot 0}{2} + 500$$
$$= 500$$

③ Give a written answer.

At the beginning of the year there are 500 frogs.

b ① Write the function.

$$P(t) = 100 \sin \frac{\pi t}{2} + 500$$

② Differentiate the function.

$$P'(t) = \frac{100\pi}{2} \cos \frac{\pi t}{2}$$
$$= 50\pi \cos \frac{\pi t}{2}$$

③ Solve $P'(t) = 0$.

For maximum or minimum $P'(t) = 0$.

$$P'(t) = 50\pi \cos \frac{\pi t}{2} = 0 \text{ when } \cos \frac{\pi t}{2} = 0.$$

④ Take the first two values to see which is the maximum and which is the minimum.

$$\frac{\pi t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$t = 1, 3$$

⑤ Use the first derivative test to check for a maximum.

t	0	1	2
$P'(t)$	50π	0	-50π
Slope	/	–	\

Maximum at $t = 1$

t	2	3	4
$P'(t)$	-50π	0	50π
Slope	\	–	/

Minimum at $t = 3$

⑥ Answer the question.

The maximum population first occurs after 1 month.

remember

1. Differentiating the function will find the gradient of the tangent to the curve at any point.
2. If m is the gradient of the tangent then $\frac{-1}{m}$ is the gradient of the normal.
3. The equation of a straight line passing through the point (x_1, y_1) and having a gradient m is:
   $y - y_1 = m(x - x_1)$.
4. Problems involving maximums and minimums are solved by differentiating the function and solving the derivative equal to zero.

EXERCISE 6C Applications of differentiation

WORKED Example 5

1 Find the equation of the tangent to the curve $y = \cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

2 Find the equation of the tangent and the normal to the curve $y = 2 \sin x$ at the point $(\pi, 0)$.

3 For each of the following functions, find the equation of:

i the tangent
ii the normal at the given value of x.

a $y = \sin 2x, x = \frac{\pi}{3}$

b $y = 3 \cos \left(\frac{x}{2}\right), x = \pi$

c $y = \sin \left(2x + \frac{\pi}{4}\right), x = 0$

d $y = 2 \cos (x - \pi)$ at $x = \frac{3\pi}{4}$

4 Find the equation of the tangent to the curve $y = 3 \cos 2x$ at the point where $x = \frac{\pi}{4}$. Find the points A and B where the tangent cuts the x- and y-axis respectively.

5 Find the equation of the tangent and normal to the curve $y = x \sin x$ at $x = \frac{\pi}{2}$.

6 Find the points on the curve $y = \sin \left(\frac{x}{2}\right)$ where the tangent is parallel to the line $y = x$.

WORKED Example 6

7 The number of visitors to a holiday resort in each month can be found using the equation $V = 500 + 200 \cos \frac{\pi t}{6}$, where t is the number of months since the beginning of the year.

a Find the initial number of visitors.
b Find when the minimum number of visitors will occur.

8 The depth of water in an inlet can be given by the formula $D = 2.5 + 0.5 \sin \frac{\pi t}{3}$ where t is the number of hours since midnight.

a Find the depth of water in the inlet at midnight.
b Find the time at which the depth of water is greatest.
c Find the minimum depth of water in the inlet.

9 The velocity, v cm/s, of an engine's piston t seconds after the engine is started is approximated by $v = 0.8 \sin 2t$.

a Find the maximum velocity and the time at which it first occurs.
b Find the minimum velocity and the time at which it first occurs.

10 The length of a snake, L cm, at any time t weeks after it is born is modelled as:

$$L = 12 + 6t + 2 \sin \frac{\pi t}{4}, \qquad 0 \leq t \leq 20.$$

Find:

a the length at **i** birth **ii** 20 weeks
b R, the rate of growth, at any time, t
c the maximum and minimum growth rate.

Halley's comet

Halley's comet is a comet that revolves around the sun in an elliptical path.

1 Use an encyclopedia or the internet to find when Halley's comet was last visible and when it will next be visible from Earth. This becomes the period of the function.

2 Find the distance of the comet from the sun when it is at the closest point of its orbit, and the comet's distance from the sun when it is at the furthest point of its orbit. Use these to find the amplitude and the equilibrium position of the function.

3 Model the distance function of Halley's comet from the sun using the form $y = A \sin (Bx + C) + D$.

4 Use your graphics calculator to graph the function.

Kinematics

eBook*plus*

Interactivity:
Motion graphs (kinematics)
int-0267

Consider the case of a body that is moving in a straight line. The displacement, x, is the distance that the body is at any time, t, from its starting point. The velocity, v, is the rate at which the displacement is changing. For example a velocity of 2 m/s means that there is a change of 2 m in the displacement for every second of time that has elapsed.

Now because velocity is the change in x per unit of t, we can say that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$.

Similarly, the acceleration, a, of the body is the rate of change in the velocity with respect to t. We can therefore say that $a = \frac{\mathrm{d}v}{\mathrm{d}t}$.

Many objects move in such a way that they can be described as periodic in nature, so the results just described allow us to solve problems that relate to them. Examples of this type of motion are tides, the swing of a pendulum and the way an object will bob up and down when attached to a spring.

This type of motion is called *simple harmonic motion* (SHM). Under SHM the displacement, x, of a body from its equilibrium (or central) position is given by:

$$x = A \sin Bt \qquad \text{or} \qquad x = A \cos Bt$$

Three preliminary characteristics can be used to describe the motion. They are:

1. The period: The time interval, T, required for the motion to repeat itself is called the *period* and is generally measured in seconds.
2. The frequency: The number of complete oscillations per unit of time is called the *frequency*, f. The unit for frequency is 'per second' or hertz (Hz). The frequency is related to the period in a simple way since if it takes $\frac{1}{10}$ of a second for a motion to repeat itself then the motion will occur 10 times per second; that is, the frequency is the reciprocal of the period. Thus:

$$f = \frac{1}{T}$$

3. The amplitude: The magnitude of the maximum displacement from the fixed or mean position, A, is called the *amplitude* of the motion. The size of the vibration is described by this quantity.

For example, a mass on the end of a vertically mounted spring may execute 42 vibrations in 60 seconds with the highest point 80 cm above the lowest point. In this case the frequency would be 42 oscillations in 60 seconds: $f = 0.7$ Hz, which is a period of $\frac{1}{0.7} \cong 1.43$ seconds. The amplitude of the motion would be 40 cm or half the range of the motion.

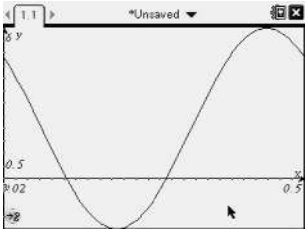
WORKED Example 7

A body is executing simple harmonic motion with a position $x(t)$ given by the equation:

$$x(t) = 4\cos\left(4\pi t + \frac{\pi}{4}\right) + 2.$$

a **What is the x-value for the mean or centre of motion?**
b **What are the amplitude and range of the motion?**
c **What are the period and frequency of the motion?**
d **What is the initial position of the body (that is, at $t = 0$)?**
e **Using a graphics calculator, draw $x(t)$ for one cycle.**

THINK	WRITE/DISPLAY
a A trigonometric function has an equilibrium position of 0. However, this function is translated vertically 2 units.	**a** The centre of the motion is at $x = 2$.
b ① The amplitude is the coefficient of the trigonometric function.	**b** The amplitude is 4 units.
② The range of the motion is found by adding and subtracting the amplitude to the centre of the motion.	Lower limit $= 2 - 4$ $= -2$ Upper limit $= 2 + 4$ $= 6$ Range $= -2 \le x \le 6$
c ① Write the formula for the period.	**c** $T = \frac{2\pi}{B}$
② Substitute $B = 4\pi$.	$T = \frac{2\pi}{4\pi}$

THINK	**WRITE**
③ Calculate.	$= \frac{1}{2}$
④ Write the formula for frequency.	$f = \frac{1}{T}$
⑤ Substitute $T = \frac{1}{2}$.	$f = \frac{1}{\frac{1}{2}}$
⑥ Calculate.	$f = 2$
⑦ Write the answer.	Period is $\frac{1}{2}$ and frequency is 2.
d ① Write the equation.	**d** $x(t) = 4\cos\left(4\pi t + \frac{\pi}{4}\right) + 2$
② Substitute $t = 0$.	$x(0) = 4\cos\left(\frac{\pi}{4}\right) + 2$
③ Calculate.	$= 2\sqrt{2} + 2$
④ Give a written answer.	The initial position of the particle is at $x = 2\sqrt{2} + 2$.
e Use a graphics calculator to draw the function.	**e** 

To find the answer in the previous worked example we had only to use our knowledge of periodic functions. In many other cases we will need to apply calculus as well.

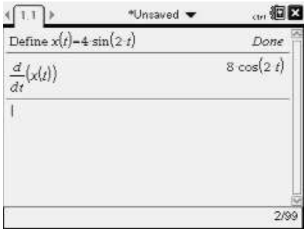
WORKED Example 8

The displacement (in metres) of a particle after t seconds is given by the equation $x = 4 \sin 2t$. Complete the following: a by hand b using the TI-Nspire CAS calculator.

- **i Find an expression for the velocity of the particle v.**
- **ii Find the time at which the particle is first at rest (that is, $v = 0$).**
- **iii Find an expression for the acceleration of the particle.**
- **iv Find the first two occasions when the acceleration is equal to 0.**

THINK	**WRITE/DISPLAY**
a i ① Write the function.	**i** $x = 4 \sin 2t$
② To find the velocity, differentiate the displacement function.	$v = 8 \cos 2t$

Continued over page

THINK	**WRITE/DISPLAY**
ii ① Set the velocity equation equal to 0 and find the smallest positive solution.	**ii** $v = 8\cos 2t = 0$ $\cos 2t = 0$ $2t = \dfrac{\pi}{2}$ $t = \dfrac{\pi}{4}$
② Give a written answer.	The particle first comes to rest after $\dfrac{\pi}{4}$ seconds.
iii ① Write the velocity equation.	**iii** $v = 8\cos 2t$
② To find the acceleration, differentiate the velocity equation.	$a = -16\sin 2t$
iv ① Set the acceleration equation equal to 0 and solve finding the first two solutions.	**iv** $a = -16\sin 2t = 0$ $\sin 2t = 0$ $2t = 0, \pi$ $t = 0, \dfrac{\pi}{2}$
② Give a written answer.	The acceleration is 0 initially and again after $\dfrac{\pi}{2}$ seconds.
For the TI-Nspire CAS	
b i ① On a Calculator page, press: • MENU menu • 1: Actions 1 • 1: Define 1. Complete the entry line as: Define $x(t) = 4\sin(2t)$. Press: • ENTER enter • MENU menu • 4: Calculus 4 • 1: Derivative 1. Complete the entry line as: $\dfrac{d}{dx}(x(t))$ then press ENTER enter.	**b i**  Define $x(t)=4\cdot\sin(2\cdot t)$ Done $\dfrac{d}{dt}(x(t))$ $8\cdot\cos(2\cdot t)$
② Write the solution.	$v(t) = 8\cos(2t)$

THINK | **WRITE/DISPLAY**

ii ① To solve $v(t) = 0$, press:

- MENU (menu)
- 3: Algebra ⟨3⟩
- 1: Solve ⟨1⟩.

Complete the entry line as:

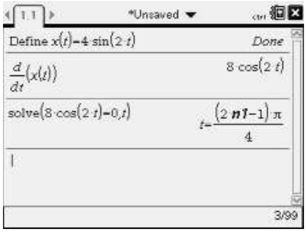
solve(8 cos(2t) = 0, t)

then press ENTER (enter).

The calculator has given the general solution to this equation.

② Write the solution.

ii



Define x(t)=4·sin(2·t) Done

$\frac{d}{dt}(x(t))$ 8·cos(2·t)

solve(8·cos(2·t)=0,t) $t=\frac{(2\cdot n1-1)\cdot\pi}{4}$

3/99

Solve 8 cos(2t) = 0 for t.

$$t = \frac{(2k-1)\pi}{4}$$

When $k = 1$, $t = \frac{\pi}{4}$.

Therefore the particle first comes to rest after $\frac{\pi}{4}$ seconds.

iii ① To find the acceleration, press:

- MENU (menu)
- 4: Calculus ⟨4⟩
- 1: Derivative ⟨1⟩.

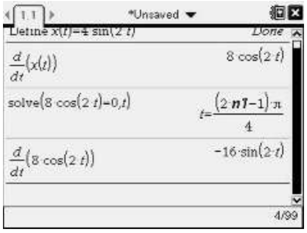
Complete the entry line as:

$$\frac{d}{dt}(8\cos(2t))$$

then press ENTER (enter).

② Write the solution.

iii



Define x(t)=4·sin(2·t) Done

$\frac{d}{dt}(x(t))$ 8·cos(2·t)

solve(8·cos(2·t)=0,t) $t=\frac{(2\cdot n1-1)\cdot\pi}{4}$

$\frac{d}{dt}(8\cdot\cos(2\cdot t))$ −16·sin(2·t)

4/99

$a(t) = {}^{-}16 \sin(2t)$

iv ① To solve $a(t) = 0$ for t, press:

- MENU (menu)
- 3: Algebra ⟨3⟩
- 1: Solve ⟨1⟩.

Complete the entry line as:

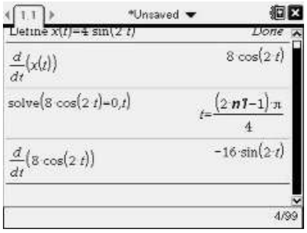
solve($^{-}$16 sin(2t) = 0, t)

then press ENTER (enter).

The calculator has given the general solution to this equation.

② Write the solution.

iv



Define x(t)=4·sin(2·t) Done

$\frac{d}{dt}(x(t))$ 8·cos(2·t)

solve(8·cos(2·t)=0,t) $t=\frac{(2\cdot n1-1)\cdot\pi}{4}$

$\frac{d}{dt}(8\cdot\cos(2\cdot t))$ −16·sin(2·t)

4/99

Solve $^{-}16 \sin(2t) = 0$ for t.

$$t = \frac{k\pi}{2}$$

When $k = 0$, $t = 0$.

When $k = 1$, $t = \frac{\pi}{2}$.

The acceleration is 0 when $t = 0$ and again after $t = \frac{\pi}{2}$ seconds.

Because acceleration is the derivative of velocity we can say that the maximum and minimum velocities will occur when the acceleration is equal to 0. Generally, the minimum velocity occurs when the particle is moving fastest in a negative direction. In such cases you may be asked for the maximum speed.

Note that velocity has direction as well as magnitude. For example, a velocity of −4 m/s means that the particle is travelling at 4 m/s in the negative direction. Speed has no direction and so this particle has a speed of 4 m/s.

WORKED Example 9

A body moves such that at time t seconds its position in metres is given by $x(t) = 5 \sin 4t$.

a Find the maximum speed of the body.

b Determine the first four times at which the body is moving with a speed of 10 m/s.

THINK	WRITE
a ① Write the equation.	**a** $x(t) = 5 \sin 4t$
② Find the velocity equation by differentiating the displacement equation.	$v(t) = 20 \cos 4t$
③ Find the acceleration equation by differentiating the velocity equation.	$a(t) = -80 \sin 4t$
④ The maximum speed occurs when acceleration equals 0. (We need the first solution only.)	For maximum speed, $a(t) = 0$ $-80 \sin 4t = 0$ $\sin 4t = 0$ $4t = 0$ $t = 0$
⑤ Substitute $t = 0$ into the velocity equation to find this maximum speed. (If v is negative, speed is positive.)	$v(t) = 20 \cos 4t$ $v(0) = 20 \cos 4(0)$ $= 20$
⑥ Give a written answer.	The maximum speed is 20 m/s.
b Set the velocity equation equal to ±10 and solve, finding the first four solutions.	**b** $v(t) = 20 \cos 4t$ $= \pm 10$ $\cos 4t = \pm\frac{1}{2}$ $4t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ $t = \frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{12}$

remember

1. A particle that moves in an oscillating fashion is said to move in *simple harmonic motion.*
2. The displacement, x, of a particle can be expressed as an equation in terms of t.
3. The velocity of the particle is $\frac{dx}{dt}$ and the acceleration of the particle is $\frac{dv}{dt}$.

EXERCISE 6D Kinematics

WORKED Example 7

1 A body is executing simple harmonic motion with a position $x(t)$ given by the equation:

$$x(t) = 4 \sin\left(4\pi t + \frac{\pi}{4}\right) + 2.$$

- a What is the x-value for the mean or centre of motion?
- b What are the amplitude and range of the motion?
- c What are the period and frequency of the motion?
- d What is the initial position of the body (that is, at $t = 0$)?
- e Using a graphics calculator or suitable graphing software, draw $x(t)$ for one cycle.

2 A body is executing simple harmonic motion with the following displacement equations:

i $x(t) = 4 \sin 3t + 1$ ii $x(t) = 5 \cos\left(\frac{t}{4} + \pi\right)$ iii $x(t) = 3 \cos(2t + 3) - 6$

In each case:

- a Sketch each of the three functions using a suitable graphing application or a graphics calculator.
- b Calculate the period.
- c Calculate the frequency.
- d Determine the centre of motion.
- e Find the amplitude of the motion.
- f Determine the initial position of the body.

3 A body is executing simple harmonic motion with the following displacement equations:

i $x(t) = 4 \sin(1.5t) + 1$ ii $x(t) = 5 \cos\left(\frac{t}{2} + \pi\right)$ iii $x(t) = 3 \sin(2t + 3)$.

In each case find expressions for the velocity and acceleration of the body as functions of time and sketch on the same axis $x(t)$, $v(t)$ and $a(t)$ using a graphing application or a graphics calculator.

WORKED Example 8

4 The displacement (in metres) of a particle after t seconds is given by the equation $x = 2 \cos 4t$.

- a Find an expression for the velocity, v, of the particle.
- b Find the time at which the particle is first at rest (that is, $v = 0$).
- c Find an expression for the acceleration of the particle.
- d Find the first two occasions when the acceleration is equal to 0.

5 The position of a particle $x(t)$ cm from the origin, O, moving in a straight line at any time, t seconds, is given by $x(t) = \sin 4t$. Find:

- a the initial position
- b the position after 3 seconds
- c the position after 4 seconds
- d the average velocity during the fourth second
- e the velocity at any time, t
- f the initial velocity
- g the velocity after 3 seconds
- h when and where the particle is stationary.

6 A mass is attached to the end of a spring. The spring is extended and the mass on the end of the spring begins to bob up and down in simple harmonic motion where its displacement at any time, t, from O, the equilibrium position, is given by the equation $x(t) = -10 \cos \pi t$.

a Find the initial position of the mass.
b Find equations for the velocity and acceleration of the mass.
c Find the times at which the particle is stationary.
d Find when the velocity is:
  i a maximum **ii** a minimum.
e Find the points where the acceleration is 0. What do you notice about the velocity at these points?

7 An object travelling in a straight line has its displacement (in metres) after t seconds given by:

$$x(t) = 2 \cos (3t - 1) + 3.$$

a Find the maximum and minimum displacement.
b Find when the velocity is first equal to 0.
c How long after it is first at rest is it next at rest?
d Find an expression for the acceleration.

8 Find the velocity and acceleration at any time, t, for each of the following.

a $x = 8 \sin \dfrac{\pi t}{4}$ **b** $x = 2t \cos 2t$

WORKED Example 9

9 A body moves such that at time, t seconds, its position in metres is given by $x(t) = 5 \cos 4t$.

a Find the maximum speed of the body.
b Determine the first four times the body is moving with a speed of 10 m/s.

10 A small buoy bobbing in the water moves in such a way that SMH is a good approximation to its vertical motion. The height of the buoy above the seabed varies from 1.4 to 1.8 m and the period of the motion is 3.5 s. Take the buoy to be at its mean position at $t = 0$.

a What is the maximum acceleration of the buoy?
b What is the maximum speed of the buoy and what is the height of the buoy above the seabed at this time?
c At what time is the speed of the buoy first equal to half its maximum speed?

11 A body is executing SHM about the origin of the x-axis. The equation for its motion is given by:

$$x(t) = 2.5 \sin (3.6t)$$

a State the amplitude and the period of the motion.
b What is the velocity when $t = 2.4$ s?
c What is the acceleration when $t = 4.7$ s?
d What is the acceleration when $x = 1.25$ m?
e What is the velocity when $x = 1.25$ m?
f What is/are the position/s of the body when it is travelling at a velocity of 1.0 m/s?
g When the acceleration of a body is -5.5 m/s^2, what is its speed?
h Sketch, using either a graphics calculator or a graphing program for a computer, $v(x)$ and $a(x)$ on the one axis. State the domain and range for both functions.

12 The height of water (in metres) at the entrance to a bay t hours after high tide is:

$$H = 10 + 2 \cos \frac{\pi t}{12}$$

Find:

a the rate of change of H at any time, t

b the rate of change of H:

  i 6 hours after high tide **ii** 15 hours after high tide **iii** 20 hours after high tide

c the minimum and maximum values of H and the times when they first occur after the initial high tide.

eBook*plus*

Digital doc:
WorkSHEET 6.2

Period and amplitude

1 Use your graphics calculator to graph the following functions.

$Y1 = \sin x \qquad Y2 = \frac{1}{2}\sin 2x \qquad Y3 = \frac{1}{3}\sin 3x$

2 By looking at the addition of ordinates, try to sketch the graph of

$$y = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x.$$

Check your result on your graphics calculator.

3 What are the period and amplitude of the resulting function?

4 If this series were to be continued, what would the resulting function look like? What would be the period and amplitude of this function?

Warehouse security

The REAC warehouse — checking its security

At the beginning of this chapter we looked at the Rastenburg Electrical Appliances Company, whose owners installed a searchlight to assist the night guard in protecting the warehouse. The front of the warehouse is 100 metres long and the searchlight takes 2 minutes to complete one full oscillation.

We are now going to examine the function that represents the movement of the searchlight across the front of the warehouse.

Take 0 to be the centre of oscillation, which will be the centre of the warehouse front; take x to be the displacement in metres from 0; and take t to be the time in minutes from when the searchlight is first switched on each night.

1 Find the amplitude of the function.

2 Remembering that the searchlight takes 2 minutes to complete one full oscillation, find the period of the oscillation.

3 Write a function in the form $x = A \sin Bt$ to describe the displacement of the light at any time, t.

4 Find functions for the velocity and the acceleration of the searchlight at any time, t.

5 For a potential intruder, the shortest path from the fence to the door of the warehouse is at the point where $x = 50$ metres. Find the times at which the searchlight is at this point and hence the maximum amount of time that an intruder would have to enter the building.

6 An intruder decides to try to cross into the warehouse at the point where the speed of the searchlight is greatest as this gives him the greatest chance of not being seen by the night guard. At what time does he attempt the crossing?

7 In your opinion, is the warehouse secure? When and where would you consider the warehouse to be most vulnerable to a break-in?

summary

The derivatives of sin x and cos x

- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\sin ax) = a\cos ax$
- $\dfrac{d}{dx}(\cos ax) = -a\sin ax$
- $\dfrac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$
- $\dfrac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$

Applications of the derivative of trigonometric functions

- The derivative of a function can be used to find the gradient of the tangent to the curve at any point.
- At the point (x_1, y_1) the gradient of the tangent $m = f'(x_1)$ and the gradient of the normal is $\dfrac{-1}{m}$.
- The equation of the tangent to the curve is found using $y - y_1 = m(x - x_1)$.
- Problems involving maximums and minimums are solved by differentiating the function and solving the derivative equal to zero.

Kinematics

- A particle that moves in an oscillating fashion is said to move in *simple harmonic motion* (SHM).
- The displacement of a particle from its equilibrium position, x, can be expressed in terms of t.
- The velocity of the particle at any time, t, is equal to $\dfrac{dx}{dt}$ and the acceleration of the particle is $\dfrac{dv}{dt}$.

CHAPTER review

1 If $y = \cos 8x$ then find $\frac{dy}{dx}$. **6A**

2 If $y = 2 \sin (2x + 3)$ then find $\frac{dy}{dx}$. **6A**

3 Find y' given that: **6A**

- **a** $y = \sin x$
- **b** $y = 4 \cos x$
- **c** $y = 3 \sin 4x$
- **d** $y = 4 \cos 3x$
- **e** $y = \sin \frac{3x}{2}$
- **f** $y = \frac{1}{2} \cos \frac{x}{2}$

4 Find $f'(x)$ if $f(x) = 3 \sin 2x$ and hence find $f'(2\pi)$. **6A**

5 Differentiate $e^{\sin 2x}$. **6A**

6 Find $\frac{d}{dx}(3x \sin 3x)$. **6B**

7 Find $f'(x)$ if $f(x) = \frac{\cos x}{x}$. **6B**

8 Find the derivative of: **6B**

- **a** $\frac{x}{\sin x}$
- **b** $3x \cos 2x$
- **c** $\log_e x \sin x$
- **d** $e^x \cos x$
- **e** $\frac{e^x}{\sin x}$
- **f** $\frac{\cos x}{\log_e x}$

9 Find the derivative of $\sin x \cos x$. **6B**

10 Find the equation of the tangent to the curve $y = \sin x$ at the point $(0, 0)$. **6C**

11 Find the equation of the tangent and the normal to the curve $y = 3 \cos x$ at the point $(\pi, -3)$. **6C**

12 Find the equation of the tangent to the curve $y = 4 \cos 2x$ at the point where $x = \frac{\pi}{4}$. Find the coordinates of the points A and B where the tangent cuts the x- and y-axis respectively. **6C**

Modelling and problem solving

1 The number of fish in an inlet is estimated using the equation $N = 500 - 250 \cos \frac{\pi t}{6}$ where t is the number of hours since low tide.

- **a** Find the number of fish in the inlet at low tide.
- **b** Find the maximum number of fish in the inlet and the time at which this maximum will occur.
- **c** Find the time at which there are approximately 500 fish in the inlet.

2 A body is executing simple harmonic motion with a position $x(t)$ given by the equation:

$$x(t) = 2 \sin \left(2\pi t + \frac{\pi}{2}\right) + 2.$$

a What is the x-value for the mean or centre of motion?
b What are the amplitude and range of the motion?
c What are the period and frequency of the motion?
d What is the initial position of the body (that is, at $t = 0$)?
e Using a graphics calculator or suitable graphing software, draw $x(t)$.

3 A body executing simple harmonic motion has the displacement equation $x(t) = 5 \sin (2t - 1) + 6$. Find the equations for the velocity and acceleration as a function of time.

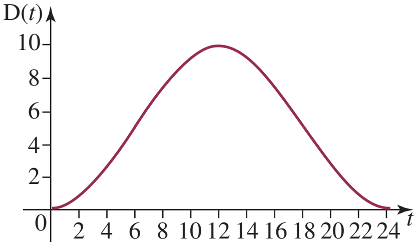
4 The displacement of a particle is given by the equation $x = 2 \sin \frac{\pi t}{2}$.
a Find an expression for the velocity of the particle, v.
b Find the times at which the particle is at rest (that is, $v = 0$).
c Find an expression for the acceleration of the particle.
d Find the maximum speed of the particle.

5 A mass is attached to the end of a spring. The spring is extended and the mass on the end of the spring begins to bob up and down in simple harmonic motion where its displacement at any time, t, from O, the equilibrium position, is given by the equation $x(t) = 10 - 10 \cos \pi t$.
a Find the initial position of the mass.
b Find equations for the velocity and acceleration of the mass.
c Find the times at which the particle is stationary.
d Find when the velocity is:
  i a maximum **ii** a minimum.
e Find the points where the acceleration is 0. What do you notice about the velocity at these points?

6 At the South Pole in midsummer on the planet Marus, the red sun of its solar system does not set. It dips towards the horizon until its lower rim just touches it, then rises until its lowest point is at an angle of D(t)° to the horizontal before sinking again. It continues in this pattern. The angle above the horizontal can be modelled by the following relation:

D(t) = $a - b \sin n(t + c)$ where t is the time in hours after midnight and a, b, c and n are positive constants.

The graph of D(t) for 24 hours is shown on the axes in the figure below right.



a State the values of a, b, c and n and hence write the rule for D(t).

b What would be the angle above the horizon at 6.00 am and at 9.00 pm? Give an exact answer where possible, otherwise give your answer correct to 2 decimal places.

c Use your graph to find at what times the angle to the horizontal is 8°. When does the rim of the sun reach this angle again?

d By using an appropriate equation, check your answer and account for any difference in your two solutions.

e If a spot on the surface of the sun is 5° above the horizon at midnight, what would be the relation G(t) which models its path?

7 Nathan and Rachel are competing in the National Ballroom Dancing Championships. The judges are evenly spaced around the circular dance floor, standing just outside the edge. As Nathan and Rachel waltz around the circular floor, their distance (in metres) from judge Maya can be described by the function $d = 10.5 - 9 \cos \frac{\pi}{30} t$, where t is time (in seconds) from the beginning of the dance.

a How far is the couple from judge Maya when they start dancing?

b What is the couple's maximum distance from the judge?

c Assuming that, while dancing, Rachel and Nathan trace a perfect circle, what is its diameter?

d How long does it take for the couple to complete one full circle around the dance floor?

e What is the couple's average speed (in m/s)? Give your answer **i** in exact form and **ii** correct to 2 decimal places.

f If the duration of the waltz is 2.5 minutes, draw the graph of $d = 10.5 - 9 \cos \frac{\pi}{30} t$ over the domain, showing the full length of this dance.

g Judge Joseph is positioned further down the dance floor, so that Nathan and Rachel are closest to him 6 seconds after the waltz begins. Write the equation describing the couple's distance from judge Joseph at any time, t, from the beginning of the dance.

h How far is the couple from judge Joseph when they finish the waltz?

eBook*plus*

Digital doc:
Test Yourself
Chapter 6

eBook*plus* ACTIVITIES

6A The derivatives of sin *x* and cos *x*

Tutorial

- **WE2** Int-0555: Watch a tutorial on using the chain rule (*page 212*)

6B Further differentiation of trigonometric functions

Digital doc

- WorkSHEET 6.1: Differentiate trigonometric expressions using the chain, product and quotient rules (*page 217*)

Interactivity

- Mixed problems on differentiation int-0252: Consolidate your understanding of differentiation (*page 215*)

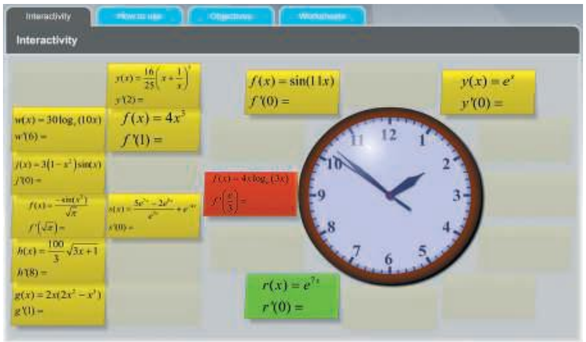
6D Kinematics

Digital doc

- WorkSHEET 6.2: Determine equations of tangents and normals to sinusoidal curves and apply the calculus of periodic functions to application questions (*page 229*)

Interactivity

- Kinematics int-0267: Consolidate your understanding of kinematics (*page 221*)



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 233*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

7

Introduction to integration

syllabus reference

Introduction to integration

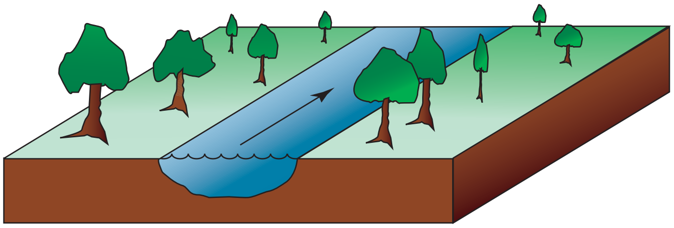
In this chapter

Introduction

The Werisie River problem

A farmer living along the Werisie River wants to draw water from it to irrigate his crops, and he applies to the local council for permission. The council sends a technical officer to measure the flow of water in the river so that an informed decision can be made about the farmer's application for a water allocation.

The technical officer begins by measuring the speed of the water and forming a profile of the cross-section of the river.



The speed of the river is 2.5 m/s and it is 4 metres wide.

It is found that the cross-section of the river can be modelled by the equation

$$y = \tfrac{1}{2}x(x - 4)$$

If the technical officer can find the area of the cross-section, then he can find the volume of water carried by the river each second because the volume would be given by:

Volume per second = area of cross-section · speed of flow.

eBook*plus*

Digital doc:

SkillSHEET 7.1

Area and perimeter of composite functions

In this section we shall consider this problem and examine different methods for finding the area of shapes bounded by curves. The problem of calculating an area bounded by a curve is difficult (except in the case of the circle) and is usually solved by approximating the curved shape by constructing a number of smaller figures made from straight lines.

Approximating areas enclosed by functions

eBook*plus*

Interactivity:

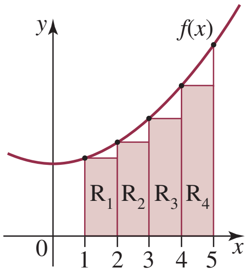
Approximating areas enclosed by functions

int-0254

There are several ways of finding an approximation to the area between a graph and the x-axis. We shall look at three methods:

1. the lower rectangle method
2. the upper rectangle method
3. the trapezoidal method.

The lower rectangle method



Consider the area between the curve $f(x)$ shown at right, the x-axis and the lines $x = 1$ and $x = 5$.

If the area is approximated by 'lower' rectangles whose width is 1 unit, then the top of each rectangle lies below the graph but touches the curve at one point. (In this case the left-hand corner of the rectangle touches the graph.)

So, the height of rectangle R_1 is $f(1)$ units
and the area of $R_1 = 1 \cdot f(1)$ square units (area of a rectangle = height · width).
Similarly, the area of $R_2 = 1 \cdot f(2)$ square units,
the area of $R_3 = 1 \cdot f(3)$ square units,
the area of $R_4 = 1 \cdot f(4)$ square units.

Therefore, the approximate area under the graph between the curve $f(x)$, the x-axis and the lines $x = 1$ to $x = 5$ is $1[f(1) + f(2) + f(3) + f(4)]$ square units, (the sum of the area of the four rectangles).

If the same area was approximated using rectangle widths of 0.5 there would be 8 rectangles and the sum of their areas would be:
$0.5[f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5)]$ square units.

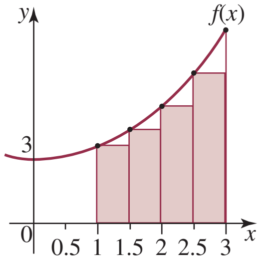
From the diagram it can be seen that the lower rectangle approximation is less than the actual area.

eBook*plus*

Tutorial:
Worked example 1
int-0565

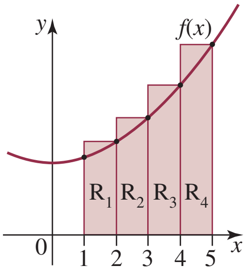
WORKED Example 1

Find an approximation for the area between the curve $f(x)$ shown and the x-axis from $x = 1$ to $x = 3$ using lower rectangles of width 0.5 units. $f(x) = 0.2x^2 + 3$



THINK	**WRITE**
① Write the number of rectangles and their width.	There are 4 rectangles of width 0.5 units.
② Find the height of each rectangle (left) by substituting the appropriate x-value into the $f(x)$ equation.	$h_1 = f(1) = 0.2(1)^2 + 3 = 3.2$ $h_2 = f(1.5) = 0.2(1.5)^2 + 3 = 3.45$ $h_3 = f(2) = 0.2(2)^2 + 3 = 3.8$ $h_4 = f(2.5) = 0.2(2.5)^2 + 3 = 4.25$
③ Area equals the width multiplied by the sum of the heights.	Area = width · (sum of heights of 4 rectangles) = 0.5(3.2 + 3.45 + 3.8 + 4.25) = 0.5(14.7)
④ Calculate this area.	= 7.35
⑤ State the solution.	The approximate area is 7.35 square units.

The upper rectangle method



Consider the area between the curve $f(x)$ shown at right, the x-axis and the lines $x = 1$ and $x = 5$.

If the area is approximated by ‘upper rectangles’ that are 1 unit wide, then the top of each rectangle is above the graph and touches the curve at one point. (In this case the top right-hand corner of the rectangle touches the graph.)

So, the height of R_1 is $f(2)$ units
and the area of R_1 is $1 \cdot f(2)$ square units.
Similarly, the area of $R_2 = 1 \cdot f(3)$ square units,
the area of $R_3 = 1 \cdot f(4)$ square units
the area of $R_4 = 1 \cdot f(5)$ square units.

Therefore, the approximate area between the curve $f(x)$, the x-axis and the lines $x = 1$ to $x = 5$ is $(R_1 + R_2 + R_3 + R_4) = 1[f(2) + f(3) + f(4) + f(5)]$ square units.

If the same area was approximated with upper rectangle widths of 0.5 units, the sum of their areas would equal:
$0.5[f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) + f(4.5) + f(5)]$ square units.

From the diagram it can be seen that the upper rectangle approximation is greater than the actual area.

Lower rectangle approximation ≤ actual area ≤ upper rectangle approximation

WORKED Example 2

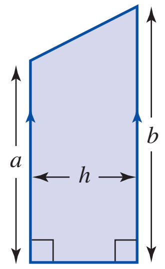
Find an approximation for the area in the diagram in worked example 1 using upper rectangles that are 0.5 units wide. $f(x) = 0.2x^2 + 3$

THINK	WRITE
1 Find the number of rectangles and the height of each one (from left to right).	There are 4 rectangles: $h_1 = f(1.5) = 0.2(1.5)^2 + 3 = 3.45$ $h_2 = f(2) = 0.2(2)^2 + 3 = 3.8$ $h_3 = f(2.5) = 0.2(2.5)^2 + 3 = 4.25$ $h_4 = f(3) = 0.2(3)^2 + 3 = 4.8$
2 The area is the width of the interval multiplied by the sum of the heights.	Area $= 0.5(3.45 + 3.8 + 4.25 + 4.8)$
3 Calculate the area.	$= 0.5(16.3)$ $= 8.15$
4 State the solution.	The approximate area is 8.15 square units.

It can be seen that the lower rectangle approximation (7.35 units) is less than the upper rectangle approximation (8.15 units).

If the area is divided into narrower strips, the estimate of the area would be closer to the true value. Use one of the Mathcad files to investigate the effect of a greater number of strips in a given interval.

The trapezoidal method



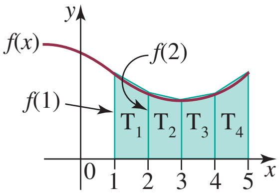
Recall that the area of a trapezium $= \frac{h}{2}(a + b)$.

The trapezoidal method involves a series of straight line approximations to the curve which generates strips in the shape of trapeziums.

Consider the area under the graph of $f(x)$ between the x-axis and the lines $x = 1$ to $x = 5$.

For each trapezium the width, or height, $h = 1$ unit.

For T_1, $a = f(1)$ and $b = f(2)$
For T_2, $a = f(2)$ and $b = f(3)$
For T_3, $a = f(3)$ and $b = f(4)$
For T_4, $a = f(4)$ and $b = f(5)$



The area of $T_1 = \frac{1}{2}[f(1) + f(2)]$

The area of $T_2 = \frac{1}{2}[f(2) + f(3)]$ and so on.

The total area of the trapeziums is:

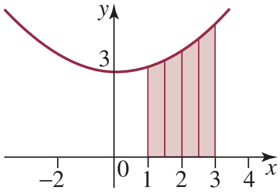
$\frac{1}{2}[f(1) + f(2) + f(2) + f(3) + f(3) + f(4) + f(4) + f(5)]$
$= \frac{1}{2}[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)]$ square units.

The first and last terms are counted only once but all others are counted twice.

WORKED Example 3

Find an approximation for the area enclosed by the graph of $f(x) = 0.2x^2 + 3$, the x-axis and the lines $x = 1$ to $x = 3$ using interval widths of 0.5 units and using the trapezoidal method:

a by hand
b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a ① Sketch the graph of $f(x)$. ② Draw trapeziums of width 0.5 unit from $x = 1$ to $x = 3$.	**a** 
③ Evaluate the height of each vertical side of the trapeziums by substituting the appropriate x-value into $f(x)$.	$f(1) = 0.2(1)^2 + 3 = 3.2$ $f(1.5) = 0.2(1.5)^2 + 3 = 3.45$ $f(2) = 0.2(2)^2 + 3 = 3.8$ $f(2.5) = 0.2(2.5)^2 + 3 = 4.25$ $f(3) = 0.2(3)^2 + 3 = 4.8$
④ Calculate the area by using the formula for the area of a trapezium where h is the width of the interval.	Total area of trapeziums $= \frac{0.5}{2}(3.2 + 2 \cdot\ 3.45 + 2 \cdot\ 3.8 + 2 \cdot\ 4.25 + 4.8)$ $= 0.25 \cdot\ 31$ $= 7.75$ Therefore, the area under the curve is approximately 7.75 units.

Continued over page →

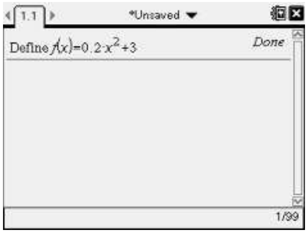
THINK

WRITE/DISPLAY

For the TI-Nspire CAS

b ① To find the sum of the areas between the lines $x = 1$, $x = 3$, the x-axis and under the curve $f(x) = 0.2x^2 + 3$, on a Calculator page, complete the entry line as:
Define $f(x) = 0.2x^2 + 3$
then press ENTER enter.

b



1.1 *Unsaved

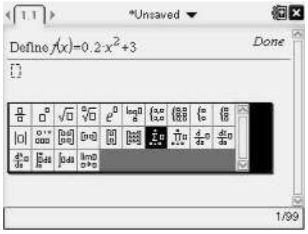
Define $f(x)=0.2\cdot x^2+3$ Done

1/99

② To use sigma notation to calculate the sum of the areas, on a Calculator page, press:

- Ctrl ctrl
- · .

This will open the maths expression template. Use the NavPad to move to the symbol sigma, Σ, then press ENTER enter.



1.1 *Unsaved

Define $f(x)=0.2\cdot x^2+3$ Done

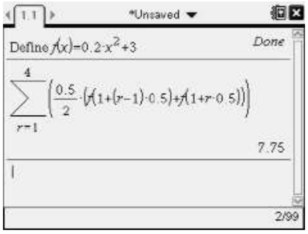
1/99

③ Complete the entry line as:

$$\sum_{r=1}^{4}\left(\frac{0.5}{2}(f(1+(r-1)\cdot\ 0.5)\right.$$

$$\left.+f(1+r\cdot\ 0.5))\right)$$

then press ENTER enter·



1.1 *Unsaved

Define $f(x)=0.2\cdot x^2+3$ Done

$\sum_{r=1}^{4}\left(\frac{0.5}{2}\cdot(f(1+(r-1)\cdot 0.5)+f(1+r\cdot 0.5))\right)$

7.75

2/99

④ Write the answer.

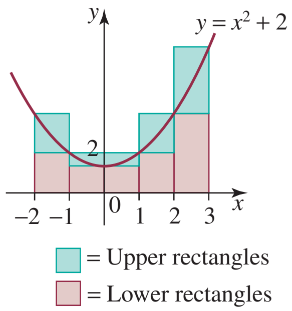
$$\sum_{r=1}^{4}\left(\frac{0.5}{2}(f(1+(r-1)\cdot\ 0.5)+f(1+r\cdot\ 0.5))\right)$$

$= 7.75$

Note that the lower rectangle approximation found in Worked example 1 was 7.35 units and the upper rectangle approximation found in Worked example 2 was 8.15 units. The average of these two approximations is $\frac{7.35 + 8.15}{2}$ or 7.75 units which is the same as the trapezoidal approximation for the area.

WORKED Example 4

Employ width intervals of 1 unit to calculate an approximation for the area between the graph of $f(x) = x^2 + 2$ and the x-axis from $x = -2$ to $x = 3$. Use:
a lower rectangles b upper rectangles c averaging of the lower and upper rectangle areas.

THINK	**WRITE/DRAW**
① Sketch the graph of $f(x)$ over a domain that exceeds the width of the required area. ② Draw the lower and upper rectangles.	
a ① Calculate the height of the lower rectangles by substituting the appropriate values of x into the equation for $f(x)$. Note that the two rectangles to the right and left of the origin have the same height and are equal in area.	**a** Lower rectangle heights: $f(-1) = (-1)^2 + 2$ $= 3$ $f(0) = 0^2 + 2$ $= 2$ $f(0) = 2$ $f(1) = 1^2 + 2$ $= 3$ $f(2) = 2^2 + 2$ $= 6$
② Find the area by multiplying the width by the sum of the heights.	Area $= 1(3 + 2 + 2 + 3 + 6)$ $= 16$ Using lower rectangles, the approximate area is 16 square units.
b ① Calculate the height of the upper rectangles by substituting the appropriate values of x into the equation for $f(x)$.	**b** Upper rectangle heights: $f(-2) = (-2)^2 + 2$ $= 6$ $f(-1) = 3$ (from above) $f(1) = 3$ $f(2) = 6$ $f(3) = 3^2 + 2$ $= 11$
② Find the area by multiplying the width by the sum of the heights.	Area $= 1(6 + 3 + 3 + 6 + 11)$ $= 29$ Using upper rectangles, the approximate area is 29 square units.
c Find the average by adding the areas of the upper rectangles and the lower rectangles, and then dividing by 2.	**c** Average of the areas $= \dfrac{16 + 29}{2}$ $= 22.5$ The approximate area is 22.5 square units when averaging the upper and lower rectangle areas and using widths of 1 unit.

Note that this average is between the area of the upper rectangles and the area of the lower rectangles and is closer to the actual area.

remember

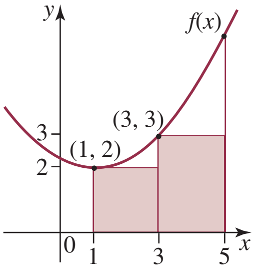
1. An approximation to the area between a curve and the x-axis can be found by dividing the area into a series of rectangles or trapeziums which are all the same width. The approximation is found by finding the sum of all the areas of the rectangles or trapeziums.
2. Lower rectangle approximation ≤ actual area ≤ upper rectangle approximation
3. Trapezoidal approximation =

$$\frac{\text{lower rectangle approximation} + \text{upper rectangle approximation}}{2}$$

EXERCISE 7A Approximating areas enclosed by functions

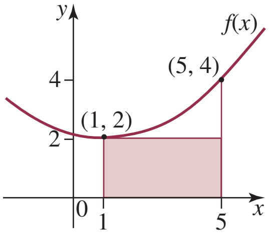
WORKED Example 1

1 Find an approximation for the area between the curve $f(x)$ at right and the x-axis from $x = 1$ to $x = 5$. Use a lower rectangle with a width of 2 units.

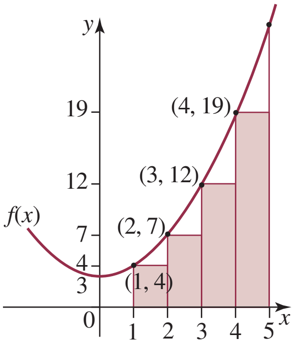


2 Find an approximation for the area between the curves below and the x-axis, from $x = 1$ to $x = 5$, by calculating the area of the shaded rectangles.

a

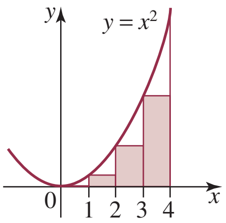


b



3 **multiple choice**

Consider the graph of $y = x^2$ from $x = 0$ to $x = 4$ (at right).



a The width of each rectangle is:

A 1 unit **B** 2 units **C** 3 units

D 4 units **E** varying

b The height of the right-hand rectangle is:

A 9 units **B** 4 units **C** 16 units

D 12 units **E** 1 unit

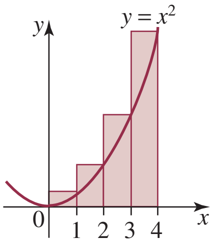
c The area between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 4$ can be approximated, by the area of the lower rectangles, as:

A 20 sq. units **B** 14 sq. units **C** 18 sq. units **D** 15 sq. units **E** 30 sq. units

WORKED Example 2

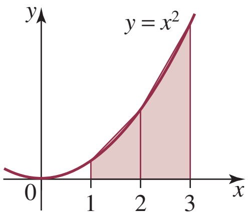
4 **a** Find an approximation for the area in the diagram at right using upper rectangles 1 unit wide.

b A better approximation for the area under this curve can be found by averaging the upper and lower rectangle areas. State this approximate value.



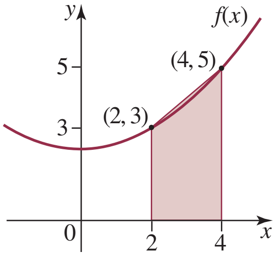
WORKED Example 3

5 Find an approximation for the area enclosed by the graph of $f(x) = x^2$, the x-axis and the lines $x = 1$ and $x = 3$ with interval widths of 1 unit. Use the trapezoidal method.

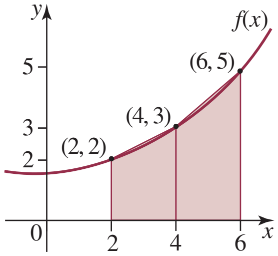


6 Find the approximate area between the curves below and the x-axis, over the interval indicated, by calculating the areas of the shaded trapeziums.

a

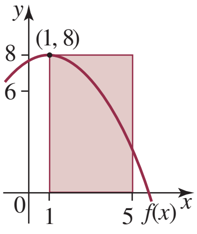


b

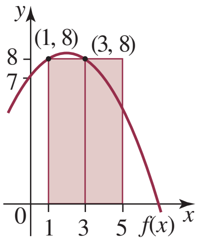


7 Find an approximation for the area between the curves below and the x-axis, from $x = 1$ to $x = 5$, by calculating the area of the shaded rectangles.

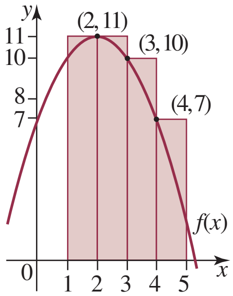
a



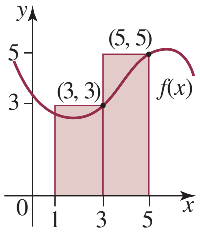
b



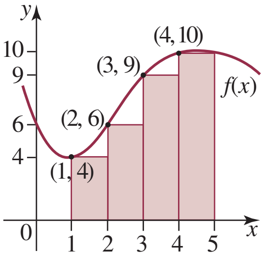
c



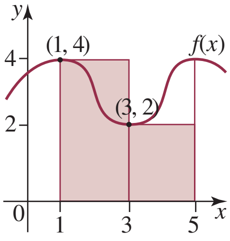
d



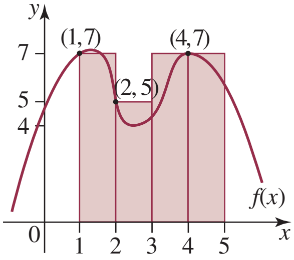
e



f



g



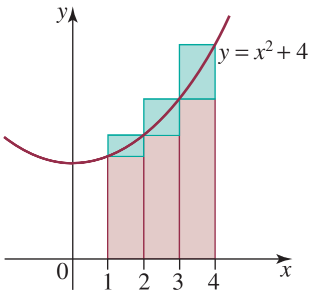
WORKED Example 4

8 With width intervals of 1 unit, calculate an approximation for the area between the graph of $f(x) = x^2 + 4$ and the x-axis from $x = 1$ to $x = 4$ using:

a lower rectangles

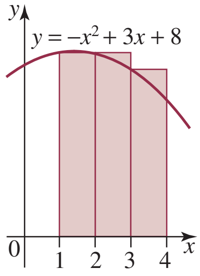
b upper rectangles

c averaging of the lower and upper rectangle areas.



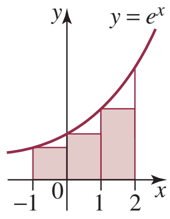
9 In the figures below, find the approximate area between the curves and the x-axis, over the interval indicated, by calculating the area of the shaded rectangles. Give exact answers.

a



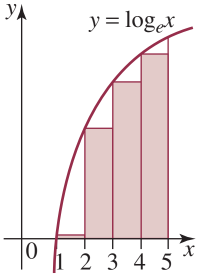
$x = 1$ to $x = 4$

b



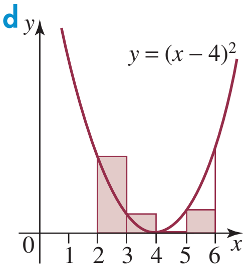
$x = -1$ to $x = 2$

c



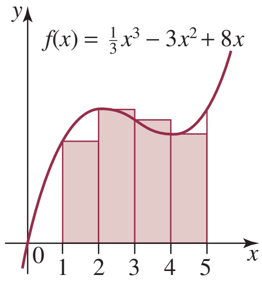
$x = 1$ to $x = 5$

d



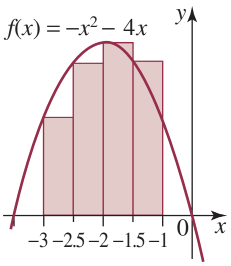
$x = 2$ to $x = 6$

e



$x = 1$ to $x = 5$

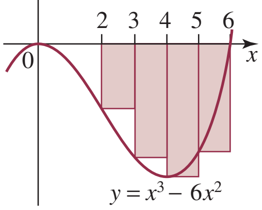
f



$x = -3$ to $x = -1$

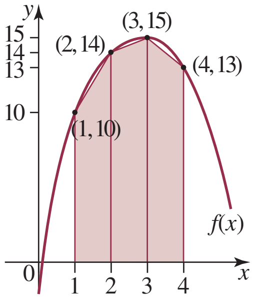
g

$x = 2$ to $x = 6$

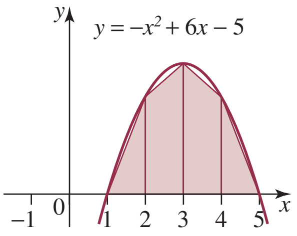


10 In the figures below, find an approximation for the area between the curve, the x-axis, and the lines $x = 1$ and $x = 5$. Use interval widths as shown. Give exact answers.

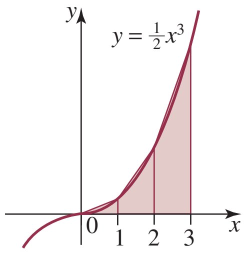
a



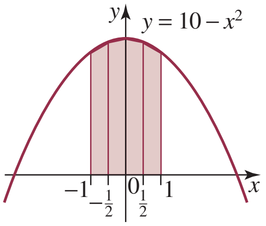
b



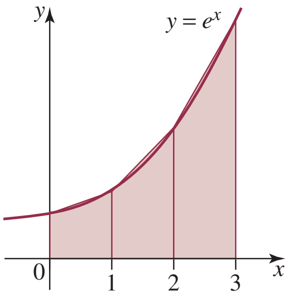
c



d



e



11 Calculate approximations for the area between the graph of $y = x(4 - x)$, the x-axis and the lines $x = 1$ and $x = 4$, using interval widths of 1 unit and:

a lower rectangles

b upper rectangles

c averaging the lower and upper rectangle areas.

12 Calculate approximations for the area under the graph of $y = x^2 - 4x + 5$ to the x-axis between $x = 0$ and $x = 3$, using interval widths of 0.5 units and:

a lower rectangles

b upper rectangles

c averaging the lower and upper rectangle areas.

13 Find an approximation for the area under the graph of $y = 2^x$ between $x = 0$ and $x = 3$, using interval widths of 1 unit, by averaging the lower and upper rectangle areas.

14 Find approximations for the area between the graph of $f(x) = (x - 1)^3$ and the x-axis, between $x = 1$ and $x = 4$, using the trapezoidal rule and:

a interval widths of 1 unit

b interval widths of 0.5 units.

Give answers correct to 1 decimal place.

15 Calculate approximations for the area under the graph of $y = \frac{1}{x}$ between $x = 0.5$ and $x = 2.5$, using the trapezoidal rule and:

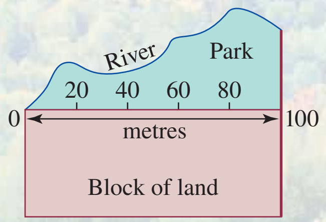
a interval widths of 1 unit

b interval widths of 0.5 units

c interval widths of 0.25 units.

Give answers correct to 2 decimal places.

16 Calculate an approximation for the area under the graph of $y = 2\log_e(x - 1)$ between $x = 2$ and $x = 6$ using the trapezoidal rule and interval widths of 1 unit.



17 At the back of a rectangular block of land, 100 metres long, is a park and a river. The distance to the river from the top of the rectangular block is shown in the table below.

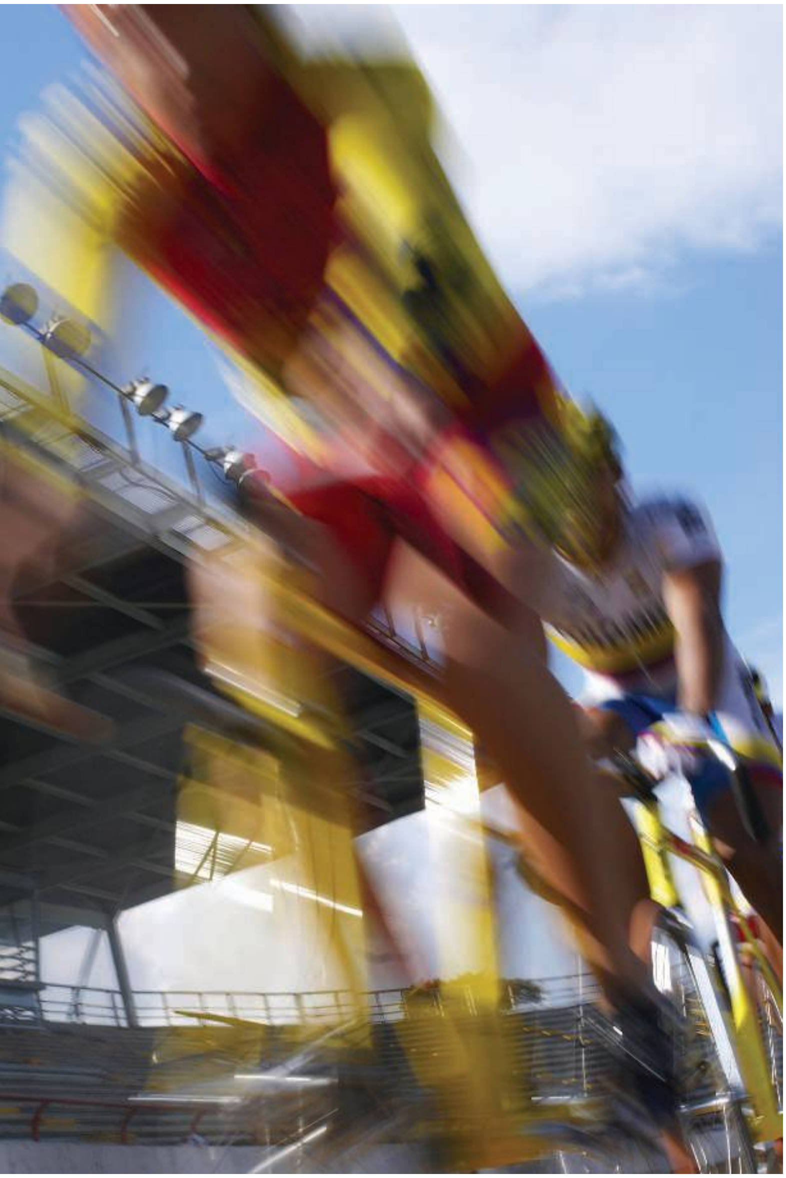
Distance across rectangular block in metres	0	20	40	60	80	100
Distance of river from the block in metres	0	30	20	40	60	50

Calculate approximations for the area of parkland between the rectangular block and the river by:

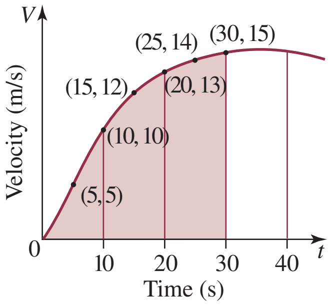
a using the area of the ‘upper’ rectangles.

b using the trapezoidal rule (use intervals of width 20 metres).

18 Calculate an approximate area under the graph of $f(x) = \sin x$, between $x = 0$ and $x = \pi$, using the trapezoidal rule and interval widths of $\frac{\pi}{6}$ units. Give your answer correct to 2 decimal places.



19 The graph below shows the velocity of a cyclist (in metres per second) at time t seconds after commencing a race.



a What does the shaded area represent?

b Find the approximate distance travelled by the cyclist in the first 30 seconds using the trapezoidal rule and interval widths of 5 seconds.

20 Answer the following statements concerning approximate areas under graphs as True or False.

a An approximation for the area can be found quickly if very small interval widths are used.

b The smaller the interval width used, the more accurate the approximation for the area.

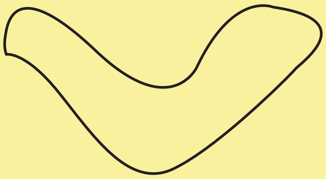
c The upper rectangle method is always more accurate than the lower rectangle method.

d Averaging the upper rectangle area and the lower rectangle area is more accurate than using the 'upper' or 'lower' approximations on their own.

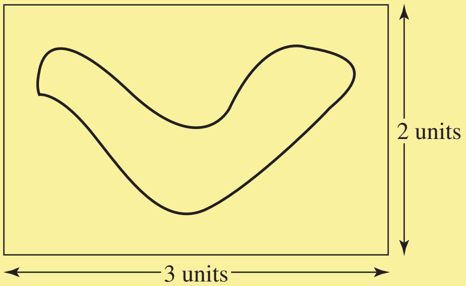
The Monte Carlo method

Monte Carlo is a small principality in Europe and its name has become synonymous with casinos and gambling. It is not surprising that the mathematical technique for approximating area that relies on the random generation of test points is called the *Monte Carlo* method.

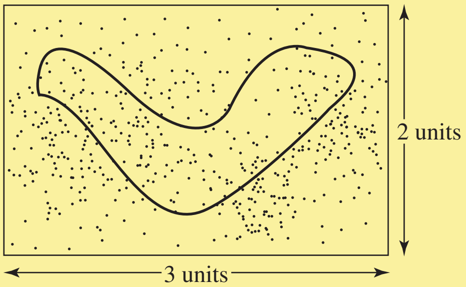
Suppose we want to find the area of the figure shown at right.



Step 1 Surround the figure by a rectangle of known dimensions.



Step 2 Randomly generate a number of points, say 500, lying within the rectangle.



Step 3 Count the number of points that lie within the curved figure. Suppose 150 lie within the figure.

$$\frac{\text{Area of curved figure}}{\text{Area of rectangle}} = \frac{\text{Number of points inside curved figure}}{\text{Total number of points}}$$

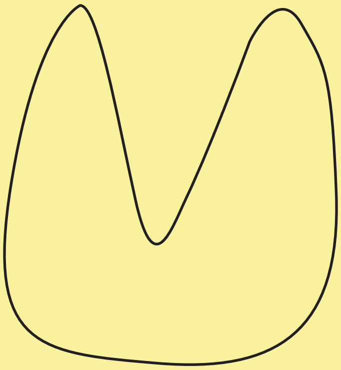
$$\frac{\text{Area of curved figure}}{6\text{ square units}} = \frac{150}{500}$$

$$\text{Area of curved figure} = 6 \cdot \frac{150}{500}$$

$$= 1.8\text{ square units.}$$

Now you try

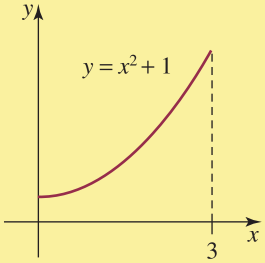
Trace the shape at right on your paper and use the Monte Carlo method to approximate its area.

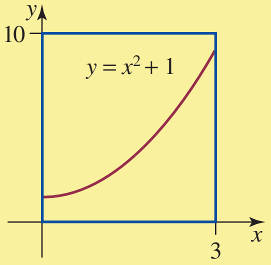


Area enclosed by a function and the Monte Carlo method

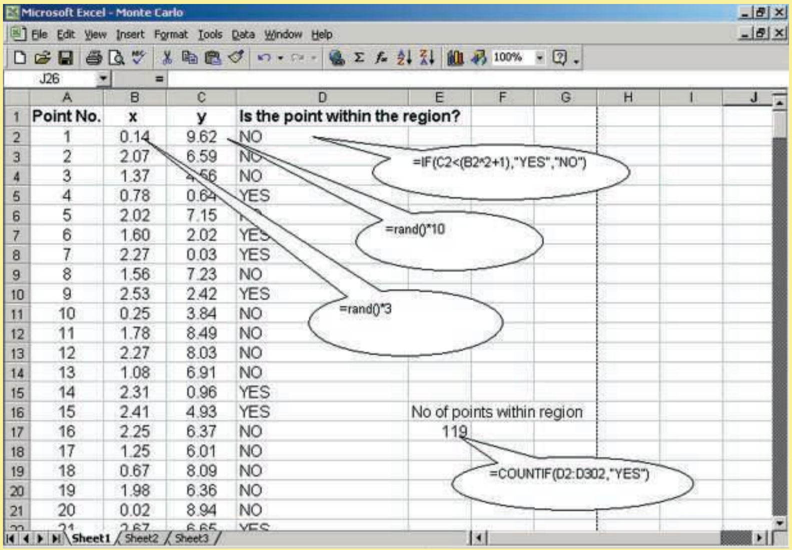
We can use a spreadsheet to apply the Monte Carlo method to an area bounded by a function. For example, approximate the area shown below left.

Step 1 Surround the area with a rectangle of known dimensions (below right).





Step 2 Generate points at random within the rectangle. This is done using a spreadsheet, generating an x-coordinate between 0 and 3 and a y-coordinate between 0 and 10.



	A	B	C	D
1	Point No.	x	y	Is the point within the region?
2	1	0.14	9.62	NO
3	2	2.07	6.59	NO
4	3	1.37	4.56	NO
5	4	0.78	0.64	YES
6	5	2.02	7.15	[illegible]
7	6	1.60	2.02	YES
8	7	2.27	0.03	YES
9	8	1.56	7.23	NO
10	9	2.53	2.42	YES
11	10	0.25	3.84	NO
12	11	1.78	8.49	NO
13	12	2.27	8.03	NO
14	13	1.08	6.91	NO
15	14	2.31	0.96	YES
16	15	2.41	4.93	YES
17	16	2.25	6.37	NO
18	17	1.25	6.01	NO
19	18	0.67	8.09	NO
20	19	1.98	6.36	NO
21	20	0.02	8.94	NO
22	21	2.67	6.65	YES

=IF(C2<(B2^2+1),"YES","NO")

=rand()*10

=rand()*3

No of points within region

119

=COUNTIF(D2:D302,"YES")

Note that a point with coordinates (s, t), say, will lie within the curved region if $t < f(s)$ where $y = f(x)$ is the equation of the curved boundary.

Step 3
$$\frac{\text{Area of enclosed figure}}{\text{Area of rectangle}} = \frac{\text{Number of points in enclosed region}}{\text{Total number of points}}$$

$$\frac{\text{Area of enclosed figure}}{\text{Area of rectangle}} = \frac{114}{300}$$

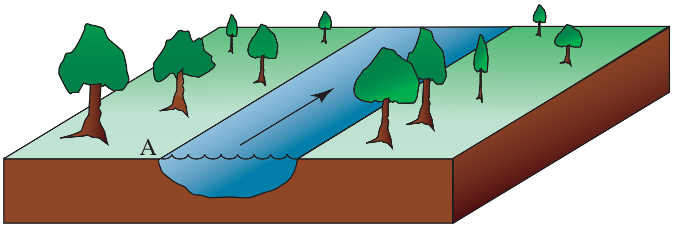
$$\text{Area of enclosed figure} = \frac{114}{300} \cdot 30$$

$$= 11.4 \text{ square units}$$

Use the Monte Carlo method and a spreadsheet similar to the one above to calculate the area enclosed by $y = 4 - x^2$ and the x-axis between $x = -2$ and $x = 2$.

The Werisie River problem — a model

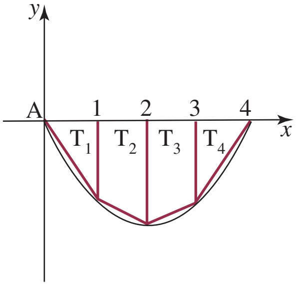
Let us return to the problem posed at the beginning of this chapter: finding an approximation to the area of a cross-section of the Werisie River; that is, finding the area enclosed by a curve.



The technical officer was able to develop a model for the curve of the river bed. He found that the equation $y = \frac{1}{2}x(x - 4)$ fitted the curve closely. The river is 4 metres wide.

Using the trapezoidal rule, we can find a close approximation to the area of the cross-section.

Using trapeziums that are 1 unit wide, we calculate:



$$\text{Area of } T_1 = \frac{1}{2} \cdot 1 \cdot \left(0 + \frac{3}{2}\right) = \frac{3}{4}$$

$$\text{Area of } T_2 = \frac{1}{2} \cdot 1 \cdot \left(\frac{3}{2} + 2\right) = \frac{7}{4}$$

$$\text{Area of } T_3 = \frac{1}{2} \cdot 1 \cdot \left(2 + \frac{3}{2}\right) = \frac{7}{4}$$

$$\text{Area of } T_4 = \frac{1}{2} \cdot 1 \cdot \left(\frac{3}{2} + 0\right) = \frac{3}{4}$$

Total area $= \frac{20}{4} = 5$ square units (or in this case, 5 m^2).

Thus the rate of flow water in the river is $5 \cdot$ (speed of flow) m^3/s

$$= 5 \cdot 2.5\ \text{m}^3/\text{s}$$

$$= 12.5\ \text{m}^3/\text{s}.$$

The local council is now in a position to decide whether, given this flow of water in the river, the farmer’s application for a water allocation is reasonable.

Antidifferentiation (integration)

The problem of calculating the area bounded by a curve has now been solved successfully or has it? The methods that we have employed so far are time consuming and tedious, and have produced only an approximate result. A more complete solution to the question of finding the area bounded by a curve lies in the domain of calculus. You are already familiar with differential calculus, or finding the rate of change of a function. Another branch of calculus, integral calculus, is concerned with finding the area bounded by a curve. In the section that follows, an exact method of calculating the area of curved figures will be developed.

As we have seen, the process of differentiation enables us to find the gradient of a function. The reverse process, antidifferentiation (or integration), will find the function for a particular gradient.

Integration has wider applications including calculation of areas, volumes, energy, probability and many more quantities in science and business.

Note that $\frac{d}{dx} f(x)$ means differentiate $f(x)$ with respect to x; that is, $\frac{d}{dx} f(x) = f'(x)$.

So $f(x)$ is the antiderivative of $f'(x)$, denoted as $f(x) = \int f'(x)\ dx$

where $\int$ means *antidifferentiate*, or *integrate*, or *find an indefinite integral* and dx indicates that the integration of the function is with respect to x.

Since $\qquad \frac{d}{dx}(ax + c) = a$, where a and c are constants

then $\qquad \int a\ dx = ax + c$.

Since $\qquad \frac{d}{dx}\left(\frac{ax^{n+1}}{n+1}\right) = ax^n$

then

$$\int ax^n\ dx = \frac{ax^{n+1}}{n+1} + c,\ n \neq -1.$$

Note: We must add a constant, c, when we are finding general antiderivatives.

However, if we have to find an antiderivative, the c is to be allocated an actual number and for convenience the number chosen is zero. That is, an antiderivative means 'let $c = 0$', or 'do not add on the c'.

For example, the antiderivative of $3x^2 + 4x + 5$ is $x^3 + 2x^2 + 5x + c$. An antiderivative of $3x^2 + 4x + 5$ is $x^3 + 2x^2 + 5x$.

Properties of integrals

Since $\frac{d}{dx}$ is a linear operator, so too is its inverse, $\int$. Therefore,

$$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

That is, each term can be integrated separately, and

$$\int k\, f(x)\ dx = k \int f(x)\ dx$$

That is, a 'constant' factor of the function can be taken to the front of the integral.

WORKED Example 5

Antidifferentiate each of the following, giving answers with positive indices.

a $2x^7$ **b** $4x^{-3}$ **c** $\frac{3}{\sqrt{x}}$

THINK	WRITE
a ① Integrate by rule; that is, add 1 to the index and divide by the new index.	**a** $\int 2x^7\ \mathrm{d}x = \frac{2x^8}{8} + c$
② Simplify.	$= \frac{x^8}{4} + c$
b ① Integrate by rule.	**b** $\int 4x^{-3}\ \mathrm{d}x = \frac{4x^{-2}}{-2} + c$
② Simplify.	$= -2x^{-2} + c$
③ Express the answer with a positive index.	$= -\frac{2}{x^2} + c$
c ① When a square root is involved, replace it with a fractional index.	**c** $\int \frac{3}{\sqrt{x}}\mathrm{d}x = \int \frac{3}{x^{\frac{1}{2}}}\ \mathrm{d}x$
② Bring the *x* to the numerator and change the sign of the index.	$= 3\int x^{-\frac{1}{2}}\ \mathrm{d}x$
③ Integrate by rule.	$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$
④ Simplify.	$= 6x^{\frac{1}{2}} + c$
⑤ Write the answer in the form in which it has been given.	$= 6\sqrt{x} + c$

WORKED Example 6

Find the following indefinite integral.

$$\int (x - 1)(3x + 5)\ \mathrm{d}x$$

THINK	WRITE
① Expand the expression.	$\int (x-1)(3x+5)\ \mathrm{d}x = \int (3x^2 - 3x + 5x - 5)\ \mathrm{d}x$
② Collect like terms.	$= \int (3x^2 + 2x - 5)\ \mathrm{d}x$
③ Integrate each term separately.	$= \frac{3x^3}{3} + \frac{2x^2}{2} - 5x + c$
④ Simplify each term.	$= x^3 + x^2 - 5x + c$

Integration of $(ax + b)^n$

By applying the chain rule for differentiation:

$$\frac{d}{dx}(ax+b)^{n+1} = a(n+1)(ax+b)^n$$

so $$\int a(n+1)(ax+b)^n \, dx = (ax+b)^{n+1} + c$$

or $$a(n+1)\int (ax+b)^n \, dx = (ax+b)^{n+1} + c$$

or $$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

WORKED Example 7

Antidifferentiate $4(5x - 2)^3$ by rule.

THINK	WRITE
1 Express as an integral and take 4 out as a factor.	$\int 4(5x-2)^3 \, dx = 4\int (5x-2)^3 \, dx$
2 Apply the rule where $a = 5$ and $n = 3$.	$= \frac{4(5x-2)^4}{5(4)} + c$
3 Simplify the antiderivative by cancelling the fraction.	$= \frac{1}{5}(5x-2)^4 + c$

Integration of $\frac{1}{x}$

Since $$\frac{d}{dx}\log_e x = \frac{1}{x}$$

then $$\int \frac{1}{x} \, dx = \log_e x + c, \text{ where } x > 0$$

or $$\int x^{-1} \, dx = \log_e x + c.$$

WORKED Example 8

Antidifferentiate $\frac{4}{7x}$.

THINK	WRITE
1 Take $\frac{4}{7}$ out as a factor.	$\int \frac{4}{7x} \, dx = \int \left(\frac{4}{7} \cdot \frac{1}{x}\right) dx$ $= \frac{4}{7}\int \frac{1}{x} \, dx$
2 Integrate by rule.	$= \frac{4}{7}\log_e x + c$

Integration of $(ax + b)^{-1}$

By applying the chain rule for differentiation:

$$\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b},$$

where a and b are constants

and $$\frac{1}{a}\cdot\frac{d}{dx}\log_e(ax+b) = \frac{a}{ax+b}\cdot\frac{1}{a}$$
$$= \frac{1}{ax+b}$$

So $$\int\frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b)+c$$

or $$\int(ax+b)^{-1}\,dx = \frac{1}{a}\log_e(ax+b)+c$$

Note that the a in the fraction $\frac{1}{a}$ is the derivative of the linear function $ax + b$.

$$\int\frac{1}{ax^2+bx+c}\,dx = \frac{1}{2ax+b}\log_e(ax^2+bx+c)+c_1$$

WORKED Example 9

Antidifferentiate $\frac{5}{2x+3}$:

a **by hand**

b **using the TI-Nspire CAS calculator.**

THINK

WRITE/DISPLAY

a 1 Express as an integral and take 5 out as a factor.

$$\int\frac{5}{2x+3}\,dx = 5\int\frac{1}{2x+3}\,dx$$

2 Integrate by rule where $a = 2$.

$$= \frac{5}{2}\log_e(2x+3)+c$$

For the TI-Nspire CAS

b 1 To antidifferentiate $\frac{5}{2x+3}$, on a Calculator page, press:

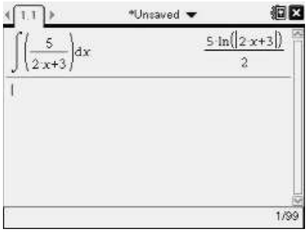
- MENU menu
- 4: Calculus 4
- 3: Integral 3.

Complete the entry line as:

$$\int\frac{5}{2x+3}\,dx$$

then press ENTER enter.

b



$\int\left(\frac{5}{2\cdot x+3}\right)dx$ $\frac{5\cdot\ln(|2\cdot x+3|)}{2}$

2 Write the answer.
Note: The TI-Nspire CAS calculator does not include the constant of integration in the integral.

$$\int\frac{5}{2x+3}\,dx = \frac{5}{2}\log_e(2x+3)+c$$

WORKED Example 10

Find $\int \frac{6x+5}{x^2}\,dx$.

THINK	WRITE
① Express as separate fractions.	$\int \frac{6x+5}{x^2}\,\mathrm{d}x = \int\left(\frac{6x}{x^2}+\frac{5}{x^2}\right)\mathrm{d}x$
② Simplify each fraction.	$= \int(6x^{-1}+5x^{-2})\,\mathrm{d}x$
③ Integrate each term separately by rule.	$= 6\log_e x + \frac{5x^{-1}}{-1} + c$
④ Simplify, leaving the answer with positive indices.	$= 6\log_e x - 5x^{-1} + c$ $= 6\log_e x - \frac{5}{x} + c$

WORKED Example 11

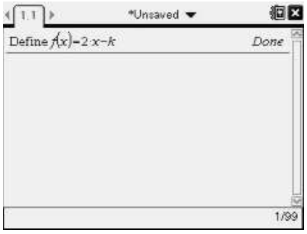
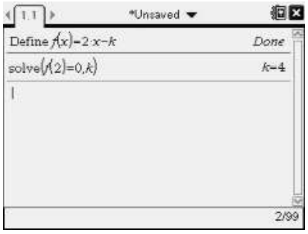
Find the equation of the curve $g(x)$ given that $g'(x) = 3\sqrt{x}+2$ and the curve passes through (1, 2).

THINK	WRITE
① Write the rule for $g'(x)$.	$g'(x) = 3\sqrt{x}+2$
② Rewrite $g'(x)$ in index form.	$g'(x) = 3x^{\frac{1}{2}}+2$
③ Express $g(x)$ in integral notation.	$g(x) = \int\left(3x^{\frac{1}{2}}+2\right)\mathrm{d}x$
④ Antidifferentiate to obtain a general rule for $g(x)$.	$= 3x^{\frac{3}{2}} \mid \frac{3}{2} + 2x + c$
⑤ Simplify.	$= \frac{3x^{\frac{3}{2}}}{1} \cdot \frac{2}{3} + 2x + c$ $g(x) = 2x^{\frac{3}{2}} + 2x + c$
⑥ Substitute coordinates of the given point into $g(x)$.	As $g(1) = 2$, $2(1)^{\frac{3}{2}} + 2(1) + c = 2$
⑦ Find the constant of antidifferentiation, c.	$2 + 2 + c = 2$ so $c = -2$
⑧ State the rule for $g(x)$ in the form in which it has been given.	$g(x) = 2x^{\frac{3}{2}} + 2x - 2$ $= 2\sqrt{x^3} + 2x - 2$

WORKED Example 12

If a curve has a stationary point (2, 3), and a gradient of 2*x* – *k*, where *k* is a constant, find:

i the value of *k*   ii *y* when *x* = 1:   a by hand   b using the TI-Nspire CAS calculator.

THINK	WRITE/DISPLAY
a i ① The gradient is $\frac{dy}{dx}$ so write the rule for the gradient.	**a i** $\frac{dy}{dx} = 2x - k$
② Let $\frac{dy}{dx} = 0$ (as stationary points occur when the derivative is zero) and substitute the value of *x* into this equation.	For stationary points, $\frac{dy}{dx} = 0$, so $2x - k = 0$ $2(2) - k = 0$ as $x = 2$
③ Solve for *k*.	$4 - k = 0$ so $k = 4$
ii ① Rewrite the rule for the gradient function, using the value of *k* found in **a** above.	**ii** $\frac{dy}{dx} = 2x - 4$
② Integrate to obtain the general rule for *y*.	$y = \int (2x - 4)\,dx$ $= x^2 - 4x + c$
③ Substitute the coordinates of the given point on the curve to find the value of *c*.	Since curve passes through (2, 3), $3 = 2^2 - 4(2) + c$ $3 = 4 - 8 + c$ $c = 7$
④ State the rule for *y*.	So $y = x^2 - 4x + 7$
⑤ Substitute the given value of *x* and calculate *y*.	When $x = 1$, $y = (1)^2 - 4(1) + 7$ $= 4$
For the TI-Nspire CAS	
b i ① To define the gradient function, on a Calculator page, press: • MENU (menu) • 1: Actions (1) • 1: Define (1). Complete the entry line as: Define *f*(*x*) = 2*x* – *k* then press ENTER (enter).	**b i**  Define f(x)=2·x−k Done 1/99
② To find the value of *k*, on a Calculator page, complete the entry line as: solve (*f*(2) = 0, *k*) then press ENTER (enter).	 Define f(x)=2·x−k Done solve(f(2)=0,k) k=4 2/99
③ Write the answer.	solve(*f*(2) = 0, *k*) *k* = 4

Continued over page →

THINK

ii ① To find y when $x = 1$, on a Calculator page, complete the entry lines as shown.

$\int 2x - 4\,dx$

solve $(x^2 - 4x + c = 3, c) \mid x = 2$

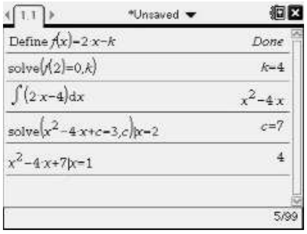
$x^2 - 4x + 7 \mid x = 1$

pressing ENTER after each line.

② Write the solution.

WRITE/DISPLAY

ii



```
Define f(x)=2·x−k                       Done
solve(f(2)=0,k)                          k=4
∫(2·x−4)dx                            x²−4·x
solve(x²−4·x+c=3,c)|x=2                  c=7
x²−4·x+7|x=1                               4
                                        5/99
```

$\int 2x - 4\,dx = x^2 - 4x + c$

Solving $x^2 - 4x + c = 3$ for c when $x = 2$, gives $c = 7$

$y = x^2 - 4x + 7$

$y(1) = 4$

remember

1. $\dfrac{d}{dx} f(x) = f'(x)$
2. $f(x) = \int f'(x)\,dx$
3. $\int a\,dx = ax + c$
4. $\int ax^n\,dx = \dfrac{ax^{n+1}}{n+1} + c,\ n \neq -1$
5. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
6. $\int k\,f(x)\,dx = k\int f(x)\,dx$
7. $\int (ax+b)^n\,dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c,\quad n \neq -1$
8. $\int \dfrac{1}{x}\,dx = \log_e x + c$, where $x > 0$ or $\int x^{-1}\,dx = \log_e x + c$
9. $\int \dfrac{1}{ax+b}\,dx = \dfrac{1}{a}\log_e(ax+b) + c$ or $\int (ax+b)^{-1}\,dx = \dfrac{1}{a}\log_e(ax+b) + c$
10. Write the answer in the same form as the question unless otherwise stated.

EXERCISE 7B Antidifferentiation (integration)

WORKED Example 5

1 Antidifferentiate each of the following, giving answers with positive indices.

a x **b** x^4 **c** x^7 **d** $3x^5$ **e** $5x^{-2}$

f $-2x^4$ **g** $-6x^{-4}$ **h** $2\sqrt{x}$ **i** $\dfrac{x^4}{5}$ **j** $\dfrac{x^3}{2}$

k $\dfrac{x^{-4}}{3}$ **l** $\sqrt{x}$ **m** $x^{\frac{2}{3}}$ **n** $4x^{\frac{3}{4}}$ **o** $x^{-\frac{3}{7}}$

p $\dfrac{5}{x^3}$ **q** $\dfrac{9}{x^2}$ **r** $\dfrac{-10}{x^6}$ **s** $\dfrac{8}{\sqrt{x}}$ **t** $\dfrac{-6}{(x\sqrt{x})}$

WORKED Example 6

2 Find the following indefinite integrals.

a $\int(2x+5)\,dx$ **b** $\int(3x^2+4x-10)\,dx$

c $\int(10x^4+6x^3+2)\,dx$ **d** $\int(-4x^5+x^3-6x^2+2x)\,dx$

e $\int(x^3+12-x^2)\,dx$ **f** $\int(x+3)(x-7)\,dx$

g $\int 5(x^2+2x-1)\,dx$ **h** $\int(x^2+4)(x-7)\,dx$

i $\int x(x-1)(x+4)\,dx$

3 **multiple choice**

$\int(x^2+x+2)\,dx$ is equal to:

A $\int x^2\,dx+x+2$ **B** $\int x^2\,dx+\int x\,dx+\int 2\,dx$

C $\int(x^2+x)\,dx+2$ **D** $x^2+\int(x+2)\,dx$

E $x^2+x+\int 2\,dx$

4 **multiple choice**

$\int x(x+3)\,dx$ is equal to:

A $\int x\,dx\int(x+3)\,dx$ **B** $x\int(x+3)\,dx$

C $(x+1)\int x\,dx$ **D** $\int x\,dx+\int(x+3)\,dx$

E $\int(x^2+3x)\,dx$

WORKED Example 7

5 Antidifferentiate each of the following by rule.

a $(x+3)^2$ **b** $(x-5)^3$ **c** $2(2x+1)^4$ **d** $-2(3x-4)^5$

e $(6x+5)^4$ **f** $3(4x-1)^2$ **g** $(4-x)^3$ **h** $(7-x)^4$

i $4(8-3x)^4$ **j** $-3(8-9x)^{10}$ **k** $(2x+3)^{-2}$ **l** $(6x+5)^{-3}$

m $6(4x-7)^{-4}$ **n** $(3x-8)^{-6}$ **o** $(6-5x)^{-3}$ **p** $-10(7-5x)^{-4}$

6 **multiple choice**

$\int 3(x+2)^4\,dx$ is equal to:

A $3+\int(x+2)^4\,dx$ **B** $\int 3\,dx+\int(x+2)^4\,dx$

C $3\int(x+2)^4\,dx$ **D** $3\int dx\cdot\int(x+2)^4\,dx$

E $(x+2)^4\int 3\,dx$

WORKED Example 8

7 Antidifferentiate the following.

a $\int \frac{3}{x}\,dx$ **b** $\int \frac{8}{x}\,dx$ **c** $\int \frac{6}{5x}\,dx$

d $\int \frac{7}{3x}\,dx$ **e** $\int \frac{4}{7x}\,dx$ **f** $\int \frac{1}{x+3}\,dx$

WORKED Example 9

g $\int \frac{3}{x+3}\,dx$ **h** $\int \frac{-2}{x+4}\,dx$ **i** $\int \frac{-6}{x+5}\,dx$

j $\int \frac{4}{3x+2}\,dx$ **k** $\int \frac{8}{5x+6}\,dx$ **l** $\int \frac{3}{2x-5}\,dx$

m $\int \frac{-5}{3+2x}\,dx$ **n** $\int \frac{-2}{6+7x}\,dx$ **o** $\int \frac{1}{5-x}\,dx$

p $\int \frac{3}{6-11x}\,dx$ **q** $\int \frac{-2}{4-3x}\,dx$ **r** $\int \frac{-8}{5-2x}\,dx$

8 **multiple choice**

$\int \frac{6}{x+5}\,dx$ is equal to:

A $6\int \frac{1}{x+5}\,dx$ **B** $\int 6\,dx \int \frac{1}{x+5}\,dx$ **C** $\int 6\,dx + \int \frac{1}{x+5}\,dx$

D $\dfrac{\int 6\,dx}{\int (x+5)\,dx}$ **E** $\dfrac{6}{\int (x+5)\,dx}$

WORKED Example 10

9 Find $\int \frac{(2x+7)}{x}\,dx$.

10 For the following mixed sets, find:

a $\int \left(x^4 + 2x + \frac{1}{x}\right)$ **b** $\int (3x+1)^5\,dx$ **c** $\int \frac{3x^2+2x-1}{x^2}\,dx$

d $\int \frac{3}{2x+1}\,dx$ **e** $\int \frac{-5}{6-10x}\,dx$ **f** $\int 4(2x-5)^5\,dx$

g $\int 3(4x+1)^{-3}\,dx$ **h** $\int \frac{(x+4)^2}{2x}\,dx$ **i** $\int \frac{(x-5)(x+3)}{x^3}\,dx$

j $\int \left(\sqrt{x} + \frac{2}{3-x}\right)dx$ **k** $\int \left(5x^{\frac{3}{2}} - 2x + 3x^{-\frac{1}{3}}\right)dx$ **l** $\int \frac{x^2+x^4}{x}\,dx$

m $\int \frac{x^2+2x-1}{\sqrt{x}}\,dx$ **n** $\int \frac{10-x+2x^4}{x^3}\,dx$

WORKED Example 11

11 Find the equation of the curve $f(x)$ given that:

a $f'(x) = 4x + 1$ and the curve passes through (0, 2)

b $f'(x) = 5 - 2x$ and the curve passes through (1, –1)

c $f'(x) = x^{-2} + 3$ and the curve passes through (1, 4)

d $f'(x) = x + \sqrt{x}$ and $f(4) = 10$

e $f'(x) = x^{\frac{1}{3}} - 3x^2 + 50$ and $f(8) = -100$

f $f'(x) = \frac{1}{\sqrt{x}} - 2x$ and $f(1) = -5$

g $f'(x) = (x + 4)^3$ and the curve passes through (–2, 5)

h $f'(x) = 8(1 - 2x)^{-5}$ and $f(1) = 3$

eBook*plus*

Digital docs:

SkillSHEET 7.2
Substitution and evaluation

WorkSHEET 7.1

i $f'(x) = (x + 5)^{-1}$ and the curve passes through $(-4, 2)$

j $f'(x) = \dfrac{8}{7 - 2x}$ and $f(3) = 7$

WORKED Example 12

12 If a curve has a stationary point $(1, 5)$ and a gradient of $8x + k$, where k is a constant, find:

a the value of k **b** y when $x = -2$.

13 A curve $g(x)$ has $g'(x) = \dfrac{kx + \sqrt{x}}{x^2}$, where k is a constant, and a stationary point $(1, 2)$. Find:

a the value of k **b** $g(x)$ **c** $g(4)$.

Integration of e^x, sin x and cos x

There are several important functions whose integration provides special cases. We shall now examine three of these functions: e^x, sin x and cos x.

Integration of the exponential function e^x

Since $\quad \dfrac{d}{dx}(e^x) = e^x$

then $\quad \displaystyle\int e^x\,dx = e^x + c$

and $\quad \dfrac{d}{dx}(e^{kx}) = ke^{kx}$, where k is a constant

or
$$\frac{d}{dx}\left(\frac{1}{k}\cdot\ e^{kx}\right) = \frac{1}{k}\cdot\ ke^{kx}$$
$$= e^{kx}$$

Therefore, $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c.$

WORKED Example 13

Antidifferentiate each of the following. **a** $3e^{4x}$ **b** $\dfrac{e^{-5x}}{4}$ **c** $(e^x - 1)^2$

		THINK	WRITE
a	1	Integrate by rule where $k = 4$.	**a** $\displaystyle\int 3e^{4x}\,dx = \frac{3e^{4x}}{4} + c$
	2	Simplify.	$= \frac{3}{4}e^{4x} + c$
b	1	Rewrite the function to be integrated so that the coefficient of the e term is clear.	**b** $\displaystyle\int \frac{e^{-5x}}{4}dx = \int \frac{1}{4}e^{-5x}\,dx$
	2	Integrate by rule where $k = -5$.	$= \dfrac{\frac{1}{4}e^{-5x}}{-5} + c$
	3	Simplify the antiderivative.	$= \frac{1}{4}e^{-5x}\cdot\ \frac{1}{-5} + c$ $= -\frac{1}{20}e^{-5x} + c$
c	1	Expand the function to be integrated.	**c** $\displaystyle\int (e^x - 1)^2\,dx = \int (e^{2x} - 2e^x + 1)\,dx$
	2	Integrate each term by the rule.	$= \frac{1}{2}e^{2x} - 2e^x + x + c$

Integration of trigonometric functions

Since $\frac{d}{dx}(\sin ax) = a\cos ax$ and $\frac{d}{dx}(\cos ax) = -a\sin ax$

it follows that

$$\boldsymbol{\int \sin ax\,dx = -\frac{1}{a}\cos ax + c}$$

$$\boldsymbol{\int \cos ax\,dx = \frac{1}{a}\sin ax + c}$$

WORKED Example 14

Antidifferentiate the following.

a $\sin 6x$ **b** $8\cos 4x$ **c** $3\sin\left(-\frac{x}{2}\right)$

THINK	**WRITE**
a Integrate by rule.	**a** $\int \sin 6x\,dx = -\frac{1}{6}\cos 6x + c$
b ① Integrate by rule.	**b** $\int 8\cos 4x\,dx = \frac{8}{4}\sin 4x + c$
② Simplify the result.	$= 2\sin 4x + c$
c ① Integrate by rule.	**c** $\int 3\sin\left(-\frac{x}{2}\right)dx = \frac{-3}{-\frac{1}{2}}\cos\left(-\frac{x}{2}\right) + c$
② Simplify the result.	$= 6\cos\left(-\frac{x}{2}\right) + c$

WORKED Example 15

Find $\int(2e^{4x} - 5\sin 2x + 4x)\,dx$:

a **by hand**

b **using the TI-Nspire CAS calculator.**

THINK	**WRITE/DISPLAY**
a ① Integrate each term separately.	**a** $\int(2e^{4x} - 5\sin 2x + 4x)\,dx$ $= \frac{2}{4}e^{4x} - \frac{-5}{2}\cos 2x + \frac{4}{2}x^2 + c$
② Simplify each term where appropriate.	$= \frac{1}{2}e^{4x} + \frac{5}{2}\cos 2x + 2x^2 + c$

THINK

b ① To find $\int(2e^{4x} - 5\sin 2x + 4x)\,dx$, on a Calculator page, press:

- MENU (menu)
- 4: Calculus ⟨4⟩
- 3: Integral ⟨3⟩.

Complete the entry line as:

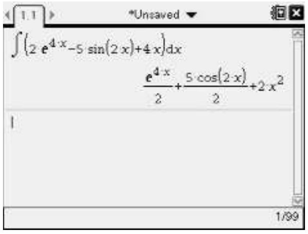
$\int(2e^{4x} - 5\sin 2x + 4x)\,dx$

then press ENTER ⟨enter⟩.

② Write the answer, including the constant of integration.

WRITE/DISPLAY

b



$\int\left(2\cdot e^{4\cdot x}-5\cdot\sin(2\cdot x)+4\cdot x\right)dx$

$\frac{e^{4\cdot x}}{2}+\frac{5\cdot\cos(2\cdot x)}{2}+2\cdot x^2$

$$\int(2e^{4x} - 5\sin 2x + 4x)\,dx$$
$$= \frac{1}{2}e^{4x} + \frac{5}{2}\cos 2x + 2x^2 + c$$

remember

1. $\int e^x\,dx = e^x + c$ and $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$
2. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$ and $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$

EXERCISE 7C Integration of e^x, sin x and cos x

WORKED Example 13

1 Antidifferentiate each of the following.

a e^{2x} **b** e^{4x} **c** e^{-x}

d e^{-3x} **e** $5e^{5x}$ **f** $7e^{4x}$

g $\frac{e^{6x}}{2}$ **h** $\frac{2e^{3x}}{3}$ **i** $-3e^{6x}$

j $-8e^{-2x}$ **k** $e^{\frac{x}{3}}$ **l** $0.1e^{\frac{x}{4}}$

m $3e^{\frac{x}{2}}$ **n** $3e^{\frac{-x}{3}}$ **o** $e^x + e^{-x}$

p $\frac{e^x - e^{-x}}{2}$

2 Find an antiderivative of $(1 + e^x)^2$.

3 Find an antiderivative of $(e^x - 1)^3$.

4 Find an antiderivative of $x^3 - 3x^2 + 6e^{3x}$.

5 **multiple choice**

If $f'(x) = e^{2x} + k$ and $f(x)$ has a stationary point (0, 2), where k is a constant, then:

a k is equal to:

A e **B** e^2 **C** 1 **D** -1 **E** $-e$

b $f(1)$ is equal to:

A $e^2 - 1$ **B** $e^2 + \frac{1}{2}$ **C** $\frac{1}{2}e^2 + \frac{1}{2}$ **D** e^4 **E** $1\frac{1}{2}$

WORKED Example 14

6 Antidifferentiate the following.

a $\sin 3x$ **b** $\sin 4x$ **c** $\cos 7x$

d $\dfrac{\cos 2x}{3}$ **e** $\sin(-2x)$ **f** $\cos(-3x)$

g $\dfrac{4 \sin 6x}{3}$ **h** $8 \cos 4x$ **i** $-6 \sin 3x$

j $-2 \cos(-x)$ **k** $\sin \dfrac{x}{3}$ **l** $\cos \dfrac{x}{2}$

m $3 \sin\left(-\dfrac{x}{4}\right)$ **n** $-2 \sin \dfrac{x}{5}$ **o** $4 \cos \dfrac{x}{4}$

p $-6 \cos\left(-\dfrac{x}{2}\right)$ **q** $4 \sin \dfrac{2x}{3}$ **r** $6 \cos \dfrac{3x}{4}$

s $-2 \sin \dfrac{5x}{2}$ **t** $-3 \cos \dfrac{7x}{4}$ **u** $5 \sin \pi x$

v $3 \cos \dfrac{\pi x}{2}$ **w** $-2 \cos \dfrac{\pi x}{3}$ **x** $-\sin\left(-\dfrac{4x}{\pi}\right)$

WORKED Example 15

7 Find:

a $\int(\sin x + \cos x)\,dx$ **b** $\int(\sin 2x - \cos x)\,dx$

c $\int(\cos 4x + \sin 2x)\,dx$ **d** $\int\left(\sin \dfrac{x}{2} - \cos 2x\right)dx$

e $\int(4 \cos 4x - \frac{1}{3} \sin 2x)\,dx$ **f** $\int(5x + 2 \sin x)\,dx$

g $\int\left(3 \sin \dfrac{\pi x}{2} + 2 \cos \dfrac{\pi x}{3}\right)dx$ **h** $\int(3e^{6x} - 4 \sin 8x + 7)\,dx$

8 Find the antiderivative of $e^{4x} + \sin 2x + x^3$.

9 Find an antiderivative of $3x^2 - 2 \cos 2x + 6e^{3x}$.

10 Antidifferentiate each of the following.

a $x^3 - \dfrac{1}{2x + 3} + e^{2x}$ **b** $x^2 + 4 \cos 2x - e^{-x}$

c $\sin \dfrac{x}{3} + e^{\frac{x}{2}} - (3x - 1)^4$ **d** $\dfrac{1}{3x - 2} + e^{4x} + \cos \dfrac{x}{5}$

e $3 \sin \dfrac{x}{2} - 2 \cos \dfrac{x}{3} - e^{\frac{-x}{5}}$ **f** $\sqrt{x} + 2x - 2 \sin \dfrac{\pi x}{3} + 5$

11 In each of the following find $f(x)$ if:

a $f'(x) = \cos x$ and $f\left(\frac{\pi}{2}\right) = 5$

b $f'(x) = 4 \sin 2x$ and $f(0) = -1$

c $f'(x) = 3 \cos \frac{x}{4}$ and $f(\pi) = 9\sqrt{2}$

d $f'(x) = \cos \frac{x}{4} - \sin \frac{x}{2}$ and $f(2\pi) = -2$.

12 If $\frac{dy}{dx} = \sin \frac{\pi x}{6} + k$, where k is a constant, and y has a stationary point (3, 4), find:

a the value of k

b the equation of the curve

c y when $x = 6$.

13 A curve has a gradient function $f'(x) = 4 \cos 2x + ke^x$, where k is a constant, and a stationary point (0, −1). Find:

a the value of k

b the equation of the curve $f(x)$

c $f\left(\frac{\pi}{6}\right)$ correct to 2 decimal places.

Integration by recognition

As we have seen, if $\frac{d}{dx}[f(x)] = g(x)$

then $$\int g(x)\,dx = f(x) + c, \text{ where } g(x) = f'(x).$$

This result can be used to determine integrals of functions that are too difficult to antidifferentiate, by differentiating a related function instead.

WORKED Example 16

a **Find the derivative of the function $y = (5x + 1)^3$.**

b **Use this result to deduce the antiderivative of $3(5x + 1)^2$.**

eBook plus

Tutorial:
Worked example 16
int-0564

THINK	WRITE
a ① Write the function and recognise that the chain rule can be used.	**a** $y = (5x + 1)^3$
② Let u equal the function inside the brackets.	Let $u = 5x + 1$
③ Find $\frac{du}{dx}$.	$\frac{du}{dx} = 5$
④ Express y in terms of u.	$y = u^3$
⑤ Find $\frac{dy}{du}$.	$\frac{dy}{du} = 3u^2$

Continued over page

THINK	**WRITE**
6 Write the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
7 Find $\frac{dy}{dx}$ using the chain rule.	$\frac{dy}{dx} = 3u^2 \cdot 5$
8 Replace u with the expression inside the brackets and simplify where applicable.	$= 15(5x + 1)^2$
b 1 Since $\int \frac{dy}{dx}\,dx = y + c_1$, express the relationship in integral notation.	**b** $\int 15(5x+1)^2\,dx = (5x+1)^3 + c_1$
2 Remove a factor from $\frac{dy}{dx}$ so that it resembles the integral required.	$5\int 3(5x+1)^2\,dx = (5x+1)^3 + c_1$
3 Divide both sides by the factor in order to obtain the integral required.	$\int 3(5x+1)^2\,dx = \frac{1}{5}(5x+1)^3 + c$, where $c = \frac{c_1}{5}$ Therefore, the antiderivative of $3(5x+1)^2$ is $\frac{1}{5}(5x+1)^3 + c$.

Note that the shorter form of the chain rule below can be used to differentiate.

If $f(x) = [g(x)]^n$ then $f'(x) = ng'(x)[g(x)]^{n-1}$

WORKED Example 17

a **Differentiate e^{x^3}.** **b** **Hence, antidifferentiate $6x^2e^{x^3}$.**

THINK	**WRITE**
a 1 Write the equation and apply the chain rule to differentiate y.	**a** $y = e^{x^3}$
2 Let u equal the index of e.	Let $u = x^3$
3 Find $\frac{du}{dx}$.	$\frac{du}{dx} = 3x^2$
4 Express y in terms of u.	$y = e^u$
5 Find $\frac{dy}{du}$.	$\frac{dy}{du} = e^u$
6 Find $\frac{dy}{dx}$ using the chain rule and replace u.	$\frac{dy}{dx} = 3x^2e^u$, applying the chain rule $= 3x^2e^{x^3}$
b 1 Express $\frac{dy}{dx}$ in integral notation.	**b** $\int 3x^2e^{x^3}\,dx = e^{x^3} + c_1$
2 Multiply both sides by a constant to obtain the integral required.	$2\int 3x^2e^{x^3}\,dx = 2e^{x^3} + 2c_1$ $\int 6x^2e^{x^3}\,dx = 2e^{x^3} + c$, where $c = 2c_1$. Therefore, the antiderivative of $6x^2e^{x^3}$ is $2e^{x^3} + c$.

Note that the shorter form of the chain rule below can be used to differentiate.

$$\textbf{If } y = e^{f(x)} \textbf{ then } \frac{dy}{dx} = f'(x)e^{f(x)}$$

WORKED Example 18

a **Find the derivative of sin (2x + 1) and use this result to deduce the antiderivative of 8 cos (2x + 1).**

b **Differentiate $\log_e(5x^2 - 2)$ and hence antidifferentiate $\frac{x}{5x^2-2}$.**

THINK	WRITE
a ① Let $f(x)$ equal the rule.	**a** $f(x) = \sin(2x + 1)$
② Differentiate using $f'(x) = g'(x)\cos[g(x)]$ where $f(x) = \sin[g(x)]$.	$f'(x) = 2\cos(2x + 1)$
③ Express $f(x)$ using integral notation.	$\int 2\cos(2x+1)\,dx = \sin(2x+1) + c_1$
④ Multiply both sides by whatever is necessary for it to resemble the integral required.	$4\int 2\cos(2x+1)\,dx = 4\sin(2x+1) + c$
⑤ Write the integral in the form in which the question has been asked.	$\int 8\cos(2x+1)\,dx = 4\sin(2x+1) + c$
b ① Let $f(x)$ equal the rule.	**b** $f(x) = \log_e(5x^2 - 2)$
② Differentiate using $f'(x) = \frac{g'(x)}{g(x)}$ where $f(x) = \log_e[g(x)]$.	$f'(x) = \frac{10x}{5x^2-2}$
③ Express $f(x)$ using integral notation.	$\int \frac{10x}{5x^2-2}\,dx = \log_e(5x^2-2) + c_1$
④ Take out a factor so that $f'(x)$ resembles the integral required.	$10\int \frac{x}{5x^2-2}\,dx = \log_e(5x^2-2) + c_1$
⑤ Divide both sides by the factor to obtain the required integral.	$\int \frac{x}{5x^2-2}\,dx = \frac{1}{10}\log_e(5x^2-2) + c$ The antiderivative of $\frac{x}{5x^2-2}$ is $\frac{1}{10}\log_e(5x^2-2) + c$

WORKED Example 19

Differentiate x cos x and hence find an antiderivative of x sin x.

THINK	WRITE
① Write the rule.	Let $y = x\cos x$
② Apply the product rule to differentiate $x\cos x$.	$\frac{dy}{dx} = x(-\sin x) + (\cos x)(1)$ $= -x\sin x + \cos x$ $= \cos x - x\sin x$

Continued over page ➔

THINK	WRITE
3 Express the result in integral notation. (Do not add c, as an antiderivative is required.)	$\therefore \int (\cos x - x \sin x)\,dx = x \cos x$
4 Express the integral as two separate integrals.	$\int \cos x\,dx - \int x \sin x\,dx = x \cos x$
5 Simplify by integrating. (Do not add c.)	$\sin x - \int x \sin x\,dx = x \cos x$
6 Make the expression to be integrated the subject of the equation.	$-\int x \sin x\,dx = x \cos x - \sin x$
7 Simplify.	$\int x \sin x\,dx = \sin x - x \cos x$ Therefore, an antiderivative of $x \sin x$ is $\sin x - x \cos x$.

WORKED Example 20

a **Show that $\dfrac{5x+1}{x+1} = 5 - \dfrac{4}{x+1}$.** **b** **Hence, find $\displaystyle\int \frac{5x+1}{x+1}\,dx$.**

THINK	WRITE
a Use algebraic long division to divide the denominator into the numerator.	**a** $\begin{array}{r} 5 \\ x+1\,\overline{)\,5x+1} \\ \underline{5x+5} \\ -4 \end{array}$ So $\dfrac{5x+1}{x+1} = 5 - \dfrac{4}{x+1}$
b 1 Write the expression using integral notation.	**b** $\displaystyle\int \frac{5x+1}{x+1}\,dx = \int \left(5 - \frac{4}{x+1}\right)$
2 Express as two separate integrals.	$\displaystyle = \int 5\,dx - \int \frac{4}{x+1}\,dx$
3 Antidifferentiate each part.	$= 5x - 4 \log_e (x+1) + c$

remember

1. To differentiate using the chain rule, use one of the following rules.
   (a) If $f(x) = [g(x)]^n$ then $f'(x) = ng'(x)[g(x)]^{n-1}$
   (b) If $y = e^{f(x)}$, $\dfrac{dy}{dx} = f'(x)e^{f(x)}$
   (c) $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
   (d) $f'(x) = g'(x) \cos [g(x)]$ where $f(x) = \sin [g(x)]$
   $f'(x) = -g'(x) \sin [g(x)]$ where $f(x) = \cos [g(x)]$
   (e) $f'(x) = \dfrac{g'(x)}{g(x)}$ where $f(x) = \log_e[g(x)]$.
2. To antidifferentiate use $\int g(x)\,dx = f(x) + c$ where $g(x) = f'(x)$.

EXERCISE 7D Integration by recognition

WORKED Example 16

1 For each of the following, find the derivative of the function in **i** and use this result to deduce the antiderivative of the function in **ii**.

a **i** $(3x - 2)^8$ **ii** $12(3x - 2)^7$

b **i** $(x^2 + 1)^5$ **ii** $5x(x^2 + 1)^4$

c **i** $\sqrt{2x - 5}$ **ii** $\dfrac{1}{\sqrt{2x - 5}}$

d **i** $\sqrt{4x + 3}$ **ii** $\dfrac{2}{\sqrt{4x + 3}}$

e **i** $(x^2 + 3x - 7)^4$ **ii** $(2x + 3)(x^2 + 3x - 7)^3$

f **i** $\dfrac{1}{x^2 - 1}$ **ii** $\dfrac{4x}{(x^2 - 1)^2}$

2 **multiple choice**

The derivative of $(x + 7)^4$ is $4(x + 7)^3$.

a Therefore, the antiderivative of $4(x + 7)^3$ is:

A $(x + 7)^4 + c$ **B** $\frac{1}{4}(x + 7)^4 + c$ **C** $4(x + 7)^4 + c$

D $3(x + 7)^4 + c$ **E** $12(x + 7)^4 + c$

b The antiderivative of $(x + 7)^3$ is:

A $(x + 7)^4 + c$ **B** $\frac{1}{4}(x + 7)^4 + c$ **C** $4(x + 7)^4 + c$

D $3(x + 7)^4 + c$ **E** $12(x + 7)^4 + c$

3 **multiple choice**

If the derivative of $(2x - 3)^6$ is $12(2x - 3)^5$, then $\int 6(2x - 3)^5\,dx$ is:

A $2(2x - 3)^6 + c$ **B** $4(2x - 3)^6 + c$ **C** $(2x - 3)^6 + c$

D $6(2x - 3)^6 + c$ **E** $\frac{1}{2}(2x - 3)^6 + c$

WORKED Example 17

4 For each of the following, differentiate **i** and hence antidifferentiate **ii**.

a **i** $e^{4x - 5}$ **ii** $2e^{4x - 5}$

b **i** $e^{6 - 5x}$ **ii** $10e^{6 - 5x}$

c **i** e^{x^2} **ii** $x\ e^{x^2}$

d **i** $e^{x - x^2}$ **ii** $(1 - 2x)e^{x - x^2}$

WORKED Example 18

5 For each of the following, find the derivative of the function in **i** and use this result to deduce the antiderivative of the function in **ii**.

a **i** $\sin(x - 5)$ **ii** $\cos(x - 5)$

b **i** $\sin(3x + 2)$ **ii** $6\cos(3x + 2)$

c **i** $\cos(4x - 7)$ **ii** $\sin(4x - 7)$

d **i** $\cos(6x - 3)$ **ii** $3\sin(6x - 3)$

e **i** $\sin(2 - 5x)$ **ii** $10\cos(2 - 5x)$

f **i** $\cos(3 - 4x)$ **ii** $-2\sin(3 - 4x)$

g **i** $\log_e(5x + 2)$ **ii** $\dfrac{20}{5x + 2}$

h **i** $\log_e(x^2 + 3)$ **ii** $\dfrac{12x}{x^2 + 3}$

i **i** $\log_e(x^2 - 4x)$ **ii** $\dfrac{x - 2}{x^2 - 4x}$

WORKED Example 19

6 Differentiate **i** and hence find an antiderivative of **ii**.

a **i** $x \cos x + 2 \sin x$ **ii** $x \sin x$

b **i** $\dfrac{\sin x}{x}$ **ii** $\dfrac{2(x \cos x - \sin x)}{x^2}$

c **i** $e^x \sin x$ **ii** $3e^x (\sin x + \cos x)$

d **i** $x \sin x$ **ii** $x \cos x$

e **i** $x\, e^x$ **ii** $x\, e^x$

7 For each of the following, differentiate **i** and use this result to antidifferentiate **ii**.

a **i** $(2x - 3x^2)^6$ **ii** $6x^5(1 - 3x)(2 - 3x)^5$

b **i** $\sqrt{x^3 + 2x}$ **ii** $\dfrac{3x^2 + 2}{\sqrt{x^3 + 2x}}$

WORKED Example 20

8 **a** Show that $\dfrac{3x-2}{x-1} = 3 + \dfrac{1}{x-1}$. **b** Hence, find $\displaystyle\int \frac{3x-2}{x-1}\, dx$.

9 **a** Show that $\dfrac{5x+8}{x+2} = 5 - \dfrac{2}{x+2}$. **b** Hence, find $\displaystyle\int \frac{5x+8}{x+2}\, dx$.

10 **a** Show that $\dfrac{8x-7}{2x-3} = 4 + \dfrac{5}{2x-3}$. **b** Hence, find $\displaystyle\int \frac{8x-7}{2x-3}\, dx$.

11 **a** Show that $\dfrac{6x-5}{3-2x} = -3 + \dfrac{4}{3-2x}$. **b** Hence, find $\displaystyle\int \frac{6x-5}{3-2x}\, dx$.

12 If $y = \log_e (\cos x)$:

a find $\dfrac{dy}{dx}$ **b** hence find $\displaystyle\int \tan x\, dx$.

13 Differentiate $\dfrac{\cos x}{\sin x}$ and hence find an antiderivative of $\dfrac{1}{\sin^2 x}$.

14 Differentiate $\log_e (3x^2 - 4)$ and hence find an antiderivative of $\dfrac{x}{3x^2 - 4}$.

15 Differentiate $\sin (ax + b)$ and hence find an antiderivative of $\cos (ax + b)$. (Here, a and b are constants.)

16 Differentiate $\cos (ax + b)$ and hence find an antiderivative of $\sin (ax + b)$. (Here, a and b are constants.)

17 Differentiate $e^{ax + b}$ and hence find an antiderivative of $e^{ax + b}$. (Here, a and b are constants.)

eBook plus

Digital doc: WorkSHEET 7.2

18 Antidifferentiate each of the following.

a $\sin (3\pi x + 1)$ **b** $\cos (1 - 4\pi x)$ **c** $e^{\pi x + 3}$

d $\sin \left(2 + \dfrac{\pi x}{3}\right)$ **e** $3 \cos \left(\dfrac{\pi x}{2} + 5\right)$ **f** $\cos x\, e^{\sin x}$

summary

Approximating areas under curves

- An approximation to the area between a curve and the x-axis can be found by dividing the area into a series of rectangles or trapeziums which are all the same width. The approximation is found by finding the sum of all the areas of the rectangles or trapeziums.
- Lower rectangle approximation $\le$ actual area $\le$ upper rectangle approximation
- Trapezoidal approximation =

$$\frac{\text{lower rectangle approximation + upper rectangle approximation}}{2}$$

- Trapezoidal rule is:
  The area between a curve, $y = f(x)$, and the x-axis from

$$x = a \text{ to } x = b \cong \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \quad + 2f(b-h) + f(b)]$$

  where h is the interval width.

Monte Carlo method

- The Monte Carlo method is used to approximate area by randomly generating a number of points. The following procedure is used.
  Step 1 Surround the curved region by a rectangle of known dimensions.
  Step 2 Randomly generate a number of points within the rectangle.
  Step 3 $\dfrac{\text{Area of curved region}}{\text{Area of rectangle}} = \dfrac{\text{Number of points within curved region}}{\text{Total number of points}}$

Antidifferentiation rules

- The relationships between $f(x)$ and $\int f(x)\,dx$ are:

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
a	$ax + c$	e^x	$e^x + c$
ax^n	$\frac{ax^{n+1}}{n+1} + c$	e^{kx}	$\frac{1}{k}e^{kx} + c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$	$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\frac{1}{x}$	$\log_e x + c$	$\cos ax$	$\frac{1}{a}\sin ax + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\log(ax+b) + c$		

- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int kf(x)\,dx = k\int f(x)\,dx$
- $\int g(x)dx = f(x) + c$, where $g(x) = f'(x)$ dx
- $\int f(x)\,dx$ is the indefinite integral

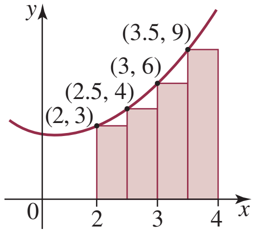
CHAPTER review

7A

1 **multiple choice**

In the figure at right, the approximation for the area under the curve from $x = 2$ to $x = 4$, using the 'lower rectangles' is:

A 22 sq. units **B** 14 sq. units **C** 11 sq. units
D 10 sq. units **E** 20 sq. units

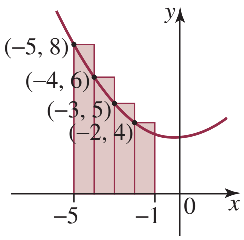


7A

2 **multiple choice**

In the figure at right, the area under the curve from $x = -5$ to $x = -1$ can be approximated by the area of the 'upper rectangles' and is equal to:

A 20 sq. units **B** 21 sq. units **C** 23 sq. units
D $11\frac{1}{2}$ sq. units **E** 10 sq. units

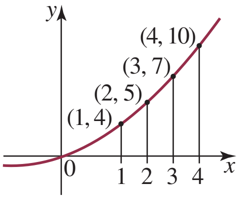


7A

3 **multiple choice**

A student is using the trapezoidal rule to find the area under the curve at right from $x = 1$ to $x = 4$. His answer should be approximately equal to:

A 26 sq. units **B** 16 sq. units **C** 22 sq. units
D 17 sq. units **E** 19 sq. units



7A

4 Apply the trapezoidal method to find the area between $y = e^{2x-1}$ and the x-axis, from $x = 0$ to $x = 4$, using intervals that are 1 unit wide. (Give an exact answer.)

7A

5 Calculate an approximation for the area under the curve $y = \log_e x$ from $x = 2$ to $x = 4$, using interval widths of 0.5 units.

7B

6 **multiple choice**

The antiderivative of $4x^3 - \frac{1}{1-x}$ is:

A $x^4 - \log_e(1 - x) + c$ **B** $x^4 + \log_e(1 - x) + c$ **C** $16x^4 - \log_e(1 - x) + c$
D $16x^4 + \log_e(1 - x) + c$ **E** $x^4 - \frac{1}{(1-x)^2} + c$

7 **multiple choice** 7B

The indefinite integral $\int (5x-4)^4\,dx$ is equal to:

A $25(5x-4)^5 + c$ **B** $5(5x-4)^5 + c$ **C** $(5x-4)^5 + c$

D $\frac{1}{5}(5x-4)^5 + c$ **E** $\frac{1}{25}(5x-4)^5 + c$

8 **multiple choice** 7B

An antiderivative of $2(3x+4)^{-4}$ is:

A $-\frac{2}{3}(3x+4)^{-3}$ **B** $-\frac{2}{3}(3x+4)^{-3} + 5$ **C** $-\frac{2}{9}(3x+4)^{-3}$

D $-\frac{2}{9}(3x+4)^{-5}$ **E** $-8(3x+4)^{-3}$

9 Find the equation of the curve $f(x)$ if it passes through $(1, -3)$ and $f'(x) = \dfrac{3x^3 - 2x^2}{x}$. 7B

10 **multiple choice** 7C

The antiderivative of $6e^{-3x}$ is:

A $-2e^{-3x} + c$ **B** $-3e^{-3x} + c$ **C** $-18e^{-3x} + c$

D $-2e^{-3x+1} + c$ **E** $-\frac{1}{2}e^{-3x} + c$

11 **multiple choice** 7C

The indefinite integral $\int \left(\cos\frac{x}{3} - 3\sin 3x\right)dx$ is equal to:

A $\sin\frac{x}{3} + \cos 3x + c$ **B** $\frac{1}{3}\sin\frac{x}{3} + 3\cos 3x + c$ **C** $-3\sin\frac{x}{3} + \cos 3x + c$

D $3\sin\frac{x}{3} + \cos 3x + c$ **E** $3\sin\frac{x}{3} - \cos 3x + c$

12 **multiple choice** 7C

An antiderivative of $x^3 + \sin 4x + e^{4x}$ is:

A $4(x^4 - \cos 4x + e^{4x})$ **B** $\frac{1}{4}x^4 + \cos 4x + \frac{1}{4}e^{4x}$ **C** $\frac{1}{4}x^4 - 4\cos 4x + \frac{1}{4}e^{4x}$

D $\frac{1}{4}(x^4 - \cos x + e^{4x})$ **E** $\frac{1}{4}(x^4 - \cos 4x + e^{4x})$

13 **multiple choice** 7C

If $f(x)$ has a stationary point at $(0, 3)$ and $f'(x) = e^x + k$, where k is a constant, then $f(x)$ is:

A $e^x - 2x + 2$ **B** $-e^x - x + 2$ **C** $e^x - x + 2$

D $e^{-x} + x + 2$ **E** $e^x + 2x + 1$

14 A particular curve has $\dfrac{dy}{dx} = \cos\dfrac{\pi x}{4} + k$, where k is a constant. It also has a stationary point $(2, 1)$. Find: 7C

a the value of k

b the equation of the curve

c the value of y when $x = 6$.

7D

15 **multiple choice**

If the derivative of $(x - x^2)^8$ is $8(1 - 2x)(x - x^2)^7$ then an antiderivative of $24(1 - 2x)(x - x^2)^7$ is:

A $2(x - x^2)^8$ **B** $3(x - x^2)^8$ **C** $\frac{1}{2}(x - x^2)^8$
D $\frac{1}{3}(x - x^2)^8$ **E** $8(x - x^2)^8$

7D

16 **multiple choice**

If the derivative of $e^{x^3 + 3x}$ is $3(x^2 + 1)e^{x^3 + 3x}$, then the antiderivative of $(x^2 + 1)e^{x^3 + 3x}$ is:

A $3e^{x^3 + 3x} + c$ **B** $\dfrac{1}{3(x^2 + 1)} + c$ **C** $e^{3x^2 + 3} + c$
D $\frac{1}{3}e^{x^3 + 3x} + c$ **E** undefined

7D

17 **multiple choice**

If the derivative of $\log_e(5 - x^2)$ is $\dfrac{-2x}{5 - x^2}$ then the antiderivative of $\dfrac{x}{5 - x^2}$ is:

A $-\frac{1}{2}\log_e(5 - x^2) + c$ **B** $-2\log_e(5 - x^2) + c$ **C** $\frac{1}{2}\log_e(5 - x^2) + c$
D $2\log_e(5 - x^2) + c$ **E** undefined

7D

18 If $y = \sin(x^2 + 2x)$, find $\dfrac{dy}{dx}$ and hence antidifferentiate $(x + 1)\cos(x^2 + 2x)$.

Modelling and problem solving

1 From past records it has been found that the cost rate of maintaining a certain car is $\dfrac{dC}{dt} = 75t^2 + 50t + 800$, where C is the accumulated cost in dollars and t is the time in years since the car was first used. Find:

a the initial maintenance cost
b C as a function of t
c the total maintenance cost during the first 5 years of use of the car
d the total maintenance cost from 3 to 5 years
e the maintenance cost for the second year.

2 Over a 24-hour period on a particular March day, starting at 12 am, the rate of change of the temperature for Brisbane was approximately $\dfrac{dT}{dt} = -\dfrac{5\pi}{12}\cos\dfrac{\pi t}{12}$, where T is the temperature in °C and t is the number of hours since midnight (when the temperature was 20 °C). Find:

a the temperature at any time, t
b whether the temperature reaches 13 °C at any time during the day
c the maximum temperature and the time at which it occurs
d the minimum temperature and the time at which it occurs

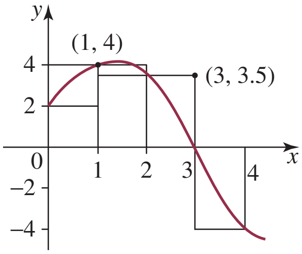
- **e** the temperature at
  - **i** 2 am
  - **ii** 3 pm
- **f** the time when the temperature first reaches 22.3 °C.

3 An infection is transferred such that the rate of the number of people infected, N, can be modelled by $\frac{dN}{dt} = 0.16t$ where t is the number of days after exposure to the infection. In a school of 800 students, how many students will not be infected after 20 days?

4 A mothball, assumed to be a sphere with diameter 20 mm, evaporates at such a rate that the radius decreases by 0.2 mm per day.

- **a** Find an expression for the radius of the mothball t days after manufacture.
- **b** Hence find an expression for the volume of the mothball t days after manufacture.
- **c** Find the rate at which the volume decreases 30 days after it was manufactured.

5 Determine an approximation for the area between the curve and the x-axis over the interval indicated in the diagram at right using the average of the upper and lower rectangles.



6 An oil slick is found to radiate outwards at a rate modelled by $\frac{dr}{dt} = \frac{5}{\sqrt{t}}$, $t \geq 1$, where t is the time measured in hours and r is the radius of the slick in metres. If the slick is 16 metres wide after one hour, how long, to the nearest hour, will it take to be 100 metres wide?

eBook*plus*

Digital doc:
Test Yourself
Chapter 7

eBook*plus* ACTIVITIES

7A Approximating areas enclosed by functions

Digital doc

- SkillSHEET 7.1: Practise calculations of area and perimeter of composite functions (*page 236*)

Interactivity

- Approximating areas enclosed by functions int-0254: Consolidate your understanding of approximating areas enclosed by functions (*page 236*)

Tutorial

- **WE1** Int-0565: Watch a tutorial on the approximation of area under a curve (*page 237*)

7B Antidifferentiation (integration)

Digital docs

- SkillSHEET 7.2: Practise substitution and evaluation (*page 258*)
- WorkSHEET 7.1: Approximate areas under curves using various techniques, antidifferentiate expressions and calculate definite integrals (*page 258*)

7D Integration by recognition

Digital doc

- WorkSHEET 7.2: Antidifferentiate exponential, logarithmic and trigonometric expressions and use integration by recognition (*page 268*)

Tutorial

- **WE16** Int-0564: Watch a tutorial on using integration by recognition (*page 263*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 273*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

Techniques of integration

8

syllabus reference

Introduction to integration

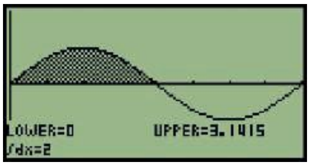
In this chapter

Introduction

The graphics calculator — a discrepancy?

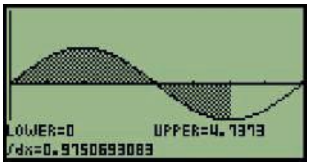
The graphics calculator can be used to calculate the area bounded by a curve. The exact details of how the calculator can be used will be discussed later.

If we use the calculator to calculate the area bounded by the curve $y = \sin(x)$ and the x-axis between $x = 0$ and $x = \pi$ (see shaded area), the calculator returns the value 2.



```
LOWER=0          UPPER=3.1415
∫dx=2
```

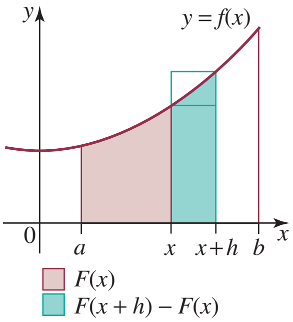
If we use the calculator to find the area bounded by the curve $y = \sin(x)$ and the x-axis between $x = 0$ and $x = \dfrac{3\pi}{2}$, the calculator returns the value 1 (or very close to this). Common sense would tell us that the second area should be greater. How can this be?



```
LOWER=0          UPPER=4.7373
∫dx=0.9750693083
```

In the following sections we shall see the reason for this apparent discrepancy.

The fundamental theorem of integral calculus



Consider the region under the curve $f(x)$ between $x = a$ and $x = b$, where $f(x) \geq 0$ and is continuous for $a \leq x \leq b$.

Let $F(x)$ be the function that is the measure of the area under the curve between a and x.

$F(x + h)$ is the area under the curve between a and $x + h$ and $F(x + h) - F(x)$ is the area of the strip indicated on the graph at right.

The area of the strip is between the areas of the upper and lower rectangles; that is, $f(x)h < F(x+h) - F(x) < f(x+h)h$

or $$f(x) < \frac{F(x+h) - F(x)}{h} < f(x+h),\ h \neq 0 \text{ (dividing by } h\text{)}$$

As $h \to 0$, $f(x + h) \to f(x)$

or $$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

that is, $F'(x) = f(x)$ (differentiation from first principles).

Therefore, $$F(x) = \int f(x)\,\mathrm{d}x$$

that is, $F(x)$ is an antiderivative of $f(x)$

or $$\int f(x)\,\mathrm{d}x = F(x) + c$$

but when $x = a$,

$$\int f(x)\,\mathrm{d}x = F(a) + c$$

$$= 0 \text{ (as the area defined is zero at } x = a\text{)}$$

or $$c = -F(a).$$

Therefore, $\int f(x)\,\mathrm{d}x = F(x) - F(a)$

and when $x = b$, $\int f(x)\,\mathrm{d}x = F(b) - F(a)$.

That is, the area under the graph of $f(x)$ between $x = a$ and $x = b$ is $F(b) - F(a)$.

This is *the fundamental theorem of integral calculus* and it enables areas under graphs to be calculated exactly. It can be stated as:

$$\begin{aligned}\text{Area} &= \int_a^b f(x)\,\mathrm{d}x \\ &= [F(x)]_a^b \quad \text{[do not add } c \text{ as } F(x) \text{ is an antiderivative of } f(x)\text{]} \\ &= F(b) - F(a)\end{aligned}$$

a and b are called the *terminals* of this definite integral.

$\int_a^b f(x)\,\mathrm{d}x$ is called the *definite integral* because it can be expressed in terms of its terminals a and b, which are usually real numbers. In this case the definite integral evaluates as a real number and not a function.

The function being integrated, $f(x)$, is called the *integrand*.

Definite integrals

1 Evaluate $\int_1^1 2x\,\mathrm{d}x$.

2 Evaluate **a** $\int_1^4 2x\,\mathrm{d}x$ **b** $\int_1^2 2x\,\mathrm{d}x + \int_2^4 2x\,\mathrm{d}x$.

c Compare answers **a** and **b**.

3 **a** Evaluate $2\int_1^4 x\,\mathrm{d}x$. **b** Compare the answer to the answer to **2a**.

4 Evaluate **a** $\int_0^3 [2x + x^2]\,\mathrm{d}x$ **b** $\int_0^3 2x\,\mathrm{d}x + \int_0^3 x^2\,\mathrm{d}x$.

c Compare answers **a** and **b**.

5 Evaluate **a** $\int_1^3 2x\,\mathrm{d}x$ **b** $\int_3^1 2x\,\mathrm{d}x$.

c Compare answers **a** and **b**.

Properties of definite integrals

Definite integrals have the following five properties.

eBook plus

Digital doc:
Investigation
Definite integrals

1. $\int_a^a f(x)\,\mathrm{d}x = 0$

2. $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x,\ a < c < b$

3. $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$

4. $\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

5. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

WORKED Example 1

eBook*plus*

Tutorial:
Worked example 1
int-0566

Evaluate the following definite integrals.

a $\int_0^3 (3x^2+4x-1)\,\mathrm{d}x$

b $\int_{-1}^2 \frac{4}{(2x+1)^3}\,\mathrm{d}x$

THINK	WRITE
a ① Antidifferentiate each term of the integrand and write in the form $[F(x)]_a^b$.	**a** $\int_0^3 (3x^2+4x-1)\,\mathrm{d}x$ $= [x^3+2x^2-x]_0^3$
② Substitute values of *a* and *b* into *F*(*b*) – *F*(*a*).	$= [3^3+2(3)^2-3]-[0^3+2(0)^2-0]$
③ Evaluate the integral.	$= 42 - 0$ $= 42$
b ① Express the integrand in simplest index form.	**b** $\int_{-1}^2 \frac{4}{(2x+1)^3}\,\mathrm{d}x = \int_{-1}^2 4(2x+1)^{-3}\,\mathrm{d}x$
② Antidifferentiate by rule.	$= \left[\frac{4(2x+1)^{-2}}{2\cdot\ -2}\right]_{-1}^{2}$ $= [-(2x+1)^{-2}]_{-1}^{2}$
③ Express the integral with a positive index number.	$= \left[\frac{-1}{(2x+1)^2}\right]_{-1}^{2}$
④ Substitute values of *a* and *b* into *F*(*b*) – *F*(*a*).	$= \left[-\frac{1}{5^2}\right]-\left[\frac{-1}{(-1)^2}\right]$
⑤ Evaluate the integral.	$= -\frac{1}{25}+1$ $= \frac{24}{25}$

WORKED Example 2

Find the exact value of each of the following definite integrals.

a $\int_{\pi}^{2\pi} \sin \frac{x}{6}\, dx$ **b** $\int_{-1}^{1} (e^{3x} - e^{-3x})\, dx$

THINK	WRITE
a ① Antidifferentiate the integrand, writing it in the form $[F(x)]_a^b$.	**a** $\int_{\pi}^{2\pi} \sin \frac{x}{6}\, dx = \left[-6 \cos \frac{x}{6}\right]_{\pi}^{2\pi}$
② Substitute values of *a* and *b* into $F(b) - F(a)$.	$= \left(-6 \cos \frac{2\pi}{6}\right) - \left(-6 \cos \frac{\pi}{6}\right)$ $= \left(-6 \cos \frac{\pi}{3}\right) - \left(-6 \cos \frac{\pi}{6}\right)$
③ Evaluate the integral.	$= \left(-6 \cdot \frac{1}{2}\right) - \left(-6 \cdot \frac{\sqrt{3}}{2}\right)$ $= (-3) - (-3\sqrt{3})$ $= -3 + 3\sqrt{3}$
b ① Antidifferentiate the integrand, using $[F(x)]_a^b$.	**b** $\int_{-1}^{1} (e^{3x} - e^{-3x})\, dx$ $= [\frac{1}{3}e^{3x} + \frac{1}{3}e^{-3x}]_{-1}^{1}$
② Substitute values of *a* and *b* into $F(a) - F(b)$.	$= (\frac{1}{3}e^{3} + \frac{1}{3}e^{-3}) - (\frac{1}{3}e^{-3} + \frac{1}{3}e^{3})$
③ Evaluate.	$= \frac{1}{3}e^{3} + \frac{1}{3}e^{-3} - \frac{1}{3}e^{-3} - \frac{1}{3}e^{3}$ $= 0$

Graphics Calculator tip! Finding definite integrals

It is possible to find the numerical value for the definite integral using the graphics calculator. Use the following steps to evaluate $\int_{\pi}^{2\pi} \sin\left(\frac{x}{6}\right) dx$.

For the Casio *fx*-9860G AU

1. Ensure that the calculator is in radian mode, then press:
   - MENU
   - 1: RUN
   - OPTN
   - F4 (CALC)
   - F4 ($\int dx$).

   Complete the entry line as:

   $\int\left(\sin\left(\frac{x}{6}\right), \pi, 2\pi\right)$

   then press EXE.

```
∫(sin (X÷6),π,2π)
                2.196152423
```

2. Write the answer.

$$\int_{\pi}^{2\pi} \sin\left(\frac{x}{6}\right) dx = 2.196$$

For the TI-Nspire CAS

1. On a Calculator page, press:
   - MENU menu
   - 4: Calculus 4
   - 3: Integral 3.

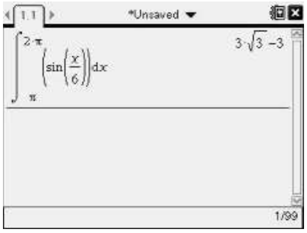
   Complete the entry line as:

   $$\int_{\pi}^{2\pi}\left(\sin\left(\frac{\pi}{6}\right)\right)dx .$$

   Note: To move between fields, press Tab tab.

   Then press ENTER enter.

   To obtain an approximate value for the integral, press:
   - Ctrl ctrl
   - ENTER enter.



$\int_{\pi}^{2\cdot\pi}\left(\sin\left(\frac{x}{6}\right)\right)dx$ $3\cdot\sqrt{3}-3$

2. Write the answer.

$$\int_{\pi}^{2\pi} \sin\left(\frac{x}{6}\right) dx = 2.196$$

WORKED Example 3

If $\int_0^k 8x \, dx = 36$, find k: a by hand b using the TI-Nspire CAS calculator.

THINK

WRITE/DISPLAY

a ① Antidifferentiate the integrand, using $[F(x)]_a^b$.

$$\int_0^k 8x\,dx = [4x^2]_0^k$$

② Substitute the values of a and b into $F(a) - F(b)$.

$$\text{So } [4x^2]_0^k = 36$$

$$4k^2 - 4(0)^2 = 36$$

③ Simplify the integral.

$$4k^2 - 0 = 36$$

$$4k^2 = 36$$

$$k^2 = 9$$

$$k = \pm\sqrt{9}$$

④ Solve the equation.

$$k = 3 \text{ or } -3$$

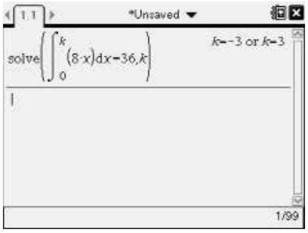
b ① On a Calculator page, press:
- MENU menu
- 3: Algebra 3
- 1: Solve 1.

Complete the entry line as:

$$\text{solve}(\int_0^k (8x)dx = 36, k)$$

then press ENTER enter.

b



$\text{solve}\left(\int_0^k (8\cdot x)dx=36,k\right)$ $k=-3 \text{ or } k=3$

② Write the answer.

Solving $\int_0^k (8x)dx = 36$ for k gives

$k = \pm 3$.

remember

1. The fundamental theorem of calculus is $\int_a^b f(x)\,dx = [F(x)]_a^b$
$= F(b) - F(a)$
where $F(x)$ is an antiderivative of $f(x)$.
2. The expression $\int_a^b f(x)\,dx$ is called the *definite integral* where a and b are the terminals and represent the lower and upper values of x.
3. Properties of definite integrals
   (a) $\int_a^a f(x)\,dx = 0$
   (b) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $a < c < b$
   (c) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
   (d) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
   (e) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

EXERCISE 8A The fundamental theorem of integral calculus

WORKED Example 1

eBook plus

Digital doc:
SkillSHEET 8.1
Subtracting function values

1 Evaluate the following definite integrals.

a $\int_0^1 x^2\,dx$ **b** $\int_0^3 x^3\,dx$ **c** $\int_3^4 (x^2 - 2x)\,dx$

d $\int_2^6 \frac{1}{x^2}\,dx$ **e** $\int_0^2 (x^3 + 3x^2 - 2x)\,dx$ **f** $\int_1^4 (3x^{\frac{1}{2}} + 2x^2)\,dx$

g $\int_{-1}^1 (6 - 2x + x^2)\,dx$ **h** $\int_{-4}^{-2} (x^3 + x - 4)\,dx$ **i** $\int_4^9 3\sqrt{x}\,dx$

j $\int_1^2 (4x^{-2} + 2x - 6)\,dx$ **k** $\int_0^3 2(x+4)^4\,dx$ **l** $\int_{-1}^2 3(5x-2)^4\,dx$

m $\int_{-2}^0 -4(2-3x)^3\,dx$ **n** $\int_0^3 \frac{5}{(2x-7)^3}\,dx$ **o** $\int_1^4 \left(2x^{\frac{3}{2}} - 3x^{-1}\right)dx$

p $\int_1^3 \frac{2x^3 + 5x^2}{x}\,dx$ **q** $\int_1^5 \frac{3}{5x}\,dx$ **r** $\int_0^2 \frac{-4}{(3x-4)^5}\,dx$

s $\int_3^7 \frac{1}{\sqrt{2x-5}}\,dx$ **t** $\int_{-2}^0 \frac{6}{\sqrt{8-3x}}\,dx$

WORKED Example 2

2 Find the exact value of each of the following definite integrals.

a $\int_0^2 e^{4x}\,dx$ **b** $\int_{-2}^0 e^{\frac{x}{3}}\,dx$ **c** $\int_{-1}^1 -4e^{-2x}\,dx$

d $\int_1^2 (3e^{6x} + 5x)\,dx$ **e** $\int_1^4 \left(\frac{5}{x} + e^{\frac{x}{2}}\right)dx$ **f** $\int_{-3}^{-1} (e^{2x} - e^{-2x})\,dx$

g $\int_0^{\frac{\pi}{2}} \sin x \, dx$ **h** $\int_{\frac{\pi}{2}}^{\pi} 3 \sin 4x \, dx$ **i** $\int_0^{\pi} 5 \sin \frac{x}{4} \, dx$

j $\int_{\pi}^{2\pi} -2 \sin \frac{x}{3} \, dx$ **k** $\int_{-\pi}^{0} \cos 2x \, dx$ **l** $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8 \cos 4x \, dx$

m $\int_{\pi}^{2\pi} 3 \cos \frac{x}{6} \, dx$ **n** $\int_{-2\pi}^{0} -7 \cos \frac{x}{2} \, dx$ **o** $\int_{-\pi}^{-\frac{\pi}{2}} -4 \cos 3x \, dx$

p $\int_0^{\pi} \pi\left(2 + \sin \frac{x}{4}\right) dx$ **q** $\int_1^3 (x^2 + 3 - 6 \sin 3x) \, dx$ **r** $\int_1^{\pi}\left(\frac{1}{x} + 3 \cos \frac{x}{2}\right) dx$

3 If $\int_1^5 f(x) \, dx = 6$, find the value of $\int_1^5 3f(x) \, dx$.

4 **multiple choice**

Given that $\int_1^5 f(x) \, dx = 6$,

a $\int_1^5 [f(x) + 1] \, dx$ is equal to:

A 16 **B** 10 **C** 11 **D** 19 **E** 22

b $\int_5^1 f(x) \, dx$ is equal to:

A –6 **B** 5 **C** –5 **D** 6 **E** 0

5 Evaluate the following.

a $\int_0^3 (t^2 - 4t) \, dt$ **b** $\int_0^{\frac{\pi}{2}} 2 \cos 3t \, dt$ **c** $\int_4^7 \frac{3}{(p-3)^{\frac{3}{2}}} \, dp$

d $\int_{5\pi}^{10\pi}\left(-\sin \frac{x}{5}\right) dx$ **e** $\int_0^{\pi}\left(e^{\frac{x}{4}} - \cos 2x\right) dx$ **f** $\int_1^2 \frac{8}{4m-3} \, dm$

g $\int_1^4 \frac{-3}{t\sqrt{t}} \, dt$ **h** $\int_{-1}^{1} (3 \sin 2x - e^{-3x}) \, dx$

WORKED Example 3

6 If $\int_0^k (2x + 3) \, dx = 4$, find k.

7 If $\int_0^k 3x^2 \, dx = 8$, find k.

8 If $\int_1^k \frac{2}{x} \, dx = \log_e 9$, find k.

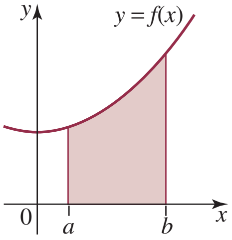
9 If $\int_0^a e^{\frac{x}{2}} \, dx = 4$, find the value of a.

10 If $\int_k^{\pi} \cos 2x \, dx = -\frac{\sqrt{3}}{4}$, find the value of k given that $0 < k < \frac{\pi}{2}$.

Signed areas

When we are calculating areas between the graph of a function $f(x)$ and the x-axis using the definite integral $\int_a^b f(x)\,dx$, the area is signed; that is, it is positive or negative. If $f(x) > 0$, the region is above the x-axis; if $f(x) < 0$ it is below the axis. We shall now examine these two situations and look at how we calculate the area of regions that include both.

Region above axis

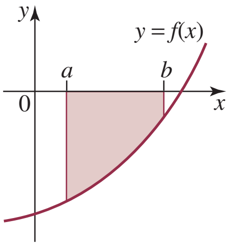


If $f(x) > 0$, that is, the region is above the x-axis, then

$\int_a^b f(x)\,dx > 0$, so the value of the definite integral is positive.

For example, if $f(x) > 0$, then the area $= \int_a^b f(x)\,dx$.

Region below axis



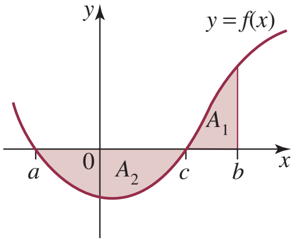
If $f(x) < 0$, that is, the region is below the x-axis, then

$\int_a^b f(x)\,dx < 0$, so the value of the definite integral is negative.

For example, if $f(x) < 0$, then the area $= -\int_a^b f(x)\,dx$, as the region is below the x-axis or area $= \int_b^a f(x)\,dx$ (reversing the terminals changes the sign).

Therefore, for areas below the x-axis, the sign of the definite integral must be made negative, or the terminals reversed, to ensure that the area has a positive value. (Areas cannot be negative.)

Combining regions



For regions which are combinations of areas above and below the x-axis, each area has to be calculated by separate integrals — one for each area above and one for each area below the x-axis.

For example, from the diagram,

$$\text{Area} = A_1 + A_2$$

However, $\int_a^b f(x)\,dx = A_1 - A_2$, because the areas are signed.

To overcome this difficulty we find the correct area as:

$$\text{Area} = \int_c^b f(x)\,dx - \int_a^c f(x)\,dx \quad (= A_1 - -A_2 = A_1 + A_2)$$

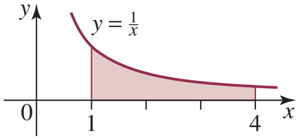
$$\text{or} \quad = \int_c^b f(x)\,dx + \left|\int_a^c f(x)\,dx\right|$$

Note: When we are calculating the area between a curve and the x-axis it is essential that the x-intercepts are determined and a graph of the curve is sketched over the interval required. The term $|x|$ means make the value of x positive even if it is negative.

WORKED Example 4

a **Express the shaded area as a definite integral.**

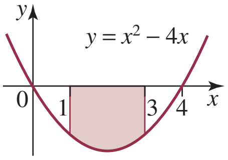
b **Evaluate the definite integral to find the shaded area, giving your answer as an exact value.**



THINK	WRITE
a Express the area in definite integral notation where $a = 1$ and $b = 4$.	**a** Area $= \int_1^4 \frac{1}{x}\,dx$
b ① Antidifferentiate the integrand.	**b** Area $= [\log_e x]_1^4$
② Evaluate.	$= \log_e 4 - \log_e 1$ $= \log_e 4 - 0$ $= \log_e 4$
③ State the solution as an exact answer.	The area is $\log_e 4$.

WORKED Example 5

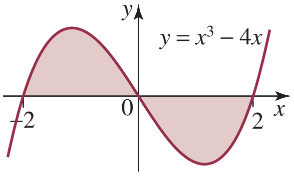
Calculate the shaded area.



THINK	WRITE
① Express the area in definite integral notation, with a negative sign in front of the integral as the region is below the x-axis.	Area $= -\int_1^3 (x^2 - 4x)\,dx$
② Antidifferentiate the integrand. ③ Evaluate.	$= -[\frac{1}{3}x^3 - 2x^2]_1^3$ $= -\{[\frac{1}{3}(3)^3 - 2(3)^2] - [\frac{1}{3}(1)^3 - 2(1)^2]\}$ $= -\{[9 - 18] - [\frac{1}{3} - 2]\}$ $= -[-9 - (-1\frac{2}{3})]$ $= -[-9 + 1\frac{2}{3}]$ $= -(-7\frac{1}{3})$ $= 7\frac{1}{3}$
④ State the solution.	The area is $7\frac{1}{3}$ square units.

WORKED Example 6

a **Express the shaded area as a definite integral.**
b **Calculate the area.**



THINK	WRITE
a Express the area above the x-axis as an integral and the area below the x-axis as an integral. For the area below the x-axis, take the negative of the integral from 0 to 2.	**a** Area $= \int_{-2}^{0} (x^3 - 4x)\ dx - \int_{0}^{2} (x^3 - 4x)\ dx$
b ① Antidifferentiate the integrands.	**b** $= [\frac{1}{4}x^4 - 2x^2]_{-2}^{0} - [\frac{1}{4}x^4 - 2x^2]_{0}^{2}$
② Evaluate.	$= \{[\frac{1}{4}(0)^4 - 2(0)^2] - [\frac{1}{4}(-2)^4 - 2(-2)^2]\}$ $- \{[\frac{1}{4}(2)^4 - 2(2)^2] - [\frac{1}{4}(0)^4 - 2(0)^2]\}$
③ Simplify.	$= [0 - (4 - 8)] - [(4 - 8) - 0]$ $= 4 - (-4)$ $= 8$
④ State the solution.	The area is 8 square units.

Graphics Calculator tip! Evaluating signed areas

To evaluate $\int_{-2}^{0} (x^3 - 4x)\ dx - \int_{0}^{2} (x^3 - 4x)\ dx$ using a graphics calculator, proceed as follows.

For the Casio *fx*-9860G AU

1. Press:
   - MENU
   - 1: RUN
   - OPTN
   - F4 (CALC)
   - F4 ($\int dx$).

   Complete the entry line as:
   $\int(x\text{^}3 - 4x, -2, 0) - \int(x\text{^}3 - 4x, -0, 2)$

   then press EXE.
2. Write the answer.

```
∫(X^3-4X,-2,0)-∫(X^3-
4X,0,2)
                     8
```

$$\int_{-2}^{0} (x^3 - 4x)\ dx - \int_{0}^{2} (x^3 - 4x)\ dx = 8$$

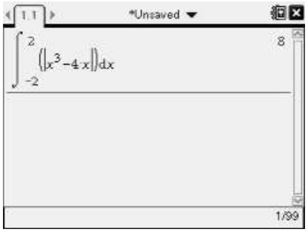
For the TI-Nspire CAS

1. On a Calculator page, press:
   - MENU menu
   - 4: Calculus 4
   - 3: Integral 3.

   Complete the entry line as:

   $\int_{-2}^{2} abs(x^3-4x)\ \mathrm{d}x$

   then press ENTER enter.
2. Write the answer.



$\int_{-2}^{2}(|x^3-4\cdot x|)dx$ 8

$\int_{-2}^{0}(x^3-4x)\,\mathrm{d}x-\int_{0}^{2}(x^3-4x)\,\mathrm{d}x=8$

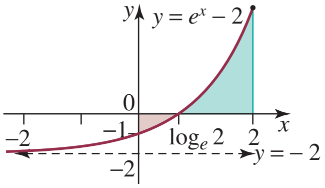
WORKED Example 7

eBook*plus*

Tutorial:
Worked example 7
int-0567

a **Sketch the graph of $y = e^x - 2$ showing all intercepts and using exact values for all key features.**

b **Find the area between the curve and the x-axis from $x = 0$ to $x = 2$.**

THINK	WRITE
a ① Find the x-intercept by letting $y = 0$ and solving for x. Take $\log_e$ of both sides.	**a** When $y = 0$, $e^x - 2 = 0$ $e^x = 2$ $\log_e e^x = \log_e 2$ $x = \log_e 2$ (or approximately 0.69) so the x-intercept is $\log_e 2$.
② Find the y-intercept by letting $x = 0$.	When $x = 0$, $y = e^0 - 2$ $= 1 - 2$ $= -1$ so the y-intercept is -1.
③ Note the vertical translation and hence sketch the graph showing the appropriate horizontal asymptote and intercept. ④ Shade the region required.	
b ① Express the area above the x-axis as an integral and the area below the x-axis as an integral. Subtract the area below the x-axis from the area above the x-axis.	**b** Area $= \int_{\log_e 2}^{2}(e^x - 2)\,\mathrm{d}x - \int_{0}^{\log_e 2}(e^x - 2)\,\mathrm{d}x$
② Antidifferentiate the integrands.	$= [e^x - 2x]_{\log_e 2}^{2} - [e^x - 2x]_{0}^{\log_e 2}$
③ Evaluate. (Remember: $e^{\log_e a} = a$)	$= [e^2 - 2(2)] - [e^{\log 2} - 2\log_e 2]$ $- \{[e^{\log_e 2} - 2(\log_e 2)] - [e^0 - 2(0)]\}$ $= [e^2 - 4] - [2 - 2\log_e 2] - \{[2 - 2\log_e 2]$ $- [1 - 0]\}$
④ Simplify.	$= e^2 - 4 - 2 + 2\log_e 2 - 2 + 2\ \log_e 2 + 1$ $= e^2 - 7 + 4\log_e 2$
⑤ State the solution.	The area is $(e^2 - 7 + 4\log_e 2)$ or approximately 3.162 square units.

remember

1. If $f(x) > 0$, area $= \int_a^b f(x)\,dx$.

2. If $f(x) < 0$, area $= -\int_a^b f(x)\,dx$, as the region is below the x-axis or

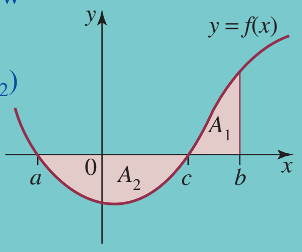
$$= \int_b^a f(x)\,dx \text{ (reversing the terminals changes the sign).}$$

3. If the required area lies above and below the x-axis:
   (a) find the intercepts and sketch the graph
   (b) calculate the integrals and subtract the area below the x-axis from the area above the x-axis.

4. Area $= \int_c^b f(x)\,dx - \int_a^c f(x)\,dx$ $(= A_1 - -A_2 = A_1 + A_2)$

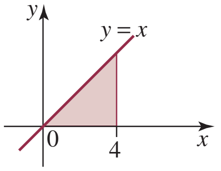
   or $= \int_c^b f(x)\,dx + \left|\int_a^c f(x)\,dx\right|$

5. $e^{\log_e a} = a$

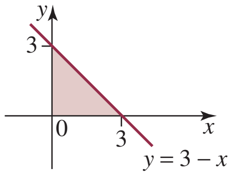


EXERCISE 8B Signed areas

1 Find the area of the triangle at right:
- **a** geometrically
- **b** using integration.



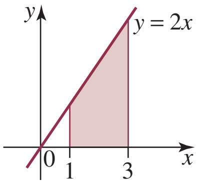
2 Find the area of the triangle at right:
- **a** geometrically
- **b** using integration.



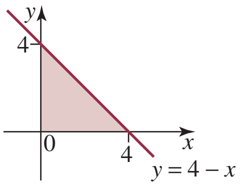
WORKED Example 4a

3 Express the following shaded areas as definite integrals.

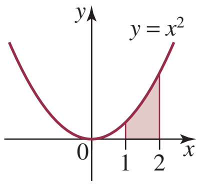
a



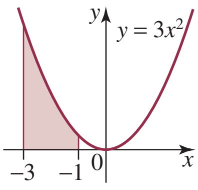
b



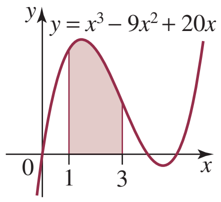
c



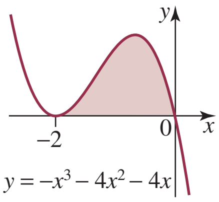
d



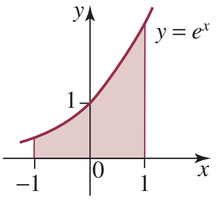
e



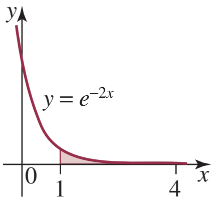
f



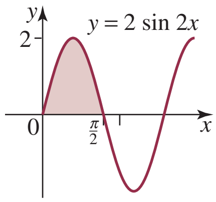
g



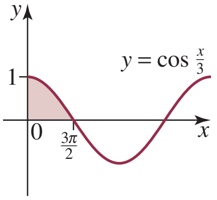
h



i



j



WORKED Example 4b

4 Evaluate each of the definite integrals in question **3** to find the shaded area. Give your answer as an exact value.

5 For each of the following, sketch a graph to illustrate the region for which the definite integral gives the area.

a $\int_0^3 4x \, dx$

b $\int_1^2 (6 - x) \, dx$

c $\int_1^3 x^2 \, dx$

d $\int_{-1}^1 (4 - x^2) \, dx$

e $\int_1^4 \sqrt{x} \, dx$

f $\int_{-3}^0 2e^x \, dx$

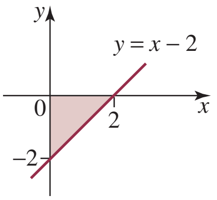
g $\int_2^4 \log_e \frac{x}{2} \, dx$

h $\int_0^{\frac{\pi}{2}} 3 \sin 2x \, dx$

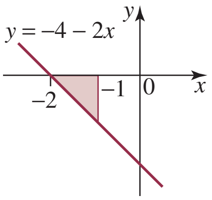
WORKED Example 5

6 Calculate each of the shaded areas below.

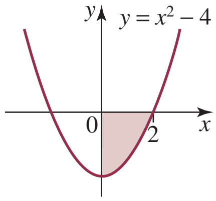
a



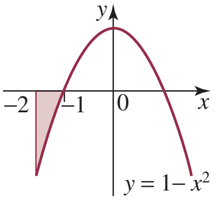
b



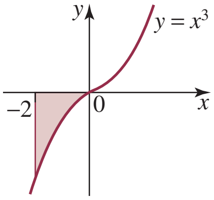
c



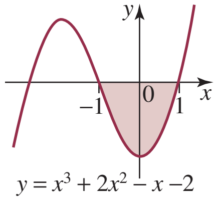
d



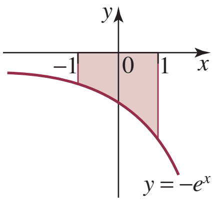
e



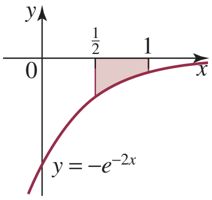
f



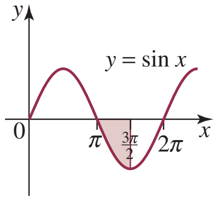
g



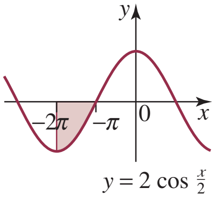
h



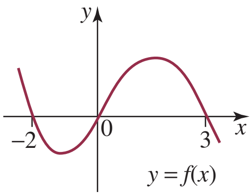
i



j



7 **multiple choice**



a The area between the graph, the x-axis and the lines $x = -2$ and $x = -1$ is equal to:

A $\int_1^2 f(x)\,dx$ **B** $\int_{-2}^{-1} f(x)\,dx$ **C** $\int_{-1}^{0} f(x)\,dx$

D $\int_2^1 f(x)\,dx$ **E** $-\int_{-2}^{-1} f(x)\,dx$

b The area between the graph, the x-axis and the lines $x = -2$ and $x = 3$ is equal to:

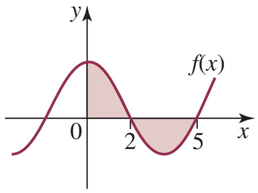
A $\int_0^3 f(x)\,dx + \int_0^{-2} f(x)\,dx$ **B** $\int_{-2}^{3} f(x)\,dx$ **C** $\int_{-2}^{1} f(x)\,dx + \int_1^3 f(x)\,dx$

D $\int_3^{-2} f(x)\,dx$ **E** $\int_{-2}^{0} f(x)\,dx - \int_0^3 f(x)\,dx$

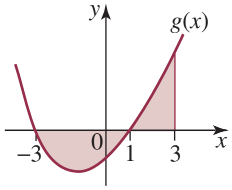
WORKED Example 6a

8 Express the following shaded areas as definite integrals which give the correct area.

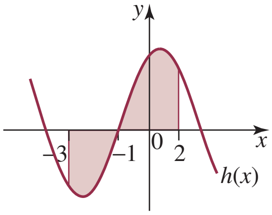
a



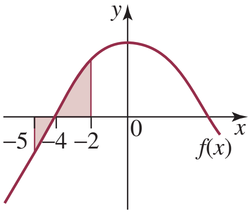
b



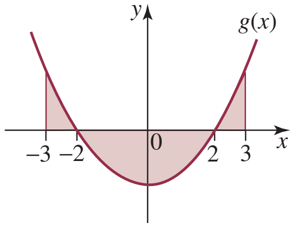
c



d

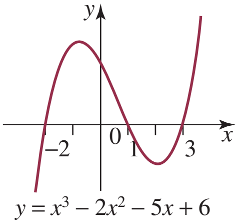


e



9 **multiple choice**

Examine the graph.



a The area between the curve and the x-axis from $x = -2$ and $x = 1$ is equal to:

A $17\frac{1}{12}$ sq. units **B** $15\frac{3}{4}$ sq. units **C** $-17\frac{1}{12}$ sq. units

D $-15\frac{3}{4}$ sq. units **E** 10 sq. units

b The area between the curve and the x-axis from $x = 1$ and $x = 3$ is equal to:

A $-6\frac{2}{3}$ sq. units **B** 2 sq. units **C** $-5\frac{1}{3}$ sq. units

D $5\frac{1}{3}$ sq. units **E** $6\frac{2}{3}$ sq. units

c The area between the curve and the x-axis from $x = -2$ and $x = 3$ is equal to:

A $10\frac{5}{12}$ sq. units **B** $11\frac{3}{4}$ sq. units **C** $22\frac{5}{12}$ sq. units

D 12 sq. units **E** $21\frac{1}{12}$ sq. units

WORKED Example 7

10 Sketch the graph of the curve $y = x^2 - 4$, showing all intercepts and using exact values for all key features. Find the area between the curve and the x-axis:

a from $x = 0$ to $x = 2$

b from $x = 2$ to $x = 4$

c from $x = 0$ to $x = 4$.

11 Sketch the graph of the curve $y = x^3 + x^2 - 2x$, showing all intercepts. Find the area between the curve and the x-axis between the lines:

a $x = -2$ and $x = 0$

b $x = 0$ and $x = 1$

c $x = -2$ and $x = 1$.

12 Sketch the graph of the curve $y = 1 + 3 \cos 2x$ for $0 \le x \le \pi$. Find the exact area between the curve and the x-axis from:

a $x = 0$ to $x = \dfrac{\pi}{4}$

b $x = \dfrac{3\pi}{4}$ to $x = \pi$.

eBook*plus*

Digital doc:
WorkSHEET 8.1

13 Sketch the graph of $f(x) = \sqrt{x} - 1$ and find the area between the curve and the x-axis and the lines $x = 2$ and $x = 3$. Give both an exact answer and an approximation to 3 decimal places.

14 Find the exact area between the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = \frac{1}{2}$ and $x = 2$.

15 Find the exact area bounded by the curve $g(x) = e^x + 2$, the x-axis and the lines $x = -1$ and $x = 3$.

Further areas

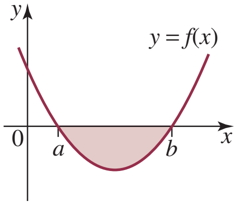
We shall now examine further situations in which areas are to be determined. These are when the area is enclosed between the curve and the x-axis, and where graphs are not easily sketched.

Areas bounded by a curve and the x-axis

For graphs with two or more x-intercepts, there is an enclosed region (or regions) between the graph and the x-axis.

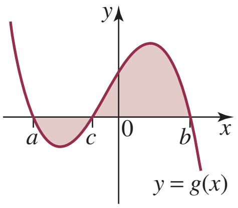
The area bounded by the graph of $f(x)$ and the x-axis is:

$-\int_a^b f(x)\,dx$ (negative because the area is below the x-axis).



The area bounded by the graph of $g(x)$ and the x-axis is:

$$\int_c^b g(x)\,dx - \int_a^c g(x)\,dx$$



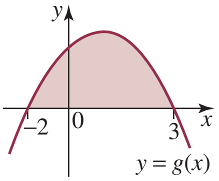
That is, if the graph has two x-intercepts then one integrand is required.

If the graph has three x-intercepts then two integrands are required, and so on.

Note: Wherever possible it is good practice to use sketch graphs to assist in any problems involving the calculation of areas under curves.

WORKED Example 8

a **Sketch the graph of the function $g(x) = (3 - x)(2 + x)$.**
b **Find the area bounded by the x-axis and the graph of the function.**

THINK	**WRITE**
a ① Determine the type of graph by looking at the number of brackets and the sign of the x-terms.	**a** $g(x) = (3 - x)(2 + x)$ is an inverted parabola.
② Solve $g(x) = 0$ to find the x-intercepts.	For x-intercepts, $g(x) = 0$ $(3 - x)(2 + x) = 0$ $x = 3$ and $x = -2$. The x-intercepts are -2 and 3.
③ Sketch the graph.	
b ① Shade the region bounded by $g(x)$ and the x-axis.	**b** 
② Express the area as an integral.	$\text{Area} = \int_{-2}^{3} (6 + x - x^2)\, dx$
③ Evaluate.	$= [6x + \frac{1}{2}x^2 - \frac{1}{3}x^3]_{-2}^{3}$ $= [6(3) + \frac{1}{2}(3)^2 - \frac{1}{3}(3)^3] - [6(-2) + \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3]$ $= (18 + \frac{9}{2} - 9) - (-12 + 2 + \frac{8}{3})$ $= 13\frac{1}{2} - (-7\frac{1}{3})$ $= 13\frac{1}{2} + 7\frac{1}{3}$ $= 20\frac{5}{6}$
④ State the solution.	The area bounded by $g(x)$ and the x-axis is $20\frac{5}{6}$ square units.

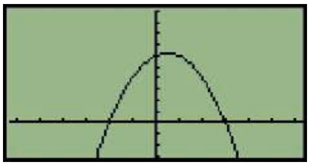
Graphics Calculator tip! Finding the area bounded by a graph and the x-axis

The area bounded by the graph and the x-axis can be drawn and evaluated using the graphics calculator.

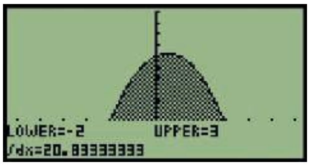
For the Casio *fx*-9860G AU

1. Press:
   - MENU
   - 5 (GRAPH).

   Complete the entry line as:
   $Y1 = (3 - x) \cdot (x + 2)$.
   Adjust the V-Window and then press:
   - F6 (DRAW).



2. Press:
   - SHIFT
   - F5 (G-Solv)
   - F6
   - F3 ($\int dx$).



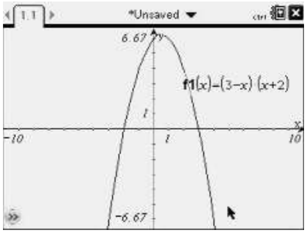
```
LOWER=-2          UPPER=3
∫dx=20.83333333
```

Use the arrow to move the cursor to the lower boundary ($x = -2$), then press EXE. Use the right arrow to move the cursor to the upper boundary ($x = 3$), then press EXE.

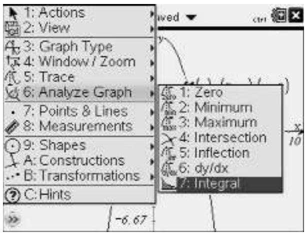
The required area will be shaded and the numeric value of the area will be displayed.

For the TI-Nspire CAS

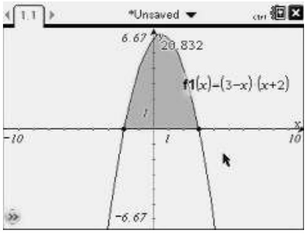
1. On a Graphs page, complete the entry line as:
   $f1(x) = (3 - x) \cdot \ (x + 2)$
   then press ENTER enter.



2. Press:
   - MENU menu
   - 6: Analyze Graph 6
   - 7: Integral 7.



3. Move the cursor until it hovers over the curve and the lower boundary ($x = -2$). Press the Click button twice. Use the NavPad to move the dotted line to the upper boundary ($x = 3$), then press Click. The required area will be shaded and the numeric value of the area will be displayed.



Finding areas without sketching graphs

When we are finding areas under curves that have functions whose graphs are not easily sketched, the area can be calculated providing the x-intercepts can be determined.

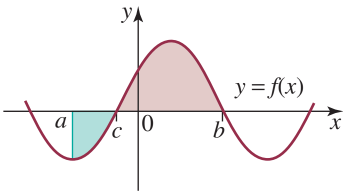
Note: The symbol $|f(x)|$ is known as the *absolute value* of $f(x)$ which means that we must make the $f(x)$ function or value positive whenever it is negative. For example,

$$|-5| = 5$$
$$|3| = 3$$
$$\left|-\tfrac{2}{3}\right| = \tfrac{2}{3}$$

As areas cannot be negative, taking the absolute value of the integrands involved in a problem will ensure that all areas are made positive. For example,

The shaded area at right $= \left|\int_c^b f(x)\,dx\right| + \left|\int_a^c f(x)\,dx\right|$

or $= \int_c^b f(x)\,dx + \left|\int_a^c f(x)\,dx\right|$

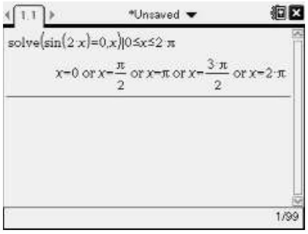


WORKED Example 9

Complete the following: **a** **by hand** **b** **using the TI-Nspire CAS calculator.**

i **Find the x-intercepts of $y = \sin 2x$ over the domain $0 \le x \le 2\pi$.**

ii **Calculate the area between the curve, the x-axis and $x = 0$ and $x = \pi$.**

THINK	**WRITE/DISPLAY**
a i ① To find the x-intercepts, let $y = 0$.	**a i** For x-intercepts, $y = 0$
② Solve for x over the given domain.	$\sin 2x = 0$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$, etc. $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$
ii ① Pick the x-intercepts that are between the given end points of the area.	**ii** $x = \frac{\pi}{2}$ is the only x-intercept between 0 and π.
② State the regions for which it is necessary to calculate the area.	Area$= \left\|\int_0^{\frac{\pi}{2}} \sin 2x\,dx\right\| + \left\|\int_{\frac{\pi}{2}}^{\pi} \sin 2x\,dx\right\|$
③ Evaluate the absolute value of the integral for each region.	$= \left\|[-\frac{1}{2}\cos 2x]_0^{\frac{\pi}{2}}\right\| + \left\|[-\frac{1}{2}\cos 2x]_{\frac{\pi}{2}}^{\pi}\right\|$ $= \left\|[-\frac{1}{2}\cos \pi - (-\frac{1}{2}\cos 0)]\right\|$ $+ \left\|[-\frac{1}{2}\cos 2\pi - (-\frac{1}{2}\cos \pi)]\right\|$ $= \left\|[\frac{1}{2} - (-\frac{1}{2})]\right\| + \left\|[-\frac{1}{2} - (\frac{1}{2})]\right\|$ $= \|1\| + \|-1\|$
④ Add the result to give the total area.	$= 1 + 1$ $= 2$
⑤ State the solution.	The area is 2 square units.
b i ① To find the x-intercepts of $y = \sin 2x$ over the domain $0 \le x \le 2\pi$, on a Calculator page, complete the entry line as: solve(sin(2x) = 0, x)|$0 \le x \le 2\pi$ then press ENTER ⏎. *Note:* Calculator is in Radian Mode.	**b i**  solve(sin(2·x)=0,x)|0≤x≤2·π $x=0$ or $x=\frac{\pi}{2}$ or $x=\pi$ or $x=\frac{3\cdot\pi}{2}$ or $x=2\cdot\pi$

Continued over page

THINK **WRITE/DISPLAY**

② Write the solution.

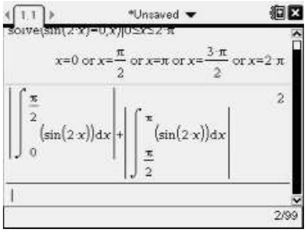
Solving $\sin 2x = 0$ for x over the domain $0 \le x \le 2\pi$ gives

$$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

ii ① To calculate the area between the curve, the x-axis, $x = 0$ and $x = \pi$, on a Calculator page, complete the entry line as:

$$\text{abs}\left(\int_0^{\frac{\pi}{2}} (\sin(2x))\text{d}x\right) + \text{abs}\left(\int_{\frac{\pi}{2}}^{\pi} (\sin(2x))\text{d}x\right).$$

ii



solve(sin(2·x)=0,x)|0≤x≤2·π

$x=0$ or $x=\frac{\pi}{2}$ or $x=\pi$ or $x=\frac{3\cdot\pi}{2}$ or $x=2\cdot\pi$

$\left|\int_0^{\frac{\pi}{2}} (\sin(2\cdot x))\text{d}x\right| + \left|\int_{\frac{\pi}{2}}^{\pi} (\sin(2\cdot x))\text{d}x\right|$ 2

2/99

② Write the answer.

$$\left|\int_0^{\frac{\pi}{2}} (\sin(2x))\text{d}x\right| + \left|\int_{\frac{\pi}{2}}^{\pi} (\sin(2x))\text{d}x\right|$$

$= 2$ square units.

WORKED Example 10

a **Differentiate $\log_e(x^2 - 1)$.**

b **Hence, find an antiderivative of $\frac{x}{x^2 - 1}$.**

c **Find the area between the graph of $\frac{x}{x^2 - 1}$, the x-axis, $x = 2$ and $x = 3$, giving your answer correct to 2 decimal places.**

THINK **WRITE**

a ① Let y equal the expression to be differentiated.

a Let $y = \log_e(x^2 - 1)$.

② Express u as a function of x in order to apply the chain rule for differentiation. (Let u equal the function inside the brackets.)

Let $u = x^2 - 1$.

③ Find $\frac{\text{d}u}{\text{d}x}$.

$$\frac{\text{d}u}{\text{d}x} = 2x$$

④ Write y in terms of u.

$$y = \log_e u$$

⑤ Find $\frac{\text{d}y}{\text{d}u}$.

$$\frac{\text{d}y}{\text{d}u} = \frac{1}{u}$$

⑥ Find $\frac{\text{d}y}{\text{d}x}$ using the chain rule.

$$\text{So } \frac{\text{d}y}{\text{d}x} = \frac{1}{u} \cdot 2x$$

$$= \frac{2x}{x^2 - 1}$$

THINK	**WRITE**
b ① Since $\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y + c$, express the relationship in integral notation.	**b** $\int \frac{2x}{x^2 - 1}\,\mathrm{d}x = \log_e(x^2 - 1) + c$
② Remove a factor from $\frac{\mathrm{d}y}{\mathrm{d}x}$ so that it resembles the integral required.	$2\int \frac{x}{x^2 - 1}\,\mathrm{d}x = \log_e(x^2 - 1) + c$
③ Divide both sides by the factor in order to obtain the required integral.	$\int \frac{x}{x^2 - 1}\,\mathrm{d}x = \frac{1}{2}\log(x^2 - 1) + c$ An antiderivative of $\frac{x}{x^2 - 1}$ is $\frac{1}{2}\log(x^2 - 1)$.
c ① Find the x-intercepts. (For $\frac{x}{x^2 - 1} = 0$, the numerator = 0.)	**c** For x-intercepts, $\frac{x}{x^2 - 1} = 0$ $x = 0$
② If the x-intercepts are not between the terminals of the area, find the area by evaluating the integrand.	Area $= \left\lvert \int_2^3 \frac{x}{x^2 - 1}\,\mathrm{d}x \right\rvert$ $= \left\lvert [\frac{1}{2}\log_e(x^2 - 1)]_2^3 \right\rvert$ $= \left\lvert [\frac{1}{2}\log_e(3^2 - 1)] - [\frac{1}{2}\log_e(2^2 - 1)] \right\rvert$ $= \left\lvert \frac{1}{2}\log_e 8 - \frac{1}{2}\log_e 3 \right\rvert$ $= \left\lvert \frac{1}{2}\log_e \frac{8}{3} \right\rvert$ $= \frac{1}{2}\log_e \frac{8}{3}$
③ State the solution.	The area is $\frac{1}{2}\log_e \frac{8}{3}$ or approximately 0.49 square units.

remember

1. For graphs with two or more intercepts, there is an enclosed region (or regions) between the graph and the x-axis.
2. The number of regions is one less than the number of intercepts.
3. Where possible, sketch graphs to make it easier to calculate the areas under curves, or use a graphics calculator.
4. As areas can't be negative, take the absolute values of the integrals.
5. When graphs are not easily drawn, areas can be calculated by finding the x-intercepts and determining whether they are within the bounds of the required area.

EXERCISE 8C Further areas

In the following exercise give all answers correct to 2 decimal places where appropriate, unless otherwise stated.

WORKED Example 8

1 **i** Sketch the graph of each of the following functions.
ii Find the area bounded by the x-axis and the graph of each function.
Use a graphics calculator to assist.

a $f(x) = x^2 - 3x$
b $g(x) = (2 - x)(4 + x)$

2 Find the area bounded by the x-axis and the graph of each of the following functions.

a $h(x) = (x + 3)(5 - x)$
b $h(x) = x^2 + 5x - 6$
c $g(x) = 8 - x^2$
d $g(x) = x^3 - 4x^2$
e $f(x) = x(x - 2)(x - 3)$
f $f(x) = x^3 - 4x^2 - 4x + 16$
g $g(x) = x^3 + 3x^2 - x - 3$
h $h(x) = (x - 1)(x + 2)(x + 5)$

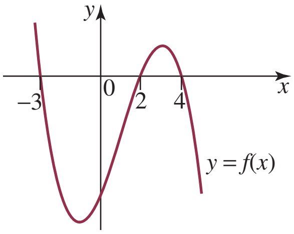
3 **multiple choice**

The area bounded by the curve with equation $y = x^2 - 6x + 8$ and the x-axis is equal to:

A $1\frac{1}{3}$ sq. units
B $6\frac{2}{3}$ sq. units
C 12 sq. units
D 3 sq. units
E $-1\frac{1}{3}$ sq. units

4 **multiple choice**

The area between the curve at right, the x-axis and the lines $x = -3$ and $x = 4$ is equal to:



A $\int_{-3}^{4} f(x)\,\mathrm{d}x$
B $\int_{-3}^{2} f(x)\,\mathrm{d}x + \int_{2}^{4} f(x)\,\mathrm{d}x$
C $\int_{4}^{-3} f(x)\,\mathrm{d}x$
D $\int_{2}^{4} f(x)\,\mathrm{d}x - \int_{-3}^{2} f(x)\,\mathrm{d}x$
E $\int_{-3}^{2} f(x)\,\mathrm{d}x - \int_{2}^{4} f(x)\,\mathrm{d}x$

5 **multiple choice**

The area between the curve $y = x^2 - x - 6$, the x-axis and the lines $x = 2$ and $x = 4$ is equal to:

A $2\frac{5}{6}$ sq. units
B $\frac{2}{3}$ sq. units
C 5 sq. units
D $2\frac{3}{4}$ sq. units
E $4\frac{1}{2}$ sq. units

6 For each of the following (**a** to **j**):

i sketch the graph of the curve over an appropriate domain, clearly labelling any x-intercepts in the interval required (use a graphics calculator to assist).

ii find the area between the curve, the x-axis and the lines indicated below.

a $y = 3 - 3x^2$, $x = 0$ and $x = 2$

b $y = \frac{2}{x}$, $x = 1$ and $x = 3$

c $y = -\frac{1}{x^2}$, $x = 1$ and $x = 2$

d $y = x^3 - 4x$, $x = -2$ and $x = 1$

e $y = e^{2x}$, $x = -2$ and $x = 0$

f $y = e^{-x}$, $x = 0$ and $x = 2$

g $y = 2 \sin x$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$

h $y = \cos \frac{x}{2}$, $x = \frac{\pi}{2}$ and $x = 2\pi$

i $y = \sin 3x$, $x = -\frac{\pi}{2}$ and $x = -\frac{\pi}{6}$

j $y = x\sqrt{x}$, $x \geq 0$, $x = 0$ and $x = 4$

WORKED Example 9

7 For each of the following functions (**a** to **g**):

i find the x-intercepts over the given domain

ii calculate the area between the curve, the x-axis and the given lines.

Use sketch graphs to assist your workings.

a $y = x - 4x^{-1}$, $x \neq 0$, $x = 1$ and $x = 3$

b $y = \sin x - \cos x$ for $0 \leq x \leq \pi$, $x = 0$ and $x = \pi$

c $y = e^x - e$, $x = 0$ and $x = 3$

d $y = x - \frac{1}{x^2}$, $x \neq 0$, $x = \frac{1}{2}$ and $x = 2$

e $y = e^{\frac{x}{2}}$, $x = -2$ and $x = 2$

f $y = x^4 - 3x^2 - 4$, $x = 1$ and $x = 4$

g $y = (x - 2)^4$, $x = 1$ and $x = 3$

8 Find the exact area of the region enclosed by the x-axis, $y = e^{3x}$ and the lines $x = 1$ and $x = 2$.

9 Find the exact area of the region enclosed by the x-axis, $y = -\cos x$ and the lines $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$. (Use a sketch graph to assist your calculation.)

10 Find the area bounded by $y = (x - 1)^3$, the x-axis and the y-axis.

11 **a** Sketch the graph of $y = \frac{1}{(x-3)^2}$ showing all asymptotes and intercepts.

b Find the area under the curve between $x = -1$ and $x = 1$.

12 **a** Give the equation of the asymptotes for the function $f(x) = (x + 2)^{-3}$.

b Find the area between the curve, the x-axis, $x = -1$ and $x = 1$.

13 Find the area bounded by the curve $y = 3 - e^{2x}$, the x-axis, $x = -2$ and $x = 0$. (Find the x-intercepts first.)

14 Find the area bounded by the curve $y = 4 - \sin 2x$, the x-axis, $x = -\frac{\pi}{2}$ and $x = \pi$. (Check the x-intercepts first.)

WORKED Example 10

15 **a** Differentiate $x \log_e x$. $(x > 0)$

b Hence, find an antiderivative of $\log_e x$.

c Find the area bounded by the graph of $\log_e x$, the x-axis, $x = 1$ and $x = 4$ giving exact answers.

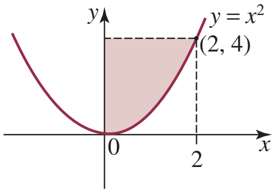
16 **a** Differentiate $\log_e (x^2 + 2)$.

b Hence, find an antiderivative of $\frac{x}{x^2 + 2}$.

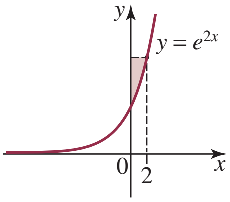
c Find the area between $\frac{x}{x^2 + 2}$, the x-axis, $x = -1$ and $x = 1$.

17 **a** Find the area between the graph of $y = x^2$, the x-axis, $x = 0$ and $x = 2$.

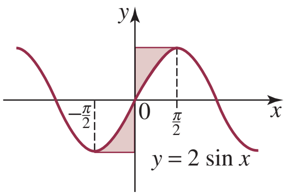
b Use this result to calculate the area between the graph, the y-axis and the line $y = 4$.



18 Find the exact area of the shaded region on the graph $y = e^{2x}$ below.



19 Find the shaded area below.



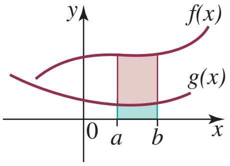
Areas between two curves

We shall now consider the area between two functions, $f(x)$ and $g(x)$, over an interval $a \leq x \leq b$. Our approach depends on whether the curves intersect or do not intersect over this interval.

If the two curves $f(x)$ and $g(x)$ do not intersect over the interval $a \leq x \leq b$

Here, we may look at three circumstances: when the region is above the x-axis, when it is below the x-axis, and when it crosses the x-axis.

Region above x-axis

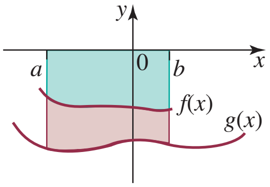


$$\text{The red shaded area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b [f(x) - g(x)]\,dx\,.$$

Note: The lower function is subtracted from the higher function to ensure a positive answer.

Region below x-axis

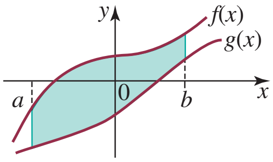


Again, the lower function is subtracted from the higher function to ensure a positive answer.

Red shaded area = $\int_a^b [f(x) - g(x)]\,dx$, as $f(x)$ is above $g(x)$

over the interval $a \leq x \leq b$.

Region crosses x-axis

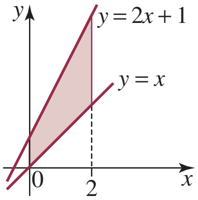


Shaded area = $\int_a^b [f(x) - g(x)]\,dx$

WORKED Example 11

a **State the definite integral which describes the shaded area on the graph at right.**

b **Find the area.**



THINK	WRITE
a 1 State the two functions $f(x)$ and $g(x)$.	**a** $f(x) = 2x + 1$ and $g(x) = x$
2 Subtract the equation of the lower function from the equation of the upper function and simplify.	$f(x) - g(x) = 2x + 1 - x$ $= x + 1$
3 Write as a definite integral between the given values of x.	Area = $\int_0^2 [f(x) - g(x)]$ $= \int_0^2 (x + 1)\,dx$

Continued over page →

THINK	WRITE
b ① Antidifferentiate.	**b** Area = $[\frac{1}{2}x^2 + x]_0^2$
② Evaluate the integral.	$= [\frac{1}{2}(2)^2 + 2] - [\frac{1}{2}(0)^2 + 0]$ $= (2 + 2) - (0)$ $= 4$
③ State the area.	The area is 4 square units.

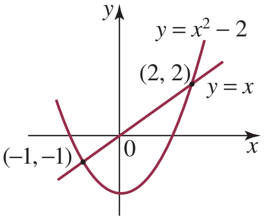
WORKED Example 12

Complete the following: a by hand b using the TI-Nspire graphics calculator.

i Find the values of x where the functions $y = x$ and $y = x^2 - 2$ intersect.

ii Sketch the graphs on the same axes. (Check using a graphics calculator.)

iii Hence, find the area bounded by the curves.

THINK	WRITE/DISPLAY
a i ① State the two functions.	**a i** $y = x$ and $y = x^2 - 2$
② Find where the curves intersect.	For points of intersection: $x = x^2 - 2$
③ Solve for x.	$x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ or $x = -1$
ii Find the key points of each function and sketch.	**ii** For $y = x$, when $x = 0, y = 0$ when $x = 2, y = 2$ when $x = -1, y = -1$ Line passes through (0, 0), (2, 2) and (−1, −1) For $y = x^2 - 2$, when $x = 0, y = -2$ Hence y-intercept is −2. Parabola also passes through (2, 2) and (−1, −1). 
iii ① Define $f(x)$ and $g(x)$.	**iii** Let $f(x) = x$ and $g(x) = x^2 - 2$
② Write the area as a definite integral between the values of x at the points of intersection.	Area $= \int_{-1}^{2} [f(x) - g(x)]\ dx$ $= \int_{-1}^{2} [x - (x^2 - 2)]\ dx$ $= \int_{-1}^{2} (x - x^2 + 2)\ dx$

THINK | **WRITE/DISPLAY**

③ Antidifferentiate.

$= [\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x]_{-1}^{2}$

④ Evaluate the integral.

$= [\frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + 2(2)] - [\frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 + 2(-1)]$

$= (2 - \frac{8}{3} + 4) - (\frac{1}{2} + \frac{1}{3} - 2)$

$= (3\frac{1}{3}) - (-1\frac{1}{6})$

$= 3\frac{1}{3} + 1\frac{1}{6}$

$= 4\frac{1}{2}$

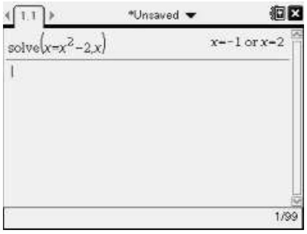
⑤ State the area.

The area is $4\frac{1}{2}$ square units.

b i ① To find where the two curves intersect, on a Calculator page, press:
- MENU (menu)
- 3: Algebra (3)
- 1: Solve (1).

Complete the entry line as:
solve($x = x^2 - 2, x$)
then press ENTER (enter).

b i



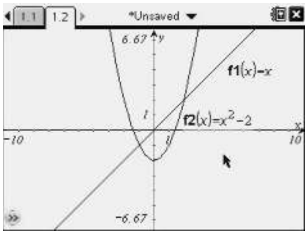
```
solve(x=x²−2,x)          x=−1 or x=2
                                1/99
```

② Write the answer.

Solving $x = x^2 - 2$ for x gives $x = -1$ or $x = 2$.

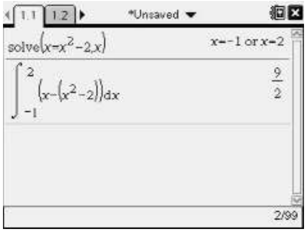
ii To sketch the graphs of $y = x$ and $y = x^2 - 2$, on a Graphs page, complete the entry lines as:
$f1(x) = x$
$f2(x) = x^2 - 2$
pressing ENTER (enter) after each line.

ii



iii ① To find the area bounded by the curves, on a Calculator page, complete the entry line as:
$\int_{-1}^{2} (x - (x^2 - 2))\mathrm{d}x$
then press ENTER (enter).

iii

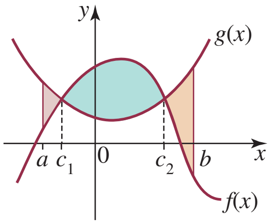


```
solve(x=x²−2,x)          x=−1 or x=2
∫_{-1}^{2} (x−(x²−2))dx          9/2
                                2/99
```

② Write the answer.

$\int_{-1}^{2} (x - (x^2 - 2))\mathrm{d}x = \frac{9}{2}$ square units.

If the two curves intersect over the interval $a \le x \le b$



Where c_1 and c_2 are the values of x where $f(x)$ and $g(x)$ intersect over the interval $a \le x \le b$, the area is found by considering the intervals $a \le x \le c_1$, $c_1 \le x \le c_2$ and $c_2 \le x \le b$ separately. For each interval care must be taken to make sure the integrand is the higher function. Subtract the lower function.

So the shaded area equals:

$$\int_a^{c_1} [g(x) - f(x)]\, dx + \int_{c_1}^{c_2} [f(x) - g(x)]\, dx + \int_{c_2}^{b} [g(x) - f(x)]\, dx$$

Therefore, when finding areas between two curves over an interval, it must be determined whether the curves intersect within that interval. If they do, the area is broken into sub-intervals as shown above.

As with areas under curves, sketch graphs should be used to assist in finding areas between curves.

If sketch graphs are not used, the absolute value of each integral, for each sub-interval, should be taken to ensure the correct value is obtained.

WORKED Example 13

eBook*plus*

Tutorial:

Worked example 13

int-0568

a **Find the values of x where the graph of the functions $f(x) = \dfrac{4}{x}$ and $g(x) = x$ intersect.**

b **Sketch the graphs on the same axes (check using a graphics calculator) and shade the region between the two curves and $x = 1$ and $x = 3$.**

c **Find the area between $f(x)$ and $g(x)$ from $x = 1$ to $x = 3$.**

THINK

WRITE

a 1 State the two functions.

a $f(x) = \dfrac{4}{x}$, $g(x) = x$

2 Let $f(x) = g(x)$ to find the values of x where the graphs intersect.

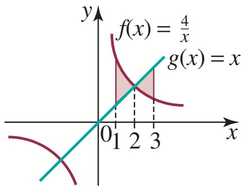
3 Solve for x.

For points of intersection, $x = \dfrac{4}{x}$.

$$x^2 = 4$$
$$x^2 - 4 = 0$$
$$(x + 2)(x - 2) = 0$$
$$x = -2 \text{ and } x = 2$$

b Sketch $f(x)$ and $g(x)$ on the same axes and shade the region between the two curves from $x = 1$ to $x = 3$.

b



c 1 State the area as the sum of two integrals for the two sub-intervals.

c Area $= \displaystyle\int_1^2 \left(\frac{4}{x} - x\right) dx + \int_2^3 \left(x - \frac{4}{x}\right) dx$

THINK	**WRITE**
2 Antidifferentiate.	$= [4\log_e x - \frac{1}{2}x^2]_1^2 + [\frac{1}{2}x^2 - 4\log_e x]_2^3$
3 Evaluate the two integrals.	$= [4\log_e 2 - \frac{1}{2}(2)^2] - [4\log_e 1 - \frac{1}{2}(1)^2]$ $+ \{[\frac{1}{2}(3)^2 - 4\log_e 3] - [\frac{1}{2}(2)^2 - 4\log_e 2]\}$ $= [4\log_e 2 - 2] - [4\log_e 1 - \frac{1}{2}]$ $+ \{[\frac{9}{2} - 4\log_e 3] - [2 - 4\log_e 2]\}$ $= 4\log_e 2 - 2 - 0 + \frac{1}{2} + \frac{9}{2} - 4\log_e 3 - 2 + 4\log_e 2$
4 Simplify.	$= 4\log_e \frac{4}{3} + 1$
5 State the area.	The area is $4\log_e \frac{4}{3} + 1$ or approximately 2.151 square units.

Note: If Worked example 13 was calculated without a graph, the area would be found by evaluating:

$$\left|\int_1^2 \left(\frac{4}{x} - x\right) dx\right| + \left|\int_2^3 \left(\frac{4}{x} - x\right) dx\right| \quad \text{or} \quad \left|\int_1^2 \left(x - \frac{4}{x}\right) dx\right| + \left|\int_2^3 \left(x - \frac{4}{x}\right) dx\right|$$

as without a graph it would not be known which function was above the other for either interval.

To find the area between two curves using a graphics calculator:

1. Find the area between the upper curve and the x-axis, and record or store the answer.
2. Find the area between the lower curve and the x-axis, and record or store the answer.
3. Subtract the two answers, giving a positive value for the area.

remember

1. If two curves $f(x)$ and $g(x)$ do not intersect over the interval $a \le x \le b$ and $f(x) > g(x)$ then the area enclosed by the two curves and the lines $x = a$ and $x = b$ is found by using the formula:

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

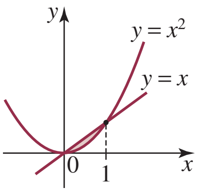
2. If two curves $f(x)$ and $g(x)$ intersect over the interval $a \le x \le b$ it is necessary to find the points of intersection and hence find the area of each section because sometimes $f(x) > g(x)$ and sometimes $g(x) > f(x)$.
3. If sketch graphs are not used to determine which is the upper curve, then it is necessary to take the absolute value or positive value of each integral.

EXERCISE 8D Areas between two curves

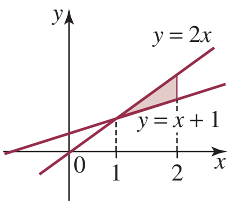
WORKED Example 11a

1 State the definite integral which will find the shaded areas on each graph below.

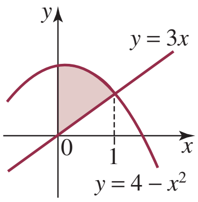
a



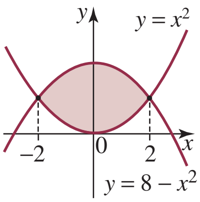
b



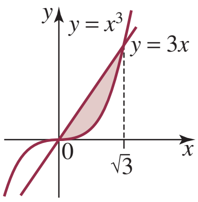
c



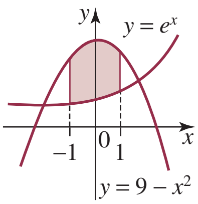
d



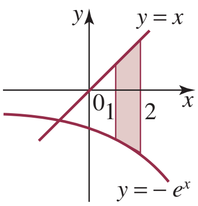
e



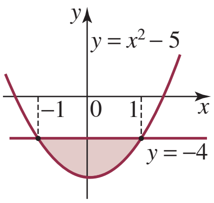
f



g



h

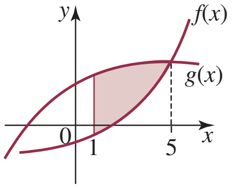


WORKED Example 11b

2 Find each of the areas in question 1.

3 **multiple choice**

Which one of the following does not equal the shaded area?



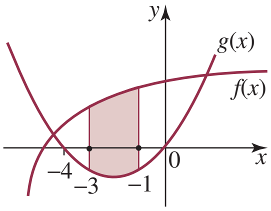
A $\int_1^5 g(x)\,dx - \int_1^5 f(x)\,dx$ **B** $\int_1^5 g(x)\,dx + \int_5^1 f(x)\,dx$

C $\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$ **D** $\int_1^5 [g(x) - f(x)]\,dx$

E $\int_5^1 [f(x) - g(x)]\,dx$

4 **multiple choice**

The area bounded by the curves $f(x)$, $g(x)$ and the lines $x = -3$ and $x = 1$ at right is equal to:



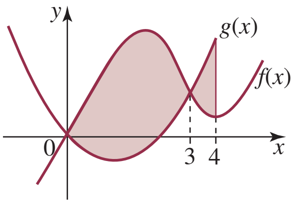
A $\int_{-1}^{-3} [f(x) - g(x)]\,dx$ **B** $\int_{-3}^{-1} [f(x) + g(x)]\,dx$

C $\int_{-3}^{-1} [g(x) - f(x)]\,dx$ **D** $\int_{-3}^{-1} [f(x) - g(x)]\,dx$

E $\int_{-1}^{-3} [f(x) + g(x)]\,dx$

5 **multiple choice**

The shaded area at right is equal to:



A $\int_0^4 [f(x) - g(x)]\, dx$

B $\int_0^3 [g(x) - f(x)]\, dx + \int_3^4 [f(x) - g(x)]\, dx$

C $\int_0^4 [g(x) - f(x)]\, dx$

D $\int_0^3 [f(x) - g(x)]\, dx$

E $\int_0^3 [f(x) - g(x)]\, dx + \int_3^4 [g(x) - f(x)]$

WORKED Example 12

6 In each of the following:

- **i** find the values of x where the functions intersect
- **ii** sketch the graphs on the same axes (check using a graphics calculator)
- **iii** hence, find the area bounded by the curves.
  - **a** $y = 4x$ and $y = x^2$
  - **b** $y = 2x$ and $y = 3 - x^2$
  - **c** $y = x^2 - 1$ and $y = 1 - x^2$
  - **d** $y = x^2 - 4$ and $y = 4 - x^2$
  - **e** $y = (x + 1)^2$ and $y = 1 - x^2$
  - **f** $y = \sqrt{x}$ and $y = x^2$

WORKED Example 13

7
- **i** Find the values of x where the functions intersect.
- **ii** Sketch the graphs on the same axes. (Check using a graphics calculator.)
- **iii** Find the area between $f(x)$ and $g(x)$ giving an exact answer.
  - **a** $y = x^3$ and $y = x$
  - **b** $y = 3x^2$ and $y = x^3 + 2x$

8 Find the area between the pairs of curves below, over the given interval. (Use a graph and graphics calculator to assist if necessary.)

- **a** $y = x^3,\ y = x^2,\ -1 \le x \le 1$
- **b** $y = \sin x,\ y = \cos x,\ 0 \le x \le \pi$
- **c** $y = (x - 1)^2,\ y = (x + 1)^2,\ -1 \le x \le 1$
- **d** $y = x^3 - 5x,\ y = 6 - 2x^2,\ 0 \le x \le 3$
- **e** $y = \dfrac{1}{x},\ y = 4x,\ \frac{1}{4} \le x \le 1$
- **f** $y = e^x,\ y = e^{-x},\ 0 \le x \le 1$
- **g** $y = 2\cos x,\ y = x - \dfrac{\pi}{2},\ 0 \le x \le \dfrac{\pi}{2}$
- **h** $y = e^x,\ y = -e^x,\ -2 \le x \le 1$

Use sketch graphs and a graphics calculator to assist in solving the following problems.

9 Find the area between the curve $y = e^x$ and the lines $y = x$, $x = 1$ and $x = 3$.

10 Find the area between the curve $y = x^2$ and the lines $y = \dfrac{x}{2} + 3$, $x = 1$ and $x = 3$.

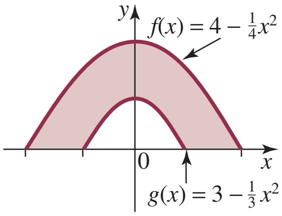
11 Calculate the area between the curves $y = \sin 2x$ and $y = \cos x$ from $x = 0$ to $x = \dfrac{\pi}{2}$.

12 Calculate the area between the curves $y = \sqrt{3} - \sin 2x$ and $y = \sin 2x$ from $x = 0$ to $x = \dfrac{\pi}{4}$.

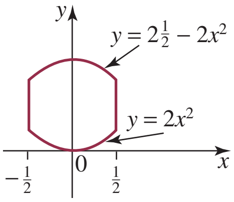
13 Find the exact area bounded by the curves $y = e^x$ and $y = 3 - 2e^{-x}$.

14 The graph at right shows the cross-section of a bricked archway. (All measurements are in metres.)

- **a** Find the x-intercepts of $f(x)$.
- **b** Find the x-intercepts of $g(x)$.
- **c** Find the cross-sectional area of the brickwork.

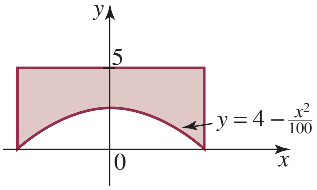


15 The diagram at right shows the outline of a window frame. If all measurements are in metres, what is the area of glass which fits into the frame?



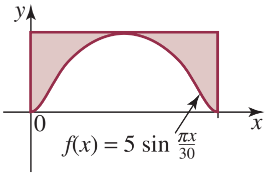
16 The diagram at right shows the side view of a concrete bridge. (All measurements are in metres.) Find:

a the x-intercepts of the curve
b the length of the bridge
c the area of the side of the bridge
d the volume of concrete used to build the bridge if the bridge is 9 metres wide.



17 The cross-section of a road tunnel entrance is shown at right. (All measurements are in metres.) The shaded area is to be concreted. Find:

a the exact area, above the entrance, which is to be concreted
b the exact volume of concrete required to build this tunnel if it is 200 metres long.



18 A section of a river can be modelled by the equation $y = 40 \sin \frac{\pi x}{120}$, where $0 \le x \le 120$ and x is in metres. On the same model a proposed section of road obeys the rule $y = \frac{x}{5}$. The area bounded by the road and the river is to have one tree planted per 12 square metres. How many trees will be planted?

Further applications of integration

EXERCISE 8E Further applications of integration — modelling and problem solving

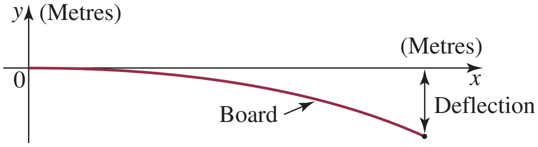
eBook plus

Interactivity:
Applications of antidifferentiation
int-0269

1 If $f'(x) = (2 - x)^2$ and the y-intercept of $f(x)$ is $\frac{4}{3}$, find the rule for $f(x)$.

2 If $\frac{dy}{dx} = 1 - 4\cos 2x$ and the y-intercept is 2, find the exact value of y when $x = \frac{\pi}{12}$.

3 The rate of deflection from a horizontal position of a 3-metre diving board when an 80 kg person is x metres from its fixed end is given by $\frac{dy}{dx} = -0.03(x + 1)^2 + 0.03$, where y is the deflection in metres.



- **a** What is the deflection when $x = 0$?
- **b** Determine the equation which measures the deflection.
- **c** Hence, find the maximum deflection.

4 On any day the cost per item for a machine producing n items is given by $\frac{dC}{dn} = 40 - 2e^{0.01n}$, where $n \in [0, 200]$ and C is the cost in dollars.

- **a** Use the rate to find the cost of producing the 100th item.
- **b** Express C as a function of n.
- **c** What is the total cost of producing the first 100 items?
- **d** Find the average cost of production for: **i** the first 100 items **ii** the second 100 items.

5 The rate of change of position (velocity) of a racing car travelling down a straight stretch of road is given by $\frac{dx}{dt} = t(16 - t)$, where x is measured in metres and t in seconds.

- **a** Find the velocity when: **i** $t = 0$ **ii** $t = 4$.
- **b** Determine: **i** when the maximum velocity occurs **ii** the maximum velocity.
- **c** Sketch the graph of $\frac{dx}{dt}$ against t for $0 \le t \le 16$.
- **d** Find the area under the graph between $t = 0$ and $t = 10$.
- **e** What does this area represent?

6 The rate at which water is pumped out of a dam, in L/min, t minutes after the pump is started is $\frac{dV}{dt} = 5 + \cos\frac{\pi t}{40}$.

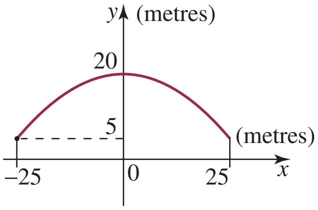
a How much water is pumped out in the 40th minute?
b Find the volume of water pumped out at any time, t, after the pump is started.
c How much water is pumped out after 40 minutes?
d Find the average rate at which water is pumped in the first hour.
e How long would it take to fill a tank holding 1600 litres?

7 The rate of flow of water into a hot water system during a 12-hour period on a certain day is thought to be $\frac{dV}{dt} = 10 + \cos\frac{\pi t}{2}$, where V is in litres and t is the number of hours after 8 am.

a Sketch the graph of $\frac{dV}{dt}$ against t.
b Find the length of time for which the rate is above 10.5 L/h.
c Find the volume of water that has flowed into the system between:
i 8 am and 2 pm **ii** 3 pm and 8 pm.

8 The roof of a stadium has the shape given by the function $f(x) = 20 - 0.024x^2$ for $-25 \le x \le 25$.

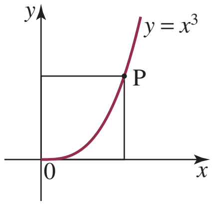
The stadium is 75 metres long and its cross-section is shown at right.



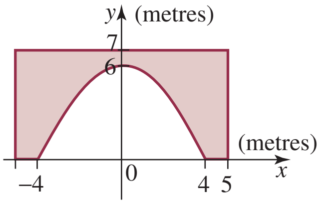
a Find the volume of the stadium.
b The stadium is to have several airconditioners strategically placed around it. Each airconditioner can service a volume of 11 250 m^3. How many airconditioners are required?

9 The cross-section of a channel is parabolic. It is 3 metres wide at the top and 2 metres deep. Find the depth of water, to the nearest cm, when the channel is half full.

10 For any point P on the curve $y = x^3$, prove that the area under the curve is one quarter of the area of the rectangle.



11 The arch of a concrete bridge has the shape of a parabola. It is 6 metres high and 8 metres long.



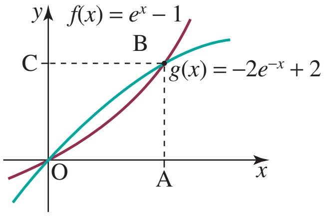
a Find the rule for the function corresponding to the arch of the bridge.
b Find the area of the shaded region.
c If the bridge is 10 metres wide, find the volume of concrete in the bridge.

12 In the figure at right $f(x)$ and $g(x)$ intersect at O and B.

a Show that the coordinate of B is $(\log_e 2, 1)$.

b Find the exact area of the region bounded by $f(x)$ and $g(x)$.

c Show that the sum of the areas under $f(x)$ and $g(x)$, from $x = 0$ to $x = \log_e 2$, is equal to the area of the rectangle OABC.



eBook plus

Digital doc:
WorkSHEET 8.2

13 The population of kangaroos on an island is increasing at a rate given by $P(t) = 12\log_e(t + 1)$, where t is the number of years since 1 January 1955.

a Find the rate of growth when $t = 0$, $t = 5$ and $t = 40$.

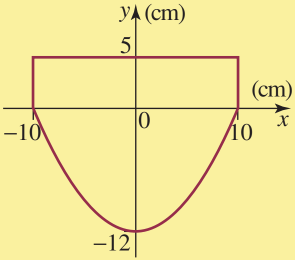
b Sketch the graph of $P(t)$.

c Determine the inverse function $P^{-1}(t)$.

d Use the inverse function to assist in finding the area under the graph of $P(t)$ between $t = 0$ and $t = 40$.

e What does this area represent?

Concrete chute



Concrete is poured from a mixer down a chute which has a cross-sectional shape as shown.
The curved bottom has the shape given by the function $f(x) = 0.12x^2 - 12$ for $-10 \le x \le 10$.

a Find the area of the cross-section.

Concrete can flow down the chute at 1.6 m/s.

b What volume of concrete can be poured in one minute?

c How long does it take to empty a mixer holding 12 m^3 of concrete?

summary

Definite integrals

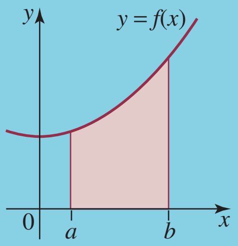
- The fundamental theorem of integral calculus:

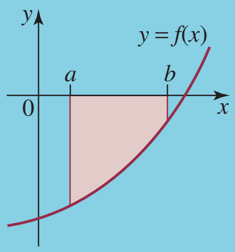
$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a) \quad \text{where } F(x) \text{ is an antiderivative of } f(x).$$

- $\int_a^b f(x)\,\mathrm{d}x$ is the definite integral.
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, $a < c < b$
- $\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$
- $\int_a^b [f(x) \pm g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

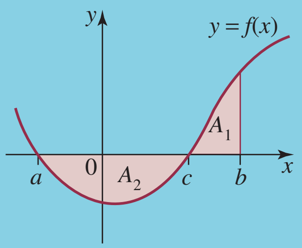
Area under curves

- Area $= \int_a^b f(x)\,\mathrm{d}x$, if $f(x) > 0$ for $a \le x \le b$





- Area $= -\int_a^b f(x)\,\mathrm{d}x$, if $f(x) < 0$ for $a \le x \le b$



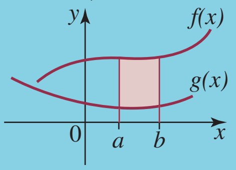
- Area $= \int_c^b f(x)\,\mathrm{d}x - \int_a^c f(x)\,\mathrm{d}x$

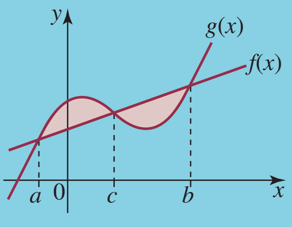
$$= \int_c^b f(x)\,\mathrm{d}x + \left|\int_a^c f(x)\,\mathrm{d}x\right|, \text{ if } f(x) > 0 \text{ for } c \le x \le b$$

and $f(x) < 0$ for $a \le x \le c$

Area between curves

- Area $= \int_a^b [f(x) - g(x)]\,\mathrm{d}x$, if $f(x) > g(x)$ for $a \le x \le b$





- Area $= \int_a^c [g(x) - f(x)]\,\mathrm{d}x + \int_c^b [f(x) - g(x)]\,\mathrm{d}x$,

  if $g(x) > f(x)$ for $a \le x \le c$
  and $f(x) > g(x)$ for $c \le x \le b$

CHAPTER review

1 **multiple choice** 8A

The expression $\int_0^4 (3\sqrt{x} - x)\ dx$ is equal to:

A 2 **B** 8 **C** $-2\frac{1}{2}$ **D** 20 **E** 16

2 **multiple choice** 8A

The exact value of the definite integral $\int_{-2}^{2} (4e^{2x} - 2e^{-2x})\ dx$ is:

A $3e^4 - e^{-4}$ **B** $2e^4 - e^{-4}$ **C** $e^4 - 2e^{-4}$ **D** $e^4 + e^{-4}$ **E** $e^4 - e^{-4}$

3 **multiple choice** 8A

The exact value of $\int_0^{\pi} \left(-2 \cos \frac{x}{3}\right) dx$ is:

A −3 **B** 3 **C** $-3\sqrt{3}$ **D** $3\sqrt{3}$ **E** $3\sqrt{2}$

4 Evaluate each of the following definite integrals. 8A

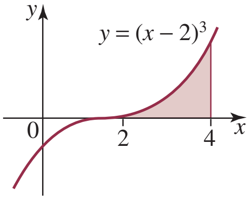
a $\int_{-1}^{0} -\frac{9}{(2x+3)^4}\ dx$ **b** $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos 2x\ dx$

5 Given that $\int_0^k (4x - 5)\ dx = -2$, find two possible values for k. 8A

6 **multiple choice** 8B

The shaded area on the graph at right is equal to:

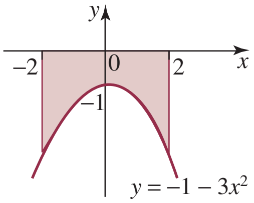
A 12 sq. units **B** 16 sq. units **C** 10 sq. units
D 4 sq. units **E** 8 sq. units



7 **multiple choice** 8B

The shaded area on the graph at right is:

A 20 sq. units **B** −20 sq. units **C** −16 sq. units
D 16 sq. units **E** −18 sq. units



8 **a** Sketch the graph of the function $f(x) = \frac{1}{x-2}$. 8B

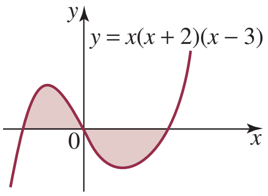
b Find the exact area between the graph of $f(x)$, the x-axis and the lines $x = 3$ and $x = 6$.

9 **multiple choice** 8C

The area bounded by the curve on the graph at right and the x-axis is equal to:

A $20\frac{5}{12}$ sq. units **B** $21\frac{1}{12}$ sq. units **C** $10\frac{5}{12}$ sq. units

D $-10\frac{5}{12}$ sq. units **E** $20\frac{7}{12}$ sq. units



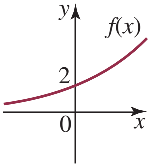
Questions **10** to **12** apply to the curve with equation $f(x) = e^x - 1$.

8C

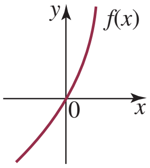
10 **multiple choice**

The graph of $f(x)$ is best represented by:

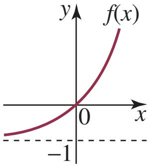
A



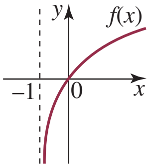
B



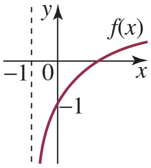
C



D



E



8C

11 **multiple choice**

The area bounded by the graph of $f(x)$, the x-axis and the line $x = 2$ is equal to:

A $e^2 - 1$ **B** $e^2 - 2$ **C** $e^2 + 1$ **D** $e^2 + 2$ **E** $e^2 - 3$

8C

12 **multiple choice**

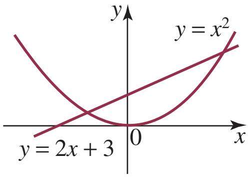
The area bounded by the graph of $f(x)$, the y-axis and the line $y = e^2 - 1$ is equal to:

A $e^2 - 5$ **B** $e^2 - 3$ **C** $e^2 + 2$ **D** $e^2 + 1$ **E** $5 - e^2$

8C

13 Find the area bounded by the curve $g(x) = (4 - x)(6 + x)$ and the x-axis.

Use the graph at right to answer questions **14** and **15**.



8D

14 **multiple choice**

The two graphs intersect where x is equal to:

A 1 and −3 **B** −1 and 3 **C** 1 and 2 **D** −1 and −2 **E** 1 and 3

8D

15 **multiple choice**

The area bounded by the two graphs is equal to:

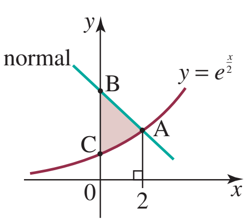
A $10\frac{2}{3}$ sq. units **B** $7\frac{1}{3}$ sq. units **C** $-7\frac{1}{3}$ sq. units

D $11\frac{1}{3}$ sq. units **E** $6\frac{2}{3}$ sq. units

8D

16 Calculate the area between the curve $y = 2 \cos x$ and the lines $y = -x$, $x = 0$ and $x = \frac{\pi}{2}$.

17 The diagram at right shows part of the curve with equation $y = e^{\frac{x}{2}}$. Find:



a the coordinate of point A

b the equation of the normal to the curve at point A

c the coordinate of point B

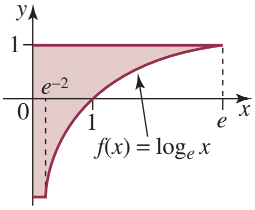
d the coordinate of point C

e the area bounded by the curve and the lines AB and BC.

18 **a** Find the derivative of $x \log_e x$.

b Hence, find an antiderivative of $\log_e x$.

The cross-section of a platform is shown at right. (All measurements are in metres.)



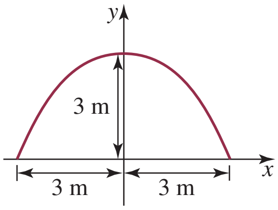
c Find the height of the platform.

d Find the cross-sectional area of the platform.

e Find the volume of concrete required to build this platform if it is 20 metres long.

Modelling and problem solving

1 The diagram at right shows an arched window with width 6 metres and height 3 metres. The arch can be approximated to a parabola.



- **a** Find the equation of the parabola.
- **b** Calculate the area of the window.
- **c** Show that the area of the window is $\frac{2}{3}$ the base of the arch times the height.
- **d** Hence find the area of a similar arched window with width 8 metres and height 4.5 metres.

2 **a** A ground-cover plant can cover the ground at a rate modelled by $\frac{dA}{dt} + 2t + 6t^2 - \frac{1}{4}t^3$, where A is the area in square centimetres and t is time in weeks.

If the plant initially covers 10 cm^2, how long, to the nearest week, will it take to cover 0.6 m^2?

- **b** What is the maximum area covered?

3 The value $\$V$ of an antique table is increasing at a rate modelled by $\frac{dV}{dt} = 300e^{0.3t}$, where t is the number of years after the table was first valued. If the table was first valued at $1000, how much would it be worth after 6 years? Round your answer to the nearest dollar.

4 A disease spreads through a community until a cure is found. Then the rate by which the number of cases of the disease is reduced is modelled by $\frac{dN}{dt} = Ae^{-0.2t}$, where t is the time in years.

- **a** Find an equation for N in terms of A that can be used to model the number of people with the disease at any time t years after the drug is first administered.
- **b** If A is −2000, find the number of people with the disease when the drug is first used.
- **c** Find the number of people with the disease after 2 years.
- **d** Find the number of people with the disease after 10 years.
- **e** According to this model, will the disease ever be eradicated? Give reasons.
- **f** When will there be as few as 100 cases?

Digital doc:
Test Yourself
Chapter 8

eBookplus ACTIVITIES

8A The fundamental theorem of integral calculus

Digital docs

- Investigation: Definite integrals (*page 277*)
- SkillSHEET 8.1: Practise subtracting function values (*page 281*)

Tutorial

WE1 Int-0566: Watch a tutorial on evaluating definite integrals (*page 278*)

8B Signed areas

Digital doc

- WorkSHEET 8.1: Evaluate definite integrals and unknown terminals and calculate areas between curves and linear graphs (*page 290*)

Tutorial

- **WE7** Int-0567: Watch a tutorial on finding the area bound by an exponential curve above and below the x-axis (*page 286*)

8D Areas between two curves

Tutorial

WE13 Int-0568: Watch a tutorial on calculating the area between two curves using a CAS calculator (*page 302*)

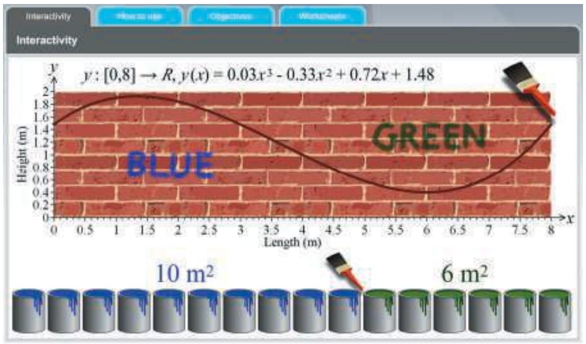
8E Further applications of integration

Digital doc

- WorkSHEET 8.2: Calculate areas enclosed by curves (*page 309*)

Interactivity

- Applications of antidifferentiation int-0269: Consolidate your understanding of applications of integration (*page 307*)



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 313*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

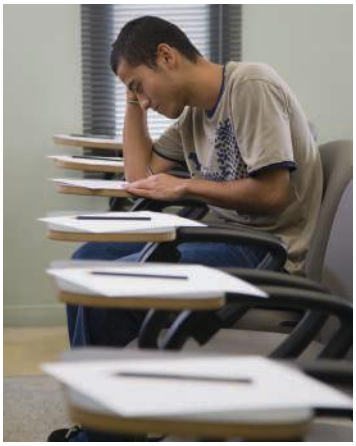
9

Probability distributions

syllabus reference

Applied statistical analysis

In this chapter



Introduction

Gary's test — did he pass?

Gary is sitting an aptitude test for a career in the armed forces. The test consists of 20 multiple-choice questions, each question with 4 alternative answers. To be accepted, Gary must answer at least 16 questions correctly. Gary is confident that he has answered 13 questions correctly but is unsure of the other 7, so he has a guess at each of these answers. What is the probability that Gary is accepted into the armed forces? This situation, and others like it, will be examined further in this chapter

Discrete random variables

A random variable is one whose value cannot be predicted but is determined by the outcome of an experiment. For example, two dice are rolled simultaneously a number of times. The sum of the numbers appearing uppermost is recorded. The possible outcomes we could expect are {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Since the possible outcomes may vary each time the dice are rolled, the sum of the numbers appearing uppermost is a random variable.

Random variables are expressed as capital letters, usually from the end of the alphabet (for example, X, Y, Z) and the value they can take on is represented by lower-case letters (for example, x, y, z respectively).

The above situation with dice illustrates an example of a discrete random variable since the possible outcomes were able to be counted. Discrete random variables generally deal with number or size.

A random variable which can take on any value is defined as a *continuous random variable*. Continuous random variables generally deal with quantities which can be measured, such as mass, height or time.

WORKED Example 1

Which of the following represent discrete random variables?

a **The number of goals scored at a football match**
b **The height of students in a Maths B class**
c **Shoe sizes**
d **The number of girls in a five-child family**
e **The time taken to run a distance of 10 kilometres in minutes**

THINK	WRITE
Determine whether the variable can be counted or needs to be measured.	
a Goals can be counted.	**a** Discrete.
b Height must be measured.	**b** Continuous.
c The number of shoe sizes can be counted.	**c** Discrete.
d The number of girls can be counted.	**d** Discrete.
e Time must be measured.	**e** Continuous.

Discrete probability distributions

When we are dealing with random variables, we often need to know the probabilities associated with them.

WORKED Example 2

Let X represent the variable 'number of Tails' obtained in three tosses. Draw up a table which displays the values the discrete random variable can assume (x) and the corresponding probabilities.

THINK

1. List all of the possible outcomes.
2. Draw up a table with two columns: one labelled 'Number of Tails', the other 'Probability'.
3. Enter the information into the table.

WRITE

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

Number of Tails (x)	Probability P(x)
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

The table above displays the probability distribution of the total number of Tails obtained in three tosses of a fair coin. Since the variable in this case is discrete, the table displays a discrete probability distribution.

In Worked example 2, we used X to denote the random variable and x the value which the random variable could take. Thus the probability can be denoted by $p(x)$ or $P(X = x)$. Hence the above table could be presented as shown below.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Close inspection of this table shows important characteristics which satisfy all discrete probability distributions.

1. Each probability lies in a restricted interval $0 \le P(X = x) \le 1$.
2. The probabilities of a particular experiment sum to 1, that is,

$$\Sigma P(X = x) = 1$$

If these two characteristics are not satisfied, then there is no discrete probability distribution.

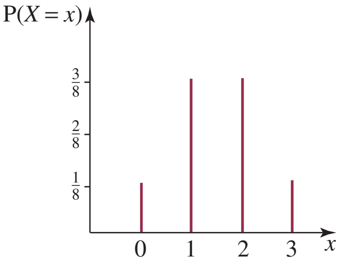
WORKED Example 3

Draw a probability distribution graph of the outcomes in Worked example 2.

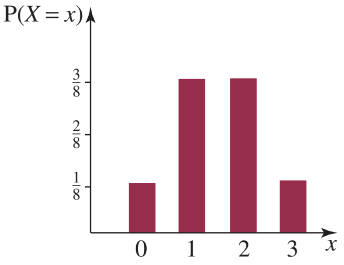
THINK

1. Draw a set of axes in the first quadrant only. Label the horizontal axis x and the vertical axis $P(X = x)$.
2. Mark graduations evenly along the horizontal and vertical axes, and label them with appropriate values.
3. Draw a straight line from each x-value to its corresponding probability.

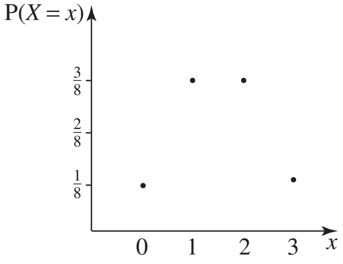
WRITE/DRAW



Note: The probability distribution graph may also be drawn as follows.



A column graph



A dot graph

WORKED Example 4

Which of the following tables represent a discrete probability distribution?

a

x	0	1	2	3
$P(X = x)$	0.2	0.5	0.2	0.1

b

x	0	2	4	6
$P(X = x)$	0.5	0.3	0.1	0.1

c

x	−1	0	1	2
$P(X = x)$	0.2	0.1	0.3	0.3

d

x	−2	0	5	7
$P(X = x)$	−0.2	0.3	0.5	0.4

THINK

a
1. Check whether each of the probabilities lies within the restricted interval $0 \le P(X = x) \le 1$.
2. Do the probabilities sum to 1?
3. Answer the question.

WRITE

a All probabilities lie between 0 and 1.

$0.2 + 0.5 + 0.2 + 0.1 = 1$

Yes, this is a discrete probability distribution since both requirements have been met.

THINK	WRITE
b ① Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.	**b** All probabilities lie between 0 and 1.
② Do the probabilities sum to 1?	$0.5 + 0.3 + 0.1 + 0.1 = 1$
③ Answer the question.	Yes, this is a discrete probability distribution since both requirements have been met.
c ① Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.	**c** All probabilities lie between 0 and 1.
② Do the probabilities sum to 1?	$0.2 + 0.1 + 0.3 + 0.3 \neq 1$ (total is 0.9)
③ Answer the question.	No, this is not a discrete probability distribution since both requirements have not been met.
d ① Check whether each of the probabilities lies within the restricted interval $0 \leq P(X = x) \leq 1$.	**d** The first probability is a negative value, so not all probabilities lie between 0 and 1.
② Answer the question.	No, this is not a discrete probability distribution since both requirements have not been met.

WORKED Example 5

Find the value of *k* for each of the following discrete probability distributions.

a

x	1	3	5	7	9
$P(X = x)$	0.2	k	0.2	0.3	0.1

b

x	0	1	2	3	4
$P(X = x)$	$5k$	$6k$	$4k$	$3k$	$2k$

THINK	WRITE
a ① Add up each of the given probabilities. This should sum to 1.	**a** $\Sigma P(X = x) = 1$ $0.2 + k + 0.2 + 0.3 + 0.1 = 1$
② Simplify.	$0.8 + k = 1$
③ Solve to find k.	$k = 1 - 0.8$ $= 0.2$
b ① Add up each of the given probabilities. This should sum to 1.	**b** $5k + 6k + 4k + 3k + 2k = 1$
② Simplify.	$20k = 1$
③ Solve to find k.	$k = \frac{1}{20}$

WORKED Example 6

a **Show that the function $p(x) = \frac{1}{42}(5x + 3)$, where $x = 0, 1, 2, 3$ is a probability function.**

b **Show that the function $p(x) = \frac{1}{100}x^2(6 - x)$, where $x = 2, 3, 4, 5$ is a probability function.**

THINK	WRITE
a 1 Substitute each of the x-values into the equation and obtain the corresponding probability.	**a** When $x = 0$, $p(x) = \frac{3}{42}$ $= \frac{1}{14}$
2 Simplify where possible.	When $x = 1$, $p(x) = \frac{8}{42}$ $= \frac{4}{21}$ When $x = 2$, $p(x) = \frac{13}{42}$ When $x = 3$, $p(x) = \frac{18}{42}$ $= \frac{3}{7}$
3 Check whether each of the probabilities lies within the restricted interval $0 \le P(X = x) \le 1$.	All probabilities lie between 0 and 1.
4 Check whether the probabilities sum to 1.	$\frac{1}{14} + \frac{4}{21} + \frac{13}{42} + \frac{3}{7} = 1$
5 Answer the question.	Yes, this is a probability function since both requirements have been met.
b 1 Substitute each of the x-values into the equation and obtain the corresponding probability.	**b** When $x = 2$, $p(x) = \frac{16}{100}$ $= \frac{4}{25}$
2 Simplify where possible.	When $x = 3$, $p(x) = \frac{27}{100}$ When $x = 4$, $p(x) = \frac{32}{100}$ $= \frac{8}{25}$ When $x = 5$, $p(x) = \frac{25}{100}$ $= \frac{1}{4}$
3 Check whether each of the probabilities lies within the restricted interval $0 \le P(X = x) \le 1$.	All probabilities lie between 0 and 1.
4 Check whether the probabilities sum to 1.	$\frac{4}{25} + \frac{27}{100} + \frac{8}{25} + \frac{1}{4} = 1$
5 Answer the question.	Yes, this is a probability function since both requirements have been met.

Consider the case of a probability experiment where we know part of the outcome. Suppose your friend Brett comes from a family of four children. What is the probability that there are three boys in Brett's family? Because you know Brett, you know that at least one of the four children is male.

Normally, the probability distribution of four children can be represented by the table shown below.

x	0	1	2	3	4
$P(X = x)$	0.0625	0.25	0.375	0.25	0.0625

Therefore, $P(X = 3) = 0.25$ but we know that the number of males in the family is greater than 0. From the table, $P(X > 0) = 0.9375$. We can say that the probability that there are three males in the family, given that at least one is male is $\frac{0.25}{0.9375}$, is 0.266.

This is known as *conditional* probability. The rule for conditional probability is written:

$$P(X = x \mid X > n) = \frac{P(X = x \cap X > n)}{P(X > n)}.$$

WORKED Example 7

Three balls are selected from a box containing 6 blue balls and 4 yellow balls. If the ball chosen after each selection is replaced before the next selection, find:

a **the probability distribution for the number of blue balls drawn:**
  i **0 blue balls** **ii** **1 blue ball** **iii** **2 blue balls** **iv** **3 blue balls**

b **the probability that 2 blue balls are chosen, given that at least one ball was blue.**

THINK	WRITE
a **i** ① Define the random variable.	**a** **i** Let X = the number of blue balls.
② Assign values which x can take on.	$x = 0, 1, 2, 3$
③ Determine the probability of each outcome.	$P(\text{blue}) = \frac{6}{10} = \frac{3}{5}$ $\quad$ $P(\text{yellow}) = \frac{4}{10} = \frac{2}{5}$
④ Simplify where possible.	$P(X = 0) \Rightarrow$ no blue, three yellow $= P(YYY)$ $P(X = 0) = \frac{2}{5} \cdot \frac{2}{5} \cdot \frac{2}{5}$ $= 0.064$
ii Simplify where possible.	**ii** $P(X = 1) \Rightarrow$ one blue, two yellow $= P(BYY) + P(YBY) + P(YYB)$ $P(X = 1) = 3 \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{2}{5}$ $= 0.288$

Continued over page ➔

THINK	**WRITE**
iii Simplify where possible.	**iii** $P(X = 2) \Rightarrow$ two blue, one yellow $= P(BBY) + P(BYB) + P(YBB)$ $P(X = 2) = 3 \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5}$ $= 0.432$
iv Simplify where possible.	**iv** $P(X = 3) \Rightarrow$ three blue, no yellow $= P(BBB)$ $P(X = 3) = \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{3}{5}$ $= 0.216$
5 Place all of the information in a table.	see table below
6 Check that the probabilities sum to 1.	$\Sigma P(X = x) = 0.064 + 0.288 + 0.432 + 0.216$ $= 1$
b **1** Define the rule for conditional probability.	**b** $P(X = 3 \vert X > 1) = \frac{P(X = 3 \cap X > 1)}{P(X > 1)}$
2 Determine each of the probabilities.	$P(X = 3 \cap X > 1) = P(X = 3)$ $= 0.216$ $P(X > 1) = 0.432 + 0.216$ $= 0.648$
3 Substitute values into the rule.	$P(X = 3 \vert X > 1) = \frac{0.216}{0.648}$
4 Evaluate and simplify.	$= 0.3333$ (or $\frac{1}{3}$)

x	0	1	2	3
P(*X* = *x*)	0.064	0.288	0.432	0.216

remember

1. Discrete random variables generally deal with number or size and are able to be counted.
2. A discrete probability distribution exists only if the following two characteristics are satisfied.
   (a) Each probability lies in a restricted interval $0 \leq P(X = x) \leq 1$.
   (b) The probabilities of a particular experiment sum to 1, that is, $\Sigma P(X = x) = 1$.
3. The rule for conditional probability is:

$$P(X = x \mid X > n) = \frac{P(X = x \cap X > n)}{P(X > n)}$$

EXERCISE 9A Discrete random variables

WORKED Example 1

1 Which of the following represent discrete random variables?

- **a** The number of people at a tennis match
- **b** The time taken to read this question
- **c** The length of the left arms of students in your class
- **d** The shoe sizes of twenty people
- **e** The weights of babies at a maternity ward
- **f** The number of grains in ten 250-gram packets of rice
- **g** The height of jockeys competing in a certain race
- **h** The number of books in Brisbane libraries

eBook*plus*

Digital doc:
Spreadsheet
048 Probability distribution

WORKED Example 2, 3

2 **a** If X represents the number of Heads obtained in two tosses of a coin, draw up a table which displays the values that the discrete random variable can assume and the corresponding probabilities.

b Draw a probability distribution graph of the outcomes in part **a**.

3 A fair coin is tossed three times and a note is taken of the number of Tails.

- **a** List the possible outcomes.
- **b** List the possible values of the random variable X, representing the number of Tails obtained in the three tosses.
- **c** Find the probability distribution of X.
- **d** Find $P(X \le 2)$.

4 Draw graphs for each of the following probability distributions.

a

x	1	2	3	4	5
$P(X = x)$	0.05	0.2	0.5	0.2	0.05

b

x	5	10	15	20
$P(X = x)$	0.5	0.3	0.15	0.05

c

x	2	4	6	8	10
$P(X = x)$	0.1	0.2	0.4	0.2	0.1

d

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.3	0.4

WORKED Example 4

5 Which of the following tables represent a discrete probability distribution?

a

x	1	3	5	7	9
$P(X = x)$	0.2	0.3	0.2	0.2	0.1

b

x	1	2	3	4	5
$P(X = x)$	0.1	0.1	0.1	0.1	0.1

c

x	3	6	9	12	15
$P(X = x)$	0.3	0.2	0.4	0.2	−0.1

d

x	−4	−3	−1	1	2
$P(X = x)$	0.1	0.1	0.4	0.2	0.2

WORKED Example 5

6 Find the value of k for each of the following discrete probability distributions.

a

x	1	2	3	4	5
$P(X = x)$	0.3	0.2	0.2	k	0.1

b

x	2	4	6	8	10
$P(X = x)$	0.1	0.1	0.1	0.1	k

c

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$4k$	k

d

x	−2	−1	0	1	2
$P(X = x)$	k	0.2	$3k$	0.3	0.1

7 Two fair dice are rolled simultaneously, and X, the sum of the two numbers appearing uppermost, is recorded.

a Draw up a table which displays the probability distribution of X, and find:
b $P(X > 9)$
c $P(X < 6)$
d $P(4 \leq X < 6)$
e $P(3 \leq X \leq 9)$
f $P(X < 12)$
g $P(6 \leq X < 10)$.

8 A spinner is numbered from 1 to 5, with each number being equally likely to come up. If X is the random variable representing the number showing on the spinner, find:

a the probability distribution of X
b the probability of getting an even number
c $P(X > 2)$.

9 A fair die is rolled and X is the square of the number appearing uppermost.

a Draw up a table which displays the probability distribution of X, and find:
b $P(X < 30)$
c $P(X > 10)$.

10 A fair die is altered so that the 1 is changed to a 5. If X is the random variable representing the number uppermost on the die, find:

a the probability distribution of X

b the probability of a number bigger than 2 appearing uppermost

c $P(X = 5 \mid X > 2)$.

WORKED Example 6a

11 Show that the function $p(x) = \frac{1}{90}(8x + 2)$, where $x = 0, 1, 2, 3, 4$ is a probability function.

WORKED Example 6b

12 Show that the function $p(x) = \frac{1}{160}x^2(x + 2)$, where $x = 1, 2, 3, 4$ is a probability function.

WORKED Example 7

13 Three balls are selected from a box containing 4 red balls and 5 blue balls. If the ball chosen after each selection is replaced before the next selection, find:

a the probability distribution for the following number of red balls drawn:

- **i** 0 red balls **ii** 1 red ball **iii** 2 red balls **iv** 3 red balls

b the probability that three reds are chosen, given that at least one ball is red.

14 A carton contains 12 light bulbs of which 3 are defective. If a random sample of 4 light bulbs is chosen without replacement, what is the probability distribution of defective light bulbs?

15 A biased coin is tossed twice. If the probability of obtaining a Head is $\frac{3}{5}$:

a find the probability distribution of the number of Heads in 2 tosses

b show that the sum of the probabilities is 1.

16 A discrete random variable has the following probability distribution:

x	1	2	3	4	5	6	7
$P(X = x)$	0.2	0.11	0.15	0.09	0.17	0.13	0.15

Find:

a $P(X > 3)$

b $P(X \le 4)$

c $P(3 \le X \le 6)$

d $P(2 < X < 5)$

e $P(X < 3 \mid X < 5)$

f $\{x\colon P(X < x) = 0.46\}$

g $\{x\colon P(X \ge x) = 0.54\}$.

17 **multiple choice**

Which one of the following random variables is not discrete?

A The price, in cents, of a loaf of bread at the local supermarket
B The number of runs scored by a batsman in each innings over a season
C The weight of a baby as he grows over a one-year period
D The number of houses sold by a real estate agent each month for a year
E The number of newspapers recycled by a family each month.

18 **multiple choice**

What is the value of k which will make this table a probability distribution table?

x	1	2	3	4
$P(X = x)$	$2k$	$3k$	$4k$	k

A 0 **B** 1 **C** 0.1 **D** $\frac{1}{9}$ **E** −0.1

19 **multiple choice**

Examine the following probability distribution table.

x	4	9	16	25	36
$P(X = x)$	0.16	0.21	0.35	0.08	0.2

$P(X \geq 10)$ is equal to:

A 0.38 **B** 0.84 **C** 0.35 **D** 0.28 **E** 0.63

20 **multiple choice**

The following table represents a discrete probability distribution for a random variable, Y.

x	4	7	10	13
$P(X = x)$	d	$4d$	$5d$	$2d$

The value of d is:

A $\frac{1}{9}$ **B** $\frac{1}{10}$ **C** $\frac{1}{11}$ **D** $\frac{1}{12}$ **E** $\frac{1}{13}$

21 **multiple choice**

A coin is biased so that the probability of obtaining a Head is $\frac{3}{7}$. If the coin is tossed 3 times, the probability of obtaining exactly 2 Heads is:

A $\frac{27}{343}$ **B** $\frac{108}{343}$ **C** $\frac{144}{343}$ **D** $\frac{135}{343}$ **E** $\frac{64}{343}$

22 **multiple choice**

Which of the following is a probability function?

A $p(x) = 0.1, 0.3, 0.4, 0.2, 0.1, x = 0, 1, 2, 3, 4$

B $p(x) = \frac{1}{66}(3x + 7), x = 0, 1, 2, 3, 4$

C $p(x) = \frac{1}{40}(5x - 1), x = 1, 2, 3, 4$

D $p(x) = \frac{1}{20}x^2(4 - x), x = 1, 2, 3$

E $p(x) = \frac{x^2}{20}(3x - 1), x = 1, 2, 3$

eBook*plus*

Digital doc:
WorkSHEET 9.1

23 If the random variable X represents the number of boys in a four-child family:

a write down the values which X may take

b assuming that the P(boy) = $\frac{1}{2}$, find the probability distribution of X.

Expected value of discrete random distributions

In past studies of statistics, the mean $(\bar{x})$ was defined as the average of a set of data or values. It was determined by the rule $\bar{x} = \frac{\sum xf}{\sum f}$, where x represented the value a variable could assume and f the frequency (that is, the number of times the variable occurred).

When dealing with discrete random variables, the mean is called the *expected value* or *expectation*. Since the expected value signifies the average outcome of an experiment, it could be used to determine the feasibility of a situation.

Consider the following example. John tosses two coins. If two Heads are obtained, he wins \$20. If one Head is obtained, he wins \$10. If no Heads are obtained, he loses \$25. John must consider his options and decide whether it is in his best interest to play. Determining the expected value (that is, the average outcome) may help John in his decision making process.

Allowing X to represent the number of Heads obtained, the above information is summarised in the table below.

Outcome	TT	TH or HT	HH
x	0	1	2
P($X = x$)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Win (\$)	−25	10	20

$$\begin{aligned}\text{The expectation or expected gain} &= \tfrac{1}{4}\cdot -25 + \tfrac{1}{2}\cdot 10 + \tfrac{1}{4}\cdot 20\\ &= -6.25 + 5 + 5\\ &= \$3.75\end{aligned}$$

The average outcome or expected gain is \$3.75 per toss. This might seem appealing; however, if there is a charge of \$5 per game played, it would not be in John's best interest to participate because he would lose \$1.25 per game on average. The above game would not be considered fair since the cost to play does not equal the expected gain.

The expected value of a discrete random variable, X, is denoted by E(X) or the symbol μ (mu). It is defined as the sum of each value of X multiplied by its respective probability; that is,

$$\begin{aligned}E(X) &= x_1P(X = x_1) + x_2P(X = x_2) + x_3P(X = x_3) + \ldots + x_nP(X = x_n)\\ &= \sum xP(X = x)\end{aligned}$$

Note: **The expected value will not always assume a discrete value.**

WORKED Example 8

Find the expected value of a random variable which has the following probability distribution.

x	1	2	3	4	5
P($X = x$)	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

THINK	WRITE
1 Write the rule for the expected value.	$E(X) = \sum xP(X = x)$
2 Substitute the values into the rule.	$E(X) = 1\cdot\frac{2}{5} + 2\cdot\frac{1}{10} + 3\cdot\frac{3}{10} + 4\cdot\frac{1}{10} + 5\cdot\frac{1}{10}$
3 Evaluate.	$= \frac{2}{5} + \frac{2}{10} + \frac{9}{10} + \frac{4}{10} + \frac{5}{10}$ $= 2\frac{2}{5}$

WORKED Example 9

Find the unknown probability, *a*, and hence determine the expected value of a random variable which has the following probability distribution.

x	2	4	6	8	10
$P(X = x)$	0.2	0.4	a	0.1	0.1

THINK	WRITE
① Determine the unknown value of a using the knowledge that the sum of the probabilities must total 1.	$0.2 + 0.4 + a + 0.1 + 0.1 = 1$ $0.8 + a = 1$ $a = 1 - 0.8$ $= 0.2$
② Write the rule for the expected value.	$E(X) = \sum x P(X = x)$
③ Substitute the values into the rule.	$E(X) = 2 \cdot 0.2 + 4 \cdot 0.4 + 6 \cdot 0.2 + 8 \cdot 0.1 + 10 \cdot 0.1$
④ Evaluate.	$= 0.4 + 1.6 + 1.2 + 0.8 + 1$ $= 5$

WORKED Example 10

Find the values of *a* and *b* of the following probability distribution if E(*X*) = 4.29.

x	1	2	3	4	5	6	7
$P(X = x)$	0.1	0.1	a	0.3	0.2	b	0.2

THINK	WRITE
① Write an equation for the unknown values of a and b using the knowledge that the sum of the probabilities must total 1. Call this equation [1].	$0.1 + 0.1 + a + 0.3 + 0.2 + b + 0.2 = 1$ $0.9 + a + b = 1$ $a + b = 1 - 0.9$ $a + b = 0.1 \quad [1]$
② Write the rule for the expected value.	$E(X) = \sum x P(X = x)$

Continued over page →

THINK | **WRITE**

3 Substitute the values into the rule.

$$4.29 = 1 \cdot 0.1 + 2 \cdot 0.1 + 3 \cdot a + 4 \cdot 0.3 + 5 \cdot 0.2 + 6 \cdot b + 7 \cdot 0.2$$

4 Evaluate and call this equation [2].

$$= 0.1 + 0.2 + 3a + 1.2 + 1 + 6b + 1.4$$

$$4.29 - 3.9 = 3a + 6b$$

$$3a + 6b = 0.39 \qquad [2]$$

5 Solve equations simultaneously.

$$a + b = 0.1 \qquad [1]$$

$$3a + 6b = 0.39 \qquad [2]$$

Multiply equation [1] by 3 and call it equation [3].

$$3 \cdot (a + b = 0.1)$$

$$3a + 3b = 0.3 \qquad [3]$$

Subtract equation [3] from equation [2]. Solve for *b*.

[2] − [3]:

$$3b = 0.09$$

$$b = 0.03$$

Substitute $b = 0.03$ into equation [1]. Solve for *a*.

$$a + 0.03 = 0.1$$

$$a = 0.1 - 0.03$$

$$= 0.07$$

6 Answer the question.

$a = 0.07$ and

$b = 0.03$

WORKED Example 11

Niki and Melanie devise a gambling game based on tossing three coins simultaneously. If three Heads or three Tails are obtained, the player wins $20. Otherwise the player loses $5. In order to make a profit they charge each person two dollars to play.

a What is the expected gain to the player? **b Do Niki and Melanie make a profit?**

c Is this a fair game?

THINK | **WRITE**

a 1 Define the random variable. Place all of the information in a table.

a Let X = the number of Heads obtained.

x	0	1	2	3
P($X = x$)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
Gain ($)	20	−5	−5	20

2 Write the rule for the expected value.

$$E(X) = \sum x\,P(X = x)$$

3 Substitute the values into the rule.

$$= 20 \cdot \tfrac{1}{8} + -5 \cdot \tfrac{3}{8} + -5 \cdot \tfrac{3}{8} + 20 \cdot \tfrac{1}{8}$$

4 Evaluate.

$$= \tfrac{20}{8} - \tfrac{15}{8} - \tfrac{15}{8} + \tfrac{20}{8}$$

$$= \tfrac{10}{8}$$

$$= \$1.25$$

5 Answer the question.

The player’s expected gain per game is $1.25; however, as each game incurs a cost of $2, the player in fact loses 75c per game.

THINK	WRITE
b Answer question using results from **a**.	**b** The girls make a profit of 75c per game.
c Answer question using results from **a**.	**c** No, this is not a fair game, since the cost to play each game does not equal the expected gain of each game.

WORKED Example 12

A random variable has the following probability distribution.

x	1	2	3	4
P($X = x$)	0.25	0.26	0.14	0.35

Find: a E(X) b E($3X$) c E($2X - 4$) d E(X^2).

THINK	WRITE
a ① Write the rule for the expected value.	**a** $E(X) = \sum x\,P(X = x)$
② Substitute the values into the rule.	$E(X) = 1 \cdot\ 0.25 + 2 \cdot\ 0.26 + 3 \cdot\ 0.14 + 4 \cdot\ 0.35$
③ Evaluate.	$= 0.25 + 0.52 + 0.42 + 1.4$ $= 2.59$
b ① Write the rule for the expected value.	**b** $E(3X) = \sum 3x\,P(X = x)$
② Substitute the values into the rule.	$E(X) = (3 \cdot\ 1) \cdot\ 0.25 + (3 \cdot\ 2) \cdot\ 0.26 + (3 \cdot\ 3) \cdot\ 0.14 + (3 \cdot\ 4) \cdot\ 0.35$
③ Evaluate. *Notes:* 1. The probability remains the same. 2. Each x-value is multiplied by 3 because of the new function, which is $3x$.	$= 3 \cdot\ 0.25 + 6 \cdot\ 0.26 + 9 \cdot\ 0.14 + 12 \cdot\ 0.35$ $= 0.75 + 1.56 + 1.26 + 4.2$ $= 7.77$
c ① Write the rule for the expected value.	**c** $E(2X - 4) = \sum (2x - 4)P(X = x)$
② Substitute the values into the rule.	$= (2 \cdot\ 1 - 4) \cdot\ 0.25 + (2 \cdot\ 2 - 4) \cdot\ 0.26 + (2 \cdot\ 3 - 4) \cdot\ 0.14 + (2 \cdot\ 4 - 4) \cdot\ 0.35$
③ Evaluate. *Notes:* 1. The probability remains the same. 2. Each x-value is multiplied by 2 and then 4 is subtracted from the result, because of the new function, which is $2x - 4$.	$= -2 \cdot\ 0.25 + 0 \cdot\ 0.26 + 2 \cdot\ 0.14 + 4 \cdot\ 0.35$ $= -0.5 + 0 + 0.28 + 1.4$ $= 1.18$

Continued over page

THINK	**WRITE**
d ① Write the rule for the expected value.	**d** $E(X^2) = \sum x^2 P(X = x)$
② Substitute the values into the rule.	$= (1^2) \cdot 0.25 + (2^2) \cdot 0.26 + (3^2) \cdot 0.14 + (4^2) \cdot 0.35$
③ Evaluate.	$= 1 \cdot 0.25 + 4 \cdot 0.26 + 9 \cdot 0.14 + 16 \cdot 0.35$
Notes: 1. The probability remains the same. 2. Each x-value is squared because of the new function, which is x^2.	$= 0.25 + 1.04 + 1.26 + 5.6$ $= 8.15$

The above worked example displays some important points that will now be investigated.

For this example, $E(X) = 2.59$

from **b** $E(3X) = 7.77$

note that $3E(X) = 3 \cdot 2.59$

$= 7.77$

from **c** $E(2X - 4) = 1.18$

note that $2E(X) - 4 = 2 \cdot 2.59 - 4$

$= 1.18$

Hence if X is a random variable and a is a constant, its expected value is defined by $E(aX) = aE(X)$. Furthermore, if X is a random variable where a and b are constants, then the expected value of a linear function in the form $f(X) = aX + b$ is defined by

$$E(aX + b) = aE(X) + b$$

If $a = 0$ then $E(aX + b) = aE(X) + b$

becomes $E(0X + b) = 0E(X) + b$

$= b$

These rules are called *expectation theorems* and are summarised below.

$\mathbf{E(aX) = aE(X)}$	**where X is a random variable and a is a constant.**
$\mathbf{E(aX + b) = aE(X) + b}$	**where X is a random variable a and b are constants.**
$\mathbf{E(b) = b}$	**where b is a constant.**
$\mathbf{E(X + Y) = E(X) + E(Y)}$	**where X and Y are both random variables.**

These theorems make it easier to calculate the expected values.

WORKED Example 13

Casey decides to apply for a job selling mobile phones. She receives a base salary of $200 per month and $15 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, *x*, being sold per month. What would be the expected salary Casey would receive each month?

x	50	100	150	200	250
$P(X = x)$	0.48	0.32	0.1	0.06	0.04

THINK	**WRITE**
Method 1	
1 Define a random variable.	Let X = the number of mobile phones sold by Casey in a month.
2 Write the rule for the expected salary.	$E(15X + 200) = \sum (15x + 200)P(X = x)$
3 Substitute the values into the rule.	$= (15 \cdot 50 + 200) \cdot 0.48 +$ $(15 \cdot 100 + 200) \cdot 0.32 +$ $(15 \cdot 150 + 200) \cdot 0.1 +$ $(15 \cdot 200 + 200) \cdot 0.06 +$ $(15 \cdot 250 + 200) \cdot 0.04$
4 Evaluate.	$= 950 \cdot 0.48 + 1700 \cdot 0.32 +$ $2450 \cdot 0.1 + 3200 \cdot 0.06 +$ $3950 \cdot 0.04$ $= 456 + 544 + 245 + 192 + 158$ $= 1595$
5 Answer the question.	The expected salary Casey would receive each month would be \$1595.
Method 2 Using the expectation theorem:	
1 Write the rule for the expected salary.	$E(X) = \sum xP(X = x)$
2 Substitute the values into the rule.	$= 50 \cdot 0.48 + 100 \cdot 0.32 + 150 \cdot 0.1 +$ $200 \cdot 0.06 + 250 \cdot 0.04$
3 Evaluate.	$= 24 + 32 + 15 + 12 + 10$ $= 93$
4 Using the fact that $E(aX + b) = aE(X) + b$ find $E(15X + 200)$.	$E(15X + 200) = 15E(X) + 200$ $= 15 \cdot 93 + 200$ $= 1595$

Note: Using the expectation theorem is quicker because it is easier to evaluate $aE(X) + b$ than $E(aX + b)$.

remember

1. The expected value of a discrete random variable, X, is defined by the rule $E(X) = \sum x\,P(X = x)$. Also $E(X^2) = \sum x^2\,P(X = x)$.
2. A game is considered fair if the cost to play the game is equal to the expected gain.
3. The expected value of a linear function can be calculated using the expectation theorems:

$$E(aX) = aE(X) \qquad E(aX + b) = aE(X) + b$$
$$E(b) = b \qquad E(X + Y) = E(X) + E(Y)$$

EXERCISE 9B Expected value of discrete random distributions

WORKED Example 8

1 Find the expected value of a random variable which has the following probability distribution.

x	1	2	3	4
$P(X = x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{3}{16}$

2 Find the expected value of a random variable which has the following probability distribution.

x	–2	–1	0	1	2	3	4
$P(X = x)$	$\frac{1}{18}$	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{9}$

3 Find the expected value of a random variable which has the following probability distribution.

x	–4	–2	0	2	4	6
$P(X = x)$	0.15	0.18	0.06	0.23	0.31	0.07

4 Find the expected value of a random variable which has the following probability distribution.

x	0	3	6	9	12
$P(X = x)$	0.21	0.08	0.19	0.17	0.35

5 Find the expected value of a random variable which has the following probability distribution.

x	–3	0	2	7	11	13
$P(X = x)$	0.2	0.05	0.26	0.3	0.01	0.18

WORKED Example 9

6 Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	1	3	5	7	9	11
$P(X = x)$	0.11	0.3	0.15	0.25	a	0.1

7 Find the unknown probability, a, and hence determine the expected value of a random variable which has the following probability distribution.

x	–2	1	4	7	10	13
$P(X = x)$	$\frac{5}{18}$	a	$\frac{1}{9}$	$\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$

8 Find the unknown probability, *b*, and hence determine the expected value of a random variable which has the following probability distribution.

x	0	1	2	3	4	5
$P(X = x)$	b	0.2	0.02	$3b$	0.1	0.08

9 Find the value of *k*, and hence determine the expected value of a random variable which has the following probability distribution.

x	2	4	6	8
$P(X = x)$	k	$2k$	$3k$	$4k$

10 Find the value of *k*, and hence determine the expected value of a random variable which has the following probability distribution.

x	4	8	12	16	20
$P(X = x)$	$6k$	$2k$	k	$3k$	$8k$

11 If X represents the outcome of a fair die being rolled, find:

a the probability distribution of each outcome **b** $E(X)$.

12 Two fair dice are rolled simultaneously. If X represents the sum of the two numbers appearing uppermost, find:

a the probability distribution of each outcome **b** $E(X)$.

13 If X represents the number of Heads obtained when a fair coin is tossed twice, find:

a the probability distribution of each outcome **b** $E(X)$.

14 A fair coin is tossed 4 times. If X represents the number of Tails obtained, find:

a the probability distribution of each outcome **b** $E(X)$.

WORKED Example 10

15 Find the values of *a* and *b* of the following distribution if $E(X) = 1.91$.

x	0	1	2	3	4	5	6
$P(X = x)$	0.2	0.32	a	0.18	b	0.05	0.05

16 Find the values of *a* and *b* of the following distribution if $E(X) = 2.41$.

x	0	1	2	3	4	5
$P(X = x)$	0.2	a	0.23	0.15	b	0.12

WORKED Example 11

17 Lucas contemplates playing a new game which involves tossing three coins simultaneously. He will receive \$15 if he obtains 3 heads, \$10 if he obtains 2 heads and \$5 if he obtains 1 head. However, if he obtains no heads he must pay \$30. He must also pay \$5 for each game he plays.

a What is Lucas' expected gain? **b** Should he play the game? Why?

c Is this a fair game? Why?

18 X is a discrete random variable with the following probability distribution.

x	2	4	7	k
$P(X = x)$	0.3	0.2	0.4	0.1

Find the value of *k* if the mean is 5.3.

19 X is a discrete random variable with the following probability distribution.

x	-2	3	8	10	14	k
$P(X = x)$	0.1	0.08	0.07	0.27	0.16	0.32

Find the value of k if the mean is 10.98.

20 A coin is biased such that the probability of obtaining a Tail is 0.6. If X represents the number of Tails in three tosses of the coin, find:

a the probability distribution of X **b** $E(X)$.

WORKED Example 12

eBook plus

Digital doc:
SkillSHEET 9.1
Expected value of a function of a random variable

21 A random variable has the following probability distribution.

x	1	2	3	4
$P(X = x)$	$\frac{2}{15}$	$\frac{7}{15}$	$\frac{1}{3}$	$\frac{1}{15}$

Find:

a $E(X)$ **b** $E(4X)$ **c** $E(2X + 1)$ **d** $E(X^2)$.

22 A random variable has the following probability distribution.

x	1	2	3	4
$P(X = x)$	0.33	0.25	0.27	0.15

Find:

a $E(X)$ **b** $E(4X - 6)$
c $E(X^2 + 1)$ **d** $E(3X^2)$.

WORKED Example 13

23 Christian decides to apply for a job selling mobile phones. He receives a base salary of $180 per month and $12 for every mobile phone sold. The following table shows the probability of a particular number of mobile phones, x, being sold per month. What would be the expected salary Christian would receive each month?

x	50	100	150	200	250
$P(X = x)$	0.32	0.38	0.2	0.06	0.04

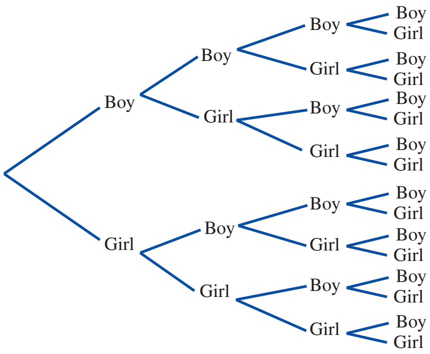
The binomial distribution

The binomial distribution is an example of a particular type of discrete probability distribution. It has relevance in many real life applications. This particular branch of mathematics moves away from the textbook and the classroom and into the areas of medical research, simulation activities and business applications such as quality control.

For a distribution to be defined as a binomial distribution, each of the following characteristics must be satisfied by the trials.

1. n independent trials must be conducted.
2. Only two possible outcomes must exist for each trial — success and failure.
3. The probability for success, p, is fixed for each trial; the probability of failure, q, is given by ($q = 1 - p$).

Consider the case of a family that has four children. Now suppose that we have been asked to find the probability that exactly two of the children are girls. This can be calculated by using the tree diagram drawn at right.



As can be seen, there are 16 ways in which the children can be born; six of these combinations consist of exactly two girls. The probability of this occurring can be found by letting X be the number of girls in a family of four children.

$$\begin{aligned} P(X = 2) &= 6 \cdot \ P(\text{Girl, Girl, Boy, Boy}) \\ &= 6 \cdot \ \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} \\ &= \tfrac{6}{16} \end{aligned}$$

We do not need to draw a tree diagram for each example, however, if we can work out the number of ways the desired outcome can occur. We are able to do this using Pascal's triangle.

Pascal's triangle is shown below. The triangle begins with a single 1 and two 1's in the next row. After that, each row begins and ends with a 1 and all other numbers are the sum of the two numbers above it.

Each row is numbered beginning with row 0. Each row then indicates the number of trials, and then the position in the row corresponds to the number of successes, beginning again with 0.

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1
Row 6	1 6 15 20 15 6 1
Row 7	1 7 21 35 35 21 7 1
Row 8	1 8 28 56 70 56 28 8 1
Row 9	1 9 36 84 126 126 84 36 9 1
Row 10	1 10 45 120 210 252 210 120 45 10 1

Consider the previous example. Four children are born, so we consider row 4 of the triangle.

$$\begin{array}{lccccc} & 1 & 4 & 6 & 4 & 1 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Number of successes} & 0 & 1 & 2 & 3 & 4 \end{array}$$

It can be seen here that exactly two girls (2 successes) from 4 children (4 trials) can occur in six ways.

On the calculator, this value can be found using the ${}^nC_r$ function. In the example, we would look to evaluate ${}^4C_2$.

In the above example, if we consider the birth of a girl a success and the birth of a boy a failure, the probability of success and failure are equal. Now we will consider an example where the probability of success and failure are not equal.

Consider the experiment where a fair die is rolled four times. If X represents the number of times a 3 appears uppermost, then X is a binomial variable. Obtaining a 3 will represent a success and all other values will represent a failure. The die is rolled four times so the number of trials, n, equals 4 and the probability, p, of obtaining a 3 is equal to $\frac{1}{6}$.

We shall now determine the probability of a 3 appearing uppermost 0, 1, 2, 3 and 4 times. Obtaining 3 is defined as a success and is denoted by S. All other numbers are defined as a failure and are denoted by F. The possible outcomes are listed in the table below.

Occurrence of 3	Possible outcomes	Probability	
0	*FFFF*	$1 \cdot \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$	q^4
1	*SFFF FSFF FFSF FFFS*	$4 \cdot \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) = \frac{500}{1296}$	$4q^3p$
2	*SSFF SFSF SFFS FSSF FSFS FFSS*	$6 \cdot \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right)^2 = \frac{150}{1296}$	$6q^2p^2$
3	*SSSF SSFS SFSS FSSS*	$4 \cdot \left(\frac{5}{6}\right) \left(\frac{1}{6}\right)^3 = \frac{20}{1296}$	$4qp^3$
4	*SSSS*	$1 \cdot \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$	p^4

This procedure for determining the individual probabilities can become tedious, particularly once the number of trials increases. Hence if X is a binomial random variable, its probability is defined as follows.

$$\mathrm{P}(X = x) = {}^nC_x p^x q^{n-x} \text{ where } x = 0, 1, 2, \ldots n.$$

x = the occurrence of the successful outcome
p = the probability of success
q = the probability of failure

Since this is a probability distribution, we would expect that the sum of the probabilities is 1. Therefore, for the above example:

$$\begin{aligned}P(X = x) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)\\ &= \frac{625}{1296} + \frac{500}{1296} + \frac{150}{1296} + \frac{20}{1296} + \frac{1}{1296}\\ &= 1\end{aligned}$$

WORKED Example 14

A binomial variable, X, has the probability function $P(X = x) = {}^6C_x(0.4)^x(0.6)^{6-x}$ where $x = 0, 1, \ldots 6$. Find:

a **n, the number of trials**

b **p, the probability of success**

c **the probability distribution for x as a table.**

Tutorial:
Worked example 14
int-0576

THINK	WRITE
a Obtain the relevant information from the given functions. The number of trials, n, is the value of the number located at the top left-hand corner of C.	**a** $n = 6$
b Obtain the relevant information from the given function. The probability of success, p, is the value in the first bracket.	**b** $p = 0.4$
c 1 Write down the rule of the given probability function.	**c** $P(X = x) = {}^6C_x(0.4)^x(0.6)^{6-x}$
2 Substitute $x = 0$ into the rule.	$P(X = 0) = {}^6C_0(0.4)^0(0.6)^6$
3 Evaluate.	$= 1 \cdot 1 \cdot 0.046\,656$ $= 0.046\,656$
4 Substitute $x = 1$ into the rule.	$P(X = 1) = {}^6C_1(0.4)(0.6)^5$
5 Evaluate.	$= 6 \cdot 0.4 \cdot 0.077\,76$ $= 0.186\,624$
6 Substitute $x = 2$ into the rule.	$P(X = 2) = {}^6C_2(0.4)^2(0.6)^4$
7 Evaluate.	$= 15 \cdot 0.16 \cdot 0.1296$ $= 0.311\,04$
8 Substitute $x = 3$ into the rule.	$P(X = 3) = {}^6C_3(0.4)^3(0.6)^3$
9 Evaluate.	$= 20 \cdot 0.064 \cdot 0.216$ $= 0.276\,48$
10 Substitute $x = 4$ into the rule.	$P(X = 4) = {}^6C_4(0.4)^4(0.6)^2$
11 Evaluate.	$= 15 \cdot 0.0256 \cdot 0.36$ $= 0.138\,24$

Continued over page →

THINK

12 Substitute $x = 5$ into the rule.

13 Evaluate.

14 Substitute $x = 6$ into the rule.

15 Evaluate.

16 Place values in a table.

WRITE

$$\begin{aligned} P(X = 5) &= {}^6C_5(0.4)^5(0.6) \\ &= 6 \cdot 0.010\,24 \cdot 0.6 \\ &= 0.036\,864 \end{aligned}$$

$$\begin{aligned} P(X = 6) &= {}^6C_6(0.4)^6(0.6)^0 \\ &= 1 \cdot 0.004\,096 \cdot 1 \\ &= 0.004\,096 \end{aligned}$$

x	$P(X = x)$
$P(X = 0)$	0.046 656
$P(X = 1)$	0.186 624
$P(X = 2)$	0.311 04
$P(X = 3)$	0.276 48
$P(X = 4)$	0.138 24
$P(X = 5)$	0.036 864
$P(X = 6)$	0.004 096

Graphics Calculator tip!

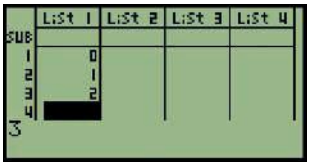
Binomial probabilities

To calculate the probabilities of a binomial distribution X, where the number of trials (n) is 4 and the probability of a 'success' in each trial (p) is 0.5, use the following steps.

For the Casio *fx*-9860G AU

1. To enter the possible values of x, press:
   - MENU
   - 2 (STAT).

   Enter the possible values for x: 0, 1, 2, 3 and 4 in List 1.



```
    List 1  List 2  List 3  List 4
SUB
1        0
2        1
3        2
4
3
```

2. To calculate the $P(X = x)$, press:
   - F5 (DIST)
   - F5 (BINM)
   - F1 (Bpd).

   Enter the fields as shown.

   Ensure the cursor is on Execute and then press F1 (CALC).

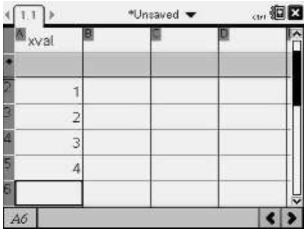
```
Binomial P.D
Data     :List
List     :List1
Numtrial:4
p        :0.5
Save Res:None
Execute
CALC
```

3. The table shows the probabilities of each outcome starting with $x = 0$.

```
Binomial P.D
1[0.0625]
2  0.25
3  0.375
4  0.25
5 0.0625
                0.0625
```

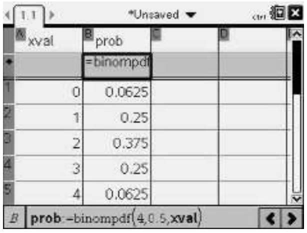
For the TI-Nspire CAS

1. To enter the possible values of x, open a Lists & Spreadsheet page.
   Label Column A ‘xval’ and enter the possible values for x: 0, 1, 2, 3 and 4 in this column.



	A xval	B	C	D
◆				
2	1			
3	2			
4	3			
5	4			
6				

2. To calculate the $P(X = x)$, label Column B ‘prob’ and press ENTER (enter).
   The grey header cell below B is now highlighted.
   Complete the entry line as:
   prob: = binompdf(4, 0.5, xval)
   then press ENTER (enter).



	A xval	B prob	C	D
◆		=binompdf		
1	0	0.0625		
2	1	0.25		
3	2	0.375		
4	3	0.25		
5	4	0.0625		

B prob:=binompdf(4,0.5,xval)

WORKED Example 15

A fair die is rolled five times. Find the probability of obtaining:

a exactly four 5s

b exactly two even numbers

c all results greater than 3

d a 5 on the first roll only.

Check parts a to c with a graphics calcuator.

THINK	WRITE
a 1 Check that all the characteristics have been satisfied for a binomial distribution.	**a** Binomial distribution — n independent trials and two outcomes, fixed p and q.
2 Write down the rule for the binomial probability distribution.	$P(X = x) = {}^nC_x p^x q^{n-x}$
3 Define and assign values to variables. The number of 5s obtained is exactly four.	$n = 5$ Let X = the number of 5s obtained. That is, $x = 4$. p = probability of a 5 $= \frac{1}{6}$ $q = \frac{5}{6}$
4 Substitute the values into the rule.	$P(X = 4) = {}^5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)$
5 Evaluate.	$= 5 \cdot \frac{1}{1296} \cdot \frac{5}{6}$ $= \frac{25}{7776}$
6 Check your answer with a graphics calculator.	

Continued over page

THINK	**WRITE**
b ① Define and assign values to variables. Two even numbers means we have $x = 2$.	**b** $n = 5$ Let X = the number of even numbers. That is, $x = 2$. p = probability of an even number $= \frac{1}{2}$ $q = \frac{1}{2}$
② Substitute the values into the rule.	$\text{P}(X = 2) = {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$
③ Evaluate.	$= 10 \cdot \frac{1}{4} \cdot \frac{1}{8}$ $= \frac{10}{32}$
④ Simplify.	$= \frac{5}{16}$
⑤ Check your answer with a graphics calculator.	
c ① Define and assign values to variables. We require five occasions when results are greater than 3.	**c** $n = 5$ Let X = values greater than 3. That is, $x = 5$. Three outcomes — 4, 5, 6 out of six are greater than 3. p = probability of number greater than 3 $= \frac{1}{2}$ $q = \frac{1}{2}$
② Substitute the values into the rule.	$\text{P}(X > 3) = {}^5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0$
③ Evaluate.	$= 1 \cdot \frac{1}{32} \cdot 1$
④ Check your answer with a graphics calculator.	$= \frac{1}{32}$
d Since a specific order is required here, the binomial rule is not required.	**d** P(5 on the first toss only) $= \text{P}(SFFFF)$ $= \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$ $= \frac{625}{7776}$

Note: If the rule for the binomial probability distribution were to be used in part **d**, it would provide an answer of $5 \cdot \frac{625}{7776} = \frac{3125}{7776}$. This answer gives the probability of obtaining a 5 once on any of the five trials, not necessarily on the first roll only. Hence, if a specific order is required the rule for the binomial probability distribution should not be used.

WORKED Example 16

A new drug for hay fever is known to be successful in 40% of cases. Ten hay fever sufferers take part in the testing of the drug. Find the probability that:

a **four people are cured** **b** **no people are cured** **c** **all ten are cured.**

THINK	**WRITE**
a ① Check that all the characteristics have been satisfied for a binomial distribution.	**a** This is a binomial distribution with n independent trials and two outcomes, p and q.
② Write down the rule for the binomial probability distribution.	$P(X = x) = {}^nC_x p^x q^{n-x}$
③ Define and assign values to variables.	$n = 10$ Let X = the number of people cured, therefore $x = 4$. $p = 0.4$ $q = 0.6$
④ Substitute the values into the rule.	$P(X = 4) = {}^{10}C_4(0.4)^4\ (0.6)^6$
⑤ Evaluate.	$= 210 \cdot\ 0.0256 \cdot\ 0.046\,656$ $= 0.250\,822\,656$
⑥ Round off the answer to 4 decimal places.	≈ 0.2508
⑦ Answer the question.	The probability that 4 people are cured is 0.2508.
b ① Define and assign values to variables.	**b** $n = 10$ Let X = the number of people cured, therefore $x = 0$. $p = 0.4$ $q = 0.6$
② Substitute the values into the rule.	$P(X = 0) = {}^{10}C_0(0.4)^0\ (0.6)^{10}$
③ Evaluate.	$= 1 \cdot\ 1 \cdot\ 0.006\,046\,617\,6$ $= 0.006\,046\,617\,6$
④ Round off the answer to 4 decimal places.	$= 0.0060$
⑤ Answer the question.	The probability that no people are cured is 0.0060.
c ① Define and assign values to variables.	**c** $n = 10$ Let X = the number of people cured, therefore $x = 10$. $p = 0.4$ $q = 0.6$
② Substitute the values into the rule.	$P(X = 10) = {}^{10}C_{10}(0.4)^{10}\ (0.6)^0$
③ Evaluate.	$= 1 \cdot\ 0.000\,104\,857\,6 \cdot\ 1$ $= 0.000\,104\,857\,6$
④ Round off the answer to 4 decimal places.	≈ 0.0001
⑤ Answer the question.	The probability that all 10 people are cured is 0.0001.

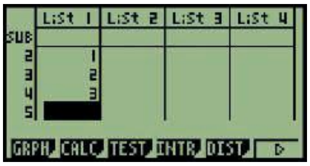
Graphics Calculator tip! Binomial probability for a range of x-values

The following steps show how to calculate the $P(X \le x)$ for all x-values in a binomial distribution involving 3 trials, with the probability of 'success' in each trial of 0.75.

For the Casio *fx*-9860G AU

1. To enter the possible values of x, press:
   - MENU
   - 2 (STAT).

   Enter the possible values for x: 0, 1, 2 and 3 in List 1.



	List 1	List 2	List 3	List 4
SUB				
2	1			
3	2			
4	3			
5				

2. To calculate the $P(X = x)$, press:
   - F5 (DIST)
   - F5 (BINM)
   - F2 (Bcd).

   Enter the fields as shown.
   Ensure the cursor is on Execute and then press F1 (CALC).

Binomial C.D
Data :List
List :List1
Numtrial:3
p :0.75
Save Res:None
Execute
CALC

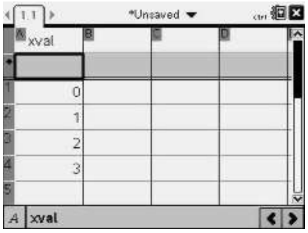
3. The table shows:
   $P(X \le 0) = 0.015625$
   $P(X \le 1) = 0.15625$
   $P(X \le 2) = 0.578125$
   $P(X \le 3) = 1.$

Binomial C.D
1 0.0156
2 0.1562
3 0.5781
4 1
0.015625

For the TI-Nspire CAS

1. To enter the possible values of x, open a Lists & Spreadsheet page.

   Label Column A 'xval' and enter the possible values for x: 0, 1, 2 and 3 in this column.



	A xval	B	C	D
◆				
1	0			
2	1			
3	2			
4	3			
5				

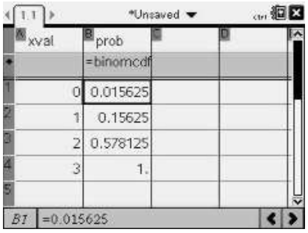
2. To calculate the $P(X \le x)$, call Column B 'prob' and press ENTER enter.
   The grey header cell below B is now highlighted.
   Complete the entry line as:

   prob: = binomcdf(3, 0.75, xval)

   then press ENTER enter.

   The table shows:

   $P(X \le 0) = 0.015625$
   $P(X \le 1) = 0.15625$
   $P(X \le 2) = 0.578125$
   $P(X \le 3) = 1.$



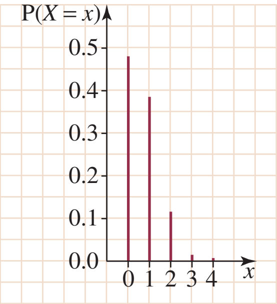
	A xval	B prob	C	D
◆		=binomcdf		
1	0	0.015625		
2	1	0.15625		
3	2	0.578125		
4	3	1.		
5				

B1 =0.015625

Graphs of the binomial distribution

We shall now consider the graph of a binomial distribution. If we refer to the example of obtaining a 3 when rolling a die four times (see table on page 338) we note that $X \sim \text{Bi}(4, \frac{1}{6})$. The probability distribution of the random variable, X, is given in the table and graph below.

x	$P(X = x)$
0	0.4823
1	0.3858
2	0.1157
3	0.0154
4	0.0008



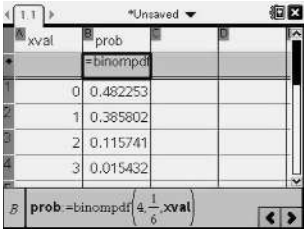
Graphics Calculator tip!

Using the TI-Nspire CAS to graph the binomial distribution $X \sim \text{Bi}(4, \frac{1}{6})$

1. To create a spreadsheet of values, open a Lists & Spreadsheet page.
   Label Column A ‘xval’ and enter the possible values for x: 0, 1, 2, 3 and 4 in this column.
   Label Column B ‘prob’ and in the grey header cell below, complete the entry line as:

   $$\text{prob: } = \text{binompdf}(4, \frac{1}{6}, \text{xval})$$

   then press ENTER.

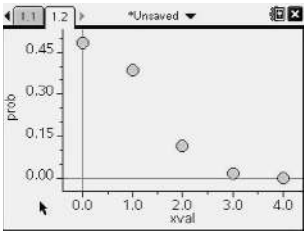


	A xval	B prob
◆		=binompdf
1	0	0.482253
2	1	0.385802
3	2	0.115741
4	3	0.015432

B prob:=binompdf$(4,\frac{1}{6},$xval$)$

2. To graph these probabilities, open a Data & Statistics page.
   Tab to the horizontal axis and label the horizontal axis with the independent variable ‘xval’; then tab to the vertical axis and label the vertical axis with the dependent variable ‘prob’.

   This gives a graph of the probability distribution.



Binomial distribution graphs

eBookplus

Digital doc:
Spreadsheet
221 Binomial probabilities

Use a graphics calculator or other technology to study graphs of the following:

1 $P(X = x)$ versus x for $X \sim \text{Bi}(10, 0.3)$; that is, binomial distribution with $n = 10$, $p = 0.3$

2 $P(X = x)$ versus x for $X \sim \text{Bi}(10, 0.5)$

3 $P(X = x)$ versus x for $X \sim \text{Bi}(10, 0.7)$

4 What effect does p have on the graph?

Now study graphs of the following:

5 P($X = x$) versus x for $X \sim \text{Bi}(10, 0.7)$

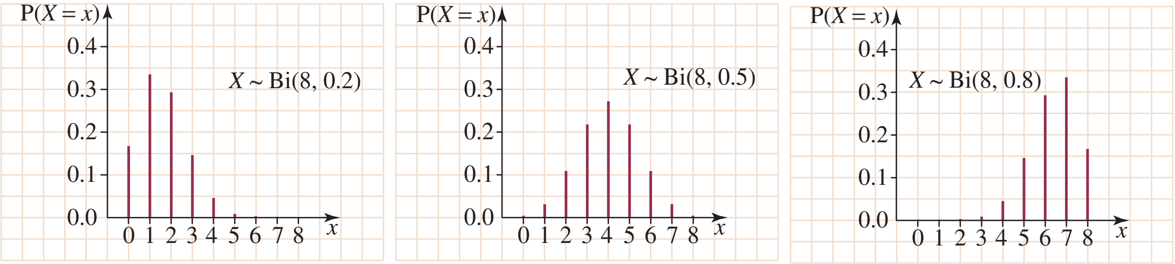
6 P($X = x$) versus x for $X \sim \text{Bi}(20, 0.7)$

7 P($X = x$) versus x for $X \sim \text{Bi}(100, 0.7)$

8 What effect does n have on the graph?

The effect of changing *n* and *p* on binomial distribution graphs

The effect of increasing p can be seen in the series of graphs below.

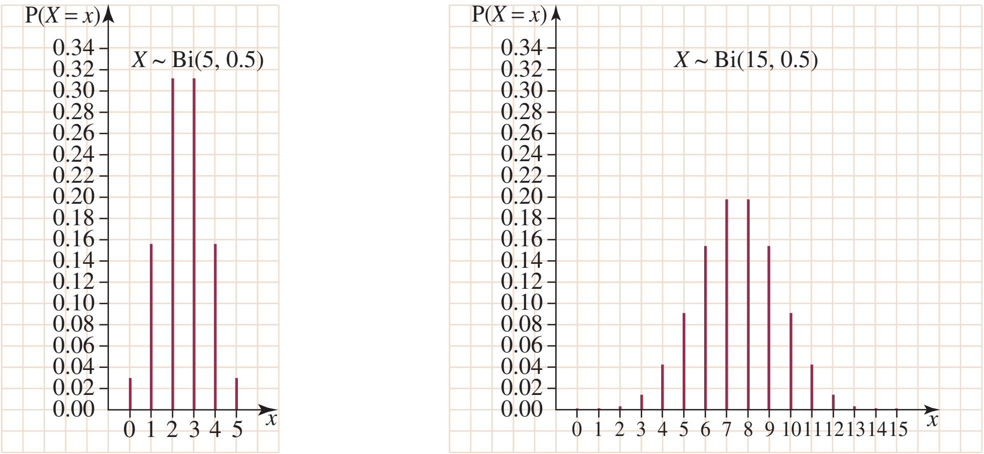


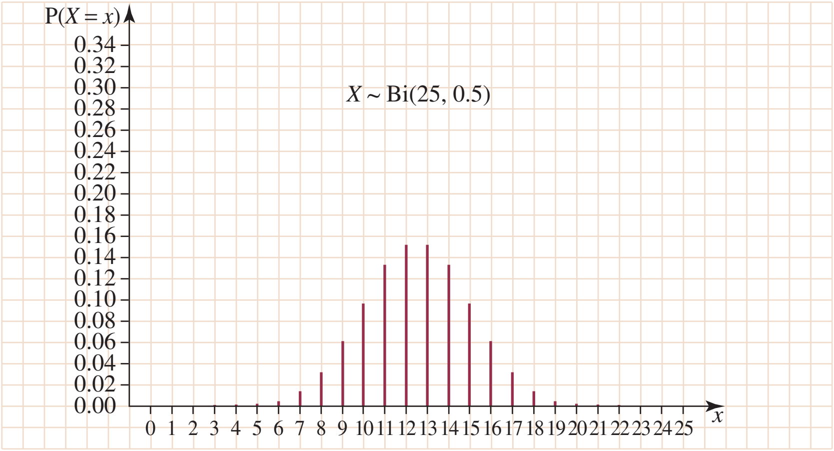
The graph is positively skewed or skewed to the right.

The graph is symmetrical. It is also called a normal distribution curve.

The graph is negatively skewed or skewed to the left.

We will now keep p fixed and vary n.



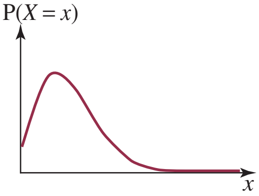


From the preceding graphs it can be seen that:

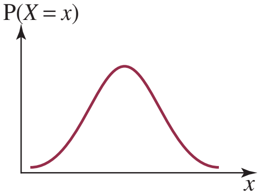
1. when $p = 0.5$, the graph is symmetrical.
2. as n increases, and the interval between the vertical columns decreases, the graph approximates a smooth hump or bell shape.

The effects that the parameters n and p have on the binomial probability distribution curve can be summarised in the following way.

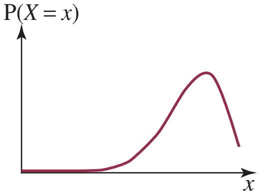
If $p < 0.5$, the graph is positively skewed, or skewed to the right.



If $p = 0.5$, the graph is symmetrical.



If $p > 0.5$, the graph is negatively skewed, or skewed to the left.



When n is large and $p = 0.5$, the vertical columns are closer together and the line graph becomes a bell shaped curve or a normal distribution curve.

remember

Before commencing binomial probability distribution problems:

1. Check that each of the characteristics has been satisfied for a binomial distribution:
   - (a) there are n independent trials
   - (b) there are two possible outcomes: success and failure
   - (c) the probability of success, p, is fixed.
2. Write down the rule for the binomial probability distribution.

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x} \text{ where } x = 0, 1, 2, \quad , n.$$

EXERCISE 9C The binomial distribution

1 Which of the following constitutes a binomial probability distribution?

- **a** Rolling a die 10 times and recording the number that comes up
- **b** Rolling a die 10 times and recording the number of 3s that come up
- **c** Spinning a spinner numbered 1 to 10 and recording the number that is obtained
- **d** Tossing a coin 15 times and recording the number of Tails obtained
- **e** Drawing a card from a fair deck, without replacement, and recording the number of picture cards
- **f** Drawing a card from a fair deck, with replacement, and recording the number of black cards
- **g** Selecting 3 marbles from a jar containing 3 yellow marbles and 2 black marbles, without replacement

2 Evalute the following:

- **a** ${}^{7}C_{2}\,(0.4)^{2}\,(0.6)^{5}$
- **b** ${}^{9}C_{3}\,(0.1)^{3}\,(0.9)^{6}$
- **c** ${}^{10}C_{5}\,(0.5)^{5}\,(0.5)^{5}$
- **d** ${}^{8}C_{5}\,(0.2)^{5}\,(0.8)^{3}$
- **e** ${}^{9}C_{7}\left(\frac{1}{3}\right)^{7}\left(\frac{2}{3}\right)^{2}$
- **f** ${}^{10}C_{0}\,(0.15)^{0}\,(0.85)^{10}$.

WORKED Example 14

3 A binomial variable, X, has the probability function $P(X = x) = {}^5C_x(0.3)^x(0.7)^{5-x}$ where $x = 0, 1, \ldots, 5$. Find:

- a n, the number of trials
- b p, the probability of success
- c the probability distribution for x as a table.

4 Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. What is the probability of 5 items being defective in a box?

5 Nadia has a 40% chance of getting a red light on the way to work. What is the probability of Nadia getting a red light on 4 out of 5 days?

6 Peter has four chances to knock an empty can off a stand by throwing a ball. On each throw, the probability of success is $\frac{1}{5}$. Find the probability that he will knock the empty can off the stand:

- a once
- b twice
- c at least once.

WORKED Example 15

7 A fair coin is tossed four times. Find the probability of obtaining:

- a Heads on the first two tosses and Tails on the second two
- b Heads on every roll
- c two Heads and two Tails.

8 A weighted coin is biased such that a Tail comes up 60% of the time. The coin is tossed five times. Find the probability of obtaining:

- a Tails on the first four tosses only
- b four Tails.

WORKED Example 16

9 Fifty-five per cent of the local municipality support the local council. If eight people are selected at random, find the probability that:

- a half support the council
- b all eight support the council
- c five support the council
- d three oppose the council.

10 The probability of Colin beating Maria at golf is 0.4. If they play once a week throughout the entire year and the outcome of each game is independent of any other, find the probability that they will have won the same number of matches.

11 It is known that 5 out of every 8 people eat Superflakes for breakfast. Find the probability that half of a random sample of 20 people surveyed eat Superflakes.

12 On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6, while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Find the probability that exactly:

- a six watched Channel 6
- b four watched Channel 8.

13 Three per cent of items produced by a certain machine are defective. A random sample of 10 items is taken. Find the probability that exactly 10% are defective.

14 A new drug being trialled gives 8% of the patients a violent reaction. If 200 patients trial the drug, find the probability that 12 patients have a violent reaction to the drug.

15 A pen company produces 30 000 pens per week. However, of these 30 000 pens, 600 are defective. Pens are sold in boxes of 20. Find the probability that:

- a two defective pens are found in one box
- b a box is free from defective pens.

16 A box contains 5 red marbles, 3 blue marbles and 2 yellow marbles. A marble is chosen at random and replaced. This selection process is completed eight times. Find the probability that:

a exactly 4 red marbles are selected **b** exactly 2 blue marbles are selected

c no yellow marbles are selected.

17 **multiple choice**

Which of the following does not represent a binomial distribution?

A Rolling a die four times and recording the number of 2s

B Tossing a coin 10 times and recording the number of Heads

C Rolling two dice simultaneously 20 times and recording the outcomes

D Drawing a card with replacement and recording the number of aces obtained

E Rolling two dice 6 times and recording the number of times a sum of 7 is obtained

18 **multiple choice**

The probability that the temperature in Brisbane will rise above 30°C on any given summer day, independent of any other summer day, is 0.6. The probability that four days in a week reach in excess of 30°C is:

A $0.6^4 \cdot 0.4^3$ **B** $7 \cdot 0.6^4 \cdot 0.4^3$ **C** $\frac{4}{7} \cdot 0.4^3$ **D** 0.6^4 **E** $35 \cdot 0.6^4 \cdot 0.4^3$

19 **multiple choice**

Rachel sits a multiple-choice test containing 20 questions, each with four possible answers. If she guesses every answer, the probability of Rachel getting 11 questions correct is:

A ${}^{20}C_{11}\left(\frac{1}{4}\right)^{20}\left(\frac{3}{4}\right)^{9}$ **B** ${}^{20}C_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$ **C** ${}^{20}C_{10}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^{9}$

D ${}^{20}C_{11}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{11}$ **E** ${}^{20}C_{10}\left(\frac{1}{4}\right)^{9}\left(\frac{3}{4}\right)^{11}$

20 **multiple choice**

A fair die is rolled 10 times. The probability of obtaining exactly two 6s is:

A 0.000 02 **B** 0.000 07 **C** 0.043 95

D 0.290 71 **E** 0.301 99

21 **multiple choice**

It is found that 3 out of every 10 cars are unroadworthy. Ten cars are selected at random. The probability that 3 are unroadworthy is:

A 0.009 **B** 0.2601

C 0.2668 **D** 0.5

E 1

eBook*plus*

Digital doc:

SkillSHEET 9.2

Solving indicial equations

22 A darts player knows that her chance of scoring a bullseye on any one throw is 0.1. Find the number of turns she would need to ensure a probability of 0.9 of scoring at least one bullseye.

23 From the following binomial distribution tables:

i draw a graph of the probability distribution

ii describe the skewness of each graph.

a $X \sim \text{Bi}(10, 0.3)$

x	$P(X = x)$
0	0.028 25
1	0.121 06
2	0.233 47
3	0.266 83
4	0.200 12
5	0.102 92
6	0.036 76
7	0.009 00
8	0.001 45
9	0.000 01
10	$5.9 \cdot 10^{-6}$

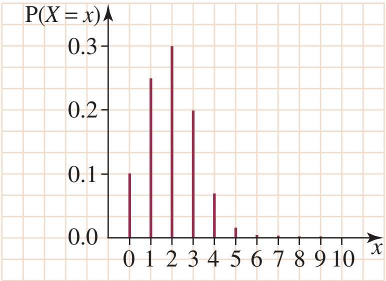
b $X \sim \text{Bi}(10, 0.5)$

x	$P(X = x)$
0	0.000 98
1	0.009 77
2	0.043 95
3	0.117 19
4	0.205 08
5	0.246 09
6	0.205 08
7	0.117 19
8	0.043 95
9	0.009 77
10	0.000 98

c $X \sim \text{Bi}(10, 0.8)$

x	$P(X = x)$
0	$1 \cdot 10^{-7}$
1	$4.1 \cdot 10^{-6}$
2	0.000 07
3	0.000 79
4	0.005 51
5	0.026 42
6	0.088 08
7	0.201 33
8	0.301 99
9	0.268 44
10	0.107 37

24 The following probability distribution is for $p = 0.2$ and $n = 10$.



a Find the most likely outcome for x.

b Describe the plot.

25 **a** Describe the plots of the following binomial probability distributions, without drawing the graphs.

i $n = 25, p = 0.1$ **ii** $n = 50, p = 0.5$ **iii** $n = 30, p = 0.9$

b What effect does p have on the graph of a binomial probability distribution?

26 **a** Describe the plot of the binomial probability distribution, $X \sim \text{Bi}(60, 0.5)$, without drawing the graph.

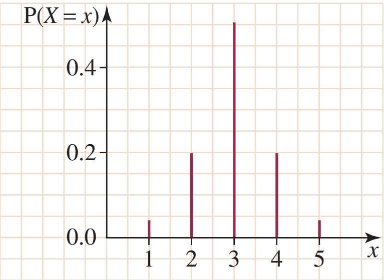
b Suggest how the graph might look for a binomial probability distribution with the same p, but double the value of n.

27 **a** Describe the plot of the binomial probability distribution, $X \sim \text{Bi}(100, 0.4)$, without drawing the graph.

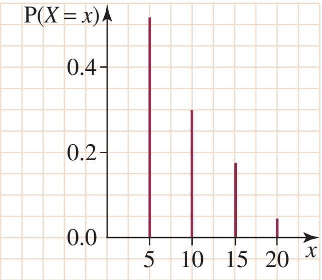
b Suggest how the graph might look for a binomial probability distribution with the same n, but double the value of p.

28 Describe the skewness of the graphs of the following binomial probability distributions.

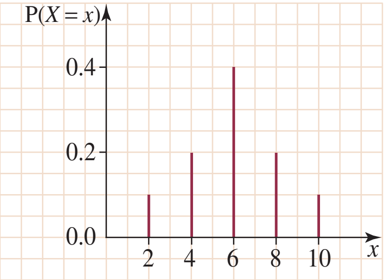
a



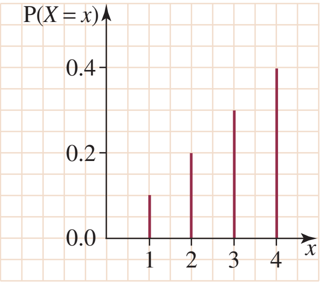
b



c



d



29 In Gold Lotto, your chance of winning first division is $\frac{1}{8\,145\,060}$. Find:

a the number of games you would need to play if you wanted to ensure a 50% chance of winning first division at least once

b the number of tickets you would need to buy for part **a** if there are 16 games on each ticket

c the cost of buying these tickets, if they cost $4.10 each.

Problems involving the binomial distribution for multiple probabilities

We shall now work with problems involving the binomial distribution for multiple probabilities.

WORKED Example 17

The binomial variable, X, has the following probability table.

x	0	1	2	3	4	5
$P(X = x)$	0.2311	0.3147	0.3321	0.1061	0.0112	0.0048

Find **a** $P(X > 3)$ **b** $P(X \le 4)$.

THINK	WRITE
a 1 $P(X > 3)$ means $P(X = 4)$ or $P(X = 5)$. Add these probabilities.	**a** $P(X > 3) = P(X = 4) + P(X = 5)$
2 Evaluate.	$= 0.0112 + 0.0048$ $= 0.0160$

Continued over page ➔

THINK	WRITE
b ① $P(X \le 4)$ would involve adding the probabilities from $P(X = 0)$ to $P(X = 4)$. Using the fact that $P(X \le 4) = 1 - P(X > 4)$ allows us to solve the problem using fewer terms.	**b** $P(X \le 4) = P(X = 0) + \ldots + P(X = 4)$ $= 1 - P(X = 5)$
② Substitute the value into the rule.	$= 1 - 0.0048$
③ Evaluate.	$= 0.9952$

WORKED Example 18

Find $P(X \ge 3)$ if X has a binomial distribution with the probability of success, p, and the number of trials, n, given by $p = 0.3$, $n = 5$.

THINK	WRITE
① Write down the rule for the binomial probability distribution.	$P(X = x) = {}^nC_x p^x q^{n-x}$
② Write down what is required.	$P(X \ge 3) = P(X = 3) + P(X = 4) + P(X = 5)$
③ Substitute the values for n, p and q into the rule.	$= {}^5C_3(0.3)^3(0.7)^2 + {}^5C_4(0.3)^4(0.7) + {}^5C_5(0.3)^5(0.7)^0$
④ Evaluate.	$= 10 \cdot 0.027 \cdot 0.49 + 5 \cdot 0.0081 \cdot 0.7 + 1 \cdot 0.002\,43 \cdot 1$ $= 0.1323 + 0.028\,35 + 0.002\,43$ $= 0.163\,08$

WORKED Example 19

A bag contains 4 red and 3 blue marbles. A marble is selected at random and replaced. The experiment is repeated 7 times. Find the probability that:

a **all 7 selections are red** **b** **at least 5 are red** **c** **no more than 2 are red.**

THINK	WRITE
① Write down the rule for the binomial probability distribution.	$P(X = x) = {}^nC_x\, p^x q^{n-x}$
② Define and assign values to variables.	$n = 7$ Let X = number of red marbles selected $p = \frac{4}{7}, q = \frac{3}{7}$
a ① Write down what is required.	**a** P(all 7 selections are red) $= P(X = 7)$
② Substitute the values for n, p and q into the rule.	$= {}^7C_7\left(\frac{4}{7}\right)^7\left(\frac{3}{7}\right)^0$
③ Evaluate.	$= 1 \cdot 0.0199 \cdot 1$ $= 0.0199$

THINK	WRITE
b ① Write down what is required.	**b** P(at least 5 are red) $= \mathrm{P}(X \geq 5)$ $= \mathrm{P}(X = 5) + \mathrm{P}(X = 6) + \mathrm{P}(X = 7)$
② Substitute the values for n, p and q into the rule.	$= {}^7C_5\left(\frac{4}{7}\right)^5\left(\frac{3}{7}\right)^2 + {}^7C_6\left(\frac{4}{7}\right)^6\left(\frac{3}{7}\right) + {}^7C_7\left(\frac{4}{7}\right)^7\left(\frac{3}{7}\right)^0$
③ Evaluate.	$= 21 \cdot 0.060\,93 \cdot 0.183\,67 +$ $7 \cdot 0.034\,82 \cdot 0.428\,57 +$ $1 \cdot 0.0199 \cdot 1$ $= 0.2350 + 0.1044 + 0.0199$ $= 0.3593$
c ① Write down what is required.	**c** P(no more than 2 are red) $= \mathrm{P}(X \leq 2)$ $= \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2)$
② Substitute the values for n, p and q into the rule.	$= {}^7C_0\left(\frac{4}{7}\right)^0\left(\frac{3}{7}\right)^7 + {}^7C_1\left(\frac{4}{7}\right)\left(\frac{3}{7}\right)^6 + {}^7C_2\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^5$
③ Evaluate.	$= 1 \cdot 1 \cdot 0.0027 + 7 \cdot 0.5714 \cdot$ $0.0062 + 21 \cdot 0.3265 \cdot 0.0145$ $= 0.0027 + 0.0248 + 0.0991$ $= 0.1266$

WORKED Example 20

X follows a binomial distribution with $n = 9$, $p = 0.4$. Find:

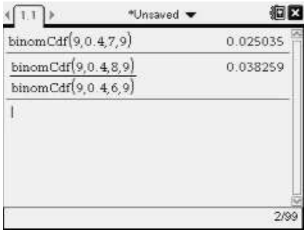
i P($X \geq 7$)

ii the probability that X is greater than 7 given it is greater than 5; that is, P($X > 7 \mid X > 5$)

a by hand **b using the TI-Nspire CAS calculator.**

THINK	WRITE
a i ① Write down what is required.	**a i** $\mathrm{P}(X \geq 7) = \mathrm{P}(X = 7) + \mathrm{P}(X = 8) + \mathrm{P}(X = 9)$
② Write down the rule for the binomial probability distribution.	$\mathrm{P}(X = x) = {}^nC_x p^x q^{n-x}$
③ Substitute the values for n, p and q into the rule.	$\mathrm{P}(X \geq 7) = {}^9C_7(0.4)^7(0.6)^2 +$ ${}^9C_8(0.4)^8(0.6) +$ ${}^9C_9(0.4)^9(0.6)^0$
④ Evaluate.	$= 36 \cdot 1.6384 \cdot 10^{-3} \cdot 0.36$ $+ 9 \cdot 6.5536 \cdot 10^{-4} \cdot 0.6$ $+ 1 \cdot 2.621\,44 \cdot 10^{-4} \cdot 1$ $= 0.021\,23 + 0.003\,54 +$ $2.621\,44 \cdot 10^{-4}$ $= 0.025\,034\,752$
⑤ Round off the answer to 4 decimal places.	$= 0.0250$

Continued over page →

THINK	**WRITE**
ii ① Write down what is required.	**ii** $P(X>7\mid X>5)=\dfrac{P(X>7\cap X>5)}{P(X>5)}$ $=\dfrac{P(X>7)}{P(X>5)}$
② Evaluate each probability individually. P($X > 7$) can be obtained from results in **i**.	$P(X>7)=P(X=8)+P(X=9)$ $=3.801\,088\cdot\ 10^{-3}$ $P(X>5)=P(X=6)+P(X=7)+$ $P(X=8)+P(X=9)$ $={}^9C_6(0.4)^6(0.6)^3+0.025\,034\,752$ $=84\cdot\ 4.096\cdot\ 10^{-3}\cdot\ 0.216+$ $0.025\,034\,752$ $=0.993\,525\,76$
③ Evaluate $\dfrac{P(X>7)}{P(X>5)}$.	$\dfrac{P(X>7)}{P(X>5)}=\dfrac{3.801\,088\cdot\ 10^{-3}}{0.993\,525\,76}$ $=0.038\,258\,75$ ≈ 0.0383
④ Round off the answer to 4 decimal places.	
For the TI-Nspire CAS	
b i ① To calculate P($X \geq 7$), complete the entry line as: binomCdf(9,0.4,7,9) then press ENTER (enter).	**b**  binomCdf(9,0.4,7,9) 0.025035 binomCdf(9,0.4,8,9)/binomCdf(9,0.4,6,9) 0.038259
ii ① To calculate P($X > 7$)|($X > 5$), complete the entry line as: binomCdf(9,0.4,8,9)/ binomCdf(9,0.4,6,9) then press ENTER (enter).	
② Write the solutions.	**i** $P(X\geq 7)=0.0250$ **ii** $P(X>7)\mid(X>5)=0.0383$

Note: Individual probabilities may be obtained using the graphics calculator and then added to arrive at the final answer.

remember

When solving problems dealing with the binomial distribution for multiple probabilities always:

1. write down what is required
2. write down the rule for the binomial probability distribution
3. substitute the values into the given rule and evaluate.

EXERCISE 9D

Problems involving the binomial distribution for multiple probabilities

eBook plus

Interactivity:
The binomial distribution
int-0256

1 Find:

a $^4C_3(0.4)^3\,(0.6) + {}^4C_4(0.4)^4\,(0.6)^0$

b $^5C_3(0.6)^3\,(0.4)^2 + {}^5C_4(0.6)^4\,(0.4) + {}^5C_5(0.6)^5\,(0.4)^0$.

WORKED Example 17

2 The binomial variable, X, has the following probability table.

x	0	1	2	3	4	5
$P(X = x)$	0.116 03	0.312 39	0.336 42	0.181 15	0.048 77	0.005 25

eBook plus

Digital doc:
SkillSHEET 9.3
Multiple probabilities

Find:

a $P(X \geq 4)$ **b** $P(X > 0)$ **c** $P(X \leq 4)$ **d** $P(X < 2)$.

3 A binomial variable, X, has the probability function:

$$P(X = x) = {}^5C_x(0.3)^x\,(0.7)^{5-x}$$

where x is the probability of success and $x = 0, 1, \ldots, 5$.

Find:

a $P(X \geq 2)$ **b** $P(X < 4)$.

WORKED Example 18

4 Find $P(X \geq 4)$ if X has a binomial distribution with the probability of success, p, and the number of trials, n, given by:

a $p = 0.6, n = 5$ **b** $p = 0.5, n = 6$ **c** $p = 0.2, n = 7$.

5 A fair die is rolled six times. Find the probability of obtaining:

a at least three 6s **b** at least four even numbers.

6 A fair coin is tossed eight times. Find the probability of obtaining:

a three Heads **b** at least seven Heads

c six or more Tails **d** no more than two Tails.

WORKED Example 19

7 A bag contains 4 red and 2 blue marbles. A marble is selected at random and replaced. The experiment is repeated 6 times. Find the probability that:

a all 6 selections are red **b** at least 2 are red

c not more than 1 is red.

8 It is known that 40% of Queenslanders play sport regularly. Ten people are selected at random. Find the probability that:

a at least half play sport regularly **b** at least nine don't play sport regularly.

9 Surveys have shown that 8 out of 10 Year-12 students study every night. Six Year-12 students are selected at random. Find the probability that, on any one day:

a at least 50% of these students study every night

b fewer than 3 students study every night.

10 A die is weighted such that

$P(X = 6) = \frac{1}{2}$, $P(X = 2) = P(X = 4) = \frac{1}{6}$ and $P(X = 1) = P(X = 3) = P(X = 5) = \frac{1}{18}$.

The die is rolled five times. Find the probability of obtaining:

a at least three 6s **b** at least two even numbers

c a maximum of two odd numbers.

WORKED Example 20

11 If X is binomially distributed with $n = 8$ and $p = 0.7$, find:

a $P(X \geq 7)$ **b** $P(X > 7 \mid X > 5)$.

12 A survey shows that 49% of the public support the current government. Twelve people are selected at random. Find:

a the probability that at least 8 support the government

b the probability that at least 10 support the government, given that at least 8 do.

13 **multiple choice**

When Graeme kicks for goal, the probability of his kicking a goal is 0.7. If he has five kicks at goal, the probability that he will score fewer than two goals is:

A $^5C_1(0.7)^1\,(0.3)^4 + (0.3)^5$
B $^5C_2(0.7)^2\,(0.3)^3$
C $^5C_2(0.7)^2\,(0.3)^3 + {}^5C_1(0.7)^1\,(0.3)^4$
D $^5C_2(0.7)^2\,(0.3)^3 + {}^5C_1(0.7)^1\,(0.3)^4 + (0.3)^5$
E $1 - {}^5C_2(0.7)^2\,(0.3)^3$

14 **multiple choice**

If X is a random variable, binomially distributed with $n = 10$ and $p = \frac{1}{4}$, $\text{P}(X \geq 1)$ is:

A $\left(\frac{1}{4}\right)^{10}$ **B** $\left(\frac{3}{4}\right)^{10}$ **C** $10\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^9$ **D** $1 - \left(\frac{3}{4}\right)^{10}$ **E** $1 - \left(\frac{1}{4}\right)^{10}$

15 **multiple choice**

A fair die is rolled six times. The probability of obtaining more than four even numbers is:

A 0.1094 **B** 0.2344 **C** 0.3438 **D** 0.6563 **E** 0.8906

16 **multiple choice**

Three per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 10. If 3 or more are defective, the box can be returned and money refunded. The chance of being eligible for a refund is:

A 0 **B** 0.0002 **C** 0.0036 **D** 0.0028 **E** 0.9972

17 **multiple choice**

Long-term statistics show that Silvana wins 60% of her tennis matches. The probability that she will win at least 80% of her next 10 matches is:

A 0.0061 **B** 0.0464 **C** 0.1673
D 0.8327 **E** 0.9536

18 **multiple choice**

Nineteen out of every 20 cricketers prefer ‘Boundary’ cricket gear. A squad of 12 cricketers train together. The probability that at least 11 use Boundary gear, given that at least 10 use it, is:

A 0.5404 **B** 0.6129 **C** 0.8816 **D** 0.8992 **E** 0.9804

19 A school council, comprising 15 members of the school community, requires a minimum two-thirds majority to pass a motion. It is known that 50% of the school community favour a new uniform. Find the probability that the school council will pass a motion in favour of a new uniform.

20 A car insurance salesman knows that he has a good chance of finding customers in the age group from 18 to 20, as people often buy their first car at this age. Five per cent of all people in this age group are looking to purchase a car. The salesman questions 30 people in this age group. Find the probability that he will get:

a no more than 3 sales **b** a least 3 sales.

21 Police radar camera tests have shown that 1% of all cars drive at over 30 km/h above the speed limit, 2% between 10 km/h and 30 km/h above the limit and 4% below 10 km/h over the limit. In one particular hour, a radar camera tests 50 cars. Find the probability that:

a at most, one car is over 30 km/h above the limit
b at most, two cars are between 10 km/h and 30 km/h above the limit
c at most, two cars are below 10 km/h above the limit
d at most, three cars are above the limit.

22 An Australian cricketer scores 50 or more in one-third of all his test match innings. The Australian selectors are aiming to predict his next 10 innings. Find the probability that he will score 50 or more on:

a no occasions
b exactly four occasions
c at least two occasions.

23 Two dice are rolled simultaneously and their difference is recorded. Find the probability that in 5 rolls:

a a difference of zero occurs at least once
b a difference of 1 occurs at least twice
c a difference of 5 occurs at least once.

24 Eighty per cent of all scheduled trains through Roma Street station arrive on time. If 10 trains go through the station every day, find:

a the probability that at least 8 trains are on time
b the probability that at least 8 trains are on time for 9 out of the next 10 days.

25 An experiment involves rolling a die 6 times. Find:

a the probability of obtaining at least four prime numbers
b the probability of obtaining at least four prime numbers on 5 occasions if the experiment is repeated 8 times.

26 Tennis balls are packed in cans of 6. Five per cent of all balls are made too flat (that is, they don’t bounce high enough). The cans are then packed in boxes of two dozen. Find the probability that:

a a can contains, at most, one flat ball
b a box contains at least 22 cans with a maximum of one flat ball.

Binomial probability table

A binomial probability can be calculated using a table. In such a table the number of trials (n) is given and we can then look up the probability of the number of successes. The table of binomial probabilities below shows the probabilities for 10 trials.

	p											
	0.01	**0.05**	**0.1**	**0.15**	**0.2**	**0.25**	**0.3**	**0.35**	**0.4**	**0.45**	**0.5**	
x												
0	0.9044	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010	10
1	0.0914	0.3151	0.3874	0.3474	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098	9
2	0.0042	0.0746	0.1937	0.2759	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439	8
3	0.0001	0.0105	0.0574	0.1298	0.2013	0.2503	0.2668	0.2522	0.2150	0.1665	0.1172	7
4	0.0000	0.0010	0.0112	0.0401	0.0881	**0.1460**	0.2001	0.2377	0.2508	0.2384	0.2051	6
5	0.0000	0.0001	0.0015	0.0085	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461	5
6	0.0000	0.0000	0.0001	0.0012	**0.0055**	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051	4
7	0.0000	0.0000	0.0000	0.0001	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172	3
8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0014	0.0043	0.0106	0.0229	0.0439	2
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098	1
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0
												x
	0.99	**0.95**	**0.9**	**0.85**	**0.8**	**0.75**	**0.7**	**0.65**	**0.6**	**0.55**	**0.5**	
	p											

Suppose an event has a probability (p) of 0.25 of occurring in a single trial. What is the probability of getting four successes (x) in 10 trials? From the table (here, we must select the table for 10 trials), $P(X = 4) = 0.1460$ (see yellow screen).

For events with probabilities greater than 0.5, the value of x is read from the right-hand column. Suppose an event has a probability of 0.8 of occurring in a single trial. The probability of getting four successes in 10 trials is now $P(X = 4) = 0.0055$ (see blue screen).

eBook plus

Digital doc:
Spreadsheet
221 Binomial probabilities

Go to www.jacplus.com.au and navigate to the spreadsheet icon *Binomial probabilities* to see a spreadsheet that details all the binomial probabilities for up to 20 trials.

Use the spreadsheet to check your answers to exercise 9D.

Expected value, variance and standard deviation of the binomial distribution

When we are working with the binomial probability distribution (like other distributions), it is very useful to know the expected value (mean), variance and the standard deviation.

The random variable, X, is such that $X \sim \text{Bi}(8, 0.3)$ and has the following probability distribution.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.057 65	0.197 65	0.296 48	0.254 12	0.136 14	0.046 68	0.010 00	0.001 22	0.000 07

We saw earlier that the expected value, $E(X)$ was defined as $E(X) = \sum x\, P(X = x)$. Hence, the expected value from this table is:

$$\begin{aligned} E(X) &= \sum x\, P(X = x) \\ &= 0 \cdot 0.057\,65 + 1 \cdot 0.197\,65 + 2 \cdot 0.296\,48 + 3 \cdot 0.254\,12 + 4 \cdot 0.136\,14 + \\ &\quad 5 \cdot 0.046\,68 + 6 \cdot 0.010\,00 + 7 \cdot 0.001\,22 + 8 \cdot 0.000\,07 \\ &= 0 + 0.197\,65 + 0.592\,96 + 0.762\,36 + 0.544\,56 + 0.233\,400 + 0.060\,00 + \\ &\quad 0.008\,54 + 0.000\,56 \\ &= 2.4 \end{aligned}$$

This means that in eight trials the average number of successes will be 2.4. If this is repeated many times we will sometimes have more than 2.4 successes, sometimes less. The variance and standard deviation measure the range within which we can reasonably expect the outcome to lie.

The variance is defined by the rule $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$. Hence, the variance for the table is:

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 0^2 \cdot 0.057\,65 + 1^2 \cdot 0.19765 + 2^2 \cdot 0.296\,48 + 3^2 \cdot 0.254\,12 + 4^2 \cdot 0.136\,14 + \\ &\quad 5^2 \cdot 0.046\,68 + 6^2 \cdot 0.010\,00 + 7^2 \cdot 0.001\,22 + 8^2 \cdot 0.000\,07 - (2.4)^2 \\ &= 0 + 0.197\,65 + 1.185\,92 + 2.287\,08 + 2.178\,24 + 1.167\,00 + 0.360\,00 + \\ &\quad 0.059\,78 + 0.004\,48 - (2.4)^2 \\ &= 7.440\,15 - (2.4)^2 \\ &= 1.68 \end{aligned}$$

The standard deviation is defined by the rule $SD(X) = \sqrt{\mathrm{Var}(X)}$. Hence the standard deviation for the table is:

$$SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{1.68} = 1.30$$

Since the above method for obtaining the expected value, the variance and the standard deviation is tedious and time consuming, a quicker method has been developed to calculate these terms. It can be shown that

if $X \sim \mathrm{Bi}(n, p)$ then:

$$\begin{aligned} E(X) &= np \\ \mathrm{Var}(X) &= npq \\ SD(X) &= \sqrt{npq} \end{aligned}$$

To check that these agree with the previous example, we can substitute the values into the given rules. When $X \sim \mathrm{Bi}(8, 0.3)$, we obtain the following.

The expected value:

$$\begin{aligned} E(X) &= np \\ &= 8 \cdot 0.3 \\ &= 2.4 \end{aligned}$$

The variance:

$$\begin{aligned} \mathrm{Var}(X) &= npq \\ &= 8 \cdot 0.3 \cdot 0.7 \\ &= 1.68 \end{aligned}$$

The standard deviation:

$$\begin{aligned} SD(X) &= \sqrt{npq} \\ &= \sqrt{1.68} \\ &= 1.30 \end{aligned}$$

As can be seen, these values correspond to those obtained earlier. A great deal of time is saved using these rules and the margin for making mistakes is greatly reduced.

Hence, if X is a random variable and $X \sim \text{Bi}(n, p)$ then:

$$\text{E}(X) = np \qquad \text{Var}(X) = npq \qquad \text{SD}(X) = \sqrt{npq}$$

Note: The distribution must be binomial for these rules to apply.

WORKED Example 21

The random variable X follows a binomial distribution such that $X \sim \text{Bi}(40, 0.25)$. Determine the:

a **expected value** **b** **variance and standard deviation.**

THINK	WRITE
a 1 Write the rule for the expected value.	**a** $\text{E}(X) = np$
2 List the values for n and p.	$n = 40, p = 0.25$
3 Substitute the values into the rule.	$\text{E}(X) = 40 \cdot 0.25$
4 Evaluate.	$= 10$
b 1 Write the rule for the variance.	**b** $\text{Var}(X) = npq$
2 Write down the values for n, p and q.	$n = 10, p = 0.25, q = 0.75$
3 Substitute the values into the rule.	$\text{Var}(X) = 40 \cdot 0.25 \cdot 0.75$
4 Evaluate.	$= 7.5$
5 Write the rule for the standard deviation.	$\text{SD}(X) = \sqrt{npq}$
6 Substitute the value obtained for the variance and take the square root.	$= \sqrt{7.5}$
7 Evaluate.	$= 2.74$

WORKED Example 22

A fair die is rolled 90 times. Find:

a **the expected number of even numbers**

b **the variance of even numbers**

c **the standard deviation of even numbers.**

THINK	WRITE
a 1 Write the rule for the expected value.	**a** $\text{E}(X) = np$
2 Write down the values for n and p.	$n = 90, p = \frac{1}{2}$
3 Substitute the values into the rule.	$\text{E}(X) = 90 \cdot \frac{1}{2}$
4 Evaluate.	$= 45$
b 1 Write the rule for the variance.	**b** $\text{Var}(X) = npq$
2 Write down the values for n, p and q.	$n = 90, p = \frac{1}{2}, q = \frac{1}{2}$
3 Substitute the values into the rule.	$\text{Var}(X) = 90 \cdot \frac{1}{2} \cdot \frac{1}{2}$
4 Evaluate.	$= 22.5$

THINK	WRITE
c ① Write the rule for the standard deviation.	**c** $\text{SD}(X) = \sqrt{npq}$
② Substitute the value obtained for the variance.	$= \sqrt{22.5}$
③ Evaluate and round off the answer to two decimal places.	$= 4.74$

WORKED Example 23

A binomial random variable has an expected value of 14.4 and a variance of 8.64. Find:

a **the probability of success, *p*** **b** **the number of trials, *n*.**

THINK	WRITE
a ① Write down what is known and what is required.	**a** $\text{E}(X) = 14.4$ so $np = 14.4$ $\text{Var}(X) = 8.64$ so $npq = 8.64$ $p = ?$
② Substitute the value of *np* into the variance equation.	$npq = 8.64$ $14.4q = 8.64$
③ Transpose the equation to make *q* the subject.	$q = \dfrac{8.64}{14.4}$
④ Evaluate.	$q = 0.6$
⑤ Solve for *p* using the relationship $q = 1 - p$.	$q = 1 - p$ $0.6 = 1 - p$ $p = 0.4$
b ① Write down what is known and what is required.	**b** $np = 14.4$ where $p = 0.4$ $n = ?$
② Substitute the values into the equation.	$n \cdot 0.4 = 14.4$
③ Transpose the equation to make *n* the subject.	$n = \dfrac{14.4}{0.4}$ $= 36$

remember

If X is a random variable and $X \sim \text{Bi}(n, p)$, then:

$$\text{E}(X) = np$$
$$\text{Var}(X) = npq \text{ where } q = 1 - p$$
$$\text{SD}(X) = \sqrt{npq}$$

EXERCISE 9E

Expected value, variance and standard deviation of the binomial distribution

In questions **1**, **2** and **3**, assume we have a binomial distribution with number of trials, n and probability of success, p, as given.

eBook plus

Digital doc:
Spreadsheet
221 Binomial probabilities

WORKED Example 21a

1 Determine the mean if:

a $n = 10$ and $p = 0.6$
b $n = 8$ and $p = 0.2$
c $n = 100$ and $p = 0.5$
d $n = 50$ and $p = \frac{3}{4}$

WORKED Example 21b

2 Determine the variance if:

a $n = 20$ and $p = 0.6$
b $n = 15$ and $p = 0.9$
c $n = 25$ and $p = 0.4$
d $n = 20$ and $p = \frac{1}{4}$

3 Determine the standard deviation if:

a $n = 10$ and $p = 0.2$
b $n = 30$ and $p = 0.5$
c $n = 50$ and $p = 0.7$
d $n = 72$ and $p = \frac{2}{5}$

WORKED Example 22

4 A fair coin is tossed 10 times. Find:

a the expected number of Heads
b the variance for the number of Heads
c the standard deviation for the number of Heads.

5 A card is selected at random from a standard playing pack of 52 and then replaced. This procedure is completed 20 times. Find:

a the expected number of picture cards
b the variance for the number of picture cards
c the standard deviation for the number of picture cards.

6 Six out of every 10 cars manufactured are white. Twenty cars are randomly selected. Find:

a the expected number of white cars
b the variance for the number of white cars
c the standard deviation for the number of white cars.

7 A fair die is rolled 10 times. Find:

a the expected number of 2s rolled
b the probability of obtaining more than the expected number of 2s.

8 Eighty per cent of rabbits that contract a certain disease will die. If a group of 120 rabbits contract the disease, how many would you expect to:

a die? **b** live?

WORKED Example 23

9 A binomial random variable has a mean of 10 and a variance of 5. Find:

a the probability of success, p
b the number of trials, n.

10 A binomial random variable has a mean of 12 and a variance of 3. Find:

a the probability of success, p
b the number of trials, n.

11 X is a binomial random variable with a mean of 9 and a variance of 6. Find:

a the probability of success, p **b** the number of trials, n

c $P(X = 10)$.

12 If X is a binomial random variable with $E(X) = 3$ and $Var(X) = 2.4$, find:

a the probability of success, p **b** the number of trials, n

c $P(X = 10)$ **d** $P(X \leq 2)$.

13 **multiple choice**

The expected number of Heads in 20 tosses of a fair coin is:

A $\frac{1}{2}$ **B** 5 **C** 10 **D** 15 **E** 20

14 **multiple choice**

Jenny is a snooker player who knows from experience that 7 out of every 10 shots she plays will result in a ball being potted. If she has 40 shots, the number of balls she expects to pot is:

A 7 **B** 14 **C** 21 **D** 25 **E** 28

15 **multiple choice**

The variance of the number of balls that Jenny pots from her 40 shots in question **14** is:

A 2.898 **B** 7.3 **C** 8.4 **D** 22.2 **E** 28

16 **multiple choice**

Eighty per cent of children are immunised against a certain disease. A sample of 200 children is taken. The mean and variance of the number of immunised children is:

A 80 and 5.66 respectively **B** 80 and 32 respectively

C 100 and 50 respectively **D** 160 and 5.66 respectively

E 160 and 32 respectively

17 **multiple choice**

A binomial random variable has a mean of 10 and a variance of 6. The values of n and p respectively are:

A 5 and $\frac{2}{5}$ **B** 5 and $\frac{3}{5}$ **C** 20 and $\frac{3}{5}$ **D** 25 and $\frac{2}{5}$ **E** 25 and $\frac{3}{5}$

18 **multiple choice**

A binomial random variable process has a probability of success, p. If 5 trials are conducted, the probability of three successes is:

A p^3q^2 **B** p^2q^3 **C** $10p^2q^3$ **D** $10p^3(1 - p)^2$ **E** p^3

19 A binomial experiment is completed 20 times, with the expected number of successes being 16. Find:

a the probability of success, p

b the variance

c the standard deviation.

20 A multiple-choice test has 20 questions with five different choices of answer for each question. If the answers to each question are guessed, find:

a the probability of getting 50% of the questions correct

b the probability of getting at least three correct.

21 Four per cent of pens made at a certain factory do not work. If pens are sold in boxes of 25, find the probability that a box contains more than the expected number of faulty pens.

22 A biased coin is tossed 10 times. Let X be the random variable representing the number of tails obtained. If X has an expectation of 3, find:

a the probability of obtaining exactly two tails

b the probability of obtaining no more than two tails.

23 Eighty per cent of Brisbane households have video recorders. A random sample of 500 households is taken. How many would you expect to have videos?

24 The executive committee of a certain company contains 15 members. Find the probability that more females than males will hold positions if:

a males and females are equally likely to fill any position

b females have a 55% chance of holding any position

c females have a 45% chance of holding any position.

25 A statistician estimates the probability that a spectator at a Brisbane Lions versus Collingwood AFL match barracks for Brisbane is $\frac{1}{2}$. At an AFL grand final between these two teams there are 100 000 spectators. Find:

a the expected number of Brisbane supporters

b the variance of the number of Brisbane supporters

c the standard deviation of the number of Brisbane supporters.

26 Thirty children are given five different yoghurts to try. The yoghurts are marked A to E, and each child has to select his or her preferred yoghurt. Each child is equally likely to select any brand. The company running the tests manufactures yoghurt B.

a How many children would the company expect to pick yoghurt B?

b The tests showed that half of the children selected yoghurt B as their favourite. What does this tell the company manufacturing this product?

27 The proportion of defective fuses made by a certain company is 0.02. A sample of 30 fuses is taken for quality control inspection.

a Find the probability that there are no defective fuses in the sample.

b Find the probability that there is only one defective fuse in the sample.

c How many defective fuses would you expect in the sample?

d The hardware chain that sells the fuses will accept the latest batch for sale only if, upon inspection, there is at most one defective fuse in the sample of 30. What is the probability that they accept the batch?

e Ten quality control inspections are conducted monthly for the hardware chain. Find the probability that all of these inspections will result in acceptable batches.

eBook*plus*

Digital doc:
WorkSHEET 9.2

Gary’s test — some answers

At the beginning of the chapter we looked at Gary who was sitting for an aptitude test for the armed forces. Gary had to correctly answer at least 16 out of 20 multiple-choice questions to be accepted. Gary was certain that he had answered 13 questions correctly but was unsure of the other 7, so he had to guess each answer.

1 Let X = number of correct answers guessed. Copy and complete the table below.

x	$P(X = x)$
$P(X = 0)$	
$P(X = 1)$	
$P(X = 2)$	
$P(X = 3)$	
$P(X = 4)$	
$P(X = 5)$	
$P(X = 6)$	
$P(X = 7)$	

2 Find the probability that Gary is accepted into the armed forces (that is, $P(X \geq 4)$).

3 Suppose that in each case, Gary was able to eliminate one possible answer from each question to which he did not know the answer. Find the probability that Gary is *now* accepted into the armed forces.

4 Suppose that Gary had been able to eliminate *two* incorrect answers from each of the questions about which he was unsure. What would his chance of being accepted be equal to then?

The binomial theorem

Winning at racquetball!

Denis and Glenn play racquetball each week. In any one game, Denis beats Glenn in the ratio of 5:2. The outcome of any one game between the two is independent of the outcome of previous games played.

1 a If Denis and Glenn play 8 games, what type of distribution has been described?

b What is the probability, to 4 decimal places, that Denis wins 5 of the 8 games?

c What is the probability, to 4 decimal places, that Denis wins at most 6 of the 8 games?

d What is the probability, to 4 decimal places, that Glenn wins at least 6 of the 8 games?

2 a Assuming the above conditions remain the same, what is the probability, to 4 decimal places, that Denis beats Glenn in the second, fourth, fifth, seventh and eighth games?

b Compare the value obtained in part 1b with that of 2a. Explain the result.

summary

Discrete random variables

- A random variable is one whose value is determined by the outcome of an experiment.
- Discrete random variables generally deal with number or size and are able to be counted.
- Two important characteristics satisfy all discrete probability distributions:
  1. Each probability lies in a restricted interval $0 \leq P(X = x) \leq 1$.
  2. The probabilities of a particular experiment sum to 1; that is,

$$\sum P(X = x) = 1$$

  If these two characteristics are not satisfied, then there is no discrete probability distribution.

Expected value of discrete random distributions

- The expected value of a discrete random variable, X, is denoted by E(X). It is defined by the rule:

$$E(X) = \sum x\, P(X = x)$$

- The expected value of a linear function can be calculated using the expectation theorems:

$$E(aX) = aE(X)$$
$$E(aX + b) = aE(X) + b$$
$$E(b) = b$$
$$E(X + Y) = E(X) + E(Y)$$

The binomial distribution

- The binomial distribution is an example of a specific type of discrete probability distribution. It has the distinct characteristics:
  1. A fixed number of n identical trials are conducted.
  2. Each trial is independent.
  3. There are only two possible outcomes for each trial: a success, p, and a failure, q.
- If X represents a random variable which has a binomial distribution then it can be expressed as $X \sim \text{Bi}(n, p)$ or $X \sim \text{B}(n, p)$. This means that X follows a binomial distribution with parameters n (the number of trials) and p (the probability of success).
- If X is a binomial random variable its probability is defined as:

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x} \text{ where } x = 0, 1, 2, \ldots, n$$

The effects of *n* and *p* on binomial distribution graphs

- The parameters n and p affect the binomial probability distribution curve as follows.
  1. If $p < 0.5$, the graph is positively skewed, or skewed to the right.
  2. If $p = 0.5$, the graph is symmetrical or is a normal distribution curve.
  3. If $p > 0.5$, the graph is negatively skewed, or skewed to the left.
  4. When n is large and $p = 0.5$, the interval between the vertical columns decreases and the graph approximates a smooth hump or bell shape.

Problems involving the binomial distribution for multiple probabilities

- When solving problems dealing with the binomial distribution for multiple probabilities always:
  1. Write down what is required.
  2. Write down the rule for the binomial probability distribution.
  3. Substitute the values into the given rule and evaluate.

Expected value, variance and standard deviation of the binomial distribution

- If X is a random variable and $X \sim \text{Bi}(n, p)$ then:

$$\text{E}(X) = np$$

$$\text{Var}(X) = npq$$

$$\text{SD}(X) = \sqrt{npq}$$

  This applies only for a binomial distribution.

CHAPTER review

9A/B

1 The probability distribution of X is given by the formula, $P(X = x) = \frac{x^2}{30}$, where $x = 1, 2, 3, 4$.

a Write the probability distribution of X as a table.
b Find the expected value of X.

9A/B

2 A player rolls a fair die. If the player gets a 1 on the first roll, she rolls again and her score is the sum of the two results; otherwise, her score is the result of the first roll. The die cannot be thrown more than twice. Find:

a the probability distribution
b the expected score
c $P(X < \propto)$.

9B

3 A game is played where two dice are rolled and the sum of the two numbers showing uppermost is recorded. If players get a sum of 7, they win \$10. If they get a sum of 2 or 12, they win \$5. For any other sum, they must pay \$2.50. Is it a fair game?

9B

4 A card game involves the selection of one card from a standard pack. If an ace is selected the player wins \$5, while a picture card results in a \$2 win. In order for the game to be fair, how much should a player pay if any other card is selected?

9B

5 A game of 'three-up' is played where three coins are tossed simultaneously. A player must pay \$2 to play the game. If three Heads come up, the player collects \$6. If two Heads come up, the player collects \$3. Is it a fair game?

9B

6 A door-to-door vacuum cleaner salesman has recorded his day-by-day sales figures over a long period of time. He knows that his probabilities of selling X vacuum cleaners on any one day follow the probability distribution shown in the table.

x	0	1	2	3	4	5	>5
$P(X = x)$	0.125	0.25	0.315	0.2	0.1	0.01	0

a Find the probability that he sells more than three vacuum cleaners on any one day.
b Find the number of vacuum cleaners he can expect to sell each day.
c If the salesman receives a commission of \$20 per vacuum cleaner sold, and a bonus of \$100 if he sells five or more vacuum cleaners on one day, find his expected daily earnings from commissions and bonuses.
d Given that the salesman will sell at least two vacuum cleaners tomorrow, find the probability that he will get his \$100 bonus.

9B

7 At Fast Eddy's Drive-In Theatre the cost is \$10 per car, plus \$3 per occupant. The variable X represents the number of people in any car and is known to follow the probability distribution above right. Find:

x	2	3	4	5
$P(X = x)$	0.4	0.2	0.3	0.1

a the expected cost per car
b Fast Eddy's expected profit if 100 cars enter, and costs for wages, electricity, etc. are \$500.

8 $P(X = x) = \dfrac{x(x+1)}{40}$ for $0 \le x \le n$ 9B

a Find the value of n.

b Write the probability distribution for X as a table.

c Find the expected value of X.

9 A fair coin is tossed 7 times. Find: 9C

a the probability of obtaining 3 heads

b the probability of obtaining 7 heads.

10 A cricketer knows that for every 20 balls faced, she hits a 4. If she faces two overs (12 balls), find the probability that she hits: 9C

a two 4s

b half the balls for 4

c every ball for 4.

11 A fair die is rolled five times. Find the probability of obtaining: 9D

a more than two 5s

b at least two even numbers

c a maximum of four results greater than 3.

12 Forty per cent of all Australians have poor diets. A random survey of 20 Australians is taken. Find the probability that: 9D

a fewer than 5 have poor diets

b at least 15 have poor diets.

13 *Superway* supermarkets claim that 8 out of every 10 customers will be served within 5 minutes of lining up at the checkout. Ten random *Superway* customers have their serving times recorded. Find the probability that: 9D

a at least five are served within 5 minutes

b at least eight are served within 5 minutes.

14 Find: 9E

i the mean

ii the variance

iii the standard deviation

for the binomial random variables with n and p given by:

a $n = 100$ and $p = 0.5$

b $n = 50$ and $p = 0.7$

c $n = 80$ and $p = 0.2$.

15 A binomial random variable has a mean of 10 and variance of 8. Find: 9E

a the probability of success, p

b the number of trials, n.

Modelling and problem solving

1 Maria works at a car wash from 4.00 pm to 5.00 pm each afternoon after school. For this, she is paid \$15 an hour. She knows that, on average, drivers give her a tip of 50c per car. The number of cars through the car wash at this time each afternoon follows the probability distribution below.

x	0	1	2	3	4	5
$P(X = x)$	0.1	0.15	0.2	0.3	0.2	0.05

Find:

a the amount of money Maria can expect to earn in tips each afternoon

b the total amount of money Maria can expect to earn each afternoon

c the total amount of money Maria can expect to earn each week.

2 One in every 100 new cars is returned with faulty steering. A survey is taken of 300 buyers. Find the probability that:

a none have cars with faulty steering

b one has a car with faulty steering.

3 Three out of five people read the *Bugle* newspaper, while one out of five reads the *Headline*. Forty people are surveyed at random. Find the probability that 25% of these people read:

a the *Bugle*

b the *Headline*.

4 One-fifth of Australia's population are of British background. Fifty Australians are randomly selected and questioned about their ancestry. Find the probability that:

a at least four come from a British background

b more than 30% of those surveyed come from a British background.

5 A bag contains two dozen lollies of which 8 are red, 6 are green and the remainder are yellow. A lolly is selected, has its colour recorded and is then replaced. If 10 lollies are chosen in this manner, find:

a the expected number of **i** red lollies **ii** green lollies **iii** yellow lollies

b the variance of the number of red lollies

c the standard deviation of the number of red lollies.

6 Amina plays roulette, a game where a wheel containing 37 slots numbered 0–36 is spun and the winning number is the one in which a ball lodges when the wheel stops spinning.

Amina plays three different games:

- **a** First she bets \$20 on her favourite number coming up at casino-nominated odds of 35 : 1 against.
  - **i** How much would Amina collect if her number came up?
  - **ii** Find her expected win or loss for the game.
  - **iii** Is this game fair?
- **b** In the second game Amina bets \$20 on an even number coming up at casino-nominated odds of 1 : 1 (even money chance).
  - **i** How much would Amina collect if an even number came up?
  - **ii** Find her expected win or loss for the game.
  - **iii** Is this game fair?
- **c** In game number three, Amina bets \$20 on a line of 12; that is, if numbers 1–12 come up, she wins. The casino-nominated odds for this game are 2 : 1 against.
  - **i** How much would Amina collect if one of her numbers came up?
  - **ii** Find her expected win or loss for the game.
  - **iii** Is this game fair?
- **d** What is the house percentage for these games?

7 Speedy Saverio's Pizza House claims to cook and deliver 90% of pizzas within 15 minutes of the order being placed. If your pizza is not delivered within this time, it is free. On one busy Saturday night, Saverio has to make 150 deliveries.

- **a** How many deliveries are expected to be made within 15 minutes of placing the order?
- **b** What is the probability of receiving a free pizza on this night?
- **c** If Saverio loses an average of \$4 for every late delivery, how much would he expect to lose on late deliveries this night?

8 Ten per cent of all Olympic athletes are tested for drugs at the conclusion of their event. One per cent of all athletes use performance-enhancing drugs. Of the 1000 Olympic wrestlers competing from all over the world, Australia sends 10. Find:

- **a** the expected number of Australian wrestlers who are tested for drugs
- **b** the probability that half the Australian wrestlers are tested for drugs
- **c** the probability that at least two Australian wrestlers are tested for drugs
- **d** the expected number of drug users among all wrestlers.

9 Five per cent of watches made at a certain factory are defective. Watches are sold to retailers in boxes of 20. Find:

- **a** the expected number of defective watches in each box
- **b** the probability that a box contains more than the expected number of defective watches per box
- **c** the probability of a 'bad batch', if a 'bad batch' entails more than a quarter of the box being defective.

eBook*plus*

Digital doc:
Test Yourself
Chapter 9

eBook*plus* ACTIVITIES

9A Discrete random variables

Digital docs

- Spreadsheet 048: Investigate probability distributions (*page 323*)
- WorkSHEET 9.1: Identify discrete random variables, show that a set of probabilities is a probability distribution and calculate expected values (*page 327*)

9B Expected value of discrete random distributions

Digital doc

- SkillSHEET 9.1: Practise the expected value of a function of a random variable (*page 336*)

9C The binomial distribution

Digital docs

- Spreadsheet 221: Investigate binomial probabilities (*page 345*)
- SkillSHEET 9.2: Practise solving indicial equations (*page 349*)

Tutorial

- **WE14** Int-0576: Watch a tutorial on identifying the number of trials in a probability experiment (*page 339*)

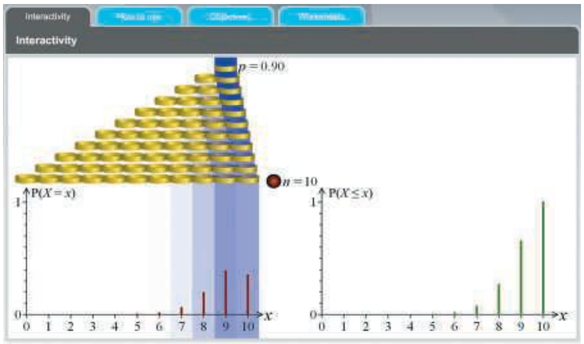
9D Problems involving the binomial distribution for multiple probabilities

Digital docs

- SkillSHEET 9.3: Practise multiple probabilities (*page 355*)
- Spreadsheet 221: Investigate binomial probabilities (*page 358*)

Interactivity

- The binomial distribution int-0256: Consolidate your understanding of the binomial distribution (*page 355*)



9E Expected value, variance and standard deviation of the binomial distribution

Digital docs

- Spreadsheet 221: Investigate binomial probabilities (*page 362*)
- WorkSHEET 9.2: Calculate probabilities for binomial distributions (*page 364*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 371*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

10

The normal distribution

syllabus reference

Applied statistical analysis

In this chapter

Introduction

The Randhill vaccine — does it work?

The team at Randhill Medical Research Centre believe that they have discovered a new vaccine against a strain of the flu virus. It is known that the probability of catching the flu when exposed to the virus is 0.4. A sample of 20 patients trial the new vaccine and it is found that when these patients are exposed to the virus only two of them develop flu symptoms.

Is this result significant enough to determine whether the new flu vaccine is effective or can this good result be written off as being by chance? In this chapter we shall study probability further, so that we are able to assess whether this result is significant or whether it is probable that the result is obtained by chance.

The normal distribution

In the previous chapter we worked with discrete random variables and the probability distributions associated with them. We shall now focus on continuous random variables and the most important distribution associated with them — the normal distribution.

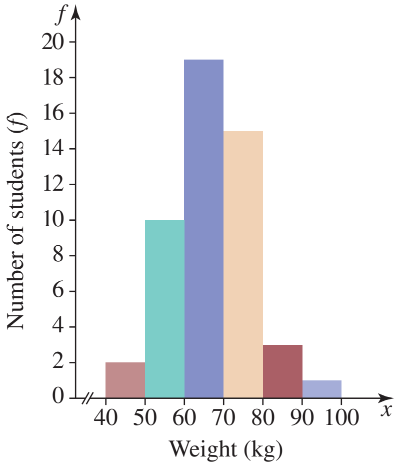
Continuous random variables

A discrete variable can only take certain values, usually whole numbers within a given range.

Continuous random variables represent quantities which can be measured and thus may assume any value in a given range. They include such variables as time, height and weight.

The weights of 50 Year-12 students are displayed in the table below, alongside a histogram.

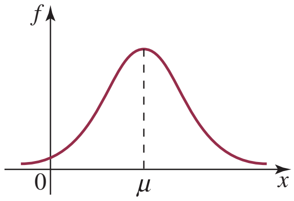
Weight (kg) x	Frequency f
40–< 50	2
50–< 60	10
60–< 70	19
70–< 80	15
80–< 90	3
90–< 100	1
	Total = 50



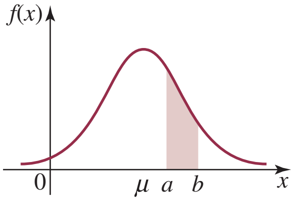
The frequency of individual weights cannot be determined due to the fact that the weights have been grouped into class intervals. This limits the information we are able to extract from the histogram. For example, to determine the number of students weighing less than 60 kg we simply add the frequencies of the 40–< 50 and 50–< 60 class intervals; that is, 10 + 2 = 12 students. However, we would not be able to determine the weight of students below 75 kg as this value lies within a class interval rather than being an end point.

Since the value that a continuous random variable can assume is *measured* in some way, the exact value cannot be obtained. Hence a weight of 60 kg, if we measure in whole numbers, is actually between 59.5 and 60.5 kg. Therefore, the probability of a continuous random variable assuming an exact value is zero. In order to determine the probability of continuous random variables, a new method must be employed.

If we were to increase the size of the sample of students' weights and make the class intervals very small, then the histogram would become a smooth frequency curve as shown at right.



A special scaled version of a smooth frequency curve is the *probability density function* or *pdf*. The scale is such that the probability of the random variable lying between certain values is given by the area between the pdf and the horizontal axis. Hence, sections such as the shaded region between $x = a$ and $x = b$, as shown on the lower graph, may be regarded as probabilities.



The curve is positioned above the x-axis since it represents a probability distribution in which individual probabilities assume a value between 0 and 1.

Hence, the probability of X, a continuous variable, falling between $x = a$ and $x = b$ is represented by the shaded area between a probability density function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The area is determined by integrating $f(x)$ from $x = a$ to $x = b$. Using mathematical notation, this may be summarised as

$$\mathbf{P(a < x < b) = \int_a^b f(x)\,dx.}$$

Properties of a probability density function

1. $f(x) \geq 0$ for all x, (that is, we never have negative probabilities) and
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (that is, the sum of all probabilities is equal to one).

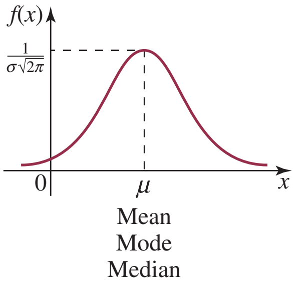
The normal distribution

eBook*plus*
Interactivity:
The normal distribution
int-0257

The normal distribution is an important tool for dealing with the probability distribution of a continuous random variable. The frequency curve of the normal distribution is characterised by the symmetrical bell shape called the *normal distribution curve* or *normal curve*. The normal curve realistically models many observed frequency distributions such as heights and weights of infants, examination results, the intelligence quotient of children in a particular age group, the lengths of battery lives, the diameters of steel cans, etc.

If X is a continuous random variable which follows a normal distribution with a mean of $\propto$ and a variance of σ^2, it is written as $X \sim N(\propto, \sigma^2)$.

Properties of the normal distribution



1. The normal probability distribution is characterised by a bell-shaped curve which is symmetrical about the mean.
2. The equation of the normal curve is given by the probability distribution function (pdf)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\propto}{\sigma}\right)^2}$$

   where $x \in R$.

3. From the graph, a maximum value of $\frac{1}{\sigma\sqrt{2\pi}}$ is obtained when $x = \propto$.
4. For a normal distribution:
   - (a) the mean, mode and median are the same
   - (b) many of the frequencies are situated near the mean
   - (c) the graph extends indefinitely to the right and left of the mean, but never touches or goes below the x-axis
   - (d) the area between the normal curve and the x-axis is equal to 1 unit square, that is,

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\propto}{\sigma}\right)^2} = 1$$

   - (e) $\mathrm{P}(a < x < b) = \int_a^b f(x)\mathrm{d}x$.

The standard normal distribution

As seen earlier, the position and shape of the normal curve depend on the parameters $\propto$ and σ respectively. Thus, the area below the normal curve and hence the probability for a given interval would vary for each different value of $\propto$ and σ. Integrating the equation for the normal distribution each time a value is required can become tedious. This problem can be quickly rectified by introducing a *standard* normal distribution where $\propto = 0$ and $\sigma = 1$. The standard normal variable is denoted by the letter z in order to distinguish it from the normal variable, x.

The equation of the normal distribution curve $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\propto}{\sigma}\right)^2}$ when converted to the standard normal curve becomes

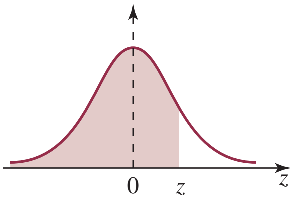
$$g(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

where $z = \frac{x-\propto}{\sigma}$ and $g(z) = \sigma f(x)$.

The standard normal distribution is written as $Z \sim \mathrm{N}(0, 1^2)$.

To convert a normal distribution into a standard normal distribution, the mean, $\propto$, is subtracted from the observed value, x, and the result is divided by the standard deviation, $\sigma \left(z = \frac{x-\propto}{\sigma}\right)$. Once the z-value is obtained, the corresponding probability may be found using the cumulative normal distribution (CND) table on page 392. This table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z; that is, $\mathrm{P}(Z < z)$.

If this were to be represented graphically it would correspond to the shaded region as illustrated below.

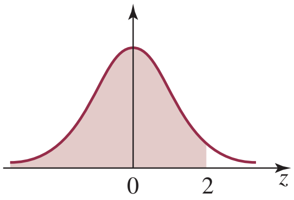
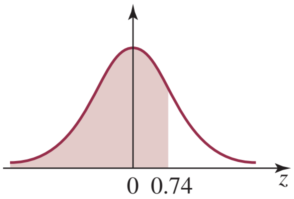
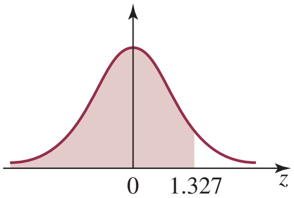


Note: A graphics calculator may also be used to determine these values.

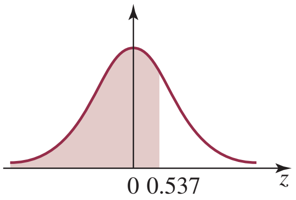
WORKED Example 1

Using the cumulative normal distribution (CND) table on page 392, find the value of the following.

a **P($Z < 2$)** **b** **P($Z < 0.74$)** **c** **P($Z \leq 1.327$)** **d** **P($Z \leq 0.5369$).**

THINK	**WRITE**
a ① Draw a diagram and shade the region required.	**a** 
② Using the CND table, go down to the row containing 2.0 and move across to the column headed 0.	
③ Write down the required value.	$P(Z < 2) = 0.9772$
b ① Draw a diagram and shade the region required.	**b** 
② Using the CND table, go down to the row containing 0.7 and move across to the column headed 4.	
③ Write down the required value.	$P(Z < 0.74) = 0.7703$
c ① Draw a diagram and shade the region required.	**c** 
② Using the CND table, go down to the row containing 1.3 and move across to the column headed 2. Write down the corresponding value.	
③ Move across again to the mean difference column headed 7. Add the value from this final column into the last two digits of the previous answer. *Note:* The value of 11 really represents 0.0011.	$P(Z \leq 1.327) = 0.9066 + 0.0011$ $= 0.9077$
④ Write down the required value.	$P(Z \leq 1.327) = 0.9077$

Continued over page →

THINK	**WRITE**
d ① Round off the z-value to 3 decimal places since this is the limit of the CND table.	**d** $P(Z \le 0.5369) = P(Z \le 0.537)$
② Draw a diagram and shade the region required.	
③ Using the CND table, go down to the row containing 0.5 and move across to the column headed 3. Write down the corresponding value.	
④ Move across again to the mean difference column headed 7. Add the value from this final column onto the last two digits of the previous answer. *Note:* The value of 24 really represents 0.0024.	$P(Z \le 0.537) = 0.7019 + 0.0024$ $= 0.7043$
⑤ Write down the required value.	$P(Z \le 0.537) = 0.7043$

Graphics Calculator tip!

Calculating probabilities for standard normal distributions

Alternatively, for part **d** of Worked example 1, the normal cumulative distribution function of a graphics calculator may be used as follows.

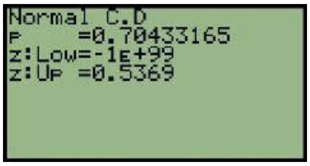
For the Casio *fx*-9860G AU

1. To calculate $P(Z \le 0.5369)$, press:
   - MENU
   - 2 (STAT)
   - F5 (DIST)
   - F1 (NORM)
   - F2 (Ncd).

   Enter the values in the fields as shown.

```
Normal C.D
Lower    :-1E+99
Upper    :0.5369
σ        :1
μ        :0
Save Res:None
Execute
CALC
```

2. Ensure cursor is on Execute and then press F1 (CALC).
   $P(Z \le 0.5369) = 0.7043$



```
Normal C.D
p      =0.70433165
z:Low=-1E+99
z:Up  =0.5369
```

For the TI-Nspire CAS

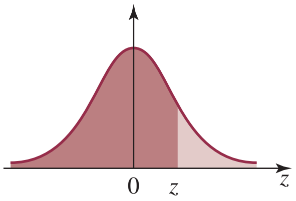
To calculate $P(Z \le 0.5369)$, on a Calculator page, complete the entry line as:
normCdf(–∞,0.5369,0,1)
then press ENTER enter.

```
normCdf(-∞,0.5369,0,1)          0.704332
```

Note: ∞ can be found in the symbol palette.
$P(Z \le 0.5369) = 0.7043$

It is important to note that $P(Z < z) = P(Z \le z)$ since $P(Z = z) = 0$ for all values of z. This is because we cannot measure, exactly, a continuous random variable.

As mentioned previously, the CND table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z; that is, $P(Z < z)$. However, it does not directly allow us to observe a result from a particular positive value of the variable z to $+\infty$; that is, $P(Z > z)$. This problem can easily be solved by drawing a diagram of the situation as shown below and using the fact that the area between the graph and the horizontal axis is one.



From the graph it can be seen that $P(Z > z) + P(Z < z) = 1$.
Transposing the equation we obtain $P(Z > z) = 1 - P(Z < z)$.

WORKED Example 2

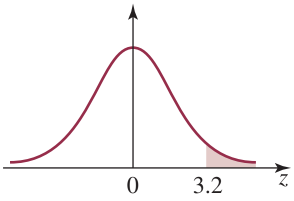
Using the cumulative normal distribution (CND) table on page 392, find the value of the following.

a $P(Z > 3.2)$ **b** $P(Z \geq 2.3741)$

THINK **WRITE**

a ① Draw a diagram and shade the region required.

a

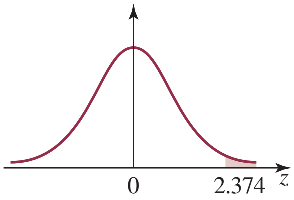


② Write down the rule for obtaining $P(Z > z)$.
③ Use the CND table to obtain the value required.
④ Evaluate.

$$\begin{aligned} P(Z > 3.2) &= 1 - P(Z < 3.2) \\ &= 1 - 0.9993 \\ &= 0.0007 \end{aligned}$$

b ① Draw a diagram and shade the region required.

b

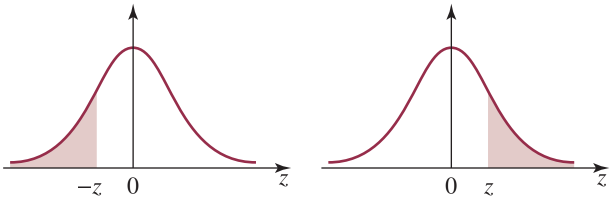


② Round off the z-value to 3 decimal places since this is the limit of the CND table. Write down the rule for obtaining $P(Z \geq z)$.
③ Use the CND table to obtain the value required.
④ Evaluate.

$$\begin{aligned} P(Z \geq 2.3741) &= P(Z \geq 2.374) \\ &= 1 - P(Z < 2.374) \\ &= 1 - (0.9911 + 0.0001) \\ &= 1 - 0.9912 \\ &= 0.0088 \end{aligned}$$

Again, $P(Z > z) = P(Z \geq z)$ since $P(Z = z) = 0$ for all values of z.

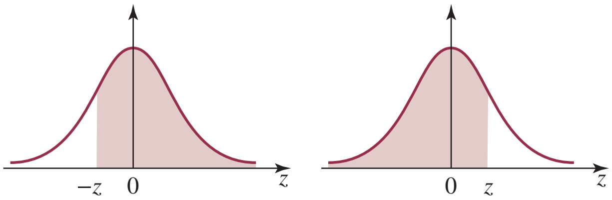
So far we have observed probabilities relating to the positive values of z; that is, values of z greater than the mean. Now we will investigate probabilities associated with negative values of z; that is, values of z less than the mean.



From the figure above, and using the fact that the curve is symmetrical, it can be seen that:

$$P(Z < -z) = P(Z > z)$$
$$= 1 - P(Z < z) \quad \text{(from previous calculations)}$$

while the figure below clearly shows that $P(Z > -z) = P(Z < z)$.

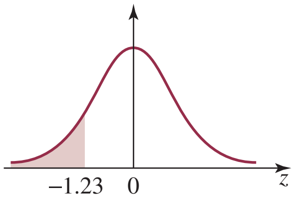
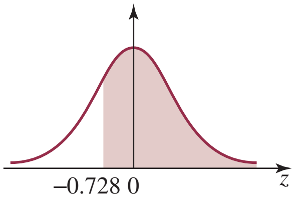


WORKED Example 3

Using the cumulative normal distribution (CND) table on page 392, find the value of the following.

a **$P(Z < -1.23)$**

b **$P(Z \geq -0.728)$**

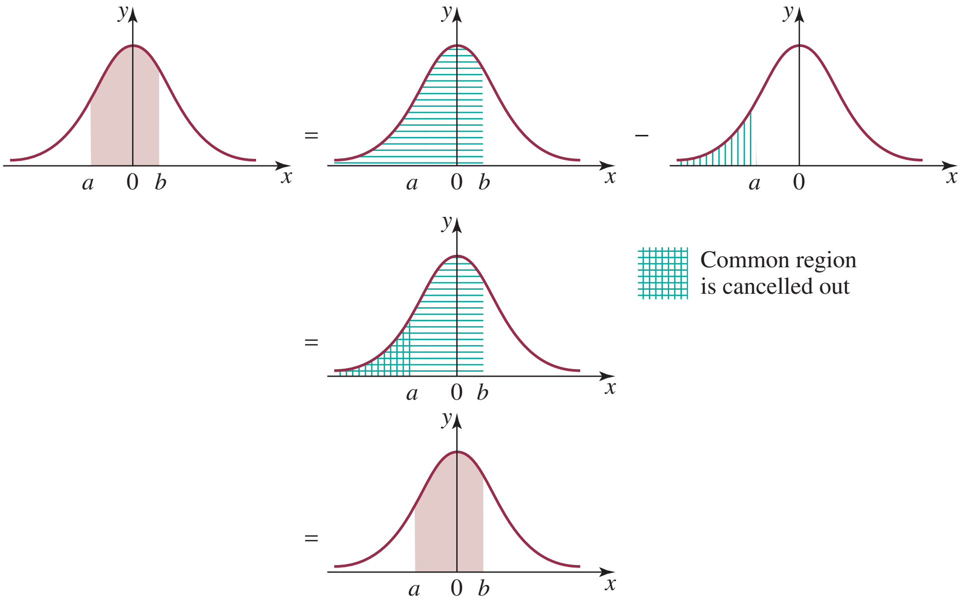
THINK	WRITE
a ① Draw a diagram and shade the region required.	**a** 
② Write down the rule for obtaining $P(Z < -z)$.	$P(Z < -1.23) = P(Z > 1.23)$ $= 1 - P(Z < 1.23)$
③ Use the CND table to obtain the value required.	$= 1 - 0.8907$
④ Evaluate.	$= 0.1093$
b ① Draw a diagram and shade the region required.	**b** 
② Write down the rule.	$P(Z \geq -0.728) = P(Z < 0.728)$
③ Use the CND table to obtain the value required.	$= 0.7642 + 0.0024$
④ Evaluate.	$= 0.7666$

Note: The inequality signs > and ≥ can be interchanged as they give the same probability.

It is also important to be able to determine the probability of z falling between two values, say a and b. As with all of these types of problems, a diagram is essential as it allows us to see the situation clearly and hence solve the problem.

Consider the equation $P(a < Z < b) = P(z < b) - P(z < a)$

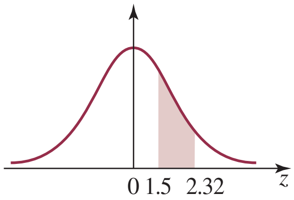
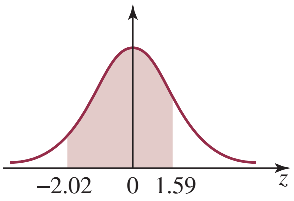
The figure below clearly demonstrates this situation.



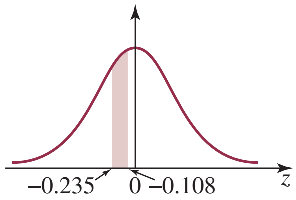
WORKED Example 4

Using the cumulative normal distribution (CND) table, find the value of the following.

a $P(1.5 < Z < 2.32)$ **b** $P(-2.02 \le Z \le 1.59)$ **c** $P(-0.235 < Z < -0.108)$

THINK	WRITE
a ① Draw a diagram and shade the region required.	**a** 
② Write down the rule required. ③ Use the CND table to obtain the value required and evaluate.	$P(1.5 < Z < 2.32) = P(Z < 2.32) - P(Z < 1.5)$ $= 0.9898 - 0.9332$ $= 0.0566$
b ① Draw a diagram and shade the region required.	**b** 

Continued over page

THINK	WRITE
② Write down the rule for obtaining P($a \le Z \le b$).	$P(-2.02 \le Z \le 1.59)$ $= P(Z < 1.59) - P(Z < -2.02)$
③ Using the symmetry of the graph, obtain P($z < -2.02$).	$= P(Z < 1.59) - P(Z > 2.02)$ $= P(Z < 1.59) - [1 - P(Z < 2.02)]$
④ Use the CND table to obtain the value required and evaluate.	$= 0.9941 - [1 - 0.9783]$ $= 0.9941 - 0.0217$ $= 0.9724$
c ① Draw a diagram and shade the region required.	**c** 
② Write down the rule.	$P(-0.235 \le Z \le -0.108)$ $= P(Z < -0.108) - P(Z < -0.235)$
③ Using the symmetry of the graph, obtain P($z < -0.108$) and P($Z < -0.235$).	$= P(Z > 0.108) - P(Z > 0.235)$ $= [1 - P(Z < 0.108)] - [1 - P(Z < 0.235)]$
④ Use the CND table to obtain the value required and evaluate.	$= [1 - (0.5398 + 0.0032)] -$ $[1 - (0.5910 + 0.0019)]$ $= [1 - 0.5430] - [1 - 0.5929]$ $= 0.4570 - 0.4071$ $= 0.0499$

Graphics Calculator tip!

Calculating probabilities for standard normal distributions

Alternatively, for part **c** of Worked example 4, the normal cumulative distribution function of a graphics calculator may be used as follows.

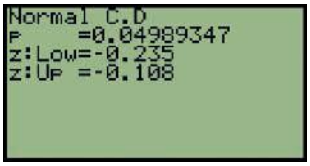
For the Casio *fx*-9860G AU

1. To calculate $P(-0.235 \le Z \le -0.108)$, press:
   - MENU
   - 2 (STAT)
   - F5 (DIST)
   - F1 (NORM)
   - F2 (Ncd).

   Enter the values in the fields as shown.

```
Normal C.D
Lower    :-0.235
Upper    :-0.108
σ        :1
μ        :0
Save Res:None
Execute
CALC
```

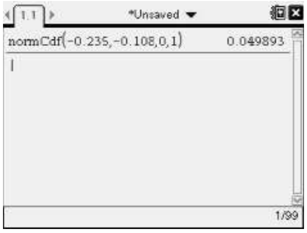
2. Ensure cursor is on Execute and then press (F1) (CALC).
$P(-0.235 \le Z \le -0.108) = 0.0499$



```
Normal C.D
p    =0.04989347
z:Low=-0.235
z:Up =-0.108
```

For the TI-Nspire CAS

To calculate $P(-0.235 \le Z \le -0.108)$, on a Calculator page, complete the entry line as:
normCdf(–0.235,–0.108,0,1)
then press ENTER (enter).
$P(-0.235 \le Z \le -0.108) = 0.0499$



```
normCdf(-0.235,-0.108,0,1)     0.049893
```

WORKED Example 5

If X is normally distributed with $\propto = 50$ and $\sigma = 8$, find:

a **$P(X > 55)$**

b **$P(28 \le X \le 65)$**

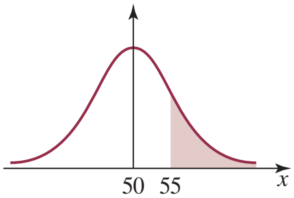
c **$P(X < 40 \mid X < 70)$.**

THINK

a ① Draw a diagram and shade the region required.

WRITE

a



② Convert the given variable to the standard normal variable using the rule $z = \frac{x - \propto}{\sigma}$.

$$z = \frac{x - \propto}{\sigma}$$

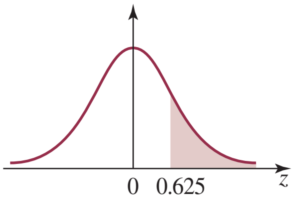
$$= \frac{55 - 50}{8}$$

③ Evaluate.

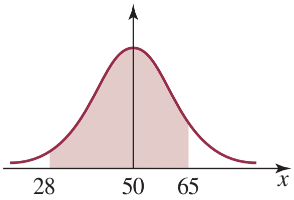
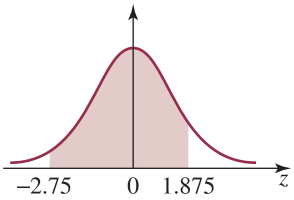
$$= \frac{5}{8}$$

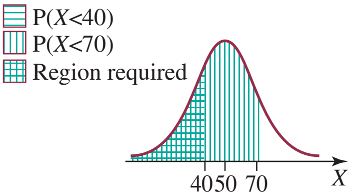
$$= 0.625$$

④ Draw a diagram and shade the region now required.



Continued over page →

THINK	WRITE
5 Write down the rule for obtaining P($Z > 0.625$).	$P(X > 55) = P(Z > 0.625)$ $= 1 - P(Z < 0.625)$
6 Use the CND table to obtain the values required and evaluate.	$= 1 - (0.7324 + 0.0016)$ $= 1 - 0.7340$ $= 0.2660$
b 1 Draw a diagram and shade the region required.	**b** 
2 Convert the given variables to the standard normal variable using the rule $z = \frac{x - \propto}{\sigma}$.	$z_1 = \frac{x - \propto}{\sigma} = \frac{28 - 50}{8}$ $z_2 = \frac{x - \propto}{\sigma} = \frac{65 - 50}{8}$
3 Evaluate.	$= -\frac{22}{8} = -2.75$ $= \frac{15}{8} = 1.875$
4 Draw a diagram and shade the region now required.	
5 Write down the rule for obtaining P($-2.75 < Z < 1.875$).	$P(28 \le X \le 65)$ $= P(-2.75 < Z < 1.875)$
6 Using the symmetry of the graph, obtain P($Z < -2.75$).	$= P(Z < 1.875) - P(Z < -2.75)$ $= P(Z < 1.875) - P(Z > 2.75)$ $= P(Z < 1.875) - [1 - P(Z < 2.75)]$
7 Use the CND table to obtain the values required and evaluate.	$= [0.9693 + 0.0004] - [1 - 0.9970]$ $= 0.9797 - 0.0030$ $= 0.9667$
c 1 Convert the given variables to the standard normal variable using the rule $z = \frac{x - \propto}{\sigma}$.	**c** $z_1 = \frac{x - \propto}{\sigma} = \frac{40 - 50}{8}$ $z_2 = \frac{x - \propto}{\sigma} = \frac{70 - 50}{8}$

THINK	**WRITE**
2 Evaluate.	$= -\frac{10}{8} \qquad = \frac{20}{8}$ $= -1.25 \qquad = 2.5$
3 Write down the rule for conditional probability. *Note:* $P(X < 40 \cap X < 70) = P(X < 40)$. This is given by the overlapping region in the diagram below.  	$P(X < 40 \mid X < 70)$ $= \frac{P(X < 40 \cap X < 70)}{P(X < 70)}$ $= \frac{P(X < 40)}{P(X < 70)}$ $= \frac{P(Z < -1.25)}{P(Z < 2.5)}$
4 Using the symmetry of the graph, obtain $P(Z < -1.25)$.	$= \frac{P(Z > 1.25)}{P(Z < 2.5)}$ $= \frac{1 - P(Z < 1.25)}{P(Z < 2.5)}$
5 Use the CND table to obtain the values required.	$= \frac{1 - 0.8944}{0.9938}$
6 Evaluate and round off the answer to 4 decimal places.	$= \frac{0.1056}{0.9938}$ $= 0.1063$

Graphics Calculator tip!

Calculating probabilities for normal distributions

Alternatively, for part **b** of Worked example 5, the normal cumulative distribution function of a graphics calculator may be used as follows.

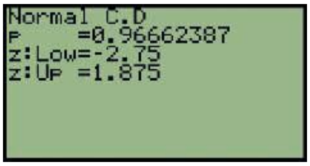
For the Casio *fx*-9860G AU

1. To calculate $P(28 \leq X \leq 65)$ with $\mu = 50$ and $\sigma = 8$, press:
   - MENU
   - 2 (STAT)
   - F5 (DIST)
   - F1 (NORM)
   - F2 (Ncd).

   Enter the values in the fields as shown.

```
Normal C.D
Lower    :28
Upper    :65
σ        :8
μ        :50
Save Res:None
Execute
CALC
```

2. Ensure cursor is on Execute and then press (F1) (CALC).
   $P(28 \leq X \leq 65) = 0.9667$



```
Normal C.D
p    =0.96662387
z:Low=-2.75
z:Up =1.875
```

For the TI-Nspire CAS

To calculate $P(28 \leq X \leq 65)$ with $\mu = 50$ and $\sigma = 8$, on a Calculator page, complete the entry line as:
normCdf(28,65,50,8)
then press ENTER.
$P(28 \leq X \leq 65) = 0.9667$

```
normCdf(28,65,50,8)          0.966624
```

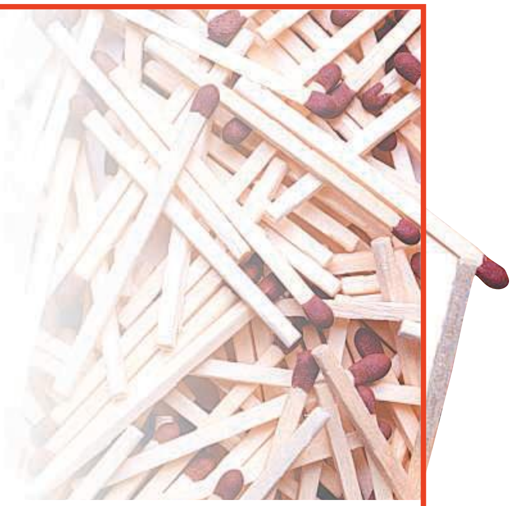
WORKED Example 6

The lengths of matches made at a certain factory are normally distributed with mean 4.1 cm and standard deviation 0.05 cm. Find the probability that a randomly selected match is:

a **longer than 4.1 cm**

b **shorter than 4.13 cm**

c **between 4.08 cm and 4.14 cm.**



THINK

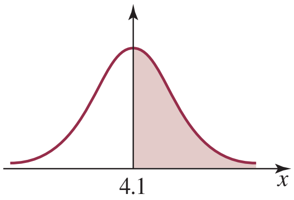
a ① Draw a diagram and shade the region required.

② Convert the given variable to the standard normal variable using the rule $z = \dfrac{x - \propto}{\sigma}$.

③ Evaluate.

WRITE

a

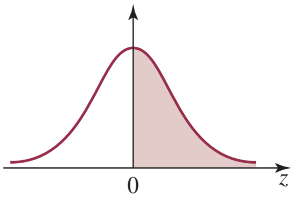
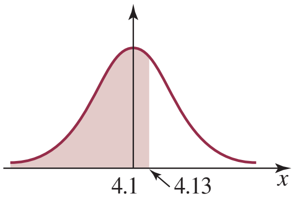
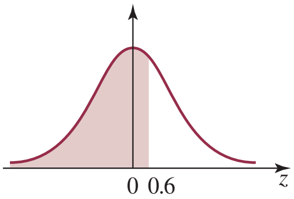
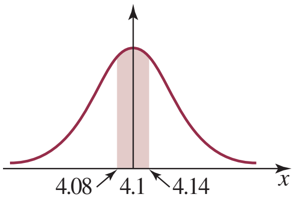


$$z = \frac{x - \propto}{\sigma}$$

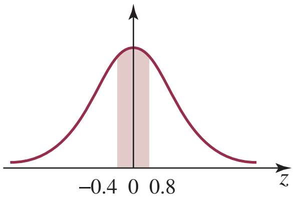
$$= \frac{4.1 - 4.1}{0.05}$$

$$= \frac{0}{0.05}$$

$$= 0$$

THINK	**WRITE**
4 Draw a diagram and shade the region now required.	
5 Write down the rule for obtaining P($Z > 0$).	$P(X > 4.1) = P(Z > 0)$ $= 1 - P(Z < 0)$
6 Use the CND table to obtain the values required and evaluate.	$= 1 - 0.5000$ $= 0.5000$
b 1 Draw a diagram and shade the region required.	**b** 
2 Convert the given variable to the standard normal variable using the rule $z = \frac{x - \propto}{\sigma}$.	$z = \frac{x - \propto}{\sigma}$ $= \frac{4.13 - 4.1}{0.05}$
3 Evaluate.	$= \frac{0.03}{0.05}$ $= 0.6$
4 Draw a diagram and shade the region now required.	
5 Using the symmetry of the graph, obtain P($Z < 0.6$).	$P(X < 4.13) = P(Z < 0.6)$
6 Use the CND table to obtain the values required.	$P(Z < 0.6) = 0.7257$
c 1 Draw a diagram and shade the region required.	**c** 

Continued over page →

THINK	**WRITE**
2 Convert each of the given variables to the standard normal variable using the rule $z = \frac{x - \propto}{\sigma}$.	$z_1 = \frac{x - \propto}{\sigma}$ $= \frac{4.08 - 4.1}{0.05}$ $z_2 = \frac{x - \propto}{\sigma}$ $= \frac{4.14 - 4.1}{0.05}$
3 Evaluate.	$= -\frac{0.02}{0.05}$ $= -0.4$ $= \frac{0.04}{0.05}$ $= 0.8$
4 Draw a diagram and shade the region now required.	
5 Write down the rule for obtaining $P(-0.4 < Z < 0.8)$.	$P(4.08 \le X \le 4.14)$ $= P(-0.4 < Z < 0.8)$ $= P(Z < 0.8) - P(Z < -0.4)$
6 Using the symmetry of the graph, obtain $P(Z < -0.4)$.	$= P(Z < 0.8) - P(Z > 0.4)$ $= P(Z < 0.8) - [1 - P(Z < 0.4)]$
7 Use the CND table to obtain the values required and evaluate.	$= 0.7881 - [1 - 0.6554]$ $= 0.7881 - 0.3446$ $= 0.4435$

remember

1. The cumulative normal distribution (CND) table represents the probability of observing a result from $-\infty$ to a particular positive value of the variable z; that is, $P(Z < z)$.
2. The probability can also be found using a graphics calculator.
   (a) For the Casio *fx*-9860G AU, press:
   - MENU
   - 2 (STAT)
   - F5 (DIST)
   - F1 (NORM)
   - F2 (Ncd) then enter lower and upper limits and σ and μ.

   (b) For the TI-Nspire CAS, on a Calculator page, complete the entry line as: normcdf(lower limit, upper limit, μ, σ) then press ENTER.
3. Using the symmetrical nature of the standard normal curve, it can be seen that:
   $P(Z > z) = 1 - P(Z < z)$
   $P(Z < -z) = 1 - P(Z < z)$
   $P(Z > -z) = P(Z < z)$.
4. To convert a given normal variable, x, to the standard normal variable, z, use the rule $z = \frac{x - \propto}{\sigma}$.
5. With all normal distribution problems, $>$ is equivalent to $\ge$ and $<$ is equivalent to $\le$, since $P(Z = z) = 0$.

EXERCISE 10A The standard normal distribution

For this exercise, probabilities may be found by using the cumulative normal distribution table on page 392 or a graphics calculator (see the graphics calculator tip on page 378, following Worked example 1).

eBook plus

Digital doc:
SkillSHEET 10.1
Using the CND table

WORKED Example 1

1 Find:

a $P(Z \leq 1)$ **b** $P(Z < 2.3)$ **c** $P(Z \leq 1.52)$
d $P(Z \leq 0.74)$ **e** $P(Z < 1.234)$ **f** $P(Z < 2.681)$

WORKED Example 2

2 Find:

a $P(Z > 2)$ **b** $P(Z \geq 1.5)$ **c** $P(Z \geq 1.22)$
d $P(Z > 0.16)$ **e** $P(Z > 1.111)$ **f** $P(Z \geq 2.632)$

WORKED Example 3a

3 Find:

a $P(Z \leq -2)$ **b** $P(Z < -1.3)$ **c** $P(Z < -1.75)$
d $P(Z < -2.23)$ **e** $P(Z \leq -2.317)$ **f** $P(Z \leq -0.669)$

WORKED Example 3b

4 Find:

a $P(Z \geq -3)$ **b** $P(Z \geq -2.1)$ **c** $P(Z > -1.77)$
d $P(Z > -2.71)$ **e** $P(Z \geq -1.139)$ **f** $P(Z > -0.642)$

WORKED Example 4a

5 Find:

a $P(1 < Z < 2)$ **b** $P(1.6 \leq Z \leq 2.5)$ **c** $P(0.42 < Z < 1.513)$

WORKED Example 4b, c

6 Find:

a $P(-1.6 \leq Z \leq 1.4)$ **b** $P(-2.21 < Z < 0.34)$ **c** $P(-0.645 \leq Z \leq 0.645)$
d $P(-0.72 \leq Z \leq -0.41)$

7 Standardise the following X-values to Z-values:

a $X = 40$ if $X \sim N(25, 25)$ **b** $X = 12$ if $X \sim N(17, 9)$
c $X = 15$ if $X \sim N(12, 6.25)$

8 The variable X is normally distributed with mean $\propto = 9$ and standard deviation $\sigma = 3$. Standardise the following X-values:

a $X = 10$ **b** $X = 7.5$ **c** $X = 12.4$

WORKED Example 5

9 If X is normally distributed with $\propto = 40$ and $\sigma = 7$, find:

a $P(X > 42)$ **b** $P(X \geq 30)$ **c** $P(X < 45)$
d $P(X \leq 27)$ **e** $P(25 \leq X \leq 45)$

10 If $X \sim N(20, 25)$, find:

a $P(X > 27)$ **b** $P(X \geq 18)$ **c** $P(X \leq 8)$
d $P(7 \leq X \leq 12)$ **e** $P(X < 17 \mid X \leq 25)$ **f** $P(X < 17 \mid X < \propto)$

WORKED Example 6

11 Light bulbs have a mean life of 125 hours and a standard deviation of 11 hours. Find the probability that a randomly selected light bulb lasts:

a longer than 140 hours **b** less than 100 hours **c** between 100 and 140 hours.

12 The heights jumped by Year-9 high-jump contestants follow a normal distribution with a mean jump height of 152 cm and a variance of 49 cm. Find the probability that a competitor jumps:

a at least 159 cm **b** less than 150 cm **c** between 145 cm and 159 cm
d between 140 cm and 160 cm
e between 145 cm and 150 cm, given that she jumped over 140 cm.

13 **multiple choice**

If Z has a standard normal distribution, then $P(Z > 1.251)$ is:

A 0.1054 **B** 0.3945 **C** 0.6055 **D** 0.6623 **E** 0.8945

14 **multiple choice**

If $Z \sim N(0, 1)$, then $P(Z < -0.25)$ is:

A 0.0987 **B** 0.4013 **C** 0.5987 **D** 0.7124 **E** 0.9013

15 **multiple choice**

The variable X is normally distributed with mean $\propto = 20$ and standard deviation $\sigma = 6$. The standardised value of Z, for $X = 29$ is:

A −1.5 **B** −1.15 **C** 0.483 **D** 1.15 **E** 1.5

16 **multiple choice**

Tennis balls are dropped from a height of 2 metres. The rebound height of the balls is normally distributed with a mean of 1.4 metres and a standard deviation of 0.1 metres. The probability that a ball rebounds more than 1.25 metres is:

A 0.0668 **B** 0.2826 **C** 0.4332 **D** 0.7174 **E** 0.9332

17 **multiple choice**

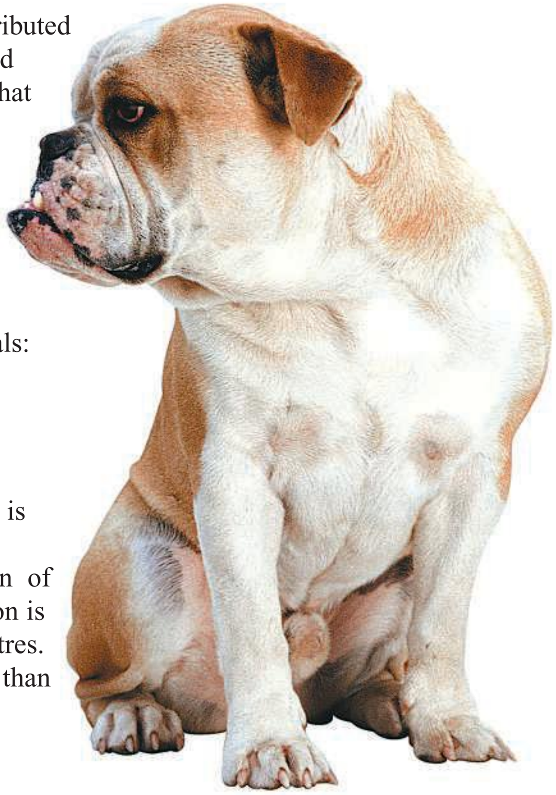
The variable X is normally distributed with mean 70 and variance 12. The probability that X is greater than 77 is:

A 0.0217 **B** 0.0429 **C** 0.0909 **D** 0.9091 **E** 0.9783

18 **multiple choice**

The life span of dogs is normally distributed with a mean of 12 years and a standard deviation of 2 years. The probability that a dog lives for less than 9 years is:

A 0.0668 **B** 0.2826
C 0.2926 **D** 0.4332
E 0.9332



19 **multiple choice**

If $X \sim N(16, 4)$, then $P(X > 11.5)$ equals:

A 0.0122 **B** 0.0836
C 0.7217 **D** 0.9164
E 0.9878

20 The volume of milk in a 1-litre carton is normally distributed with a mean of 1.000 litres and a standard deviation of 0.006 litres. A randomly selected carton is known to have more than 1.004 litres. Find the probability that it has less than 1.011 litres.

21 Eye fillet steaks are cut with a mean weight of 82 grams and a standard deviation of 5 grams. Steaks are sold at different prices according to their weights, as shown in the table below.

Weight (g)	< 70	70–80	80–90	> 90
Cost ($)	1.40	1.60	1.80	2.00

a Find the probability that a randomly selected steak weighs between 80 grams and 90 grams.

b Find the probability that a randomly selected steak costs $2.

c Copy and complete the following table.

Cost ($)	1.40	1.60	1.80	2.00
Probability				

d Using your answers from part **c**, find the average price of an eye fillet steak.

22 The length of 6-cm nails is normally distributed with a mean length of 6 cm and a standard deviation of 0.03 cm. Only nails that are between 5.93 cm and 6.07 cm are acceptable and packaged accordingly. Find:

a the probability of a randomly selected nail being an acceptable length

b the expected number of acceptable nails in a batch of 1000.

eBook*plus*

Digital doc:
WorkSHEET 10.1

23 The length of fish caught in a certain river follows a normal distribution, with mean 32 cm and standard deviation 4 cm. Fish that are less than 27 cm are considered to be undersized and must be returned to the river. Find:

a the probability that a fish is undersized

b the expected number of fish that a fisherman could take home if he catches 20 fish in one afternoon and follows the rules for undersized fish.

Cumulative normal distribution table (CND table)

z	0	1	2	3	4	5	6	7	8	9	Mean differences								
											1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8330	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

The inverse cumulative normal distribution

Up to now, we have obtained the probability of observing a result from $-\infty$ to a particular positive value of the variable z; that is, $P(Z < z)$. Sometimes, however, we are given the probability and must determine the value of the variable; that is, we are required to work in the reverse order. This can be done in a number of ways:

1. using an inverse CND table
2. using a regular CND table
3. using a graphics calculator.

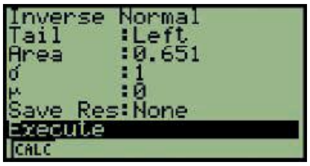
The regular CND table on page 392 has been used in the following worked examples; however, a graphics calculator can also be used throughout.

Graphics Calculator tip!

The inverse cumulative normal distribution

A graphics calculator can be used to determine the value of z if the probability is known.

For the Casio *fx*-9860G AU



```
Inverse Normal
Tail     :Left
Area     :0.651
σ        :1
μ        :0
Save Res:None
Execute
CALC
```

```
Inverse Normal
    x=0.38802166
```

1. To find c if $P(Z \le c) = 0.651$, press:
   - MENU
   - 2 (STAT)
   - F1 (NORM)
   - F3 (InvN).

   Enter the values in the fields as shown.
2. Ensure cursor is on Execute and then press F1 (CALC).
   For $P(Z \le c) = 0.651$, $c = 0.3880$.

For the TI-Nspire CAS

```
1.1          *Unsaved
invNorm(0.651,0,1)          0.388022
                                1/99
```

To find c if $P(Z \le c) = 0.651$, on a Calculator page, complete the entry line as:
invNorm(0.651, 0, 1)
then press ENTER (enter).
For $P(Z \le c) = 0.651$, $c = 0.3880$.

WORKED Example 7

eBook plus

Tutorial:
Worked example 7
int-0182

Find the value of c in the following.

a $P(Z < c) = 0.57$ **b** $P(Z \le c) = 0.25$ **c** $P(Z \ge c) = 0.91$

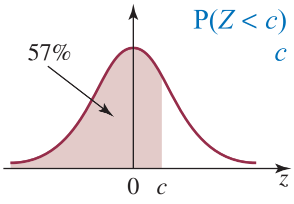
THINK

a ① Draw a diagram of the situation.

② Using the CND table on page 392, look up the z-value corresponding to the given probability.
Obtain the closest probability value to 0.57 (0.5675 when $c = 0.17$) and then go to the mean differences column headed 6 (gives 0.0024).

WRITE

a



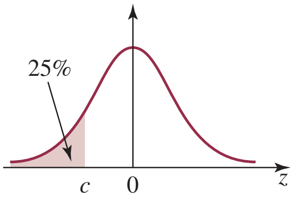
$P(Z < c) = 0.57$
$c = 0.17 + 0.006$
$= 0.176$

Continued over page

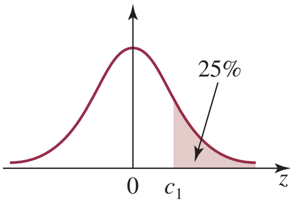
THINK | **WRITE**

b ① Draw a diagram of the situation.

b



② This value is not contained in the table, so we must use the symmetry of the standard normal distribution.
Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$.



③ Determine the unshaded section of the curve.

25% is shaded, therefore 75% is unshaded.

④ Use the CND table to determine the z-value of this probability.

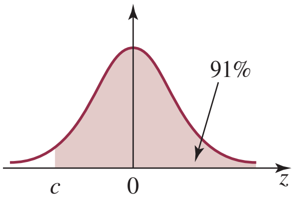
$$P(Z < c_1) = 0.75$$
$$c_1 = 0.67 + 0.004$$
$$c_1 = 0.674$$

⑤ Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

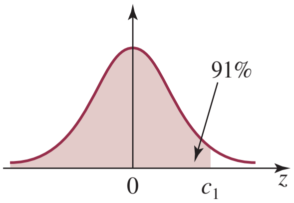
For $P(Z \le c) = 0.25$
$$c = -0.674$$

c ① Draw a diagram of the situation.

c



② Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$.



③ Using the CND table, it is possible to determine the value from $-\infty$ to the positive z-value which corresponds to a probability of 0.91.

$$P(Z < c_1) = 0.91$$
$$c_1 = 1.34$$

④ Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

For $P(Z \ge c) = 0.91$
$$c = -1.34$$

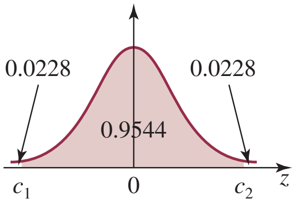
WORKED Example 8

Find the value of c in $P(-c < Z < c) = 0.9544$.

THINK

1. Draw a diagram of the situation. Label the unknown values as c_1 and c_2.
Note: $c_1 = -c_2$.

WRITE



2. Determine the probability of each of the unshaded areas by subtracting the given probability from 1 and dividing the result by 2.

$$\text{Unshaded area} = \frac{1 - 0.9544}{2}$$
$$= \frac{0.0456}{2}$$
$$= 0.0228$$

3. Using the fact that $P(Z < c_2) = 0.9772$ (that is, $0.9544 + 0.0228$) and using the CND table (page 392), determine c_2.

$$P(Z < c_2) = 0.9772$$
$$c_2 = 2$$

4. Using the symmetry of the standard normal distribution, determine the value of c_1.

$$c_1 = -2$$

5. Answer the question.

$$P(-2 < Z < 2) = 0.9544$$

Percentiles and quantiles

Percentiles and *quantiles* are two new terms which will be used in the following problems. Percentiles and quantiles both define the value below which a given proportion of the distribution falls. A percentile is a probability value expressed as a percentage while a quantile is a probability value expressed as a decimal; for example, $P(Z < c) = 0.7$ means that c is the 70th percentile or the 0.70 quantile.

WORKED Example 9

If $Z \sim N(0, 1^2)$, find:

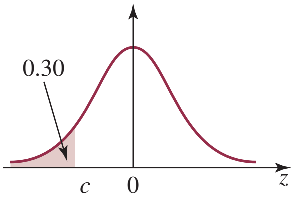
a **the 0.30 quantile** **b** **the 80th percentile.**

THINK

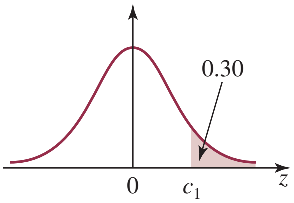
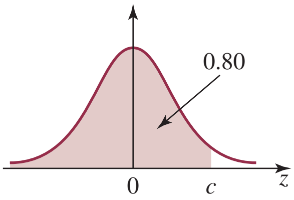
a 1. Draw a diagram illustrating the information.

WRITE

a



Continued over page →

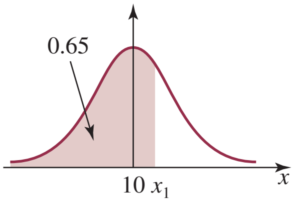
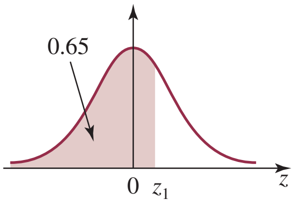
THINK	**WRITE**
2 This value is not contained in the table, so we must use the symmetry of the standard normal distribution. Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section. *Note:* $c = -c_1$.	
3 Determine the unshaded section of the curve.	30% is shaded, therefore 70% is unshaded.
4 Use the CND table (page 392) to determine the z-value of this probability.	$P(Z < c_1) = 0.70$ $c_1 = 0.524$
5 Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.	$P(Z < c) = 0.3$ $c = -0.524$
b 1 Draw a diagram of the situation.	**b** The 80th percentile is the 0.80 quantile. 
2 Using the CND table, look up the z-value corresponding to the given probability.	$P(Z < c) = 0.80$ $c = 0.842$

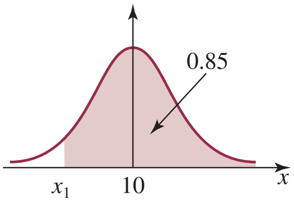
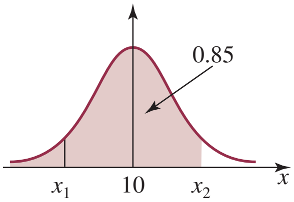
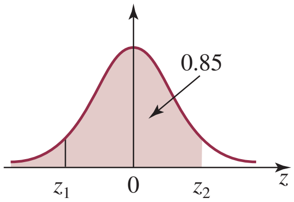
WORKED Example 10

X is normally distributed with a mean of 10 and a standard deviation of 2. Find x_1 if:

a **$P(X \leq x_1) = 0.65$**

b **$P(X > x_1) = 0.85$.**

THINK	**WRITE**
a 1 Draw a diagram illustrating the information.	**a** 
2 Redraw the graph as a standard normal graph.	

THINK	**WRITE**
3 Use the CND table (page 392) to determine z_1, where $P(Z \le z_1) = 0.65$.	$P(Z \le z_1) = 0.65$ $z_1 = 0.385$
4 To solve for x, substitute $x = x_1$ and $z = z_1 = 0.385$ into $z = \frac{x - \propto}{\sigma}$.	$z = \frac{x - \propto}{\sigma}$ $z_1 = \frac{x_1 - \propto}{\sigma}$ $0.385 = \frac{x_1 - 10}{2}$
5 Transpose the equation to make x_1 the subject.	$0.385 \cdot 2 = x_1 - 10$ $0.77 + 10 = x_1$ $x_1 = 10.77$
b 1 Draw a diagram illustrating the information.	**b** 
2 This z-value corresponding to x_1 is not contained in the table (as x_1 is to the left of the mean), so we must use the symmetry of the standard normal distribution. Redraw the graph, displaying x_2 on the opposite side of the mean (such that $P(x < x_2) = 0.85$) and shade the equivalent section.	
3 Redraw the graph as a standard normal graph.	
4 Use the CND table to determine the z-value of this probability.	$P(Z < z_2) = 0.85$ $z_2 = 1.036$
5 Using the symmetry of the standard normal distribution, determine the value of z_1. As z_1 is to the left of the mean, it will be negative. *Note:* $z_1 = -z_2$.	For $P(Z > z_1) = 0.85$ $z_1 = -1.036$

Continued over page

THINK	WRITE
⑥ To solve for x, substitute $x = x_1$ and $z = z_1 = -1.036$ into $z = \dfrac{x - \propto}{\sigma}$.	$z = \dfrac{x - \propto}{\sigma}$ $z_1 = \dfrac{x_1 - \propto}{\sigma}$ $-1.036 = \dfrac{x_1 - 10}{2}$
⑦ Transpose the equation to make x_1 the subject.	$-1.036 \cdot 2 = x_1 - 10$ $-2.072 + 10 = x_1$ $x_1 = 7.928$

remember

1. To find the value of c such that $P(Z < c) = 0.96$, look for the value 0.96 from within the CND table.
   (If this value does not appear exactly — that is, a value of 0.9599 is obtained, which corresponds to a c value of 1.75 — then go to the mean differences column headed 1 where the difference of 0.0001 can be made up.)
   Hence, $P(Z < c) = 0.96$
   $c = 1.751$
2. A percentile is a probability value expressed as a percentage while a quantile is a probability value expressed as a decimal.

EXERCISE 10B The inverse cumulative normal distribution

For this exercise, the CND table on page 392 or a graphics calculator may be used.

WORKED Example 7a

1 Find the value of c in the following.
- **a** $P(Z < c) = 0.9$
- **b** $P(Z < c) = 0.6$
- **c** $P(Z \leq c) = 0.75$
- **d** $P(Z \leq c) = 0.52$

WORKED Example 7b

2 Find the value of c in the following.
- **a** $P(Z \leq c) = 0.3$
- **b** $P(Z < c) = 0.2$
- **c** $P(Z \leq c) = 0.35$
- **d** $P(Z < c) = 0.42$

WORKED Example 7c

3 Find the value of c in the following.
- **a** $P(Z \geq c) = 0.8$
- **b** $P(Z \geq c) = 0.65$
- **c** $P(Z > c) = 0.54$
- **d** $P(Z > c) = 0.72$

4 Find the value of c in the following.
- **a** $P(Z > c) = 0.3$
- **b** $P(Z \geq c) = 0.45$
- **c** $P(Z > c) = 0.22$

WORKED Example 8

5 Find the value of c in the following.

a $P(-c < Z < c) = 0.6826$ **b** $P(-c \le Z \le c) = 0.5$

c $P(-c < Z < c) = 0.2$ **d** $P(-c \le Z \le c) = 0.38$

WORKED Example 9

6 If $Z \sim N(0, 1)$, find:

a the 0.25 quantile **b** the 40th percentile

c the 0.72 quantile **d** the 0.995 quantile.

WORKED Example 10

7 X is normally distributed with a mean of 10 and a standard deviation of 2. Find x_1 if:

a $P(X \le x_1) = 0.72$ **b** $P(X < x_1) = 0.4$

c $P(X > x_1) = 0.63$ **d** $P(X \ge x_1) = 0.2$.

8 X is normally distributed with a mean of 34 and a standard deviation of 16. Find c if:

a $P(X > c) = 0.31$ **b** $P(X \le c) = 0.75$

c $P(X < c) = 0.21$ **d** $P(X \ge c) = 0.55$.

9 Let $X \sim N(22, 25)$. Find k if:

a $P(22 - k \le X \le 22 + k) = 0.7$ **b** $P(22 - k < X < 22 + k) = 0.24$

c $P(X < k \mid X < 23) = 0.32$.

10 **multiple choice**

If $P(Z \le c) = 0.8$, then c equals:

A −0.842 **B** −0.253 **C** 0.253 **D** 0.524 **E** 0.842

11 **multiple choice**

If $P(Z > c) = 0.7$, then c equals:

A −0.524 **B** −0.496 **C** 0.496 **D** 0.524 **E** 0.553

12 **multiple choice**

If $P(-1.2 < Z < k) = 0.4$, then k equals:

A −0.885 **B** −0.253 **C** 0.038 **D** 0.253 **E** 0.885

13 **multiple choice**

If $Z \sim N(0, 1)$, then the 0.35 quantile is:

A −0.675 **B** −0.385 **C** 0.350 **D** 0.385 **E** 0.675

14 **multiple choice**

If $P(Z < k \mid Z < 0.5) = 0.6$, then k equals:

A −0.215 **B** −0.253 **C** 0 **D** 0.253 **E** 0.215

15 **multiple choice**

If X is normally distributed with a mean of 20 and a standard deviation of 4 and $P(X < k) = 0.6$, then k equals:

A 18.99 **B** 19.49 **C** 20.51 **D** 21.01 **E** 21.79

16 **multiple choice**

If $X \sim N(12, 4)$ and $P(X > k) = 0.82$, then k equals:

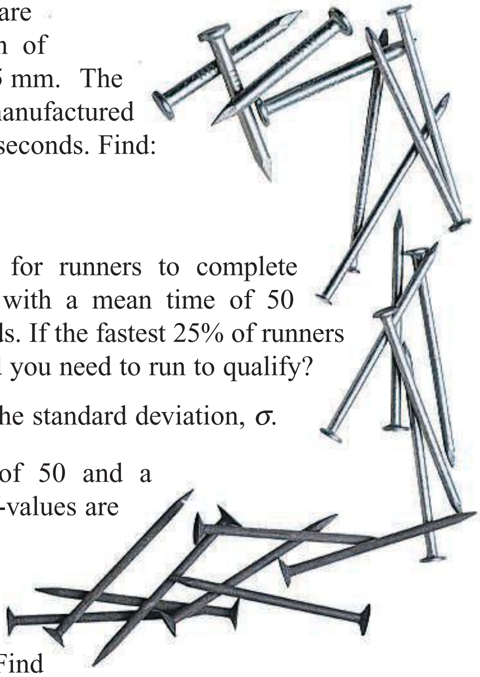
A 8.34 **B** 10.17 **C** 13.83 **D** 14.27 **E** 15.66

17 The height of Year-9 students is known to be normally distributed with a mean of 160 cm and a standard deviation of 8 cm.

- **a** How tall is Theo if he is taller than 95% of Year-9 students?
- **b** How tall is Luisa if she is shorter than 80% of Year-9 students?

18 The length of nails manufactured as 45 mm are actually normally distributed with a mean of 45 mm and a standard deviation of 0.5 mm. The shortest 20% and longest 20% of nails manufactured are discarded before packaging and sold as seconds. Find:

- **a** the minimum length of *packaged* nails
- **b** the maximum length of *packaged* nails.



19 At a qualifying meeting, the time taken for runners to complete 400 metres follows a normal distribution, with a mean time of 50 seconds and a standard deviation of 2 seconds. If the fastest 25% of runners qualify for the next meeting, how fast would you need to run to qualify?

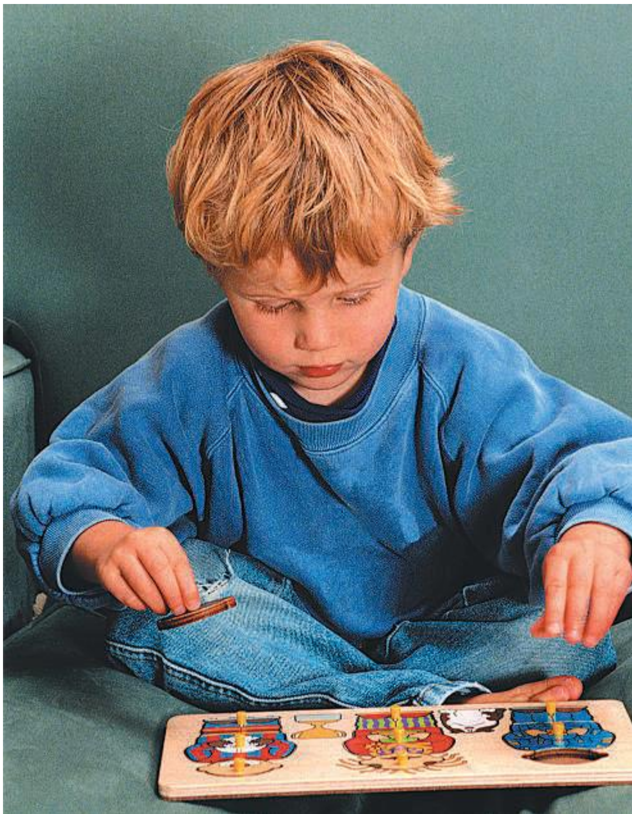
20 If $X \sim \mathrm{N}(20, \sigma^2)$ and $\mathrm{P}(X \geq 19) = 0.7$, find the standard deviation, σ.

21 X is normally distributed with a mean of 50 and a standard deviation of σ. Forty per cent of X-values are less than 48. Find σ.

22 Weights of packaged rice are normally distributed with a mean of 500 grams. Ten per cent of packages are under 485 grams. Find the standard deviation.

23 If $X \sim \mathrm{N}(\propto, 16)$ and $\mathrm{P}(X \geq 17) = 0.99$, find the mean, $\propto$.

24 X is normally distributed with a mean of $\propto$ and a standard deviation of 3. If 35% of X-values are at least 27, find the mean.



25 The time taken for Grade 4 students to complete a small jigsaw puzzle follows a normal distribution with a standard deviation of 30 seconds. If 70% of Grade 4 students complete the puzzle in 4 minutes or less, find the mean completion time for Grade 4 students.

Assessment methods

The results of a mathematics test taken by 120 students show a mean of 60% and a standard deviation of 10%. Grades are to be reported as A, B, C, D and E.

1. Design two different criteria that could be used to decide grade cut-offs.
2. Use the terms 'percentiles' and 'outcomes' to compare the two methods.
3. Use both methods to report the achievement of a student scoring 75%.
4. Which method do you prefer? Why?
5. Do you still agree with your answer to **4** if the student's score was 51%?

Sunflower stems

The lengths of certain sunflower stems follow a normal distribution with a mean of 75 cm and a standard deviation of 8 cm. Stems are measured and awarded grades depending on their lengths. The top 10% receive an A grade, the next 10% a B grade and the third 10% a C grade. Give a range of sunflower stem lengths to 2 decimal places for which:

a an A grade is awarded **b** a B grade is awarded **c** a C grade is awarded.

The normal approximation to the binomial distribution

As we saw in chapter 9, the binomial distribution deals with discrete random variables. An important observation was that as the number of trials, n, increased and the probability of a success, p, was close to 0.5, and if the interval between the vertical columns decreased, then the vertical columns of the histogram collectively resembled or approximated a normal distribution curve. In fact, when n is large, a binomial distribution can be approximated to a normal distribution. This information is useful, particularly if we were required to determine, say, $P(X \geq 40)$ given $X \sim \text{Bi}(400,\ 0.5)$, which would become an onerous task.

Hence, if X, which follows a binomial distribution with n trials and a probability of success, p, is approximated to a normal distribution, then the mean is defined by $\propto = np$ with variance $\sigma^2 = npq$ and standard deviation $\sigma = \sqrt{npq}$. A rule of thumb to use before approximating the binomial distribution to the normal distribution is that the following conditions should apply: $n \geq 30$, $np \geq 10$ and $nq \geq 10$. These values are to be used as a guide only.

The above information may be summarised as follows:

If $n \geq 30$, $np \geq 10$ and $nq \geq 10$ then $X \sim \text{Bi}(n, p)$ may be approximated by

$X \sim \text{N}(np, npq)$.

WORKED Example 11

Write the binomial distribution $X \sim \text{Bi}(200, 0.48)$ as an approximated normal distribution.

THINK	WRITE
1 Write down the notation for the normal distribution.	$X \sim \text{Bi}(200, 0.48) \Rightarrow X \sim \text{N}(np, npq)$
2 Determine $\propto$ and σ^2.	$\propto = np$ $= 200 \cdot 0.48$ $= 96$ $\sigma^2 = npq$ $= 200 \cdot 0.48 \cdot 0.52$ $= 49.92$
3 Check that the criteria for the normal approximation have been satisfied.	$n = 200$ (that is, ≥ 30), $np = 96$ (that is, ≥ 10) and $nq = 104$ (that is, ≥ 10). The criteria have been satisfied.
4 Substitute the values into the notation.	$X \sim \text{Bi}(200, 0.48) \Rightarrow X \sim \text{N}(96, 49.92)$

The major difference between the binomial and normal distributions is that the binomial distribution deals with discrete random variables while the normal distribution deals with continuous random variables.

To compensate for the differences between the binomial distribution and the normal distribution the concept of a *continuity correction* can be introduced.

However, the concept of a continuity correction is not required in this course and will not be considered in the following calculations.

WORKED Example 12

a **If $X \sim \text{Bi}(120, 0.52)$, use a normal approximation to find $\text{P}(X < 50)$.**

b **If $X \sim \text{Bi}(30, 0.51)$, use a normal approximation to find $\text{P}(12 \le X \le 18)$.**

THINK / **WRITE**

a

1. Write down the notation for the normal distribution.

$X \sim \text{Bi}(120, 0.52) \Rightarrow X \sim \text{N}(np, npq)$

2. Determine ∝ and σ^2.

$$\begin{aligned} \propto &= np \\ &= 120 \cdot\ 0.52 \\ &= 62.4 \end{aligned} \qquad \begin{aligned} \sigma^2 &= npq \\ &= 120 \cdot\ 0.52 \cdot\ 0.48 \\ &= 29.952 \end{aligned}$$

3. Check that the criteria for the normal approximation have been satisfied.

$n = 120$ (that is, ≥ 30), $np = 62.4$ (that is, ≥ 10) and $nq = 57.6$ (that is, ≥ 10). The criteria have been satisfied.

4. Substitute the values into the notation.

$X \sim \text{Bi}(120, 0.52) \Rightarrow X \sim \text{N}(62.4, 29.952)$

5. Approximate the binomial variable to a normal variable and convert the normal variable to a standard normal variable, z, using the rule $z = \dfrac{x - \propto}{\sigma}$.

$$\begin{aligned} \text{P}(X < 50) &= \text{P}\left(Z < \frac{50 - 62.4}{\sqrt{29.952}}\right) \\ &= \text{P}(Z < -2.267) \\ &= \text{P}(Z > 2.267) \\ &= 1 - \text{P}(Z < 2.267) \\ &= 1 - 0.9883 \\ &= 0.0117 \end{aligned}$$

b

1. Write down the notation for the normal distribution.

$X \sim \text{Bi}(30, 0.51) \Rightarrow X \sim \text{N}(np, npq)$

2. Determine ∝ and σ^2.

$$\begin{aligned} \propto &= np \\ &= 30 \cdot\ 0.51 \\ &= 15.3 \end{aligned} \qquad \begin{aligned} \sigma^2 &= npq \\ &= 30 \cdot\ 0.51 \cdot\ 0.49 \\ &= 7.497 \end{aligned}$$

3. Check that the criteria for the normal approximation have been satisfied.

$n = 30$ (that is, ≥ 30), $np = 15.3$ (that is, ≥ 10) and $nq = 14.7$ (that is, ≥ 10). The criteria have been satisfied.

4. Substitute the values into the notation.

$X \sim \text{Bi}(30, 0.51) \Rightarrow X \sim \text{N}(15.3, 7.497)$

5. Convert each of the normal variables to a standard normal variable, z, using the rule $z = \dfrac{x - \propto}{\sigma}$.

$$\begin{aligned} &\text{P}(12 \le X \le 18) \\ &\quad = \text{P}\left(\frac{12 - 15.3}{\sqrt{7.497}} \le Z \le \frac{18 - 15.3}{\sqrt{7.497}}\right) \\ &\quad = \text{P}(-0.840 \le Z \le 0.986) \\ &\quad = \text{P}(Z \le 0.986) - \text{P}(Z \le -0.840) \\ &\quad = \text{P}(Z \le 0.986) - \text{P}(Z \ge 0.840) \\ &\quad = \text{P}(Z \le 0.986) - [1 - \text{P}(Z \le 0.840)] \\ &\quad = 0.8380 - [1 - 0.7995] \\ &\quad = 0.8380 - 0.2005 \\ &\quad = 0.6375 \end{aligned}$$

WORKED Example 13

David draws a card at random from a standard pack, records the result and then replaces the card. He performs the experiment a total of 200 times. Use a normal approximation to estimate the probability of obtaining:

a **more than 105 red cards**
b **fewer than 98 black cards.**

THINK	WRITE
a ① Write down the notation for the normal distribution.	**a** $X \sim \text{Bi}(200, 0.50) \Rightarrow X \sim \text{N}(np, npq)$
② Determine $\propto$ and σ^2.	$\propto = np$ $= 200 \cdot 0.50$ $= 100$ $\sigma^2 = npq$ $= 200 \cdot 0.50 \cdot 0.50$ $= 50$
③ Check that the criteria for the normal approximation have been satisfied.	$n = 200$ (that is, ≥ 30), $np = 100$ (that is, ≥ 10) and $nq = 100$ (that is, ≥ 10). The criteria have been satisfied.
④ Substitute the values into the notation.	$X \sim \text{Bi}(200, 0.50) \Rightarrow X \sim \text{N}(100, 50)$
⑤ Approximate the binomial variable to a normal variable.	$\text{P}(X > 105)$
⑥ Convert the normal variable to a standard normal variable, z, using the rule $z = \frac{x - \propto}{\sigma}$.	$= \text{P}\left(Z > \frac{105 - 100}{\sqrt{50}}\right)$ $= \text{P}(Z > 0.707)$ $= 1 - \text{P}(Z < 0.707)$ $= 1 - 0.7601$ $= 0.2399$
⑦ Answer the question.	The probability of obtaining more than 105 red cards is 0.2399.
b ① Repeat steps 1 to 4 from part **a** since the probability of obtaining a black card is equal to the probability of obtaining a red card.	**b** $X \sim \text{Bi}(200, 0.50) \Rightarrow X \sim N(100, 50)$
② Approximate the binomial variable to a normal variable.	$\text{P}(X < 98) = \text{P}\left(Z < \frac{98 - 100}{\sqrt{50}}\right)$
③ Convert the normal variable to a standard normal variable, z, using the rule $z = \frac{x - \propto}{\sigma}$.	$= \text{P}(Z < -0.283)$ $= \text{P}(Z > 0.283)$ $= 1 - \text{P}(Z < 0.283)$ $= 1 - 0.6115$ $= 0.3885$
④ Answer the question.	The probability of obtaining fewer than 98 black cards is 0.3885.

remember

The binomial distribution may be approximated by the normal distribution if $n \geq 30$, $np \geq 10$ and $nq \geq 10$; then $X \sim \text{Bi}(n, p)$ may be approximated by $X \sim \text{N}(np, npq)$.

EXERCISE 10C The normal approximation to the binomial distribution

eBook *plus*

Digital doc: WorkSHEET 10.2

WORKED Example 11

1 Write the following binomial distributions as approximated normal distributions.

a $X \sim \text{Bi}(500, 0.52)$ **b** $X \sim \text{Bi}(400, 0.48)$

c $X \sim \text{Bi}(300, 0.5)$ **d** $X \sim \text{Bi}(250, 0.55)$

2 If $X \sim \text{Bi}(15, 0.5)$:

a use a normal approximation to the binomial distribution, to approximate the following probabilities:

- **i** $P(X < 8)$
- **ii** $P(X \geq 12)$

b calculate the exact probabilities using the binomial probability function and compare the approximations.

WORKED Example 12a

3 If $X \sim \text{Bi}(120, 0.47)$, use a normal approximation to find $P(X < 50)$.

4 If $X \sim \text{Bi}(200, 0.52)$, use a normal approximation to find $P(X \geq 110)$.

WORKED Example 12b

5 If $X \sim \text{Bi}(30, 0.51)$, use a normal approximation to find $P(14 \leq X \leq 16)$.

6 If $X \sim \text{Bi}(350, 0.45)$, use a normal approximation to find $P(150 \leq X \leq 170)$.

7 If $X \sim \text{Bi}(400, 0.55)$, use a normal approximation to find $P(X > 230 \mid X > 200)$.

8 If $X \sim \text{Bi}(650, 0.51)$, use a normal approximation to find $P(X > 310 \mid X < 350)$.

9 **multiple choice**

If $X \sim \text{Bi}(170, 0.48)$ is approximated as a normal distribution it would be:

A $X \sim \text{N}(9.03, 42.43)$ **B** $X \sim \text{N}(81.6, 6.51)$ **C** $X \sim \text{N}(81.6, 42.43)$

D $X \sim \text{N}(170, 0.48)$ **E** $X \sim \text{N}(170, 42.43)$

10 **multiple choice**

If $X \sim \text{Bi}(200, 0.5)$, $P(X < 90)$ is approximated, using normal distribution, as:

A 0.5793 **B** 0.0787 **C** 0.4207 **D** 0.9213 **E** 0.9997

11 **multiple choice**

A maternity hospital delivers 200 new babies over a three-month period. The chances of having a boy or a girl are equally likely. When a normal approximation is used, the probability of more than 105 girls being born out of the 200 is:

A 0.0807 **B** 0.2098 **C** 0.2399 **D** 0.2182 **E** 0.7601

12 **multiple choice**

A fair die is rolled 200 times. Which of the following probabilities would be best approximated using a normal approximation?

A The probability of obtaining at least 50 sixes
B The probability of obtaining at least 90 prime numbers
C The probability of obtaining at least 20 square numbers
D The probability of obtaining less than 40 fives
E The probability of obtaining at least 10 numbers greater than six

WORKED Example 13

13 A fair coin is tossed one hundred times. Use a normal approximation to find the probability that:

a more than 48 heads are obtained
b fewer than 51 heads are obtained
c the number of heads obtained is between 45 and 55 inclusive.

14 An archery contestant knows that she will hit the bullseye 49% of the time. If she fires 500 shots at practice, find (using a normal approximation) the probability that she will obtain:

a at least 240 bullseyes
b fewer than 260 bullseyes.

15 Fifty-two per cent of batteries have a lifetime exceeding 100 hours. Julia buys a large box of 300 such batteries. Use a normal approximation to estimate the probability that:

a at least 160 batteries last over 100 hours
b fewer than 130 batteries *don't* last at least 100 hours.

16 A die is weighted such that $P(X = 1) = P(X = 3) = P(X = 5) = 0.16$ and $P(X = 2) = P(X = 4) = P(X = 6)$. If the die is rolled 200 times, use normal approximation to estimate the probability of obtaining:

a more than 95 odd numbers
b fewer than 100 even numbers.

Supporting the proposal

The Student Representative Council (SRC) estimates that 75% of the students will support the committee's recommendations.

a If 20 students are interviewed, use a normal approximation to approximate the probability of:
  i fewer than 9 students supporting the proposal, that is, $P(X < 9)$
  ii more than 15 students supporting the proposal, that is, $P(X > 15)$.

b Calculate the exact probability of part **a ii** using the binomial distribution and compare the approximations.

Hypothesis testing*

Consider the case of a medical research team who believe that they have found a cure for a disease. This cure will need to be tested to see whether or not there is a greater likelihood that a patient will be cured. The testing of the cure involves a binomial probability.

Hypothesis testing is an important application of the binomial probability distributions looked at in the previous section.

In testing a hypothesis, we need to consider the *null* hypothesis (H_0). The null hypothesis states that there is no effect in what is being tested. In the above example the null hypothesis states that the cure will have no effect on patients.

The alternative belief (that an effect exists) is called the *alternative hypothesis* (H_1). In the above example the alternative hypothesis states that the cure works.

The convention is that we test the null hypothesis and examine the data to determine the truth or otherwise of the null hypothesis.

WORKED Example 14

Andrea begins to receive one hour of private tuition each week in mathematics. With reference to Andrea's marks in her next exam, state the null hypothesis and the alternative hypothesis.

THINK	WRITE
1 The null hypothesis states that there will be no effect.	H_0: The tuition will have no effect on Andrea's marks in Mathematics.
2 The alternative hypothesis is that there will be an effect.	H_1: The tuition will improve Andrea's marks in Mathematics.

In other cases we may need to form a null hypothesis based on a comparison with previous results. Consider the case where a medical treatment cures 45% of patients, on average, within a period of 1 week. When a new treatment is tested:

H_0: The new treatment will cure 45% of patients within 1 week.

H_1: The new treatment will cure more than 45% of patients within 1 week.

Now suppose that in a trial of 20 patients the new treatment cures 12 patients; that is, the cure rate is 60%. We need to decide if this is a significant increase on the previous percentage or whether the increase was simply due to chance with this particular sample.

We need to consider the probability distribution for 20 trials with $p = 0.45$.

Let X = number of patients that are cured.

x	0	1	2	3	4	5	6	7
$P(X = x)$	0.0002	0.0020	0.0100	0.0323	0.0738	0.1272	0.1712	0.1844

x	8	9	10	11	12	13	14	15
$P(X = x)$	0.1614	0.1158	0.0686	0.0336	0.0136	0.0045	0.0012	0.0003

x	16	17	18	19	20
$P(X = x)$	0.0000	0.0000	0.0000	0.0000	0.0000

---

[*] This is not a part of the Maths Quest 12 Maths B course, but may be useful as extension.

If the null hypothesis is true, the probability that this result was obtained by chance is only 0.0136. It would therefore appear that the new treatment is significantly better than the previous treatment and as such H_0 is rejected and H_1 accepted.

Now consider:

$P(X \geq 12)$

$= 0.0136 + 0.0045 + 0.0012 + 0.0003 + 0.0000 + 0.0000 + 0.0000 + 0.0000 + 0.0000$

$= 0.0196$

The probability that this result was obtained by chance is 0.0196 or 1.96%. This is the probability that the null hypothesis is incorrectly rejected. We could say that at $X = 12$ the significance level is approximately 2%.

We can consider a hypothesis at different levels of significance. This is the level that we set as the chance we are prepared to take that we will incorrectly reject a null hypothesis. The table below shows the rejection level for H_0 at various levels of significance.

Significance level	1%	2%	5%	10%	20%	25%
No. of patients that need to be cured	13	12	12	11	10	10
$P(X \geq x)$	0.0060	0.0196	0.0196	0.0532	0.1218	0.2376

In each case $P(X \geq x)$ is the greatest probability less than the stated degree of significance.

WORKED Example 15

The probability that a particular student passes an exam is found to be 0.6. A sample of 20 students are put through a training program before the exam and 15 of them pass the exam. Is this result significant at the 10% level?

THINK	WRITE
1 State the null hypothesis.	H_0: The training program has no effect. Let X = number of students that pass the exam.
2 Use the binomial probabilities table on page 421 (or the Excel spreadsheet on eBookPLUS), for $n = 20$ to find $P(X \geq 15)$. *Note:* Instructions for reading these tables are given on page 358.	$P(X \geq 15)$ $= P(X = 15) + P(X = 16) + P(X = 17) + P(X = 18) + P(X = 19) + P(X = 20)$ $= 0.0746 + 0.0350 + 0.0123 + 0.0031 + 0.0005 + 0.0000$ $= 0.1255$
3 If $P(X \geq 15)$ is greater than 0.1, accept the null hypothesis; otherwise reject it.	As $P(X \geq 15) \geq 0.1$ we accept H_0 that the training program has had no effect.

eBook*plus*

Digital doc:
Spreadsheet
002 Binomial probabilities

We can apply this procedure to test the accuracy of stated data.

Consider the case of a bottle of soft drink that is labelled as having contents of 1 litre. Assume that customers have complained that many bottles of the soft drink are under volume. We would test a sample, determine the number of bottles that are under volume and then calculate whether this is below the desired level of significance.

WORKED Example 16

A bottle of soft drink is labelled as having contents of 1 litre. The drink bottles are tested after complaints that the bottles are under volume. A sample of 15 bottles of soft drink are tested and the contents are listed (in millilitres):

995 998 990 1002 1005 995 997 999 992 985 990 1000 1010 995 990

Determine whether the bottles are under volume at the 10% level of significance.

THINK	WRITE
1 State the null hypothesis.	H_0: The average bottle contains at least 1 litre. Let X = number of bottles that contain at least 1 litre.
2 As there are 4 bottles with at least 1 litre, use the binomial probabilities table on page 419 to find $P(X \le 4)$ with $p = 0.5$ and $n = 15$. Alternatively, you may use the Excel spreadsheet found at www.jacplus.com.au.	$P(X \le 4)$ $= P(X = 4) + P(X = 3) + P(X = 2)$ $+ P(X = 1) + P(X = 0)$ $= 0.0417 + 0.0139 + 0.0032 + 0.0005 + 0.0000$ $= 0.0593$
3 If $P(X \le 4)$ is greater than 0.1 (10%) accept the null hypothesis; otherwise reject it.	As $P(X \le 4)$ is less than 0.1 we reject H_0 and agree that the contents of the soft drink bottles are under volume.

remember

1. The null hypothesis (H_0) is the statement that the factor being tested has no effect.
2. The opposite of the null hypothesis is the alternative hypothesis (H_1).
3. The steps in testing a hypothesis are:
   - State the null hypothesis.
   - Select the desired level of significance.
   - Select a sample for the testing of the hypothesis.
   - Use the table of binomial probabilities to calculate the probability of a result as unlikely as the one obtained.
   - If the probability is greater than the level of significance, accept H_0; if it is less, reject it.

EXERCISE 10D Hypothesis testing

For questions dealing with probability, refer to the binomial probability tables on pages 417–21.

WORKED Example 14

1 Michael attempts to improve his time in the 100 metres by undertaking an extra training session each week. In this case, state the null and alternative hypotheses.

2 In each of the following experiments, state the null hypothesis (H_0).

a Sales of a particular brand of dishwashing liquid will increase with a television advertising campaign.

b Marks in an exam will increase with more study.
c The shelf-life of milk in the bottle is less than 4 days.
d The average mass of a packet of chips will be at least 250 g.

3 For each of the experiments in question **2** state the alternative hypothesis.

WORKED Example 15

4 The probability that a particular student passes an exam is found to be 0.5. A sample of 10 students are put through a training program before the exam and 7 of them pass the exam. Is this result significant at the 10% level?

5 The probability that a patient will recover from an illness without drugs during the period of one week is 0.2. A sample of 15 patients trial a new drug and it is found that 8 of them recover within a week. Is this result significant at the 5% level?

6 Each year it is found that 70% of Year-12 students will pass the end-of-year examination in Maths B. A sample of 20 students who are privately tutored take the exam and it is found that 17 pass the exam. Is this result significant at:
a the 20% level? **b** the 10% level? **c** the 5% level?

WORKED Example 16

7 Boxes of matches are labelled ‘Average contents 50 matches’. A sample of 12 boxes are tested to see if they contain fewer than the labelled number of matches. The results are:
48 49 52 49 49 45 52 53 44 48 49 50
Determine whether the boxes contain less than the labelled contents at the 10% level of significance.

8 A group of 8 people are asked to try ‘Big Kev’s’ laundry products. Of these, 6 people say they prefer ‘Big Kev’s’ products to the other leading brand. Is this result significant at the 20% level of significance?

9 A tyre manufacturer guarantees that its brand of tyre will withstand 20 000 km of normal motoring. The results of a sample of 16 worn tyres are shown below.
23 000 21 000 17 000 19 000 25 000 30 000 29 000 22 000
21 000 22 000 27 000 19 000 15 000 32 000 26 000 27 000

To maintain this guarantee the manufacturer must be sure at the 5% level of significance that the guarantee is valid. Will the guarantee be maintained?

10 Over a period of time the electricity bill in the Robinson household has averaged \$200 per quarter. Mr Robinson feels that over the past two years the electricity bills have been increasing.

a State the null and alternative hypotheses.

b The electricity bills over the past two years have been:

\$239 \$187 \$250 \$284

\$175 \$190 \$212 \$240

Find the probability of obtaining at least 6 results that are greater than average.

c State the level of significance at which Mr Robinson's hypothesis has been established.

The Randhill vaccine

Consider again our problem at the start of this chapter. The team at the Randhill Medical Research Centre believe they have found a vaccine for a strain of the flu. The probability that a patient exposed to the flu virus develops symptoms is 0.7. After testing the vaccine on 20 patients it is found that only 8 develop symptoms.

eBook*plus*

Digital doc:
Spreadsheet
002 Binomial probabilities

1. State the null and alternative hypotheses.
2. Let X = the number of patients developing symptoms. Use the table of binomial probabilities on page 421 (or on eBookPLUS) to determine $P(X \le 8)$ where $n = 20$ and $p = 0.7$.
3. The research centre is applying for funding to develop this vaccine further. They are able to receive this funding only if the result obtained is significant at the 5% level. Will they receive funding?
4. A bonus grant will be given if the result is significant at the 2% level. If a second batch of 15 patients are trialled, what is the maximum number of cases of the flu that can be developed, which will allow the extra funding to be received?

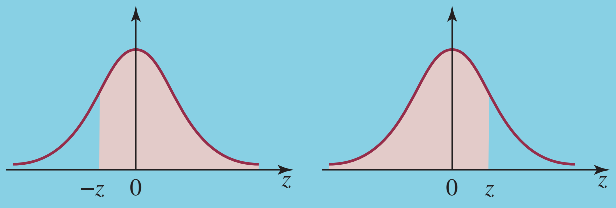
summary

Continuous random variables

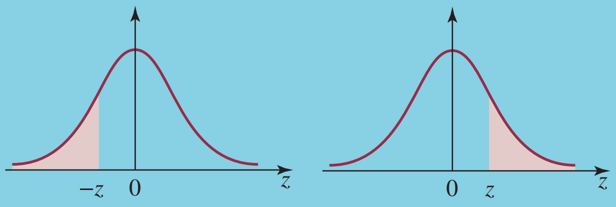
- Continuous random variables represent quantities which can be measured and thus may assume any value in a given range.

The standard normal distribution

- The standard normal distribution is written as $Z \sim \mathrm{N}(0, 1^2)$.
- To convert a given normal variable, x, to the standard normal variable, z, use the rule $z = \dfrac{x - \propto}{\sigma}$.
- The cumulative normal distribution (CND) table represents the probability of observing a result from $-\infty$ to a particular positive value of the variable, z; that is, $\mathrm{P}(Z < z)$.
- Using the symmetrical nature of the standard normal curve, it can be seen that:
  $\mathrm{P}(Z > z) = 1 - \mathrm{P}(Z < z)$
  $\mathrm{P}(Z < -z) = 1 - \mathrm{P}(Z < z)$



- $\mathrm{P}(Z > -z) = \mathrm{P}(Z < z)$



- With all normal distribution problems, > is equivalent to ≥ and < is equivalent to ≤, since $\mathrm{P}(Z = z) = 0$.

The inverse cumulative normal distribution

- To find the value of c such that $\mathrm{P}(Z < c) = 0.96$, look for the value 0.96 from within the CND table.

  Though the exact value does not appear, a value of 0.9599 is obtained which corresponds to a c value of 1.75. Go to the mean differences column headed 1 where the difference of 0.0001 can be made up. Hence,

  $$\mathrm{P}(Z < c) = 0.96$$
  $$c = 1.751$$
- A percentile is a probability value expressed as a percentage while a quantile is a probability value expressed as a decimal.

The normal approximation to the binomial distribution

- The binomial distribution may be approximated by the normal distribution if $n \geq 30$, $np \geq 10$ and $nq \geq 10$. $X \sim \mathrm{Bi}(n, p)$ may be approximated by $X \sim \mathrm{N}(np, npq)$.

Hypothesis testing*

- The null hypothesis (H_0) is the statement that the factor being tested has no effect.
- The opposite of the null hypothesis is the alternative hypothesis (H_1).
- The steps in testing a hypothesis are:
  1. State the null hypothesis.
  2. Select the desired level of significance.
  3. Select a sample for the testing of the hypothesis.
  4. Use the table of binomial probabilities to calculate the probability of a result as unlikely as the one obtained.
  5. If the probability is greater than the level of significance, accept H_0; if it is less, reject it.

---

* This is not a part of the Maths Quest 12 Maths B course, but may be useful as extension.

CHAPTER review

For questions dealing with probability, refer to the binomial probability tables on pages 417–21.

1 Find: **10A**

a $P(Z \leq 2.6)$ **b** $P(Z < 1.1)$ **c** $P(Z \leq 1.34)$ **d** $P(Z \leq 0.98)$
e $P(Z > 1.5)$ **f** $P(Z \geq 2.3)$ **g** $P(Z \geq 1.18)$ **h** $P(Z > 0.73)$
i $P(Z \leq -1)$ **j** $P(Z < -1.8)$ **k** $P(Z < -2.18)$ **l** $P(Z < -2.39)$
m $P(Z \geq -2)$ **n** $P(Z \geq -2.5)$ **o** $P(Z > -1.61)$ **p** $P(Z > -2.03)$

2 Find: **10A**

a $P(Z < 1.2 \mid Z < 1.5)$ **b** $P(Z < -0.53 \mid Z < 0.265)$
c $P(Z \geq 2.19 \mid Z \geq 1.2)$ **d** $P(Z > 1.9 \mid Z < 2.368)$

3 Standardise the following X-values to Z-values. **10A**

a $X = 26$ if $X \sim N(22, 36)$ **b** $X = 18$ if $X \sim N(16, 16)$ **c** $X = 56$ if $X \sim N(50.9, 100)$

4 If X is normally distributed with $\propto = 22$ and $\sigma = 6$, find: **10A**

a $P(X > 23)$ **b** $P(X \geq 11)$ **c** $P(X < 31)$
d $P(X \leq 19)$ **e** $P(20 \leq X \leq 26)$.

5 A jeweller knows that the diameter of wedding rings follows a normal distribution, with a mean of 18 mm and a standard deviation of 1 mm. **10A**

Find the probability that a customer requires a ring with a diameter that is:

a greater than 20.5 mm
b less than 19 mm
c greater than 19 mm, given that it is less than 20.5 mm.

6 Find the value of c in the following. **10B**

a $P(Z < c) = 0.7$ **b** $P(Z \leq c) = 0.95$ **c** $P(Z \leq c) = 0.1$
d $P(Z < c) = 0.28$ **e** $P(Z \geq c) = 0.71$ **f** $P(Z > c) = 0.89$
g $P(Z > c) = 0.4$ **h** $P(Z \geq c) = 0.27$

7 X is normally distributed with a mean of 31 and a standard deviation of 9. Find k if: **10B**

a $P(X < k) = 0.3$ **b** $P(X > k) = 0.83$
c $P(X \geq k) = 0.11$ **d** $P(X \leq k) = 0.997$.

8 Lengths of fish caught in a particular river are normally distributed with a mean of 28 cm. Ninety per cent of fish caught are more than 25 cm long. Find: **10B**

a the standard deviation for the length of fish caught in the river
b the probability of catching a fish which is greater than 30 cm from this river
c the maximum length of fish which must be thrown back, if the lowest 30% must be returned to the river.

9 Write the following binomial distributions as approximated normal distributions. **10C**

a $X \sim Bi(220, 0.5)$ **b** $X \sim Bi(600, 0.49)$
c $X \sim Bi(325, 0.51)$ **d** $X \sim Bi(1000, 0.47)$

10 If $X \sim Bi(250, 0.52)$, use a normal approximation to find $P(X < 120)$. **10C**

10C **11** If $X \sim \text{Bi}(100, 0.49)$, use a normal approximation to find $P(47 \leq X \leq 50)$.

10C **12** A fair die is rolled 300 times. Use a normal approximation to find the probability of obtaining at least 160 even numbers.

10D ***13** For each of the following, state the null and alternative hypotheses.

a Brent says that by travelling a different route to work each day he can make the travelling time shorter.

b Fiona complains that too many overweight vehicles are using the backstreets of her neighbourhood.

c A computer software company claims that students who use their software will increase their marks in Mathematics by up to 20%.

10D ***14** The probability of a coin landing heads is believed to be 0.5. After 20 tosses of the coin, there have been 16 tails. It is claimed that the coin is biased.

a State the null and alternative hypotheses.

b Find $P(X \geq 16)$.

c Based on the above result, would you say that there is any evidence to support the claim that the coin is biased?

10D ***15** The average shift that a truck driver works is 9 hours. Department of Roads officials are concerned that the average shift is becoming much longer. Sixteen drivers have their log books checked and their average shift length in hours is:

9 11 10 11 12 9 8 6 12 10 11 14 12 8 10 11

Are the concerns of the officials justified at the 10% confidence level?

eBook*plus*

Digital doc:
Test Yourself
Chapter 10

---

*This is not a part of the Maths Quest 12 Maths B course, but may be useful as extension.

Binomial probability tables (selected)

These samples are provided to enable students to answer questions and complete exercises found in the textbook. A complete set of tables (from $n = 1$ to $n = 20$) is found on eBookPLUS. (Instructions for reading the tables may be found on page 358.)

$n = 6$

	p											
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.9415	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156	6
1	0.0571	0.2321	0.3543	0.3993	0.3932	0.3560	0.3025	0.2437	0.1866	0.1359	0.0938	5
2	0.0014	0.0305	0.0984	0.1762	0.2458	0.2966	0.3241	0.3280	0.3110	0.2780	0.2344	4
3	0.0000	0.0021	0.0146	0.0415	0.0819	0.1318	0.1852	0.2355	0.2765	0.3032	0.3125	3
4	0.0000	0.0001	0.0012	0.0055	0.0154	0.0330	0.0595	0.0951	0.1382	0.1861	0.2344	2
5	0.0000	0.0000	0.0001	0.0004	0.0015	0.0044	0.0102	0.0205	0.0369	0.0609	0.0938	1
6	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0007	0.0018	0.0041	0.0083	0.0156	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
	p											

$n = 8$

	p											
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.9227	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039	8
1	0.0746	0.2793	0.3826	0.3847	0.3355	0.2670	0.1977	0.1373	0.0896	0.0548	0.0313	7
2	0.0026	0.0515	0.1488	0.2376	0.2936	0.3115	0.2965	0.2587	0.2090	0.1569	0.1094	6
3	0.0001	0.0054	0.0331	0.0839	0.1468	0.2076	0.2541	0.2786	0.2787	0.2568	0.2188	5
4	0.0000	0.0004	0.0046	0.0185	0.0459	0.0865	0.1361	0.1875	0.2322	0.2627	0.2734	4
5	0.0000	0.0000	0.0004	0.0026	0.0092	0.0231	0.0467	0.0808	0.1239	0.1719	0.2188	3
6	0.0000	0.0000	0.0000	0.0002	0.0011	0.0038	0.0100	0.0217	0.0413	0.0703	0.1094	2
7	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0012	0.0033	0.0079	0.0164	0.0313	1
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0007	0.0017	0.0039	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
	p											

n **= 10**

						p						
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.9044	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010	10
1	0.0914	0.3151	0.3874	0.3474	0.2684	0.1877	0.1211	0.0725	0.0403	0.0207	0.0098	9
2	0.0042	0.0746	0.1937	0.2759	0.3020	0.2816	0.2335	0.1757	0.1209	0.0763	0.0439	8
3	0.0001	0.0105	0.0574	0.1298	0.2013	0.2503	0.2668	0.2522	0.2150	0.1665	0.1172	7
4	0.0000	0.0010	0.0112	0.0401	0.0881	0.1460	0.2001	0.2377	0.2508	0.2384	0.2051	6
5	0.0000	0.0001	0.0015	0.0085	0.0264	0.0584	0.1029	0.1536	0.2007	0.2340	0.2461	5
6	0.0000	0.0000	0.0001	0.0012	0.0055	0.0162	0.0368	0.0689	0.1115	0.1596	0.2051	4
7	0.0000	0.0000	0.0000	0.0001	0.0008	0.0031	0.0090	0.0212	0.0425	0.0746	0.1172	3
8	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0014	0.0043	0.0106	0.0229	0.0439	2
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0016	0.0042	0.0098	1
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
						p						

n **= 12**

						p						
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.8864	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002	12
1	0.1074	0.3413	0.3766	0.3012	0.2062	0.1267	0.0712	0.0368	0.0174	0.0075	0.0029	11
2	0.0060	0.0988	0.2301	0.2924	0.2835	0.2323	0.1678	0.1088	0.0639	0.0339	0.0161	10
3	0.0002	0.0173	0.0852	0.1720	0.2362	0.2581	0.2397	0.1954	0.1419	0.0923	0.0537	9
4	0.0000	0.0021	0.0213	0.0683	0.1329	0.1936	0.2311	0.2367	0.2128	0.1700	0.1208	8
5	0.0000	0.0002	0.0038	0.0193	0.0532	0.1032	0.1585	0.2039	0.2270	0.2225	0.1934	7
6	0.0000	0.0000	0.0005	0.0040	0.0155	0.0401	0.0792	0.1281	0.1766	0.2124	0.2256	6
7	0.0000	0.0000	0.0000	0.0006	0.0033	0.0115	0.0291	0.0591	0.1009	0.1489	0.1934	5
8	0.0000	0.0000	0.0000	0.0001	0.0005	0.0024	0.0078	0.0199	0.0420	0.0762	0.1208	4
9	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0015	0.0048	0.0125	0.0277	0.0537	3
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0008	0.0025	0.0068	0.0161	2
11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0.0029	1
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
						p						

$n = 15$

	p											
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.8601	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0016	0.0005	0.0001	0.0000	15
1	0.1303	0.3658	0.3432	0.2312	0.1319	0.0668	0.0305	0.0126	0.0047	0.0016	0.0005	14
2	0.0092	0.1348	0.2669	0.2856	0.2309	0.1559	0.0916	0.0476	0.0219	0.0090	0.0032	13
3	0.0004	0.0307	0.1285	0.2184	0.2501	0.2252	0.1700	0.1110	0.0634	0.0318	0.0139	12
4	0.0000	0.0049	0.0428	0.1156	0.1876	0.2252	0.2186	0.1792	0.1268	0.0780	0.0417	11
5	0.0000	0.0006	0.0105	0.0449	0.1032	0.1651	0.2061	0.2123	0.1859	0.1404	0.0916	10
6	0.0000	0.0000	0.0019	0.0132	0.0430	0.0917	0.1472	0.1906	0.2066	0.1914	0.1527	9
7	0.0000	0.0000	0.0003	0.0030	0.0138	0.0393	0.0811	0.1319	0.1771	0.2013	0.1964	8
8	0.0000	0.0000	0.0000	0.0005	0.0035	0.0131	0.0348	0.0710	0.1181	0.1647	0.1964	7
9	0.0000	0.0000	0.0000	0.0001	0.0007	0.0034	0.0116	0.0298	0.0612	0.1048	0.1527	6
10	0.0000	0.0000	0.0000	0.0000	0.0001	0.0007	0.0030	0.0096	0.0245	0.0515	0.0916	5
11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0006	0.0024	0.0074	0.0191	0.0417	4
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0004	0.0016	0.0052	0.0139	3
13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0003	0.0010	0.0032	2
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	1
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
	p											

n = 16

	p											
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.8515	0.4401	0.1853	0.0743	0.0281	0.0100	0.0033	0.0010	0.0003	0.0001	0.0000	16
1	0.1376	0.3706	0.3294	0.2097	0.1126	0.0535	0.0228	0.0087	0.0030	0.0009	0.0002	15
2	0.0104	0.1463	0.2745	0.2775	0.2111	0.1336	0.0732	0.0353	0.0150	0.0056	0.0018	14
3	0.0005	0.0359	0.1423	0.2285	0.2463	0.2079	0.1465	0.0888	0.0468	0.0215	0.0085	13
4	0.0000	0.0061	0.0514	0.1311	0.2001	0.2252	0.2040	0.1553	0.1014	0.0572	0.0278	12
5	0.0000	0.0008	0.0137	0.0555	0.1201	0.1802	0.2099	0.2008	0.1623	0.1123	0.0667	11
6	0.0000	0.0001	0.0028	0.0180	0.0550	0.1101	0.1649	0.1982	0.1983	0.1684	0.1222	10
7	0.0000	0.0000	0.0004	0.0045	0.0197	0.0524	0.1010	0.1524	0.1889	0.1969	0.1746	9
8	0.0000	0.0000	0.0001	0.0009	0.0055	0.0197	0.0487	0.0923	0.1417	0.1812	0.1964	8
9	0.0000	0.0000	0.0000	0.0001	0.0012	0.0058	0.0185	0.0442	0.0840	0.1318	0.1746	7
10	0.0000	0.0000	0.0000	0.0000	0.0002	0.0014	0.0056	0.0167	0.0392	0.0755	0.1222	6
11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0013	0.0049	0.0142	0.0337	0.0667	5
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	0.0040	0.0115	0.0278	4
13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0008	0.0029	0.0085	3
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0005	0.0018	2
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	1
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
	p											

$n = 20$

						p						
	0.01	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	
x												
0	0.8179	0.3585	0.1216	0.0388	0.0115	0.0032	0.0008	0.0002	0.0000	0.0000	0.0000	20
1	0.1652	0.3774	0.2702	0.1368	0.0576	0.0211	0.0068	0.0020	0.0005	0.0001	0.0000	19
2	0.0159	0.1887	0.2852	0.2293	0.1369	0.0669	0.0278	0.0100	0.0031	0.0008	0.0002	18
3	0.0010	0.0596	0.1901	0.2428	0.2054	0.1339	0.0716	0.0323	0.0123	0.0040	0.0011	17
4	0.0000	0.0133	0.0898	0.1821	0.2182	0.1897	0.1304	0.0738	0.0350	0.0139	0.0046	16
5	0.0000	0.0022	0.0319	0.1028	0.1746	0.2023	0.1789	0.1272	0.0746	0.0365	0.0148	15
6	0.0000	0.0003	0.0089	0.0454	0.1091	0.1686	0.1916	0.1712	0.1244	0.0746	0.0370	14
7	0.0000	0.0000	0.0020	0.0160	0.0545	0.1124	0.1643	0.1844	0.1659	0.1221	0.0739	13
8	0.0000	0.0000	0.0004	0.0046	0.0222	0.0609	0.1144	0.1614	0.1797	0.1623	0.1201	12
9	0.0000	0.0000	0.0001	0.0011	0.0074	0.0271	0.0654	0.1158	0.1597	0.1771	0.1602	11
10	0.0000	0.0000	0.0000	0.0002	0.0020	0.0099	0.0308	0.0686	0.1171	0.1593	0.1762	10
11	0.0000	0.0000	0.0000	0.0000	0.0005	0.0030	0.0120	0.0336	0.0710	0.1185	0.1602	9
12	0.0000	0.0000	0.0000	0.0000	0.0001	0.0008	0.0039	0.0136	0.0355	0.0727	0.1201	8
13	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0010	0.0045	0.0146	0.0366	0.0739	7
14	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0012	0.0049	0.0150	0.0370	6
15	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0013	0.0049	0.0148	5
16	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0003	0.0013	0.0046	4
17	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0011	3
18	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	2
19	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1
20	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0
												x
	0.99	0.95	0.9	0.85	0.8	0.75	0.7	0.65	0.6	0.55	0.5	
						p						

eBook*plus* ACTIVITIES

10A The standard normal distribution

Digital docs

- SkillSHEET 10.1: Practise using the CND table (*page 389*)
- WorkSHEET 10.1: Use the CND table to determine probabilities for standard and non standard normal distributions (*page 391*)

Interactivity

- The normal distribution int-0257: Consolidate your understanding of the normal distribution (*page 375*)

10B The inverse cumulative normal distribution

Digital doc

- WorkSHEET 10.2: Approximate binomial distributions to normal distributions and calculate probabilities (*page 405*)

Tutorial

- **WE 7** Int-0182: Watch a tutorial on inverse normal calculations using a CAS calculator (*page 393*)

10D Hypothesis testing

Digital doc

- Spreadsheet 002: Investigate binomial probabilities (*pages 408, 412*)

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress (*page 416*).

To access eBookPLUS activities, log on to

www.jacplus.com.au

Appendix

Instructions for the TI-89 Titanium graphics calculator

Chapter 1 — Modelling change and rates of change

Chapter 2 — Applications of differentiation

Chapter 3 — Exponential and logarithmic functions

Chapter 4 — Derivatives of exponential and logarithmic functions

Chapter 5 — Periodic functions

Chapter 6 — The calculus of periodic functions

Chapter 7 — Introduction to integration

Chapter 8 — Techniques of integration

Chapter 9 — Probability distributions

Chapter 10 — The normal distribution

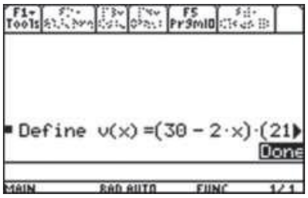
Chapter 1 page 4

WORKED Example 3

THINK | **WRITE/DISPLAY**

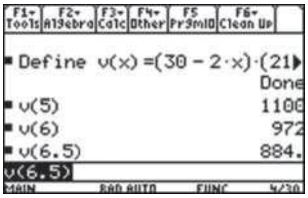
① To calculate the given values of V, we need to use the CATALOG menu, Press:
- HOME
- CATALOG.

Define $v(x) = (30 - 2x) \times (21 - 2x) \times x$ then press ENTER.



```
■Define v(x) =(30 - 2·x)·(21▸
                          Done
```

② Enter:
$v(5)$
$v(6)$
$v(6.5)$
pressing ENTER after each line.



```
■Define v(x) =(30 - 2·x)·(21▸
                          Done
■v(5)                     1100
■v(6)                      972
■v(6.5)                   884.
v(6.5)
```

③ Write the answers.

a $V(5) = 1100 \text{ cm}^3$
b $V(6) = 972 \text{ cm}^3$
c $V(6.5) = 884 \text{ cm}^3$

Chapter 1 page 5

WORKED Example 4

THINK

WRITE/DISPLAY

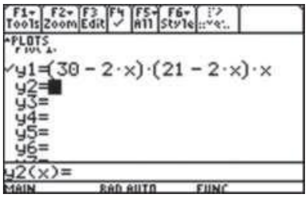
① To solve $V(x) = 900$, on a HOME page, press:

- F4 (Other)
- 1: (Define).

Complete the entry line as:

Define $V(x) = (30 - 2x) \times (21 - 2x) \times x$

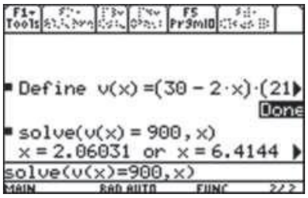
then press ENTER.



```
✓y1=(30 - 2·x)·(21 - 2·x)·x
 y2=█
 y3=
 y4=
 y5=
 y6=
y2(x)=
```

② Complete the entry line as:

solve($V(x) = 900$, x).



```
■ Define v(x) =(30 - 2·x)·(21▶
                          Done
■ solve(v(x) = 900, x)
  x = 2.06031  or  x = 6.4144 ▶
solve(v(x)=900,x)
```

③ There are three solutions; however, we are concerned only with those between 0 and 15, so write only the first two.

Note: The TI-89 Titanium can solve the equation with one part of the restricted domain. To solve for $x \leq 15$, enter the equation as:

solve($V(x) = 900$, x) | $x \leq 15$.

Solving $V(x) = 900$, for $0 \leq x \leq 15$

$x = 2.06031$ or $x = 6.4144$

Chapter 1 page 6

WORKED Example 5

THINK

WRITE/DISPLAY

① To find the value of x for which $V(x)$ is a maximum, where $0 \leq x \leq 10.5$, on a HOME page, complete the entry line as:

Define $V(x) = (30 - 2x) \times (21 - 2x) \times x$

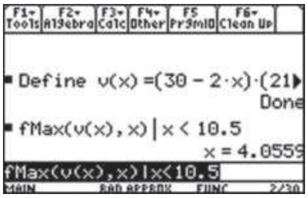
then press (ENTER).

Complete the entry line as:

fMax$(v(x), x) \,|\, x \leq 10.5$

then press (ENTER).

If your calculator gives an exact value you will need to change the mode to Approx.

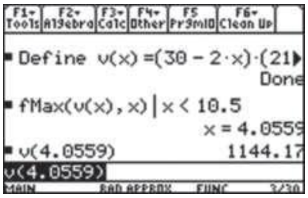


```
■Define v(x) =(30 − 2·x)·(21▸
                          Done
■fMax(v(x), x)|x < 10.5
                      x = 4.0559
fMax(v(x),x)|x<10.5
MAIN        RAD APPROX    FUNC    2/30
```

② To find the maximum value of $V(x)$ when $x = 4.0559$, complete the entry line as:

$V(4.0559)$

then press (ENTER).



```
■Define v(x) =(30 − 2·x)·(21▸
                          Done
■fMax(v(x), x)|x < 10.5
                      x = 4.0559
■v(4.0559)            1144.17
v(4.0559)
MAIN        RAD APPROX    FUNC    3/30
```

③ Write the answer.

The maximum value of $V(x)$ for $0 \leq x \leq 10.5$ is 1144.17 cm^3 when $x = 4.0559$ cm.

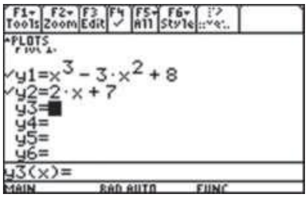
Chapter 1 page 7

WORKED Example 6

THINK

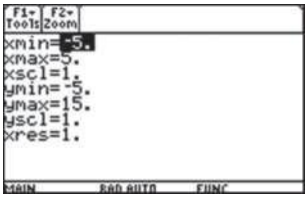
WRITE/DISPLAY

① To sketch the curve, press APPS and select ◆ [Y=].
Complete the entry lines as:
$y1 = x^3 - 3x^2 + 8$
$y2 = 2x + 7$.



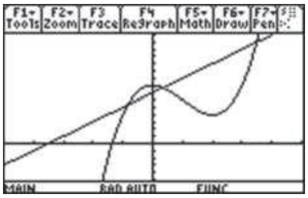
```
✓y1=x^3-3·x^2+8
✓y2=2·x+7
 y3=
 y4=
 y5=
 y6=
y3(x)=
```

② Press ◆ [WINDOW] and enter the settings as shown.



```
xmin=-5.
xmax=5.
xscl=1.
ymin=-5.
ymax=15.
yscl=1.
xres=1.
```

③ Press ◆ [GRAPH] to draw the graph of the two curves.

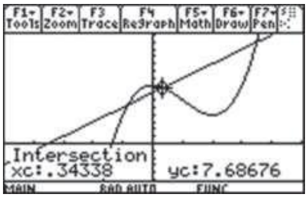


④ To find the intersection of the curves, press:
- F5 (Math)
- 5: (Intersection).

Press ENTER at any point on the first curve and again at any point on the second curve.

Select a point just before the point of intersection to select the lower bound and a point just above to select the upper bound.

The intersection will then be displayed.



```
Intersection
xc:.34338        yc:7.68676
```

⑤ Write the answer.

Note: The TI-89 Titanium calculator can find the x-values of these points of intersection algebraically. On a HOME page, complete the entry line as:

solve($x^3 - 3x^2 + 8 = 2x + 7, x$)

then press ENTER.

The intersections of the curves $y = x^3 - 3x^2 + 8$ and $y = 2x + 7$ occur at (–0.83, 5.33), (0.34, 7.69) and (3.49, 13.98).

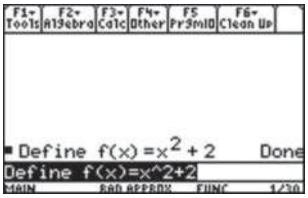
Chapter 1 page 24

WORKED Example 13b

THINK

① On a HOME page, press CATALOG and select the Define function.
Complete the entry line as:
Define $f(x) = x^2 + 2$
then press (ENTER).

WRITE/DISPLAY



■Define $f(x) = x^2 + 2$ Done

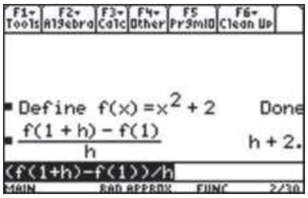
Define f(x)=x^2+2

MAIN RAD APPROX FUNC 1/30

② To find the gradient of the chord PQ, complete the entry line as:

$$\frac{f(1+h) - f(1)}{h}$$

then press (ENTER).



■Define $f(x) = x^2 + 2$ Done

■$\frac{f(1+h) - f(1)}{h}$ h + 2.

(f(1+h)-f(1))/h

MAIN RAD APPROX FUNC 2/30

③ Write the answer.

The gradient of the chord PQ is $2 + h$.

Chapter 1 page 26

WORKED Example 16b

THINK

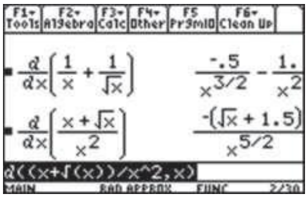
① On a HOME page, press CATALOG and select the *d(differentiate* function.
Complete the entry lines as:

$$\mathrm{d}\left(\frac{1}{x}+\frac{1}{\sqrt{x}}, x\right)$$

$$\mathrm{d}\left(\frac{x+\sqrt{x}}{x^2}, x\right)$$

pressing ENTER after each line.

WRITE/DISPLAY



$\blacksquare\ \frac{d}{dx}\left(\frac{1}{x}+\frac{1}{\sqrt{x}}\right) \qquad \frac{^{-}.5}{x^{3/2}}-\frac{1.}{x^2}$

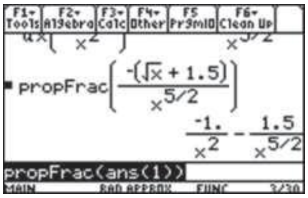
$\blacksquare\ \frac{d}{dx}\left(\frac{x+\sqrt{x}}{x^2}\right) \qquad \frac{^{-}(\sqrt{x}+1.5)}{x^{5/2}}$

```
d((x+√(x))/x^2,x)
```

② To separate the answer to part **ii** into two separate fractions, press CATALOG and select the propFrac function.
Complete the entry line as:

propfrac **2ND ANS**

then press ENTER.



$\blacksquare\ \text{propFrac}\left(\frac{^{-}(\sqrt{x}+1.5)}{x^{5/2}}\right) \qquad \frac{^{-}1.}{x^2}-\frac{1.5}{x^{5/2}}$

```
propFrac(ans(1))
```

③ Write the answer.

i $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}+\frac{1}{\sqrt{x}}\right)=\frac{^{-}1}{2x^{\frac{3}{2}}}-\frac{1}{x^2}$

ii $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+\sqrt{x}}{x^2}\right)=\frac{^{-}1}{x^2}-\frac{3}{2x^{\frac{5}{2}}}$

Chapter 1 page 34

Graphics Calculator tip!

Using the TI-89 Titanium calculator

The **TI-89 Titanium** calculator can be used to do almost all the calculations in this section on calculus. This does not necessarily mean the by-hand skills associated with derivatives are redundant. Rather, your understanding can be reinforced with appropriate calculator use.

The following examples showcase some of the calculus that can be done with the **TI-89 Titanium** calculator.

1. To calculate the derivative of

   $y(x) = \dfrac{1}{\sqrt{3x-4}}$, on a HOME page,

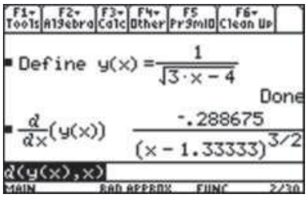
   complete the entry line as:

   Define $y(x) = \dfrac{1}{\sqrt{3x-4}}$

   then from the CATALOG menu select the *d(differentiate* function and complete the entry line as:

   d($y(x)$,x)

   then press **ENTER**.



■Define $y(x) = \dfrac{1}{\sqrt{3\cdot x-4}}$ Done

■$\dfrac{d}{dx}(y(x))$ $\dfrac{-.288675}{(x-1.33333)^{3/2}}$

d(y(x),x)

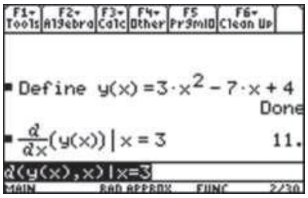
2. To calculate the derivative of $y = 3x^2 - 7x + 4$ at the point $x = 3$, on a Home page, complete the entry line as:

   Define $y(x) = 3x^2 - 7x + 4$

   then from the CATALOG menu select the *d(differentiate* function and complete the entry line as:

   d($y(x)$,x) | $x = 3$

   then press **ENTER**.



■Define $y(x) = 3\cdot x^2 - 7\cdot x + 4$ Done

■$\dfrac{d}{dx}(y(x)) \mid x = 3$ 11.

d(y(x),x)|x=3

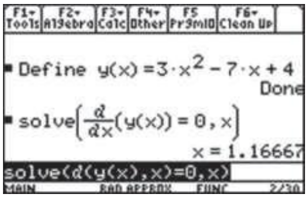
3. To find the point where the derivative of $y = 3x^2 - 7x + 4$ is equal to 0, on a Home page, complete the entry line as:

   Define $y(x) = 3x^2 - 7x + 4$

   then from the CATALOG menu select the *d(differentiate* function and complete the entry line as:

   solve(d($y(x)$,x) = 0, x

   then press **ENTER**.



■Define $y(x) = 3\cdot x^2 - 7\cdot x + 4$ Done

■solve$\left(\dfrac{d}{dx}(y(x)) = 0, x\right)$ $x = 1.16667$

solve(d(y(x),x)=0,x)

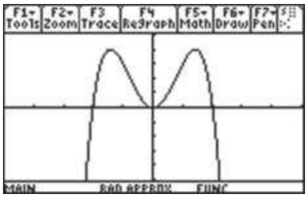
Chapter 2 page 49

WORKED Example 2

THINK

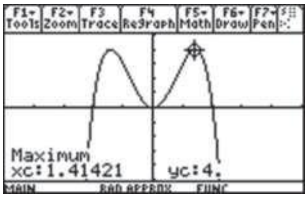
WRITE/DISPLAY

① To sketch the graph of $g(x) = x^2(4 - x^2)$, press APPS and select ♦ [Y=].
Complete the entry line as:
$y1 = x^2(4 - x^2)$
then press ENTER.
Press ♦ [GRAPH].
You may need to adjust the window settings.



② To identify important features, press:
- F5 (Math)
- 2: (Maximum).

Press ENTER when the cursor is a little to the left of the maximum value and again when the cursor is a little to the right of the maximum value. The maximum value will then be displayed. Repeat these steps for the minimum value and the x intercepts (zeros).



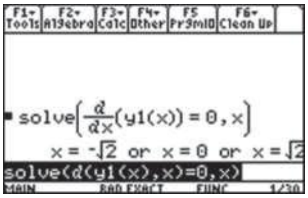
Maximum
xc:1.41421 yc:4.

③ To find the turning points in exact form, check that the calculator is in exact mode and on a new HOME screen press:
- F2 (Algebra)
- 1: (Solve).

Complete the entry line as:

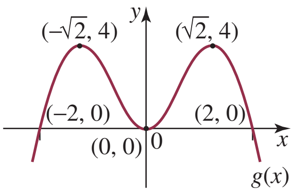
$$\text{solve}\left(\frac{d}{dx}(y1(x), x) = 0, x\right)$$

then press ENTER.
The y-coordinates of the turning points can be found by substitution.



```
▪ solve(d/dx(y1(x)) = 0, x)
        x = -√2 or x = 0 or x = √2
solve(d(y1(x),x)=0,x)
```

④ Sketch the graph, showing all the important features.



Chapter 2 page 59

WORKED Example 7

THINK

i To find the initial population when $P(t) = 20t(9 - t^2) + 300$, on a HOME page, press CATALOG and select Define.
Complete the entry lines as:

Define $P(t) = 20t \times (9 - t^2) + 300$
$p(0)$

pressing ENTER after each line.

ii ① To find the time when the number of birds is a maximum, complete the entry line as:

fMax($P(t)$,t)|$t \geq 0$

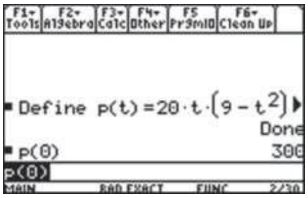
then press ENTER.
To find the maximum number of birds, complete the entry line as:

$p(1.73205)$

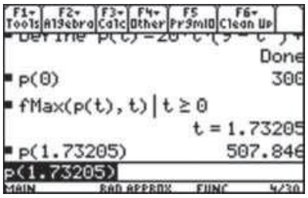
then press ENTER.

② Write the answer.
(Remember the number of birds must be an integer.)

WRITE/DISPLAY



```
■ Define p(t)=20·t·(9-t²)▸
                        Done
■ p(0)                   300
p(0)
```



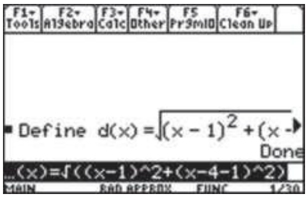
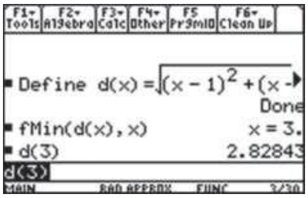
```
                        Done
■ p(0)                   300
■ fMax(p(t),t)|t≥0
                 t=1.73205
■ p(1.73205)         507.846
p(1.73205)
```

i For $P(t) = 20t(9 - t^2) + 300$, the initial population of birds is $P(0) = 300$.

ii Solve $P'(t) = 0$ for $0 \leq t \leq 3$.
The maximum number of birds is 507 after 1.73 months.

Chapter 2 page 66

WORKED Example 11

THINK	WRITE/DISPLAY
1 The minimum distance between a straight line and a point is a perpendicular line from a point on the straight line, Q (x, y) to the point P (1, 1).	Let Q be the point (x, y).
2 The y-coordinate of Q can be expressed in terms of x, as Q is on the line $y = x - 4$.	P is (1, 1). Q is $(x, x - 4)$.
3 The distance between P and Q can be found using the formula for distance between two points.	$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $d(x) = \sqrt{(x - 1)^2 + (x - 4 - 1)^2}$
4 On a HOME page, complete the entry line as: Define $d(x) = \sqrt{(x - 1)^2 + (x - 4 - 1)^2}$ then press ENTER.	 Define d(x) = √((x − 1)² + (x - Done …(x)=√((x-1)^2+(x-4-1)^2)
5 To find the value of x for which $d(x)$ is a minimum, complete the entry lines as: fMin($d(x)$, x) $d(3)$ pressing ENTER after each line. If the calculator is in exact mode, an exact answer of $2\sqrt{2}$ will be given.	 Define d(x) = √((x − 1)² + (x - Done fMin(d(x), x) x = 3. d(3) 2.82843 d(3)
6 Write the answer.	Using a CAS calculator, $d(x)$ is a minimum when $x = 3$. Therefore the minimum distance is approximately 2.83 units.

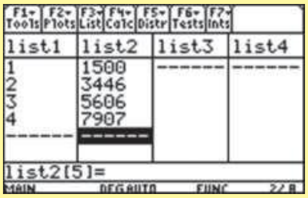
Chapter 3 page 82

Cell growth — seeking a model

1 From the **APPS** menu select Stats/List Editor, then press:

- 3: (New).

Put the Day data in List 1 and the Number of Cells data in List 2.



```
list1  list2  list3  list4
1      1500   ------ ------
2      3446
3      5606
4      7907
------ ------
list2[5]=
```

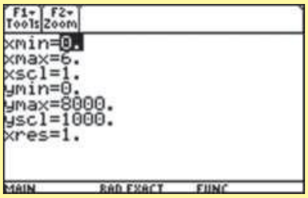
2 To graph this data on a scatterplot, press ◆ [Y=] and clear any existing functions. Return to the Stats/List Editor page and press:

- **F2** (Plot)
- 1: (Plot Setup)
- **F1** (Define).

Complete the fields as shown.

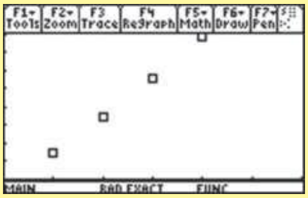
Plot Type Scatter →
Mark Box →
x list1
y list2
Use Freq and Categories? NO →

3 Press ◆ [WINDOW] and enter the settings shown at right.



```
xmin=0.
xmax=6.
xscl=1.
ymin=0.
ymax=8000.
yscl=1000.
xres=1.
```

4 Press ◆ [GRAPH] to display the graph.



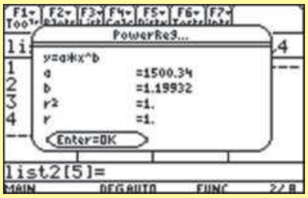
5 This scatterplot could fit an exponential curve. A number of different models can be tested. The model that best fits this data is the power model: $y = a \times x^b$. To find the values of a and b, return to the Stats/List Editor page and press:

- **F4** (Calc)
- 3: (Regressions)
- 9: (Power Reg).

Complete the fields as shown, then press **ENTER**.

X List: list1
Y List: list2
Store RegEqn to: none →
Freq: 1
Category List:
Include Categories: {}

6 These procedures yield the relationship: $N = 1500 \times D^{1.2}$.



```
y=a*x^b
a      =1500.34
b      =1.19932
r²     =1.
r      =1.
list2[5]=
```

Chapter 3 page 87

Graphics Calculator tip! Simplifying expressions involving indices

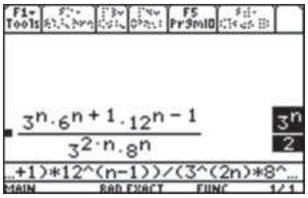
The **TI-89 Titanium** can be used to simplify expressions of the form

$$\frac{3^n \cdot 6^{n+1} \cdot 12^{n-1}}{3^{2n} \cdot 8^n}.$$

On a HOME page, complete the entry line as:

$$\frac{3^n \times 6^{n+1} \times 12^{n-1}}{3^{2n} \times 8^n}$$

then press ENTER.



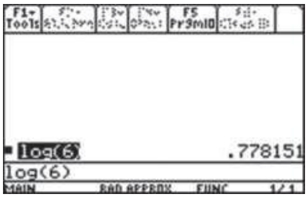
$\frac{3^{n}\cdot 6^{n+1}\cdot 12^{n-1}}{3^{2\cdot n}\cdot 8^{n}}$ $\frac{3^n}{2}$

```
…+1)*12^(n-1))/(3^(2n)*8^…
MAIN    RAD EXACT    FUNC    1/1
```

Chapter 3 page 93

Graphics Calculator tip! Evaluating logarithms to base 10

The TI-89 Titanium can be used to evaluate logarithms to base 10. To find the approximate value of $\log_{10}6$, on a HOME page, press CATALOG and select the log function and enter the number 6. Close the brackets. You may need to make sure that your calculator is in Approx mode.



```
▪ log(6)                    .778151
log(6)
MAIN    RAD APPROX    FUNC    1/1
```

Chapter 3 page 99

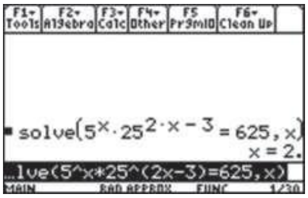
WORKED Example 15

THINK

1. To solve $5^x \times 25^{2x-3} = 625$ for x, on a HOME page, complete the entry line as: solve($5^x \times 25^{2x-3} = 625$, x) then press ENTER.

2. Write the answer.

WRITE/DISPLAY



▪ solve($5^{x}\cdot 25^{2\cdot x-3}=625$, x) x = 2.

```
…lve(5^x*25^(2x-3)=625,x)
MAIN    RAD APPROX    FUNC    1/30
```

Solve $5^x \times 25^{2x-3} = 625$ for x.
$x = 2$.

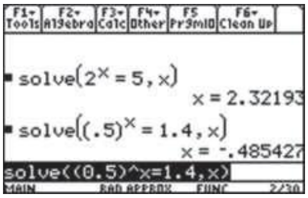
Chapter 3 page 102

WORKED Example 19

THINK

WRITE/DISPLAY

1. On a HOME page, complete the entry lines as:
   solve($2^x = 5,x$)
   solve($(0.5)^x = 1.4,x$)
   pressing ENTER after each line.



```
■ solve(2^x = 5, x)
                x = 2.32193
■ solve((.5)^x = 1.4, x)
                x = -.485427
solve((0.5)^x=1.4,x)
```

2. The TI-89 Titanium does not allow us to solve inequalities, so we solve the equality and re-insert the inequality sign in the answer.

i Solve $2^x > 5$ for x
$x > 2.322$ (3 decimal places).

ii Solve $0.5^x \leq 1.4$ for x
$x \geq -0.485$(3 decimal places).

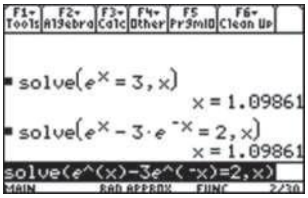
Chapter 3 page 115

WORKED Example 27

THINK

WRITE/DISPLAY

1. On a HOME page, complete the entry lines as:
   solve($e^x = 3,x$)
   solve($e^x - 3e^{-x} = 2,x$)
   pressing ENTER after each line.



```
■ solve(e^x = 3, x)
                x = 1.09861
■ solve(e^x - 3·e^-x = 2, x)
                x = 1.09861
solve(e^(x)-3e^(-x)=2,x)
```

2. Write the answer.

i Solve $e^x = 3$ for x
$x = 1.099$ (3 decimal places).

ii Solve $e^x - 3e^{-x} = 2$ for x
$x = 1.099$ (3 decimal places).

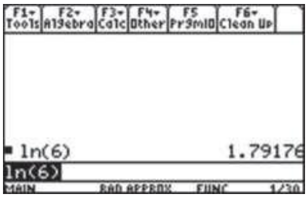
Chapter 3 page 118

Graphics Calculator tip! Evaluating natural logarithms (base *e*)

The **TI-89 Titanium** can be used to evaluate logarithms to base *e*.

1. To find the approximate value of $\log_e 6$, on a HOME page, press CATALOG and select the ln function. Enter the number 6 and close the brackets.

   Press ENTER.

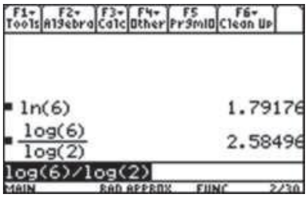


```
■ ln(6)                 1.79176
ln(6)
MAIN       RAD APPROX    FUNC     1/30
```

2. For other bases we can use the change of base law.

$$\log_a b = \frac{\log b}{\log a}$$

   For example, the calculation at right shows $\log_2 6$.



```
■ ln(6)                 1.79176
■ log(6)/log(2)         2.58496
log(6)/log(2)
MAIN       RAD APPROX    FUNC     2/30
```

Chapter 4 page 135

WORKED Example 2

THINK

① Interchange x and y to write the inverse equation.

② To rearrange the equation of the inverse to make y the subject, on a HOME page complete the entry line as:

solve(x = 2 ln(y – 1) + 1,y)

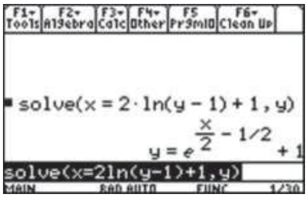
then press ENTER.

③ Write the answer.

WRITE/DISPLAY

$$y = 2\log_e(x - 1) + 1$$

Inverse is $x = 2\log_e(y - 1) + 1$



■ solve(x = 2·ln(y – 1) + 1, y)

$$y = e^{\frac{x}{2} - 1/2} + 1$$

```
solve(x=2ln(y-1)+1,y)
```

$$y = e^{\frac{x-1}{2}} + 1$$

Chapter 4 page 138

WORKED Example 5

THINK

① To calculate the derivatives, on a HOME page, press F3 (Calc) and select the *d*(*differentiate* function. Complete the entry lines as:

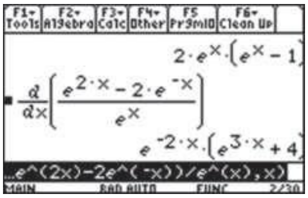
$d(e^x(e^x - 2),x)$

$$d\left(\left(\frac{e^{2x} - 2e^{-x}}{e^x}\right), x\right)$$

then press ENTER.

② Write the answer.

WRITE/DISPLAY



$$2\cdot e^{x}\cdot(e^{x} - 1)$$

$$■\ \frac{d}{dx}\left(\frac{e^{2\cdot x} - 2\cdot e^{-x}}{e^{x}}\right) \qquad e^{-2\cdot x}\cdot(e^{3\cdot x} + 4)$$

```
…e^(2x)-2e^(-x))/e^(x),x)
```

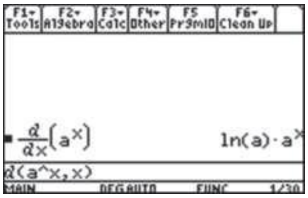
i $\frac{d}{dx}(e^x(e^x - 2)) = 2e^x(e^x - 1)$

ii $\frac{d}{dx}\left(\frac{e^{2x} - 2e^{-x}}{e^x}\right) = e^{-2x}(e^{3x} + 4)$

Chapter 4 page 140

Graphics Calculator tip! **Finding the derivative of $y = a^x$**

To find the derivative of $y = a^x$, on a HOME page, complete the entry line as:
$d(a^x,x)$.



■ $\frac{d}{dx}(a^x)$ ln(a)·a^x

d(a^x,x)

MAIN DEG AUTO FUNC 1/30

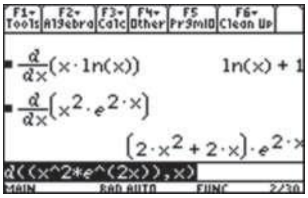
Chapter 4 page 147

WORKED Example 13

THINK

1. To calculate the derivatives, on a HOME page, complete the entry lines as:
   $d(x \times \ln(x),x)$
   $d((x^2 \times e^{2x}),x)$
   pressing ENTER after each line.
2. Write the answer.

WRITE/DISPLAY



■ $\frac{d}{dx}(x \cdot \ln(x))$ ln(x) + 1

■ $\frac{d}{dx}(x^2 \cdot e^{2 \cdot x})$

$(2 \cdot x^2 + 2 \cdot x) \cdot e^{2 \cdot x}$

d((x^2*e^(2x)),x)

MAIN RAD AUTO FUNC 2/30

i $\frac{\mathrm{d}}{\mathrm{d}x}(x \times \log_e(x)) = \log_e x + 1$

ii $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \times e^{2x}) = e^{2x}(2x^2 + 2x)$

Chapter 5 page 172

WORKED Example 7

THINK

① On a HOME page, complete the entry lines as:

$\sin\left(\frac{{}^{-}4\pi}{3}\right)$

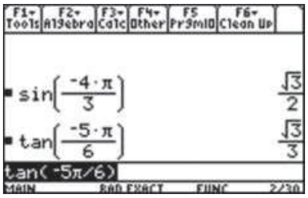
$\tan\left(\frac{{}^{-}5\pi}{6}\right)$

pressing (ENTER) after each line.

Ensure your calculator is in Exact mode.

② Write the exact value.

WRITE/DISPLAY



```
■ sin(-4·π/3)          √3/2
■ tan(-5·π/6)          √3/3
tan(-5π/6)
```

i $\frac{\sqrt{3}}{2}$

ii $\frac{\sqrt{3}}{3}$

Chapter 5 page 179

WORKED Example 13

THINK

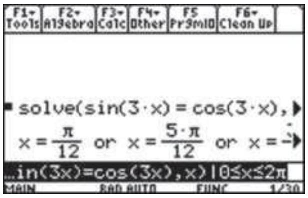
① On a HOME page, select the solve function and complete the entry line as:

solve(sin(3x) = cos(3x),x) | $0 \le x \le 2\pi$

then press (ENTER).

② Write the solutions.

WRITE/DISPLAY



```
■ solve(sin(3·x) = cos(3·x),▶
  x = π/12 or x = 5·π/12 or x =▶
…in(3x)=cos(3x),x)|0≤x≤2π
```

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$$

Chapter 5 page 186

Graphics Calculator tip! Graphing periodic functions

Worked example 14 can be done on a graphics calculator.

THINK

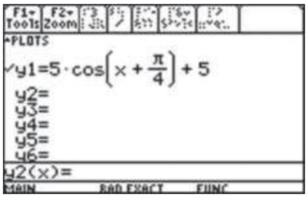
DISPLAY

1. To sketch the graph of $y = 5\cos\left(x + \frac{\pi}{4}\right) + 5$ for $0 \le x \le 2\pi$, press **APPS** and select Y= Editor.

   Complete the entry line as:

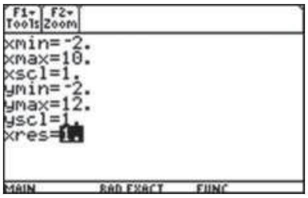
   $$y1 = 5\cos\left(x + \frac{\pi}{4}\right) + 5 \,|\, 0 \le x \le 2\pi$$

   then press **ENTER**.



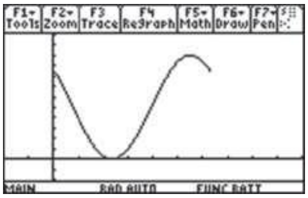
```
✓y1=5·cos(x+π/4)+5
 y2=
 y3=
 y4=
 y5=
 y6=
y2(x)=
```

2. To adjust the window, press ◆ [WINDOW]. Complete the fields as shown.

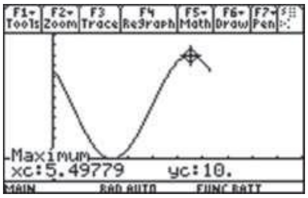


```
xmin=-2.
xmax=10.
xscl=1.
ymin=-2.
ymax=12.
yscl=1.
xres=1.
```

3. Press ◆ [GRAPH] to draw the graph of the function.



4. To find significant points on the graph, press **F5** and select maximum or minimum.



```
Maximum
xc:5.49779     yc:10.
```

Chapter 6 page 208

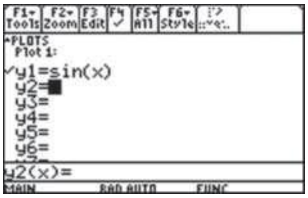
The derivatives of sin x and cos x

We shall now examine the derivatives of sin x and cos x using a graphics calculator.

1. To draw the graph of sin x, press **APPS** and select Y= Editor.

   Complete the entry line as:

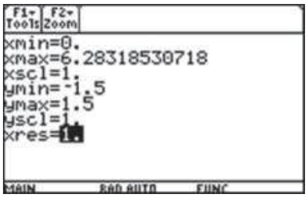
   $y1 = \sin(x)$.



```
y1=sin(x)
y2=
y3=
y4=
y5=
y6=
y2(x)=
```

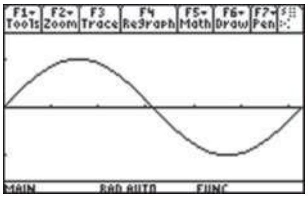
2. Press ◆ [WINDOW] and enter the window settings shown at right.

   For xmax, enter 2π.



```
xmin=0.
xmax=6.28318530718
xscl=1.
ymin=-1.5
ymax=1.5
yscl=1.
xres=1.
```

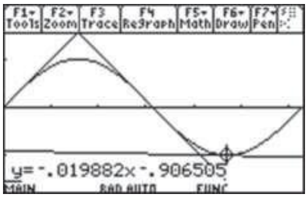
3. Press ◆ [GRAPH] to draw the graph.



4. To draw a tangent to the curve at $x = 0$, press:
   - **F5** (Math)
   - A: (Tangent).

   Move the cursor to the point (0, 0) and then press **ENTER**; a tangent line will appear.

   Repeat this step to draw tangents at $x = \pi$ and $x = \frac{3\pi}{2}$.

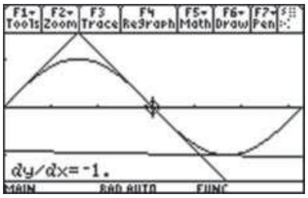


```
y=-.019882x-.906505
```

5. To find the gradient of these tangents, press:
   - **F5**
   - 6: (Derivatives).

   Move the cursor to each point where a tangent has been drawn. Press **ENTER**. The value of the gradient will be displayed.

   Each tangent will need to be done separately.



```
dy/dx=-1.
```

6. Repeat steps 4 and 5 for differing values of x to complete the table at the right.

X	$\frac{dy}{dx}$
0	
$\frac{\pi}{4}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	
π	
$\frac{5\pi}{4}$	
$\frac{3\pi}{2}$	
$\frac{7\pi}{4}$	
2π	

7. To draw a scatterplot of this data, press **APPS** and select CellSheet.

   Put the x values in Column A and the values of $\frac{dy}{dx}$ in Column B, then press:

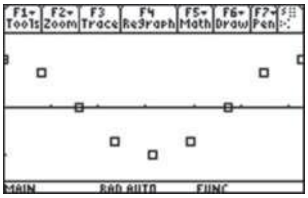
   - **F2** (Plot)
   - 1: (Plot Setup)
   - 1: (Define).

   Enter the settings shown at right.

Define Plot 1

Plot Type........ Scatter →
Mark.............. Box →
xRange.......... a1:a9
yRange.......... b1:b9
Use Freq and Categories? YES →
Category........
Include Categories {}

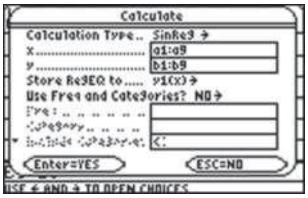
8. Press ◆ [GRAPH] to draw the scatterplot.



9. To obtain an equation for this curve, return to the CellSheet and press:

   - **F7** (Stat)
   - 1: (Calculate).

   Complete the fields as shown.



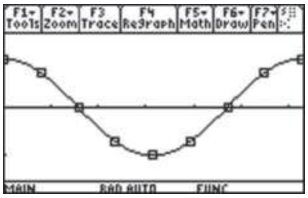
Calculate

Calculation Type.. SinReg →
x....................... a1:a9
y....................... b1:b9
Store RegEQ to..... y1(x) →
Use Freq and Categories? NO →

10. The equation in the form $y = a\ \sin(bx + c) + d$ will be shown, with the values of a, b, c and d given.

```
STAT VARS
y=a·sin(b·x+c)+d
a          =1.
b          =1.
c          =1.570796
d          =2.495667E-15
```

11. Press ◆ [GRAPH] to draw the graph.



Chapter 6 page 212

WORKED Example 2

THINK

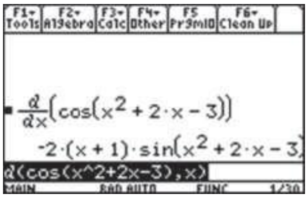
① On a HOME page, press:
- F3 (Calc)
- 1: (d(differentiate).

Complete the entry line as:

$d(\cos(x^2 + 2x - 3),x)$

then press ENTER.

WRITE/DISPLAY



$$\frac{d}{dx}(\cos(x^2+2\cdot x-3))$$

$$^-2\cdot(x+1)\cdot\sin(x^2+2\cdot x-3$$

```
d(cos(x^2+2x-3),x)
```

② Write the solution.

$$\frac{dy}{dx} = -2(x + 1)\sin(x^2 + 2x - 3)$$

Chapter 6 page 215

WORKED Example 4

THINK

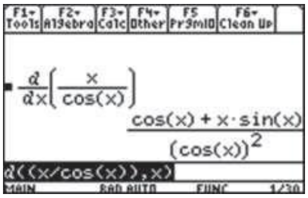
① On a HOME screen, press:
- F3 (Calc)
- 1: (d(differentiate).

Complete the entry line as:

$$d\left(\frac{x}{\cos(x)}, x\right)$$

then press ENTER.

WRITE/DISPLAY



$$\frac{d}{dx}\left(\frac{x}{\cos(x)}\right)$$

$$\frac{\cos(x)+x\cdot\sin(x)}{(\cos(x))^2}$$

```
d((x/cos(x)),x)
```

② Write the solution.

$$\frac{dy}{dx} = \frac{\cos x + x\sin x}{\cos^2 x}$$

Chapter 6 page 218

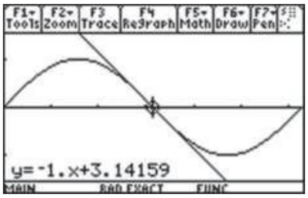
WORKED Example 5

THINK

1. On a GRAPH page, draw the graph of $y = \sin x$, then press:
   - F5 (Math)
   - A: (Tangent).

   Move the cursor to the point where $x = \pi$. The equation of the tangent is displayed.
2. Write the solution.

WRITE/DISPLAY



```
y=-1.x+3.14159
```

$y = \pi - x$

Chapter 6 page 222

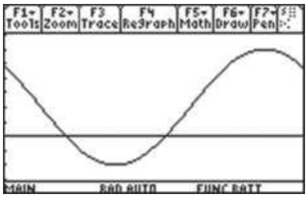
WORKED Example 7

THINK

On a GRAPH page, draw the graph of

$y1 = 4\cos\left(4\pi t + \frac{\pi}{4}\right) + 2\,.$

WRITE/DISPLAY



Chapter 6 page 223

WORKED Example 8

THINK | **WRITE/DISPLAY**

i ① On a HOME page, press CATALOG and select define. Complete the entry line as:

Define $x(t) = 4\ \sin(2t)$

then press (ENTER).

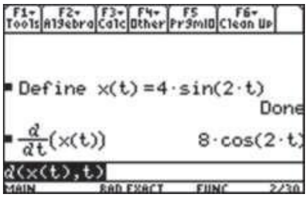
Press:

- (F3) (Calc)
- 1: d(differentiate.

Complete the entry line as:

$d(x(t),t)$

then press (ENTER).

i 

```
■Define x(t)=4·sin(2·t)
                      Done
■ d/dt(x(t))      8·cos(2·t
d(x(t),t)
MAIN        RAD EXACT     FUNC     2/30
```

② Write the solution.

$v(t) = 8\ \cos(2t)$

ii ① To solve $v(t) = 0$, press:

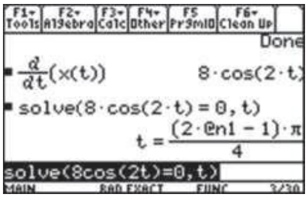
- (F2) (Algebra)
- 1: (Solve).

Complete the entry line as:

solve(8 cos(2t) = 0,t)

then press (ENTER).

The calculator has given the general solution to this equation.

ii 

```
                          Done
■ d/dt(x(t))      8·cos(2·t
■solve(8·cos(2·t)=0,t)
             t = (2·@n1 - 1)·π
                 ------------
                      4
solve(8cos(2t)=0,t)
MAIN        RAD EXACT     FUNC     3/30
```

② Write the solution.

Solve 8 cos(2t) = 0 for t.

$$t = \frac{(2k-1)\pi}{4}$$

When $k = 1$, $t = \frac{\pi}{4}$.

Therefore the particle first comes to rest after $\frac{\pi}{4}$ seconds.

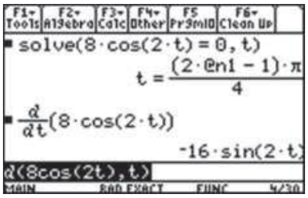
iii ① To find the acceleration, press:

- (F3) (Calc)
- 1: d(differentiate.

Complete the entry line as:

$d(8\ \cos(2t),t)$

then press (ENTER).

iii 

```
■solve(8·cos(2·t)=0,t)
             t = (2·@n1 - 1)·π
                 ------------
                      4
■ d/dt(8·cos(2·t))
                 -16·sin(2·t
d(8cos(2t),t)
MAIN        RAD EXACT     FUNC     4/30
```

② Write the solution.

$a(t) = -16\ \sin(2t)$

THINK

iv ① To solve $a(t) = 0$ for t, again use the solve function and complete the entry line as:

solve(–16 sin(2t) = 0,t)

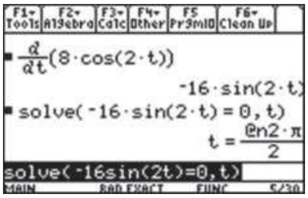
then press **ENTER**.

The calculator has given the general solution to this equation.

② Write the solution.

WRITE/DISPLAY

iv



```
■ d/dt(8·cos(2·t))
                    -16·sin(2·t)
■ solve(-16·sin(2·t) = 0, t)
                    t = @n2·π / 2
solve(-16sin(2t)=0,t)
```

Solve –16 sin(2t) = 0 for t.

$$t = \frac{k\pi}{2}$$

When $k = 0$, $t = 0$.

When $k = 1$, $t = \frac{\pi}{2}$.

The acceleration is 0 when $t = 0$ and again after $t = \frac{\pi}{2}$ seconds.

Chapter 7 page 239

WORKED Example 3

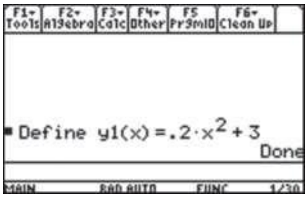
THINK

① To find the sum of the areas between the lines $x = 1$, $x = 3$, the x-axis and under the curve $f(x) = 0.2x^2 + 3$, on a HOME page, complete the entry line as:

Define $y1(x) = 0.2x^2 + 3$

then press ENTER.

WRITE/DISPLAY

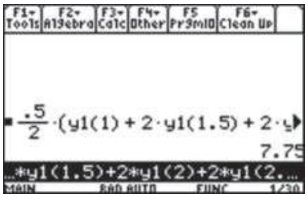


```
■Define y1(x)=.2·x²+3
                         Done
```

② To find the value of the area, complete the entry line as:

$$\frac{0.5}{2}(y1(1) + 2 \times y1(1.5) + 2 \times y1(2) + 2 \times y1(2.5) + y1(3))$$

then press ENTER.



```
■ .5/2·(y1(1)+2·y1(1.5)+2·y▸
                         7.75
...*y1(1.5)+2*y1(2)+2*y1(2...
```

③ Write the answer.

$$\begin{aligned}\text{Area} &= \frac{0.5}{2}(y1(1) + 2 \times y1(1.5) + 2 \times y1(2) \\ &\quad + 2 \times y1(2.5) + y1(3)) \\ &= 7.75\end{aligned}$$

Chapter 7 page 253

WORKED Example 9

THINK

① To antidifferentiate $\frac{5}{2x+3}$, on a HOME page, press:

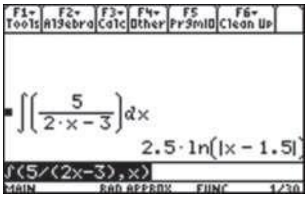
- F3 (Calc)
- 2: ∫(integrate.

Complete the entry line as:

$\int((5/(2x - 3)),x)$

then press ENTER.

WRITE/DISPLAY



```
■∫(5/(2·x-3))dx
                 2.5·ln(|x-1.5|)
∫(5/(2x-3),x)
```

② Write the answer.

$$\int \frac{5}{2x+3}\,dx = \frac{5}{2}\log_e(2x+3) + c$$

Note: The TI-89 Titanium calculator does not include the constant of integration in the integral.

Chapter 7 page 255

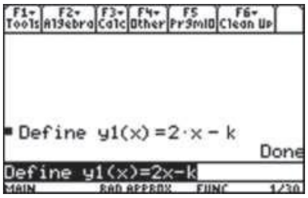
WORKED Example 12

THINK

WRITE/DISPLAY

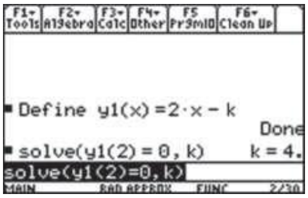
i ① To define the gradient function, on a HOME page, press CATALOG and select define.
Complete the entry line as:
Define $y1(x) = 2x - k$
then press ENTER.

i



```
■Define y1(x)=2·x−k
                      Done
Define y1(x)=2x-k
```

② To find the value of k, on a HOME page, complete the entry line as:
solve($y1(2) = 0,k$)
then press ENTER.



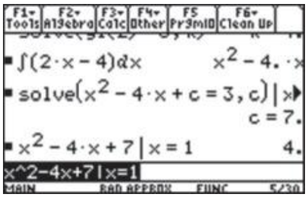
```
■Define y1(x)=2·x−k
                      Done
■solve(y1(2)=0,k)    k=4.
solve(y1(2)=0,k)
```

③ Write the answer.

Let $f(x) = 2x - k$.
Then solve($f(2) = 0, k$))
$k = 4$

ii ① To find y when $x = 1$, on a HOME page, complete the entry lines as shown.
$\int 2\text{x} - 4 \text{ dx}$
solve($x^2 - 4x + c = 3, c$) | $x = 2$
$x^2 - 4x + 7$ | $x = 1$
pressing ENTER after each line.

ii



```
■∫(2·x−4)dx          x²−4.·x
■solve(x²−4·x+c=3,c)|x▸
                         c=7.
■x²−4·x+7|x=1            4.
x^2-4x+7|x=1
```

② Write the solution.

$\int 2x - 4 \text{ dx} = x^2 - 4x + c$
Solving $x^2 - 4x + c$ for c when $x = 2$, gives $c = 7$
$y = x^2 - 4x + 7$
$y(1) = 4$

Chapter 7 page 260

WORKED Example 15

THINK

WRITE/DISPLAY

① To find $\int 2e^{4x} - 5\sin 2x + 4x)\,dx$, on a HOME page, press:

- F3 (Calc)
- 2: $\int$(integrate.

Complete the entry line as:

$\int((2e\hat{}(4x) - 5\sin(2x) + 4x),x)$

then press ENTER.

```
∫(2·e^(4·x) − 5·sin(2·x) + 4·▸
.5·e^(4·x) + 2.5·cos(2.·x) + ▸
…(2e^(4x)−5sin(2x)+4x),x)
```

② Write the answer, including the constant of integration.

$$\int 2e^{4x} - 5\sin 2x + 4x)\,dx$$
$$= \frac{1}{2}e^{4x} + \frac{5}{2}\cos 2x + 2x^2 + c$$

Chapter 8 page 279

Graphics Calculator tip! Finding definite integrals

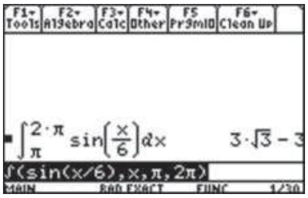
It is possible to find the numerical value for the definite integral using the graphics calculator. Use the following steps to evaluate $\int_{\pi}^{2\pi} \sin\left(\frac{x}{6}\right) \mathrm{d}x$.

1. On a HOME page, press:
   - (F3) (Calc)
   - 2: $\int$(integrate.

   Complete the entry line as:

   $\int$(sin(x/6),x,π,2π)

   then press (ENTER).
   To obtain an approximate value for the integral, press;
   - 2nd
   - (ENTER).
2. Write the answer.



```
∫(sin(x/6),x,π,2π)
```

$\int_{\pi}^{2\cdot\pi} \sin\left(\frac{x}{6}\right) dx$ $\quad 3\cdot\sqrt{3}-3$

$$\int_{\pi}^{2\pi} \sin\left(\frac{x}{6}\right) \mathrm{d}x = 2.196$$

Chapter 8 page 280

WORKED Example 3

THINK

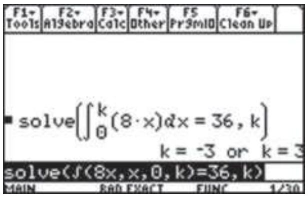
① On a HOME page, press:
- (F2) (Algebra)
- 1: (Solve)
- (F3) (Calc)
- 2: $\int$(integrate.

Complete the entry line as:

solve($\int$(8,x,0,k) = 36,k)

then press (ENTER).

WRITE/DISPLAY



```
solve(∫(8x,x,0,k)=36,k)
```

$\text{solve}\left(\int_{0}^{k}(8\cdot x)dx = 36, k\right)$

k = ⁻3 or k = 3

② Write the answer.

Solving $\int_{0}^{k} (8x)\ \mathrm{d}x = 36$ for k gives $k = \pm 3$.

Chapter 8 page 285

Graphics Calculator tip! Evaluating signed areas

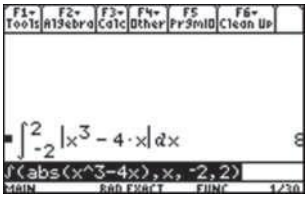
To evaluate $\int_{-2}^{0}(x^3-4x)\,dx-\int_{0}^{2}(x^3-4x)\,dx$ using a graphics calculator, proceed as follows.

1. On a HOME page, press:
   - F3 (Calc)
   - 2: $\int$(integrate.

   Complete the entry line as:

   $\int(abs(x^3-4x),x,-2,2)$

   then press ENTER.



```
■ ∫ -2..2 |x^3 - 4·x| dx        8
∫(abs(x^3-4x),x,-2,2)
```

2. Write the answer.

$$\int_{-2}^{0}(x^3-4x)\,dx-\int_{0}^{2}(x^3-4x)\,dx=8$$

Chapter 8 page 291

Graphics Calculator tip! Finding the area bounded by a graph and the x-axis

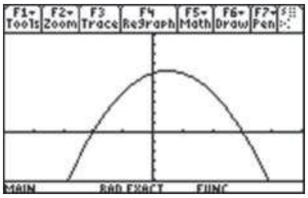
The area bounded by the graph and the x-axis can be drawn and evaluated using the graphics calculator.

1. Press APPS and select ♦ [Y=] . Complete the entry line as:

   $y1 = (3 - x)(x + 2)$

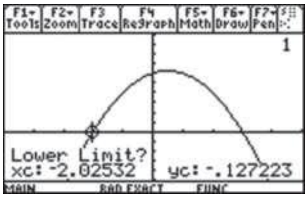
   then press ENTER.

   Set appropriate window settings, then press ♦ [GRAPH].



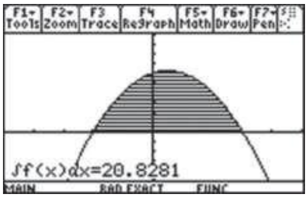
2. Press:
   - F5 (Math)
   - 7: $\int(f(x)$ dx.

   Use the arrow keys to move to the lower limit and then press ENTER. Repeat for the upper limit. Note that you may be only able to get close to the limits of integration, as shown at right.



```
Lower Limit?
xc: -2.02532    yc: -.127223
```

3. The required area will be shaded and the numeric value of the area will be displayed.



```
∫f(x)dx=20.8281
```

Chapter 8 page 293

WORKED Example 9

THINK

i ① To find the x-intercepts of $y = \sin 2x$ over the domain $0 \le x \le 2\pi$, on a HOME page, complete the entry line as:

solve(sin(2x) = 0,x | $x \ge 0$ and $x \le 2\pi$

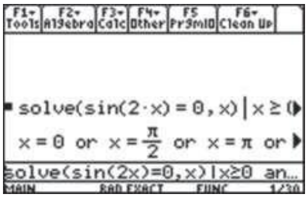
then press ENTER.

Note: Calculator is in Radian Mode.

② Write the solution.

WRITE/DISPLAY

i



■ solve(sin(2·x) = 0, x) | x ≥ 0

x = 0 or x = $\frac{\pi}{2}$ or x = π or

solve(sin(2x)=0,x)|x≥0 an…

MAIN RAD EXACT FUNC 1/30

Solving $\sin 2x = 0$ for x over the domain $0 \le x \le 2\pi$ gives

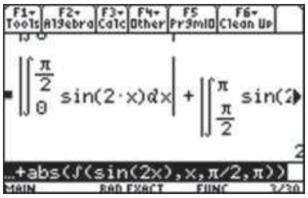
$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$

ii ① To calculate the area between the curve, the x-axis and $x = 0$ and $x = \pi$, on a HOME page, complete the entry line as:

$$\text{abs}\int_0^{\frac{\pi}{2}}(\sin(2x))\,dx) + \text{abs}\int_{\frac{\pi}{2}}^{\pi}(\sin(2x))\,dx.$$

② Write the answer.

ii



■ $\left|\int_0^{\frac{\pi}{2}} \sin(2\cdot x)dx\right| + \left|\int_{\frac{\pi}{2}}^{\pi}\right.$ sin(2

2

…+abs(∫(sin(2x),x,π/2,π))

MAIN RAD EXACT FUNC 2/30

$$\left|\int_0^{\frac{\pi}{2}}(\sin(2x))\,dx\right| + \left|\int_{\frac{\pi}{2}}^{\pi}(\sin(2x))\,dx\right|$$

= 2 square units.

Chapter 8 page 300

WORKED Example 12

THINK

i ① To find where the two curves intersect, on a HOME page, press:

- (F2) (Algebra)
- 1: (Solve).

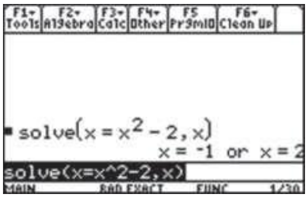
Complete the entry line as:

solve($x = x^2 - 2, x$)

then press (ENTER).

WRITE/DISPLAY

i



```
■ solve(x = x² - 2, x)
                  x = ⁻1 or x = 2
solve(x=x^2-2,x)
```

② Write the answer.

Solving $x = x^2 - 2$ for x gives $x = -1$ or $x = 2$.

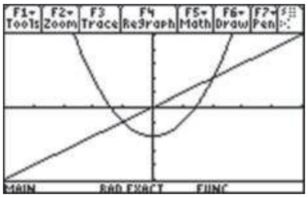
ii To sketch the graphs of $y = x$ and $y = x^2 - 2$ on a GRAPHS page, complete the entry lines as:

$f1(x) = x$

$f2(x) = x^2 - 2$

pressing (ENTER) after each line.

ii

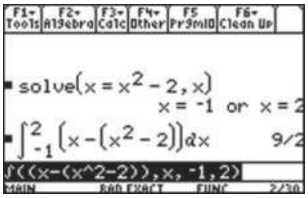


iii ① To find the area bounded by the curves, on a HOME page, complete the entry line as:

$\int(x - (x^2 - 2), x, -1, 2)$

then press (ENTER).

iii



```
■ solve(x = x² - 2, x)
                  x = ⁻1 or x = 2
■ ∫²₋₁(x-(x² - 2))dx           9/2
∫((x-(x^2-2)),x,⁻1,2)
```

② Write the answer.

$\int_{-1}^{2} (x - (x^2 - 2))\ \mathrm{d}x = \frac{9}{2}$ square units.

Chapter 9 page 340

Graphics Calculator tip! Binomial probabilities

To calculate the probabilities of a binomial distribution X, where the number of trials (n) is 4 and the probability of a 'success' in each trial (p) is 0.5, use the following steps.

1. To enter the possible values of x, on a Stats/List Editor page, enter the values for x: 0, 1, 2, 3 and 4 in List 1.

list1	list2	list3	list4
0	------	------	------
1			
2			
3			
4			
------			

list2[1]=

2. To calculate $P(X = x)$, press:
   - F5 (Distr)
   - B: (Binomial Pdf)

   Complete the fields as shown and press ENTER and then ENTER.

Binomial Pdf...

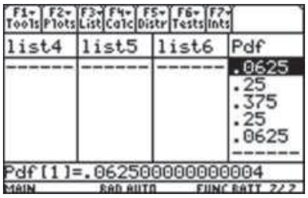
Num Trials, n: 4

Prob Success, p: .5

X Value:

list2[1]=

3. The probability distribution will appear in a column labeled Pdf.



list4	list5	list6	Pdf
------	------	------	.0625
			.25
			.375
			.25
			.0625

Pdf[1]=.062500000000004

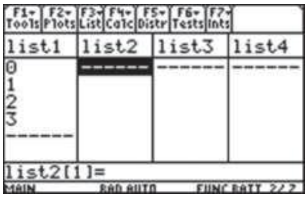
Chapter 9 page 344

Graphics Calculator tip!

Binomial probability for a range of *x*-values

The following steps show how to calculate the P($X \leq x$) for all x-values in a binomial distribution involving 3 trials, with the probability of ‘success’ in each trial of 0.75.

1. To enter the possible values of x, on a Stats/List Editor page, enter the values for x: 0, 1, 2 and 3 in List 1.

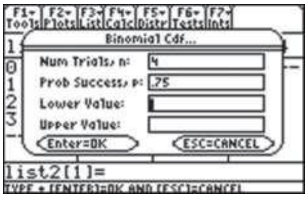


list1	list2	list3	list4
0			
1			
2			
3			

list2[1]=

2. To calculate $P(X \leq x)$, press:
   - F5 (Distr)
   - C: (Binomial Cdf).

   Complete the fields as shown and press ENTER and then ENTER.



Binomial Cdf...

Num Trials, n: 4

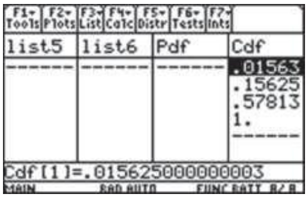
Prob Success, p: .75

Lower Value:

Upper Value:

list2[1]=

3. The probability distribution will appear in a column labeled Cdf.



list5	list6	Pdf	Cdf
			.01563
			.15625
			.57813
			1.

Cdf[1]=.015625000000003

Chapter 9 page 353

WORKED Example 20

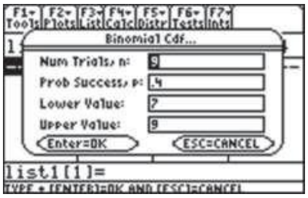
THINK | **WRITE/DISPLAY**

i ① To calculate P(X ≥ 7), on a Stats/List Editor page, press:

- F5 (Distr)
- C: (Binomial Cdf).

Complete the fields as shown, then press ENTER.

i



Binomial Cdf...

Num Trials, n: 9

Prob Success, p: .4

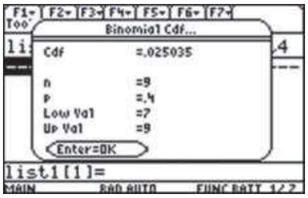
Lower Value: 7

Upper Value: 9

list1[1]=

TYPE + [ENTER]=OK AND [ESC]=CANCEL

② The value is displayed.



Binomial Cdf...

Cdf =.025035

n =9

p =.4

Low Val =7

Up Val =9

list1[1]=

MAIN RAD AUTO FUNC BATT 1/2

③ Write the solution.

$P(X \geq 7) = 0.0250$

ii ① To calculate $P(X > 7 \mid X > 5) = \frac{P(X > 7)}{P(X > 5)}$, on a Stats/List Editor page, press:

- F5 (Distr)
- C: (Binomial Cdf).

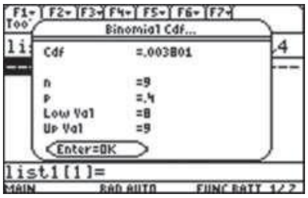
Complete the fields with Low Val = 8 and Up Val = 9, then press ENTER.

$P(X > 7) = 0.003801$

Repeat with Low Val = 6 and Up Val = 9.

$P(X > 5) = 0.099353$

ii



Binomial Cdf...

Cdf =.003801

n =9

p =.4

Low Val =8

Up Val =9

list1[1]=

MAIN RAD AUTO FUNC BATT 1/2

② Write the solution.

$$P(X > 7 \mid X > 5) = \frac{P(X > 7)}{P(X > 5)}$$

$$= \frac{0.003801}{0.099353} = 0.0383$$

Chapter 10 page 378

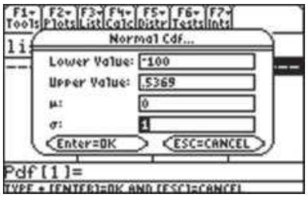
Graphics Calculator tip!

Calculating probabilities for standard normal distributions

Alternatively, for part **d** of Worked example 1, the normal cumulative distribution function of a graphics calculator may be used as follows.

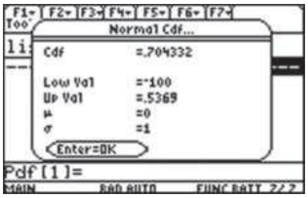
1. To calculate P($Z \leq 0.5369$), on a Stats/List Editor page, press:
   - (F5) (Distr)
   - 4: (Normal Cdf).

   Complete the fields as shown.



```
Lower Value: ⁻100
Upper Value: .5369
μ: 0
σ: 1
Pdf[1]=
TYPE + [ENTER]=OK AND [ESC]=CANCEL
```

2. Press (ENTER).



```
Cdf       =.704332

Low Val   =⁻100
Up Val    =.5369
μ         =0
σ         =1
Pdf[1]=
MAIN      RAD AUTO      FUNC BATT 2/2
```

3. Write the solution.

P($Z \leq 0.5369$) = 0.7043

Chapter 10 page 382

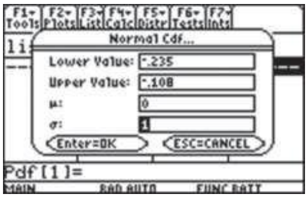
Graphics Calculator tip!

Calculating probabilities for standard normal distributions

Alternatively, for part **c** of Worked example 4, the normal cumulative distribution function of a graphics calculator may be used as follows.

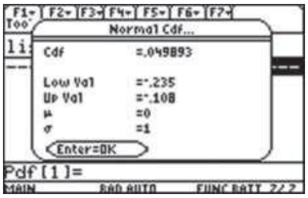
1. To calculate P(–0.235 ≤ Z ≤ –0.108), on a Stats/List Editor page, press:
   - (F5) (Distr)
   - 4: (Normal Cdf).

   Complete the fields as shown.



```
Lower Value: ⁻.235
Upper Value: ⁻.108
μ:           0
σ:           1
Pdf[1]=
```

2. Press (ENTER).



```
Cdf          =.049893

Low Val      =⁻.235
Up Val       =⁻.108
μ            =0
σ            =1
Pdf[1]=
```

3. Write the solution.

P(–0.235 ≤ Z ≤ –0.108) = 0.0499

Chapter 10 page 385

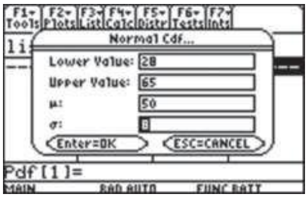
Graphics Calculator tip!

Calculating probabilities for normal distributions

Alternatively, for part **b** of Worked example 5, the normal cumulative distribution function of a graphics calculator may be used as follows.

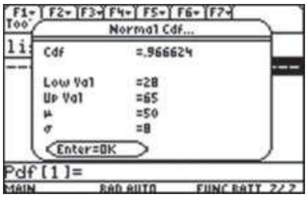
1. To calculate P($28 \le X \le 65$), with $\mu = 50$ and $\sigma = 8$, on a Stats/List Editor page, press:
   - F5 (Distr)
   - 4: (Normal Cdf).

   Complete the fields as shown.



Lower Value: 28
Upper Value: 65
μ: 50
σ: 8
Pdf[1]=

2. Press ENTER.



Cdf =.966624
Low Val =28
Up Val =65
μ =50
σ =8
Pdf[1]=

3. Write the solution.

P($28 \le X \le 65$) = 0.9666
with $\mu = 50$ and $\sigma = 8$

Chapter 10 page 393

Graphics Calculator tip!

The inverse cumulative normal distribution

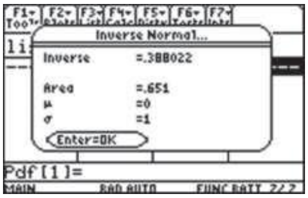
A graphics calculator can be used to determine the value of z if the probability is known.

1. To find c if $P(Z \leq c) = 0.651$, on a Stats/List Editor page, press:
   - F5 (Distr)
   - 2: (Inverse)
   - 1: (Inverse Normal).

   Complete the fields as shown.

Area: .651
μ: 0
σ: 1

2. Press ENTER.



Inverse =.388022

Area =.651
μ =0
σ =1

3. Write the solution.

For $P(Z \leq c) = 0.651$,
$c = 0.3880$

Answers

CHAPTER 1 Modelling change and rates of change

Exercise 1A — Using functions to model change

1 **a** 2 **b** $(3x - 5)^2 + 1$
c 10 **d** $3x^2 - 2$

2 **a** 36 **b** 82
c $(1 + t)^4 + 1$ **d** $(t^2 + 2)^2$

3 **a** 5 **b** $\frac{4 - y}{2}$

4 **a** $\frac{2}{3}$ **b** $\frac{2}{3}$ **c** $\frac{2}{y}$

5 **a** 24 **b** 38.375 **c** 58

6 **a** 3.55 **b** −1.61

7 **a** 0.00 **b** 2.00

8 (−0.73, 2.73) (2.73, −0.73)

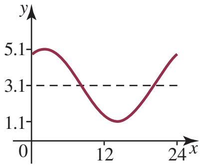
9 **a**

4	80
5	75
6	60
7	35

b 0.26, 7.74.

10 **a** $7.5t - 40$ **c** Yes, at 6.6 s

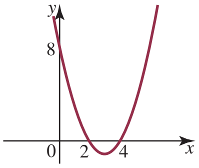
11 **a**



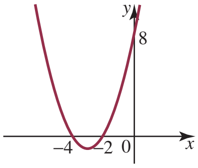
b Between 9.34 and 19.02 hours after midnight

Exercise 1B — Graphing polynomial functions

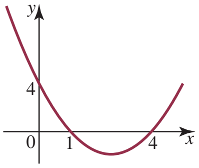
1 **a**



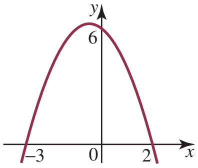
b



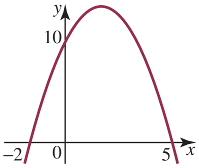
c



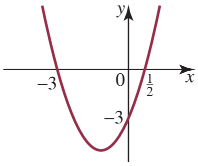
d



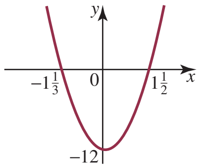
e



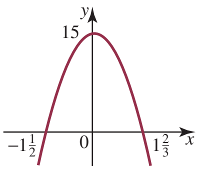
f



g



h



2 **a** (3, −1) **b** (−3, −1)
c $(\frac{5}{2}, -\frac{9}{4})$ **d** $(-\frac{1}{2}, 6\frac{1}{4})$
e $(1\frac{1}{2}, 12\frac{1}{4})$ **f** $(-1\frac{1}{4}, -6\frac{1}{8})$
g $(\frac{1}{12}, -12\frac{1}{24})$ **h** $(\frac{1}{12}, 15\frac{1}{24})$

3 **a** $y = x^2 - 2x - 1$ **b** $y = -x^2 - 6x - 7$
c $y = x^2 - 10x + 25$ **d** $y = 6 - x^2$
e $y = x^2 - 4x + 1$

4 **a** B **b** C

5 **a** 8 m^3 **b** 80 m^3

6 **a** 48 m **b** 6 s

7 **a** $y = x(x + 6)(x - 5)$ **b** $y = (x + 2)^2(x - 1)$

8 **a** **v** **b** **iv** **c** **ii** **d** **i**
e **vi** **f** **viii** **g** **vii** **h** **iii**

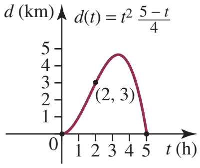
9 C **10** A

11 $a = 6, b = -24$

12 **a** $a = \frac{1}{4}, b = 5$

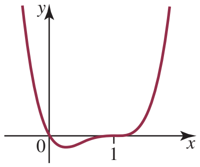
b $d(t) = t^2\frac{(5 - t)}{4}$, domain = [0, 5]

c

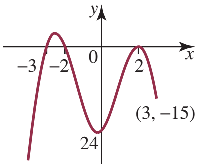


d Max. $d \approx 4.6$ km when $t = 3$ h 20 min

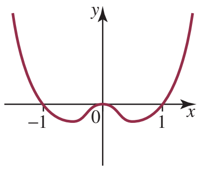
13 **a**



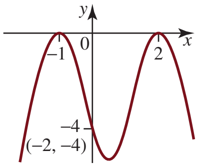
b



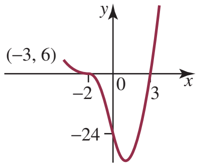
c



d



e

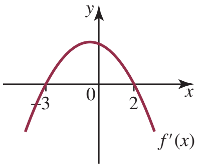


14 B

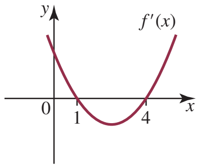
15 $a = 4, b = -19$

Exercise 1C — Review of differentiation

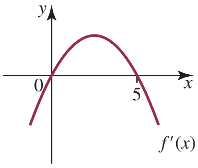
1 **a**



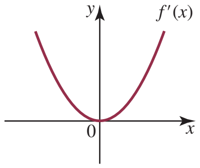
b



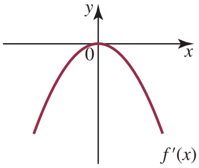
c



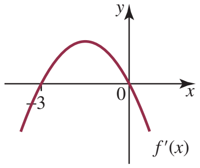
d



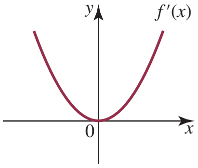
e



f



g



2 $4 + h$
3 E
4 **a** h **b** 0
5 C
6 **a** 3 **b** $2x$ **c** $2x + 6$
d $2x - 2$ **e** $-6x$ **f** $3x^2$
7 **a** -4 **b** $2x + 3$ **c** $2x - 2$
d $6x + 8$ **e** $3x^2 - 4$ **f** $5 - 6x^2$
g $\frac{2x}{3}$ **h** $-2x - 2$
8 **a** $\frac{dy}{dx} = 6x^5$ **b** $\frac{dy}{dx} = 6x$
c $\frac{dy}{dx} = 20x^3$ **d** $\frac{dy}{dx} = 20x^{19}$
e $\frac{dy}{dx} = -12x^2$ **f** $\frac{dy}{dx} = -5$
g $\frac{dy}{dx} = \frac{3}{2}x^2$ **h** $\frac{dy}{dx} = \frac{4}{3}x^3$
i $\frac{dy}{dx} = 0$ **j** $\frac{dy}{dx} = 40x^4$
9 **a** $f'(x) = 12x^2 + 5$ **b** $g'(x) = -10x + 6$
c $h'(x) = \frac{3}{5}x^2$ **d** $h'(x) = -3 + 12x + 3x^2$
e $g'(x) = 77x^{10} + 30x^4$ **f** $f'(x) = 2x^4 + x^2$
g $f'(x) = -6 + 6x - 12x^2$ **h** $g'(x) = 14x - 4$
i $h'(x) = 2x + 3$ **j** $f'(x) = 9x^2 - 12$
10 **a** $-\frac{6}{x^4}$ **b** $\frac{3}{2\sqrt{x}}$
c $\frac{1}{3x^{\frac{2}{3}}}$ **d** $5x^{\frac{1}{4}}$
e $\frac{1}{2\sqrt{x}} - 4x$ **f** $-\frac{1}{x^2} + 2x$
g $-\frac{1}{2}x^{-\frac{3}{2}} + \frac{2}{3}x^{-\frac{1}{3}}$ **h** $-\frac{3}{x^2}$
i $1 + 2x$ **j** $-\frac{3}{4x^2}$
k $-\frac{4}{5x^3}$ **l** $-\frac{1}{x^{\frac{3}{2}}} - \frac{6}{x^3}$
m $x^2 - 4 - 3x^{-4}$ **n** $\frac{1}{2\sqrt{x}} - \frac{1}{4x^{\frac{5}{4}}}$
11 **a** $10x^4 - 10$ **b** 150
12 D
13 C
14 $3\frac{1}{4}$
15 **a** $-\frac{5}{8}$ **b** Undefined
16 **a** $3\frac{1}{2}$ **b** $87\frac{15}{16}$ **c** $467\frac{53}{54}$
17 **a** $g'(x) = \frac{1}{3x^{\frac{2}{3}}} + 4$ **b** $4\frac{1}{3}$
c $4\frac{1}{12}$ **d** $4\frac{1}{12}$
19 **a** **i** $x^2 + 2x + 1$ **ii** $2x + 2$
iii $2(x + 1)$
b **i** $x^3 + 3x^2 + 3x + 1$ **ii** $3x^2 + 6x + 3$
iii $3(x + 1)^2$
c **i** $4x^2 + 4x + 1$ **ii** $8x + 4$
iii $4(2x + 1)$
d **i** $8x^3 + 12x^2 + 6x + 1$ **ii** $24x^2 + 24x + 6$
iii $6(2x + 1)^2$
e **i** $9x^2 + 6x + 1$ **ii** $18x + 6$
iii $6(3x + 1)$
f **i** $27x^3 + 27x^2 + 9x + 1$ **ii** $81x^2 + 54x + 9$
iii $9(3x + 1)^2$
20 $na(ax + b)^{n-1}$

Exercise 1D — Rules for differentiation

1 **a** **i** $5x - 4$ **ii** 3
b **i** $3x + 1$ **ii** $\frac{1}{2}$
c **i** $2x + 3$ **ii** -4
d **i** $7 - 4x$ **ii** -1
e **i** $5x + 3$ **ii** -6
f **i** $4 - 3x$ **ii** $\frac{4}{3}$
2 B
3 **a** **i** $2u$ **ii** 3 **iii** $6(3x + 2)$
b **i** $3u^2$ **ii** -1 **iii** $-3(7 - x)^2$
c **i** $\frac{-1}{u^2}$ **ii** 2 **iii** $-\frac{2}{(2x-5)^2}$
d **i** $\frac{-4}{u^5}$ **ii** -2 **iii** $\frac{8}{(4-2x)^5}$
e **i** $\frac{1}{2\sqrt{u}}$ **ii** 5 **iii** $\frac{5}{2\sqrt{5x+2}}$
f **i** $\frac{-3}{2u^{\frac{3}{2}}}$ **ii** 3 **iii** $\frac{-9}{2(3x-2)^{\frac{3}{2}}}$

g **i** $15u^4$ **ii** $4x + 5$ **iii** $15(4x + 5)(2x^2 + 5x)^4$

h **i** $-2u^{-3}$ **ii** $4 - 6x$ **iii** $-4(2 - 3x)(4x - 3x^2)^{-3}$

i **i** $6u^5$ **ii** $\dfrac{x^2 - 1}{x^2}$ **iii** $\dfrac{6(x^2 - 1)\left(x + \frac{1}{x}\right)^5}{x^2}$

j **i** $-16u^{-5}$ **ii** -6 **iii** $96(5 - 6x)^{-5}$

4 C **5** A **6** B

7 **a** $32(8x + 3)^3$ **b** $6(2x - 5)^2$

c $-15(4 - 3x)^4$ **d** $\dfrac{3x}{\sqrt{3x^2 - 4}}$

e $\frac{2}{3}(x - 2)(x^2 - 4x)^{-\frac{2}{3}}$ **f** $-2(6x^2 + 1)(2x^3 + x)^{-3}$

g $6\left(1 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)^5$ **h** $-(2x - 3)(x^2 - 3x)^{-2}$

8 $\dfrac{-2}{\sqrt{(4x + 7)^3}}$

9 **a** $\dfrac{-3}{(6x - 5)^{\frac{3}{2}}}$ **b** $3x\sqrt{x^2 + 2}$

10 **a** $8x^7(2x + 5)(x + 5)^7$ **b** $2x(3x^2 - 2)(x^2 - 2)$

c $\dfrac{3x^2 + 4x}{5(x^3 + 2x^2 - 7)^{\frac{4}{5}}}$

d $3x(4x^2 - 3)\sqrt{2x^4 - 3x^2 + 1}$

11 37 500

12 $\frac{7}{500}$ or 0.014

13 **a** 2 **b** $\dfrac{x - 1}{\sqrt{x^2 - 2x + 1}}$

c 1 **d** 1

14 $\frac{7}{4}$

15 $\frac{2}{9}$

16 **a** $u = x + 3,\ v = 2x^2 - 5x$

b $\dfrac{du}{dx} = 1, \dfrac{dv}{dx} = 4x - 5$

c $\dfrac{dy}{dx} = 6x^2 + 2x - 15$

17 **a** $x(5x + 2)(x + 1)^2$

b $(x + 1)(5x + 3)x^2$

c $\dfrac{1}{2\sqrt{x}}(x + 1)^4(11x + 1)$

d $\frac{3}{2}\sqrt{x}(x - 2)^2(3x - 2)$

e $-(x + 1)(x - 1)^{-3}$

f $\dfrac{1}{2\sqrt{x + 1}}(3x + 2)$

18 **a** $u = x + 3,\ v = x + 7$

b $\dfrac{du}{dx} = 1\ , \dfrac{dv}{dx} = 1$

c $\dfrac{dy}{dx} = \dfrac{4}{(x + 7)^2}$

19 **a** $u(x) = x^2 + 2x,\ v(x) = 5 - x$

b $u'(x) = 2x + 2,\ v'(x) = -1$

c $f'(x) = \dfrac{-x^2 + 10x + 10}{(5 - x)^2}$

20 **a** $\dfrac{-2}{(x - 4)^2}$ **b** $\dfrac{3x^2 + 4x - 4}{(3x + 2)^2}$

c $\dfrac{33}{(10 - x)^2}$ **d** $-\dfrac{(x^2 + 15)}{2x^{\frac{5}{2}}}$

21 D

22 **a** $(x^2 + 1)^2(7x^2 + 1)$ **b** $\dfrac{-1}{\sqrt{x}(\sqrt{x} - 1)^2}$

c $\dfrac{(x^2 + 1)(5x^2 - 1)}{x^2}$ **d** $\dfrac{-10x}{(x^2 - 3)^6}$

e $\dfrac{(x + 1)^2(5x^2 - 8x - 1)}{2\sqrt{x}(x - 1)^2}$

Chapter review

1 **a** 996 **b** $(4x + 2)^3 - 4$

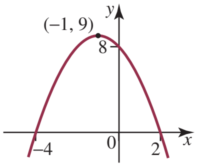
c 18 **d** $4x^3 - 14$

2 C

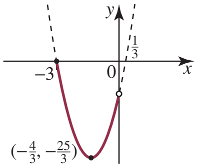
3 **a** 2 **b** $\dfrac{y + 2}{4}$

4 A

5



6



7 E

8 D

9 A

10 **a** $3x^2 + 2$ **b** 5

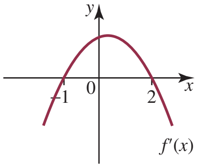
11 **a** $x^2 - 4$ **b** 5

12 **a** $6x^3 + \frac{3}{4}x^2 - 3$ **b** **i** $-8\frac{1}{4}$ **ii** 48

13 A

14 C

15



16 **a** $-6x \times (4 - x^2)^2$ **b** $6x(x + 1)(x + 3)^3$

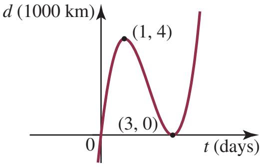
c $\dfrac{x^4 + 3x^2}{(x^2 + 1)^2}$

17 D **18** B **19** E **20** D

Modelling and problem solving

1 a 0 km

b



c $d = t^3 - 6t^2 + 9t$

d The satellite by 40 000 km

e 10 days

f Domain = [0, 10], range = [0, 490]

2 a C is (3, 0) and D is (2.25, −8.54)

b $y = x^4 - 3x^3$

c 2 km

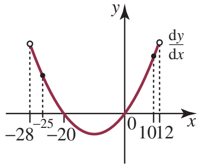
d Yes, because a straight route from O to D to C is approximately 17.4 km and the river course is longer than this.

3 a i $x = -20$ and $x = 0$

ii (−28, −20)∪(0, 12)

iii (−20, 0)

b



c $x = 10$

d $\frac{dy}{dx} = \frac{3x}{200}(x + 20)$

e i 1.875

ii −1.5

iii 4.5

f Yes, the largest absolute value of the gradient is 4.5, that is, the steepest section.

g 30.24 m (at $x = 12$)

CHAPTER 2 Applications of differentiation

Exercise 2A — Sketching curves

1 a (0, 8) a local max.

b (−1, 2) a local max., (1, −2) a local min.

c (2, −8) a local min.

d (−2, −8) a local min., $(\frac{2}{3}, 1\frac{13}{27})$ a local max.

e (0, 0) a positive point of inflection, (1, 1) a local max.

f (−2, 4) a local max., (0, 0) a local min.

g (3, −4) a local min.

h (0, 8) a positive point of inflection

i $(-\frac{1}{2}, 6\frac{1}{4})$ a local max.

j (0, 5) a local min., (1, 6) a positive point of inflection

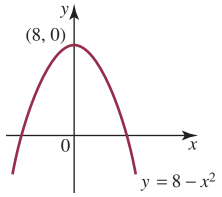
k (−3, 54) a local max., (3, −54) a local min.

l (−3, 16) a local max., $(\frac{1}{3}, -2\frac{14}{27})$ a local min.

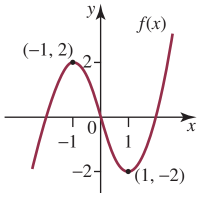
m (0, 12) a negative point of inflection

n (0, 0) a negative point of inflection, (3, −27) a local min.

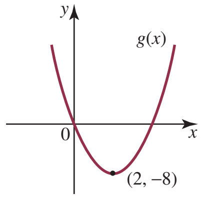
2 a



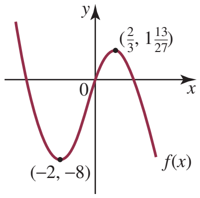
b



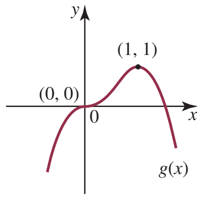
c



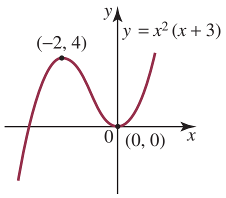
d



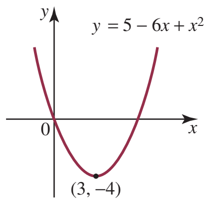
e



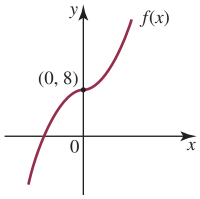
f



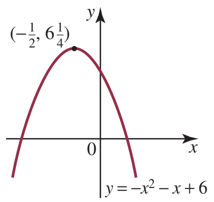
g



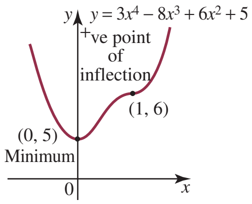
h



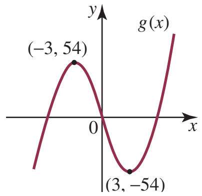
i



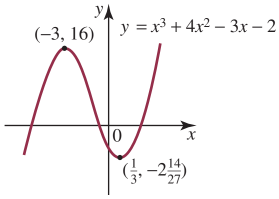
j



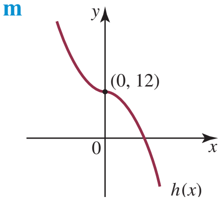
k



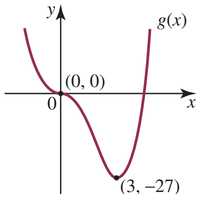
l



m

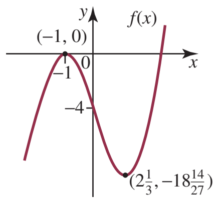


n



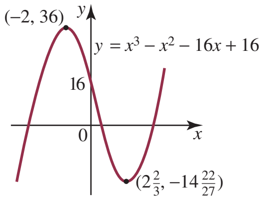
3 a (−1, 0) a local max., $(2\frac{1}{3}, -18\frac{14}{27})$ a local min.

b



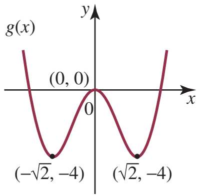
4 **a** $(-2, 36)$ a local max. and $(2\frac{2}{3}, -14\frac{22}{27})$ a local min.

b



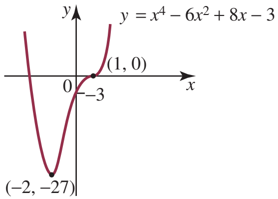
5 **a** $(-\sqrt{2}, -4)$ a local min., $(0, 0)$ a local max., $(\sqrt{2}, -4)$ a local min.

b



6 **a** $(-2, -27)$ a local min., $(1, 0)$ a positive point of inflection

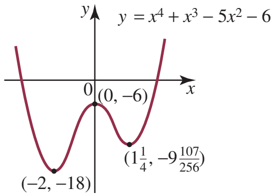
c



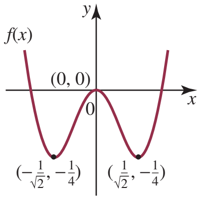
7 **a** $(-2, -18)$ a local min., $(0, -6)$ a local max., $(1\frac{1}{4}, -9\frac{107}{256})$ a local min.

b $(0, -6)$

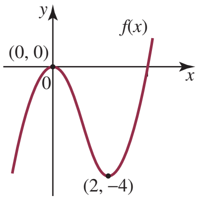
c



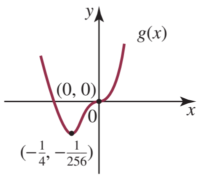
8 **a**



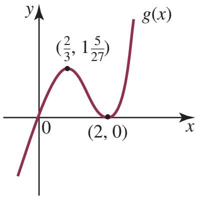
b



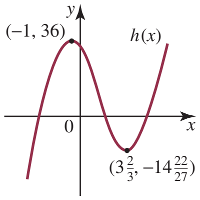
c



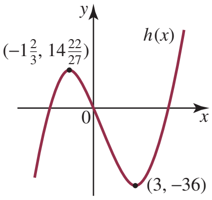
d



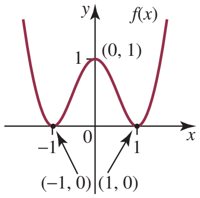
e



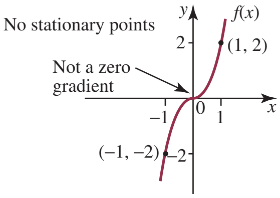
f



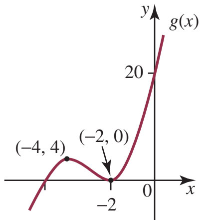
g



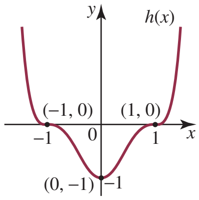
h



i



j



9 B **10** A **11** C **12** E

13 **a** $x = -3$ a local min., $x = 0$ a local max.

b $x = -2$ a local max., $x = 1$ a local min., $x = 4$ a local max.

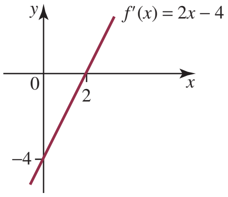
c $x = -2$ a negative point of inflection, $x = 3$ a local min.

d $x = -5$ a local min., $x = 2$ a positive point of inflection

e $x = -3$ a local max., $x = 0$ a local min., $x = 2$ a local max.

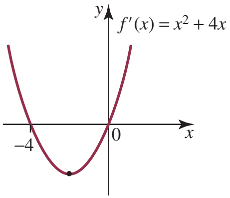
f $x = 1$ a local max., $x = 5$ a local min.

14



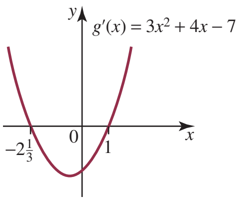
$f'(x) < 0$ if $x < 2$ and $f'(x) > 0$ if $x > 2$

15 **a** **i**



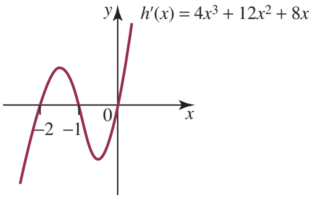
ii $x < -4$ and $x > 0$ **iii** $-4 < x < 0$

b **i**



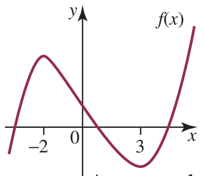
ii $x < -2\frac{1}{3}$ and $x > 1$ **iii** $-2\frac{1}{3} < x < 1$

c **i**



ii $-2 < x < -1$ and $x > 0$ **iii** $x < -2$ and $-1 < x < 0$

16



An example of the answer.

17 **a** $a = -1$ and $b = -8$

b At $x = -\frac{4}{3}$ a local max. and at $x = 2$ a local min.

18 **a** $a = -2$ and $b = 5$

b $(-1, 4)$ and $(0, 5)$

c $(-1, 4)$ a local min., $(0, 5)$ a local max. and $(1, 4)$ a local min.

Exercise 2B — Equations of tangents and normals

1 $y = 5x - 4$

2 At $(-6, 0)$ $y + 7x + 42 = 0$ and at $(1, 0)$ $y = 7x - 7$

3 $x + y = 3$

4 $x + 3y + 21 = 0$

5 **a** **i** $y = 2x$ **ii** $x + 2y = 5$

b **i** $y = 6x + 16$ **ii** $x + 6y = 22$

c **i** $x + 4y = 4$ **ii** $2y = 8x - 15$

d **i** $y = 7x + 1$ **ii** $x + 7y + 43 = 0$

e **i** $4y = x + 4$ **ii** $4x + y = 18$

f **i** $3y = x + 6$ **ii** $y + 3x = 12$

g **i** $x + y = 1$ **ii** $y = x + 3$

h **i** $y = 4x$ **ii** $x + 4y = 0$

i **i** $y = 2x - 3$ **ii** $x + 2y + 1 = 0$

6 **a** B **b** C

7 $y = 2x$

8 $y = -1$

Exercise 2C — Maximum and minimum problems when the function is known

1 **a** $\frac{dC}{dx} = 5 - 4x$

b $\frac{dP}{dn} = 3n^2 + 4n - \frac{5}{2\sqrt{n}}$

c $\frac{dV}{dh} = h^3 - 2h^2$

2 D

3 C

4 **a** 8 workers **b** \$1112

5 20 metres

6 $x = 1$ hour for max. and $x = 7$ hours for min.

7 **a** $x = -5$°C for min. and $x = 15$°C for max.

b Minimum is $N = 291$ rabbits (round down) and maximum is $N = 1625$ rabbits.

8 **a** **i** 16 cm **ii** 132 cm

b $R = 0.02t + 5.6$

c Max. = 6, min. = 5.6

9 **a** 75 **b** \$50 **c** \$3750

Exercise 2D — Maximum and minimum problems when the function is unknown

1 Both numbers are 5.

2 2 and 6

3 **b** $A = 60x - x^2$

c Length and width are each 30 cm.

d Max. area is 900 cm^2.

4 10 000 m^2

5 Radius $= \frac{40}{\pi}$ cm (or 12.73 cm), length = 40 cm, circumference = 80 cm

6 **a** $L = 4.5 - 3x$

b $V = 9x^2 - 6x^3$

c Edges are 1 m, 1.5 m and 2 m, for max.

d $V = 3$ m^3

7 **b** $V = 50x - \frac{1}{2}x^3$ **c** 192 cm^3

8 6298.9 cm^2

9 Approx. 6564 cm^3

10 $x = 1.5$ km

11 **a** $\frac{800}{v}$ hours

b $C = 40\,000\,v^{-1} + 0.8\,v$

c 223.61 km/h

12 Approx. 13.86 cm

13 1239 cm^3

14 $\sqrt{5}$ units

15 4.06 units

16 $\sqrt{7}$ million km.

Investigation — Cross-country run

1 $\sqrt{x^2 + 2700^2}$

2 $\frac{\sqrt{x^2 + 2700^2}}{2}$

3 $3800 - x + \sqrt{x^2 + 2700^2}$

4 $\frac{3800 - x}{5} + \frac{\sqrt{x^2 + 2700^2}}{2}$

5 1178.38 m

6 33.29 minutes

7 76 seconds longer

Exercise 2E — Rates of change

1 **a** $\frac{dV}{dr}$ **b** $\frac{dS}{dh}$ **c** $\frac{dA}{dt}$

d $\frac{dC}{dx}$ **e** $\frac{dI}{dp}$ **f** $\frac{dv}{dt}$

2 **a** $\frac{dA}{dr} = 2\pi r$ **b** 20π m^2/m

3 **a** $\frac{dV}{dr} = -4\pi r^2$ **b** -100π cm^3/cm

4 6 cm^2/cm

5 **a** 10 m **b** 5 m/s

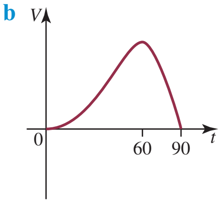
c $\frac{dh}{dt} = 20 - 10t$

d **i** 10 m/s **ii** −10 m/s

When $t = 1$ the projectile is rising but when $t = 3$ it is falling.

6 **a** $\frac{dv}{dt} = \frac{1}{100}(90t^2 - t^3)$

b



c $t = 60$ min

7 a $v = \dfrac{3t}{(3t^2+4)^{\frac{1}{2}}} = \dfrac{3t}{\sqrt{3t^2+4}}$

b $a = \dfrac{3}{\sqrt{3t^2+4}} - \dfrac{9t^2}{(\sqrt{3t^2+4})^3} = \dfrac{12}{(3t^2+4)^{\frac{3}{2}}}$

c $v = 1.5$, $a = \frac{3}{16}$

8 a $v = 3t^2 - 24t + 36$

b $t = 6$ $x = 0$ and $t = 2$ $x = 32$

c $a = 12$ and -12

9 a D **b** A

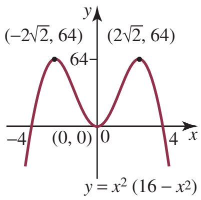
10 0.2 mL/h

Chapter review

1 C **2** E **3** A **4** D

5 (–3, 86) a local max. and (2, –39) a local min.

6



7 D **8** D

9 $2x + y = 6$

10 $y = 3x - 14$

11 $8x + 6y + 31 = 0$ and $8x + 6y = 31$

12 C **13** B **14** A **15** D

16 A **17** E **18** B **19** D

Modelling and problem solving

1 1430 bees

2 a $x = 12$ units **b** 16 sq. units

3 c $\dfrac{2000\pi\sqrt{3}}{27}$

4 a $A = 2\pi r^2 + \dfrac{100}{r}$ **b** $r = 2$ cm

c $A = 75.1$ cm^2 **d** \$3004

5 a A is (0.5, 1) and B is (–0.5, 1)

b $y = 4x - 1$ **c** (0, –1)

d 1 cm **e** 1.125 cm

CHAPTER 3 Exponential and logarithmic functions

Exercise 3A — The index laws

1 a x^7 **b** x^5 **c** x^{10} **d** $\dfrac{1}{x^6}$

e x^6 **f** x **g** $5x^4y^3$ **h** $6x^2y^3$

i $\dfrac{10x^4y^2}{3}$ **j** $\dfrac{27x^2y^{14}}{2}$

2 a 9 **b** 8 **c** $\frac{1}{125}$ **d** $\frac{1}{1000}$

e 3 **f** 216 **g** $\frac{3}{7}$ **h** $\frac{9}{16}$

i $\frac{8}{27}$ **j** $\frac{27}{64}$

3 a $\dfrac{3}{x^{11}y^2}$ **b** $\dfrac{x^6}{y^4}$ **c** $6x^{\frac{5}{4}}y^{\frac{5}{3}}$ **d** $\dfrac{20x}{y^{\frac{1}{4}}}$

e $\dfrac{243x^{\frac{5}{2}}}{y^2}$ **f** $8y^{\frac{3}{8}}$ **g** $\dfrac{3}{a^9b^3}$ **h** $\dfrac{2a^{\frac{15}{2}}b^{\frac{7}{2}}}{3}$

4 a 2^{6n-1} **b** 3^{6n+1} **c** $2^{2n} \times 3^{2n+1}$

d $2^n \times 3^{3n+1}$ **e** $\frac{2}{3}$ **f** 1

5 a $\dfrac{1+x^2}{x}$ **b** $\dfrac{(x+1)^2}{x^4}$ **c** $\dfrac{2x}{1-x^2}$ **d** $\dfrac{1}{x+y}$

6 a $\frac{4}{3}$ **b** $\frac{1}{243}$

7 B

8 a 1000 **b** 1414

9 E

10 a 10 m **b** 4.10 m **c** 49.04 m

Exercise 3B — Logarithms and laws of logarithms

1 a 0 **b** 0 **c** 1 **d** 1

e Undefined **f** Undefined

2 a $2^4 = 16$ **b** $3^3 = 27$ **c** $x^2 = 25$

d $x^6 = 64$ **e** $5^x = 125$ **f** $10^x = 10\,000$

g $3^5 = x$ **h** $2^7 = x$ **i** $5^{-1} = \frac{1}{5}$

j $7^{-2} = \frac{1}{49}$

3 a $\log_2 8 = 3$ **b** $\log_2 32 = 5$ **c** $\log_3 81 = 4$

d $\log_3 9 = 2$ **e** $\log_4 x = 3$ **f** $\log_4 x = 5$

g $\log_5 125 = x$ **h** $\log_5 25 = x$ **i** $\log_2\left(\frac{1}{2}\right) = -1$

j $\log_3 \frac{1}{9} = -2$ **k** $\log_x 27 = 3$ **l** $\log_p 16 = 4$

4 a 0.477 **b** 0.602 **c** −0.301 **d** −0.097

e 1.301 **f** 1.778 **g** 2.477 **h** 2.699

5 a 1 **b** 1 **c** 1 **d** 1

e 5 **f** 4 **g** −1 **h** −3

6 a $\frac{1}{2}\log_2 x$ **b** $\frac{1}{3}\log_3 x$ **c** $\log_3 x$

d $\log_4 x$ **e** $\log_2\left(\dfrac{x^2}{y}\right)$ **f** $\log_3\left(\dfrac{x}{y^2}\right)$

7 a 1 **b** 1 **c** 1 **d** 0 **e** 2

f 1 **g** 1 **h** 1 **i** 9 **j** $\frac{1}{4}$

8 a 4 **b** −3 **c** 3 **d** 1 **e** $\log_2 10$

f $\log_3 54$ **g** 2 **h** $\frac{3}{2}$ **i** $\frac{1}{2}$ **j** $\frac{2}{3}$

9 a 0 **b** 0 **c** 0

d 0 **e** $3\log_{10} x$ **f** $5\log_{10} x$

g $\log_5 (x+1)^3$ or $3\log_5 (x+1)$ **h** $\log_4 (x-2)$

10 a 5.322 **b** 2.570 **c** 0.849

11 C

12 E

13 31.623

14 0.699

15 7.8

Exercise 3C — Indicial equations

1 a P **b** S **c** R **d** Q

2 a S **b** B **c** Q **d** P

3 a 4 **b** 6 **c** −2 **d** −3 **e** −5

f −3 **g** −2 **h** −3 **i** $\frac{1}{5}$ **j** $\frac{1}{4}$

4 **a** 4 **b** 3 **c** 2 **d** 5 **e** –1

f –1 **g** 2 **h** 3

5 **a** 3 **b** 1 **c** 2 **d** $-\frac{1}{3}$ **e** –1

f $\frac{1}{6}$ **g** $-1\frac{1}{2}$ **h** $\frac{1}{7}$

6 **a** 2, 0 **b** 3, 1 **c** 2, 1 **d** 1, 0 **e** 0

f 3 **g** 1, 2 **h** 2, 3 **i** 5, 0 **j** 1

7 **a** 3 **b** 5 **c** 16 **d** 27 **e** 3

f 1, –2 **g** 2, –6 **h** 5

8 **a** 2.322 **b** 2.262 **c** –0.912 **d** 5.336 **e** –3.325

f 2.535 **g** 1.365 **h** 1.730 **i** –0.827 **j** –0.774

k 0.672 **l** 2.301

9 **a** $x > 1.465$ **b** $x < 0.431$ **c** $x \leq 1.404$

d $x \geq 0.431$ **e** $x < -0.683$ **f** $x > -1.151$

g $x \geq -0.356$ **h** $x \leq -0.285$ **i** $x < 1.756$

j $x > 0.152$

10 D

11 E

12 **a** \$13 382 **b** 8 **c** 11 years 11 months

13 **a** \$21 250

b 85%

c $V = 25\,000\,(0.85)^n$

d \$9428.74

e 24 years

f No. As n increases $(0.85)^n$ becomes smaller and smaller but never reaches zero. Value is always 85% of the value the previous year.

14 **a** $D = 20 \times 10^{0.04t}$ **b** 2 years and 5 months

c 5 years

Exercise 3D — Logarithmic equations using any base

1 **a** 25 **b** 81 **c** $\frac{1}{8}$ **d** $\frac{1}{16}$ **e** 100, –100

f 16 **g** 26 **h** 127 **i** 2 **j** 0

k $-\frac{1}{32}$ **l** $-\frac{1}{9}$ **m** –624 **n** –2.5

2 **a** 3 **b** 2 **c** 125 **d** 625 **e** 2

f 8 **g** 6 **h** 4

3 **a** 3 **b** 2 **c** –1 **d** –2 **e** $\frac{1}{2}$

f $\frac{1}{3}$ **g** 0 **h** 0 **i** –1 **j** –2

4 **a** 5 **b** 6 **c** 10 **d** 8 **e** 4

f 2 **g** 9 **h** $\frac{2}{5}$ **i** 500 **j** 128

5 **a** 5 **b** 6 **c** 1 **d** 2 **e** 3

f 2 **g** 8 **h** 5 **i** 4 **j** 3

6 B

7 D

8 **a** 10, $\frac{1}{100}$ **b** 1000, $\frac{1}{10}$ **c** 16, $\frac{1}{4}$ **d** 2, $\frac{1}{16}$

e 3, 27 **f** 25, 5 **g** 1, 16 **h** 1, 27

i –5, 3 **j** –2, 5

9 16

10 **a** $N = 200t^{1.5}$ **b** $N = 840t^{\frac{1}{2}}$

c $N = 283t^{1.188}$

11 **a** $A = 4.7$, $n = 2$, so $d = 4.7t^2$

b 230.3 m

12 **a** $A = 3000$, $n = -1.5$, so $H = 3000d^{-1.5}$

b $d = 4.64$ m

13 **a** 120 **b** 130

c 0.001 **d** 3 decibels are added.

e 10 decibels are added. **f** 100

Exercise 3E — Exponential equations (base *e*)

1 **a** 7.389 **b** 54.598 **c** 1.649 **d** 1.396

e 1.284 **f** 1.221 **g** 1.386 **h** 1.609

i 0.405 **j** 1.281

2 **a** 1 **b** 2 **c** 6 **d** $-\frac{1}{2}$

e 2 **f** 0 **g** $-\frac{11}{6}$ **h** $1\frac{1}{4}$

3 **a** 0.693 **b** 1.609 **c** –0.693 **d** –1.386

e 0.262 **f** 0.956 **g** 1.099 **h** 1.386

i 0.875 **j** 0.916

4 **a** 0 **b** 0 **c** –0.693, 0.549

d 0.405, –0.693 **e** 1.386 **f** –0.405

g 0 **h** 1 **i** 0.693, 1.609

j 0.916 **k** –0.405, 1.099 **l** 0.251, 0.693

5 **a** 0.693 **b** 1.609 **c** 1.609

d 1.099 **e** –0.405, 1.099 **f** –0.288, 0.693

g 0, 0.693 **h** 1.609

6 **a** $x > 0$ **b** $x < 1$ **c** $x < 0.693$

d $x > 1.609$ **e** $x \geq 0.693$ **f** $x \leq 0.792$

g $x \geq -1.303$ **h** $x \leq -1.708$ **i** $x < 0.288$

j $x > 1.347$ **k** $x < -0.458$ **l** $x > 1.251$

7 20.00, 0.01 **8** 0.04

Exercise 3F — Equations with natural (base *e*) logarithms

1 **a** e **b** e^2 **c** $\frac{1}{e^2}$, or ≈ 0.135

d $\frac{1}{e}$ or ≈ 0.368 **e** ≈ 1.350 **f** ≈ 1.649

g ≈ 0.861 **h** ≈ 0.502

2 **a** 3.695 **b** 0.906 **c** e **d** e

e 1.221 **f** 1.350 **g** 1.368 **h** –0.432

i –1.433 **j** –2.054 **k** 0.417 **l** –1.564

3 **a** 2 **b** 5 **c** 3 **d** 4

e 10 **f** 12 **g** 2.207 **h** 0.388

i 2 **j** $\frac{1}{5}$

4 **a** 6 **b** 12 **c** $\frac{2}{3}$ **d** 2.5

e 3 **f** 6 **g** 1 **h** 1.5

i 2 **j** 1.5

5 B **6** A **7** ex^2 **8** $y = x$ **9** 2

10 **a** 2.5 **b** $e^{-kt} = \frac{1}{2}$ **c** $t = \frac{\ln 2}{k}$ **d** 139 days

Investigation — An earthquake formula

1 $8.5 = 17A + B$ **2** $3.8 = 10A + B$

3 $A = 0.67$, $B = -2.91$ **4** $M = 0.67 \log_e E - 2.91$

5 10^{16} Joules **6** 5.14

Exercise 3G — Exponential and logarithmic modelling

1 **a** 200 **b** $y = 200e^{-0.6t}$ **c** 110 g

d $2\frac{1}{4}$ hours

2 **a** 10 g **b** $y = 10e^{-0.18t}$ **c** 8 g

d 13 days

3 **a** 3, 2 **b** $y = 3 + 2\log_e x$

c 5.197 **d** 10

4 **a** 1000, 0.05 **b** $A = 1000e^{0.05t}$ **c** 1051, 1649
d 14

5 **a** 500 **b** 0.03 **c** $P = 500e^{0.03t}$
d 2240 **e** 1896

Chapter review

1 E **2** $\frac{x^{\frac{33}{5}}}{y^{\frac{14}{15}}}$ **3** A **4** C

5 –0.849 **6** $1\frac{2}{3}$ **7** A

8 $\{x: x < -1.27\}$ **9** C **10** 6

11 D **12** B **13** 2.59 **14** B

15 50, 0.18

16 **a** $\frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}$

b $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720}$

c $e \approx 2.718\,06$

17 –0.41 **18** E **19** 4 **20** B

Modelling and problem solving

1 **a** 15°C, 90°C, 0.05 **b** $T = 15 + 75e^{-0.05t}$
c 60°C **d** 22 **e** 17

2 **a** 50, 0.09 **b** $D = 50e^{0.09t}$ **c** 78 cm
d 8 years

3 **a** 100 **b** 0.002 **c** $M = 100e^{-0.002t}$
d 90 g **e** 347 years

4 **a** 0.23 **b** $N = N_0e^{0.23t}$ **c** 1000
d $N = 1000e^{0.23t}$ **e** 6000

5 **a** 0.1386 **b** $I = I_0e^{-0.1386d}$ **c** 25%
d 16.6 metres

6 **a** 30, 0.4055 **b** $N = 30e^{0.4055t}$ **c** 228
d 9 years **e** 1000, 0.1054 **f** $E = 1000e^{-0.1054t}$
g 590 **h** 7 years **i** 485

CHAPTER 4 Derivatives of exponential and logarithmic functions

Exercise 4A — Inverses

1 **a** $y = \log_e\left(\frac{x}{2}\right)$ **b** $y = \log_e\left(\frac{x}{3}\right)$ **c** $y = \log_e x - 1$

d $y = \log_e x + 1$ **e** $y = \frac{1 + \log_e x}{2}$ **f** $y = \frac{\log_e x - 1}{3}$

g $y = 1 - \log_e x$ **h** $y = 2 - \log_e x$ **i** $y = \frac{2 - \log_e x}{3}$

j $y = \frac{1 - \log_e x}{2}$

2 **a** $y = \log_e(x - 1)$ **b** $y = \log_e(x - 2)$
c $y = \log_e(2 - x)$ **d** $y = \log_e(1 - x)$

e $y = \log_e\left(\frac{1 - x}{2}\right)$ **f** $y = \log_e\left(\frac{2 - x}{3}\right)$

g $y = \log_e(x - 2) - 1$ **h** $y = \log_e(x - 1) + 2$

i $y = 2 + \log_e\left(\frac{3 - x}{2}\right)$ **j** $y = \log_e\left(\frac{2 - x}{3}\right) - 1$

3 **a** $y = e^{\frac{x}{2}}$ **b** $y = e^{\frac{x}{3}}$ **c** $y = e^x - 1$

d $y = e^x + 1$ **e** $y = \frac{e^x + 1}{2}$ **f** $y = \frac{e^x - 1}{3}$

g $y = 1 - e^x$ **h** $y = 2 - e^x$ **i** $y = \frac{2 - e^x}{3}$

j $y = \frac{1 - e^x}{2}$

4 **a** $y = e^{x - 1}$ **b** $y = e^{x - 2}$ **c** $y = e^{2 - x}$

d $y = e^{1 - x}$ **e** $y = e^{\frac{x - 2}{3}}$ **f** $y = e^{\frac{x - 3}{5}}$

g $y = e^{1 - x} - 2$ **h** $y = e^{2 - x} + 1$ **i** $y = 1 + e^{\frac{x - 3}{2}}$

j $y = e^{\frac{1 - x}{3}} - 2$

5 C **6** E

Exercise 4B — The derivative of e^x

1 **a** $10e^{10x}$ **b** $\frac{1}{3}e^{\frac{x}{3}}$ **c** $\frac{1}{4}e^{\frac{x}{4}}$

d $-e^{-x}$ **e** $6e^{3x}$ **f** $-20e^{-5x}$

g $12e^{-2x}$ **h** $e^{0.2x}$ **i** $22e^{-11x}$

2 **a** $6e^{6x - 2}$ **b** $-6e^{8 - 6x}$ **c** $10e^{5x + 3}$
d $-8e^{7 - 2x}$ **e** $-24e^{8x + 1}$ **f** $10e^{6 - 5x}$
g $-90e^{6 - 9x}$ **h** $-15e^{3x + 4}$ **i** $-42e^{-7x}$

j $e^{\frac{x}{2} + 1}$ **k** $-e^{2 - \frac{x}{3}}$ **l** $-e^{\frac{x}{4} + 5}$

3 A

4 **a** $2e^x$ **b** $9e^{3x} + 6e^{2x}$ **c** $-20e^{-4x} + 10$
d $3e^x - 2e^{-x}$ **e** $6e^{2x} - 7e^{-7x}$ **f** $36e^{9x} - 2e^x$
g e^x **h** $20e^{5x} + 4x$

5 **a** $(2x + 3)e^{x^2 + 3x}$ **b** $(2x - 3)e^{x^2 - 3x - 1}$

c $2(x - 1)e^{x^2 - 2x}$ **d** $-5e^{2 - 5x}$

e $(2x - 3)e^{6 - 3x + x^2}$ **f** $3(x^2 + 1)e^{x^2 + 3x - 2}$

g $3(8x - 7)e^{4x^2 - 7x}$ **h** $10(1 + 3x)e^{1 - 2x - 3x^2}$

i $6(2x + 1)^2e^{(2x + 1)^3}$ **j** $-4(4 - x)^3e^{(4 - x)^4}$

k $-2(x + 2)^{-3}e^{(x + 2)^{-2}}$ **l** $\frac{3e^{\sqrt{3x + 4}}}{2\sqrt{3x + 4}}$

m $\frac{e^{(x + 1)^{\frac{1}{3}}}}{3(x + 1)^{\frac{2}{3}}}$ **n** $2x(2x + 3)(x + 3)e^{(x^2 + 3x)^2}$

6 B

7 **a** $\ln 4 \times 4^x$ **b** $\ln 10 \times 10^x$ **c** $-\frac{\ln 2}{2x}$

8 $-20e$ **9** $-6e^2$ **10** $-25e^{-3}$

Exercise 4C — The derivative of $\log_e x$

1 **a** $4x$ **b** 4 **c** $\frac{1}{u}$ **d** $\frac{1}{x}$

2 **a** $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{1}{x}$ **d** $\frac{1}{x}$

e $\frac{3}{x}$ **f** $-\frac{6}{x}$ **g** $\frac{1}{x}$ **h** $\frac{1}{x}$

i $\frac{4}{x}$ **j** $-\frac{5}{x}$

3 D

4 **a** A **b** D **c** C

5 **a** $\frac{2}{2x+5}$ **b** $\frac{6}{6x+1}$ **c** $\frac{3}{3x-4}$

d $\frac{8}{8x-1}$ **e** $\frac{-5}{3-5x}$ or $\frac{5}{5x-3}$

f $\frac{-1}{2-x}$ or $\frac{1}{x-2}$

g $\frac{-7}{4-7x}$ or $\frac{7}{7x-4}$ **h** $\frac{30}{5x+2}$

i $\frac{16}{2x-1}$ **j** $\frac{-48}{12x+5}$ **k** $\frac{63}{8-9x}$

6 **a** $\frac{4}{x}$ **b** $\frac{2x}{x^2+3}$ **c** $\frac{2x+4}{x^2+4x}$

d $\frac{2x-3}{x^2-3x+2}$ **e** $\frac{3x^2+4x-7}{x^3+2x^2-7x}$ **f** $\frac{4x^3-6x^2+2x}{x^4-2x^3+x^2}$

g $\frac{1}{2x+1}$ **h** $\frac{2}{4x-3}$ **i** $\frac{x}{x^2+2}$

j $\frac{1}{4(x+3)}$ **k** $\frac{5}{3(5x+2)}$ **l** $\frac{3}{5(3x-2)}$

m $\frac{-1}{x+3}$ **n** $\frac{12}{3x-2}$ **o** $\frac{-10}{5x+8}$

p $\frac{-3}{4+3x}$

7 E

8 **a** $\frac{1}{x\log_e 2}$ **b** $\frac{1}{x\log_e 4}$ **c** $\frac{2}{x\log_e 10}$

9 $-\frac{18}{7}$ **10** $\frac{9}{5}$ **11** $15\frac{1}{3}$

12 **a** $\frac{1}{x}e^{\log_e x}$ or 1

b **i** 1 **ii** 1 **iii** 1 **iv** 1

Gradient is always one since $e^{\log_e x} = x$.

13 **a** $\frac{2}{x}e^{\log_e x^2}$ or $2x$

b **i** 2 **ii** 10 **iii** –4

Exercise 4D — Derivatives of exponential and logarithmic functions

1 **a** $2xe^{x^2}$ **b** $\frac{2x}{x^2+1}$ **c** $\frac{e^x}{2\sqrt{1+e^x}}$

d $\frac{e^x}{e^x+1}$ **e** 1

2 **a** $e^x(x^2+2x)$ **b** $1+\log_e x$

c $\frac{e^x}{x}+e^x\log_e x$ **d** $(1-x)e^{-x}$

3 **a** $\frac{-e^x}{(e^x+1)^2}$ **b** $\frac{\log_e x-1}{(\log_e x)^2}$

c $\frac{e^x}{(e^x+1)^2}$ **d** $\frac{xe^x-e^x}{x^2}$

4 D

5 **a** $\frac{e^x+1}{e^x+x}$ **b** $\frac{x}{x^2+1}$

c $e^{\frac{1}{x}}(2x-1)$ **d** $2x$

e $\frac{2x}{(x^2+1)\log_e(x^2+1)}$

Exercise 4E — Applications of derivatives of exponential functions

1 $y = 30e^{0.196x}$

2 $y = 20e^{-0.087x}$

3 $N = 100e^{0.078t}$

4 E

5 **a** $V = 10\,000e^{0.147t}$ **b** \$43 490 **c** 10.95 years

6 **a** $N = 5e^{0.85t}$ **b** 1923 **c** 8.94 days

7 **a** $N = 50e^{-0.173t}$ **b** 6.27 g **c** 9.3 days

8 D

9 **a** $N = 30e^{-0.012t}$ **b** 150 min

10 **a** $\frac{dP}{dt} = 0.02P$ **b** $P = 6e^{0.02t}$ **c** 7.7 years

11 8.56 min

Chapter review

1 B **2** D

3 $y = \frac{\ln\frac{x}{2}+1}{2}$ **4** $y = \frac{e^x-4}{2}$

5 A **6** D

7 **a** $3e^{3x-1}$ **b** $2\ln 3 \times 3^{2x}$ **c** $-2e^{-2x}$

8 –9.96

9 E **10** A

11 **a** $\frac{1}{x}$ **b** $\frac{-4}{3-4x}$ **c** $\frac{1}{\log_e 10} \times \frac{1}{x}$

12 Because $\log_e 100x = \log_e 100 + \log_e x$ and $\log_e 100$ is constant.

13 A **14** B **15** C **16** D

17 **a** $y = 20e^{3x}$ **b** 65 380 347 **c** 1.00

Modelling and problem solving

1 **a** $V = 4500e^{0.231t}$ **b** 5.2 years

2 **a** $\frac{d\theta}{dt} = -k\theta$ **b** $\theta = 155e^{-0.0726t}$

c 32.2 min **d** 37.2°

3 **a** 1 January 2022 **b** 941 cheetahs

c **i** 824 **ii** 507

4 **a** 90 kg

b Approximately 77.3 kg after 3 months

5 **a** 3000 people **b** –371.6 people/day

c $\frac{dN}{dt} = -\frac{4000}{8t+1}$ **d** –97.56 people/day

6 **a** 7 weeks

b **i** 98.25 rabbits/week **ii** 44.15 rabbits/week

c $P_0 = 267$ (rounding down)

d 782 rabbits

e **i** 33 rabbits/week **ii** 44 rabbits/week

f 39 weeks

CHAPTER 5 Periodic functions

Exercise 5A — Revision of radians and the unit circle

1 **a** 171.89° **b** 286.48° **c** 275.02°
d 146.68° **e** 63° **f** 54°
g 150° **h** 225°

2 **a** $\left(\frac{\pi}{36}\right)^c$ **b** $\left(\frac{\pi}{12}\right)^c$ **c** $\left(\frac{2\pi}{3}\right)^c$
d $\left(\frac{13\pi}{8}\right)^c$ **e** 1.12^c **f** 1.38^c
g 4.10^c **h** 4.54^c **i** 5.41^c
j 6.11^c

3 **a** 0.389 **b** 0.717 **c** 0.170
d –0.129 **e** –0.246 **f** –0.916

4 **a** 0.966 **b** 0.927 **c** –0.940
d –0.996 **e** 11.430 **f** 1.732

5 **a** 0 **b** 0 **c** 1
d –1 **e** Undefined **f** Undefined

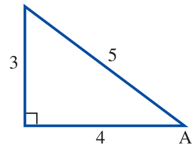
6 **a** 1 **b** 0 **c** –1
d 1 **e** Undefined **f** 0

7 **a** 1 **b** 1 **c** 1
d 1 **e** 1 **f** 1
g 2 **h** 5

8 **a** sin 240°, sin 150°, sin 35°, sin 120°, sin 70°
b cos 3.34, cos 1.5, cos 5.3, cos 0.2, cos 6.3

9 $\frac{8}{15}$

10 $\frac{3}{4}$



11 D

12 B

13 13.5°C, 15.8°C

Exercise 5B — Symmetry and exact values

1 **a** $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{2}}{2}$ **c** $-\frac{\sqrt{3}}{3}$
d $-\frac{\sqrt{2}}{2}$ **e** $-\frac{1}{2}$ **f** $-\frac{\sqrt{3}}{3}$
g $-\frac{\sqrt{2}}{2}$ **h** $\frac{1}{2}$ **i** 1
j $\frac{\sqrt{3}}{2}$ **k** $\frac{\sqrt{2}}{2}$ **l** $\sqrt{3}$

2 **a** $\frac{\sqrt{2}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** $-\sqrt{3}$
d $-\frac{1}{2}$ **e** $-\frac{\sqrt{2}}{2}$ **f** $\frac{\sqrt{3}}{3}$
g $-\frac{1}{2}$ **h** $\frac{1}{2}$ **i** –1
j $\frac{\sqrt{2}}{2}$ **k** $\frac{1}{2}$ **l** $\frac{\sqrt{3}}{3}$

3 **a** 0.3 **b** –0.5 **c** –2.4
d $\frac{\sqrt{91}}{10}$ **e** $\frac{\sqrt{3}}{2}$ **f** 2.4
g –0.3 **h** 0.5 **i** –2.4
j $\frac{\sqrt{91}}{10}$ **k** $\frac{\sqrt{3}}{2}$ **l** 2.4
m 0.3 **n** –0.5 **o** –2.4

4 $\frac{7}{24}$ $\left(\frac{7}{24}\right)^2 + \left(\frac{24}{25}\right)^2 = \frac{49}{625} + \frac{576}{625} = \frac{625}{625} = 1$

5 **a** $-\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{3}$ **b** $-\frac{1}{\sqrt{2}}$ or $-\frac{\sqrt{2}}{2}, 1$
c $\frac{1}{2}, -\sqrt{3}$ **d** $-\frac{\sqrt{3}}{2}, -\frac{1}{2}$

6 **a** $-\frac{1}{2}$ **b** $\frac{\sqrt{2}}{2}$ **c** $-\sqrt{3}$
d $-\frac{\sqrt{3}}{2}$ **e** $-\frac{\sqrt{3}}{2}$ **f** 1
g $\frac{\sqrt{2}}{2}$ **h** $-\frac{\sqrt{3}}{2}$ **i** $-\sqrt{3}$
j $\frac{\sqrt{3}}{2}$ **k** $\frac{\sqrt{2}}{2}$ **l** $\sqrt{3}$
m $-\frac{\sqrt{3}}{2}$ **n** $\frac{\sqrt{3}}{2}$ **o** –1

7 2

8 **a** $-\frac{\sqrt{3}}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** –1
d $-\frac{\sqrt{2}}{2}$ **e** $-\frac{\sqrt{3}}{2}$ **f** $\frac{\sqrt{3}}{3}$
g $\frac{1}{2}$ **h** $-\frac{\sqrt{2}}{2}$ **i** $-\sqrt{3}$
j $\frac{1}{2}$ **k** $-\frac{1}{2}$ **l** –1

9 $\left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$

10 **a** 0.7 **b** 0.3 **c** 2.5
d –0.3 **e** –0.7 **f** –2.5
g 0.7 **h** –0.3 **i** 2.5
j 0.3 **k** –0.7 **l** –2.5

11 C **12** D

13 **a** 10 cm/s **b** 11 cm/s **c** 12 cm/s

14 **a** 0.9 m **b** 0.3 m **c** 0.7 m

Exercise 5C — Further trigonometric equations

1 **a** $\frac{\pi}{2}, \frac{3\pi}{2}$ **b** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$
d $\frac{3\pi}{2}$ **e** $\frac{5\pi}{6}, \frac{7\pi}{6}$

2 **a** 90° **b** 60°, 300° **c** 60°, 120°
d 180° **e** 45°, 135°

3 **a** 2.2904, 3.9928, 8.5736, 10.2760
b 1.1442, 1.9973, 7.4274, 8.2805
c 1.0701, 5.2130, 7.3533, 11.4962
d 3.5217, 5.9031, 9.8049, 12.1863

4 **a** 15.58°, 164.42° **b** 137.91°, 222.09°
c 212.90°, 327.10° **d** 78.83°, 281.17°

5 **a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{2}, \frac{3\pi}{2}$

c $\frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $\frac{\pi}{4}, \frac{7\pi}{4}$

6 **a** 0°, 180°, 360°

b 105°, 165°, 285°, 345°

c 45°, 75°, 165°, 195°, 285°, 315°

d 20°, 40°, 140°, 160°, 260°, 280°

e 62.40°, 117.60°, 182.40°, 237.60°, 302.40°, 357.60°

f 39.44°, 140.56°, 219.44°, 320.56°

g 26.39°, 333.61°

h 101.22°, 258.78°

7 **a** $\frac{\pi}{2}$ **b** 0, 2π **c** 2.7184

d 0.9772

e $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

f $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{15\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

g $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

h 1.2579, 5.0253

8 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$

b $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

c $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

d $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

e 0.6781, 1.7253, 2.7725, 3.8197, 4.8669, 5.9141

f 1.8925, 5.0341

9 **a** The particle is 3 metres from O.

b It takes the particle 1.99 seconds to reach O for the first time.

Exercise 5D — Further trigonometric graphs

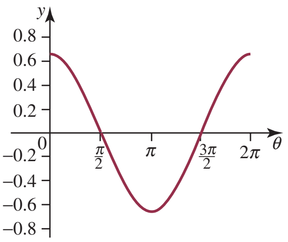
1 **a** 2π, 1 **b** 2π, 1 **c** 2π, 4

d 2π, $\frac{1}{3}$ **e** $\frac{2\pi}{3}$, 2 **f** π, 3

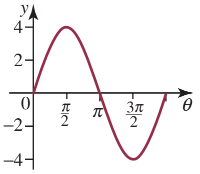
g 4π, 3 **h** 6π, 2 **i** π, $\frac{1}{3}$

j $\frac{2\pi}{3}$, 4

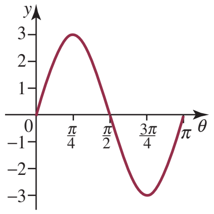
2 **a** $\frac{2}{3}$, 2π, $-\frac{2}{3}$ to $\frac{2}{3}$



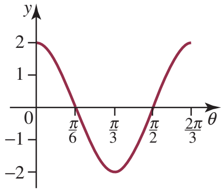
b 4, 2π, −4 to 4



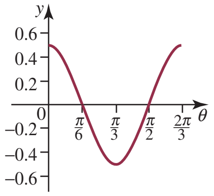
c 3, π, −3 to 3



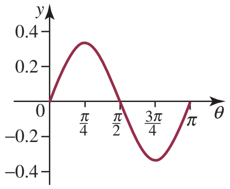
d 2, $\frac{2\pi}{3}$, −2 to 2



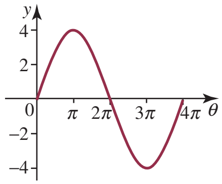
e $\frac{1}{2}$, $\frac{2\pi}{3}$, $-\frac{1}{2}$ to $\frac{1}{2}$



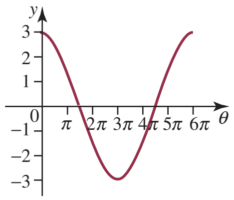
f $\frac{1}{3}$, π, $-\frac{1}{3}$ to $\frac{1}{3}$



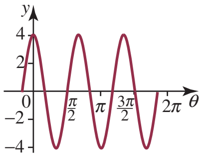
g 4, 4π, −4 to 4



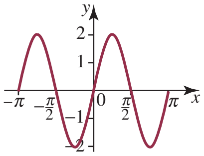
h 3, 6π, −3 to 3



3 4, $\frac{2\pi}{3}$, –4 to 4



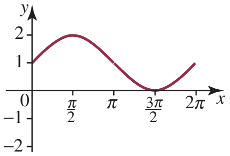
4 2, π, –2 to 2



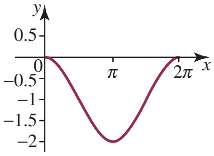
5 **a** $\frac{\pi}{3}$ to the left, up 3 **b** $\frac{\pi}{2}$ to the right, up 1

c $\frac{\pi}{4}$ to the right, down 2 **d** $\frac{\pi}{3}$ to the left, down 1

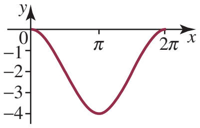
6 **a** 1, 2π, 0 to 2



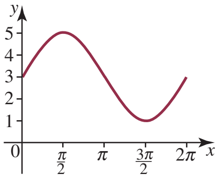
b 1, 2π, –2 to 0



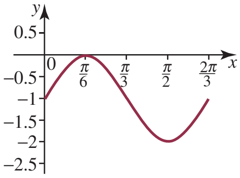
c 2, 2π, –4 to 0



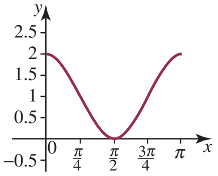
d 2, 2π, 1 to 5



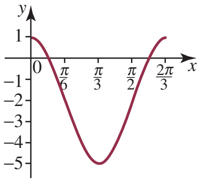
e 1, $\frac{2\pi}{3}$, –2 to 0



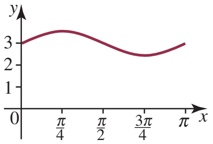
f 1, π, 0 to 2



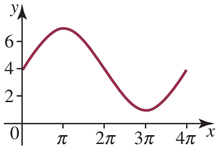
g 3, $\frac{2\pi}{3}$, –5 to 1



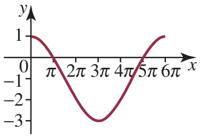
h $\frac{1}{2}$, π, 2.5 to 3.5



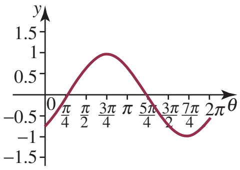
i 3, 4π, 1 to 7



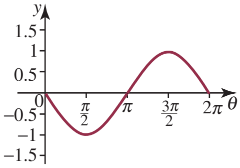
j 2, 6π, –3 to 1



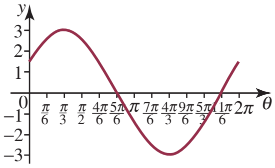
7 **a** 2π, 1



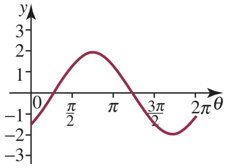
b 2π, 1



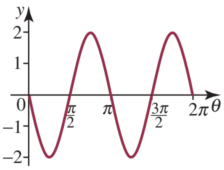
c 2π, 3



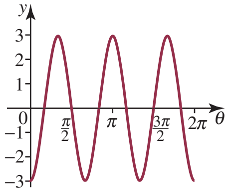
d 2π, 2



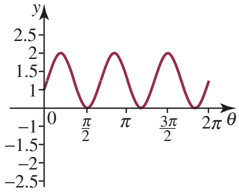
e π, 2



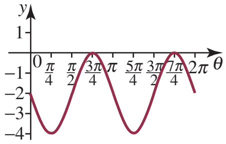
f $\frac{2\pi}{3}$, 3



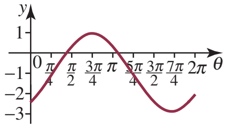
g $\frac{2\pi}{3}$, 1



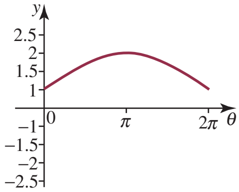
h π, 2



i 2π, 2



j 4π, 1



8 **a** 4, π, −4 to 4 **b** 3, 4π, −3 to 3

c 2, $\frac{2\pi}{3}$, −2 to 2 **d** 1, 6π, −1 to 1

e 3, π, −2 to 4 **f** 5, $\frac{\pi}{2}$, −7 to 3

g 1, 2π, −1 to 1 **h** 1, 2π, −1 to 1

9 **a** 1, −1 **b** 1, −1 **c** 3, −3

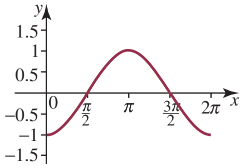
d 2, −2 **e** 2, −2 **f** 3, −3

g 5, −3 **h** −1, −3 **i** −1, −5

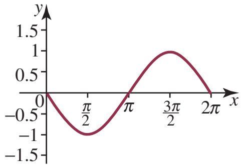
j 5, −3 **k** $2\frac{1}{2}$, $1\frac{1}{2}$

l $-3\frac{2}{3}$, $-4\frac{1}{3}$

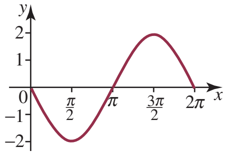
10 **a** 2π, 1, −1 to 1



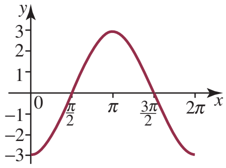
b 2π, 1, −1 to 1



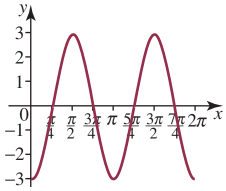
c 2π, 2, −2 to 2



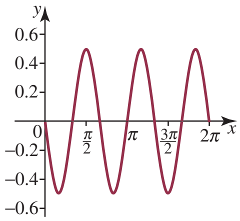
d 2π, 3, −3 to 3



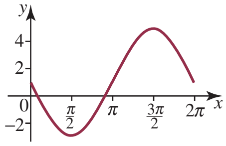
e π, 3, −3 to 3



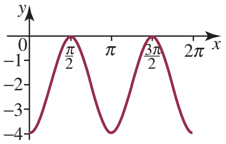
f $\frac{2\pi}{3}$, $\frac{1}{2}$, $-\frac{1}{2}$ to $\frac{1}{2}$



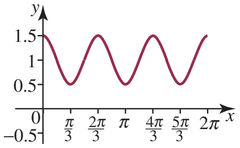
g 2π, 4, −3 to 5



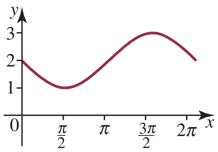
h π, 2, -4 to 0



i $\frac{2\pi}{3}$, $\frac{1}{2}$, $\frac{1}{2}$ to $1\frac{1}{2}$



j 2π, 1, 1 to 3



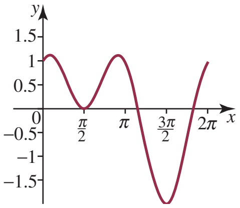
11 $y = \sin\left(x + \frac{\pi}{3}\right) + 1$

12 $y = 2\cos\left(3x - \frac{3\pi}{4}\right) + 1$

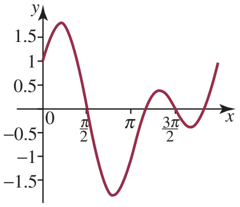
13 $y = 3\sin\left(x - \frac{2\pi}{3}\right) - 2$

14 a 1 **b** 12 **c** 3 metres
d 1 metre **e** 3.00 am and 3.00 pm
f 9.00 am and 9.00 pm **g** $y = \sin\frac{\pi x}{6} + 2$

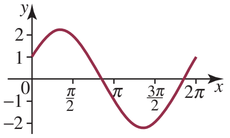
15 a



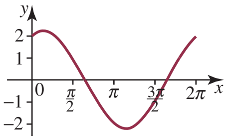
b



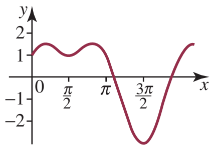
c



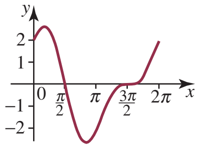
d



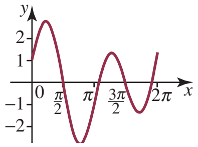
e



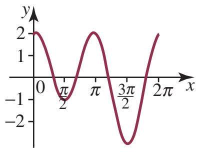
f



g



h



Exercise 5E — Finding equations of trigonometric graphs

1 a 3, 2, $y = 3\sin 2x$ **b** 2, $\frac{\pi}{6}$, $y = 2\sin\frac{\pi}{6}x$

2 a 2, 4, $y = 2\cos 4x$ **b** 2, $\frac{\pi}{3}$, $y = 2\cos\frac{\pi}{3}x$

3 a $\frac{1}{2}$, 1, 1, $y = \frac{1}{2}\sin x + 1$

b 3, $\frac{\pi}{4}$, -2, $y = 3\sin\frac{\pi}{4}x - 2$

4 a 4, 2, $-\pi$, $y = 4\cos(2x - \pi)$

b 5, 2, $\frac{\pi}{2}$, $y = 5\cos\left(2x - \frac{\pi}{2}\right)$

5 a 2, 1, $\frac{\pi}{3}$, -1, $y = 2\sin\left(x + \frac{\pi}{3}\right) - 1$

b 3, 1, $\frac{\pi}{2}$, 2, $y = 3\sin\left(x + \frac{\pi}{2}\right) + 2$

6 a -1, $\frac{\pi}{6}$, 3, $y = 3 - \cos\frac{\pi}{6}x$

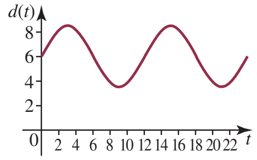
b -1, $\frac{\pi}{4}$, 2, $y = 2 - \cos\frac{\pi}{4}x$

7 D **8** E

Exercise 5F — Trigonometric modelling and problem solving

1 a 8.5 m, 3.00 pm **b** 12 hours **c** 3.5 m

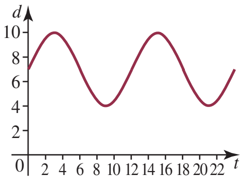
d



e 1.00 pm and 5.00 pm, 1.00 am and 5.00 am

2 a 10 m, 4 m

b



c 3.00 am

d 1.00 am to 5.00 am Saturday, 1.00 pm to 5.00 pm Saturday, 1.00 am to 5.00 am Sunday morning

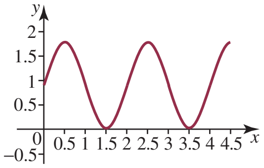
e 3.00 am to 5.00 am Sunday

3 a 18°C, 14.5° **b** 25°C, January, December

c 21.5°C **d** 8 months

4 a 0.9, π, 0.9, $h = 0.9 \sin \pi t + 0.9$

b



c 1.3 seconds

5 a 2, 1 **b** 8 m, 8

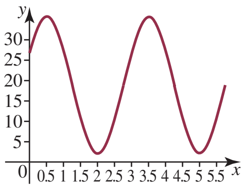
6 a 18, 16, $\frac{2\pi}{3}$

b $h(t) = 18 - 16 \cos \frac{2\pi}{3} t$ **c** 5, 34 m **d** 2 m

e 3 minutes **f** 26 m

g $h(t) = 18 - 16 \cos \left(\frac{2\pi}{3} t + \frac{2\pi}{3}\right)$

h



7 a 10 **b** 24 **c** 20

d 20 **e** 3.00 am, 3.00 pm

f $T = 20 - 10 \cos \left(\frac{\pi}{12} t - \frac{\pi}{4}\right)$ or

$T = 20 - 10 \cos \left(\frac{\pi}{12} t - \frac{7\pi}{4}\right)$

or $T = 20 + 10 \cos \left(\frac{\pi}{12} t - \frac{5\pi}{4}\right)$

or $T = 20 + 10 \cos \left(\frac{\pi}{12} t - \frac{3\pi}{4}\right)$

g 10°C, 20°C, 30°C, 20°C **h** 13°C

Chapter review

1 $\frac{16\pi}{9}$ **2** 390°

3 a $\frac{-\sqrt{161}}{15}$ **b** $\frac{8}{\sqrt{161}}$

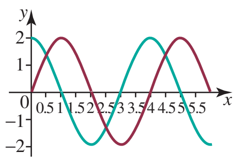
4 $-\frac{\sqrt{3}}{2}$ **5** $\frac{5\sqrt{2}}{2}$ **6** $\frac{4\pi}{3}$

7 0.273, 1.297

8 a 3 **b** π **c** $-4 \le y \le 2$

9 $\frac{2-3\sqrt{3}}{2}$ **10** 3π

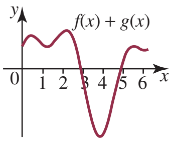
11



12 a Dilation 2 parallel to y-axis, period change to $\frac{2\pi}{3}$, translate graph $\frac{\pi}{6}$ to the left and 1 unit down.

b $f(x) = -2 \cos (3x + \frac{\pi}{2}) + 1$

13



14 Amplitude = 2.5, $T = 3$, range = $\{y: -1.5 \le y \le 3.5\}$

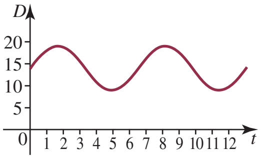
15 $y = -\cos \left(x - \frac{\pi}{3}\right)$

Modelling and problem solving

1 90, 30, $\frac{\pi}{30}$; that is, $y = 90 + 30 \sin \frac{\pi}{30} x$

2 a 19 metres **b** 9 metres **c** 6.5 hours

d 5 **e**



CHAPTER 6 The calculus of periodic functions

Exercise 6A — The derivatives of sin x and cos x

1 a $8 \cos 8x$ **b** $-6 \cos (-6x)$ **c** $\cos x$

d $\frac{1}{3} \cos \frac{x}{3}$ **e** $-\frac{1}{2} \cos \left(-\frac{x}{2}\right)$ **f** $\frac{2}{3} \cos \frac{2x}{3}$

2 a $-3 \sin 3x$ **b** $2 \sin (-2x)$ **c** $-\frac{1}{3} \sin \frac{x}{3}$

d $-21 \sin 21x$ **e** $7 \sin (-7x)$ **f** $-\frac{\pi}{4} \sin \frac{\pi x}{4}$

g $\frac{1}{8} \sin \left(-\frac{x}{8}\right)$ **h** $-\frac{2}{5} \sin \frac{2x}{5}$

3 a A **b** E **c** D **d** B **e** C

4 a $\cos u$ **b** 4 **c** $4 \cos (4x + 3)$

5 a $-\sin u$ **b** 3 **c** $-3 \sin (3x + 1)$

6 a $2 \cos (2x + 3)$ **b** $-7 \cos (6 - 7x)$

c $5 \cos (5x - 4)$ **d** $\frac{3}{4} \cos \left(\frac{3x+2}{4}\right)$

e $-\frac{7}{3} \cos \left(\frac{8-7x}{3}\right)$ **f** $10\pi^2 \cos 2\pi x$

g $-\frac{3}{2} \cos \frac{3x}{8}$

7 **a** $-3\sin(3x-2)$ **b** $-4\sin(4x+7)$

c $\sin(8-x)$ **d** $5\sin(6-5x)$

e $-\tfrac{2}{3}\sin\left(\dfrac{2x+3}{3}\right)$ **f** $-\tfrac{4}{5}\sin\left(\dfrac{4x-1}{5}\right)$

g $-40\pi^2\sin 10\pi x$ **h** $-12\sin(-2x)$

8 **a** $2(2-x)\sin(x^2-4x+3)$

b $(2x-5)\cos(10-5x+x^2)$

c $e^x\cos(e^x)$

d $-(2x+7)\sin(x^2+7x)$

e $2(x-2)\sin(4x-x^2)$

f $(2x+3)\cos(x^2+3x)$

g $-\dfrac{1}{x}\sin(\log_e x)$

h $4e^{4x}\cos(e^{4x})$

i $\dfrac{1}{x}\cos(\log_e 3x)$

j $\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)$

k $\dfrac{2}{2x-1}\cos(\log_e(2x-1))$

l $6e^{2x}\cos(3e^{2x})$

m $-6e^{3x}\sin(2e^{3x})$

n $-\dfrac{3}{x}\sin(\log_e 10x)$

o $4x(3x+4)\cos(x^3+2x^2)$

p $\tfrac{24}{5}\cos\left(-\dfrac{3x}{5}\right)$

q $\tfrac{1}{2}\sin\dfrac{x}{4}$

r $-2(x+1)\sin(x^2+2x)+3\cos(3x-9)$

s $2x\cos(x^2-4)-9\sin(8-3x)$

t $\tfrac{3}{2}\sin\left(\dfrac{3x+7}{10}\right)-8\cos\left(\dfrac{5-4x}{3}\right)+12x^2$

9 -3.745

10 0

11 **a** **i** $\cos x\, e^{\sin x}$ **ii** $\dfrac{\sqrt{3}}{2}e^{\frac{1}{2}}$ or $\dfrac{\sqrt{3e}}{2}$

b **i** $-\sin x\, e^{\cos x}$ **ii** $-\tfrac{1}{2}e^{\frac{\sqrt{3}}{2}}$

c **i** $\dfrac{\cos x}{\sin x}$ or $\cot x$ **ii** $\sqrt{3}$

d **i** $-\dfrac{\sin x}{\cos x}$ or $-\tan x$ **ii** $-\dfrac{1}{\sqrt{3}}$

Exercise 6B — Further differentiation of trigonometric functions

1 $\cos x - x\sin x$

2 **a** $x^2\cos x + 2x\sin x$ **b** $3\sin x + 3x\cos x$

c $5x^4\cos(3x+1)-3x^5\sin(3x+1)$

d $\cos^2 x-\sin^2 x$

e $\dfrac{8\sin 5x}{x}+40\cos 5x\log_e 5x$

f $5\cos 2x\cos x-10\sin 2x\sin x$

g $\dfrac{4}{3}\cos\dfrac{4x}{3}\cos x-\sin\dfrac{4x}{3}\sin x$

h $4x^{-3}\cos(2x+3)-6x^{-4}\sin(2x+3)$

i $-4e^{-5x}\cos(2-x)-20e^{-5x}\sin(2-x)$

j $-\dfrac{1}{2\sqrt{x^3}}\cos 6x-\dfrac{6}{\sqrt{x}}\sin 6x$

k $\dfrac{\sin x}{x}+\cos x\log_e x$

l $\pi\cos 2\pi x-2\pi^2 x\sin 2\pi x$

3 D

4 10

5 17.279

6 $\dfrac{\sin x - x\cos x}{\sin^2 x}$

7 **a** $\dfrac{x\cos x-\sin x}{x^2}$

b $\dfrac{4\cos 2x\cos 4x+2\sin 4x\sin 2x}{\cos^2 2x}$

c $\dfrac{-x\sin x-\cos x}{x^2}$ **d** $\dfrac{-e^x\sin x-e^x\cos x}{e^{2x}}$

e $\dfrac{\frac{1}{2}\sqrt{x}\cos\sqrt{x}-\sin\sqrt{x}}{x^2}$

f $\dfrac{4x\sin(3-2x)-4\cos(3-2x)}{x^3}$

8 A **9** $\dfrac{1}{\cos^2 x}$ or $\sec^2 x$

10 **a** $f'(-\pi)$ is undefined, $f'(0)$ is undefined, $f'(\pi)$ is undefined. Function exists only when $\sin x > 0$ and stationary points at $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2}$ etc.

b $f'(-\pi)$ and $f'(0)$ undefined, $f'(\pi) = 0.13$. Function exists only for $x \geq 0$.

c $f'(-\pi) = -0.04$, $f'(0) = 1$, $f'(\pi) = -23.14$. Function intercepts at $-\pi$.

d $f'(-\pi)$ and $f'(0)$ undefined, $f'(\pi) = -1.14$. Function exists only for $x > 0$.

11 **a** $\dfrac{dy}{dx}=x\cos 2x+\sin x\cos x$

b and **c** Check with your teacher.

Exercise 6C — Applications of differentiation

1 $x+y-\dfrac{\pi}{2}=0$

2 Tangent: $2x+y-2\pi=0$ normal: $x-2y-\pi=0$

3 **a** **i** $x+y-\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{3}=0$ **ii** $x-y+\dfrac{\sqrt{3}}{2}-\dfrac{\pi}{3}=0$

b **i** $3x+2y-3\pi=0$ **ii** $2x-3y-2\pi=0$

c **i** $2x-\sqrt{2}\,y+1=0$ **ii** $x+\sqrt{2}\,y-1=0$

d **i** $2x-\sqrt{2}\,y+2-\dfrac{3\pi}{2}=0$

ii $x+\sqrt{2}\,y-2-\dfrac{3\pi}{4}=0$

4 $6x + y - \frac{3\pi}{2} = 0$ A$\left(\frac{\pi}{4}, 0\right)$ B$\left(0, \frac{3\pi}{2}\right)$

5 Tangent: $x - y = 0$ normal: $x + y - \pi = 0$

6 Tangent is never parallel to $y = x$.

7 **a** 700 **b** 6 months

8 **a** 2.5 metres **b** 1:30 am **c** 2 metres

9 **a** 0.8 cm/s after $\frac{\pi}{4}$ seconds

b –0.8 cm/s after $\frac{3\pi}{4}$ seconds

10 **a** **i** 12 cm **ii** 132 cm **b** $6 + \frac{\pi}{2} \cos \frac{\pi t}{4}$

c Max rate = $6 + \frac{\pi}{2}$, min rate = $6 - \frac{\pi}{2}$

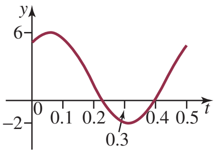
Exercise 6D — Kinematics

1 **a** 2

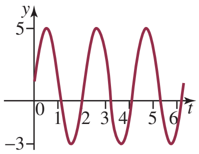
b Amplitude = 4, range = 8

c $T = \frac{1}{2}, f = 2$ **d** $2\sqrt{2} + 2$

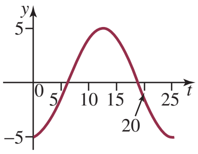
e



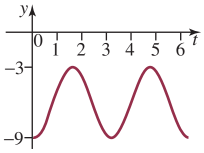
2 **a** **i**



ii



iii



b **i** $\frac{2\pi}{3}$ **ii** 8π **iii** π

c **i** $\frac{3}{2\pi}$ **ii** $\frac{1}{8\pi}$ **iii** $\frac{1}{\pi}$

d **i** $x = 1$ **ii** $x = 0$ **iii** $x = -6$

e **i** 4 **ii** 5 **iii** 3

f **i** $x = 1$ **ii** $x = -5$ **iii** $x = -8.97$

3 **i** $v(t) = 6 \cos 1.5t$
$a(t) = -9 \sin 1.5t$

ii $v(t) = \frac{5}{2} \sin \frac{t}{2}$

$a(t) = \frac{5}{4} \cos \frac{t}{2}$

iii $v(t) = -6 \sin (2t + 3)$
$a(t) = -12 \cos (2t + 3)$

4 **a** $v = -8 \sin 4t$ **b** $t = 0$

c $a = -32 \cos 4t$

d $t = \frac{\pi}{8}, \frac{3\pi}{8}$ seconds

5 **a** $x = 0$ **b** $x = -0.54$

c $x = -0.28$ **d** 0.26 cm/s

e $v(t) = 4 \cos 4t$ **f** 4 cm/s

g 3.37 cm/s

h $t = \frac{\pi}{8}, x = 1, t = \frac{3\pi}{8}, x = -1$

6 **a** $x = -10$

b $v(t) = 10\pi \sin \pi t$ $a(t) = 10\pi^2 \cos \pi t$

c $t = 0, 1, 2 \ldots.$

d **i** $t = \frac{1}{2}$ **ii** $t = \frac{3}{2}$

e $t = \frac{1}{2}, \frac{3}{2}$ v is a max or min

7 **a** Max displacement = 5 m, min displacement = 1 m

b $t = \frac{1}{3}$ s **c** After $\frac{\pi}{3}$ seconds

d $a(t) = -18 \cos (3t - 1)$

8 **a** $v = 2\pi \cos \frac{\pi t}{4}$

$a = \frac{-\pi^2}{2} \sin \frac{\pi t}{4}$

b $v = 2 \cos 2t - 4t \sin 2t$
$a = -8t \cos 2t - 8 \sin 2t$

9 **a** 20 m/s **b** $\frac{\pi}{24}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{11\pi}{24}$ seconds

10 **a** $\frac{16\pi^2}{245}$ m/s^2

b Max speed = $\frac{4\pi}{35}$ m/s when $x = 1.6$ m

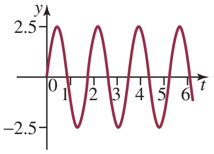
c $t = \frac{7}{12}$ s

11 **a** Amp = 2.5, $T = \frac{5\pi}{9}$ **b** –6.36 m/s

c 30.34 m/s^2 **d** –16.2 m/s^2

e $\frac{9\sqrt{3}}{2}$ m/s **f** 2.48 m

g 8.87 m/s **h**



12 **a** $H' = \frac{-\pi}{6} \sin \frac{\pi t}{12}$

b **i** $\frac{-\pi}{6}$ m/h **ii** 0.37 m/h **iii** 0.45 m/h

c Min = 8 m, 12 hours after high tide,
max = 12 m, 24 hours after high tide

Chapter review

1 $-8 \sin 8x$

2 $4 \cos (2x + 3)$

3 **a** $\cos x$ **b** $-4 \sin x$ **c** $12 \cos 4x$

d $-12 \sin 3x$ **e** $\frac{3}{2} \cos \frac{3x}{2}$ **f** $-\frac{1}{4} \sin \frac{x}{2}$

4 $f'(x) = 6 \cos 2x$ $f'(2\pi) = 6$

5 $2 \cos 2x \, e^{\sin 2x}$

6 $9x \cos 3x + 3 \sin 3x$

7 $\frac{-x \sin x - \cos x}{x^2}$

8 **a** $\dfrac{\sin x - x\cos x}{\sin^2 x}$ **b** $-6x\sin 2x + 3\cos 2x$

c $\log x \,.\, \cos x + \dfrac{\sin x}{x}$ **d** $e^x\cos x - e^x\sin x$

e $\dfrac{e^x\sin x - e^x\cos x}{\sin^2 x}$ **f** $\dfrac{-\log x\sin x - \dfrac{\cos x}{x}}{(\log x)^2}$

9 $\cos^2 x - \sin^2 x$

10 $y = x$

11 $y = -3 \quad x = \pi$

12 $8x + y - 2\pi = 0 \quad \text{A}\left(\dfrac{\pi}{4}, 0\right) \quad \text{B}\,(0, 2\pi)$

Modelling and problem solving

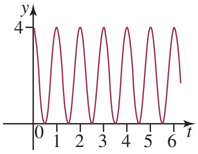
1 **a** 250 **b** 750 after 6 h

c 3 h, 9 h etc.

2 **a** $x = 2$ **b** Amp = 2, range 0 to 4

c $T = 1, f = 1$ **d** $x = 4$

e



3 $v(t) = 10\cos(2t - 1) \quad a(t) = -20\sin(2t - 1)$

4 **a** $v = \pi\cos\dfrac{\pi t}{2}$ **b** $t = 0, 2, 4, \ldots$

c $a = \dfrac{-\pi^2}{2}\sin\dfrac{\pi t}{2}$ **d** π m/s

5 **a** $x = 0$

b $v(t) = 10\pi\sin\pi t \quad a(t) = 10\pi^2\cos\pi t$

c $t = 0, 1, 2,$ etc.

d **i** $t = \frac{1}{2}, \frac{5}{2},$ etc. **ii** $t = \frac{3}{2}, \frac{7}{2},$ etc.

e $t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ etc. Velocity is a max or a min at these points.

6 **a** 5, 5, 6 and $\dfrac{\pi}{12}$, $D(t) = 5 - 5\sin\dfrac{\pi}{12}(t + 6)$

b 5°, 1.46°

c 8.30 am and 3.30 pm. 8.30 am the next day

d 8.27 am or 8.46 hours after midnight and 3.32 pm or 15.54 hours after midnight. Differences occur due to inaccuracy of graphical methods.

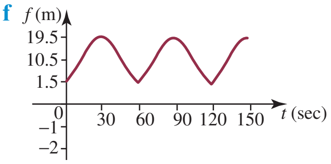
e $D(t) = 10 - 5\sin\dfrac{\pi}{12}(t + 6)$

7 **a** 1.5 m **b** 19.5 m

c 18 m **d** 60 s

e **i** $\dfrac{3\pi}{10}$ m/s **ii** 0.94 m/s

f



g $d = 10.5 - 9\cos\dfrac{\pi}{30}(t - 6)$

h 17.78 m

CHAPTER 7 Introduction to integration

Exercise 7A — Approximating areas enclosed by functions

All answers for areas in questions **1** to **15** are in square units.

1 10

2 **a** 8 **b** 42

3 **a** A **b** A **c** B

4 **a** 30 **b** 22

5 9

6 **a** 8 **b** 13

7 **a** 32 **b** 32 **c** 39 **d** 16

e 29 **f** 12 **g** 26

8 **a** 26 **b** 41 **c** $33\frac{1}{2}$

9 **a** 28 **b** $e^{-1} + 1 + e$ **c** $\log_e 24$ **d** 6

e $23\frac{1}{3}$ **f** 7.25 **g** 100

10 **a** $40\frac{1}{2}$ **b** 10 **c** $11\frac{1}{4}$

d $19\frac{1}{4}$ **e** $\dfrac{1 + 2e + 2e^2 + e^3}{2}$

11 **a** 6 **b** 11 **c** 8.5

12 **a** $4\frac{7}{8}$ or 4.875 **b** $7\frac{3}{8}$ or 7.375 **c** $6\frac{1}{8}$ or 6.125

13 10.5

14 **a** 22.5 **b** 20.8

15 **a** 1.87 **b** 1.68 **c** 1.63

16 $\log_e 2880$ or approx. 7.96 (2 d.p.)

17 **a** 4400 sq. metres **b** 3500 sq. metres

18 $\dfrac{\pi(2 + \sqrt{3})}{6}$ or approx. 1.95 sq. units

19 **a** Distance travelled by the cyclist in the first 30 seconds.

b 307.5 metres

20 **a** False **b** True **c** False **d** True

Exercise 7B — Antidifferentiation (integration)

1 **a** $\frac{1}{2}x^2 + c$ **b** $\frac{1}{5}x^5 + c$ **c** $\frac{1}{8}x^8 + c$

d $\frac{1}{2}x^6 + c$ **e** $-\dfrac{5}{x} + c$ **f** $-\frac{2}{5}x^5 + c$

g $\dfrac{2}{x^3} + c$ **h** $\frac{4}{3}x^{\frac{3}{2}} + c$ **i** $\frac{1}{25}x^5 + c$

j $\frac{1}{8}x^4 + c$ **k** $-\dfrac{1}{9x^3} + c$ **l** $\frac{2}{3}x^{\frac{3}{2}} + c$

m $\frac{3}{5}x^{\frac{5}{3}} + c$ **n** $\frac{16}{7}x^{\frac{7}{4}} + c$ **o** $\frac{7}{4}x^{\frac{4}{7}} + c$

p $-\dfrac{5}{2x^2} + c$ **q** $-\dfrac{9}{x} + c$ **r** $\dfrac{2}{x^5} + c$

s $16\sqrt{x} + c$ **t** $\dfrac{12}{\sqrt{x}} + c$

2 **a** $x^2 + 5x + c$

b $x^3 + 2x^2 - 10x + c$

c $2x^5 + \frac{3}{2}x^4 + 2x + c$

d $-\frac{2}{3}x^6 + \frac{1}{4}x^4 - 2x^3 + x^2 + c$

e $\frac{1}{4}x^4 + 12x - \frac{1}{3}x^3 + c$

f $\frac{1}{3}x^3 - 2x^2 - 21x + c$

g $\frac{5}{3}x^3 + 5x^2 - 5x + c$

h $\frac{1}{4}x^4 - \frac{7}{3}x^3 + 2x^2 - 28x + c$

i $\frac{1}{4}x^4 + x^3 - 2x^2 + c$

3 B

4 E

5 **a** $\frac{1}{3}(x+3)^3 + c$ **b** $\frac{1}{4}(x-5)^4 + c$

c $\frac{1}{5}(2x+1)^5 + c$ **d** $-\frac{1}{9}(3x-4)^6 + c$

e $\frac{1}{30}(6x+5)^5 + c$ **f** $\frac{1}{4}(4x-1)^3 + c$

g $-\frac{1}{4}(4-x)^4 + c$ **h** $-\frac{1}{5}(7-x)^5 + c$

i $-\frac{4}{15}(8-3x)^5 + c$ **j** $\frac{1}{33}(8-9x)^{11} + c$

k $-\frac{1}{2}(2x+3)^{-1} + c$ **l** $-\frac{1}{12}(6x+5)^{-2} + c$

m $-\frac{1}{2}(4x-7)^{-3} + c$ **n** $-\frac{1}{15}(3x-8)^{-5} + c$

o $\frac{1}{10}(6-5x)^{-2} + c$ **p** $-\frac{2}{3}(7-5x)^{-3} + c$

6 C

7 **a** $3\log_e x + c$ **b** $8\log_e x + c$

c $\frac{6}{5}\log_e x + c$ **d** $\frac{7}{3}\log_e x + c$

e $\frac{4}{7}\log_e x + c$ **f** $\log_e(x+3) + c$

g $3\log_e(x+3) + c$ **h** $-2\log_e(x+4) + c$

i $-6\log_e(x+5) + c$ **j** $\frac{4}{3}\log_e(3x+2) + c$

k $\frac{8}{5}\log_e(5x+6) + c$ **l** $\frac{3}{2}\log_e(2x-5) + c$

m $-\frac{5}{2}\log_e(3+2x) + c$ **n** $-\frac{2}{7}\log_e(6+7x) + c$

o $-\log_e(5-x) + c$ **p** $-\frac{3}{11}\log_e(6-11x) + c$

q $\frac{2}{3}\log_e(4-3x) + c$ **r** $4\log_e(5-2x) + c$

8 A

9 $2x + 7\log_e x + c$

10 **a** $\frac{1}{5}x^5 + x^2 + \log_e x + c$

b $\frac{1}{18}(3x+1)^6 + c$

c $3x + 2\log_e x + x^{-1} + c$

d $\frac{3}{2}\log_e(2x+1) + c$

e $\frac{1}{2}\log_e(6-10x) + c$

f $\frac{1}{3}(2x-5)^6 + c$

g $-\frac{3}{8}(4x+1)^{-2} + c$

h $\frac{1}{4}x^2 + 4x + 8\log_e x + c$

i $\log_e x + 2x^{-1} + \frac{15}{2}x^{-2} + c$

j $\frac{2}{3}x^{\frac{3}{2}} - 2\log_e(3-x) + c$

k $2x^{\frac{5}{2}} - x^2 + \frac{9}{2}x^{\frac{2}{3}} + c$

l $\frac{1}{2}x^2 + \frac{1}{4}x^4 + c$

m $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

n $-5x^{-2} + x^{-1} + x^2 + c$

11 **a** $f(x) = 2x^2 + x + 2$

b $f(x) = 5x - x^2 - 5$

c $f(x) = 3x + 2 - \frac{1}{x}$

d $f(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} - \frac{10}{3}$

e $f(x) = \frac{3}{4}x^{\frac{4}{3}} - x^3 + 50x$

f $f(x) = 2\sqrt{x} - x^2 - 6$

g $f(x) = \frac{1}{4}(x+4)^4 + 1$

h $f(x) = (1-2x)^{-4} + 2$

i $f(x) = \log_e(x+5) + 2$

j $f(x) = -4\log_e(7-2x) + 7$

12 **a** $k = -8$

b $y = 41$

13 **a** $k = -1$

b $g(x) = 4 - \frac{2}{\sqrt{x}} - \log_e x$

c $3 - \log_e 4$

Exercise 7C — Integration of e^x, sin x and cos x

1 **a** $\frac{1}{2}e^{2x} + c$ **b** $\frac{1}{4}e^{4x} + c$ **c** $-e^{-x} + c$

d $-\frac{1}{3}e^{-3x} + c$ **e** $e^{5x} + c$ **f** $\frac{7}{4}e^{4x} + c$

g $\frac{1}{12}e^{6x} + c$ **h** $\frac{2}{9}e^{3x} + c$ **i** $-\frac{1}{2}e^{6x} + c$

j $4e^{-2x} + c$ **k** $3e^{\frac{x}{3}} + c$ **l** $0.4e^{\frac{x}{4}} + c$

m $6e^{\frac{x}{2}} + c$ **n** $-9e^{-\frac{x}{3}} + c$ **o** $e^x - e^{-x} + c$

p $\frac{e^x + e^{-x}}{2} + c$

2 $x + 2e^x + \frac{1}{2}e^{2x}$

3 $\frac{1}{3}e^{3x} - \frac{3}{2}e^{2x} + 3e^x - x$

4 $\frac{1}{4}x^4 - x^3 + 2e^{3x}$

5 **a** D **b** C

6 **a** $-\frac{1}{3}\cos 3x + c$ **b** $-\frac{1}{4}\cos 4x + c$

c $\frac{1}{7}\sin 7x + c$ **d** $\frac{1}{6}\sin 2x + c$

e $\frac{1}{2}\cos(-2x) + c$ **f** $-\frac{1}{3}\sin(-3x) + c$

g $-\frac{2}{9}\cos 6x + c$

h $2\sin 4x + c$

i $2\cos 3x + c$

j $2\sin(-x) + c$

k $-3\cos\frac{x}{3} + c$

l $2\sin\frac{x}{2} + c$

m $12\cos\left(\frac{-x}{4}\right) + c$

n $10\cos\frac{x}{5} + c$

o $16\sin\frac{x}{4} + c$

p $12\sin\left(\frac{-x}{2}\right) + c$

q $-6\cos\frac{2x}{3} + c$

r $8\sin\frac{3x}{4} + c$

s $\frac{4}{5}\cos\frac{5x}{2} + c$

t $-\frac{12}{7}\sin\frac{7x}{4} + c$

u $-\frac{5}{\pi}\cos\pi x + c$

v $\frac{6}{\pi}\sin\frac{\pi x}{2} + c$

w $-\frac{6}{\pi}\sin\frac{\pi x}{3} + c$

x $-\frac{\pi}{4}\cos\left(\frac{-4x}{\pi}\right) + c$

7 a $\sin x - \cos x + c$

b $-\frac{1}{2}\cos 2x - \sin x + c$

c $\frac{1}{4}\sin 4x - \frac{1}{2}\cos 2x + c$

d $-2\cos\frac{x}{2} - \frac{1}{2}\sin 2x + c$

e $\sin 4x + \frac{1}{6}\cos 2x + c$

f $\frac{5}{2}x^2 - 2\cos x + c$

g $\frac{6}{\pi}\left(\sin\frac{\pi x}{3} - \cos\frac{\pi x}{2}\right) + c$

h $\frac{1}{2}e^{6x} + \frac{1}{2}\cos 8x + 7x + c$

8 $\frac{1}{4}e^{4x} - \frac{1}{2}\cos 2x + \frac{1}{4}x^4 + c$

9 $x^3 - \sin 2x + 2e^{3x}$

10 a $\frac{1}{4}x^4 - \frac{1}{2}\log_e(2x + 3) + \frac{1}{2}e^{2x} + c$

b $\frac{1}{3}x^3 + 2\sin 2x + e^{-x} + c$

c $-3\cos\frac{x}{3} + 2e^{\frac{x}{2}} - \frac{1}{15}(3x - 1)^5 + c$

d $\frac{1}{3}\log_e(3x - 2) + \frac{1}{4}e^{4x} + 5\sin\frac{x}{5} + c$

e $-6\cos\frac{x}{2} - 6\sin\frac{x}{3} + 5e^{-\frac{x}{5}} + c$

f $\frac{2}{3}x^{\frac{3}{2}} + x^2 + \frac{6}{\pi}\cos\frac{\pi x}{3} + 5x + c$

11 a $f(x) = 4 + \sin x$

b $f(x) = 1 - 2\cos 2x$

c $f(x) = 3\sqrt{2} + 12\sin\frac{x}{4}$

d $f(x) = 4\sin\frac{x}{4} + 2\cos\frac{x}{2} - 4$

12 a $k = -1$

b $y = -\frac{6}{\pi}\cos\frac{\pi x}{6} - x + 7$

c $y = 1 + \frac{6}{\pi}$

13 a $k = -4$

b $f(x) = 2\sin 2x - 4e^x + 3$

c $f\left(\frac{\pi}{6}\right) = -2.02$

Exercise 7D — Integration by recognition

1 a i $24(3x - 2)^7$ **ii** $\frac{1}{2}(3x - 2)^8 + c$

b i $10x(x^2 + 1)^4$ **ii** $\frac{1}{2}(x^2 + 1)^5 + c$

c i $\frac{1}{\sqrt{2x - 5}}$ **ii** $\sqrt{2x - 5} + c$

d i $\frac{2}{\sqrt{4x + 3}}$ **ii** $\sqrt{4x + 3} + c$

e i $4(2x + 3)(x^2 + 3x - 7)^3$ **ii** $\frac{1}{4}(x^2 + 3x - 7)^4 + c$

f i $-\frac{2x}{(x^2 - 1)^2}$ **ii** $-\frac{2}{x^2 - 1} + c$

2 a A **b** B

3 E

4 a i $4e^{4x - 5}$ **ii** $\frac{1}{2}e^{4x - 5} + c$

b i $-5e^{6 - 5x}$ **ii** $-2e^{6 - 5x} + c$

c i $2xe^{x^2}$ **ii** $\frac{1}{2}e^{x^2} + c$

d i $(1 - 2x)e^{x - x^2}$ **ii** $e^{x - x^2} + c$

5 a i $\cos(x - 5)$ **ii** $\sin(x - 5) + c$

b i $3\cos(3x + 2)$ **ii** $2\sin(3x + 2) + c$

c i $-4\sin(4x - 7)$ **ii** $-\frac{1}{4}\cos(4x - 7) + c$

d i $-6\sin(6x - 3)$ **ii** $-\frac{1}{2}\cos(6x - 3) + c$

e i $-5\cos(2 - 5x)$ **ii** $-2\sin(2 - 5x) + c$

f i $4\sin(3 - 4x)$ **ii** $-\frac{1}{2}\cos(3 - 4x) + c$

g i $\frac{5}{5x + 2}$ **ii** $4\log_e(5x + 2) + c$

h i $\frac{2x}{x^2 + 3}$ **ii** $6\log_e(x^2 + 3) + c$

i i $\frac{2x - 4}{x^2 - 4x}$ **ii** $\frac{1}{2}\log_e(x^2 - 4x) + c$

6 a i $3\cos x - x\sin x$ **ii** $\sin x - x\cos x$

b i $\frac{x\cos x - \sin x}{x^2}$ **ii** $\frac{2\sin x}{x}$

c i $e^x(\sin x + \cos x)$ **ii** $3e^x\sin x$

d i $\sin x + x\cos x$ **ii** $x\sin x + \cos x$

e i $e^x + xe^x$ **ii** $xe^x - e^x$

7 a i $12x^5(1 - 3x)(2 - 3x)^5$ **ii** $\frac{x^6}{2}(2 - 3x)^6 + c$

b i $\frac{3x^2 + 2}{2\sqrt{x^3 + 2x}}$ **ii** $2\sqrt{x^3 + 2x} + c$

8 b $3x + \log_e(x-1) + c$

9 b $5x - 2\log_e(x+2) + c$

10 b $4x + \frac{5}{2}\log_e(2x-3) + c$

11 b $-3x - 2\log_e(3-2x) + c$

12 a $\frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x$

b $-\log_e(\cos x) + c$

13 $-\frac{\cos x}{\sin x}$

14 $\frac{1}{6}\log_e(3x^2 - 4)$

15 $\frac{1}{a}\sin(ax+b)$

16 $-\frac{1}{a}\cos(ax+b)$

17 $\frac{1}{a}\ e^{ax+b}$

18 a $-\frac{1}{3\pi}\cos(3\pi x + 1) + c$

b $-\frac{1}{4\pi}\sin(1 - 4\pi x) + c$

c $\frac{1}{\pi}e^{\pi x + 3} + c$

d $-\frac{3}{\pi}\cos\left(2 + \frac{\pi x}{3}\right) + c$

e $\frac{6}{\pi}\sin\left(\frac{\pi x}{2} + 5\right) + c$

f $e^{\sin x} + c$

Chapter review

1 C **2** C **3** E

4 $\frac{1}{2}(e^7 + 2e^5 + 2e^3 + 2e + e^{-1})$ sq. units

5 Approx. 2.15 sq. units

6 B **7** E **8** C

9 $f(x) = x^3 - x^2 - 3$

10 A **11** D **12** E **13** C

14 a $k = 0$

b $\frac{4}{\pi}\sin\frac{\pi x}{4} + 1 - \frac{4}{\pi}$

c $1 - \frac{8}{\pi}$

15 B **16** D **17** A

18 $\frac{dy}{dx} = 2(x+1)\cos(x^2 + 2x)$ and antiderivative is $\frac{1}{2}\sin(x^2 + 2x)$

Modelling and problem solving

1 a \$0

b $C = 25t^3 + 25t^2 + 800t$

c \$7750

d \$4450

e \$1050

2 a $T = 20 - 5\sin\frac{\pi t}{12}$

b No

c 25 °C at 6 pm

d 15 °C at 6 am

e i 17.5 °C **ii** Approx. 23.5 °C

f 1.50 pm

3 768 students are not infected

4 a $r = 10 - 0.2t$ **b** $V = \frac{4}{3}\pi(10 - 0.2t)^3$

c -12.8π mm^3/day

5 10.5 square units

6 27 hours

CHAPTER 8 Techniques of integration

Exercise 8A — The fundamental theorem of integral calculus

1 a $\frac{1}{3}$ **b** $20\frac{1}{4}$ **c** $5\frac{1}{3}$

d $\frac{1}{3}$ **e** 8 **f** 56

g $12\frac{2}{3}$ **h** -74 **i** 38

j -1 **k** 6313.2 **l** 5949

m -1360 **n** $-\frac{60}{49}$ **o** $24.8 - 3\log_e 4$ or 20.64

p $37\frac{1}{3}$ **q** $\frac{3}{5}\log_e 5$ or 0.966 **r** $\frac{5}{256}$ or 0.0195

s 2 **t** 3.65

2 a $\frac{1}{4}(e^8 - 1)$ **b** $3(1 - e^{-\frac{2}{3}})$

c $2(e^{-2} - e^2)$ **d** $\frac{1}{2}(e^{12} + 15 - e^6)$

e $5\log_e 4 + 2e^2 - 2e^{\frac{1}{2}}$ **f** $\frac{1}{2}(e^{-2} + e^2 - e^{-6} - e^6)$

g 1 **h** 0

i $20 - 10\sqrt{2}$ **j** -6

k 0 **l** 0

m $9\sqrt{3} - 9$ **n** 0

o $-\frac{4}{3}$ **p** $\pi(2\pi - 2\sqrt{2} + 4) \approx 23.4$

q $14\frac{2}{3} + 2\cos 9^c - 2\cos 3^c$ **r** $6 + \log_e \pi - 6\sin 0.5^c$

3 18

4 a B **b** A

5 a -9 **b** $-\frac{2}{3}$ **c** 3

d 10 **e** $4e^{\frac{\pi}{4}} - 4$ **f** $2\log_e 5$

g -3 **h** $\frac{1}{3}(e^{-3} - e^3)$

6 $k = 1$ or -4

7 $k = 2$

8 $k = 3$

9 $a = 2\log_e 3$

10 $k = \frac{\pi}{6}$

Exercise 8B — Signed areas

The answers to all area questions are in square units.

1 a 8 **b** 8

2 a 4.5 **b** 4.5

3 a $\int_1^3 2x\,dx$ **b** $\int_0^4 (4 - x)\,dx$

c $\int_{1}^{2} x^2\,dx$ **d** $\int_{-3}^{-1} 3x^2\,dx$

e $\int_{1}^{3} (x^3 - 9x^2 + 20x)\,dx$ **f** $\int_{-2}^{0} (-x^3 - 4x^2 - 4x)\,dx$

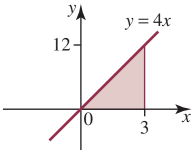
g $\int_{-1}^{1} e^x\,dx$ **h** $\int_{1}^{4} e^{-2x}\,dx$

i $\int_{0}^{\frac{\pi}{2}} 2\sin 2x\,dx$ **j** $\int_{0}^{\frac{3\pi}{2}} \cos\frac{x}{3}\,dx$

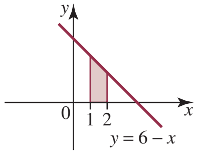
4 a 8 **b** 8 **c** $\frac{7}{3}$ **d** 26 **e** 22

f $1\frac{1}{3}$ **g** $e - e^{-1}$ **h** $-\frac{1}{2}(e^{-8} - e^{-2})$ **i** 2 **j** 3

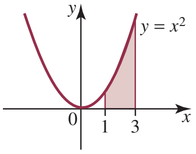
5 a



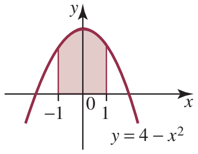
b



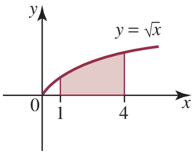
c



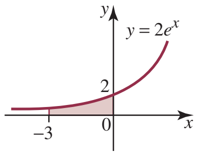
d



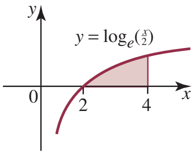
e



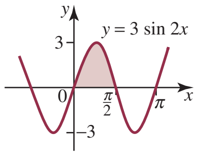
f



g



h



6 a 2 **b** 1 **c** $5\frac{1}{3}$ **d** $\frac{4}{3}$

e 4 **f** $2\frac{2}{3}$ **g** $e - e^{-1}$ **h** $\frac{1}{2}(e^{-1} - e^{-2})$

i 1 **j** 4

7 a E **b** A

8 a $\int_{0}^{2} f(x)\,dx - \int_{2}^{5} f(x)\,dx$

b $\int_{1}^{3} g(x)\,dx - \int_{-3}^{1} g(x)\,dx$

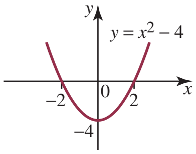
c $\int_{-1}^{2} h(x)\,dx - \int_{-3}^{-1} h(x)\,dx$

d $\int_{-4}^{-2} f(x)\,dx - \int_{-5}^{-4} f(x)\,dx$

e $\int_{2}^{3} g(x)\,dx - \int_{-2}^{2} g(x)\,dx + \int_{-3}^{-2} g(x)\,dx$

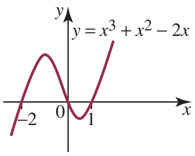
9 a B **b** D **c** E

10



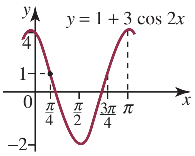
a $5\frac{1}{3}$ **b** $10\frac{2}{3}$ **c** 16

11



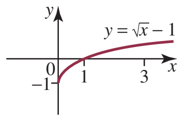
a $2\frac{2}{3}$ **b** $\frac{5}{12}$ **c** $3\frac{1}{12}$

12



a $\frac{\pi}{4} + \frac{3}{2}$ **b** $\frac{\pi}{4} + \frac{3}{2}$

13



Exact area is $2\sqrt{3} - 1 - \frac{4\sqrt{2}}{3}$ or approx. 0.578.

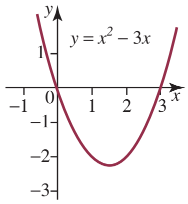
14 $2\log_e 2$

15 $8 + e^3 - e^{-1}$

Exercise 8C — Further areas

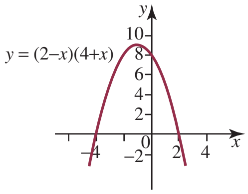
All answers for areas are exact and in square units unless stated otherwise.

1 a i



ii $4\frac{1}{2}$

b i



ii 36

2 a $85\frac{1}{3}$ **b** $57\frac{1}{6}$ **c** $\frac{64\sqrt{2}}{3}$ **d** $21\frac{1}{3}$ **e** $3\frac{1}{12}$

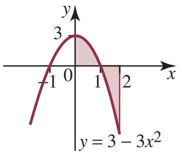
f $49\frac{1}{3}$ **g** 8 **h** $40\frac{1}{2}$

3 A

4 D

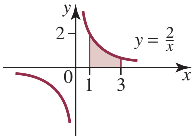
5 C

6 **a** **i**



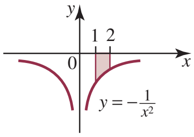
ii 6

b **i**



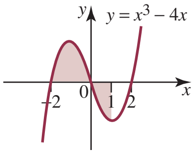
ii $2\log_e 3$

c **i**



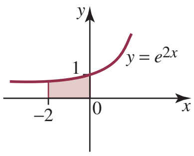
ii $\frac{1}{2}$

d **i**



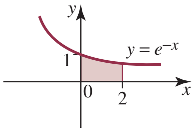
ii $5\frac{3}{4}$

e **i**



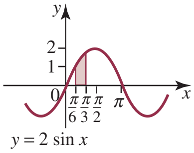
ii $\frac{1}{2}\ (1 - e^{-4})$

f **i**



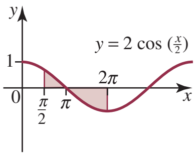
ii $1 - e^{-2}$

g **i**



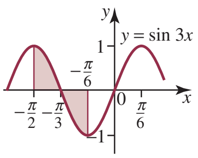
ii $\sqrt{3} - 1$

h **i**



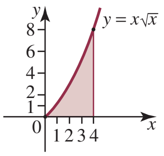
ii $4 - \sqrt{2}$

i **i**



ii $\frac{2}{3}$

j **i**



ii $12\frac{4}{5}$

7 **a** **i** $x = -2$ and 2 **ii** $1 - 4\log_e \frac{3}{4}$ (or approx. 2.15)

b **i** $x = \dfrac{\pi}{4}$ **ii** $2\sqrt{2}$

c **i** $x = 1$ **ii** $e^3 + 1 - 3e$ (or approx. 12.93)

d **i** $x = 1$ **ii** $1\frac{5}{8}$

e **i** No x-intercepts **ii** $2e - 2e^{-1}$ (or approx. 4.7)

f **i** $x = 2$ and -2 **ii** 139.2

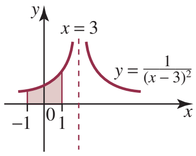
g **i** $x = 2$ **ii** $\frac{2}{5}$

8 $\frac{1}{3}\ (e^6 - e^3)$

9 $\dfrac{3 - \sqrt{3}}{2}$

10 $\frac{1}{4}$

11 **a**



b $\frac{1}{4}$

12 **a** $x = -2$ and $y = 0$ **b** $\frac{4}{9}$

13 x-intercept is $\frac{1}{2}\log_e 3$ (or 0.55), area is $\dfrac{11 + e^{-4}}{2}$

14 No x-intercepts, area is $6\pi + 1$.

15 **a** $1 + \log_e x$

b $x\log_e x - x$

c $4\log_e 4 - 3$ (or approx. 2.55)

16 **a** $\dfrac{2x}{x^2 + 2}$

b $\frac{1}{2}\log_e (x^2 + 2)$

c $\log_e 1.5$ (or approx. 0.41)

17 **a** $2\frac{2}{3}$ **b** $5\frac{1}{3}$

18 $\dfrac{(3e^4 + 1)}{2}$ (or approx. 82.40)

19 $2\pi - 4$ (or approx. 2.28)

Exercise 8D — Areas between two curves

All answers are in square units unless stated otherwise.

1 **a** $\displaystyle\int_0^1 (x - x^2)\, dx$ **b** $\displaystyle\int_1^2 (x - 1)\, dx$

c $\displaystyle\int_0^1 (4 - x^2 - 3x)\, dx$ **d** $\displaystyle\int_{-2}^2 (8 - 2x^2)\, dx$

e $\displaystyle\int_0^{\sqrt{3}} (3x - x^3)\, dx$ **f** $\displaystyle\int_{-1}^1 (9 - x^2 - e^x)\, dx$

g $\displaystyle\int_1^2 (x + e^x)\, dx$ **h** $\displaystyle\int_{-1}^1 (1 - x^2)\, dx$

2 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** $2\frac{1}{6}$ **d** $21\frac{1}{3}$ **e** $2\frac{1}{4}$

f $17\frac{1}{3} - e + e^{-1}$ (approx. 14.98)

g $1\frac{1}{2} + e^2 - e$ (or approx. 6.17)

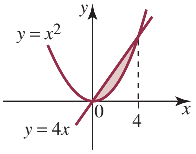
h $\frac{4}{3}$

3 C

4 D

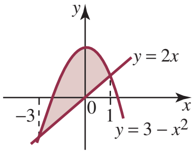
5 E

6 **a** **i** $x = 0$ and 4 **ii**



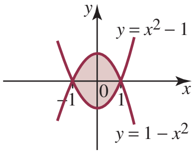
iii $10\frac{2}{3}$

b **i** $x = -3$ and 1 **ii**



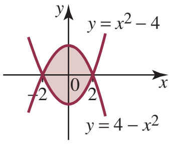
iii $10\frac{2}{3}$

c **i** $x = -1$ and 1 **ii**



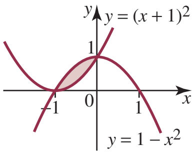
iii $2\frac{2}{3}$

d **i** $x = -2$ and 2 **ii**



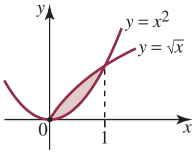
iii $21\frac{1}{3}$

e **i** $x = 0$ and -1 **ii**



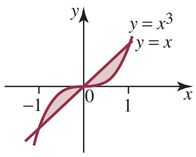
iii $\frac{1}{3}$

f **i** $x = 0$ and 1 **ii**



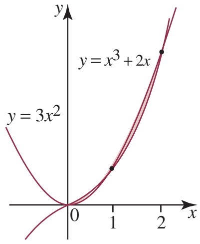
iii $\frac{1}{3}$

7 **a** **i** $x = -1$, 0 and 1 **ii**



iii $\frac{1}{2}$

b **i** $x = 0$, 1 and 2 **ii**



iii $\frac{1}{2}$

8 **a** $\frac{2}{3}$ **b** $2\sqrt{2}$ **c** 4 **d** $23\frac{1}{12}$

e $1\frac{1}{8}$

f $e - e^{-1} - 2$ (or approx. 1.09)

g $2 + \frac{\pi^2}{8}$ (or approx. 3.23)

h $2e - 2e^{-2}$ (or approx. 5.17)

9 $e^3 - e - 4$ (or approx. 13.37)

10 $3\frac{1}{2}$ **11** $\frac{1}{2}$

12 $\frac{\pi\sqrt{3}}{12}$ (or approx. 0.45)

13 $3 \log_e 2 - 2$

14 **a** $x = -4$ and 4

b $x = -3$ and 3

c $9\frac{1}{3}$

15 $2\frac{1}{6}$ m^2

16 **a** $x = -20$ and 20

b 40 m

c $93\frac{1}{3}$ m^2

d 840 m^3

17 **a** $\left(150 - \frac{300}{\pi}\right)$ m^2

b $\left(30\,000 - \frac{60\,000}{\pi}\right)$ m^3

18 154 trees

Exercise 8E — Further applications of integration

1 $f(x) = -\frac{(2-x)^3}{3} + 4$

2 $1 + \frac{\pi}{12}$

3 **a** 0 m

b $y = -0.01(x + 1)^3 + 0.03x + 0.01$ or $y = -0.01x^3 - 0.03x^2$

c 54 cm

4 **a** \$34.56

b $C = 40n - 200e^{0.01n} + 200$

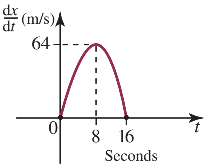
c \$3656.34

d **i** \$36.56 **ii** \$30.66

5 **a** **i** 0 m/s **ii** 48 m/s

b **i** $t = 8$ **ii** 64 m/s

c



d $466\frac{2}{3}$ m

e The distance travelled in 10 seconds

6 **a** 4 L

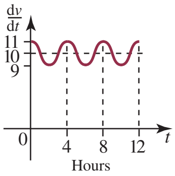
b $V = 5t + \frac{40}{\pi} \sin \frac{\pi t}{40}$

c 200 L

d 4.79 L/min

e 5 h 20 min

7 **a**



b 4 h

c **i** 60 L **ii** 50.6 L

8 **a** 56 250 m^3 **b** 5

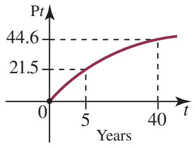
9 1.26 m

11 **a** $y = -\frac{3}{8}x^2 + 6$ **b** 38 m^2 **c** 380 m^3

12 **b** $3\log_e 2 - 2$

13 **a** 0, 21.5 and 44.6 kangaroos per year.

b



c $P^{-1}(t) = e^{\frac{t}{12}} - 1$

d 1347

e The population of kangaroos on 1 January 1995.

Investigation — Concrete chute

a 260 cm^2 **b** 2.496 m^3 **c** 4 min 48 s

Chapter review

1 B

2 E

3 C

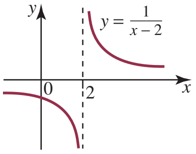
4 **a** $-1\frac{4}{9}$ **b** $-\frac{\sqrt{3}}{2}$

5 $k = \frac{1}{2}$ or 2

6 D

7 A

8 **a**



b $\log_e 4$

9 B **10** C **11** E **12** D

13 $166\frac{2}{3}$ sq. units

14 B **15** A

16 $\left(\frac{\pi^2}{8} + 2\right)$ sq. units (or approx. 3.23)

17 **a** $(2, e)$

b $y = (-2e^{-1})x + e + 4e^{-1}$

c $(0, e + 4e^{-1})$

d $(0, 1)$

e $(2 + 4e^{-1})$ sq. units

18 **a** $1 + \log_e x$

b $x\log_e x - x$

c 3 m

d $(e - e^{-2})$ m^2 (or approx. 2.58 m^2)

e $20(e - e^{-2})$ m^3 (or approx. 51.66 m^3)

Modelling and problem solving

1 **a** $y = -\frac{1}{3}x^2 + 3$

b 12 square metres

d 24 square metres

2 **a** 19 weeks

b 0.75 square metres

3 $6050

4 **a** $N = \frac{A}{-0.2}e^{-0.2t}$

b $N = 10\,000$

c There are 6703 people with the disease after 2 years.

d After 10 years there are 1353 people with the disease.

e No, because it is an exponential curve and therefore the number of people with the disease approaches zero but never reaches it.

f It takes about 23 years to reduce the number of cases to 100.

CHAPTER 9 Probability distributions

Exercise 9A — Discrete random variables

1 **a** Discrete **b** Continuous

c Continuous **d** Discrete

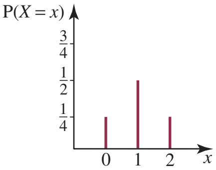
e Continuous **f** Discrete

g Continuous **h** Discrete

2 **a**

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b



3 **a** HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

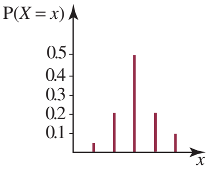
b $x = 0, 1, 2, 3$

c

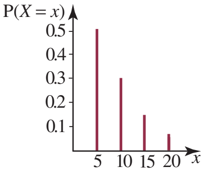
x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

d $\frac{7}{8}$

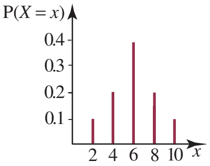
4 **a**



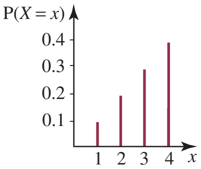
b



c



d



5 **a, d**

6 **a** 0.2 **b** 0.6 **c** $\frac{1}{11}$ **d** 0.1

7 **a**

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b $\frac{1}{6}$ **c** $\frac{5}{18}$ **d** $\frac{7}{36}$ **e** $\frac{29}{36}$

f $\frac{35}{36}$ **g** $\frac{5}{9}$

8 **a**

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

b $\frac{2}{5}$ **c** $\frac{3}{5}$

9 **a**

x	1	4	9	16	25	36
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b $\frac{5}{6}$ **c** $\frac{1}{2}$

10 **a**

x	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

b $\frac{5}{6}$ **c** $\frac{2}{5}$

11

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{45}$	$\frac{1}{9}$	$\frac{1}{5}$	$\frac{13}{45}$	$\frac{17}{45}$

12

x	1	2	3	4
$P(X = x)$	$\frac{3}{160}$	$\frac{1}{10}$	$\frac{9}{32}$	$\frac{3}{5}$

Note: For all $X = x$, **i** $0 \le P(X = x) \le 1$ and **ii** $\Sigma P(X = x) = 1$.

13 **a**

x	0	1	2	3
$P(X = x)$	0.1715	0.4115	0.3292	0.0878

Note: For all $X = x$, **i** $0 \le P(X = x) \le 1$ and **ii** $\Sigma P(X = x) = 1$.

b 0.1060

14

x	0	1	2	3
$P(X = x)$	$\frac{14}{55}$	$\frac{28}{55}$	$\frac{12}{55}$	$\frac{1}{55}$

15 **a**

x	0	1	2
$P(X = x)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{9}{25}$

b $\frac{4}{25} + \frac{12}{25} + \frac{9}{25} = 1$

16 **a** 0.54 **b** 0.55 **c** 0.54 **d** 0.24

e 0.5636 **f** 4 **g** 4

17 C **18** C **19** E **20** D

21 B **22** D

23 **a** $x = 0, 1, 2, 3, 4$

b

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Exercise 9B — Expected value of discrete random distributions

1 $2\frac{7}{16}$ **2** $\frac{13}{18}$ **3** 1.16 **4** 7.11 **5** 4.47

6 $a = 0.09$, $E(X) = 5.42$

7 $a = \frac{1}{18}$, $E(X) = 5\frac{1}{3}$

8 $b = 0.15$, $E(X) = 2.39$

9 $k = 0.1$, $E(X) = 6$

10 $k = 0.05$, $E(X) = 13$

11 **a**

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

b $3\frac{1}{2}$

12 **a**

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

b 7

13 **a**

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

b 1

14 **a**

x	0	1	2	3	4
$P(X = x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

b 2

15 $a = 0.15$, $b = 0.05$

16 $a = 0.1$, $b = 0.2$

17 **a** \$3.75

b No, because although his expected gain is \$3.75 per game, he must pay \$5 to play each game. Therefore his loss per game will be \$1.25.

c No, because the expected gain is not equal to the initial cost of the game.

18 11

19 17

20 **a**

x	0	1	2	3
$P(X = x)$	$\frac{8}{125}$	$\frac{36}{125}$	$\frac{54}{125}$	$\frac{27}{125}$

b $1\frac{4}{5}$

21 **a** $2\frac{1}{3}$ **b** $9\frac{1}{3}$ **c** $5\frac{2}{3}$ **d** $6\frac{1}{15}$

22 **a** 2.24 **b** 2.96 **c** 7.16 **d** 18.48

23 \$1452

Exercise 9C — The binomial distribution

1 **b**, **d** and **f** constitute a binomial probability distribution.

2 **a** 0.2613 **b** 0.0446 **c** 0.2461

d 0.0092 **e** 0.0073 **f** 0.1969

3 **a** 5 **b** 0.3

c

x	0	1	2	3	4	5
$P(X = x)$	0.16807	0.36015	0.3087	0.1323	0.02835	0.00243

4 0.135 93

5 0.0768

6 **a** $\frac{256}{625}$ **b** $\frac{96}{625}$ **c** $\frac{369}{625}$

7 **a** 0.0625 **b** 0.0625 **c** 0.375

8 **a** 0.0518 **b** 0.2592

9 **a** 0.2627 **b** 0.0084 **c** 0.2568 **d** 0.2568

10 0.0381

11 0.0924

12 **a** 0.1023 **b** 0.2001

13 0.2281

14 0.0653

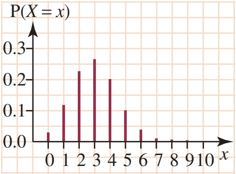
15 **a** 0.0528 **b** 0.6676

16 **a** 0.2734 **b** 0.2965 **c** 0.1678

17 C **18** E **19** B **20** D **21** C

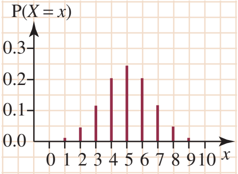
22 22

23 **a** **i**



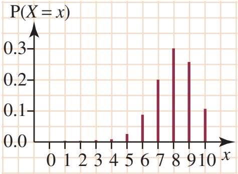
ii is positively skewed

b **i**



ii is symmetrical or normally distributed

c **i**



ii is negatively skewed

24 **a** 2 **b** positively skewed

25 **a** **i** Positively skewed

ii Symmetrical or is normally distributed

iii Negatively skewed

b Controls skewness

26 **a** Symmetrical or is normally distributed

b The curve would resemble more of a bell shape which is tall and narrow.

27 **a** Positively skewed

b Negatively skewed

28 **a** Symmetrical or is normally distributed

b Positively skewed

c Symmetrical or is normally distributed

d Negatively skewed

29 **a** 5 645 717 **b** 352 858 **c** $1 446 717

Exercise 9D — Problems involving the binomial distribution for multiple probabilities

1 **a** 0.1792 **b** 0.6826

2 **a** 0.0540 **b** 0.8840

c 0.9948 **d** 0.4284

3 **a** 0.4718 **b** 0.9692

4 **a** 0.3370 **b** 0.3438

c 0.0333

5 **a** 0.0623 **b** 0.3438

6 **a** 0.2188 **b** 0.0352

c 0.1445 **d** 0.1445

7 **a** 0.0878 **b** 0.9822

c 0.0178

8 **a** 0.3669 **b** 0.0464

9 **a** 0.9830 **b** 0.0170

10 **a** 0.5000 **b** 0.9967

c 0.9645

11 **a** 0.2553 **b** 0.1045

12 **a** 0.1751 **b** 0.0930

13 A

14 D

15 A

16 D

17 C

18 D

19 0.1509

20 **a** 0.9392 **b** 0.1878

21 **a** 0.9106 **b** 0.9216

c 0.6767 **d** 0.5327

22 **a** 0.0173 **b** 0.2276

c 0.8960

23 **a** 0.5981 **b** 0.4256

c 0.2486

24 **a** 0.6778 **b** 0.0973

25 **a** 0.3438 **b** 0.0760

26 **a** 0.9672 **b** 0.9573

Exercise 9E — Expected value, variance and standard deviation of the binomial distribution

1 **a** 6 **b** 1.6

c 50 **d** $37\frac{1}{2}$

2 **a** 4.8 **b** 1.35

c 6 **d** $3\frac{3}{4}$

3 **a** 1.26 **b** 2.74

c 3.24 **d** 4.16

4 **a** 5 **b** 2.5 **c** 1.58

5 **a** 4.62 **b** 3.55 **c** 1.88

6 **a** 12 **b** 4.8 **c** 2.19

7 **a** 1.67 **b** 0.2248

8 **a** 96 **b** 24

9 **a** $\frac{1}{2}$ **b** 20

10 **a** $\frac{3}{4}$ **b** 16

11 **a** $\frac{1}{3}$ **b** 27 **c** 0.1450

12 **a** $\frac{1}{5}$ **b** 15 **c** 0.0001 **d** 0.3980

13 C

14 E

15 C

16 E

17 D

18 D

19 **a** $\frac{4}{5}$ **b** 3.2 **c** 1.79
20 **a** 0.0020 **b** 0.7939
21 0.2642
22 **a** 0.2335 **b** 0.3828
23 400
24 **a** 0.5000 **b** 0.6535 **c** 0.3465
25 **a** 50 000 **b** 25 000 **c** 158.11
26 **a** 6
b The company has an extremely popular product.
27 **a** 0.5455 **b** 0.3340 **c** 0.6
d 0.8795 **e** 0.2769

Investigation — The binomial theorem

1 **a** A binomial distribution has been described, that is $X \sim \text{Bi}(8, \frac{5}{7})$.
b 0.2429
c 0.7154
d 0.0087
2 **a** 0.0043
b The answer in **1b** incorporates all the probabilities of having 5 wins and 3 losses. The answer in **2a** is for a specific combination.

Chapter review

1 **a**

x	1	2	3	4
$P(X = x)$	$\frac{1}{30}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{8}{15}$

b $3\frac{1}{3}$

2 **a**

x	2	3	4	5	6	7
$P(X = x)$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{7}{36}$	$\frac{1}{36}$

b $4\frac{1}{12}$ **c** $\frac{7}{12}$

3 Yes
4 $1.22
5 No
6 **a** 0.11 **b** 1.93 **c** $39.60 **d** 0.016
7 **a** $19.30 **b** $1430
8 **a** 4
b

x	0	1	2	3	4
$P(X = x)$	0	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{3}{10}$	$\frac{1}{2}$

c $3\frac{1}{4}$

9 **a** 0.2734 **b** 0.0078
10 **a** 0.0988 **b** 1.0613×10^{-5} **c** 2.4414×10^{-16}
11 **a** 0.0355 **b** 0.8125 **c** 0.9688
12 **a** 0.0510 **b** 0.0016
13 **a** 0.9936 **b** 0.6778
14 **a** **i** 50 **ii** 25 **iii** 5
b **i** 35 **ii** 10.5 **iii** 3.24
c **i** 16 **ii** 12.8 **iii** 3.58
15 **a** 0.2 **b** 50

Modelling and problem solving

1 **a** $1.25 **b** $16.25
c $81.25 (*Note:* Maria works only 5 days.)
2 **a** 0.0490 **b** 0.1486
3 **a** 5.9093×10^{-6} **b** 0.1075
4 **a** 0.9943 **b** 0.0308
5 **a** **i** 3.33 **ii** 2.5 **iii** 4.17
b 2.22 **c** 1.49
6 **a** **i** $720 **ii** Loss of 54c **iii** No
b **i** $40 **ii** Loss of 54c **iii** No
c **i** $60 **ii** Loss of 54c **iii** No
d 2.7%
7 **a** 135 **b** 0.1 **c** $60
8 **a** 1 **b** 0.0015
c 0.2639 **d** 10
9 **a** 1 **b** 0.2642 **c** 0.0003

CHAPTER 10 The normal distribution

Exercise 10A — The standard normal distribution

1 **a** 0.8413 **b** 0.9893 **c** 0.9357
d 0.7703 **e** 0.8914 **f** 0.9963
2 **a** 0.0228 **b** 0.0668 **c** 0.1112
d 0.4364 **e** 0.1333 **f** 0.0043
3 **a** 0.0228 **b** 0.0968 **c** 0.0401
d 0.0129 **e** 0.0102 **f** 0.2517
4 **a** 0.9987 **b** 0.9821 **c** 0.9616
d 0.9966 **e** 0.8727 **f** 0.7395
5 **a** 0.1359 **b** 0.0486 **c** 0.2721
6 **a** 0.8644 **b** 0.6195 **c** 0.4810
d 0.1051
7 **a** 3 **b** −1.667 **c** 1.2
8 **a** 0.333 **b** −0.5 **c** 1.133
9 **a** 0.3874 **b** 0.9235 **c** 0.7623
d 0.0317 **e** 0.7462
10 **a** 0.0808 **b** 0.6554 **c** 0.0082
d 0.0501 **e** 0.3260 **f** 0.5486
11 **a** 0.0863 **b** 0.0115 **c** 0.9022
12 **a** 0.1587 **b** 0.3874 **c** 0.6826
d 0.8302 **e** 0.2391
13 A **14** B **15** E **16** E
17 A **18** A **19** E **20** 0.8676
21 **a** 0.6006 **b** 0.0548
c

Cost ($)	1.40	1.60	1.80	2.00
Probability	0.0082	0.3364	0.6006	0.0548

d $1.74
22 **a** 0.9804 **b** 980
23 **a** 0.1056 **b** 17

Exercise 10B — The inverse cumulative normal distribution

1 **a** 1.282 **b** 0.253 **c** 0.675
d 0.050
2 **a** −0.524 **b** −0.842 **c** −0.385
d −0.202
3 **a** −0.842 **b** −0.385 **c** −0.100
d −0.583

4 a 0.524 b 0.126 c 0.772
5 a 1 b 0.675 c 0.253
d 0.496
6 a −0.675 b −0.253 c 0.583
d 2.576
7 a 11.166 b 9.493 c 9.336
d 11.683
8 a 35.984 b 36.698 c 30.774
d 33.497
9 a 5.182 b 1.528 c 17.525
10 E 11 A 12 C 13 B
14 A 15 D 16 B
17 a 173.16 cm b 153.27 cm
18 a 44.58 mm b 45.42 mm
19 48.65 seconds
20 1.907
21 7.896
22 11.704 grams
23 26.305
24 25.844
25 3 minutes 44.27 seconds

Investigation — Sunflower stems

a Stem lengths ≥ 85.26 cm
b 81.74 cm ≤ stem lengths < 85.26 cm
c 79.19 cm ≤ stem lengths < 81.74 cm

Exercise 10C — The normal approximation to the binomial distribution

1 a $X \sim N(260, 124.8)$ b $X \sim N(192, 99.84)$
c $X \sim N(150, 75)$ d $X \sim N(137.5, 61.875)$
2 a i 0.60 ii 0.0194
b i 0.5 ii 0.0176
3 0.1208 4 0.1980 5 0.2836 6 0.7002
7 0.1611 8 0.9506 9 C 10 B
11 C 12 B
13 a 0.6554 b 0.5793 c 0.6826
14 a 0.6725 b 0.9102
15 a 0.3221 b 0.0529
16 a 0.5565 b 0.2856

Investigation — Supporting the proposal

a i 0.0010
ii 0.5000
b Binomial: 0.4148
Difference = 0.5000 − 0.4148
= 0.0852
Approximated value compares well with actual value.

Exercise 10D — Hypothesis testing[*]

1 H_0: The extra training has no effect.
H_1: The extra training improves Michael's times.
2 a The television campaign will have no effect on sales.
b Extra study will have no effect on exam marks.
c The life of milk in a bottle is not less than 4 days.
d The average mass of a packet of chips is not at least 250 g.
3 a The television campaign will increase sales.
b Extra study will improve exam marks.
c The life of milk in a bottle is less than 4 days.
d The average mass of a packet of chips is at least 250 g.
4 No, $P(X \geq 7) = 0.1719$
5 Yes, $P(X \geq 8) = 0.0043$
6 $P(X \geq 17) = 0.107$
a Yes b No c No
7 No, $P(X \leq 4) = 0.1937$
8 Yes, $P(X \leq 2) = 0.1446$
9 Yes, $P(X \leq 4) = 0.2891$
10 a H_0: The electricity bill is not increasing.
H_1: The electricity bill is increasing.
b $P(X \geq 6) = 0.1446$
c 15% level of significance

Chapter review

1 a 0.9953 b 0.8643 c 0.9099
d 0.8365 e 0.0668 f 0.0107
g 0.1190 h 0.2327 i 0.1587
j 0.0359 k 0.0146 l 0.0084
m 0.9772 n 0.9938 o 0.9463
p 0.9788
2 a 0.9482 b 0.4931 c 0.1242
d 0.0200
3 a 0.6667 b 0.5 c 0.51
4 a 0.4336 b 0.9666 c 0.9332
d 0.3085 e 0.3781
5 a 0.0062 b 0.8413 c 0.1535
6 a 0.524 b 1.645 c −1.282
d −0.583 e −0.553 f −1.227
g 0.253 h 0.613
7 a 26.28 b 22.41 c 42.04
d 55.73
8 a 2.34 cm b 0.1963 c 26.77 cm
9 a $X \sim N(110, 55)$ b $X \sim N(294, 149.94)$
c $X \sim N(165.75, 81.2175)$ d $X \sim N(470, 249.1)$
10 0.1028
11 0.2347
12 0.1241
13 a H_0: The alternative route makes no difference to travelling time.
H_1: The alternative route shortens travelling time.
b H_0: Not many overweight vehicles use Fiona's street.
H_1: Too many overweight vehicles are using Fiona's street.
c H_0: Using the software will make no impact on Mathematics marks.
H_1: Using the software will increase Mathematics marks.
14 a H_0: The coin is not biased.
H_1: The coin is biased.
b $P(X \geq 16) = 0.0059$
c Significant at the 6% level of confidence
15 $P(X \leq 5) = 0.1051$, not significant at 10% level of confidence

---

[*] This is not a part of the Maths Quest 12 Maths B course, but may be useful as extension.

Index