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ISBN 978-1-009-65470-8

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CambridgeMATHS NSW STAGE 6

SECOND EDITION

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INCLUDES INTERACTIVE
TEXTBOOK POWERED BY
CAMBRIDGE HOTMATHS





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First published 2019

Second Edition 2025

20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Cover designed by Sardine Design

Text designed by diacriTech

Typeset by diacriTech

Printed in China by C & C Offset Printing Co. Ltd.

A catalogue record for this book is available from the National Library of Australia at www.nla.gov.au

ISBN 978-1-009-65470-8

Additional resources for this publication at www.cambridge.edu.au/GO

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Introduction and overview

CambridgeMATHS for NSW Mathematics Advanced Year 11 Second Edition provides complete and aligned coverage of the NESA syllabus to be implemented 2026, and forms part of a continuum of learning from Year 7 to Year 12. Its four components – the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS) – contain a huge range of resources, including worked solutions and additional worksheets, available to schools in a single package (the OTS is included with class adoptions, conditions apply).

New features in the second edition:

- **Learning intentions** at the start of each section complemented by a downloadable Skills Checklist for each chapter that allows students to check understanding and tick off your achievements,
- **Auto-marked quiz questions** in the interactive textbook for every section to let students test understanding in a self directed way.
- **Foundation Worksheets** available in the interactive textbook give additional practice at core skills within the chapter. Linked closely to examples and learning intentions to help guide students towards mastery.

The Second Edition also features significantly revised and updated material from the first edition, including:

Review of Years 9 and 10: The first two chapters in particular can be considered as a review of material covered in Years 9 and 10. The topics covered in these chapters are essential knowledge before proceeding with the rest of the book.

Grading of exercises: Exercises are graded based on difficulty into three groups, Foundation, Development and Challenge to give students more paths for progressing through the content. Foundation questions provide a gentle start to each exercise and cover the core skills required for the syllabus. Development is typically the longest section and ramp up from straightforward to more complex problems like the ones you may see towards the end of the HSC exam. Challenge questions are designed to be more difficult and often require a deeper understanding of the concepts as well as a mix of other topics. These questions are meant to enrich students, typically beyond the level that is required by the syllabus. They don't need to always be attempted and should serve as a challenge for content that is already well understood.

Worked examples and Summary boxes: Plenty of worked examples throughout that cover every key skill from the syllabus in a clear and concise manner. They are coloured with titles to make them easy to find and refer back to when studying. To accompany the worked examples, spread throughout each section there are also numbered and coloured boxes that highlight and summarise the key concepts within a section. To help assist note taking and make later revision straightforward.

Assessment practice and review: Every question in the book has been carefully crafted and refined to give students the widest coverage and number of HSC style questions possible, in order to prepare for the exam. chapter review exercises at the end of every chapter also serve to assess learning and harder practice of key syllabus concepts needed for HSC success. Additional practice questions, including Exam style multiple-choice questions, are also available in the Interactive Textbook.

Interactive Textbook (ITB)

The Interactive Textbook (ITB) is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

Updated and revised for the new syllabus, the Interactive Textbook includes:

- **Video demonstrations** of all worked examples to encourage independent learning.
- **Quick quizzes** containing auto-marked multiple-choice questions have been added to every section, enabling students to quickly check their understanding of a section.
- **Chapter quizzes** provide a more comprehensive review of the chapter with auto-marked HSC style multiple-choice questions.
- A **success criteria checklist** at the end of each chapter with linked Worksheet questions and examples available for download, allows students to check their understanding of the syllabus as they progress through the course in a self-directed way.
- Comprehensive **worked solutions** for all questions are provided in the Interactive Textbook as an option that teacher can choose to enable for their students.
- **Downloadable Worksheets** containing additional foundation level questions, can be used for homework or in class to focus on the learning intention skills. Questions are linked to sections and modelled on the learning intentions and worked examples to help with follow-up.
- **Interactive Widgets** based on Desmos technology, allow students to explore and visualise key concepts in a dynamic way. Desmos tools are also embedded in the Interactive Textbook and can be used to complete calculations, create custom graphs, shapes and other visual representations.

The Online Teaching Suite (OTS)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

- A teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.
- The **task manager** allowing to direct students on a custom activity sequence based on their scores in measurable activities
- Quickly create customised tests from a bank of multiple-choice questions using the **test generator**. Tests are auto-marked in the Interactive Textbook or can be printed and used for homework or assessment practice.
- An expanded and revised suite of **chapter tests** and **assessment**
- Editable **curriculum grids** and **teaching programs**.
- A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite, available from 2025, will include a comprehensive bank of NESA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

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1

Algebra review

Chapter introduction

Fluency in algebra, particularly in factoring, is absolutely vital for everything in mathematics. This chapter is intended as a review of earlier algebraic techniques, and readers should do as much or as little of it as necessary. Some readers will work steadily through it. Others will remember that this review is here, and work through parts of it as necessary.

The course proper begins with Chapter 2.

The word ‘algebra’ has an interesting history. Al-Khwarizmi was a famous and influential Persian mathematician who worked in Baghdad during the early ninth century when the Baghdad Caliphate excelled in science and mathematics. The Arabic word ‘algebra’ comes from *al-jabr*, a word in the title of his most important work, and means ‘the restoration of broken parts’ — a reference to the balancing of terms on both sides of an equation. Al-Khwarizmi’s own name came into modern European languages as ‘algorithm’.

1A Expanding brackets

Learning intentions

- Work with like and unlike terms in algebraic expressions.
- Expand brackets in algebraic expressions.
- Develop the three special expansions, and work with them.

This chapter reviews the manipulation of algebraic expressions involving pronumerals, where a *pronumeral* is a letter or symbol that stands ‘for a number’. The pronumeral may stand for a known or unknown constant, or for an unknown number in an equation, or it may be a *variable* and stand for any one of a whole set of possible numbers.

Pronumerals are numbers, so they can take part in all the normal operations of arithmetic, including addition, subtraction, multiplication, division (except by zero), and forming powers.

Algebraic expressions — like and unlike terms

An *algebraic expression* consists of pronumerals, numbers, and the operations of arithmetic. Here an algebraic expression is simplified by collecting like terms:

$$x^2 + 2xy + 3x^2 - 4xy - 3 = 4x^2 - 2xy - 3$$

This particular algebraic expression can be *simplified* by combining *like terms*.

- The two like terms x^2 and $3x^2$ can be combined to give $4x^2$.
- Another pair of like terms $2xy$ and $-4xy$ can be combined to give $-2xy$.
- This yields three *unlike terms*, $4x^2$, $-2xy$ and -3 , which cannot be combined.

Assuming that these very basic foundations are well known, we now begin reviewing further methods that have been developed to handle algebraic expressions.

Expanding brackets in arithmetic and algebra

Expanding brackets is routine in arithmetic. For example, calculate 7×61 as

$$7 \times (60 + 1) = 7 \times 60 + 7 \times 1,$$

which quickly gives the result $7 \times 61 = 420 + 7 = 427$. The algebraic version of this procedure can be written as:

1 Expanding brackets in algebraic expressions

$$a(x + y) = ax + ay \quad \text{and} \quad (x + y)a = xa + ya$$

In longer expressions, there may be like terms to collect after the expansion.



Example 1 Expanding brackets and simplifying

Expand and simplify each expression.

a $3x(4x - 7)$

b $5a(3 - b) - 3b(1 - 5a)$

Solution

a $3x(4x - 7) = 12x^2 - 21x$

b $5a(3 - b) - 3b(1 - 5a) = 15a - 5ab - 3b + 15ab$
 $= 15a + 10ab - 3b$

Expanding the product of two bracketed terms

Expand one pair of brackets, then expand the other pair of brackets. Then collect any like terms.



Example 2 Expanding more than one set of brackets

Expand and simplify each expression.

a $(x + 3)(x - 5)$

b $(3 + x)(9 + 3x + x^2)$

Solution

a $(x + 3)(x - 5)$

$$= x(x - 5) + 3(x - 5)$$

$$= x^2 - 5x + 3x - 15$$

$$= x^2 - 2x - 15$$

b $(3 + x)(9 + 3x + x^2)$

$$= 3(9 + 3x + x^2) + x(9 + 3x + x^2)$$

$$= 27 + 9x + 3x^2 + 9x + 3x^2 + x^3$$

$$= 27 + 18x + 6x^2 + x^3$$

Special expansions

These three identities are important and must be memorised. Examples of these expansions occur constantly, and knowing the formulae greatly simplifies the working. They are proven in the following exercise.

2 Special expansions

- Square of a sum:

$$(A + B)^2 = A^2 + 2AB + B^2$$

- Square of a difference:

$$(A - B)^2 = A^2 - 2AB + B^2$$

- Difference of squares:

$$(A + B)(A - B) = A^2 - B^2$$



Example 3 Using the three special expansions

Use the three special expansions above to simplify:

a $(x + 4)^2$

b $(s - 3t)^2$

c $(x + 3y)(x - 3y)$

Solution

a $(x + 4)^2 = x^2 + 8x + 16$

(the square of a sum)

b $(s - 3t)^2 = s^2 - 6st + 9t^2$

(the square of a difference)

c $(x + 3y)(x - 3y) = x^2 - 9y^2$

(the difference of squares)

Exercise 1A

FOUNDATION

1 Expand:

a $3(x - 2)$

b $2(x - 3)$

c $-3(x - 2)$

d $-2(x - 3)$

e $-3(x + 2)$

f $-2(x + 3)$

g $-(x - 2)$

h $-(2 - x)$

i $-(x + 3)$

2 Expand:

a $3(x + y)$

b $-2(p - q)$

c $4(a + 2b)$

d $x(x - 7)$

e $-x(x - 3)$

f $-a(a + 4)$

g $5(a + 3b - 2c)$

h $-3(2x - 3y + 5z)$

i $xy(2x - 3y)$

3 Expand and simplify:

a $2(x + 1) - x$

d $-3(a + 2) + 10$

g $(2x - 3y) - (3x - 2y)$

j $4(s - t) - 6(s + t)$

b $3a + 5 + 4(a - 2)$

e $3 - (x + 1)$

h $3(x - 2) - 2(x - 5)$

k $2x(x + 6y) - x(x - 5y)$

c $2 + 2(x - 3)$

f $b + c - (b - c)$

i $4(2a - 3b) - 3(a + 2b)$

l $5(2a - 5b) - 6(-a - 4b)$

4 Expand and simplify:

a $(x + 2)(x + 3)$

d $(x - 4)(x + 2)$

g $(u - 4)(3u + 2)$

j $(5a - 2)(3a + 1)$

b $(y + 4)(y + 7)$

e $(t - 1)(t - 3)$

h $(4p + 5)(2p - 3)$

k $(6 - c)(c - 3)$

c $(t + 6)(t - 3)$

f $(2a + 3)(a + 5)$

i $(2b - 7)(b - 3)$

l $(2d - 3)(4 + d)$

DEVELOPMENT

5 a By expanding $(A + B)(A + B)$, prove the special expansion $(A + B)^2 = A^2 + 2AB + B^2$.

b Similarly, prove the special expansions:

i $(A - B)^2 = A^2 - 2AB + B^2$

ii $(A - B)(A + B) = A^2 - B^2$

6 Use the special expansions to expand:

a $(x + y)^2$

d $(a + 3)^2$

g $(d - 6)(d + 6)$

j $(9 - g)^2$

m $(2a + 1)^2$

p $(2d + 3e)^2$

s $(5j + 4)^2$

v $(5 - 3n)^2$

b $(x - y)^2$

e $(b - 4)^2$

h $(7 + e)(7 - e)$

k $(h + 10)(h - 10)$

n $(2b - 3)^2$

q $(2f + 3g)(2f - 3g)$

t $(4k - 5\ell)^2$

w $(7p + 4q)^2$

c $(x - y)(x + y)$

f $(c + 5)^2$

i $(8 + f)^2$

l $(i + 11)^2$

o $(3c + 2)^2$

r $(3h - 2i)(3h + 2i)$

u $(4 + 5m)(4 - 5m)$

x $(8 - 3r)^2$

CHALLENGE

7 Expand and simplify:

a $\left(t + \frac{1}{t}\right)^2$

b $\left(t - \frac{1}{t}\right)^2$

c $\left(t + \frac{1}{t}\right)\left(t - \frac{1}{t}\right)$

8 By writing 102 as $(100 + 2)$, and adopting a similar approach for the other two parts, use the special expansions to find (without using a calculator) the value of:

a 102^2

b 999^2

c 203×197

9 Expand and simplify:

a $(a - b)(a^2 + ab + b^2)$

c $(a - 3)^2 - (a - 3)(a + 3)$

e $(x - 2)^3$

b $(x + 2)^2 - (x + 1)^2$

d $(2x + 3)(x - 1) - (x - 2)(x + 1)$

f $(p + q + r)^2 - 2(pq + qr + rp)$

1B Factoring

Learning intentions

- Factor using HCF, difference of squares, quadratic factoring, and grouping.
- Recognise quadratic expressions that are perfect squares.

Factoring is the reverse process of expanding brackets. It is needed routinely throughout the course, and very often is the key step in solving a problem.

Four standard methods of factoring

There are four standard methods, but in every situation, common factors should always be taken out first.

3 The four basic methods of factoring

- **Highest common factor:** Always try this first.
- **Difference of squares:** This involves two square terms separated by minus.
- **Quadratics:** This involves three terms.
 - ▷ **Quadratics that are perfect squares:** Recognise a crucial special case.
- **Grouping:** This involves four or more terms.

Factoring should continue until each factor is *irreducible*, meaning that it cannot be factored further.

Factoring by taking out the highest common factor

Always look first for any common factors of all the terms. Then take out the HCF (highest common factor), or take out common factors in successive steps.



Example 4 Factoring by taking out the highest common factor

Factor each expression by taking out the highest common factor.

a $4x^3 + 3x^2$

b $9a^2b^3 - 15b^3$

Solution

a The HCF of $4x^3$ and $3x^2$ is x^2 ,
so $4x^3 + 3x^2 = x^2(4x + 3)$.

b The HCF of $9a^2b^3$ and $15b^3$ is $3b^3$,
so $9a^2b^3 - 15b^3 = 3b^3(3a^2 - 5)$.

Factoring using the difference of squares

If the expression has two terms, both squares, with a minus between them, reverse the special expansion from Section 1A labelled 'Difference of squares':

$$A^2 - B^2 = (A + B)(A - B)$$

Sometimes a common factor must be taken out first.


Example 5 Factoring using the difference of squares

Use the difference of squares to factor each expression.

a $a^2 - 36$

b $80x^2 - 5y^2$

Solution

a $a^2 - 36 = (a + 6)(a - 6)$

b $80x^2 - 5y^2 = 5(16x^2 - y^2)$ (take out the highest common factor)
 $= 5(4x - y)(4x + y)$ (use the difference of squares)

Factoring monic quadratics

A quadratic is called *monic* if the coefficient of x^2 is 1. Suppose that we want to factor the monic quadratic expression $x^2 - 13x + 36$. Look for two numbers:

- whose sum is -13 (the coefficient of x), and
- whose product is $+36$ (the constant term).


Example 6 Factoring monic quadratics

Factor these monic quadratics.

a $x^2 - 13x + 36$

b $a^2 + 12a - 28$

Solution

a The numbers with sum -13 and product $+36$ are -9 and -4 ,
 so $x^2 - 13x + 36 = (x - 9)(x - 4)$.

b The numbers with sum $+12$ and product -28 are $+14$ and -2 ,
 so $a^2 + 12a - 28 = (a + 14)(a - 2)$.

Factoring non-monic quadratics

In a *non-monic* quadratic such as $2x^2 + 11x + 12$, where the coefficient of x^2 is not 1, look for two numbers:

- whose sum is 11 (the coefficient of x), and
- whose product is $12 \times 2 = 24$ (the constant times the coefficient of x^2).

Then split the middle term into two terms.


Example 7 Factoring non-monic quadratics

Factor these non-monic quadratics.

a $2x^2 + 11x + 12$

b $6s^2 - 11s - 10$

Solution

a The numbers with sum 11 and product $12 \times 2 = 24$ are 8 and 3,
 so $2x^2 + 11x + 12 = (2x^2 + 8x) + (3x + 12)$ (split $11x$ into $8x + 3x$)
 $= 2x(x + 4) + 3(x + 4)$ (take out the HCF of each group)
 $= (2x + 3)(x + 4)$ ($x + 4$ is a common factor)

- b** The numbers with sum -11 and product $-10 \times 6 = -60$ are -15 and 4 ,
 so $6s^2 - 11s - 10 = (6s^2 - 15s) + (4s - 10)$ (split $-11s$ into $-15s + 4s$)
 $= 3s(2s - 5) + 2(2s - 5)$ (take out the HCF of each group)
 $= (3s + 2)(2s - 5)$. ($2s - 5$ is a common factor)

Recognising quadratics that are perfect squares

A quadratic is called a *perfect square* if it is the square of a binomial expression. The special expansions in Section 1A gave the perfect square formula in two forms:

$$\text{Square of a sum: } A^2 + 2AB + B^2 = (A + B)^2$$

$$\text{Square of a difference: } A^2 - 2AB + B^2 = (A - B)^2$$

where we have reversed the sides to make them into identities for factoring:

$$x^2 + 6x + 9 = (x + 3)^2 \quad \text{and} \quad x^2 - 18xy + 81y^2 = (x - 9y)^2$$

There is a test. To recognise that $x^2 + 6x + 9$ is a perfect square:

Halve the coefficient 6 of x to get 3, then square it to get 9.

And 9 is the constant term.

The previous method of factoring a quadratic will still work, but recognising perfect squares is quicker, and is an important skill in itself. More in Section 1D.

Factoring by grouping

When there are four or more terms, it is sometimes possible to factor the expression by *grouping*.

- Split the expression into groups.
- Then factor each group in turn.
- Then factor the whole expression by taking out a common factor or by some other method.

Example 8 Factoring by grouping

Factor each expression by grouping.

a $12xy - 9x - 16y + 12$

b $s^2 - t^2 + s - t$

Solution

a $12xy - 9x - 16y + 12 = 3x(4y - 3) - 4(4y - 3)$ (take out the HCF of each pair)
 $= (3x - 4)(4y - 3)$ ($4y - 3$ is a common factor)

b $s^2 - t^2 + s - t = (s + t)(s - t) + (s - t)$ (factor $s^2 - t^2$ using difference of squares)
 $= (s - t)(s + t + 1)$ ($s - t$ is a common factor)

Exercise 1B

FOUNDATION

- 1** Factor by taking out any common factors.

a $2x + 8$

b $6a - 15$

c $ax - ay$

d $20ab - 15ac$

e $x^2 + 3x$

f $p^2 + 2pq$

g $3a^2 - 6ab$

h $12x^2 + 18x$

i $20cd - 32c$

j $a^2b + b^2a$

k $6a^2 + 2a^3$

l $7x^3y - 14x^2y^2$

2 Factor by grouping in pairs.

a $mp + mq + np + nq$

d $a^2 + ab + ac + bc$

g $pu - qu - pv + qv$

j $2ax - bx - 2ay + by$

m $a^3 - 3a^2 - 2a + 6$

b $ax - ay + bx - by$

e $z^3 - z^2 + z - 1$

h $x^2 - 3x - xy + 3y$

k $ab + ac - b - c$

n $2t^3 + 5t^2 - 10t - 25$

c $ax + 3a + 2x + 6$

f $ac + bc - ad - bd$

i $5p - 5q - px + qx$

l $x^3 + 4x^2 - 3x - 12$

o $2x^3 - 6x^2 - ax + 3a$

3 Factor using the difference of squares.

a $a^2 - 1$

c $c^2 - 9$

e $25 - y^2$

g $49 - x^2$

i $4c^2 - 9$

k $25x^2 - 16$

m $x^2 - 4y^2$

o $25m^2 - 36n^2$

b $b^2 - 4$

d $d^2 - 100$

f $1 - n^2$

h $144 - p^2$

j $9u^2 - 1$

l $1 - 49k^2$

n $9a^2 - b^2$

p $81a^2b^2 - 64$

4 Factor each quadratic expression. They are all monic quadratics.

a $a^2 + 3a + 2$

d $x^2 + 8x + 15$

g $x^2 - 4x + 3$

j $b^2 - 8b + 12$

m $w^2 - 2w - 8$

p $y^2 + 3y - 28$

s $x^2 - x - 90$

v $p^2 + 9p - 36$

b $k^2 + 5k + 6$

e $y^2 + 9y + 20$

h $c^2 - 7c + 10$

k $t^2 + t - 2$

n $a^2 + 2a - 8$

q $c^2 - 12c + 27$

t $x^2 + 3x - 40$

w $u^2 - 16u - 80$

c $m^2 + 7m + 6$

f $t^2 + 12t + 20$

i $a^2 - 7a + 12$

l $u^2 - u - 2$

o $p^2 - 2p - 15$

r $u^2 - 13u + 42$

u $t^2 - 4t - 32$

x $t^2 + 23t - 50$

5 Factor these perfect squares.

a $x^2 + 4x + 4$

c $p^2 + 14p + 49$

e $t^2 - 16t + 64$

g $u^2 - 40u + 400$

b $y^2 + 2y + 1$

d $m^2 - 12m + 36$

f $x^2 + 20x + 100$

h $a^2 - 24a + 144$

DEVELOPMENT**6** Copy and complete:

a $x^2 + 6x + \dots = (x + \dots)^2$

c $a^2 - 20a + \dots = (a - \dots)^2$

e $u^2 + u + \dots = (u + \dots)^2$

g $m^2 + 50m + \dots = (m + \dots)^2$

b $y^2 + 8y + \dots = (y + \dots)^2$

d $b^2 - 100b + \dots = (b - \dots)^2$

f $t^2 - 7t + \dots = (t - \dots)^2$

h $c^2 - 13c + \dots = (c - \dots)^2$

7 Factor each quadratic expression. They are all non-monic quadratics.

a $3x^2 + 4x + 1$

d $3x^2 + 8x + 4$

g $5x^2 - 11x + 6$

j $2x^2 + 3x - 5$

m $2x^2 - 7x - 15$

p $6x^2 - 7x - 3$

s $5x^2 + 4x - 12$

v $5x^2 + 28x - 12$

b $2x^2 + 5x + 2$

e $2x^2 - 3x + 1$

h $6x^2 - 11x + 3$

k $3x^2 + 2x - 5$

n $2x^2 + x - 15$

q $6x^2 + 5x - 6$

t $5x^2 - 19x + 12$

w $9x^2 - 6x - 8$

c $3x^2 + 16x + 5$

f $5x^2 - 13x + 6$

i $2x^2 - x - 3$

l $3x^2 + 14x - 5$

o $6x^2 + 17x - 3$

r $5x^2 + 23x + 12$

u $5x^2 - 11x - 12$

x $3x^2 + 13x - 30$

8 Use the techniques of the previous questions to factor each expression.

a $a^2 - 25$

c $c^2 - 25c + 100$

e $e^3 + 5e^2 + 5e + 25$

g $16g^2 - g^3$

i $i^2 - 16i - 36$

k $4k^2 - 16k - 9$

m $2a^2 + ab - 4a - 2b$

o $49p^2 - 121q^2$

q $3t^2 + 2t - 40$

s $5t^2 + 33t + 40$

u $u^2 + 15u - 54$

w $(p + q)^2 - r^2$

b $b^2 - 25b$

d $2d^2 + 25d + 50$

f $16 - f^2$

h $h^2 + 16h + 64$

j $5j^2 + 16j - 16$

l $2k^3 - 16k^2 - 3k + 24$

n $6m^3n^4 + 9m^2n^5$

p $t^2 - 14t + 40$

r $5t^2 + 54t + 40$

t $5t^3 + 10t^2 + 15t$

v $3x^3 - 2x^2y - 15x + 10y$

x $4a^2 - 12a + 9$

CHALLENGE

9 Factor each expression as fully as possible. (Take out any common factors first.)

a $3a^2 - 12$

c $x^3 - x$

e $25y - y^3$

g $4x^2 + 14x - 30$

i $c^3 + 9c^2 - c - 9$

k $x^4 - 3x^2 - 4$

b $x^4 - y^4$

d $5x^2 - 5x - 30$

f $16 - a^4$

h $a^4 + a^3 + a^2 + a$

j $x^3 - 8x^2 + 7x$

l $ax^2 - a - 2x^2 + 2$

1C Algebraic fractions

Learning intentions

- Simplify algebraic fractions using the HCF of numerator and denominator.
- Add, subtract, multiply, and divide algebraic fractions.
- Simplify compound algebraic fractions.

An *algebraic fraction* is a fraction that contains pronumerals. Algebraic fractions are manipulated in the same way as arithmetic fractions, and factoring may play a major role.

Cancelling algebraic fractions

The key step here is to factor the numerator and denominator completely before cancelling factors.

4 Cancelling algebraic fractions

- First factor the numerator and denominator.
- Then cancel all common factors.

Always cancel each algebraic fraction in the expression before combining them.



Example 9 Factoring and cancelling algebraic fractions

Simplify each algebraic fraction.

a $\frac{6x+8}{6}$

Solution

$$\begin{aligned} \text{a } \frac{6x+8}{6} &= \frac{2(3x+4)}{6} \\ &= \frac{3x+4}{3} \quad (\text{or } x + \frac{4}{3}) \end{aligned}$$

b $\frac{x^2-x}{x^2-1}$

$$\begin{aligned} \text{b } \frac{x^2-x}{x^2-1} &= \frac{x(x-1)}{(x+1)(x-1)} \\ &= \frac{x}{x+1} \end{aligned}$$

Adding and subtracting algebraic fractions

A common denominator is needed. Finding the *lowest common denominator* may involve factoring each denominator.

5 Adding and subtracting algebraic fractions

- First factor each denominator.
- Then work with the *lowest common denominator*.

If necessary, cancel the resulting algebraic fraction.



Example 10 Adding and subtracting algebraic fractions

Use a common denominator to add and subtract these algebraic fractions.

a $\frac{x}{2} - \frac{x}{3}$

b $\frac{5x}{6} + \frac{11x}{4}$

c $\frac{2}{3x} - \frac{3}{5x} + \frac{2}{15x}$

d $\frac{1}{x-4} - \frac{1}{x}$

Solution

$$\text{a } \frac{x}{2} - \frac{x}{3} = \frac{3x}{6} - \frac{2x}{6} \\ = \frac{x}{6}$$

$$\text{c } \frac{2}{3x} - \frac{3}{5x} + \frac{2}{15x} = \frac{10}{15x} - \frac{9}{15x} + \frac{2}{15x} \\ = \frac{3}{15x} \\ = \frac{1}{5x} \quad (\text{There was a final cancellation.})$$

$$\text{b } \frac{5x}{6} + \frac{11x}{4} = \frac{10x}{12} + \frac{33x}{12} \\ = \frac{43x}{12}$$

$$\text{d } \frac{1}{x-4} - \frac{1}{x} = \frac{x - (x-4)}{x(x-4)} \\ = \frac{4}{x(x-4)}$$

**Example 11 Factoring before adding and subtracting**

Factor the denominators of $\frac{2+x}{x^2-x} - \frac{5}{x-1}$, then simplify the expression.

Solution

$$\frac{2+x}{x^2-x} - \frac{5}{x-1} = \frac{2+x}{x(x-1)} - \frac{5}{x-1} \\ = \frac{2+x-5x}{x(x-1)} \\ = \frac{2-4x}{x(x-1)}$$

Multiplying and dividing with algebraic fractions

These processes are done exactly as for arithmetic fractions.

6 Multiplying and dividing with algebraic fractions**Multiplying algebraic fractions:**

- First factor all numerators and denominators completely.
- Then cancel common factors.

Dividing by an algebraic fraction:

- To divide by an algebraic fraction, multiply by its reciprocal. For example:

$$\frac{3}{x} \div \frac{4}{y} = \frac{3}{x} \times \frac{y}{4}$$

- The reciprocal of the fraction $\frac{4}{y}$ is $\frac{y}{4}$.


Example 12 Multiplying and dividing algebraic fractions

Simplify these products and quotients of algebraic fractions.

a $\frac{2a}{a^2 - 9} \times \frac{a - 3}{5a}$

b $\frac{12x}{x + 1} \div \frac{6x}{x^2 + 2x + 1}$

Solution

a $\frac{2a}{a^2 - 9} \times \frac{a - 3}{5a} = \frac{2a}{(a - 3)(a + 3)} \times \frac{a - 3}{5a}$ (factor $a^2 - 9$)

$= \frac{2}{5(a + 3)}$ (cancel $a - 3$ and a)

b $\frac{12x}{x + 1} \div \frac{6x}{x^2 + 2x + 1} = \frac{12x}{x + 1} \times \frac{x^2 + 2x + 1}{6x}$ (multiply by the reciprocal)

$= \frac{12x}{x + 1} \times \frac{(x + 1)^2}{6x}$ (factor $x^2 + 2x + 1$)

$= 2(x + 1)$ (cancel $x + 1$ and $6x$)

Simplifying compound fractions

A *compound fraction* is a fraction in which either the numerator or the denominator is itself a fraction.

7 Simplifying compound fractions

- Find the lowest common multiple of the denominators on the top and bottom.
- Multiply top and bottom by this lowest common multiple.

This will clear all the fractions from the top and bottom together.


Example 13 Simplifying compound arithmetic and algebraic fractions

Simplify each compound fraction.

a $\frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}}$

b $\frac{\frac{1}{t} + 1}{\frac{1}{t} - 1}$

Solution

a $\frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} = \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4} + \frac{1}{6}} \times \frac{12}{12}$

$= \frac{6 - 4}{3 + 2}$

$= \frac{2}{5}$

b $\frac{\frac{1}{t} + 1}{\frac{1}{t} - 1} = \frac{\frac{1}{t} + 1}{\frac{1}{t} - 1} \times \frac{t}{t}$

$= \frac{1 + t}{1 - t}$

Exercise 1C

FOUNDATION

1 Simplify:

a $\frac{x}{x}$

b $\frac{2x}{x}$

c $\frac{x}{2x}$

d $\frac{a}{a^2}$

e $\frac{3x^2}{9xy}$

f $\frac{12ab}{4a^2b}$

2 Simplify:

a $\frac{x}{3} \times \frac{3}{x}$

b $\frac{a}{4} \div \frac{a}{2}$

c $x^2 \times \frac{3}{x}$

d $\frac{1}{2b} \times b^2$

e $\frac{3x}{4} \times \frac{2}{x^2}$

f $\frac{5}{a} \div 10$

g $\frac{2ab}{3} \times \frac{6}{ab^2}$

h $\frac{8ab}{5} \div \frac{4ab}{15}$

3 Write as a single fraction:

a $x + \frac{x}{2}$

b $\frac{y}{4} + \frac{y}{2}$

c $\frac{m}{3} - \frac{m}{9}$

d $\frac{n}{2} + \frac{n}{5}$

e $\frac{x}{8} - \frac{y}{12}$

f $\frac{2a}{3} + \frac{3a}{2}$

g $\frac{7b}{10} - \frac{19b}{30}$

h $\frac{xy}{30} - \frac{xy}{12}$

4 Write as a single fraction:

a $\frac{1}{a} + \frac{1}{a}$

b $\frac{1}{x} - \frac{2}{x}$

c $\frac{1}{a} + \frac{1}{2a}$

d $\frac{1}{2x} - \frac{1}{3x}$

e $\frac{3}{4a} + \frac{4}{3a}$

f $\frac{5}{6x} - \frac{1}{3x}$

DEVELOPMENT

5 Simplify:

a $\frac{x+1}{2} + \frac{x+2}{3}$

b $\frac{2x-1}{5} + \frac{2x+3}{4}$

c $\frac{x+3}{6} + \frac{x-3}{12}$

d $\frac{x+2}{2} - \frac{x+3}{3}$

e $\frac{2x+1}{4} - \frac{2x-3}{5}$

f $\frac{2x-1}{3} - \frac{2x+1}{6}$

6 Factor where possible and then simplify:

a $\frac{2p+2q}{p+q}$

b $\frac{3t-12}{2t-8}$

c $\frac{x^2+3x}{3x+9}$

d $\frac{a}{ax+ay}$

e $\frac{3a^2-6ab}{2a^2b-4ab^2}$

f $\frac{x^2+2x}{x^2-4}$

g $\frac{a^2-9}{a^2+a-12}$

h $\frac{x^2+2x+1}{x^2-1}$

i $\frac{x^2+10x+25}{x^2+9x+20}$

7 Simplify:

a $\frac{1}{x} + \frac{1}{x+1}$

b $\frac{1}{x} - \frac{1}{x+1}$

c $\frac{1}{x+1} + \frac{1}{x-1}$

d $\frac{2}{x-3} + \frac{3}{x-2}$

e $\frac{3}{x+1} - \frac{2}{x-1}$

f $\frac{2}{x-2} - \frac{2}{x+3}$

8 Simplify:

a $\frac{3x+3}{2x} \times \frac{x^2}{x^2-1}$

b $\frac{a^2+a-2}{a+2} \times \frac{a^2-3a}{a^2-4a+3}$

c $\frac{c^2+5c+6}{c^2-16} \div \frac{c+3}{c-4}$

d $\frac{x^2-x-20}{x^2-25} \times \frac{x^2-x-2}{x^2+2x-8} \div \frac{x+1}{x^2+5x}$

CHALLENGE

9 Simplify:

a $\frac{b-a}{a-b}$

b $\frac{1}{a-b} - \frac{1}{b-a}$

c $\frac{m}{m-n} + \frac{n}{n-m}$

d $\frac{x^2-5x+6}{2-x}$

10 Simplify:

a $\frac{1}{x^2+x} + \frac{1}{x^2-x}$

b $\frac{1}{x-y} + \frac{2x-y}{x^2-y^2}$

c $\frac{3}{x^2+2x-8} - \frac{2}{x^2+x-6}$

d $\frac{1}{x^2-4x+3} + \frac{1}{x^2-5x+6} - \frac{1}{x^2-3x+2}$

11 Study the exercise on compound fractions and then simplify:

a $\frac{1-\frac{1}{2}}{1+\frac{1}{2}}$

b $\frac{2+\frac{1}{3}}{5-\frac{2}{3}}$

c $\frac{\frac{1}{2}-\frac{1}{5}}{1+\frac{1}{10}}$

d $\frac{\frac{17}{20}-\frac{3}{4}}{\frac{4}{5}-\frac{3}{10}}$

e $\frac{\frac{1}{x}}{1+\frac{2}{x}}$

f $\frac{t-\frac{1}{t}}{t+\frac{1}{t}}$

g $\frac{1}{\frac{1}{b}+\frac{1}{a}}$

h $\frac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}$

i $\frac{1-\frac{1}{x+1}}{\frac{1}{x}+\frac{1}{x+1}}$

j $\frac{\frac{3}{x+2}-\frac{2}{x+1}}{\frac{5}{x+2}-\frac{4}{x+1}}$

1D Solving quadratic equations

Learning intentions

- Solve a quadratic equation by factoring.
- Solve a quadratic equation by completing the square.
- Solve a quadratic equation by the formula.

This section will review the three approaches to solving a *quadratic equation*:

- Factoring — this only works in special cases.
- Completing the square — this works for every quadratic equation.
- The quadratic formula — developed from completing the square.

Solving linear equations

It is assumed that readers can solve linear equations, using the two principles:

- The same term can be added to or subtracted from both sides of an equation.
- Both sides can be multiplied or divided by the same expression, *with the exception that this expression cannot be zero*.

These procedures will be used throughout the book, but are not reviewed here.

Solving a quadratic equation by factoring

This method is the simplest, but it only works in special cases.

8 Solving a quadratic equation by factoring

- Get all the terms on the left, then factor the left-hand side.
- Then use the principle that if $A \times B = 0$, then $A = 0$ or $B = 0$.



Example 14 Solving a quadratic equation by factoring

Solve the quadratic equation $5x^2 + 34x - 7 = 0$ by factoring.

Solution

$$5x^2 + 34x - 7 = 0$$

$$5x^2 + 35x - x - 7 = 0 \quad (35 \text{ and } -1 \text{ have sum } 34 \text{ and product } -7 \times 5 = -35.)$$

$$5x(x + 7) - (x + 7) = 0$$

$$(5x - 1)(x + 7) = 0$$

(the LHS is now factored)

$$5x - 1 = 0 \text{ or } x + 7 = 0$$

(one of the factors must be zero)

$$x = \frac{1}{5} \text{ or } x = -7$$

(there are two solutions)

Completing the square in an expression $x^2 + bx + \dots$

In section 1B, we expanded the quadratic $(x + 3)^2$,

$$(x + 3)^2 = x^2 + 6x + 9,$$

and noticed that the coefficient of x is twice 3, and the constant is 3 squared.

Reversing the process, the constant term in a perfect square can be found by taking half the coefficient of x and squaring the result.

9 Completing the square in an expression $x^2 + bx + \dots$

Halve the coefficient b of x , and square the result — this is the constant term.



Example 15 Completing the square in an expression $x^2 + bx + \dots$

Complete the square in each expression:

a $x^2 + 16x + \dots$

b $x^2 - 3x + \dots$

Solution

a The coefficient of x is 16, half of 16 is 8, and $8^2 = 64$,
so $x^2 + 16x + 64 = (x + 8)^2$.

b The coefficient of x is -3 , half of -3 is $-1\frac{1}{2}$, and $(-1\frac{1}{2})^2 = 2\frac{1}{4}$,
so $x^2 - 3x + 2\frac{1}{4} = (x - 1\frac{1}{2})^2$.

Solving a quadratic equation by completing the square

Solving a quadratic equation by factoring is only possible in special cases. But *completing the square* gives a way to solve every quadratic equation.

10 Solving a quadratic equation by completing the square

- Divide both sides by the coefficient of x^2 .
- Then complete the square in the quadratic by adding the same to both sides.



Example 16 Solving a quadratic equation by completing the square

Solve each quadratic equation by completing the square.

a $3t^2 + 24t = 60$

b $x^2 - x - 1 = 0$

c $x^2 - 14x + 49 = 0$

d $x^2 + x + 1 = 0$

Solution

a $3t^2 + 24t = 60$

$\boxed{\div 3}$ $t^2 + 8t = 20$

$\boxed{+16}$ $t^2 + 8t + 16 = 36$

$(t + 4)^2 = 36$

$t + 4 = 6$ or $t + 4 = -6$

$t = 2$ or -10

b $x^2 - x - 1 = 0$

$\boxed{+1}$ $x^2 - x = 1$

$\boxed{+\frac{1}{4}}$ $x^2 - x + \frac{1}{4} = 1\frac{1}{4}$

$(x - \frac{1}{2})^2 = \frac{5}{4}$

$x - \frac{1}{2} = \frac{1}{2}\sqrt{5}$ or $x - \frac{1}{2} = -\frac{1}{2}\sqrt{5}$

$x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}$

c $x^2 - 14x + 49 = 0$

This is already a perfect square.

$$(x - 7)^2 = 0$$

$$x = 7$$

The equation has just one solution.

d $x^2 + x + 1 = 0$

$$x^2 + x + \frac{1}{4} = -\frac{3}{4}$$

$$(x + \frac{1}{2})^2 = -\frac{3}{4},$$

But a square can't be negative,
so the equation has no solutions.

Solving a quadratic equation by the formula

This method works whether the solutions are rational numbers or involve surds. It will be proven in the last question of Exercise 3E.

11 The quadratic formula

- The solutions of $ax^2 + bx + c = 0$ are:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- Always calculate $b^2 - 4ac$ first.

(Later, this quantity will be called the *discriminant* and given the symbol Δ .)



Example 17 Solving a quadratic equation by the formula

Solve each quadratic equation using the quadratic formula.

a $5x^2 + 2x - 7 = 0$

b $3x^2 + 4x - 1 = 0$

c $x^2 - 8x + 16 = 0$

d $x^2 - 8x + 10 = 0$

Solution

a For $5x^2 + 2x - 7 = 0$,

$$a = 5, b = 2 \text{ and } c = -7.$$

$$\text{Hence } b^2 - 4ac = 2^2 + 140$$

$$= 144$$

$$= 12^2,$$

$$\text{so } x = \frac{-2 + 12}{10} \text{ or } \frac{-2 - 12}{10}$$

$$= 1 \text{ or } -1\frac{2}{5}.$$

c For $x^2 - 8x + 16 = 0$,

$$a = 1, b = -8 \text{ and } c = 16.$$

$$\text{Hence } b^2 - 4ac = (-8)^2 - 64$$

$$= 0,$$

$$\text{so } x = \frac{8}{2} = 4 \text{ is the only solution.}$$

The LHS is a perfect square $(x - 4)^2$.

b For $3x^2 + 4x - 1 = 0$,

$$a = 3, b = 4 \text{ and } c = -1.$$

$$\text{Hence } b^2 - 4ac = 4^2 + 12$$

$$= 28$$

$$= 4 \times 7,$$

$$\text{so } x = \frac{-4 + 2\sqrt{7}}{6} \text{ or } \frac{-4 - 2\sqrt{7}}{6}$$

$$= \frac{-2 + \sqrt{7}}{3} \text{ or } \frac{-2 - \sqrt{7}}{3}.$$

d For $x^2 - 8x + 10 = 0$,

$$a = 1, b = -8 \text{ and } c = 10.$$

$$\text{Hence } b^2 - 4ac = (-8)^2 - 40$$

$$= -24.$$

There are no square roots of -24 ,
so the equation has no solutions.

Exercise 1D**FOUNDATION****1** Solve:

a $x^2 = 9$

d $c^2 - 36 = 0$

g $4x^2 - 1 = 0$

b $y^2 = 25$

e $1 - t^2 = 0$

h $9a^2 - 64 = 0$

c $a^2 - 4 = 0$

f $x^2 = \frac{9}{4}$

i $25y^2 = 16$

2 Solve by factoring:

a $x^2 - 5x = 0$

c $c^2 + 2c = 0$

e $t^2 = t$

g $2b^2 - b = 0$

i $4x^2 + 3x = 0$

k $3y^2 = 2y$

b $y^2 + y = 0$

d $k^2 - 7k = 0$

f $3a = a^2$

h $3u^2 + u = 0$

j $2a^2 = 5a$

l $3n + 5n^2 = 0$

3 Solve by factoring:

a $x^2 + 4x + 3 = 0$

c $x^2 + 6x + 8 = 0$

e $t^2 - 4t - 12 = 0$

g $n^2 - 9n + 8 = 0$

i $a^2 - 10a - 24 = 0$

k $p^2 = p + 6$

m $c^2 + 18 = 9c$

o $u^2 + u = 56$

q $50 + 27h + h^2 = 0$

b $x^2 - 3x + 2 = 0$

d $a^2 - 7a + 10 = 0$

f $c^2 - 10c + 25 = 0$

h $p^2 + 2p - 15 = 0$

j $y^2 + 4y = 5$

l $a^2 = a + 132$

n $8t + 20 = t^2$

p $k^2 = 24 + 2k$

r $a^2 + 20a = 44$

DEVELOPMENT**4** Solve by factoring:

a $2x^2 + 3x + 1 = 0$

c $4y^2 - 5y + 1 = 0$

e $2x^2 + x - 3 = 0$

g $3b^2 - 4b - 4 = 0$

i $2y^2 - y - 15 = 0$

k $5x^2 - 26x + 5 = 0$

m $13t + 6 = 5t^2$

o $25x^2 + 1 = 10x$

q $12b^2 + 3 + 20b = 0$

b $3a^2 - 7a + 2 = 0$

d $2x^2 + 11x + 5 = 0$

f $3n^2 - 2n - 5 = 0$

h $2a^2 + 7a - 15 = 0$

j $3y^2 + 10y = 8$

l $4t^2 + 9 = 15t$

n $10u^2 + 3u - 4 = 0$

p $6x^2 + 13x + 6 = 0$

r $6k^2 + 13k = 8$

- 5** Solve each equation using the quadratic formula. Give exact answers, followed by approximations correct to four significant figures where appropriate.

a $x^2 - x - 1 = 0$

c $a^2 + 12 = 7a$

e $c^2 - 6c + 2 = 0$

g $2a^2 + 1 = 4a$

i $2b^2 + 3b = 1$

k $4t^2 = 2t + 1$

b $y^2 + y = 3$

d $u^2 + 2u - 2 = 0$

f $4x^2 + 4x + 1 = 0$

h $5x^2 + 13x - 6 = 0$

j $3c^2 = 4c + 3$

l $x^2 + x + 1 = 0$

- 6** Solve each quadratic equation by completing the square.

a $x^2 - 2x = 3$

b $x^2 - 6x = 0$

c $a^2 + 6a + 8 = 0$

d $x^2 + 4x + 1 = 0$

e $x^2 - 10x + 20 = 0$

f $y^2 + 3y = 10$

g $b^2 - 5b - 14 = 0$

h $y^2 - y + 2 = 0$

i $a^2 + 7a + 7 = 0$

- 7** Solve by factoring:

a $x = \frac{x+2}{x}$

b $a + \frac{10}{a} = 7$

c $y + \frac{2}{y} = \frac{9}{2}$

d $(5b-3)(3b+1) = 1$

- 8** Find the exact solutions of:

a $x = \frac{1}{x} + 2$

b $\frac{4x-1}{x} = x$

c $a = \frac{a+4}{a-1}$

d $\frac{5m}{2} = 2 + \frac{1}{m}$

CHALLENGE

- 9** Solve, by dividing both sides by the coefficient of x^2 and then completing the square:

a $2x^2 - 4x - 1 = 0$

b $2x^2 + 8x + 3 = 0$

c $3x^2 + 6x + 5 = 0$

d $4x^2 + 4x - 3 = 0$

e $4x^2 - 2x - 1 = 0$

f $2x^2 - 10x + 7 = 0$

- 10** Solve each equation.

a $\frac{5k+7}{k-1} = 3k+2$

b $\frac{u+3}{2u-7} = \frac{2u-1}{u-3}$

c $\frac{y+1}{y+2} = \frac{3-y}{y-4}$

d $2(k-1) = \frac{4-5k}{k+1}$

e $\frac{2}{a+3} + \frac{a+3}{2} = \frac{10}{3}$

f $\frac{k+10}{k-5} - \frac{10}{k} = \frac{11}{6}$

g $\frac{3t}{t^2-6} = \sqrt{3}$

h $\frac{3m+1}{3m-1} - \frac{3m-1}{3m+1} = 2$

1E Solving simultaneous equations

Learning intentions

- Solve simultaneous equations by substitution.
- Solve simultaneous equations by elimination.

There are two algebraic approaches to solving *simultaneous equations* — substitution and elimination. They can be applied to both linear and non-linear simultaneous equations.

Solving by substitution

This method can be applied whenever one of the equations can be solved for one of the variables.

12 Solving simultaneous equations by substitution

- Solve one of the equations for one of the variables.
- Then substitute it into the other equation.



Example 18 Solving simultaneous equations by substitution

Solve each pair of simultaneous equations by substitution.

a $3x - 2y = 29$ (1)

$4x + y = 24$ (2)

b $y = x^2$ (1)

$y = x + 2$ (2)

Solution

a Solving (2) for y , $y = 24 - 4x$. (2A)

Substituting (2A) into (1), $3x - 2(24 - 4x) = 29$

$$x = 7.$$

Substituting $x = 7$ into (1), $21 - 2y = 29$

$$y = -4.$$

Hence $x = 7$ and $y = -4$. (This should be checked in the original equations.)

b Substituting (1) into (2), $x^2 = x + 2$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \text{ or } -1.$$

From (1), when $x = 2$, $y = 4$, and when $x = -1$, $y = 1$.

Hence $x = 2$ and $y = 4$, or $x = -1$ and $y = 1$. (Check in the original equations.)

Solving by elimination

This method, when it can be used, is more elegant, and usually involves less algebraic manipulation.

13 Solving simultaneous equations by elimination

Take suitable multiples of the equations so that one variable is eliminated when the equations are added or subtracted.


Example 19 Solving simultaneous equations by elimination

Solve each pair of simultaneous equations by elimination.

a $3x - 2y = 29$ (1)

$4x + 5y = 8$ (2)

Solution
a Taking $4 \times (1)$ and $3 \times (2)$,

$12x - 8y = 116$ (1A)

$12x + 15y = 24$ (2A)

Subtracting (1A) from (2A),

$23y = -92$

$\div 23$ $y = -4$.

Substituting into (1),

$3x + 8 = 29$

$x = 7$.

 Hence $x = 7$ and $y = -4$.

b $x^2 + y^2 = 53$ (1)

$x^2 - y^2 = 45$ (2)

b Adding (1) and (2),

$2x^2 = 98$

$x^2 = 49$.

Subtracting (2) from (1),

$2y^2 = 8$

$y^2 = 4$.

 Hence $x = 7$ and $y = 2$,

 or $x = 7$ and $y = -2$,

 or $x = -7$ and $y = 2$,

 or $x = -7$ and $y = -2$.

Exercise 1E
FOUNDATION
1 Solve by substituting the first equation into the second.

a $y = x$ and $2x + y = 9$

c $y = x - 1$ and $2x + y = 5$

e $p = 2 - q$ and $p - q = 4$

b $y = 2x$ and $3x - y = 2$

d $a = 2b + 1$ and $a - 3b = 3$

f $v = 1 - 3u$ and $2u + v = 0$

2 Solve by either adding or subtracting the two equations.

a $x + y = 5$ and $x - y = 1$

c $2x + y = 9$ and $x + y = 5$

e $4c - d = 6$ and $2c - d = 2$

b $3x - 2y = 7$ and $x + 2y = -3$

d $a + 3b = 8$ and $a + 2b = 5$

f $p - 2q = 4$ and $3p - 2q = 0$

DEVELOPMENT
3 Solve by substitution:

a $y = 2x$ and $3x + 2y = 14$

c $y = 4 - x$ and $x + 3y = 8$

e $2x + y = 10$ and $7x + 8y = 53$

g $4x - 5y = 2$ and $x + 10y = 41$

b $y = -3x$ and $2x + 5y = 13$

d $x = 5y + 4$ and $3x - y = 26$

f $2x - y = 9$ and $3x - 7y = 19$

h $2x + 3y = 47$ and $4x - y = 45$

4 Solve by elimination:

a $2x + y = 1$ and $x - y = -4$

c $3x + 2y = -6$ and $x - 2y = -10$

e $3x + 2y = 7$ and $5x + y = 7$

g $15x + 2y = 27$ and $3x + 7y = 45$

i $2x + 3y = 28$ and $3x + 2y = 27$

b $2x + 3y = 16$ and $2x + 7y = 24$

d $5x - 3y = 28$ and $2x - 3y = 22$

f $3x + 2y = 0$ and $2x - y = 56$

h $7x - 3y = 41$ and $3x - y = 17$

j $3x - 2y = 11$ and $4x + 3y = 43$

5 Solve by substitution:

a $y = 2 - x$ and $y = x^2$

c $y = 3x^2$ and $y = 4x - x^2$

e $x - y = 2$ and $xy = 15$

b $y = 2x - 3$ and $y = x^2 - 4x + 5$

d $x - y = 5$ and $y = x^2 - 11$

f $3x + y = 9$ and $xy = 6$

CHALLENGE

6 Solve simultaneously:

a $x + y = 15$ and $x^2 + y^2 = 125$

c $2x + y = 5$ and $4x^2 + y^2 = 17$

e $x + 2y = 5$ and $2xy - x^2 = 3$

b $x - y = 3$ and $x^2 + y^2 = 185$

d $x + y = 9$ and $x^2 + xy + y^2 = 61$

f $3x + 2y = 16$ and $xy = 10$

Chapter 1 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 1 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

1 Expand and simplify:

a $4(x + 3) + 5(2x - 3)$

c $-(a - b) - (a + b)$

e $(n + 7)(2n - 3)$

g $(y - 5)(y + 5)$

i $(t - 8)^2$

k $(4p + 1)^2$

b $8(a - 2b) - 6(2a - 3b)$

d $-4x^2(x + 3) - 2x^2(x - 1)$

f $(r + 3)^2$

h $(3x - 5)(2x - 3)$

j $(2c + 7)(2c - 7)$

l $(3u - 2)^2$

2 Factor:

a $18a + 36$

d $d^2 - 36$

g $36 - 25g^2$

j $2j^2 + 11j + 12$

m $4m^2 + 4m - 15$

p $qt - rt - 5q + 5r$

b $20b - 36$

e $e^2 + 13e + 36$

h $h^2 - 9h - 36$

k $3k^2 - 7k - 6$

n $mn + m + pn + p$

q $u^2w + vw - u^2x - vx$

c $9c^2 + 36c$

f $f^2 - 12f + 36$

i $i^2 + 5i - 36$

l $5\ell^2 - 14\ell + 8$

o $p^3 + 9p^2 + 4p + 36$

r $x^2 - y^2 + 2x - 2y$

3 Simplify:

a $\frac{x}{2} + \frac{x}{4}$

e $\frac{3a}{2b} + \frac{2a}{3b}$

i $\frac{x}{y} + \frac{y}{x}$

b $\frac{x}{2} - \frac{x}{4}$

f $\frac{3a}{2b} - \frac{2a}{3b}$

j $\frac{x}{y} - \frac{y}{x}$

c $\frac{x}{2} \times \frac{x}{4}$

g $\frac{3a}{2b} \times \frac{2a}{3b}$

k $\frac{x}{y} \times \frac{y}{x}$

d $\frac{x}{2} \div \frac{x}{4}$

h $\frac{3a}{2b} \div \frac{2a}{3b}$

l $\frac{x}{y} \div \frac{y}{x}$

4 Simplify:

a $\frac{x+4}{5} + \frac{x-5}{3}$

c $\frac{x+1}{2} - \frac{x-4}{5}$

e $\frac{x}{2} - \frac{x+3}{4}$

b $\frac{5}{x+4} + \frac{3}{x-5}$

d $\frac{2}{x+1} - \frac{5}{x-4}$

f $\frac{2}{x} - \frac{4}{x+3}$

- 5 Factor numerator and denominator where possible, then simplify:

a $\frac{6a + 3b}{10a + 5b}$

c $\frac{x^2 + 2x - 3}{x^2 - 5x + 4}$

e $\frac{a + b}{a^2 + 2ab + b^2}$

b $\frac{2x - 2y}{x^2 - y^2}$

d $\frac{2x^2 + 3x + 1}{2x^3 + x^2 + 2x + 1}$

f $\frac{3x^2 - 19x - 14}{9x^2 - 4}$

- 6 Solve each quadratic equation by factoring the left-hand side.

a $a^2 - 49 = 0$

c $c^2 + 7c + 6 = 0$

e $e^2 - 5e + 6 = 0$

g $2g^2 - 13g + 6 = 0$

b $b^2 + 7b = 0$

d $d^2 + 6d - 7 = 0$

f $2f^2 - f - 6 = 0$

h $3h^2 + 2h - 8 = 0$

- 7 Solve, using the quadratic formula. Write the solutions in simplest exact form.

a $x^2 - 4x + 1 = 0$

c $t^2 + 6t + 4 = 0$

e $2a^2 + 5a - 5 = 0$

b $y^2 + 3y - 3 = 0$

d $3x^2 - 2x - 2 = 0$

f $4k^2 - 6k - 1 = 0$

- 8 Solve each quadratic by completing the square on the left-hand side.

a $x^2 + 4x = 6$

c $x^2 - 2x = 12$

b $y^2 - 6y + 3 = 0$

d $y^2 + 10y + 7 = 0$

Chapter introduction

Arithmetic is the study of the properties of numbers and of operations on them. This short chapter reviews the four number systems that we will be dealing with:

whole numbers, integers, rational numbers, real numbers.

Real numbers are central to this course, and almost all the functions we study act on real numbers. The arithmetic of surds also needs attention, because they arise all the time in quadratics and trigonometry.

Most of this material will be familiar from earlier years, but the alternative notation for intervals on the number line is probably new.

2A Real numbers and intervals

Learning intentions

- Relate whole numbers, integers, and rational numbers to counting and rhythms.
- Show that there are irrational numbers, and define real numbers geometrically.
- Distinguish various types of intervals on the real number line.
- Write intervals as inequalities, and also using bracket interval notation.

Our ideas about numbers arise from the two quite distinct sources:

- The *whole numbers*, *integers*, and *rational numbers* arise from counting and rhythms.
- The *real numbers* are developed from geometry and the number line.

Real numbers are the subject of this section. But first, we briefly review the whole numbers, the integers, and the rational numbers.

The whole numbers

Counting is the first operation in arithmetic. Counting things such as people in a room requires *zero* (if the room is empty) and then the successive numbers 1, 2, 3, ..., generating all the *whole numbers*:

0, 1, 2, 3, 4, 5, 6, ...

There is no last number, because every number is followed by another number, distinct from all previous numbers. The list is therefore called *in-finite*, which means that it never ‘finishes’. The symbol $\mathbf{N}$ is used for the set of whole numbers.

A non-zero whole number can be factored, in one and only one way, into the product of *prime numbers*, which are the whole numbers greater than 1 whose only divisors are itself and 1. The primes form a sequence whose distinctive pattern has confused every mathematician since Greek times:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, ...

The whole numbers greater than 1 and not prime are called *composite numbers*. The whole numbers 0 and 1 are special cases, being neither prime nor composite. The number 0 is divisible by every whole number except zero.

1 The set $\mathbf{N}$ of whole numbers

- The *whole numbers* $\mathbf{N}$ are 0, 1, 2, 3, 4, 5, 6, ...
- Every whole number except 0 and 1 is either *prime* or *composite*, and every composite number can be factored into primes in one and only one way.
- When whole numbers are added or multiplied, the result is a whole number.

The integers

Any two whole numbers can be *added* or *multiplied*, and the result is another whole number. *Subtraction*, however, requires the *negative integers* as well:

..., -6, -5, -4, -3, -2, -1

so that calculations such as $5 - 7 = -2$ can be completed. The symbol $\mathbf{Z}$ (from German ‘Zahlen’ meaning ‘numbers’) is used for the set of integers.

2 The set $\mathbb{Z}$ of integers

- The integers $\mathbb{Z}$ are $\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots$
- When integers are added, subtracted, or multiplied, the result is an integer.

The rational numbers

A problem such as, 'Divide 7 cakes into 3 equal parts', leads naturally to *fractions*, where the whole is 'fractured' or 'broken' into pieces. Thus we have the system of *rational numbers*, which are numbers that can be written as the 'ratio' of two integers. Here are some examples of rational numbers written as single fractions:

$$2\frac{1}{3} = \frac{7}{3} \quad -\frac{1}{3} = \frac{-1}{3} \quad 30 \div 24 = \frac{5}{4} \quad 3.72 = \frac{372}{100} \quad 4 = \frac{4}{1}$$

The symbol $\mathbb{Q}$ for 'quotient' is used for the set of rational numbers.

Operations on the rational numbers

Addition, multiplication, subtraction and division (except by 0) can all be carried out within the rational numbers, as is well known from earlier years.

- Rational numbers are *simplified* by dividing top and bottom by their HCF (highest common factor).

$$\frac{21}{35} = \frac{21 \div 7}{35 \div 7} = \frac{3}{5}$$

- Rational numbers are *added and subtracted* using the LCM (lowest common multiple) of their denominators.

For example, 6 and 8 have LCM 24, so

$$\frac{1}{6} + \frac{5}{8} = \frac{1 \times 4}{24} + \frac{5 \times 3}{24} = \frac{19}{24} \quad \text{and} \quad \frac{1}{6} - \frac{5}{8} = \frac{1 \times 4}{24} - \frac{5 \times 3}{24} = -\frac{11}{24}$$

- Fractions are *multiplied* by multiplying the numerators and multiplying the denominators, after first cancelling out any common factors. To *divide* by a fraction, multiply by its reciprocal.

$$\frac{10}{21} \times \frac{9}{25} = \frac{2}{7} \times \frac{3}{5} = \frac{6}{35} \quad \text{and} \quad \frac{8}{21} \div \frac{3}{4} = \frac{8}{21} \times \frac{4}{3} = \frac{32}{63}$$

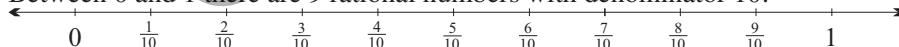
3 The set $\mathbb{Q}$ of rational numbers

- The *rational numbers* $\mathbb{Q}$ are the numbers that can be written as *fractions* $\frac{a}{b}$, where a and b are integers and $b \neq 0$.
- Every integer a can be written as a fraction $\frac{a}{1}$, so is a rational number.
- When rational numbers are added, subtracted, multiplied, and divided (but not by zero), the result is a rational number.

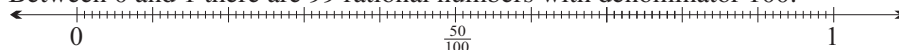
There are numbers that are not rational

At first glance, it would seem reasonable to believe that all the numbers on the number line are rational, because the rational numbers are clearly spread 'as finely as we like' along the whole number line.

Between 0 and 1 there are 9 rational numbers with denominator 10:



Between 0 and 1 there are 99 rational numbers with denominator 100:



Most points on the number line, however, represent numbers that cannot be written as fractions, and are called *irrational numbers*. Some of the most important numbers in this course are irrational, such as $\sqrt{2}$ and π , and Euler's number e that will be introduced in Chapter 10.

The square root of 2 is irrational

The number $\sqrt{2}$ is particularly important, because by Pythagoras' theorem, $\sqrt{2}$ is the diagonal of a unit square. Here is a proof by contradiction that $\sqrt{2}$ is an irrational number.

Suppose that $\sqrt{2}$ were a rational number.

Then $\sqrt{2}$ could be written as a fraction $\frac{a}{b}$ in lowest terms.

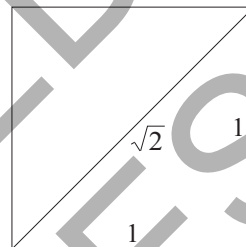
That is, $\sqrt{2} = \frac{a}{b}$, where a and b have no common factor, and we know that $b > 1$ because $\sqrt{2}$ is not a whole number.

Squaring, $2 = \frac{a^2}{b^2}$, where $b^2 > 1$ because $b > 1$.

Because $\frac{a}{b}$ is in lowest terms, $\frac{a^2}{b^2}$ is also in lowest terms, which is impossible, because $\frac{a^2}{b^2} = 2$, but $b^2 > 1$.

This is a contradiction, so $\sqrt{2}$ cannot be a rational number.

The Greek mathematicians were greatly troubled by the existence of irrational numbers. Their concerns can still be seen in modern English, where the word 'irrational' means both 'not a fraction' and 'not reasonable'.

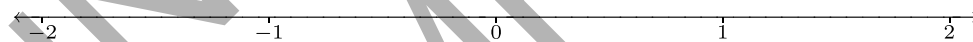


The real numbers and the number line

The whole numbers, the integers, and the rational numbers are based on *counting*. The existence of irrational numbers, however, means that this approach to arithmetic is inadequate, and a more general idea of number is needed. We have to turn away from counting now, and *make an appeal to geometry*.

4 Definition of the set $\mathbf{R}$ of real numbers

- The *real numbers* $\mathbf{R}$ are defined to be all the points on the number line.
- All rational numbers are real, but real numbers such as $\sqrt{2}$ and π are irrational.



At this point, geometry replaces counting as the basis of our arithmetic.

An irrational real number cannot be written as a fraction, or as a terminating or recurring decimal. In this course, an irrational number is usually specified in exact form, such as $x = \sqrt{2}$ or $x = \pi$, or as a decimal approximation correct to a certain number of significant figures, such as $x \doteq 1.4142$ or $x \doteq 3.1416$. Very occasionally a fractional approximation is useful or traditional, such as $\pi \doteq 3\frac{1}{7}$.

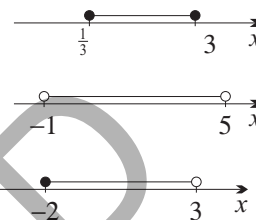
The real numbers are often referred to as the *continuum*, because the rationals, despite being dense, are scattered along the number line like specks of dust, but do not 'join up'. For example, the rational multiples of $\sqrt{2}$, which are all irrational, are just as dense on the number line as the rational numbers. It is only the real line itself that is completely joined up, to be the continuous line of geometry rather than falling apart into an infinitude of discrete points.

The real line looks exactly the same under any magnification. But zoom in on physical matter, and eventually you get to discrete atoms, then nuclei, then quarks — zoom way beyond that, and space itself breaks up into a 'quantum foam', where distance has no meaning.

Open and closed intervals

Any connected part of the real number line is called an *interval*.

- An interval such as $\frac{1}{3} \leq x \leq 3$ is called a *closed interval* because it contains all its endpoints.
- An interval such as $-1 < x < 5$ is called an *open interval* because it does not contain any of its endpoints.
- An interval such as $-2 \leq x < 3$ is closed on the left, but open on the right (the word *half-closed* is often used).

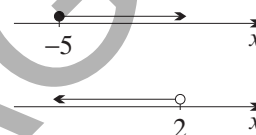


In diagrams, an endpoint is represented by a *closed circle* • if it is contained in the interval, and by an *open circle* ○ if it is not contained in the interval.

Bounded and unbounded intervals

The three intervals above are *bounded* because they have two endpoints, which bound the interval. An *unbounded interval* in contrast is either open or closed, and the direction that continues towards ∞ or $-\infty$ is represented by an arrow.

- The unbounded interval $x \geq -5$ is a *closed interval* because it contains all its endpoints (it only has one).
- The unbounded interval $x < 2$ is an *open interval* because it does not contain its single endpoint.
- The real line itself is an unbounded interval without any endpoints.



5 Intervals

- An *interval* is a connected part of the number line.
- A *closed interval* such as $\frac{1}{3} \leq x \leq 3$ contains all its endpoints.
- An *open interval* such as $-1 < x < 5$ does not contain any of its endpoints.
- An interval such as $-2 \leq x < 3$ is neither open nor closed.
- A *bounded interval* has two endpoints, which bound the interval.
- An *unbounded interval* such as $x \geq -5$ continues towards ∞ or $-\infty$ (or both).

A single point is regarded as a *degenerate interval*, and is closed.

The empty set is sometimes also regarded as a degenerate interval.

(If you continue with mathematics, you will learn that both the whole line $\mathbf{R}$, and the empty set, are both open and closed — each contains all its endpoints, and each contains none of its endpoints, because they have none! These are not concerns of our course, however.)

Bracket interval notation

The three bounded intervals above can be written in an alternative notation:

$\frac{1}{3} \leq x \leq 3$ can also be written as $[\frac{1}{3}, 3]$.

$-1 < x < 5$ can also be written as $(-1, 5)$.

$-2 \leq x < 3$ can also be written as $[-2, 3)$.

This notation places the endpoints of the interval inside brackets, using a square bracket when the endpoint is included, and a round bracket otherwise.

The two unbounded intervals above are written using the symbols ∞ and $-\infty$:

$x \geq -5$ can also be written as $[-5, \infty)$.

$x < 2$ can also be written as $(-\infty, 2)$.

Always use a round bracket for $-\infty$ or ∞ , because the symbols ∞ and $-\infty$ are just notation, and are not numbers at all. Also the whole real line is $(-\infty, \infty)$.

Exercise 2A

FOUNDATION

- 1** Classify these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$ in lowest terms, where a and b are integers.

a -3

b $1\frac{1}{2}$

c $\sqrt{3}$

d $\sqrt{4}$

e $\sqrt[3]{27}$

f $\sqrt[4]{8}$

g $\sqrt{\frac{4}{9}}$

h 0.45

i 12%

j 0.333

k $0.\dot{3}$

l $3\frac{1}{7}$

m π

n 3.14

o 0

- 2** Write these intervals using inequalities.

a The real numbers greater than 4.

b The real numbers less than or equal to 3.

c The real numbers greater than -2 and less than 2.

d The real numbers greater than or equal to -3 and less than or equal to zero.

- 3** Write these intervals using bracket interval notation.

a The real numbers greater than 0 and less than 7.

b The real numbers greater than or equal to 2 and less than or equal to 5.

c The real numbers greater than or equal to 4.

d The real numbers less than -4 .

- 4** Graph these intervals on the real number line.

a $x \leq -1$

b $x > -2$

c $-3 < x \leq 4$

- 5** Graph these intervals on the real number line.

a $[3, \infty)$

b $(-\infty, 1)$

c $[-5, 5)$

- 6** Write these intervals using bracket interval notation.

a $-3 \leq x \leq 2$

b $-3 < x < 2$

c $-3 < x \leq 2$

d $-3 \leq x < 2$

e $x \leq 2$

f $x > -3$

- 7** Write these intervals using inequalities.

a $(-6, 6)$

b $[-6, 6]$

c $[-6, 6)$

d $(-6, 6]$

e $(6, \infty)$

f $(-\infty, -6]$

DEVELOPMENT

- 8** Between which two consecutive integers does each of these irrational numbers lie? (Do NOT use a calculator.)

a $\sqrt{15}$

b $\sqrt{150}$

c $\sqrt[3]{15}$

d π

- 9 a** Classify each interval as open or closed or neither (that is, half-closed).

i $0 \leq x \leq 7$

ii $x > 5$

iii $x \leq 7$

iv $5 < x \leq 15$

v $x < -1$

vi $-4 < x < 10$

vii $x \geq 6$

viii $-4 \leq x < -3$

- b** Classify each interval in part (a) as bounded or unbounded.

- 10 a** Continue the list of primes at the start of the section to all primes less than 100.

- b** Factor into primes:

i 30

ii 98

iii 144

iv 365

v 2025

vi 1 000 000 000

- 11 a** Write down all the integer factors of:

i 6

ii -28

- b** Write down three distinct integers whose product is:

i -6

ii 3

- 12 a** Write down all rational numbers in the interval $[0, \pi]$ with denominators less than 4.

- b** What is the lowest common denominator required to calculate $\frac{1}{12} + \frac{1}{30} + \frac{1}{75}$?

CHALLENGE

- 13** Write each of these sets of real numbers in simplest form using bracket interval notation.

a $x > -1$ or $x < 1$

b $x \geq -1$ and $x \leq 1$

- 14** Prove that $\sqrt{3}$ is irrational. (Adapt the given proof that $\sqrt{2}$ is irrational.)

2B Surds and their arithmetic

Learning intentions

- Understand the meaning of the term surd, and of the notation $\sqrt{x}$.
- Simplify surds by taking out square divisors and collecting like terms.

Numbers such as $\sqrt{2}$ and $\sqrt{3}$ occur constantly in this course because they occur when solving quadratic equations and when using Pythagoras' theorem. These last three sections of the chapter review various methods of dealing with them.

Square roots and positive square roots

The square of any real number is positive, except that $0^2 = 0$. Hence a negative number cannot have a square root, and the only square root of 0 is 0 itself.

A positive number, however, has two square roots, which are the opposites of each other. For example, the square roots of 9 are 3 and -3 .

Warning: The well-known symbol $\sqrt{x}$ does not mean 'the square root of x '. It is defined to mean *the positive square root of x* (or zero, if $x = 0$).

6 Definition of the symbol $\sqrt{x}$

- For $x > 0$, $\sqrt{x}$ means the *positive square root* of x .
- For $x = 0$, $\sqrt{0} = 0$.
- For $x < 0$, $\sqrt{x}$ is not defined.

For example, $\sqrt{25} = 5$, even though 25 has two square roots, -5 and 5 . The symbol for the negative square root of 25 is $-\sqrt{25}$.

Cube roots

Cube roots are less complicated. Every number has exactly one cube root, so the symbol $\sqrt[3]{x}$ simply means 'the cube root of x '. For example,

$$\sqrt[3]{8} = 2 \quad \text{and} \quad \sqrt[3]{-8} = -2 \quad \text{and} \quad \sqrt[3]{0} = 0.$$

What is a surd?

The word *surd* is very commonly used to refer to any expression involving a square or higher root. More precisely, however, surds do not include expressions such as $\sqrt{\frac{4}{9}}$ and $\sqrt[3]{8}$ that can be simplified to rational numbers.

7 Surds

An expression $\sqrt[n]{x}$, where x is a real number and $n \geq 2$ is a whole number, is called a *surd* if it is not itself a rational number.

The word 'surd' is related to 'absurd' — surds are irrational.

It was proven in the last section that $\sqrt{2}$ is irrational, and in the same way, ‘most’ roots of rational numbers are irrational. Here is the precise result for square roots, which won’t be proven formally, and applies similarly to higher roots:

‘If $b \neq 0$ and a are whole numbers with no common factor, then $\sqrt{a/b}$ is rational if and only if both a and b are squares of whole numbers.’

Simplifying expressions involving surds

Here are some laws from earlier years for simplifying expressions involving square roots. The first pair restate the definition of the square root, and the second pair are easily proven by squaring.

8 Laws concerning surds

Let $a \geq 0$ and $b \geq 0$ be real numbers. Then:

$$\sqrt{a^2} = a$$

$$(\sqrt{a})^2 = a$$

and

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}, \text{ provided that } b \neq 0.$$

Taking out square divisors

A surd such as $\sqrt{500}$ is not regarded as being simplified, because 500 is divisible by the square number 100, so $\sqrt{500}$ can be written as $10\sqrt{5}$:

$$\sqrt{500} = \sqrt{100 \times 5} = \sqrt{100} \times \sqrt{5} = 10\sqrt{5}.$$

9 Simplifying a surd

- Check the number inside the square root for divisibility by one of the squares
4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...
- Continue until the number inside the square root sign has no more square divisors (apart from 1).

Example 1 Simplifying surds by taking out squares

Simplify these expressions involving surds.

a $\sqrt{108}$

b $5\sqrt{27}$

c $\sqrt{216}$

Solution

$$\begin{aligned} \text{a } \sqrt{108} &= \sqrt{36 \times 3} \\ &= \sqrt{36} \times \sqrt{3} \\ &= 6\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{b } 5\sqrt{27} &= 5\sqrt{9 \times 3} \\ &= 5 \times \sqrt{9} \times \sqrt{3} \\ &= 15\sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{c } \sqrt{216} &= \sqrt{4 \times 54} \\ &= \sqrt{4} \times \sqrt{54} \\ &= 2 \times \sqrt{9 \times 6} \\ &= 2 \times 3\sqrt{6} \\ &= 6\sqrt{6} \end{aligned}$$

Example 2 Simplifying surds by collecting like terms

Simplify the surds in these expressions, then collect like terms.

a $\sqrt{44} + \sqrt{99}$

b $\sqrt{72} - \sqrt{50} + \sqrt{12}$

Solution

$$\begin{aligned} \text{a } \sqrt{44} + \sqrt{99} &= 2\sqrt{11} + 3\sqrt{11} \\ &= 5\sqrt{11} \end{aligned}$$

$$\begin{aligned} \text{b } \sqrt{72} - \sqrt{50} + \sqrt{12} &= 6\sqrt{2} - 5\sqrt{2} + 2\sqrt{3} \\ &= \sqrt{2} + 2\sqrt{3} \end{aligned}$$

Exercise 2B**FOUNDATION****1** Write down the value of:

a $\sqrt{16}$

b $\sqrt{36}$

c $\sqrt{81}$

d $\sqrt{121}$

e $\sqrt{144}$

f $\sqrt{400}$

g $\sqrt{2500}$

h $\sqrt{10\,000}$

2 Simplify:

a $\sqrt{12}$

b $\sqrt{18}$

c $\sqrt{20}$

d $\sqrt{27}$

e $\sqrt{28}$

f $\sqrt{40}$

g $\sqrt{32}$

h $\sqrt{99}$

i $\sqrt{54}$

j $\sqrt{200}$

k $\sqrt{60}$

l $\sqrt{75}$

m $\sqrt{80}$

n $\sqrt{98}$

o $\sqrt{800}$

p $\sqrt{1000}$

3 Simplify:

a $\sqrt{3} + \sqrt{3}$

b $5\sqrt{7} - 3\sqrt{7}$

c $2\sqrt{5} - \sqrt{5}$

d $-3\sqrt{2} + \sqrt{2}$

e $4\sqrt{3} + 3\sqrt{2} - 2\sqrt{3}$

f $-5\sqrt{5} - 2\sqrt{7} + 6\sqrt{5}$

g $7\sqrt{6} + 5\sqrt{3} - 4\sqrt{6} - 7\sqrt{3}$

h $-6\sqrt{2} - 4\sqrt{5} + 3\sqrt{2} - 2\sqrt{5}$

i $3\sqrt{10} - 8\sqrt{5} - 7\sqrt{10} + 10\sqrt{5}$

DEVELOPMENT**4** Simplify:

a $3\sqrt{8}$

b $5\sqrt{12}$

c $2\sqrt{24}$

d $4\sqrt{44}$

e $3\sqrt{45}$

f $6\sqrt{52}$

g $2\sqrt{300}$

h $2\sqrt{96}$

5 Write each expression as a single square root. For example, $3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}$.

a $2\sqrt{5}$

b $5\sqrt{2}$

c $8\sqrt{2}$

d $6\sqrt{3}$

e $5\sqrt{5}$

f $4\sqrt{7}$

g $2\sqrt{17}$

h $7\sqrt{10}$

6 Simplify fully:

a $\sqrt{8} + \sqrt{2}$

b $\sqrt{12} - \sqrt{3}$

c $\sqrt{50} - \sqrt{18}$

d $\sqrt{54} + \sqrt{24}$

e $\sqrt{45} - \sqrt{20}$

f $\sqrt{90} - \sqrt{40} + \sqrt{10}$

g $\sqrt{27} + \sqrt{75} - \sqrt{48}$

h $\sqrt{45} + \sqrt{80} - \sqrt{125}$

i $\sqrt{2} + \sqrt{32} + \sqrt{72}$

CHALLENGE**7** Simplify fully:

a $\sqrt{600} + \sqrt{300} - \sqrt{216}$

b $4\sqrt{18} + 3\sqrt{12} - 2\sqrt{50}$

c $2\sqrt{175} - 5\sqrt{140} - 3\sqrt{28}$

8 Find the value of x if:

a $\sqrt{63} - \sqrt{28} = \sqrt{x}$

b $\sqrt{80} - \sqrt{20} = \sqrt{x}$

c $2\sqrt{150} - 3\sqrt{24} = \sqrt{x}$

2C Further simplification of surds

Learning intentions

- Expand and simplify surdic expressions using algebraic techniques.

This section deals with the simplification of more complicated surdic expressions.

Simplifying products of surds

The product of two surds is found using the identity

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}.$$

It is important to check whether the answer needs further simplification.



Example 3 Simplifying products of surds

Simplify each product.

a $\sqrt{15} \times \sqrt{5}$

b $5\sqrt{6} \times 7\sqrt{10}$

Solution

$$\begin{aligned} \mathbf{a} \quad \sqrt{15} \times \sqrt{5} &= \sqrt{75} \\ &= \sqrt{25 \times 3} \\ &= 5\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 5\sqrt{6} \times 7\sqrt{10} &= 35\sqrt{60} \\ &= 35\sqrt{4 \times 15} \\ &= 35 \times 2\sqrt{15} \\ &= 70\sqrt{15} \end{aligned}$$

Simplifying surds using binomial expansions

Expanding binomial products can be applied to surdic expressions.



Example 4 Simplifying surds using binomial expansions

Expand these products and then simplify them.

a $(\sqrt{15} + 2)(\sqrt{3} - 3)$

b $(\sqrt{15} - \sqrt{6})^2$

Solution

$$\begin{aligned} \mathbf{a} \quad (\sqrt{15} + 2)(\sqrt{3} - 3) &= \sqrt{15}(\sqrt{3} - 3) + 2(\sqrt{3} - 3) \\ &= \sqrt{45} - 3\sqrt{15} + 2\sqrt{3} - 6 \\ &= 3\sqrt{5} - 3\sqrt{15} + 2\sqrt{3} - 6 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (\sqrt{15} - \sqrt{6})^2 &= 15 - 2\sqrt{90} + 6 \quad (\text{using the identity } (A - B)^2 = A^2 - 2AB + B^2) \\ &= 21 - 2 \times 3\sqrt{10} \\ &= 21 - 6\sqrt{10} \end{aligned}$$

Exercise 2C

FOUNDATION

1 Simplify:

a $(\sqrt{3})^2$

c $\sqrt{7} \times \sqrt{7}$

e $2 \times 3\sqrt{2}$

g $2\sqrt{3} \times 3\sqrt{5}$

i $(2\sqrt{3})^2$

k $5\sqrt{2} \times 3\sqrt{2}$

b $\sqrt{2} \times \sqrt{3}$

d $\sqrt{6} \times \sqrt{5}$

f $2\sqrt{5} \times 5$

h $6\sqrt{2} \times 5\sqrt{7}$

j $(3\sqrt{7})^2$

l $6\sqrt{10} \times 4\sqrt{10}$

2 Simplify:

a $\sqrt{15} \div \sqrt{3}$

c $3\sqrt{5} \div 3$

e $3\sqrt{10} \div \sqrt{5}$

g $10\sqrt{14} \div 5\sqrt{2}$

b $\sqrt{42} \div \sqrt{6}$

d $2\sqrt{7} \div \sqrt{7}$

f $6\sqrt{33} \div 6\sqrt{11}$

h $15\sqrt{35} \div 3\sqrt{7}$

3 Expand:

a $\sqrt{5}(\sqrt{5} + 1)$

d $2\sqrt{2}(\sqrt{5} - \sqrt{2})$

b $\sqrt{2}(\sqrt{3} - 1)$

e $\sqrt{7}(7 - 2\sqrt{7})$

c $\sqrt{3}(2 - \sqrt{3})$

f $\sqrt{6}(3\sqrt{6} - 2\sqrt{5})$

DEVELOPMENT

4 Simplify fully:

a $\sqrt{6} \times \sqrt{2}$

d $\sqrt{2} \times 2\sqrt{22}$

b $\sqrt{5} \times \sqrt{10}$

e $4\sqrt{12} \times \sqrt{3}$

c $\sqrt{3} \times \sqrt{15}$

f $3\sqrt{8} \times 2\sqrt{5}$

5 Expand and simplify:

a $\sqrt{2}(\sqrt{10} - \sqrt{2})$

d $\sqrt{6}(\sqrt{8} - 2)$

b $\sqrt{6}(3 + \sqrt{3})$

e $3\sqrt{3}(9 - \sqrt{21})$

c $\sqrt{5}(\sqrt{15} + 4)$

f $3\sqrt{7}(\sqrt{14} - 2\sqrt{7})$

6 Expand and simplify:

a $(\sqrt{3} + 1)(\sqrt{2} - 1)$

d $(\sqrt{6} - 1)(\sqrt{6} - 2)$

b $(\sqrt{5} - 2)(\sqrt{7} + 3)$

e $(\sqrt{7} - 2)(2\sqrt{7} + 5)$

c $(\sqrt{5} + \sqrt{2})(\sqrt{3} + \sqrt{2})$

f $(3\sqrt{2} - 1)(\sqrt{6} - \sqrt{3})$

7 Use the special expansion $(a + b)(a - b) = a^2 - b^2$ to expand and simplify:

a $(\sqrt{5} + 1)(\sqrt{5} - 1)$

b $(3 - \sqrt{7})(3 + \sqrt{7})$

c $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

d $(3\sqrt{2} - \sqrt{11})(3\sqrt{2} + \sqrt{11})$

e $(2\sqrt{6} + 3)(2\sqrt{6} - 3)$

f $(7 - 2\sqrt{5})(7 + 2\sqrt{5})$

- 8** Expand and simplify the following, using the special expansions

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \text{and} \quad (a - b)^2 = a^2 - 2ab + b^2.$$

a $(\sqrt{3} + 1)^2$

b $(\sqrt{5} - 1)^2$

c $(\sqrt{3} + \sqrt{2})^2$

d $(\sqrt{7} - \sqrt{5})^2$

e $(2\sqrt{3} - 1)^2$

f $(2\sqrt{5} + 3)^2$

g $(2\sqrt{7} + \sqrt{5})^2$

h $(3\sqrt{2} - 2\sqrt{3})^2$

i $(3\sqrt{5} + \sqrt{10})^2$

CHALLENGE

- 9** Simplify fully:

a $\frac{\sqrt{40}}{\sqrt{10}}$

c $\frac{2\sqrt{6} \times \sqrt{5}}{\sqrt{10}}$

e $\frac{\sqrt{15} \times \sqrt{20}}{\sqrt{12}}$

b $\frac{\sqrt{18}}{\sqrt{50}}$

d $\frac{5\sqrt{7} \times \sqrt{3}}{\sqrt{28}}$

f $\frac{6\sqrt{3} \times 8\sqrt{2}}{\sqrt{32} \times \sqrt{27}}$

- 10** Use Pythagoras' theorem to find the hypotenuse of the right-angled triangle in which the lengths of the other two sides are:

a $\sqrt{2}$ and $\sqrt{7}$

b $\sqrt{5}$ and $2\sqrt{5}$

c $\sqrt{7} + 1$ and $\sqrt{7} - 1$

d $2\sqrt{3} - \sqrt{6}$ and $2\sqrt{3} + \sqrt{6}$

- 11** Simplify by forming the lowest common denominator:

a $\frac{1}{\sqrt{3} + 1} + \frac{1}{\sqrt{3} - 1}$

b $\frac{3}{2\sqrt{5} - \sqrt{7}} - \frac{3}{2\sqrt{5} + \sqrt{7}}$

- 12 a** Write down the expansion of $(a + b)^2$.

b Use the expansion in part (a) to square $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.

c Hence simplify $\sqrt{6 + \sqrt{11}} - \sqrt{6 - \sqrt{11}}$.

2D Rationalising the denominator

Learning intentions

- Simplify surdic expressions by rationalising the denominator.
- Use conjugates where necessary when rationalising the denominator.

When dealing with surdic expressions, it is usual to remove any surds from the denominator, a process called *rationalising the denominator*. There are two cases.

The denominator has a single term

In the first case, the denominator is a surd or a multiple of a surd.

10 Rationalising a single-term denominator

In an expression such as $\frac{\sqrt{7}}{2\sqrt{3}}$, multiply top and bottom by $\sqrt{3}$.



Example 5 Simplifying surds by rationalising the denominator

a $\frac{\sqrt{7}}{2\sqrt{3}}$

Solution

$$\begin{aligned}\text{a } \frac{\sqrt{7}}{2\sqrt{3}} &= \frac{\sqrt{7}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{\sqrt{21}}{2 \times 3} \\ &= \frac{\sqrt{21}}{6}\end{aligned}$$

b $\frac{55}{\sqrt{11}}$

$$\begin{aligned}\text{b } \frac{55}{\sqrt{11}} &= \frac{55}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{11}} \\ &= \frac{55\sqrt{11}}{11} \\ &= 5\sqrt{11}\end{aligned}$$

The denominator has two terms

The second case involves a denominator with two terms, one or both of which contain a surd. The method uses the difference of squares identity

$$(A + B)(A - B) = A^2 - B^2$$

to square the unwanted surds in the denominator and convert them to integers.

11 Rationalising a binomial denominator

- In an expression such as $\frac{3}{5 + \sqrt{3}}$, multiply top and bottom by $5 - \sqrt{3}$.
- Then use the difference of squares.


Example 6 Rationalising a binomial surd denominator

a $\frac{3}{5 + \sqrt{3}}$

b $\frac{1}{2\sqrt{3} - 3\sqrt{2}}$

Solution

$$\begin{aligned} \text{a } \frac{3}{5 + \sqrt{3}} &= \frac{3}{5 + \sqrt{3}} \times \frac{5 - \sqrt{3}}{5 - \sqrt{3}} \\ &= \frac{15 - 3\sqrt{3}}{25 - 3} \\ &= \frac{15 - 3\sqrt{3}}{22} \end{aligned}$$

Using the difference of squares,
 $(5 + \sqrt{3})(5 - \sqrt{3}) = 5^2 - (\sqrt{3})^2$
 $= 25 - 3.$
 $= 22.$

$$\begin{aligned} \text{b } \frac{1}{2\sqrt{3} - 3\sqrt{2}} &= \frac{1}{2\sqrt{3} - 3\sqrt{2}} \times \frac{2\sqrt{3} + 3\sqrt{2}}{2\sqrt{3} + 3\sqrt{2}} \\ &= \frac{2\sqrt{3} + 3\sqrt{2}}{4 \times 3 - 9 \times 2} \\ &= -\frac{2\sqrt{3} + 3\sqrt{2}}{6} \end{aligned}$$

Using the difference of squares,
 $(2\sqrt{3} - 3\sqrt{2})(2\sqrt{3} + 3\sqrt{2}) = (2\sqrt{3})^2 - (3\sqrt{2})^2$
 $= 4 \times 3 - 9 \times 2.$
 $= -6.$

Conjugates of surdic expressions

A new word is often used when describing rationalising the denominator.

The expressions $5 + \sqrt{3}$ and $5 - \sqrt{3}$ are called *conjugates* of each other, and

$$\begin{aligned} (5 + \sqrt{3})(5 - \sqrt{3}) &= 5^2 - 3, \text{ by the difference of squares,} \\ &= 22. \end{aligned}$$

12 Conjugates of surdic expressions

- The expressions $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$ are *conjugates* of each other.
- The product of a surdic expression and its conjugate has no surds:

$$(a\sqrt{b} + c\sqrt{d})(a\sqrt{b} - c\sqrt{d}) = a^2b - c^2d, \text{ by the difference of squares.}$$

‘Conjugate’ is a Latin word meaning ‘joined together’.

Exercise 2D
FOUNDATION

1 Write down the conjugate of:

a $\sqrt{3} - 1$

b $2\sqrt{5} + 3$

c $\sqrt{7} - \sqrt{2}$

d $3\sqrt{11} + 4\sqrt{6}$

2 Write down the result when each expression in the previous question is multiplied by its conjugate.

a $\sqrt{3} - 1$

b $2\sqrt{5} + 3$

c $\sqrt{7} - \sqrt{2}$

d $3\sqrt{11} + 4\sqrt{6}$

3 Rewrite each fraction with a rational denominator.

a $\frac{1}{\sqrt{3}}$

b $\frac{1}{\sqrt{7}}$

c $\frac{3}{\sqrt{5}}$

d $\frac{5}{\sqrt{2}}$

e $\frac{\sqrt{2}}{\sqrt{3}}$

f $\frac{\sqrt{5}}{\sqrt{7}}$

g $\frac{2\sqrt{11}}{\sqrt{5}}$

h $\frac{3\sqrt{7}}{\sqrt{2}}$

4 Rewrite each fraction with a rational denominator.

a $\frac{1}{\sqrt{3}-1}$

b $\frac{1}{\sqrt{7}+2}$

c $\frac{1}{3+\sqrt{5}}$

d $\frac{1}{4-\sqrt{7}}$

e $\frac{1}{\sqrt{5}-\sqrt{2}}$

f $\frac{1}{\sqrt{10}+\sqrt{6}}$

g $\frac{1}{2\sqrt{3}+1}$

h $\frac{1}{5-3\sqrt{2}}$

DEVELOPMENT

5 Simplify each expression by rationalising the denominator.

a $\frac{2}{\sqrt{2}}$

b $\frac{5}{\sqrt{5}}$

c $\frac{6}{\sqrt{3}}$

d $\frac{21}{\sqrt{7}}$

e $\frac{3}{\sqrt{6}}$

f $\frac{5}{\sqrt{15}}$

g $\frac{8}{\sqrt{6}}$

h $\frac{14}{\sqrt{10}}$

6 Rewrite each fraction with a rational denominator.

a $\frac{1}{2\sqrt{5}}$

b $\frac{1}{3\sqrt{7}}$

c $\frac{3}{5\sqrt{2}}$

d $\frac{2}{7\sqrt{3}}$

e $\frac{10}{3\sqrt{2}}$

f $\frac{9}{4\sqrt{3}}$

g $\frac{\sqrt{3}}{2\sqrt{10}}$

h $\frac{2\sqrt{11}}{5\sqrt{7}}$

7 Rewrite each fraction with a rational denominator.

a $\frac{3}{\sqrt{5}+1}$

b $\frac{4}{2\sqrt{2}-\sqrt{3}}$

c $\frac{\sqrt{7}}{5-\sqrt{7}}$

d $\frac{3\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

e $\frac{2\sqrt{7}-5}{2\sqrt{7}+5}$

f $\frac{\sqrt{5}}{\sqrt{10}-\sqrt{5}}$

g $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

h $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

i $\frac{3-\sqrt{7}}{3+\sqrt{7}}$

j $\frac{3\sqrt{2}+\sqrt{5}}{3\sqrt{2}-\sqrt{5}}$

k $\frac{\sqrt{10}-\sqrt{6}}{\sqrt{10}+\sqrt{6}}$

l $\frac{7+2\sqrt{11}}{7-2\sqrt{11}}$

CHALLENGE

8 Simplify each expression by rationalising the denominator.

a $\frac{\sqrt{3}-1}{2-\sqrt{3}}$

b $\frac{2\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

9 Show that each expression is rational by first rationalising the denominators.

a $\frac{3}{\sqrt{2}} + \frac{3}{2+\sqrt{2}}$

b $\frac{1}{3+\sqrt{6}} + \frac{2}{\sqrt{6}}$

c $\frac{4}{2+\sqrt{2}} + \frac{1}{3-2\sqrt{2}}$

d $\frac{8}{3-\sqrt{7}} - \frac{6}{2\sqrt{7}-5}$

10 If $x = \frac{\sqrt{5}+1}{2}$, show that $1 + \frac{1}{x} = x$.

11 The expression $\frac{\sqrt{6}+1}{\sqrt{3}+\sqrt{2}}$ can be written in the form $a\sqrt{3} + b\sqrt{2}$. Find a and b .

12 a Expand $\left(x + \frac{1}{x}\right)^2$.

b Suppose that $x = \sqrt{7} + \sqrt{6}$.

i Show that $x + \frac{1}{x} = 2\sqrt{7}$.

ii Use the result in part (a) to find the value of $x^2 + \frac{1}{x^2}$.

Chapter 2 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 2 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Classify each of these real numbers as rational or irrational. Express those that are rational in the form $\frac{a}{b}$, where a and b are integers.

a 7	b $-2\frac{1}{4}$	c $\sqrt{9}$	d $\sqrt{10}$
e $\sqrt[3]{15}$	f $\sqrt[4]{16}$	g -0.16	h π
- Graph these intervals on the real number line.

a $x \leq 2$	b $x > -1$	c $-2 < x \leq 5$
- Graph these intervals on the real number line.

a $[-2, \infty)$	b $(-\infty, 0)$	c $[-3, 3)$
- Write these intervals using bracket interval notation.

a $-4 \leq x \leq 6$	b $-4 < x < 6$	c $-4 < x \leq 6$
d $-4 \leq x < 6$	e $x \leq 6$	f $x > -4$
- Write these intervals using inequalities.

a $(-5, 3)$	b $[-5, 3]$	c $[-5, 3)$
d $(-5, 3]$	e $(3, \infty)$	f $(-\infty, -5]$
- Simplify:

a $\sqrt{24}$	b $\sqrt{45}$	c $\sqrt{50}$
d $\sqrt{500}$	e $3\sqrt{18}$	f $2\sqrt{40}$
- Simplify:

a $\sqrt{5} + \sqrt{5}$	b $\sqrt{5} \times \sqrt{5}$	c $(2\sqrt{7})^2$
d $2\sqrt{5} + \sqrt{7} - 3\sqrt{5}$	e $\sqrt{35} \div \sqrt{5}$	f $6\sqrt{55} \div 2\sqrt{11}$
g $\sqrt{8} \times \sqrt{2}$	h $\sqrt{10} \times \sqrt{2}$	i $2\sqrt{6} \times 4\sqrt{15}$

8 Simplify:

a $\sqrt{27} - \sqrt{12}$

c $3\sqrt{2} + 3\sqrt{8} - \sqrt{50}$

b $\sqrt{18} + \sqrt{32}$

d $\sqrt{54} - \sqrt{20} + \sqrt{150} - \sqrt{80}$

9 Expand:

a $\sqrt{7}(3 - \sqrt{7})$

c $\sqrt{15}(\sqrt{3} - 5)$

b $\sqrt{5}(2\sqrt{6} + 3\sqrt{2})$

d $\sqrt{3}(\sqrt{6} + 2\sqrt{3})$

10 Expand and simplify:

a $(\sqrt{5} + 2)(3 - \sqrt{5})$

c $(\sqrt{7} - 3)(2\sqrt{5} + 4)$

e $(2\sqrt{6} + \sqrt{11})(2\sqrt{6} - \sqrt{11})$

g $(\sqrt{5} + \sqrt{2})^2$

b $(2\sqrt{3} - 1)(3\sqrt{3} + 5)$

d $(\sqrt{10} - 3)(\sqrt{10} + 3)$

f $(\sqrt{7} - 2)^2$

h $(4 - 3\sqrt{2})^2$

11 Write with a rational denominator:

a $\frac{1}{\sqrt{5}}$

d $\frac{1}{5\sqrt{3}}$

b $\frac{3}{\sqrt{2}}$

e $\frac{5}{2\sqrt{7}}$

c $\frac{\sqrt{3}}{\sqrt{11}}$

f $\frac{\sqrt{2}}{3\sqrt{10}}$

12 Write with a rational denominator:

a $\frac{1}{\sqrt{5} + \sqrt{2}}$

d $\frac{\sqrt{3}}{\sqrt{3} + 1}$

b $\frac{1}{3 - \sqrt{7}}$

e $\frac{3}{\sqrt{11} + \sqrt{5}}$

c $\frac{1}{2\sqrt{6} - \sqrt{3}}$

f $\frac{3\sqrt{7}}{2\sqrt{5} - \sqrt{7}}$

13 Rationalise the denominator of each fraction.

a $\frac{\sqrt{7} - \sqrt{2}}{\sqrt{7} + \sqrt{2}}$

b $\frac{3\sqrt{3} + 5}{3\sqrt{3} - 5}$

14 Find the value of x if $\sqrt{18} + \sqrt{8} = \sqrt{x}$.

15 Simplify $\frac{3}{\sqrt{5} - 2} + \frac{2}{\sqrt{5} + 2}$ by forming the lowest common denominator.

16 Find the values of p and q such that $\frac{\sqrt{5}}{\sqrt{5} - 2} = p + q\sqrt{5}$.

17 Show that $\frac{2}{6 - 3\sqrt{3}} - \frac{1}{2\sqrt{3} + 3}$ is rational by first rationalising each denominator.

3

Functions and graphs

Chapter introduction

The principal purpose of our course is the study of functions. Now that the real numbers have been reviewed, this chapter develops the idea of functions and relations involving real numbers, and their graphs. A variety of known graphs are then discussed, with particular emphasis on quadratics and their parabolic graphs, on circles, powers, and factored cubics, and on two graphs that have asymptotes.

The final section deals with direct and inverse variation, which involves practical applications of some of the graphs in the chapter.

Curve-sketching software is useful in emphasising the basic idea that a function has a graph, and is very useful in sketching quickly a large number of graphs and recognising their relationships — everyone with access to the technology should be doing this. But the ability to sketch the graph of a given function without technology is a central concern of this course, and will require a large variety of methods, which will be developed gradually as the course progresses.

Most readers will already know many things in this chapter. Few readers will have mastered the details. Be very careful.

3A Functions and function notation

Learning intentions

- Understand a function as a rule, displayed using a table of values.
- Identify independent and dependent variables of a function written as $y = \dots$
- Identify any natural restrictions on the variables of a function.
- Work fluently with function notation $f(x) = \dots$
- Substitute numbers and expressions into a function.

Most of the graphs studied in previous years are examples of functions. We need to make the idea of functions more precise, and introduce some new notation.

Functions

Here is a situation that naturally leads to a function. An electrician charges \$100 to visit a home, and then charges \$40 for each power point that he installs.

Let x be the number of power points that he installs.

Let y be the total cost, in dollars.

Then $y = 100 + 40x$.

This is an example of a *function*. We say that y is a *function of x* because the value of y is determined by the value of x , and we call x and y *variables* because they take many different values. The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because its value depends on x .

Thus a function is a *rule*. We input a value of x , and the rule produces an output y . In this example, x must be a whole number 0, 1, 2, ..., and we can add this restriction to the rule, describing the function as:

‘The function $y = 100 + 40x$, where x is a whole number.’

A *table of values* is a useful tool — a few values of the function are selected and arranged in a table. Here is a representative table of values showing the total cost y dollars of installing x power points:

x	0	1	2	3	4	5	6
y	100	140	180	220	260	300	340

1 A function is a rule

- A variable y is a *function of a variable x* when y is completely determined by x as a result of some rule.
- The variable x is called the *independent variable* of the function, and the variable y is called the *dependent variable* because it depends on x .
- The function rule is almost always an *equation*, possibly with restrictions. For example:

‘The function $y = 100 + 40x$, where x is a whole number.’

The function machine

The function and its rule can be regarded as a ‘machine’ with inputs and outputs. For example, the numbers in the right-hand column are the outputs from the function $y = 100 + 40x$, when the numbers 0, 1, 2, 3 and 4 are the inputs.

This model of a function has become far more intuitive in the last few decades because computers and calculators routinely produce output from a given input.

Input x		Output y
0	→	100
1	→	140
2	→	180
3	→	220
4	→	260
...		...
x	→	$100 + 40x$

A question in the following exercise presents a variation of this diagram.

Function notation

In the diagram above, we gave the name f to our function. We can now write the results of the input–output routines as follows:

$f(0) = 100, \quad f(1) = 140, \quad f(2) = 180, \quad f(3) = 220, \quad \dots$

This is read aloud as, ‘ f of zero is equal to one hundred’, and so on. When x is the input, the output is $100 + 40x$, so using the well-known notation introduced by Euler in 1735, we can write the *function rule* as

$f(x) = 100 + 40x,$ where x is a whole number.

This $f(x)$ notation will be used throughout the course alongside $y = \dots$ notation. The previous table of values can therefore also be written as

x	0	1	2	3	4	5	6
$f(x)$	100	140	180	220	260	300	340



Example 1 Substituting into a function

A function is defined by the rule $f(x) = x^2 + 5x$. Find $f(3)$, $f(0)$, and $f(-3)$.

Solution

$f(3) = 3^2 + 5 \times 3$
 $= 9 + 15$
 $= 24$

$f(0) = 0^2 + 0 \times 3$
 $= 0 + 0$
 $= 0$

$f(-3) = (-3)^2 + 5 \times (-3)$
 $= 9 - 15$
 $= -6$



Example 2 Finding the formula for a function described verbally

A function $g(x)$ is defined by the rule, ‘Cube the number and subtract 7’. Write down its function rule as an equation, then draw up a table of values for -2 , -1 , 0 , 1 and 2 .

Solution

The function rule is $g(x) = x^3 - 7$.

x	-2	-1	0	1	2
$g(x)$	-15	-8	-7	-6	1


Example 3 Substituting algebraic expressions into a function

If $f(x) = x^2 + 5$, find and simplify $f(a)$, $f(a + 1)$, and $f(a + h)$.

Solution

$$\begin{aligned} f(a) &= a^2 + 5 & f(a + 1) &= (a + 1)^2 + 5 & f(a + h) &= (a + h)^2 + 5 \\ & & &= a^2 + 2a + 1 + 5 & &= a^2 + 2ah + h^2 + 5 \\ & & &= a^2 + 2a + 6 & & \end{aligned}$$

Restrictions on the variables of a function

With many functions that describe practical situations, the variables have restrictions arising from the situation. Any such restriction should be made part of the function by adding the restriction after the formula, as in the following example.


Example 4 Restricting the variables of a function

Sadie the snail is crawling at a steady 10 cm per minute vertically up a wall 3 metres high, starting at the bottom. Write down the height y metres as a function of the time x minutes of climbing, adding the restriction on x .

Solution

The snail's height y after x minutes is $\frac{1}{10}x$ metres, and the snail will take 30 minutes to get to the top, so the function is

$$y = \frac{1}{10}x, \quad \text{where } 0 \leq x \leq 30.$$

Exercise 3A
FOUNDATION

- Let $f(x) = 2x + 3$. Calculate the following function values.

a $f(1)$	b $f(0)$	c $f(-2)$	d $f(4)$
- Let $p(x) = x^2 - 2x - 3$. Determine the following.

a $p(0)$	b $p(4)$	c $p(3)$	d $p(-2)$
- Find the value of the function $y = 5 + 2x - x^2$ at each value of x .

a 0	b 5	c -2	d -1
- Find $h(-3)$, $h(1)$ and $h(5)$ for each function.

a $h(x) = 2x + 2$	b $h(x) = \frac{1}{x}$
c $h(x) = 3x - x^2$	d $h(x) = \sqrt{x + 4}$
- Write down $g(t)$ for each of the following functions.

a $g(x) = 3x + 1$	
b $g(x) = x^2 - 2x + 8$	
c $g(x) = x - \sqrt{x}$	

6 In each case do the given substitution then expand and simplify your answer.

a $y = 2 - x$ at $x = p - 1$

b $y = 2x - 5$ at $x = q + 2$

c $y = x^2 - x$ at $x = a + 1$

d $y = x^2 - 3x - 4$ at $x = b - 2$

DEVELOPMENT

7 Copy and complete the table of values for each function.

a $y = 2x + 1$

x	-1	0	1
y			

c $f(x) = 1 - x^2$

x	-2	-1	0	1	2
$f(x)$					

b $y = x^2 - 2x$

x	-1	0	1	2	3
y					

d $f(x) = x^3 - 4x$

x	-3	-2	-1	0	1	2	3
$f(x)$							

8 I think of a number, then subtract the square of that number from 6. Let x be the number and let $f(x)$ be the final answer.

a Write down the function rule for this arithmetic.

b Copy and complete the table of values for your function.

x	-2	-1	0	1	2
$f(x)$					

9 For the function $L(x) = 3x + 1$, determine:

a $L(1) - 2$

b $3L(-1)$

c $L(1) + L(2)$

d $L(9) \div L(2)$

10 Given that $P(x) = x^2 - 2x - 4$, find the value of:

a $P(\sqrt{2})$,

b $P(\sqrt{7})$.

11 **a** Write the equation $3x + 4y + 5 = 0$ as a function with independent variable x .

b Write the equation $3x + 4y + 5 = 0$ as a function with independent variable y .

c Write the equation $4 + xy = 0$ as a function with dependent variable y .

d The volume of a cube with side length s is $V = s^3$, and its surface area is $A = 6s^2$. Write each formula as a function with dependent variable s .

e If a rectangle has area 100 m^2 and sides ℓ and b , then $\ell b = 100$. Write this formula as a function with:

i ℓ as the dependent variable,

ii ℓ as the independent variable.

- 12** A restaurant offers a special deal to groups by charging a cover fee of \$50, then \$20 per person. Write down C , the total cost of the meal in dollars, as a function of x , the number of people in the group.
- 13** In each case explain why the function value cannot be found.
- a** $F(0)$, where $F(x) = \sqrt{x-4}$.
 - b** $H(3)$, where $H(x) = \sqrt{1-x^2}$.
 - c** $g(-2)$, where $g(x) = \frac{1}{2+x}$.
 - d** $f(0)$, where $f(x) = \frac{1}{x}$.
- 14** Find $g(a)$, $g(-a)$ and $g(a+1)$ for each function.
- a** $g(x) = 2x - 4$
 - b** $g(x) = 2 - x$
 - c** $g(x) = x^2$
 - d** $g(x) = \frac{1}{x-1}$
- 15** Find $F(t-2)$ and $F(t-2)$ for each function.
- a** $F(x) = 5x + 2$
 - b** $F(x) = \sqrt{x}$
 - c** $F(x) = x^2 + 2x$
 - d** $F(x) = 2 - x^2$

CHALLENGE

- 16** Again using $P(x) = x^2 - 2x - 4$, find the value of:
- a** $P(1 + \sqrt{5})$,
 - b** $P(\sqrt{3} - 1)$.
- 17** If $f(x) = x^2 + 5x$, find in simplest form:
- a** $\frac{f(1+h) - f(1)}{h}$
 - b** $\frac{f(p) - f(q)}{p-q}$
 - c** $\frac{f(x+h) - f(x)}{h}$
- 18**
- a** If $f(x) = x^4 + 2x^2 + 3$, show that $f(-x) = f(x)$ for all values of x .
 - b** If $g(x) = x^3 + \frac{4}{x}$, show that $g(-x) = -g(x)$ whenever $x \neq 0$.
 - c** If $h(x) = \frac{x}{x^2 + 1}$, show that $h(\frac{1}{x}) = h(x)$ whenever $x \neq 0$.

3B Functions, relations, and graphs

Learning intentions

- Draw the graph of a function from a table of values.
- Understand a function as a set of ordered pairs satisfying a condition.
- Understand a relation as any set of ordered pairs, and draw its graph.
- Find the domain and range of a function or relation.

A *graph* is the most important and helpful way to represent a function or relation. Most things that we will discover about a function or relation can be seen on its graph, so the sketching of graphs is central to the course.

A function and its graph

A ball is thrown vertically upwards. While the ball is in the air, its height y metres, x seconds after it is thrown, is

$$y = 5x(6 - x).$$

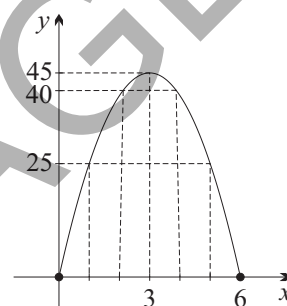
Here is a representative table of values of the function.

x (time)	0	1	2	3	4	5	6
y (height)	0	25	40	45	40	25	0

Each x value and its corresponding y value can be put into an ordered pair ready to plot on a graph of the function. The seven ordered pairs calculated here are:

$$(0, 0), (1, 25), (2, 40), (3, 45), (4, 40), (5, 25), (6, 0)$$

and the graph is sketched opposite. The seven representative points have been plotted, but there are infinitely many such ordered pairs, and they all join up to make the nice smooth curve shown to the right.



Like all graphs of functions, this graph has a crucial property — *no two points have the same x -coordinate*. This is because at any one time, the ball can only be in one position. In function-machine language, *no input can have two outputs*.

2 The graph of a function

- The *graph of a function* consists of all the ordered pairs (x, y) plotted on a pair of axes, where x and y are the values of the input and output variables.
- No two points on the graph ever have the same x -coordinate.
- In most graphs in this course, the points join up to make a smooth curve.

A function as a set of ordered pairs

These ideas of the graph and its ordered pairs allow a more formal definition of a function — a function can be defined simply as a set of ordered pairs satisfying the crucial property mentioned above.

3 More formally, a function is a set of ordered pairs satisfying a condition

A *function* is a set of ordered pairs (x, y) in which:

- no two ordered pairs have the same x -coordinate.

Domain and range

There are two restrictions on the time x in our example above:

- The time x cannot be negative, because the ball had not been thrown then.
- The time cannot be greater than 6, because the ball hits the ground after 6 seconds.

The *domain* of the function is the set of possible x -values, so the domain is the closed interval $0 \leq x \leq 6$. Thus the function is more correctly written as

$$y = 5x(6 - x), \quad \text{where } 0 \leq x \leq 6.$$

The endpoints of the graph are marked with *closed (filled-in) circles* • to indicate that these endpoints are included in the graph. If they were not included, they would be marked with *open circles* ○. These are the same conventions that were used with intervals in Section 2A.

From the graph, we can see that the height of the ball ranges from 0 on the ground to 45 metres. The *range* is the set of possible y -values, so the range is the interval $0 \leq y \leq 45$, which includes the two endpoints $y = 0$ and $y = 45$.

Domain and range here are intervals, so we can also use bracket interval notation:

$$\text{domain} = [0, 6] \quad \text{and} \quad \text{range} = [0, 45].$$

But be careful when using this notation! The domain here is an interval on the x -axis, whereas the range is an interval on the y -axis.

4 The domain and range of a function

- The *domain* of a function is the set of all possible x -coordinates.
- The *range* of a function is the set of all possible y -coordinates.

It is usually easier to find the range after the graph has been drawn.

Reading the domain and range from the graph

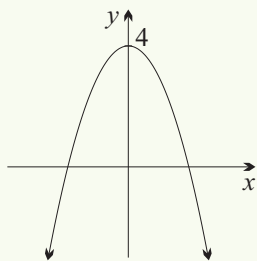
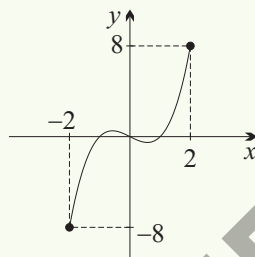
Sometimes we have the graph of a function, but not its equation or rule. We can usually read the domain and range off such a graph.

5 Reading the domain and range from the graph

- **Domain:** All values on the x -axis that have graph points above or below.
- **Range:** All values on the y -axis that have graph points to the left or right.


Example 5 Reading the domain and range off the graph

Write down the domain and the range of the functions whose graphs are sketched below.

a

b

Solution

- a** Domain: all real x , range: $y \leq 4$,
 OR domain = $(-\infty, \infty)$, range = $(-\infty, 4]$.
b Domain: $-2 \leq x \leq 2$, range: $-8 \leq y \leq 8$,
 OR domain = $[-2, 2]$, range = $[-8, 8]$.

The natural domain

When a function is given as an equation with no restrictions, we assume as a *convention* that the domain is all the x -values that can validly be substituted into the equation. This is called the *natural domain*.

6 The natural domain

If no restrictions are given, the domain is all x -values that can validly be substituted into the equation. This is called the *natural domain*.

There are many reasons why a number cannot be substituted into an equation. So far, the two most common reasons are:

- We cannot divide by zero.
- We cannot take square roots of negative numbers.


Example 6 Finding the domain of a function, given its formula

Find the natural domain of each function:

a $y = \frac{1}{x-2}$

b $y = \sqrt{x-2}$

Solution

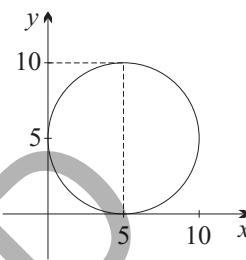
- a** We cannot divide by zero.
 Hence the domain is $x - 2 \neq 0$
 that is, $x \neq 2$.
b We cannot take square roots of negative numbers.
 Hence the domain is $x - 2 \geq 0$
 that is, $x \geq 2$.

Relations

We shall often be dealing with graphs such as the circle sketched to the right. This graph is a set of ordered pairs. But it is not a function, because, for example:

The points $(5, 0)$ and $(5, 10)$ have the same x -coordinate, so that the input $x = 5$ has the two outputs $y = 0$ and $y = 10$. The graph thus fails the crucial property that no input can have more than one output.

The more general word 'relation' is used for any graph in the plane, whether a function or not. In terms of ordered pairs:



7 Relations

A *relation* is any set of ordered pairs.

- Like a function, a relation has a graph.
- Like a function, a relation has a *domain* and a *range*.
- Unlike a function, a relation may have two or more points with the same x -coordinate.

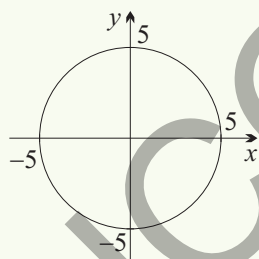
A function is thus a special type of relation, just as a square is a special type of rectangle.



Example 7 Identifying why a particular relation is not a function

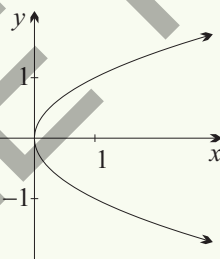
In each part, show that the relation is **not** a function by writing down two ordered pairs on the graph with the same x -coordinate. Illustrate this by connecting the two points by a vertical line. Then write down the domain and range.

a



$$x^2 + y^2 = 25$$

b



$$y^2 = x$$

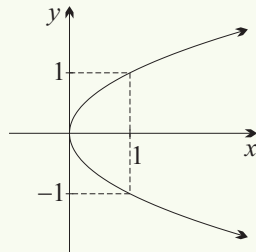
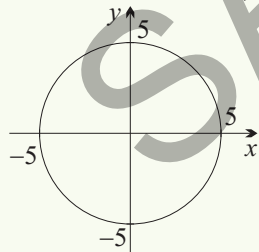
Solution

a The points $(0, 5)$ and $(0, -5)$ on the graph have the same x -coordinate $x = 0$.

Thus when $x = 0$ is the input, there are two outputs, $y = 5$ and $y = -5$.

The vertical line $x = 0$ meets the graph at $(0, 5)$ and at $(0, -5)$.

Domain: $-5 \leq x \leq 5$, range: $-5 \leq y \leq 5$ OR domain = $[-5, 5]$, range = $[-5, 5]$.



b The points $(1, 1)$ and $(1, -1)$ on the graph have the same x -coordinate $x = 1$.

Thus when $x = 1$ is the input, there are two outputs $y = -1$ and $y = 1$.

The vertical line $x = 1$ meets the graph at $(1, -1)$ and $(1, 1)$.

Domain: $x \geq 0$, range: all real y OR domain = $[0, \infty)$, range = $(-\infty, \infty)$.

The vertical line test

The example above shows that we can easily use vertical lines on any graph to see whether or not it is a function.

To show that the graph is not a function, we need to identify just two points with the same x -coordinate. That means, we need to draw just one vertical line that crosses the graph twice.

8 The vertical line test

If at least one vertical line meets the graph of a relation more than once, then the relation is not a function.

The word 'map'

Functions (but not relations in general) are also called *maps* or *mappings*. The word *map* may also be used as a verb. Thus in the function $f(x) = 2^x$, we may say that '3 is mapped to 8'.

In a map of NSW, every point on the ground in NSW corresponds to a unique point on the map.

A relation really can be any finite or infinite set of ordered pairs

Most relations that arise in this course are like the circles and sideways parabolas in the examples above.

The following worked example pushes the definition of a relation in unexpected directions, so it may be a bit unsettling.



Example 8 Examining unfamiliar examples of relations

a State whether or not each set of ordered pairs, or set of points in a graph, is a relation. If it is a relation, state whether or not it is a function, giving reasons if it is not a function.

i $\{(1, 3), (2, 7), (3, 5), (5, 3), (3, 1), (0, 0)\}$

ii $\{(-1, 4), (10, -6), (-6, 10), (2\frac{1}{2}, 3\frac{1}{2})\}$

iii The region inside the circle $x^2 + y^2 = 25$.

iv The region between the lines $x = 1$ and $x = 4$.

v The region inside a cube.

b State the domain and range of those that are relations.

Solution

a i A relation, but not a function — it contains $(3, 5)$ and $(3, 1)$.

ii A relation, and a function.

iii A relation, but not a function — it contains $(0, 0)$ and $(0, 4)$.

iv A relation, but not a function — it contains $(2, 0)$ and $(2, 5)$.

v Not a relation, because points in 3D space are not ordered pairs.

b i Domain: 0, 1, 2, 3, 5. Range: 0, 1, 3, 5, 7.

ii Domain: $-6, -1, 2\frac{1}{2}, 10$. Range: $-6, 3\frac{1}{2}, 4, 10$.

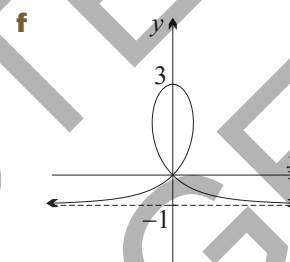
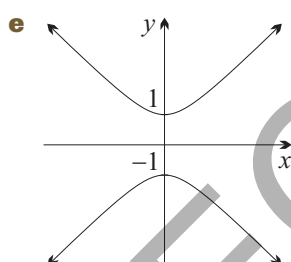
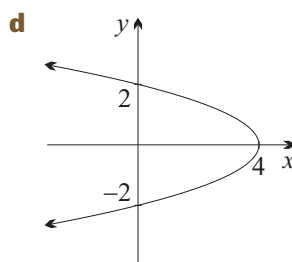
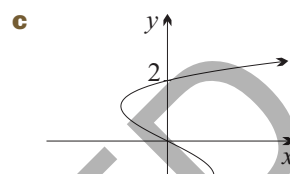
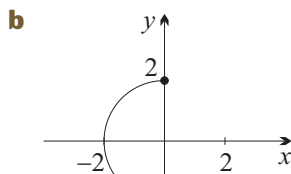
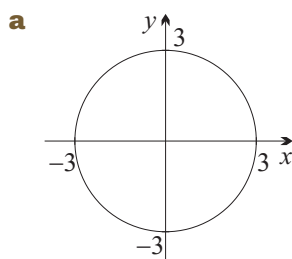
iii Domain: $-5 < x < 5$. Range: $-5 < y < 5$.

iv Domain: $1 < x < 4$. Range: all real y .

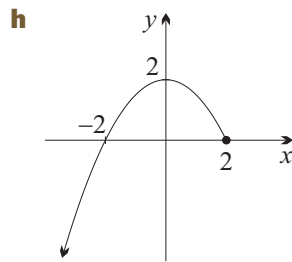
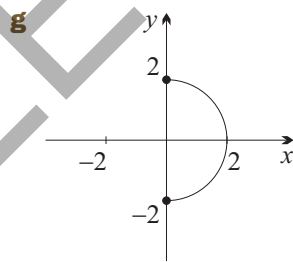
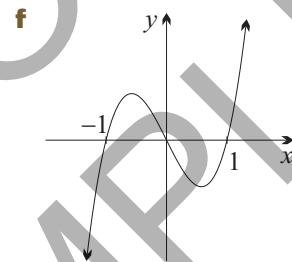
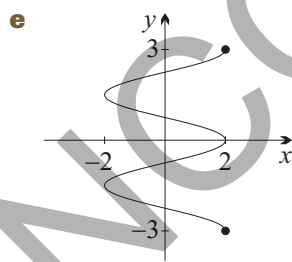
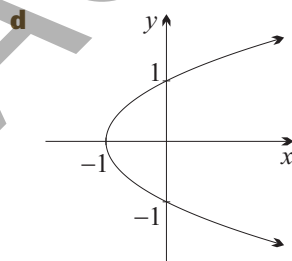
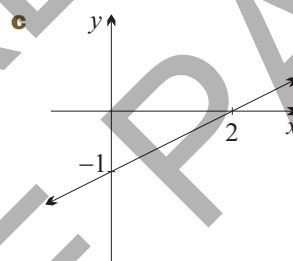
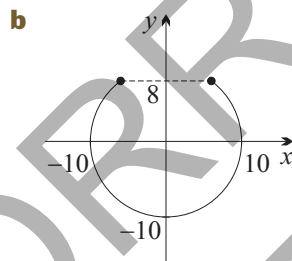
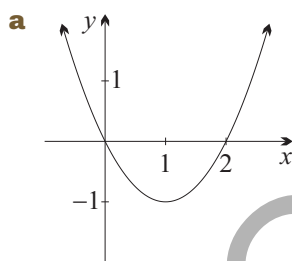
Exercise 3B

FOUNDATION

- 1 In each case, copy the graph, then draw a vertical line to show that the curve does not represent a function.



- 2 Use the vertical line test to find which of the following graphs represent functions.



- 3 What are the domain and range of each relation in Question 2?

- 4 For each of the following functions:

- Copy and complete the table of values.
- Plot the points in the table and hence sketch the function.
- Then write down the domain and range.

a $y = -2x + 3$

x	0	1	2
y			

b $y = x^2 + 2x + 1$

x	-3	-2	-1	0	1
y					

- 5 Use the fact that division by zero is undefined to find the natural domain of each function.

a $f(x) = \frac{1}{x}$

b $f(x) = \frac{1}{x-3}$

c $f(x) = \frac{1}{x+1}$

d $f(x) = \frac{1}{2+x}$

- 6 Use the fact that a negative number does not have a square root to find the natural domain of each function.

a $f(x) = \sqrt{x}$

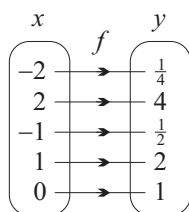
b $f(x) = \sqrt{x-2}$

c $f(x) = \sqrt{x+3}$

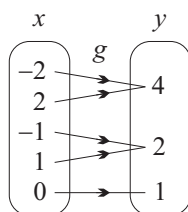
d $f(x) = \sqrt{5+x}$

- 7 Which of these relations are also functions? Explain your answers.

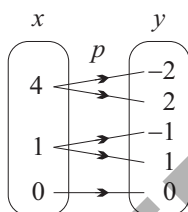
a



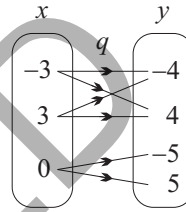
b



c



d

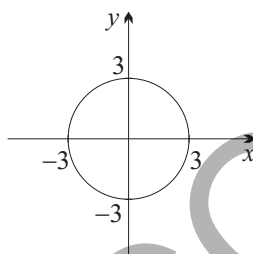


- 8 **a** Consider the set of points $\{(1, 2), (2, 3), (3, 0), (4, 2)\}$. Is this a function or relation?
b Two coloured dice are rolled. The value on the red die is used as the x -coordinate and the green die is used as the y -coordinate.
i List the set of points possible if the numbers either add or multiply to give 3.
ii Is this set a relation or function?

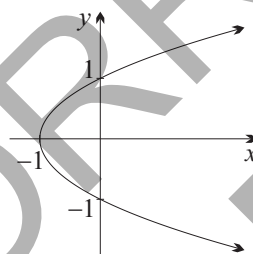
DEVELOPMENT

- 9 The following relations are not functions. Write down the coordinates of two points on each graph that have the same x -coordinate.

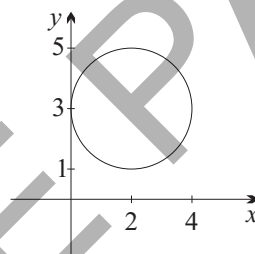
a



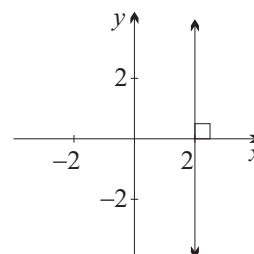
b



c



d



- 10 Find the natural domain of each function.

a $f(x) = 4x$

b $f(x) = 7 - 3x$

c $f(x) = \frac{1}{4-x}$

d $f(x) = \frac{3}{2x-1}$

e $f(x) = \sqrt{x+4}$

f $f(x) = \sqrt{2x+1}$

g $f(x) = \sqrt{5-x}$

h $f(x) = \sqrt{4-2x}$

i $f(x) = \frac{1}{\sqrt{x}}$

j $f(x) = \frac{1}{\sqrt{x+1}}$

k $f(x) = \frac{2}{\sqrt{1-x}}$

l $f(x) = \frac{1}{\sqrt{2x-3}}$

- 11 For each of the following functions:

i copy and complete the table of values,

ii plot the points in the table and hence sketch the function,

iii then write down the domain, and in part (b) also write down the range.

a $y = 2^x$

x	-2	-1	0	1	2
y					

b $y = \frac{1}{3}(4x - x^3)$

x	-3	-2	-1	0	1	2	3
y							

12 Let $R(x) = \sqrt{x}$.

a What is the natural domain of $R(x)$?

b Copy and complete the table of values. Use a calculator to approximate values correct to one decimal place where necessary.

x	0	$\frac{1}{2}$	1	2	3	4	5
$R(x)$							

c Plot these points and join them with a smooth curve starting at the origin. This curve may look similar to a curve you know. Describe it.

13 Let $h(x) = \frac{2}{x}$.

a What is the natural domain of $h(x)$?

b Copy and complete the table of values. Why is there a star for the value where $x = 0$?

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
$h(x)$					*				

c Plot these points and join them with a smooth curve in two parts. This curve is called a *rectangular hyperbola*.

14 Jordan is playing with a 20 cm piece of copper wire, which he bends into the shape of a rectangle, as shown. Let x be the length of a side of the rectangle.

a What is the length of the opposite side?

b What is the length of each of the remaining sides?

c Find the area A of the rectangle as a function of x ?

d Use the fact that lengths must be positive to find the domain of A .

e Use a table of values to graph A for the domain you found in part (d).



15 Solve each equation for y and hence show that it represents a function.

a $2x - y + 3 = 0$

b $xy = 4$

c $xy - 2y = 3$

d $y + 2 = \sqrt{9 - x^2}$

CHALLENGE

16 State the natural domain of each function.

a $f(x) = \frac{x}{\sqrt{x+2}}$

b $f(x) = \frac{2}{x^2 - 4}$

c $f(x) = \frac{1}{x^2 + x}$

d $f(x) = \frac{1}{x^2 - 5x + 6}$

e $f(x) = \sqrt{x^2 - 4}$

f $f(x) = \frac{1}{\sqrt{1 - x^2}}$

3C Review of linear graphs

Learning intentions

- Work with linear functions and relations and their graphs.
- Solve simultaneous linear equations graphically and algebraically.

The next few sections will review some functions and relations that have been introduced in previous years, and the sketching of their graphs. Linear graphs are briefly reviewed in this section, and are the main subject of Chapter 7.

Linear functions

A function is called *linear* if its graph is a straight line. Its equation is then something like

$$y = 2x - 3, \quad \text{or in function form,} \quad f(x) = 2x - 3,$$

with a term in x and a constant term. We often write a linear function with all its terms on the left — the equation of the function above then becomes

$$2x - y - 3 = 0,$$

where the coefficient of y cannot be zero, because we must be able to solve for y .

9 Linear functions

- A *linear function* has a graph that is a straight line.
- The equation of a linear function can be written in *gradient–intercept form*,

$$y = mx + b, \quad \text{or in function form,} \quad f(x) = mx + b.$$
- Alternatively, the equation of a linear function can be written in *general form*,

$$ax + by + c = 0, \quad \text{where the coefficient of } y \text{ is non-zero.}$$

Sketching linear functions

When all three terms of the equation are non-zero, the easiest way to sketch a linear function is to find the two intercepts with the axes.

10 Sketching a linear function whose equation has three non-zero terms

- Find the x -intercept by putting $y = 0$.
- Find the y -intercept by putting $x = 0$.


Example 9 Solving simultaneous linear equations graphically

a Sketch these linear functions on one set of axes by finding the x -intercept and the y -intercept.

i $x + 2y - 6 = 0$

ii $y = x - 3$

b Estimate from the graph where the two lines intersect.

Solution

a i Consider $x + 2y - 6 = 0$.

When $y = 0$, $x - 6 = 0$

$x = 6$.

When $x = 0$, $2y - 6 = 0$

$y = 3$.

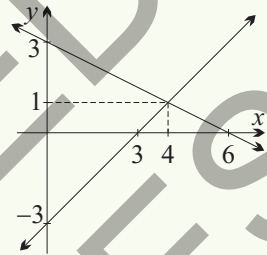
ii Consider $y = x - 3$.

When $y = 0$, $x - 3 = 0$

$x = 3$.

When $x = 0$, $y = -3$.

b From the diagram, the lines appear to meet at $(4, 1)$.



Using simultaneous equations

The methods of solving simultaneous equations were reviewed in Section 1E.


Example 10 Solving simultaneous linear equations algebraically

Solve the simultaneous equations in the previous example, and check that the solution agrees with the estimate from the graph.

Solution

The equations are $x + 2y - 6 = 0$ (1)

$x - y - 3 = 0$. (2)

Subtracting (2) from (1), $3y - 3 = 0$

$y = 1$.

Substituting into (1), $x + 2 - 6 = 0$

$x = 4$. Check this by substitution into (2).

Thus the lines meet at $(4, 1)$, as seen in the graph in the previous example.

Linear relations

A *linear relation* is a relation whose graph is a straight line. By the vertical line test, every linear relation is a function except for vertical lines, which fail the vertical line test dramatically. The equation of such a relation has no term in y , such as the relation:

$$x = 3.$$

Its graph consists of all points whose x -coordinate is 3, giving a vertical line, which is sketched in the next example.

Three special cases

The two-intercept method won't work for sketching the graph of $ax + by + c = 0$ if any of the constants a , b or c is zero.

11 Sketching special cases of linear graphs $ax + by + c = 0$

Horizontal lines: If $a = 0$, the equation can be put into the form $y = k$.

- Its graph is a horizontal line with y -intercept k .

Vertical lines: If $b = 0$, the equation can be put into the form $x = \ell$.

- Its graph is a vertical line with x -intercept ℓ .
- This is the only type of linear graph that is a relation, but not a function.

Lines through the origin: If $c = 0$, and the line is neither horizontal nor vertical, then the equation can be put into the form $y = mx$, where $m \neq 0$.

- Both intercepts are zero, so the graph passes through the origin.
- Find one more point on it, by substituting a value such as $x = 1$.



Example 11 Sketching three special cases of linear relations

Sketch these three lines. Which of the three is not a function?

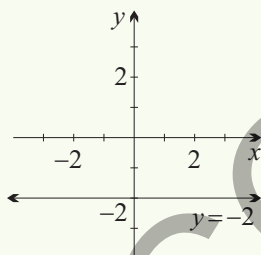
a $y + 2 = 0$

b $x - 3 = 0$

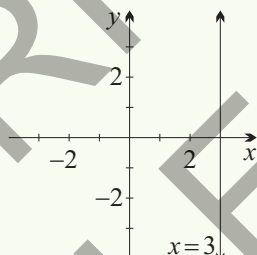
c $x + 2y = 0$

Solution

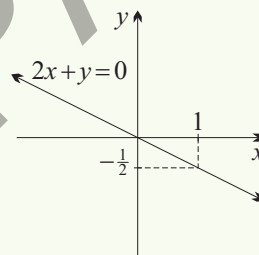
a



b



c



a The line $y + 2 = 0$, or $y = -2$, is horizontal with y -intercept -2 .

b The line $x - 3 = 0$, or $x = 3$, is vertical with x -intercept 3 .

c The line $x + 2y = 0$ passes through the origin, and when $x = 1$, $y = -\frac{1}{2}$.

The vertical line in part (b) is a relation, but not a function — it fails the vertical line test.

Domain and range of linear functions and relations

The following remarks will be clear from the diagram above, and from the other sketches in the section.

- A horizontal line $f(x) = c$ has domain all real x and range $y = c$.
- A vertical line $x = k$, which is a relation but not a function, has domain $x = k$ and range all real y .
- An oblique line $f(x) = mx + b$, where $m \neq 0$, has domain all real x and range all real y .

Exercise 3C

FOUNDATION

- 1 Follow these steps for the linear function $y = 2x - 2$.
 - a Find the y -intercept by putting $x = 0$.
 - b Find the x -intercept by putting $y = 0$.
 - c Plot these intercepts and hence sketch the line.
- 2 Carry out the following steps for the straight line $x + 2y - 4 = 0$.
 - a Find the y -intercept by putting $x = 0$.
 - b Find the x -intercept by putting $y = 0$.
 - c Plot these intercepts and hence sketch the line.
- 3 Repeat the steps in the previous two questions for each line.

a $y = x + 1$	b $y = 4 - 2x$	c $y = \frac{1}{2}x - 3$
d $x + y - 1 = 0$	e $2x - y + 2 = 0$	f $x - 3y - 3 = 0$
g $2x - 3y - 12 = 0$	h $x + 4y + 6 = 0$	i $5x + 2y - 10 = 0$
- 4 A linear function has equation $y = -2x$.
 - a Show that the x -intercept and the y -intercept are both zero.
 - b Substitute $x = 1$ to find a second point on the line, then sketch the line.
- 5 Repeat the steps in the previous question for each line.

a $y = x$	b $y = 3x$
c $y = -4x$	d $x + y = 0$
e $x - 2y = 0$	f $3x + 2y = 0$
- 6 Graph $y = f(x)$ where:
 - a $f(x) = -3x - 6$
 - b $f(x) = \frac{1}{2}x - 2$
 - c $f(x) = -\frac{1}{2}x$
- 7 Sketch the following vertical and horizontal lines.

a $x = 1$	b $y = 2$
c $x = -2$	d $y = 0$
e $2y = -3$	f $3x = 5$
- 8
 - a State which lines in the previous question are not functions.
 - b For each line that is not a function, write down the coordinates of two points on it with the same x -coordinate.

DEVELOPMENT

- 9** For each of parts (d) to (i) in Question 3, solve the given equation for y and hence write the equation of the line using function notation.
- 10** Determine, by substitution, whether or not the given point lies on the given line.
- a** $(3, 1)$ $y = x - 2$
 - b** $(7, 4)$ $y = 20 - 2x$
 - c** $(1, -2)$ $y = -3x + 1$
 - d** $(-5, 3)$ $2x + 3y + 1 = 0$
 - e** $(-1, -4)$ $3x - 2y - 5 = 0$
 - f** $(-6, -4)$ $4x - 5y - 4 = 0$
- 11** Consider the lines $x + y = 5$ and $x - y = 1$.
- a** Graph the lines on a number plane, using a scale of 1 cm to 1 unit on each axis.
 - b** Read off the point of intersection of the two lines.
 - c** Confirm your answer to part (b) by solving the two equations simultaneously.
- 12** Repeat the previous question for the following pairs of lines.
- a** $x + y = 2$
 $x - y = -4$
 - b** $x - y = 3$
 $2x + y = 0$
 - c** $x + 2y = -4$
 $2x - y = -3$
- 13** Looksmart Shirts charges \$60 for one shirt and \$50 for each shirt after this.
- a** Find, as a function of n , the cost C in dollars of n shirts.
 - b** Delivery costs \$10 for one shirt and \$2 for each subsequent shirt.
 - i** Find, as a function of n , the cost D in dollars of delivering n shirts.
 - ii** Find, as a function of n , the total cost T in dollars of buying n shirts and having them delivered.

CHALLENGE

- 14** Consider the linear equation $y = \frac{1}{2}x + c$
- a** Sketch on one number plane the four lines corresponding to the following values of c :
 - i** $c = -2$
 - ii** $c = -1$
 - iii** $c = 1$
 - iv** $c = 2$
 - b** What do you notice about all these lines?
- 15** Consider the linear equation $y - 2 = m(x - 1)$
- a** Sketch on one number plane the four lines corresponding to the following values of m :
 - i** $m = 1$
 - ii** $m = 2$
 - iii** $m = -\frac{1}{2}$
 - iv** $m = 0$
 - b** Which point in the number plane do all these lines pass through?
 - c** Now prove that the line $y - 2 = m(x - 1)$ passes through the point found in the previous part, regardless of the value of m .

3D Quadratic functions — factoring and the graph

Learning intentions

- Work with quadratic functions and their terminology.
- Use factoring to find the zeroes, vertex, and axis of symmetry, and sketch the graph.
- Use correctly the language of zeroes and roots.
- Use correctly the language of double and simple zeroes and roots, and perfect squares.
- Use correctly the language of equal and distinct zeroes and roots.

The material in Sections 3D–3F will be partly review and partly new work, with the emphasis on the graph. Readers confident with quadratics from earlier years will not need to work so thoroughly through the earlier questions in the exercises.

Quadratic functions

A *quadratic function* is a function that can be written in the form:

$$f(x) = ax^2 + bx + c, \quad \text{where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

A *quadratic equation* is an equation that can be written in the form

$$ax^2 + bx + c = 0, \quad \text{where } a, b \text{ and } c \text{ are constants, and } a \neq 0.$$

The requirement that $a \neq 0$ means that the term in x^2 cannot vanish. Thus linear functions and equations are not regarded as special cases of quadratics.

The word ‘quadratic’ comes from the Latin word *quadratus*, meaning ‘square’, and reminds us that quadratics tend to arise as the areas of regions in the plane.

Monic quadratics

A quadratic function $f(x) = ax^2 + bx + c$ is called *monic* if $a = 1$. Calculations are usually easier in monic quadratics than in non-monic quadratics.

12 Monic quadratics

A quadratic is called *monic* if the coefficient of x^2 is 1. For example,

$$y = x^2 - 8x + 15 \text{ is monic} \quad \text{and} \quad y = -x^2 + 8x - 15 \text{ is non-monic.}$$

Zeroes and roots

The solutions of a quadratic equation are called the *roots* of the equation, and the x -intercepts of a quadratic function are called the *zeroes* of the function. This distinction is often not strictly observed, however, because questions about quadratic equations are so closely related to questions about quadratic functions and their graphs.

Five questions about the graph of a quadratic

The graph of any quadratic function $y = ax^2 + bx + c$ is a *parabola*, as seen in earlier years. Before sketching the parabola, five questions need to be answered:

13 Five questions about the parabola $y = ax^2 + bx + c$

- 1** Which way up is the parabola? **ANSWER:** Look at the sign of a .
- 2** What is the y -intercept? **ANSWER:** Put $x = 0$, and then $y = c$.
- 3** What are the x -intercepts, or zeroes, if there are any?
- 4** What is the axis of symmetry?
- 5** What is the vertex? **METHOD:** Substitute the axis back into the quadratic.

The first two questions are easy to answer:

- If a is positive, the curve is concave up. If a is negative, it is concave down.
- To find the y -intercept, put $x = 0$, then $y = c$.
- And once the axis of symmetry is found, the y -coordinate of the vertex can be found by substituting back into the quadratic.

But finding the x -intercepts, and finding the axis of symmetry, need careful work — the three standard approaches are discussed in this and the next two sections:

- Section 3D: Factoring
- Section 3E: Completing the square
- Section 3F: Formulae

Factoring and the zeroes

Factoring of monic and of non-monic quadratics were reviewed in Section 1B.

Most quadratics cannot easily be factored, or can't be factored at all, but when straightforward factoring is possible, this is usually the quickest approach. After factoring, the zeroes can be found by putting $y = 0$ and using the principle:

$$\text{If } A \times B = 0 \quad \text{then} \quad A = 0 \text{ or } B = 0.$$

For example, $y = x^2 - 2x - 3$ is a quadratic function.

- 1** Its graph is concave up because $a = 1$ is positive. (Next heading for the sketch.)
- 2** Substituting $x = 0$ gives $y = 0 + 0 - 3$
so the y -intercept is $y = -3$.
- 3** Factoring, $y = (x + 1)(x - 3)$,
Substituting $y = 0$ gives $0 = (x + 1)(x - 3)$
 $x + 1 = 0$ or $x - 3 = 0$
so the x -intercepts are $x = -1$ and $x = 3$.

Finding the axis of symmetry and the vertex from the zeroes

The axis of symmetry is always the vertical line midway between the x -intercepts. Thus its x -intercept is the average of the zeroes.

Continuing with our example of $y = x^2 - 2x - 3$, which factors as $y = (x + 1)(x - 3)$, and so has zeroes -1 and 3 :

- 4 Taking the average of the zeroes, the axis of symmetry is

$$x = \frac{1}{2}(-1 + 3)$$

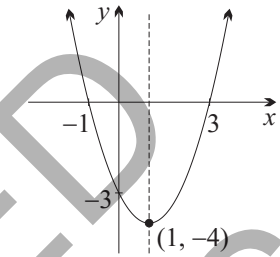
$$x = 1.$$

- 5 Substituting $x = 1$ into the factored quadratic,

$$y = (1 + 1)(1 - 3)$$

$$= -4,$$

so the vertex is $(1, -4)$.



14 The zeroes and intercepts of a factored quadratic

Suppose that we have managed to factor a quadratic as $y = a(x - \alpha)(x - \beta)$.

- Its x -intercepts (zeroes) are $x = \alpha$ and $x = \beta$.
- Its axis of symmetry is the line $x = \frac{1}{2}(\alpha + \beta)$. Take the average of the zeroes.
- Substitute the axis into the factored form of the quadratic to find the y -coordinate of the vertex.



Example 12 Sketching a non-monic quadratic

Sketch the curve $y = -x^2 - 2x + 3$.

Solution

- 1 Because $a < 0$, the curve is concave down.

- 2 When $x = 0$, $y = 3$.

- 3 Factoring, $y = -(x^2 + 2x - 3)$
 $= -(x + 3)(x - 1).$

When $y = 0$, $x + 3 = 0$ or $x - 1 = 0$,
 so the zeroes are $x = -3$ and $x = 1$.

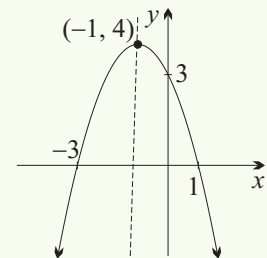
- 4 Taking the average of the zeroes, the axis of symmetry is

$$x = \frac{1}{2}(-3 + 1)$$

$$x = -1.$$

- 5 When $x = -1$, $y = -(-1 + 3) \times (-1 - 1)$ (substitute into the factored form)
 $= 4,$

so the vertex is $(-1, 4)$.



Double zeroes, double roots, and perfect squares

The quadratic $y = -2x^2 + 4x - 2$ factors as

$$y = -2(x - 1)^2,$$

which is -2 times the *perfect square* $(x - 1)^2$.

To find the zeroes, put $y = 0$

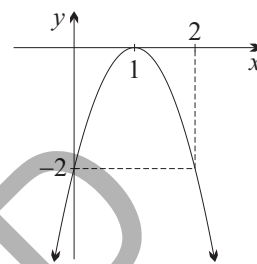
$$-2(x - 1)^2 = 0$$

so $x = 1$.

Should we write the last line as ' $x = 1$ or 1 '? That could be confusing.

Instead, we say that $x = 1$ is a *double zero of the function*, and that $x = 1$ is a *double root of the quadratic equation* $-2(x - 1)^2 = 0$.

The graph of the quadratic is tangent to the x -axis at $x = 1$.



15 Double zeroes, double roots, and perfect squares

- The quadratic function $y = -2(x - 1)^2$ is said to have a *double zero* at $x = 1$.
- The quadratic equation $-2(x - 1)^2 = 0$ is said to have a *double root* at $x = 1$.
- Square expressions such as $(x - 1)^2$ and $(x + 6)^2$ are called *perfect squares*.

Some other terminology used only with quadratics: 'Equal roots' and 'distinct roots'

Warning: Use this terminology only with quadratics. Never use it at all with cubics, quartics, and higher degrees, or you will end up with the most dreadful verbal confusion.

- The quadratic equation $(x - 4)^2 = 0$ is often said to have *equal roots*, (and the quadratic function $f(x) = (x - 4)^2$ is often said to have *equal zeroes*).
- The quadratic equation $(x - 3)(x - 5) = 0$ is often said to have *distinct roots*, (and the function $f(x) = (x - 3)(x - 5)$ is often said to have *distinct zeroes*).

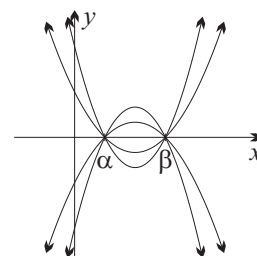
Care: These plural forms 'equal zeroes' and 'equal roots' can be misleading, because there is still only one zero or root.

Quadratics with given zeroes

If a quadratic $f(x)$ has zeroes at $x = \alpha$ and $x = \beta$, then its equation has the form

$$f(x) = a(x - \alpha)(x - \beta),$$

where a is the coefficient of x^2 . By taking different values of the coefficient a , this equation forms a *family of quadratics*, all with the same x -intercepts. The sketch to the right shows four of the curves in the family, two concave up with positive values of a , and two concave down with negative values of a .



16 The family of quadratics with given zeroes

The quadratics with zeroes at $x = \alpha$ and $x = \beta$ form a *family* of parabolas with equation

$$y = a(x - \alpha)(x - \beta), \quad \text{for some non-zero value of } a.$$


Example 13 Finding a family of quadratics with given zeroes

- a** Write down the family of quadratics with zeroes $x = -2$ and $x = 4$.
b Then find the equation of such a quadratic if:
- i** the y -intercept is 16,
 - ii** the curve passes through $(5, 1)$.

Solution

a The family of quadratics with zeroes -2 and 4 is $y = a(x + 2)(x - 4)$.

- b i** Substituting the point $(0, 16)$ gives $16 = a \times 2 \times (-4)$
 $a = -2$,
 so the quadratic is $y = -2(x + 2)(x - 4)$.
ii Substituting the point $(5, 1)$ gives $1 = a \times 7 \times 1$
 $a = \frac{1}{7}$,
 so the quadratic is $y = \frac{1}{7}(x + 2)(x - 4)$.

Domains and ranges of quadratic functions

As with linear function, the diagrams above should make it clear that for a quadratic function $f(x)$ with vertex (h, k) :

- If $f(x)$ is concave up, the domain is all real x , and the range is $y \geq k$.
- If $f(x)$ is concave down, the domain is still all real x , but the range is $y \leq k$.

Exercise 3D
FOUNDATION

- 1 a** The parabola with equation $y = (x - 1)(x - 3)$ is concave up.
- i** Explain why the parabola is concave up.
 - ii** Write down its y -intercept.
 - iii** Put $y = 0$ to find the x -intercepts.
 - iv** Hence determine the equation of the axis of symmetry.
 - v** Use the axis of symmetry to find the coordinates of the vertex.
 - vi** Sketch the parabola, showing these features.
 - vii** Does the quadratic function have equal zeroes or are the zeroes distinct?
- b** Follow the steps of part (a) to sketch these concave-up parabolas.
- i** $y = (x - 1)(x + 3)$
 - ii** $y = (x - 1)(x + 1)$
- 2 a** The parabola $y = -x(x - 2)$ is concave down.
- i** Explain why the parabola is concave down.
 - ii** Write down its y -intercept.
 - iii** Put $y = 0$ to find the x -intercepts.
 - iv** Hence determine the equation of the axis of symmetry.
 - v** Use the axis of symmetry to find the coordinates of the vertex.
 - vi** Sketch the parabola, showing these features.
 - vii** Does the quadratic function have equal zeroes or are the zeroes distinct?
- b** Follow the steps of part (a) to sketch these concave-down parabolas.
- i** $y = (2 + x)(2 - x)$
 - ii** $y = (x + 2)(4 - x)$

- 3 a** The parabola $y = (x - 1)^2$ is a perfect square.

- Explain whether the parabola is concave up or concave down.
- Write down its y -intercept.
- Put $y = 0$ to find its single x -intercept at the vertex.
- Use symmetry and the y -intercept to find another point on the parabola.
- Sketch the parabola showing these points.
- Does the quadratic function have equal zeroes or are the zeroes distinct?

- b** Follow similar steps to part (a) to sketch these parabolas involving perfect squares.

i $y = (x + 1)^2$

ii $y = -(x - 2)^2$

- 4 a** The graph of $y = \frac{1}{2}x^2$ passes through the origin and has no other x -intercept.

- Plot the point corresponding to $x = 2$ on this parabola.
- What other point must lie on the parabola, and why?
- Hence graph the parabola $y = \frac{1}{2}x^2$.

- b** Use a similar approach to graph these parabolas.

i $y = -x^2$

ii $y = \frac{1}{3}x^2$

- 5** Write down, in factored form, the equation of the monic quadratic function with zeroes:

a 4 and 6

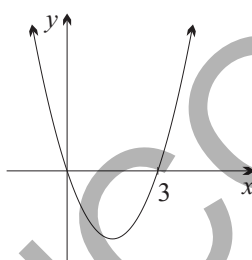
b 0 and 3

c -3 and 5

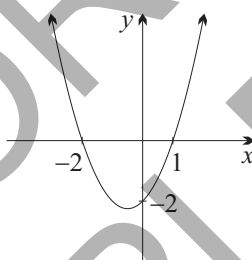
d -6 and -1

- 6** Write down, in factored form, the equation of each quadratic function $y = f(x)$ sketched below. In each case, the coefficient of x^2 is either 1 or -1.

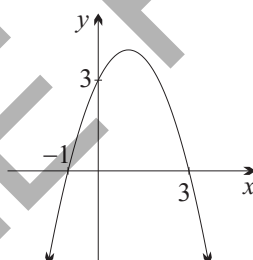
a



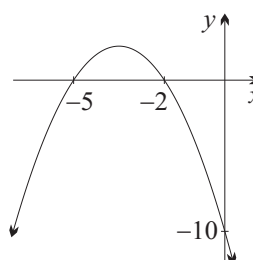
b



c



d



DEVELOPMENT

- 7** Use factoring to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex.

a $f(x) = x^2 - 9$

b $f(x) = x^2 + 4x - 5$

c $f(x) = x^2 + 4x - 12$

- 8** Use factoring to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex.

a $f(x) = 4x - x^2$

b $f(x) = -x^2 + 2x + 3$

c $f(x) = 8 - 2x - x^2$

- 9** Sketch these parabolas involving perfect squares. Begin by factoring the quadratic. In each case, symmetry will be needed to find a third point on the parabola.

a $y = x^2 - 6x + 9$

b $y = -x^2 + 2x - 1$

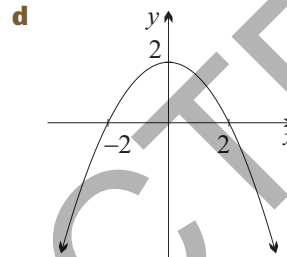
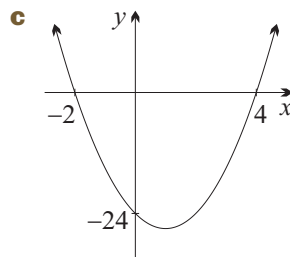
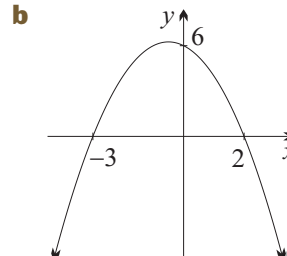
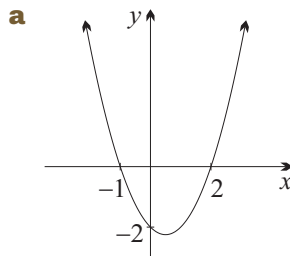
- 10** Use factoring to sketch the graphs of the following non-monic quadratic functions, clearly indicating the vertex and the intercepts with the axes.

a $y = 2x^2 + 7x + 5$

b $y = 2x^2 + 5x - 3$

c $y = 3x^2 + 2x - 8$

11 Find the equations of the quadratic functions sketched below.



12 Find, in factored form, the equations of the parabolas with the given intercepts.

a $x = 1, 3$
 $y = 6$

b $x = -2, 1$
 $y = 4$

c $x = -1, 5$
 $y = 15$

d $x = -2, -4$
 $y = 2$

13 [Technology]

a Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)(x - 3)$ for the following values of a .

i $a = 2$

ii $a = 1$

iii $a = -1$

iv $a = -2$

b Which two points do all these parabolas pass through?

CHALLENGE

14 Use factoring to find the zeroes of each quadratic function. Sketch a graph of the function, clearly indicating all intercepts and the coordinates of the vertex.

a $y = 2x^2 - 18$

b $y = 3x^2 + x - 4$

c $y = 7x - 3 - 4x^2$

15 The general form of a quadratic with zeroes $x = 2$ and $x = 8$ is $y = a(x - 2)(x - 8)$. Find the equation of such a quadratic for which:

a the coefficient of x^2 is 3,

b the y -intercept is -16 ,

c the vertex is $(5, -12)$

d the curve passes through $(1, -20)$.

16 Consider the quadratic function $f(x) = x^2 - 2x - 8$.

a Factor the quadratic and hence find the equation of the axis of symmetry.

b i Expand and simplify $f(1 + h)$ and $f(1 - h)$. What do you notice?

ii What geometric feature of the parabola does this result demonstrate?

3E Completing the square and the graph

Learning intentions

- Complete the square in monic and non-monic quadratics.
- Use completing the square to find the axis of symmetry, the vertex, and the zeroes.

Completing the square is the most general method of dealing with quadratics. It works in every case, whereas factoring only works in exceptional cases.

We first review the algebra of completing the square, and extend it to non-monic quadratics. Once this has been done, sketching the graph follows easily.

Completing the square in a monic quadratic

The quadratic $f(x) = x^2 + 6x + 5$ is not a perfect square, but it can be made the sum of a perfect square and a constant. Adapting the procedure developed in Section 1D to a function rather than an equation:

- Look just at the two terms in x , that is, $x^2 + 6x$.
- Halve the coefficient 6 of x to get 3, then square to get 9.
- Add and subtract 9 on the RHS to produce a perfect square plus a constant.

$$\begin{aligned} f(x) &= x^2 + 6x + 5 \\ &= (x^2 + 6x + 9) - 9 + 5 && \text{(add and subtract 9)} \\ &= (x + 3)^2 - 4 \end{aligned}$$

17 Completing the square in a monic quadratic

- Take the coefficient of x , halve it, then square the result.
- Add and subtract this number to produce a perfect square plus a constant.



Example 14 Completing the square in a monic quadratic

Complete the square in each quadratic.

a $f(x) = x^2 - 4x - 5$

b $f(x) = x^2 + x + 1$

Solution

a Here $f(x) = x^2 - 4x - 5$.

The coefficient of x is -4 . Halve it to get -2 , then square to get $(-2)^2 = 4$.

$$\begin{aligned} \text{Hence } f(x) &= (x^2 - 4x + 4) - 4 - 5 && \text{(add and subtract 4)} \\ &= (x - 2)^2 - 9. \end{aligned}$$

b Here $f(x) = x^2 + x + 1$.

The coefficient of x is 1. Halve it to get $\frac{1}{2}$, then square to get $(\frac{1}{2})^2 = \frac{1}{4}$.

$$\begin{aligned} \text{Hence } f(x) &= (x^2 + x + \frac{1}{4}) - \frac{1}{4} + 1 && \text{(add and subtract } \frac{1}{4}) \\ &= (x + \frac{1}{2})^2 + \frac{3}{4}. \end{aligned}$$

Completing the square in a non-monic quadratic

For a *non-monic quadratic* such as $y = 2x^2 - 12x + 16$, where the coefficient of x^2 is not 1, divide through by the coefficient of x^2 before completing the square. This slightly more difficult procedure was not covered in Chapter 1.

18 Completing the square in a non-monic quadratic

- Divide through by the coefficient of x^2 so that the coefficient of x^2 is 1.
 - Complete the square in the resulting monic quadratic.
- ▷ This method runs more cleanly using $y = \dots$ rather than $f(x) = \dots$



Example 15 Completing the square in a non-monic quadratic

Complete the square in each quadratic.

a $y = 2x^2 - 12x + 16$

b $y = -x^2 + 8x - 15$

Solution

a $y = 2x^2 - 12x + 16$

$$\frac{y}{2} = x^2 - 6x + 8$$

(divide through by the coefficient 2 of x^2)

$$\frac{y}{2} = (x^2 - 6x + 9) - 9 + 8$$

(complete the square on the RHS)

$$\frac{y}{2} = (x - 3)^2 - 1$$

$$y = 2(x - 3)^2 - 2$$

(multiply by 2 to make y the subject again)

b $y = -x^2 + 8x - 15$

$$-y = x^2 - 8x + 15$$

(divide through by the coefficient -1 of x^2)

$$-y = (x^2 - 8x + 16) - 16 + 15$$

(complete the square on the RHS)

$$-y = (x - 4)^2 - 1$$

$$y = -(x - 4)^2 + 1$$

(multiply by -1 to make y the subject again)

Finding the vertex from the completed square

The completed square allows the vertex to be found using one fundamental fact about squares:

19 A square can never be negative

- $x^2 = 0$, when $x = 0$,
- $x^2 > 0$, when $x \neq 0$.

Thus in part (a) of the Example above, where

$$y = 2(x - 3)^2 - 2,$$

the term $2(x - 3)^2$ can never be negative.

$$\begin{aligned} \text{Substituting } x = 3 \text{ gives } y &= 0 - 2 \\ &= -2, \end{aligned}$$

$$\text{but if } x \neq 3, \text{ then } y > -2.$$

Hence the graph passes through the point $V(3, -2)$, but never goes below it. This means that $V(3, -2)$ is the vertex of the parabola, and $x = 3$ is its axis of symmetry.

But in part (b) of the example above, where

$$y = -(x - 4)^2 + 1,$$

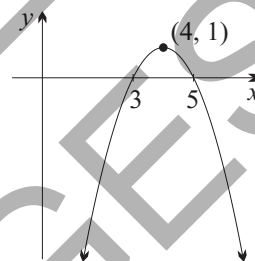
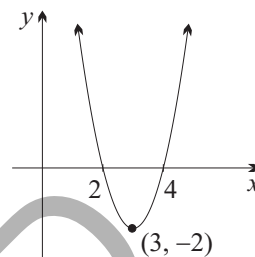
the term $-(x - 4)^2$ can never be positive.

$$\begin{aligned} \text{Substituting } x = 4 \text{ gives } y &= 0 + 1 \\ &= 1, \end{aligned}$$

$$\text{but if } x \neq 4, \text{ then } y < 1$$

Hence the graph passes through the point $V(4, 1)$, but never goes above it. This means that $V(4, 1)$ is the vertex of the parabola, and $x = 4$ is its axis of symmetry.

There is no need to repeat this argument every time. The result is simple:



20 Finding the axis and vertex from the completed square

For the quadratic $y = a(x - h)^2 + k$,

the axis is $x = h$ and the vertex is $V(h, k)$.

In Chapter 5, we will interpret this result as the effect of translations and dilations on $y = x^2$. Some readers may have done this already in earlier years.

Finding the zeroes from the completed square

The completed square also allows the zeroes to be found in the usual way:

21 Finding the zeroes from the completed square

To find the x -intercepts from the completed square, put $y = 0$.

- There may be two zeroes, in which case they may or may not involve surds.
- There may be no zeroes.
- There may be exactly one zero, in which case the quadratic is a *perfect square*. The x -axis is a tangent to the graph, and the zero is called a *double zero* of the quadratic.

These methods are illustrated in the worked examples below.



Example 16 Completing the square with and without surds

Use completing the square to sketch the graphs of these quadratic functions.

a $f(x) = x^2 - 4x - 5$

b $f(x) = x^2 - 4x - 1$

Solution

a The curve $y = x^2 - 4x - 5$ is concave up, with y-intercept -5 .

The square was completed earlier in this section,

$$y = (x - 2)^2 - 9,$$

so the axis is $x = 2$ and the vertex is $(2, -9)$.

Put $y = 0$, then $(x - 2)^2 = 9$

$$x - 2 = 3 \text{ or } x - 2 = -3$$

$$x = 5 \text{ or } x = -1.$$

b The curve $y = x^2 - 4x - 1$ is concave up, with y-intercept -1 .

Completing the square, $y = (x^2 - 4x + 4) - 4 - 1$,

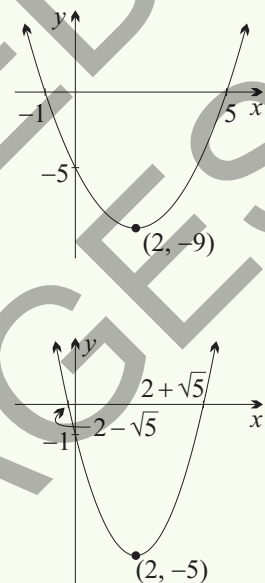
$$= (x - 2)^2 - 5,$$

so the axis is $x = 2$ and the vertex is $(2, -5)$.

Put $y = 0$, then $(x - 2)^2 = 5$

$$x - 2 = \sqrt{5} \text{ or } x - 2 = -\sqrt{5}$$

$$x = 2 + \sqrt{5} \text{ or } x = 2 - \sqrt{5}.$$



Note: With this method, the axis of symmetry and the vertex are read directly off the completed-square form — the zeroes are then calculated afterwards.

Compare this with factoring, where the zeroes are found first and the axis and vertex can then be calculated from them.



Example 17 Completing the square with no zeroes, and with a double zero

Use completing the square to sketch the graphs of these quadratics. Then write down the domain and range.

a $y = x^2 + x + 1$

b $y = x^2 + 6x + 9$

Solution

a The curve $y = x^2 + x + 1$ is concave up, with y-intercept 1 .

The square was completed earlier in this section,

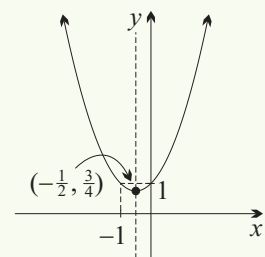
$$y = (x + \frac{1}{2})^2 + \frac{3}{4},$$

so the axis is $x = -\frac{1}{2}$, and the vertex is $(-\frac{1}{2}, \frac{3}{4})$.

Put $y = 0$, then $(x + \frac{1}{2})^2 = -\frac{3}{4}$.

Because negative numbers do not have square roots, this equation has no solutions, so there are no x -intercepts.

Domain: All real x OR $(-\infty, \infty)$. Range: $y \geq \frac{3}{4}$ OR $[\frac{3}{4}, \infty)$.



- b** $y = x^2 + 6x + 9$ is concave up, with y -intercept 9.

This quadratic is already a perfect square,

$$y = (x + 3)^2 + 0,$$

so the axis is $x = -3$ and the vertex is $(-3, 0)$.

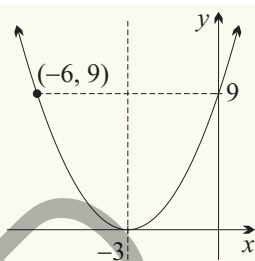
Put $y = 0$, then $(x + 3)^2 = 0$

$$x + 3 = 0$$

$$x = -3 \quad (\text{a double zero}).$$

Domain: All real x OR $(-\infty, \infty)$. Range: $y \geq 0$ OR $[0, \infty)$.

The symmetric point $(-6, 9)$ has been plotted so that the parabolic graph has at least three points on it.



Note: The symmetric point $(-6, 9)$ has been plotted so that the parabolic graph has at least three points on it.



Example 18 Sketching the graph from the completed square

- a** Use the previous completed square to sketch $y = -x^2 + 8x - 15$.

- b** Then write down its domain and range, in both notations

Solution

- a** $y = -x^2 + 8x - 15$ is concave down, with y -intercept -15 .

Completing the square, $y = -(x - 4)^2 + 1$,

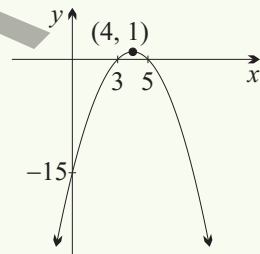
so the axis is $x = 4$ and the vertex is $(4, 1)$.

Put $y = 0$, then $(x - 4)^2 = 1$

$$x - 4 = 1 \text{ or } x - 4 = -1$$

$$x = 5 \text{ or } x = 3.$$

- b** Domain: All real x OR $(-\infty, \infty)$. Range: $y \leq 1$ OR $(-\infty, 1]$.

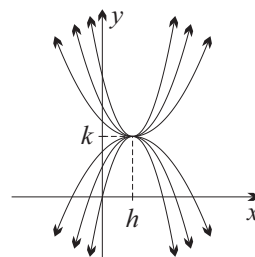


The family of quadratics with a common vertex

A quadratic with vertex (h, k) has an equation of the form

$$y = a(x - h)^2 + k$$

where a is the coefficient of x^2 . This equation gives a *family* of parabolas all with vertex (h, k) , as different values of a are taken. The sketch to the right shows six curves in the family, three with a positive, and three with a negative.



22 The family of quadratics with a common vertex

The quadratics with vertex (h, k) form a family of parabolas, all with equation

$$y = a(x - h)^2 + k, \quad \text{for some non-zero value of } a.$$


Example 19 Finding the family of quadratics with common vertex

Write down the family of quadratics with vertex $(-3, 2)$. Then find the equation of such a quadratic:

a if $x = 5$ is one of its zeroes,

b if the coefficient of x is equal to 1.

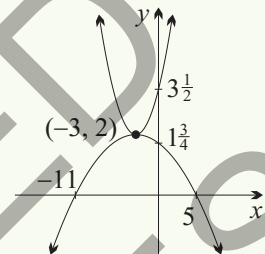
Solution

The family of quadratics with vertex $(-3, 2)$ is

$$y = a(x + 3)^2 + 2. \quad (*)$$

a Substituting $(5, 0)$ into $(*)$ gives $0 = a \times 64 + 2$,
so $a = -\frac{1}{32}$, and the quadratic is $y = -\frac{1}{32}(x + 3)^2 + 2$.

b Expanding the equation $(*)$, $y = ax^2 + 6ax + (9a + 2)$,
so $6a = 1$.
Hence $a = \frac{1}{6}$, and the quadratic is $y = \frac{1}{6}(x + 3)^2 + 2$.


Exercise 3E
FOUNDATION

- 1 a** The equation of a parabola in completed square form is $y = (x - 2)^2 - 1$.
 - i** What is the concavity of this parabola?
 - ii** Substitute $x = 0$ to find the y -intercept.
 - iii** Use the results of Box 19 to write down the equation of the axis of symmetry and the coordinates of the vertex.
 - iv** Use your answers to parts (i) and (iii) to explain why there must be x -intercepts, then find them by putting $y = 0$.
 - v** Hence sketch this parabola, showing this information.
- b** Repeat the steps of part (a) in order to sketch the graphs of these quadratic functions.
 - i** $y = (x + 1)^2 - 4$
 - ii** $y = (x - 1)^2 - 9$
- 2 a** A concave-down parabola has the completed square form $y = -(x - 1)^2 + 1$.
 - i** Why is the parabola concave down?
 - ii** Substitute $x = 0$ to find the y -intercept.
 - iii** Use the results of Box 19 to write down the equation of the axis of symmetry and the coordinates of the vertex.
 - iv** Use your answers to part (iii) to explain why there must be x -intercepts, then find them by putting $y = 0$.
 - v** Hence sketch this parabola, showing this information.
- b** Repeat the steps of part (a) in order to sketch the graphs of these parabolas.
 - i** $y = -(x + 2)^2 + 4$
 - ii** $y = -(x - 2)^2 + 9$
- 3** Complete the square in each monic quadratic function.

a $f(x) = x^2 - 4x + 5$	b $f(x) = x^2 + 6x + 11$	c $f(x) = x^2 - 2x + 8$
d $f(x) = x^2 - 10x + 1$	e $f(x) = x^2 + 2x - 5$	f $f(x) = x^2 + 4x - 1$

- 4 Follow the steps in Question 1 to sketch these quadratics. The x -intercepts involve surds.

a $y = (x + 1)^2 - 3$

b $y = (x - 4)^2 - 7$

c $y = (x - 3)^2 - 2$

DEVELOPMENT

- 5 Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 - 2x$

b $y = x^2 - 4x + 3$

c $y = x^2 - 2x - 5$

d $y = x^2 + 2x - 1$

e $y = x^2 + 2x + 2$

f $y = x^2 - 3x + 4$

- 6 Find the zeroes of each quadratic functions by completing the square. Then show that the same answer is obtained by factoring.

a $f(x) = x^2 - 4x + 3$

b $f(x) = x^2 + 2x - 3$

c $f(x) = x^2 - x - 2$

- 7 Complete the square of each quadratic function in order to find its zeroes. Each zero involves a surd.

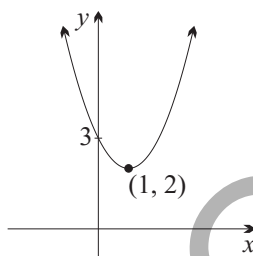
a $f(x) = x^2 - 4x + 1$

b $f(x) = 2x^2 - 4x - 2$

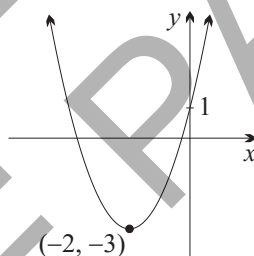
c $f(x) = -x^2 - 6x + 1$

- 8 Use the formula in Box 21 to write down the equation of each of the quadratic functions sketched below. In each case, the coefficient of x^2 is either 1 or -1 .

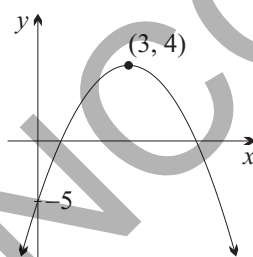
a



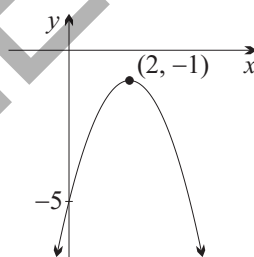
b



c



d



- 9 Write down the equation of the monic quadratic with vertex:

a $(2, 5)$

b $(0, -3)$

c $(-1, 7)$

d $(3, -11)$

- 10 [Technology]

- a** Use computer graphing software to plot accurately on the one number plane the parabola $y = a(x - 1)^2 - 2$ for the following values of a .

i $a = 2$

ii $a = 1$

iii $a = -1$

iv $a = -2$

- b** Which point do all these parabolas pass through?

- c** For which values of a does the parabola have x -intercepts?

- d** Explain your answer to part (c) geometrically.

11 Consider the general quadratic function $y = ax^2 + bx + c$.

a Divide through by a and then complete the square in x in order to show that

$$\frac{y}{a} = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}.$$

b Now make y the subject, and hence write down the coordinates of the vertex and the equation of the axis of symmetry.

c Put $y = 0$ in order to find the x -intercepts.

CHALLENGE

12 Write down the coordinates of the vertex and the concavity for each parabola. Hence determine the number of x -intercepts.

a $y = 2(x - 3)^2 - 5$

b $y = -3(x + 2)^2 - 1$

c $y = 4(x + 1)^2$

13 Complete the square for these non-monic quadratics. (In each case, notice that the coefficient of x^2 is not 1.) Then sketch each curve, showing the vertex and any intercepts.

a $y = -x^2 + 4x + 1$

b $y = 2x^2 - 4x + 3$

c $y = 2x^2 + 6x + 2$

14 Complete the square for each quadratic function. Hence write each quadratic in factored form. Your answers will involve surds.

a $f(x) = x^2 + 2x - 1$

b $f(x) = x^2 - 4x + 1$

c $f(x) = -x^2 - 2x + 4$

15 Explain why $y = a(x + 4)^2 + 2$ is the general form of a quadratic with vertex $(-4, 2)$. Then find the equation of such a quadratic for which:

a the quadratic is monic,

b the coefficient of x^2 is 3,

c the y -intercept is 16,

d the curve passes through the origin,

3F The quadratic formulae and the graph

Learning intentions

- Use the discriminant to work with the formula for the zeroes of a quadratic.
- Use the formula for the axis of symmetry, and hence find the vertex.
- Use the discriminant and leading coefficient to answer questions about the quadratic.
- Use the discriminant to ‘discriminate’ (predict) the number of zeroes.

Completing the square in the general quadratic yields formulae for its axis of symmetry and for its zeroes. These formulae are extremely useful, and like factoring and completing the square, allow the graph to be sketched.

The algebra of completing the square in a general quadratic is presented in the structured Question 11 at the end of Development in the previous Exercise 3E.

The formula for the axis of symmetry

Question 11 in the previous Exercise 3E gives the axis of symmetry:

23 The axis of symmetry of $y = ax^2 + bx + c$

- The axis of symmetry is the line $x = -\frac{b}{2a}$.
- Substitute back into the quadratic to find the y coordinate of the vertex.

Readers should remember only the formula for the axis of symmetry, and find the y-coordinate of the vertex by substituting back into the quadratic.

The formula for the zeroes

Further working in the structured question of Exercise 3E shows that putting $y = 0$ into $y = ax^2 + bx + c$ gives

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac$$

The quantity $\Delta = b^2 - 4ac$ is very important in the theory of quadratics. It is called the *discriminant* because it discriminates, and has the symbol Δ (Greek uppercase letter *delta*, corresponding to the Latin letter ‘D’). Using Δ in this formula makes it much easier to deal with and to remember.

24 The zeroes of the quadratic function $y = ax^2 + bx + c$

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac.$$

- Always calculate the discriminant first when finding the zeroes of a quadratic.
- If $\Delta > 0$, there are two zeroes, because positives have two square roots.
- If $\Delta = 0$, there is one *double zero*, because 0 is the only square root of 0.
 - ▷ In this situation, the quadratic is a *perfect square*,
 - ▷ and the x-axis is tangent to the graph.
- If $\Delta < 0$, there are no zeroes, because negatives don’t have square roots.
Using the alternative terminology for quadratics, we can say instead:
 - If $\Delta > 0$, the quadratic has distinct zeroes (meaning ‘two distinct zeroes’).
 - If $\Delta = 0$, the quadratic has two equal zeroes (noting that there is only zero).


Example 20 Using the quadratic formulae to sketch a quadratic

Use the quadratic formulae to sketch each quadratic. Give any irrational zeroes first in simplified surd form, then approximated correct to three decimal places.

a $y = -x^2 + 6x + 1$

b $y = 3x^2 - 6x + 4$

Solution

a The curve $y = -x^2 + 6x + 1$ is concave down, with y -intercept 1.

The formulae are now applied with $a = -1$, $b = 6$, and $c = 1$.

First, the axis of symmetry is $x = -\frac{b}{2a}$

$$x = -\left(\frac{6}{-2}\right)$$

$$x = 3.$$

When $x = 3$, $y = -9 + 18 + 1$

$$= 10,$$

so the vertex is $(3, 10)$.

Secondly, $\Delta = b^2 - 4ac$

$$= 40$$

$$= 4 \times 10 \quad (\text{take out square factors}),$$

$$\begin{aligned} \text{so } y = 0 \text{ when } x &= \frac{-b + \sqrt{\Delta}}{2a} \text{ or } \frac{-b - \sqrt{\Delta}}{2a} \\ &= \frac{-6 + 2\sqrt{10}}{-2} \text{ or } \frac{-6 - 2\sqrt{10}}{-2} \\ &= 3 - \sqrt{10} \text{ or } 3 + \sqrt{10} \\ &\div -0.162 \text{ or } 6.162. \end{aligned}$$

b The curve $y = 3x^2 - 6x + 4$ is concave up, with y -intercept 4.

Apply the formulae with $a = 3$, $b = -6$ and $c = 4$.

First, the axis of symmetry is $x = -\frac{b}{2a}$

$$x = -\left(\frac{-6}{6}\right)$$

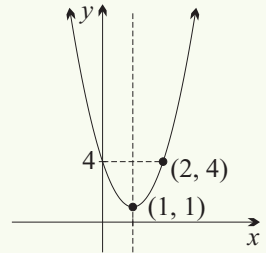
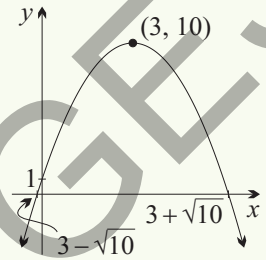
$$x = 1,$$

and substituting $x = 1$, the vertex is $(1, 1)$.

Secondly, $\Delta = 36 - 48$

$$= -12$$

which is negative, so there are no zeroes, because negative numbers do not have square roots.



Notice again that the symmetric point $(2, 4)$ has been plotted so that the parabolic graph has at least three points on it.


Example 21 Using the discriminant to find the number of zeroes

- a** Use the discriminant to find the number of zeroes of $f(x) = 5x^2 - 20x + 20$.
b What would be a better approach to this question?

Solution

a $f(x) = 5x^2 - 20x + 20$

$$\Delta = 20^2 - 4 \times 5 \times 20$$

$$= 0,$$

so there is exactly one zero.

- b** Better, take out the common factor:

$$f(x) = 5x^2 - 20x + 20$$

$$= 5(x^2 - 4x + 4)$$

$$= 5(x - 2)^2, \text{ giving } 5 \times \text{a perfect square.}$$


Example 22 Using the discriminant to test for a double zero

Find m if the quadratic $f(x) = 2x^2 - 4mx + 10m$ has a double zero (or 'two equal zeroes'), and write down the resulting quadratic function(s) in factored form.

Solution

Here $a = 2$, $b = -4m$, and $c = 10m$, so $\Delta = 16m^2 - 80m$
 $= 16m(m - 5).$

The condition for a double root is

$$\Delta = 0,$$

that is,

$$m = 0 \text{ or } 5.$$

Hence there are two quadratics $f(x) = 2x^2$ and $f(x) = 2(x - 5)^2$.

Exercise 3F
FOUNDATION

- 1** Answer the following questions for the parabola $y = x^2 - 2x - 1$.
 - a** **i** Use the value of a to determine the concavity of the parabola.
 - ii** Write down the value of the y -intercept.
 - iii** Use the formula $x = \frac{-b}{2a}$ to find the axis of symmetry.
 - iv** Use the axis of symmetry to find the y -coordinate of the vertex.
 - v** Calculate the discriminant $\Delta = b^2 - 4ac$.
 - vi** Explain why this parabola must have x -intercepts.
 - vii** Find the x -intercepts by the formula $x = \frac{-b + \sqrt{\Delta}}{2a}$ or $\frac{-b - \sqrt{\Delta}}{2a}$.
- b** Sketch the parabola, showing these features.
- 2** Follow the steps of Question 1 to sketch these parabolas.
 - a** $y = x^2 + 4x + 1$
 - b** $y = -x^2 + 2x + 5$

- ## DEVELOPMENT

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3G Powers, cubics, and circles

Learning intentions

- Sketch graphs involving $y = \sqrt{x}$, and understand their properties.
- Sketch cubic graphs, and understand some of their properties.
- Sketch circles and semicircles.

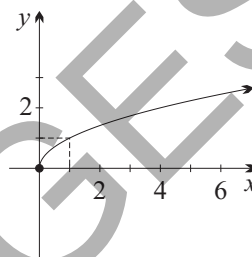
This section deals with the graphs of cubes and some higher powers of x , and of the square root of x , then sketches cubic functions that have been factored into linear factors. Finally, it reviews circles and semicircles.

The function $y = \sqrt{x}$

The graph of $y = \sqrt{x}$ is the upper half of a parabola on its side, as can be seen by squaring both sides to give $y^2 = x$. Remember that the symbol $\sqrt{x}$ means the *positive or zero* square root of x , so the lower half of the parabola $y^2 = x$ is excluded:

$y = \sqrt{x}$	x	0	$\frac{1}{4}$	1	2	4
	y	0	$\frac{1}{2}$	1	$\sqrt{2}$	2

The domain is $x \geq 0$, and the range is $y \geq 0$.

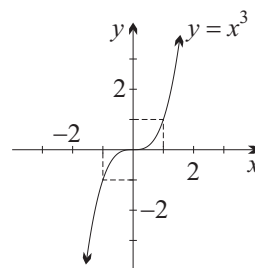


The cube of x

Graphed to the right is the cubic function $f(x) = x^3$. It has a zero at $x = 0$, it is positive when x is positive, and is negative when x is negative.

$y = x^3$	x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
	y	-8	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	8

The curve becomes flat at the origin. You can see this by substituting a small number such as $x = 0.1$. The corresponding value of y is $y = 0.001$, which is far smaller.



The origin is called a *horizontal inflection* of the curve — ‘inflection’ means that the curve ‘flexes’ smoothly from concave down on the left to concave up on the right, and ‘horizontal’ means that the curve is momentarily horizontal. Inflections in general will be studied in Year 12.

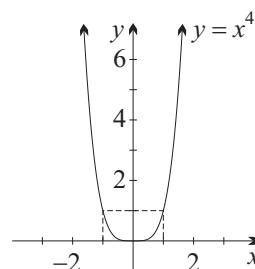
The graphs of odd powers $f(x) = x^3$, $f(x) = x^5$, $f(x) = x^7$, ..., look similar. The origin is always a horizontal inflection. As the index increases, the curves become flatter near the origin, and steeper further away.

The fourth power of x

The graph to the right shows $f(x) = x^4$. It has a zero at $x = 0$, and is positive elsewhere.

$y = x^4$	x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
	y	16	1	$\frac{1}{16}$	0	$\frac{1}{16}$	1	16

All even powers $y = x^6$, $y = x^8$, ... look similar. Like odd powers, as the index increases they become flatter near the origin and steeper further away.



The origin is called a *turning point* of the curve because at the origin, the curve turns smoothly from decreasing on the left to increasing on the right. Turning points in general are an important topic in Year 12.

Polynomials

Polynomials are expressions such as

$$3x^4 + 4x^2 - 2x \quad \text{and} \quad x^3 - x^2 \quad \text{and} \quad 7 - 2x^3 + \frac{3}{4}x^6$$

that can be written as the sum of multiples of x , x^2 , x^3 , ... and a constant.

A polynomial is usually written with the powers in descending or ascending order, as above. The index of the power with highest index (and non-zero coefficient) is called the *degree* of the polynomial, so the polynomials above have degrees 4, 3, and 6 respectively. Thus familiar quadratics such as $3x^2 + 4x + 5$ are polynomials of degree 2.

This section introduces *cubics* such as $x^3 - x^2$, but the discussion is limited here only to cubics that have been factored completely, or to cubics that are easily factored such as $x^3 - x^2 = x^2(x - 1)$.

Sketching a cubic factored into linear factors

At this stage, little systematic graphing can be done of polynomial functions of degree 3 and higher. If, however, the polynomial has already been factored into linear factors, its basic shape can be established. This is done by drawing up a table of values to test its sign. For example, consider the cubic

$$y = (x + 1)(x - 2)(x - 4).$$

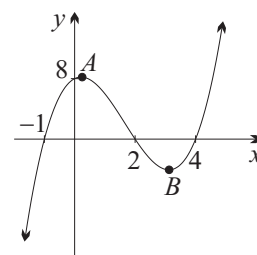
The function has zeroes at $x = -1$, $x = 2$ and $x = 4$.

Its domain is all real x , and cubic curves don't have any breaks, so the three zeroes are the only places where it can change sign.

We can draw up a table of values dodging these zeroes,

x	-2	-1	0	2	3	4	5
y	-24	0	8	0	-4	0	18
sign	-	0	+	0	-	0	+

We also know that, as with any cubic, y becomes very large positive or negative for large positive or negative values of x . Combining this with the table above allows us to draw a sketch.



- Be careful! We cannot yet find the two turning points marked A and B on the curve, because that requires calculus. Notice that if we draw tangents there, the tangents are horizontal. Don't ever think that those points A and B are midway between the zeroes — this curve is not a parabola. Mark the zeroes and the y-intercept, and nothing else.
- Even more importantly, there is a *point of inflection* on the curve midway between A and B. We will have more to say on all this in Year 12.

Cubics with square factors

The cubic in the next example has a square factor, so that there are only two zeroes, one of which is a double zero.



Example 23 Sketching cubics with square factors

Sketch $y = -(x - 1)^2(x - 4)$. What is happening at $x = 1$?

Solution

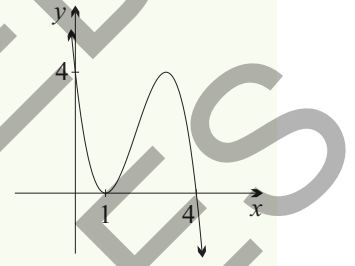
The cubic has zeroes at $x = 1$ and $x = 4$.

Here is a table of test values dodging around these zeroes,

x	0	1	2	4	5
y	4	0	2	0	-16
sign	+	0	+	0	-

At $x = 1$, the cubic has a double zero, but does not cross the x -axis.

Instead, the x -axis is a tangent to the curve at $(1, 0)$.



Circles

The equation of a circle, centre $O(0, 0)$ and radius r , is found by Pythagoras' theorem, in the form of the distance formula.

A point $P(x, y)$ in the plane lies on this circle

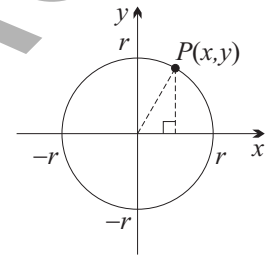
if and only if

$$OP = r$$

$$OP^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = r^2 \quad (\text{distance formula})$$

$$x^2 + y^2 = r^2.$$



To put it very briefly, the equation of a circle is Pythagoras' theorem.

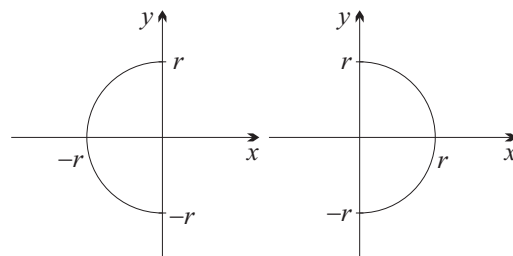
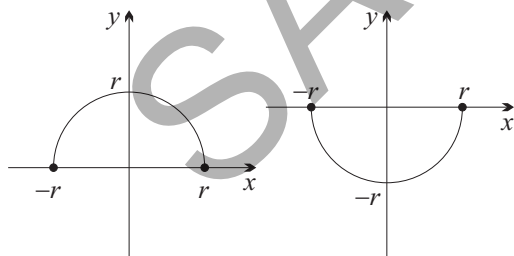
Four important semicircles

The circle graph fails the vertical line test, so is not a function. This can also be seen algebraically — solving for y gives two values of y for $-r < x < r$:

$$y = \sqrt{r^2 - x^2} \quad \text{or} \quad y = -\sqrt{r^2 - x^2}.$$

These two graphs are the first two graphs below — the upper and lower semicircles — and in contrast to the circle, they are clearly both functions.

Their domains are both $-r \leq x \leq r$, and ranges are $0 \leq y \leq r$ and $-r \leq y \leq 0$.



Solving

instead for x , and again separating the positive and negative square roots,

$$x = \sqrt{r^2 - y^2} \quad \text{or} \quad x = -\sqrt{r^2 - y^2},$$

These two graphs are the last two graphs above — the left and right semicircles. They are relations, but not functions, because they clearly fail the vertical line test. Their domains are $-r \leq x \leq 0$ and $0 \leq x \leq r$, and ranges are both $-r \leq y \leq r$.

Exercise 3G

FOUNDATION

- 1 Write down the coordinates of the centre and the radius of each circle.

a $x^2 + y^2 = 16$

b $x^2 + y^2 = 49$

c $x^2 + y^2 = \frac{1}{9}$

d $x^2 + y^2 = 1.44$

- 2 Sketch graphs of these circles, marking all intercepts with the axes, then write down the domain and range of each.

a $x^2 + y^2 = 1$

b $x^2 + y^2 = 9$

c $x^2 + y^2 = \frac{1}{4}$

d $x^2 + y^2 = \frac{9}{4}$

- 3 Consider the curve $y = x^3$.

- a** Copy and complete the following table of values:

x	-1.5	-1	-0.5	0	0.5	1	1.5
y							

- b** Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

- 4 Repeat the previous question for the curve $y = x^4$.

- 5 [Technology]

- a** Use computer graphing software to plot accurately on the one number plane $y = x$, $y = x^3$ and $y = x^5$.

- b** Which three points do all these graphs pass through?

- c** Which curve is nearest the x -axis for:

i $0 < x < 1$,

ii $x > 1$?

- d** Which curve is nearest the x -axis for:

i $-1 < x < 0$,

ii $x < -1$?

- e** Rotate each curve by 180° about the origin. What do you notice?

- f** Try finding other powers of x which have the same feature found in part (e). What do you notice about the index of these functions?

- 6 [Technology]

- a** Use computer graphing software to plot accurately on the one number plane $y = x^2$, $y = x^4$ and $y = x^6$.

- b** Which three points do all these graphs pass through?

- c** Which curve is nearest the x -axis for:

i $0 < x < 1$,

ii $x > 1$?

- d** Which curve is nearest the x -axis for:

i $-1 < x < 0$,

ii $x < -1$?

- e** Reflect each curve in the y -axis. What do you notice?

- f** Try finding other powers of x which have the same feature found in part (e). What do you notice about the index of these functions?

- 7 Consider the curve $y = \sqrt{x}$.

- a** Copy and complete the following table of values:

x	0	0.25	1	2.25	4	6.25
y						

- b** Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

- 8** Write down the zeroes of each cubic, use a table of values to test its sign, then sketch it, showing the y-intercept.

a $y = (x - 1)(x - 3)(x - 5)$

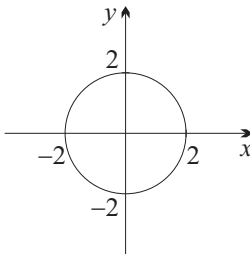
b $y = -3(x + 4)x(x - 2)$

c $y = 2x^2(3 - x)$

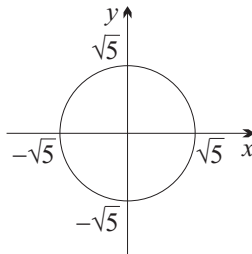
DEVELOPMENT

- 9** Write down the equation of each circle.

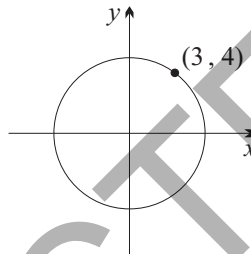
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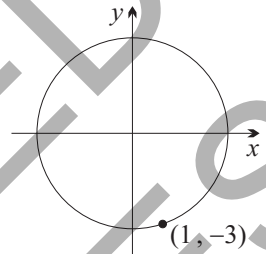
b



c



d



- 10** Consider the circle $x^2 + y^2 = 25$.

- a** Copy and complete the following table of values, correct to one decimal place where necessary. (Remember that a positive number has two square roots.)

x	0	1	2	3	4	5
$y \geq 0$						
$y \leq 0$						

- b** Plot the points in the table, using a scale of 1 cm to 1 unit on each axis.

- c** Reflect the points plotted in part (b) in the y-axis, and so sketch the entire circle.

- 11** Sketch each semicircle, and state the domain and range:

a $y = \sqrt{4 - x^2}$

b $y = -\sqrt{4 - x^2}$

c $y = -\sqrt{1 - x^2}$

d $y = \sqrt{\frac{25}{4} - x^2}$

- 12** Sketch these semicircles which are not functions. State the domain and range of each.

a $x = \sqrt{1 - y^2}$

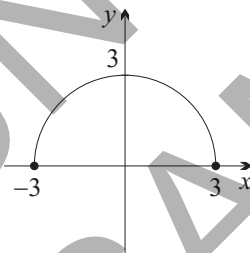
b $x = -\sqrt{4 - y^2}$

c $x = -\sqrt{9 - y^2}$

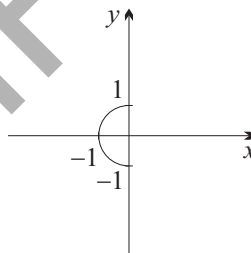
d $x = \sqrt{\frac{9}{4} - y^2}$

- 13** Write down the equation of each semicircle.

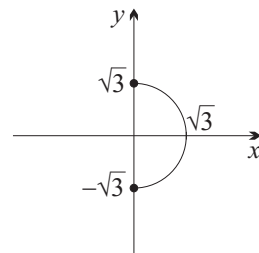
a



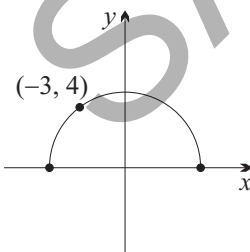
b



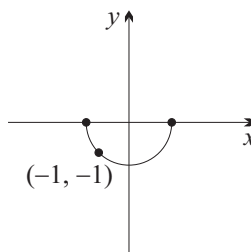
c



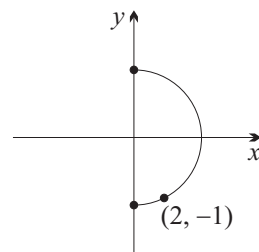
d



e



f



14 Consider the curve $y = -\sqrt{x}$.

a Copy and complete the following table of values:

x	0	0.25	1	2.25	4	6.25
y						

b Plot the points in the table, using a scale of 2 cm to 1 unit on each axis, and then join the points with a smooth curve.

15 [Technology]

a Graph these cubic polynomials accurately using computer graphing software.

i $y = \frac{1}{4}x^3 + 2$

ii $y = \frac{1}{2}(x^3 - 6x^2 + 9x)$

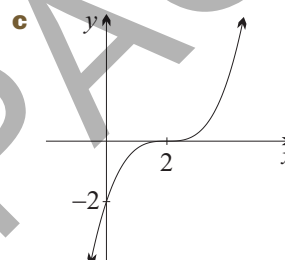
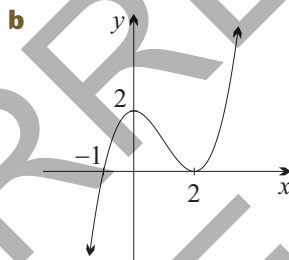
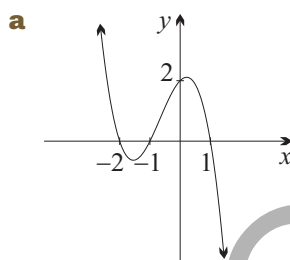
iii $y = \frac{1}{2}(x^3 - 2x^2 - 5x + 6)$

b In each case, check that the y -intercept is equal to the constant term.

c Read the x -intercepts from the screen and then check these values by substituting them into the corresponding polynomial.

CHALLENGE

16 Each graph shows a cubic of the form $y = a(x - \alpha)(x - \beta)(x - \gamma)$ for some constant a and where $\alpha \leq \beta \leq \gamma$. Use the graph to determine these values and hence write down the equation of the cubic.



17 a Use the results of Question 7 to sketch the graph of $y = \sqrt{x}$.

b On the same number plane, sketch the graph of $y = -\sqrt{x}$, by using Question 14.

c What shape has been formed?

d Explain why this has happened.

18 [Technology] As discovered in the previous sections, the graphs of quadratic functions fall into one of two categories, concave up and concave down. In contrast, cubic functions have six basic types.

a Use computer graphing software to plot accurately the cubic $y = c(x)$ for:

i $c(x) = x^3 + x$

ii $c(x) = x^3$

iii $c(x) = x^3 - x$

b These three curves have some similarities.

i For large positive and negative values of x , which two quadrants does the graph of $y = c(x)$ lie in?

ii Rotate each curve by 180° about the origin. What do you notice?

iii By considering your investigation in Question 5, explain why that might be.

c Look carefully at each curve as it passes through the origin. What distinguishes each graph there?

d Conclude this investigation by accurately graphing $y = -c(x)$ for each cubic function in part (a). Then look for any similarities and differences.

3H Two graphs that have asymptotes

Learning intentions

- Sketch exponential functions, and identify their asymptotes.
- Sketch rectangular hyperbolas and identify their asymptotes.
- Use limit notation for exponential graphs and rectangular hyperbolas.

This section reviews simple exponential graphs and rectangular hyperbolas. They are grouped together because both types of graphs have asymptotes, which need further discussion.

Exponential functions

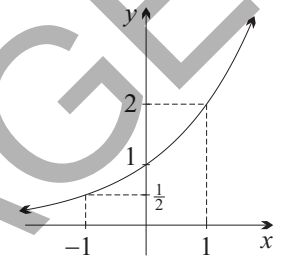
Functions of the form $f(x) = a^x$, where the base a is positive and $a \neq 1$, are called *exponential functions*, because the variable x is in the *exponent* or *index*.

Here is a sketch of the function $f(x) = 2^x$.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Three key features should be shown when sketching this graph:

- The y -intercept is $y = 1$, because $2^0 = 1$.
- When $x = 1$, $y = 2$, which is the base 2, because $2^1 = 2$.
- The x -axis is a horizontal asymptote, as discussed below.



The domain of an exponential function is all real x , and the range is $y > 0$.

Limits and asymptotes of exponential functions

On the far left of the graph, as x is becoming a very large negative number, $y = 2^x$ becomes very small. Indeed, we can make y 'as small as we like' by choosing sufficiently large negative values of x . We say that 'as x approaches negative infinity, y approaches the limit zero', and write

$$\text{As } x \rightarrow -\infty, y \rightarrow 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} y = 0.$$

The x -axis is called an *asymptote* of the curve (from the Greek word *asymptotos*, meaning 'apt to fall together'), because the curve gets 'as close as we like' to the x -axis for sufficiently large negative values of x .

In the right-hand direction, there is no limit, because 'As $x \rightarrow \infty$, $y \rightarrow \infty$ '.

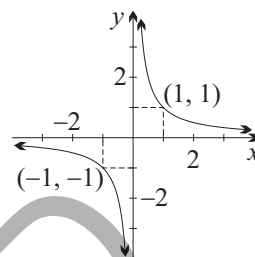
Thus $y = 2^x$ has domain all real x , and range $y > 0$, or in bracket interval notation:

Domain: $(-\infty, \infty)$. Range: $(0, \infty)$.

Rectangular hyperbolas

The *reciprocal function* $y = \frac{1}{x}$ can also be written as $xy = 1$, and has a graph that is called a *rectangular hyperbola*. This graph has two parts, called *branches*, which in this case are disconnected.

$y = \frac{1}{x}$	x	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	2	5	10
	y	*	10	5	2	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$
	x	-10	-5	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$	
	y	$-\frac{1}{10}$	$-\frac{1}{5}$	$-\frac{1}{2}$	-1	-2	-5	-10	



The star (*) at $x = 0$ means that the function is not defined there.

Limits and asymptotes of rectangular hyperbolas

The x -axis is an asymptote to this curve on both sides of the graph. We can make y ‘as small as we like’ by choosing sufficiently large positive or negative values of x . We say that ‘as x approaches ∞ , y approaches zero’, and ‘as x approaches $-\infty$, y approaches zero’, and we write

As $x \rightarrow \infty, y \rightarrow 0$ and As $x \rightarrow -\infty, y \rightarrow 0$
or $\lim_{x \rightarrow \infty} y = 0$ or $\lim_{x \rightarrow -\infty} y = 0$

The y-axis is a second asymptote to the graph. On the right-hand side of the origin, when x is a very small positive number, y becomes very large. We can make y ‘as large as we like’ by taking sufficiently small but still positive values of x . We say that ‘as x approaches zero from the right, y approaches ∞ ’, and write

As $x \rightarrow 0^+$, $y \rightarrow \infty$.

On the left-hand side of the origin, y is negative and can be made ‘as large negative as we like’ by taking sufficiently small negative values of x . We say that ‘as x approaches zero from the left, y approaches $-\infty$ ’, and write

As $x \rightarrow 0^-$, $y \rightarrow -\infty$.

Domain and range

From the sketch, it is clear that $y = \frac{1}{x}$ has domain $x \neq 0$, and range $y \neq 0$.

Neither domain nor range is an interval, so bracket interval notation cannot be used here as an alternative.

Exercise 3H

FOUNDATION

Note: If computer graphing software is not available, the two technology questions can be completed using tables of values.

- 1 a** Copy and complete the following table of values of the hyperbola $y = \frac{2}{x}$.

x	-4	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	4
y					*				

- b** Plot the points, using a scale of 1 cm to 1 unit on each axis, then sketch the hyperbola.
 - c** Which two quadrants do the branches of the curve lie in?
 - d** Write down the equations of the two asymptotes of the hyperbola.
 - e** Write down the domain and range of the function.
- 2** Construct a table of values for each hyperbola, then sketch it. Write down the domain and range of each hyperbola and the equations of the two asymptotes. Also state which quadrants the branches lie in.
 - a** $y = \frac{4}{x}$
 - b** $y = \frac{3}{x}$

- 3** [Technology] Use computer graphing software (if unavailable, tables of values will also do) to plot accurately on the one number plane

$$y = \frac{1}{x} \quad \text{and} \quad y = \frac{4}{x} \quad \text{and} \quad y = \frac{9}{x}.$$

- Which two quadrants do the branches of each hyperbola lie in?
- Write down the equations of the two asymptotes of each hyperbola.
- Write down the domain and range of each hyperbola.
- Write down the coordinates of the points on each hyperbola closest to the origin. What do you notice?
- Add the line $y = x$ to the graph and observe that the graphs are symmetric in this line. What other line are the graphs symmetric in?

- 4** Consider the exponential curve $y = 3^x$.

- a** Copy and complete the following table of values of the exponential function $y = 3^x$. Give answers correct to one decimal place where necessary.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y									

- Plot these points, using scales of 1 cm to 1 unit on both axes, then sketch the curve.
 - What is the y -intercept of the function?
 - What is the y -coordinate when $x = 1$?
 - Write down the equation of the asymptote.
 - Write down the domain and range of the function.
- 5** Construct a table of values for each exponential function, then sketch its graph. Write down the domain and range in each part and the equation of the asymptote. Also state the y -intercept and the y -coordinate at $x = 1$.
- $y = 4^x$
 - $y = 1.5^x$
- 6** [Technology] Use computer graphing software to plot accurately on the one number plane $y = 2^x$, $y = 3^x$ and $y = 4^x$.
- Which point is common to all three graphs?
 - Write down the equation of the asymptote of each exponential curve.
 - Write down the domain and range of each exponential function.
 - Confirm by observation that at $x = 1$, the y -coordinate equals the base.
 - Which curve increases more rapidly to the right of the y -axis, and why?
 - Which curve approaches the asymptote more quickly to the left of the y -axis? Why?

DEVELOPMENT

- 7 a** Construct a table of values for each hyperbola then sketch it.

i $y = -\frac{2}{x}$

ii $y = -\frac{4}{x}$

iii $y = -\frac{3}{x}$

- Compare these equations and graphs with those in Questions 1 and 2.
 - Which quadrants do the graphs of part (a) lie in?
 - What has changed in the equation to cause this difference?
- The domain and range of each graph in part (a) are the same.
 - Write down the domain and range.
 - Are they the same as in Questions 1 and 2?

8 Rewrite each equation with x as the independent variable, then sketch its graph.

a $xy = \frac{1}{2}$

b $xy = -6$

9 a Construct a table of values for each exponential function then sketch it.

i $y = 3^{-x}$

ii $y = 4^{-x}$

iii $y = 1.5^{-x}$

b Compare these equations and graphs with those in Question 4 and 5.

i Has the y -intercept changed?

ii Has the asymptote changed?

iii For the graphs in part (a), at what value of x is the y -coordinate equal to the base?

iv Heading to the right along each of these curves, describe how the y -coordinate changes.

v What has changed in the equation to cause these differences?

c The domain and range of each graph in part (a) are the same.

i Write down the domain and range.

ii Are they the same as in Questions 4 and 5?

10 Sketch these exponential graphs without resorting to a table of values. Ensure the key features are shown.

a $y = 5^x$

b $y = 2^{-x}$

11 This question requires the language of limits from the first two pages of this section.

a In Question 4 above, the line $y = 0$ is an asymptote to $y = 3^x$. Write a statement using limits to justify this.

b In Question 9(a)(ii), the line $y = 0$ is an asymptote to the exponential curve $y = 4^{-x}$. Write a statement using limits to justify this.

c In Question 1 above, the lines $y = 0$ and $x = 0$ are asymptotes to the hyperbola $y = \frac{2}{x}$. Write four statements using limits to justify this.

CHALLENGE

12 Does the equation $xy = 0$ represent a hyperbola? Explain your answer.

13 a Construct a table of values, and hence sketch the graph of the function $y = (\frac{1}{2})^x$.

b Use the index laws to explain why $y = (\frac{1}{2})^x$ has the same graph as $y = 2^{-x}$.

14 a Where does the hyperbola $xy = c^2$ intersect the line $y = x$?

b Confirm your answer by plotting the situation when $c = 2$.

15 Some curves can cross their asymptote.

a Complete the following table of values for $y = \frac{2x}{x^2 + 1}$.

x	-8	-4	-2	-1	-0.5	0	0.5	1	2	4	8
y											

b Plot the points, using a scale of 1 cm to 1 unit on each axis, and join them with a smooth curve.

c What is the horizontal asymptote of this curve?

d Where does the curve cross its asymptote?

3I Direct and inverse variation

Learning intentions

- Understand ‘varies with’ as a synonym for ‘is proportional to’.
- Understand variation with higher powers of x , and with square roots.
- Work with the functions arising from variation in scientific and practical examples.

Direct variation and inverse variation were introduced in previous years, where we usually used the language of ‘is proportional to’ rather than ‘varies with’

Direct and inverse variation

25 Direct and inverse variation

Direct variation: A variable y *varies directly* with a variable x if

$$y = kx, \quad \text{for some non-zero constant } k \text{ of proportionality.}$$

- The graph of y as a function of x is thus a line through the origin.

Inverse variation: A variable y *varies inversely* with a variable x if

$$y = \frac{k}{x}, \quad \text{for some non-zero constant } k \text{ of proportionality.}$$

- The graph of y as a function of x is thus a rectangular hyperbola whose asymptotes are the x -axis and the y -axis.

The term ‘varies with’ is often used as a synonym for ‘is proportional to’.

Inverse variation is more commonly called ‘indirect variation’, and common usage often says ‘is indirectly proportional to’ rather than ‘varies inversely with’.

Most applications begin by finding the constant of proportionality, as in the next two examples.



Example 24 Using direct variation in a practical situation

- a** Garden mulch is sold in bulk, with the cost C proportional to the volume V in cubic metres. Write this algebraically.
- b** The shop quotes \$270 for 7.5 m^3 . Find the constant of proportionality, and graph the function.
- c** How much does 12 m^3 cost?
- d** How much can I buy for \$600?

Solution

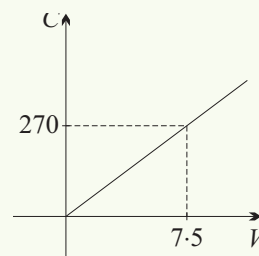
a $C = kV$, for some constant k .

b Substituting, $270 = k \times 7.5$
 $k = 36.$

Adding the units, $k = \$36/\text{m}^3.$

c $C = 36 \times 12$
 $= \$432$

d $600 = 36V$
 $V = 16\frac{2}{3} \text{ m}^3.$



Note: The functions $y = kx$ associated with direct variation with x are not just linear functions, but *linear functions through the origin*. Think $y = mx + b$, where $b = 0$, so that we only need to find the single constant m (the gradient), conventionally written k in variation.

The constant of proportionality may or may not have units

In the example above, we can solve the equation $C = kV$ for k to give $k = \frac{C}{V}$. Thus the constant k has units of Cost (in dollars) over Volume (in cubic metres). Hence the units of k are ‘dollars per cubic metre’, so we wrote that $k = \$36/m^3$.

But the constant may have no units — in a circle, the circumference C varies directly with the radius r , and the radius r varies directly with the circumference:

$$C = 2\pi r \quad \text{and} \quad r = \frac{1}{2\pi} C.$$

Both C and r are lengths, so the two constants 2π and $\frac{1}{2\pi}$ are dimensionless.



Example 25 Using inverse variation in a practical situation

- The wavelength λ in metres of a musical tone is inversely proportional to its frequency f in vibrations per second. Write this algebraically.
- The frequency of middle C is about 260 s^{-1} (‘260 vibrations per second’), and its wavelength is about 1.319 m. Find the constant of proportionality.
- Find the wavelength of a sound wave with frequency 440 s^{-1} .
- Find the frequency of a sound wave with wave length 1 metre.
- What is the approximate speed of sound in air, and why?

Solution

a $\lambda = \frac{k}{f}$, for some constant k .

b Substituting, $1.319 = \frac{k}{260}$

$$k \doteq 343$$

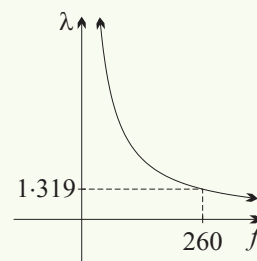
Adding the units, $k \doteq 343 \text{ m/s}$.

c $\lambda = \frac{k}{440}$
 $\doteq 0.779 \text{ m}$

d $1 = \frac{k}{f}$
 $f \doteq 343 \text{ s}^{-1}$

- e** About 343 m/s, because 343 waves, each 1 metre long, go past in 1 second.

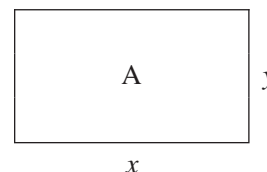
(Very roughly, this is 3 seconds per kilometre, which is extremely useful for estimating how far away a loud lightning strike is.)



Direct and inverse variation are closely related

Direct and inverse variation are closely related, despite their contrasting graphs. For example, if a rectangle has area A and adjacent sides x and y , then

$$A = xy \quad \text{and} \quad y = \frac{A}{x}.$$



- If the side y is constant, the area A is directly proportional to the other side x .
- If the area A is constant, the side y is inversely proportional to the side x .

Direct and inverse variation with other powers

A quantity can vary directly and indirectly with powers of x , typically x^2 and x^3 .

It can also vary with the square root $\sqrt{x}$ and the cube root $\sqrt[3]{x}$ of x .

In other situations that we will not consider until later chapters, a quantity Q varies with other functions of time t .

- With rates, Q typically varies with an exponential function such as 2^t .
- With periodic phenomena such as waves and air temperature and tides, a quantity Q typically varies with say $\sin t$ or $\cos t$.



Example 26 Varying with the cube root of a quantity

Spherical balls are being made solid, from just one material.

- Explain why the mass of the ball is proportional to the cube of the radius.
- Write part (a) as a formula, and hence show the radius is proportional to the cube root of the mass.
- Write part (b) as a formula, and hence find (nearest millimetre) the radius of a ball with mass 19 kg if a ball of mass 0.9 kg has a radius of 5 cm.

Solution

Let the ball have mass m kg, radius r cm, and volume V cm³.

- We know that $V = \frac{4}{3}\pi r^3$, so volume is proportional to the cube of the radius.
Also mass is proportional to volume, because only one material is used.
Hence the mass is also proportional to the cube of the radius.

- Thus $M = kr^3$, for some constant $k > 0$,

so $r = \ell \sqrt[3]{M}$, where $\ell = \frac{1}{\sqrt[3]{k}}$ is also a positive constant.

- Substituting, $5 = \ell \sqrt[3]{0.9}$

$$\ell = \frac{5}{\sqrt[3]{0.9}}$$

$$= 5.178720 \dots \text{ (store in the calculator).}$$

When $M = 19$, $r = \ell \sqrt[3]{19}$

$$\doteq 13.8 \text{ cm.}$$

Gravitational force involves four different proportions

In our universe (and on Earth), bodies are attracted to each other by gravity. The force F of attraction between two bodies is known to be:

- directly proportional to each mass m_1 and m_2 of the two bodies, and
- inversely proportional to the square of the distance r between them.

This can be written as a single function:

$$F = \frac{Gm_1m_2}{r^2},$$

where G is a constant called the *universal gravitational constant* (ignore the units).

We can solve this equation for each of the other variables:

$$m_1 = \frac{Fr^2}{Gm_2} \quad \text{and} \quad m_2 = \frac{Fr^2}{Gm_1} \quad \text{and} \quad r = \sqrt{\frac{Gm_1m_2}{F}}$$

Hence m_1 varies directly with the force F and with the square of the distance r , and varies inversely with the other mass m_2 . Similarly m_2 .

Also r varies directly with the square root of each mass m_1 and m_2 , and varies indirectly with the square root of the force F .

This equation is one of the most famous in all of physics, and goes back to Sir Isaac Newton (1635–1727). At the time, this law of universal gravity was more often thought of in terms of proportionality than as an equation.

Exercise 3I

FOUNDATION

- 1 It is known that y varies directly with x .
 - a Write this algebraically using k as the constant of proportionality.
 - b Find the value of k given that:
 - i $y = 10$ when $x = 2$
 - ii $y = -6$ when $x = 3$
 - iii $y = 6$ when $x = 4$
- 2 It is known that y is inversely proportional to x .
 - a Write this algebraically using k as the constant of proportionality.
 - b Find the value of k given that:
 - i $y = 2$ when $x = 3$
 - ii $y = \frac{1}{2}$ when $x = -3$
 - iii $y = -2$ when $x = 4$
- 3 a The value of y is in proportion with x . What happens to the value of y when the value of x is:
 - i doubled
 - ii halved
 b The value of y varies inversely with x . What happens to the value of y when the value of x is:
 - i doubled
 - ii halved
- 4 The quantity y varies inversely with the square of x .
 - a
 - i Write this algebraically using k as the constant of proportionality.
 - ii It is found that $y = 2$ when $x = 3$. Determine the constant k .
 - iii Evaluate y when $x = 6$.
 - b What happens to the value of y when the value of x is:
 - i doubled,
 - ii halved?
- 5 The area $A \text{ cm}^2$ of a triangle varies directly with its height $h \text{ cm}$.
 - a Write this algebraically using k as the constant of proportionality.
 - b Determine k if $A = 12 \text{ cm}^2$ when $h = 3 \text{ cm}$.
 - c What are the units of k .
 - d Explain the units of k using the formula for the area of a triangle.
- 6 In a scale model of an Spitfire (aeroplane), each length m on the model is in varies directly with the corresponding length s on the Spitfire.
 - a Write this algebraically.
 - b The length of the Spitfire is 9.12 m and the model has length 0.38 m. What is the constant of proportionality?
 - c The height of the Spitfire on the runway is 3.48 m. What is the height of the model.
 - d The wing span of the model is 468 mm. What is the wingspan of the Spitfire in metres.

- 7 a** The amount of paint P , in litres, used to paint a building is directly proportional to the area A , in square metres, to be covered. Write this algebraically.
- b** A certain building requires 48 L to cover an area of 576 m^2 . Find the constant of proportionality, and graph the function.
- c** A larger building has an area of 668 m^2 to be painted. How many litres of paint will this require?
- d** A paint supplier sells 40 L buckets and 4 L tins of paint. How many buckets and tins must be bought in order to paint the building completely?
- 8** The owners of the Fizgig Manufacturing Company use a uniformly elastic demand curve to model their sales. Thus if the price of a Fizgig is p and the quantity sold per year is q then the turnover from sales is constant. That is, $pq = T$, for some constant T .
- a** Last year, the price of a Fizgig was $p = \$6$ and the quantity sold was $q = 400\,000$. Find T .
- b** The company's board of directors want to raise the price to $p = \$8$ next year. How many Fizgigs can the company expect to sell if this happens?
- c** Under this model, what will happen to the sales if the price is doubled instead?
- d** Sketch the graph of the demand curve with q on the horizontal axis and p on the vertical axis.

DEVELOPMENT

- 9** If a gas behaves in an ideal way then the relationship between pressure, volume and temperature can be approximated with simple equations.
- a** When the gas is kept at a constant pressure, the volume V it occupies varies directly with its temperature T when that is expressed in Kelvin instead of Celsius.
- i** Write down an equation relating V and T .
- ii** A certain amount of gas occupies 2 m^3 when the temperature is $T = 273 \text{ Kelvin}$ ($= 0^\circ\text{C}$). What volume will it occupy when the temperature is $T = 298 \text{ Kelvin}$ ($= 25^\circ\text{C}$)? Give your answer correct to the nearest cm^3 .
- b** When the gas is kept at a constant temperature, the pressure P varies inversely with the volume.
- i** Write down an equation relating P and V .
- ii** A certain amount of gas is at twice atmospheric pressure when it occupies 11.2 L. What is the pressure when the volume is 30 L. Express your answer as a multiple of the atmospheric pressure correct to two decimal places.
- 10** An object is initially stationary and accelerates. The acceleration is constant.
- a** It is found that the velocity $v \text{ m/s}$ varies directly with the time t seconds.
- i** After 4 seconds the speed is 5 m/s. Find the constant of proportionality.
- ii** What is the speed of the object at $t = 12 \text{ s}$?
- iii** When will the object be travelling at 12.5 m/s?
- b** The distance x metres that the object travels is proportional to the square of the time.
- c** Write this algebraically using k as the constant of proportionality.
- i** After 4 seconds the distance is $x = 10 \text{ m}$. Show that the constant of proportionality is half that in part (a).
- ii** How far has the object travelled at $t = 12 \text{ s}$?
- iii** How much time has elapsed when $x = 160 \text{ m}$?
- iv** What happens to x when the value of t is doubled? Check your answer by comparing x when $t = 8 \text{ s}$ with part (i).

- 11** The acceleration a due to gravity varies inversely with the square of the distance r to the centre of the earth.
- Write this algebraically using k as the constant of proportionality.
 - At the surface of the earth the acceleration due to gravity is approximately 9.81 m s^{-2} . The radius of the earth at sea level is approximately 6 378 137 m. Find the value of k correct to 3 significant figures.
 - Galileo dropped a cannonball from the tower of Pisa Cathedral, approximately 55 m above sea level. Show that the acceleration due to gravity at this height is still 9.81 m s^{-2} correct to two decimal places.
 - A satellite is in geosynchronous orbit above the equator at an altitude of 35 786 km above sea level. What is the acceleration due to gravity at this height? Give your answer correct to 2 decimal places.
 - What happens to a when the distance from the centre of the earth is doubled?
- 12** The formula for the simple interest $\$I$ paid when a principal $\$P$ is invested at an interest rate R for n years is $I = nPR$.
- Suppose that R and n are fixed. What is the relationship between I and P ?
 - Suppose that I and P are fixed. What is the relationship between n and R ?
 - Suppose that $PR = k$ is constant so that $I = kn$. What are the units of k ?
- 13** In simple cases, the time t it takes for a tank to empty is found to be proportional to a power of the initial depth h of the water.
- The time taken to empty a cylindrical tank is proportional to the square-root of the initial depth.
 - Write this algebraically using k as the constant of proportionality.
 - When the initial depth was 9 m the time taken was 8448 s. How long will it take to empty this tank if it is filled to a depth of 4 m?
 - The time taken to empty a conical tank is proportional to the square-root of the fifth power of the initial depth.
 - Write this algebraically using k as the constant of proportionality.
 - When the initial depth was 3 m the time taken was 956 s. How long will it take to empty this tank if it is filled to a depth of 4 m? Give your answer correct to the nearest second.
- 14** The lateral force F required to keep a car on the road when driving around a bend with constant radius is proportional to the square of the speed V .
- Write this algebraically using k as the constant of proportionality.
 - A formula one car drives at 302.4 km/h around the corner named 130R at Suzuka in Japan. The force keeping the car on the road is 15, 120 newtons. Find k correct to 6 decimal places.
 - On the next lap the car takes the corner at 288 km/h. What is the force for this lap, correct to the nearest newton.
 - The driver goes slowly around the corner on the third lap. On the fourth lap the car travels around 130R at twice the speed of lap 3. By what factor has the force increased from lap 3 to lap 4?
- 15** In a special type of pendulum, called a conical pendulum, the time taken to complete one oscillation, which is called the period P , varies directly with the square-root of the length L of the pendulum.
- Write this algebraically using k as the constant of proportionality.
 - When the length of the pendulum is $L = 0.29 \text{ m}$ the period is $P = 1.005 \text{ s}$. What will be the period of a conical pendulum with length $L = 2 \text{ m}$? Give your answer in seconds rounded to two decimal places.

CHALLENGE

- 16** [Inverse Variation] An architect is designing a building. The client has insisted that one of the rooms in the building must have an area of 48 m^2 . For ease of design, the length ℓ and the breadth b must each be a whole number of metres. Because of the furniture that must go into the room, no wall may be less than 4 m. What are the possible dimensions of the room?

Chapter 3 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 3 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

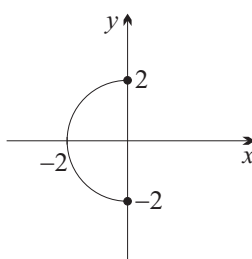


- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

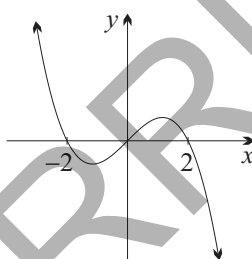
Chapter Review Exercise

- Determine which of the following relations are functions.

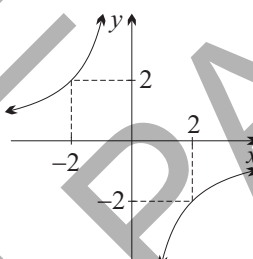
a



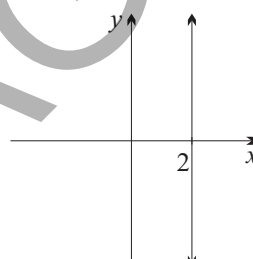
b



c



d



- Write down the domain and range of each relation in the previous question.

- Find $f(3)$ and $f(-2)$ for each function.

a $f(x) = x^2 + 4x$

b $f(x) = x^3 - 3x^2 + 5$

- Find the natural domain of each function.

a $f(x) = \frac{1}{x-2}$

b $f(x) = \sqrt{x-1}$

c $f(x) = \sqrt{3x+2}$

d $f(x) = \frac{1}{\sqrt{2-x}}$

- Find $F(a) - 1$ and $F(a - 1)$ for each function.

a $F(x) = 2x + 3$

b $F(x) = x^2 - 3x - 7$

- Find the x -intercept and the y -intercept of each line, then sketch it.

a $y = 2x + 2$

b $x - 3y + 6 = 0$

- Sketch each of these lines through the origin.

a $y = 2x$

b $x + 2y = 0$

- Sketch each vertical or horizontal line.

a $y = -1$

b $x - 3 = 0$

- Use factoring where necessary to find the zeroes of each quadratic function. Hence sketch the graph of $y = f(x)$, showing all intercepts and the coordinates of the vertex. Also state the domain and range.

a $f(x) = 16 - x^2$

b $f(x) = x(x + 2)$

c $f(x) = (x - 2)(x - 6)$

d $f(x) = -(x + 5)(x - 1)$

e $f(x) = x^2 + x - 6$

f $f(x) = -x^2 + 2x + 8$

- 10** Complete the square in each quadratic. Then sketch the graph of each function, showing the vertex and the intercepts with the axes.

a $y = x^2 + 2x - 5$

b $y = -x^2 + 6x - 6$

c $y = -x^2 + 2x - 3$

d $y = x^2 + 6x + 10$

- 11** Sketch the graph of each parabola. Use the discriminant to determine whether or not there are any x -intercepts. Show the vertex and the intercepts with the axes.

a $y = -x^2 - 2x + 1$

b $y = x^2 - 4x + 2$

c $y = x^2 - 4x + 8$

d $y = -x^2 + 6x - 15$

- 12** Use a table of values to test the sign, and then sketch each cubic polynomial.

a $y = (x - 1)(x - 3)(x - 6)$

b $y = x^2(x - 3)$

- 13** Sketch each circle.

a $x^2 + y^2 = 9$

b $x^2 + y^2 = 25$

- 14** Sketch each semicircle, and write down the domain and range in each case.

a $y = \sqrt{16 - x^2}$

b $x = \sqrt{9 - y^2}$

c $x = -\sqrt{4 - y^2}$

d $y = -\sqrt{25 - x^2}$

- 15** Construct a table of values for each hyperbola, then sketch it. Write down the domain and range in each case.

a $y = \frac{8}{x}$

b $y = -\frac{4}{x}$

- 16** Construct a table of values for each exponential function, then sketch it. Write down the domain and range in each case.

a $y = 2^x$

b $y = 3^{-x}$

- 17** [A revision medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. A table of values may help in certain cases.

a $y = 2x,$ $y = 2x + 3,$ $y = 2x - 1$

b $y = -\frac{1}{2}x,$ $y = -\frac{1}{2}x + 1,$ $y = -\frac{1}{2}x - 2$

c $y = x^2,$ $y = (x + 2)^2,$ $y = (x - 1)^2$

d $x + y = 0,$ $x + y = 2,$ $x + y = -3$

e $y = x^2,$ $y = 2x^2,$ $y = \frac{1}{2}x^2$

f $x - y = 0,$ $x - y = 1,$ $x - y = -2$

g $x^2 + y^2 = 4,$ $x^2 = 1 - y^2,$ $y^2 = 25 - x^2$

h $y = 3x,$ $x = 3y,$ $y = 3x + 1,$ $x = 3y + 1$

i $y = 2^x,$ $y = 3^x,$ $y = 4^x$

j $y = -x,$ $y = 4 - x,$ $y = x - 4,$ $x = -4 - y$

k $y = x^2 - x,$ $y = x^2 - 4x,$ $y = x^2 + 3x$

l $y = x^2 - 1,$ $y = 1 - x^2,$ $y = 4 - x^2,$ $y = -1 - x^2$

m $y = (x + 2)^2,$ $y = (x + 2)^2 - 4,$ $y = (x + 2)^2 + 1$

n $y = x^2 - 1,$ $y = x^2 - 4x + 3,$ $y = x^2 - 8x + 15$

o $y = \sqrt{9 - x^2},$ $y = -\sqrt{4 - x^2},$ $y = \sqrt{1 - x^2}$

p $y = \frac{1}{x},$ $y = \frac{2}{x},$ $y = -\frac{3}{x}$

q $y = \sqrt{x},$ $y = 2 - \sqrt{x},$ $y = \sqrt{1 - x}$

$$\begin{array}{lll}
 \mathbf{r} & y = x^3, & y = x^3 + 1, & y = (x + 1)^3 \\
 \mathbf{s} & y = x^4, & y = (x - 1)^4, & y = x^4 - 1 \\
 \mathbf{t} & y = 2^{-x}, & y = \left(\frac{1}{2}\right)^x, & y = \frac{1}{2^x}
 \end{array}$$

- 18** In simple spring theory, the change in length x of a spring varies directly with the mass m hung from it.
- Write this algebraically using k as the constant of proportionality.
 - For a certain spring, $x = 2$ cm when $m = 3$ kg. Evaluate k , giving its units in your answer.
 - What will be the change in length of the spring when $m = 4.5$ kg?
 - What happens x when the mass is double?
- 19** A common formula learnt in science is $D = S \times T$, that is, distance equals the product of speed and time.
- How does distance vary with time when S is constant?
 - Give an example of inverse variation using this equation.
 - In your example, how will one variable change when the other is doubled?
- 20** When an object is thrown up a hill, its maximum range R is the greatest distance it can reach. The maximum range of an object varies directly with the square of the speed V it is thrown at.
- Write this algebraically using k as the constant of proportionality.
 - It is found that $R = 5.505$ m, correct to four significant figures, when $V = 9 \text{ m s}^{-1}$. Evaluate k correct to four significant figures.
 - What is R when $V = 15 \text{ m s}^{-1}$. Give your answer correct to the nearest centimetre.

4

Equations and inequations

Chapter introduction

This short chapter introduces some further approaches to linear inequations, and to quadratic equations and inequations. The discriminant of a quadratic is developed further, and quadratic identities are discussed with a clear statement of the relevant theorem leading the discussion.

Practical applications particularly are developed in the exercises. The algebraic ideas are in the foreground, but the parabolic graphs of quadratics are always in the background of every discussion, and should be drawn whenever there are any questions about what it all means.

4A Linear equations and inequations

Learning intentions

- Solve linear inequations, reversing the inequality sign when necessary.

Most of this chapter involves solving inequations. The necessary algebraic operations on an inequation are the same as for equations, with a single extra rule — reverse the inequality sign when multiplying or dividing by a negative.

In this section, the linear inequations can be solved by these operations alone.

Algebraic manipulation of equations and inequations

The rules for operations on equations are well known, and have been used without comment in earlier chapters. One extra rule is needed for inequations.

1 Algebraic operations on equations and inequations

- Equations:** Add or subtract the same number on both sides.
 - ▷ Multiply or divide both sides by the same nonzero number.
- The extra rule for inequations:**
 - ▷ Reverse the inequality when multiplying or dividing by a negative number.

The extra rule for inequations is easily justified by two simple examples:

We know that $6 \geq -4$.

$$\boxed{\times (-1)} \quad -6 \leq 4.$$

We know that $8 < 12$.

$$\boxed{\div (-4)} \quad -2 > -3.$$

Writing the operation on the left draws attention to the needed reversal.



Example 1 Solving a linear inequation and graphing the solution

Solve these linear inequations, then graph their solutions on a number line.

a $x - 12 < 5 + 3x$

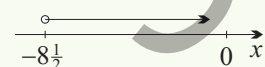
Solution

a $x - 12 < 5 + 3x$

$$\boxed{-3x} \quad -2x - 12 < 5$$

$$\boxed{+12} \quad -2x < 17$$

$$\boxed{\div (-2)} \quad x > -8\frac{1}{2}$$



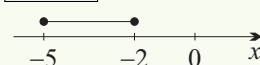
b $13 \geq \frac{4-7x}{3} \geq 6$

b $13 \geq \frac{4-7x}{3} \geq 6$

$$\boxed{\times 3} \quad 39 \geq 4 - 7x \geq 18$$

$$\boxed{-4} \quad 35 \geq -7x \geq 14$$

$$\boxed{\div (-7)} \quad -5 \leq x \leq -2$$



In each case, the inequalities were reversed when we divided by a negative.

**Example 2 Solving a practical problem using a linear inequation**

A school wants to buy some PCs from a technology fund of \$30 000.

- a** If the n PCs cost \$820 each, how much money will remain in the fund?
b If at least \$14 500 must remain, how many computers can they buy?

Solution

a Total cost = $820n$,
 so amount remaining = $30\,000 - 820n$.

b Put $30\,000 - 820n \geq 14\,500$

$$\begin{array}{rcl} -30\,000 & & -820n \geq -15\,500 \\ \div (-820) & & n \leq 18.9024 \dots \end{array}$$

So they can buy 18 PCs.

Exercise 4A**FOUNDATION**

- Graph each set of real numbers on the number line.

a $x > 1$	b $x \leq 2$	c $x \geq -3$
d $0 < x < 3$	e $-2 < x \leq 1$	f $-5 \leq x < 10$
- Solve each inequation, and graph your solution on the number line.

a $x - 2 < 3$	b $3x \geq -6$	c $5x + 5 > 15$
d $4x - 3 \leq -7$	e $2x + 3 \geq x + 7$	f $6x - 5 < 3x - 17$
- Solve each inequation.

a $-2x < 6$	b $-5x \geq -50$	c $-x \leq 1$
d $3 - 2x > 7$	e $11 - 3x < 2$	f $-4 - x \geq 1$
g $2 - x > 2x - 4$	h $3 - 3x \leq 19 + x$	i $12 - 7x > -2x - 18$

DEVELOPMENT

- Solve each double inequation, and graph your solution on the number line.

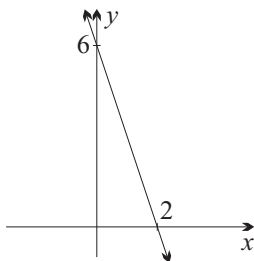
a $3 < x + 2 < 6$	b $-5 < x - 3 \leq 4$	c $-8 \leq 4x < 12$
d $-1 \leq 2x \leq 3$	e $-10 < 3x - 1 < 2$	f $-7 < 5x + 3 \leq 3$
- Solve each double inequation. Write your answers using bracket interval notation.

a $-4 < -2x < 8$	b $-2 \leq -x \leq 1$
c $-7 \leq 5 - 3x < 4$	d $-4 < 1 - \frac{1}{3}x \leq 3$
- Solve each inequation. Write your answers using bracket interval notation.

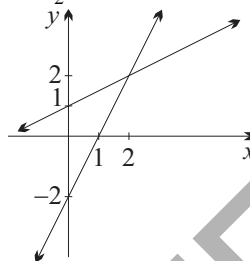
a $\frac{x}{5} < 3 + \frac{x}{2}$	b $\frac{1}{4}x \geq \frac{1}{2}x - 1$
c $\frac{2x - 1}{3} \geq \frac{x + 1}{4} - 1$	d $\frac{1}{6}(2 - x) - \frac{1}{3}(2 + x) > 2$

- 7** In each case, use the given graph to help solve the inequation. Then confirm your answer by solving it algebraically.

a $6 - 3x > 0$.



b $1 + \frac{1}{2}x \leq 2x - 2$



- 8 a i** Graph the line $y = 2x - 3$.
ii How can this graph be used to solve $2x - 3 \geq 0$? Write down the solution.
- b i** Graph the two lines $y = 2x - 1$ and $y = 2 - x$ on the same number plane.
ii How can this graph be used to solve $2x - 1 < 2 - x$? Write down the solution.
- 9** Write down and solve a suitable inequation to find the values of x for which the line $y = 5x - 4$ is below the line $y = 7 - \frac{1}{2}x$.
- 10 a** Sketch the lines $y = 1 - x$, $y = 2$ and $y = -1$ on a number plane and find the points of intersection.
b Solve the inequation $-1 < 1 - x \leq 2$ and relate the answer to the graph in part (a).
c Write down similar graphical explanations of Questions 4(a) and 5(a).
- 11** An investment of \$200 000 made on the 2nd of January is earning \$823 per month in simple interest. The interest is paid into the account on the 1st of each month.
a Write down an equation for the balance B of the account on the 1st of the n^{th} month.
b Find the last time that the total balance of the account is no more than \$205 000
c Find the first time that the balance is greater than \$215 000.
- 12** An electrical company is installing new lights in a new house. The company will charge the customer a call-out fee of \$ 211 plus \$ 55 per light. This fee covers both the purchase of the lights and their installation. The customer has a budget of \$650. What is the maximum number of lights that can be fitted?
- 13** Jack and Jill were running for president of the Hill Climb Club. There were 21 members of the club who voted. The number x of members that voted for Jack was more than twice the number y that voted for Jill. What was the minimum number of votes that Jack received. Show how this can be determined graphically.
- 14** When Jack's and Jill's daughter Janine was born they made two investments for her. They put \$600 in a term deposit earning 5% simple interest and another \$500 in a managed fund earning 7% simple interest. In each case the interest is paid on Janine's birthday each year.
a Write down an expression for each investment giving the balance y after x years.
b For how many years is the balance of the term deposit greater than that of the managed fund? Show this graphically.

- 15** [Break-even point] A certain business has fixed costs of \$900 plus costs of \$30 per item sold. The sale price of each item is \$50. If enough items are sold then the company is just able to pay its total costs. That point is called the break-even point. Companies may use several different methods to graph this information. Here are two such methods. In each case, let n be the number of items sold.
- a**
- i** The gross profit per item is $\$50 - \$30 = \$20$. Sketch the graph of the gross profit for n items $y = \$20 \times n$.
 - ii** On the same graph, sketch the fixed costs $y = \$900$.
 - iii** The point where these two lines cross is the break-even point. How many items need to be sold to break even?
- b**
- i** On a new graph, draw $y = \$50 \times n$, the total sales for n items.
 - ii** On the same graph, draw $y = \$900 + \$30 \times n$, the total cost for n items.
 - iii** Does the break-even point for this graph agree with the result of part (a)?
- c** The net profit is the income from sales minus the cost. A positive net profit indicates that the business makes a profit and a negative net profit indicates a loss.
- i** Write down a formula for the net profit $\$P$ in terms of n .
 - ii** The company has sufficient funds to cover a loss of up to \$400 and still stay in business. How many items are sold if the company does indeed remain in business?
 - iii** The company has a profit target of \$750. How many items are sold given that the company exceeds the profit target?

CHALLENGE

- 16** If $-1 \leq t < 3$, what is the range of values of:
- | | | |
|-------------------|-------------------------------|--------------------------------|
| a $4t$ | b $-t$ | c $t + 7$ |
| d $2t - 1$ | e $\frac{1}{2}(t + 1)$ | f $\frac{1}{2}(3t - 1)$ |
- 17** Cartons are stored in a large room. In order to allow access to the boxes, the available space in the room is a 7 m by 7 m square. Cartons cannot be stacked on top of each other.
- a**
- i** Some cartons have a base 3 m by 2 m. By comparing areas, what is the maximum number of these cartons that can be stored in this room?
 - ii** Draw a diagram to show how this can be achieved.
- b**
- i** Other cartons have a base 4 m by 2 m. By comparing areas, what is the maximum number of these cartons that can be stored in this room?
 - ii** It is not possible to store the number of boxes found in part (i). What is the actual maximum number of these boxes that can be stored in this room?

4B Quadratic equations and inequations

Learning intentions

- Solve quadratic equations by the three standard methods.
- Solve quadratic inequations using the graph.
- Solve quadratic inequations using a table of test points.

This section is mostly about the inequations. First, we quickly review from Chapter 3 the three methods of solving quadratic equations.

Three ways to solve a quadratic equation

For comparison of the methods, here is $x^2 - 9x + 14 = 0$ solved in the three ways. Factoring is quick when it works, but is only available in special cases:

FACTORING:

$$\begin{aligned}x^2 - 9x + 14 &= 0 \\(x - 7)(x - 2) &= 0 \\ \text{so } x &= 7 \text{ or } 2.\end{aligned}$$

COMPLETING THE SQUARE:

$$\begin{aligned}x^2 - 9x + 14 &= 0 \\x^2 - 9x + \frac{81}{4} &= -14 + \frac{81}{4} \\(x - \frac{9}{2})^2 &= \frac{25}{4} \\x - 4\frac{1}{2} &= 2\frac{1}{2} \text{ or } -2\frac{1}{2} \\x &= 7 \text{ or } 2\end{aligned}$$

FORMULA:

$$\begin{aligned}x^2 - 9x + 14 &= 0 \\ \Delta &= 81 - 56 \\ &= 5^2 \\ x &= \frac{9+5}{2} \text{ or } \frac{9-5}{2} \\ &= 7 \text{ or } 2\end{aligned}$$

We have also seen in Chapter 3 how all three methods can identify a *double root*, otherwise called *two equal roots*, and how completing the square and the formula both show when an equation has no roots.

Solving a quadratic inequation — graphical method

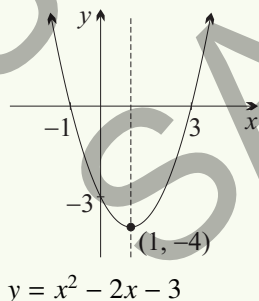
The clearest way to solve a quadratic inequation is to read its solution off the graph. First, put the inequation in the form $ax^2 + bx + c > 0$, then graph $y = \text{LHS}$, and see where the curve is above the x -axis. And similarly for the other three inequality signs.



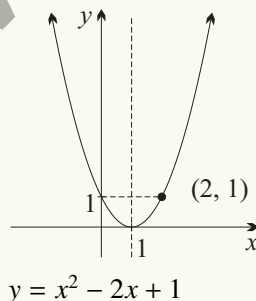
Example 3 Solving a quadratic inequation graphically

Use these graphs — drawn in Chapter 3 and copied below — to solve:

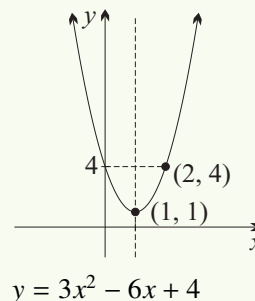
a $x^2 - 2x - 3 \geq 0$



b $x^2 + 1 \leq 2x$



c $6x - 4 > x^2$



Solution

a $x^2 - 2x - 3 \geq 0$

$(x - 3)(x + 1) \geq 0$

 $y = \text{LHS}$ is graph 1,so $x \leq -1$ or $x \geq 3$.

b $x^2 + 1 \leq 2x$

$\boxed{-2x} \quad x^2 - 2x + 1 \leq 0$

$(x - 1)^2 \leq 0$

 $y = \text{LHS}$ is graph 2,so $x = 1$ is the only solution.

c $6x - 4 > 3x^2$

$3x^2 - 6x + 4 < 0$

Here $\Delta = -12 < 0$. $y = \text{LHS}$ is graph 3,

so there are no solutions.

2 Graphical solution of a quadratic inequation

- Get the inequation into the form $ax^2 + bx + c > 0$ (or $<$, or $\geq$, or $\leq$).
- Graph $y = \text{LHS}$, and observe where y is positive, zero, and negative.
- Hence read the solution off the graph.

Solving a quadratic inequation — zeroes and test points

- The graphs show that everything depends on finding the zeroes of the LHS.
- Once the zeroes are found, a *table of test points dodging around the zeroes* can be used to establish the sign of the function for different values of x — because zeroes are the only places where a quadratic function can change sign.

3 Solution of a quadratic inequation using zeroes and test points

- With the inequation in the form $ax^2 + bx + c > 0$, find the zeroes of LHS.
- Take a table of test points of the LHS dodging around the zeroes.
- This is because a quadratic function can only change sign at a zero.

**Example 4 Solving quadratic inequations using zeroes and test points**

Solve the inequations in Example 3 using their known zeroes and a table of values:

a $x^2 - 2x - 3 \geq 0$

b $x^2 - 2x + 1 \leq 0$

c $3x^2 - 6x + 4 \leq 0$

Solution**a** The zeroes are -1 and 3 .

x	-2	-1	0	3	4
y	5	0	-3	0	5

sign $+$ 0 $-$ 0 $+$ Hence $x \leq -1$ or $x \geq 3$ **b** The only zero is 1 .

x	0	1	2
y	1	0	1

sign $+$ 0 $+$ Hence $x = 1$.**c** There are no zeroes.

x	0
y	4

sign $+$

There are no solutions.



Example 5 A practical application of quadratic inequations

A ball is thrown vertically upwards from the ground, and its height h metres after t seconds is given by $h = -5t^2 + 30t$. When is the ball:

- a** on the ground again, **b** above 40 metres, **c** in the air and below 25 metres?

Solution

a Put $h = 0$

$$-5t^2 + 30t = 0$$

$$-5t(t - 6) = 0$$

$$t = 0 \text{ or } 6.$$

Hence the ball hits the ground again after 6 seconds.

Thus it is in the air for $0 < t < 6$.

b Solve $-5t^2 + 30t > 40$

$$\div (-5) \quad t^2 - 6t + 8 < 0$$

$$(t - 4)(t - 2) < 0$$

The zeroes of the LHS are 2 and 4.

t	0	2	3	4	5
$(t - 4)(t - 2)$	8	0	-1	0	3
sign	+	0	-	0	+

Hence $2 < t < 4$.

c Solve $-5t^2 + 30t < 25$

$$\div (-5) \quad t^2 - 6t + 5 \leq 0$$

$$(t - 1)(t - 5) \leq 0$$

The zeroes of the LHS are 1 and 5.

Hence $0 < t < 1$ or $5 < t < 6$.

t	0	1	3	5	6
$(t - 1)(t - 5)$	5	0	-4	0	5
sign	+	0	-	0	+

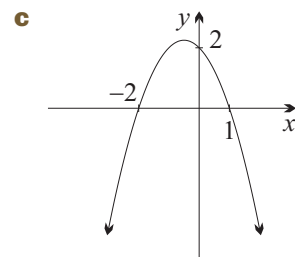
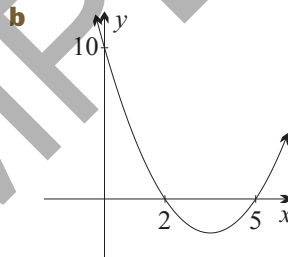
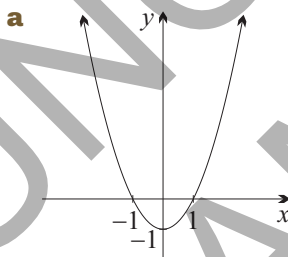
Exercise 4B

FOUNDATION

1 Write down the values of x for which each parabola is:

i below the x -axis,

ii above the x -axis.

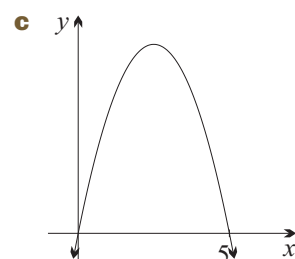
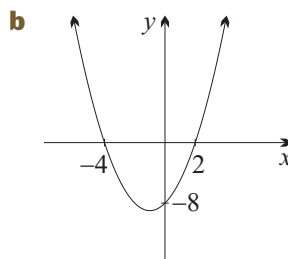
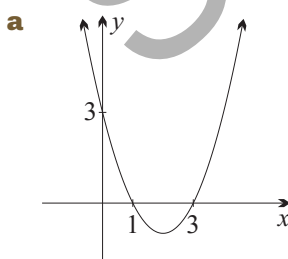


2 For each parabola, state the values of x for which:

i $y = 0$,

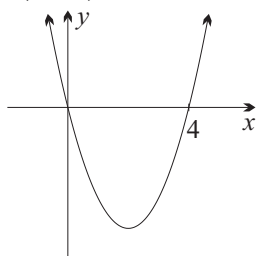
ii $y > 0$,

iii $y < 0$.

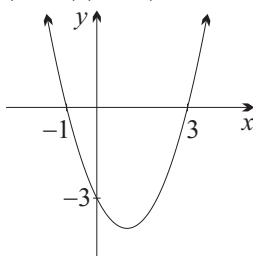


- 3 a** Use the given graph of the LHS to help solve each inequation.

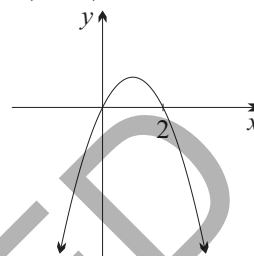
i $x(x - 4) < 0$



ii $(x - 3)(x + 1) \geq 0$



iii $x(2 - x) \leq 0$



- b** Confirm your answers to part (a) by using tables of values.

- 4 a** Sketch the associated parabola and hence solve:

i $(x + 2)(x - 4) < 0$

ii $(3 - x)(x + 1) > 0$

iii $(2 - x)(x - 5) \leq 0$

iv $(x + 1)(x + 3) \geq 0$

v $(x + 1)(1 - x) < 0$

vi $(x + 1)(x + 2) \leq 0$

- b** Confirm your answers to part (a) by using tables of values.

DEVELOPMENT

- 5** Factor the LHS, then sketch an appropriate parabola in order to solve:

a $x^2 + 2x - 3 < 0$

b $x^2 - 5x + 4 \geq 0$

c $x^2 + 6x + 8 > 0$

d $x^2 - x - 6 \leq 0$

e $2x^2 - x - 3 \leq 0$

f $4 + 3x - x^2 > 0$

- 6** Solve:

a $x^2 \leq 1$

b $x^2 > 3x$

c $x^2 \geq 144$

d $x^2 > 0$

e $x^2 + 9 \leq 6x$

f $4x - 3 \geq x^2$

- 7** Solve these inequations with quadratics on both sides.

a $2x^2 + x - 1 < x^2 + 4x + 3$

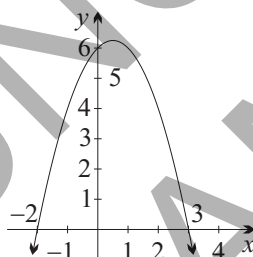
b $x^2 + 5x + 6 \leq -x^2 + x + 12$

c $x^2 + 4x + 3 \geq -x^2 + 5x + 6$

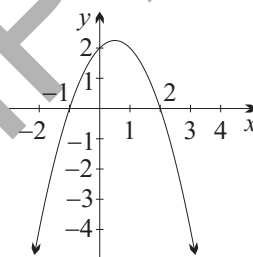
d $4x^2 - 7x - 2 > x^2 - 2x$

- 8** In each case solve the quadratic inequation by using the given graph of the LHS.

a $6 + x - x^2 > 4$



b $2 + x - x^2 \leq -4$



- c** How could the graph in part (a) be used to solve the problem in part (b)?

- 9 a** Graph the parabola $y = x^2 - x$ and the line $y = 1 - x$ on the same number plane.

- b** How can this graph be used to solve $x^2 - x < 1 - x$? Write down the solution.

- 10** Write down and solve a suitable inequation to find the values of x for which the line $y = 2x - 4$ is below the parabola $y = x^2 - 3x + 2$.

- 11** Two positive numbers have a sum of 30. Let one of the numbers be x .
- a** Find an expression for the product P of the two numbers.
 - b** Find the values of x for which $P \geq 200$.
 - c** List the distinct pairs of integers that have a sum of 30 and a product of at least 200.
- 12** A rectangle has perimeter 180 mm. Let one side have length x m.
- a** Write down an expression for the area A of this rectangle in terms of x .
 - b** What values may x take if $A > 20 \text{ cm}^2$? Take care with the units.
- 13** A farmer needs to build a rectangular holding pen for some sheep. The side of a barn will be used for one side of the pen and there is 56 m of fencing available for the remaining three sides. Let x be the length of each side perpendicular to the barn wall.
- a** Find an expression for the third side in terms of x and draw a diagram showing all this information.
 - b** The area A of the pen must be at least 392 m^2 so that the sheep are not overcrowded. What are the possible dimensions of the pen?
- 14** The surface area of a cylinder with fixed height 3 cm is given by $SA_{\text{cyl}} = 2\pi r(r + 3)$. What values may r take if SA_{cyl} is no greater than $8\pi \text{ cm}^2$?

CHALLENGE

- 15** What range of values may $x^2 + 1$ take if:
- a** $2 < x < 4$
 - b** $-1 < x \leq 3$
- 16** A business-owner leases a shop in a shopping centre. There is only one type of product sold by this business.
- a** It is found that the number n of items sold each week varies directly with the difference between \$84 and the price $\$p$. That is, when $p = 84$ there are no items sold. When the price is \$60 there are 192 items sold. Find n as a function of p .
 - b** The turnover T for the product is the number sold times the price. Find T as a function of p .
 - c** In order to stay in business, the turnover must be at least \$11 800. What price may be charged for the product in order to stay in business.
 - d** Because of the business-owner's lease agreement, if turnover is greater than \$14 080 then extra rent must be paid. What prices can be charged without paying extra rent but still staying in business?

4C The discriminant

Learning intentions

- Understand and use various ways in which the discriminant discriminates.
- Apply the discriminant to solve problems.

Chapter 3 established that the zeroes of a quadratic $y = ax^2 + bx + c$ are

$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}, \quad \text{where } \Delta = b^2 - 4ac.$$

This section returns to the discriminant $\Delta = b^2 - 4ac$, and develops further methods of analysing quadratics using the power of the discriminant to discriminate.

The discriminant tells us the number of zeroes

The most obvious purpose of the discriminant is to find the zeroes, but even more importantly, it tells us how many zeroes there are. This is because:

- Only positive numbers have two square roots.
- Zero has just one square root, namely zero.
- Negative numbers have no square roots.

4 The Zeroes (x -intercepts) of the quadratic $y = ax^2 + bx + c$

Always find the discriminant $\Delta = b^2 - 4ac$ before finding the zeroes.

- If $\Delta > 0$, there are two zeroes, $x = \frac{-b + \sqrt{\Delta}}{2a}$ and $x = \frac{-b - \sqrt{\Delta}}{2a}$.
- If $\Delta = 0$, there is just one *double zero*, $x = -\frac{b}{2a}$.
 - ▷ In this situation, the quadratic is a *perfect square*,
 - ▷ and the x -axis is tangent to the graph.
- If $\Delta < 0$, there are no zeroes.

We can say ‘equal zeroes’ when $\Delta = 0$, and ‘distinct zeroes’ when $\Delta > 0$.

The discriminant tells us whether the zeroes are rational or irrational

Suppose now that all three coefficients of $y = ax^2 + bx + c$ are rational numbers. Then because we need to take the square root of Δ , the zeroes will also be rational numbers if Δ is a square. But if Δ is not a square and $\Delta > 0$, the zeroes will involve a surd and be irrational. So the discriminant discriminates again.

5 The discriminant and rational zeroes

Suppose that $y = ax^2 + bx + c$, where a , b , and c are rational.

- If Δ is a square, then the zeroes are rational.
- If Δ is positive and not a square, then the zeroes are irrational.


Example 6 Using the discriminant to describe the zeroes of a quadratic

Use the discriminant to describe the zeroes of each quadratic (do not solve):

a $y = 5x^2 - 2x - 3$

b $y = 3x^2 - 12x + 12$

c $y = 8 + 3x - 2x^2$

Solution

a For $y = 5x^2 - 2x - 3$,

$$\Delta = 2^2 - 4 \times 5 \times (-3)$$

$= 64$, which is a square,
so there are two rational zeroes.

b For $y = 3x^2 - 12x + 12$,

$$\Delta = 12^2 - 4 \times 3 \times 12$$

$= 0$,
so there is one rational double zero.

c For $y = 8 + 3x - 2x^2$,

$$\Delta = 3^2 - 4 \times (-2) \times 8$$

$= 73$, which is positive, but not a square,
so there are two irrational zeroes.


Example 7 Using the discriminant to investigate tangents

Use the discriminant to find the values of m for which the line $y = mx$ is tangent to the circle $(x - 10)^2 + y^2 = 25$.

Solution

Solve the line and circle simultaneously by substituting the line into the circle:

$$(x - 10)^2 + m^2x^2 = 25$$

$$(1 + m^2)x^2 - 20x + 75 = 0$$

$$\Delta = 20^2 - 4 \times 75(1 + m^2)$$

$$= 100 - 300m^2$$

The line is a tangent when $\Delta = 0$, because then line and circle meet in just one point.

Hence $100 - 300m^2 = 0$

$$m = \frac{1}{3}\sqrt{3} \text{ or } -\frac{1}{3}\sqrt{3}$$


Example 8 Using the condition for a quadratic to have zeroes

Find the values of k for which $y = x^2 + kx + 9$ has zeroes.

Solution

Here the discriminant is

$$\Delta = k^2 - 4 \times 9$$

$$= k^2 - 36.$$

The quadratic has zeroes when

$$\Delta \geq 0$$

$$k^2 - 36 \geq 0$$

$$(k - 6)(k + 6) \geq 0.$$

The LHS has zeroes at $x = 6$ and $x = -6$, so we construct a table of test points: Hence $k \leq -6$ or $k \geq 6$.

k	-7	-6	0	6	7
$(k - 6)(k + 6)$	13	0	-36	0	13
sign	+	0	-	0	+


Example 9 Using the condition for a quadratic to have zeroes

For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have:

a equal roots,

b no roots?

Solution

Here the discriminant is $\Delta = (\lambda + 5)^2 - 36$.

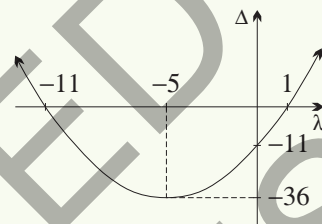
a $\Delta = 0$ when $(\lambda + 5)^2 = 36$

$$\lambda + 5 = 6 \text{ or } \lambda + 5 = -6$$

$$\lambda = 1 \text{ or } -11,$$

so there are equal roots when $\lambda = 1$ and when $\lambda = -11$.

b There are no roots when Δ is negative, so from the graph of Δ as a function of λ , there are no roots for $-11 < \lambda < 1$.


Exercise 4C
FOUNDATION

1 In each part below, the discriminant of a quadratic function with rational coefficients is given. State whether there are any zeroes. If there are zeroes then state whether they are rational or irrational, and whether they equal or are distinct.

a $\Delta = 2$

b $\Delta = -2$

c $\Delta = 0$

d $\Delta = 1$

e $\Delta = -3$

f $\Delta = 4$

2 Find the discriminant Δ for each equation. Hence state how many roots there are, and whether or not they are rational.

a $x^2 - 4x + 3 = 0$

b $x^2 + 2x - 7 = 0$

c $x^2 + 3x + 4 = 0$

d $4x^2 + 4x + 1 = 0$

e $x^2 - 3x + 5 = 0$

f $6x^2 + 11x - 10 = 0$

3 In each case the discriminant Δ of the quadratic function involves the pronumeral k . Put $\Delta = 0$ in order to find the values of k for which the quadratic has exactly one zero.

a $y = x^2 + 10x + k$

b $y = kx^2 - 4x + 1$

c $y = 2x^2 - 3x + (k + 1)$

d $y = (k - 2)x^2 + 6x + 1$

4 In each quadratic equation the discriminant Δ involves the pronumeral p . Solve $\Delta \geq 0$ to find the values of p for which the roots are real numbers.

a $x^2 + 2x + p = 0$

b $px^2 - 8x + 2 = 0$

c $3x^2 - 4x + (p + 1) = 0$

d $(2p - 1)x^2 - 5x + 2 = 0$

5 In each equation the discriminant Δ of the quadratic involves λ . Solve $\Delta < 0$ to find the values of λ for which each quadratic equation has no zeroes.

a $x^2 + 6x + \lambda = 0$

b $\lambda x^2 - 10x + 1 = 0$

c $2x^2 - 5x + (\lambda + 3) = 0$

d $(\lambda - 1)x^2 + 6x + 3 = 0$

DEVELOPMENT

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- 12 a** Use the discriminant of a suitable quadratic equation to determine whether or not the parabola $y = x^2 + 5x + 1$ intersects the parabola $y = -x^2 + x - 2$. If they do intersect then how many points are there and what are the x -coordinates?
- b** Use a similar method to investigate these parabolas lines.
- i** $y = x^2 + 3x + 1$
 $y = -x^2 - 5x - 1$
- ii** $y = \frac{1}{2}x^2 - x + 2$
 $y = -\frac{1}{2}x^2 - 3x + 1$
- 13 a** Consider the circle $x^2 + y^2 = 4$ and the line $y = x + 1$.
- i** Use substitution to obtain a quadratic equation in x .
- ii** Find the discriminant for this quadratic and hence state how many times the line intersects the circle. (Do NOT solve the quadratic equation.)
- b** Likewise, determine how many times the line intersects the circle in each case.
- i** $x^2 + y^2 = 1, y = x + 2$
- ii** $x^2 + y^2 = 5, y = -2x + 5$
- 14 a** Use substitution to show that $y = x + b$ and $y = 2x^2 - 7x + 4$ intersect when $2x^2 - 8x + (4 - b) = 0$.
- b** Solve $\Delta = 0$ to find the value of b for which the line is a tangent to the curve.
- 15** The straight line function $y = mx$ passes through the origin and is a secant of the circle $x^2 + (y - 2)^2 = 1$, intersecting it twice.
- a** What values may m take?
- b** There is one secant of the circle which passes through the origin but is not a function. Write down its equation.
- 16** The line $y = mx + b$ is tangent to the parabola $y = x^2$.
- a** Show that $b = -\frac{1}{4}m^2$.
- b** Hence find the coordinates of the point of contact in terms of m .

CHALLENGE

- 17** There are two lines $y = mx + b$ that pass through $(1, -3)$ and are tangent to $y = x^2$. Find the equations of these lines.
- 18** The line $y = mx + b$ passes through $(-4, -4)$. What values may m take if this line does not intersect the hyperbola $y = \frac{-9}{x}$?
- 19 a** Prove that the roots of each equation are real and distinct for all real values of λ . (Hint: Find Δ and write it in such a way that it is obviously positive.)
- i** $x^2 + \lambda x - 1 = 0$
- ii** $x^2 + (\lambda + 1)x + (\lambda - 2) = 0$
- b** In the case of part (a)(i), use the concavity and y -intercept of the associated parabola to give an alternative explanation.
- 20** [Other Methods] Each problem in Question 13 could be easily solved by graphing the line and circle. How could parts (a) and (b)(i) be solved using geometry?

4D Quadratic identities

Learning intentions

- Distinguish identities and equations.
- Show that the graph of a quadratic determines its quadratic coefficients.
- Find coefficients of quadratics, given sufficient data.

This section continues the attention on quadratics, and particularly on quadratic identities and the coefficients.

Identities and equations

When we are doing algebra, we are constantly using *identities*, such as:

$$(x-5)^2 = x^2 - 10x + 25 \quad \text{and} \quad \frac{3}{x} - \frac{2}{x^3} = \frac{3x^2 - 2}{x^3}, \text{ for } x \neq 0$$

These are algebraic statements true for all values of x (apart from troublesome values of x that we specifically exclude, or are understood to be excluded).

Compare these with *equations*, which have *solutions*, meaning all the values of x for which they are true. For example:

$$3x + 4 = 7 \text{ has solution } x = 1 \quad \text{and} \quad x^2 = 49 \text{ has solutions } x = 7 \text{ and } x = -7.$$

6 Identities and equations

- A statement of equality between two algebraic expressions is called an *identity* if it is true for all values of the variables (apart from troublesome values that we specifically exclude, or are understood to be excluded).
 ▷ We solve equations, and we prove identities.
- Any statement of equality between two algebraic expressions is called an *equation*, and an equation may or may not have one or more *solutions*.

Thus we prove identities, and we solve equations.

The formula for a quadratic is unique

We have tacitly assumed so far that any quadratic function $f(x)$ has exactly one expression in the form $f(x) = ax^2 + bx + c$, and that if you change the coefficients, you change the function. This is actually a theorem that needs to be proven:

Theorem. Suppose that two quadratics are equal for all values of x , that is,

$$ax^2 + bx + c = Ax^2 + Bx + C, \text{ for all values of } x, \quad (*)$$

where a, b, c, A, B , and C are constants. Then $a = A$, $b = B$, and $c = C$.

Proof We only need to substitute three values of x into (*) to prove the result.

$$\text{Substitute } x = 0, \text{ then } c = C \quad (1)$$

$$\text{Substitute } x = 1, \text{ then } a + b + c = A + B + C \quad (2)$$

$$\text{Substitute } x = -1, \text{ then } a - b + c = A - B + C \quad (3)$$

Substituting (1) into (2) and (3), and cancelling:

$$a + b = A + B \quad (2A)$$

$$a - b = A - B \quad (3A)$$

Then adding and subtracting (2A) and (3A) gives $a = A$ and $b = B$, as required.

Application of this theorem to finding coefficients

Because of this theorem, if two quadratic expressions are identically equal, there are now two ways of generating equations for finding unknown coefficients.

7 Two methods of generating equations for finding coefficients

- Equate coefficients of like terms.
- Substitute carefully chosen values of x .

We actually used the second method three times in the proof of the theorem above. Now that theorem justifies the first method.



Example 10 Finding coefficients of a quadratic function

a Express n^2 as a quadratic in $(n - 3)$.

b Find r , s , and t if $3(x - 1)^2 + 2(x - r) + 6 = sx^2 + tx + 7$, for all x .

Solution

a Let $n^2 = a(n - 3)^2 + b(n - 3) + c$

$$\text{Equate coefficients of } n^2, \quad 1 = a, \quad (1)$$

$$\text{substitute } n = 3, \text{ then } n - 3 = 0, \text{ so } 9 = c, \quad (2)$$

$$\text{substitute } n = 0, \quad 0 = 9a - 3b + c. \quad (3)$$

$$\text{Now substituting (1) and (2) into (3), } 0 = 9 - 3b + 9$$

$$b = 6,$$

$$\text{so } n^2 = (n - 3)^2 + 6(n - 3) + 9.$$

b We know that for all x , $3(x - 1)^2 + 2(x - r) + 6 = sx^2 + tx + 7$.

$$\text{Equate coefficients of } x^2, \quad 3 = s, \quad (1)$$

$$\text{substitute } x = 0, \quad 3 - 2r + 6 = 7$$

$$r = 1, \quad (2)$$

$$\text{substitute } x = 1, \quad 0 + 2(1 - r) + 6 = s + t + 7. \quad (3)$$

$$\text{Now substituting (1) and (2) into (3), } 6 = 3 + t + 7$$

$$t = -4.$$

Exercise 4D

FOUNDATION

- 1 It is known that $a(x + 2) + b(x - 1) = 5x + 1$ for all values of x .
 - a Substitute $x = 1$ to find a .
 - b Substitute $x = -2$ to find b .
 - c Why were the values $x = 1$ and $x = -2$ chosen?
 - d Equate the coefficients of like terms to get a pair of simultaneous equations in a and b , then solve them. Do you get the same answer?
- 2 It is given that $a(x + 3) + b(x - 2) = 3x - 1$ for all values of x .
 - a Substitute $x = 2$ to find a .
 - b Substitute $x = -3$ to find b .
 - c Why were the values $x = 2$ and $x = -3$ chosen?
 - d Equate the coefficients of like terms to get a pair of simultaneous equations in a and b , then solve them. Do you get the same answer?
- 3 Let $2x^2 - 5x + 3 = a(x - 2)^2 + b(x - 2) + c$.
 - a Equate the coefficients of x^2 to find the value of a .
 - b Substitute $x = 2$ into the identity to find the value of c .
 - c Substitute $x = 3$ into the identity to find the value of b .
 - d Why was the value $x = 3$ chosen in part (c)?
- 4 Suppose that $x^2 + x + 1 = a(x - 1)^2 + b(x - 1) + c$ for all values of x .
 - a Equate the coefficients of x^2 to find the value of a .
 - b Substitute $x = 1$ into the identity to find the value of c .
 - c Substitute $x = 2$ into the identity to find the value of b .
 - d Why was the value $x = 2$ chosen in part (c)?
- 5 The quadratic $f(x) = a(x + 1)^2 + b(x + 1) + c$ has the same graph as $g(x) = 2x^2 + 3x - 1$.
 - a Compare the coefficients of x^2 to find the value of a .
 - b Substitute $x = -1$ to find the value of c .
 - c Substitute another suitable value of x to determine b .
 - d Write down the resulting quadratic identity.

DEVELOPMENT

- 6 Use similar methods to Questions 3 and 4 to solve these problems.
 - a Find a , b and c if $n^2 - n = a(n - 4)^2 + b(n - 4) + c$ for all values of n .
 - b Find a , b and c if $2x^2 + 4x + 5 = a(x - 3)^2 + b(x - 3) + c$ for all values of x .
- 7 Express x^2 in the form:
 - a $A(x + 1)^2 + B(x + 1) + C$
 - b $A(x - 4)^2 + B(x - 4) + C$
 - c $A(x + 2)^2 + B(x + 2) + C$
- 8
 - a Express $2x^2 + 3x - 6$ in the form $A(x + 1)^2 + B(x + 1) + C$.
 - b Express $3x^2 - 2x - 1$ in the form $p(x - 1)^2 + q(x - 1) + r$.
- 9 Determine the parabola $y = ax^2 + bx + c$ passing through $A(-1, 0)$, $B(0, 4)$ and $C(1, 6)$.

- 10** Find the values of a , b and c for which $x^2 = a(x-1)^2 + b(x-2)^2 + c(x-3)^2$.
- 11** Find the unknown constants if $P(x) = ax^2 + b(x+1)^2 + c(x+2)^2$ and $Q(x) = -x^2 + 3x$ have the same graph.
- 12** **a** Find A and B such that $Am^2 + B(m-1)^2 = 2m - 1$.
b Hence find the sum of $1 + 3 + 5 + \cdots + 61$.
- 13** Let $13x^2 - 7x + 6 = a(2-x)^2 + b(2-x) + c$.
a Evaluate the constants a , b and c .
b Hence evaluate $13x^2 - 7x + 6$ when $x = 1.98$ without using a calculator.

CHALLENGE

- 14** **a** Find A and B such that $A(m+1)^2 + B(m-1)^2 = 4m$.
b Hence find the sum of the first eleven multiples of 4.
- 15** Let p , q and r be distinct real numbers. Substitute three appropriate values of x in order to show that
- $$\frac{(p-x)(q-x)}{(p-r)(q-r)} + \frac{(q-x)(r-x)}{(q-p)(r-p)} + \frac{(r-x)(p-x)}{(r-q)(p-q)} = 1$$
- for all values of x .
- 16** [Another Approach]
- a** Express x^2 as a quadratic in $(x-a)$. Begin by writing $x^2 = ((x-a) + a)^2$, then expand only the outer brackets of the RHS.
- b** Use the formula you found in part (a) to quickly find x^2 as a quadratic in:
- i** $(x-1)$ **ii** $(x+3)$ **iii** $(x-11)$
- c** The formula can be used to express $2x^2 - 3x + 4$ in terms of $(x-3)$ as follows:
- $$\begin{aligned} 2x^2 - 3x + 4 &= 2((x-3)^2 + 6(x-3) + 9) - 3((x-3) + 3) + 4 \\ &= 2(x-3)^2 + (2 \times 6 - 3)(x-3) + (2 \times 9 - 3 \times 3 + 4) \\ &= 2(x-3)^2 + 9(x-3) + 13 \end{aligned}$$
- Redo Question 8 using this method.

Chapter 4 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 4 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Solve each linear inequation, and graph your solution on the number line.

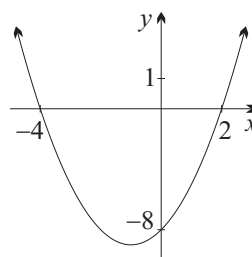
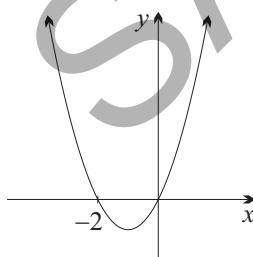
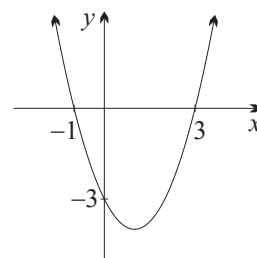
a $2x + 5 < 11$	b $3x + 2 > 17$	c $4 - 3x \geq 16$
d $2 - 3x \leq 5$	e $6 - x < 3x - 2$	f $7x + 5 \leq 3x + 1$
- Solve each double inequation then give your answer using bracket interval notation.

a $4 < x + 1 \leq 9$	b $1 \leq 2x + 3 < 11$	c $-1 < 5 - 3x < 14$
- For what values of x does the line $y = \frac{1}{2}x + 1$ lie between $y = -1$ and $y = 2$?
- Find the discriminant Δ of each equation. Hence state how many zeroes there are, and whether or not they are rational.

a $x^2 - 6x + 9 = 0$	b $-x^2 - 4x + 2 = 0$	c $9x^2 - 1 = 0$
- The parabola $y = x^2 - 2x - 3$ is sketched on the right. By considering where the curve is in relation to the x -axis, write down the values of x for which:

a $y = 0$
b $y > 0$
c $y < 0$
- Solve each quadratic inequation, using the graph provided.

a $x^2 + 2x < 0$	b $x^2 + 2x - 8 \geq 0$



- 7** Solve each quadratic inequation using a suitable sketch or table of values.
- a** $(x+4)(x-3) < 0$ **b** $(x-1)(x+5) \geq 0$ **c** $x^2 - 4x > 0$
d $x^2 - 9 \leq 0$ **e** $x^2 - 4x - 12 \geq 0$ **f** $2x^2 + x - 1 < 0$
- 8** Find the values of k for which the quadratic equation $x^2 + 6x + k = 0$ has:
a real roots, **b** equal roots, **c** unreal roots.
- 9** Find the values of m for which the quadratic equation $x^2 - 3mx + 9 = 0$ has:
a real roots, **b** equal roots, **c** unreal roots.
- 10** Find the values of ℓ that will make the quadratic $y = (\ell + 6)x^2 - 2\ell x + 3$ a perfect square.
- 11** [Break-even point] A certain business has fixed costs of \$10 500 and costs of \$121 per item sold. Each item sells for \$416. Using either a graph or suitable inequation, find the minimum number n of items that must be sold in order to make a profit.
- 12** A rectangle has perimeter 22 cm. Let one side have length x cm.
a Write down an expression for the area A of this rectangle in terms of x .
b What values may x take if $A > 28 \text{ cm}^2$?
- 13** A farmer needs to build a rectangular holding pen for some sheep. The side of a barn will be used for one side of the pen and there is 60 m of fencing available for the remaining three sides. Let x be the length of each side perpendicular to the barn wall.
a Find an expression for the third side in terms of x and draw a diagram showing all this information.
b The area A of the pen must be at least 400 m^2 so that the sheep are not overcrowded. What is the least value of x for these requirements?
- 14** **a** Find the values of a , b and c so that $3x^2 - 5x + 7 = a(x-1)^2 + b(x-1) + c$ for all x .
b Hence evaluate $3x^2 - 5x + 7$ when $x = 1.12$ without using a calculator.
- 15** Determine the values of p , q and r so that $3x^2 - 2x - 5 = px^2 + q(x-1)^2 + r(x-2)^2$ for all values of x .

5

Transformations and symmetry

Chapter introduction

In the previous two chapters, various graphs of functions and relations were introduced or reviewed. Sections 5A–5D of this chapter deal with *transformations* of these graphs under horizontal and vertical translations, under reflections in the y -axis and the x -axis, under rotations of 180° about the origin, and under horizontal dilations from the y -axis and vertical dilations from the x -axis. These procedures allow a wide variety of new graphs to be obtained quickly and simply, and relationships amongst different graphs to be discovered.

Many graphs are unchanged under one or more of these transformations, which means that they are *symmetric* in some way. Section 5C deals with line symmetry under reflection in the y -axis, and point symmetry under rotation of 180° about the origin. All these transformations and symmetries are described geometrically and algebraically — the theme remains, as always, the interrelationship between the algebra and the graphs.

Section 5E introduces the absolute value function together with its own transformations and reflection symmetry. Section 5F generalises the transformations earlier in this chapter to composite functions, and Section 5G examines the composition of transformations rather than of functions. The final section 5H introduces continuity at a point and its importance with piecewise-defined functions.

As always, computer sketching of curves can demonstrate quickly how the geometric features of a graph are related to the algebraic properties of its equation, and can help in becoming familiar with the variety of graphs and their interrelationships. The reader should apply any curve-sketching software particularly to transformations.

5A Translations of known graphs

Learning intentions

- Transform an equation so that its graph shifts vertically or horizontally.
- Show that the order of performing a vertical and a horizontal translation is irrelevant.
- Apply vertical and horizontal translations particularly to parabolas and circles.

Once a graph is drawn, it can be *shifted* (or more formally *translated*) vertically or horizontally to produce further graphs. These procedures work generally on all functions and relations, and greatly extend the range of functions and relations whose graphs can be quickly recognised and drawn.

In particular, translations are particularly helpful when dealing with parabolas and circles, where they are closely related to completing the square.

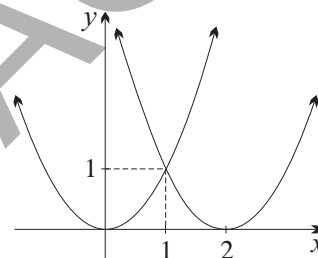
Shifting right and left

The graphs of $y = x^2$ and $y = (x - 2)^2$ are sketched from their tables of values.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$(x - 2)^2$	16	9	4	1	0	1	4

The values for $(x - 2)^2$ in the third row are the values of x^2 in the second row shifted 2 steps to the right.

Hence the graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ to the right by 2 units.



1 Shifting (or translating) right and left

- To shift a graph a units *right*, replace x by $x - a$.
- Thus for a function $y = f(x)$, the new function rule is $y = f(x - a)$.

Shifting a units left means shifting $-a$ units right, so replace x by $x - (-a) = x + a$.

Thus for a function $y = f(x)$, the new function rule is $y = f(x + a)$.


Example 1 Shifting (or translating) a graph right and left

- a** Draw up tables of values for $y = \frac{1}{x}$ and $y = \frac{1}{x+1}$.
- b** Sketch the two graphs, and state the asymptotes of each graph.
- c** What transformation maps $y = \frac{1}{x}$ to $y = \frac{1}{x+1}$?

Solution

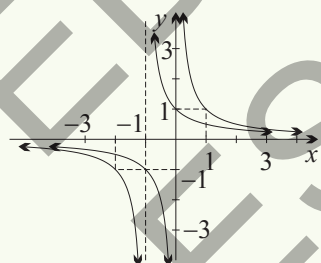
a

x	-3	-2	-1	0	1	2	3
$\frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{1}{x+1}$	$-\frac{1}{2}$	-1	*	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

- b** $y = \frac{1}{x}$ has asymptotes $x = 0$ and $y = 0$.

$y = \frac{1}{x+1}$ has asymptotes $x = -1$ and $y = 0$.

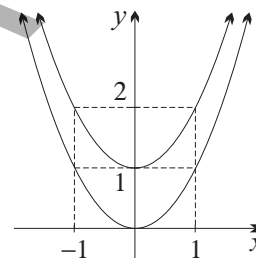
- c** Because x is replaced by $x+1 = x - (-1)$, it is a shift left of 1 unit.


Shifting up and down

The graphs of $y = x^2$ and $y = x^2 + 1$ are sketched from their tables of values:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$x^2 + 1$	10	5	2	1	2	5	10

- The values for $x^2 + 1$ in the third row are each 1 more than the corresponding values of x^2 in the second row.
- Hence the graph of $y = x^2 + 1$ is produced by shifting the graph of $y = x^2$ upwards 1 unit.



Rewriting the transformed graph as $y - 1 = x^2$ makes it clear that the shifting has been obtained by replacing y by $y - 1$, giving a rule that is completely analogous to that for horizontal shifting:

2 Shifting (or translating) up and down

- To shift a graph b units *upwards*, replace y by $y - b$.
- For a function $y = f(x)$, the new function rule is $y - b = f(x)$, or $y = f(x) + b$.

Shifting b units down means shifting $-b$ units up, so replace y by $y - (-b) = y + b$.

Thus for a function $y = f(x)$, the new function rule is $y + b = f(x)$, or $y = f(x) - b$.


Example 2 Shifting (or translating) a graph up and down

The graph of $y = 2^x$ is shifted down 2 units.

- a** Write down the equation of the shifted graph.
- b** Construct tables of values, and sketch the two graphs.
- c** State the asymptotes of the two graphs, and their domains and ranges.

Solution

a Replace y by $y - (-2) = y + 2$, so the new function is

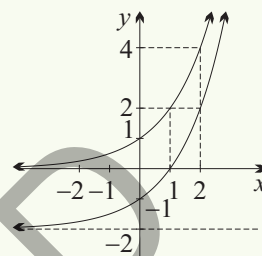
$$y + 2 = 2^x, \quad \text{that is,} \quad y = 2^x - 2.$$

b

x	-2	-1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$2^x - 2$	$-1\frac{3}{4}$	$-1\frac{1}{2}$	-1	0	2

c $y = 2^x$ has asymptote $y = 0$. Domain: all real x , range: $y > 0$.

$y = 2^x - 2$ has asymptote $y = -2$. Domain: all real x , range: $y > -2$.

**Combining horizontal and vertical translations**

When a graph is shifted horizontally and vertically, the order in which the translations are applied makes no difference. This is because:

If $y = f(x)$ is shifted a units right, then b units up,

$$y = f(x) \longrightarrow y = f(x - a) \longrightarrow y - b = f(x - a),$$

and if $y = f(x)$ is shifted b units up, then then a units right,

$$y = f(x) \longrightarrow y - b = f(x) \longrightarrow y - b = f(x - a).$$

3 The order of two perpendicular translations does not matter

- If two perpendicular translations are applied to a graph, order does not matter.

**Example 3 Combining two translations of a parabola**

- a** How is the graph of $y = (x + 2)^3 - 4$ obtained from the graph of $y = x^3$ by a horizontal translation followed by a vertical translation?
- b** Draw up a table of values for the two functions and the intermediate function.
- c** Sketch the two curves, together with the intermediate graph.
- d** Perform this by a vertical translation followed by a horizontal translation.

Solution

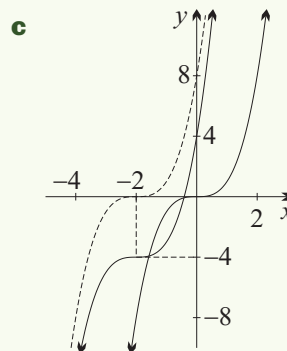
a Shifting $y = x^3$ left 2 gives $y = (x + 2)^3$.

Shifting $y = (x + 2)^3$ down 4 gives $y + 4 = (x + 2)^3$,

which can be written as $y = (x + 2)^3 - 4$.

b

x	-4	-3	-2	-1	0	1	2
x^3	-64	-27	-8	-1	0	1	8
$(x + 2)^3$	-8	-1	0	1	8	27	64
$(x + 2)^3 - 4$	-12	-5	-4	-3	4	23	60



d Translate $y = x^3$ down 4. The result is $y + 4 = x^3$, which is $y = x^3 - 4$.

Then translate left 2. The result is the same: $y = (x + 2)^3 - 4$.

(We have already shown above in Box 3 that order is not important here.)

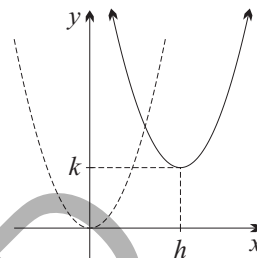
Translations and the vertex of a parabola

When we complete the square in a quadratic, it has the form

$$y = a(x - h)^2 + k, \quad \text{that is,} \quad y - k = a(x - h)^2.$$

This is a translation of the quadratic $y = ax^2$. The parabola has been shifted h units right and k units up.

This gives a clear and straightforward motivation for completing the square.



4 The completed square and the vertex of a parabola

- The completed square form of a quadratic

$$y = a(x - h)^2 + k \quad \text{or} \quad y - k = a(x - h)^2$$

displays its graph as the parabola $y = ax^2$ shifted right h units and up k units.

- Thus the vertex of the parabola is (h, k) .



Example 4 Shifting $y = x^2$ to any parabola

In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin. Sketch its graph, and state its range.

a $y = x^2 - 4x + 5$

b $y = -2x^2 - 4x$

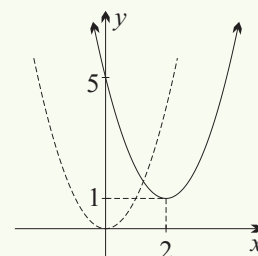
Solution

a $y = x^2 - 4x + 5$

$$y = (x^2 - 4x + 4) - 4 + 5$$

$$y = (x - 2)^2 + 1, \quad \text{that is,} \quad y - 1 = (x - 2)^2.$$

This is $y = x^2$ shifted right 2 and up 1. Range: $y \geq 1$.



b $y = -2x^2 - 4x$

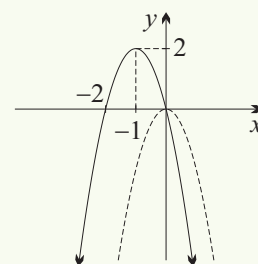
$$\div (-2) \quad -\frac{y}{2} = x^2 + 2x$$

$$-\frac{y}{2} = (x^2 + 2x + 1) - 1$$

$$-\frac{y}{2} = (x + 1)^2 - 1$$

$$\times (-2) \quad y = -2(x + 1)^2 + 2 \quad \text{or} \quad y - 2 = -2(x + 1)^2$$

This is $y = -2x^2$ shifted left 1 and up 2. Range: $y \leq 2$.



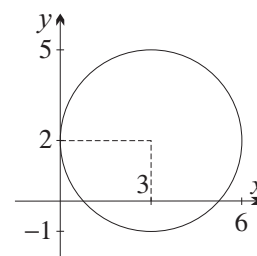
Translations and the centre of a circle

The circle drawn to the right has centre $(3, 2)$ and radius 3. To find its equation, we start with the circle with centre the origin and radius 3,

$$x^2 + y^2 = 9,$$

then translate it 3 to the right and 2 up, to give

$$(x - 3)^2 + (y - 2)^2 = 9.$$



This formula can also be established directly by Pythagoras' theorem in the form of the distance formula, but as we saw with parabolas, translations make things clearer and more straightforward.

Completing the squares to identify the translations

When the equation of a circle is presented with any binomial squares expanded, the centre and radius can be found by completing the squares in x and in y .



Example 5 Shifting any circle

- Complete the squares in x and in y of the relation $x^2 + y^2 - 6x + 8y = 0$.
- Identify the circle with centre the origin that can be translated to it, and state the translations.
- Sketch both circles on the same diagram, and explain why each circle passes through the centre of the other circle.
- Write down the domain and range of each circle.

Solution

- Completing the squares in x and in y ,

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = 9 + 16$$

$$(x - 3)^2 + (y + 4)^2 = 25.$$

- It is the circle $x^2 + y^2 = 5^2$ shifted right 3 and down 4, so its centre is $Z(3, -4)$ and its radius is 5.

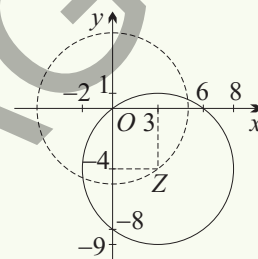
- Using the distance formula,

$$OZ^2 = 3^2 + 4^2$$

$$OZ = 5,$$

which is the radius of each circle.

- For the circle with centre the origin, domain: $-5 \leq x \leq 5$, range: $-5 \leq y \leq 5$.
For the other circle, domain: $-2 \leq x \leq 8$, range: $-9 \leq y \leq 1$.



Exercise 5A

FOUNDATION

- Copy and complete the table of values for $y = x^2$ and $y = (x - 1)^2$.

x	-2	-1	0	1	2	3
x^2						
$(x - 1)^2$						

- Sketch the two graphs and state the vertex of each.
- What transformation maps $y = x^2$ to $y = (x - 1)^2$?

- Copy and complete the table of values for $y = \frac{1}{4}x^3$ and $y = \frac{1}{4}x^3 + 2$.

x	-3	-2	-1	0	1	2	3
$\frac{1}{4}x^3$							
$\frac{1}{4}x^3 + 2$							

- Sketch the two graphs and state the y -intercept of each.
- What transformation maps $y = \frac{1}{4}x^3$ to $y = \frac{1}{4}x^3 + 2$?

- How far and in which direction has the parabola $y = x^2$ been shifted to produce each of these parabolas?

$$\mathbf{a} \quad y = x^2 + 2$$

$$\mathbf{b} \quad y = x^2 - 5$$

$$\mathbf{c} \quad y = (x + 4)^2$$

$$\mathbf{d} \quad y = (x - 3)^2$$

- 4 How far and in which direction has the hyperbola $y = \frac{1}{x}$ been shifted to produce each of these hyperbolas?

a $y = \frac{1}{x-2}$

b $y = \frac{1}{x+3}$

c $y = \frac{1}{x} - 4$

d $y = \frac{1}{x} + 5$

- 5 Sketch each parabola by shifting $y = x^2$ either horizontally or vertically. Mark all intercepts with the axes.

a $y = x^2 + 1$

b $y = x^2 - 1$

c $y = (x-1)^2$

d $y = (x+1)^2$

- 6 Sketch each hyperbola by shifting $y = \frac{1}{x}$ either horizontally or vertically. Mark any intercepts with the axes.

a $y = \frac{1}{x+1}$

b $y = \frac{1}{x-1}$

c $y = \frac{1}{x} + 1$

d $y = \frac{1}{x} - 1$

- 7 Sketch each circle by shifting $x^2 + y^2 = 1$ either horizontally or vertically. Mark all intercepts with the axes.

a $(x-1)^2 + y^2 = 1$

b $x^2 + (y-1)^2 = 1$

c $x^2 + (y+1)^2 = 1$

d $(x+1)^2 + y^2 = 1$

DEVELOPMENT

- 8 Write down the new equation for each function or relation after the given translation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: right 2 unit

b $y = 2^x$: down 2 units

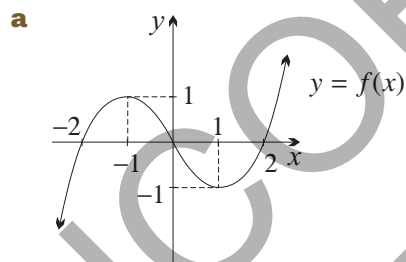
c $y = x^3$: left 1 unit

d $y = \frac{1}{x}$: right 3 units

e $x^2 + y^2 = 4$: up 1 unit

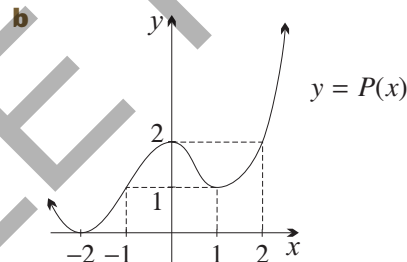
f $y = x^2 - 4$: left 1 unit

- 9 In each case an unknown function has been drawn. Draw the functions specified below it.



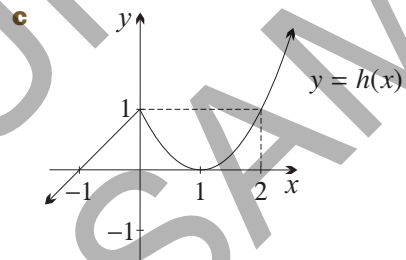
i $y = f(x-2)$

ii $y = f(x+1)$



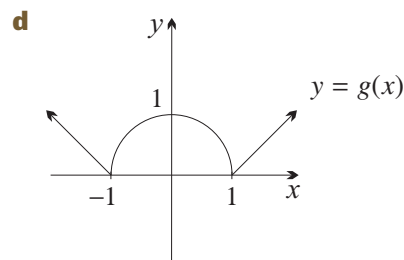
i $y = P(x+2)$

ii $y = P(x+1)$



i $y - 1 = h(x)$

ii $y = h(x) - 1$



i $y - 1 = g(x)$

ii $y = g(x-1)$

- 10 In each part, complete the square in the quadratic. Then identify its graph as a translation of either $y = x^2$ or $y = -x^2$. Finally, sketch its graph.

a $y = x^2 + 2x + 3$

b $y = x^2 - 2x - 2$

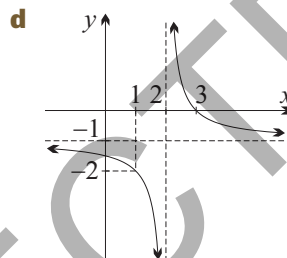
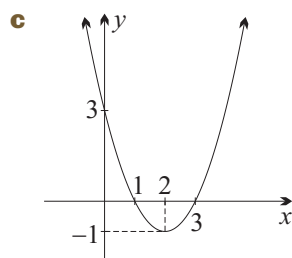
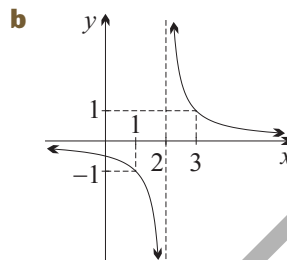
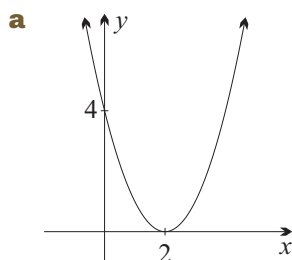
c $y = -x^2 + 4x + 1$

d $y = -x^2 - 4x - 5$

e $y = x^2 - 2x - 1$

f $y = x^2 - 2x - 4$

- 11** Describe each graph below as the parabola $y = x^2$ or the hyperbola $xy = 1$ transformed by shifts, and hence write down its equation:



- 12** Use shifting, and completion of squares where necessary, to determine the centre and radius of each circle.

a $(x + 1)^2 + y^2 = 4$

b $(x - 1)^2 + (y - 2)^2 = 1$

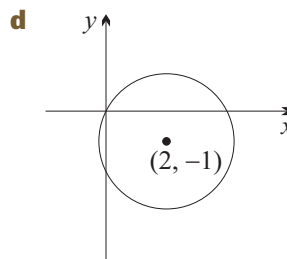
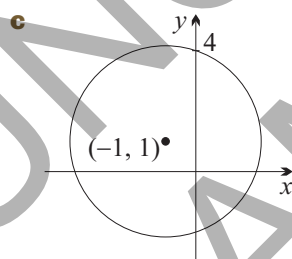
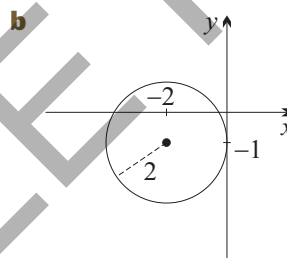
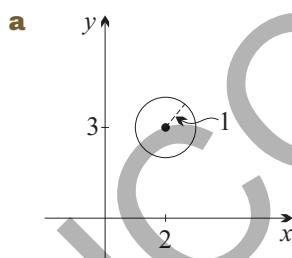
c $x^2 - 2x + y^2 - 4y - 4 = 0$

d $x^2 + 6x + y^2 - 8y = 0$

e $x^2 - 10x + y^2 + 8y + 32 = 0$

f $x^2 + 14x + 14 + y^2 - 2y = 0$

- 13** Describe each graph below as the circle $x^2 + y^2 = r^2$ transformed by shifts, and hence write down its equation.



- 14 a** Use a table of values to sketch $y = \frac{1}{4}x^3$. Then use translations to sketch:

i $y = \frac{1}{4}x^3 - 2$

ii $y = \frac{1}{4}(x - 2)^3$

iii $y = \frac{1}{4}(x + 3)^3 + 1$

- b** Use a table of values to sketch $y = -2x^3$. Then use translations to sketch:

i $y = 3 - 2x^3$

ii $y = -2(x + 3)^3$

iii $y = -2(x - 1)^3 - 2$

CHALLENGE

15 Consider the straight line equation $x + 2y - 4 = 0$.

- a** The line is translated 2 units left. Find the equation of the new line.
- b** The original line is translated 1 unit down. Find the equation of this third line.
- c** Comment on your answers. Drawing the situation may help.

16 Sketch $y = \frac{1}{x}$, then use shifting to sketch the following graphs. Find any x -intercepts and y -intercepts, and mark them on your graphs.

a $y = \frac{1}{x-2}$

b $y = 1 + \frac{1}{x-2}$

c $y = \frac{1}{x-2} - 2$

d $y = \frac{1}{x+1} - 1$

e $y = 3 + \frac{1}{x+2}$

f $y = \frac{1}{x-3} + 4$

17 In each part, explain how the graph of each subsequent equation is a translation of the first graph (there may be more than one answer), then sketch each function.

a From $y = 2x$:

i $y = 2x + 4$

ii $y = 2x - 4$

b From $y = x^2$:

i $y = x^2 + 9$

ii $y = x^2 - 9$

iii $y = (x-3)^2$

c From $y = -x^2$:

i $y = 1 - x^2$

ii $y = -(x+1)^2$

iii $y = -(x+1)^2 + 2$

d From $y = \sqrt{x}$:

i $y = \sqrt{x+4}$

ii $y = \sqrt{x} + 4$

iii $y = \sqrt{x+4} - 2$

e From $y = \frac{2}{x}$:

i $y = \frac{2}{x} + 1$

ii $y = \frac{2}{x+2}$

iii $y = \frac{2}{x+2} + 1$

18 a The circle $x^2 + y^2 = r^2$ has centre the origin and radius r . This circle is shifted so that its centre is at $C(h, k)$. Write down its equation.

b The point $P(x, y)$ lies on the circle with centre $C(h, k)$ and radius r . That is, P lies on the shifted circle in part (a). This time use the distance formula for the radius PC to obtain the equation of the circle.

19 Explain the point-gradient formula $y - y_1 = m(x - x_1)$ for a straight line in terms of a translation of the line $y = mx$.

5B Reflection in the y -axis and x -axis

Learning intentions

- Transform an equation so that its graph is reflected in the y -axis or x -axis.
- Show that the order of performing reflections in the x -axis and y -axis is irrelevant.
- Apply these reflections to familiar curves.

Reflecting only in the y -axis and the x -axis may seem an unnecessary restriction, but in fact these two transformations are the key to understanding many significant properties of functions, particularly the symmetry of graphs.

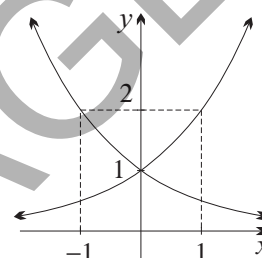
When these two reflections are combined, they produce a rotation of 180° about the origin, which again is the key to the symmetry of many graphs.

Reflection in the y -axis

The graphs of $y = 2^x$ and $y = 2^{-x}$ are sketched from their tables of values:

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

- The second and third rows are the reverse of each other.
- Hence the graphs are reflections of each other in the y -axis.



5 Reflection in the y -axis

- To reflect a graph in the y -axis, replace x by $-x$.
- Thus for a function $y = f(x)$, the new function rule is $y = f(-x)$.

Reflection in the y -axis is mutual — it maps each graph to the other graph.

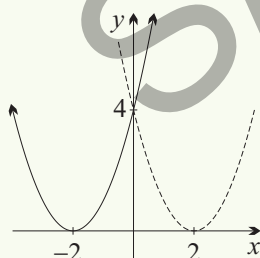


Example 6 Reflecting in the y -axis

- Sketch the parabola $y = (x - 2)^2$, and on the same set of axes, sketch its reflection in the y -axis.
- Use the rule in Box 5 above to write down the equation of the reflected graph.
- Why can this equation be written as $y = (x + 2)^2$?
- What are the vertices of the two parabolas?
- What other transformation moves the first parabola to the second, and why?

Solution

a



b Replacing x by $-x$ gives the equation

$$y = (-x - 2)^2.$$

c Taking out the factor -1 from the brackets,

$$y = (-x - 2)^2$$

$$y = (-1)^2 \times (x + 2)^2$$

$$y = (x + 2)^2.$$

d The vertices are $(2, 0)$ and $(-2, 0)$.

- e The second parabola is also the first parabola shifted left 4 units.

This replaces x by $x + 4$, so the new equation is the same as before,

$$y = ((x + 4) - 2)^2, \quad \text{that is,} \quad y = (x + 2)^2.$$

The reason why there are two possible transformations is that every parabola has line symmetry in its axis of symmetry.

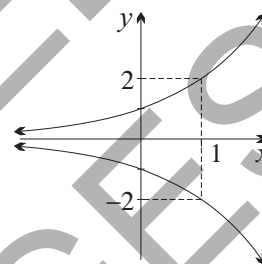
Reflection in the x -axis

The graphs of $y = 2^x$ and $y = -2^x$ are sketched from their tables of values.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
-2^x	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{2}$	-1	-2	-4	-8

- The values in the second and third rows are the opposites of each other.
- Hence the graphs are reflections of each other in the x -axis.

Rewriting the transformed graph as $-y = 2^x$ makes it clear that the reflection has been obtained by replacing y by $-y$, giving a rule that is completely analogous to that for reflection in the y -axis.



6 Reflection in the x -axis

- To reflect a graph in the x -axis, replace y by $-y$.
- For a function $y = f(x)$, the new function rule is $-y = f(x)$, or $y = -f(x)$.

Again, reflection in the x -axis is mutual — it maps each graph to the other graph.



Example 7 Reflecting in the x -axis

The graph of $y = (x - 2)^2$ is reflected in the x -axis.

- Write down the equation of the reflected graph.
- Construct tables of values, and sketch the two graphs.
- What are the vertices of the two parabolas?
- What are their domains and ranges?

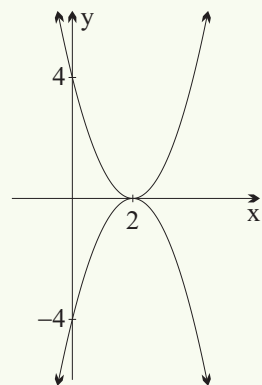
Solution

- a Replace y by $-y$, so the new function is

$$-y = (x - 2)^2, \quad \text{that is,} \quad y = -(x - 2)^2.$$

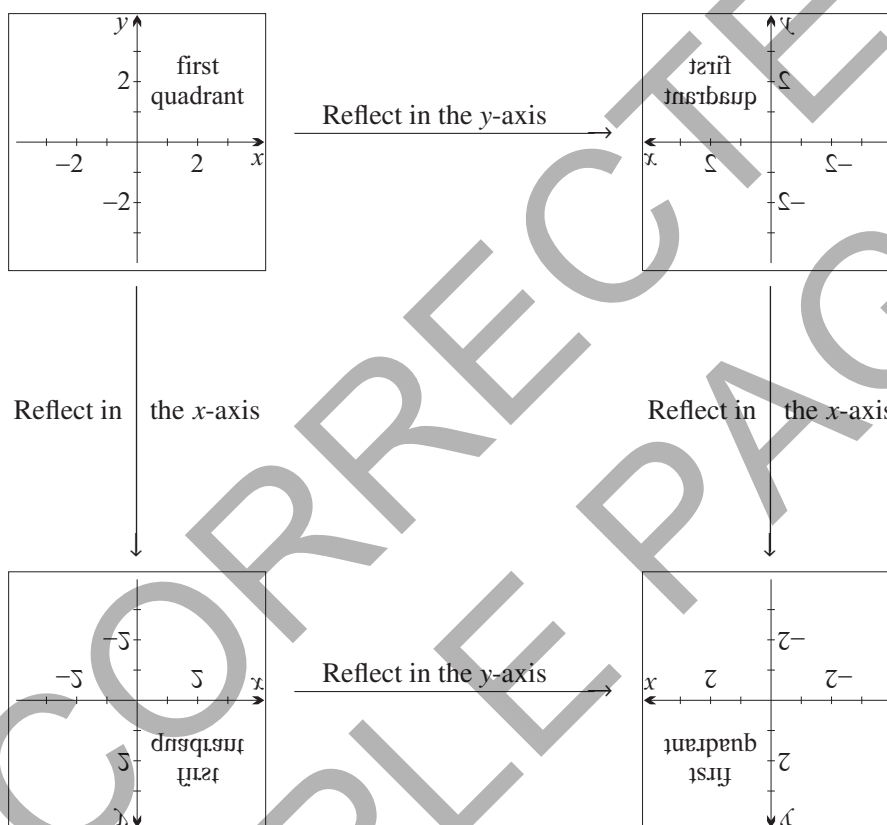
x	0	1	2	3	4
$(x - 2)^2$	4	1	0	1	4
$-(x - 2)^2$	-4	-1	0	-1	-4

- They both have vertex $(2, 0)$.
- For $y = (x - 2)^2$, domain: all real x , range: $y \geq 0$.
For $y = -(x - 2)^2$, domain: all real x , range: $y \leq 0$.



Combining the two reflections — rotating 180° about the origin

- Draw an x -axis and y -axis on a thin, semi-transparent sheet of paper.
- Hold the paper out flat, and regard it as a two-dimensional object.
- Reflect in the x -axis — do this by holding the sheet steady at the two ends of the x -axis and rotating it 180° so that you are now looking at the back of the sheet. Then reflect in the y -axis — do this by holding the sheet at the two ends of the y -axis and rotating it 180° so that you are looking at the front of the sheet again. What has happened?
- Reflect it in the y -axis, then in the x -axis. What happens?



This little experiment should convince you of two things:

- Performing successive reflections in the x -axis and in the y -axis results in a rotation of 180° about the origin.
- The order in which these two reflections are done does not matter.

This rotation of 180° about the origin is sometimes called *reflection in the origin*, because every point in the plane is moved along a line through the origin to a point the same distance from the origin on the opposite side.

7 Rotation of 180° about the origin

- To rotate a graph 180° about the origin, replace x by $-x$ and y by $-y$.
- Successive reflections in the x -axis and the y -axis are the same as a rotation of 180° about the origin.
- The order in which these two successive reflections are done does not matter.
- Rotation of 180° about O is also called *reflection in O* , because every point is moved through O to a point the same distance from O on the opposite side.

Rotation of 180° about O is also mutual — it maps each graph to the other graph.


Example 8 Applying two perpendicular reflections

- a** From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$ using reflections.
b What single transformation maps each graph to the other?
c What are the domains and ranges of the two functions?

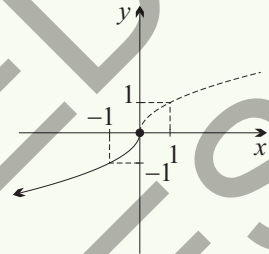
Solution

- a** The equation $y = -\sqrt{-x}$ can be rewritten as

$$-y = \sqrt{-x}$$

so the graph is obtained from the graph of $y = \sqrt{x}$ by successive reflections in the x -axis and the y -axis, where the reflections may be done in either order.

- b** This is the same as rotation of 180° about the origin.
c For $y = \sqrt{x}$, domain: $x \geq 0$, range: $y \geq 0$.
 For $y = -\sqrt{-x}$, domain: $x \leq 0$, range: $y \leq 0$.


Exercise 5B
FOUNDATION

- 1** Consider the parabola $y = x^2 - 2x$.
a Show that when y is replaced by $-y$, the equation becomes $y = 2x - x^2$.
b Copy and complete the table of values for $y = x^2 - 2x$ and $y = 2x - x^2$.

x	-2	-1	0	1	2	3	4
$x^2 - 2x$							
$2x - x^2$							

- c** Sketch the two parabolas and state the vertex of each.
d What transformation maps $y = x^2 - 2x$ to $y = 2x - x^2$?
2 Consider the hyperbola $y = \frac{2}{x-2}$.
a Show that when x is replaced by $-x$ the equation becomes $y = -\frac{2}{x+2}$.
b Copy and complete the table of values for $y = \frac{2}{x-2}$ and $y = -\frac{2}{x+2}$.

x	-4	-3	-2	-1	0	1	2	3	4
$\frac{2}{x-2}$							*		
$-\frac{2}{x+2}$			*						

- c** Sketch the two hyperbolas and state the vertical asymptote of each.
d What transformation maps $y = \frac{2}{x-2}$ to $y = -\frac{2}{x+2}$?
3 **a** Sketch the graph of the quadratic function $y = x^2 - 2x - 3$, showing the intercepts and vertex.
b In each case, determine the equation of the result when this parabola is reflected as indicated. Then draw a sketch of the new function.
 i in the y -axis *ii* in the x -axis *iii* in both axes

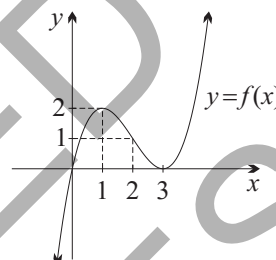
- 4 a** Sketch the graph of the exponential function $y = 2^{-x}$, showing the y -intercept and the coordinates at $x = -1$, and clearly indicating the asymptote.
- b** In each case, find the equation of the curve when this exponential is reflected as indicated. Then draw a sketch of the new function.

i in the x -axisii in the y -axis

iii in both axes

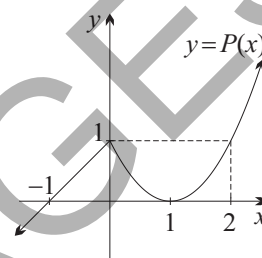
- 5 a** The graph of an unknown function $y = f(x)$ has been drawn on the right.

- i Which reflection produces $y = f(-x)$? Draw the graph of this function.
- ii Which reflection produces $y = -f(x)$? Draw the graph of this function.
- iii Compare the zeroes of $f(x)$, $f(-x)$ and $-f(x)$. Which pair have the same zeroes, and why?



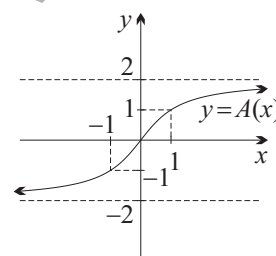
- b** The graph of an unknown function $y = P(x)$ has been drawn on the right.

- i Which transformation produces $y = -P(x)$? Draw the graph of this function.
- ii Which transformation produces $y = -P(-x)$? Draw the graph of this function.
- iii In this case, $P(x)$ and $-P(-x)$ both have the same zeroes. Why might that be?



- c** The graph of an unknown function $y = A(x)$ has been drawn on the right.

- i Which transformation produces $y = A(-x)$? Draw the graph of this function.
- ii Which transformation produces $y = -A(-x)$? Draw the graph of this function.
- iii Explain why the zeroes of all three functions are the same in this instance.
- iv What do you notice about the graphs of $y = A(x)$ and $y = -A(-x)$?



DEVELOPMENT

- 6** Write down the new equation for each function or relation after the given transformation has been applied. Then sketch the graph of the new curve.

a $y = x^2$: reflect in the x -axis

c $y = 2^x$: rotate by 180°

e $x^2 + y^2 = 9$: reflect in the y -axis

b $y = x^3$: reflect in the y -axis

d $y = 2x - x^2$: rotate by 180°

f $y = \frac{1}{x}$: reflect in the x -axis

- 7** Consider the hyperbola $y = \frac{1}{x+2} - 1$.

a Sketch this hyperbola.

b In each case, determine the reflection or rotation required to achieve the specified result. Then write down the equation of the new hyperbola and sketch it.

- i The vertical asymptote is unchanged, but the horizontal asymptote changes sign.
- ii The intercepts with the axes are positive.

- 8 a** Sketch the circles $(x - 3)^2 + y^2 = 4$ and $(x + 3)^2 + y^2 = 4$.
b What transformation maps each circle onto the other?
c Confirm your answer by making an appropriate substitution into the first equation.
- 9** Consider $x^2 + y^2 = r^2$, the circle with centre the origin and radius r .
a Show that this equation is unchanged when reflected in either the x -axis or the y -axis.
b Explain this result geometrically.

CHALLENGE

- 10** In each part, explain how the graph of each subsequent equation is a reflection of the first graph or a rotation of 180° , then sketch each one.

a From $y = \frac{1}{2}x + 1$:

i $y = -\frac{1}{2}x + 1$

ii $y = -\frac{1}{2}x - 1$

iii $y = \frac{1}{2}x - 1$

b From $y = 4 - x$:

i $y = x - 4$

ii $y = x + 4$

iii $y = -x - 4$

c From $y = (x - 1)^2$:

i $y = -(x + 1)^2$

ii $y = (x + 1)^2$

iii $y = -(x - 1)^2$

d From $y = \sqrt{x}$:

i $y = -\sqrt{-x}$

ii $y = -\sqrt{x}$

iii $y = \sqrt{-x}$

e From $y = 3^x$:

i $y = -3^x$

ii $y = -3^{-x}$

iii $y = 3^{-x}$

f From $y = 1 + \frac{1}{x-1}$:

i $y = 1 - \frac{1}{x+1}$

ii $y = -1 + \frac{1}{x+1}$

iii $y = -1 + \frac{1}{1-x}$

- 11** Consider the two parabolas $y = x^2 - 4x + 3$ and $y = x^2 + 4x + 3$.

- a** Sketch both quadratic functions on the same set of axes.
b What reflection maps each parabola onto the other?
c How can the second parabola be obtained by shifting the first?
d Confirm your answer to part (c) algebraically.
e Investigate which parts of the previous question could also have been achieved by shifting instead.

- 12 a** Let $c(x) = \frac{2^x + 2^{-x}}{2}$. Show that $c(-x) = c(x)$, and explain this geometrically.

b Let $t(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$. Show that $-t(-x) = t(x)$, and explain this geometrically.

c [Technology] Confirm your observations in parts (a) and (b) by plotting each function using graphing software.

5C Even and odd symmetry

Learning intentions

- Understand even and odd functions and relations geometrically and algebraically.
- Test the equations of functions and relations for evenness and oddness.

It is often said that all mathematics is the study of symmetry. Two simple types of symmetry occur so often in the functions of this course that every function should be tested routinely for them.

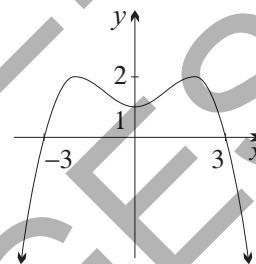
Even functions and line symmetry in the y-axis

A relation or function is called *even* if its graph has *line symmetry in the y-axis*. This means the graph is unchanged by reflection in the y-axis, as with the graph to the right.

As explained in Section 5B, to reflect the graph of a function $y = f(x)$ in the y-axis, the new curve has equation $y = f(-x)$. Hence for a function to be *even*, the graph of $y = f(x)$ and the graph of $y = f(-x)$ must coincide, that is,

$$f(-x) = f(x), \quad \text{for all } x \text{ in the domain.}$$

More generally, to reflect a relation in the y-axis, replace x by $-x$ everywhere in its equation. Thus the relation is even when its equation is unchanged after replacing x by $-x$.



8 Even functions and line symmetry in the y-axis

- A relation is called *even* if its graph has *line symmetry in the y-axis*.
- Algebraically, this means that a relation is even if its equation is unchanged when x is replaced by $-x$.
- In particular, a function $f(x)$ is even if

$$f(-x) = f(x), \quad \text{for all } x \text{ in the domain.}$$

Odd functions and point symmetry in the origin

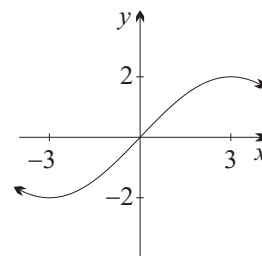
A relation or function is called *odd* if its graph has *point symmetry in the origin*. ‘Point symmetry in the origin’ means that the graph is unchanged by a rotation of 180° about the origin.

Because a rotation of 180° about the origin is the same as successive reflections in the y-axis and x-axis (in any order), oddness means also that the graph is unchanged by successive reflections in the y-axis and the x-axis.

Hence a relation is odd when its equation is unchanged after replacing x by $-x$ and y by $-y$.

In particular, reflecting a function $y = f(x)$ in both axes gives $-y = f(-x)$. Replacing y by $f(x)$, and swapping sides, this condition for $f(x)$ to be odd is

$$f(-x) = -f(x), \quad \text{for all } x \text{ in the domain.}$$



9 Odd functions and point symmetry in the origin

- *Point symmetry in the origin* means that the graph is mapped onto itself by a rotation of 180° about the origin.
 - ▷ Equivalently, it means that the graph is mapped onto itself by successive reflections in the x -axis and the y -axis (where order does not matter).
- A relation is called *odd* if its graph has *point symmetry in the origin*.
- Algebraically, this means that a relation is odd if its equation is unchanged when x is replaced by $-x$ and y is replaced by $-y$.
- In particular, a function $f(x)$ is odd if

$$f(-x) = -f(x), \text{ for all } x \text{ in the domain.}$$

Testing functions algebraically for evenness and oddness

A single test will pick up both these types of symmetry in functions:

10 Testing a function for evenness and oddness (or neither)

- Simplify $f(-x)$ and note whether it is $f(x)$, $-f(x)$, or neither.

Most functions are neither even nor odd.



Example 9 Testing a function for evenness and oddness

Test each function for evenness or oddness, then sketch it.

a $f(x) = x^4 - 3$

b $f(x) = x^3$

c $f(x) = x^2 - 2x$

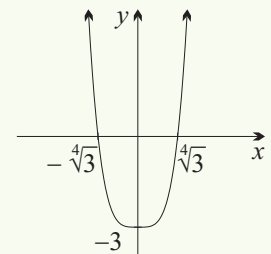
Solution

a Here

$$f(x) = x^4 - 3.$$

$$\begin{aligned} \text{Substituting } -x \text{ for } x, \quad f(-x) &= (-x)^4 - 3 \\ &= x^4 - 3 \\ &= f(x). \end{aligned}$$

Hence $f(x)$ is an even function.

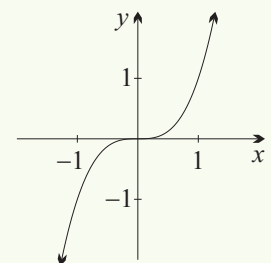


b Here

$$f(x) = x^3.$$

$$\begin{aligned} \text{Substituting } -x \text{ for } x, \quad f(-x) &= (-x)^3 \\ &= -x^3 \\ &= -f(x). \end{aligned}$$

Hence $f(x)$ is an odd function.



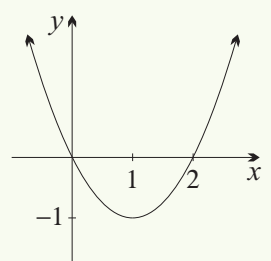
c Here

$$f(x) = x^2 - 2x.$$

$$\begin{aligned} \text{Substituting } -x \text{ for } x, \quad f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x. \end{aligned}$$

Because $f(-x)$ is equal neither to $f(x)$ nor to $-f(x)$, the function is neither even nor odd.

(The parabola does, however, have line symmetry, not in the y -axis, but in its axis of symmetry $x = 1$.)



**Example 10** The symmetries of a circle

- a** Test the circle $x^2 + y^2 = r^2$ for evenness and oddness.
b What other rotation and reflection symmetries does the circle have?

Solution

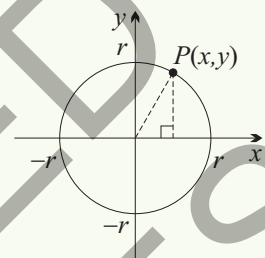
- a** Replacing x by $-x$, $(-x)^2 + y^2 = r^2$
 $x^2 + y^2 = r^2$,

so the circle is even.

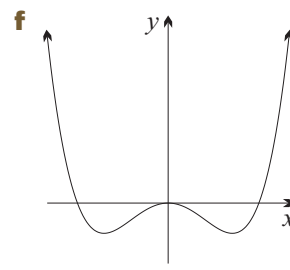
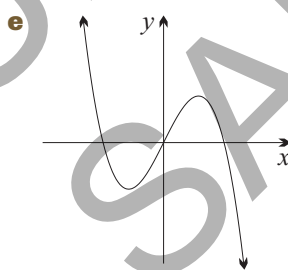
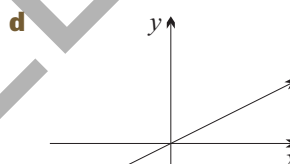
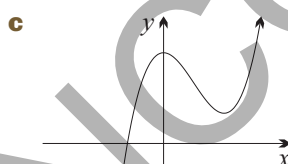
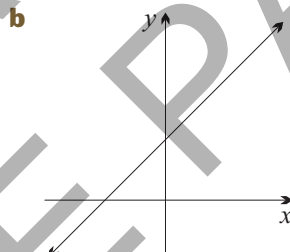
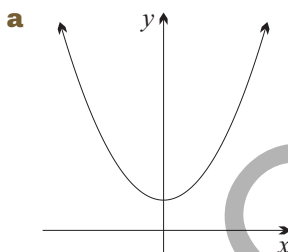
Replacing x by $-x$ and y by $-y$, $(-x)^2 + (-y)^2 = r^2$
 $x^2 + y^2 = r^2$,

so the circle is also odd.

- b** From the diagram, the circle has reflection symmetry in every diameter, and has rotation symmetry when rotated about the centre through any angle.

**Exercise 5C****FOUNDATION**

- 1** Classify each function $y = f(x)$ as even, odd or neither.

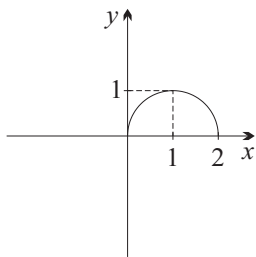


2 In each diagram below, complete the graph of $y = f(x)$ so that $f(x)$ is:

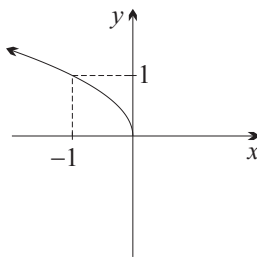
i even,

ii odd.

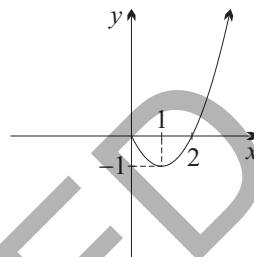
a



b



c



3 Consider the function $f(x) = x^4 - 2x^2 + 1$.

a Simplify $f(-x)$.

b Hence show that $f(x)$ is an even function.

4 Consider the function $g(x) = x^3 - 3x$.

a Simplify $g(-x)$.

b Hence show that $g(x)$ is an odd function.

5 Consider the function $h(x) = x^3 + 3x^2 - 2$.

a Simplify $h(-x)$.

b Hence show that $h(x)$ is neither even nor odd.

6 Simplify $f(-x)$ for each function, and hence determine whether it is even, odd or neither.

a $f(x) = x^2 - 9$

b $f(x) = x^2 - 6x + 5$

c $f(x) = x^3 - 25x$

d $f(x) = x^4 - 4x^2$

e $f(x) = x^3 + 5x^2$

f $f(x) = x^5 - 16x$

7 On the basis of the previous questions, copy and complete these sentences:

a 'A polynomial function is odd if ...'.

b 'A polynomial function is even if ...'.

DEVELOPMENT

8 Factor each polynomial in Question 6 above and write down its zeroes (that is, its x -intercepts). Then use a table of values to sketch its graph. Graphing software might also be used. Confirm that the graph exhibits the symmetry established above.

9 [Algebra and Technology] In Questions 3 to 8, the odd and even functions were all polynomials. Other functions can also be classified as odd or even. In each case following, simplify $f(-x)$ and compare it with $f(x)$ and $-f(x)$ to determine whether it is odd or even. Then confirm your answer by plotting the function on appropriate graphing software.

a $f(x) = \frac{2^x + 2^{-x}}{2}$

b $f(x) = \frac{2^x - 2^{-x}}{2}$

c $f(x) = \sqrt[3]{x}$

d $f(x) = (\sqrt[3]{x})^2$

e $f(x) = \frac{x}{x^2 - 4}$

f $f(x) = \frac{2}{x^2 - 4}$

g $f(x) = \sqrt{9 - x^2}$

h $f(x) = x\sqrt{9 - x^2}$

10 Determine whether each function is even, odd or neither.

a $f(x) = 2^x$

b $f(x) = 2^{-x}$

c $f(x) = \sqrt{3 - x^2}$

d $f(x) = \frac{1}{x^2 + 1}$

e $f(x) = \frac{4x}{x^2 + 4}$

f $f(x) = 3^x + 3^{-x}$

g $f(x) = 3^x - 3^{-x}$

h $f(x) = 3^x + x^3$

CHALLENGE

11 a Show that $f(x) = x^2 - 2x - 3$ is neither even nor odd.

b Let $g(x) = \frac{1}{2}(f(x) + f(-x))$. Determine $g(x)$ and show that it is even.

c Let $h(x) = \frac{1}{2}(f(x) - f(-x))$. Determine $h(x)$ and show that it is odd.

d Suppose that $f(x)$ is replaced with the cubic $f(x) = ax^3 + bx^2 + cx + d$. Will $g(x)$ still be even and will $h(x)$ still be odd?

e Try to generalise the result for any polynomial.

12 a Prove that if $f(x)$ is an odd function defined at $x = 0$, then $y = f(x)$ passes through the origin. (Hint: Apply the condition for a function to be odd to $f(0)$.)

b If $f(x)$ is an even function defined at $x = 0$, does the graph of $y = f(x)$ have to pass through the origin? Either prove that it does or give a counter-example.

13 a Suppose that $h(x) = f(x) \times g(x)$.

i Show that if f and g are both even or both odd, then $h(x)$ is even.

ii Show that if one of $f(x)$ and $g(x)$ is even and the other odd, then $h(x)$ is odd.

b Suppose that $h(x) = f(x) + g(x)$.

i Show that if $f(x)$ and $g(x)$ are both even, then $h(x)$ is even.

ii Show that if $f(x)$ and $g(x)$ are both odd, then $h(x)$ is odd.

5D Horizontal and vertical dilations

Learning intentions

- Transform an equation so that its graph is dilated horizontally or vertically.
- Show that the order of performing a horizontal and a vertical dilation is irrelevant.
- Apply horizontal and vertical dilations to familiar curves.
- Obtain a given function formula by successive translations, reflections and rotations.

Dilations stretch a curve so that, for example, a circle becomes an ellipse. These are the third and last of our standard transformations.

Most of the functions and relations in the course can be reduced to very simple functions using a combination of translations, reflections, and dilations.

Stretching a graph horizontally away from the y-axis

In the graphs sketched opposite, x has been replaced by $\frac{x}{3}$:

$$y = x(x - 2) \quad \text{and} \quad y = \frac{x}{3}\left(\frac{x}{3} - 2\right) = \frac{1}{9}x(x - 6),$$

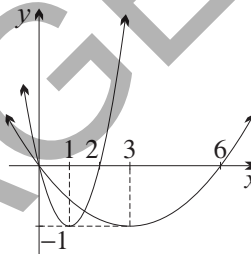
and the tables of values are:

x	-2	-1	0	1	2	3	4
$x(x - 2)$	8	3	0	-1	0	3	8

x	-6	-3	0	3	6	9	12
$\frac{x}{3}\left(\frac{x}{3} - 2\right)$	8	3	0	-1	0	3	8

The y -coordinates in each table are the same, but we needed to treble the x -coordinates to produce those same y -coordinates in the second table.

Geometrically, this means that all points on the graph triple their distance from the y -axis, and the graph is *stretched or dilated away from the y -axis in the horizontal direction by a dilation factor of 3*. The points on the y -axis do not move.



11 Horizontal dilations — stretching a graph horizontally away from the y -axis

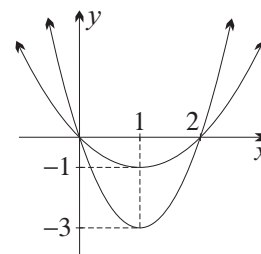
- To stretch a graph horizontally from the y -axis with factor k , replace x by $\frac{x}{k}$.
- For a function $y = f(x)$, the new function rule is $y = f\left(\frac{x}{k}\right)$.

Stretching a graph vertically away from the x -axis

Compare the graphs of $y = x(x - 2)$ and $\frac{y}{3} = x(x - 2)$ or $y = 3x(x - 2)$.

Each value in the table below for $y = 3x(x - 2)$ is three times the corresponding value in the table for $y = x(x - 2)$. This means that the graph of $y = 3x(x - 2)$ is obtained from the graph of $y = x(x - 2)$ by *stretching away from the x -axis in the vertical direction* by a factor of 3:

x	-2	-1	0	1	2	3	4
$x(x - 2)$	8	3	0	-1	0	3	8
$3x(x - 2)$	24	9	0	-3	0	9	24



This time the points on the x -axis do not move, and all other points on the graph triple their distance from the x -axis.

12 Vertical dilations — stretching a graph vertically away from the x -axis

- To stretch a graph vertically from the x -axis by a factor of ℓ , replace y by $\frac{y}{\ell}$.
- For a function $y = f(x)$, the new function rule is $\frac{y}{\ell} = f(x)$, or $y = \ell f(x)$.

Combining horizontal and vertical dilations

When a graph is dilated horizontally and vertically, the order of applying the dilations doesn't matter. The argument is almost the same as for translations.

If $y = f(x)$ is dilated k horizontally and then ℓ vertically,

$$y = f(x) \longrightarrow y = f\left(\frac{x}{k}\right) \longrightarrow \frac{y}{\ell} = f\left(\frac{x}{k}\right),$$

and if $y = f(x)$ is dilated ℓ vertically and then k horizontally,

$$y = f(x) \longrightarrow \frac{y}{\ell} = f(x) \longrightarrow \frac{y}{\ell} = f\left(\frac{x}{k}\right).$$

13 The order of two perpendicular dilations does not matter

- If two dilations are applied to a graph, the order does not matter.



Example 11 Identifying dilations of a circle

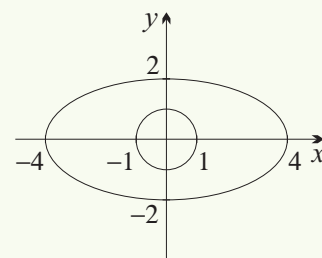
Obtain the graph of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ from the graph of the circle $x^2 + y^2 = 1$.

Solution

The equation can be rewritten as

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1,$$

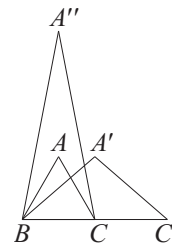
which is the unit circle stretched vertically by a factor of 2, and horizontally by a factor of 4.



Note: Any curve of the form $\frac{x^2}{k^2} + \frac{y^2}{\ell^2} = 1$ is called an *ellipse*. It can be obtained from the unit circle $x^2 + y^2 = 1$ by stretching horizontally by a factor of k and vertically by a factor of ℓ , so that its x -intercepts are k and $-k$ and its y -intercepts are ℓ and $-\ell$.

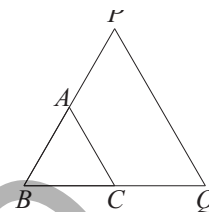
Enlargements

A dilation of a figure is usually not similar to the original. For example, the equilateral triangle ABC in the figure to the right with its base on the x -axis is stretched to the squat isosceles triangle $A'BC'$ by a horizontal dilation with factor 2. And it is stretched to the skinny isosceles triangle $A''BC$ by a vertical dilation with factor 3.



But if two dilations with the same factor 2, one horizontal and the other vertical, are applied in order to the equilateral triangle ABC — the order does not matter — the result is the similar equilateral triangle PBQ . Such a combined transformation is called an *enlargement*, and the factor 2 is called the *enlargement factor* or *similarity factor*.

In the coordinate plane, the *centre* of an enlargement is normally taken as the origin.



14 Enlargements

- An *enlargement* of a figure is similar to the original. In particular, matching angles are equal, and the ratios of matching lengths are equal.
- The composition of two dilations with the same factor, one horizontal and one vertical, is an *enlargement* with centre the origin.
- To apply an enlargement with factor k , centre O , replace x by $\frac{x}{k}$ and y by $\frac{y}{k}$.
- For a function, the new function rule is $\frac{y}{k} = f\left(\frac{x}{k}\right)$, or $y = kf\left(\frac{x}{k}\right)$.



Example 12 Enlarging a circle

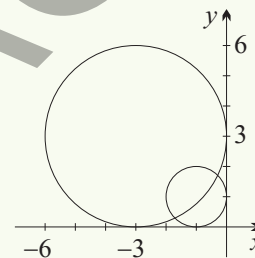
Apply an enlargement with centre the origin and factor 3 to the circle $(x + 1)^2 + (y - 1)^2 = 1$. Write down the new function, then sketch both curves.

Solution

The new function is $\left(\frac{x}{3} + 1\right)^2 + \left(\frac{y}{3} - 1\right)^2 = 1$

$$\boxed{\times 3^2 = 9} \quad (x + 3)^2 + (y - 3)^2 = 9.$$

The two circles are sketched to the right.



Stretching with a fractional or negative factor

In the upper diagram to the right, a vertical dilation with factor $\frac{1}{2}$ has been applied to the parabola $y = x^2 + 2$ to yield the parabola

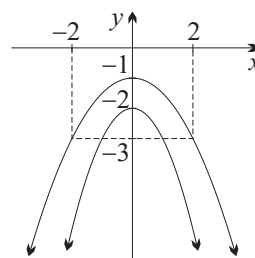
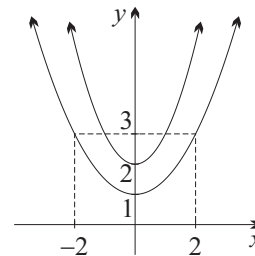
$$\frac{y}{1/2} = x^2 + 2, \quad \text{that is,} \quad y = \frac{1}{2}x^2 + 1.$$

The result is a compression, but we still call it a dilation.

In the lower diagram to the right, vertical dilations with factors -1 and $-\frac{1}{2}$ have been applied to the same parabola $y = x^2 + 2$. The results are the parabolas

$$y = -x^2 - 2 \quad \text{and} \quad y = -\frac{1}{2}x^2 - 1.$$

The first parabola is the reflection of the original in the x -axis. The second parabola is the reflection in the x -axis of the compressed image.



15 Dilations with a fractional or negative factor

- If the dilation factor is between 0 and 1, the graph is compressed.
- If the dilation factor is negative, the dilation is the composition of a dilation with the opposite positive factor and a reflection — the order does not matter.
- In particular:
 - ▷ A reflection is a dilation with factor -1 .
 - ▷ A rotation of 180° about the origin is an enlargement with factor -1 , and is often called a *reflection in the origin*.



Example 13 Applying dilations with negative and fractional factors

Write down the new functions when each dilation is applied to the parabola $y = (x - 3)(x - 5)$. Then sketch the four curves on one set of axes.

- A horizontal dilation with factor -2 .
- A horizontal dilation with factor $-\frac{1}{2}$.
- A vertical dilation with factor -1 . (What other transformation is this?)
- Which domains and ranges are changed by these transformations?

Solution

- Replacing x by $\frac{x}{-2}$,

$$y = (-\frac{x}{2} - 3)(-\frac{x}{2} - 5)$$

$$y = \frac{1}{4}(x + 6)(x + 10).$$

- Replacing x by $\frac{x}{-1/2} = -2x$,

$$y = (-2x - 3)(-2x - 5)$$

$$y = 4(x + 1\frac{1}{2})(x + 2\frac{1}{2}).$$

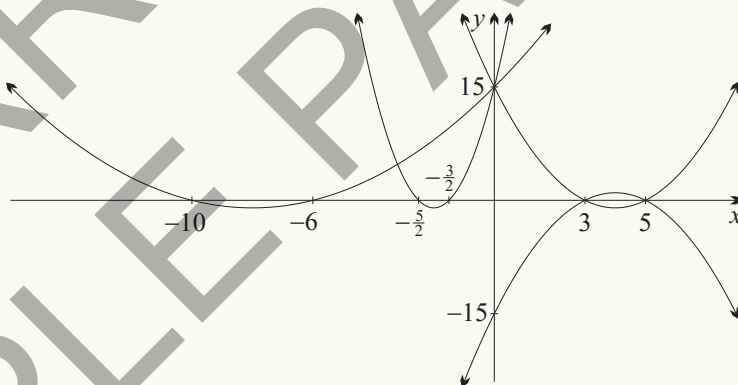
- Replacing y by $\frac{y}{-1} = -y$,

$$-y = (x - 3)(x - 5)$$

$$y = -(x - 3)(x - 5).$$

This is a vertical dilation with factor -1 , so it is also a reflection in the x -axis.

- In part (c), the original range $y \geq -1$ is changed to the range $y \leq 1$.



Warning: The word ‘dilation’ is often used to mean ‘enlargement’ rather than a stretching in one direction. Be careful when looking at other sources on the web.

Combining different types of transformations — the order usually matters

Be careful, because with different types of transformations, order usually matters.

These examples also show how many complex-looking functions are generated by applying a sequence of different transformations to a very simple function.

Example 14 Describing a function as the result of a sequence of transformations

Describe how $y = x^2$ can be transformed to give $y = 2(3x + 5)^2 - 7$. Write down the result of each successive step.

Solution

Shift left 5, then dilate horizontally with factor $\frac{1}{3}$, then dilate vertically with factor 2, then shift down 7.

$$y = x^2 \rightarrow y = (x + 5)^2 \rightarrow y = (3x + 5)^2 \rightarrow y = 2(3x + 5)^2 \rightarrow y = 2(3x + 5)^2 - 7$$

Example 15 The order in which transformations are applied usually matters

Transform $y = x^3$ in the two different orders below, showing steps.

- a** Shift $y = x^3$ right 3, then dilate horizontally with factor 5, showing steps.
b Dilate $y = x^3$ horizontally with factor 5, then shift right 3, showing steps.

Solution

a $y = x^3 \rightarrow y = (x - 3)^3 \rightarrow y = \left(\frac{x}{5} - 3\right)^3$, that is, $y = \frac{1}{125}(x - 15)^3$

b $y = x^3 \rightarrow y = \left(\frac{x}{5}\right)^3 \rightarrow y = \left(\frac{x - 3}{5}\right)^3$, that is, $y = \frac{1}{125}(x - 3)^3$

Example 16 Identifying the asymptotes of a transformed function

- a** Show that $\frac{21x - 62}{3x - 9} = \frac{1}{3x - 9} + 7$.
b Hence identify, showing steps, a sequence of transformations that will move $y = \frac{1}{x}$ to $\frac{21x - 62}{3x - 9}$.
c Identify the domain, range, and asymptotes of the resulting function.

Solution

a $\text{RHS} = \frac{1}{3x - 9} + \frac{21x - 63}{3x - 9}$
 $= \text{LHS}.$

b Shift right 9: $y = \frac{1}{x} \rightarrow \frac{1}{x - 9}$
Dilate horizontally factor $\frac{1}{3}$: $y = \frac{1}{x - 9} \rightarrow \frac{1}{3x - 9}$
Shift up 7: $y = \frac{1}{3x - 9} \rightarrow \frac{1}{3x - 9} + 7$

- c** Domain: $x \neq 3$. Range: $y \neq 7$. Asymptotes: $x = 3$ and $y = 7$.

Exercise 5D**FOUNDATION**

- 1** Write down the new equation for each function or relation after the given dilation has been applied. Draw a graph of the image after the shift:

a $y = x^2$: horizontally by $\frac{1}{2}$

b $y = 2^x$: vertically by 2

c $y = x^2 - 1$: vertically by -1

d $y = \frac{1}{x}$: horizontally by 2

e $x^2 + y^2 = 4$: vertically by $\frac{1}{3}$

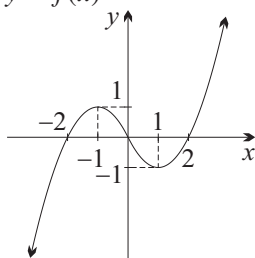
f $y = \sqrt{1 - x}$: horizontally by -1

g $y = \sqrt{4 - x^2}$: horizontally by $\frac{1}{2}$

h $y = \sqrt{x}$: vertically by -2

- 2** In each case an unknown function has been drawn. Use dilations to draw the new functions indicated beneath each, then write down the y-intercept and zeroes of the dilated function..

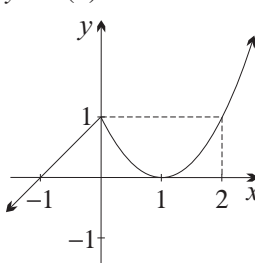
a $y = f(x)$



i $y = f(2x)$

ii $y = 2f(x)$

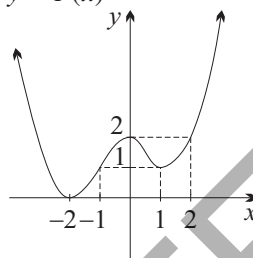
c $y = h(x)$



i $\frac{y}{2} = h(x)$

ii $y = h\left(\frac{x}{2}\right)$

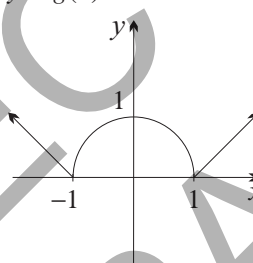
b $y = P(x)$



i $y = P\left(\frac{x}{2}\right)$

ii $y = \frac{1}{2}P(x)$

d $y = g(x)$



i $2y = g(x)$

ii $y = g(2x)$

- 3** [Dilations and Intercepts] On the basis of the previous question, copy and complete these two sentences about the intercepts of the graph of a function with the axes.
- a** A zero of a function is unchanged by a ... dilation.
- b** The value of $f(0)$ is unchanged by a ... dilation.
- 4** Sketch $x + y = 1$. Then explain how each graph below may be obtained by dilations of the first graph (there may be more than one answer), and sketch it.
- a** $\frac{x}{2} + y = 1$ **b** $\frac{x}{2} + \frac{y}{4} = 1$ **c** $2x + y = 1$
- 5** **a** The circle $(x - 3)^2 + y^2 = 4$ is enlarged by factor $\frac{1}{3}$ with centre the origin. Write down the new equation and draw both circles on the one set of axes.
- b** The hyperbola $y = \frac{1}{x}$ is enlarged by factor $\sqrt{3}$ with centre the origin. Write down the new equation and draw both hyperbolae on the one set of axes.
- 6** In each case graph the three given equations on one set of axes by using dilations:
- a** $y = x(4 + x)$, $y = 2x(4 + x)$, and $y = \frac{x}{2}(4 + \frac{x}{2})$.
- b** $x^2 + y^2 = 36$, $(\frac{x}{2})^2 + (\frac{y}{3})^2 = 36$, and $(2x)^2 + (3y)^2 = 36$.

7 [The Order of Dilations]**a** Graph the function $y = x(x + 2)$

- i** The graph in part (a) is stretched horizontally by factor 2. Write down the equation of this new function and graph it on the same number plane.
- ii** This graph is then stretched vertically by factor 3. Write down the equation of this new function and graph it on the same number plane.

b On a new number plane, draw another copy of the parabola $y = x(x + 2)$.

- i** The graph in part (b) is stretched vertically by factor 3. Write down the equation of this new function and graph it on the same number plane.
- ii** This graph is then stretched horizontally by factor 2. Write down the equation of this new function and graph it on the same number plane.

c Compare your graphs and equations in parts (a)(ii) and (b)(ii). Did the order of the two dilations change the final results?**8** [Dilations and Symmetry]**a i** Show that $y = x^3 - x$ is an odd function.**ii** This function is stretched horizontally by factor 2. Write down the equation of this new function.**iii** Is the new function odd?**b i** Show that $y = 1 - x^2$ is an even function.**ii** This function is stretched vertically by factor 3. Write down the equation of this new function.**iii** Is the new function even?**c** What do you conclude about dilations and symmetry from these two examples?**DEVELOPMENT****9** Sketch each group of three functions on the one set of axes.**a** $y = \sqrt{4 + x}$, $y = 2\sqrt{4 + x}$, $y = 2\sqrt{4 + 2x}$ **b** $y = 2^x$, $y = 2^{-x}$, $y = 3 \times 2^{-x}$ **10** Answer the following questions about the cubic $y = x^3 - 3x^2 + 4$, which can be factored as $y = (x - 2)^2(x + 1)$.**a** Graph this cubic, using graphing software if needed. Notice that the curve is momentarily horizontal at $(0, 4)$ and $(2, 0)$.**b** The cubic is dilated vertically by factor 2.

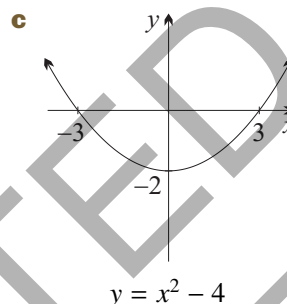
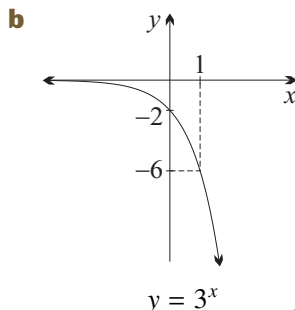
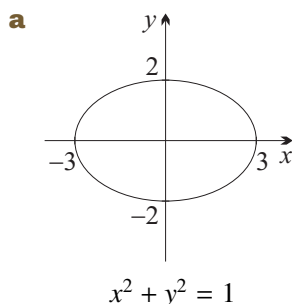
- i** Write down the equation of this new cubic and graph it.
- ii** Where is this new curve horizontal? Have the x -coordinates changed?

c The original cubic is dilated horizontally by factor 3.

- i** Write down the equation of this third cubic and graph it.
- ii** Where is this third curve horizontal? Have the y -coordinates changed?

- 11 a** The cubic $y = x^3$ is transformed to obtain $y = \frac{1}{8}x^3$.
- Explain the transformation as a horizontal stretch.
 - Explain the transformation as a vertical stretch.
 - Graph both functions on the same number plane.
- b** The cubic $y = x^3$ is transformed to obtain $y = -x^3$.
- Explain the transformation as a horizontal stretch.
 - Explain the transformation as a vertical stretch.
 - Graph both functions on the same number plane.
- c** Give two possible interpretations of transforming $y = x^3$ to get the graph of $y = \frac{1}{4}x^3$, then graph both functions on the same number plane.
- 12** In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.
- a** $y = x^2 - 2x$ and $y = 3x^2 - 6x$
- b** $y = \frac{1}{x-4}$ and $y = \frac{1}{2x-4}$
- c** $y = 3^x$ and $y = 3^{\frac{1}{4}x}$
- d** $y = \frac{1}{x+1}$ and $y = \frac{2}{x+1}$
- 13** Consider the hyperbola $y = \frac{1}{x}$.
- The hyperbola is stretched horizontally by factor 2. Write down its equation.
 - The original hyperbola is stretched vertically by factor 2. Write down its equation.
 - What do you notice about the answers to parts (a) and (b)?
 - Can the hyperbolae in parts (a) or (b) be achieved by an enlargement?
 - Investigate whether there are any other functions that exhibit similar behaviour.
- 14** Consider the parabola $y = x^2$.
- The parabola is dilated horizontally by factor $\frac{1}{2}$. Write down its equation.
 - The original parabola is dilated vertically by factor 4. Write down its equation.
 - What do you notice about the answers to parts (a) and (b)?
 - Can the parabolas in parts (a) or (b) be achieved by an enlargement?
 - Investigate whether there are any other functions that exhibit similar behaviour.
- 15** The mass M grams of a certain radioactive substance after t years is modeled by the formula $M = 3 \times 2^{-\frac{1}{53}t}$.
- Find the initial mass.
 - Find the time taken for the mass to halve, called the *half-life*.
 - Suppose now that the initial mass is doubled.
 - Explain this in terms of a dilation and hence write down the new equation for M .
 - Show that the dilation does not change the value of the half-life.

- 16** Show that the equation $y = mx$ of a straight line through the origin is unchanged by any enlargement with centre the origin.
- 17** Describe each graph below as the given function transformed by dilations, and hence write down its equation.



CHALLENGE

- 18** Explain the gradient-intercept formula $y = mx + b$ for a straight line in terms of a dilation of the line $y = x + b$.
- 19 a** For each pair of curves, suggest two simple and distinct transformations by which the second equation may be obtained from the first:
- i** $y = 2^x$, $y = 2^{x+1}$ **ii** $y = \frac{1}{x}$, $y = \frac{k^2}{x}$ **iii** $x^2 + (y - 2)^2 = 1$, $x^2 + (y + 2)^2 = 1$
- b** Investigate other combinations of curves and transformations with similar ambiguity.
- 20** The parabola $y = x^2$ is stretched horizontally by factor a . Clearly a horizontal stretch by factor $\frac{1}{a}$ will restore the original parabola. What other stretch will produce a new parabola that appears identical to the original parabola $y = x^2$?

5E The absolute value function

Learning intentions

- Define absolute value geometrically, and apply it.
- Understand absolute value as a piecewise-defined function, and apply it.
- Understand absolute value as the positive square root of a square, and apply it.
- Sketch graphs and solve simple equations using these approaches.

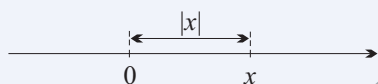
Often it is the size or magnitude of a number that is significant, rather than whether it is positive or negative. *Absolute value* is the mathematical name for this concept. We begin the discussion with its geometric definition.

Absolute value as distance

Distance is the clearest way to define absolute value.

16 Absolute value as distance

- The absolute value $|x|$ of a number x is the distance from x to the origin on the number line.



For example, $|-5| = 5$ and $|0| = 0$ and $|5| = 5$.

- Distance is always positive or zero, so

$$|x| \geq 0, \quad \text{for all real numbers } x.$$

- The numbers x and $-x$ are equally distant from the origin, so

$$|-x| = |x|, \quad \text{for all real numbers } x.$$

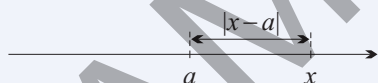
Thus absolute value is a measure of the *size* or *magnitude* of a number. In the examples above, the numbers -5 and $+5$ both have the same magnitude 5, and differ only in their signs.

Distance between numbers

Replacing x by $x - a$ in the previous definition gives a measure of the distance from x to a on the number line.

17 Distance between numbers

- The distance from x to a on the number line is $|x - a|$.



For example, the distance between 5 and -2 is $|5 - (-2)| = 7$.

- By Box 16 above, $|x - a| = |a - x|$, for all real numbers x and a .

Absolute value defined piecewise, using cases

If x is a negative number, then the absolute value of x is $-x$, the opposite of x . This gives an alternative *piecewise definition* of $|x|$ using cases:

18 Absolute value defined piecewise, using cases

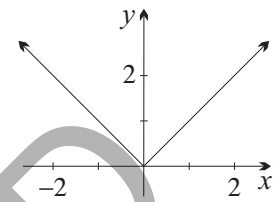
$$\text{If } x \text{ is any real number, then } |x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

A piecewise definition naturally divides the graph into *pieces*, also called *branches*.

Both branches here are linear functions, resulting in a sharp point at the origin where the two branches meet at right angles:

x	-2	-1	0	1	2
$ x $	2	1	0	1	2

- The domain is all real numbers, and the range is $y \geq 0$.
- The function is even, because the graph has line symmetry in the y -axis.
- The function has a zero at $x = 0$, and is positive for all other values of x .



Graphing functions with absolute value

Transformations can now be applied to the graph of $y = |x|$ to sketch many functions involving absolute value. More complicated functions, however, require the approach involving cases.

Example 17 Using transformations to graph a function with absolute value

- Sketch $y = |x - 2|$ using shifting.
- Check the graph using a table of values.
- Write down the equations of the two branches.

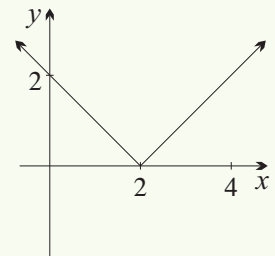
Solution

- This is $y = |x|$ shifted 2 units to the right.

x	0	1	2	3	4
y	2	1	0	1	2

- From the expression using cases, or from the graph:

$$y = \begin{cases} x - 2, & \text{for } x \geq 2, \\ -x + 2, & \text{for } x < 2. \end{cases}$$



Example 18 Using cases to graph a function with absolute value

- Use cases to sketch $y = |x| - x$.
- Check using a table of values, and state its domain and range.

Solution

- Considering separately the cases $x \geq 0$ and $x < 0$,

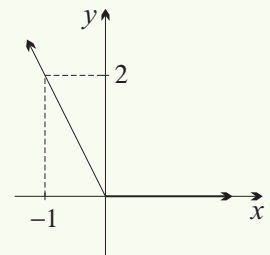
$$y = \begin{cases} x - x, & \text{for } x \geq 0, \\ -x - x, & \text{for } x < 0, \end{cases}$$

$$\text{that is, } y = \begin{cases} 0, & \text{for } x \geq 0, \\ -2x, & \text{for } x < 0. \end{cases}$$

- Checking using a table of values:

x	-2	-1	0	1	2
y	4	2	0	0	0

Domain: all real x , range: $y \geq 0$.



Solving absolute value equations of the form $|ax + b| = k$

Three observations should make everything clear:

- An equation such as $|3x + 6| = -21$ has no solutions, because an absolute value can never be negative.
- An equation such as $|3x + 6| = 0$ is true when $3x + 6 = 0$.
- An equation such as $|3x + 6| = 21$ can be solved by realising that:

$$|3x + 6| = 21 \quad \text{is true when} \quad 3x + 6 = 21 \text{ or } 3x + 6 = -21.$$

19 To solve the equation $|ax + b| = k$

- If $k < 0$, the equation has no solutions.
- If $k = 0$, the equation has one solution, found by solving

$$ax + b = 0.$$

- If $k > 0$, the equation has two solutions, found by solving

$$ax + b = k \quad \text{or} \quad ax + b = -k.$$



Example 19 Solving simple absolute value equations

Solve these absolute value equations. Then check using the graph drawn below.

a $|3x + 6| = -21$

b $|3x + 6| = 0$

c $|3x + 6| = 6$

Solution

a $|3x + 6| = -21$ has no solutions, because an absolute value cannot be negative.

b $|3x + 6| = 0$

$$3x + 6 = 0$$

$$\boxed{-6}$$

$$3x = -6$$

$$\boxed{\div 3}$$

$$x = -2$$

c $|3x + 6| = 6$

$$3x + 6 = 6 \quad \text{or} \quad 3x + 6 = -6$$

$$\boxed{-6}$$

$$3x = 0 \quad \text{or} \quad 3x = -12$$

$$\boxed{\div 3}$$

$$x = 0 \quad \text{or} \quad x = -4$$



Example 20 Solving simple absolute value equations

Solve each absolute value equation.

a $|x - 2| = 3$

b $|7 - \frac{1}{4}x| = 3$

Solution

a $|x - 2| = 3$

$$x - 2 = 3 \quad \text{or} \quad x - 2 = -3$$

$$\boxed{+2}$$

$$x = 5 \quad \text{or} \quad x = -1$$

b $|7 - \frac{1}{4}x| = 3$

$$7 - \frac{1}{4}x = 3 \quad \text{or} \quad 7 - \frac{1}{4}x = -3$$

$$\boxed{-7}$$

$$-\frac{1}{4}x = -4 \quad \text{or} \quad -\frac{1}{4}x = -10$$

$$\boxed{\times (-4)}$$

$$x = 16 \quad \text{or} \quad x = 40$$

Sketching $y = |3x + 6|$ (or any function of the form $y = |ax + b|$)

Method 1: Sketch using cases, as in Example 18 above.

Method 2: Find and plot the x -intercept and the y -intercept.

$$\text{Put } y = 0, \text{ then } |3x + 6| = 0 \quad \text{Put } x = 0.$$

$$3x + 6 = 0 \quad \text{Then } y = |0 + 6|$$

$$x = -2. \quad = 6.$$

The graph is symmetric about the vertical line $x = -2$, so the point $(-4, 6)$ also lies on the curve.

Now join up the points in the characteristic V shape.

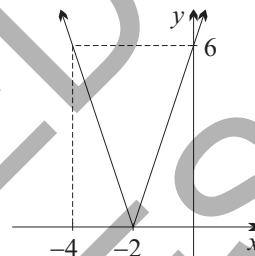
Method 3: Draw up a small table of values — the pieces are linear:

x	-4	-3	-2	-1	0
y	6	3	0	3	6

A good final check: The two branches of the graph should have gradients 3 and -3 .

Method 4: Use transformations of $y = |x|$.

- Shift the graph left 6: $y = |x| \rightarrow y = |x + 6|$
- Then dilate horizontally with factor $\frac{1}{3}$: $y = |x + 6| \rightarrow y = |3x + 6|$.



Absolute value as the square root of the square

Taking the absolute value of a number means stripping any negative sign from the number. We have algebraic functions that can do this job — square the number, then apply the function $\sqrt{}$ that says ‘take the positive square root (or zero)’.

20 Absolute value as the positive square root of the square

- For all real numbers x , $|x|^2 = x^2$ and $|x| = \sqrt{x^2}$.

For example, $|-3|^2 = 9 = (-3)^2$ and $|-3| = \sqrt{9} = \sqrt{(-3)^2}$.

Note: We have now defined the absolute value function $y = |x|$ using distance, using cases, and using a formula. The phrase ‘piecewise-defined function’ does not characterise a function, but only the way that it is currently being defined.

Identities involving absolute value

Here are some standard identities involving absolute value.

21 Identities involving absolute value

- $|-x| = |x|$, for all x .
- $|x - y| = |y - x|$, for all x and y .
- $|xy| = |x||y|$, for all x and y .
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, for all x , and for all $y \neq 0$.

Substitution of some positive and negative values for x and y should be sufficient to demonstrate these results.

Exercise 5E

FOUNDATION

1 Evaluate:

a $|3|$

b $|-3|$

c $|4 - 7|$

d $|7 - 4|$

e $|14 - 9 - 12|$

f $|-7 + 8|$

g $|3^2 - 5^2|$

h $|11 - 16| - 8$

2 In each case, use the given values of x and y to verify that:

i $|x| = |-x|$

ii $|x - y| = |y - x|$

iii $|xy| = |x||y|$

iv $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

a $x = 4, y = -2$

b $x = -4, y = 2$

c $x = -4, y = -2$

3 Solve each absolute value equation, then graph the solution on a number line.

a $|x| = 1$

b $|x| = 3$

c $|4x| = 8$

d $|2x| = 10$

e $|2x| = 6$

f $|3x| = 12$

4 Solve each equation and graph the solution on a number line.

a $|x - 4| = 1$

b $|x - 3| = 7$

c $|x - 3| = 3$

d $|x - 7| = 2$

e $|x + 5| = 2$

f $|x + 2| = 2$

g $|x + 1| = 6$

h $|x + 3| = 1$

5 **a** Copy and complete the tables of values for the functions $y = |x - 1|$ and $y = |x| - 1$.

x	-2	-1	0	1	2	3
$ x - 1 $						

x	-2	-1	0	1	2	3
$ x - 1$						

b Draw the graphs of the two functions on separate number planes, and observe the similarities and differences between them.**c** Explain how each graph is obtained by shifting $y = |x|$.**d** Write down the domain and range of each function.6 Show that each statement is true when $x = -3$.

a $|5x| = 5|x|$

b $|-x| = |x|$

c $|x|^2 = x^2$

d $|x - 7| = |7 - x|$

e $x \leq |x|$

f $-|x| \leq x$

7 Show that each statement is false when $x = -2$.

a $|x| = x$

b $|-x| = x$

c $|x + 2| = |x| + 2$

d $|x + 1| = x + 1$

e $|x - 1| < |x| - 1$

f $|x|^3 = x^3$

DEVELOPMENT

8 In each case, use the rules of Box 19 to solve the equation for x .

a $|7x| = 35$

b $|2x + 1| = 3$

c $|2x - 1| = 11$

d $|7x - 3| = -11$

e $|3x + 2| = -8$

f $|5x + 2| = 0$

g $|3x - 5| = 0$

h $|6x - 7| = 5$

i $|5x + 4| = 6$

9 **a** Consider the equation $|1 - 2x| = 3$.**i** Explain why this equation has the same solution as $|2x - 1| = 3$. (Refer to Box 21.)**ii** Hence, or otherwise, solve the equation.**b** Likewise, solve these equations.

i $|3 - 2x| = 1$

ii $|1 - 3x| = 2$

- 10 a** Use cases to help sketch the branches of $y = |x|$.
- b** In each part, identify the shift or shifts of $y = |x|$ and hence sketch the graph. Then write down the equations of the two branches.
- i** $y = |x - 3|$ **ii** $y = |x + 2|$ **iii** $y = |x| - 2$
iv $y = |x| + 3$ **v** $y = |x - 2| - 1$ **vi** $y = |x + 1| - 1$
- c** Write down the domain and range for each graph in part (b).
- 11 a** Sketch each function using cases. Check the graph with a table of values.
- i** $y = |2x|$ **ii** $y = |\frac{1}{2}x|$
- b** Now sketch each function using a suitable vertical dilation of $y = |x|$, and check that you get the same graph.
- 12** Find and plot the x -intercept and y -intercept of each function. Then complete each graph using symmetry. Finally, check each graph with a small table of values.
- a** $y = |2x - 6|$ **b** $y = |6 - 3x|$ **c** $y = |5x + 5|$
- 13** Use cases to sketch each of these functions. Then check each graph with a suitable table of values.
- a** $y = -|3x|$ **b** $y = |2x + 4|$ **c** $y = -|x + 3|$
- 14** When the graph of $y = |x|$ is shifted horizontally by a the result is $y = |x - a|$. When this undergoes a horizontal stretch with factor $\frac{1}{h}$ the final result is $y = |hx - a|$. Use this approach to sketch the following functions. Then check each graph with a suitable table of values.
- a** $y = |2x - 4|$ **b** $y = |3x + 3|$ **c** $y = |\frac{1}{2}x - 1|$
- 15** [Technology] Use suitable graphing software to help solve these problems.
- a i** Sketch $y = |x - 4|$ and $y = 1$ on the same set of axes, clearly showing the points of intersection.
ii Hence write down the solution of $|x - 4| = 1$.
- b** Now use similar graphical methods to solve each of the following.
- i** $|x + 3| = 1$ **ii** $|2x + 1| = 3$ **iii** $|3x - 3| = -2$ **iv** $|2x - 5| = 0$
- 16** Consider the absolute value function $f(x) = |x|$.
- a** Use the result $f(x) = \sqrt{x^2}$ given in Box 20 to help prove that the absolute value function is even.
b Why was this result obvious from the graph of $y = |x|$?

CHALLENGE

- 17 a** For what values of x is $y = \frac{|x|}{x}$ undefined?
b Use a table of values of x from -3 to 3 to sketch the graph.
c Hence write down the equations of the two branches of $y = \frac{|x|}{x}$.
- 18** Prove that $f(|x|)$ is an even function, for all functions $f(x)$.
- 19** Sketch each graph by drawing up a table of values for $-3 \leq x \leq 3$. Then use cases to determine the equation of each branch of the function.
- a** $y = |x| + x$ **b** $y = |x| - x$
c $y = 2(x + 1) - |x + 1|$ **d** $y = x^2 - |2x|$

5F Composite functions

Learning intentions

- Form composite functions, and work with them.
- Discover many composites that are transformations of one of the functions.

Shifting, reflecting, dilating, and taking absolute value, are all examples of a far more general procedure of creating a composite function from a chain of two given functions. The first example below shows how a translation left 3 of $y = x^2$, and a translation up 3 of $y = x^2$, are both obtained using composites. In this section, however, attention is on the algebra rather than on the final graph.

Composition of functions

Suppose that we have the two functions $f(x) = x + 3$ and $g(x) = x^2$. We can put them together in a *chain* by placing their function machines so that the output of the first function is the input of the second function

x		$f(x)$		$g(f(x))$
-2	→	→ 1	→	→ 1
-1	→	→ 2	→	→ 4
0	→	→ 3	→	→ 9
1	→	→ 4	→	→ 16
2	→	→ 5	→	→ 25
x	→	→ $x + 3$	→	→ $(x + 3)^2$

The middle column is the output of the first function 'Add 3'. This output is then the input of the second function 'Square'. The result is the *composite function*

$$g(f(x)) = (x + 3)^2 \quad \text{'Add 3, then square.'}$$

If the two functions are composed the other way around, the result is different:

$$f(g(x)) = x^2 + 3 \quad \text{'Square, then add 3.'}$$

Notice how in this example, both ways around are examples of translations:

- The composite graph $y = g(f(x))$ is $y = g(x)$ shifted left 3.
- The composite graph $y = f(g(x))$ is $y = g(x)$ shifted up 3.

The domains and ranges of these composites are easily read off their equations:

- For $g(f(x)) = (x + 3)^2$, the domain is all real x , and the range is $y \geq 0$.
- For $f(g(x)) = x^2 + 3$, the domain is all real x , and the range is $y \geq 3$.

22 Composition of functions

- A composite function $g(f(x))$ is formed by first applying the function f to x , and then applying the function g to the output $f(x)$.
- The composite function $f(g(x))$ is formed by first applying the function g to x , and then applying the function f to the output $g(x)$.

Care: A function is written on the left of the number it acts on. Thus the order in which the functions are *written* is the reverse of the order in which they are *applied*.

We often call these successive functions a *chain of functions*. Later, in calculus, we will develop a rule called *the chain rule* for handling composite functions.

This chapter has been concerned with transformations. Many composites are transformations of one of the functions, particularly when one function is linear, as the examples below and the following exercises will make clear — but this is to be observed, not learnt.



Example 21 Evaluating and simplifying composite functions

Let $f(x) = x^2$ and $g(x) = x + 3$.

a Evaluate:

i $g(f(5))$

ii $g(f(-2))$

iii $f(g(5))$

iv $f(g(-2))$

b i Find and simplify $g(f(x))$, and write down its domain and range.

ii Explain how $y = g(f(x))$ is a transformation of $y = f(x)$.

c Repeat part (b) for the other composite function $f(g(x))$.

Solution

a i $g(f(5)) = g(25)$
 $= 28$

ii $g(f(-2)) = g(4)$
 $= 7$

iii $f(g(5)) = f(8)$
 $= 64$

iv $f(g(-2)) = f(1)$
 $= 1$

b i $g(f(x)) = g(x^2) = x^2 + 3$. The domain is all real x , and the range is $y \geq 3$.

ii $g(f(x)) = x^2 + 3$ is the function $f(x) = x^2$ shifted up 3.

c i $f(g(x)) = f(x + 3) = (x + 3)^2$. The domain is all real x , and the range is $y \geq 0$.

ii $f(g(x)) = (x + 3)^2$ is the function $f(x) = x^2$ shifted left 3.



Example 22 Taking composites of an exponential function and a linear function

Let $f(x) = 2^x$ and $g(x) = 4x - 7$.

a Simplify $g(f(x))$, write down its asymptotes and its domain and range, and characterise it as transformation of $y = f(x)$.

b Repeat for $f(g(x))$.

Solution

a $g(f(x)) = 4 \times 2^x - 7$. The line $y = -7$ is a horizontal asymptote on the left.
The domain is all real x , and the range is $y > -7$.

The composite $g(f(x)) = 4 \times 2^x - 7$ is the function $f(x) = 2^x$ dilated vertically with factor 4 to give $y = 4 \times 2^x$, then shifted down 7

b $f(g(x)) = 2^{4x-7}$. The line $y = 0$ is a horizontal asymptote on the left.
The domain is all real x , and the range is $y > 0$.

The composite $f(g(x)) = 2^{4x-7}$ is the function $f(x) = 2^x$ shifted right 7 to give $y = 2^{x-7}$, then dilated horizontally with factor $\frac{1}{4}$.

Domain and range of a composite function

It is elaborate to formulate rules for the domain and range of a composite function. The best approach is to write down the formula for the composite function, and determine its domain and range in the usual ways.

See the examples above for this approach used in various situations.

The empty function — for clarification only

Let $f(x) = -x^2 - 1$ and $g(x) = \sqrt{x}$. Then

$$g(f(x)) = \sqrt{-x^2 - 1}.$$

We have a problem here, because $\sqrt{-x^2 - 1}$ is undefined, whatever the value of x . Thus $g(f(x))$ has domain the empty set, and its range is therefore also the empty set. This is *the empty function*, whose domain and range are both the empty set.

Exercise 5F

FOUNDATION

- 1 Often a pattern emerges when a function is composed with itself. Let $f(x) = x + 2$.
 - a Evaluate the following.
 - i $f(0)$
 - ii $f(f(0))$
 - iii $f(f(f(0)))$
 - b What number pattern is formed in part (a)?
 - c Now evaluate these.
 - i $f(1)$
 - ii $f(f(1))$
 - iii $f(f(f(1)))$
 - d What number pattern is formed in part (c)?
- 2 Once again, let $f(x) = x + 2$.
 - a Find the values of:
 - i $f(f(3))$
 - ii $f(f(-1))$
 - iii $f(f(-8))$
 - b Find expressions for:
 - i $f(f(x))$
 - ii $f(f(f(x)))$
 - c Find the value of x for which $f(f(x)) = 0$.
- 3 Consider the function $g(x) = 2 - x$.
 - a Find the values of $g(g(0))$, $g(g(4))$, $g(g(-2))$ and $g(g(-9))$.
 - b Show that $g(g(x)) = x$.
 - c Show that $g(g(g(x))) = g(x)$.
- 4 Consider the function $h(x) = 3x - 5$.
 - a Find the values of $h(h(0))$, $h(h(5))$, $h(h(-1))$ and $h(h(-5))$.
 - b Find expressions for $h(h(x))$ and for $h(h(h(x)))$.
- 5 When a function $f(x)$ is composed with a linear function $g(x)$ the result is a transformation of $f(x)$. Let $f(x) = x^2 - 4$ and $g(x) = x + 1$.
 - a Write down the domain and range of $y = f(x)$. A sketch may help.
 - b Find the values of:
 - i $f(g(3))$
 - ii $g(f(3))$
 - c Find expressions for:
 - i $f(g(x))$
 - ii $g(f(x))$
 - d
 - i What transformation maps the graph of $y = f(x)$ to the graph of $y = f(g(x))$?
 - ii Hence write down the domain and range of $y = f(g(x))$.
 - e
 - i What transformation maps the graph of $y = f(x)$ to the graph of $y = g(f(x))$?
 - ii Hence write down the domain and range of $y = g(f(x))$.

- 6** Let $p(x) = x(x - 2)$ and $d(x) = 2x$. When $p(x)$ is composed with $d(x)$ the result is a transformation since $d(x)$ is linear.
- Write down the domain and range of $y = p(x)$. A sketch may help.
 - Find the values of:
 - $p(d(1))$
 - $d(p(1))$
 - Find expressions for:
 - $p(d(x))$
 - $d(p(x))$
 - What transformation maps the graph of $y = p(x)$ to the graph of $y = p(d(x))$?
 - Hence write down the domain and range of $y = p(d(x))$.
 - What transformation maps the graph of $y = p(x)$ to the graph of $y = d(p(x))$?
 - Hence write down the domain and range of $y = d(p(x))$.
- 7** When $Q(x) = (x - 1)^2$ is composed with $R(x) = -x$ the result is a reflection.
- Write down the domain and range of $y = q(x)$. A sketch may help.
 - Find the values of:
 - $Q(R(2))$
 - $R(Q(2))$
 - Find expressions for:
 - $Q(R(x))$
 - $R(Q(x))$
 - What transformation maps the graph of $y = Q(x)$ to the graph of $y = Q(R(x))$?
 - Hence write down the domain and range of $y = Q(R(x))$.
 - What transformation maps the graph of $y = Q(x)$ to the graph of $y = R(Q(x))$?
 - Hence write down the domain and range of $y = R(Q(x))$.

DEVELOPMENT

- 8** Suppose that $F(x) = 4x$ and $G(x) = \sqrt{x}$.
- Find the values of $F(G(25))$, $G(F(25))$, $F(F(25))$ and $G(G(25))$.
 - Find $F(G(x))$.
 - Find $G(F(x))$.
 - Hence show that $F(G(x)) = 2G(F(x))$.
 - State the domain and range of $F(G(x))$.
- 9** Two functions f and h are defined by $f(x) = -x$ and $h(x) = \frac{1}{x}$.
- Find the values of $f(h(-\frac{1}{4}))$, $h(f(-\frac{1}{4}))$, $f(f(-\frac{1}{4}))$ and $h(h(-\frac{1}{4}))$.
 - Show that for all $x \neq 0$:
 - $f(h(x)) = h(f(x))$
 - $f(f(x)) = h(h(x))$
 - Write down the domain and range of $f(h(x))$.
 - Describe how the graph of $h(x)$ is transformed to obtain the graph of $h(f(x))$.
- 10** Suppose that $f(x) = -5 - |x|$ and $g(x) = \sqrt{x}$.
- Find $f(g(x))$, state its domain and range, and sketch its graph.
 - Explain why $g(f(x))$ is the empty function.

- 11** Let $H(x) = x - 2$, $R(x) = -x$ and $g(x) = \sqrt{4 + x}$.
- Draw the graph of $y = g(x)$.
 - Use the graph in part (a) to help answer the following.
 - Find an expression for $y = g(R(x))$ and interpret the result as a transformation.
 - Hence sketch $y = g(R(x))$ and state its domain and range.
 - Use your results in part (b) to help answer the following.
 - Find an expression for $y = H(g(R(x)))$ and interpret this as a transformation of $y = g(R(x))$.
 - Hence sketch $y = H(g(R(x)))$ and state its domain and range.
- 12** **a** Show that if $f(x)$ and $g(x)$ are odd functions then $g(f(x))$ is odd.
b Show that if $f(x)$ is an odd function and $g(x)$ is even then $g(f(x))$ is even.
c Show that if $f(x)$ is an even function then $g(f(x))$ is even, regardless of $g(x)$.
- 13** Find the composite functions $g(f(x))$ and $f(g(x))$.
- $f(x) = 4$, for all x , and $g(x) = 7$, for all x .
 - $f(x) = x$, $g(x)$ any function.

CHALLENGE

- 14** Let $f(x) = 2x + 3$ and $g(x) = 5x + b$, where b is a constant. Find the value of b so that $g(f(x)) = f(g(x))$, for all x .
- 15** Let $f(x) = 2x + 3$ and $g(x) = ax + b$, where b is a constant.
- Find expressions for $g(f(x))$ and $f(g(x))$.
 - Hence find the values of a and b so that $g(f(x)) = x$, for all x .
 - Show that if a and b have these values, then $f(g(x)) = x$, for all x .
- 16** **a** Let $f(x)$ be any function, and let $g(x) = x - a$, where a is a constant. Describe each composite graph as a transformation of the graph of $y = f(x)$.
- $y = g(f(x))$
 - $y = f(g(x))$
- b** Let $f(x)$ be any function, and let $g(x) = -x$. Describe each composite graph as a transformation of the graph of $y = f(x)$.
- $y = g(f(x))$
 - $y = f(g(x))$
- c** Let $f(x)$ be any function, and let $g(x) = kx$, where k is a constant. Describe each composite graph as a transformation of the graph of $y = f(x)$.
- $y = g(f(x))$
 - $y = f(g(x))$
- 17** In some cases there are connections between familiar functions via composite functions. Let $p(x) = 4 - x^2$ and $r(x) = \sqrt{x}$.
- Graph $y = p(x)$ and $y = r(x)$, writing down their domains and ranges.
 - Simplify $c(x) = r(p(x))$ and graph the result. What familiar curve is this? What is the domain and range of $c(x)$?
 - Simplify $a(x) = p(r(x))$ and graph the result. Explain the graph in terms of transformations of a familiar function. What is the domain and range of $a(x)$?

5G Combining transformations

Learning intentions

- Review the transformations, understanding reflections also as dilations with factor -1 .
- Examine where order does or doesn't matter when combining transformations.

We can apply transformations in succession to a graph. This immediately raises the question, 'Does it matter in which order the transformations are applied?'

A summary of transformations

First, here are the procedures for applying translations and dilations to a graph:

23 A summary of translations and dilations

Transformation	By replacement	By function rule
Shift horizontally right h	Replace x by $x - h$	$y = f(x) \longrightarrow y = f(x - h)$
Shift vertically up k	Replace y by $y - k$	$y = f(x) \longrightarrow y = f(x) + k$
Stretch horizontally factor a	Replace x by $\frac{x}{a}$	$y = f(x) \longrightarrow y = f\left(\frac{x}{a}\right)$
Stretch vertically factor b	Replace y by $\frac{y}{b}$	$y = f(x) \longrightarrow y = bf(x)$

Secondly, we can regard reflections as dilations with factor -1 , as we saw in Section 5D (Box 15). And rotating 180° about the origin is the combination of reflections in the x -axis and y -axis (in any order). This means that when we are examining whether order matters, we can consider just four transformations — horizontal and vertical translations, and horizontal and vertical dilations.

24 A summary of reflections and 180° rotations

Transformation	By replacement	By function rule
Reflect in the y -axis ▷ A reflection in the y -axis is a horizontal dilation with factor -1 .	Replace x by $-x$	$y = f(x) \longrightarrow y = f(-x)$
Reflect in the x -axis ▷ A reflection in the x -axis is a vertical dilation with factor -1 .	Replace y by $-y$	$y = f(x) \longrightarrow y = -f(x)$
Rotate 180° about O ▷ A rotation of 180° about the origin is the combination of reflections in the y -axis and x -axis (in any order). Hence it is also the combination of horizontal and vertical dilations each with factor -1 (in any order).	Replace x by $-x$, y by $-y$	$y = f(x) \longrightarrow y = -f(-x)$

The word 'commute'

The word 'commute' is not in the course, but it is a standard and very useful word that makes sentences shorter and easier to read. We will use it in this section.

25 Commuting transformations

Two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

Several results about commuting transformations were established in Section 5A (Box 3), Section 5B (Box 7) and Section 5D (Box 13). To summarise:

26 Commuting transformations

- Any two translations commute.
- Any two dilations commute (including reflections and rotating 180° about O).
- A translation and a dilation commute if one is vertical and the other horizontal.

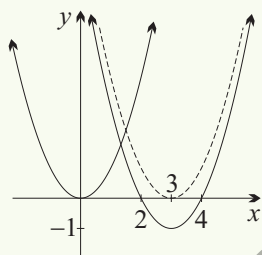
The following examples are intended to illustrate these three results:



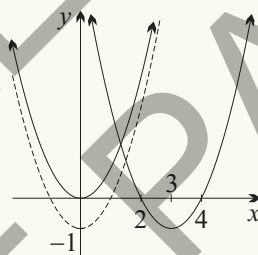
Example 23 Illustrating that two translations always commute

- a** On one set of axes, draw the three graphs to illustrate the parabola $y = x^2$ shifted right 3, then shifted down 1.
- b** On one set of axes, draw the three graphs to illustrate the parabola $y = x^2$ shifted down 1, then shifted right 3.

Solution



- a** The three graphs are $y = x^2$, $y = (x - 3)^2$, and $y = (x - 3)^2 - 1$.



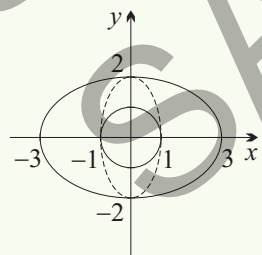
- b** The three graphs are $y = x^2$, $y = x^2 - 1$, and $y = (x - 3)^2 - 1$.



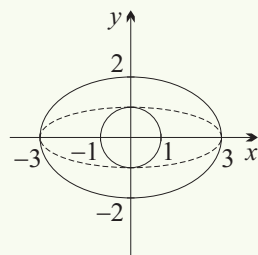
Example 24 Illustrating that two dilations always commute

- a** On one set of axes, draw the three graphs to illustrate the circle $x^2 + y^2 = 1$ stretched vertically with factor 2, then horizontally with factor 3.
- b** On one set of axes, draw the three graphs to illustrate the circle $x^2 + y^2 = 1$ stretched horizontally with factor 3, then vertically with factor 2.

Solution



- a** The three graphs are $x^2 + y^2 = 1$, $x^2 + \frac{y^2}{4} = 1$, and $\frac{x^2}{9} + \frac{y^2}{4} = 1$.



- b** The three graphs are $x^2 + y^2 = 1$, $\frac{x^2}{9} + y^2 = 1$, and $\frac{x^2}{9} + \frac{y^2}{4} = 1$.


Example 25 A horizontal dilation and a vertical translation commute

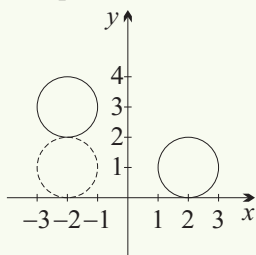
- a** Apply a reflection in the y -axis (that is, a horizontal dilation with factor -1), then a shift up 2, successively to the circle $(x - 2)^2 + (y - 1)^2 = 1$.
- b** Apply the transformations in the reverse order.

Solution

- a** The three graphs are

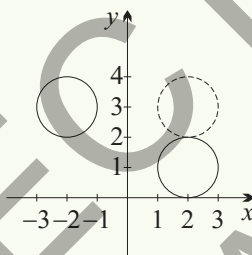
$$\begin{aligned}(x - 2)^2 + (y - 1)^2 &= 1, \\(x + 2)^2 + (y - 1)^2 &= 1, \\ \text{and } (x + 2)^2 + (y - 3)^2 &= 1.\end{aligned}$$

In both parts, note that $(-x - 2)^2 = (x + 2)^2$.



- b** The three graphs are

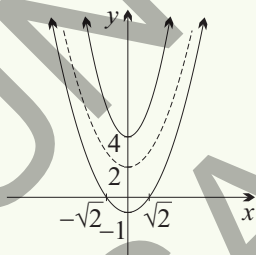
$$\begin{aligned}(x - 2)^2 + (y - 1)^2 &= 1, \\(x - 2)^2 + (y - 3)^2 &= 1, \\ \text{and } (x + 2)^2 + (y - 3)^2 &= 1.\end{aligned}$$


Transformations that do not commute

Here are some situations where the transformations do not commute. They demonstrate a general result that a translation and a dilation do not commute when they are both horizontal or both vertical.

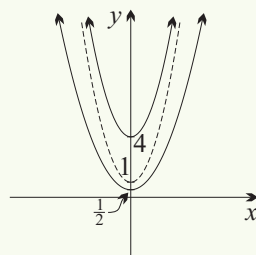

Example 26 A vertical dilation and a vertical translation don't commute

- a** Apply a vertical dilation with factor $\frac{1}{2}$, then a translation down 3, to the parabola $y = x^2 + 4$.
- b** Apply the transformations in the reverse order

Solution


- a** The three graphs are

$$\begin{aligned}y &= x^2 + 4, \\y &= \frac{1}{2}x^2 + 2, \\y &= \frac{1}{2}x^2 - 1.\end{aligned}$$



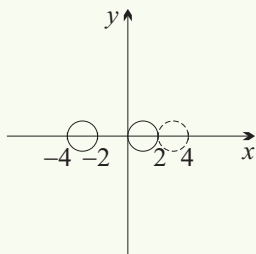
- b** The three graphs are

$$\begin{aligned}y &= x^2 + 4, \\y &= x^2 + 1, \\y &= \frac{1}{2}x^2 + \frac{1}{2}.\end{aligned}$$

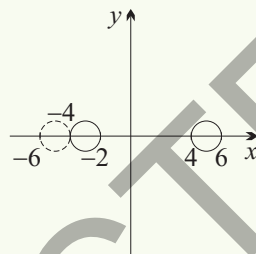

Example 27 A horizontal dilation and horizontal translation don't commute

- a** Apply a reflection in the y -axis (that is, a horizontal dilation with factor -1), then a translation left 2, to the circle $(x + 3)^2 + y^2 = 1$.
- b** Apply the transformations in the reverse order.

Solution



- a** The three graphs are
- $$(x + 3)^2 + y^2 = 1,$$
- $$(x - 3)^2 + y^2 = 1,$$
- $$(x - 1)^2 + y^2 = 1.$$
- Note that $(-x + 3)^2 = (x - 3)^2$.



- b** The three graphs are
- $$(x + 3)^2 + y^2 = 1,$$
- $$(x + 5)^2 + y^2 = 1,$$
- $$(x - 5)^2 + y^2 = 1.$$
- Note that $(-x + 5)^2 = (x - 5)^2$.

27 Transformations that do not commute

- A vertical translation and a vertical dilation do not commute.
- A horizontal translation and a horizontal dilation do not commute.

Exercise 5G
FOUNDATION

Note: The word 'commute' is not in the course, but it is standard and very useful, because it makes sentences shorter and easier to read. Two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

- Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.
 - The parabola is shifted right 1 unit. Sketch the situation and find its equation, expanding any brackets.
 - The parabola in part (i) is then shifted up 2 units. Sketch the new graph and find its equation.
 - The original parabola $y = x^2 - 2x$ is translated up 2 units. Sketch the result and find its equation.
 - The parabola in part (i) is then translated 1 unit right. Sketch the situation and find its equation, expanding any brackets.
 - Parts (a) and (b) used the same translations, 1 unit right and 2 units up, but in a different order. Do these transformations commute?
- As in Question 1, start with the parabola $y = x^2 - 2x$.
 - The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.
 - The parabola in part (i) is then dilated by factor 3 vertically. Sketch the new graph and find its equation.
 - The original parabola $y = x^2 - 2x$ is stretched by 3 vertically. Sketch the result and find its equation.
 - The parabola in part (i) is then stretched by 2 horizontally. Sketch the situation and find its equation.
 - Parts (a) and (b) used the same dilations, factor 2 horizontally and factor 3 vertically, but in a different order. Do these transformations commute?

- 3** Once again, start with the parabola $y = x^2 - 2x$.
- a i** The parabola is dilated by factor 2 horizontally. Sketch the situation and find its equation.
 - ii** The parabola in part (i) is then translated 1 unit up. Sketch the new graph and find its equation.
 - b i** The original parabola $y = x^2 - 2x$ is shifted 1 unit up. Sketch the result and find its equation.
 - ii** The parabola in part (i) is then stretched by 2 horizontally. Sketch the situation and find its equation.
 - c** Parts (a) and (b) used the same transformations, stretch by factor 2 horizontally and shift 1 unit up, but in a different order. Do these transformations commute?
- 4** Let $y = x^2 - 2x$. Sketch the graph of this function showing the intercepts and vertex.
- a i** The parabola is shifted right by 1 unit. Sketch the situation and find its equation, expanding any brackets.
 - ii** The shifted parabola is then reflected in the y -axis. Sketch the new graph and find its equation.
 - b i** The original parabola $y = x^2 - 2x$ is reflected in the y -axis. Sketch the result and find its equation.
 - ii** The reflected parabola is then shifted 1 unit right. Sketch the situation and find its equation, expanding any brackets.
 - c** Parts (a) and (b) used a shift 1 unit right and reflection in the y -axis, but in a different order. Do these transformations commute?

DEVELOPMENT

- 5** Which of the following pairs of transformations commute?
- a** reflection in the y -axis and horizontal translation
 - b** vertical dilation and vertical translation
 - c** vertical dilation and reflection in the x -axis
 - d** horizontal translation and vertical translation
 - e** horizontal dilation and horizontal translation
 - f** reflection in the x -axis and horizontal translation
- 6** Write down the new equation for each function or relation after the given transformations have been applied. Draw a graph of the image.
- a** $y = x^2$: right 2 units,
then dilate by factor $\frac{1}{2}$ horizontally
 - b** $y = 2^x$: down 1 unit
then reflect in the y -axis
 - c** $y = x^2 - 1$: down 3 units,
then dilate by factor -1 vertically
 - d** $y = \frac{1}{x}$: right 3 units
then dilate by factor 2 vertically
 - e** $x^2 + y^2 = 4$: up 2 units
then dilate by factor $\frac{1}{2}$ vertically
 - f** $y = \sqrt{x}$: up 2 units
then dilate by factor -1 horizontally

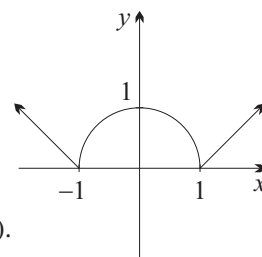
- 7** The table shows pairs of transformations with V for vertical, H for horizontal, T for translation and D for dilation.

	HT	VT	HD	VD
HT	C			
VT		C		
HD	N		C	
VD			C	C

- a** Copy the table and complete the entries below the diagonal. Use C if the transformations commute and N if they do not.
- b** The table does not directly include reflections. How can the table be used to decide whether a reflection commutes with another transformation?
- c** Check that the table agrees with your answers to Question 5.
- 8** Determine the equation of the curve after the given transformations have been applied in the order stated.
- a** $y = x^2$: left 1, down 4, dilate horizontally by 2,
- b** $y = x^2$: down 4, dilate horizontally by 2, left 1,
- c** $y = 2^x$: down 1, right 1, dilate vertically by -2 ,
- d** $y = \frac{1}{x}$: right 2, dilate by 2 vertically, up 1.

CHALLENGE

- 9** The parabola $y = (x - 1)^2$ is shifted 2 left and then reflected in the y -axis.
- a** Show that the new parabola has the same equation.
- b** Investigate why this has happened.
- 10** Let $f(x)$ be any function and let $\ell(x) = ax + b$, a linear function. The graph of $y = f(\ell(x)) = f(ax + b)$ can be obtained by two transformations of $y = f(x)$. This was investigated briefly in Questions 5, 6 and 7 of the previous exercise. Here the situation is investigated further. Although $f(x)$ can be any function, the one graphed on the right is used for parts (a) to (c).



- a**
- i** Let $g(x) = f(x + 1)$. Describe the transformation of $f(x)$ and so graph $y = g(x)$.
 - ii** Let $h(x) = g(2x)$. Describe the transformation of $g(x)$ and so graph $y = h(x)$.
 - iii** Hence describe $h(x)$ as a transformation of $f(x)$, then write $h(x)$ in terms of the function f .
- b**
- i** Let $p(x) = f(2x)$. Describe the transformation of $f(x)$ and so graph $y = p(x)$.
 - ii** Let $q(x) = p(x + \frac{1}{2})$. Describe the transformation of $p(x)$ and so graph $y = q(x)$.
 - iii** Hence describe $q(x)$ as a transformation of $f(x)$, then write $q(x)$ in terms of the function f .
- c** The functions $h(x)$ and $q(x)$ are identical. In parts (a) and (b), the horizontal stretches of f are the same but the shifts left differ. Why?
- d** Describe $y = f(ax + b)$ as transformations of $y = f(x)$ in two different ways.
- e** Now complete the situation by describing $y = \ell(f(x)) = af(x) + b$ as transformations of $y = f(x)$ in two different ways.

5H Continuity and piecewise-defined functions

Learning intentions

- Understand and apply the informal definition of continuity at a point.
- Work with piecewise-defined functions and their possible discontinuities.

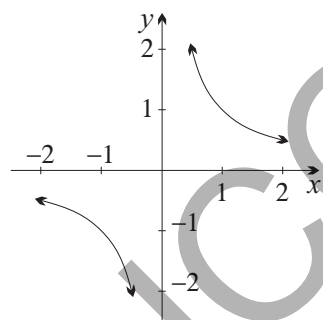
A curve such as $y = 1/x$, sketched below, has a break in the curve at its asymptote $x = 0$, and there are many other such curves in the course, particularly when we deal with trigonometry. We need a workable definition of *continuity at a point*, and we also need to test the continuity of functions defined piecewise using cases.

Continuity at a point

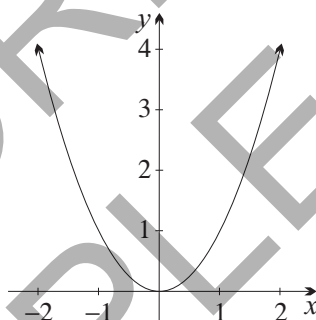
Continuity at a point means that there is no break in the curve at that point. This is an informal definition, but it will be quite enough for our purposes.

28 Continuity at a point — informal definition

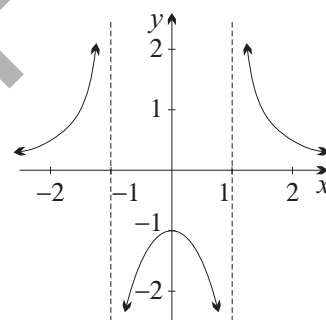
- A function $f(x)$ is called *continuous at $x = a$* if its graph can be drawn *through the point where $x = a$, on both sides*, without lifting the pencil off the paper.
 - If there is a break in the curve on either side of $x = a$, the curve is *not continuous* (or is *discontinuous*) at $x = a$.
- ▷ We say then that there is a *discontinuity* at $x = a$.



$y = \frac{1}{x}$ has a discontinuity at $x = 0$, and is continuous for all other values of x .



$y = x^2$ is continuous for all values of x .

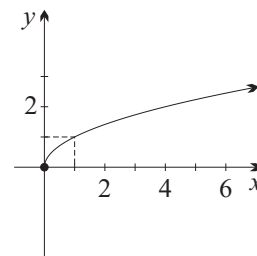


$y = \frac{1}{x^2 - 1}$ has discontinuities at $x = 1$ and at $x = -1$, and is continuous for all other values of x .

A function is never continuous at an endpoint of its domain

For a function $f(x)$ be continuous at $x = a$, we must be able to *draw the function through the point, on both sides*.

For example, the function $f(x) = \sqrt{x}$ has domain $x \geq 0$, but is not continuous at its endpoint at $x = 0$.



An assumption of continuity

It is intuitively obvious that a function such as $y = x^2$ is continuous for every value of x . Without further formality, we will make a general assumption of continuity, loosely stated as follows:

29 Assumption about continuity at a point

The functions in this course are continuous for every value of x in their domain, except where there is an obvious problem.

Piecewise-defined functions

When discussing $y = |x|$ in Section 5E, we introduced the term *piecewise-defined function* — often referred to as a *function defined using cases* — to mean a function defined differently in different parts of its domain, for example:

$$f(x) = \begin{cases} 4 - x^2, & \text{for } x \leq 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$$

A piecewise definition naturally divides the graph into *pieces*, also called *branches*. The two branches here are a parabola and line, and our first question is, ‘Do these two pieces join up, like $y = |x|$, or leave a gap?’. The most obvious approach is to construct two tables of values, one for each line of the function’s definition.



Example 28 Do the branches of a piecewise-defined function join up?

Let $f(x) = \begin{cases} 4 - x^2, & \text{for } x \leq 0, \\ 4 + x, & \text{for } x > 0. \end{cases}$

- a** Find $f(0)$, then construct tables of values of $4 - x^2$ on the left of $x = 0$, and of $4 + x$ on the right, to establish whether the graph is continuous at $x = 0$.
- b** Sketch the graph, and hence write down its domain and range.

Solution

- a** First, $f(0) = 4 - 0^2 = 4$ (using the top line of the definition).

The values of $4 - x^2$ on the left of $x = 0$, and of $4 + x$ on the right of $x = 0$, are

x	-2	-1	0
$4 - x^2$	0	3	4

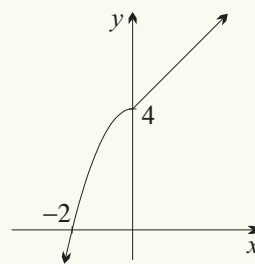
x	0	1	2
$4 + x$	4	5	6

Hence the curve is *continuous* at $x = 0$ (where $f(0) = 4$).

Reason: When $x = 0$, the values of $4 - x^2$ and $4 + x$ are both 4, which is the same as $f(0)$, so the two branches join up.

- b** The graph is made up of the parabolic piece on the left, and the linear piece on the right.

The domain is all real x , and the range is all real y .



Testing whether piecewise-defined functions join up

A straightforward way of testing this should now be clear:

30 Testing whether the branches of a piecewise-defined functions join up

Suppose that the definition of $f(x)$ changes at $x = a$.

- Find $f(a)$.
- Substitute $x = a$ into the formula on the left.
- Substitute $x = a$ into the formula on the right.

These three values must all exist, and all agree, for $f(x)$ to be continuous at $x = a$.

This method is quick, but the tables of values above may be useful when sketching.



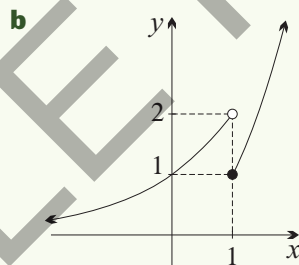
Example 29 Testing for continuity where the definition changes

$$\text{Let } f(x) = \begin{cases} 2^x, & \text{for } x < 1, \\ 1, & \text{for } x = 1, \\ x^2, & \text{for } x > 1. \end{cases}$$

- Establish whether the graph is continuous at $x = 1$.
- Sketch the graph.
- Hence write down its domain and range, and any asymptotes.

Solution

- First, $f(x) = 1$.
Secondly, $2^x = 2$, when $x = 1$.
Thirdly, $x^2 = 1$, when $x = 1$.
Hence the function has a *discontinuity* at $x = 1$.



- The domain is all real x , and the range is $y > 0$, and the x -axis is an asymptote on the left.

Note: When $x = 1$, the value of $f(x)$ is 1, so the endpoint $(1, 1)$ of the right-hand branch is included in the graph, as indicated by the closed circle. But the endpoint $(1, 2)$ of the left-hand branch is not included in the graph, as indicated by the open circle there.

31 Open and closed circles, and arrows

Use the same conventions for endpoints of curves as for the intervals in Section 2A.

- A *closed circle* marks an endpoint that is included in the graph.
- An *open circle* marks an endpoint that is not included in the graph.
- An *arrow* marks a curve that continues forever.

Note: The test given in Box 30 does not cover some rather subtle cases — for example, in worked Example 29, replace the formula in the third line of the function definition by $\frac{x^2(x-1)}{x-1}$. But in the absence of a definition involving limits, it is the best that we can do.

Testing piecewise-defined functions for evenness and oddness

- It is best to draw the graph. Then it is usually clear whether the function is even or odd or neither. For example, the two graphs in worked Examples 27 and 28 clearly have neither line symmetry in the y -axis, nor point symmetry in the origin — if required, a single counter-example can settle the matter:
 - In worked Example 27: $f(-2) = 0$ and $f(2) = 6$.
 - In worked Example 28: $f(-1) = \frac{1}{2}$ and $f(1) = 1$.
- With no graph, when the intervals where the pieces are defined are not balanced around $x = 0$, it is usually easy to construct a counter-example.
- With no graph, and with the intervals balanced around $x = 0$, then the usual test for oddness and evenness can be applied, as in the next worked example.



Example 30 Testing piecewise-defined functions for evenness and oddness

Test these piecewise-defined functions for evenness and oddness:

$$\mathbf{a} \quad f(x) = \begin{cases} x - \frac{5}{x}, & \text{for } x > 0, \\ \frac{5}{x} - x, & \text{for } x < 0. \end{cases}$$

$$\mathbf{b} \quad g(x) = \begin{cases} 2 - 3x^2, & \text{for } x \geq 0, \\ 3x^2 - 2, & \text{for } x < 0. \end{cases}$$

Solution

a Suppose $x > 0$. Then $-x$ is negative,

$$\begin{aligned} \text{so } f(-x) &= \frac{5}{-x} - (-x) \\ &= x - \frac{5}{x} \\ &= f(x), \text{ because } x > 0. \end{aligned}$$

Hence $f(-x) = f(x)$ for all non-zero values of x , so the function $f(x)$ is even.

b Suppose $x \geq 0$. Then $-x \leq 0$,

$$\begin{aligned} \text{so } g(-x) &= 3(-x)^2 - 2 \\ &= 3x^2 - 2 \\ &= -g(x), \text{ because } x \geq 0. \end{aligned}$$

This proves that $g(-x) = -g(x)$ for all values of x , so $g(x)$ is odd.

Suppose $x < 0$. Then $-x$ is positive,

$$\begin{aligned} \text{so } f(-x) &= -x - \frac{5}{-x} \\ &= \frac{5}{x} - x \\ &= f(x), \text{ because } x < 0. \end{aligned}$$

Suppose $x < 0$. Then $-x > 0$,

$$\begin{aligned} \text{so } g(-x) &= 2 - 3(-x)^2 \\ &= 2 - 3x^2 \\ &= -g(x), \text{ because } x < 0. \end{aligned}$$

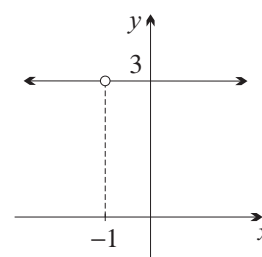
Functions in which numerator and denominator have the same zero

The function $f(x) = \frac{3x+3}{x+1}$ is tricky, because $x = -1$ is a zero of the numerator and of the denominator. The function is therefore not defined at $x = -1$, so it is not continuous there. But what does it look like elsewhere?

In this case, we can factor top and bottom. The factors will cancel everywhere except at the common zero:

$$\begin{aligned} f(x) &= \frac{3(x+1)}{x+1} \\ &= 3, \text{ provided that } x \neq -1. \end{aligned}$$

The graph is drawn to the right. It is the horizontal line $y = 3$, with the single point $(-1, 3)$ removed. Its domain is $x \neq -1$, and its range is $y = 3$.

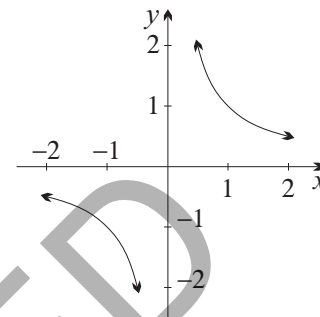


Continuous functions — for clarification only

A function $f(x)$ is called *continuous* if it is continuous at every point in its domain. This concept, however, is rather unhelpful for our purposes.

For example, the function $y = 1/x$ is continuous at every value of x except $x = 0$, which is outside its domain. Hence $y = 1/x$ is a continuous function with a discontinuity at $x = 0$!!!

The emphasis in this course will thus be on ‘continuity at a point’ rather than on ‘continuous function’.



Exercise 5H

FOUNDATION

- 1 State the zeroes and discontinuities of each function.

a $f(x) = \frac{5}{6-x}$

b $f(x) = \frac{3x}{(x-1)(x-3)(x-5)}$

c $f(x) = \frac{x(x+1)}{(x+2)(x+3)}$

2 Let $f(x) = \begin{cases} 1-x, & \text{for } x < 0, \\ 1+x^2, & \text{for } x \geq 0. \end{cases}$

- a** Find $f(0)$. Then evaluate the left branch $(1-x)$ and right branch $(1+x^2)$ at $x = 0$.

- b** Is $f(x)$ continuous at $x = 0$?

- c** Sketch the graph, and write down its domain and range.

3 Let $g(x) = \begin{cases} 2-x, & \text{for } x \leq 1, \\ x-1, & \text{for } x > 1. \end{cases}$

- a** Find $g(1)$. Then evaluate the left branch $(2-x)$ and right branch $(x-1)$ at $x = 1$.

- b** Is the function continuous at $x = 1$?

- c** Sketch the graph, and write down its domain and range.

- 4 Follow similar steps to Questions 2 and 3 in order to decide whether or not each function is continuous at $x = 0$. Then use the graph to decide whether the function is odd, even or neither.

a $f(x) = \begin{cases} x^2 + 2x, & \text{for } x < 0 \\ x^2 - 2x, & \text{for } x \geq 0 \end{cases}$

b $f(x) = \begin{cases} -1 - \sqrt{-x}, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 + \sqrt{x}, & \text{for } x > 0 \end{cases}$

c $f(x) = \begin{cases} \frac{1}{2} - x, & \text{for } x < 0 \\ x - \frac{1}{2}, & \text{for } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2 + 2x, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 2x - x^2, & \text{for } x > 0 \end{cases}$

DEVELOPMENT

- 5 Factor numerator and denominator, and hence write down the zeroes and discontinuities.

a $f(x) = \frac{1}{x^2 - 5x}$

b $f(x) = \frac{x}{x^2 - 5x + 6}$

c $f(x) = \frac{x^2 - 16}{x^2 - 9}$

- 6 Draw up a table of values for $y = \frac{|x|}{x}$, and explain whether the function is continuous at $x = 0$. Sketch the curve, and write down its domain and range.

- 7** Find whether each function is continuous at $x = 2$. Then sketch the curve and state the domain and range.

$$\mathbf{a} \quad f(x) = \begin{cases} x^3, & \text{for } x \leq 2, \\ 10 - x, & \text{for } x > 2. \end{cases} \quad \mathbf{b} \quad f(x) = \begin{cases} 3^x, & \text{for } x < 2, \\ 13 - x^2, & \text{for } x > 2, \\ 4, & \text{for } x = 2. \end{cases}$$

$$\mathbf{c} \quad f(x) = \begin{cases} \frac{1}{x}, & \text{for } 0 < x < 2, \\ 1 - \frac{1}{4}x, & \text{for } x > 2, \\ \frac{1}{2}, & \text{for } x = 2. \end{cases} \quad \mathbf{d} \quad f(x) = \begin{cases} x, & \text{for } x < 2, \\ 2 - x, & \text{for } x > 2, \\ 2, & \text{for } x = 2. \end{cases}$$

8 Consider the function $f(x) = \begin{cases} x + 2, & \text{for } x < -1 \\ -x, & \text{for } -1 \leq x \leq 1 \\ x - 2, & \text{for } x > 1 \end{cases}$

- a** Show that $f(x)$ is continuous at $x = -1$ and $x = 1$.
b Sketch $y = f(x)$.
c Simplify $f(-x)$ and hence show that $f(x)$ is an odd function.

9 Consider the function $f(x) = \begin{cases} -2x - 1, & \text{for } x < -1 \\ x^2, & \text{for } -1 \leq x \leq 1 \\ 2x - 1, & \text{for } x > 1 \end{cases}$

- a** Show that $f(x)$ is continuous at $x = -1$ and $x = 1$.
b Sketch $y = f(x)$.
c Simplify $f(-x)$ and hence show that $f(x)$ is an even function.

- 10** In Question 6 and 7 of Exercise 5C you discovered that a polynomial with only odd powers is an odd function, and a polynomial with only even powers is an even function. These conclusions may not be valid for piecewise functions. Follow similar procedures to Questions 8 and 9 above to determine whether these functions are odd, even or neither. Graph each function to confirm your answer.

$$\mathbf{a} \quad f(x) = \begin{cases} x - x^3, & \text{for } x < 0 \\ x^3 - x, & \text{for } x \geq 0 \end{cases} \quad \mathbf{b} \quad g(x) = \begin{cases} x^2 + 1, & \text{for } x < 0 \\ x^2 - 1, & \text{for } x \geq 0 \end{cases}$$

- 11** Cancel the algebraic fraction in each function, noting first the values of x for which the function is undefined. Then sketch the curve and state its domain and range:

$$\mathbf{a} \quad y = \frac{x^2 + 2x + 1}{x + 1} \quad \mathbf{b} \quad y = \frac{x^4 - x^2}{x^2 - 1} \quad \mathbf{c} \quad y = \frac{x - 3}{x^2 - 4x + 3} \quad \mathbf{d} \quad y = \frac{3x + 3}{x + 1}$$

- 12** In each case find the value of a that makes the function continuous.

$$\mathbf{a} \quad f(x) = \begin{cases} ax^2, & \text{for } x \leq 1, \\ 6 - x, & \text{for } x > 1. \end{cases} \quad \mathbf{b} \quad g(x) = \begin{cases} \frac{a(x^2 - 9)}{x + 3}, & \text{for } x \neq -3, \\ 12, & \text{for } x = -3. \end{cases}$$

CHALLENGE

- 13 a** Show that the function $h(x)$ below is discontinuous at $x = 1$.

$$h(x) = \begin{cases} 2^x, & \text{for } x < 1 \\ 4, & \text{for } x = 1 \\ 3 - x, & \text{for } x > 1 \end{cases}$$

- b** What simple change to $h(x)$ would make it continuous.

- 14 a** Suppose that $f(x)$ is discontinuous at $x = a$, but continuous at $x = -a$. Explain why $f(x)$ can be neither even nor odd.

- b** Use your answer to decide which of the functions in Question 11 are neither even nor odd.

Chapter 5 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 5 Multiple-choice quiz

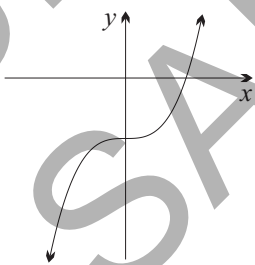
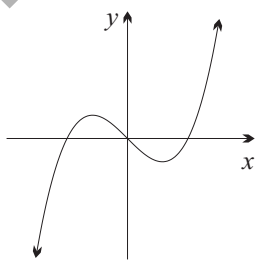
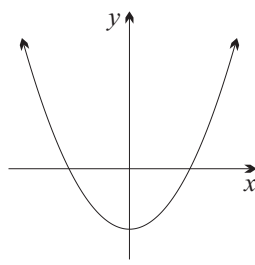
- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



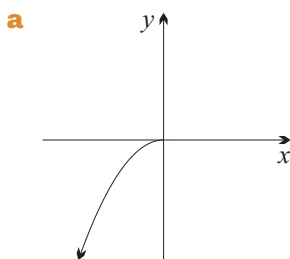
- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

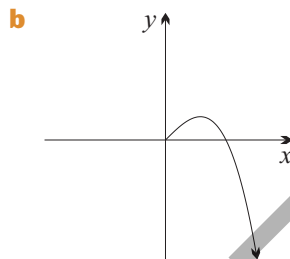
- Evaluate:
 - $|4|$
 - $|3 - 8|$
 - $|-2 - (-5)|$
 - $|-2| - |-5|$
- Solve for x :
 - $|3x| = 18$
 - $|x - 2| = 4$
 - $|2x - 3| = 5$
 - $|3x - 4| = 7$
- Explain how to shift the graph of $y = x^2$ to obtain each function.
 - $y = x^2 + 5$
 - $y = x^2 - 1$
 - $y = (x - 3)^2$
 - $y = (x + 4)^2 + 7$
- Write down the centre and radius of each circle. Shifting may help locate the centre.
 - $x^2 + y^2 = 1$
 - $(x + 1)^2 + y^2 = 4$
 - $(x - 2)^2 + (y + 3)^2 = 5$
 - $x^2 + (y - 4)^2 = 64$
- In each case, find the function obtained by the given reflection or rotation.
 - $y = x^3 - 2x + 1$: reflect in the y -axis
 - $y = x^2 - 3x - 4$: reflect in the x -axis
 - $y = 2^x - x$: rotate 180° about the origin
 - $y = \sqrt{9 - x^2}$: reflect in the y -axis
- Classify each function $y = f(x)$ as odd, even or neither.
 - 
 - 
 - 

7 In each diagram below, complete the graph so that:

i $f(x)$ is odd,



ii $f(x)$ is even.



8 Sketch each graph by shifting $y = |x|$, or by using a table of values.

a $y = |x| - 2$

b $y = |x - 2|$

c $y = |x + 2|$

d $y = |x| + 2$

9 Sketch each graph, showing the x -intercept and y -intercept.

a $y = |3x + 9|$

b $y = -|2x - 8|$

c $y = |4x + 13|$

10 Solve these absolute value equations:

a $|3x| = 15$

b $|x + 4| = -5$

c $|2x + 7| = 9$

d $|7x + 2| = 0$

11 Find $f(-x)$ for each function, and then decide whether the function is odd, even or neither.

a $f(x) = x + 3$

b $f(x) = 2x^2 - 5$

c $f(x) = \frac{1}{x}$

d $f(x) = \frac{x}{x^2 + 1}$

12 Complete the square and so use shifting to help sketch the graph of each quadratic function. Indicate the vertex and all intercepts with the axes.

a $y = x^2 + 2x + 3$

b $y = x^2 - 4x + 1$

c $y = 2 + 2x - x^2$

d $y = x^2 - x - 1$

13 Complete the squares to find the centre and radius of each circle.

a $x^2 + y^2 - 2y = 3$

b $x^2 + 6x + y^2 + 8 = 0$

c $x^2 - 4x + y^2 + 6y - 3 = 0$

d $x^2 + y^2 - 8x + 14y = 35$

14 Write down the equation for each function after the given translations have been applied.

a $y = x^2$: right 2 units, up 1 unit

b $y = \frac{1}{x}$: left 2 units, down 3 units

15 Given that $f(x) = 5x - 2$ and $g(x) = x^2 + 3$, find:

a $f(g(0))$

b $g(f(0))$

c $f(g(4))$

d $g(f(4))$

e $f(g(a))$

f $g(f(a))$

16 Find the domain and range of the composite functions $f(g(x))$ and $g(f(x))$ given that:

a $f(x) = x - 1$ and $g(x) = \sqrt{x}$

b $f(x) = \frac{1}{x}$ and $g(x) = x^2 + 1$

17 The graph drawn to the right shows the curve $y = f(x)$. Use this graph to sketch the following.

a $y = f(x + 1)$

b $y = f(x) + 1$

c $y = f(x - 1)$

d $y = f(x) - 1$

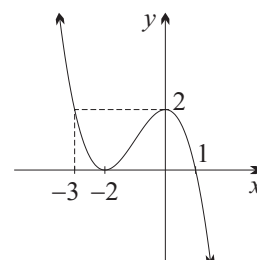
e $y = f(-x)$

f $y = -f(x)$

g $y = -f(-x)$

h $y = 2f(x)$

i $y = f(\frac{1}{2}x)$



18 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

a $y = x^2 - 2x$ and $y = \frac{1}{4}x^2 - x$

b $y = \frac{1}{x-4}$ and $y = \frac{1}{2x-8}$

- 19** [A revision medley of curve sketches] Sketch each set of graphs on a single pair of axes, showing all significant points. Use transformations, tables of values, or any other convenient method.

a	$y = 2x,$	$y = 2x + 3,$	$y = 2x - 2$
b	$y = -\frac{1}{2}x,$	$y = -\frac{1}{2}x + 1,$	$y = -\frac{1}{2}x - 2$
c	$y = x + 3,$	$y = 3 - x,$	$y = -x - 3$
d	$y = (x - 2)^2 - 1,$	$y = (x + 2)^2 - 1,$	$y = -(x + 2)^2 + 1$
e	$y = x^2,$	$y = (x + 2)^2,$	$y = (x - 1)^2$
f	$(x - 1)^2 + y^2 = 1,$	$(x + 1)^2 + y^2 = 1,$	$x^2 + (y - 1)^2 = 1$
g	$y = x^2 - 1,$	$y = 1 - x^2,$	$y = 4 - x^2$
h	$y = (x + 2)^2,$	$y = (x + 2)^2 - 4,$	$y = (x + 2)^2 + 1$
i	$y = - x ,$	$y = - x + 1,$	$y = - x - 2 $
j	$y = \sqrt{x},$	$y = \sqrt{x} + 1,$	$y = \sqrt{x + 1}$
k	$y = 2^x,$	$y = 2^x - 1,$	$y = 2^{x-1}$
l	$y = \frac{1}{x},$	$y = \frac{1}{x-2},$	$y = \frac{1}{x+1}$
m	$y = x^3,$	$y = x^3 - 1,$	$y = (x - 1)^3$
n	$y = x^4,$	$y = (x - 1)^4,$	$y = x^4 + 1$
o	$y = \sqrt{x},$	$y = -\sqrt{x},$	$y = 2 - \sqrt{x}$
p	$y = 2^{-x},$	$y = 2^{-x} - 2,$	$y = -2^x + 2$

$$\mathbf{b} \quad y = -\frac{1}{2}x, \quad y = -\frac{1}{2}x + 1, \quad y = -\frac{1}{2}x - 2$$

C $y = x + 3$, $y = 3 - x$, $y = -x - 3$

d $y = (x - 2)^2 - 1$, $y = (x + 2)^2 - 1$, $y = -(x + 2)^2 + 1$

e $y = x^2$, $y = (x + 2)^2$, $y = (x - 1)^2$

f $(x-1)^2 + y^2 = 1, \quad (x+1)^2 + y^2 = 1, \quad x^2 + (y-1)^2 = 1$

$$\text{b) } y = x^2 - 1, \quad y = 1 - x^2, \quad y = 4 - x^2$$

h

$y = (x + 2)^2,$

$y = (x + 2)^2 - 4,$

$y = (x + 2)^2 + 1$

i $y = -|x|$, $y = -|x| + 1$, $y = -|x - 2|$

j $y = \sqrt{x}$, $y = \sqrt{x} + 1$, $y = \sqrt{x+1}$

$$\mathbf{k} \quad y = 2^x, \quad y = 2^x - 1, \quad y = 2^{x-1}$$

$$y = \frac{1}{x}, \quad y = \frac{1}{x-2}, \quad y = \frac{1}{x+1}$$

$$\text{m} \quad y = x^3, \quad y = x^3 - 1, \quad y = (x - 1)^3$$

n $y = x^4$, $y = (x-1)^4$, $y = x^4 + 1$

$$\bullet \quad y = \sqrt{x}, \quad y = -\sqrt{x}, \quad y = 2 - \sqrt{x}$$

P $y = 2^{-x}$, $y = 2^{-x} - 2$, $y = -2^x + 2$

Note: In the next two questions, two transformations are said to *commute* if the order in which they are applied does not matter, whatever graph they are applied to.

- 20** Which of the following pairs of transformations commute?

- a** reflection in the y -axis and reflection in the x -axis,
- b** vertical reflection and vertical translation,
- c** horizontal translation and horizontal dilation,
- d** vertical translation and horizontal dilation.

- 21** Identify the various transformations of the standard functions and hence graph each. Make sure the transformations are applied in the correct order when they do not commute.

a $y = 4 - x^2$ **b** $y = \frac{1}{2}(x-2)^2 - 1$

- 22** Let $f(x) = \begin{cases} x^2, & \text{for } x \leq 0, \\ x^2 + 1, & \text{for } x > 0. \end{cases}$

- a** Is the curve continuous at $x = 0$?
b Write down the domain and range of the function.

- 23** Let $f(x) = \begin{cases} x^2 - 4, & \text{for } x < 2, \\ x - 2, & \text{for } x \geq 2. \end{cases}$

- a** Is the curve continuous at $x = 2$?
- b** What are the domain and range of the function?

6

Trigonometry

Chapter introduction

Trigonometry is important in modern science principally because the graphs of the sine and cosine functions are waves. Waves appear everywhere in the natural world, for example as water waves, as sound waves, or as the electromagnetic waves that are responsible for radio, heat, light, ultraviolet radiation, X-rays and gamma rays. In quantum mechanics, a wave is associated with every particle.

Trigonometry began in classical times, however, in practical situations such as building, surveying, navigation and astronomy. It used the relationships between the angles and the side lengths in a triangle, and its name comes from the Greek words *trigonon*, ‘triangle’, and *metron*, ‘measure’. This chapter develops the trigonometric functions and their graphs from the geometry of triangles and circles, and applies the trigonometric functions in practical problems.

Some of this chapter will be new to most readers, in particular the extension of the trigonometric functions to angles of any magnitude, the graphs of these functions, and trigonometric identities and equations.

The graphs of $y = \operatorname{cosec} x$, $y = \sec x$ and $y = \cot x$ are not required in this course, but their algebraic properties are required. So that the story is complete, we have included their graphs in Section 6C, but without their constructions. Then in the Challenge section of Exercise 6C, the standard properties of these graphs — domain, range, oddness or evenness, periodicity, asymptotes — are to be read off the graph. Reading such properties off a given graph is, of course, one of the standard tasks in the course.

Radian measure is essential later for the calculus of the trigonometric functions. This chapter, however, is already long, and we have placed radian measure a little later in Chapter 11, after differentiation, when its relevance can be explained.

6A Trigonometry with right-angled triangles

Learning intentions

- Understand the role of Pythagoras' theorem and similarity in trigonometry.
- Know the definitions of the six trigonometric functions for acute angles.
- Evaluate the trigonometric functions of the special angles 30° , 45° and 60° .
- Find sides and angles of a right-angled triangle, given sufficient information.

This section and the next will review the definitions of the trigonometric functions for acute angles, and apply them to problems involving right-angled triangles.

Pythagoras' theorem

You will know from earlier years that the trigonometry of triangles begins with two fundamental ideas — *Pythagoras' theorem* and *similarity-congruence*.

Pythagoras' theorem tells us how to find the third side of a right-angled triangle. The theorem is the best-known theorem in all of mathematics, and has been mentioned several times already in earlier chapters. Here is what it says:

The square on the hypotenuse of a right-angled triangle is the sum of the squares on the other two sides.

The diagrams below provide a very simple proof. Can you work out how the four shaded triangles have been pushed around inside the square to prove the theorem?



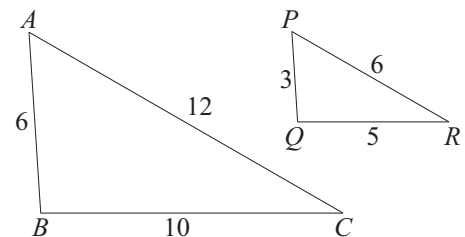
Similarity and congruence

Similarity is required even to define the trigonometric functions, because each function is defined as the ratio of two sides of a triangle.

- Two figures are called *congruent* if one can be obtained from the other by translations, rotations and reflections.
- They are called *similar* if enlargements are allowed as well.

In two similar figures:

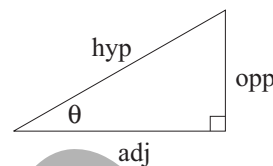
matching angles are equal and matching sides are in ratio.



The trigonometric functions for acute angles

Let θ be any acute angle, that is, $0^\circ < \theta < 90^\circ$. Construct a right-angled triangle with an acute angle θ , and label the sides:

- hyp — the *hypotenuse*, the side opposite the right angle,
- opp — the side *opposite* the angle θ ,
- adj — the third side, *adjacent* to θ but not the hypotenuse.



1 The trigonometric functions for an acute angle θ

$$\begin{array}{lll} \sin \theta = \frac{\text{opp}}{\text{hyp}} & \cos \theta = \frac{\text{adj}}{\text{hyp}} & \tan \theta = \frac{\text{opp}}{\text{adj}} \\ \text{cosec } \theta = \frac{\text{hyp}}{\text{opp}} & \sec \theta = \frac{\text{hyp}}{\text{adj}} & \cot \theta = \frac{\text{adj}}{\text{opp}} \end{array}$$

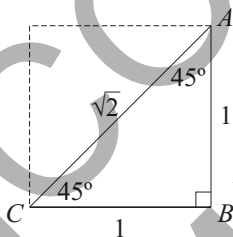
Any two triangles with angles of 90° and θ are *similar*, by the AA similarity test. Hence the values of the six trigonometric functions are the same, whatever the size of the triangle. The full names of the six trigonometric functions are:

sine, cosine, tangent, cosecant, secant, cotangent.

Question 14 in the next exercise gives some clues about these names, and the way in which the functions were originally defined.

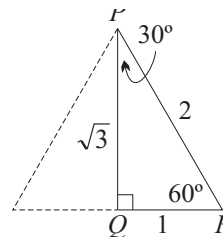
Special angles

The values of the trigonometric functions for the three acute angles 30° , 45° and 60° can be calculated exactly, using half a square and half an equilateral triangle, and applying *Pythagoras' theorem*.



Take half a square with side length 1.
The resulting right-angled triangle ABC has two angles of 45° .
By *Pythagoras' theorem*, the hypotenuse AC has length $\sqrt{2}$.

Applying the definitions on the previous page gives the values in the table below.



Take half an equilateral triangle with side length 2 by dropping an altitude.
The resulting right-angled $\triangle PQR$ has angles of 60° and 30° .
By *Pythagoras' theorem*, $PQ = \sqrt{3}$.

2 A table of exact values

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\operatorname{cosec} \theta$	$\sec \theta$	$\cot \theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

Trigonometric functions of other angles

The calculator is usually used to approximate trigonometric functions of other angles. Always check first that the calculator is in degrees mode — there is usually a key labelled (mode) or (DRG) or something similar. Later, you will be swapping between degrees mode and radian mode (ignore the ‘grads’ unit).

Make sure also that you can enter angles in degrees, minutes and seconds, and that you can convert decimal output to degrees, minutes and seconds — there is usually a key labelled (° ' ") or (DMS) or something similar.

Check that you can perform these two procedures:

$$\sin 53^\circ 47' \doteq 0.8068 \quad \text{and} \quad \sin \theta = \frac{5}{8}, \quad \text{so } \theta \doteq 38^\circ 41'.$$

The reciprocal trigonometric functions

The functions cosecant, secant and cotangent can mostly be avoided in simple trigonometric problems by using the sine, cosine and tangent functions.

3 Avoiding the reciprocal trigonometric functions

The three reciprocal trigonometric functions can mostly be avoided because

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \cot \theta = \frac{1}{\tan \theta}.$$

Finding an unknown side of a right-angled triangle

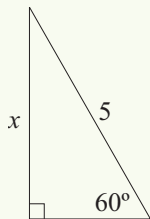
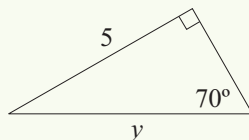
Calculators only have the sine, cosine and tangent functions, so it is best to use only these three functions in problems.

4 To find an unknown side of a right-angled triangle

- 1 Start by writing $\frac{\text{unknown side}}{\text{known side}} = \dots\dots$ (place the unknown at top left)
- 2 Complete the RHS with sin, cos or tan, or the reciprocal of one of these.


Example 1 Finding an unknown side of a right-angled triangle

Find the side marked with a pronumeral in each triangle. Give the answer in exact form if possible, or else correct to five significant figures.

a**b****Solution**

$$\mathbf{a} \quad \frac{x}{5} = \sin 60^\circ \quad \left(\frac{\text{opposite}}{\text{hypotenuse}} \right)$$

$$\boxed{\times 5} \quad x = 5 \sin 60^\circ$$

$$= \frac{5\sqrt{3}}{2}.$$

b

$$\frac{y}{5} = \frac{1}{\sin 70^\circ} \quad \left(\frac{\text{hypotenuse}}{\text{opposite}} \right)$$

$$\boxed{\times 5} \quad y = \frac{5}{\sin 70^\circ}$$

$$\doteq 5.3209.$$

Finding an unknown angle of a right-angled triangle

As before, use only sin, cos and tan.

5 Finding an unknown angle, given two sides of a right-angled triangle

Work out from the known sides which one of $\cos \theta$, $\sin \theta$ or $\tan \theta$ to use.


Example 2 Finding an unknown angle of a right-angled triangle

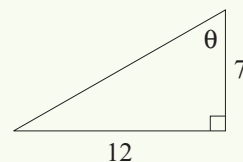
Find θ in the triangle drawn to the right.

Solution

The given sides are the opposite and the adjacent sides, so $\tan \theta$ is known.

$$\tan \theta = \frac{12}{7} \quad \left(\frac{\text{opposite}}{\text{adjacent}} \right)$$

$$\theta \doteq 59^\circ 45'.$$


Exercise 6A
FOUNDATION

1 From the diagram to the right, write down the values of:

a $\cos \alpha$

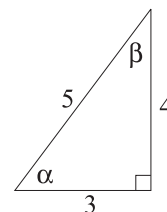
b $\tan \beta$

c $\sin \alpha$

d $\cos \beta$

e $\sin \beta$

f $\tan \alpha$



2 Use your calculator to find, correct to four decimal places:

a $\sin 24^\circ$

b $\cos 61^\circ$

c $\tan 35^\circ$

d $\sin 87^\circ$

e $\tan 2^\circ$

f $\cos 33^\circ$

g $\sin 1^\circ$

h $\cos 3^\circ$

3 Use your calculator to find, correct to four decimal places:

a $\tan 57^\circ 30'$

b $\cos 32^\circ 24'$

c $\tan 78^\circ 40'$

d $\cos 16^\circ 51'$

e $\sin 43^\circ 6'$

f $\sin 5^\circ 50'$

g $\sin 8'$

h $\tan 57'$

4 Use your calculator to find the acute angle θ , correct to the nearest degree, if:

a $\tan \theta = 4$

b $\cos \theta = 0.7$

c $\sin \theta = \frac{1}{5}$

d $\sin \theta = 0.456$

e $\cos \theta = 2$

f $\cos \theta = \frac{7}{9}$

g $\tan \theta = 1\frac{3}{4}$

h $\sin \theta = 1.1$

5 Use your calculator to find the acute angle α , correct to the nearest minute, if:

a $\cos \alpha = \frac{3}{4}$

b $\tan \alpha = 2$

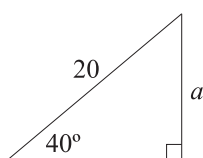
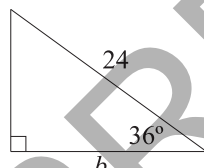
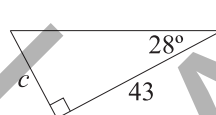
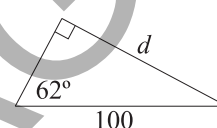
c $\sin \alpha = 0.1$

d $\tan \alpha = 0.3$

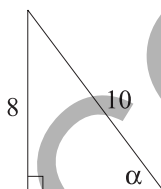
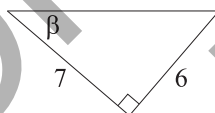
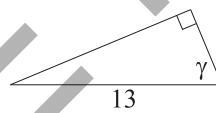
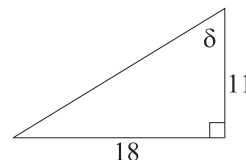
e $\sin \alpha = 0.7251$

f $\cos \alpha = \frac{7}{13}$

6 Find, correct to the nearest whole number, the value of each pronumeral.

a**b****c****d**

7 Find, correct to the nearest degree, the size of each angle marked with a pronumeral.

a**b****c****d**

DEVELOPMENT

8 From the diagram opposite, write down the values of:

a $\sin \alpha$

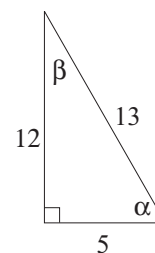
b $\tan \beta$

c $\sec \beta$

d $\cot \alpha$

e $\operatorname{cosec} \alpha$

f $\sec \alpha$



9 a Use Pythagoras' theorem to find the unknown side in each of the two right-angled triangles in the diagram opposite.

b Write down the values of:

i $\cos y$

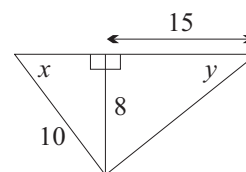
ii $\sin x$

iii $\cot x$

iv $\operatorname{cosec} y$

v $\sec x$

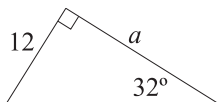
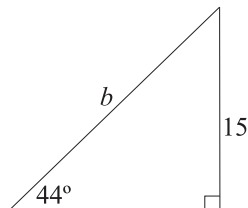
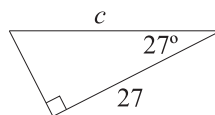
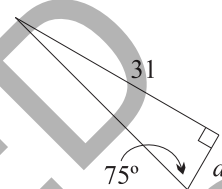
vi $\cot y$



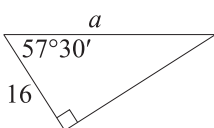
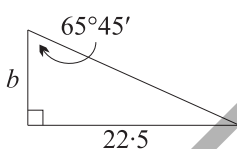
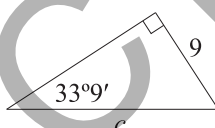
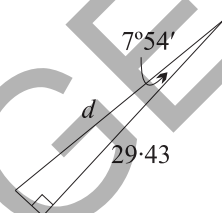
- 10** Draw the two special triangles containing the acute angles 30° , 60° and 45° . Hence write down the exact values of:

a $\sin 60^\circ$ **b** $\tan 30^\circ$ **c** $\cos 45^\circ$ **d** $\sec 60^\circ$ **e** $\operatorname{cosec} 45^\circ$ **f** $\cot 30^\circ$

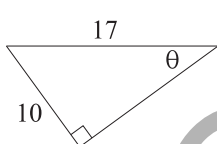
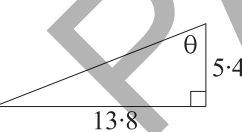
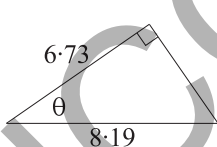
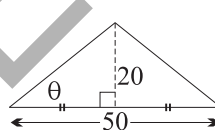
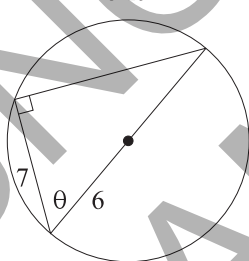
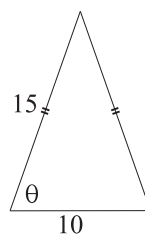
- 11** Find, correct to one decimal place, the lengths of the sides marked with pronumerals.

a**b****c****d**

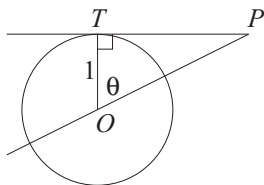
- 12** Find, correct to two decimal places, the lengths of the sides marked with pronumerals.

a**b****c****d**

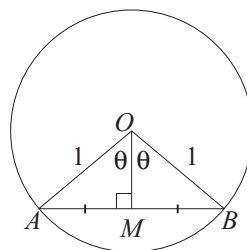
- 13** Find θ , correct to the nearest minute, in each diagram below.

a**b****c****d****e****f**

14



- a** The names *tangent* and *secant* refer to the lengths of the tangent TP and secant OP in the diagram above, where the tangent TP subtends an angle θ at the centre O of the circle of radius 1. Show that $TP = \tan \theta$ and $OP = \sec \theta$.



- b** In classical times, the sine function was usually defined to be the length of the semi-chord AM in the diagram above, where the semi-chord AM subtends an angle θ at the centre O of a circle of radius 1. Show that $AM = \sin \theta$.

CHALLENGE

- 15** It is given that α is an acute angle and that $\tan \alpha = \frac{\sqrt{5}}{2}$.

- Draw a right-angled triangle (it may be of any size), one of whose angles is α , and show this information.
- Use Pythagoras' theorem to find the length of the unknown side.
- Hence write down the exact values of $\sin \alpha$ and $\cos \alpha$.
- Show that $\sin^2 \alpha + \cos^2 \alpha = 1$.

- 16** Suppose that β is an acute angle and $\sec \beta = \frac{\sqrt{11}}{3}$.

- a** Find the exact values of:

i $\operatorname{cosec} \beta$

ii $\cot \beta$

- b** Show that $\operatorname{cosec}^2 \beta - \cot^2 \beta = 1$.

- 17** Find, without using a calculator, the value of:

- $\sin 45^\circ \cos 45^\circ + \sin 30^\circ$
- $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ$
- $1 + \tan^2 60^\circ$
- $\operatorname{cosec}^2 30^\circ - \cot^2 30^\circ$

- 18** Without using a calculator, show that:

- $1 + \tan^2 45^\circ = \sec^2 45^\circ$
- $2 \sin 30^\circ \cos 30^\circ = \sin 60^\circ$
- $\cos^2 60^\circ - \cos^2 30^\circ = -\frac{1}{2}$
- $\frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} = \tan 60^\circ$

6B Problems involving right-angled triangles

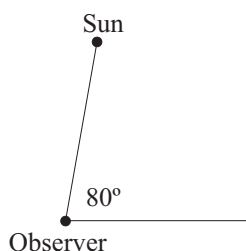
Learning intentions

- Solve problems involving right-angled triangles.
- Work with compass bearings and with angles of elevation and depression.

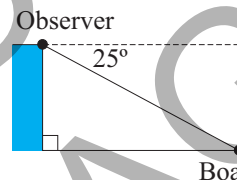
Trigonometry developed in the ancient world solving practical problems with right-angled triangles, typically involving compass bearings and angles of elevation or depression. It was also used extensively in ancient astronomy.

Angles of elevation and depression

Angles of elevation and depression are always measured from the horizontal, and are always acute angles.



The *angle of elevation* of the sun in the diagram above is 80° , because the angle at the observer between the sun and the horizontal is 80° .



For an observer on top of the cliff, the *angle of depression* of the boat is 25° , because the angle at the observer between boat and horizontal is 25° .



Example 3 Angle of depression

From a plane flying at 9000 metres above level ground, I can see a church at an angle of depression of 35° from the cabin of the plane. Find how far the church is from the plane, correct to the nearest 100 metres:

a measured along the ground,

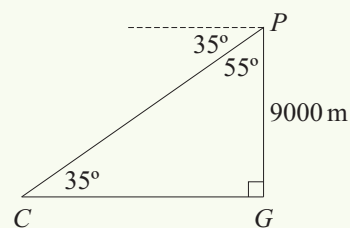
b measured along the line of sight.

Solution

The situation is illustrated in the diagram by $\triangle PGC$.

$$\begin{aligned} \text{a } \frac{GC}{9000} &= \tan 55^\circ \\ GC &= 9000 \tan 55^\circ \\ &\doteq 12\,900 \text{ metres.} \end{aligned}$$

$$\begin{aligned} \text{b } \frac{PC}{9000} &= \frac{1}{\cos 55^\circ} \\ PC &= \frac{9000}{\cos 55^\circ} \\ &\doteq 15\,700 \text{ metres.} \end{aligned}$$



Example 4 Angle of elevation

A walker on level ground is 1 kilometre from the base of a 300-metre vertical cliff.

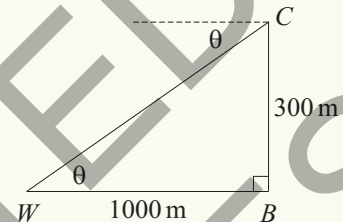
- a** Find, correct to the nearest minute, the angle of elevation of the top.
b Find, correct to the nearest metre, the line-of-sight distance to the top.

Solution

The situation is illustrated by $\triangle CWB$ in the diagram to the right.

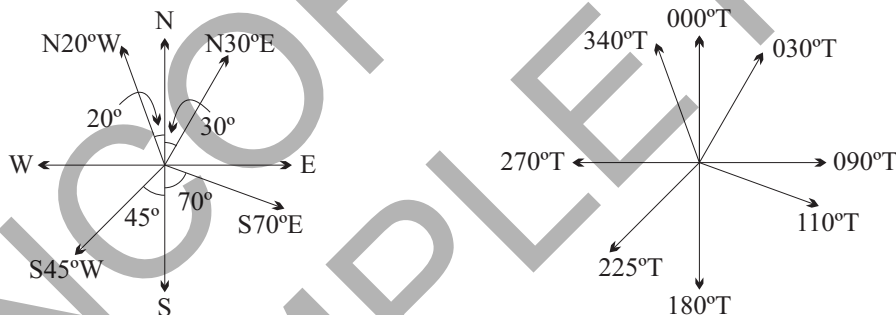
$$\begin{aligned}\mathbf{a} \quad \tan \theta &= \frac{CB}{WB} \\ &= \frac{300}{1000} \\ &= \frac{3}{10} \\ \theta &\doteq 16^\circ 42' .\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \text{Using Pythagoras' theorem,} \\ CW^2 &= CB^2 + WB^2 \\ &= 1000^2 + 300^2 \\ &= 1\,090\,000 \\ CW &\doteq 1044 \text{ metres.}\end{aligned}$$

**Compass bearings and true bearings**

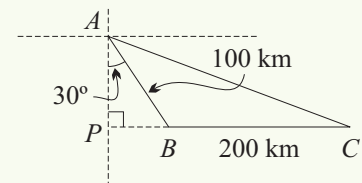
Compass bearings are based on north, south, east and west. Any other direction is specified by the deviation from north or south towards the east or west. The diagram to the left below gives four examples. Note that $S45^\circ W$ can also be written simply as SW (that is, south west).

True bearings are measured clockwise from north (not anti-clockwise as in the coordinate plane). The diagram to the right below gives the same four directions expressed as true bearings. Three digits are used, even for angles less than 100° .

**Example 5** Compass bearings and true bearings

A plane flies at 400 km per hour, and flies from A to B in the direction $S30^\circ E$ for 15 minutes. The plane then turns sharply to fly due east for 30 minutes to C .

- a** Find how far south and east of A the point B is.
b Find the true bearing of C from A , correct to the nearest degree.



Continued on next page.

Solution

- a** The distances AB and BC are 100 km and 200 km respectively.

Working in $\triangle PAB$,

$$\frac{PB}{100} = \sin 30^\circ$$

$$PB = 100 \sin 30^\circ$$

$$= 50 \text{ km,}$$

$$\text{and } AP = 100 \cos 30^\circ$$

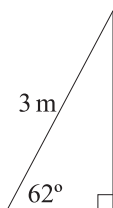
$$= 50\sqrt{3} \text{ km.}$$

- b** Using opposite over adjacent in $\triangle PAC$,

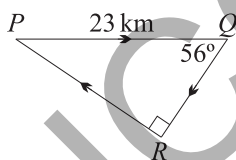
$$\begin{aligned} \tan \angle PAC &= \frac{PC}{AP} \\ &= \frac{50 + 200}{50\sqrt{3}} \\ &= \frac{5}{\sqrt{3}} \end{aligned}$$

$$\angle PAC \doteq 71^\circ.$$

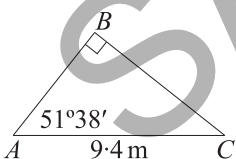
Hence the bearing of C from A is about 109° T.

Exercise 6B**FOUNDATION****1**

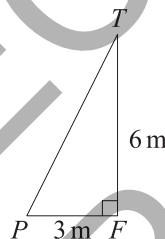
A ladder of length 3 metres is leaning against a wall and is inclined at 62° to the ground. How far does it reach up the wall? (Answer in metres correct to two decimal places.)

3

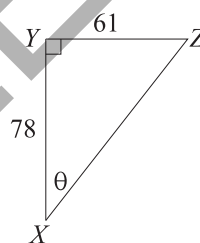
Ben cycles from P to Q to R and then back to P in a road-race. Find, correct to the nearest kilometre, the distance he has ridden.

5

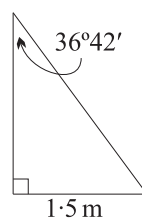
A tree snapped into two sections AB and BC in high winds and then fell. The section BA is inclined at $51^\circ 38'$ to the horizontal and AC is 9.4 metres long. Find the height of the original tree, in metres correct to one decimal place.

2

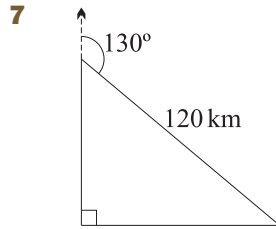
Determine, correct to the nearest degree, the angle of elevation of the top T of a 6 metre flagpole FT from a point P on level ground 3 metres from F .

4

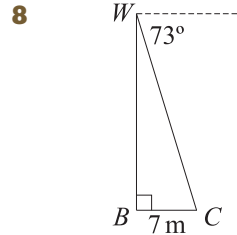
A ship sails 78 nautical miles due north from X to Y , then 61 nautical miles due east from Y to Z . Find θ , the bearing of Z from X , correct to the nearest degree.

6

A ladder makes an angle of $36^\circ 42'$ with a wall, and its foot is 1.5 metres out from the base of the wall. Find the length of the ladder, in metres correct to one decimal place.



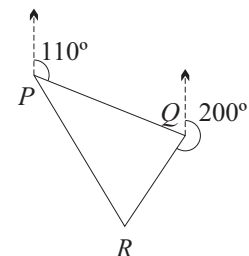
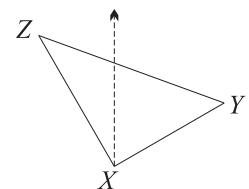
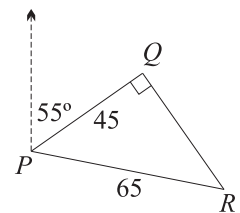
Eleni drives 120 km on a bearing of 130°T . She then drives due west until she is due south of her starting point. How far is she from her starting point, correct to the nearest kilometre?



John is looking out a window W at a car C parked on the street below. If the angle of depression of C from W is 73° and the car is 7 metres from the base B of the building, find the height WB of the window, correct to the nearest metre.

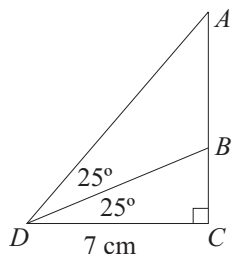
DEVELOPMENT

- 9** A ladder of length 5 metres is placed on level ground against a vertical wall. If the foot of the ladder is 1.5 metres from the base of the wall, find, correct to the nearest degree, the angle at which the ladder is inclined to the ground.
- 10** Find, correct to the nearest tenth of a metre, the height of a tower, if the angle of elevation of the top of the tower is $64^\circ 48'$ from a point on horizontal ground 10 metres from the base of the tower.
- 11** A boat is 200 metres out to sea from a vertical cliff of height 40 metres. Find, correct to the nearest degree, the angle of depression of the boat from the top of the cliff.
- 12** Port Q is 45 nautical miles from port P on a bearing of 055°T . Port R is 65 nautical miles from port P , and $\angle PQR = 90^\circ$.
- Find $\angle QPR$ to the nearest degree.
 - Hence find the bearing of R from P , correct to the nearest degree.
- 13** The bearings of towns Y and Z from town X are 060°T and 330°T respectively.
- Show that $\angle ZXY = 90^\circ$.
 - Given that town Z is 80 km from town X and that $\angle XYZ = 50^\circ$, find, correct to the nearest kilometre, how far town Y is from town X .
- 14** A ship leaves port P and travels 150 nautical miles to port Q on a bearing of 110°T . It then travels 120 nautical miles to port R on a bearing of 200°T .
- Explain why $\angle PQR = 90^\circ$.
 - Find, correct to the nearest degree, the bearing of port R from port P .



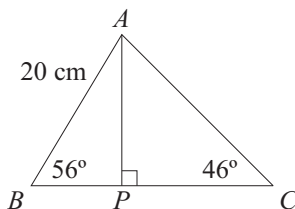
CHALLENGE

15 a



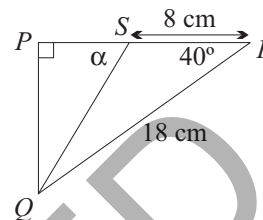
Show that $AC = 7 \tan 50^\circ$ and $BC = 7 \tan 25^\circ$, and hence find the length AB , correct to 1 mm.

b



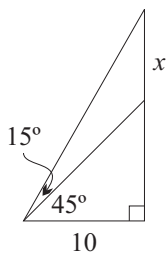
Show that $AP = 20 \sin 56^\circ$, and hence find the length of PC , giving your answer correct to 1 cm.

c



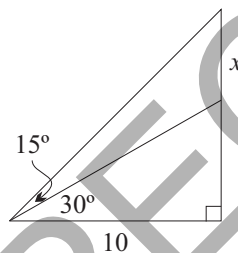
Show that $PR = 18 \cos 40^\circ$, find an expression for PQ , and hence find the angle α , correct to the nearest minute.

16 a



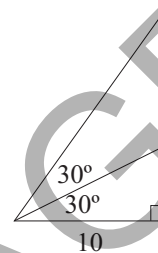
Show that $x = 10(\sqrt{3} - 1)$.

b



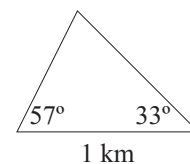
Show that $x = \frac{10}{3}(3 - \sqrt{3})$.

c



Show that $x = \frac{20}{3}\sqrt{3}$.

- 17 From the ends of a straight horizontal road 1 km long, a balloon directly above the road is observed to have angles of elevation of 57° and 33° respectively. Find, correct to the nearest metre, the height of the balloon above the road.



6C Trigonometric functions of a general angle

Learning intentions

- Use circles to extend the six trigonometric functions to angles of any size.
- Work with boundary angles, and determine the domains of the trig functions.
- Sketch $y = \sin x$, $y = \cos x$, and $y = \tan x$ using their circle definitions.
- Identify standard properties of all six functions from their graphs.

The definitions of the trigonometric functions in Section 6A only apply to acute angles, because in a right-angled triangle, both other angles are acute angles.

This section introduces more general definitions based on circles in the coordinate plane (whose equations are Pythagoras' theorem, as we saw in Section 3G). The new definitions will apply to any angle, but will, of course, give the same values as the previous definitions for acute angles.

Putting a general angle on the coordinate plane

Let θ be any angle — possibly negative, possibly obtuse or reflex, possibly greater than 360° . We shall associate with θ a ray with vertex at the origin.

6 The ray corresponding to θ

- The positive direction of the x -axis is the ray representing the angle 0° .
- For all other angles, rotate this ray anti-clockwise through an angle θ .

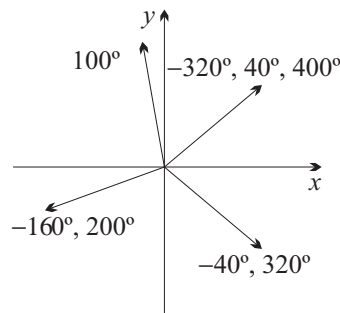
If the angle is negative, the ray is rotated backwards, which means clockwise.

Here are some angles and corresponding rays. The angles are written at the ends of the arrows representing the rays.

Notice that one ray can correspond to many angles. For example, all the following angles have the same ray as 40° :

..., -680° , -320° , 40° , 400° , 760° , ...

A given ray thus corresponds to infinitely many angles, all differing by multiples of 360° .



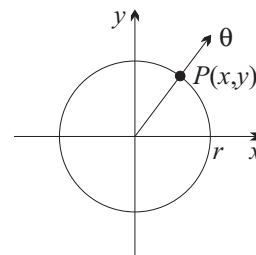
7 Corresponding angles and rays

- To each angle, there corresponds exactly one ray.
- To each ray, there correspond infinitely many angles, all differing from each other by multiples of 360° .

Defining the trigonometric functions for general angles

Construct a circle with centre the origin and any positive radius r . Let the ray corresponding to θ meet the circle at the point $P(x, y)$.

The six trigonometric functions are now defined in terms of x , y and r as follows:



8 Definitions of the six trigonometric functions

$$\sin \theta = \frac{y}{r}$$
$$\operatorname{cosec} \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r}$$
$$\sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x}$$
$$\cot \theta = \frac{x}{y}$$

Note: We chose r to be ‘any positive radius’. If a different radius were chosen, the two figures would be *similar*, so the lengths x , y and r would stay in the same ratio, so the values of the trigonometric functions would not change.

In particular, one may use a circle of radius 1 in the definitions. When this is done, however, we lose the intuition that a trigonometric function is not a length, but is the ratio of two lengths.

Agreement with the earlier definition

Let θ be an acute angle, and construct the ray corresponding to θ . Let the perpendicular from P meet the x -axis at M . Then $\theta = \angle POM$, so relating the sides to the angle θ ,

hyp = $OP = r$, opp = $PM = y$, adj = $OM = x$.

Hence the old and the new definitions are in agreement.

Note: Most people find that the diagram above is the easiest way to learn the new definitions of the trigonometric functions. Take the old definitions in terms of hypotenuse, opposite and adjacent sides, and make the replacements

hyp $\longleftrightarrow r$, opp $\longleftrightarrow y$, adj $\longleftrightarrow x$.

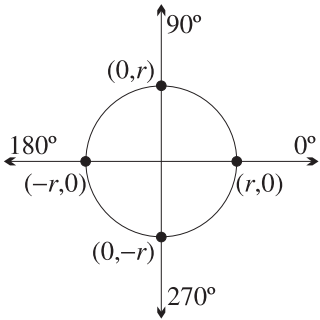
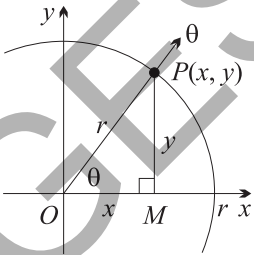
Boundary angles

Integer multiples of 90° , that is

$\dots, -90^\circ, 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ, 450^\circ, \dots$

are called *boundary angles* because they lie on the boundaries between quadrants.

The values of the trigonometric functions at these boundary angles are not always defined, and are 0, 1 or -1 when they are defined. The diagram to the right can be used to calculate them, and the results are shown in the table below.



9 The boundary angles

θ	0°	90°	180°	270°
x	r	0	$-r$	0
y	0	r	0	$-r$
r	r	r	r	r
$\sin \theta$	0	1	0	-1
$\cos \theta$	1	0	-1	0
$\tan \theta$	0	undefined	0	undefined
$\operatorname{cosec} \theta$	undefined	1	undefined	-1
$\sec \theta$	1	undefined	-1	undefined
$\cot \theta$	undefined	0	undefined	0

In practice, read the values of the trigonometric functions at boundary angles off the graphs of the sine, cosine and tangent functions. These graphs need to be known very well indeed.

A warning about asymptotes and zeroes

You may have concluded from the definitions $\tan \theta = \frac{y}{x}$ and $\cot \theta = \frac{x}{y}$ that

$$\cot \theta = \frac{1}{\tan \theta} \quad \text{and} \quad \tan \theta = \frac{1}{\cot \theta}.$$

This is true everywhere except at boundary angles, because:

- When $\theta = 0^\circ$, $\tan \theta = 0$ and $\cot \theta$ is undefined.
- When $\theta = 90^\circ$, $\tan \theta$ is undefined and $\cot \theta = 0$.

Always be aware that the number 0 has no reciprocal. Also ‘infinity’ is a mathematical idea, but not a number, so it cannot possibly have a reciprocal either.

The domains of the six trigonometric functions

The trig functions are defined everywhere except where the denominator is zero.

10 Domains of the trigonometric functions

- $\sin \theta$ and $\cos \theta$ are defined for all angles θ :
Their domains are all angles θ .
- $\tan \theta$ and $\sec \theta$ are undefined when $x = 0$:
Their domains are $\theta \neq \dots, -270^\circ, -90^\circ, 90^\circ, 270^\circ, 450^\circ, \dots$
- $\cot \theta$ and $\operatorname{cosec} \theta$ are undefined when $y = 0$:
Their domains are $\theta \neq \dots, -360^\circ, -180^\circ, 0^\circ, 180^\circ, 360^\circ, \dots$

The graphs of the trigonometric functions

The diagrams on the next page show the graphs of all six trigonometric functions over an interval extending beyond $-360^\circ \leq x \leq 360^\circ$. This extended interval makes clear how the graphs are built up by infinite repetition of a simple element.

But readers need to construct the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ by themselves from the definitions. Question 9 below generates these graphs using approximations from the circle diagram drawn on graph paper.

Note: The last three graphs of $y = \operatorname{cosec} x$, $y = \sec x$ and $y = \cot x$ are not required in this course, but are included to complete the picture. Challenge questions ask for properties to be read off the graphs — a standard exercise.

Period and amplitude

The graphs of $y = \sin x$ and $y = \cos x$ repeat every 360° , so the graphs are called *periodic*. The *period* is 360° , the length of the smallest repeating piece. The periods of the other four graphs will be identified in the exercise.

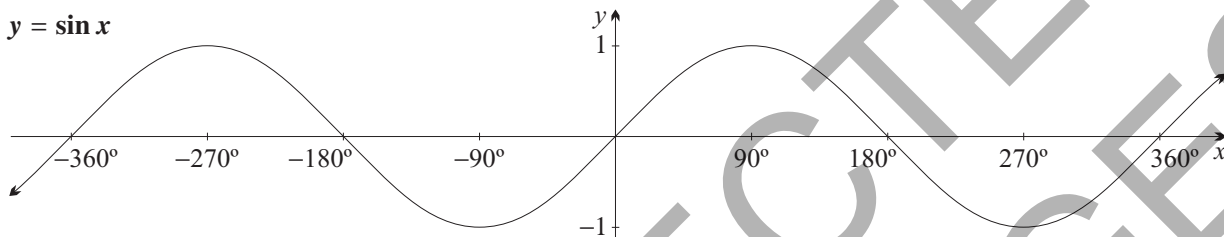
The graphs of $y = \sin x$ and $y = \cos x$ are waves. They each rise to a maximum height of 1 above the x -axis, then fall to a minimum of -1 below the x -axis, which is the *mean position*. This difference of 1 is called the *amplitude*.

11 Period and amplitude

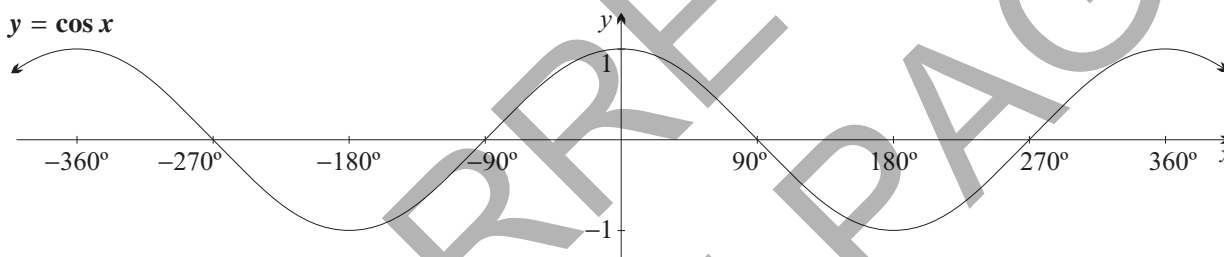
- The graphs of $y = \sin x$ and $y = \cos x$ each consist of a small piece infinitely repeated, so the functions are called *periodic*.
 - They each have period 360° , the length of the smallest repeating piece.
- The graphs of $y = \sin x$ and $y = \cos x$ are waves, rising and falling 1 unit above and below the *mean position*, so these functions are said to have *amplitude 1*.

The wave graphs of $y = \sin x$ and $y = \cos x$ are the basis of all mathematics that deals with waves. The later trigonometry in this course will mostly concern these wave properties.

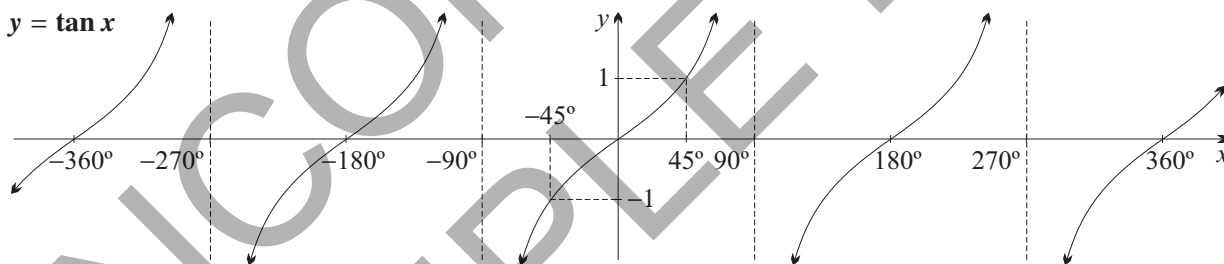
$$y = \sin x$$



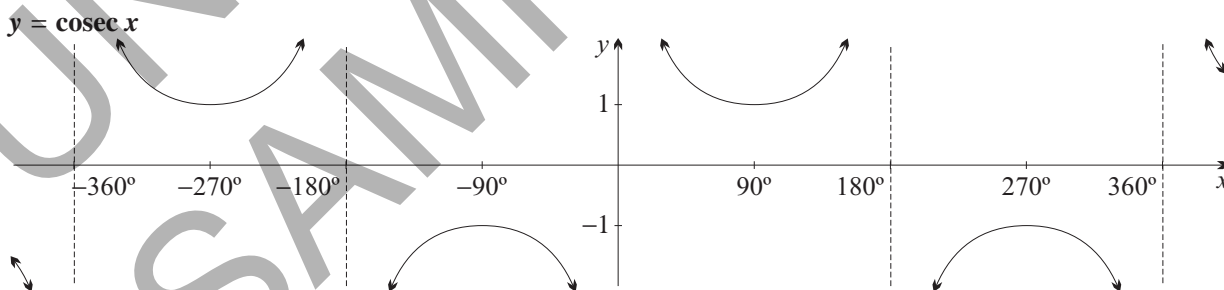
$$y = \cos x$$



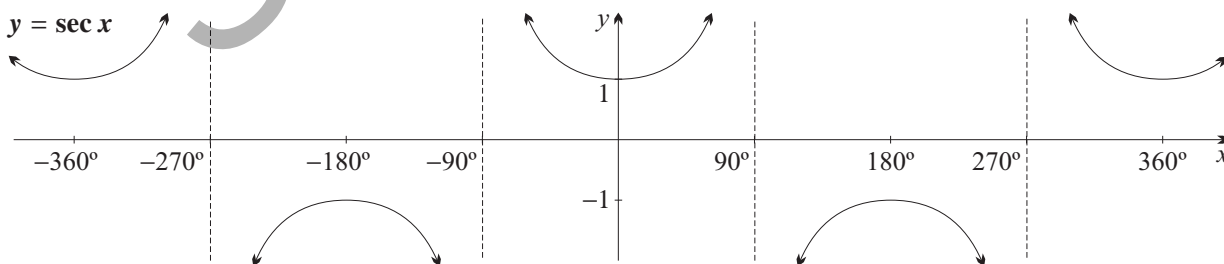
$$y = \tan x$$

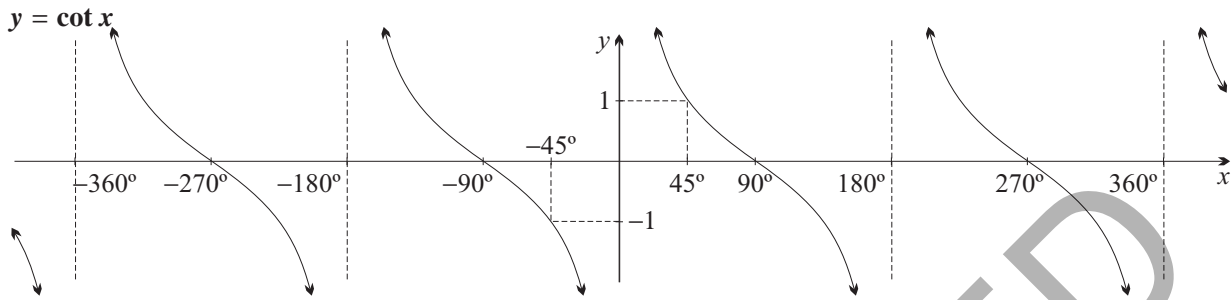


$$y = \operatorname{cosec} x$$



$$y = \sec x$$





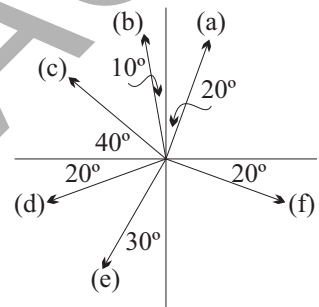
Exercise 6C

FOUNDATION

- On a number plane, draw rays representing the following angles.

a 40°	b 110°	c 190°
d 290°	e 420°	f 500°
- On another number plane, draw rays representing the following angles.

a -50°	b -130°	c -250°
d -350°	e -440°	f -550°
- For each of the angles in question 1, write down the size of the negative angle between -360° and 0° that is represented by the same ray.
- For each of the angles in question 2, write down the size of the positive angle between 0° and 360° that is represented by the same ray.
- Write down two positive angles between 0° and 720° and two negative angles between -720° and 0° that are represented by each of the rays in the diagram to the right.

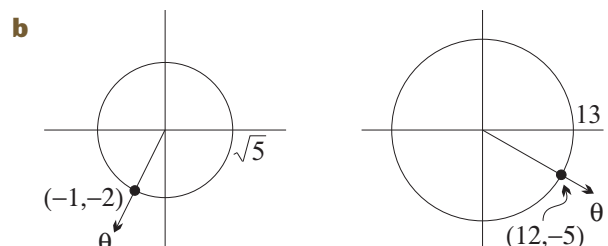
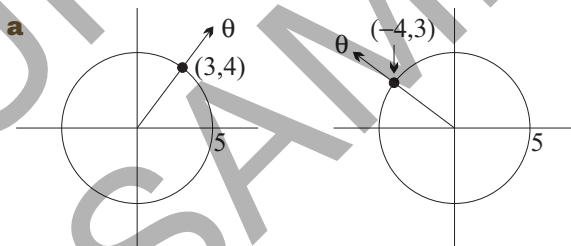


DEVELOPMENT

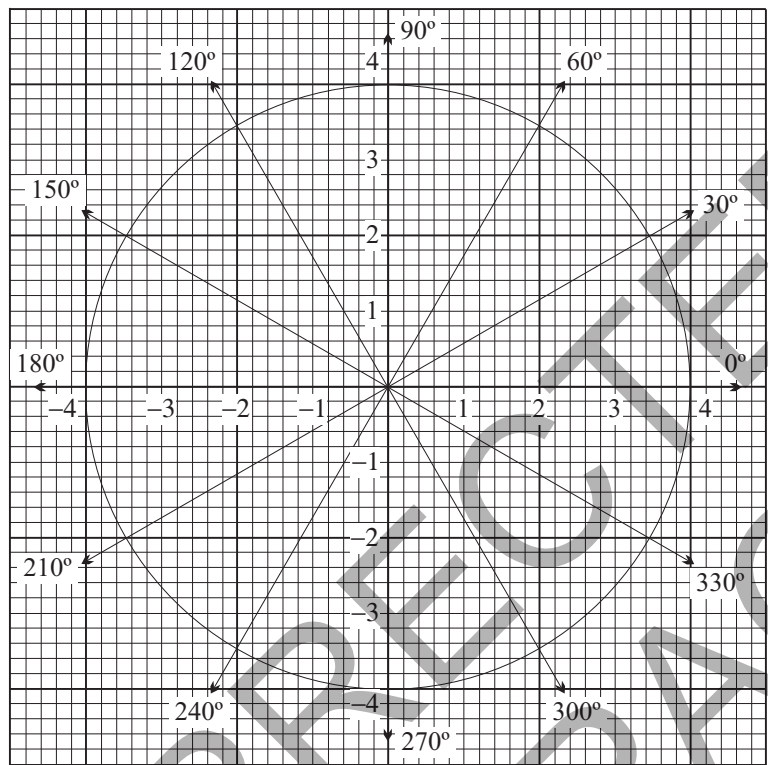
- Use the definitions

$$\sin \theta = \frac{y}{r} \quad \text{and} \quad \cos \theta = \frac{x}{r} \quad \text{and} \quad \tan \theta = \frac{y}{x}$$

to write down the values of the six trigonometric ratios of the angle θ in each diagram.



7 [The graphs of $\sin \theta$, $\cos \theta$ and $\tan \theta$] The diagram shows angles from 0° to 360° at 30° intervals. The circle has radius 4 units.

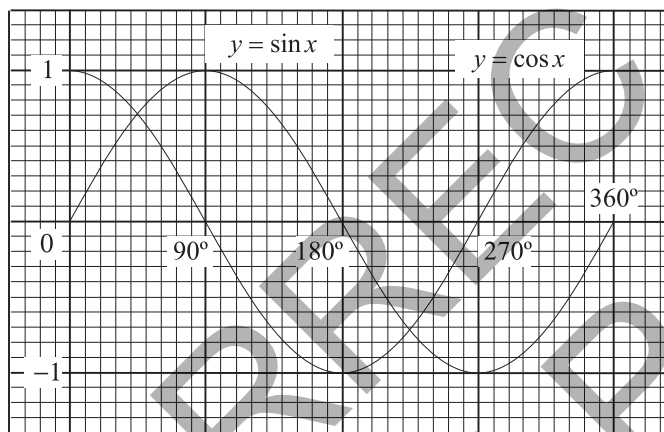


a Use the diagram and the definitions of the three trigonometric ratios to complete the following table. Measure the values of x and y correct to two decimal places, and use your calculator only to perform the necessary divisions.

θ	-30°	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°	390°
x															
y															
r															
$\sin \theta$															
$\cos \theta$															
$\tan \theta$															

- b Use your calculator to check the accuracy of the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ that you obtained in part (a).
- c Using the table of values in part (a), graph the curves $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$ as accurately as possible on graph paper. Use the following scales:
- On the horizontal axis: let 2 mm represent 10° .
- On the vertical axis: let 2 cm represent 1 unit.

- 8** By referring to the graph of the function $y = \sin x$ drawn in degrees in the text, write down its:
a domain, **b** range, **c** period, **d** amplitude.
- 9** By referring to the graph of the function $y = \cos x$, write down its:
a domain, **b** range, **c** period, **d** amplitude.
- 10** By referring to the graph of the function $y = \tan x$, write down:
a its domain, **b** its range, **c** its period,
d the equations of its asymptotes for $-360^\circ \leq x \leq 360^\circ$.
- 11** From the graphs, classify $y = \sin x$, $y = \cos x$, and $y = \tan x$ as even, odd, or neither.

CHALLENGE**12**

- a** Read off the diagram, correct to two decimal places where necessary, the values of:
- | | | | | |
|---------------------------|-----------------------------|-----------------------------|----------------------------|---------------------------|
| i $\cos 60^\circ$ | ii $\sin 210^\circ$ | iii $\sin 72^\circ$ | iv $\cos 18^\circ$ | v $\sin 144^\circ$ |
| vi $\cos 36^\circ$ | vii $\cos 153^\circ$ | viii $\sin 27^\circ$ | ix $\sin 234^\circ$ | x $\cos 306^\circ$ |
- b** Find from the graphs two values of x between 0° and 360° for which:
- | | | | |
|-------------------------|---------------------------|----------------------------|-----------------------------|
| i $\sin x = 0.5$ | ii $\cos x = -0.5$ | iii $\sin x = 0.9$ | iv $\cos x = 0.6$ |
| v $\sin x = 0.8$ | vi $\cos x = -0.8$ | vii $\sin x = -0.4$ | viii $\cos x = -0.3$ |
- c** Find two values of x between 0° and 360° for which $\sin x = \cos x$.
- 13** By referring to the graph of the function $y = \operatorname{cosec} x$, write down:
a its domain, **b** its range, **c** its period,
d the equations of its asymptotes for $-360^\circ \leq x \leq 360^\circ$.
- 14** By referring to the graph of the function $y = \sec x$, write down:
a its domain, **b** its range, **c** its period,
d the equations of its asymptotes for $-360^\circ \leq x \leq 360^\circ$.
- 15** By referring to the graph of the function $y = \cot x$, write down:
a its domain, **b** its range, **c** its period,
d the equations of its asymptotes for $-360^\circ \leq x \leq 360^\circ$.
- 16** From the graphs, classify $y = \operatorname{cosec} x$, $y = \sec x$, and $y = \cot x$ as even, odd, or neither.

6D Quadrant, sign, and related acute angle

Learning intentions

- Recognise the quadrant and the related angle of a general angle.
- Understand how to find the sign of a trig function from the ASTC diagram.
- Use quadrant and related angle to find the trig functions of a general angle.

Symmetry is an essential aspect of trigonometric functions. This section uses symmetry to express the values of the trigonometric functions of any angle in terms of trigonometric functions of acute angles.

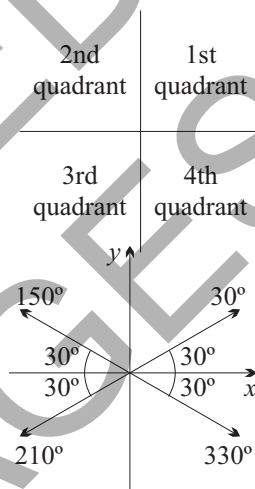
The diagram shows the conventional anti-clockwise numbering of the four quadrants of the coordinate plane. Acute angles are in the first quadrant, obtuse angles are in the second, and reflex angles are in the third and fourth.

The quadrant and the related acute angle

The diagram to the right shows the four rays corresponding to the four angles

30° , 150° , 210° and 330° .

These four rays lie in the four quadrants of the plane, and they all make the same acute angle of 30° with the x -axis. They are thus the reflections of each other in the two axes.



12 Quadrant and related acute angle

Let θ be any angle.

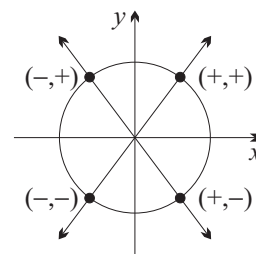
- The *quadrant* of θ is the quadrant (1, 2, 3 or 4) in which the ray lies.
- The *related acute angle* of θ is the acute angle between the ray and the x -axis.

Each of the four angles above has the same related acute angle 30° . Notice that θ and its related angle are the same only when θ is an acute angle.

The signs of the trigonometric functions

The signs of the trigonometric functions depend only on the signs of x and y . (The radius r is always positive.) The signs of x and y depend in turn only on the quadrant in which the ray lies. Thus we can easily compute the signs of the trigonometric functions from the accompanying diagram and the definitions.

quadrant	1st	2nd	3rd	4th	
x	+	-	-	+	
y	+	+	-	-	
r	+	+	+	+	
$\sin \theta$	+	+	-	-	
$\cos \theta$	+	-	-	+	
$\tan \theta$	+	-	+	-	
$\operatorname{cosec} \theta$	+	+	-	-	(same as $\sin \theta$)
$\sec \theta$	+	-	-	+	(same as $\cos \theta$)
$\cot \theta$	+	-	+	-	(same as $\tan \theta$)



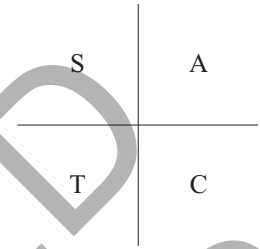
In NSW, these results are usually remembered by the phrase:

13 Signs of the trigonometric functions

‘All Stations To Central’

indicating that the four letters A, S, T and C are placed successively in the four quadrants as shown. The significance of the letters is:

- A means *All* six functions are positive,
- S means only *Sine* (and cosecant) are positive,
- T means only *Tangent* (and cotangent) are positive,
- C means only *Cosine* (and secant) are positive.



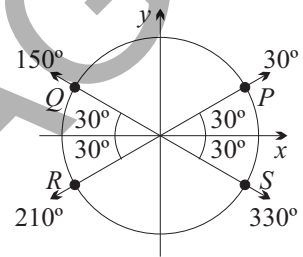
The graphs of $y = \sin \theta$, $y = \cos \theta$, and $y = \tan \theta$ were constructed in the previous exercise, and all six trigonometric functions were drawn on the one page at the end of Section 6C. Study each of the six graphs to see how the table of signs above, and the ASTC rule, agree with your observations about when the graph is above the x -axis and when it is below.

The angle and the related acute angle

In the diagram to the right, a circle of radius r has been added to the earlier diagram showing the four angles 30° , 150° , 210° and 330° .

The four points P , Q , R and S where the four rays meet the circle are all reflections of each other in the x -axis and y -axis. Because of this symmetry, the coordinates of these four points are identical, apart from their signs.

Hence the trigonometric functions of these angles will all be the same too, except that the signs may be different.



14 The angle and the related acute angle

- The trigonometric functions of any angle θ are the same as the trigonometric functions of its related acute angle, apart from a possible change of sign.
- The sign is best found using the ASTC diagram.

Evaluating the trigonometric functions of any angle

This gives a two-step way to evaluate the trigonometric functions of any angle.

15 Trigonometric functions of any angle

Draw a quadrants diagram, then:

- Place the ray in the correct quadrant, and use the ASTC rule to work out the sign of the answer.
- Find the related acute angle, and work out the value of the trigonometric function at this related angle.


Example 6 Using the ASTC diagram and the related angle

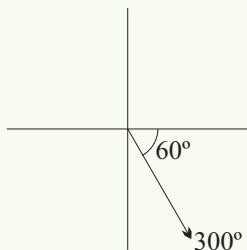
Find the exact value of:

a $\tan 300^\circ$

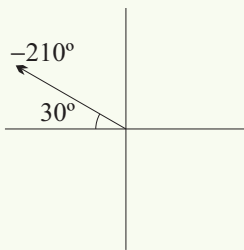
b $\sin(-210^\circ)$

c $\cos 570^\circ$

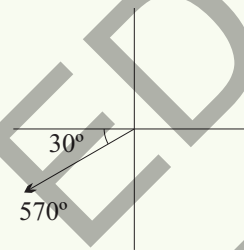
Solution



a 300° is in quadrant 4, the related angle is 60° , so
 $\tan 300^\circ = -\tan 60^\circ$
 $= -\sqrt{3}.$



b -210° is in quadrant 2, the related angle is 30° , so
 $\sin(-210^\circ) = +\sin 30^\circ$
 $= \frac{1}{2}.$



c 570° is in quadrant 3, the related angle is 30° , so
 $\cos 570^\circ = -\cos 30^\circ$
 $= -\frac{1}{2}\sqrt{3}.$

Note: The calculator will give approximate values of the trigonometric functions without any need to find the related acute angle. It will *not* give exact values, however, when these values involve surds.

General angles with pronumerals

This quadrants-diagram method generates formulae for expressions such as:

$$\sin(180^\circ + A) \quad \text{or} \quad \cot(360^\circ - A).$$

The trick is to place A on the quadrants diagram as *if it were acute*.

16 Some formulae with general angles

$$\sin(180^\circ - A) = \sin A$$

$$\sin(180^\circ + A) = -\sin A$$

$$\sin(360^\circ - A) = -\sin A$$

$$\cos(180^\circ - A) = -\cos A$$

$$\cos(180^\circ + A) = -\cos A$$

$$\cos(360^\circ - A) = \cos A$$

$$\tan(180^\circ - A) = -\tan A$$

$$\tan(180^\circ + A) = \tan A$$

$$\tan(360^\circ - A) = -\tan A$$

Some people prefer to learn this list of identities to evaluate trigonometric functions, but this seems unnecessary when the quadrants-diagram method is so clear.

Exercise 6D

FOUNDATION

1 Use the ASTC rule to determine the sign (+ or -) of each trigonometric ratio.

a $\sin 20^\circ$

b $\cos 50^\circ$

c $\cos 100^\circ$

d $\tan 140^\circ$

e $\tan 250^\circ$

f $\sin 310^\circ$

g $\sin 200^\circ$

h $\cos 280^\circ$

i $\sin 340^\circ$

j $\cos 350^\circ$

k $\tan 290^\circ$

l $\cos 190^\circ$

m $\tan 170^\circ$

n $\sin 110^\circ$

o $\tan 80^\circ$

p $\cos 170^\circ$

2 Find the related acute angle of each angle.

a 10°	b 150°	c 310°	d 200°	e 80°
f 250°	g 290°	h 100°	i 350°	j 160°

3 Write each trigonometric ratio as the ratio of an acute angle with the correct sign attached.

a $\tan 130^\circ$	b $\cos 310^\circ$	c $\sin 220^\circ$
d $\tan 260^\circ$	e $\cos 170^\circ$	f $\sin 320^\circ$
g $\cos 185^\circ$	h $\sin 125^\circ$	i $\tan 325^\circ$
j $\sin 85^\circ$	k $\cos 95^\circ$	l $\tan 205^\circ$

4 Use the trigonometric graphs to find the values, if they exist, of these trigonometric ratios of boundary angles.

a $\sin 0^\circ$	b $\cos 180^\circ$	c $\cos 90^\circ$
d $\tan 0^\circ$	e $\sin 90^\circ$	f $\cos 0^\circ$
g $\sin 270^\circ$	h $\tan 270^\circ$	i $\sin 180^\circ$
j $\cos 270^\circ$	k $\tan 90^\circ$	l $\tan 180^\circ$

5 Find the exact value of:

a $\sin 60^\circ$	b $\sin 120^\circ$	c $\sin 240^\circ$
d $\sin 300^\circ$	e $\cos 45^\circ$	f $\cos 135^\circ$
g $\cos 225^\circ$	h $\cos 315^\circ$	i $\tan 30^\circ$
j $\tan 150^\circ$	k $\tan 210^\circ$	l $\tan 330^\circ$

DEVELOPMENT

6 Find the exact value of:

a $\cos 120^\circ$	b $\tan 225^\circ$	c $\sin 330^\circ$
d $\sin 135^\circ$	e $\tan 240^\circ$	f $\cos 210^\circ$
g $\tan 315^\circ$	h $\cos 300^\circ$	i $\sin 225^\circ$
j $\cos 150^\circ$	k $\sin 210^\circ$	l $\tan 300^\circ$

7 Find the exact value of:

a $\operatorname{cosec} 150^\circ$	b $\sec 225^\circ$	c $\cot 120^\circ$
d $\cot 210^\circ$	e $\sec 330^\circ$	f $\operatorname{cosec} 300^\circ$

8 a If θ is acute, use the ASTC diagram to write as trigonometric functions of θ :

i $\sin(-\theta)$	ii $\cos(-\theta)$	iii $\tan(-\theta)$
b Hence find the exact values of:		
i $\sin(-60^\circ)$	ii $\cos(-45^\circ)$	iii $\tan(-30^\circ)$
iv $\sin(-30^\circ)$	v $\cos(-60^\circ)$	vi $\tan(-45^\circ)$

9 Use the trigonometric graphs to find, if they exist:

a $\sec 0^\circ$	b $\operatorname{cosec} 270^\circ$
c $\sec 90^\circ$	d $\operatorname{cosec} 180^\circ$
e $\cot 90^\circ$	f $\cot 180^\circ$

10 Find the related acute angle of each angle.

a -60°	b -200°
c -150°	d -300°
e 430°	f 530°
g 590°	h 680°

11 Find the exact value of:

a $\cos(-60^\circ)$	b $\sin(-120^\circ)$	c $\tan(-120^\circ)$
d $\sin(-315^\circ)$	e $\tan(-210^\circ)$	f $\cos(-225^\circ)$
g $\tan 420^\circ$	h $\cos 510^\circ$	i $\sin 495^\circ$
j $\sin 690^\circ$	k $\cos 600^\circ$	l $\tan 585^\circ$

CHALLENGE

12 Given that $\sin 25^\circ \div 0.42$ and $\cos 25^\circ \div 0.91$, write down approximate values, without using a calculator, for:

a $\sin 155^\circ$	b $\cos 205^\circ$	c $\cos 335^\circ$
d $\sin 335^\circ$	e $\sin 205^\circ - \cos 155^\circ$	f $\cos 385^\circ - \sin 515^\circ$

13 Given that $\tan 35^\circ \div 0.70$ and $\sec 35^\circ \div 1.22$, write down approximate values, without using a calculator, for:

a $\tan 145^\circ$	b $\sec 215^\circ$	c $\tan 325^\circ$
d $\tan 215^\circ + \sec 145^\circ$	e $\sec 325^\circ + \tan 395^\circ$	f $\sec(-145^\circ) - \tan(-215^\circ)$

14 Show by substitution into LHS and RHS that each trigonometric identity is satisfied by the given values of the angles.

- a** Show that $\sin 2\theta = 2 \sin \theta \cos \theta$, when $\theta = 150^\circ$.
b Show that $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$, when $\theta = 225^\circ$.
c Show that $\sin(A + B) = \sin A \cos B + \cos A \sin B$, when $A = 300^\circ$ and $B = 240^\circ$.

15 Write as a trigonometric ratio of θ , with the correct sign attached:

a $\sin(-\theta)$	b $\cos(-\theta)$
c $\tan(-\theta)$	d $\sec(-\theta)$
e $\sin(180^\circ - \theta)$	f $\sin(360^\circ - \theta)$
g $\cos(180^\circ - \theta)$	h $\tan(180^\circ + \theta)$

6E Given one trigonometric function, find another

Learning intentions

- Find, given a value of a trig function, the exact values of other trig functions there.

When the exact value of one trigonometric function is known for an angle, the exact values of the other trigonometric functions can easily be found using the circle diagram and Pythagoras' theorem.

Given one trigonometric function, find another

17 Given one trigonometric function, find another

- Place a ray or rays on a circle diagram in the quadrants allowed in the question.
- Complete the triangle and use *Pythagoras' theorem* to find whichever of x , y and r is missing.



Example 7 Given one trigonometric function, find another

It is known that $\sin \theta = \frac{1}{5}$.

- Find the possible values of $\cos \theta$.
- Find $\cos \theta$ if it is also known that $\tan \theta$ is negative.

Solution

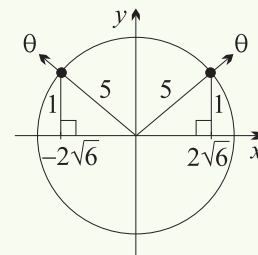
- The angle must be in quadrant 1 or 2, because $\sin \theta$ is positive.

Because $\sin \theta = \frac{y}{r} = \frac{1}{5}$, we can take $y = 1$ and $r = 5$,
so by *Pythagoras' theorem*, $x = \sqrt{24}$ or $-\sqrt{24}$,

so $\cos \theta = \frac{2\sqrt{6}}{5}$ or $-\frac{2\sqrt{6}}{5}$.

- Because $\tan \theta$ is negative, θ can only be in quadrant 2,

so $\cos \theta$ is negative,
 $\cos \theta = -\frac{2\sqrt{6}}{5}$.

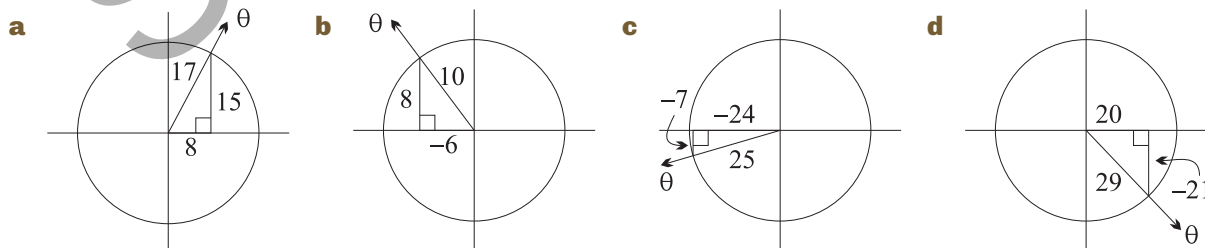


Exercise 6E

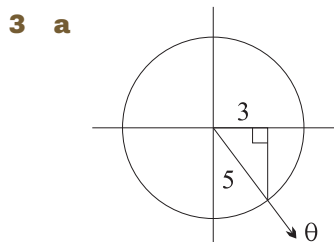
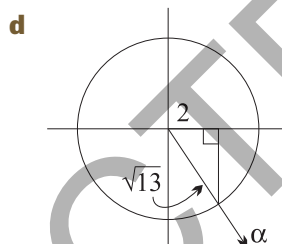
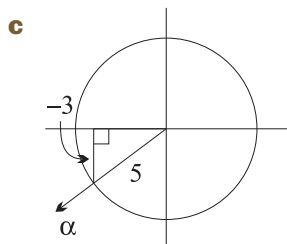
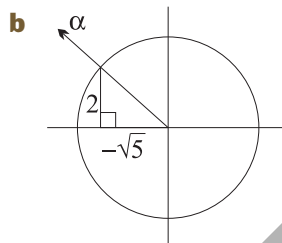
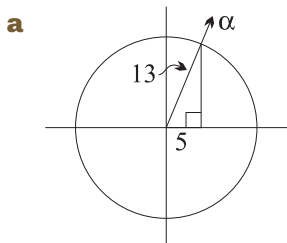
FOUNDATION

Note: Diagrams have been drawn for questions 1–4, and similar diagrams should be drawn for the subsequent questions. Many answers will involve surds, but it is not important to rationalise denominators. Do not use the calculator at all in this exercise, because you are looking for exact values, not approximations.

- Write down the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ in each part.

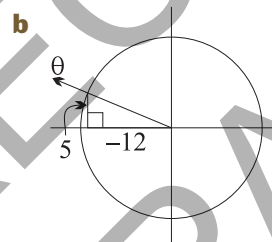


- 2** In each part use Pythagoras' theorem to find whichever of x , y or r is unknown. Then write down the values of $\sin \alpha$, $\cos \alpha$ and $\tan \alpha$.



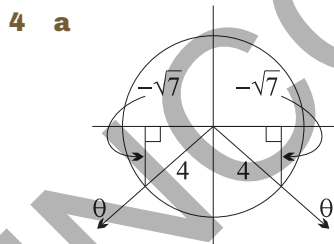
Let $\cos \theta = \frac{3}{5}$, where $270^\circ < \theta < 360^\circ$.

- i Find $\sin \theta$.
- ii Find $\tan \theta$.



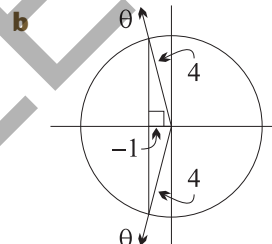
Let $\tan \theta = -\frac{5}{12}$, where θ is obtuse.

- i Find $\sin \theta$.
- ii Find $\cos \theta$.



Suppose that $\sin \theta = -\frac{\sqrt{7}}{4}$.

- i Find the possible values of $\cos \theta$.
- ii Find the possible values of $\tan \theta$.



Suppose that $\cos \theta = -\frac{1}{4}$.

- i Find the possible values of $\sin \theta$.
- ii Find the possible values of $\tan \theta$.

- 5** Draw similar diagrams for this question and subsequent questions.

- a** If $\cos \theta = \frac{1}{3}$ and θ is acute, find $\tan \theta$.
- c** If $\cos \theta = \frac{1}{2}$ and θ is reflex, find $\sin \theta$.
- e** Find $\sin \theta$, given that $\cos \theta = -\frac{40}{41}$ and $0^\circ \leq \theta \leq 180^\circ$.
- b** If $\cos \theta = -\frac{4}{5}$ and θ is obtuse, find $\tan \theta$.
- d** Find $\cos \theta$ if $\tan \theta = -\frac{2}{3}$ and θ is reflex.
- f** Find $\tan \theta$, given that $\sin \theta = \frac{1}{\sqrt{5}}$ and $-90^\circ \leq \theta \leq 90^\circ$.

DEVELOPMENT

- 6** In this question, each part has two possible answers.
- a** If $\tan \alpha = \frac{1}{3}$, find $\sin \alpha$. **b** If $\cos \theta = \frac{2}{\sqrt{5}}$, find $\sin \theta$.
- c** If $\sin \theta = \frac{3}{5}$, find $\cos \theta$. **d** Find $\tan \theta$, given that $\cos \theta = -\frac{2}{3}$.
- e** Find $\tan \theta$, given that $\sin \theta = -\frac{12}{13}$. **f** Find $\cos \theta$, given that $\tan \theta = -\frac{2}{\sqrt{3}}$.
- 7** **a** If $\cos \alpha = \frac{4}{5}$ and $\sin \alpha < 0$, find $\tan \alpha$.
b If $\tan \theta = -\frac{8}{15}$ and $\sin \theta > 0$, find $\cos \theta$.
c Find $\cos \theta$, given that $\sin \theta = \frac{1}{4}$ and $\tan \theta < 0$.
d Find $\sin \theta$, given that $\tan \theta = \frac{35}{12}$ and $\cos \theta > 0$.
e Find $\tan \theta$, given that $\sin \theta = -\frac{21}{29}$ and $\cos \theta > 0$.
f Find $\sin \theta$, given that $\cos \theta = -\frac{5}{6}$ and $\tan \theta < 0$.
- 8** **a** Find $\sec \theta$, given that $\sin \theta = \frac{1}{\sqrt{2}}$.
b Find $\tan \theta$, given that $\sec \theta = -\frac{17}{8}$.
c If $\sec C = -\frac{\sqrt{7}}{\sqrt{3}}$, find $\cot C$.
d If $\cot D = \frac{\sqrt{11}}{5}$, find $\operatorname{cosec} D$.
- 9** **a** Find $\sec \theta$, given that $\operatorname{cosec} \theta = \frac{3}{2}$ and θ is obtuse.
b Find $\sec \theta$, given that $\cot \theta = \frac{9}{40}$ and θ is reflex.
c Find $\tan \theta$, given that $\sec \theta = -\frac{17}{8}$ and $0^\circ \leq \theta \leq 180^\circ$.
d Find $\operatorname{cosec} \theta$, given that $\cot \theta = \frac{2}{\sqrt{3}}$ and $-90^\circ \leq \theta \leq 90^\circ$.
- 10** **a** If $\sin A = -\frac{1}{3}$ and $\tan A < 0$, find $\sec A$.
b If $\operatorname{cosec} B = \frac{7}{3}$ and $\cos B < 0$, find $\tan B$.
c Find $\cot \theta$, given that $\sec \theta = -\sqrt{2}$ and $\operatorname{cosec} \theta < 0$.
d Find $\cos \theta$, given that $\operatorname{cosec} \theta = -\frac{13}{5}$ and $\cot \theta < 0$.

CHALLENGE

- 11** Given that $\sin \theta = \frac{p}{q}$, with θ obtuse and p and q both positive, find $\cos \theta$ and $\tan \theta$.
- 12** If $\tan \alpha = k$, where $k > 0$, find the possible values of $\sin \alpha$ and $\sec \alpha$.
- 13** **a** Prove the algebraic identity $(1 - t^2)^2 + (2t)^2 = (1 + t^2)^2$.
b If $\cos x = \frac{1 - t^2}{1 + t^2}$, where x is acute and t is positive, find expressions for $\sin x$ and $\tan x$.

6F Trigonometric identities

Learning intentions

- Prove the reciprocal identities, ratio identities, and Pythagorean identities.
- Prove the complementary angle identities, and interpret the function names.
- Use these identities to prove further identities.

Working with the trigonometric functions requires knowledge of a number of formulae called *trigonometric identities*, which relate trigonometric functions to each other. This section introduces eleven trigonometric identities in four groups:

- the three *reciprocal identities*,
- the two *ratio identities*,
- the three *Pythagorean identities*,
- the three *identities concerning complementary angles*.

The three reciprocal identities

It follows immediately from the definitions of the trigonometric functions in terms of x , y and r that:

18 The reciprocal identities

For any angle θ :

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} \quad (\text{provided that } \sin \theta \neq 0)$$

$$\sec \theta = \frac{1}{\cos \theta} \quad (\text{provided that } \cos \theta \neq 0)$$

$$\cot \theta = \frac{1}{\tan \theta} \quad (\text{provided that } \tan \theta \neq 0 \text{ and } \cot \theta \neq 0)$$

Note: We cannot use a calculator to find $\cot 90^\circ$ or $\cot 270^\circ$ by first finding $\tan 90^\circ$ or $\tan 270^\circ$, because both are undefined. We already know, however, that

$$\cot 90^\circ = \cot 270^\circ = 0.$$

The two ratio identities

Again using the definitions of the trigonometric functions:

19 The ratio identities

For any angle θ :

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (\text{provided that } \cos \theta \neq 0)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{provided that } \sin \theta \neq 0)$$

The three Pythagorean identities

The point $P(x, y)$ lies on the circle with centre O and radius r , so its coordinates satisfy

$$x^2 + y^2 = r^2.$$

Dividing through by r^2 , $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$,

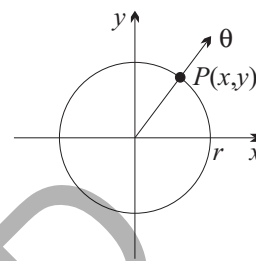
that is, $\sin^2 \theta + \cos^2 \theta = 1$.

Dividing through by $\cos^2 \theta$ and using the ratio and reciprocal identities,

$$\tan^2 \theta + 1 = \sec^2 \theta, \quad \text{provided that } \cos \theta \neq 0.$$

Dividing instead by $\sin^2 \theta$, $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$, provided that $\sin \theta \neq 0$.

These identities are called the *Pythagorean identities* because they rely on the circle equation $x^2 + y^2 = r^2$, which is a restatement of Pythagoras' theorem.



20 The Pythagorean identities

For any angle θ :

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta \quad (\text{provided that } \cos \theta \neq 0)$$

$$\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta \quad (\text{provided that } \sin \theta \neq 0)$$

The three identities for complementary angles

The angles θ and $90^\circ - \theta$ are called *complementary angles* because they add to a right angle. Three trigonometric identities relate the values of the trigonometric functions at an angle θ and at the complementary angle $90^\circ - \theta$.

21 The complementary angle identities

For any angle θ :

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\cot(90^\circ - \theta) = \tan \theta \quad (\text{provided that } \tan \theta \text{ is defined})$$

$$\operatorname{cosec}(90^\circ - \theta) = \sec \theta \quad (\text{provided that } \sec \theta \text{ is defined})$$

For example, $\cos 20^\circ = \sin 70^\circ$,

$$\operatorname{cosec} 20^\circ = \sec 70^\circ,$$

$$\cot 20^\circ = \tan 70^\circ.$$

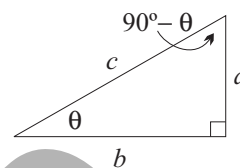
Proof

- A** [Acute angles] The triangle to the right shows that when a right-angled triangle is viewed from $90^\circ - \theta$ instead of from θ , then the opposite side and the adjacent side are exchanged. Hence

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin \theta,$$

$$\cot(90^\circ - \theta) = \frac{a}{b} = \tan \theta,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{c}{b} = \sec \theta.$$



- B** [General angles] For general angles, we take the full circle diagram, and reflect it in the diagonal line $y = x$. Let P' be the image of P under this reflection.

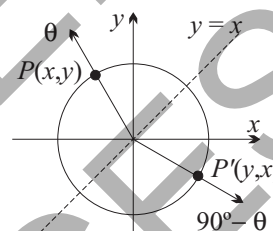
- 1 The image OP' of the ray OP corresponds to the angle $90^\circ - \theta$.
- 2 The image P' of $P(x, y)$ has coordinates $P'(y, x)$, because reflection in the line $y = x$ reverses the coordinates of each point.

Applying the definitions of the trigonometric functions to the angle $90^\circ - \theta$:

$$\cos(90^\circ - \theta) = \frac{y}{r} = \sin \theta,$$

$$\cot(90^\circ - \theta) = \frac{y}{x} = \tan \theta, \quad \text{provided that } x \neq 0,$$

$$\operatorname{cosec}(90^\circ - \theta) = \frac{r}{x} = \sec \theta, \quad \text{provided that } x \neq 0.$$



Cosine, cosecant and cotangent

The complementary identities are the origin of the names ‘cosine’, ‘cosecant’ and ‘cotangent’ — the prefix ‘co-’ has the same meaning as the prefix ‘com-’ of ‘complementary’ angle.

22 Cosine, cosecant and cotangent

$$\underline{\text{cosine}} \theta = \text{sine}(\underline{\text{complement of}} \theta)$$

$$\underline{\text{cotangent}} \theta = \text{tangent}(\underline{\text{complement of}} \theta)$$

$$\underline{\text{cosecant}} \theta = \text{secant}(\underline{\text{complement of}} \theta)$$

Proving identities

An *identity* is a statement that is true for all values of θ for which both sides are defined. An identity needs to be proven — it is quite different from an *equation*, which needs to be solved and to have its solutions listed.

23 Proving a trigonometric identity

- Work separately on the LHS and the RHS until they are the same.
- Use the four sets of identities in Boxes 18–21 above.
- *Never treat it as an equation, moving terms from one side to the other.*

Mostly it is only necessary to work on one of the two sides.

**Example 8** Proving an identity using the Pythagorean identities

Prove that $(1 - \cos \theta)(1 + \cos \theta) = \sin^2 \theta$.

Solution

$$\begin{aligned} \text{LHS} &= 1 - \cos^2 \theta && \text{(use the difference of squares identity, from algebra)} \\ &= \sin^2 \theta && \text{(use the Pythagorean identities in Box 20)} \\ &= \text{RHS.} \end{aligned}$$

**Example 9** Using the reciprocal and ratio identities

Prove that $\sin A \sec A = \tan A$.

Solution

$$\begin{aligned} \text{LHS} &= \sin A \times \frac{1}{\cos A} && \text{(use the reciprocal identities in Box 18)} \\ &= \tan A && \text{(use the ratio identities in Box 19)} \\ &= \text{RHS.} \end{aligned}$$

**Example 10** Using the Pythagorean and reciprocal identities

Prove that $\frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \sec^2 \theta \operatorname{cosec}^2 \theta$.

Solution

$$\begin{aligned} \text{LHS} &= \frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \\ &= \frac{\cos^2 \theta + \sin^2 \theta}{\sin^2 \theta \cos^2 \theta} && \text{(use a common denominator)} \\ &= \frac{1}{\sin^2 \theta \cos^2 \theta} && \text{(use the Pythagorean identities in Box 20)} \\ &= \sec^2 \theta \operatorname{cosec}^2 \theta && \text{(use the reciprocal identities in Box 18)} \\ &= \text{RHS.} \end{aligned}$$

Exercise 6F**FOUNDATION**

1 Use your calculator to verify that:

a $\sin 16^\circ = \cos 74^\circ$

b $\tan 63^\circ = \cot 27^\circ$

c $\sec 7^\circ = \operatorname{cosec} 83^\circ$

d $\sin^2 23^\circ + \cos^2 23^\circ = 1$

e $1 + \tan^2 55^\circ = \sec^2 55^\circ$

f $\operatorname{cosec}^2 32^\circ - 1 = \cot^2 32^\circ$

2 Simplify:

a $\frac{1}{\sin \theta}$

b $\frac{1}{\tan \alpha}$

c $\frac{\sin \beta}{\cos \beta}$

d $\frac{\cos \varphi}{\sin \varphi}$

3 Simplify:

a $\sin \alpha \operatorname{cosec} \alpha$

b $\cot \beta \tan \beta$

c $\cos \theta \sec \theta$

4 Prove:

a $\tan \theta \cos \theta = \sin \theta$

b $\cot \alpha \sin \alpha = \cos \alpha$

c $\sin \beta \sec \beta = \tan \beta$

5 Use the complementary identities to simplify:

a $\sin(90^\circ - \theta)$

b $\sec(90^\circ - \alpha)$

c $\frac{1}{\cot(90^\circ - \beta)}$

d $\frac{\cos(90^\circ - \varphi)}{\sin(90^\circ - \varphi)}$

6 Use the Pythagorean identities to simplify:

a $\sin^2 \alpha + \cos^2 \alpha$

b $1 - \cos^2 \beta$

c $1 + \tan^2 \varphi$

d $\sec^2 x - \tan^2 x$

7 Use the Pythagorean identities to simplify:

a $1 - \sin^2 \beta$

b $1 + \cot^2 \varphi$

c $\operatorname{cosec}^2 A - 1$

d $\cot^2 \theta - \operatorname{cosec}^2 \theta$

DEVELOPMENT

8 Use the reciprocal and ratio identities to simplify:

a $\frac{1}{\sec^2 \theta}$

b $\frac{\sin^2 \beta}{\cos^2 \beta}$

c $\frac{\cos^2 A}{\sin^2 A}$

d $\sin^2 \alpha \operatorname{cosec}^2 \alpha$

9 Use the reciprocal and ratio identities to prove:

a $\cos A \operatorname{cosec} A = \cot A$

b $\operatorname{cosec} x \cos x \tan x = 1$

c $\sin y \cot y \sec y = 1$

10 Use the reciprocal and ratio identities to simplify:

a $\frac{\cos \alpha}{\sec \alpha}$

b $\frac{\sin \alpha}{\operatorname{cosec} \alpha}$

c $\frac{\tan A}{\sec A}$

d $\frac{\cot A}{\operatorname{cosec} A}$

11 Prove the identities:

a $(1 - \sin \theta)(1 + \sin \theta) = \cos^2 \theta$

b $(1 + \tan^2 \alpha) \cos^2 \alpha = 1$

c $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$

d $\cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$

e $\tan^2 \varphi \cos^2 \varphi + \cot^2 \varphi \sin^2 \varphi = 1$

f $3 \cos^2 \theta - 2 = 1 - 3 \sin^2 \theta$

g $2 \tan^2 A - 1 = 2 \sec^2 A - 3$

h $1 - \tan^2 \alpha + \sec^2 \alpha = 2$

i $\cos^4 x + \cos^2 x \sin^2 x = \cos^2 x$

j $\cot \theta (\sec^2 \theta - 1) = \tan \theta$

12 Prove the identities:

a $\tan \alpha \operatorname{cosec} \alpha = \sec \alpha$

b $\cot \beta \sec \beta = \operatorname{cosec} \beta$

c $\operatorname{cosec}^2 \gamma + \sec^2 \gamma = \operatorname{cosec}^2 \gamma \sec^2 \gamma$

d $\tan \delta + \cot \delta = \operatorname{cosec} \delta \sec \delta$

e $\operatorname{cosec} \varphi - \sin \varphi = \cos \varphi \cot \varphi$

f $\sec \theta - \cos \theta = \tan \theta \sin \theta$

CHALLENGE

13 Prove the identities:

a $\sin \theta \cos \theta \operatorname{cosec}^2 \theta = \cot \theta$

b $(\cos \varphi + \cot \varphi) \sec \varphi = 1 + \operatorname{cosec} \varphi$

c $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$

d $\sin \beta + \cot \beta \cos \beta = \operatorname{cosec} \beta$

e $\frac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

f $\frac{1 + \cot x}{1 + \tan x} = \cot x$

g $\frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} = 2 \sec^2 \theta$

h $\frac{1}{\sec \varphi - \tan \varphi} - \frac{1}{\sec \varphi + \tan \varphi} = 2 \tan \varphi$

6G Trigonometric equations

Learning intentions

- Solve various types of trig equations.

This piece of work is absolutely vital, because so many problems in later work result in a trigonometric equation that has to be solved.

There are many small details and qualifications in the methods, and the subject needs a great deal of careful study.

Pay attention to any restriction on the angle

To begin with a warning, before any other details:

24 Restrictions on the angle

Pay careful attention to any restriction on the interval in which the angle can lie.

Equations involving boundary angles

Boundary angles are a special case because they do not lie in any quadrant.

25 The boundary angles

If the solutions are boundary angles, read the solutions off a sketch of the graph.



Example 11 Solving trig equations involving boundary angles

a Solve $\sin x = -1$, for $0^\circ \leq x \leq 720^\circ$.

b Solve $\sin x = 0$, for $0^\circ \leq x \leq 720^\circ$.

Solution

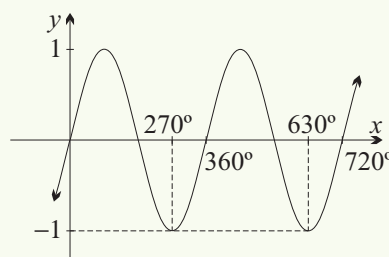
a The graph of $y = \sin x$ is drawn to the right.

Examine where the curve touches the line $y = -1$, and read off the x -coordinates of these points.

The solution is $x = 270^\circ$ or 630° .

b Examine where the graph crosses the x -axis.

The solution is $x = 0^\circ, 180^\circ, 360^\circ, 540^\circ$ or 720° .



The standard method — quadrants and the related acute angle

Most trigonometric equations eventually come down to one or more equations such as

$$\sin x = -\frac{1}{2}, \text{ where } -180^\circ \leq x \leq 180^\circ.$$

Provided that the angle is not a boundary angle, the method is:

26 The quadrants-and-related-angle method

- 1** Draw a quadrant diagram, then draw a ray in each quadrant that the angle could be in.
- 2** Find the related acute angle — only work with positive numbers here:
 - a** using special angles, or
 - b** using the calculator to find an approximation.

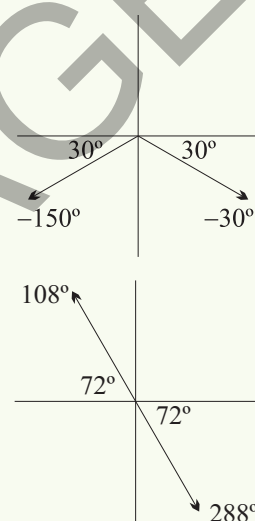
Never enter a negative number into the calculator at this point.
- 3** Mark the angles on the ends of the rays, taking account of any restrictions on x , and write a conclusion.

**Example 12 Solving trig equations — the standard method**

- a** Solve the equation $\sin x = -\frac{1}{2}$, for $-180^\circ \leq x \leq 180^\circ$.
- b** Solve $\tan x = -3$, for $0^\circ \leq x \leq 360^\circ$, correct to the nearest degree.

Solution

- a** Here $\sin x = -\frac{1}{2}$, where $-180^\circ \leq x \leq 180^\circ$.
 Because $\sin x$ is negative, x is in quadrant 3 or 4.
 The sine of the related acute angle is $+\frac{1}{2}$, so the related angle is 30° .
 Hence $x = -150^\circ$ or -30° .
- b** Here $\tan x = -3$, where $0^\circ \leq x \leq 360^\circ$.
 Because $\tan x$ is negative, x is in quadrant 2 or 4.
 The tangent of the related acute angle is $+3$, so the related angle is about 72° .
 Hence $x \doteq 108^\circ$ or 288° .



Note: When using the calculator, *never enter a negative number and take an inverse trigonometric function of it.*

In the example above, the calculator was used to find the *related acute angle* whose $\tan$ is 3, which is $71^\circ 34'$, correct to the nearest minute. The positive number 3 was entered, not -3 .

The three reciprocal functions

The calculator doesn't have specific keys for secant, cosecant and cotangent. Convert these functions to sine, cosine and tangent as quickly as possible.

27 The reciprocal functions

Take reciprocals to convert the three reciprocal functions secant, cosecant and cotangent to the three more common functions.


Example 13 Solving trig equations with the three reciprocal functions

a Solve $\operatorname{cosec} x = -2$, for $-180^\circ \leq x \leq 180^\circ$.

b Solve $\sec x = 0.7$, for $-180^\circ \leq x \leq 180^\circ$.

Solution

a Taking reciprocals of both sides gives

$$\sin x = -\frac{1}{2},$$

which was solved in the previous example,

so $x = -150^\circ$ or -30° .

b Taking reciprocals of both sides gives

$$\cos x = \frac{10}{7},$$

which has no solutions, because $\cos \theta$ can never be greater than 1.

Equations with compound angles

In some equations, the angle is a function of x rather than simply x itself. For example,

$$\tan 2x = \sqrt{3}, \text{ where } 0^\circ \leq x \leq 360^\circ, \text{ or}$$

$$\sin(x - 250^\circ) = \frac{\sqrt{3}}{2}, \text{ where } 0^\circ \leq x \leq 360^\circ.$$

These equations are really trigonometric equations in the *compound angles* $2x$ and $(x - 250^\circ)$ respectively. The secret lies in solving for the *compound angle*, and in *first calculating any restrictions for that compound angle*.

28 Equations with compound angles

- 1** Let u be the compound angle.
- 2** Find any restrictions on u from the given restrictions on x .
- 3** Solve the trigonometric equation for u .
- 4** Hence solve for x .


Example 14 Solving trig equations with compound angles

a Solve $\tan 2x = \sqrt{3}$, where $0^\circ \leq x \leq 360^\circ$.

b Solve $\sin(x - 250^\circ) = \frac{\sqrt{3}}{2}$, where $0^\circ \leq x \leq 360^\circ$.

Solution

a Let $u = 2x$.

Then $\tan u = \sqrt{3}$.

The restriction on x is $0^\circ \leq x \leq 360^\circ$

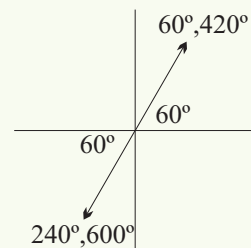
$$\boxed{\times 2} \quad 0^\circ \leq 2x \leq 720^\circ$$

and replacing $2x$ by u , $0^\circ \leq u \leq 720^\circ$.

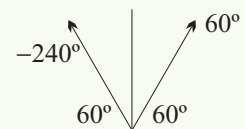
(The restriction on u is the key step here.)

Hence from the diagram, $u = 60^\circ, 240^\circ, 420^\circ$ or 600° .

Because $x = \frac{1}{2}u$, $x = 30^\circ, 120^\circ, 210^\circ$ or 300° .



b Let $u = x - 250^\circ$.
 Then $\sin u = \frac{\sqrt{3}}{2}$.
 The restriction on x is $0^\circ \leq x \leq 360^\circ$
 $-250^\circ \leq x - 250^\circ \leq 110^\circ$
 and replacing $x - 250^\circ$ by u , $-250^\circ \leq u \leq 110^\circ$.
 (Again, the restriction on u is the key step here.)
 Hence from the diagram, $u = -240^\circ$ or 60° .
 Because $x = u + 250^\circ$, $x = 10^\circ$ or 310° .



Equations with more than one trigonometric function

Some trigonometric equations involve more than one trigonometric function, as in

$$\sin x + \sqrt{3} \cos x = 0.$$

The general approach is to use trigonometric identities to produce an equation in only one trigonometric function.



Example 15 Trig equations with more than one trig function

Solve $\sin x + \sqrt{3} \cos x = 0$, where $0^\circ \leq x \leq 360^\circ$.

Solution

$$\begin{aligned} \sin x + \sqrt{3} \cos x &= 0 \\ \div \cos x \quad \tan x + \sqrt{3} &= 0 \\ \tan x &= -\sqrt{3}, \quad \text{where } 0^\circ \leq x \leq 360^\circ. \end{aligned}$$

Because $\tan x$ is negative, x is in quadrants 2 or 4.

The $\tan$ of the related acute angle is $\sqrt{3}$, so the related angle is 60° .

Hence $x = 120^\circ$ or 300° .

Exercise 6G

FOUNDATION

1 Solve each equation for $0^\circ \leq \theta \leq 360^\circ$. (Each related acute angle is 30° , 45° or 60° .)

a $\sin \theta = \frac{\sqrt{3}}{2}$

b $\sin \theta = \frac{1}{2}$

c $\tan \theta = 1$

d $\tan \theta = \sqrt{3}$

e $\cos \theta = -\frac{1}{\sqrt{2}}$

f $\tan \theta = -\sqrt{3}$

g $\sin \theta = -\frac{1}{2}$

h $\cos \theta = -\frac{\sqrt{3}}{2}$

2 Solve each equation for $0^\circ \leq \theta \leq 360^\circ$. (The trigonometric graphs are helpful here.)

a $\sin \theta = 1$

b $\cos \theta = 1$

c $\cos \theta = 0$

d $\cos \theta = -1$

e $\tan \theta = 0$

f $\sin \theta = -1$

3 Solve each equation for $0^\circ \leq x \leq 360^\circ$. Use your calculator to find the related acute angle in each case, and give solutions correct to the nearest degree.

a $\cos x = \frac{3}{7}$

b $\sin x = 0.1234$

c $\tan x = 7$

d $\sin x = -\frac{2}{3}$

e $\tan x = -\frac{20}{9}$

f $\cos x = -0.77$

DEVELOPMENT

- 4 Solve for $0^\circ \leq \alpha \leq 360^\circ$. Give solutions correct to the nearest minute where necessary.
- | | |
|---|---|
| a $\sin \alpha = 0.1$ | b $\cos \alpha = -0.1$ |
| c $\tan \alpha = -1$ | d $\operatorname{cosec} \alpha = -1$ |
| e $\sin \alpha = 3$ | f $\sec \alpha = -2$ |
| g $\sqrt{3} \tan \alpha + 1 = 0$ | h $\cot \alpha = 3$ |
- 5 Solve for $-180^\circ \leq x \leq 180^\circ$. Give solutions correct to the nearest minute where necessary.
- a** $\tan x = -0.3$
b $\cos x = 0$
c $\sec x = \sqrt{2}$
d $\sin x = -0.7$
- 6 Solve each equation for $0^\circ \leq \theta \leq 720^\circ$.
- a** $2 \cos \theta - 1 = 0$
b $\cot \theta = 0$
c $\operatorname{cosec} \theta + 2 = 0$
d $\tan \theta = \sqrt{2} - 1$

CHALLENGE

- 7 Solve each equation for $0^\circ \leq x \leq 360^\circ$. (Let $u = 2x$.)
- a** $\sin 2x = \frac{1}{2}$
b $\tan 2x = \sqrt{3}$
c $\cos 2x = -\frac{1}{\sqrt{2}}$
d $\sin 2x = -1$
- 8 Solve each equation for $0^\circ \leq \alpha \leq 360^\circ$. (Let u be the compound angle.)
- a** $\tan(\alpha - 45^\circ) = \frac{1}{\sqrt{3}}$
b $\sin(\alpha + 30^\circ) = -\frac{\sqrt{3}}{2}$
c $\cos(\alpha + 60^\circ) = 1$
d $\cos(\alpha - 75^\circ) = -\frac{1}{\sqrt{2}}$
- 9 Solve each equation for $0^\circ \leq \theta \leq 360^\circ$.
- a** $\sin \theta = \cos \theta$
b $\sin \theta + \cos \theta = 0$
c $\sin \theta = \sqrt{3} \cos \theta$
d $\sqrt{3} \sin \theta + \cos \theta = 0$

6H The sine rule and the area formula

Learning intentions

- Prove the sine rule and the area formula.
- Use the sine rule and the area formula to solve non-right-angled triangles.
- Work with the ambiguities with these formulae in the ASS situation.

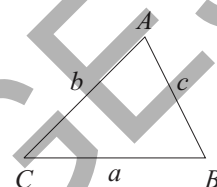
These last three sections review the sine rule, the area formula and the cosine rule. These three rules extend trigonometry to non-right-angled triangles.

The three formulae are closely related to the standard congruence tests:

SSS, SAS, AAS, RHS

from Euclidean geometry, and to the spurious 'ASS non-test' and its ambiguities.

The usual statement of all three rules uses the convention shown in the diagram to the right. The vertices are named with upper-case letters, then each side takes its name from the lower-case letter of the opposite vertex.



Statement of the sine rule

The sine rule states that the ratio of each side of a triangle to the sine of its opposite angle is constant for the triangle:

29 The sine rule

In any triangle ABC ,

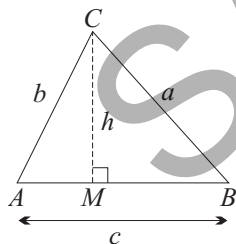
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

'The ratio of each side to the sine of the opposite angle is constant.'

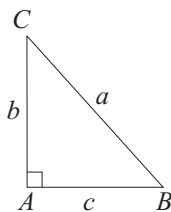
Proving the sine rule by constructing an altitude

Any proof of the sine rule must involve a construction with a right angle, because trigonometry so far has been restricted to right-angled triangles. Thus we construct an *altitude*, which is the perpendicular from one vertex to the opposite side. This breaks the triangle into two right-angled triangles, where previous methods can be used.

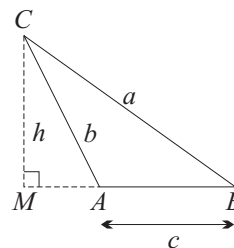
Given: Let ABC be any triangle. There are three cases, depending on whether $\angle A$ is an acute angle, a right angle, or an obtuse angle.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90^\circ$



Case 3: $\angle A$ is obtuse

Aim: To prove that $\frac{a}{\sin A} = \frac{b}{\sin B}$.

In case 2, $\sin A = \sin 90^\circ = 1$, and $\sin B = \frac{b}{a}$, so the result is clear.

Construction: In the remaining cases 1 and 3, construct the altitude from C , meeting AB (produced if necessary) at M . Let h be the length of CM .

Proof

Case 1 — Suppose that $\angle A$ is acute.

$$\begin{array}{l} \text{In } \triangle ACM, \quad \frac{h}{b} = \sin A \\ \boxed{\times b} \quad \quad \quad h = b \sin A. \end{array}$$

$$\begin{array}{l} \text{In } \triangle BCM, \quad \frac{h}{a} = \sin B \\ \boxed{\times a} \quad \quad \quad h = a \sin B. \end{array}$$

$$\begin{array}{l} \text{Equating these,} \quad b \sin A = a \sin B \\ \quad \quad \quad \frac{b}{\sin B} = \frac{a}{\sin A}. \end{array}$$

Case 3 — Suppose that $\angle A$ is obtuse.

$$\begin{array}{l} \text{In } \triangle ACM, \quad \frac{h}{b} = \sin(180^\circ - A), \\ \text{and because} \quad \sin(180^\circ - A) = \sin A, \end{array}$$

$$\begin{array}{l} \boxed{\times b} \quad \quad \quad h = b \sin A. \\ \frac{h}{a} = \sin B \\ h = a \sin B. \end{array}$$

$$\begin{array}{l} \boxed{\times a} \quad \quad \quad b \sin A = a \sin B \\ \text{Equating these,} \quad \quad \quad \frac{b}{\sin B} = \frac{a}{\sin A}. \end{array}$$

Using the sine rule to find a side — the AAS congruence situation

When using the sine rule to find a side, one side and two angles must be known. This is the situation described by the *AAS congruence test* from geometry, so we know that there will only be one solution.

30 Using the sine rule to find a side

In the AAS congruence situation:

$$\frac{\text{unknown side}}{\text{sine of its opposite angle}} = \frac{\text{known side}}{\text{sine of its opposite angle}}.$$

Always place the unknown side at the top left of the equation.

If two angles of a triangle are known, so is the third, because the angles add to 180° .

Example 16 Using the sine rule to find a side length

Find the side x in the triangle drawn to the right.

Solution

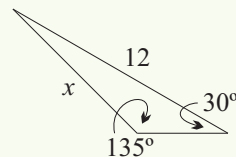
Using the sine rule, and placing the unknown at the top left,

$$\begin{array}{l} \frac{x}{\sin 30^\circ} = \frac{12}{\sin 135^\circ} \\ \boxed{\times \sin 30^\circ} \quad \quad x = \frac{12 \sin 30^\circ}{\sin 135^\circ}. \end{array}$$

$$\text{Using special angles,} \quad \sin 30^\circ = \frac{1}{2},$$

$$\begin{array}{l} \text{and} \quad \sin 135^\circ = +\sin 45^\circ \quad \quad \quad (\text{sine is positive for obtuse angles}) \\ \quad \quad = \frac{1}{\sqrt{2}}. \quad \quad \quad (\text{the related acute angle is } 45^\circ) \end{array}$$

$$\begin{array}{l} \text{Hence} \quad x = 12 \times \frac{1}{2} \times \frac{\sqrt{2}}{1} \\ \quad \quad = 6\sqrt{2}. \end{array}$$



Using the sine rule to find an angle — the ambiguous ASS situation

The SAS congruence test requires that the angle be included between the two sides. When two sides and a non-included angle are known, the resulting triangle may not be determined up to congruence, and two triangles may be possible. This situation may be referred to as ‘the ambiguous ASS situation’.

When the sine rule is applied in the ambiguous ASS situation, there is only one answer for the sine of an angle. There may be two possible solutions for the angle itself, however, one acute and one obtuse.

31 Using the sine rule to find an angle

In the ambiguous ASS situation, where two sides and a non-included angle of the triangle are known,

$$\frac{\text{sine of unknown angle}}{\text{its opposite side}} = \frac{\text{sine of known angle}}{\text{its opposite side}}.$$

Always check the angle sum to see whether both answers are possible.

Example 17 Using the sine rule in the ambiguous ASS situation

Find the angle θ in the triangle drawn to the right.

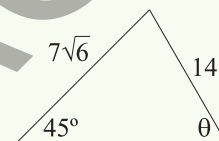
Solution

$$\frac{\sin \theta}{7\sqrt{6}} = \frac{\sin 45^\circ}{14} \quad (\text{always place the unknown at the top left})$$

$$\sin \theta = 7 \times \sqrt{6} \times \frac{1}{14} \times \frac{1}{\sqrt{2}}, \quad \text{because } \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ \text{ or } 120^\circ.$$

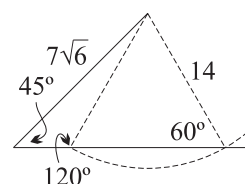


Note: There are two angles whose sine is $\frac{\sqrt{3}}{2}$, one of them acute and the other obtuse. Moreover,

$$120^\circ + 45^\circ = 165^\circ,$$

leaving just 15° for the third angle in the obtuse case, so it all seems to work.

Opposite is the ruler-and-compasses construction of the triangle, showing how two different triangles can be produced from the same given ASS measurements.



In many examples, however, the obtuse angle solution can be excluded using the fact that the angle sum of a triangle cannot exceed 180° .

Example 18 Finding only one result in the ambiguous ASS situation

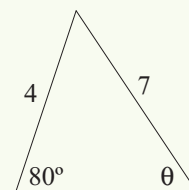
Find the angle θ in the triangle drawn to the right.

Solution

$$\frac{\sin \theta}{4} = \frac{\sin 80^\circ}{7} \quad (\text{always place the unknown at the top left})$$

$$\sin \theta = \frac{4 \sin 80^\circ}{7}$$

$$\theta \doteq 34^\circ 15' \text{ or } 145^\circ 45'$$



But $\theta \doteq 145^\circ 45'$ is impossible, because the angle sum would then exceed 180° , so $\theta \doteq 34^\circ 15'$ is the only solution.

The area formula

The standard formula for the area of a triangle is well known:

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}.$$

This can be developed to a formula involving two sides and the included angle.

32 The area formula for a triangle

In any triangle ABC ,

$$\text{area } \triangle ABC = \frac{1}{2}bc \sin A.$$

‘The area of a triangle is half the product of any two sides times the sine of the included angle.’

Proof Use the same three diagrams as in the proof of the sine rule.

In case 2, $\angle A = 90^\circ$ and $\sin A = 1$, so $\text{area} = \frac{1}{2}bc = \frac{1}{2}bc \sin A$, as required.

$$\begin{aligned} \text{Otherwise, area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times AB \times h \\ &= \frac{1}{2} \times c \times b \sin A, \quad \text{because we proved before that } h = b \sin A. \end{aligned}$$

Using the area formula — the SAS congruence situation

The area formula requires the *SAS congruence situation* in which two sides and the included angle are known.

33 Using the area formula

In the SAS congruence situation:

$$\text{area} = (\text{half the product of two sides}) \times (\text{sine of the included angle}).$$



Example 19 Finding the area of a triangle

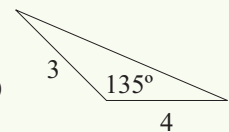
Find the area of the triangle drawn to the right.

Solution

Using the formula, $\text{area} = \frac{1}{2} \times 3 \times 4 \times \sin 135^\circ$.

Because $\sin 135^\circ = \frac{1}{\sqrt{2}}$, (as in the previous worked exercise)

$$\begin{aligned} \text{area} &= 6 \times \frac{1}{\sqrt{2}} \\ &= 6 \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \quad (\text{rationalise the denominator}) \\ &= 3\sqrt{2} \text{ square units.} \end{aligned}$$



Using the area formula to find a side

Substituting a known area into the area formula may allow an unknown side to be found.



Example 20 Using the area formula to find a side

Find x , correct to four significant figures, given that the triangle to the right has area 72 m^2 .

Solution

Substituting into the area formula,

$$72 = \frac{1}{2} \times 24 \times x \times \sin 67^\circ$$

$$72 = 12 \times x \times \sin 67^\circ$$

$$\begin{aligned} \div 12 \sin 67^\circ \quad x &= \frac{6}{\sin 67^\circ} \\ &\doteq 6.518 \text{ metres.} \end{aligned}$$



Using the area formula to find an angle

Substituting a known area into the area formula when two sides are known, will allow $\sin \theta$ to be found uniquely, where θ is the included angle. But unless $\theta = 90^\circ$, there will be two answers for θ , supplementary to each other.



Example 21 Using the area formula to find an angle

Find θ , correct to the nearest minute, given that the triangle to the right has area 60 cm^2 .

Solution

Substituting into the area formula,

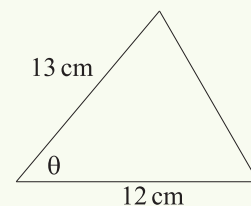
$$60 = \frac{1}{2} \times 13 \times 12 \times \sin \theta$$

$$60 = 6 \times 13 \sin \theta$$

$$\div (6 \times 13) \quad \sin \theta = \frac{10}{13}$$

$$\text{Hence} \quad \theta \doteq 50^\circ 17' \text{ or } 129^\circ 43'.$$

Notice that the second angle is the supplement of the first.



34 Two ambiguous situations

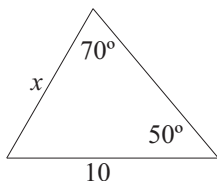
- When finding an angle in the ASS situation, $\sin \theta$ will be unique, but there may be two possible answers for θ , supplementary to each other.
- When finding an angle using the area formula, $\sin \theta$ will be unique, but unless $\theta = 90^\circ$, there will be two possible answers for θ , supplementary to each other.

Exercise 6H

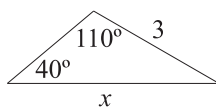
FOUNDATION

1 Find x in each triangle, correct to one decimal place.

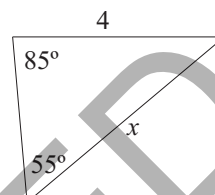
a



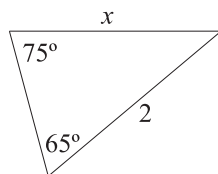
b



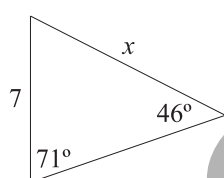
c



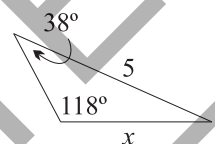
d



e

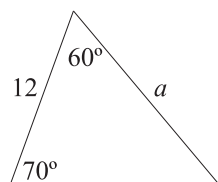


f

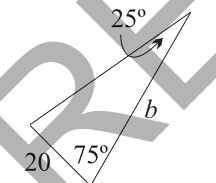


2 Find the value of the pronumeral in each triangle, correct to two decimal places.

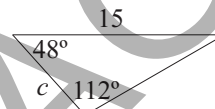
a



b

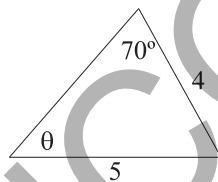


c

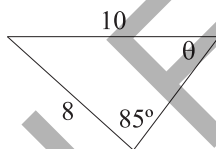


3 Find θ in each triangle, correct to the nearest degree.

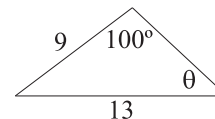
a



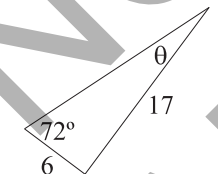
b



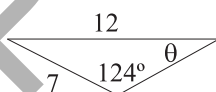
c



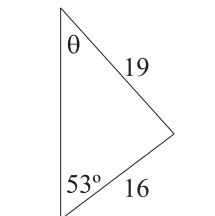
d



e

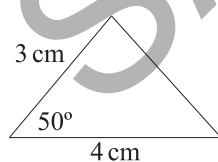


f

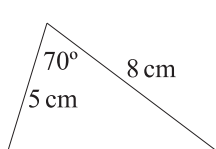


4 Find the area of each triangle, correct to the nearest square centimetre.

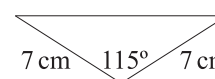
a



b



c



DEVELOPMENT

- 5 a Sketch $\triangle ABC$ in which $A = 43^\circ$, $B = 101^\circ$ and $a = 7.5$ cm.

b Find b and c , in cm correct to two decimal places.

- 6 a Sketch $\triangle XYZ$ in which $y = 32$ cm, $Y = 58^\circ$ and $Z = 52^\circ$.

b Find the perimeter of $\triangle XYZ$, correct to the nearest centimetre.

- 7 Sketch $\triangle ABC$ in which $a = 2.8$ cm, $b = 2.7$ cm and $A = 52^\circ 21'$.

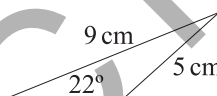
a Find B , correct to the nearest minute.

b Hence find C , correct to the nearest minute.

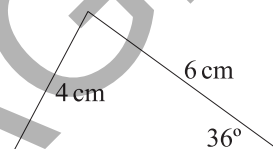
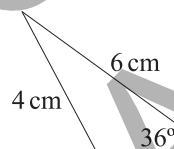
c Hence find the area of $\triangle ABC$ in cm^2 , correct to two decimal places.

- 8 There are two triangles that have sides 9 cm and 5 cm, and in which the angle opposite the 5 cm side is 22° .

Find, in each case, the size of the angle opposite the 9 cm side, correct to the nearest degree.



- 9 Two triangles are shown, with sides 6 cm and 4 cm, in which the angle opposite the 4 cm side is 36° . Find, in each case, the angle opposite the 6 cm side, correct to the nearest degree.



- 10 Sketch $\triangle PQR$ in which $p = 7$ cm, $q = 15$ cm and $\angle P = 25^\circ 50'$.

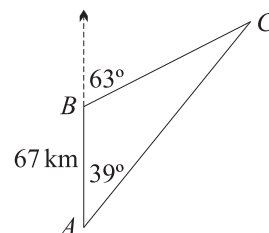
a Find the two possible sizes of $\angle Q$, correct to the nearest minute.

b For each possible size of $\angle Q$, find r in cm, correct to one decimal place.

- 11 A travelling salesman drove from town A to town B, then to town C, and finally directly home to town A.

Town B is 67 km north of town A, and the bearings of town C from towns A and B are 039°T and 063°T respectively.

Find, correct to the nearest kilometre, how far the salesman drove.

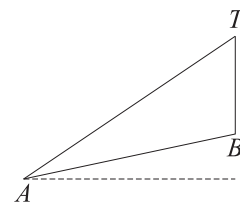


- 12 Melissa is standing at A on a path that leads to the base B of a vertical flagpole.

The path is inclined at 12° to the horizontal and the angle of elevation of the top T of the flagpole from A is 34° .

a Explain why $\angle TAB = 22^\circ$ and $\angle ABT = 102^\circ$.

b Given that $AB = 20$ metres, find the height of the flagpole, correct to the nearest metre.

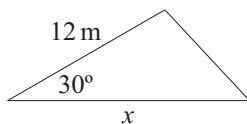


- 13 a In $\triangle ABC$, $\sin A = \frac{1}{4}$, $\sin B = \frac{2}{3}$ and $a = 12$. Find the value of b .

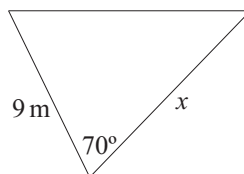
b In $\triangle PQR$, $p = 25$, $q = 21$ and $\sin Q = \frac{3}{5}$. Find the value of $\sin P$.

- 14** Substitute into the area formula to find the side length x , given that each triangle has area 48 m^2 . Give your answers in exact form, or correct to the nearest centimetre.

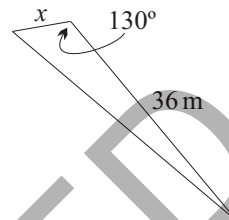
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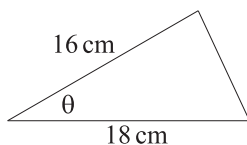


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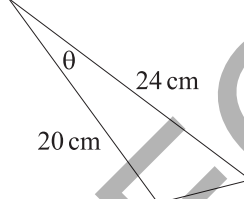


- 15** Substitute into the area formula to find the angle θ , given that each triangle has area 72 cm^2 . Give answers correct to the nearest minute, where appropriate.

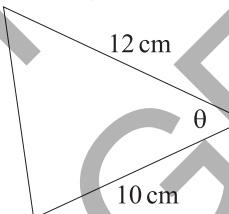
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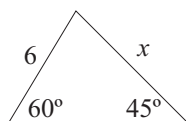
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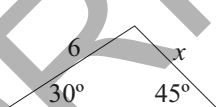
CHALLENGE

- 16** Find the exact value of x in each diagram.

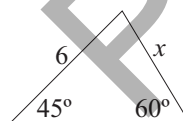
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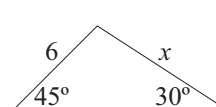
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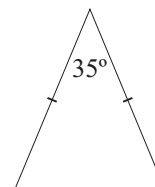
c



d



- 17** The diagram to the right shows an isosceles triangle in which the apex angle is 35° . Its area is 35 cm^2 . Find the length of the equal sides, correct to the nearest millimetre.



6I The cosine rule

Learning intentions

- Prove the cosine rule using Pythagoras' theorem.
- Understand the cosine rule as Pythagoras' theorem with correction term.
- Use the cosine rule to find a side or an angle of a triangle.

The cosine rule is a generalisation of *Pythagoras' theorem* to non-right-angled triangles. It gives a formula for the square of any side in terms of the squares of the other two sides and the cosine of the opposite angle.

The cosine rule and Pythagoras' theorem

35 The cosine rule

In any triangle ABC ,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

'The square of any side of a triangle equals:
the sum of the squares of the other two sides, minus
twice the product of those sides and the cosine of their included angle.'

The formula should be understood as Pythagoras' theorem with an error term $-2bc \cos A$ that disappears when $\angle A = 90^\circ$, leaving Pythagoras theorem.

36 The cosine rule and Pythagoras' theorem

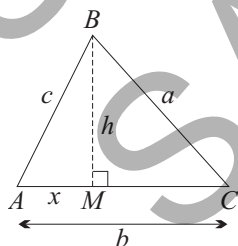
- When $\angle A = 90^\circ$, then $\cos A = 0$ and the cosine rule is Pythagoras' theorem.
- The last term is thus a *correction to Pythagoras' theorem* when $\angle A \neq 90^\circ$.
- When $\angle A < 90^\circ$, then $\cos A$ is positive, so $a^2 < b^2 + c^2$.

When $\angle A > 90^\circ$, then $\cos A$ is negative, so $a^2 > b^2 + c^2$.

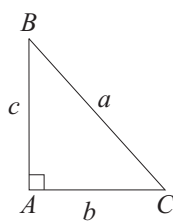
Proof of the cosine rule

The proof is based on Pythagoras' theorem, and again begins with the construction of an altitude.

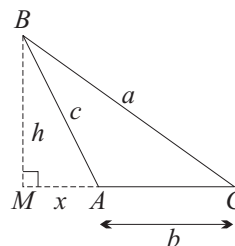
Given: Let ABC be any triangle. Again, there are three cases, according as to whether $\angle A$ is acute, obtuse, or a right angle.



Case 1: $\angle A$ is acute



Case 2: $\angle A = 90^\circ$



Case 3: $\angle A$ is obtuse

Aim: To prove that $a^2 = b^2 + c^2 - 2bc \cos A$.

In case 2, $\cos A = 0$, and this is just Pythagoras' theorem.

Construction: In the remaining cases 1 and 3, construct the altitude from B , meeting AC , produced if necessary, at M . Let $BM = h$ and $AM = x$.

Proof

Case 1 — Suppose that $\angle A$ is acute.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b - x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\begin{aligned}\text{so } a^2 &= c^2 - x^2 + (b - x)^2 \\ &= c^2 - x^2 + b^2 - 2bx + x^2 \\ &= b^2 + c^2 - 2bx. \quad (*)\end{aligned}$$

Using trigonometry in $\triangle ABM$,

$$x = c \cos A.$$

$$\text{So } a^2 = b^2 + c^2 - 2bc \cos A.$$

Case 3 — Suppose that $\angle A$ is obtuse.

By Pythagoras' theorem in $\triangle BMC$,

$$a^2 = h^2 + (b + x)^2.$$

By Pythagoras' theorem in $\triangle BMA$,

$$h^2 = c^2 - x^2,$$

$$\begin{aligned}\text{so } a^2 &= c^2 - x^2 + (b + x)^2 \\ &= c^2 - x^2 + b^2 + 2bx + x^2 \\ &= b^2 + c^2 + 2bx. \quad (*)\end{aligned}$$

Using trigonometry in $\triangle ABM$,

$$x = c \cos(180^\circ - A)$$

$$= -c \cos A.$$

$$\text{So } a^2 = b^2 + c^2 - 2bc \cos A.$$

Note: The identity $\cos(180^\circ - A) = -\cos A$ is the key step in Case 3 of the proof. The cosine rule appears in Euclid's geometry book, but without any mention of the cosine ratio — the form given there is approximately the two statements in the proof marked with (*).

Using the cosine rule to find a side — the SAS situation

For the cosine rule to be applied to find a side, the other two sides and their included angle must be known. This is the *SAS congruence situation*.

37 Using the cosine rule to find a side

In the SAS congruence situation:

(square of any side) = (sum of squares of other two sides)

– (twice the product of those sides) $\times$ (cosine of their included angle).

Example 22 Using the cosine rule to find a side

Find x in the triangle drawn to the right.

Solution

Applying the cosine rule to the triangle,

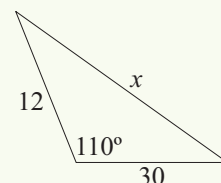
$$\begin{aligned}x^2 &= 12^2 + 30^2 - 2 \times 12 \times 30 \times \cos 110^\circ \\ &= 144 + 900 - 720 \cos 110^\circ \\ &= 1044 - 720 \cos 110^\circ,\end{aligned}$$

and because $\cos 110^\circ = -\cos 70^\circ$, (cosine is negative in the second quadrant)

$$x^2 = 1044 + 720 \cos 70^\circ \quad (\text{until this point, all calculations have been exact})$$

Using the calculator to approximate x^2 , and then to take the square root,

$$x \doteq 35.92.$$



Using the cosine rule to find an angle — the SSS situation

To use the cosine rule to find an angle, all three sides need to be known, which is the *SSS congruence test*. Finding the angle is done most straightforwardly by substituting into the usual form of the cosine rule:

38 Using the cosine rule to find an angle

In the SSS congruence situation:

- Substitute into the cosine rule and solve for $\cos \theta$.

There is an alternative approach. Solving the cosine rule for $\cos A$ gives a formula for $\cos A$. Some readers may prefer to remember and apply this second form of the cosine rule — but the triangle may then need to be relabelled.

39 The cosine rule with $\cos A$ as subject

In any triangle ABC ,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Notice that $\cos \theta$ is positive when θ is acute, and is negative when θ is obtuse. Hence there is only ever one solution for the unknown angle, unlike the situation for the sine rule, where there are often two possible angles.



Example 23 Using the cosine rule to find an angle

Find θ in the triangle drawn to the right.

Solution

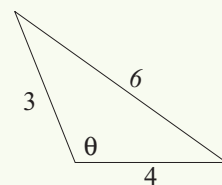
Substituting into the cosine rule,

$$\begin{aligned} 6^2 &= 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos \theta \\ 24 \cos \theta &= -11 \\ \cos \theta &= -\frac{11}{24} \\ \theta &\doteq 117^\circ 17'. \end{aligned}$$

OR

Using the boxed formula,

$$\begin{aligned} \cos \theta &= \frac{3^2 + 4^2 - 6^2}{2 \times 3 \times 4} \\ &= \frac{-11}{24} \\ \theta &\doteq 117^\circ 17'. \end{aligned}$$

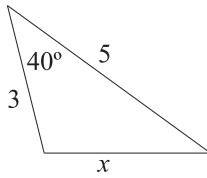


Exercise 6I

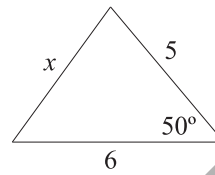
FOUNDATION

1 Find x in each triangle, correct to one decimal place.

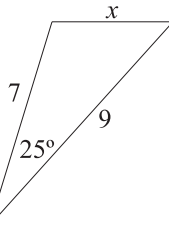
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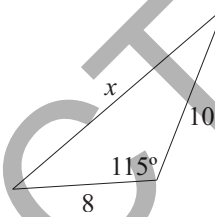
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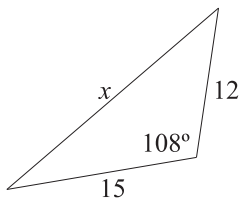
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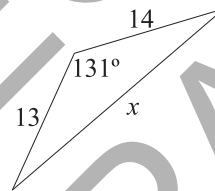
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e

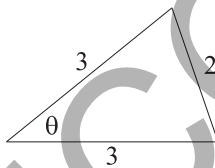


f

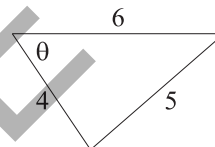


2 Find θ in each triangle, correct to the nearest degree.

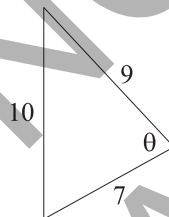
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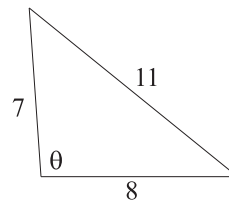
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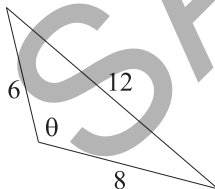
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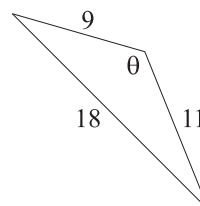
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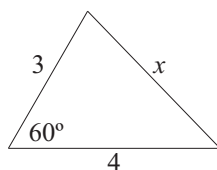
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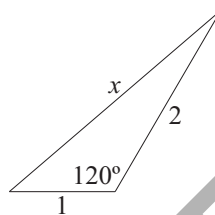
DEVELOPMENT

- 3 Using the fact that $\cos 60^\circ = \frac{1}{2}$ and $\cos 120^\circ = -\frac{1}{2}$, find x as a surd in each triangle.

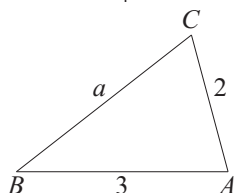
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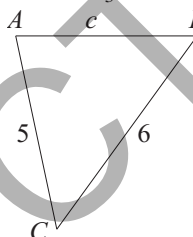
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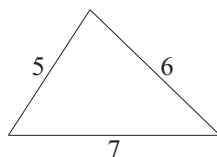
- 4 a If $\cos A = \frac{1}{4}$, find the exact value of a .



- b If $\cos C = \frac{2}{3}$, find the exact value of c .



5 a



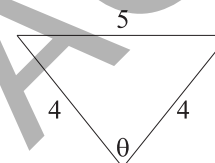
Find the smallest angle of the triangle, correct to the nearest minute.

b



Find the largest angle of the triangle, correct to the nearest minute.

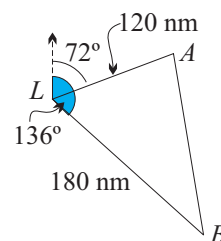
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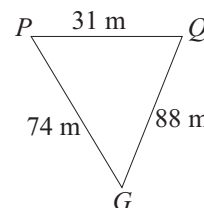
Find the value of $\cos \theta$.

- 6 There are three landmarks, P , Q and R . It is known that R is 8.7 km from P and 9.3 km from Q , and that $\angle PRQ = 79^\circ 32'$. Draw a diagram and find the distance between P and Q , in kilometres correct to one decimal place.

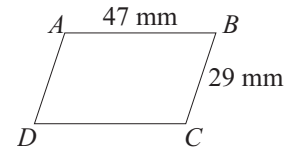
- 7 In the diagram to the right, ship A is 120 nautical miles from a lighthouse L on a bearing of 072°T , while ship B is 180 nautical miles from L on a bearing of 136°T . Calculate the distance between the two ships, correct to the nearest nautical mile.



- 8 A golfer at G wishes to hit a shot between two trees P and Q , as shown in the diagram opposite. The trees are 31 metres apart, and the golfer is 74 metres from P and 88 metres from Q . Find the angle within which the golfer must play the shot, correct to the nearest degree.



- 9** A parallelogram $ABCD$ has sides $AB = DC = 47$ mm and $AD = BC = 29$ mm. The longer diagonal BD is 60 mm.



- a** Use the cosine rule to find the size of $\angle BCD$.
b Use co-interior angles to find the size of $\angle ABC$.
 Give your answers correct to the nearest minute

- 10** The sides of a triangle are in the ratio 5 : 16 : 19. Find the smallest and largest angles of the triangle, correct to the nearest minute where necessary.

- 11** In $\triangle ABC$, $a = 31$ units, $b = 24$ units and $\cos C = \frac{59}{62}$.

- a** Show that $c = 11$ units. **b** Show that $A = 120^\circ$.

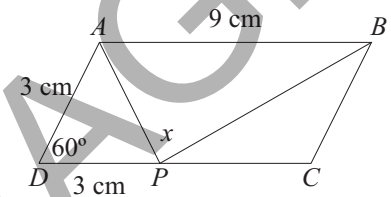
- 12** In $\triangle PQR$, $p = 5\sqrt{3}$ cm, $q = 11$ cm and $R = 150^\circ$.

- a** Find r . **b** Find $\cos P$.

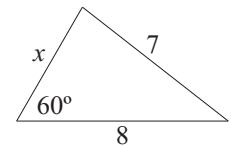
CHALLENGE

- 13** In a parallelogram $ABCD$, $\angle ADC = 60^\circ$, $AB = 9$ cm and $AD = 3$ cm. The point P lies on DC such that $DP = 3$ cm.

- a** Explain why $\triangle ADP$ is equilateral, and hence find AP .
b Use the cosine rule in $\triangle BCP$ to find BP .
c Let $\angle APB = x$. Show that $\cos x = -\frac{\sqrt{7}}{14}$.

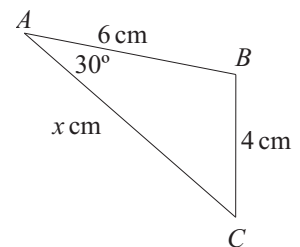


- 14** Use the cosine rule to find the two possible values of x in the diagram to the right.



- 15** The diagram shows $\triangle ABC$ in which $\angle A = 30^\circ$, $AB = 6$ cm and $BC = 4$ cm. Let $AC = x$ cm.

- a** Use the cosine rule to show that $x^2 - 6\sqrt{3}x + 20 = 0$.
b Use the quadratic formula to show that AC has length $3\sqrt{3} + \sqrt{7}$ cm or $3\sqrt{3} - \sqrt{7}$ cm.
c Copy the diagram and indicate on it (approximately) the other possible position of the point C .



6J Problems involving general triangles

Learning intentions

- Work with the relationship between congruence tests and the three formulae.

A triangle has three lengths and three angles, and most triangle problems involve using three of these six measurements to calculate some of the others. The key to deciding which formula to use is to see which congruence situation applies.

Trigonometry and the congruence tests

There are four standard congruence tests — RHS, SSS, SAS and AAS. These tests can also be seen as theorems about constructing triangles from given data.

Thus if three measurements of a triangle are known, including at least one length, then apart from the ambiguous ASS situation, any two triangles with these three measurements are *congruent*, and the other sides and angles can be calculated.

40 The standard congruence tests, and the sine, cosine, and area formulae

RHS: Use simple trigonometry and Pythagoras' theorem.

SSS: Use the cosine rule to find any angle.

SAS: Use the cosine rule to find the third side,
then the sine rule will find any angle.
Use the area formula to find the area.

AAS: Use the sine rule to find each of the other two sides.

[ASS]: [Not a congruence test — this is the ambiguous situation]
When using the sine rule or the area formula to find an unknown angle,
 $\sin \theta$ will be unique, but there may be two possible solutions for θ .

In the ambiguous ASS situation, it is also possible to use the cosine rule to find the third side. See the last two questions in the previous exercise.

Problems requiring two steps

Various situations with non-right-angled triangles require two steps for their solution. For example, two steps are needed to find the other two angles in an SAS situation, or to find the area in an AAS, ASS, or SSS situations.



Example 24 Solving a two-step problem

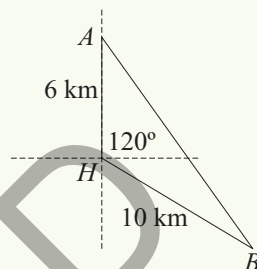
A boat sails 6 km due north from the harbour H to A , and a second boat sails 10 km from H to B on a bearing of 120°T .

- What is the distance AB ?
- What is the bearing of B from A , correct to the nearest minute?

Solution

a This is an SAS situation, so we use the cosine rule to find AB :

$$\begin{aligned} AB^2 &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ \\ &= 36 + 100 - 120 \times \left(-\frac{1}{2}\right) \\ &= 196 \\ AB &= 14 \text{ km.} \end{aligned}$$



b Because AB is now known, this is an SSS situation, so we use the cosine rule in reverse to find $\angle A$:

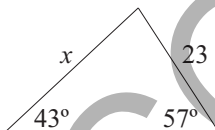
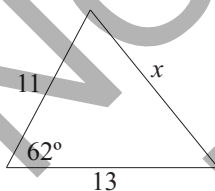
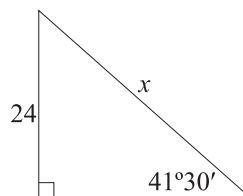
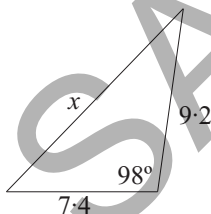
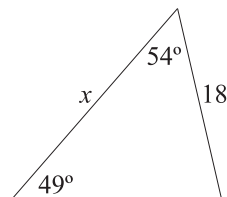
$$\begin{aligned} 10^2 &= 14^2 + 6^2 - 2 \times 14 \times 6 \times \cos A \\ 12 \times 14 \times \cos A &= 196 + 36 - 100 \end{aligned}$$

$$\begin{aligned} \cos A &= \frac{132}{12 \times 14} \\ &= \frac{11}{14} \end{aligned}$$

$$A \doteq 38^\circ 13', \quad \text{and the bearing of } B \text{ from } A \text{ is about } 141^\circ 47' \text{T.}$$

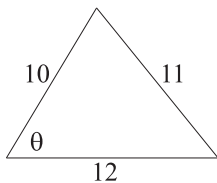
Exercise 6J**FOUNDATION**

1 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find x in each triangle, correct to one decimal place.

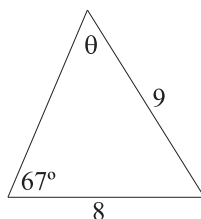
a**b****c****d****e****f**

- 2 Use right-angled triangle trigonometry, the sine rule or the cosine rule to find θ in each triangle, correct to the nearest degree.

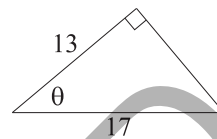
a



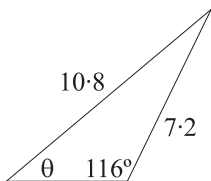
b



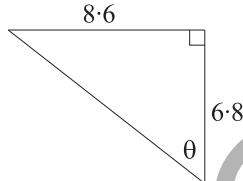
c



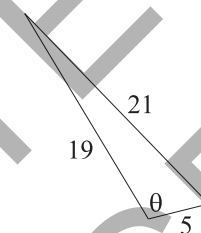
d



e



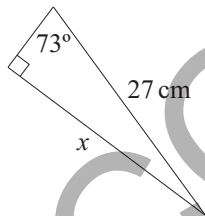
f



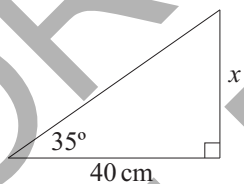
DEVELOPMENT

- 3 [This question is designed to show that the sine and cosine rules work in right-angled triangles, but are *not* the most efficient methods.] In each part find the pronumeral (correct to the nearest cm or to the nearest degree), using either the sine rule or the cosine rule. Then check your answer using right-angled triangle trigonometry.

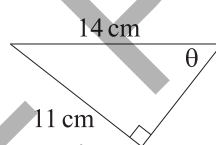
a



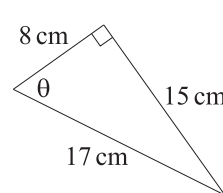
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c

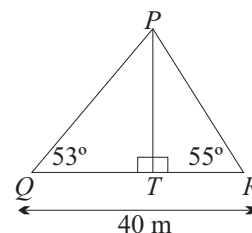


d



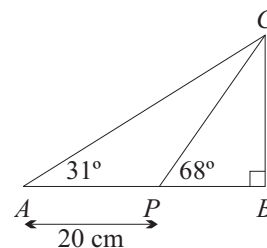
- 4 In $\triangle PQR$, $\angle Q = 53^\circ$, $\angle R = 55^\circ$ and $QR = 40$ metres. The point T lies on QR such that $PT \perp QR$.

- a Use the sine rule in $\triangle PQR$ to show that $PQ = \frac{40 \sin 55^\circ}{\sin 72^\circ}$.
- b Use $\triangle PQT$ to find PT , correct to the nearest metre.



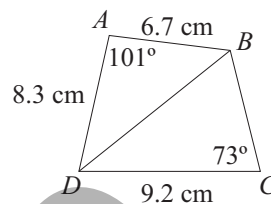
- 5 In $\triangle ABC$, $\angle B = 90^\circ$ and $\angle A = 31^\circ$. The point P lies on AB such that $AP = 20$ cm and $\angle CPB = 68^\circ$.

- a Explain why $\angle ACP = 37^\circ$.
- b Use the sine rule to show that $PC = \frac{20 \sin 31^\circ}{\sin 37^\circ}$.
- c Hence find PB , correct to the nearest centimetre.



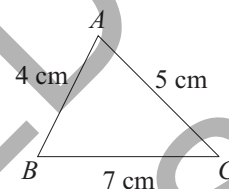
- 6** In the diagram to the right, $AB = 6.7$ cm, $AD = 8.3$ cm and $DC = 9.2$ cm. Also, $\angle A = 101^\circ$ and $\angle C = 73^\circ$.

- a** Use the cosine rule to find the diagonal BD , correct to the nearest millimetre.
b Hence use the sine rule to find $\angle CBD$, correct to the nearest degree.



- 7** In $\triangle ABC$, $AB = 4$ cm, $BC = 7$ cm and $CA = 5$ cm.

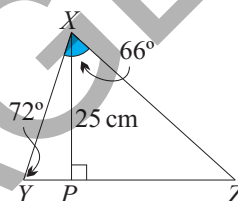
- a** Use the cosine rule to find $\angle ABC$, correct to the nearest minute.
b Hence calculate the area of $\triangle ABC$, correct to the nearest square centimetre.



- 8** A triangle has sides 13 cm, 14 cm and 15 cm. Use the cosine rule to find one of its angles, and hence show that its area is 84 cm².

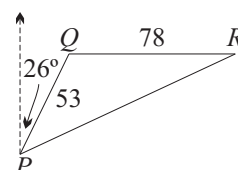
- 9** In a triangle XYZ , $\angle Y = 72^\circ$ and $\angle YXZ = 66^\circ$. $XP \perp YZ$ and $XP = 25$ cm.

- a** Use the sine ratio in $\triangle PXY$ to show that $XY \doteq 26.3$ cm.
b Hence use the sine rule in $\triangle XYZ$ to find YZ , correct to the nearest centimetre.
c Check your answer to part (b) by using the tangent ratio in triangles PXY and PXZ to find PY and PZ .



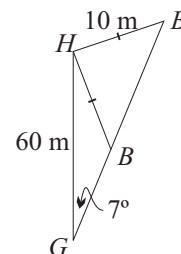
- 10** A ship sails 53 nautical miles from P to Q on a bearing of 026° T. It then sails 78 nautical miles due east from Q to R .

- a** Explain why $\angle PQR = 116^\circ$.
b How far apart are P and R , correct to the nearest nautical mile?



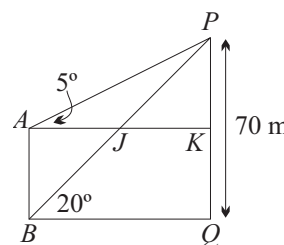
- 11** A golfer at G , 60 metres from the hole H , played a shot that landed at B , 10 metres from the hole. The direction of the shot was 7° away from the direct line between G and H .

- a** Find, correct to the nearest minute, the two possible sizes of $\angle GBH$.
b Hence find the two possible distances the ball has travelled. (Answer in metres correct to one decimal place.)



- 12** Two towers AB and PQ stand on level ground. The angles of elevation of the top of the taller tower from the top and bottom of the shorter tower are 5° and 20° respectively. The height of the taller tower is 70 metres.

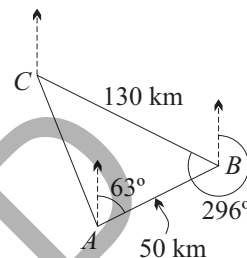
- a** Explain why $\angle APJ = 15^\circ$.
b Show that $AB = \frac{BP \sin 15^\circ}{\sin 95^\circ}$.
c Show that $BP = \frac{70}{\sin 20^\circ}$.
d Hence find the height of the shorter tower, correct to the nearest metre.



CHALLENGE

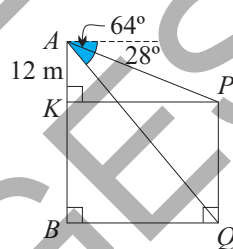
- 13** A ship sails 50 km from port A to port B on a bearing of 063°T , then sails 130 km from port B to port C on a bearing of 296°T .

- Show that $\angle ABC = 53^\circ$.
- Find, correct to the nearest km, the distance of port A from port C .
- Use the cosine rule to find $\angle ACB$, and hence find the bearing of port A from port C , correct to the nearest degree.



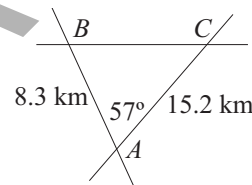
- 14** Two towers AB and PQ stand on level ground. Tower AB is 12 metres taller than tower PQ . From A , the angles of depression of P and Q are 28° and 64° respectively.

- Use $\triangle AKP$ to show that $KP = BQ = 12 \tan 62^\circ$.
- Use $\triangle ABQ$ to show that $AB = 12 \tan 62^\circ \tan 64^\circ$.
- Hence find the height of the shorter tower, correct to the nearest metre.
- Solve the problem again, by using $\triangle AKP$ to find AP , and then using the sine rule in $\triangle APQ$.

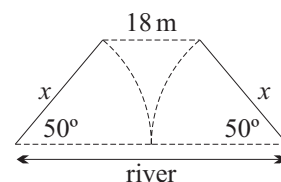


- 15** The diagram shows three straight roads, AB , BC and CA , where $AB = 8.3$ km, $AC = 15.2$ km, and the roads AB and AC intersect at 57° .

Two cars, P_1 and P_2 , leave A at the same instant. Car P_1 travels along AB and then BC at 80 km/h while P_2 travels along AC at 50 km/h. Which car reaches C first, and by how many minutes? (Answer correct to one decimal place.)



- 16** A bridge spans a river, and the two identical sections of the bridge, each of length x metres, can be raised to allow tall boats to pass. When the two sections are fully raised, they are each inclined at 50° to the horizontal, and there is an 18-metre gap between them, as shown in the diagram. Calculate the width of the river in metres, correct to one decimal place.



Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- 1 Find, correct to four decimal places:

a $\cos 73^\circ$

b $\tan 42^\circ$

c $\sin 38^\circ 24'$

d $\cos 7^\circ 56'$

- 2 Find the acute angle θ , correct to the nearest minute, given that:

a $\sin \theta = 0.3$

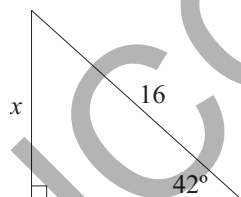
b $\tan \theta = 2.36$

c $\cos \theta = \frac{1}{4}$

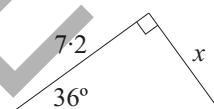
d $\tan \theta = 1\frac{1}{3}$

- 3 Find, correct to two decimal places, the side marked x in each triangle below.

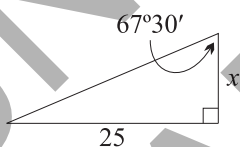
a



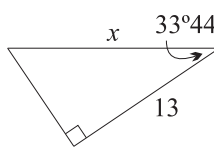
b



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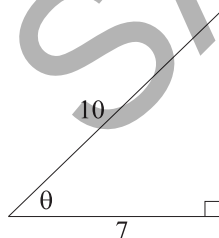


d

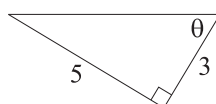


- 4 Find, correct to the nearest minute, the angle θ in each triangle below.

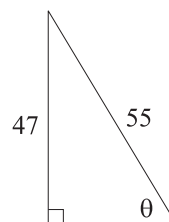
a



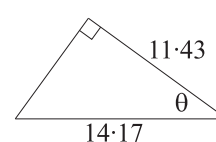
b



c



d



5 Use the special triangles to find the exact values of:

a $\tan 60^\circ$

b $\sin 45^\circ$

c $\cos 30^\circ$

d $\cot 45^\circ$

e $\sec 60^\circ$

f $\operatorname{cosec} 60^\circ$

6 A vertical pole stands on level ground. From a point on the ground 8 metres from its base, the angle of elevation of the top of the pole is 38° . Find the height of the pole, correct to the nearest centimetre.

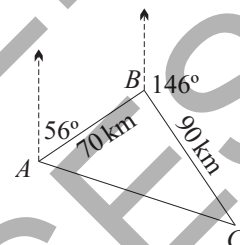
7 At what angle, correct to the nearest degree, is a 6-metre ladder inclined to the ground if its foot is 2.5 metres out from the wall?

8 A motorist drove 70 km from town A to town B on a bearing of 056°T , and then drove 90 km from town B to town C on a bearing of 146°T .

a Explain why $\angle ABC = 90^\circ$.

b How far apart are the towns A and C, correct to the nearest kilometre?

c Find $\angle BAC$, and hence find the bearing of town C from town A, correct to the nearest degree.



9 Sketch each graph for $0^\circ \leq x \leq 360^\circ$.

a $y = \sin x$

b $y = \cos x$

c $y = \tan x$

10 Write each trigonometric ratio as the ratio of its related acute angle, with the correct sign attached.

a $\cos 125^\circ$

b $\sin 312^\circ$

c $\tan 244^\circ$

d $\sin 173^\circ$

11 Find the exact value of:

a $\tan 240^\circ$

b $\sin 315^\circ$

c $\cos 330^\circ$

d $\tan 150^\circ$

12 Use the graphs of the trigonometric functions to find these values, if they exist.

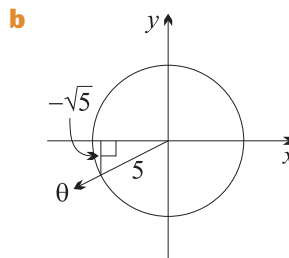
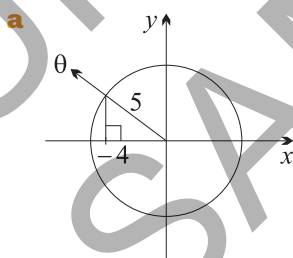
a $\sin 180^\circ$

b $\cos 180^\circ$

c $\tan 90^\circ$

d $\sin 270^\circ$

13 Use Pythagoras' theorem to find whichever of x , y or r is unknown. Then write down the values of $\sin \theta$, $\cos \theta$ and $\tan \theta$.



14 **a** If $\tan \alpha = \frac{12}{5}$ and α is acute, find the values of $\sin \alpha$ and $\cos \alpha$.

b If $\sin \beta = \frac{2\sqrt{6}}{7}$ and β is acute, find the values of $\cos \beta$ and $\tan \beta$.

c If $\tan \alpha = -\frac{9}{40}$ and $270^\circ < \alpha < 360^\circ$, find the values of $\sin \alpha$ and $\cos \alpha$.

d If $\sin \beta = \frac{2\sqrt{6}}{7}$ and $90^\circ < \beta < 180^\circ$, find the values of $\cos \beta$ and $\tan \beta$.

15 Simplify:

a $\frac{1}{\cos \theta}$

d $1 - \sin^2 \theta$

b $\frac{1}{\cot \theta}$

e $\sec^2 \theta - \tan^2 \theta$

c $\frac{\sin \theta}{\cos \theta}$

f $\operatorname{cosec}^2 \theta - 1$

16 Prove the following trigonometric identities.

a $\cos \theta \sec \theta = 1$

b $\tan \theta \operatorname{cosec} \theta = \sec \theta$

c $\frac{\cot \theta}{\cos \theta} = \operatorname{cosec} \theta$

d $2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$

e $4 \sec^2 \theta - 3 = 1 + 4 \tan^2 \theta$

f $\cos \theta + \tan \theta \sin \theta = \sec \theta$

17 Solve each trigonometric equation for $0^\circ \leq x \leq 360^\circ$.

a $\cos x = \frac{1}{2}$

b $\sin x = 1$

c $\tan x = -1$

d $\cos x = 0$

e $\sqrt{3} \tan x = 1$

f $\tan x = 0$

g $\sqrt{2} \sin x + 1 = 0$

h $2 \cos x + \sqrt{3} = 0$

i $\cos^2 x = \frac{1}{2}$

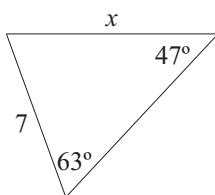
j $\cos 2x = \frac{1}{2}$

k $\cos(x - 75^\circ) = \frac{1}{2}$

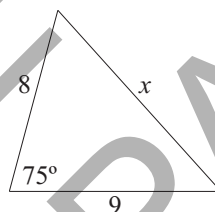
l $\sin x = -\sqrt{3} \cos x$

18 Use the sine rule or the cosine rule in each triangle to find x , correct to one decimal place.

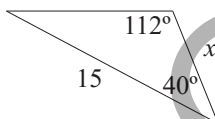
a



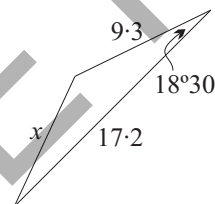
b



c

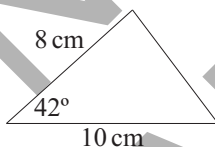


d

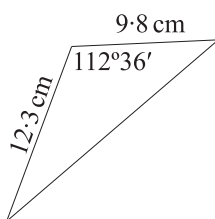


19 Calculate the area of each triangle, correct to the nearest cm^2 .

a

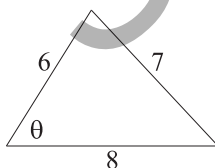


b

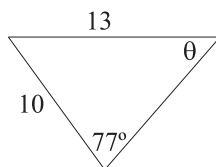


20 Use the sine rule or the cosine rule in each triangle to find θ , correct to the nearest minute.

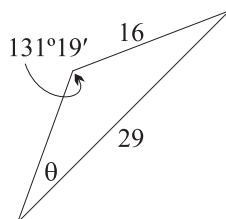
a



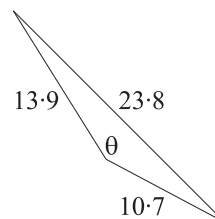
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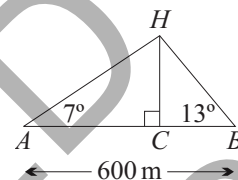
d



- 21** A triangle has sides 7 cm, 8 cm and 10 cm. Use the cosine rule to find one of its angles, and hence find the area of the triangle, correct to the nearest cm^2 .

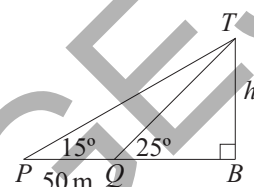
- 22** **a** Find the side a in $\triangle ABC$, where $\angle C = 60^\circ$, $b = 24$ cm and the area is 30 cm^2 .
b Find the size of $\angle B$ in $\triangle ABC$, where $a = 9$ cm, $c = 8$ cm and the area is 18 cm^2 .

- 23** A helicopter H is hovering above a straight, horizontal road AB of length 600 metres. The angles of elevation of H from A and B are 7° and 13° respectively. The point C lies on the road directly below H .



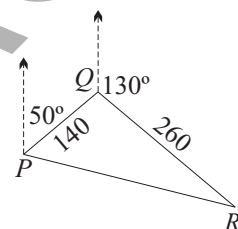
- a** Use the sine rule to show that $HB = \frac{600 \sin 7^\circ}{\sin 160^\circ}$.
b Hence find the height CH of the helicopter above the road, correct to the nearest metre.

- 24** A man is sitting in a boat at P , where the angle of elevation of the top T of a vertical cliff BT is 15° . He then rows 50 metres directly towards the cliff to Q , where the angle of elevation of T is 25° .



- a** Show that $TQ = \frac{50 \sin 15^\circ}{\sin 10^\circ}$.
b Hence find the height h of the cliff, correct to the nearest tenth of a metre.

- 25** A ship sailed 140 nautical miles from port P to port Q on a bearing of 050° T. It then sailed 260 nautical miles from port Q to port R on a bearing of 130° T.



- a** Explain why $\angle PQR = 100^\circ$.
b Find the distance between ports R and P , correct to the nearest nautical mile.
c Find the bearing of port R from port P , correct to the nearest degree.

7

Lines in the coordinate plane

Chapter introduction

The earlier chapters have used graphs to turn functions into geometric objects within the coordinate plane, thus allowing them to be visualised and studied by geometric as well as algebraic methods. In the coordinate plane:

- ▶ Points are represented by pairs of numbers.
- ▶ Lines are represented by linear equations.
- ▶ Circles, parabolas and other curves are represented by non-linear equations.

Points, lines and line segments are the main concern of this chapter. Its purpose is to prepare for Chapter 9, where the new topic of calculus will be introduced in the coordinate plane, heavily based on geometric ideas of tangents and areas.

This material may all have been covered in earlier years. Readers should do as much or as little of it as they need in order to master the skills.

7A Lengths and midpoints of line segments

Learning intentions

- Define a line segment, and recognise the unique line containing it.
- Prove and use the distance formula for a line segment.
- Prove and use the midpoint formula for a line segment.

Many of the familiar objects of geometry — triangles, squares, parallelograms, hexagons — are made up of line segments, and we need to know how to deal with line segments in the coordinate plane.

Line segments

A *line segment* is completely determined by specifying its two distinct endpoints.

1 Line segment

We know that two distinct points P and Q determine a line ℓ .

- The *line segment* PQ is the part of the line ℓ contained between these two distinct points, including the two endpoints P and Q .
- The line ℓ is *the only line containing the line segment* PQ .

The word ‘interval’ used to be used for both intervals and line segments, but the two closely related ideas now have separate words, which is more satisfactory.

The symbol PQ now has three distinct meanings. It can mean;

the line PQ , or the line segment PQ , or the distance PQ .

In practice, this rarely causes problems, but if it does, add the preceding noun.

There are simple formulae for the length and the midpoint of a line segment.

The distance formula

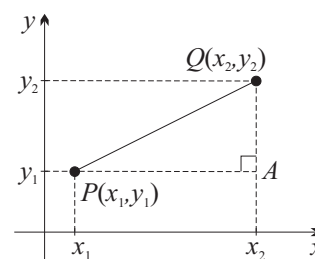
The formula for the length of a line segment PQ in the coordinate plane is just Pythagoras’ theorem in different notation.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.

Construct the right-angled triangle $\triangle PQA$, where $A(x_2, y_1)$ lies level with P and vertically above or below Q .

Then $PA = |x_2 - x_1|$ and $QA = |y_2 - y_1|$, so by Pythagoras’ theorem in $\triangle PQA$,

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$



2 Distance formula

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane. Then

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

First find the square PQ^2 of the distance, then take the square root.



Example 1 Using the distance formula

Find the lengths of the sides AB and AC of the triangle with vertices $A(1, -2)$, $B(-4, 2)$, and $C(5, -7)$, and hence say why $\triangle ABC$ is isosceles.

Solution

$$\begin{aligned}\text{First, } AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (-4 - 1)^2 + (2 - (-2))^2 \\ &= (-5)^2 + 4^2 \\ &= 41,\end{aligned}$$

$$\text{so } AB = \sqrt{41}.$$

$$\begin{aligned}\text{Secondly, } AC^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (5 - 1)^2 + (-7 - (-2))^2 \\ &= 4^2 + (-5)^2 \\ &= 41,\end{aligned}$$

$$\text{so } AC = \sqrt{41}.$$

Because the two sides AB and AC are equal, the triangle is isosceles.

The midpoint formula

The midpoint of a line segment is found by taking the averages of the coordinates of the two points. Congruence is the basis of the proof below.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane, and let $M(x, y)$ be the midpoint of PQ .

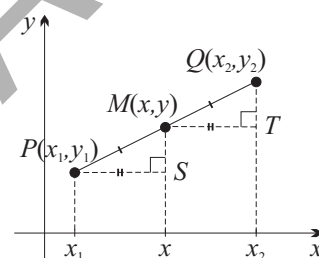
Construct $S(x, y_1)$ and $T(x_2, y)$, as shown.

Then $\triangle PMS \cong \triangle MQT$ (AAS).

Hence $PS = MT$ (matching sides of congruent triangles).

$$\begin{aligned}x - x_1 &= x_2 - x \\ 2x &= x_1 + x_2 \\ x &= \frac{x_1 + x_2}{2}, \text{ which is the average of } x_1 \text{ and } x_2.\end{aligned}$$

The calculation of the y -coordinate of M is similar.



3 Midpoint formula

Let $M(x, y)$ be the midpoint of the line segment joining $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Then $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$ (take the averages of the coordinates).-



Example 2 Using the midpoint formula and the distance formula

The line segment joining $A(3, -1)$ and $B(-7, 5)$ is a diameter of a circle. Find:

- a** the centre M , **b** the radius, **c** the equation.

Solution

- a** The centre of the circle is the midpoint $M(x, y)$ of the line segment AB .

$$\begin{aligned} \text{Using the midpoint formula, } x &= \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} \\ &= \frac{3 - 7}{2} \quad \quad \quad = \frac{-1 + 5}{2} \\ &= -2, \quad \quad \quad = 2, \end{aligned}$$

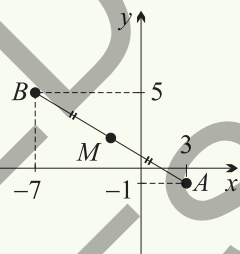
so the centre is $M(-2, 2)$.

- b** Using the distance formula, $AM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
- $$\begin{aligned} &= (-2 - 3)^2 + (2 - (-1))^2 \\ &= 34, \end{aligned}$$

$$AM = \sqrt{34}.$$

Hence the circle has radius $\sqrt{34}$.

- c** The equation of the circle is therefore $(x + 2)^2 + (y - 2)^2 = 34$.

**Exercise 7A****FOUNDATION**

Note: Diagrams should be drawn wherever possible.

- Find the midpoint of each line segment AB . Use $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$.
 - $A(3, 5)$ and $B(1, 9)$
 - $A(4, 8)$ and $B(6, 4)$
 - $A(-4, 7)$ and $B(8, -11)$
 - $A(-3, 6)$ and $B(3, 1)$
 - $A(0, -8)$ and $B(-11, -12)$
 - $A(4, -7)$ and $B(4, 7)$
- Find the length of each line segment. Use $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, then find AB .
 - $A(1, 4)$, $B(5, 1)$
 - $A(-2, 7)$, $B(3, -5)$
 - $A(-5, -2)$, $B(3, 4)$
 - $A(3, 6)$, $B(5, 4)$
 - $A(-4, -1)$, $B(4, 3)$
 - $A(5, -12)$, $B(0, 0)$
- Find the midpoint M of the line segment joining $P(-2, 1)$ and $Q(4, 9)$.
 - Find the lengths PM and MQ , and verify that $PM = MQ$.
- Find the length of each side of the triangle formed by $P(0, 3)$, $Q(1, 7)$ and $R(5, 8)$.
 - Hence show that $\triangle PQR$ is isosceles.
- The vertices of $\triangle ABC$ are given by $A(0, 0)$, $B(9, 12)$ and $C(25, 0)$.
 - Find the lengths of the three sides AB , BC , AC .
 - By checking that $AB^2 + BC^2 = AC^2$, show that $\triangle ABC$ is right-angled.
- Find the length of each side of $\triangle ABC$, where $A = (0, 5)$, $B = (3, -2)$ and $C = (-3, 4)$.
 - Find the midpoint of each side of this triangle ABC .

DEVELOPMENT

- A circle with centre $O(0, 0)$ passes through $A(5, 12)$. What is its radius?
 - A circle with centre $B(4, 5)$ passes through the origin. What is its radius?
 - Find the centre of the circle with diameter CD , where $C = (2, 1)$ and $D = (8, -7)$.
 - Show that $E(-12, -5)$ lies on the circle with centre the origin and radius 13.

- 8** In previous studies of special quadrilaterals, you should have seen the result that *A quadrilateral whose diagonals bisect is a parallelogram*, that is a quadrilateral with opposite sides parallel.
- Find the midpoint of the line segment joining $A(4, 9)$ and $C(-2, 3)$.
 - Find the midpoint of the line segment joining $B(0, 4)$ and $D(2, 8)$.
 - Explain why the quadrilateral $ABCD$ is a parallelogram.
- 9** A quadrilateral with four sides of equal lengths is a *rhombus*. The points $A(3, 1)$, $B(10, 2)$, $C(5, 7)$ and $D(-2, 6)$ are the vertices of a quadrilateral.
- Find the lengths of all four sides.
 - What can you conclude about $ABCD$?
- 10**
- Find the side lengths of the triangle with vertices $X(0, -4)$, $Y(4, 2)$ and $Z(-2, 6)$.
 - Show that $\triangle XYZ$ is a right-angled isosceles triangle by showing that its side lengths satisfy Pythagoras' theorem.
 - Hence find the area of $\triangle XYZ$.
- 11**
- Find the distance of each point $A(1, 4)$, $B(2, \sqrt{13})$, $C(3, 2\sqrt{2})$ and $D(4, 1)$ from the origin O . Hence explain why the four points lie on a circle with centre the origin.
 - What are the radius, diameter, circumference and area of this circle?
- 12** The point $M(3, 7)$ is the midpoint of the line segment joining $A(1, 12)$ and $B(x_2, y_2)$. Find the coordinates x_2 and y_2 of B by substituting into the formulae
- $$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}.$$
- 13** Solve each problem using the same methods as in the previous question.
- If $A(-1, 2)$ is the midpoint of $S(x, y)$ and $T(3, 6)$, find the coordinates of S .
 - The midpoint of the line segment PQ is $M(2, -7)$. Find the coordinates of P if:
 - $Q = (0, 0)$
 - $Q = (5, 3)$
 - $Q = (-3, -7)$
 - Find B , if AB is a diameter of a circle with centre $Q(4, 5)$, and $A = (8, 3)$.
 - Given that $P(4, 7)$ is one vertex of the square $PQRS$, and that the centre of the square is $M(8, -1)$, find the coordinates of the opposite vertex R .
- 14**
- Write down any two points A and B whose midpoint is $M(4, 6)$.
 - Write down any two points C and D that are 10 units apart.

CHALLENGE

- 15** Each set of three points given below forms a triangle of one of these types:
- | | | | |
|---------------------|-----------------------|------------------------|-------------------------|
| A isosceles, | B equilateral, | C right-angled, | D none of these. |
|---------------------|-----------------------|------------------------|-------------------------|
- Find the side lengths of each triangle below and hence determine its type.
- | | |
|--|---|
| a $A(-1, 0)$, $B(1, 0)$, $C(0, \sqrt{3})$ | b $P(-1, 1)$, $Q(0, -1)$, $R(3, 3)$ |
| c $D(1, 1)$, $E(2, -2)$, $F(-3, 0)$ | d $X(-3, -1)$, $Y(0, 0)$, $Z(-2, 2)$ |
- 16** As was discussed in Section 4A, the circle with centre (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$. By identifying the centre and radius, find the equations of:
- the circle with centre $(5, -2)$ and passing through $(-1, 1)$,
 - the circle with $K(5, 7)$ and $L(-9, -3)$ as endpoints of a diameter.

7B Gradients of line segments and lines

Learning intentions

- Develop and apply the theory of the gradient of line segments and lines.
- Develop and apply the conditions for lines to be parallel or perpendicular.
- Define and apply the angle of inclination of a line.

Gradient is the key idea that will be used in the next section to bring lines and their equations into the coordinate plane.

The gradient of a line segment

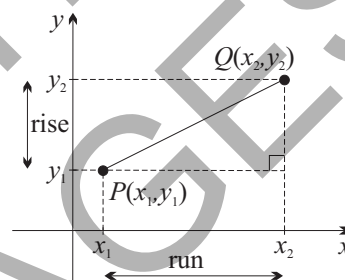
Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the plane.

The *gradient* of the line segment PQ is a measure of its steepness, as someone walks along it from P to Q .

The *rise* is the vertical difference $y_2 - y_1$, and the *run* is the horizontal difference $x_2 - x_1$.

The *gradient* of the line segment PQ is defined to be the ratio of the rise and the run:

$$\begin{aligned}\text{gradient of } PQ &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y_2 - y_1}{x_2 - x_1}.\end{aligned}$$



4 The gradient of a line segment

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points. Then

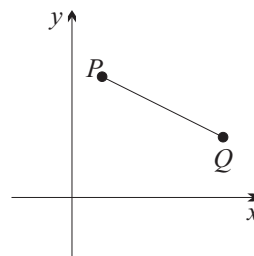
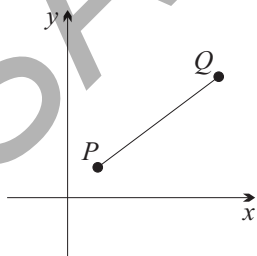
$$\text{gradient of } PQ = \frac{\text{rise}}{\text{run}}, \quad \text{that is,} \quad \text{gradient of } PQ = \frac{y_2 - y_1}{x_2 - x_1}.$$

- Horizontal line segments have gradient zero, because the rise is always zero.
- Vertical line segments don't have a gradient — the run is always zero, so the fraction is undefined.

Positive and negative gradients

If the rise and the run have the same sign, then the gradient is *positive*, as in the first diagram below. In this case the line segment slopes *upwards* as we move from left to right.

If the rise and run have opposite signs, then the gradient is *negative*, as in the second diagram. The line segment slopes *downwards* as we move from left to right.



If the points P and Q are interchanged, in either diagram, then the rise and the run both change sign, but the gradient remains the same.



Example 3 Finding gradients

Find the gradients of the sides of $\triangle XYZ$, where $X = (2, 5)$, $Y = (5, -2)$ and $Z = (-3, 4)$.

Solution

$$\begin{aligned}\text{gradient of } XY &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 5}{5 - 2} \\ &= \frac{-7}{3} \\ &= -\frac{7}{3}\end{aligned}$$

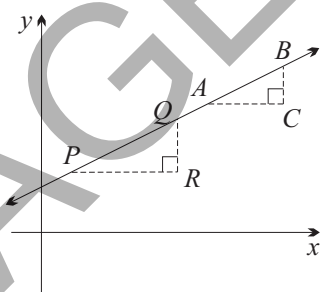
$$\begin{aligned}\text{gradient of } YZ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - (-2)}{-3 - 5} \\ &= \frac{4 + 2}{-8} \\ &= -\frac{3}{4}\end{aligned}$$

$$\begin{aligned}\text{gradient of } ZX &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 4}{2 - (-3)} \\ &= \frac{1}{2 + 3} \\ &= \frac{1}{5}\end{aligned}$$

The gradient of a line

The *gradient of a line* is defined to be the gradient of any line segment within the line. This definition makes sense because any two line segments on the same line always have the same gradient.

To prove this, suppose that PQ and AB are two line segments on the same line ℓ . Construct right triangles PQR and ABC underneath the line segments, with sides parallel to the axes. Because these two triangles are similar (by the AA similarity test), the ratios of their heights and bases are the same, which means that the two line segments AB and PQ have the same gradient.



5 The Gradient of a Line

- The *gradient of a line* is found by taking any two distinct points P and Q on the line and finding the gradient of the line segment PQ .
- It doesn't matter which two points on the line are taken, because the ratio of rise over run will be same, as seen using similar triangles.

A condition for two lines to be parallel

Any two vertical lines are parallel. Otherwise, lines are parallel when their gradients are equal.

6 Parallel lines

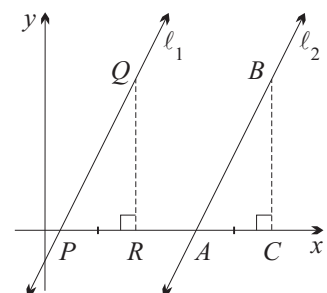
Two lines are parallel if and only if:

- they have the same gradient OR they are both vertical.

The phrase 'if and only if' means that

- if the condition holds, then the lines are parallel, and
- if the lines are parallel, then the condition holds,

and the two statements are called *converses* of each other. 'If and only if' is a very convenient abbreviation, but its logic needs attention.



Proof (assuming neither is vertical)

The two lines ℓ_1 and ℓ_2 meet the x -axis at P and A .

Construct $PR = AC$, then construct QR and BC perpendicular to the x -axis.

A If the lines ℓ_1 and ℓ_2 are parallel, then

$$\angle QPR = \angle BAC \quad (\text{corresponding angles on parallel lines}).$$

$$\text{Hence} \quad \triangle PQR \equiv \triangle ABC \quad (\text{AAS})$$

$$\text{so} \quad QR = BC \quad (\text{matching sides of congruent triangles})$$

$$\frac{QR}{PR} = \frac{BC}{AC} \quad (\text{because } PR = AC).$$

$$\text{gradient } \ell_1 = \text{gradient } \ell_2.$$

B Conversely, if the two gradients are equal, then

$$\frac{QR}{PR} = \frac{BC}{AC}$$

$$QR = BC \quad (\text{because } PR = AC).$$

$$\text{Hence} \quad \triangle PQR \equiv \triangle ABC \quad (\text{SAS})$$

$$\text{so} \quad \ell_1 \parallel \ell_2 \quad (\text{corresponding angles are equal}).$$

Example 4 Testing whether lines are parallel

Given the four points $A(3, 6)$, $B(7, -2)$, $C(4, -5)$ and $D(-1, 5)$, show that the quadrilateral $ABCD$ is a trapezium with $AB \parallel CD$.

Solution

$$\begin{aligned} \text{gradient of } AB &= \frac{-2 - 6}{7 - 3} \\ &= \frac{-8}{4} \\ &= -2, \end{aligned}$$

$$\begin{aligned} \text{gradient of } CD &= \frac{5 - (-5)}{-1 - 4} \\ &= \frac{10}{-5} \\ &= -2. \end{aligned}$$

Hence $AB \parallel CD$ because their gradients are equal, so $ABCD$ is a trapezium.

Testing for collinear points

Three points are called *collinear* if they all lie on one line.

7 Testing for collinear points

- To test whether three points A , B and C are collinear, test whether AB and BC are parallel. (Are they both vertical, or do they have the same gradient?)
- If AB and BC are parallel, then the three points are collinear, because then AB and BC are parallel lines passing through a common point B .

Example 5 Testing whether three points are collinear

Test whether the three points $A(-2, 5)$, $B(1, 3)$ and $C(7, -1)$ are collinear.

Solution

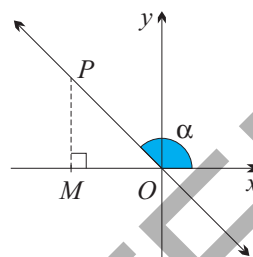
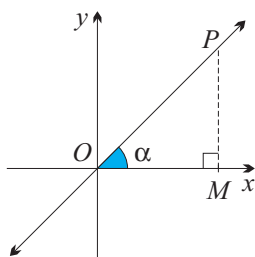
$$\begin{aligned} \text{gradient of } AB &= \frac{3 - 5}{1 - (-2)} \\ &= \frac{-2}{3}, \end{aligned}$$

$$\begin{aligned} \text{gradient of } BC &= \frac{-1 - 3}{7 - 1} \\ &= \frac{-4}{6} \\ &= -\frac{2}{3}. \end{aligned}$$

Because the gradients are equal, the points A , B and C are collinear.

Gradient and the angle of inclination

The *angle of inclination* of a line is the angle between the upward direction of the line and the positive direction of the x -axis.



The two diagrams above show that lines with positive gradients have acute angles of inclination, and lines with negative gradients have obtuse angles of inclination. They also illustrate the trigonometric relationship between the gradient and the angle of inclination α .

8 Angle of inclination

Suppose that a line has angle of inclination α . Then:

- gradient of line = $\tan \alpha$ OR the line is vertical and $\alpha = 90^\circ$.

Proof

- A** When α is acute, as in the first diagram, then the rise MP and the run OM are the opposite and adjacent sides of the triangle POM , so

$$\tan \alpha = \frac{MP}{OM} = \text{gradient of } OP.$$

- B** When α is obtuse, as in the second diagram, then $\angle POM = 180^\circ - \alpha$, so

$$\tan \alpha = -\tan \angle POM = -\frac{MP}{OM} = \text{gradient of } OP.$$

Example 6 Finding angles of inclination

- a** Given the points $A(-3, 5)$, $B(-6, 0)$ and $O(0, 0)$, find the angles of inclination of the line segments AB and AO .

- b** What sort of triangle is $\triangle ABO$?

Solution

a First,
$$\text{gradient of } AB = \frac{0 - 5}{-6 + 3} = \frac{-5}{-3} = \frac{5}{3},$$

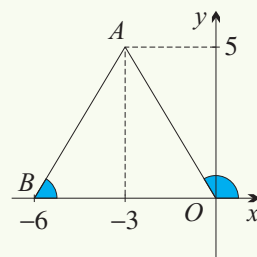
and using a calculator to solve $\tan \alpha = \frac{5}{3}$,
angle of inclination of $AB \doteq 59^\circ$.

Secondly,
$$\text{gradient of } AO = \frac{0 - 5}{0 + 3} = \frac{-5}{3} = -\frac{5}{3},$$

and using a calculator to solve $\tan \alpha = -\frac{5}{3}$,
angle of inclination of $AO \doteq 121^\circ$.

- b** Hence $\angle AOB = 59^\circ$ (straight angle).

Thus the base angles of $\triangle AOB$ are equal, so the triangle is isosceles.



A condition for lines to be perpendicular

A vertical and a horizontal line are perpendicular. Otherwise, the condition is that the product of their gradients is -1 :

9 Perpendicular lines

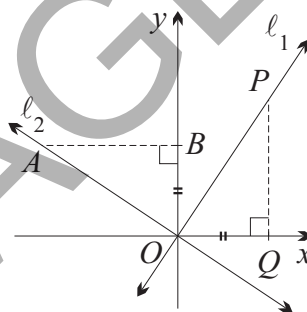
Two lines are perpendicular if and only if:

- $m_1 m_2 = -1$, where m_1 and m_2 are the gradients of the two lines.
OR
- one is vertical and the other is horizontal.

The condition $m_1 m_2 = -1$ can also be written as $m_2 = -\frac{1}{m_1}$.

Proof (assuming that neither is vertical)

Shift each line sideways, without rotating it, so that the lines intersect at the origin. We can assume that one line has positive gradient and the other negative gradient, otherwise one of the angles between them would be acute. So let ℓ_1 be a line with positive gradient through the origin, and let ℓ_2 be a line with negative gradient through the origin. Construct the two triangles POQ and AOB as shown in the diagram, with the run OQ of ℓ_1 equal to the rise OB of ℓ_2 . Then $m_1 \times m_2 = \frac{QP}{OQ} \times \left(-\frac{OB}{AB}\right) = -\frac{QP}{AB}$, because $OQ = OB$.



A If the lines are perpendicular, then $\angle AOB = \angle POQ$. (adjacent angles at O)

Hence $\triangle AOB \equiv \triangle POQ$ (AAS)

so $QP = AB$ (matching sides of congruent triangles)

so $m_1 \times m_2 = -1$.

B Conversely, if $m_1 \times m_2 = -1$, then $QP = AB$.

Hence $\triangle AOB \equiv \triangle POQ$ (SAS)

so $\angle AOB = \angle POQ$, (matching angles of congruent triangles)

and so ℓ_1 and ℓ_2 are perpendicular.

Example 7 Finding perpendicular gradients

What is the gradient of a line perpendicular to a line with gradient $\frac{2}{3}$?

Solution

Perpendicular gradient $= -\frac{3}{2}$ (take the negative of the reciprocal of $\frac{2}{3}$)


Example 8 Testing whether line segments are perpendicular

Show that the diagonals of the quadrilateral $ABCD$ are perpendicular, where the vertices are $A(3, 7)$, $B(-1, 6)$, $C(-2, -3)$ and $D(11, 0)$.

Solution

$$\begin{aligned}\text{Gradient of } AC &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 - 7}{-2 - 3} \\ &= 2,\end{aligned}$$

$$\begin{aligned}\text{gradient of } BD &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - 6}{11 - (-1)} \\ &= -\frac{1}{2}.\end{aligned}$$

Hence $AC \perp BD$, because the product of the gradients of AC and BD is -1 .


Example 9 Using the condition for lines to be perpendicular

The line segment joining the points $C(-6, 0)$ and $D(-1, a)$ is perpendicular to a line with gradient 10. Find the value of a .

Solution

$$\begin{aligned}\text{The line segment } CD \text{ has gradient} &= \frac{a - 0}{-1 - (-6)} \\ &= \frac{a}{5}.\end{aligned}$$

Because the line segment CD is perpendicular to a line with gradient 10,

$$\frac{a}{5} \times \frac{10}{1} = -1 \quad (\text{the product of the gradients is } -1)$$

$$a \times 2 = -1$$

$$\boxed{\div 2} \quad a = -\frac{1}{2}.$$

Exercise 7B
FOUNDATION

Note: Diagrams should be drawn wherever possible.

1 a Write down the gradient of a line parallel to a line with gradient:

i 2

ii $\frac{3}{4}$

iii $-1\frac{1}{2}$

b Find the gradient of a line perpendicular to a line with gradient:

i 2

ii $\frac{3}{4}$

iii $-1\frac{1}{2}$

2 Use the formula $\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient of each line segment AB below. Then find the gradient of a line perpendicular to it.

a $A(1, 4)$, $B(5, 0)$

b $A(-2, -7)$, $B(3, 3)$

c $A(-5, -2)$, $B(3, 2)$

d $A(3, 6)$, $B(5, 5)$

e $A(-1, -2)$, $B(1, 4)$

f $A(-5, 7)$, $B(15, -7)$

- 3** Each of the lines AB in this question are either horizontal (with gradient zero), vertical (with gradient undefined) or neither. State which in each case.
- a** $A(1, 2), B(1, 3)$ **b** $A(4, 3), B(7, 3)$ **c** $A(5, 1), B(4, 5)$
d $A(6, -3), B(1, -3)$ **e** $A(-2, 2), B(2, -2)$ **f** $A(0, 4), B(0, -4)$
- 4** The points $A(2, 5), B(4, 11), C(12, 15)$ and $D(10, 9)$ form a quadrilateral. We can show this by demonstrating that the opposite sides are parallel.
- a** Find the gradients of AB and DC , and hence show that $AB \parallel DC$.
b Find the gradients of BC and AD , and hence show that $BC \parallel AD$.
c What type of quadrilateral is $ABCD$?
d Calculate the lengths AB, CD and BC, DA .
e What result have you verified?
- 5** **a** Show that $A(-2, -6), B(0, -5), C(10, -7)$ and $D(8, -8)$ form a parallelogram.
b Show that $A(2, 5), B(3, 7), C(-4, -1)$ and $D(-5, 2)$ do not form a parallelogram.
- 6** Use the formula $\text{gradient} = \tan \alpha$ to find the gradient, correct to two decimal places where necessary, of a line with angle of inclination:
- a** 15° **b** 135° **c** $22\frac{1}{2}^\circ$ **d** 72°
- 7** Use the formula $\text{gradient} = \tan \alpha$ to find the angle of inclination, correct to the nearest degree where necessary, of a line with gradient:
- a** 1 **b** $-\sqrt{3}$ **c** 4 **d** $\frac{1}{\sqrt{3}}$
- 8** The gradient measures the steepness of a line. Each of the following lines AB has positive gradient. By comparing the gradient to 1, determine if the angle of inclination is less, equal to, or more than 45° .
- a** $A(1, 3), B(6, 5)$ **b** $A(4, 1), B(7, 4)$
c $A(-3, 2), B(1, 5)$ **d** $A(-3, 4), B(-7, -2)$

DEVELOPMENT

- 9** To show that a parallelogram is a rectangle, it is enough to show that one internal angle is 90° . To show that a rectangle is a square, it is enough to show that one pair of adjacent sides are equal in length. Consider the quadrilateral $ABCD$, with vertices $A(-1, 1), B(3, -1), C(5, 3)$ and $D(1, 5)$.
- a** Show that the opposite sides are parallel, and hence that $ABCD$ is a parallelogram.
b Show that $AB \perp BC$. What can you conclude about $ABCD$?
c Show that $AB = BC$. What can you conclude about $ABCD$ now?
- 10** Use gradients to show that each quadrilateral $ABCD$ below is a parallelogram. Then use the methods of the previous question to show that it is:
- a** a rhombus, for the vertices $A(2, 1), B(-1, 3), C(1, 0)$ and $D(4, -2)$,
b a rectangle, for the vertices $A(4, 0), B(-2, 3), C(-3, 1)$ and $D(3, -2)$,
c a square, for the vertices $A(3, 3), B(-1, 2), C(0, -2)$ and $D(4, -1)$.

- 11** Find the gradients of PQ and QR , and hence determine whether P , Q and R are collinear.
a $P(-2, 7)$, $Q(1, 1)$, $R(4, -6)$ **b** $P(-5, -4)$, $Q(-2, -2)$, $R(1, 0)$
- 12** Show that the four points $A(2, 5)$, $B(5, 6)$, $C(11, 8)$ and $D(-16, -1)$ are collinear.
- 13** The triangle ABC has vertices $A(-1, 0)$, $B(3, 2)$ and $C(4, 0)$. Calculate the gradient of each side, and hence show that $\triangle ABC$ is a right-angled triangle.
- 14** Similarly, show that each triangle below is right-angled. Then find the lengths of the sides enclosing the right angle, and calculate the area of each triangle.
a $P(2, -1)$, $Q(3, 3)$, $R(-1, 4)$ **b** $X(-1, -3)$, $Y(2, 4)$, $Z(-3, 2)$
- 15** **a** Write down two points A and B for which the line segment AB has gradient 3.
b Write down two points A and B for which the line segment AB is vertical.
- 16** The line segment PQ has gradient -3 . A second line passes through $A(-2, 4)$ and $B(1, k)$.
a Find k if AB is parallel to PQ . **b** Find k if AB is perpendicular to PQ .
- 17** Find the points A and B where each line below meets the x -axis and y -axis respectively. Hence find the gradient of AB and its angle of inclination α (correct to the nearest degree).
a $y = 3x + 6$ **b** $y = -\frac{1}{2}x + 1$
c $3x + 4y + 12 = 0$ **d** $\frac{x}{3} - \frac{y}{2} = 1$
- 18** The quadrilateral $ABCD$ has vertices $A(1, -4)$, $B(3, 2)$, $C(-5, 6)$ and $D(-1, -2)$.
a Find the midpoints P of AB , Q of BC , R of CD , and S of DA .
b Prove that $PQRS$ is a parallelogram by showing that $PQ \parallel RS$ and $PS \parallel QR$.
- 19** **a** Show that the points $A(-5, 0)$, $B(5, 0)$ and $C(3, 4)$ all lie on the circle $x^2 + y^2 = 25$.
b Explain why AB is a diameter of the circle.
c Show that $AC \perp BC$.

CHALLENGE

- 20** Find the gradient, correct to two decimal places where appropriate, of a line sloping upwards, if its acute angle with the y -axis is:
a 15° **b** 45° **c** $22\frac{1}{2}^\circ$ **d** 72°
- 21** Given the points $X(-1, 0)$, $Y(1, a)$ and $Z(a, 2)$, find a if $\angle YXZ = 90^\circ$.
- 22** For the four points $P(k, 1)$, $Q(-2, -3)$, $R(2, 3)$ and $S(1, k)$, it is known that PQ is parallel to RS . Find the possible values of k .

7C Equations of lines

Learning intentions

- Develop equations for horizontal and vertical lines.
- Develop gradient–intercept form and general form equations for lines.
- Develop tests for lines to be parallel and perpendicular.
- Develop and apply the various forms of the equation of a line.

In the coordinate plane, a line is represented by a linear equation in x and y . This section and the next develop various useful forms for the equation of a line.

Horizontal and vertical lines

All the points on a horizontal line through $P(a, b)$ have the same x -coordinate, but the y -coordinate can take any value. Hence the equation of the line is $y = b$.

All the points on a vertical line through $P(a, b)$ have the same x -coordinate a , but the y -coordinate can take any value. Hence the equation of the line is $x = a$.

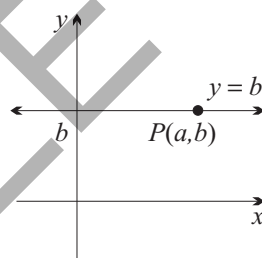
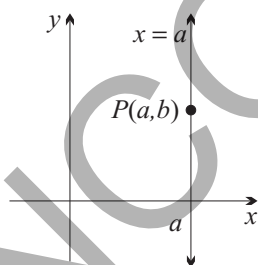
10 Horizontal and vertical lines

The horizontal line through the point $P(a, b)$ has equation

$$y = b.$$

The vertical line through the point $P(a, b)$ has equation

$$x = a.$$



Gradient–intercept form

There is a simple form of the equation of a line whose gradient and y -intercept are known.

Let ℓ have gradient m and y -intercept b , passing through the point $B(0, b)$.

Let $Q(x, y)$ be any other point in the plane.

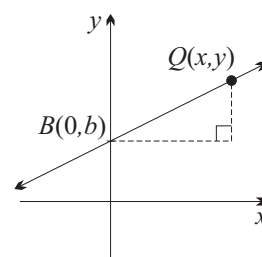
Then the condition for Q to lie on the line ℓ is

$$\text{gradient of } BQ = m,$$

that is,
$$\frac{y - b}{x - 0} = m \quad (\text{this is the formula for gradient})$$

$$y - b = mx.$$

$$y = mx + b.$$



11 Gradient–intercept form

The line with gradient m and y-intercept b is

$$y = mx + b.$$

The gradient–intercept formula is often written using c for the y-intercept as:

$$y = mx + c,$$

but such a change of pronumeral makes no difference whatsoever.

**Example 10 Using gradient–intercept form**

- a** Write down the gradient and the y-intercept of the line ℓ : $y = 3x - 2$.
- b** Find the equation of the line through $B(0, 5)$ parallel to ℓ .
- c** Find the equation of the line through $B(0, 5)$ perpendicular to ℓ .

Solution

- a** The line ℓ : $y = 3x - 2$ has gradient 3 and y-intercept -2 .
- b** The line through $B(0, 5)$ parallel to ℓ has gradient 3 and y-intercept 5, so its equation is $y = 3x + 5$.
- c** The line through $B(0, 5)$ perpendicular to ℓ has gradient $-\frac{1}{3}$ and y-intercept 5, so its equation is $y = -\frac{1}{3}x + 5$.

General form

It is often useful to write the equation of a line so that all the terms are on the LHS and only zero is on the RHS. This is called *general form*.

12 General form

The equation of a line is said to be in *general form* if it is

$$ax + by + c = 0, \quad \text{where } a, b \text{ and } c \text{ are constants.}$$

When an equation is given in general form, it should usually be *simplified* by multiplying out all fractions and dividing through by all common factors.

**Example 11 Converting from general form**

- a** Put the equation of the line $3x + 4y + 5 = 0$ into gradient–intercept form.
- b** Hence write down the gradient and y-intercept of the line $3x + 4y + 5 = 0$.

Solution

- a** Solving the equation for y , $4y = -3x - 5$

$$\div 4 \quad y = -\frac{3}{4}x - \frac{5}{4}, \quad \text{which is gradient–intercept form.}$$

- b** Hence the line has gradient $-\frac{3}{4}$ and y-intercept $-1\frac{1}{4}$.

**Example 12** Converting to general form

Find, in general form, the equation of the line passing through $B(0, -2)$ and:

- a** perpendicular to a line ℓ with gradient $\frac{2}{3}$,
b having angle of inclination 60° .

Solution

- a** The line through B perpendicular to ℓ has gradient $-\frac{3}{2}$ and y -intercept -2 ,
 so its equation is $y = -\frac{3}{2}x - 2$ (this is gradient–intercept form)

$$\begin{array}{rcl} \boxed{\times 2} & & 2y = -3x - 4 \\ & & 3x + 2y + 4 = 0. \end{array}$$

- b** The line through B with angle of inclination 60° has gradient $\tan 60^\circ = \sqrt{3}$,
 so its equation is $y = x\sqrt{3} - 2$ (this is gradient–intercept form)
 $x\sqrt{3} - y - 2 = 0.$

Exercise 7C**FOUNDATION**

- Determine, by substitution, whether the point $A(3, -2)$ lies on the line:

a $y = 4x - 10$ **b** $8x + 10y - 4 = 0$ **c** $x = 3$
- Find the x -intercept and y -intercept of each line.

a $3x + 4y = 12$ **b** $y = 4x - 6$ **c** $x - 2y = 8$
- Write down the coordinates of any three points on the line $x + 3y = 24$.
- Write down the equations of the vertical and horizontal lines through:

a $(1, 2)$ **b** $(0, -4)$ **c** $(5, 0)$
- Write down the gradient and y -intercept of each line.

a $y = 4x - 2$ **b** $y = \frac{1}{5}x - 3$ **c** $y = 2 - x$ **d** $y = -\frac{5}{7}x$
- Use the formula $y = mx + b$ to write down the equation of the line with gradient -3 and:

a y -intercept 5 , **b** y -intercept $-\frac{2}{3}$, **c** y -intercept 0 .
- Use the formula $y = mx + b$ to write down the equation of the line with y -intercept -4 and:

a gradient 5 , **b** gradient $-\frac{2}{3}$, **c** gradient 0 .
- Use the formula $y = mx + b$ to write down the equation of each line, giving your answer in general form:

a line with gradient 1 and y -intercept 3 , **b** line with gradient -2 and y -intercept 5 ,
c line with gradient $\frac{1}{5}$ and y -intercept -1 , **d** line with gradient $-\frac{1}{2}$ and y -intercept 3 .
- Solve each equation for y and hence write down its gradient m and y -intercept b .

a $x - y + 3 = 0$ **b** $y + x - 2 = 0$
c $x - 3y = 0$ **d** $3x + 4y = 5$

- 10** Write down the gradient m of each line. Then use the formula $\text{gradient} = \tan \alpha$ to find its angle of inclination α , correct to the nearest minute where appropriate.

a $y = x + 3$

b $y = -x - 16$

c $y = 2x$

d $y = -\frac{3}{4}x$

DEVELOPMENT

- 11** Substitute $y = 0$ and $x = 0$ into the equation of each line below to find the points A and B where the line crosses the x -axis and y -axis respectively. Hence sketch the line.

a $5x + 3y - 15 = 0$

b $2x - y + 6 = 0$

c $3x - 5y + 12 = 0$

- 12** Find the gradient of the line through each pair of given points. Then find its equation, using gradient-intercept form. Give your final answer in general form.

a $(0, 4), (2, 8)$

b $(0, 0), (1, -1)$

c $(-9, -1), (0, -4)$

- 13** Find the gradient of each line below. Hence find, in gradient-intercept form, the equation of a line passing through $A(0, 3)$ and:

i parallel to it,**ii** perpendicular to it.

a $2x + y + 3 = 0$

b $5x - 2y - 1 = 0$

c $3x + 4y - 5 = 0$

- 14** In each part, find the gradients of the four lines, and hence state what sort of special quadrilateral they enclose.

a $3x + y + 7 = 0, \quad x - 2y - 1 = 0, \quad 3x + y + 11 = 0, \quad x - 2y + 12 = 0$

b $4x - 3y + 10 = 0, \quad 3x + 4y + 7 = 0, \quad 4x - 3y - 7 = 0, \quad 3x + 4y + 1 = 0$

- 15** Find the gradients of the three lines $5x - 7y + 5 = 0$, $2x - 5y + 7 = 0$ and $7x + 5y + 2 = 0$. Hence show that they enclose a right-angled triangle.

- 16** Using a sketch, find the equations of the sides of:

a the rectangle with vertices $P(3, -7)$, $Q(0, -7)$, $R(0, -2)$ and $S(3, -2)$,**b** the triangle with vertices $F(3, 0)$, $G(-6, 0)$ and $H(0, 12)$.

- 17** In each part below, the angle of inclination α and the y -intercept A of a line are given. Use the formula $\text{gradient} = \tan \alpha$ to find the gradient of each line, then find its equation in general form.

a $\alpha = 45^\circ, A = (0, 3)$

b $\alpha = 60^\circ, A = (0, -1)$

c $\alpha = 30^\circ, A = (0, -2)$

d $\alpha = 135^\circ, A = (0, 1)$

CHALLENGE

- 18** A triangle is formed by the x -axis and the lines $5y = 9x$ and $5y + 9x = 45$.

a Find (correct to the nearest degree) the angles of inclination of the two lines.**b** What sort of triangle has been formed?

- 19** Consider the two lines $\ell_1: 3x - y + 4 = 0$ and $\ell_2: x + ky + \ell = 0$. Find the value of k if:

a ℓ_1 is parallel to ℓ_2 ,**b** ℓ_1 is perpendicular to ℓ_2 .

7D Further equations of lines

Learning intentions

- Develop and apply point–gradient form of the equation of a line.
- Find the line through two given points.
- Test three lines for concurrency.

This section introduces another standard form of the equation of a line, called *point–gradient form*. It also deals with lines through two given points, and the point of intersection of two lines.

Point–gradient form

Point–gradient form gives the equation of a line with gradient m passing through a particular point $P(x_1, y_1)$.

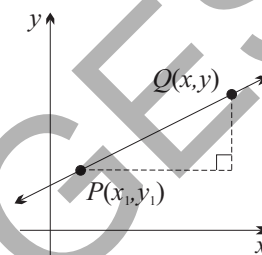
Let $Q(x, y)$ be any other point in the plane.

Then the condition for Q to lie on the line is

$$\text{gradient of } PQ = m,$$

that is, $\frac{y - y_1}{x - x_1} = m$ (this is the formula for gradient)

$$y - y_1 = m(x - x_1).$$



The proof doesn't work when the point Q coincides with P , that is, when $x = x_1$, first because PP does not have a gradient, and secondly because you can't divide by $x - x_1 = 0$. But (x_1, y_1) is still a point on the line $y - y_1 = m(x - x_1)$, as is obvious by direct substitution.

13 Point–gradient form

The line with gradient m through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

If a question doesn't specify which form to express the equation of a line in, then there are two forms that are acceptable in this situation:

- Gradient–intercept form: $y = 3x + 7$
- General form: $3x - y + 7 = 0$

If general form is used, then in most circumstances it should be simplified by multiplying out any fractions, and dividing through by any common factor.



Example 13 Using point–gradient form

- Find the equation of the line through $(-2, -5)$ and parallel to $y = 3x + 2$.
- Express the answer in gradient–intercept form, and hence find its y -intercept.

Solution

- The line $y = 3x + 2$ has gradient 3.

Hence the required line is $y - y_1 = m(x - x_1)$ (this is point–gradient form)

$$y + 5 = 3(x + 2)$$

$$y + 5 = 3x + 6$$

$$y = 3x + 1. \quad \text{(this is gradient–intercept form)}$$

- Hence the new line has y -intercept 1.

The line through two given points

Given two distinct points, there is just one line passing through them both. Its equation is best found by a two-step approach.

14 The line through two given points

- First find the gradient of the line, using the gradient formula.
- Then find the equation of the line, using point–gradient form.



Example 14 Finding the line through two given points

Find the equation of the line passing through $A(1, 5)$ and $B(4, -1)$.

Solution

First, using the gradient formula,

$$\begin{aligned}\text{gradient of } AB &= \frac{-1 - 5}{4 - 1} \\ &= -2.\end{aligned}$$

Then, using point–gradient form for a line with gradient -2 through $A(1, 5)$, the line AB is

$$\begin{aligned}y - y_1 &= m(x - x_1) && \text{(this is point–gradient form)} \\ y - 5 &= -2(x - 1) && \text{(the coordinates of } A(1, 5) \text{ have been used)} \\ y - 5 &= -2x + 2 \\ y &= -2x + 7.\end{aligned}$$

Note: Using the coordinates of $B(4, -1)$ rather than $A(1, 5)$ would give the same equation.



Example 15 Finding the perpendicular bisector

Given the points $A(6, 0)$ and $B(0, 9)$, find, in general form, the equation of the perpendicular bisector of AB .

Solution

$$\begin{aligned}\text{First, gradient of } AB &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{9 - 0}{0 - 6} \\ &= -\frac{3}{2},\end{aligned}$$

so any line perpendicular to AB has gradient $\frac{2}{3}$.

$$\begin{aligned}\text{Secondly, midpoint of } AB &= \left(\frac{6+0}{2}, \frac{0+9}{2} \right) \\ &= (3, 4\frac{1}{2}).\end{aligned}$$

Hence, using point–gradient form, the perpendicular bisector of AB is

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ y - 4\frac{1}{2} &= \frac{2}{3}(x - 3) \\ \boxed{\times 6} \quad 6y - 27 &= 4x - 12 \\ 4x - 6y + 15 &= 0.\end{aligned}$$

Intersection of lines — concurrent lines

The point where two distinct lines intersect can be found using simultaneous equations, as discussed in Sections 1E and 3C.

Three distinct lines are called *concurrent* if they all pass through the same point.

15 Testing for concurrent lines

To test whether three lines are concurrent:

- First find the point of intersection of two of them.
- Then test, by substitution, whether this point lies on the third line.



Example 16 Testing whether three lines are concurrent

Test whether these three lines are concurrent.

$$\ell_1: 5x - y - 10 = 0, \quad \ell_2: x + y - 8 = 0, \quad \ell_3: 2x - 3y + 9 = 0.$$

Solution

A Solve ℓ_1 and ℓ_2 simultaneously.

$$\begin{aligned} \text{Adding } \ell_1 \text{ and } \ell_2, \quad 6x - 18 &= 0 \\ x &= 3, \end{aligned}$$

$$\text{and substituting into } \ell_2, \quad 3 + y - 8 = 0$$

$$y = 5$$

so the lines ℓ_1 and ℓ_2 intersect at $(3, 5)$.

B Substituting the point $(3, 5)$ into the third line ℓ_3 ,

$$\text{LHS} = 6 - 15 + 9$$

$$= 0$$

$$= \text{RHS},$$

so the three lines are concurrent, meeting at $(3, 5)$.

Exercise 7D

FOUNDATION

- Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(3, 5)$ with gradient 3. Rearrange your answer into general form.
- Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through the point $P(-2, 7)$ with each given gradient. Rearrange your answer into general form.

a gradient 6	b gradient -2
c gradient $\frac{2}{3}$	d gradient $-\frac{7}{2}$
- Use point–gradient form $y - y_1 = m(x - x_1)$ to find the equation of the line through each given point with gradient $-\frac{3}{5}$. Rearrange your answer into general form.

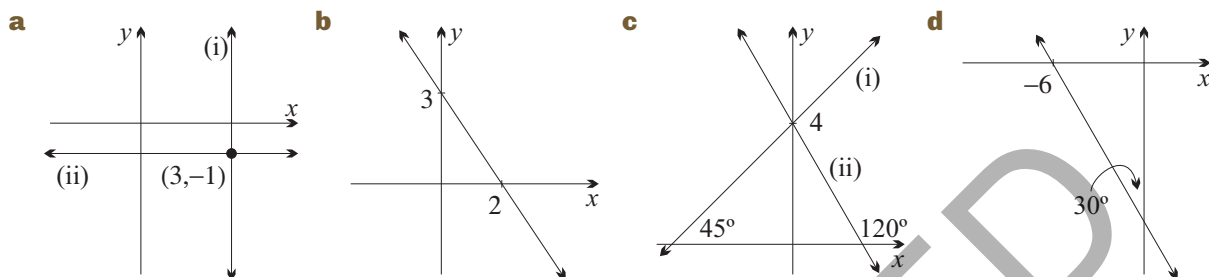
a through $(1, 2)$	b through $(6, 0)$
c through $(-5, 3)$	d through $(0, -4)$

- 4** Find, in general form, the equation of the line:
- a** through (1, 1) with gradient 2,
 - b** with gradient -1 through (3, 1),
 - c** through (0, 0) with gradient -5 ,
 - d** through $(-1, 3)$ with gradient $-\frac{1}{3}$,
 - e** with gradient $-\frac{4}{5}$ through (3, -4).
- 5** Find, in gradient–intercept form, the equation of:
- a** the line through (2, 5) and parallel to $y = 2x + 5$,
 - b** the line through (2, 5) and perpendicular to $y = 2x + 5$,
 - c** the line through (5, -7) and perpendicular to $y = -5x$,
 - d** the line through $(-7, 6)$ and parallel to $y = \frac{3}{7}x - 8$,
 - e** the line through $(-4, 0)$ and perpendicular to $y = -\frac{2}{3}x$.
- 6** **a** Find the gradient of the line through the points $A(4, 7)$ and $B(6, 13)$.
b Hence use point–gradient form to find, in general form, the equation of the line AB .
- 7** Find the gradient of the line through each pair of points, and hence find its equation.
- | | | |
|-----------------------------|---------------------------------|-----------------------------|
| a (3, 4), (5, 8) | b $(-1, 3)$, (1, -1) | c (5, 6), $(-1, 4)$ |
| d $(-1, 0)$, (0, 2) | e (0, -1), $(-4, 0)$ | f (0, -3), (3, 0) |
- 8** **a** Find the gradient of the line through $A(1, -2)$ and $B(-3, 4)$.
b Hence find, in general form, the equation of:
- i** the line AB ,
 - ii** the line through A and perpendicular to AB .
- 9** Find the equation of the line parallel to $2x - 3y + 1 = 0$ and:
- a** passing through (2, 2),
 - b** passing through (3, -1).
- 10** Find the equation of the line perpendicular to $3x + 4y - 3 = 0$ and:
- a** passing through $(-1, -4)$,
 - b** passing through $(-2, 1)$.

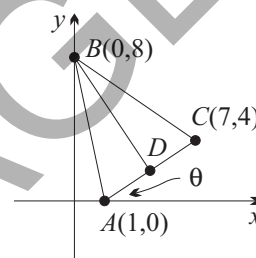
DEVELOPMENT

- 11** **a** Find the point M of intersection of the lines $\ell_1: x + y = 2$ and $\ell_2: 4x - y = 13$.
b Show that M lies on $\ell_3: 2x - 5y = 11$, and hence that ℓ_1 , ℓ_2 and ℓ_3 are concurrent.
c Use the same method to test whether each set of lines is concurrent.
- i** $2x + y = -1$, $x - 2y = -18$ and $x + 3y = 15$
 - ii** $6x - y = 26$, $5x - 4y = 9$ and $x + y = 9$
- 12** Put the equation of each line into gradient–intercept form and hence write down the gradient. Then find, in gradient–intercept form, the equation of the line that is:
- i** parallel to it through $A(3, -1)$,
 - ii** perpendicular to it through $B(-2, 5)$.
- | | | |
|---------------------------|----------------------------|----------------------------|
| a $2x + y + 3 = 0$ | b $5x - 2y - 1 = 0$ | c $4x + 3y - 5 = 0$ |
|---------------------------|----------------------------|----------------------------|
- 13** The angle of inclination α and a point A on a line are given below. Use the formula gradient $= \tan \alpha$ to find the gradient of each line, then find its equation in general form.
- | | |
|--|--|
| a $\alpha = 45^\circ$, $A = (1, 0)$ | b $\alpha = 120^\circ$, $A = (-1, 0)$ |
| c $\alpha = 30^\circ$, $A = (4, -3)$ | d $\alpha = 150^\circ$, $A = (-2, -5)$ |

- 14** Determine, in general form, the equation of each straight line sketched below.

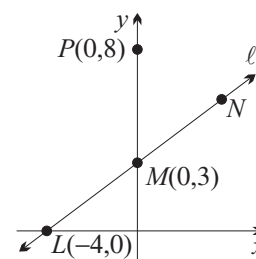


- 15** Explain why the four lines $\ell_1: y = x + 1$, $\ell_2: y = x - 3$, $\ell_3: y = 3x + 5$ and $\ell_4: y = 3x - 5$ enclose a parallelogram. Then find the vertices of this parallelogram.
- 16** Show that the triangle with vertices $A(1, 0)$, $B(6, 5)$ and $C(0, 2)$ is right-angled. Then find the equation of each side.
- 17** The three points $A(1, 0)$, $B(0, 8)$ and $C(7, 4)$ form a triangle. Let θ be the angle between AC and the x -axis.
- Find the gradient of the line AC and hence determine θ , correct to the nearest degree.
 - Derive the equation of AC .
 - Find the coordinates of the midpoint D of AC .
 - Show that AC is perpendicular to BD .
 - What type of triangle is ABC ?
 - Find the area of this triangle.
 - Write down the coordinates of a point E such that the parallelogram $ABCE$ is a rhombus.



CHALLENGE

- 18** The line ℓ crosses the x - and y -axes at $L(-4, 0)$ and $M(0, 3)$. The point N lies on ℓ , and P is the point $P(0, 8)$.
- Copy the sketch and find the equation of ℓ .
 - Find the lengths of ML and MP and hence show that LMP is an isosceles triangle.
 - If M is the midpoint of LN , find the coordinates of N .
 - Show that $\angle NPL = 90^\circ$.
 - Write down the equation of the circle through N , P and L .
- 19** Find k if the lines $\ell_1: x + 3y + 13 = 0$, $\ell_2: 4x + y - 3 = 0$ and $\ell_3: kx - y - 10 = 0$ are concurrent. (Hint: Find the point of intersection of ℓ_1 and ℓ_2 and substitute into ℓ_3 .)
- 20** Consider the two lines $\ell_1: 3x + 2y + 4 = 0$ and $\ell_2: 6x + \mu y + \lambda = 0$. Write down the value of μ if:
- ℓ_1 is parallel to ℓ_2 ,
 - ℓ_1 is perpendicular to ℓ_2 .



7E Using pronumerals in place of numbers

Learning intentions

- Work with the various formulae using pronumerals in place of numbers.
- Use these approaches to prove some general results about geometric figures.

Most problems of this chapter so far have used numbers for the coordinates of points, and for the coefficients of x and y . When constant pronumerals are used, however, far more general results can be obtained. For example, Chapter 3 used pronumeral constants in the study of quadratics and their parabolic graphs. Accordingly, this final section consolidates the methods of the last four sections, but uses pronumerals, rather than numbers, for points and coefficients. Because of the generality, most of the questions have a geometric interpretation as a result.

When the French mathematician and philosopher René Descartes introduced the coordinate plane in the 17th century, he intended it as an alternative approach to geometry, in which all the ancient theorems of Euclid would be proven again using algebraic rather than geometric arguments. In our course, however, Descartes' methods are used with the reverse purpose — we are turning functions that have been defined algebraically into geometric objects that can be visualised and interpreted using the methods of geometry. Questions here, and throughout the course, should be interpreted both algebraically and geometrically.



Example 17 Applying formulae with pronumerals rather than numbers

- a** A triangle has vertices at $A(1, -3)$, $B(3, 3)$ and $C(-3, 1)$.
- Find the coordinates of the midpoint P of AB and the midpoint Q of BC .
 - Show that $PQ \parallel AC$ and that $PQ = \frac{1}{2}AC$.
- b** Repeat this procedure for a triangle with vertices $A(2a, 0)$, $B(2b, 2c)$ and $C(0, 0)$, where $a > 0$.

Solution

- a i** Using the midpoint formula,

$$\begin{aligned} \text{For } P, \quad x &= \frac{1+3}{2} & y &= \frac{-3+3}{2} \\ &= 2 & &= 0. \end{aligned}$$

$$\begin{aligned} \text{For } Q, \quad x &= \frac{3-3}{2} & y &= \frac{3+1}{2} \\ &= 0 & &= 2. \end{aligned}$$

Hence $P = (2, 0)$ and $Q = (0, 2)$.

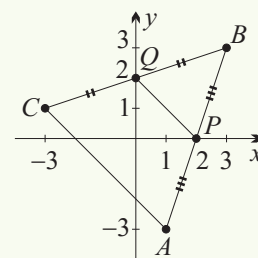
- ii** The lines PQ and AC are parallel because they both have gradient -1 :

$$\begin{aligned} \text{gradient } PQ &= \frac{2-0}{0-2} & \text{gradient } AC &= \frac{1+3}{-3-1} \\ &= -1 & &= -1. \end{aligned}$$

Using the distance formula,

$$\begin{aligned} PQ^2 &= (0-2)^2 + (2-0)^2 & AC^2 &= (-3-1)^2 + (1+3)^2 \\ &= 8 & &= 32 \end{aligned}$$

$$\text{so } PQ = 2\sqrt{2} \quad \text{and} \quad AC = 4\sqrt{2} = 2PQ.$$



b i Using the midpoint formula,

$$\begin{aligned} \text{For } P, \quad x &= \frac{2a + 2b}{2} & y &= \frac{0 + 2c}{2} \\ &= a + b & &= c. \end{aligned}$$

$$\begin{aligned} \text{For } Q, \quad x &= \frac{2b + 0}{2} & y &= \frac{2c + 0}{2} \\ &= b & &= c. \end{aligned}$$

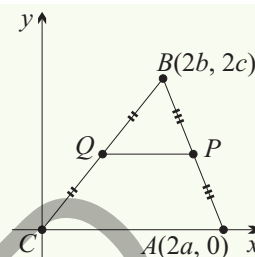
Hence $P = (a + b, c)$ and $Q = (b, c)$.

ii The lines PQ and AC are parallel because they are both horizontal.

Using the distance formula,

$$\begin{aligned} PQ^2 &= (b - a - b)^2 + (c - c)^2 & AC^2 &= (0 - 2a)^2 + (0 - 0)^2 \\ &= a^2 & &= 4a^2 \end{aligned}$$

$$\text{so } PQ = a \quad \text{and} \quad AC = 2a = 2PQ.$$



Any triangle can be translated and rotated into the position of the triangle in part (b), so this example demonstrates that the line segment joining the midpoints of two sides of a triangle is parallel to the base and half its length.

Exercise 7E

FOUNDATION

Note: This exercise involves lines with equations defined in terms of various constants k, m, b , etc. Without further definition, these are taken to be some constant real value.

- Find k for each of the following situations:
 - The line $y = 2x + k$ passes through $(3, 2)$.
 - The line $y = (2k - 1)x + 4$ has gradient 3.
 - The line $y = kx + (3 - k)$ passes through the origin.
- The two lines $y = kx + 3$ and $y = (2 - 3k)x + 7$ are defined in terms of a constant k . Find the value of k if:
 - The lines are parallel.
 - The lines are perpendicular.
- Find the area of the triangle bounded by the line $y = mx$, the x -axis and the line $x = 4$.
- Find the area of the triangle bounded by the line $y = mx$, the y -axis and the line $y = 6$.
- The triangle OAB has vertices $O(0, 0)$, $A(k + 2, 0)$ and $B(k, 3)$. It is known to have area 6. Find k .
- The line $y = 4 - kx$ cuts the x -axis at A and the y -axis at B . Find the area of triangle OAB , where O is the origin $O(0, 0)$.

DEVELOPMENT

- The line $y = mx - 4$ is tangential to the parabola $y = x^2$.
 - By solving the equations simultaneously, find a quadratic involving m .
 - Use the discriminant, and the fact the quadratic has one solution, to find m .
- The line $y = 2x + b$ is tangential to the parabola $y = x^2$.
 - Find a quadratic involving b for their point of intersection.
 - Use the discriminant to find b .
 - Where do the line and parabola intersect?

- 9 a** These three lines are concurrent:

$$y = 2x + 1$$

$$y = 3x - 4$$

$$y = mx + 7$$

Find m .

- b** These three lines are concurrent.

$$y = x + k$$

$$x = 4$$

$$y = -\frac{1}{2}x + 3k$$

Find their common point of intersection.

- 10** Consider the points $A(2, 0)$, $B(4, 2)$ and a point $P(x, y)$. It is known that P is the same distance from A and B (we say P is *equidistant* from A and B).

- a** Use the distance formula to write expressions for square distances PA^2 and PB^2 in terms of x and y .
- b** Hence find the equation of the line on which P sits.
- c** What can you state about triangle ABP and the midpoint of AB ?

- 11** Consider the points $A(2, 0)$ and $C(0, -4)$ and a point $P(x, y)$. It is known that P is equidistant from A and C .

- a** Use the methods of the previous question to find the equation of the line on which P sits.
- b** Which point in the plane is equidistant from points A , C , and also from point B in the previous question?

- 12 a** On a number plane, plot the points $O(0, 0)$, $A(6, 0)$, $B(6, 6)$ and $C(0, 6)$, which form a square.

- i** Find the gradients of the diagonals OB and AC .
- ii** Hence show that the diagonals OB and AC are perpendicular.

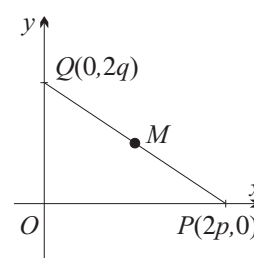
- b** Let the vertices of a square be $O(0, 0)$, $A(a, 0)$, $B(a, a)$ and $C(0, a)$.

- i** Find the gradients of the diagonals OB and AC .
- ii** Hence show that the diagonals OB and AC are perpendicular.

In this question you have shown that *the diagonals of a square are perpendicular*.

- 13 a** The points $O(0, 0)$, $P(8, 0)$ and $Q(0, 10)$ form a right-angled triangle. Let M be the midpoint of PQ .

- i** Find the coordinates of M .
- ii** Find the distances OM , PM and QM , and show that M is equidistant from each of the vertices.
- iii** Explain why a circle with centre M can be drawn through the three vertices O , P and Q .



- b** Consider a right-angled triangle with vertices at $O(0, 0)$, $P(2p, 0)$ and $Q(0, 2q)$ and repeat the steps of part (a).

In this question you have shown that *the midpoint of the hypotenuse of a right-angled triangle is the centre of a circle through all three vertices*.

- 14** The points $A(a, 0)$ and $Q(q, 0)$ are points on the positive x -axis, and the points $B(0, b)$ and $P(0, p)$ lie on the positive y -axis. Show that $AB^2 - AP^2 = QB^2 - QP^2$.

- 15** Place three vertices of a parallelogram at $A(0, 0)$, $B(2a, 2b)$ and $D(2c, 2d)$.

- a** Use gradients to show that with $C = (2a + 2c, 2b + 2d)$, the quadrilateral $ABCD$ is a parallelogram.
- b** Find the midpoints of the diagonals AC and BD .
- c** Explain why this proves that *the diagonals of a parallelogram bisect each other*.

CHALLENGE

- 16** Two circles $x^2 + y^2 = 4$ and $(x - 5)^2 + y^2 = 16$ intersect at P and Q . Solve the equations simultaneously to find the points of intersection of the two circles, and hence find the equation of the secant PQ .
Note: The *secant* is the line through two distinct points on a curve.
- 17** The point $P(a, a^2)$ lies on the parabola $y = x^2$.
- Write down the equation of the line ℓ through P with gradient m .
 - Substitute the parabola into the line to produce a quadratic equation in x (that is, begin to solve simultaneously the equations of the line and the parabola).
 - Find the discriminant of this quadratic equation, and write the discriminant as a quadratic function of m .
 - If ℓ is known to be a tangent to $y = x^2$, use this discriminant to find m in terms of a .
 - complete this sentence:
The tangent to $f(x) = x^2$ at $P(a, a^2)$ has gradient $g(a) = \dots$
 (The function g is called the gradient function for $f(x)$ and we shall investigate this more closely with better techniques in later chapters.)
- 18 a** The points $A(1, -2)$, $B(5, 6)$ and $C(-3, 2)$ are the vertices of a triangle, and P , Q and R are the midpoints of BC , CA and AB respectively.
- Find the equations of the three medians BQ , CR and AP .
 - Find the intersection of BQ and CR , and show that it lies on the third median AP .
- b** Now consider the more general situation of a triangle with vertices at $A(6a, 6b)$, $B(-6a, -6b)$ and $C(0, 6c)$, and follow these steps.
- Find the midpoints P , Q and R of BC , CA and AB respectively.
 - Show that the median through C is $x = 0$ and find the equations of the other two medians.
 - Find the point where the median through C meets the median through A , and show that this point lies on the median through B .

In this question you have shown that *the three medians of a triangle are concurrent*. (Their point of intersection is called the *centroid*.)

Chapter 7 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 7 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Let $X = (2, 9)$ and $Y = (14, 4)$. Use the standard formulae to find:
 - the midpoint of XY ,
 - the gradient of XY ,
 - the length of XY .
- A triangle has vertices $A(1, 4)$, $B(-3, 1)$ and $C(-2, 0)$.
 - Find the lengths of all three sides of $\triangle ABC$.
 - What sort of triangle is $\triangle ABC$?
- A quadrilateral has vertices $A(2, 5)$, $B(4, 9)$, $C(8, 1)$ and $D(-2, -7)$.
 - Find the midpoints P of AB , Q of BC , R of CD and S of DA .
 - Find the gradients of PQ , QR , RS and SP .
 - What sort of quadrilateral is $PQRS$?
- A circle has diameter AB , where $A = (2, -5)$ and $B = (-4, 7)$.
 - Find the centre C and radius r of the circle.
 - Use the distance formula to test whether $P(6, -1)$ lies on the circle.
- Find the gradients of the sides of $\triangle LMN$, given $L(3, 9)$, $M(8, -1)$ and $N(-1, 7)$.
 - Explain why $\triangle LMN$ is a right-angled triangle.
- Find the gradient of the line segment AB , where $A = (3, 0)$ and $B = (5, -2)$.
 - Find a if $AP \perp AB$, where $P = (a, 5)$.
 - Find the point $Q(b, c)$ if B is the midpoint of AQ .
 - Find d if the line segment AD has length 5, where $D = (6, d)$.
- Find, in general form, the equation of the line:
 - with gradient -2 and y -intercept 5,
 - with gradient $\frac{2}{3}$ through the point $A(3, 5)$,
 - through the origin and perpendicular to $y = 7x - 5$,
 - through $B(-5, 7)$ and parallel to $y = 4 - 3x$,
 - with y -intercept -2 and angle of inclination 60° .

- 8** Put the equation of each line into gradient–intercept form, and hence find its y -intercept b , its gradient m and its angle of inclination α (correct to the nearest minute when necessary).
- a** $5x - 6y - 7 = 0$ **b** $4x + 4y - 3 = 0$
- 9** Find the gradient of each line AB , then find its equation in general form.
- a** $A(3, 0)$ and $B(4, 8)$ **b** $A(5, -2)$ and $B(7, -7)$
- 10** **a** Are the points $L(7, 4)$, $M(13, 2)$ and $N(25, -3)$ collinear?
b Are the lines $2x + 5y - 29 = 0$, $4x - 2y + 2 = 0$ and $7x - 3y + 1 = 0$ concurrent?
- 11** **a** Determine whether the lines $8x + 7y + 6 = 0$, $6x - 4y + 3 = 0$ and $2x + 3y + 9 = 0$ enclose a right-angled triangle.
b Determine what sort of figure the lines $4x + 8y + 3 = 0$, $5x - 2y + 7 = 0$, $x + 2y - 6 = 0$ and $9x - 3y = 0$ enclose.
- 12** **a** Find the points where the line $5x + 4y - 30 = 0$ meets the x -axis and y -axis.
b Hence find the area of the triangle formed by the line, the x -axis and the y -axis.
- 13** A sketch is essential in this question.
- a** Find the gradient, length and midpoint M of the line segment joining $A(10, 2)$ and $B(2, 8)$.
b Show that the perpendicular bisector of the line segment AB has equation $4x - 3y - 9 = 0$.
c Find the point C where the perpendicular bisector meets the line $x - y + 2 = 0$.
d Use the distance formula to show that C is equidistant from A and B .
e Show that $CM = 15$ and hence find the area of $\triangle ABC$.
f Let $\theta = \angle ACB$. Use the area formula $\text{area} = \frac{1}{2} \times AC \times BC \times \sin \theta$ to find θ , correct to the nearest minute.

8

Exponential and logarithmic functions

Chapter introduction

Suppose that the amount of mould on a piece of cheese is doubling every day.

- ▶ In one day, the mould increases by a factor of 2.
- ▶ In two days, the mould increases by a factor of $2^2 = 4$.
- ▶ In three days, the mould increases by a factor of $2^3 = 8$.
- ▶ In four days, the mould increases by a factor of $2^4 = 16$.

And so on, until the cheese is thrown out. Applying mathematics to situations such as this requires indices and logarithms, and their graphs. Logarithms may be quite unfamiliar to many readers.

The application questions throughout, and particularly the final exercise, indicate some of the extraordinary variety of situations where exponential and logarithmic functions are needed.

Readers who have studied this material well in earlier years may be selective with the earlier questions in the exercises.

8A Indices

Learning intentions

- Use the terms power, base, and index correctly.
- Develop the index laws for powers whose indices are positive whole numbers.
- Extend to powers whose indices are zero or a negative integer.

This section reviews powers such as 2^3 and 5^{-2} , whose indices are integers. Then the various index laws for working with them are reviewed.

Power, base and index

These words ‘power’, ‘base’, and ‘index’ have three different very exact meanings.

1 Power, base and index

- An expression a^n is called a *power*.
- The number a is called the *base*.
- The number n is called the *index* or *exponent*.

Thus 2^3 is a power with base 2 and index 3.

The words *exponent* and *index* (plural *indices*) mean exactly the same thing.

Powers whose indices are positive whole numbers

Powers are defined differently depending on what sort of number the index is. When the index is a positive integer, the definition is quite straightforward.

2 Powers whose indices are positive whole numbers

For any real number a ,

$$a^1 = a, \quad a^2 = a \times a, \quad a^3 = a \times a \times a, \quad \dots$$

In general, $a^n = \underbrace{a \times a \times a \times \dots \times a}_{n \text{ factors}}$, for all positive whole numbers n .

$$\begin{array}{lll} \text{For example,} & 7^1 = 7 & 3^4 = 3 \times 3 \times 3 \times 3 & 9^5 = 9 \times 9 \times 9 \times 9 \times 9 \\ & & = 81 & = 59\,049 \end{array}$$

although with a large number such as 9^5 , the index form may be more convenient.

Scientific calculators can help to evaluate or approximate powers. The button for powers is labelled $\boxed{x^y}$ or $\boxed{\wedge}$, depending on the calculator. For example,

$$2^{26} = 67\,108\,864 \qquad 3^{50} \div 7.179 \times 10^{23} \qquad \pi^4 \div 97.41$$

Index laws — combining powers with the same base

The first three index laws show how to combine powers when the base is fixed.

3 Index laws — products, quotients and powers of powers

- To multiply powers with the same base, add the indices:

$$a^m \times a^n = a^{m+n}$$

- To divide powers with the same base, subtract the indices:

$$a^m \div a^n = a^{m-n} \quad (\text{also written as } \frac{a^m}{a^n} = a^{m-n})$$

- To raise a power to a power, multiply the indices:

$$(a^m)^n = a^{mn}$$

Demonstrating the results when $m = 5$ and $n = 3$ should make these laws clear.

$$\begin{aligned} \bullet \quad a^5 \times a^3 &= (a \times a \times a \times a \times a) \times (a \times a \times a) \\ &= a^8 \\ &= a^{5+3} \end{aligned}$$

$$\begin{aligned} \bullet \quad a^5 \div a^3 &= \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a^2 \\ &= a^{5-3} \end{aligned}$$

$$\begin{aligned} \bullet \quad (a^5)^3 &= a^5 \times a^5 \times a^5 \\ &= a^{5+5+5} \quad (\text{to multiply powers of the same base, add the indices}) \\ &= a^{5 \times 3} \end{aligned}$$



Example 1 Using the index laws

Use the index laws above to simplify each expression.

a $x^3 \times x^7$

b $3^x \times 3^{5x}$

c $w^{12} \div w^3$

d $10^{a+b} \div 10^b$

e $(y^6)^9$

f $(2^{3x})^{2y}$

Solution

a $x^3 \times x^7 = x^{10}$

b $3^x \times 3^{5x} = 3^{6x}$

c $w^{12} \div w^3 = w^9$

d $10^{a+b} \div 10^b = 10^a$

e $(y^6)^9 = y^{54}$

f $(2^{3x})^{2y} = 2^{6xy}$

Index laws — powers of products and quotients

The next two index laws show how to work with powers of products and powers of quotients.

4 Index laws — powers of products and quotients

- The power of a product is the product of the powers:

$$(ab)^n = a^n \times b^n$$

- The power of a quotient is the quotient of the powers:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Demonstrating the results when $n = 3$ should make the general results obvious.

$$\begin{aligned} \bullet (ab)^3 &= ab \times ab \times ab \\ &= a \times a \times a \times b \times b \times b \\ &= a^3 b^3 \end{aligned} \qquad \bullet \left(\frac{a}{b}\right)^3 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$$



Example 2 Powers of products and quotients

Expand the brackets in each expression.

a $(10a)^3$

b $\left(\frac{x}{3}\right)^4$

c $(3x^2y)^3$

d $\left(\frac{a^3}{3b}\right)^2$

Solution

a $(10a)^3 = 10^3 \times a^3$
 $= 1000a^3$

b $\left(\frac{x}{3}\right)^4 = \frac{x^4}{3^4}$
 $= \frac{x^4}{81}$

c $(3x^2y)^3 = 3^3 \times (x^2)^3 \times y^3$
 $= 27x^6y^3$

d $\left(\frac{a^3}{3b}\right)^2 = \frac{(a^3)^2}{3^2 \times b^2}$
 $= \frac{a^6}{9b^2}$

Zero and negative indices

All the indices were positive whole numbers in the examples above. Powers with zero and negative indices are defined so that these laws also hold for them.

Zero and negative indices involve division by the base a , so a cannot be zero.

1 We know that $a^3 \div a^3 = 1$.
If we use the index laws, however, $a^3 \div a^3 = a^{3-3} = a^0$.
Hence it is convenient to define $a^0 = 1$.

2 We know that $a^2 \div a^3 = \frac{1}{a}$.
If we used the index laws, however, $a^2 \div a^3 = a^{2-3} = a^{-1}$.
Hence it is convenient to define $a^{-1} = \frac{1}{a}$.

3 Similarly, we shall define $a^{-2} = \frac{1}{a^2}$, and $a^{-3} = \frac{1}{a^3}$, and so on.

5 Zero and negative indices

Let the base a be any non-zero real number.

- Define $a^0 = 1$.
- Define $a^{-1} = \frac{1}{a}$, define $a^{-2} = \frac{1}{a^2}$, define $a^{-3} = \frac{1}{a^3}$, ...
- In general, define $a^{-n} = \frac{1}{a^n}$, for all whole numbers n .

Thus the negative sign in the index says, 'Take the reciprocal'. Always do this first, and always write the reciprocal of $\frac{2}{3}$ as $\frac{3}{2}$, never as $\frac{1}{\frac{2}{3}}$. For example,

$$\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}, \text{ where we have dealt with the negative sign first.}$$



Example 3 Converting powers with negative indices to fractions

Write each expression using fractions instead of negative indices, then simplify it.

a 12^{-1}

b 7^{-2}

c $\left(\frac{2}{3}\right)^{-1}$

d $\left(\frac{2}{3}\right)^{-4}$

Solution

a $12^{-1} = \frac{1}{12}$

b $7^{-2} = \frac{1}{7^2}$
 $= \frac{1}{49}$

c $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$
 (the negative index says, 'take the reciprocal')

d $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^4$ (take the reciprocal first)
 $= \frac{81}{16}$ (then take the fourth power)



Example 4 Converting fractions to powers with negative indices

Write each expression using negative indices instead of fractions.

a $\frac{1}{2^{20}}$

b $\frac{1}{x^3}$

c $\frac{5}{a^7}$

d $\frac{w^2}{v^4}$

Solution

a $\frac{1}{2^{20}} = 2^{-20}$

b $\frac{1}{x^3} = x^{-3}$

c $\frac{5}{a^7} = 5 \times \frac{1}{a^7}$
 $= 5a^{-7}$

d $\frac{w^2}{v^4} = w^2 \times \frac{1}{v^4}$
 $= w^2 v^{-4}$



Example 5 Simplifying expressions with indices

Use the index laws, extended to negative indices, to simplify these expressions. Give your answers in fraction form without negative indices.

a $x^4 \times x^{-10}$

b $12a^2b^4 \div 6a^3b^4$

c $(2^n)^{-3}$

Solution

a $x^4 \times x^{-10} = x^{4-10}$
 $= x^{-6}$
 $= \frac{1}{x^6}$

b $12a^2b^4 \div 6a^3b^4 = \frac{12}{6} a^{2-3} b^{4-4}$
 $= 2a^{-1}b^0$
 $= 2a^{-1}$
 $= \frac{2}{a}$

c $(2^n)^{-3} = 2^{n \times (-3)}$
 $= 2^{-3n}$
 $= \frac{1}{2^{3n}}$

Exercise 8A

FOUNDATION

Note: Do not use a calculator in this exercise at all, except for the very last question.

- 1 The town of Goldhope had a small population in 1970, but the population has been increasing by a factor of 3 every decade since then.
 - a By what factor had the population increased from its 1970 value in 1980, 1990, 2000, 2010 and 2020?
 - b If the town's population was 10 000 in 1970, during which decade did the population pass 1 000 000?
- 2 **Preparation:** The answers to this question will be needed for all the exercises in this chapter. Keep the results in a place where you can refer to them easily.
 - a Write down all the powers of 2 from $2^0 = 1$ to $2^{12} = 4096$.
 - b In the same way, write down:
 - i the powers of 3 to $3^6 = 729$,
 - ii the powers of 5 to $5^5 = 3125$,
 - iii the powers of 6 to $6^3 = 216$,
 - iv the powers of 7 to $7^3 = 343$,
 - v the powers of 10 to $10^6 = 1\,000\,000$,
 - vi the powers of 20 to $20^6 = 64\,000\,000$.
 - c From your list of powers of 2, read off:
 - i powers of 4,
 - ii powers of 8.
 - d From your list of powers of 3, read off powers of 9.
 - e From your list of powers of 5, read off powers of 25.
- 3 Simplify:

a 2^3	b 2^6	c 3^4	d 9^3	e $\left(\frac{2}{3}\right)^2$
f $\left(\frac{2}{3}\right)^3$	g $\left(\frac{3}{10}\right)^4$	h $\left(\frac{4}{7}\right)^2$	i $\left(\frac{5}{9}\right)^1$	j 1^1
- 4 Simplify:

a 3^0	b 7^0	c 5^{-1}	d 11^{-1}	e 6^{-2}
f 10^{-2}	g 3^{-3}	h 5^{-3}	i 2^{-5}	j 10^{-6}
- 5 Simplify. With fractions, a negative index says, 'Take the reciprocal,'

a $\left(\frac{1}{5}\right)^{-1}$	b $\left(\frac{1}{11}\right)^{-1}$	c $\left(\frac{2}{7}\right)^{-1}$	d $\left(\frac{7}{2}\right)^{-1}$	e $\left(\frac{3}{4}\right)^{-1}$
f $\left(\frac{10}{23}\right)^{-1}$	g $(10)^{-1}$	h $(0.1)^{-1}$	i $(0.01)^{-1}$	j $(0.02)^{-1}$
- 6 Simplify. First take the reciprocal, then take the power.

a 5^{-2}	b $\left(\frac{1}{5}\right)^{-2}$	c $\left(\frac{1}{5}\right)^{-3}$	d $\left(\frac{1}{2}\right)^{-4}$	e $\left(\frac{1}{10}\right)^{-6}$
f $\left(\frac{2}{3}\right)^{-2}$	g $\left(\frac{2}{3}\right)^{-4}$	h $\left(\frac{3}{2}\right)^{-4}$	i $\left(\frac{2}{5}\right)^{-2}$	j $\left(\frac{3}{7}\right)^0$
- 7 Simplify, leaving your answers in index form:

a $2^9 \times 2^5$	b $a^8 \times a^7$	c $7^2 \times 7^{-10}$	d $x^7 \times x^{-5}$
e $9^6 \times 9^{-6}$	f $a^5 \times a^{-5}$	g 5×5^{-4}	h $8^4 \times 8 \times 8^{-4}$

8 Simplify, leaving your answers in index form:

a $7^8 \div 7^3$

b $a^5 \div a^7$

c $x^{10} \div x^{-2}$

d $x^{-10} \div x^2$

e $2^8 \div 2^{-8}$

f $2^8 \div 2^8$

g $y^{12} \div y$

h $y \div y^{12}$

9 Simplify, leaving your answers in index form:

a $x^5 \times x^5 \times x^5$

b $(x^5)^3$

c $(z^2)^7$

d $a^{-2} \times a^{-2} \times a^{-2}$

e $(a^{-2})^3$

f $(5^4)^{-7}$

g $(y^{-2})^{-5}$

h $(2^{-4})^{-4}$

10 Write down the solutions of these index equations.

a $3^x = 9$

b $2^x = 16$

c $5^x = 125$

d $2^x = 64$

e $7^x = \frac{1}{7}$

f $5^x = \frac{1}{5}$

g $9^x = \frac{1}{81}$

h $2^x = \frac{1}{8}$

i $\left(\frac{1}{3}\right)^x = 3$

j $\left(\frac{5}{6}\right)^x = \frac{6}{5}$

k $9^x = 1$

l $12^x = 1$

11 Expand the brackets in each expression.

a $(3x)^2$

b $(5a)^3$

c $(2c)^6$

d $(3st)^4$

e $(7xyz)^2$

f $\left(\frac{1}{x}\right)^5$

g $\left(\frac{3}{x}\right)^2$

h $\left(\frac{y}{5}\right)^2$

i $\left(\frac{7a}{5}\right)^2$

j $\left(\frac{3x}{2y}\right)^3$

DEVELOPMENT

12 A huge desert rock is entirely made up of coarse grains of sandstone, each about 1 mm^3 in size, and is very roughly a rectangular block 3 km long, 2 km wide, and 500 m high.

a Find the approximate volume of the rock in cubic kilometres.

b How many cubic millimetres are in a cubic kilometre? Answer in index notation.

c Find the approximate number of grains of sandstone in the rock.

13 Write each expression as a fraction without negative indices or brackets.

a 9^{-1}

b x^{-1}

c b^{-2}

d $-a^{-4}$

e $(7x)^{-1}$

f $7x^{-1}$

g $-9x^{-1}$

h $(3a)^{-2}$

i $3a^{-2}$

j $4x^{-3}$

14 Write each fraction in index form.

a $\frac{1}{x}$

b $-\frac{1}{x^2}$

c $-\frac{12}{x}$

d $\frac{9}{x^2}$

e $-\frac{1}{x^3}$

f $\frac{12}{x^5}$

g $\frac{7}{x^3}$

h $-\frac{6}{x}$

i $\frac{1}{6x}$

j $-\frac{1}{4x^2}$

15 First change each mixed numeral or decimal to a fraction, then simplify the expression.

a $\left(1\frac{1}{2}\right)^{-1}$

b $\left(2\frac{1}{3}\right)^{-1}$

c $\left(2\frac{2}{3}\right)^{-1}$

d $\left(2\frac{1}{2}\right)^{-2}$

e $\left(3\frac{1}{3}\right)^{-3}$

f $\left(6\frac{2}{3}\right)^{-2}$

g 0.2^{-1}

h 2.4^{-1}

i 2.25^{-1}

j 2.5^{-2}

k 2.5^{-3}

l 0.05^{-2}

16 Write down the solutions of these index equations.

a $\left(\frac{5}{8}\right)^x = \frac{25}{64}$

b $\left(\frac{5}{8}\right)^x = \frac{8}{5}$

c $\left(\frac{3}{5}\right)^x = \frac{25}{9}$

d $\left(\frac{3}{2}\right)^x = \frac{8}{27}$

e $x^2 = \frac{100}{169}$

f $x^{-2} = \frac{1}{4}$

g $x^{-3} = 27$

h $x^{-2} = \frac{64}{81}$

17 Simplify each expression, giving the answer without negative indices.

a $2^x \times 2^3$

b $3^x \times 3$

c $7^{4x} \times 7^{-5x}$

d $5^{2x} \div 5^3$

e $(10^{2x})^3$

f $(5^{4x})^{-2}$

g $(6^x)^4 \times (6^{2x})^5$

h $(2^x)^3 \div 2^4$

18 Simplify each expression, giving the answer without negative indices.

a $x^2y \times x^4y^3$

b $x^{-5}y^3 \times x^3y^{-2}$

c $3ax^{-2} \times 7a^2x$

d $\frac{1}{15}s^{-2}t^3 \times 5st^{-5}$

e $56x^2y^6 \div 8xy^8$

f $35a^{-2}b^4 \div 28a^4b^{-6}$

g $(s^2y^{-3})^3$

h $(5c^{-2}d^3)^{-1}$

i $(3x^2y^3)^3 \times (xy^4)^2$

j $(2a^{-2}y^3)^{-2} \times (2ay^{-3})^3$

k $(5st^2)^3 \div (5s^{-1}t^3)^2$

l $(10x^2y^{-2})^3 \div (2x^{-1}y^3)^2$

CHALLENGE

19 Expand and simplify, answering without using negative indices:

a $(x + x^{-1})^2$

b $(x - x^{-1})^2$

c $(x^2 - x^{-2})^2$

20 Write each expression as a single power.

a 2×2^x

b $2^x + 2^x$

c 3×3^x

d $3^x + 3^x + 3^x$

e 4×2^x

f 32×2^x

g 125×5^x

h 81×3^x

i $\frac{1}{2} \times 2^x$

j $\frac{1}{9} \times 3^x$

21 Explain why if $3^{3x-1} = 9$, then $3x - 1 = 2$, and so $x = 1$. Similarly, solve:

a $5^{x+3} = 25$

b $3^{2x-8} = 81$

c $4^{7-x} = \frac{1}{4}$

d $4^{5x+2} = \frac{1}{64}$

e $\left(\frac{1}{3}\right)^{x+1} = 27$

f $\left(\frac{1}{5}\right)^{x+1} = \left(\frac{1}{125}\right)^{x-1}$

22 A calculator is required in this question — use scientific notation throughout.

a The mass of a neutron is about 1.675×10^{-27} kg. About how many neutrons are there in 1 kg of neutrons?

b The radius of a neutron is about 1.11×10^{-15} m. Use the volume-of-a-sphere formula $V = \frac{4}{3}\pi r^3$ to find its approximate volume.

c Use the formula density = $\frac{\text{mass}}{\text{volume}}$ to find its approximate density in kg/m^3 .

Note: This question makes several extremely naive assumptions about what a neutron actually is, but the extraordinarily high density that this calculation gives is close to the density of neutron stars, which are the densest objects in the universe.

8B Fractional indices

Learning intentions

- Define and work with powers with fractional indices.

In this section, the definition of powers is extended to powers such as $4^{\frac{1}{2}}$ and $27^{-\frac{2}{3}}$, where the index is a positive or negative fraction. Again, the definitions are made so that the index laws work for all indices.

Numerical answers can be checked on the calculator using (x^y) or $(\wedge)$.

Fractional indices with numerator 1

As before, let the base a be any real number $a \geq 0$.

1 We know that $(\sqrt{a})^2 = a$.

If we use the index laws, however, $(a^{\frac{1}{2}})^2 = a^1 = a$.

Hence it is convenient to define $a^{\frac{1}{2}} = \sqrt{a}$.
(Remember that $\sqrt{a}$ means the *non-negative square root* of a .)

2 We know that $(\sqrt[3]{a})^3 = a$.

If we use the index laws, however, $(a^{\frac{1}{3}})^3 = a^1 = a$.

Hence it is convenient to define $a^{\frac{1}{3}} = \sqrt[3]{a}$.
Remember that we have restricted the base a to *non-negative numbers*.

6 Fractional indices with numerator 1

Let the base a be any real number $a \geq 0$.

- Define $a^{\frac{1}{2}} = \sqrt{a}$, define $a^{\frac{1}{3}} = \sqrt[3]{a}$, define $a^{\frac{1}{4}} = \sqrt[4]{a}$, ...
- In general, define $a^{\frac{1}{n}} = \sqrt[n]{a}$, for all positive whole numbers n , where in every case, the *non-negative* root of a is to be taken.



Example 6 Converting powers with fractional indices to surds

Write each expression using surd notation, then simplify it.

a $64^{\frac{1}{2}}$

b $27^{\frac{1}{3}}$

c $10\,000^{\frac{1}{4}}$

d $32^{\frac{1}{5}}$

Solution

a $64^{\frac{1}{2}} = \sqrt{64}$
 $= 8$

b $27^{\frac{1}{3}} = \sqrt[3]{27}$
 $= 3$

c $10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000}$
 $= 10$

d $32^{\frac{1}{5}} = \sqrt[5]{32}$
 $= 2$

General fractional indices

If the index laws are to apply to a power such as $a^{\frac{2}{3}}$, then we must be able to write

$$(a^{\frac{1}{3}})^2 = a^{\frac{2}{3}} \quad \text{and} \quad (a^2)^{\frac{1}{3}} = a^{\frac{2}{3}},$$

because to raise a power to a power, we multiply the indices.

Hence we shall define $a^{\frac{2}{3}}$ as $(a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2$, which is the same as $(a^2)^{\frac{1}{3}} = \sqrt[3]{a^2}$.

7 General fractional indices

Let the base a be any positive real number, and $n \geq 1$ and m be whole numbers.

- Define $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, which is the same as $\sqrt[n]{a^m}$.
- Define $a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$.

When there are negatives and fractions in the index, first deal with the negative sign, then deal with the fractional index:

8 Deal with complicated indices in this order

- 1 If the index is negative, take the reciprocal.
- 2 If the index has a denominator, take the corresponding root.
- 3 Finally, take the power indicated by the numerator of the index.

For example, $\left(\frac{4}{9}\right)^{-\frac{3}{2}} = \left(\frac{9}{4}\right)^{\frac{3}{2}} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$.

Example 7 Using fractional indices

Simplify each power. First take the root indicated by the denominator.

a $125^{\frac{2}{3}}$

b $100^{\frac{3}{2}}$

c $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

Solution

a $125^{\frac{2}{3}} = 5^2$
 $= 25$

b $100^{\frac{3}{2}} = 10^3$
 $= 1000$

c $\left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2}{3}\right)^3$
 $= \frac{8}{27}$

Example 8 Using negative fractional indices

Simplify each power. First take the reciprocal as indicated by the negative index.

a $25^{-\frac{3}{2}}$

b $1000^{-\frac{2}{3}}$

c $\left(\frac{8}{27}\right)^{-\frac{4}{3}}$

Solution

a $25^{-\frac{3}{2}} = \left(\frac{1}{25}\right)^{\frac{3}{2}}$
 $= \left(\frac{1}{5}\right)^3$
 $= \frac{1}{125}$

b $1000^{-\frac{2}{3}} = \left(\frac{1}{1000}\right)^{\frac{2}{3}}$
 $= \left(\frac{1}{10}\right)^2$
 $= \frac{1}{100}$

c $\left(\frac{8}{27}\right)^{-\frac{4}{3}} = \left(\frac{27}{8}\right)^{\frac{4}{3}}$
 $= \left(\frac{3}{2}\right)^4$
 $= \frac{81}{16}$

Example 9 Converting surds to fractional indices

Write each expression using a fractional index.

a $\sqrt{x^3}$

b $\frac{1}{\sqrt{x^3}}$

c $\sqrt[3]{x^2}$

d $\frac{1}{\sqrt[3]{x^2}}$

Solution

a $\sqrt{x^3} = x^{\frac{3}{2}}$

b $\frac{1}{\sqrt{x^3}} = x^{-\frac{3}{2}}$

c $\sqrt[3]{x^2} = x^{\frac{2}{3}}$

d $\frac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$

Negative and zero bases, and irrational indices

Negative numbers do not have square roots, so negative bases are impossible if a square root or fourth root or any even root is involved. For example, it may be reasonable to define $(-64)^{\frac{1}{3}} = -4$, but $(-64)^{\frac{1}{2}}$ is undefined.

Powers of zero are only defined for positive indices x , in which case $0^x = 0$ — for example, $0^3 = 0$. If the index x is negative, then 0^x is undefined, being the reciprocal of zero — for example, 0^{-3} is undefined. If the index is zero, then 0^0 is also undefined — see Question 24.

We can only define irrational indices informally. To define 2^π , for example, take the sequence 3, 3.1, 3.14, 3.141, 3.1415, ... of truncated decimals for π , then form the sequence of rational powers $2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, \dots$. Everything then works as expected.

Exercise 8B

FOUNDATION

Note: Do not use a calculator in this exercise unless the question asks for it. Make sure that you can refer easily to the list of powers of 2, 3, 5, ... from the previous exercise.

- 1 Simplify these powers, giving each answer as a whole number.

a $25^{\frac{1}{2}}$

b $36^{\frac{1}{2}}$

c $100^{\frac{1}{2}}$

d $27^{\frac{1}{3}}$

e $64^{\frac{1}{3}}$

f $1000^{\frac{1}{3}}$

g $81^{\frac{1}{4}}$

h $32^{\frac{1}{5}}$

i $1\,000\,000^{\frac{1}{6}}$

j $1\,000\,000^{\frac{1}{2}}$

- 2 Simplify these powers. First take the root indicated by the denominator.

a $25^{\frac{3}{2}}$

b $9^{\frac{3}{2}}$

c $27^{\frac{2}{3}}$

d $8^{\frac{2}{3}}$

e $16^{\frac{3}{4}}$

f $81^{\frac{3}{4}}$

g $27^{\frac{4}{3}}$

h $4^{\frac{5}{2}}$

i $32^{\frac{3}{5}}$

j $64^{\frac{2}{3}}$

- 3 Simplify these powers, giving each answer as a fraction.

a $\left(\frac{1}{49}\right)^{\frac{1}{2}}$

b $\left(\frac{1}{8}\right)^{\frac{1}{3}}$

c $\left(\frac{25}{49}\right)^{\frac{1}{2}}$

d $\left(\frac{27}{8}\right)^{\frac{1}{3}}$

e $\left(\frac{1}{4}\right)^{\frac{3}{2}}$

f $\left(\frac{1}{25}\right)^{\frac{3}{2}}$

g $\left(\frac{4}{9}\right)^{\frac{3}{2}}$

h $\left(\frac{9}{100}\right)^{\frac{3}{2}}$

- 4 Use the calculator button labelled x^y or $\wedge$ to find these powers. Give each answer exactly if possible, or else correct to four significant figures.

a 13^4

b 3.235^4

c 2^{50}

d $759\,375^{\frac{1}{5}}$

e $10^{\frac{1}{3}}$

f $17^{\frac{1}{4}}$

g 7^{-12}

h $15^{-\frac{3}{5}}$

- 5 Use the index laws to simplify these expressions, leaving your answers in index form.

a $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$

b $x^{2\frac{1}{2}} \times x^{3\frac{1}{2}}$

c $x^4 \times x^{-\frac{1}{2}}$

d $x^{4\frac{1}{2}} \div x^{3\frac{1}{2}}$

e $x \div x^{\frac{1}{2}}$

f $x^{-7} \div x^{-2\frac{1}{2}}$

g $(x^{\frac{1}{3}})^6$

h $(x^{-\frac{2}{3}})^6$

i $(x^9)^{\frac{2}{3}}$

- 6 Use the index laws to simplify these expressions, giving answers as integers or fractions.

a $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$

b $2^{\frac{1}{2}} \times 2^{-\frac{1}{2}}$

c $2^{\frac{1}{2}} \times 2^{2\frac{1}{2}}$

d $3^{4\frac{1}{2}} \div 3^{5\frac{1}{2}}$

e $25 \div 25^{\frac{1}{2}}$

f $7^{\frac{2}{3}} \div 7^{\frac{2}{3}}$

g $3^3 \times 3^4 \div 3^{10}$

h $(3^{-\frac{1}{3}})^6$

i $(9^8)^{\frac{1}{4}}$

7 Write down the solutions of these index equations.

a $9^x = 3$

b $121^x = 11$

c $81^x = 3$

d $64^x = 2$

e $\left(\frac{1}{25}\right)^x = \frac{1}{5}$

f $\left(\frac{1}{8}\right)^x = \frac{1}{2}$

8 Rewrite each expression using surds instead of fractional indices.

a $x^{\frac{1}{2}}$

b $x^{\frac{1}{3}}$

c $7x^{\frac{1}{2}}$

d $(7x)^{\frac{1}{2}}$

e $15x^{\frac{1}{4}}$

f $x^{\frac{3}{2}}$

g $6x^{\frac{5}{2}}$

h $x^{\frac{4}{3}}$

9 Rewrite each expression using fractional indices instead of surds.

a $\sqrt{x}$

b $3\sqrt{x}$

c $\sqrt{3x}$

d $12\sqrt[3]{x}$

e $9\sqrt[6]{x}$

f $\sqrt{x^3}$

g $\sqrt{x^9}$

h $25\sqrt[5]{x^6}$

DEVELOPMENT

10 Simplify these powers. First take the reciprocal as indicated by the negative index.

a $25^{-\frac{1}{2}}$

b $100^{-\frac{1}{2}}$

c $125^{-\frac{1}{3}}$

d $16^{-\frac{1}{4}}$

e $16^{-\frac{3}{4}}$

f $27^{-\frac{2}{3}}$

g $81^{-\frac{3}{4}}$

h $49^{-\frac{3}{2}}$

11 Simplify these powers.

a $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$

b $\left(\frac{1}{125}\right)^{-\frac{1}{3}}$

c $\left(\frac{1}{49}\right)^{-\frac{1}{2}}$

d $\left(\frac{1}{27}\right)^{-\frac{1}{3}}$

e $\left(\frac{1}{16}\right)^{-\frac{3}{4}}$

f $\left(\frac{1}{81}\right)^{-\frac{3}{4}}$

g $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$

h $\left(\frac{125}{8}\right)^{-\frac{2}{3}}$

12 Simplify each expression, giving the answer in index form.

a $3x^{\frac{1}{2}}y \times 3x^{\frac{1}{2}}y^2$

b $5a^{\frac{1}{3}}b^{\frac{2}{3}} \times 7a^{-\frac{1}{3}}b^{\frac{1}{3}}$

c $\frac{1}{8}s^{2\frac{1}{2}} \times 24s^{-2}$

d $x^2y^3 \div x^{\frac{1}{2}}y^{\frac{1}{2}}$

e $a^{\frac{1}{2}}b^{\frac{1}{2}} \div a^{-\frac{1}{2}}b^{\frac{1}{2}}$

f $(a^{-2}b^4)^{\frac{1}{2}}$

g $(8x^3y^{-6})^{\frac{1}{3}}$

h $(p^{\frac{1}{5}}q^{-\frac{3}{5}})^{10}$

i $(x^{\frac{3}{4}})^4 \times (x^{\frac{4}{3}})^3$

13 Rewrite these expressions, using fractional and negative indices.

a $\frac{1}{\sqrt{x}}$

b $\frac{12}{\sqrt{x}}$

c $-\frac{5}{\sqrt{x}}$

d $\frac{15}{\sqrt[3]{x}}$

e $-\frac{4}{\sqrt[3]{x^2}}$

f $x\sqrt{x}$

g $\frac{5}{x\sqrt{x}}$

h $8x^2\sqrt{x}$

14 Given that $x = 16$ and $y = 25$, evaluate:

a $x^{\frac{1}{2}} + y^{\frac{1}{2}}$

b $x^{\frac{1}{4}} - y^{\frac{1}{2}}$

c $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$

d $(y - x)^{\frac{1}{2}} \times (4y)^{-\frac{1}{2}}$

15 The cost of installing a standard swimming pool is rising exponentially according to the formula $C = \$6000 \times (1.03)^n$, where n is the number of years after 1st January 2015.

a What was the original cost on 1st January 2015?

b What was the cost on 1st January 2016?

c Use your calculator to find, correct to the nearest \$10, the cost of installing a pool on:

i 1st January 2020

ii 1st July 2015

iii 1st July 2018.

16 Use the button labelled (x^y) or $(\wedge)$ on your calculator to approximate these powers. Express your answers in scientific notation correct to four significant figures.

a 7^8

b $565^{\frac{2}{5}}$

c $5^{-0.12}$

d $(0.001)^{0.7}$

e 2^π

f $10^{\sqrt{2}}$

g $(\sqrt{7})^{-\sqrt{7}}$

h $0.001^{-\frac{\pi}{2}}$

CHALLENGE

17 Expand and simplify, giving answers in index form:

a $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$

b $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$

c $(x^{\frac{5}{2}} - x^{-\frac{5}{2}})^2$

18 Write down the solutions of these index equations.

a $49^x = \frac{1}{7}$

b $81^x = \frac{1}{3}$

c $8^x = 4$

d $8^x = \frac{1}{4}$

e $4^x = 8$

f $4^x = \frac{1}{8}$

g $81^x = 27$

h $27^x = \frac{1}{81}$

i $\left(\frac{1}{25}\right)^x = 5$

j $\left(\frac{8}{125}\right)^x = \frac{25}{4}$

19 By taking 6th powers of both sides, show that $11^{\frac{1}{3}} < 5^{\frac{1}{2}}$. Using similar methods (followed perhaps by a check on the calculator), compare:

a $3^{\frac{1}{3}}$ and $2^{\frac{1}{2}}$

b $2^{\frac{1}{2}}$ and $5^{\frac{1}{5}}$

c $7^{\frac{3}{2}}$ and 20

d $5^{\frac{1}{5}}$ and $3^{\frac{1}{3}}$

8C Logarithms

Learning intentions

- Understand what a logarithm and its base are.
- Convert a log equation to an index equation.
- Use a calculator to approximate powers and base 10 logs.
- Locate a log between two integers.

The introduction to this chapter mentioned that mould growing on some cheese was doubling every day. Suppose now that someone asks the question backwards:

'How many days does it take for the mould to increase by a factor of 8?'

The answer is 3 days, because $8 = 2^3$, so in 3 days, the mould doubles three times.

This index 3 is called the *logarithm base 2 of 8*, and is written as $\log_2 8 = 3$:

$$\log_2 8 = 3 \quad \text{means that} \quad 8 = 2^3.$$

Read this as, 'The log base 2 of 8', mentioning the base first, and then the number.

Logarithms

The logarithm is the index. More precisely:

9 The logarithm is the index

- The *logarithm base a* of a positive number x is the index, when the number x is expressed as a power of the base a :

$$y = \log_a x \quad \text{means that} \quad x = a^y.$$
- The base a must always be positive, and not equal to 1.

The basic skill with logarithms is converting between statements about indices and statements about logarithms. The next box should be learnt off by heart.

10 A sentence to commit to memory

This one sentence should fix most problems:

$$' \log_2 8 = 3 \quad \text{means that} \quad 8 = 2^3.'$$

Examine the sentence and notice that:

- The base of the log is the base of the power (in this case 2).
- The log is the index (in this case 3).



Example 10 Rewriting an index statement in log form

Rewrite each index statement in logarithmic form.

a $10^3 = 1000$

b $3^4 = 81$

Solution

a $10^3 = 1000$

b $3^4 = 81$

$\log_{10} 1000 = 3$

$\log_3 81 = 4$

**Example 11** Rewriting a log statement in index form

Rewrite each logarithmic statement in index form. Then state whether it is true or false.

a $\log_2 16 = 4$

b $\log_3 27 = 4$

Solution

a $\log_2 16 = 4$

$16 = 2^4$, which is true.

b $\log_3 27 = 4$

$27 = 3^4$, which is false.

Finding logarithms

Change a question about logarithms to a question about indices. You will usually need to introduce a pronumeral.

**Example 12** Convert a log equation to an index equation.

a Find $\log_2 32$.

b Find $\log_{10} 1\,000\,000$.

Solution

a Let $x = \log_2 32$.

Then $2^x = 32$.

Hence $x = 5$.

b Let $x = \log_{10} 1\,000\,000$.

Then $10^x = 1\,000\,000$.

Hence $x = 6$.

Negative and fractional indices

A logarithmic equation can involve a negative or a fractional index.

**Example 13** Convert a log equation to an index equation

Solve each logarithmic equation by changing it to an index equation.

a $x = \log_7 \frac{1}{7}$

b $\frac{1}{3} = \log_8 x$

c $\log_x 9 = -1$

Solution

a $x = \log_7 \frac{1}{7}$

$\frac{1}{7} = 7^x$

$x = -1$

b $\frac{1}{3} = \log_8 x$

$x = 8^{\frac{1}{3}}$

$= 2$

c $\log_x 9 = -1$

$x^{-1} = 9$

$x = \frac{1}{9}$

Logarithms on the calculator

Most scientific calculators only allow direct calculations of logarithms base 10, using the button labelled $\boxed{\log}$. (They also allow calculation of logarithms using the important mathematical constant e as the base — this is the purpose of the button marked $\boxed{\ln}$, as explained later in Chapter 10.)

The function $\boxed{10^x}$ is usually on the same key as $\boxed{\log}$, and is reached by pressing $\boxed{\text{shift}}$ followed by $\boxed{\log}$. These two functions $\boxed{\log}$ and $\boxed{10^x}$ are called *inverses* of each other — when used one after the other, the original number returns.

**Example 14** Solve an index equation by converting it to logs

Write each statement below in logarithmic form, then use the function labelled $\log$ on your calculator to approximate x , correct to four significant figures. Then check your calculation using the button labelled 10^x .

a $10^x = 750$

b $10^x = 0.003$

Solution

a $10^x = 750$

$x = \log_{10} 750$

$\doteq 2.875$, using $\log$.

Check this using the 10^x button:

$10^{2.875} \doteq 750$.

b $10^x = 0.003$

$x = \log_{10} 0.003$

$\doteq -2.523$, using $\log$.

Check this using the 10^x button:

$10^{-2.523} \doteq 0.003$.

Locating a logarithm between two integers

Consider again the cheese with a mould that doubles in amount every day. Suppose that someone asks how many days it takes for the amount of mould to increase by a factor of 10.

The answer is $\log_2 10$ days, but this is an awkward question, because the answer is not an integer.

- In three days the mould increases 8-fold.
- In four days it increases 16-fold.

Thus we can at least say that the answer lies between 3 days and 4 days.

**Example 15** Locating a logarithm between two integers

Use a list of powers of 5 to explain why $\log_5 100$ is between 2 and 3.

Solution

We know that $5^2 = 25$ and $5^3 = 125$, so $5^2 < 100 < 5^3$.

Hence taking logarithms base 5, $2 < \log_5 100 < 3$.

Note: The change-of-base formula in the next section will allow $\log_2 10$ to be approximated on a calculator.

Why can't the base a of a logarithm be 1?

We restricted discussion of powers to bases $a > 0$. With logarithms, we must also exclude $a = 1$, because all powers of 1 are 1. For example, $\log_1 4$ would be undefined because $1^x = 4$ has no solutions. And $\log_1 1$ could be any number.

Exercise 8C**FOUNDATION**

Note: Do not use a calculator in this exercise unless the question asks for it. Make sure that you can refer easily to the list of powers of 2, 3, 5, ... from Exercise 7A.

1 Copy and complete each sentence.

a ' $\log_2 8 = 3$ because ...'

b ' $\log_5 25 = 2$ because ...'

c ' $\log_{10} 1000 = 3$ because ...'

d ' $7^2 = 49$, so $\log_7 49 = \dots$ '

e ' $3^4 = 81$, so ...'

f ' $10^5 = 100\,000$, so ...'

2 Copy and complete the following statements of the meaning of logarithms.

a ' $y = \log_a x$ means that ...'

b ' $y = a^x$ means that ...'

3 Rewrite each equation in index form, then solve it for x .

a $x = \log_{10} 10\,000$

b $x = \log_{10} 1000$

c $x = \log_{10} 100$

d $x = \log_{10} 10$

e $x = \log_{10} 1$

f $x = \log_{10} \frac{1}{10}$

g $x = \log_{10} \frac{1}{100}$

h $x = \log_{10} \frac{1}{1000}$

4 Rewrite each equation in index form, then solve it for x .

a $x = \log_3 9$

b $x = \log_5 125$

c $x = \log_7 49$

d $x = \log_2 64$

e $x = \log_4 64$

f $x = \log_8 64$

g $x = \log_8 8$

h $x = \log_8 1$

i $x = \log_7 \frac{1}{7}$

j $x = \log_{12} \frac{1}{12}$

k $x = \log_{11} \frac{1}{121}$

l $x = \log_6 \frac{1}{36}$

m $x = \log_4 \frac{1}{64}$

n $x = \log_8 \frac{1}{64}$

o $x = \log_2 \frac{1}{64}$

p $x = \log_5 \frac{1}{125}$

5 Rewrite each equation in index form, then solve it for x .

a $\log_7 x = 2$

b $\log_9 x = 2$

c $\log_5 x = 3$

d $\log_2 x = 5$

e $\log_4 x = 3$

f $\log_{100} x = 3$

g $\log_7 x = 1$

h $\log_{11} x = 0$

i $\log_{13} x = -1$

j $\log_7 x = -1$

k $\log_{10} x = -2$

l $\log_{12} x = -2$

m $\log_5 x = -3$

n $\log_7 x = -3$

o $\log_2 x = -5$

p $\log_3 x = -4$

6 Rewrite each equation in index form, then solve it for x . Remember that a base must be positive and not equal to 1.

a $\log_x 49 = 2$

b $\log_x 8 = 3$

c $\log_x 27 = 3$

d $\log_x 10\,000 = 4$

e $\log_x 10\,000 = 2$

f $\log_x 64 = 6$

g $\log_x 64 = 2$

h $\log_x 125 = 1$

i $\log_x 11 = 1$

j $\log_x \frac{1}{17} = -1$

k $\log_x \frac{1}{6} = -1$

l $\log_x \frac{1}{7} = -1$

m $\log_x \frac{1}{9} = -2$

n $\log_x \frac{1}{49} = -2$

o $\log_x \frac{1}{8} = -3$

p $\log_x \frac{1}{81} = -2$

DEVELOPMENT

7 Rewrite each equation in index form and then solve it for x (where a is a constant).

a $x = \log_a a$

b $\log_a x = 1$

c $\log_x a = 1$

d $x = \log_a \frac{1}{a}$

e $\log_a x = -1$

f $\log_x \frac{1}{a} = -1$

g $x = \log_a 1$

h $\log_a x = 0$

i $\log_x 1 = 0$

8 Given that a is a positive real number not equal to 1, evaluate:

a $\log_a a$

b $\log_a \frac{1}{a}$

c $\log_a a^3$

d $\log_a \frac{1}{a^2}$

e $\log_a \frac{1}{a^5}$

f $\log_a \sqrt{a}$

g $\log_a \frac{1}{\sqrt{a}}$

h $\log_a 1$

9 Use the list of powers of 2 prepared in Exercise 7A to find which two integers each expression lies between.

a $\log_2 3$

b $\log_2 7$

c $\log_2 1.8$

d $\log_2 13$

e $\log_2 50$

f $\log_2 1000$

g $\log_2 \frac{4}{5}$

h $\log_2 \frac{1}{3}$

10 Use the tables of powers of 2, 3, 5, ... prepared in Exercise 7A to find which two integers each expression lies between.

a $\log_{10} 35$

b $\log_{10} 6$

c $\log_{10} 2000$

d $\log_3 2$

e $\log_3 50$

f $\log_3 100$

g $\log_5 100$

h $\log_7 20$

i $\log_{10} 0.4$

j $\log_{10} 0.05$

11 Use the calculator buttons $(\log)$ and (10^x) to approximate each expression. Give answers correct to three significant figures, then check each answer using the other button.

a $\log_{10} 2$

b $\log_{10} 20$

c $10^{0.301}$

d $10^{1.301}$

e $10^{0.5}$

f $10^{1.5}$

g $\log_{10} 3.16$

h $\log_{10} 31.6$

i $\log_{10} 1000$

j $\log_{10} 1\,000\,000$

k 10^3

l 10^6

m $\log_{10} 0.7$

n $\log_{10} 0.007$

o $10^{-0.155}$

p $10^{-2.15}$

12 Problems arise when a logarithm is written down with too few significant figures.

a Find $\log_{10} 45$ correct to two significant figures.

b Then find 10 to that index, again correct to two significant figures.

CHALLENGE

13 Solve each equation for x . These questions involve fractional indices.

a $x = \log_7 \sqrt{7}$

b $x = \log_{11} \sqrt{11}$

c $\log_9 x = \frac{1}{2}$

d $\log_{144} x = \frac{1}{2}$

e $\log_x 3 = \frac{1}{2}$

f $\log_x 13 = \frac{1}{2}$

g $x = \log_6 \sqrt[3]{6}$

h $x = \log_9 3$

i $\log_{64} x = \frac{1}{3}$

j $\log_{16} x = \frac{1}{4}$

k $\log_x 2 = \frac{1}{3}$

l $\log_x 2 = \frac{1}{6}$

m $x = \log_8 2$

n $x = \log_{125} 5$

o $\log_7 x = \frac{1}{2}$

p $\log_7 x = -\frac{1}{2}$

q $\log_x \frac{1}{7} = -\frac{1}{2}$

r $\log_x \frac{1}{20} = -\frac{1}{2}$

s $x = \log_4 \frac{1}{2}$

t $x = \log_{27} \frac{1}{3}$

u $\log_{121} x = -\frac{1}{2}$

v $\log_{81} x = -\frac{1}{4}$

w $\log_x \frac{1}{2} = -\frac{1}{4}$

x $\log_x 2 = -\frac{1}{4}$

8D The laws for logarithms

Learning intentions

- Use the fact that a log function is the inverse function of an exponential function.
- Use the log laws in expressions and equations.

Logarithms are indices, so the laws for manipulating indices can be rewritten as laws for manipulating logarithms. As always, the base a of a logarithm must be a positive number not equal to 1.

Logarithmic and exponential functions are inverse functions

We have already seen that the functions $\log$ and 10^x are *inverse functions*. That is, when the two functions are applied to a number one after the other, in any order, the original number returns. Check, for example, using the buttons $\log$ and 10^x on your calculator, that

$$\log_{10} 10^3 = 3 \quad \text{and} \quad 10^{\log_{10} 3} = 3.$$

These relationships are true whatever the base of the logarithm, simply because of the definition of logarithms. For example:

$$\log_2 2^6 = 6, \quad \text{because 6 is the index, when } 2^6 \text{ is written as a power of 2.}$$

$$2^{\log_2 64} = 64, \quad \text{because } \log_2 64 \text{ is the index, when 64 is written as a power of 2.}$$

The exponential and logarithmic functions to any base a are therefore called *inverse functions*.

11 The functions $y = a^x$ and $y = \log_a x$ are inverse functions

Let the base a be any positive real number not equal to 1. Then

- $\log_a a^x = x$, for all real x , and
- $a^{\log_a x} = x$, for all real $x > 0$.



Example 16 Simplify expressions using the inverse property

a Simplify $\log_7 7^{12}$.

b Simplify $5^{\log_5 11}$.

Solution

a Using the first identity above,
 $\log_7 7^{12} = 12$

b Using the second identity above,
 $5^{\log_5 11} = 11$



Example 17 Write any number as power or as a logarithm

a Write 3 as a power of 10.

b Write 7 as a logarithm base 2.

Solution

a Using the second identity above,
 $3 = 10^{\log_{10} 3}$

b Using the first identity above,
 $7 = \log_2 2^7$

Three laws for logarithms

Three laws are particularly useful when working with logarithms:

12 Three laws for logarithms

- The log of a product is the sum of the logs:

$$\log_a xy = \log_a x + \log_a y$$

- The log of a quotient is the difference of the logs:

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

- The log of a power is the multiple of the log:

$$\log_a x^n = n \log_a x$$

The simple reason for these three laws is that logarithms are indices, and these laws therefore mirror the index laws.

- When multiplying powers, add the indices.
- When dividing powers, subtract the indices.
- And when raising a power to a power, multiply the indices.

Here is a more formal proof.

Proof To prove each law, show that $a^{\text{LHS}} = a^{\text{RHS}}$, by using the second identity in the previous paragraph and the index laws for powers.

$$a^{\log_a x + \log_a y} = a^{\log_a x} \times a^{\log_a y} \quad (\text{to multiply powers, add the indices})$$

$$= xy$$

$$= a^{\log_a(xy)}$$

$$a^{\log_a x - \log_a y} = a^{\log_a x} \div a^{\log_a y} \quad (\text{to divide powers, subtract the indices})$$

$$= x \div y$$

$$= \frac{x}{y}$$

$$= a^{\log_a(\frac{x}{y})}$$

$$a^{n(\log_a x)} = (a^{\log_a x})^n \quad (\text{to raise a power to a power, multiply the indices})$$

$$= x^n$$

$$= a^{\log_a(x^n)}$$


Example 18 Working with approximations to logarithms

Suppose that it has been found that $\log_3 2 \doteq 0.63$ and $\log_3 5 \doteq 1.46$. Use the log laws to find approximations for:

a $\log_3 10$

b $\log_3 \frac{2}{5}$

c $\log_3 32$

d $\log_3 18$

Solution

a Because $10 = 2 \times 5$,
 $\log_3 10 = \log_3 2 + \log_3 5$
 $\doteq 0.63 + 1.46$
 $\doteq 2.09$.

c Because $32 = 2^5$,
 $\log_3 32 = 5 \log_3 2$
 $\doteq 3.15$.

b $\log_3 \frac{2}{5} = \log_3 2 - \log_3 5$
 $\doteq 0.63 - 1.46$
 $\doteq -0.83$.

d Because $18 = 2 \times 3^2$,
 $\log_3 18 = \log_3 2 + 2 \log_3 3$
 $= \log_3 2 + 2$, because $\log_3 3 = 1$,
 $\doteq 2.63$.

Some particular values of logarithmic functions

Some particular values of logarithmic functions occur very often, and are worth committing to memory.

13 Some particular values and identities of logarithmic functions

$\log_a 1 = 0$,	because $1 = a^0$.
$\log_a a = 1$,	because $a = a^1$.
$\log_a \sqrt{a} = \frac{1}{2}$,	because $\sqrt{a} = a^{\frac{1}{2}}$.
$\log_a \frac{1}{a} = -1$,	because $\frac{1}{a} = a^{-1}$.
$\log_a \frac{1}{x} = -\log_a x$,	because $\log_a \frac{1}{x} = \log_a x^{-1} = -\log_a x$.


Example 19 Simplifying expressions involving logarithms

Use the log laws to expand:

a $\log_3 7x^3$

b $\log_5 \frac{5}{x}$

Solution

a $\log_3 7x^3 = \log_3 7 + \log_3 x^3$ (the log of a product is the sum of the logs)
 $= \log_3 7 + 3 \log_3 x$ (the log of a power is the multiple of the log)

b $\log_5 \frac{5}{x} = \log_5 5 - \log_5 x$ (the log of a quotient is the difference of the logs)
 $= 1 - \log_5 x$ ($\log_5 5 = 1$, because $5 = 5^1$)

Exercise 8D

FOUNDATION

Note: Do not use a calculator in this exercise. Make sure that you can refer easily to the list of powers of 2, 3, 5, ... from Exercise 7A.

- 1
 - a Use the first log law in Box 12 to find $\log_6 9 + \log_6 4$.
 - b Use the second log law in Box 12 to find $\log_5 75 - \log_5 3$.
 - c Use the first and second log laws in Box 12 to find $\log_2 12 + \log_2 6 - \log_2 9$.
- 2 Use the log law $\log_a x + \log_a y = \log_a xy$ to simplify:
 - a $\log_6 2 + \log_6 3$
 - b $\log_{15} 3 + \log_{15} 5$
 - c $\log_{10} 4 + \log_{10} 25$
 - d $\log_{12} 72 + \log_{12} 2$
 - e $\log_{10} 50 + \log_{10} 20$
 - f $\log_6 18 + \log_6 2$
- 3 Use the log law $\log_a x - \log_a y = \log_a \frac{x}{y}$ to simplify:
 - a $\log_3 15 - \log_3 5$
 - b $\log_4 20 - \log_4 5$
 - c $\log_2 24 - \log_2 3$
 - d $\log_5 50 - \log_5 2$
 - e $\log_3 810 - \log_3 10$
 - f $\log_2 96 - \log_2 3$
- 4 Use the log laws to simplify:
 - a $\log_{30} 2 + \log_{30} 3 + \log_{30} 5$
 - b $\log_{12} 9 + \log_{12} 8 + \log_{12} 2$
 - c $\log_2 12 + \log_2 6 - \log_2 9$
 - d $\log_3 6 + \log_3 12 - \log_3 8$
 - e $\log_5 12 + \log_5 2 - \log_5 24$
 - f $\log_5 2 - \log_5 50$
 - g $\log_2 6 - \log_2 48$
 - h $\log_2 12 + \log_2 \frac{1}{3}$
 - i $\log_7 \frac{1}{9} + \log_7 9$
- 5 Use the log law $\log_a x^n = n \log_a x$ to write each expression in terms of $\log_a 2$.
 - a $\log_a 8$
 - b $\log_a 16$
 - c $\log_a 64$
 - d $\log_a \frac{1}{2}$
 - e $\log_a \frac{1}{8}$
 - f $\log_a \frac{1}{32}$
 - g $\log_a \sqrt{2}$
 - h $\log_a \frac{1}{\sqrt{2}}$
- 6 Express each logarithm in terms of $\log_2 3$ and $\log_2 5$. Remember that $\log_2 2 = 1$.
 - a $\log_2 9$
 - b $\log_2 25$
 - c $\log_2 6$
 - d $\log_2 10$
 - e $\log_2 18$
 - f $\log_2 20$
 - g $\log_2 \frac{2}{3}$
 - h $\log_2 2\frac{1}{2}$
- 7 Given that $\log_2 3 \div 1.58$ and $\log_2 5 \div 2.32$, use the log laws to find approximations for:
 - a $\log_2 15$
 - b $\log_2 9$
 - c $\log_2 10$
 - d $\log_2 50$
 - e $\log_2 \frac{3}{2}$
 - f $\log_2 \frac{3}{5}$
 - g $\log_2 \frac{2}{3}$
 - h $\log_2 75$

DEVELOPMENT

- 8** Use the identity $\log_a a^x = x$ to simplify:
- a** $\log_{10} 10^3$ **b** $\log_7 7^5$ **c** $\log_{12} 12^{1.3}$ **d** $\log_8 8^n$
- 9** Use the identity $a^{\log_a x} = x$ to simplify:
- a** $10^{\log_{10} 100}$ **b** $3^{\log_3 7}$ **c** $4^{\log_4 3.6}$ **d** $6^{\log_6 y}$
- 10** Use the log law $\log_a x^n = n \log_a x$ and the identity $\log_a a = 1$ to simplify:
- a** $\log_a a^2$ **b** $5 \log_a a^3$ **c** $\log_a \frac{1}{a}$ **d** $12 \log_a \sqrt{a}$
- 11** Use the log law $\log_a x^n = n \log_a x$ to write each expression in terms of $\log_a x$.
- a** $\log_a x^3$ **b** $\log_a \frac{1}{x}$ **c** $\log_a \sqrt{x}$ **d** $\log_a \frac{1}{x^2}$
e $\log_a x^3 - \log_a x^5$ **f** $\log_a x^4 + \log_a \frac{1}{x^2}$ **g** $2 \log_a a^4 - \log_a x^8$ **h** $\log_a \frac{1}{\sqrt{x}} + 3 \log_a \sqrt{x}$
- 12** Write these expressions in terms of $\log_a x$, $\log_a y$ and $\log_a z$.
- a** $\log_a yz$ **b** $\log_a \frac{z}{y}$ **c** $\log_a y^4$ **d** $\log_a \frac{1}{x^2}$
e $\log_a xy^3$ **f** $\log_a \frac{x^2 y}{z^3}$ **g** $\log_a \sqrt{y}$ **h** $\log_a \sqrt{xz}$
- 13** Using the identities $x = a^{\log_a x}$ and $x = \log_a a^x$, express:
- a** 10 as a power of 3 **b** 3 as a power of 10 **c** 0.1 as a power of 2
d 2 as a logarithm base 10 **e** -4 as a logarithm base 3 **f** $\frac{1}{2}$ as a logarithm base 7
g 8 as a power of a **h** 15 as a logarithm base a **i** y as a logarithm base 5
- 14** Given that $\log_{10} 2 \div 0.30$ and $\log_{10} 3 \div 0.48$, use the fact that $\log_{10} 5 = \log_{10} 10 - \log_{10} 2$ to find an approximation for $\log_{10} 5$. Then use the log laws to find approximations for:
- a** $\log_{10} 20$ **b** $\log_{10} 0.2$ **c** $\log_{10} 360$ **d** $\log_{10} \sqrt{2}$
e $\log_{10} \sqrt{8}$ **f** $\log_{10} \frac{1}{\sqrt{10}}$ **g** $\log_{10} \sqrt{12}$ **h** $\log_{10} \frac{1}{\sqrt{5}}$
- 15** If $x = \log_a 2$, $y = \log_a 3$ and $z = \log_a 5$, simplify:
- a** $\log_a 64$ **b** $\log_a \frac{1}{30}$ **c** $\log_a 27a^5$ **d** $\log_a \frac{100}{a}$
e $\log_a 1.5$ **f** $\log_a \frac{18}{25a}$ **g** $\log_a 0.04$ **h** $\log_a \frac{8}{15a^2}$

CHALLENGE

- 16** Simplify these expressions.
- a** $5^{-\log_5 2}$ **b** $12^{2 \log_{12} 7}$ **c** $2^{\log_2 3 + \log_2 5}$ **d** $a^{n \log_a x}$
e $7^{-\log_7 x}$ **f** $5^{x + \log_5 x}$ **g** $2^{x \log_2 x}$ **h** $3^{\frac{\log_3 x}{x}}$

8E Equations involving logarithms and indices

Learning intentions

- Apply the change-of-base formula.
- Solve index and logarithmic equations to any base.
- Solve exponential and logarithmic inequations.

The calculator allows approximations to logarithms base 10. This section explains how to obtain approximations to any other base. Index equations can then be solved approximately, whatever the base. As always, every base must be positive and not equal to 1.

The change-of-base formula

Finding approximations of logarithms to bases not on the calculator requires a formula that converts logarithms from one base to another.

14 The change-of-base formula

Let the bases a and b be positive real numbers not equal to 1.

- To write logs base a in terms of logs base b , use the formula

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Remember this as ‘the log of the number over the log of the base’.

- The calculator button $\boxed{\log}$ gives approximations for logs base 10, so to find logarithms to another base a , put $b = 10$ and use the formula

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

Proof To prove this formula, let

$$y = \log_a x.$$

Then by the definition of logs,

$$x = a^y$$

and taking logs base b of both sides,

$$\log_b x = \log_b a^y.$$

Using the log laws,

$$\log_b x = y \log_b a$$

and rearranging,

$$y = \frac{\log_b x}{\log_b a}, \text{ as required.}$$

Example 20 Finding approximations of logs to other bases

Find, correct to four significant figures:

a $\log_2 5$,

b $\log_3 0.02$.

Then check your approximations using the button labelled $\boxed{x^y}$ or $\boxed{\wedge}$.

Solution

In each part, use the change-of-base-formula to change to logs base 10.

a $\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2}$

$$\div 2.322$$

Checking, $2^{2.322} \div 5$.

b $\log_3 0.02 = \frac{\log_{10} 0.02}{\log_{10} 3}$

$$\div -3.561$$

Checking, $3^{-3.561} \div 0.02$.

Solving index equations

An index equation such as $5^x = 18$ is solved, in exact form, by rewriting it in terms of logarithms, then changing to base 10 allows approximation by calculator:

$$\begin{aligned} 5^x &= 18 \\ x &= \log_5 18 && \text{(the exact solution)} \\ &= \frac{\log_{10} 18}{\log_{10} 5} \\ &\doteq 1.796 && \text{(correct to four significant figures).} \end{aligned}$$



Example 21 Solving index equations

Solve these equations correct to four significant figures. Then check your approximations using the button labelled (x^y) or $(\wedge)$.

a $2^x = 7$

b $3^x = 0.05$

Solution

a

$$2^x = 7$$

Solving for x ,

$$x = \log_2 7$$

Changing to base 10,

$$\begin{aligned} x &= \frac{\log_{10} 7}{\log_{10} 2} \\ &\doteq 2.807 \end{aligned}$$

Checking, $2^{2.807} \doteq 7$.

b

$$3^x = 0.05$$

Solving for x ,

$$x = \log_3 0.05$$

Changing to base 10,

$$\begin{aligned} x &= \frac{\log_{10} 0.05}{\log_{10} 3} \\ &\doteq -2.727 \end{aligned}$$

Checking, $3^{-2.727} \doteq 0.05$.



Example 22 A problem with rabbits

The rabbit population on Kanin Island is doubling every year. A few years ago, there were 500 rabbits there. How many years later will the rabbit population be 10 000? Answer in exact form, then correct to the nearest tenth of a year.

Solution

Let P be the rabbit population n years after the first estimate.

Then

$$P = 500 \times 2^n.$$

Put $P = 10\,000$, then

$$10\,000 = 500 \times 2^n$$

$$2^n = 20$$

$$n = \log_2 20 \quad \text{(the exact answer)}$$

$$= \frac{\log_{10} 20}{\log_{10} 2}$$

$$\doteq 4.3 \text{ years} \quad \text{(correct to 0.1 years).}$$

Exponential equations reducible to quadratics

Some exponential equations can be reduced to quadratics with a substitution.



Example 23 Solving an exponential equation using a substitution

Use the substitution $u = 2^x$ to solve the equation $4^x - 7 \times 2^x + 12 = 0$.

Solution

Writing $4^x = (2^x)^2$, the equation becomes

$$(2^x)^2 - 7 \times 2^x + 12 = 0.$$

Substitute $u = 2^x$, $u^2 - 7u + 12 = 0$

$$(u - 4)(u - 3) = 0$$

$$u = 4 \text{ or } 3,$$

and returning to x ,

$$2^x = 4 \text{ or } 2^x = 3$$

$$x = 2 \text{ or } \log_2 3.$$

Solving exponential and logarithmic inequations

Provided that the base a is greater than 1, the exponential function $y = a^x$ and the logarithmic function $y = \log_a x$ both increase as x increases, so there are no difficulties solving inequations such as $2^x \leq 64$ and $\log_{10} x > 4$.



Example 24 Solving exponential and logarithmic inequations

Solve:

a $2^x \leq 64$

b $\log_{10} x \geq 4$

Solution

a $2^x \leq 64$

$$2^x \leq 2^6$$

$$x \leq 6$$

b $\log_{10} x \geq 4$

$$x \geq 10^4$$

$$x \geq 10\,000$$

Taking logarithms of both sides

When an approximate solution of an index equation is required, it is often quicker to take logarithms base 10 of both sides. This is particularly useful when powers with different bases are involved.



Example 25 Taking logarithms of both sides

Solve these equations correct to four significant figures by taking logarithms base 10 of both sides.

a $5^{3x} = 3 \times 5^{x+1}$

b $2^{3x} = 3^{x+1}$

Solution

a $5^{3x} = 3 \times 5^{x+1}$

Taking logs base 10 of both sides,

$$\log_{10} 5^{3x} = \log_{10} 3 + \log_{10} 5^{x+1}$$

$$3x \log_{10} 5 = \log_{10} 3 + (x + 1) \log_{10} 5$$

$$2x \log_{10} 5 = \log_{10} 3 + \log_{10} 5$$

$$x = \frac{\log_{10} 3 + \log_{10} 5}{2 \log_{10} 5}$$

$$\doteq 0.8413$$

b $2^{3x} = 3^{x+1}$

Taking logs base 10 of both sides,

$$3x \log_{10} 2 = (x + 1) \log_{10} 3$$

$$x(3 \log_{10} 2 - \log_{10} 3) = \log_{10} 3$$

$$x = \frac{\log_{10} 3}{3 \log_{10} 2 - \log_{10} 3}$$

$$\doteq 1.1201.$$

Making both sides the indices of powers with the same base

This is particularly useful in equations involving logarithms to the same base.



Example 26 Both sides become indices of powers with the same base

Solve $\log_3 x - \log_3(x - 6) = 1$

Solution

Using the base 3, and LHS and RHS of the equation as indices,

$$3^{\log_3 x - \log_3(x-6)} = 3^1$$

and using the log laws, $3^{\log_3 x} \div 3^{\log_3(x-6)} = 3$

From the log identities, $\frac{x}{x-6} = 3$

$$x = 3x - 18$$

$$x = 9.$$

Exercise 8E

FOUNDATION

- Use the change-of-base formula, $\log_a x = \frac{\log_{10} x}{\log_{10} a}$, and your calculator, to verify that:
 - $\log_2 8 = 3$
 - $\log_3 81 = 4$
 - $\log_{100} 10\,000 = 2$
- Use the change-of-base formula, $\log_a x = \frac{\log_{10} x}{\log_{10} a}$, to approximate these logarithms, correct to four significant figures. Check each answer using the button labelled (x^y) or $(\wedge)$.
 - $\log_2 7$
 - $\log_2 26$
 - $\log_2 0.07$
 - $\log_3 4690$
 - $\log_5 2$
 - $\log_7 31$
 - $\log_6 3$
 - $\log_{12} 2$
 - $\log_3 0.1$
 - $\log_3 0.0004$
 - $\log_{11} 1000$
 - $\log_{0.8} 0.2$
 - $\log_{0.03} 0.89$
 - $\log_{0.99} 0.003$
 - $\log_{0.99} 1000$
- Rewrite each equation with x as the subject, using logarithms. Then use the change-of-base formula to solve it, giving your answers correct to four significant figures. Check each answer using the button labelled (x^y) or $(\wedge)$.
 - $2^x = 15$
 - $2^x = 5$
 - $2^x = 1.45$
 - $2^x = 0.1$
 - $2^x = 0.0007$
 - $3^x = 10$
 - $3^x = 0.01$
 - $5^x = 10$
 - $12^x = 150$
 - $8^x = \frac{7}{9}$
 - $6^x = 1.4$
 - $30^x = 2$
 - $0.7^x = 0.1$
 - $0.98^x = 0.03$
 - $0.99^x = 0.01$
- Give exact solutions to these inequations. Do not use a calculator.
 - $2^x > 32$
 - $2^x \leq 32$
 - $2^x < 64$
 - $3^x \geq 81$
 - $5^x > 5$
 - $4^x \leq 1$
 - $2^x < \frac{1}{2}$
 - $10^x \leq 0.001$

DEVELOPMENT

- Give exact solutions to these inequations. Do not use a calculator.
 - $\log_2 x < 3$
 - $\log_2 x \geq 3$
 - $\log_{10} x > 3$
 - $\log_{10} x \geq 1$
 - $\log_5 x > 0$
 - $\log_6 x < 1$
 - $\log_5 x \leq 3$
 - $\log_6 x > 2$

- 6** Rewrite each inequation in terms of logarithms, with x as the subject. Then use the change-of-base formula to solve it, giving your answer correct to three significant figures.
- a** $2^x > 12$ **b** $2^x < 100$ **c** $2^x < 0.02$ **d** $2^x > 0.1$
e $5^x < 100$ **f** $3^x < 0.007$ **g** $1.2^x > 10$ **h** $1.001^x > 100$
- 7** A few years ago, Rahul and Fiona were interested in building a new garage, whose price is rising with inflation at 5% per annum. Its price then was \$12 000.
- a** Explain why the cost C in dollars is $C = 12\,000 \times 1.05^n$, where n is the years since.
b Substitute into the formula to find, correct to the nearest 0.1 years, how many years later the cost will be \$18 000.
- 8** Use the change-of-base formula $\log_a x = \frac{\log_b x}{\log_b a}$ to prove that:
- a** $\log_8 x = \frac{1}{3} \log_2 x$ **b** $\log_{a^n} x = \frac{1}{n} \log_a x$
- 9** Use the log laws, including possibly the change-of-base formula, to prove that:
- a** $\log_a \frac{b}{c} = -\log_a \frac{c}{b}$. **b** $\log_{\frac{1}{a}} x = -\log_a x$. **c** $\log_{\frac{1}{a}} x^{-1} = \log_a x$.
- 10** Solve these equation and inequations, correct to three significant figures if necessary.
- a** $2^{x+1} = 16$ **b** $3^{2x} = 81$ **c** $5^{x-1} < 1$ **d** $10^{\frac{1}{3}x} \leq 1\,000$
e $\left(\frac{1}{2}\right)^{x-3} = 8$ **f** $\left(\frac{1}{10}\right)^{5x} = \frac{1}{10}$ **g** $2^{x-2} < 7$ **h** $3^{x+5} > 10$
- 11** Solve these equations by making both sides indices of powers with the same base. Check your solutions to make sure that LHS and RHS are defined when they are substituted.
- a** $\log_2 (x+1) = \log_2 (11-x) + 1$
b $\log_3 (x+3) + \log_3 (x-3) = \log_3 (4x+3)$
c $\log_2 x + \log_2 (x-3) - \log_2 (x+2) = 2$

CHALLENGE

- 12 a** Solve $2^x < 10^{10}$. How many positive integer powers of 2 are less than 10^{10} ?
b Solve $3^x < 10^{50}$. How many positive integer powers of 3 are less than 10^{50} ?
- 13 a** Explain why $\log_{10} 300$ lies between 2 and 3.
b If x is a two-digit number, what two integers does $\log_{10} x$ lie between?
c If $\log_{10} x = 4.7$, how many digits does x have to the left of the decimal point?
d Find $\log_{10} 25^{20}$, and hence find the number of digits in 25^{20} .
e Find $\log_{10} 2^{1000}$, and hence find the number of digits in 2^{1000} .

8F Exponential and logarithmic graphs

Learning intentions

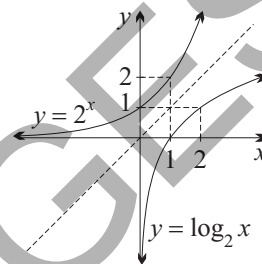
- Sketch and describe graphs of exponential and logarithmic functions.
- Apply the standard transformations to exponential and logarithmic graphs.
- Develop the interaction of transformations with the index laws and log laws.

The function $y = a^x$ is an *exponential function*, because the variable x is in the *exponent* or *index*. The function $y = \log_a x$ is a *logarithmic function*.

The graphs of $y = 2^x$ and $y = \log_2 x$

These two graphs will demonstrate the characteristic features of all exponential and logarithmic graphs. Here are their tables of values:

$y = 2^x$	x	-2	-1	0	1	2
	y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$y = \log_2 x$	x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
	y	-2	-1	0	1	2



Here are some of the properties of the two graphs:

- The two graphs are reflections of each other in the diagonal line $y = x$.
This is because they are *inverse functions of each other*, so their tables of values are the same, except that the x -values and y -values have been swapped.
- For $y = 2^x$, the domain is all real x , also written as $(-\infty, \infty)$,
and the range is $y > 0$, also written as $(0, \infty)$.
For $y = \log_2 x$, the domain is $x > 0$, also written as $(0, \infty)$,
and the range is all real y , also written as $(-\infty, \infty)$.
- For $y = 2^x$, the x -axis is a horizontal asymptote — as $x \rightarrow -\infty$, $y \rightarrow 0$.
For $y = \log_2 x$, the y -axis is a vertical asymptote — as $x \rightarrow 0^+$, $y \rightarrow -\infty$.

Notation: The symbol $x \rightarrow 0^+$ here means, 'As x approaches 0 from the right'. Elsewhere, we will use symbol $x \rightarrow 0^-$ to mean, 'As x approaches 0 from the left'.

- As $x \rightarrow \infty$, $2^x \rightarrow \infty$, and $y = 2^x$ gets steeper all the time.
As $x \rightarrow \infty$, $\log_2 x \rightarrow \infty$, but $y = \log_2 x$ gets flatter all the time.
- The graph of $y = 2^x$ is concave up, but $y = \log_2 x$ is concave down.

15 Exponential and logarithmic functions are inverse functions

Let the base a be any positive number not equal to 1.

- The functions $y = a^x$ and $y = \log_a x$ are *inverse functions*, meaning that

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a x} = x.$$

- The graphs of $y = a^x$ and $y = \log_a x$ are reflections of each other in the diagonal line $y = x$.
- The domains, the asymptotes, the steepness, and the concavity, of the two graphs are clearly seen from their graphs and this reflection property.
- We can take logarithms base 2 by reading the graph of $y = 2^x$ backwards.
Input the number on the y -axis, go horizontally across to the graph of $y = 2^x$, go vertically down to the x -axis, and read the output off the x -axis.

Reflections of exponential and logarithmic functions

Shifting, reflecting and dilating apply to exponential and logarithmic functions. Transformations of some exponential functions were done earlier, but logarithmic functions were only introduced in this chapter.



Example 27 Reflections in the x -axis and y -axis

For each pair of functions:

- i Draw up convenient tables of values for them.
- ii Sketch both functions on one set of axes.
- iii Explain what transformation transforms one graph onto the other, and why.
- iv State their domain and range.

a $y = 3^x$ and $y = 3^{-x}$

b $y = 3^x$ and $y = -3^x$

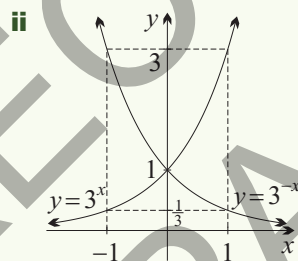
Solution

a i

x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
3^{-x}	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$

iii The graphs are reflections of each other in the y -axis, because x has been replaced by $-x$.

iv For both functions, the domain is all real x , and the range is $y > 0$.



b i

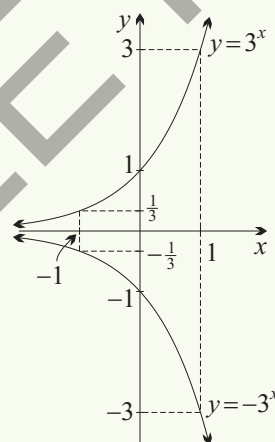
x	-2	-1	0	1	2
3^x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
-3^x	$-\frac{1}{9}$	$-\frac{1}{3}$	-1	-3	-9

iii The graphs are reflections of each other in the x -axis, because y has been replaced by $-y$.

iv Both functions have domain all real x .

For $y = 3^x$, the range is $y > 0$.

For $y = -3^x$, the range is $y < 0$.



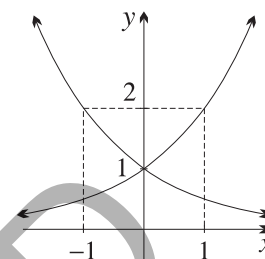
Exponential functions with bases between 0 and 1 are reflections

In the introduction, the mould on the cheese was doubling every day, so its growth is proportional to the function $y = 2^x$. But the decay of radioactive element is specified by its *half-life* — for example, approximately half a sample of thorium-231 decays every day — so its decay is proportional to the exponential function $y = (\frac{1}{2})^x$, whose base is between 0 and 1.

The surprising result here is that the decay function $y = (\frac{1}{2})^x$ and the growth function $y = 2^x$ are reflections of each other in the y -axis, by successive applications of the index laws:

$$y = (\frac{1}{2})^x \text{ is the same as } y = \frac{1}{2^x}, \text{ which is the same as } y = 2^{-x}.$$

Because of this reflection, we can see immediately that the domains and ranges of the two functions are the same, and the x -axis is horizontal asymptote for both. But for $y = (\frac{1}{2})^x$, the limiting behaviours as $x \rightarrow -\infty$ and as $x \rightarrow \infty$ are the reverse of $y = 2^x$.



Logarithmic functions with bases between 0 and 1 are reflections

Now let us look at the two logarithmic functions $y = \log_{\frac{1}{2}} x$ and $y = \log_2 x$. We start this time by applying the log laws to $y = \log_{\frac{1}{2}} x$:

The original function is

$$y = \log_{\frac{1}{2}} x,$$

using the change-of-base formula,

$$y = \frac{\log_2 x}{\log_2 \frac{1}{2}},$$

writing the denominator as a power of 2,

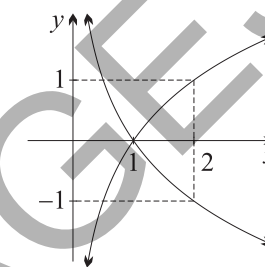
$$y = \frac{\log_2 x}{\log_2 2^{-1}},$$

using the definition of logs,

$$y = \frac{\log_2 x}{-1},$$

so $y = \log_{\frac{1}{2}} x$ is the same as

$$y = -\log_2 x.$$



The result is that this time $y = \log_{\frac{1}{2}} x$ is the reflection of $y = \log_2 x$ in the x -axis rather than the y -axis. The reader may have expected this because of the symmetry of the inverse functions $y = 2^x$ and $y = \log_2 x$ in the line $y = x$.

Once again, the domains and ranges of the two functions are the same, and now the y -axis is a horizontal asymptote for both. And again, for $y = \log_{\frac{1}{2}} x$, the limiting behaviours as $x \rightarrow 0^+$ and as $x \rightarrow \infty$ are the reverse of $y = \log_2 x$.

16 A function with base between 0 and 1 can be understood as a reflection

Let $a > 1$ be a base greater than 1.

Exponential functions:

- The function $y = (\frac{1}{a})^x$ is $y = a^{-x}$, so is $y = a^x$ reflected in the y -axis.
 - Thus $y = (\frac{1}{a})^x$ has the same domain, range, and asymptotes as $y = a^x$, but its behaviours as $x \rightarrow -\infty$ and as $x \rightarrow \infty$ are the reverse of $y = a^x$.

Logarithmic functions:

- The function $y = \log_{\frac{1}{a}} x$ is $y = -\log_a x$, so is $y = \log_a x$ reflected in the x -axis.
 - Thus $y = \log_{\frac{1}{a}} x$ has the same domain, range, and asymptotes as $\log_a x$, but its behaviours as $x \rightarrow 0^+$ and as $x \rightarrow \infty$ are the reverse of $y = \log_a x$.

Translations of exponential and logarithmic functions

Translations of exponential functions are in Section 5A, but not log functions.



Example 28 Translations horizontally and vertically

Repeat the steps of worked Example 27, and state the asymptotes, for:

a $y = \log_2 x$ and $y = \log_2(x - 1)$

b $y = \log_2 x$ and $y = 3 + \log_2 x$

Solution

a i

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2

x	$1\frac{1}{4}$	$1\frac{1}{2}$	2	3	5
$\log_2(x - 1)$	-2	-1	0	1	2

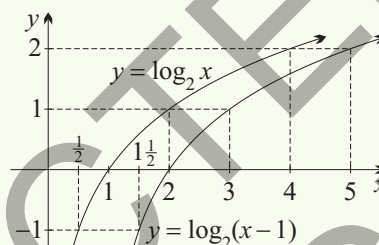
iii The second graph is the first shifted right by 1 unit, because x has been replaced by $x - 1$.

iv Both functions have range all real y .

For $y = \log_2 x$, the domain is $x > 0$, and the y -axis is a vertical asymptote.

For $y = \log_2(x - 1)$, the domain: $x > 1$, (and $x = 1$ is a vertical asymptote).

ii



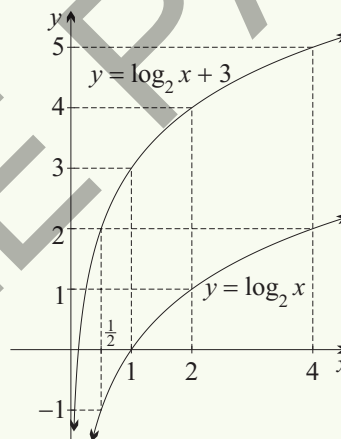
b i

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-2	-1	0	1	2
$3 + \log_2 x$	1	2	3	4	5

iii The second graph is the first shifted up by 3 units, because y has been replaced by $y - 3$ (or because 3 has been added to each value of y).

iv Both have domain $x > 0$, and range all real y .

ii



Dilations of exponential and logarithmic functions

Again, we did not have log functions when discussing dilations in Chapter 5.



Example 29 Dilations horizontally and vertically

Repeat the steps of worked Example 27, and state the asymptotes, for:

a $y = \log_2 x$ and $y = \log_2 4x$

b $y = \log_2 x$ and $y = 4 \log_2 x$

Solution

a i

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-3	-2	-1	0	1	2
$\log_2 4x$	-1	0	1	2	3	4

iii The second is the first dilated horizontally factor $\frac{1}{4}$, because x has been replaced by $4x = x/\frac{1}{4}$.

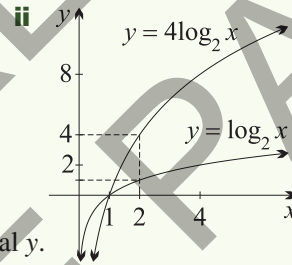
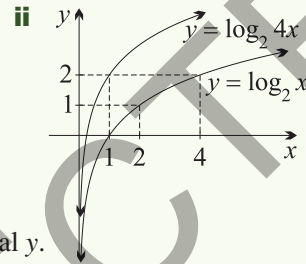
iv Both functions have domain $x > 0$, and range all real y . Both functions have the y -axis as an asymptote.

b i

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	-3	-2	-1	0	1	2
$4 \log_2 x$	-12	-8	-4	0	4	8

iii The second is the first dilated vertically factor 4, because y has been replaced by $y/4$.

iv Both functions have domain $x > 0$, and range all real y . Both functions have the y -axis as an asymptote.



Dilations with negative factors

When a dilation with a negative factor is applied to an exponential or log function, it is usually better to regard it as the composition of a dilation with positive factor, and a reflection — in either order.

In particular, the behaviour of the transformed function near asymptotes and for large positive and negative values of x often reverses with a reflection, and the situation is usually far more straightforward to follow if it is done in two steps.

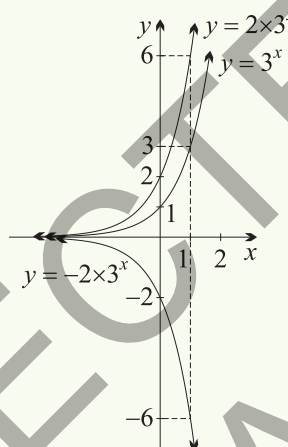
The following worked example should be sufficient to illustrate this point. There are four cases to cover, as is done in the rather difficult Q12 in Exercise 8F.


Example 30 Applying a dilation with a negative factor

- a** Interpret $y = -2 \times 3^x$ as the composition of a dilation and a reflection.
- b** Sketch $y = 3^x$, $y = 2 \times 3^x$, and $y = -2 \times 3^x$ on one set of axes.
- c** Detail the domain and range, the asymptotes, and the behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$, of the three functions.

Solution

- a** A vertical dilation with factor 2 transforms $y = 3^x$ to $y = 2 \times 3^x$.
A reflection in the x -axis then transforms $y = 2 \times 3^x$ to $y = -2 \times 3^x$.

b


- c** For all three functions, the x -axis is an asymptote, the domain is all real x , and $y \rightarrow 0$ as $x \rightarrow -\infty$. For $y = 3^x$ and $y = 2 \times 3^x$, the range is $y > 0$, and $y \rightarrow \infty$ as $x \rightarrow \infty$. For $y = -2 \times 3^x$, the range is $y < 0$, and $y \rightarrow -\infty$ as $x \rightarrow \infty$.

The word 'logarithm'

The Scottish mathematician John Napier (1550–1617) formed his new word 'log | arithm' from the Greek words 'logos', meaning 'ratio', and 'arithmos', meaning 'number'. Until the invention of calculators, the routine method for difficult calculations in arithmetic used tables of logarithms, invented by Napier, to convert products to sums, quotients to differences, and powers to multiples.

Exercise 8F
FOUNDATION

Technology: All these graphs can be confirmed and further developed using graphing software, after which experimentation with further graphs can easily be done.

- 1 a** Use the calculator button $\boxed{\log}$ to complete the following table of values, giving each entry correct to two significant figures where appropriate.

x	0.1	0.25	0.5	0.75	1	2	3	4	5	6	7	8	9	10
$\log_{10} x$														

- b** Hence sketch the graph of $y = \log_{10} x$. Use a large scale, with the same scale on both axes. Ideally use graph paper so that you can see the shape of the curve.
- c** Copy and complete:
- i** 'As $x \rightarrow \infty$, $\log_{10} x \rightarrow \dots$ '
- ii** 'As $x \rightarrow 0^+$, $\log_{10} x \rightarrow \dots$ '

- 2 a** Copy and complete these two tables of values.

i	x	-3	-2	-1	0	1	2	3
	2^x							

ii	x	-3	-2	-1	0	1	2	3
	2^{-x}							

- b** Hence sketch the graphs of $y = 2^x$ and $y = 2^{-x}$ on one set of axes.
c How are the two tables of values related to each other?
d What symmetry does the diagram of the two graphs display, and why?
e Write down the domain and range of:

i $y = 2^x$,

ii $y = 2^{-x}$.

- f** Write down the equations of the asymptotes of:

i $y = 2^x$,

ii $y = 2^{-x}$.

- g** Copy and complete:

i 'As $x \rightarrow -\infty$, $2^x \rightarrow \dots$ '

ii 'As $x \rightarrow \infty$, $2^x \rightarrow \dots$ '

- h** Copy and complete:

i 'As $x \rightarrow -\infty$, $2^{-x} \rightarrow \dots$ '

ii 'As $x \rightarrow \infty$, $2^{-x} \rightarrow \dots$ '

- 3 a** Copy and complete these two tables of values.

i	x	-2	-1	0	1	2
	3^x					

ii	x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
	$\log_3 x$					

- b** Hence sketch the graphs of $y = 3^x$ and $y = \log_3 x$ on one set of axes.
c How are the two tables of values related to each other?
d What symmetry does the diagram of the two graphs display, and why?
e Write down the domain and range of:

i $y = 3^x$,

ii $y = \log_3 x$.

Both domains and both ranges are intervals — use both notations for them.

- f** Write down the equations of the asymptotes of:

i $y = 3^x$,

ii $y = \log_3 x$.

- g** Copy and complete:

i 'As $x \rightarrow -\infty$, $3^x \rightarrow \dots$ '

ii 'As $x \rightarrow 0^+$, $\log_3 x \rightarrow \dots$ '

DEVELOPMENT

- 4 a** Sketch on one set of axes the graphs of $y = 3^x$ and $y = 3^{-x}$.

- b** Sketch on one set of axes the graphs of $y = 10^x$ and $y = 10^{-x}$.

- 5** Sketch the four graphs below on one set of axes.

a $y = 2^x$

b $y = -2^x$

c $y = 2^{-x}$

d $y = -2^{-x}$

- 6** Sketch each set of graphs on one set of axes, clearly indicating the asymptotes, the y-intercepts, and the x-intercepts if they exist. Use shifting of the graphs in the previous questions, but also use a table of values to confirm your diagram.

a $y = 2^x$

b $y = -2^x$

c $y = 2^{-x}$

$y = 2^x + 3$

$y = 2 - 2^x$

$y = 2^{-x} + 1$

$y = 2^x - 1$

$y = -2 - 2^x$

$y = 2^{-x} - 2$

- 7 a** Use reflections in the x - and y -axes to sketch the four graphs below on one set of axes.

$$y = \log_2 x$$

$$y = -\log_2 x$$

$$y = \log_2(-x)$$

$$y = -\log_2(-x)$$

- b** Copy and complete:

i 'As $x \rightarrow -\infty$, $\log_2(-x) \rightarrow \dots$ '

ii 'As $x \rightarrow 0^-$, $\log_2(-x) \rightarrow \dots$ '

- 8** Use shifting to sketch each set of graphs on one set of axes, clearly indicating the asymptote and the intercepts with the axes.

a $y = \log_2 x$

$$y = \log_2 x + 1$$

$$y = \log_2 x - 1$$

b $y = -\log_2 x$

$$y = 2 - \log_2 x$$

$$y = -2 - \log_2 x$$

- 9 a** Use dilations to sketch these graphs on the one set of axes:

$$y = 3^x$$

$$y = 2 \times 3^x$$

$$y = 3^{2x}$$

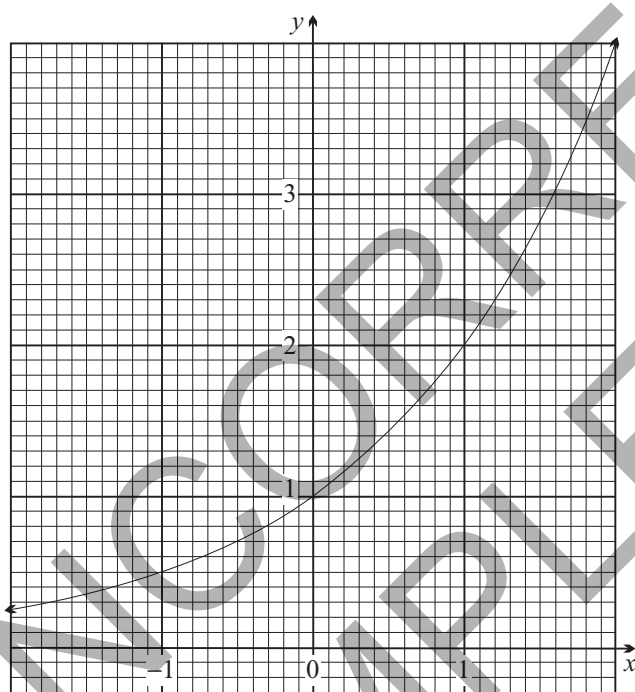
- b** Use dilations to sketch these graphs on the one set of axes:

$$y = \log_3 x$$

$$y = 2 \log_3 x$$

$$y = \log_3 2x$$

10



The diagram above shows the graph of $y = 2^x$. Use the graph to answer the questions below, giving your answers correct to no more than two decimal places.

- a** Read from the graph the values of:

i 2^2

ii 2^{-2}

iii $2^{1.5}$

iv $2^{0.4}$

v $2^{-0.6}$

- b** Find x from the graph if:

i $2^x = 2$

ii $2^x = 3$

iii $2^x = 1.2$

iv $2^x = 0.4$

- c** Find the values of x for which:

i $1 \leq 2^x \leq 4$

ii $1 \leq 2^x \leq 2$

iii $1.5 \leq 2^x \leq 3$

iv $0.5 \leq 2^x \leq 2$

- d** Read the graph backwards to find:

i $\log_2 4$

ii $\log_2 3$

iii $\log_2 1.4$

iv $\log_2 0.8$

- 11** Use the change-of-base formula to show that $\log_{\frac{1}{3}} x = -\log_3 x$, and hence sketch these graphs on one set of axes:

$$y = \log_{\frac{1}{3}} x$$

$$y = 2 \log_{\frac{1}{3}} x$$

$$y = \log_{\frac{1}{3}} 2x$$

- 12** Each function below is either $y = 5^x$ or $y = \log_5 x$ transformed by a horizontal or vertical dilation with negative factor. For each of the four functions:
- Interpret the function as the original function transformed by a dilation with positive factor, then by a reflection. Give the equation of the middle stage.
 - Then sketch all three stages of the transformation on one set of axes.
 - From the sketch, describe, for all three stages, the domain and range, the asymptotes, and the behaviour as $x \rightarrow \infty$ and as $x \rightarrow -\infty$.
- a** $y = -2 \times 5^x$ **b** $y = 5^{-2x}$ **c** $y = -2 \log_5 x$ **d** $y = \log_5(-2x)$
- 13** Characterise these graphs as single or as multiple transformations of $y = \log_5 x$:
- a** $y = 4 \log_5 x$ **b** $y = -\frac{2}{3} \log_5 x$ **c** $y = \log_5(3x)$
d $y = 3 \log_5(-x)$ **e** $y = \log_5(5x + 8)$ **f** $y = -2 \log_5(3x - 7)$
- 14** Sketch, on separate axes, the graphs of:
- a** $y = 2^{x-2}$ **b** $y = 2^{x+1}$ **c** $y = \log_2(x + 1)$ **d** $y = \log_2(x - 1)$

CHALLENGE

Note: The next two questions demonstrate how some transformations of exponential and log graphs can be done in two different ways. You will need the index and log laws.

- 15** In each part, name the initial transformation of $y = 3^x$, and the resulting transformation.
- Rewrite $y = 3^{4+x}$ to show that it is a dilation of $y = 3^x$.
 - Rewrite $y = 3^{x-1}$ to show that it is a dilation of $y = 3^x$.
 - Rewrite $y = 27 \times 3^x$ to show that it is a translation of $y = 3^x$.
 - Rewrite $y = 20 \times 3^x$ to show that it is a translation of $y = 3^x$.
- 16** For each, name the initial transformation of $y = \log_3 x$, and the resulting transformation.
- Rewrite $y = \log_3 9x$ to show that it is a translation of $y = \log_3 x$.
 - Rewrite $y = \log_3 4x$ to show that it is a translation of $y = \log_3 x$.
 - Rewrite $y = \log_3 \frac{x}{27}$ to show that it is a translation of $y = \log_3 x$.
 - Rewrite $y = 5 + \log_3 x$ to show that it is a dilation of $y = \log_3 x$.
- 17** [Super Challenge] In each part, name the transformation involved. Some parts here required the change-of-base formula summarised in Box 14.
- Show how to rewrite $y = \left(\frac{1}{3}\right)^x$ so that it is naturally a reflection of $y = 3^x$.
 - Show how to rewrite $y = \log_{\frac{1}{3}} x$ so that it is naturally a reflection of $y = \log_3 x$.
 - Show how to rewrite $y = \log_3 x^5$ so that it is naturally a dilation of $y = \log_3 x$.
 - Show how to rewrite $y = \log_3 \frac{1}{x^3}$ so that it is naturally a dilation of $y = \log_3 x$.
 - Show how to rewrite $y = \log_9 x$ so that it is naturally a dilation of $y = \log_3 x$.
 - Show how to rewrite $y = \log_{\frac{1}{9}} x$ so that it is naturally a dilation of $y = \log_3 x$.
 - Show how to rewrite $y = \log_{20} x$ so that it is naturally a dilation of $y = \log_3 x$.
- 18** [Super Challenge] Sketch, on separate axes, the graphs of:
- a** $y = \frac{1}{2}(2^x + 2^{-x})$ **b** $y = \frac{1}{2}(2^x - 2^{-x})$

8G Applications of these functions

Learning intentions

- Solve practical problems involving exponential and logarithmic functions.

Some applications of these functions have been scattered through the preceding exercises — the applications in this exercise are longer and more sustained. The intention of all the questions is to indicate how diverse the applications of exponential and logarithmic functions are in various sciences and technologies.

The questions require conversion between a statement in exponential form and a statement in logarithmic form. The pattern for this is

$$2^3 = 8 \quad \text{means that} \quad 3 = \log_2 8.$$

The change-of-base formula is also needed to convert logs to base 10,

$$\log_a x = \frac{\log_{10} x}{\log_{10} a}.$$

The later questions in this exercise are more difficult than in other exercises, but they are included because of their vital interest to readers studying computing, physics, geology and chemistry. Many of them could easily be adapted to projects that would go into the subjects in more detail.

Exercise 8G

FOUNDATION

Technology: Graphing software will display these functions, and give opportunities to answer many further questions about the situations described in the exercise.

- A quantity Q is varying over time t according to the formula $Q = 5 \times 10^{\frac{t}{5}}$. Give answers correct to four significant figures.
 - Find Q when $t = 6$, and when $t = 5.43$.
 - Rewrite the formula with t as the subject.
 - Find t when $Q = 500$, and when $Q = 256$.
- A quantity Q is varying over time t according to the formula $t = 20 \log_2 2Q$.
 - Find t when $Q = 4$, and when $Q = 6$.
 - Rewrite the formula with Q as the subject.
 - Find Q when $t = 40$, and when $t = 45$.

DEVELOPMENT

- The population of a country is doubling every 30 years, and was 3 000 000 at the last census.
 - Explain why the population P after another n years is $P = 3\,000\,000 \times 2^{\frac{n}{30}}$.
 - Find the population (nearest million) after:
 - 90 years,
 - 100 years.
 - By substituting into the equation, find (nearest year) when the population will be:
 - 48 000 000,
 - 60 000 000.

- 4 The price of a particular metal has been rising with inflation at 5% per annum, from a base price of \$100 per kilogram in 1900. Let P be the price n years since 1900, so that $P = 100 \times (1.05)^n$.

a Copy and complete the following table of values, giving values correct to the nearest whole number:

n	0	20	40	60	80	100
P	100					

b Sketch the graph of the function.

c Now copy and complete this table for $\log_{10} P$, correct to two decimal places:

n	0	20	40	60	80	100
$\log_{10} P$	2					

d Draw a graph with n on the horizontal axis and $\log P$ on the vertical axis.

e Use the log laws to prove that $\log_{10} P = 2 + n \log_{10}(1.05)$, and hence explain the shape of the second graph.

- 5 [Moore's law] Gordon Moore predicted in 1965 that over the next few decades, the number of transistors within a computer chip would very roughly double every two years — this is called *Moore's law*. Let D_0 be the density in 1975.

a Explain why the density D after n more years is predicted to be $D = D_0 2^{\frac{n}{2}}$.

b What was the predicted density for 2015?

c Substitute into the formula to find, correct to the nearest year, the prediction of the year when the density increases by a factor of 10 000 000.

- 6 [Newton's law of cooling] A container of water, originally just below boiling point at 96°C , is placed in a fridge whose temperature is 0°C . The container is known to cool in such a way that its temperature halves every 20 minutes.

a Explain why the temperature $T^\circ\text{C}$ after n hours in the fridge is given by the function $T = 96 \times (\frac{1}{2})^{3n}$.

b Draw up a table of values of the function and sketch its graph.

c What is the temperature after 2 hours?

d Write the equation with the time n in hours as the subject.

e How long does it take, correct to the nearest minute, for the temperature to fall to 1°C ?

- 7 [Decay of Uranium-235] Uranium-235 is a naturally-occurring isotope of uranium. It is particularly important because it can be made to fission, when it releases vast amounts of energy in a nuclear power station or a nuclear bomb. It has a half-life of about 700 million years, which means that if you store a mass of uranium-235 for 700 million years, half of it will then have decayed into other elements.

a Explain why the mass of uranium-235 in the Earth n years after the present is given by $M = M_0 \times (\frac{1}{2})^{\frac{n}{700\,000\,000}}$, where M_0 is the mass in the Earth now.

b The Andromeda Galaxy will collide with our Milky Way galaxy in about 4 billion years. What percentage of the present uranium-235 will still be present in the Earth?

c The Earth itself is about 4.5 billion years old. How many times more uranium-235 was present in the Earth when it was formed?

CHALLENGE

- 8** [The Richter scale] Earthquake strengths are usually reported on the Richter scale, which has several minor variants. They are all based on the log base 10 of the ‘shaking amplitude’ of the earthquake wave at some standardised distance from the epicentre. (An earthquake occurs kilometres underground — the *epicentre* is the point on the surface above the place where the earthquake occurs.)
- a** An earthquake of strength 4.0–4.9 is classified as ‘light’, and will cause only minimal damage. An earthquake of strength 7.0 or above is classified as ‘major’, and will cause damage or total collapse to most buildings. What is the ratio of the shaking amplitudes of the smallest light quake and the smallest major quake?
 - b** The energy released by the quake is proportional to the $\frac{3}{2}$ th power of the shaking amplitude. What is the ratio of the energies released by the quakes in part (a)?
 - c** An earthquake of magnitude 9.0 will cause total destruction. What are the ratios of the shaking amplitudes and the energies released of such an earthquake and the smallest light quake?
- 9** [pH of a solution] The pH of a liquid is traditionally defined on a logarithmic scale as $\text{pH} = -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in units of moles per litre (mol/L). It is not necessary to understand the units, except to know that the greater the concentration of hydrogen ions, the greater the acidity.
- a** Rewrite the formula with $[\text{H}^+]$ as the subject.
 - b** Pure water has a pH of about 7. What is its hydrogen ion concentration?
 - c** Lemon juice typically has a pH of about 2.0. What is its hydrogen ion concentration? How many times more acidic is it than pure water?
 - d** Sea water typically has a pH of about 8.1. What is its hydrogen ion concentration? How many times more alkaline (meaning ‘less acidic’) is it than pure water?

Chapter 8 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 8 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- 1 Write each expression as an integer or fraction.

a 5^3

b 2^8

c 10^9

d 17^{-1}

e 9^{-2}

f 2^{-3}

g 3^{-4}

h 27^0

i $\left(\frac{2}{3}\right)^3$

j $\left(\frac{7}{12}\right)^{-1}$

k $\left(\frac{5}{6}\right)^{-2}$

l $36^{\frac{1}{2}}$

m $27^{\frac{1}{3}}$

n $8^{\frac{2}{3}}$

o $9^{\frac{5}{2}}$

p $\left(\frac{4}{49}\right)^{\frac{1}{2}}$

q $\left(\frac{14}{59}\right)^0$

r $\left(\frac{9}{25}\right)^{-\frac{1}{2}}$

s $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

t $\left(\frac{9}{100}\right)^{-\frac{3}{2}}$

- 2 Write each expression in index form.

a $\frac{1}{x}$

b $\frac{7}{x^2}$

c $-\frac{1}{2x}$

d $\sqrt{x}$

e $5\sqrt{36x}$

f $\frac{4}{\sqrt{x}}$

g $\frac{y}{x}$

h $2y\sqrt{x}$

- 3 Simplify:

a $(x^4)^5$

b $\left(\frac{3}{a^3}\right)^4$

c $(25x^6)^{\frac{1}{2}}$

d $\left(\frac{8r^3}{t^6}\right)^{\frac{1}{3}}$

- 4 Simplify each expression, leaving the answer in index form.

a $x^2y \times y^2x$

b $15xyz \times 4y^2z^4$

c $3x^{-2}y \times 6xy^{-3}$

d $4a^{-2}bc \times a^5b^2c^{-2}$

e $x^3y \div xy^3$

f $14x^{-2}y^{-1} \div 7xy^{-2}$

g $m^{\frac{1}{2}}n^{-\frac{1}{2}} \times m^{\frac{1}{2}}n^{-\frac{1}{2}}$

h $(2st)^3 \times (3s^3)^2$

i $(4x^2y^{-2})^3 \div (2xy^{-1})^3$

- 5 Solve each equation for x .

a $3^x = 81$

b $5^x = 25$

c $7^x = \frac{1}{7}$

d $2^x = \frac{1}{32}$

e $\left(\frac{1}{3}\right)^x = \frac{1}{9}$

f $\left(\frac{2}{3}\right)^x = \frac{8}{27}$

g $25^x = 5$

h $8^x = 2$

- 6 Rewrite each logarithmic equation as an index equation and solve it for x .

a $x = \log_2 8$

b $x = \log_3 9$

c $x = \log_{10} 10\,000$

d $x = \log_5 \frac{1}{5}$

e $x = \log_7 \frac{1}{49}$

f $x = \log_{13} 1$

g $x = \log_9 3$

h $x = \log_2 \sqrt{2}$

i $2 = \log_7 x$

j $-1 = \log_{11} x$

k $\frac{1}{2} = \log_{16} x$

l $\frac{1}{3} = \log_{27} x$

m $2 = \log_x 36$

n $3 = \log_x 1000$

o $-1 = \log_x \frac{1}{7}$

p $\frac{1}{2} = \log_x 4$

- 7 Use the log laws and identities to simplify:

a $\log_{22} 2 + \log_{22} 11$

b $\log_{10} 25 + \log_{10} 4$

c $\log_7 98 - \log_7 2$

d $\log_5 6 - \log_5 150$

e $\log_3 54 + \log_3 \frac{1}{6}$

f $\log_{12} \frac{2}{7} + \log_{12} \frac{7}{2}$

- 8 Write each expression in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

a $\log_a xyz$

b $\log_a \frac{x}{y}$

c $\log_a x^3$

d $\log_a \frac{1}{z^2}$

e $\log_a x^2 y^5$

f $\log_a \frac{y^2}{xz^2}$

g $\log_a \sqrt{x}$

h $\log_a \sqrt{xyz}$

- 9 Between what two consecutive integers do the following logarithms lie?

a $\log_{10} 34$

b $\log_7 100$

c $\log_3 90$

d $\log_2 35$

e $\log_{10} 0.4$

f $\log_{10} 0.007$

g $\log_2 0.1$

h $\log_7 0.1$

- 10 Use a calculator, and the change-of-base formula if necessary, to approximate these logarithms, correct to four significant figures.

a $\log_{10} 215$

b $\log_{10} 0.0045$

c $\log_7 50$

d $\log_2 1000$

e $\log_5 0.215$

f $\log_{1.01} 2$

g $\log_{0.5} 8$

h $\log_{0.99} 0.001$

- 11 Use logarithms to solve these index equations, correct to four significant figures.

a $2^x = 11$

b $2^x = 0.04$

c $7^x = 350$

d $3^x = 0.67$

e $1.01^x = 5$

f $1.01^x = 0.2$

g $0.8^x = \frac{1}{10}$

h $0.99^x = \frac{1}{100}$

- 12 a Sketch on one set of axes $y = 3^x$, $y = 3^{-x}$, $y = -3^x$ and $y = -3^{-x}$.

b Sketch on one set of axes $y = 2^x$ and $y = \log_2 x$.

c Sketch on one set of axes $y = 3^x$, $y = 3^x - 1$, and $y = 3^x + 2$, and write down the asymptotes of the three graphs.

- 13 a Sketch on one set of axes $y = \log_5 x$, $y = \log_{\frac{1}{5}} x$, and $y = \log_5 (-x)$.

b Sketch on one set of axes $y = \log_5 x$, $y = \log_5 (x - 2)$, and $y = \log_5 (2 - x)$, and write down the asymptotes of the three graphs.

c Sketch on one set of axes $y = \log_5 x$, $y = \log_5 2x$, and $y = 2 \log_5 x - 2$.

- 14 Describe a sequence of transformations that will transform:

a $y = \log_6 x$ to $y = -4 \log_6 (3x - 7)$.

b $y = 6^x$ to $y = -6^{\frac{1}{2}x+9} + 10$.

and then write down the domain and range of the transformed function, in both notations.

- 15 The number of bacteria growing in a Petri dish is doubling every four hours. Initially there were 100 bacteria.

a Explain why the formula for the population P after n hours is $P = 100 \times 2^{\frac{n}{4}}$.

b How many bacteria are there after:

c 12 hours,

d 13 hours?

e Write the formula with n as the subject.

f How long, correct to the nearest hour, before there are 10 000 000 bacteria?

9

Differentiation

Chapter introduction

The next step in studying functions and their graphs is called *calculus*. Calculus begins with two processes called *differentiation* and *integration*.

- ▶ *Differentiation* looks at the changing steepness of a curve.
- ▶ *Integration* looks at the areas of regions bounded by curves.

Both processes involve taking limits. They were well known to the Greeks, but it was not until the late 17th century that Gottfried Leibniz in Germany and Sir Isaac Newton in England independently gave systematic accounts of them. Leibniz's notation is in standard use today.

This chapter deals with differentiation, and introduces the derivative as the gradient of the tangent to a curve. Integration will be introduced in Year 12. Section 9J on rates of change begins to show how essential calculus is for science, for economics, and for solving practical problems.

The proofs of the basic results here are difficult. Some are given in a Challenge question or in the Appendix at the end of this chapter, and some are omitted. The Challenge questions and the Appendix could perhaps be tackled during a second reading.

The final two sections are an introduction to motion, which is another form of rate. But motion is far more complicated than the rates covered in Section 9J. Readers may like to delay the treatment of motion until much later in the year.

9A Tangents and the derivative

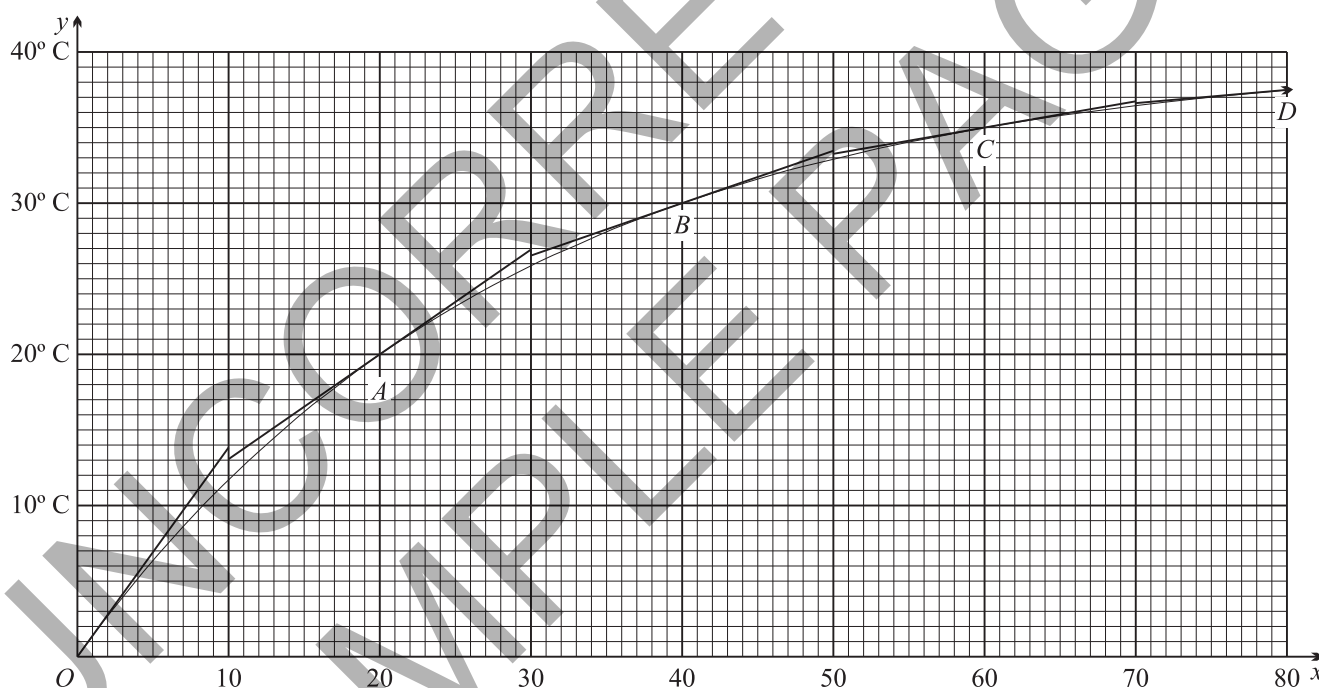
Learning intentions

- Understand the derivative as the gradient of a tangent.
- Understand the derivative as a measure of instantaneous change.
- Find the derivatives of linear and constant functions.

The formula for a function $y = f(x)$ tells us how to find the value of a quantity y given a value of a quantity x . But throughout science, engineering, and economics, we are constantly asking, ‘How does the quantity y change as x changes?’.

Differentiation answers this question very precisely. Its basic approach is to draw a tangent to the curve, and calculate its gradient. This gradient gives us a measure of how y is changing as x changes, at the exact point where the tangent is drawn.

An example of secants and tangents used to measure how a quantity is changing



A bottle of water was taken out of a fridge on a hot day when the air temperature was 40°C . The graph $y = f(x)$ above shows how the temperature increased over the next 80 minutes. The horizontal axis gives the time x in minutes, and the vertical axis gives the temperature $y^{\circ}\text{C}$.

The water temperature was originally 0°C (at O), and 20 minutes later it had risen to 20°C (at A). Thus during the first 20 minutes, the temperature was rising at an average rate of 1°C per minute. This rate is the gradient of the secant OA .

Measuring the *instantaneous rate of temperature increase*, however, requires a *tangent* to be drawn. The gradient of the tangent is the instantaneous rate of increase at the time x — such gradients are easy to measure on the graph paper by counting little divisions and using the formula $\text{gradient} = \frac{\text{rise}}{\text{run}}$.

The gradient of the tangent to the curve $y = f(x)$ at any point is called the *derivative*, which is written as $f'(x)$.

Measuring the gradients at the marked points O , A , B and C gives a table of values of the derivative:

x	0	20	40	60
$f'(x)$	1.39	0.69	0.35	0.17

This derivative $f'(x)$ is a new function. It has a table of values, and it can be sketched, just as the original function was. You will be sketching this derivative in the first question of the following exercise.

Geometric definition of the derivative

Here is the definition of the derivative expressed in geometric language.

1 The Derivative $f'(x)$ defined geometrically

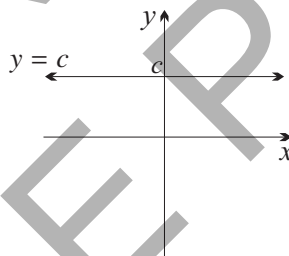
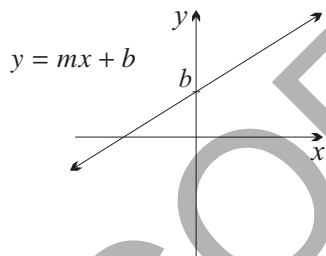
$f'(x)$ is the gradient of the tangent to $y = f(x)$ at each point on the curve.

The gradient of a tangent to a curve is often called the *gradient of the curve at the point*. Thus the derivative $f'(x)$ is also called the *gradient function* of $f(x)$.

The derivatives of linear and constant functions

When a graph is a straight line, the tangent at every point on the graph is just the line itself.

The diagram on the left below is the line $f(x) = mx + b$ with gradient m . At every point on the line, the tangent is the same line, and so also has gradient m . Hence the derivative is the constant function $f'(x) = m$.



In particular, a horizontal straight line has gradient zero, as in the graph of $y = c$ on the right above. The tangent at any point is this same horizontal line. Hence the derivative is the zero function $f'(x) = 0$.

2 The derivative of linear and constant functions

- If $f(x) = mx + b$ is a linear function, then
 $f'(x) = m$ is a constant function — it is the gradient of the line.
- In particular, if $f(x) = c$ is a constant function, then
 $f'(x) = 0$ is the zero function.



Example 1 Derivatives of linear functions

Write down the derivative $f'(x)$ of each linear function.

a $f(x) = 3x + 2$

b $f(x) = \frac{5 - 2x}{3}$

c $f(x) = -4$

Solution

In each case the derivative is just the gradient of the line:

a $f'(x) = 3$

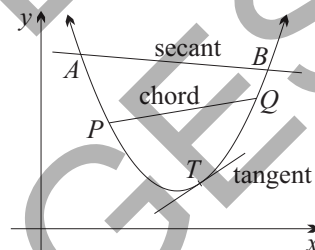
b $f'(x) = -\frac{2}{3}$

c $f'(x) = 0$

Secants, chords, and tangents

These three words were introduced in earlier years only for circles, but are now being used in the context of graphs.

- A *secant* (meaning ‘cutting’) is the line through two distinct points A and B on the graph.
- A *chord* (meaning ‘cord’, such as a bowstring) is the line segment joining two distinct points P and Q on the graph. It is part of the secant PQ , and has the same gradient.
- Informally, a *tangent* (meaning ‘touching’) at a point T on the graph is the line that continues in the direction that the curve is going at the point T — think of the headlights of a car going around a bend. The definition used with circles, ‘meets the curve only at T ’, doesn’t work for graphs. The curve may twist around so that the tangent meets the curve again at one or more other points. Section 9B has a more formal definition.



Some preliminary investigations

Questions 2–5 of the following exercise consist of four investigations that apply the definition of the derivative directly to several functions. They may be done by photocopying the graphs, or by using graphing software.

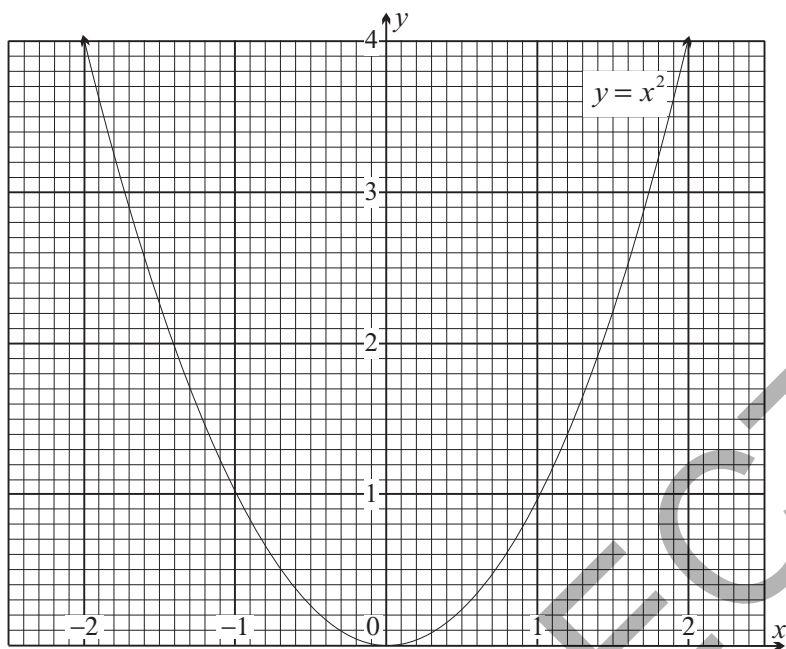
Section 9B develops these ideas into a more rigorous treatment of the derivative.

Exercise 9A

INVESTIGATION

- 1 A question about the graph of water temperature at the start of the chapter.
 - a** Look at the graph, and say in words what happens to the temperature of the water bottle as time goes on.
 - b** What is the average rate of temperature increase in the first 40 seconds, giving units?
 - c** From the table of values of $f'(x)$, draw a sketch of the derivative $f'(x)$. Label the x -axis *Time*, and the y -axis *Instantaneous rate of temperature increase*.
 - d** What is the instantaneous rate of temperature increase after 40 seconds, giving units?
 - e** From your graph, describe in words the behaviour of the rate of temperature increase.

2 [Graph paper]



- a** Photocopy the sketch above of $f(x) = x^2$.
- b** At the point $P(1, 1)$, construct the tangent. Place your pencil point on P , bring your ruler to the pencil, then rotate the ruler about P until it seems reasonably like a tangent.
- c** Use the definition $\text{gradient} = \frac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent correct to at most two decimal places. Choose the run to be 10 little divisions, and count how many vertical divisions the tangent rises as it runs across the 10 horizontal divisions.
- d** Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the nine points on the curve and measuring its gradient.

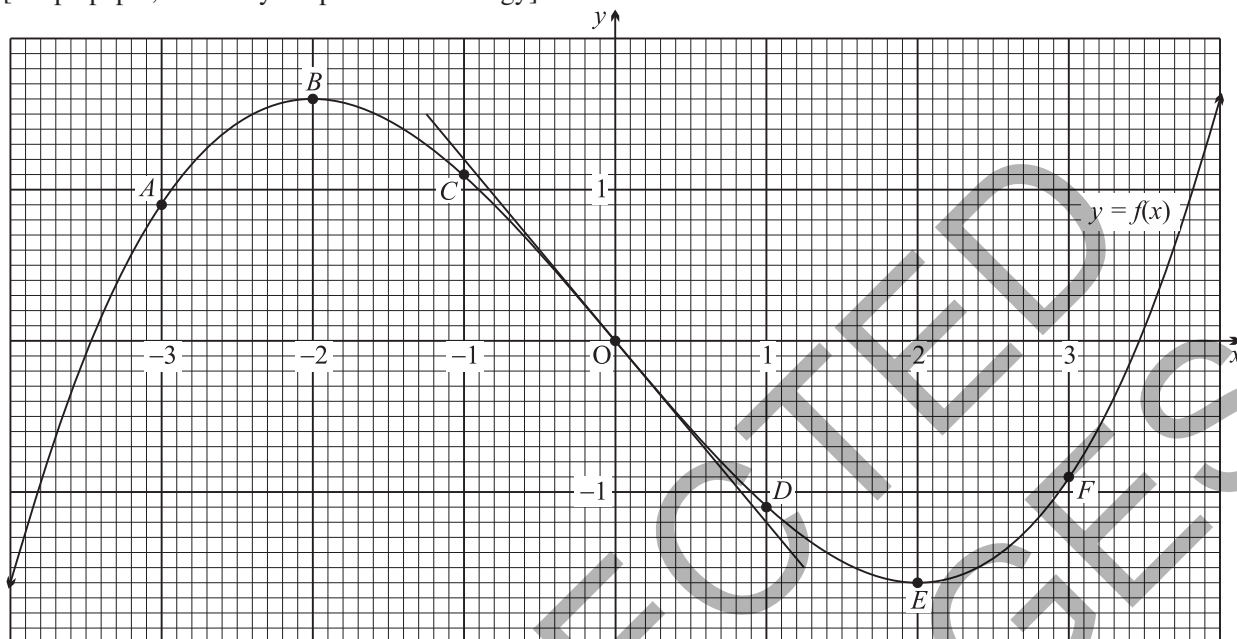
x	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$f'(x)$									

- e** On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.
- f** Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

3 [The same investigation using technology]

- a** Use graphing software to sketch the graph of $y = x^2$. Use the same scale on both axes so that gradients are represented correctly. The values on the y -axis should run from at least $y = 0$ to $y = 4$, and on the x -axis from at least $x = -2$ to $x = 2$.
- b** Construct the tangent at the point $P(1, 1)$.
- c** Find the gradient of this tangent, either by asking the software, or by producing a graph-paper background and counting little squares as in question 1.
- d** Copy and complete the table of values of $f'(x)$ as in question 1(d).
- e** On a new set of axes, plot these values of $f'(x)$, and join them up to form the graph of $y = f'(x)$.
- f** Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

4 [Graph paper, but easily adapted to technology]



- a** Photocopy the cubic graph above.
- b** The tangent at the origin $O(0, 0)$ has been drawn — notice that it crosses the curve at the origin. Use the definition $\text{gradient} = \frac{\text{rise}}{\text{run}}$ to measure the gradient of this tangent.
- c** Copy and complete the following table of values of the derivative $f'(x)$ by constructing a tangent at each of the seven points on the curve and measuring its gradient.

x	-3	-2	-1	0	1	2	3
$f'(x)$							

- d** On a separate set of axes, use your table of values to sketch the curve $y = f'(x)$.

5 [Technology, but easily done on graph paper]

- a** Use graphing software to sketch the graph of $y = x^3$, using the same scale on both axes. The values on the y -axis should run from just $y = -1$ to $y = 1$, and on the x -axis from $x = -1$ to $x = 1$. (Larger values of x will send the graph off the screen.)
- b** Using the software to construct tangents and calculate their gradients, copy and complete the following table of values of the derivative $f'(x)$.

x	-1	-0.9	-0.8	-0.5	0	0.5	0.8	0.9	1
$f'(x)$									

- c** On a new set of axes, plot the graph of $y = f'(x)$.
- d** Make a reasonable guess as to what the equation of the derivative $f'(x)$ is.

Conclusions

- Either of the first two investigations should demonstrate reasonably clearly that the derivative of the quadratic function $f(x) = x^2$ is the linear function $y = 2x$.
- Either of the last two investigations should demonstrate at least that the derivative of a cubic curve looks very much like a quadratic curve — the last investigation may have indicated that $f(x) = x^3$ has derivative $f'(x) = 3x^2$.

DEVELOPMENT

6 Write each function in the form $f(x) = mx + b$, and hence write down $f'(x)$.

a $f(x) = 2x + 3$

b $f(x) = 5 - 3x$

c $f(x) = \frac{1}{2}x - 7$

d $f(x) = -4$

e $f(x) = ax + b$

f $f(x) = \frac{2}{3}(x + 4)$

g $f(x) = \frac{3 - 5x}{4}$

h $f(x) = \frac{5}{2}(7 - \frac{4}{3}x)$

i $f(x) = \frac{1}{2} + \frac{1}{3}$

CHALLENGE

7 Write each function in the form $f(x) = mx + b$, and hence write down the derived function.

a $f(x) = \frac{3 + 5x}{2} - \frac{5 - 2x}{2}$

b $f(x) = (x + 3)^2 - (x - 3)^2$

c $f(x) = \frac{k - \ell x}{r} + \frac{k + \ell x}{r}$

8 Sketch graphs of these functions, draw tangents at the points where $x = -2, -1, 0, 1, 2$, estimate their gradients, and hence draw a reasonable sketch of the derivative.

a $f(x) = 4 - x^2$

b $f(x) = \frac{1}{x}$

c $f(x) = 2^x$

9B The derivative as a limit

Learning intentions

- Understand a tangent as the limit of secants.
- Differentiate using the limit formula for the derivative.
- Work with tangents and their gradients.
- Use a spreadsheet to model the limit definition of the derivative.

This section introduces a limiting process to find the gradient of a tangent at a point P on a curve. This will become the formal definition of the derivative.

The tangent as the limit of secants

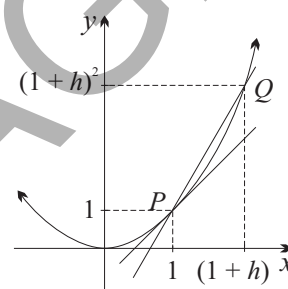
For non-linear functions, we look at secants through P that cross the curve again at another point Q near P , and then take the limit as Q moves towards P . The diagram below shows $f(x) = x^2$ and the tangent at $P(1, 1)$ on the curve.

Let Q be another point on the curve, and join the secant PQ .

Let the x -coordinate of Q be $1 + h$, where $h \neq 0$.

Then the y -coordinate of Q is $(1 + h)^2$.

$$\begin{aligned} \text{Hence gradient } PQ &= \frac{y_2 - y_1}{x_2 - x_1} \quad (\text{this is rise over run}) \\ &= \frac{(1 + h)^2 - 1}{(1 + h) - 1} \\ &= \frac{2h + h^2}{h} \\ &= 2 + h, \quad \text{where we can cancel because } h \neq 0. \end{aligned}$$



As Q moves along the curve, to the right or left of P , the secant PQ changes.

The closer Q is to the point P , the closer the secant PQ is to the tangent at P .

The gradient of the secant PQ becomes 'as close as we like' to the gradient of the tangent as Q moves sufficiently close to P .

This is called 'taking the limit as Q approaches P ', with notation $\lim_{Q \rightarrow P}$.

$$\begin{aligned} \text{gradient (tangent at } P) &= \lim_{Q \rightarrow P} (\text{gradient } PQ) \\ &= \lim_{h \rightarrow 0} (2 + h), & \text{because } h \rightarrow 0 \text{ as } Q \rightarrow P \\ &= 2, & \text{because } 2 + h \rightarrow 2 \text{ as } h \rightarrow 0. \end{aligned}$$

Thus the tangent at P has gradient 2, which means that $f'(1) = 2$.

The point Q cannot coincide with P , because there would then be no line PQ . The point Q may, however, be on the left of P ($h < 0$) or on the right of P ($h > 0$).

The derivative as a limit

This process can be applied to any function $f(x)$.

Let $P(x, f(x))$ be any point on the curve $y = f(x)$.

Let Q be any other point on the curve, left or right of P , and let Q have x -coordinate $x + h$, where $h \neq 0$, and y -coordinate $f(x + h)$.

Then using the formula $\text{gradient} = \text{rise over run}$,

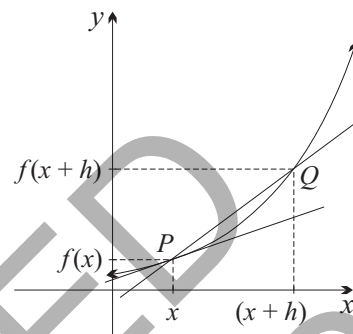
$$\text{gradient of secant } PQ = \frac{f(x + h) - f(x)}{h} \quad (\text{rise over run}).$$

As $h \rightarrow 0$, the point Q moves ‘as close as we like’ to P , and the gradient of secant PQ becomes ‘as close as we like’ to the gradient of the tangent at P .

Hence $\text{gradient of tangent at } P = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$,

that is,
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$

This last expression is the *limit formula for the derivative*, and is taken as its definition.



3 The derivative $f'(x)$ defined as a limit

For each value of x , the derivative is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \quad (\text{if the limit exists}).$$

The limit is *two-sided* — it involves both positive and negative small values of h . But h cannot be zero, because then the expression is $\frac{0}{0}$, which is not defined.

The expression $\frac{f(x + h) - f(x)}{h}$ is sometimes called a *difference quotient*, because the numerator $f(x + h) - f(x)$ is the difference between the heights at P and Q , and the denominator h is the difference between the x -coordinates of P and Q . Geometrically, the expression is the gradient of the secant PQ .

Using the definition of the derivative — first-principles differentiation

First-principles differentiation is using the limit formula to find the derivative.



Example 2 First-principles differentiation

- a** For $f(x) = x^2$, use first-principles differentiation to show that $f'(5) = 10$.
- b i** Use first-principles differentiation to find the derivative $f'(x)$.
- ii** Substitute $x = 5$ to confirm that $f'(5) = 10$, as in part (a).

Solution

$$\begin{aligned} \text{a For all } h \neq 0, \quad \frac{f(5 + h) - f(5)}{h} &= \frac{(5 + h)^2 - 5^2}{h} \\ &= \frac{25 + 10h + h^2 - 25}{h} \\ &= \frac{10h + h^2}{h} \\ &= 10 + h, \quad \text{because } h \neq 0. \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(5) = 10$.

$$\begin{aligned}
 \text{b For all } h \neq 0, \quad \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\
 &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\
 &= \frac{2xh + h^2}{h} \\
 &= 2x + h, \quad \text{because } h \neq 0.
 \end{aligned}$$

Taking the limit as $h \rightarrow 0$, $f'(x) = 2x$.

c Substituting $x = 5$, $f'(5) = 10$, as established in part (a).

Differentiable (or smooth) at a point

Two issues may raise questions, and need explanation.

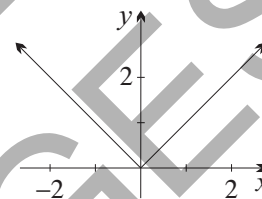
First, the graph $f(x) = |x|$ sketched to the right is continuous at the origin, but there is clearly a problem differentiating at the origin O , because the curve is not *smooth* there.

When we substitute $a = 0$ into the limit formula, there is no limit

$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$, because all the secants on the right of O have gradient 1, and on the left they all have gradient -1 .

We say that $y = |x|$ is *not differentiable* (or *smooth*) at $x = 0$.

If a curve is not continuous at a point, it is certainly not differentiable there.



What is a tangent

Secondly, the term ‘tangent’ was introduced in Section 9A without a formal definition. We can now use the limit formula for the derivative to define a tangent (provided that it is not vertical) as the line through the point with gradient $f'(x)$.

4 Differentiable at a point, and the definition of a non-vertical tangent

- A function $f(x)$ is called *differentiable* or *smooth* at $x = a$ if the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

▷ If $f(x)$ is not continuous at $x = a$, it is certainly not differentiable there.

- If $f(x)$ is differentiable at $x = a$, the *tangent* at the point P where $x = a$ is the line through P with gradient $f'(a)$.

Vertical tangents need further discussion in Year 12.

As with continuity at a point, we make a second standard assumption:

5 Assumption about differentiability at a point

The functions in this course are differentiable (smooth) for every value of x in their domains, except where there is an obvious problem.

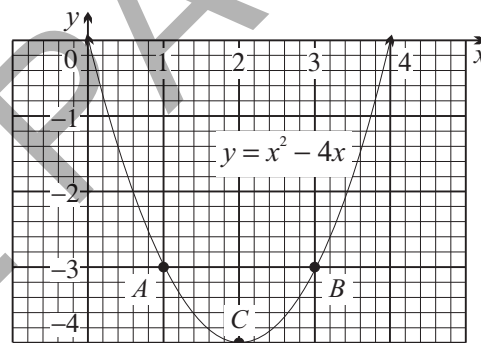
Exercise 9B

FOUNDATION

Note: The questions in this exercise use the formula for the derivative as a limit:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

- 1 Consider the function $f(x) = 5x^2$.
 - a Show that $f(1) = 5$ and $f(1+h) = 5 + 10h + 5h^2$.
 - b Hence find $f(1+h) - f(1)$.
 - c Show that $\frac{f(1+h) - f(1)}{h} = 10 + 5h$.
 - d Take the limit as $h \rightarrow 0$ to show that $f'(1) = 10$.
- 2 Consider again the function $f(x) = 5x^2$.
 - a Show that $f(x+h) = 5x^2 + 10xh + 5h^2$.
 - b Hence find $f(x+h) - f(x)$.
 - c Show that $\frac{f(x+h) - f(x)}{h} = 10x + 5h$.
 - d Take the limit as $h \rightarrow 0$ to show that $f'(x) = 10x$.
 - e Substitute $x = 1$ into $f'(x)$ to confirm that $f'(1) = 10$, as found in question 2.
- 3 Consider the function $f(x) = x^2 - 4x$.
 - a Show that $f(x+h) = x^2 + 2xh + h^2 - 4x - 4h$.
 - b Show that $\frac{f(x+h) - f(x)}{h} = 2x + h - 4$.
 - c Show that $f'(x) = 2x - 4$ by taking the limit as $h \rightarrow 0$.
 - d Evaluate $f'(1)$ to find the gradient of the tangent at $A(1, -3)$.
 - e Similarly, find the gradients of the tangents at $B(3, -3)$ and $C(2, -4)$.
 - f The function $f(x) = x^2 - 4x$ is graphed above. Place your ruler on the curve at A , B and C to check the reasonableness of the results obtained above.



DEVELOPMENT

- 4 Consider the function $f(x) = x^2 + 2$.
 - a Find $f(x+h) - f(x)$ and hence show that $\frac{f(x+h) - f(x)}{h} = 2x + h$.
 - b Take the limit as $h \rightarrow 0$ to find $f'(x)$.
 - c Evaluate $f'(0)$ to find the gradient of the tangent at the point where $x = 0$.
 - d Evaluate $f'(3)$ to find the gradient of the tangent at the point where $x = 3$.
- 5 Consider the function $f(x) = x^2 + 4x$.
 - a Simplify $\frac{f(x+h) - f(x)}{h}$, then take the limit as $h \rightarrow 0$ to find $f'(x)$.
 - b Find the gradients of the tangents to $y = x^2 + 4x$ at the points where $x = 0$ and $x = -2$.
- 6 a Find $\frac{f(x+h) - f(x)}{h}$ for the function $f(x) = x^2 - 2x$ and hence find its derivative.
 - b Hence find the gradients of the tangents at the points where $x = 0$ and $x = 2$.

- 7 a** Find $\frac{f(x+h) - f(x)}{h}$ for the function $f(x) = x^2 + 6x + 8$ and hence find its derivative.
b Hence find the gradients of the tangents at the points where $x = 0$ and $x = -3$.
- 8 a** Find $\frac{f(2+h) - f(2)}{h}$ for the function $f(x) = 1 + x^2$ and hence find $f'(2)$.
b Find $\frac{f(h) - f(0)}{h}$ for the function $f(x) = 2x^2 + 3x$ and hence find $f'(0)$.
c Find $\frac{f(-1+h) - f(-1)}{h}$ for the function $f(x) = x^2 - 4x$ and hence find $f'(-1)$.
- 9** In the previous section, you found by geometry that a linear function $f(x) = mx + b$ has derivative $f'(x) = m$. This question confirms that the limit formula for $f'(x)$ gives the same answer.
a Let $f(x) = 3x + 7$. Find $\frac{f(x+h) - f(x)}{h}$, and hence show that $f'(x) = 3$.
b Let $f(x) = mx + b$. Find $\frac{f(x+h) - f(x)}{h}$, and hence show that $f'(x) = m$.
c Let $f(x) = c$. Find $\frac{f(x+h) - f(x)}{h}$, and hence show that $f'(x) = 0$.
- 10** [Spreadsheet technology (written for Excel)] Consider the function $f(x) = x^3$.
a Use the formula $f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$ to prove that $f'(2) = 12$. You may want to use the standard expansion $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.
b In the top row of a spreadsheet, in cell A1 enter the letter h in cell A2 enter the formula $f(2+h) - f(2)$ and in cell A3 enter the word gradient
c In the second row, in cell B1 enter the number 1, in cell B2 enter the string $=(2+A2)^3 - 2^3$ and in cell B3 enter $=B2/A2$
d In the first column, in cell A3 enter 0.5, in cell A4 enter 0.1, and in cell A5 enter 0.05.
e Select cell B2, then hold the Shift key and use the arrows to highlight the cells that are in rows 2, 3, 4, and 5 and in columns B and C.
f With these cells highlighted, press Ctrl+D to Fill Down the formulae into the rows below.
g Continue down column A with ever smaller numbers, using Fill Down each time.
h Explain what this spreadsheet procedure has to do with part **a**.

CHALLENGE

- 11 a** If $f(x) = x^3$, show that $\frac{f(x+h) - f(x)}{h} = 3x^2 + 3xh + h^2$ and hence find $f'(x)$.
 You will need to use the expansion $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.
b Prove the necessary expansion of $(x+h)^3$ used in part **a**. Begin with:

$$(x+h)^3 = (x+h)(x+h)^2$$

$$= (x+h)(x^2 + 2xh + h^2)$$
- 12 a** If $f(x) = x^4$, show that $\frac{f(x+h) - f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$, then find $f'(x)$.
 You will need to use the expansion $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.
b Prove the expansion of $(x+h)^4$ used in part **a**. Use the previous question, and begin:

$$(x+h)^4 = (x+h)(x+h)^3$$

$$= (x+h)(x^3 + 3x^2h + 3xh^2 + h^3)$$

9C A rule for differentiating powers of x

Learning intentions

- Find and use the derivative of x^n for whole-number indices $n > 0$.
- Work with tangents and their gradients, and angles of inclinations.

From this limit definition, we will develop formulae for differentiation for each type of function. This section concerns a formula that applies to all powers of x .

A formula for differentiating powers of x

The long calculations of $f'(x)$ in the previous exercise had quite simple answers, as you will probably have noticed. Here is the pattern:

$f(x)$	1	x	x^2	x^3	x^4	x^5	...
$f'(x)$	0	1	$2x$	$3x^2$	$4x^3$	$5x^4$	...

These results are examples of a simple rule for differentiating any power of x :

6 The derivative of any power of x

Let $f(x) = x^n$, where n is any real number. Then the derivative is

$$f'(x) = nx^{n-1}.$$

This rule is usually memorised as:

‘Take the index as a factor, and reduce the index by 1.’

The proof is tricky, and is given in the Appendix at the end of this chapter only for whole-number indices $n > 0$. Questions in the Challenge sections of Exercises 9E and 9F develop the further proofs for $n = -1$ and $n = \frac{1}{2}$ respectively.

This course, however, will assume that the result is true for all real numbers n . Here are some examples where the index n is a positive whole number.



Example 3 Differentiating whole number powers of x

Differentiate each function.

a $f(x) = x^5$

b $f(x) = x^8$

c $f(x) = x^{37}$

Solution

a $f(x) = x^5$
 $f'(x) = 5x^4$

b $f(x) = x^8$
 $f'(x) = 8x^7$

c $f(x) = x^{37}$
 $f'(x) = 37x^{36}$

Linear combinations of functions

Functions formed by taking sums and multiples of simpler functions can be differentiated in the obvious way, one term at a time. The proofs here are reasonably straightforward, but are given in the Appendix at the end of this chapter.

7 Linear combinations of functions

- If $f(x) = u(x) + v(x)$, then $f'(x) = u'(x) + v'(x)$.
- If $f(x) = au(x)$, then $f'(x) = au'(x)$ where a is a constant.



Example 4 Differentiating linear combinations of powers of x

Differentiate each function.

a $f(x) = x^2 + 3x + 5$

c $f(x) = 3x^{10} + 4x^9$

Solution

a $f(x) = x^2 + 3x + 5$

$$f'(x) = 2x + 3 + 0$$

$$= 2x + 3$$

c $f(x) = 3x^{10} + 4x^9$

$$f'(x) = 3 \times 10x^9 + 4 \times 9x^8$$

$$= 30x^9 + 36x^8$$

b $f(x) = 4x^2 - 3x + 2$

d $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$

b $f(x) = 4x^2 - 3x + 2$

$$f'(x) = 4 \times 2x - 3 + 0$$

$$= 8x - 3$$

d $f(x) = \frac{1}{2}x^6 - \frac{1}{6}x^3$

$$f'(x) = \frac{1}{2} \times 6x^5 - \frac{1}{6} \times 3x^2$$

$$= 3x^5 - \frac{1}{2}x^2$$

Expanding products

Sometimes a product needs to be expanded before differentiating the function.



Example 5 Expanding a product before differentiating

Differentiate each function after first expanding the bracket, factoring if possible.

a $f(x) = x^3(x - 10)$

b $f(x) = (x + 2)(2x + 3)$

Solution

a $f(x) = x^3(x - 10)$

$$= x^4 - 10x^3$$

$$f'(x) = 4x^3 - 30x^2 = 2x^2(2x - 15)$$

b $f(x) = (x + 2)(2x + 3)$

$$= 2x^2 + 7x + 6$$

$$f'(x) = 4x + 7$$

Gradient of a curve, and angle of inclination of a tangent

The *gradient of a curve* at a particular point P on it is defined to be the gradient of the tangent at P — the gradient of the curve at P is thus the value of the derivative at P .

The steepness of a tangent can be expressed either by giving its gradient, or by giving its angle of inclination. As we saw in Chapter 7, negative gradients correspond to obtuse angles of inclination.

8 Gradient of a curve, and angle of inclination of a tangent

- The *gradient of a curve* at a point P on it is the gradient of the tangent at P .
- The steepness of a tangent can also be expressed by its angle of inclination.


Example 6 Differentiating to find the angle of inclination

- a** Differentiate $f(x) = x^2 + 2x$.
b Find the gradient and angle of inclination of the curve at the origin.
c Find the gradient and angle of inclination of the curve at the point $A(-2, 0)$.
d Sketch the curve and the tangents, marking their angles of inclination.

Solution

- a** Differentiating $f(x) = x^2 + 2x$ gives $f'(x) = 2x + 2$.
b At the origin $O(0, 0)$, gradient of tangent $= f'(0)$

$$= 2.$$

Let α be the angle of inclination of the tangent.

Then $\tan \alpha = 2$, where $0^\circ \leq \alpha < 180^\circ$,

so $\alpha \doteq 63^\circ 26'$.

- c** At $A(-2, 0)$, gradient of tangent $= f'(-2)$

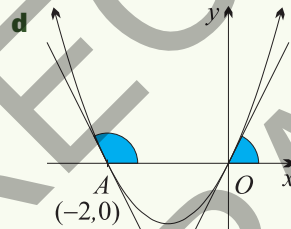
$$= -4 + 2$$

$$= -2.$$

Let β be the angle of inclination of the tangent.

Then $\tan \beta = -2$, where $0^\circ \leq \beta < 180^\circ$,

so $\beta \doteq 180^\circ - 63^\circ 26'$
 $\doteq 116^\circ 34'$.



Method: Enter 2 into the calculator, take $\tan^{-1} 2$, then find its supplement.

Finding points on a curve with a given gradient

Once the derivative has been found, we can find the points on a curve where the tangent has a particular gradient.

9 Finding points on a curve with a given gradient

- To find the points where the tangent has a given gradient m , solve the equation

$$f'(x) = m.$$

- To find the y -coordinates of the points, substitute back into $f(x)$.

The points on the curve where the tangent is horizontal are particularly important.

**Example 7** Finding points with a given gradient

Find the point on $f(x) = 6x - x^2$ where the tangent is horizontal, then sketch the curve.

Solution

Here $f(x) = 6x - x^2$,

so $f'(x) = 6 - 2x$.

Put $f'(x) = 0$ (because the tangent is horizontal).

Then $6 - 2x = 0$

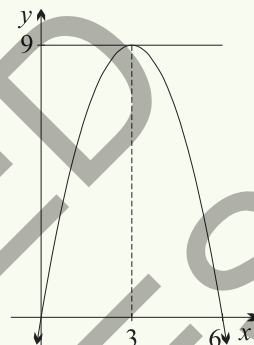
$$x = 3.$$

Substituting, $f(3) = 18 - 9$

$$= 9,$$

so the tangent is horizontal at $(3, 9)$.

(This is, of course, the vertex of the parabola.)

**Finding points on a curve with a given angle of inclination**

When the angle of inclination is given, use the $\tan \theta$ function to find the gradient.

**Example 8** Finding points with a given angle of inclination

Find the points on the graph of $f(x) = x^2 - 5x + 4$ where:

a the tangent has gradient -3 ,

b the tangent has angle of inclination 45° .

Solution

Here $f(x) = x^2 - 5x + 4$,

so $f'(x) = 2x - 5$.

a Put $f'(x) = -3$.

Then $2x - 5 = -3$

$$2x = 2$$

$$x = 1.$$

Substituting $x = 1$ into the function,

$$f(1) = 1 - 5 + 4$$

$$= 0,$$

so the tangent has gradient -3 at the point $(1, 0)$.

b First, $\tan 45^\circ = 1$,

so put $f'(x) = 1$.

Then $2x - 5 = 1$

$$2x = 6$$

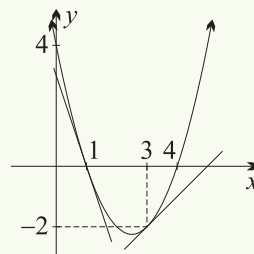
$$x = 3.$$

Substituting $x = 3$ into the function,

$$f(3) = 9 - 15 + 4$$

$$= -2,$$

so the tangent has angle of inclination 45° at the point $(3, -2)$.

**10 Finding points on a curve with a given angle θ of inclination**

- Solve $\frac{dy}{dx} = \tan \theta$,

because the gradient of the tangent is $\tan \theta$.

Exercise 9C

FOUNDATION

- 1** Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative $f'(x)$ of:
- a** $f(x) = x^7$ **b** $f(x) = x^5$ **c** $f(x) = -x^{24}$
d $f(x) = 9x^5$ **e** $f(x) = 3x^2$ **f** $f(x) = -5x^{12}$
g $f(x) = \frac{1}{3}x^6$ **h** $f(x) = \frac{1}{2}x^8$ **i** $f(x) = -\frac{2}{3}x^9$
- 2** **a** Write $f(x) = 9x$ as $f(x) = 9x^1$, and then use the rule to show that $f'(x) = 9$.
b Write $g(x) = 4$ as $g(x) = 4x^0$, and then use the rule to show that $g'(x) = 0$.
- 3** For each function, find the derivative $f'(x)$.
- a** $f(x) = 5x + 7$ **b** $f(x) = 8 - x$ **c** $f(x) = x^2 + 5x + 7$
d $f(x) = 3x^2 - 5x$ **e** $f(x) = x^4 - 5x^2 + 5$ **f** $f(x) = 2 - 3x - 5x^3$
g $f(x) = x^4 + x^3 + x^2 + x + 1$ **h** $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2$ **i** $f(x) = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2 - 2$
- 4** **a** Find the derivative of $f(x) = x^2 + 7x - 2$.
b Evaluate $f'(0)$ to find the gradient of the tangent at the point $(0, -2)$.
- 5** Differentiate each function, then find the gradient of the tangent at the point where $x = 2$.
a $f(x) = 7 - x^2$ **b** $f(x) = x^3 + 6x$ **c** $f(x) = 10x^2 - x^4$
- 6** Differentiate $f(x) = x^3$, and hence find the gradient of the tangent to $y = x^3$ at:
a $(2, 8)$ **b** $(1, 1)$ **c** $(0, 0)$ **d** $(-1, -1)$ **e** $(-2, -8)$
- 7** **a** Differentiate $f(x) = x^2 - 4$.
b Solve $f'(x) = 0$ to find the point on $y = x^2 - 4$ where the tangent is horizontal.
c Solve $f'(x) = 6$ to find the point on $y = x^2 - 4$ where the tangent has gradient 6.
d Solve $f'(x) = -6$ to find the point on $y = x^2 - 4$ where the tangent has gradient -6 .

DEVELOPMENT

- 8** Expand each product, then find the derivative.
- a** $f(x) = x(4 - x)$ **b** $f(x) = x(x^2 + 1)$ **c** $f(x) = x^2(3 - 4x^2)$
d $f(x) = (x + 4)(x - 2)$ **e** $f(x) = (2x + 1)(2x - 1)$ **f** $f(x) = (x^2 + 3)^2$
g $f(x) = (7 - x)^2$ **h** $f(x) = (x^2 + 3)(x - 5)$ **i** $f(x) = (3x - 5)^2$
- 9** **a** Find the derivative of $f(x) = x^2 + x + 1$.
b Evaluate $f'(0)$ to show that the tangent at the point $(0, 1)$ has gradient 1.
c Solve $\tan \alpha = 1$ to find the angle of inclination of the tangent at $(0, 1)$.
- 10** **a** Find the derivative of $f(x) = 9 - x + x^2$.
b Evaluate $f'(2)$ to show that the tangent at the point $(2, 11)$ has gradient 3.
c Find the angle of inclination of the tangent at $(2, 11)$, correct to the nearest minute.
- 11** Differentiate $f(x) = x^2 - 3x - 6$. Hence find the gradient and the angle of inclination (correct to the nearest minute, where appropriate) of the tangent at the point where:
- a** $x = 3$ **b** $x = 2$ **c** $x = 1\frac{1}{2}$ **d** $x = 1$ **e** $x = 0$

- 12** **a** Find the derivative of $f(x) = 8x - x^2$.
b Use the factoring $f(x) = x(8 - x)$ to find the x -intercepts of $y = f(x)$, then sketch it.
c Find the gradients of the tangents at these two x -intercepts.
d Show that the tangent is horizontal at the point where $x = 4$.
- 13** **a** Find the derivative of $f(x) = x^2 + 8x + 7$.
b Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
c Solve $f'(x) = 12$. Hence find the point on $y = f(x)$ where the tangent has gradient 12.
- 14** **a** Find the derivative of $f(x) = 3 - 2x^2$.
b Solve $f'(x) = 0$. Hence find the point on $y = f(x)$ where the tangent is horizontal.
c Solve $f'(x) = -20$. Hence find the point where the tangent has gradient -20 .
- 15** Differentiate, then find the points on each curve where the tangent is horizontal.
a $f(x) = x^2 - 2x + 7$ **b** $f(x) = x^2 + 4x - 10$ **c** $f(x) = x^2 - 10x + 15$
- 16** **a** Find the derivative of $f(x) = x^2 - 6x$.
b Use the factoring $f(x) = x(x - 6)$ to find the x -intercepts of $y = f(x)$, then sketch it.
c Find the gradients of the tangents at these two x -intercepts.
d Find the point on the curve where the tangent is horizontal.
- 17** Differentiate $f(x) = x^2 - 5x + 1$ and hence find the points on the curve where:
a the gradient is 3, **b** the gradient is -5 ,
c the angle of inclination is 45° , **d** the angle of inclination is 135° .

CHALLENGE

- 18** **a** Differentiate $f(x) = x^3 - 3x + 2$. Hence find the two points on $y = x^3 - 3x + 2$ where the tangent is horizontal.
b Differentiate $f(x) = x^4 - 18x^2$. Hence find the three points on $y = x^4 - 18x^2$ where the tangent is horizontal.
c Differentiate $f(x) = x^3 + 6$. Hence find the two points on $y = x^3 + 6$ where the tangent has gradient 75.
- 19** Sketch the graph of $f(x) = x^2 - 6x$ and find the gradient of the tangent at the point $A(a, a^2 - 6a)$ on the curve.
a Hence find the value of a if the tangent has gradient:
i 0 **ii** 2 **iii** $\frac{1}{2}$
b Find the value of a if the tangent has angle of inclination 135° .
c Find the value of a if the tangent is:
i parallel to $2x - 3y + 4 = 0$, **ii** perpendicular to $2x - 3y + 4 = 0$.

9D dy/dx notation — tangents and normals

Learning intentions

- Use dy/dx notation for the derivative.
- Work with tangents and normals.
- Find tangents through other points in the plane.

Leibniz's original notation for the derivative remains the most widely used and best-known notation, particularly in science. It is even said that Dee Why Beach was named after the derivative dy/dx . The notation is extremely flexible, and clearly expresses the fact that the derivative behaves very much like a fraction.

Small changes in x and in y

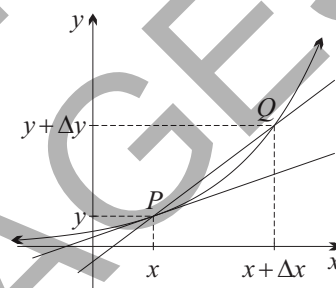
The diagram used to explain $\frac{dy}{dx}$ is exactly the same as for $f'(x)$, but the notation is a little different.

Let $P(x, y)$ be any point on the graph of a function.

Let x change by a small amount Δx to $x + \Delta x$, and let y change by a corresponding amount Δy to $y + \Delta y$.

Thus $Q(x + \Delta x, y + \Delta y)$ is another point on the curve. Then

$$\text{gradient } PQ = \frac{\Delta y}{\Delta x} \quad (\text{rise over run}).$$



When Δx is small, the secant PQ is almost the same as the tangent at P , and the derivative is the limit of $\frac{\Delta y}{\Delta x}$ as Q gets 'as close as we like' to P , that is, as $\Delta x \rightarrow 0$.

This is the basis for Leibniz's notation:

11 An alternative notation for the derivative

Let Δy be the small change in y resulting from a small change Δx in x . Then

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

(The letter Δ is the upper-case Greek letter delta, symbolising Difference.)

The object dx is intuitively understood as an 'infinitesimal change' in x , and dy is understood as the corresponding 'infinitesimal change' in y . The derivative dy/dx is then understood as the ratio of these two infinitesimal changes. 'Infinitesimal changes', however, are for the intuition only — the logic of the situation is:

12 Fraction notation and the derivative $\frac{dy}{dx}$

The derivative $\frac{dy}{dx}$ is not a fraction, but is the limit of the fraction $\frac{\Delta y}{\Delta x}$ as $\Delta x \rightarrow 0$.

The notation is very clever because the derivative is a gradient, any gradient is a fraction $\frac{\text{rise}}{\text{run}}$, and the notation $\frac{dy}{dx}$ preserves the intuition of fractions.

The words ‘differentiation’ and ‘calculus’

The *small differences* Δx and Δy , and the *infinitesimal differences* dx and dy , are the origins of the word ‘differentiation’. The symbol Δ is the upper-case Greek letter ‘Delta’, standing for difference.

Calculus is a Latin word meaning ‘stone’. An *abacus* consists of stones sliding on bars and was used routinely in the past to help with arithmetic — this is the origin of the modern word ‘calculate’. The word ‘calculus’ in English can refer to any systematic method of calculation, but is most often used for the twin theories of differentiation and integration studied in this course.

Second and higher derivatives

The derivative of a function is another function, so it can in turn be differentiated to give the *second derivative* of the original function. This is written as $f''(x)$ in $f(x)$ notation, and as $\frac{d^2y}{dx^2}$ in the new $\frac{dy}{dx}$ notation.

Thus for a function such as $x^4 + 5x^3$, the two notations are:

$$\begin{array}{ll} y = x^4 + 5x^3 & f(x) = x^4 + 5x^3 \\ \frac{dy}{dx} = 4x^3 + 15x^2 & f'(x) = 4x^3 + 15x^2 \\ \frac{d^2y}{dx^2} = 12x^2 + 30x & f''(x) = 12x^2 + 30x \end{array}$$

This process can be continued indefinitely to *third and higher derivatives*:

$$\begin{array}{ll} \frac{d^3y}{dx^3} = 24x + 30 & f'''(x) = 24x + 30 \\ \frac{d^4y}{dx^4} = 24 & f^{(4)}(x) = 24 \end{array}$$

and after that, the fifth and higher derivatives are all zero. Notice that to avoid counting multiple dashes, we usually write $f^{(4)}(x)$ rather than $f''''(x)$.

The second derivative $f''(x)$ is the *gradient function of the gradient function*. Next year, we will interpret the second derivative in terms of curvature, but there are no obvious geometric interpretations of higher derivatives.

Operator notation

The derivative $\frac{dy}{dx}$ can also be regarded as the *operator* $\frac{d}{dx}$ operating on the function y . This gives an alternative notation for the derivative:

$$\frac{d}{dx}(x^5) = 5x^4 \quad \text{and} \quad \frac{d}{dx}(x^2 + x - 1) = 2x + 1.$$

(An *operator* acts on a function to give another function. It is thus a function whose domain and range are sets of functions rather than sets of numbers.)

Thus the standard rule for differentiating powers of x can be written as

$$\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for all real numbers } n.$$

Setting out using $\frac{dy}{dx}$ notation

The remaining worked examples in this section show how the new notation is used in calculations on the geometry of a curve.


Example 9 Finding points with given gradient — dy/dx notation

- a** Differentiate $y = 4 - x^2$.
b Find the gradient of the tangent at the point $P(-1, 3)$ on the curve.
c Find the point on the curve where the gradient of the tangent is 6.

Solution

a $y = 4 - x^2$

$$\frac{dy}{dx} = -2x \quad (\text{apply the usual rule for differentiating})$$

- b** At the point $P(-1, 3)$, the gradient of the tangent is

$$\frac{dy}{dx} = (-2) \times (-1) \quad (\text{substitute } x = -1 \text{ into the formula for } \frac{dy}{dx})$$

$$= 2$$

- c** Put $\frac{dy}{dx} = 6$ (to find where the gradient is 6)

then $-2x = 6$

$$x = -3.$$

When $x = -3$, $y = 4 - 9$

$$= -5,$$

so the gradient is 6 at the point $(-3, -5)$.

Tangents and normals to a curve

The *normal* to a curve at a point P on the curve is the line through P perpendicular to the tangent at P . Provided that the tangent and normal are not parallel to the axes, the product of their gradients is -1 , so their equations are easily found.

13 Tangents and normals to a curve at a point P on the curve

- The *tangent* at P is the line through P whose gradient is the derivative at P .
- The *normal* at P is the line through P perpendicular to the tangent at P .
- If they are not parallel to the axes, their gradients m_1 and m_2 are related by

$$m_1 m_2 = -1, \quad \text{or equivalently,} \quad m_2 = -\frac{1}{m_1}.$$

- Their equations are then easily found using point–gradient form:

$$y - y_1 = m(x - x_1).$$


Example 10 Finding gradients and angles of inclination of tangents and normals

- a** Find the gradient and angle of inclination of the tangent to $y = (x + 1)^2$ at $A(0, 1)$.
b Find the gradient and angle of inclination of the normal at A , and draw a sketch.

Solution

a Expanding, $y = x^2 + 2x + 1$,

so $\frac{dy}{dx} = 2x + 2$.

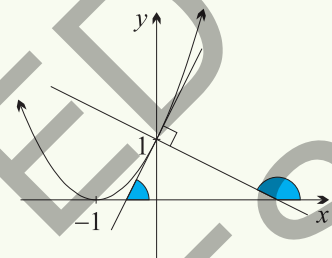
When $x = 0$, $\frac{dy}{dx} = 0 + 2$,

so the tangent at $A(0, 1)$ has gradient 2.

Solving $\tan \alpha = 2$, where $0^\circ \leq \alpha < 180^\circ$,
 its angle of inclination is about $63^\circ 26'$.

b The normal has gradient $-\frac{1}{2}$ (opposite of the reciprocal).

Solving $\tan \beta = -\frac{1}{2}$, where $0^\circ \leq \beta < 180^\circ$, its angle of inclination is about $153^\circ 26'$.


Example 11 Finding the intersection of tangents

- a** Find the equations of the tangents to $y = x^2 + x + 1$ at $P(1, 3)$ and $Q(-1, 1)$.
b At what point do these tangents intersect?

Solution

a Differentiating, $\frac{dy}{dx} = 2x + 1$.

When $x = 1$, $\frac{dy}{dx} = 3$,

so, using point-gradient form, the tangent at $P(1, 3)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 3(x - 1)$$

$$y = 3x.$$

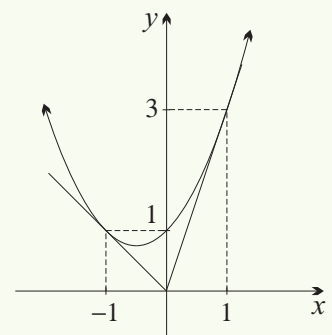
When $x = -1$, $\frac{dy}{dx} = -1$,

so the tangent at $Q(-1, 1)$ is $y - y_1 = m(x - x_1)$

$$y - 1 = -1(x + 1)$$

$$y = -x.$$

- b** Both $y = 3x$ and $y = -x$ pass through the origin, so the origin is the point of intersection of the two tangents.


Geometric figures created by tangents and normals

Tangents and normals to a graph, together with the axes, often produce geometric figures that can be studied further. The following worked exercise finds the area of a triangle formed by the tangent and normal at a point, and the y-axis.


Example 12 Working with the geometry of tangents and normals

- a** Find the equation of the tangent to $y = x^3$ at $A(1, 1)$.
b Find the equation of the normal to the curve at $A(1, 1)$.
c Find the coordinates of the y -intercepts T and N of the tangent and normal.
d Sketch the situation and find the area of $\triangle ANT$.

Solution

a Differentiating, $\frac{dy}{dx} = 3x^2$.

When $x = 1$, $\frac{dy}{dx} = 3$,
 so the tangent at A has gradient 3.

Using point–gradient form, the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 3(x - 1)$$

$$y = 3x - 2.$$

- b** The normal has gradient $-\frac{1}{3}$.

Using point–gradient form, the normal is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{3}(x - 1)$$

$$y = -\frac{1}{3}x + 1\frac{1}{3}.$$

- c** Both lines are already in gradient–intercept form, and their y -intercepts are therefore $T(0, -2)$ and $N(0, 1\frac{1}{3})$.

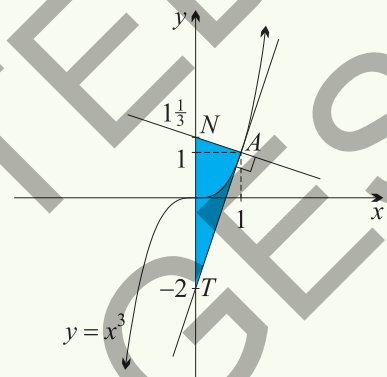
- d** The triangle ANT has base NT and perpendicular height 1,
 where

$$NT = 2 + 1\frac{1}{3}$$

$$= 3\frac{1}{3},$$

so $\text{area } \triangle ANT = \frac{1}{2} \times 3\frac{1}{3} \times 1$

$$= 1\frac{2}{3} \text{ square units.}$$


Finding the tangents passing through some fixed point in the plane

Sometimes we want to find the equation of a tangent to a curve $y = f(x)$ passing through a fixed point F in the plane. This problem can be solved by finding the equation of the tangent at a variable point $P(t, f(t))$ on the curve, then substituting the coordinates of F into this equation. The method can be applied to other similar problems.


Example 13 Finding tangents from a fixed point in the plane

- a** Find the equation of the tangent to $y = x^2 + x + 1$ at the point P on the curve where $x = t$.
b Hence find the equations of the tangents to the curve passing through the origin, and the corresponding point of contact of each tangent.

Solution

a Differentiating, $\frac{dy}{dx} = 2x + 1$,

so at P , $\frac{dy}{dx} = 2t + 1$.

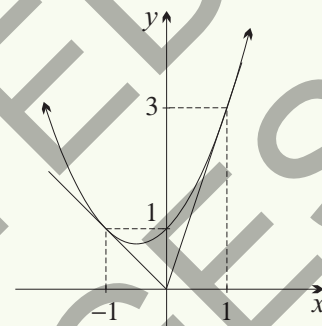
The coordinates of P are $(t, t^2 + t + 1)$,

so the tangent is $y - (t^2 + t + 1) = (2t + 1)(x - t)$
 $y = (2t + 1)x - t^2 + 1$.

b Substituting $(0, 0)$, $0 = -t^2 + 1$
 $t = 1$ or -1 .

When $t = 1$, the tangent is $y = 3x$ and the point of contact is $P = (1, 3)$.

When $t = -1$, the tangent is $y = -x$ and the point of contact is $P = (-1, 1)$.


Exercise 9D
FOUNDATION

- Find the derivative $\frac{dy}{dx}$ of each function.
 - $y = x^2$
 - $y = x^2 + 7x - 10$
 - $y = x^3 + 3x^2 + 6x + 8$
 - $y = x^4 + x^2 + 8x$
 - $y = 4x - 7$
 - $y = 7$
- Using $\frac{dy}{dx}$ notation, find the first, second and third derivatives of each function.
 - $y = x^6 + 2x$
 - $y = 5x^2 - x^5$
 - $y = 4x$
- Using $f(x)$ notation, find the 1st, 2nd, 3rd and 4th derivatives of each function.
 - $f(x) = 10x^3 + x$
 - $f(x) = 2x^4 + 2$
 - $f(x) = 5$
- Differentiate $y = x^2 + x$. Then solve $\frac{dy}{dx} = 7$ and hence find the coordinates of any points on $y = x^2 + x$ where the tangent has gradient 7.
- Differentiate $y = x^3 - 1$. Then solve $\frac{dy}{dx} = 12$ and hence find the coordinates of any points on $y = x^3 - 1$ where the tangent has gradient 12.
- Differentiate the function $y = x^2 - 3x$.
 - Substitute into $\frac{dy}{dx}$ to find the gradient of the tangent at the point P where $x = 2$.
 - Substitute into the function to find the y -coordinate of the point P .
 - Use point–gradient form to find the equation of the tangent to the curve at P .
 - Write down the gradient of the normal at P .
 - Use point–gradient form to find the equation of the normal to the curve at P .

- 7 a** Differentiate $y = x^5 + x^3 + 2x$.
b Find the values of y and $\frac{dy}{dx}$ at the point P where $x = 0$.
c Use point–gradient form to find the equation of the tangent to the curve at P .
d Write down the gradient of the normal at P .
e Use point–gradient form to find the equation of the normal to the curve at P .
f Which function below is not in the same family as the other three?

$$y = \frac{1}{2}x^4 + 3x^2 + 6, \quad y = \frac{1}{2}x^4 - 7 + 3x^2, \quad y = 3x^2 - \frac{1}{2}x^4 + 1, \quad y = 3x^2 + \frac{1}{2}x^4 - 1$$

DEVELOPMENT

- 8** Find the derivative of each function. You will need to expand the brackets first.
a $y = (x + 1)(x - 1)$ **b** $y = x^2(3 - x^3)$ **c** $y = (x - 1)(x - 2)$
- 9 a** Differentiate $y = x^3 - 2x$ and find the values of $\frac{dy}{dx}$ when $x = 1$ and when $x = 2$.
b Find, correct to the nearest degree where appropriate, the angles of inclination of the tangents at the points where $x = 1$ and $x = 2$. (Hint: You will need the formula $\text{gradient} = \tan \alpha$, where α is the angle of inclination of the tangent.)
- 10** Differentiate each function. Then solve $\frac{dy}{dx} = 0$ and hence find the coordinates of any points on each curve where the tangent is horizontal.
a $y = 3 - 2x + x^2$ **b** $y = x^4 + 18x^2$
- 11** Differentiate $y = x^2 - 8x + 15$. Hence find the equations of the tangent and normal at:
a $A(1, 8)$ **b** $B(6, 3)$ **c** $C(0, 15)$ **d** $D(4, -1)$
- 12** In each part, find the derivative $\frac{dy}{dx}$ of the function. Then find the equations of the tangent and normal to the curve at the point indicated.
a $y = x^2 - 6x$ at $O(0, 0)$ **b** $y = x^3 - 4x$ at $P(2, 0)$
c $y = x^2 - x^4$ at $Q(-1, 0)$ **d** $y = x^3 - 3x + 2$ at $R(1, 0)$
- 13 a** Find the equations of the tangent and normal to $y = x^2$ at the point $H(2, 4)$.
b Find the points A and B where the tangent and normal respectively meet the y -axis.
c Sketch the situation, then find the length AB and the area of $\triangle ABH$.
- 14 a** Find the equations of the tangent and normal to $y = 9 - x^2$ at the point $K(1, 8)$.
b Find the points A and B where the tangent and normal respectively meet the x -axis.
c Sketch the situation, then find the length AB and the area of $\triangle ABK$.
- 15 a** Find the equation of the tangent to $y = x^2 + 9$ at the point $P(p, p^2 + 9)$.
b Hence find the points on the curve where the tangents pass through the origin, and the equation of those tangents. Draw a sketch of the situation.
- 16** For each curve below, find the equation of the tangent at the point P where $x = t$. Hence find the equations of any tangents passing through the origin.
a $y = x^2 - 10x + 9$ **b** $y = x^2 + 15x + 36$

- 17 a** Find the equation of the tangent to $y = x^2 + 2x - 8$ at the point K on the curve with x -coordinate t .
b Hence find the points on the curve where the tangents from $H(2, -1)$ touch the curve.
- 18 a** Show that the line $y = -3$ meets the parabola $y = x^2 - 4$ at $D(1, -3)$ and $E(-1, -3)$.
b Find the equations of the tangents to $y = x^2 - 4$ at D and E .
c Find the point where these tangents intersect. Sketch the situation.
- 19 a** Differentiate $y = 4 - x^2$, and hence find the coordinates of the points A and B on the curve where the tangent has gradient 2 and -2 respectively. Sketch the situation.
b Find the equations of the tangents at A and B , and find their point of intersection.
c Find the equations of the normals at A and B , and find their point of intersection.
- 20** The tangent and normal to $f(x) = x^3$ at $U(1, 1)$ meet the y -axis at P and Q respectively. Find their equations, the coordinates of P and Q , and the area of $\triangle QUP$.
- 21 a** Differentiate $y = x^5$ successively until the result is zero. How many successive derivatives of $y = x^5$ are non-zero?
b How many successive derivatives of $y = x^6$ are non-zero?
c How many successive derivatives of $y = x^n$ are non-zero, where n is a whole number?

CHALLENGE

- 22 a** Differentiate the quadratic function $y = ax^2 + bx + c$, where a , b and c are constants.
b Find the x -coordinate of the point V on the curve where the tangent is horizontal.
c In Chapter 3, you used completing the square to show that this parabola has axis of symmetry $x = -\frac{b}{2a}$. Explain why parts (a) and (b) are another proof of this formula.
- 23** Let $P(a, a^2)$ be any point on the parabolic graph of $y = x^2$. Let $A(a - h, (a - h)^2)$ and $B(a + h, (a + h)^2)$ be two other points on the curve, so that the x -coordinates of A , P and B are equally spaced along the x -axis.
a Find the gradient of the tangent at P .
b Find the gradient of the chord AB , and show that the chord is parallel to the tangent.
c What property of the parabola has been proven here?
- 24** Differentiate $y = x^2 + bx + c$, and hence find b and c , given that:
a the parabola passes through the origin, and the tangent there has gradient 7,
b the parabola has y -intercept -3 and gradient -2 there,
c the parabola is tangent to the x -axis at the point $(5, 0)$,
d when $x = 3$ the gradient is 5, and $x = 2$ is a zero,
e the parabola is tangent to $3x + y - 8 = 0$ at the point $T(3, -1)$,
f the line $3x + y - 8 = 0$ is a normal at the point $T(3, -1)$.

9E Differentiating powers with negative indices

Learning intentions

- Differentiate powers with negative indices.

The standard formula $\frac{d}{dx}(x^n) = nx^{n-1}$ applies to all powers of x . This section extends differentiation of powers with negative integer indices.

Differentiate powers with negative indices

First-principles differentiation of $y = x^{-1}$ is in the Challenge section below. But we assume now that the result is true for all negative integer indices.

14 Differentiating powers with negative indices

‘Take the index as a factor and reduce the index by 1’.

For example, if $y = x^{-4}$, then $\frac{dy}{dx} = -4x^{-5}$.

Example 14 Differentiate powers with negative indices

Differentiate each function.

a $y = x^{-20}$

b $y = 6x^{-1}$

c $y = -5x^{-4}$

Solution

a $y = x^{-20}$

$$\frac{dy}{dx} = -20x^{-21}$$

b $y = 6x^{-1}$

$$\frac{dy}{dx} = -6x^{-2}$$

c $y = -5x^{-4}$

$$\frac{dy}{dx} = 20x^{-5}$$

Converting reciprocals of powers to powers with negative indices

Using the formula for differentiation often requires quick conversion between a power with a negative index and the reciprocal of a power. For example:

$$\frac{1}{x} = x^{-1} \quad \text{and} \quad 5x^{-2} = \frac{5}{x^2} \quad \text{and} \quad \frac{1}{3x^5} = \frac{1}{3}x^{-5}$$

Example 15 Converting to powers with negative indices

Differentiate each function, giving your answer without negative indices.

a $f(x) = \frac{1}{x}$

b $f(x) = \frac{5}{x^2}$

c $f(x) = \frac{1}{3x^5}$

Solution

In each example, first convert the expression to negative indices, then apply the rule, then convert back to fraction notation.

a $f(x) = \frac{1}{x}$
 $= x^{-1}$

$$f'(x) = -x^{-2}$$

$$= -\frac{1}{x^2}$$

b $f(x) = \frac{5}{x^2}$
 $= 5x^{-2}$

$$f'(x) = -10x^{-3}$$

$$= -\frac{10}{x^3}$$

c $f(x) = \frac{1}{3x^5}$
 $= \frac{1}{3}x^{-5}$

$$f'(x) = -\frac{5}{3}x^{-6}$$

$$= -\frac{5}{3x^6}$$

The result $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ occurs so often that readers may like to memorise it.

**Example 16** Using a common denominator

- a** Differentiate $f(x) = \frac{16}{x^3} - \frac{24}{x^2}$.
b Use a common denominator to write your answer as a single fraction.
c Hence find the point on the curve $y = f(x)$ where the tangent is horizontal.

Solution

a $f(x) = \frac{16}{x^3} - \frac{24}{x^2}$
 $= 16x^{-3} - 24x^{-2}$ (rewrite the function using negative indices)
 $f'(x) = -48x^{-4} + 48x^{-3}$ (apply the usual rule for differentiation)
b $f'(x) = -\frac{48}{x^4} + \frac{48}{x^3}$ (go back to fraction notation)
 $= \frac{-48 + 48x}{x^4}$ (combine the fractions, using a common denominator)
c $f'(x) = \frac{48(x-1)}{x^4}$ (factor the numerator)
Hence $f'(x) = 0$ when $\frac{48(x-1)}{x^4} = 0$
that is, when $x = 1$. (the numerator must be zero)
Since $f(1) = 16 - 24 = -8$, the tangent is horizontal at the point $(1, -8)$.

Dividing through by the denominator

If the denominator is a single term, divide through by it first, then differentiate.

**Example 17** Dividing through by the denominator

In each function, first divide through by the denominator. Then differentiate.

a $y = \frac{x^3 + x^2 + x}{x}$ **b** $y = \frac{5 - x^2 + 5x^4}{x^2}$

Solution

a $y = \frac{x^3 + x^2 + x}{x}$ **b** $y = \frac{5 - x^2 + 5x^4}{x^2}$
 $= x^2 + x + 1$ $= 5x^{-2} - 1 + 5x^2$
 $\frac{dy}{dx} = 2x + 1$ $\frac{dy}{dx} = -10x^{-3} + 10x$

Exercise 9E**FOUNDATION**

- a** Change the fraction $\frac{1}{x^3}$ to a term with a negative index.
b Hence use the rule that the derivative of x^n is nx^{n-1} to write down the derivative of $f(x) = \frac{1}{x^3}$.
c Convert the answer to part (b) back to a fraction.
- Write down the derivative $f'(x)$ of each function. Leave your answers with negative indices.

a $f(x) = x^{-1}$	b $f(x) = x^{-5}$	c $f(x) = 3x^{-1}$
d $f(x) = 5x^{-2}$	e $f(x) = -\frac{4}{3}x^{-3}$	f $f(x) = 2x^{-2} + \frac{1}{2}x^{-8}$

- 3** Rewrite each function using a negative power of x , then differentiate it, then convert the answer back to a fraction.

a $f(x) = \frac{1}{x}$

b $f(x) = \frac{1}{x^2}$

c $f(x) = \frac{1}{x^4}$

d $f(x) = \frac{3}{x}$

- 4** Find the derivative $\frac{dy}{dx}$ of each function. Then find the gradient of the tangent to the curve at the point where $x = 1$.

a $y = 3x^{-2}$

b $y = 15x^{-4}$

c $y = 2x - 2x^{-1}$

d $y = x + 5 + 5x^{-6}$

e $y = \frac{1}{2}x^{-2} + \frac{1}{3}x^{-3}$

f $y = x^6 + x^{-6}$

- 5** For each function, first divide through by the denominator. Then find the derivative. Leave your answer with negative indices where appropriate.

a $y = \frac{x^3 + x^2}{x}$

b $y = \frac{x^3 + x^2}{x^5}$

c $y = \frac{4x - 5x^4}{x^2}$

d $y = \frac{6x^2 + 6x^6}{2x^6}$

DEVELOPMENT

- 6 a** Differentiate $f(x) = \frac{1}{x}$. Hence find $f'(3)$ and $f'(\frac{1}{3})$.
- b** Find the two points on $y = \frac{1}{x}$ where the tangent has gradient -1 .
- c** Find the two points on $y = \frac{1}{x}$ where the tangent has gradient -4 .
- d** Are there any points on $y = \frac{1}{x}$ where the tangent has zero gradient?
- e** Are there any points on $y = \frac{1}{x}$ where the tangent has negative gradient?
- 7 a** Differentiate $f(x) = -\frac{3}{x}$. Hence find $f'(2)$ and $f'(6)$.
- b** Find the two points on the curve where the tangent has gradient 3 .
- 8 a** Differentiate $f(x) = \frac{12}{x}$. Hence find $f'(2)$ and $f'(6)$.
- b** Hence find the tangent and normal to the curve at $M(2, 6)$ and at $N(6, 2)$.
- c** Find the points on the curve where the tangent has gradient -12 . Sketch the situation.

- 9** Rewrite each function using negative powers of x , then differentiate it. Give each final answer in fractional form without negative indices.

a $f(x) = \frac{1}{x^6} - \frac{1}{x^8}$

b $f(x) = \frac{1}{3x}$

c $f(x) = \frac{5}{x^3}$

d $f(x) = \frac{1}{5x^4}$

e $f(x) = -\frac{7}{x}$

f $f(x) = \frac{7}{2x}$

g $f(x) = -\frac{7}{3x}$

h $f(x) = -\frac{3}{5x^5}$

- 10** Find the derivative $\frac{dy}{dx}$ of each function. Then find the value of $\frac{dy}{dx}$ when $x = 1$.

a $y = x^4 - x^2 + 1$

b $y = \frac{1}{3}x^6 - \frac{1}{2}x^4 + x^2$

c $y = \frac{1}{10}x^5 - \frac{1}{6}x^3 + \frac{1}{2}x$

d $y = (x+1)(x-1)$

e $y = x^2(3-x^3)$

f $y = (2x-1)(x-2)$

g $y = 4x^{-1}$

h $y = \frac{2}{3}x^{-3}$

i $y = x^{-1} - x^{-2}$

- 11** For each function, first divide through by the denominator, then find the derivative. Leave your answer with negative indices where appropriate.

a $y = \frac{3x^4 - 5x^2}{x}$

b $y = \frac{x^4 - 4x^2}{x^2}$

c $y = \frac{5x^6 + 4x^5}{3x^3}$

d $y = \frac{3x^2 - 1}{x}$

e $y = \frac{1 + 7x}{x^3}$

f $y = \frac{3x^5 - 5x^3 + x}{x^2}$

- 12** Differentiate $y = \frac{a}{x} - \frac{b}{cx^2}$, where a , b and c are constants, answering in fractional form.

CHALLENGE

- 13** Consider the function $f(x) = x^{-1}$.

a Find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ and $f^{(5)}(x)$.

b Find $f'(1)$, $f''(1)$, $f'''(1)$, $f^{(4)}(1)$ and $f^{(5)}(1)$.

c Describe the pattern in the results of part (b).

d Describe the pattern if -1 is substituted rather than 1 .

- 14** This question shows how to differentiate the function $f(x) = \frac{1}{x}$ by first principles.

a Show that $\frac{1}{x+h} - \frac{1}{x} = -\frac{h}{(x+h)x}$.

b Hence show that $\frac{f(x+h) - f(x)}{h} = \frac{-1}{x(x+h)}$.

c Take the limit as $h \rightarrow 0$ to show that $f'(x) = -\frac{1}{x^2}$.

- 15 a** Show that the tangent to $y = x^2 + 15x + 36$, at the point P where $x = a$, has equation $y = (2a + 15)x - a^2 + 36$.

b By substituting $O(0, 0)$ into the equation and solving it for a , find the equations of any tangents passing through the origin.

9F Differentiating powers with fractional indices

Learning intentions

- Differentiate powers with fractional indices.
- Understand that the same rules apply for powers with real indices.

The formula for the derivative of x^n applies also when n is a fraction, or indeed any real number (first-principles differentiation of $y = x^{\frac{1}{2}}$ is in Challenge below).

Differentiating powers with fractional indices

First-principles differentiation of $y = \sqrt{x}$ is in the Challenge section below, but the general proof is beyond even the Year 12 course.

15 Differentiating powers with fractional indices

‘Take the index as a factor and reduce the index by 1’.

For example, if $y = x^{\frac{5}{2}}$, then $\frac{dy}{dx} = \frac{5}{2}x^{\frac{3}{2}}$.

The standard formula also applies to negative fractional indices, as in part (c) and (d) of the worked exercise below. The formula also applies for all real indices,



Example 18 Differentiating powers with fractional indices

Differentiate each function.

a $f(x) = x^{\frac{4}{3}}$

b $f(x) = x^{\frac{1}{4}}$

c $f(x) = x^{-\frac{1}{5}}$

d $f(x) = x^{-2\frac{1}{2}}$

Solution

a $f(x) = x^{\frac{4}{3}}$
 $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$

b $f(x) = x^{\frac{1}{4}}$
 $f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$

c $f(x) = x^{-\frac{1}{5}}$
 $f'(x) = -\frac{1}{5}x^{-\frac{6}{5}}$

d $f(x) = x^{-2\frac{1}{2}}$
 $f'(x) = -\frac{5}{2}x^{-3\frac{1}{2}}$

Converting between surd notation and fractional indices

When using the formula for differentiation, we often need to convert between fractional index notation and surd notation:

$$\sqrt{x} = x^{\frac{1}{2}} \quad \text{and} \quad \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \quad \text{and} \quad x^{-1\frac{1}{2}} = \frac{1}{x\sqrt{x}}$$

Remember that $\sqrt{x}$ and $x^{\frac{1}{2}}$ mean exactly the same thing — the *positive square root* of x — and that they are not defined when x is a negative number. Thus

$$\sqrt{9} = 9^{\frac{1}{2}} = 3 \quad (\text{not } -3) \quad \text{and} \quad \sqrt{-9} \text{ and } (-9)^{\frac{1}{2}} \text{ are undefined.}$$

**Example 19** Surd notation and fractional indices

Write each function using a fractional index, then differentiate it, then convert the answer back to surd form.

a $y = 12\sqrt{x}$

b $y = \sqrt{16x}$

c $y = \frac{3}{\sqrt{x}}$

Solution

a $y = 12\sqrt{x}$
 $= 12x^{\frac{1}{2}}$

$$f'(x) = 12 \times \frac{1}{2} \times x^{-\frac{1}{2}}$$
$$= \frac{6}{\sqrt{x}}$$

b $f(x) = \sqrt{16x}$
 $= 4x^{\frac{1}{2}}$

$$f'(x) = 4 \times \frac{1}{2} \times x^{-\frac{1}{2}}$$
$$= \frac{2}{\sqrt{x}}$$

c $f(x) = \frac{3}{\sqrt{x}}$
 $= 3x^{-\frac{1}{2}}$

$$f'(x) = -\frac{1}{2} \times 3 \times x^{-1\frac{1}{2}}$$
$$= -\frac{3}{2x\sqrt{x}}$$

Sometimes surds and whole-number powers need to be combined, and the index laws applied, as in the following two more difficult worked exercises.

**Example 20** Surd notation and fractional indices

Write each function in terms of a power of x , then differentiate it.

a $f(x) = x^3\sqrt{x}$

b $f(x) = \frac{1}{x^3\sqrt{x}}$

Solution

a $f(x) = x^3\sqrt{x}$
 $= x^3 \times x^{\frac{1}{2}}$
 $= x^{3\frac{1}{2}}$

$$f'(x) = \frac{7}{2}x^{2\frac{1}{2}}$$

b $f(x) = \frac{1}{x^3\sqrt{x}}$
 $= \frac{1}{x^3 \times x^{\frac{1}{2}}}$
 $= x^{-3\frac{1}{2}}$

$$f'(x) = -\frac{7}{2}x^{-4\frac{1}{2}}$$

**Example 21** Dividing through by the denominator

Divide through by the denominator, then differentiate.

a $y = \frac{10x - 6}{\sqrt{x}}$

b $y = \frac{3x + 4\sqrt{x}}{x}$

Solution

a $y = \frac{10x^1}{x^{\frac{1}{2}}} - \frac{6}{x^{\frac{1}{2}}}$
 $= 10x^{\frac{1}{2}} - 6x^{-\frac{1}{2}}$
 $\frac{dy}{dx} = 5x^{-\frac{1}{2}} + 3x^{-1\frac{1}{2}}$

b $y = \frac{3x}{x} + \frac{4x^{\frac{1}{2}}}{x^1}$
 $= 3 + 4x^{-\frac{1}{2}}$
 $\frac{dy}{dx} = -2x^{-1\frac{1}{2}}$

Exercise 9F

FOUNDATION

- 1 **a** Rewrite the function $y = 20\sqrt{x}$ using index notation.
b Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative.
c Convert your answer back to surd and fraction notation.
- 2 Use the rule that the derivative of x^n is nx^{n-1} to write down the derivative of each function. Leave answers in index notation.

a $y = x^{\frac{1}{2}}$	b $y = x^{-\frac{1}{2}}$	c $y = x^{1\frac{1}{2}}$
d $y = 6x^{\frac{2}{3}}$	e $y = 12x^{-\frac{1}{3}}$	f $y = 4x^{\frac{1}{4}} + 8x^{-\frac{1}{4}}$
g $y = 7x^{2\frac{1}{3}}$	h $y = 5x^{-\frac{2}{3}}$	i $y = -10x^{-0.6}$
- 3 Write each function as a power with a fractional index and then differentiate it.

a $y = \sqrt{x}$	b $y = \sqrt[3]{x}$	c $y = \sqrt[4]{x}$	d $y = 10\sqrt{x}$
- 4 Find the first and second derivatives of $y = x^{\frac{1}{2}}$.
- 5 **a** Explain why $x\sqrt{x} = x^{\frac{3}{2}}$, and hence differentiate $y = x\sqrt{x}$.
b Similarly, write $y = x^2\sqrt{x}$ as a single power of x , and hence differentiate it.
c Write $y = \frac{1}{\sqrt{x}}$ as a single power of x , and hence differentiate it.
d Write $y = \frac{1}{x\sqrt{x}}$ as a single power of x , and hence differentiate it.

DEVELOPMENT

- 6 **a** Show that the derivative of $y = \sqrt{x}$ is $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.
b Find the gradients of the tangents to $y = \sqrt{x}$ at the points $A(1, 1)$ and $B(4, 2)$.
c Find the equations of the tangents to $y = \sqrt{x}$ at A and B .
d Find the gradients of the normals to $y = \sqrt{x}$ at A and B .
e Find the equations of the normals to $y = \sqrt{x}$ at A and B .
- 7 **a** Show that the derivative of $y = \frac{4}{\sqrt{x}}$ is $\frac{dy}{dx} = -\frac{2}{x\sqrt{x}}$.
b Find the equations of the tangent and normal to the curve at the point where $x = 4$.
c Explain why the domain is $x > 0$, and why every tangent has negative gradient.
- 8 **a** Differentiate $y = 2\sqrt{x}$ and find the gradients of the tangent and normal at $P(4, 4)$.
b Find the equations of the tangent and normal at P .
c Find the points A and B where the tangent and normal respectively meet the x -axis.
d Find the length AB , and hence find the area of $\triangle PAB$.
- 9 Find any points on these curves where the tangent has gradient -1 .

a $y = \frac{1}{x}$	b $y = \frac{1}{2}x^{-2}$	c $y = -\sqrt{x}$	d $y = \frac{1}{4}x^4 - \frac{1}{2}x^2 - x$

- 10** Divide each function through by the denominator, using index notation, then differentiate.

a $y = \frac{x^2 + 6x\sqrt{x} + x}{x}$

b $y = \frac{3x - 3x\sqrt{x} - 8x^2}{x}$

c $y = \frac{3x - 2\sqrt{x}}{\sqrt{x}}$

- 11** Write each function as a power with a fractional index and then differentiate it.

a $f(x) = 24\sqrt{x}$

c $f(x) = \sqrt{25x}$

e $f(x) = 12x\sqrt{x}$

g $f(x) = 24\sqrt[3]{x}$

i $f(x) = 30\sqrt[3]{x^2}$

k $f(x) = \frac{6}{\sqrt{x}}$

b $f(x) = \sqrt{64x}$

d $f(x) = 2x\sqrt{x}$

f $f(x) = 4x^2\sqrt{x}$

h $f(x) = \sqrt[3]{x^2}$

j $f(x) = \frac{1}{\sqrt{x}}$

l $f(x) = \frac{5}{x\sqrt{x}}$

CHALLENGE

- 12** This question shows how to differentiate the function $f(x) = \sqrt{x}$ by first principles.

a Multiply top and bottom of the fraction $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ by $\sqrt{x+h} + \sqrt{x}$ (use the difference of squares identity), and show that it equals $\frac{1}{\sqrt{x+h} + \sqrt{x}}$.

b Write down $\frac{f(x+h) - f(x)}{h}$ for the function $f(x) = \sqrt{x}$, then use part (a) to show that $\frac{f(x+h) - f(x)}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$.

c Take the limit as $h \rightarrow 0$ to prove that $f'(x) = \frac{1}{2\sqrt{x}}$.

- 13 a** Find the gradient of the tangent to $y = x^2 - 10x + 9$ at the point P where $x = a$.

b Find the value of y at $x = a$, and hence show that the tangent at P has equation $y = (2a - 10)x - a^2 + 9$.

c By substituting $(0, 0)$ into the tangent, find the value of a when the tangent passes through the origin, and hence find the equations of the tangents through the origin.

9G The chain rule

Learning intentions

- Use the chain rule to differentiate composite functions.

In Section 9C, we easily developed rules for differentiating sums and multiple of functions. Differentiating composite functions, and products or quotients of functions, is a little more elaborate. This section concerns composite functions,

Composition of functions — a chain of functions

Section 5F introduced composite functions, formed by putting two functions into a *chain* so that the output of the first function becomes the input of the second.

We now want to apply the process in reverse. For example, we can decompose the function $y = (x^2 - 2)^4$ into a chain of two functions — ‘square and subtract 2’, followed by ‘raise to the power 4’.

x		u		y
0	→	-2	→	16
1	→	-1	→	1
-2	→	2	→	16
x	→	$x^2 - 2$	→	$(x^2 - 2)^4$

The middle column is the output of the first function ‘Square and subtract 2’. This output is then the input of the second function ‘Take the fourth power’. The resulting decomposition of the original function $y = (x^2 - 2)^4$ into the chain of two functions may be expressed as follows:

‘Let $u = x^2 - 2$. Then $y = u^4$.’

Our original function has now been decomposed into the composition of two simpler functions, ready to be differentiated using the chain rule.

The chain rule — differentiating a composite function

Suppose then that y is a function of u , where u is a function of x .

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x} \right) \quad (\text{multiplying top and bottom by } \Delta u) \\
 &= \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \times \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \quad (\text{because } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0) \\
 &= \frac{dy}{du} \times \frac{du}{dx}.
 \end{aligned}$$

16 The chain rule

Suppose that y is a function of u , where u is a function of x . Then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Although the proof uses limits, and dy/dx is not a fraction, the usual attitude to this rule is that ‘the du ’s cancel out’.


Example 22 Differentiating using the chain rule

Use the chain rule to differentiate each function.

a $(x^2 - 2)^4$

b $7(3x + 4)^5$

Note: The working in the right-hand column is the strongly recommended setting out of the calculation.

Always begin with that working, because the first step is the decomposition of the function into a chain of two functions.

Solution

a Let $y = (x^2 - 2)^4$.
 Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
 $= 4(x^2 - 2)^3 \times 2x$
 $= 8x(x^2 - 2)^3$.

b Let $y = 7(3x + 4)^5$.
 Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
 $= 35(3x + 4)^4 \times 3$
 $= 105(3x + 4)^4$.

Let $u = x^2 - 2$.

Then $y = u^4$.

Hence $\frac{du}{dx} = 2x$

and $\frac{dy}{du} = 4u^3$.

Let $u = 3x + 4$.

Then $y = 7u^5$.

Hence $\frac{du}{dx} = 3$

and $\frac{dy}{du} = 35u^4$.

This is much easier than expanding the brackets first and then differentiating.

Powers of a linear function

Here is the derivative of $(ax + b)^n$.

Let $y = (ax + b)^n$.
 Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
 $= n(ax + b)^{n-1} \times a$
 $= an(ax + b)^{n-1}$.

Let $u = ax + b$.

Then $y = u^n$.

Hence $\frac{du}{dx} = a$

and $\frac{dy}{du} = nu^{n-1}$.

This result occurs so often that it should be remembered as a formula for differentiating any linear function of x raised to a power.

17 Differentiating a power of a linear function

$$\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$$


Example 23 Differentiating a power of a linear function

Use the formula $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate each function. Convert the answers to parts (b) and (c) back to fraction or surd form.

a $y = (4x - 1)^7$

b $y = \frac{1}{3x - 8}$

c $y = \sqrt{5x + 3}$

Solution

a $y = (4x - 1)^7$

$$\begin{aligned}\frac{dy}{dx} &= 4 \times 7 \times (4x - 1)^6 \quad (\text{here } a = 4, b = -1 \text{ and } n = 7) \\ &= 28(4x - 1)^6\end{aligned}$$

b $y = (3x - 8)^{-1}$ (convert to index form)

$$\begin{aligned}\frac{dy}{dx} &= 3 \times (-1) \times (3x - 8)^{-2} \quad (\text{here } a = 3, b = -8 \text{ and } n = -1) \\ &= -\frac{3}{(3x - 8)^2}\end{aligned}$$

c $y = (5x + 3)^{\frac{1}{2}}$ (convert to index form)

$$\begin{aligned}\frac{dy}{dx} &= 5 \times \frac{1}{2} \times (5x + 3)^{-\frac{1}{2}} \quad (\text{here } a = 5, b = 3 \text{ and } n = \frac{1}{2}) \\ &= \frac{5}{2\sqrt{5x + 3}}\end{aligned}$$

The chain rule and tangents

The usual methods of dealing with tangents apply when the chain rule is used to find the derivative.


Example 24 The chain rule and tangents

Differentiate $y = (1 - x^4)^4$, and hence find the points on the curve where the tangent is horizontal.

Solution

$$\begin{aligned}y &= (1 - x^4)^4 \\ \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4(1 - x^4)^3 \times (-4x^3) \\ &= -16x^3(1 - x^4)^3,\end{aligned}$$

which is zero when $x = -1, 0$ or 1 ,
so the points are $(-1, 0)$, $(0, 1)$ and $(1, 0)$.

Let $u = 1 - x^4$.

Then $y = u^4$.

Hence $\frac{du}{dx} = -4x^3$

and $\frac{dy}{du} = 4u^3$.

A shorter setting-out of the chain rule

It is important to practise the full setting-out.

- That is the best way to handle many tricky calculations.
- The process will be reversed in Year 12 and must be clearly understood.

Many people like to shorten the setting-out — if so, write down at least the function u on the right, and take at least one middle step in the working.

**Example 25** Shorter setting out of the chain rule

Differentiate $y = (3 - 5x^3)^7$ with a shorter setting-out.

Solution

$y = (3 - 5x^3)^7$ $\frac{dy}{dx} = 7(3 - 5x^3)^6 \times (-15x^2)$ $= -105x^2(3 - 5x^3)^6$	$\text{Let } u = 3 - 5x^3.$ (further steps if it gets tricky)

Another approach to the chain rule?

For a composite function $h(x) = f(g(x))$, it is not difficult to see that the derivative is given by the formula:

$$h'(x) = f'(g(x))g'(x)$$

Again, always begin by writing down the functions $f(x)$ and $g(x)$ on the right.

**Example 26** Using another formula for the chain rule

Differentiate $y = (3 - 5x^3)^7$ using the formula above.

Solution

$h(x) = (3 - 5x^3)^7$ $h'(x) = f'(g(x))g'(x)$ $= -105x^2(3 - 5x^3)^6$	$\text{Let } f(x) = x^7,$ and $g(x) = 3 - 5x^3.$ Then $f'(x) = 7x^6,$ and $g'(x) = -15x^2.$

We do not recommend this approach ever in this course, and certainly not now.

Exercise 9G**FOUNDATION**

Note: Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ when the function is a power of a linear function. Otherwise use the full setting-out of the chain rule, or use a shorter form if you are quite confident to do so.

- 1** Copy and complete the setting out below to differentiate $(x^2 + 9)^5$ by the chain rule.

$\text{Let } y = (x^2 + 9)^5.$ $\text{Then } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= \dots \times \dots$ $= \dots$	$\text{Let } u = x^2 + 9.$ $\text{Then } y = u^5.$ $\text{Hence } \frac{du}{dx} = \dots$ and $\frac{dy}{du} = \dots$

- 2** Use the chain rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ to differentiate each function. In each example, use the full setting-out above, first identifying u as a function of x , and y as a function of u .

a $y = (3x + 7)^4$

b $y = (5x - 9)^6$

c $y = (5 - 4x)^7$

d $y = (1 - x)^4$

e $y = (x^2 + 1)^{12}$

f $y = (x^2 - 2)^7$

3 Use the full setting-out of the chain rule to differentiate these functions.

a $y = (7x + 2)^{-1}$

b $y = 3(x - 1)^{-2}$

c $y = (x^3 - 12)^{-4}$

d $y = (5x^2 - 2)^{-3}$

e $y = 8(7 - x^2)^4$

f $y = -3(x^3 + x + 1)^6$

4 Use the standard form $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:

a $y = (5x - 7)^5$

b $y = (7x + 3)^7$

c $y = 9(5x + 3)^4$

d $y = (4 - 3x)^7$

e $y = 11(3 - x)^2$

f $y = (4x - 5)^{-7}$

g $y = 2(3x + 7)^{-5}$

h $y = (10 - 3x)^{-4}$

i $y = 3(5 - 7x)^{-4}$

5 a Differentiate $y = (x - 3)^2$ by expanding the RHS and differentiating each term.

b Differentiate $y = (x - 3)^2$, using the chain rule.

6 a Differentiate $y = 3(1 - 2x)^2$ by expanding the RHS and differentiating each term.

b Differentiate $y = 3(1 - 2x)^2$, using the chain rule.

7 Find the first and second derivatives of $f(x) = (2x + 3)^5$.

DEVELOPMENT

8 Differentiate:

a $y = (5 - x^2)^3$

b $y = (3x^2 + 7)^7$

c $y = (x^4 + 1)^4$

d $y = (3x^3 - 7)^5$

e $y = (x^3 - x^2)^{-5}$

f $y = (x^2 + 3x + 1)^{-9}$

9 Rewrite each function with negative index notation. Then differentiate it using the formula $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$. Give each answer as a fraction without negative indices.

a $y = \frac{1}{2x + 7}$

b $y = \frac{1}{2 - x}$

c $y = \frac{1}{3 + 5x}$

d $y = \frac{7}{4 - 3x}$

e $y = \frac{4}{(3x - 1)^5}$

f $y = -\frac{5}{(x + 1)^3}$

10 a Find the derivative of $y = (5x - 4)^4$.

b Find the values of y and $\frac{dy}{dx}$ at the point P where $x = 1$.

c Hence find the equations of the tangent and normal at P .

11 Find the equation of the tangent to each curve at the point P where $x = 1$.

a $y = (x^2 + 1)^3$

b $y = (3x - 2)^{-1}$

c $y = (x^2 + 1)^{-1}$

12 Differentiate each function using the chain rule. Then solve $\frac{dy}{dx} = 0$ to find the x -coordinate and y -coordinate of any points where the tangent is horizontal.

a $y = (x - 5)^4$

b $y = (x^2 - 1)^3$

c $y = (2x + x^2)^5$

d $y = \frac{1}{5x + 2}$

e $y = 4 + (x - 5)^6$

f $y = \frac{1}{1 + x^2}$

13 Find the x -coordinates of any points on $y = (4x - 7)^3$ where the tangent is:

a parallel to $y = 108x + 7$,

b perpendicular to $y = -\frac{1}{12}x + 2$.

14 Differentiate each function, then find the gradient of the tangent at the point where $x = 2$.

a $y = (6x + 4)^{\frac{1}{2}}$

b $y = \sqrt{2x + 5}$

c $y = \sqrt{x^2 - 3}$

- 15** Differentiate each function, then find the x -coordinate and y -coordinate of any points where the tangent is horizontal.

a $y = \sqrt{3 - 2x}$

b $y = \sqrt{x^2 - 2x + 5}$

c $y = \sqrt{x^2 - 2x}$

- 16 a** Find the derivative of the semicircle $y = \sqrt{25 - x^2}$, and sketch the semicircle.
b Show that the tangent at $P(3, 4)$ is $3x + 4y = 25$.
c Verify that the tangent at P is perpendicular to the radius OP . Add the tangent and radius to your sketch.

CHALLENGE

- 17** Differentiate:

a $y = (\sqrt{x} - 3)^{11}$

b $y = 3\sqrt{4 - \frac{1}{2}x}$

c $y = \frac{3}{1 - x\sqrt{2}}$

d $y = (5 - x)^{-\frac{1}{2}}$

e $y = \frac{-a}{\sqrt{1 + ax}}$

f $y = \frac{b}{\sqrt{c - x}}$

g $y = -4\left(x + \frac{1}{x}\right)^4$

h $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^6$

- 18 a** Differentiate $y = (x - a)^3$, then find the value of a if $\frac{dy}{dx} = 12$ when $x = 6$.

b Differentiate $y = \frac{1}{x + a}$, then find the value of a if $\frac{dy}{dx} = -1$ when $x = 6$.

- 19** Differentiate $y = a(x + b)^2 - 8$, then find a and b if the parabola:

a passes through the origin with gradient 16,

b has tangent $y = 2x$ at the point $P(4, 8)$.

- 20 a** Find the equation of the tangent to $y = \frac{1}{x - 4}$ at the point L where $x = b$.

b Hence find the equation of the tangent to the curve passing through:

i the origin,

ii $W(6, 0)$.

9H The product rule

Learning intentions

- Use the product rule to differentiate a product of two functions.

The product rule extends the methods for differentiation to functions that are products of two simpler functions. For example,

$$y = x(x - 10)^4 \quad \text{is the product} \quad y = x \times (x - 10)^4,$$

and we shall decompose y as

$$y = uv, \quad \text{where } u = x \quad \text{and} \quad v = (x - 10)^4.$$

The product rule — differentiating the product of two functions

The proof of the product rule is given in the Appendix at the end of this chapter.

18 The product rule

Suppose that the function

$$y = u \times v$$

is the *product* of two functions u and v , each of which is a function of x . Then

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} \quad \text{or} \quad y' = vu' + uv'.$$

The second form uses the convention of the dash ' to represent differentiation with respect to x . That is, $y' = \frac{dy}{dx}$ and $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$.

Using the product rule

The working in the right-hand column is the recommended setting out. The first step is to decompose the function into the product of two simpler functions.



Example 27 Differentiating the product of two functions

Differentiate each function, writing the result in fully factored form. Then state for what value(s) of x the derivative is zero.

a $x(x - 10)^4$

b $x^2(3x + 2)^3$

Solution

a Let $y = x(x - 10)^4$.
Then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$$= (x - 10)^4 \times 1 + x \times 4(x - 10)^3$$

$$= (x - 10)^3(x - 10 + 4x)$$

$$= (x - 10)^3(5x - 10)$$

$$= 5(x - 10)^3(x - 2),$$

so the derivative is zero for $x = 10$ and for $x = 2$.

Let $u = x$
and $v = (x - 10)^4$.
Then $\frac{du}{dx} = 1$
and $\frac{dv}{dx} = 4(x - 10)^3$
(using the chain rule).

b Let $y = x^2(3x + 2)^3$.

Then $y' = vu' + uv'$
 $= (3x + 2)^3 \times 2x + x^2 \times 9(3x + 2)^2$
 $= x(3x + 2)^2(6x + 4 + 9x)$
 $= x(3x + 2)^2(15x + 4),$

so the derivative is zero for $x = 0$, $x = -\frac{2}{3}$ and for $x = -\frac{4}{15}$.

Let $u = x^2$

and $v = (3x + 2)^3$.

Then $u' = 2x$

and $v' = 9(3x + 2)^2$.

Using the chain rule first to differentiate one of the factors

The product rule is sometimes difficult to use with the algebraic functions under consideration at present, because the calculations can easily become quite involved. The rule will seem far more straightforward later in the context of exponential and trigonometric functions.

This happens particularly when one of the factors u or v needs to be differentiated first using the chain rule. See the Development Questions 12 and 13 below.

Exercise 9H

FOUNDATION

- 1** Copy and complete the setting out below to differentiate $5x(x - 2)^4$ by the product rule.

Let $y = 5x(x - 2)^4$.

Then $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$
 $= \dots \times \dots + \dots \times \dots$
 $= 5(x - 2)^4 + 20x(x - 2)^3$
 $= \dots$ (take out the common factor)
 $= 5(x - 2)^3(5x - 2)$

Let $u = 5x$

and $v = (x - 2)^4$.

Then $\frac{du}{dx} = \dots$

and $\frac{dv}{dx} = \dots$

- 2 a** Differentiate $y = x^3(x - 2)$ by expanding the product and differentiating each term.

- b** Differentiate $y = x^3(x - 2)$ using the product rule with $u = x^3$ and $v = x - 2$.
 Use the setting out shown in the worked exercises above.

- 3 a** Differentiate $y = (2x + 1)(x - 5)$ by expanding and then differentiating each term.

- b** Differentiate $y = (2x + 1)(x - 5)$ using the product rule with $u = 2x + 1$ and $v = x - 5$.

- 4 a** Differentiate $y = (x^2 - 3)(x^2 + 3)$ by expanding and then differentiating each term.

- b** Differentiate $y = (x^2 - 3)(x^2 + 3)$ using the product rule with $u = x^2 - 3$ and $v = x^2 + 3$.

- 5 a** Given that $u = x^4$ and $v = (2x - 1)^5$, find $\frac{du}{dx}$ and $\frac{dv}{dx}$.

- b** Use the product rule and the results of part (a) to show that if $y = x^4(2x - 1)^5$, then $\frac{dy}{dx} = 4x^3(2x - 1)^5 + 10x^4(2x - 1)^4$.

- c** By taking out the common factor $2x^3(2x - 1)^4$, show that $\frac{dy}{dx} = 2x^3(2x - 1)^4(9x - 2)$.

- d** Hence find the x -coordinates of the points on the curve where the tangent is horizontal.

- 6** Differentiate these functions by the product rule, giving your answers in factored form.

a $y = x(3x + 5)^3$

b $y = x^2(x - 1)^3$

c $y = x^4(1 - 5x)^6$

- 7** Show that the derivative of $y = x(1 - x)^6$ is $\frac{dy}{dx} = (7x - 1)(x - 1)^5$, then find the tangent and normal to the curve at the origin.

DEVELOPMENT

- 8** Differentiate these functions using the product rule, identifying the factors u and v in each example. Express the answers in fully factored form, and state the values of x for which the derivative is zero.

a $y = x(x - 1)^4$

b $y = x(x + 5)^5$

c $y = x(4 - 3x)^5$

d $y = x(3 - 2x)^5$

e $y = x^3(x + 1)^4$

f $y = x^3(3x - 2)^4$

g $y = x^5(1 - x)^7$

h $y = (x - 1)(x - 2)^3$

i $y = (x + 2)(x + 5)^6$

- 9 a** Show that the derivative of $y = x^3(1 - x)^5$ is $y' = x^2(1 - x)^4(3 - 8x)$.

b Hence find the points on the curve where the tangents are horizontal.

- 10 a** Show that the derivative of $y = (2x - 1)^3(x - 2)^4$ is $\frac{dy}{dx} = 2(2x - 1)^2(x - 2)^3(7x - 8)$.

b Hence find the tangent and normal to the curve at the point $A(1, 1)$.

- 11** Show that the derivative of $y = (x + 1)(x + 3)^3$ is $\frac{dy}{dx} = 2(2x + 3)(x + 3)^2$, then find the tangent and normal to the curve at $A(-1, 0)$.

- 12 a** Differentiate $y = (x^2 + 1)^5$ using the chain rule. Then use the product rule to differentiate $y = x(x^2 + 1)^5$.

b Differentiate $y = (1 - x^2)^4$. Hence differentiate $y = 2x^3(1 - x^2)^4$.

c Differentiate $y = (x^2 + x + 1)^3$, then differentiate $y = -2(x^2 + x + 1)^3x$.

d Similarly, differentiate $y = x(4 - 9x^4)^4$.

- 13 a** Differentiate $y = (x^2 - 10)^3x^4$, using the chain rule to differentiate the first factor.

b Hence find the points on the curve where the tangent is horizontal.

CHALLENGE

- 14** Differentiate each function using the product rule. Then combine terms using a common denominator, and factor the numerator completely. State the values of x for which the derivative is zero.

a $y = 6x\sqrt{x + 1}$

b $y = -4x\sqrt{1 - 2x}$

c $y = 10x^2\sqrt{2x - 1}$

- 15** Find the derivative of each function below, giving the answer in fully factored form. Then state the values of x for which the derivative is zero.

a $y = (x + 1)^3(x + 2)^4$

b $y = (2x - 3)^4(2x + 3)^5$

c $y = x\sqrt{1 - x^2}$

- 16 a** Using the product rule differentiate $y = a(x - 1)(x - 5)$, where $a > 0$ is a constant, then sketch the curve.

b Show that the tangents at the x -intercepts $(1, 0)$ and $(5, 0)$ have opposite gradients.

c Find the equations of these tangents and their point M of intersection.

d Find the point V where the tangent is horizontal.

e Show that M is vertically below V , and twice as far from the x -axis.

9I The quotient rule

Learning intentions

- Use the quotient rule to differentiate a quotient of two functions.

The quotient rule extends the formulae for differentiation to functions that are quotients of two simpler functions. For example,

$$y = \frac{2x+1}{2x-1} \quad \text{is the quotient of the two functions } 2x+1 \text{ and } 2x-1.$$

The quotient rule — differentiating the quotient of two functions

See the Appendix at the end of this chapter for the proof of the quotient rule.

19 The quotient rule

Suppose that the function

$$y = \frac{u}{v}$$

is the *quotient* of two functions u and v , each of which is a function of x . Then

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \quad \text{or} \quad y' = \frac{vu' - uv'}{v^2}.$$

Example 28 Differentiating the quotient of two functions

Differentiate each function, stating any values of x where the derivative is zero.

a $y = \frac{2x+1}{2x-1}$

b $y = \frac{x}{x^2+1}$

Solution

a Let $y = \frac{2x+1}{2x-1}$.
 Then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$$= \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$$

$$= \frac{-4}{(2x-1)^2}, \quad \text{which is never zero.}$$

b Let $y = \frac{x}{x^2+1}$.
 Then $y' = \frac{vu' - uv'}{v^2}$

$$= \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

$$= \frac{(1-x)(1+x)}{(x^2+1)^2}, \quad \text{which is zero when } x = 1 \text{ or } x = -1.$$

Let $u = 2x+1$
 and $v = 2x-1$.
 Then $\frac{du}{dx} = 2$
 and $\frac{dv}{dx} = 2$.

Let $u = x$
 and $v = x^2+1$.
 Then $u' = 1$
 and $v' = 2x$.

Note: Both these functions could have been differentiated using the product rule after writing them as $(2x+1)(2x-1)^{-1}$ and $x(x^2+1)^{-1}$. The quotient rule, however, makes the calculations much easier.

Exercise 9I

FOUNDATION

- 1 Copy and complete the setting out below to differentiate $\frac{2x+3}{3x+2}$ by the quotient rule.

$$\begin{aligned} \text{Let } y &= \frac{2x+3}{3x+2} \\ \text{Then } \frac{dy}{dx} &= \frac{v u' - u v'}{v^2} \\ &= \frac{\dots \times \dots - \dots \times \dots}{\dots} \\ &= \frac{-5}{(3x+2)^2} \end{aligned}$$

$$\begin{aligned} \text{Let } u &= 2x+3 \\ \text{and } v &= 3x+2 \\ \text{Then } u' &= \dots \\ \text{and } v' &= \dots \end{aligned}$$

- 2 Differentiate each function using the quotient rule. Identify u and v first, and use the setting out shown in the worked exercises and in question 1.

a $y = \frac{x}{x+1}$

c $y = \frac{x}{1-3x}$

e $y = \frac{x-2}{3x-2}$

g $y = \frac{2x-3}{2x-3}$

b $y = \frac{2x}{x+2}$

d $y = \frac{x+1}{x-1}$

f $y = \frac{x+2}{5-4x}$

h $y = \frac{5-4x}{5+4x}$

- 3 Differentiate each function, using the quotient rule. Express your answer in fully factored form, then state any values of x for which the tangent is horizontal.

a $y = \frac{x^2}{x+1}$

c $y = \frac{x^2}{1-x}$

e $y = \frac{x^2-1}{x^2+1}$

b $y = \frac{x}{3-x^2}$

d $y = \frac{x}{1-x^2}$

f $y = \frac{x^2-9}{x^2-4}$

- 4 Differentiate $y = \frac{1}{3x-2}$ in two different ways:

a Use the chain rule with $u = 3x-2$ and $y = \frac{1}{u}$. This is the better method.

b Use the quotient rule with $u = 1$ and $v = 3x-2$. This method is longer.

DEVELOPMENT

- 5 a Show that the derivative of $y = \frac{x}{5-3x}$ is $\frac{dy}{dx} = \frac{5}{(5-3x)^2}$.
 b Hence find the gradient of the tangent at $K(2, -2)$ and its angle of inclination.
 c Find the equations of the tangent and normal at K .
- 6 a Show that the derivative of $y = \frac{x^2-4}{x-1}$ is $\frac{dy}{dx} = \frac{x^2-2x+4}{(x-1)^2}$.
 b Hence find the gradient of the tangent at $L(4, 4)$ and its angle of inclination.
 c Find the equations of the tangent and normal at L .

- 7 a** Find the equation of the tangent to $y = \frac{x}{x+1}$ at the origin $O(0, 0)$.
- b** Show that the tangent to $y = \frac{x}{x+1}$ at the point $P(1, \frac{1}{2})$ is $y = \frac{1}{4}x + \frac{1}{4}$.
- c** Find the points A and B where the tangent in part (b) crosses the x -axis and y -axis respectively.
- d** Find the area of the triangle ABO , where O is the origin.
- e** Use simultaneous equations to find where the tangents at O and P intersect.

CHALLENGE

- 8 a** Differentiate $y = \frac{x^2}{x+1}$, and hence find the value of c if $y' = 0$ at $x = c$.
- b** Differentiate $y = \frac{x^2+k}{x^2-k}$, and hence find the value of k if $y' = 1$ at $x = -3$.
- 9** Differentiate, using the most appropriate method. Factor each answer completely.
- | | |
|------------------------------------|---|
| a $(3x-7)^4$ | b $\frac{x^2+3x-2}{x}$ |
| c $(2x+3)(2x-3)$ | d $\frac{1}{x^2-9}$ |
| e $x(4-x)^3$ | f $\frac{3-x}{3+x}$ |
| g $(x^4-1)^5$ | h $\frac{1}{\sqrt{2-x}}$ |
| i $(x^3+5)^2$ | j $\frac{x^2+x+1}{2\sqrt{x}}$ |
| k $\frac{2}{3}x^2(x^3-1)$ | l $\frac{x}{x+5}$ |
| m $x\sqrt{x} + x^2\sqrt{x}$ | n $\left(x - \frac{1}{x}\right)^2$ |
| o $x^3(x-1)^8$ | p $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ |
- 10 a** Show that the derivative of $y = \frac{x-a}{x-b}$ is $y' = \frac{a-b}{(x-b)^2}$ (where a and b are constants).
- b** Explain why all tangents have positive gradient when $a > b$, and all tangents have negative gradient when $a < b$.

9J Rates of change

Learning intentions

- Interpret the derivative as an instantaneous rate.
- Solve practical problems involving instantaneous and average rates.
- Find where a cubic function is stationary, increasing, and decreasing.
- Determine when a cubic function of time is increasing, and decreasing.

So far, differentiation has been developed geometrically in terms of tangents to curves. We now return to the very general initial idea in Section 9A that the derivative is a measure of how some quantity y is changing when y is a function of x and x is changing.

This section and next will interpret the derivative as a *rate of change* of some quantity Q that is a function of time t . Differentiation with respect to time will give the *instantaneous rate* at which the quantity Q is changing over time. This will allow us to analyse and solve problems in various practical situations.

Two important type of rates are velocity and acceleration, but we have left a fuller discussion of motion until Year 12 because motion requires a great deal more theory than we have at the moment. A couple of questions in the next exercise, however, indicate how velocity and acceleration are particular types of rates.

A quantity Q that is a cubic function of time is accessible, because the rate is a quadratic function. A few examples of when such a quantity Q is increasing or decreasing break the ice for a general approach to curve sketching next year.

Average rates and instantaneous rates

Suppose that a quantity Q is given as a function of time t , as in the diagram to the right. There are two types of rates:

- An *average rate of change corresponds to a secant*.

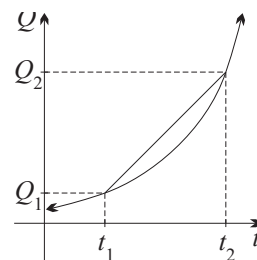
In the diagram, the quantity Q takes the value Q_1 at time t_1 , and the value Q_2 at time t_2 . The average rate of change from time t_1 to time t_2 is the usual gradient formula:

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- An *instantaneous rate of change corresponds to a tangent*.

The instantaneous rate of change at time t_1 is the value of the derivative $\frac{dQ}{dt}$ at time $t = t_1$:

$$\text{instantaneous rate} = \frac{dQ}{dt}, \text{ evaluated at } t = t_1.$$



One can see from the diagram that as the time t_2 gets closer to t_1 , the average rate from $t = t_1$ to $t = t_2$ gets closer to the instantaneous rate at t_1 (provided that the curve is continuous and smooth at $t = t_1$). This is exactly the way in which first-principles differentiation was defined using a limit in Section 9B.

20 Average and instantaneous rates of change

Suppose that a quantity Q is a function of time t .

- The *average rate of change* from the time $t = t_1$, when the value is Q_1 , to the time $t = t_2$, when the value is Q_2 , is the gradient of the secant:

$$\text{average rate} = \frac{Q_2 - Q_1}{t_2 - t_1}.$$

- The *instantaneous rate of change* at time $t = t_1$ is the gradient of the tangent, that is, the value of the derivative $\frac{dQ}{dt}$ evaluated at $t = t_1$:

$$\text{instantaneous rate} = \frac{dQ}{dt}, \text{ evaluated at } t = t_1.$$

- In this course, the unqualified phrase ‘rate of change’ will always mean the ‘instantaneous rate of change’.
- Take particular care to use the correct units for the quantity, the time, and the rate, in final answers.

In many problems, it is useful to sketch two graphs, one of the quantity Q as a function of time t , the other of the rate of change $\frac{dQ}{dt}$ also as a function of time t .

The quantity Q is usually replaced by a more convenient pronumeral, such as P for population, or V for volume, or M for mass.

**Example 29** Interpreting the derivative as a rate

A cockroach plague hit the suburb of Berrawong last year, but was gradually brought under control. The council estimated that the cockroach population P in millions, t months after 1st January, was given by

$$P = 7 + 6t - t^2.$$

- Differentiate to find the rate of change $\frac{dP}{dt}$ of the cockroach population.
- Find the cockroach population on 1st January, and the rate at which the population was increasing at that time. Be careful with the units.
- Find the cockroach population on 1st April, and the rate at which the population was increasing at that time.
- When did the council manage to stop the cockroach population increasing any further, and what was the population then?
- When were the cockroaches finally eliminated?
- What was the average rate of increase from 1st January to 1st April?
- Draw the graphs of P and $\frac{dP}{dt}$ as functions of time t , and state their domains and ranges. Add to your graph of P the tangents and secants corresponding to parts (b), (d) and (f).

Solution

- a** The population function was $P = 7 + 6t - t^2$.

Differentiating, the rate of change was $\frac{dP}{dt} = 6 - 2t$.

- b** When $t = 0$, $P = 7$,

and when $t = 0$, $\frac{dP}{dt} = 6$.

Thus on 1st January, the population was 7 million cockroaches, and the population was increasing at 6 million cockroaches per month. (The time t is in months, so the rate is ‘per month’.)

c On 1st April, $t = 3$, so $P = 16$ million cockroaches,

and $\frac{dP}{dt} = 0$ cockroaches/month.

d We found in part (b) that the population stopped increasing on 1st April, and that the population then was 16 million.

e To find the time when the population was zero,

put $P = 0$,

that is, $7 + 6t - t^2 = 0$

$\times (-1)$ $t^2 - 6t - 7 = 0$

$(t - 7)(t + 1) = 0$

$t = 7$ or -1 .

Hence $t = 7$, and the cockroaches were finally eliminated on 1st August.

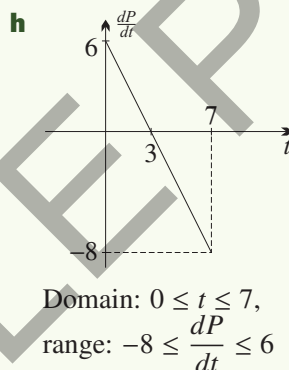
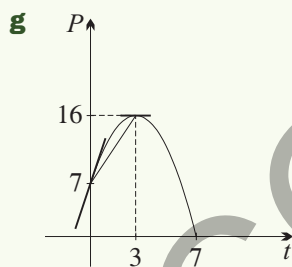
(The negative solution is irrelevant because council action started at $t = 0$.)

f From part (b), the population was 7 million on 1st January, and from part(c), the population was 16 million on 1st April.

Hence average rate of increase $= \frac{P_2 - P_1}{t_2 - t_1}$ (gradient of chord formula)

$$= \frac{16 - 7}{3}$$

$= 3$ million cockroaches per month.



Increasing, decreasing, and stationary

We can see from the left-hand graph above that the cockroach population was *increasing* when $0 < t < 3$ — every tangent to the left of $t = 3$ has positive gradient. For $3 < t < 7$, however, the cockroach population was *decreasing* — every tangent to the right of $t = 3$ has negative gradient.

At $t = 3$, the cockroach population was *stationary*, that is, it was neither increasing nor decreasing and the tangent is horizontal. These three words are used throughout calculus to describe functions, whether or not time is involved.

21 Increasing, decreasing, stationary

Suppose that $y = f(x)$ is defined at $x = a$, with $f'(a)$ also defined.

- If $f'(a)$ is positive (that is, if the tangent slopes upwards), then the curve is called *increasing* at $x = a$.
- If $f'(a)$ is negative (that is, if the tangent slopes downwards), then the curve is called *decreasing* at $x = a$.
- If $f'(a)$ is zero (that is, if the tangent is horizontal), then the curve is called *stationary* at $x = a$.

The diagram on the right above is a graph of the rate at which the population is changing. It is a straight line with negative gradient — this means that *the rate of change of the cockroach population was decreasing at a constant rate*.

Where is a cubic stationary, increasing, and decreasing?

These problems require first finding the stationary points, then drawing up table of test points dodging the zeroes of the quadratic dQ/dt . The principle being used here to solve the quadratic inequations was introduced in Chapters 3 and 4:

The quadratic dQ/dt can only change sign at zeroes.

Alternatively, sketch the graph and read the answer off the graph, as in Question 7 of the following exercise.

**Example 30 Where is a cubic stationary, increasing and decreasing?**

Consider the cubic function $Q = t^3 - 9t^2 + 15t + 1$.

- Find $\frac{dQ}{dt}$, and hence find the stationary points of the cubic.
- Solve the quadratic inequality $\frac{dQ}{dt} > 0$ by taking a table of test points of dQ/dt dodging around the stationary points.
- What mathematical principle is being used in part (b)?
- Hence state where the curve is increasing, and where is it decreasing.

Solution

$$\begin{aligned} \text{a } \frac{dQ}{dt} &= 3t^2 - 18t + 15 \\ &= 3(t-1)(t-5). \end{aligned}$$

Hence the stationary points are (1, 8) and (5, -24)

- We solve the inequations $\frac{dQ}{dt} > 0$ and $\frac{dQ}{dt} < 0$ using a table of values of dQ/dt .

t	0	1	2	5	6
dQ/dt	15	0	-9	0	15
gradient of Q	+	0	-	0	+

so $dQ/dt > 0$ for $t < 1$ and for $t > 5$, and $dQ/dt < 0$ for $1 < t < 5$.

- The quadratic dQ/dt can only change sign at zeroes..
- Hence Q is increasing for $t < 1$ and for $t > 5$, and is decreasing for $1 < t < 5$.

Questions with a diagram or a graph instead of an equation

In some problems about rates, the graph of some quantity Q over time is known, but its equation is unknown. Such problems require careful reading of the graph — pay attention to zeroes of the quantity and of the rate, and to where the quantity and rates are increasing, decreasing, and stationary.

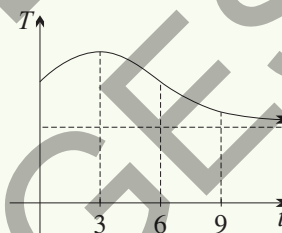
Occasionally, it may be often useful to draw an approximate sketch of the rate of change of the rate of change. This second rate-of-change graph is the derivative of the rate of change $\frac{dQ}{dt}$, and is thus the second derivative $\frac{d^2Q}{dt^2}$ of the quantity.



Example 31 Interpreting a graph of a rate with no function formula

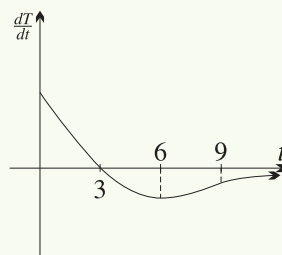
The graph to the right shows the temperature $T^\circ\text{C}$ of a patient suffering from Symond's syndrome at time t hours after her admission to hospital at midnight.

- When did her temperature reach its maximum?
- When was her temperature increasing most rapidly, when was it decreasing most rapidly, and when was it stationary?
- What happened to her temperature eventually?
- Sketch the graph of the rate $\frac{dT}{dt}$ at which the temperature is changing.
- When was the rate of change decreasing, and when was it increasing?



Solution

- The maximum temperature occurred when $t = 3$, that is, at 3:00 am.
- Her temperature was increasing most rapidly at the start when $t = 0$, that is, at midnight. It was decreasing most rapidly at 6:00 am. It was stationary at 3:00 am.
- The patient's temperature eventually stabilised.
- The graph of $\frac{dT}{dt}$ is zero at $t = 3$, and minimum at $t = 6$.
As $t \rightarrow \infty$, $\frac{dT}{dt} \rightarrow 0$, so the t -axis is a horizontal asymptote.
- The rate of change of temperature was decreasing from midnight to 6:00 am, and increasing from 6:00 am onwards.



Exercise 9J

FOUNDATION

Note: Except in questions 1–3, be careful to give the correct units in all final answers.

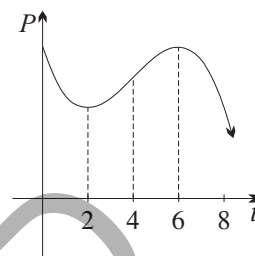
- If $Q = t^3 - 10t^2$, find the function $\frac{dQ}{dt}$ for the rate of change of Q .
 - Hence find the quantity Q and the rate $\frac{dQ}{dt}$ when $t = 2$.
- If $Q = t^2 + 6t$, find the function $\frac{dQ}{dt}$ for the rate of change of Q .
 - Hence find the quantity Q and the rate $\frac{dQ}{dt}$ when $t = 2$.
 - When is the quantity Q :
 - stationary,
 - increasing,
 - decreasing.

- 3 a** If $Q = 8t - t^2$, find the values of Q when $t = 1$ and $t = 3$.
b Hence find the average rate of change from $t = 1$ to $t = 3$.
c Similarly find the average rate of change from $t = 5$ to $t = 7$.
- 4** Orange juice is being poured at a constant rate into a glass. After t seconds there are V ml of juice in the glass, where $V = 60t$.
a How much juice is in the glass after 3 seconds?
b Show that the glass was empty to begin with.
c If the glass takes 5 seconds to fill, what is its capacity?
d Differentiate to find the rate at which the glass is being filled.
e What sort of function is the derivative in part (d)?
- 5** The volume V litres of fuel in a tanker, t minutes after it has started to empty, is given by $V = 200(400 - t^2)$. Initially the tanker was full.
a Find the initial volume of fuel in the tanker.
b Find the volume of fuel in the tanker after 15 minutes.
c Find the time taken for the tanker to empty.
d Show that $\frac{dV}{dt} = -400t$, and hence find the rate at which the tanker is emptying after 5 minutes.

DEVELOPMENT

- 6** [When is a cubic stationary, increasing or decreasing?]
 Consider the cubic function $Q = t^3 - 6t^2 + 9t$.
a Find $\frac{dQ}{dt}$ in factored form, and hence find the stationary points of the cubic.
b Solve the quadratic inequality $\frac{dQ}{dt} > 0$ by taking a table of test points of dQ/dt dodging around the stationary points.
c What principle did you use in part (b)?
d Hence state where the curve is increasing, and where is it decreasing.
- 7** [When is a cubic stationary, increasing or decreasing?]
a For $Q = t^3 - 3t^2$, find the rate of change $\frac{dQ}{dt}$, and factor it.
b Find the zeroes of Q , and the rate of change at those zeroes.
c Find the stationary points of the function Q .
d **i** Find Q when $t = -1$ and $t = 4$, and **ii** find dQ/dt when $t = -1$ and $t = 1$.
e Hence sketch the curve, and read off from it:
i When is Q positive, and when is Q negative?
ii When is Q increasing, and when is Q decreasing?

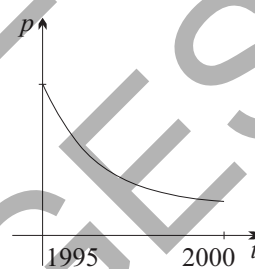
- 8** The graph to the right shows the share price P in Penn & Penn Stationery Suppliers t months after 1st January.



- a** When is the price stationary, neither increasing nor decreasing?
- b** When is the price maximum and when is it minimum?
- c** When is the price increasing and when is it decreasing?
- d** When is the share price increasing most rapidly?
- e** When is the share price increasing at an increasing rate?
- f** Sketch a possible graph of $\frac{dP}{dt}$ as a function of time t .

- 9** The graph shows the level of pollution in a river between 1995 and 2000. In 1995, the local council implemented a scheme to reduce the level of pollution in the river.

Comment briefly on whether this scheme worked and how the level of pollution changed. Include mention of the rate of change, and a sketch of the graph of the rate $\frac{dp}{dt}$.



- 10** Water surges into a rock pool, and then out again. The mass M in kilograms of water in the pool t seconds after time zero is given by the function $M = 10t - t^2$.

- a** Find the rate $\frac{dM}{dt}$ as a function of time t .
- b** Find (with units) the values of M and $\frac{dM}{dt}$ when $t = 4$.
- c** Find the value of M when $t = 2$, and hence find the average rate of change over the time interval from $t = 2$ to $t = 4$.
- d** At what times is M zero?
- e** For how long was there water in the pool?
- f** How long after time zero is $\frac{dM}{dt}$ zero?
- g** For how long was the water level in the pool rising, and for how long was it falling?
- h** Sketch the graphs of M and $\frac{dM}{dt}$ as functions of t , showing these results. Add the secant corresponding to the result in part (c) and the tangent corresponding to the result in part (f).

- 11** The share price $\$P$ of the Eastcom Bank t years after it opened on 1st January 1970 was $P = -0.4t^2 + 4t + 2$.
- What was the initial share price?
 - What was the share price after one year?
 - At what rate was the share price increasing after two years?
 - Find when the share price was stationary.
 - Explain why the maximum share price was $\$12$, at the start of 1975.
 - Explain why the rate of change of the price decreased at a constant rate.
 - The directors decided to close the bank when the share price fell back to its initial value. When did this happen?
- 12** For a certain brand of medicine, the amount M present in the blood after t hours is given by $M = 3t^2 - t^3$, for $0 \leq t \leq 3$.
- Sketch a graph of M as a function of t .
 - When is the amount of medicine in the blood a maximum?
 - Differentiate to find the rate $\frac{dM}{dt}$, and find the axis of symmetry of the resulting parabola.
 - When is the amount of medicine increasing most rapidly?
- 13** A lightning bolt hits the ground with a tremendous noise. The noise spreads out across the suburbs in a circle of area A , startling people.
- The speed of sound in air is about $\frac{1}{3}$ km/sec. Show that the area A in which people are startled as a function of the time t in seconds after the strike is $A = \frac{\pi}{9}t^2$.
 - Find the rate at which the area of startled people is increasing.
 - Find, correct to four significant figures, the time when the area is 5 km^2 , and the rate at which the area is increasing at that time.

CHALLENGE

- 14** A 3000 cm long water trough has the shape of a triangular prism. Its cross-section is an isosceles triangle with apex 90° , and the trough is built with the apex at the bottom.
- Find the area of an isosceles triangle with apex angle 90° , and height h centimetres.
 - Find the volume of water in the water trough when the water height is h cm.
 - The water trough was initially empty, and then was filled with water at a constant rate of 27 litres per minute.
 - Find the height h cm of the water after t minutes, and the rate the water was rising.
 - Find the water height, and the rate the water was rising, after 25 minutes.
 - Find when the water height was 30 cm, and the rate the water was then rising.

9K Average velocity and average speed

Learning intentions

- Understand the displacement of a particle moving on a line as a function of time.
- Calculate average velocity, and interpret it as the gradient of a secant.
- Calculate distance, distance travelled, and average speed.

Velocity is also a rate, but the ideas and language of motion are quite different from the rates of the previous section, and readers may like to delay these two sections until later in the year. Motion will be covered far more thoroughly in Year 12 after we have developed the necessary machinery.

These last two section's set up motion in one dimension, using a function or graph to relate the time and displacement of an object. Average velocity is the gradient of a secant on the graph, and instantaneous velocity is the gradient of a tangent.

Motion in one dimension — displacement and time

When a particle is moving in one dimension, turn the line that it is moving on into a number line with an origin and a positive direction. Its *displacement* x at any time t after *time zero* is then read off the number line. Thus its motion is described by giving x as a function of the *time* t , or by drawing the graph.

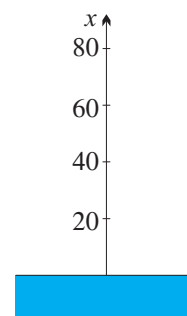
Suppose, for example, that a ball is hit vertically upwards from ground level, and lands 8 seconds later in the same place. Its motion can be described approximately by the following equation of motion and table of values,

$$x = 5t(8 - t)$$

t	0	2	4	6	8
x	0	60	80	60	0

Here x is the height in metres of the ball above the ground t seconds after it is thrown. The diagram to the right shows the path of the ball up and down along the same vertical line.

- This vertical line has thus been made into a number line, with the ground as the origin, upwards as the positive direction, and metres as the units of displacement.
- Time has also become a number line. The origin of time is when the ball is thrown, and the units of time are seconds.

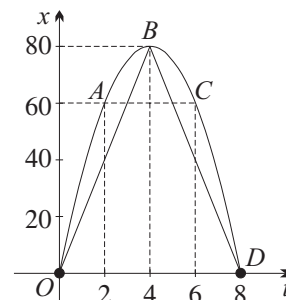


Graphing the equation of motion

The graph to the right is the resulting graph of the equation of motion $x = 5t(8 - t)$. The horizontal axis is time, and the vertical axis is displacement — *the graph must not be mistaken as a picture of the ball's path*.

The graph is part of a parabola with vertex $(4, 80)$, so the ball reaches maximum height 80 metres after 4 seconds. When $t = 8$, the height is zero, and the ball returns to the ground. The equation of motion therefore has restricted domain and range:

$$\text{Domain: } 0 \leq t \leq 8 \quad \text{and} \quad \text{Range: } 0 \leq x \leq 80.$$



Most equations of motion have this sort of restriction on the domain of t . In particular, *it is a convention of this course that negative values of time are excluded unless the question specifically allows it*.

22 Motion in one dimension

- Motion in one dimension is specified by making the line into a number line, and giving the *displacement* x as a function of time t after time zero.
- Negative values of time are excluded unless otherwise stated.



Example 32 Working with an equation of motion

Consider the example above, where $x = 5t(8 - t)$.

- a** Find the height of the ball after 1 second.
b At what other time is the ball at this same height above the ground?

Solution

- a** When $t = 1$, $x = 5 \times 1 \times 7$

$$= 35.$$

Hence the ball is 35 metres above the ground after 1 second.

- b** To find when the height is 35 metres, solve the equation $x = 35$.

Substituting into $x = 5t(8 - t)$ gives

$$5t(8 - t) = 35$$

$$\boxed{\div 5}$$

$$t(8 - t) = 7$$

$$8t - t^2 - 7 = 0$$

$$\boxed{\times (-1)}$$

$$t^2 - 8t + 7 = 0$$

$$(t - 1)(t - 7) = 0$$

$$t = 1 \text{ or } 7.$$

Hence the ball is 35 metres high after 1 second, and again after 7 seconds.

Average velocity

During its ascent, the ball in the worked example above moved 80 metres upwards. This is a change in displacement of +80 metres in 4 seconds, giving an average velocity of $\frac{80}{4} = 20$ metres per second.

Average velocity thus equals the *gradient of the secant* OB on the displacement–time graph — but be careful, because there are different scales on the two axes. Hence the formula for average velocity is the familiar gradient formula.

23 Average velocity

Suppose that a particle has displacement $x = x_1$ at time $t = t_1$, and displacement $x = x_2$ at time $t = t_2$.

Then

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}.$$

That is, on the displacement–time graph,

$$\text{average velocity} = \text{gradient of the secant}.$$

During its descent, the ball moved 80 metres downwards in 4 seconds, which is a change in displacement of $0 - 80 = -80$ metres. The average velocity is therefore -20 metres per second, which is equal to the gradient of the secant BD .



Example 33 Finding average velocities

Consider again the example $x = 5t(8 - t)$. Find the average velocities of the ball:

a during the first second,

b during the fifth second.

Solution

The first second stretches from $t = 0$ to $t = 1$, and the fifth second stretches from $t = 4$ to $t = 5$. The displacements at these times are given in the table to the right.

t	0	1	4	5
x	0	35	80	75

a Average velocity during 1st second

$$\begin{aligned}
 &= \frac{x_2 - x_1}{t_2 - t_1} \\
 &= \frac{35 - 0}{1 - 0} \\
 &= 35 \text{ m/s.}
 \end{aligned}$$

b Average velocity during 5th second

$$\begin{aligned}
 &= \frac{x_2 - x_1}{t_2 - t_1} \\
 &= \frac{75 - 80}{5 - 4} \\
 &= -5 \text{ m/s.}
 \end{aligned}$$

Distance travelled

The change in displacement can be positive, negative, or zero. *Distance*, however, is always positive or zero. In the previous example, the change in displacement during the 4 seconds from $t = 4$ to $t = 8$ is -80 metres, but the distance travelled is 80 metres.

The *distance travelled* by a particle also takes into account any journey and return. Thus the total distance travelled by the ball is $80 + 80 = 160$ metres — even though the ball's change in displacement over the first 8 seconds is zero, because the ball is back at its original position on the ground.

24 Distance travelled

- The *distance travelled* takes into account any journey and return.
- Distance travelled, like all distances, can never be negative.

Average speed

The *average speed* is the distance travelled divided by the time taken. Average speed, unlike average velocity, can never be negative.

25 Average Speed

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

Average speed, like all speeds, can never be negative.

During the 8 seconds of its flight, the change in displacement of the ball is zero, but the distance travelled is 160 metres, so

$$\begin{aligned}
 \text{average velocity} &= \frac{0 - 0}{8 - 0} \\
 &= 0 \text{ m/s,}
 \end{aligned}$$

$$\begin{aligned}
 \text{average speed} &= \frac{160}{8} \\
 &= 20 \text{ m/s.}
 \end{aligned}$$


Example 34 Contrasting average velocity and average speed

Consider once again the original example $x = 5t(8 - t)$. Find the average velocity and the average speed of the ball:

a during the eighth second,

b during the last six seconds.

Solution

The eighth second stretches from $t = 7$ to $t = 8$ and the last six seconds stretch from $t = 2$ to $t = 8$. The displacements at these times are given in the table to the right.

t	0	2	7	8
x	0	60	35	0

a During the eighth second, the ball moves 35 metres down from $x = 35$ to $x = 0$.

$$\begin{aligned}\text{Hence average velocity} &= \frac{0 - 35}{8 - 7} \\ &= -35 \text{ m/s.}\end{aligned}$$

Also distance travelled = 35 metres,

so average speed = 35 m/s.

b During the last six seconds, the ball rises 20 metres from $x = 60$ to $x = 80$, and then falls 80 metres from $x = 80$ to $x = 0$.

$$\begin{aligned}\text{Hence average velocity} &= \frac{0 - 60}{8 - 2} \\ &= -10 \text{ m/s.}\end{aligned}$$

Also distance travelled = $20 + 80$
= 100 metres,

$$\begin{aligned}\text{so average speed} &= \frac{100}{6} \\ &= 16\frac{2}{3} \text{ m/s.}\end{aligned}$$

Exercise 9K
FOUNDATION

1 A particle is moving with displacement function $x = t^2 + 2$, where time t is in seconds and displacement x is in metres.

a Find the position when $t = 0$.

b Find the position when $t = 4$.

c Find the average velocity during the first 4 seconds, using the definition:

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}.$$

2 For each displacement function below, copy and complete the table of values to the right. Hence find the average velocity during the first 2 seconds. The units in each part are seconds and centimetres.

t	0	2
x		

a $x = 12t - t^2$

b $x = (t - 2)^2$

c $x = t^3 - 4t + 3$

d $x = 2^t$

- 3** A particle moves according to the equation $x = t^3 - 4t$, where x is the displacement in centimetres from the origin O at time t seconds after time zero.

a Copy and complete the table of values to the right.

b Hence find the average velocity:

- i** during the first second (that is, from $t = 0$ to $t = 1$),
- ii** during the second second (that is, from $t = 1$ to $t = 2$),
- iii** during the third second (that is, from $t = 2$ to $t = 3$),
- iv** during the fourth second (that is, from $t = 3$ to $t = 4$).

t	0	1	2	3	4
x					

- 4** A particle moves according to the equation $x = t^2 - 4$, where x is the displacement in metres from the origin O at time t seconds after time zero.

a Copy and complete the table of values to the right.

b Hence find the average velocity:

- i** during the first second,
- ii** during the first two seconds,
- iii** during the first three seconds,
- iv** during the third second.

c Use the table of values above to sketch the displacement–time graph. Then add the secants corresponding to the average velocities calculated in part (b).

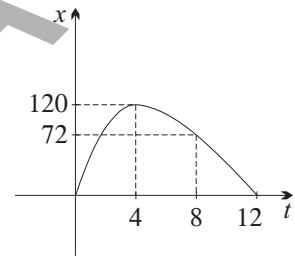
t	0	1	2	3
x				

- 5** A piece of cardboard is shot 120 metres vertically into the air by an explosion and floats back to the ground, landing at the same place. The graph to the right gives its height x metres above the ground t seconds after the explosion.

a When was the cardboard ascending, and when descending?

b Copy and complete the following table of values.

t	0	4	8	12
x				



c What is the total distance travelled by the cardboard?

d Find the average speed of the cardboard during its travels, using the formula

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}.$$

e Find the average velocity during:

- i** the ascent,
 - ii** the descent,
 - iii** the full 12 seconds.
- f** At roughly what other time was the height the same as the height when $t = 8$?
- g** What are the domain and range of the equation of motion in this graph?

- 6** A particle moves according to the equation $x = 4t - t^2$, where distance is in metres and time is in seconds.

a Copy and complete the table of values to the right.

t	0	1	2	3	4
x					

b Hence sketch the displacement–time graph.

c Find the total distance travelled during the first 4 seconds.

Then find the average speed during this time.

d Find the average velocity during the time:

i from $t = 0$ to $t = 2$,

ii from $t = 2$ to $t = 4$,

iii from $t = 0$ to $t = 4$.

e Add to your graph the secants corresponding to the average velocities in part (d).

DEVELOPMENT

- 7** Michael the mailman rides his bicycle 1 km up a hill at a constant speed of 10 km/hr. He then turns around and rides back down the hill at a constant speed of 30 km/hr.

a How many minutes does he take to travel:

i the first kilometre, when he is riding up the hill,

ii the second kilometre, when he is riding back down again?

b Use these values to draw a displacement–time graph, with the time axis in minutes.

c What is his average speed over the total 2 km journey?

d What is the average of his speeds up and down the hill?

- 8** Eleni is practising reversing in her driveway. Starting 8 metres from the gate, she reverses to the gate, and pauses. Then she drives forward 20 metres, and pauses. Then she reverses to her starting point. The graph to the right shows her distance x in metres from the front gate after t seconds.

a What is her average velocity:

i during the first 8 seconds,

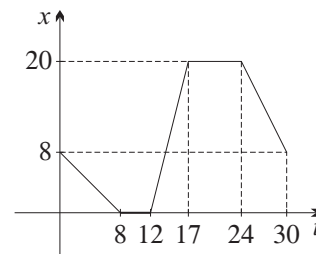
ii while she is driving forwards,

iii while she is reversing the second time?

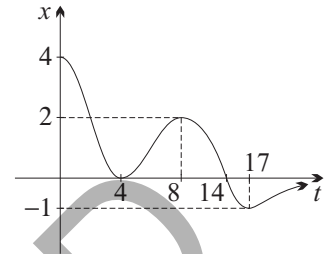
b Find the total distance she travelled, and her average speed, over the 30 seconds.

c Find her change in displacement, and her average velocity, over the 30 seconds.

d What would her average speed have been had she not paused at the gate and garage?



- 9** A girl is leaning over a bridge 4 metres above the water, playing with a weight on the end of a spring. The graph shows the height x in metres of the weight above the water as a function of time t seconds after she first drops it.



- a** How many times is the weight:
- i** at $x = 3$, **ii** at $x = 1$, **iii** at $x = -\frac{1}{2}$?
- b** At what times is the weight:
- i** at the water surface, **ii** above the water surface?
- c** How far above the water does it rise again after it first touches the water, and when does it reach this greatest height?
- d** What is the weight's greatest depth under the water and when does it occur?
- e** What happens to the weight eventually?
- f** What is its average velocity:
- i** during the first 4 seconds, **ii** from $t = 4$ to $t = 8$, **iii** from $t = 8$ to $t = 17$?
- g** What distance does it travel:
- i** over the first 4 seconds, **ii** over the first 8 seconds,
 - iii** over the first 17 seconds, **iv** eventually?
- h** What is its average speed over the first:
- i** 4, **ii** 8, **iii** 17 seconds?
- i** What are the domain and range of the equation of motion in this graph?

CHALLENGE

- 10** A particle moves according to the equation $x = 2\sqrt{t}$, for $t \geq 0$, where distance x is in centimetres and time t is in seconds.

a Copy and complete the table of values to the right, and sketch the curve.

b Hence find the average velocity as the particle moves:

- i** from $x = 0$ to $x = 2$, **ii** from $x = 2$ to $x = 4$,
- iii** from $x = 4$ to $x = 6$, **iv** from $x = 0$ to $x = 6$.

c What does the equality of the answers to parts (ii) and (iv) of part (b) tell you about the corresponding secants in part (c)?

t					
x	0	2	4	6	8

9L Instantaneous velocity and speed

Learning intentions

- Interpret instantaneous velocity as the gradient of a tangent.
- Use the derivative to solve problems about instantaneous velocity and speed.

If I drive the 160 km from Sydney to Newcastle in 2 hours, my *average velocity* is 80 km per hour. But my *instantaneous velocity* during the journey, as displayed on the speedometer, may range from zero at traffic lights to 110 km per hour on expressways. On the displacement–time graph:

- An average velocity corresponds to the gradient of a secant.
- An instantaneous velocity corresponds to the gradient of a tangent.

Instantaneous velocity as a derivative

Thus, as we have seen many times before in this chapter, we can find the instantaneous velocity by differentiating the displacement–time function. The usual symbol for instantaneous velocity is v , but it can also be written as dx/dt , or as $\dot{x}$.

26 Instantaneous velocity

- The *instantaneous velocity* v of a particle in motion is the gradient of the tangent on the displacement–time graph,

$$v = \frac{dx}{dt} \quad \text{which can also be written as} \quad v = \dot{x}.$$

- The dot over any symbol means *differentiation only with respect to time t* .

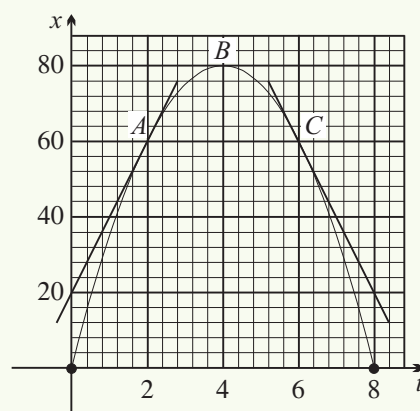
The notation $\dot{x}$, originally introduced by Newton, is yet another way of writing the derivative. The dot over the x , or over any symbol, stands for differentiation only with respect to time t . Thus the symbols v , $\frac{dx}{dt}$ and $\dot{x}$ all mean the velocity.



Example 35 Constructing the velocity–time graph

Here again is the displacement–time graph of the ball moving with equation $x = 5t(8 - t)$.

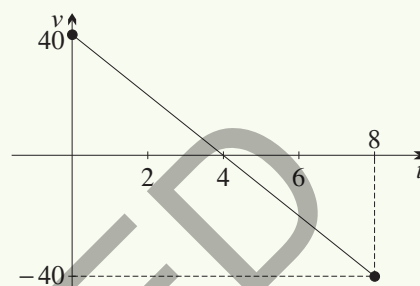
- Differentiate to find the equation for the velocity v , draw up a table of values at 2-second intervals, and sketch the velocity–time graph.
- Measure the gradients of the tangents that have been drawn at A , B , and C on the displacement–time graph, and compare your answers with the table of values in part (a).
- With what velocity was the ball originally hit?
- What is its impact speed when it hits the ground?



Solution

- a** Expanding, the motion equation is $x = 40t - 5t^2$,
and differentiating, $v = 40 - 10t$.
The graph of velocity is a straight line, with v -intercept 40 and gradient -10 .

t	0	2	4	6	8
v	40	20	0	-20	-40



- b** These values agree with the measurements of the gradients of the tangents at A where $t = 2$, at B where $t = 4$, and at C where $t = 6$.
(Be careful to take account of the different scales on the two axes.)
- c** When $t = 0$, $v = 40$, so the ball was originally hit upwards at 40 m/s.
- d** When $t = 8$, $v = -40$, so the ball hits the ground again at 40 m/s.

Instantaneous speed

The *instantaneous speed* is the absolute value $|v|$ of the velocity. This means that instantaneous speed, like average speed, can never be negative.

27 Instantaneous speed

- The *instantaneous speed* is the absolute value $|v|$ of the instantaneous velocity.
- From now on, the *unqualified words velocity and speed*, written on their own, will always mean instantaneous velocity and instantaneous speed.

Vector and scalar quantities

- Displacement and velocity are called *vector quantities*, meaning that they have a direction built into them. In the example above, a negative velocity means the ball is going downwards, and a negative displacement would mean it was below ground level.
- Distance and speed, however, are called *scalar quantities* — distance is the magnitude of the displacement, and speed is the magnitude of the velocity — and neither can be negative.

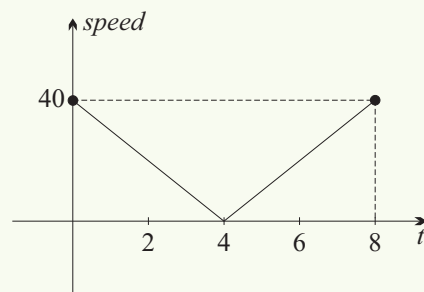
**Example 36 Constructing the graph of instantaneous speed**

From the previous worked example, construct the graph of the instantaneous speed as a function of time.

Solution

We are graphing $|v|$, so this is the graph of
speed = $|40 - 10t|$.

Section 5E explained how to graph such a function.



**Example 37** Contrasting vector quantities with scalar quantities

A particle moves with displacement $x = t^3 - 6t - 2$, in units of metres and seconds.

- a** Differentiate to find the velocity function.
b Find the displacement, distance from the origin, velocity, and speed when:
i $t = 0$, **ii** $t = 3$.

Solution

a The displacement equation is $x = t^3 - 6t - 2$,
 and differentiating, $v = 3t^2 - 6$.

b i When $t = 0$, $x = -2$
 and $v = -6$.

Thus when $t = 0$, the displacement is -2 metres, the particle is 2 metres from the origin, the velocity is -6 m/s, and the speed is 6 m/s.

ii When $t = 3$, $x = 27 - 18 - 2$
 $= 7$,
 and $v = 27 - 6$
 $= 21$.

Thus when $t = 3$, the displacement is 7 metres, the particle is 7 metres from the origin, the velocity is 21 m/s, and the speed is 21 m/s.

Finding when a particle is stationary

A particle is said to be *stationary* when its velocity v is zero, that is, when $\frac{dx}{dt} = 0$. This is the origin of the term ‘stationary point’ that we have been using to describe a point on a graph where the derivative is zero. For example, the ball in the first example was stationary for an instant at the top of its flight when $t = 4$, because the velocity was zero at the instant when its motion changed from upwards to downwards.

28 Finding when a Particle is Stationary

- A particle is *stationary* when its velocity is zero.
- To find when a particle is stationary, put $v = 0$ and solve for t .

**Example 38** Finding when a particle is stationary

A particle moves so that its distance in metres from the origin at time t seconds is given by $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$.

- a** Find the times when the particle is stationary.
b Find its distance from the origin at these times.

Solution

a The displacement function is $x = \frac{1}{3}t^3 - 6t^2 + 27t - 18$,
 and differentiating, $v = t^2 - 12t + 27$
 $= (t - 3)(t - 9)$,

so the particle is stationary after 3 seconds and after 9 seconds.

b When $t = 3$, $x = 9 - 54 + 81 - 18$
 $= 18$,

and when $t = 9$, $x = 243 - 486 + 243 - 18$
 $= -18$.

Thus the particle is 18 metres from the origin on both occasions.

Exercise 9L

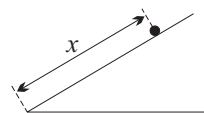
FOUNDATION

Note: Most questions in this exercise are long in order to illustrate how the physical situation of the particle's motion is related to the mathematics and the graph. The calculations should now be well known, but the physical interpretations can be confusing.

- 1** A particle is moving with displacement function $x = 20 - t^2$, in units of metres and seconds.
 - a** Differentiate to find the velocity v as a function of time t .
 - b** Find the displacement and velocity when $t = 3$.
 - c** What are the distance from the origin and the speed when $t = 3$?
- 2** For each displacement function below, differentiate to find the velocity function v . Then find the displacement and velocity when $t = 1$. The units are metres and seconds.
 - a** $x = 5t^2 - 10t$
 - b** $x = 3t - 2t^3$
 - c** $x = t^4 - t^2 + 4$
- 3** A particle's displacement function is $x = t^2 - 10t$, in units of centimetres and seconds.
 - a** Differentiate to find the velocity $v = \dot{x}$ as a function of t .
 - b** What are the displacement, the distance from the origin, the velocity, and the speed after 3 seconds?
 - c** When is the particle stationary, and where is it then?
- 4** A particle moves on a horizontal line so that its displacement x cm to the right of the origin at time t seconds is $x = t^3 - 6t^2$.
 - a** Differentiate to find $\dot{x}$ as a function of t .
 - b** Where is the particle initially, and what is its speed then?
 - c** At time $t = 3$, is the particle to the left or to the right of the origin?
 - d** At time $t = 3$, is the particle travelling to the left or to the right?
 - e** Show that the particle is stationary when $t = 4$, and find where it is at this time.
 - f** Show that the particle is at the origin when $t = 6$, and find its velocity and speed at this time.
- 5** A cricket ball is thrown vertically upwards. Its height x in metres at time t seconds after it is thrown is given by $x = 20t - 5t^2$.
 - a** Find v as a function of t . Then sketch graphs of x , and of v , as functions of t .
 - b** Write down the domain and range of the displacement and velocity functions.
 - c** Find the speed at which the ball was thrown.
 - d** Find when it returns to the ground (that is, when $x = 0$), and show that its speed then is equal to the initial speed.
 - e** Find its maximum height above the ground, and the time taken to reach this height.

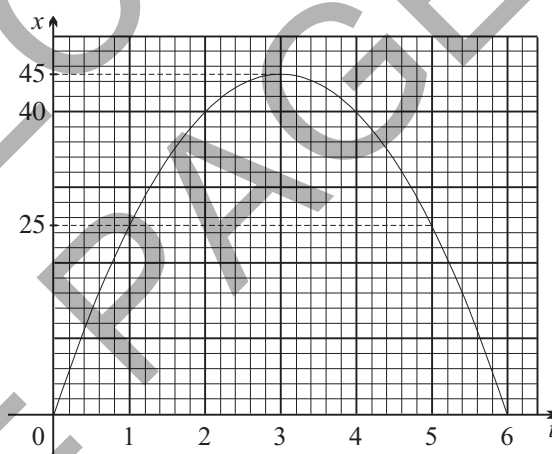
DEVELOPMENT

- 6** A smooth piece of ice is projected up a smooth inclined surface, as shown to the right. Its distance x in metres up the surface at time t seconds is $x = 6t - t^2$.

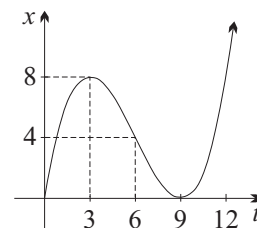


- Find the velocity v as a function of t .
- Sketch the graphs of displacement x and velocity v .
- In which direction is the ice moving:
 - when $t = 2$?
 - when $t = 4$?
- When is the ice stationary, for how long is it stationary, and where is it then?
- Show that the average velocity over the first 2 seconds is 4 m/s. Then find the time and place at which the instantaneous velocity equals this average velocity.
- Show that the average speed during the first 3 seconds, the next 3 seconds, and the first 6 seconds are all the same.

- 7** A stone was thrown vertically upwards. The graph to the right shows its height x metres at time t seconds after it was thrown.

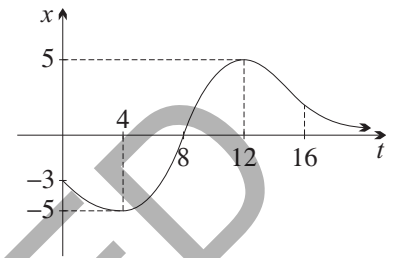


- What was the stone's maximum height, how long did it take to reach it, and what was its average speed during this time?
 - Draw tangents and measure their gradients to find the velocity of the stone at times $t = 0, 1, 2, 3, 4, 5$ and 6.
 - For what length of time was the stone stationary at the top of its flight?
 - Estimate the speed at $t = 5$.
 - Estimate the average speed from $t = 1.5$ to $t = 4.5$.
 - The graph is concave down everywhere. How is this relevant to the motion?
 - Draw graphs of the velocity, and the speed, of the stone from $t = 0$ to $t = 6$.
 - What does velocity–time graph tell you about what happened to the velocity during these 6 seconds?
- 8** A particle is moving horizontally so that its displacement x metres to the right of the origin at time t seconds is given by the graph to the right.
- In the first 10 seconds, what is its maximum distance from the origin and when does it occur?
 - By examining the gradient, find when the particle is:
 - stationary,
 - moving to the right,
 - moving to the left.
 - When does it return to the origin, and what is its velocity then?
 - At about what times are:
 - the displacement,
 - the velocity,
 about the same as those at $t = 2$?
 - Sketch (roughly) the graphs of the velocity v and the speed.



CHALLENGE

- 9 A particle is moving vertically according to the graph shown to the right, where upwards has been taken as positive.



- a** At what times is this particle:
- i** below the origin,
 - ii** moving downwards.
- b** At about what time is its speed greatest?
- c** At about what times are:
- i** the distance from the origin,
 - ii** the velocity,
- about the same as those at $t = 3$?
- d** How many times between $t = 4$ and $t = 12$ is the instantaneous velocity equal to the average velocity during this time?
- e** How far will the particle eventually travel?
- f** Draw an approximate sketch of the graph of v as a function of time.

Chapter 9 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 9 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Use the definition of the derivative $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of each function by first principles.
 - $f(x) = x^2 + 5x$
 - $f(x) = 6 - x^2$
 - $f(x) = 3x^2 - 2x + 7$
- Write down the derivative of each function. You will need to expand any brackets.
 - $y = x^3 - 2x^2 + 3x - 4$
 - $y = x^6 - 4x^4$
 - $y = 3x^2(x - 2x^3)$
 - $y = (x + 3)(x - 2)$
 - $y = (2x - 1)(2 - 3x)$
 - $y = 3x^{-2} - 2x^{-1}$
 - $y = 4x^3 - 4x^{-3}$
 - $y = 3x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$
 - $y = x^{-2}(x^2 - x + 1)$
- Find the first and second derivatives of each function:
 - $f(x) = x^4 + x^3 + x^2 + x + 1$
 - $f(x) = 5x^{-2}$
 - $f(x) = 8x^{-\frac{1}{2}}$
- Find the family of curves whose derivative is:
 - $\frac{dy}{dx} = 3x^2 + 4$
 - $\frac{dy}{dx} = 7 - 12x - 12x^2$
 - $\frac{dy}{dx} = 20x^4 - 12x^2 + 4$
- Rewrite each function in index notation and then differentiate it. Write each final answer without negative or fractional indices.
 - $y = \frac{3}{x}$
 - $y = \frac{1}{6x^2}$
 - $y = 7\sqrt{x}$
 - $y = \sqrt{144x}$
 - $y = -3x\sqrt{x}$
 - $y = \frac{6}{\sqrt{x}}$
- Divide each function through by the numerator and then differentiate it. Give each final answer without negative or fractional indices.
 - $y = \frac{3x^4 - 2x^3}{x^2}$
 - $y = \frac{x^3 - x^2 + 7x}{2x}$
 - $y = \frac{5x^3 - 7}{x}$
 - $y = \frac{x^2 + 2x + 1}{x^2}$
 - $y = \frac{4x + 5\sqrt{x}}{\sqrt{x}}$
 - $y = \frac{2x^2\sqrt{x} + 3x\sqrt{x}}{x}$

- 7** Use the formula $\frac{d}{dx}(ax + b)^n = an(ax + b)^{n-1}$ to differentiate:
- a** $y = (3x + 7)^3$ **b** $y = (5 - 2x)^2$ **c** $y = \frac{1}{5x - 1}$
d $y = \frac{1}{(2 - 7x)^2}$ **e** $y = \sqrt{5x + 1}$ **f** $y = \frac{1}{\sqrt{1 - x}}$
- 8** Use the chain rule to differentiate:
- a** $y = (7x^2 - 1)^3$ **b** $y = (1 + x^3)^{-5}$ **c** $y = (1 + x - x^2)^8$
d $y = \frac{1}{(x^2 - 1)^3}$ **e** $y = \sqrt{9 - x^2}$ **f** $y = \frac{1}{\sqrt{9 - x^2}}$
- 9** Differentiate each function, using the product or quotient rule. Factor each answer completely.
- a** $y = x^9(x + 1)^7$ **b** $y = \frac{x^2}{1 - x}$ **c** $y = x^2(4x^2 + 1)^4$
d $y = \frac{2x - 3}{2x + 3}$ **e** $y = (x + 1)^5(x - 1)^4$ **f** $y = \frac{x^2 + 5}{x - 2}$
- 10** Differentiate $y = x^2 + 3x + 2$. Hence find the gradient and the angle of inclination, correct to the nearest minute when appropriate, of the tangents at the points where:
- a** $x = 0$ **b** $x = -1$ **c** $x = -2$
- 11** **a** Find the equations of the tangent and normal to $y = x^3 - 3x$ at the origin.
b Find the equation of the tangent and normal to the curve at the point $P(1, -2)$.
c By solving $f'(x) = 0$, find the points on the curve where the tangent is horizontal.
d Find the points on the curve where the tangent has gradient 9.
- 12** **a** Find the equations of the tangent and normal to $y = x^2 - 5x$ at the point $P(2, -6)$.
b The tangent and normal at P meet the x -axis at A and B respectively. Find the coordinates of A and B .
c Find the length of the line segment AB and the area of the triangle PAB .
- 13** Find the tangents to the curve $y = x^4 - 4x^3 + 4x^2 + x$ at the origin and at the point where $x = 2$, and show that these two lines are the same.
- 14** Find any points on each curve where the tangent has the given angle of inclination.
- a** $y = \frac{1}{3}x^3 - 7$, 45° **b** $y = x^2 + \frac{1}{3}x^3$, 135°
- 15** **a** Find the points on $y = x^3 - 4x$ where the tangents are parallel to $\ell: x + y + 2 = 0$.
b Find the equations of these two tangents, and show that ℓ is one of them.
- 16** Find the x -coordinates of any points on $y = (4x - 7)^3$ where the tangent is:
- a** parallel to $y = 108x + 7$ **b** perpendicular to $x + 12y + 6 = 0$
- 17** **a** Find the values of t where $Q = t^3 - 15t^2 + 27t$ has stationary points.
b Draw up a table of test points of dQ/dt to solve $dQ/dt < 0$.
c Hence say where Q is increasing and where it is decreasing.

- 18** A ball is thrown vertically upwards from the ground, and its height h metres after t seconds is given by $h = -5t^2 + 50t$.
- a** When does the ball hit the ground again?
 - b** Differentiate to find the velocity function v of the ball.
 - c** At what speed did the ball leave the thrower's hand?
 - d** When was the ball at the top of its flight, and how high does it go?
 - e** How far in total did the ball travel, and what was its average speed?
 - f** What was its average speed during the first 3 seconds?
 - g** At what times was it travelling at a speed less than 20 m/s?
 - h** At what times was it higher than 80 m above the ground?
- 19** A huge fireball high in the air is exploding at the speed of sound, which is about $\frac{1}{3}$ km/s.
- a** Using the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere, show that the volume V of the fireball after t seconds is $V = \frac{4\pi}{81}t^3$.
 - b** Differentiate to find the rate at which the volume is expanding.
 - c** Find, correct to two significant figures, the volume and the rate after 0.1 seconds.
 - d** Find, correct to two significant figures, when the volume is increasing by $1 \text{ km}^3/\text{s}$.

Appendix — proving differentiation rules

This appendix explains some proofs that were needed in the course of this chapter.

Differentiating powers of x

The rule for differentiating powers of x is very simple:

Theorem: Let $y = x^n$, where n is any real number. Then $\frac{dy}{dx} = nx^{n-1}$.

Unfortunately, the proof is complicated, and the general proof is only developed below in the case where n is a positive whole number. Challenge questions in Exercise 9E and 9F develop the proofs for $n = -1$ and $n = \frac{1}{2}$ respectively.

This proof is *recursive*, meaning that the proof for each whole number uses the result for the previous whole numbers. Each step uses the product rule, and we start with the fact that $\frac{d}{dx}(x^1) = 1$, because $y = x$ is a straight line with gradient 1.

For $y = x^2$, write $y = x \times x$,
Then $y' = v u' + u v'$
 $= x \times 1 + x \times 1$
 $= 2x$, as required.

Let $u = x$
and $v = x$
Hence $u' = 1$, as mentioned above,
and $v' = 1$.

For $y = x^3$, write $y = x \times x^2$,
Then $y' = v u' + u v'$
 $= x^2 \times 1 + x \times 2x$
 $= 3x^2$, as required.

Let $u = x$
and $v = x^2$
Hence $u' = 1$
and $v' = 2x$, proven in the step above.

For $y = x^4$, write $y = x \times x^3$,
Then $y' = v u' + u v'$
 $= x^3 \times 1 + x \times 3x^2$
 $= 4x^3$, as required.

Let $u = x$
and $v = x^3$
Hence $u' = 1$
and $v' = 3x^2$, proven in the step above.

and so on, through all the positive whole numbers. This recursive proof is quite convincing, but it is given greater formality in the Extension 1 course by the *principle of mathematical induction*.

Linear combinations of functions

Two results need to be proven:

DERIVATIVE OF A SUM: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.

DERIVATIVE OF A MULTIPLE: If $f(x) = ag(x)$, then $f'(x) = ag'(x)$.

Proof For the first:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) + h(x+h) - g(x) - h(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} \frac{h(x+h) - h(x)}{h} \\ &= g'(x) + h'(x). \end{aligned}$$

For the second:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{ag(x+h) - ag(x)}{h} \\ &= a \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= ag'(x). \end{aligned}$$

Proof of the product rule

The product rule states that:

Theorem: Let $y = uv$, where u and v are functions of x . Then

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}.$$

Proof Suppose that x changes to $x + \Delta x$.

Suppose that as a result, u changes to $u + \Delta u$ and v changes to $v + \Delta v$, and that finally y changes to $y + \Delta y$.

Then

$$y = uv,$$

and

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

and expanding

$$y + \Delta y = uv + v\Delta u + u\Delta v + \Delta u\Delta v,$$

so subtracting $y = uv$,

$$\Delta y = v\Delta u + u\Delta v + \Delta u\Delta v.$$

Hence, dividing by Δx ,

$$\frac{\Delta y}{\Delta x} = v \frac{\Delta u}{\Delta x} + u \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x} \times \frac{\Delta v}{\Delta x} \times \Delta x,$$

and taking limits as $\Delta x \rightarrow 0$,

$$\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx} + 0, \text{ as required.}$$

Proof of quotient rule

The quotient rule states that:

Theorem: Let $y = \frac{u}{v}$, where u and v are functions of x . Then

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

Proof Differentiate uv^{-1} using the product rule.

Let

$$y = uv^{-1}.$$

Then

$$\begin{aligned} \frac{dy}{dx} &= v \frac{dU}{dx} + U \frac{dV}{dx} \\ &= v^{-1} \frac{du}{dx} - uv^{-2} \frac{dv}{dx} \\ &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\ &\quad \left(\text{after multiplying by } \frac{v^2}{v^2} \right). \end{aligned}$$

Let

$$U = u$$

and

$$V = v^{-1}.$$

Then

$$\frac{dU}{dx} = \frac{du}{dx}$$

and by the chain rule,

$$\begin{aligned} \frac{dV}{dx} &= \frac{dV}{dv} \times \frac{dv}{dx} \\ &= -v^{-2} \frac{dv}{dx}. \end{aligned}$$

10

Euler's number

Chapter introduction

Chapter 8 dealt with *exponential functions* such as $y = 2^x$ and $y = 10^x$, where the variable is in the exponent (index). We saw there how vitally important these functions are when modelling so many very common natural phenomena — where something is dying away, such as radioactive decay or the noise of a plucked guitar string, or where something is growing, such as populations or inflation.

But we cannot yet differentiate these functions, so we cannot work with the *rates* associated with these phenomena. Thus some of the most important functions in science are still beyond our reach, because so far, calculus has only been developed for algebraic functions such as

$$f(x) = x^3 + 8x \quad \text{and} \quad f(x) = \sqrt{x} \quad \text{and} \quad f(x) = x^2 - \frac{1}{x^2}$$

that involve powers and the four operations of arithmetic.

This short chapter explains only the very first steps in extending calculus to exponential functions. Next year we will develop all this much further, applying the standard methods of calculus. Later, we shall apply calculus also to the trigonometric functions so that we can model wavelike phenomena.

Differentiating exponential functions requires replacing our usual bases such as 2 and 10 by a new base $e \doteq 2.7183$. This new number e is called *Euler's number*, because it was introduced by the great 18th century Swiss mathematician Leonhard Euler. The number e is just as important in mathematics as π , but only makes its first appearance in this chapter, in the exponential function $y = e^x$.

Both numbers π and e are real numbers. But neither number is a fraction, although the proofs of this are difficult.

10A The exponential function base e

Learning intentions

- Define Euler's number using tangents to exponential functions.
- Understand that $\frac{d}{dx}(e^x) = e^x$ — that e^x is its own derivative..
- Graph $y = e^x$ and identify its characteristic tangent property..

As indicated in the Introduction, differentiating exponential functions requires a new base $e \doteq 2.7183$. The fundamental result of this section is surprisingly simple — the function $y = e^x$ is its own derivative:

$$\frac{d}{dx} e^x = e^x.$$

An investigation of the graph of $y = 2^x$

We begin by examining the tangents to the familiar exponential function $y = 2^x$. Its curve is drawn below, and we will need our assumption in Section 9B that, there being no obvious problem, the curve is differentiable for every value of x .

Before reading the following argument, carry out the investigation in Question 1 of Exercise 10A. The investigation looks at the tangent to $y = 2^x$ at each of several points on the curve, and relates its gradient to the height of the curve at the point. The procedures demonstrate graphically what the argument proves.

Differentiating $y = 2^x$

Below is a sketch of $y = 2^x$, with the tangent drawn at its y -intercept $A(0, 1)$. Differentiating $y = 2^x$ requires first-principles differentiation, because the theory so far hasn't provided any rule for differentiating 2^x .

The formula for first-principles differentiation is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Applying this formula to the function $f(x) = 2^x$,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{2^{x+h} - 2^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2^x \times 2^h - 2^x}{h}, \quad \text{because } 2^x \times 2^h = 2^{x+h}, \end{aligned}$$

and taking out the common factor 2^x in the numerator,

$$\begin{aligned} f'(x) &= 2^x \times \lim_{h \rightarrow 0} \frac{2^h - 1}{h} \\ f'(x) &= 2^x \times m, \quad \text{where } m = \lim_{h \rightarrow 0} \frac{2^h - 1}{h}. \end{aligned}$$

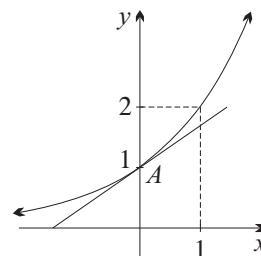
We assumed in Section 9B that all our functions are differentiable at every point, unless there is an obvious problem, so we are assuming $y = 2^x$ is differentiable at x .

Hence this limit $m = \lim_{h \rightarrow 0} \frac{2^h - 1}{h}$ exists, but it cannot be found algebraically.

Substituting $x = 0$, however, gives a very simple geometric interpretation of the limit:

$$\begin{aligned} f'(0) &= 2^0 \times m \\ &= m, \quad \text{because } 2^0 = 1, \end{aligned}$$

so m is just the gradient of the tangent to $y = 2^x$ at its y -intercept $(0, 1)$.



The conclusion of all this is that

$$\frac{d}{dx} 2^x = 2^x \times m, \quad \text{where } m \text{ is the gradient of } y = 2^x \text{ at its y-intercept.}$$

The investigation in Question 1 of Exercise 10A below shows that $m \doteq 0.7$

This argument applied to the base 2. But exactly the same argument can be applied to any exponential function $y = a^x$, whatever the base a , simply by replacing 2 by a in the argument above:

1 Differentiating $y = a^x$

For all real numbers $a > 0$,

$$\frac{d}{dx} a^x = a^x \times m, \quad \text{where } m \text{ is the gradient of } y = a^x \text{ at its y-intercept.}$$

That is, the derivative of an exponential function is a multiple of itself.

The gradient m of $y = a^x$ at its y-intercept changes as the base a changes.

- When $a = 1$, the function is the constant function $y = 1$, so $m = 0$,
- As the base a increases, the curve $y = a^x$ tilts up more and more steeply at its y-intercept $(0, 1)$ — see the investigations in the following exercise.

The definition of Euler's number e

It now makes sense to choose the base that will make the gradient exactly 1 at the y-intercept, because the value of m will then be exactly 1. This base is called *Euler's number* — it is given the symbol e , and has the value $e \doteq 2.7183$.

2 The definition of e

- Define *Euler's number* e to be the number such that the exponential function $y = e^x$ has gradient exactly 1 at its y-intercept. Then

$$e \doteq 2.7183.$$

- The function $y = e^x$ is called the exponential function to distinguish it from all other exponential functions $y = a^x$.

Investigations to approximate e

Question 5 of Exercise 10A uses a simple argument to prove that e is between 2 and 4. The investigation in Question 6 uses technology to find ever closer approximations to e . Both questions are based on the definition of e given above.

The derivative of e^x

The fundamental result of this section then follows immediately by combining the two boxed results developed so far in this chapter. By box 1,

$$\begin{aligned} \frac{d}{dx} e^x &= e^x \times m, \quad \text{where } m \text{ is the gradient of } y = e^x \text{ at its y-intercept,} \\ &= e^x \times 1, \quad \text{by the definition of } e, \text{ in Box 2} \\ &= e^x. \end{aligned}$$

3 The exponential function $y = e^x$ is its own derivative

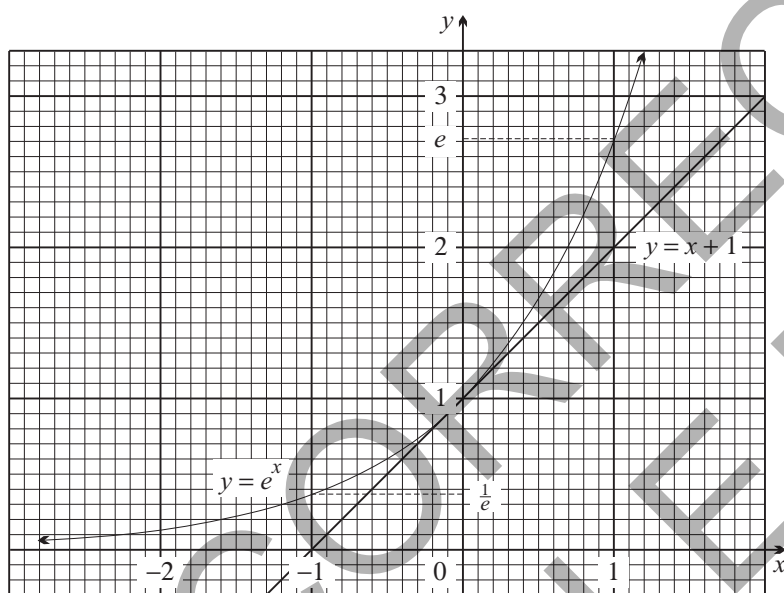
$$\frac{d}{dx}(e^x) = e^x.$$

We will not develop the calculus of e^x any further until next year, when $y = e^x$ will be the foundation function for a large number of very common rate situations.

The graph of e^x

The graph of $y = e^x$ has been drawn below on graph paper. The tangent has been drawn at the y -intercept $(0, 1)$ — it has gradient exactly 1.

This graph of $y = e^x$ is one of the most important graphs in the whole course, and its shape and properties need to be memorised.



Some vitally important properties of the graph of $y = e^x$

- The domain is all real x , or we can write domain $= (-\infty, \infty)$.
The range is $y > 0$, or we can write range $= (0, \infty)$.
- There are no zeroes.
The curve is always above the x -axis.
- The x -axis $y = 0$ is a horizontal asymptote to the curve on the left.
That is, as $x \rightarrow -\infty$, $y \rightarrow 0$ and $\frac{dy}{dx} \rightarrow 0$.
- On the right-hand side, the curve rises steeply at an increasing rate.
That is, as $x \rightarrow \infty$, $y \rightarrow \infty$ and $\frac{dy}{dx} \rightarrow \infty$.
- The curve has gradient 1 at its y -intercept $(0, 1)$.
- The curve is always increasing — increasing at an increasing rate — and is always concave up.

Gradient equals height

The fact that the derivative of the exponential function e^x is the same function has a striking geometrical interpretation in terms of its graph. If $y = e^x$, then $\frac{dy}{dx} = e^x$, which means that for this function

$$\frac{dy}{dx} = y, \quad \text{that is,} \quad \text{gradient} = \text{height}.$$

Thus at each point on the curve $y = e^x$, the gradient $\frac{dy}{dx}$ of the curve is equal to the height y of the curve above the x -axis. We have already seen this happening at the y -intercept $(0, 1)$, where the gradient is 1 and the height is also 1.

4 Gradient equals height

At each point on the graph of the exponential function $y = e^x$,

$$\frac{dy}{dx} = y.$$

That is, the gradient of the curve is always equal to its height above the x -axis.

We shall see next year why this property of the exponential function is the reason why the function is so important in calculus.

An investigation of the graph of $y = e^x$

The investigation in Question 2 of Exercise 10A confirms these properties of the graph of $y = e^x$, particularly the fact that the gradient and the height are equal at every point on the curve.

Exercise 10A

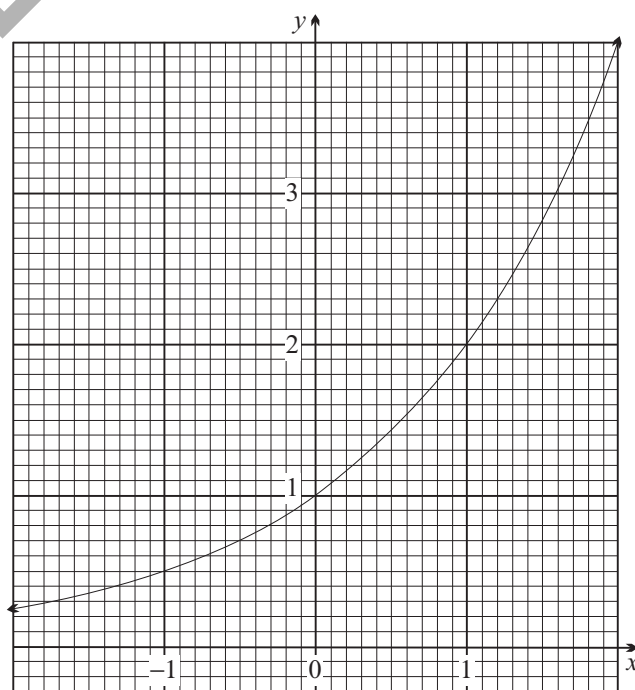
INVESTIGATION

Technology: The first two investigations are written as graph-paper exercises, so that they are independent of any device. The instructions are identical if graphing software is used. When using the graph-paper method, the diagrams should first be photocopied.

- 1** [Graph paper, but easily transferred to graphing software]

The graph is the function $y = 2^x$.

- a** Photocopy the graph of $y = 2^x$, and on it draw the tangent at the point $(0, 1)$. Extend the tangent across the diagram.
- b** Use ‘rise over run’ to measure the gradient $\frac{dy}{dx}$ of this tangent. The run should be chosen as 10 or 20 little divisions, then count how many little divisions the rise is.
- c** Similarly, draw tangents at four more points on the curve where $x = -2, -1, 1$ and 2 . Measure the gradient $\frac{dy}{dx}$ of each tangent.
- d** Copy and complete the table of values to the right (the values will only be rough).

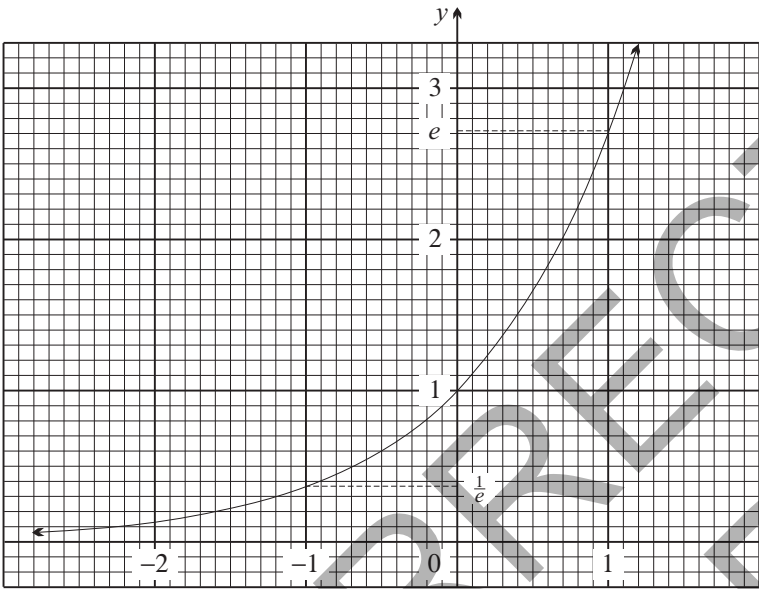


- e What do you notice about the ratios of gradient to height?
- f Hence copy and complete:

$\frac{dy}{dx} = \dots y$

x	-2	-1	0	1	2
height y					
gradient $\frac{dy}{dx}$					
$\frac{\text{gradient}}{\text{height}}$					

2 [Graph paper, but easily transferred to graphing software]



These questions refer to the graph of $y = e^x$ drawn above.

- a Photocopy the graph of $y = e^x$ above and on it draw the tangent at the point $(0, 1)$ where the height is 1. Extend the tangent across the diagram.
 - b Measure the gradient of this tangent and confirm that it is equal to the height of the exponential graph at the point of contact.
 - c Copy and complete the table of values to the right by measuring the gradient $\frac{dy}{dx}$ of the tangent at the points where the height y is $\frac{1}{2}$, 1, 2 and 3.
 - d What do you notice about the ratios of gradient to height?
 - e What does this tell you about the derivative of $y = e^x$?
- 3
- a Photocopy again the graph of $y = e^x$ in Question 2 (or use graphing software).
 - b Draw the tangents at the points where $x = -2, -1, 0$ and 1 , extending each tangent down to the x -axis.
 - c Measure the gradient of each tangent and confirm that it is equal to the height of the graph at the point.
 - d What do you notice about the x -intercepts of all the tangents?

height y	$\frac{1}{2}$	1	2	3
gradient $\frac{dy}{dx}$				
$\frac{\text{gradient}}{\text{height}}$				

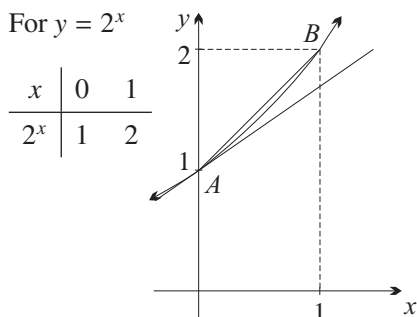
DEVELOPMENT

- 4 This will Confirm the result in Q3(d). Let $P(a, e^a)$ lie on the curve $y = e^x$.
- a Use the fact that $y' = e^x$ to find the gradient of the tangent at P .
 - b Find the equation of the tangent at P .
 - c Show that the x -intercept of the tangent is 1 unit to the left of P .

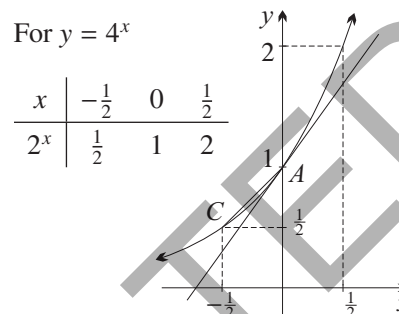
- 5** Approximating e to many significant figures is difficult. The argument below, however, at least shows very quickly that the number e lies between 2 and 4.

Here are tables of values and sketches of $y = 2^x$ and $y = 4^x$. On each graph, the tangent at the y -intercept $A(0, 1)$ has been drawn.

For $y = 2^x$



For $y = 4^x$



- a**
 - i** In the first diagram, find the gradient of the chord AB .
 - ii** Hence explain why the tangent at A has gradient less than 1.
 - b**
 - i** In the second diagram, find the gradient of the chord CA .
 - ii** Hence explain why the tangent at A has gradient greater than 1.
 - c** Use these two results to explain why $2 < e < 4$.
- 6** [Technology] This question requires graphing software.
- a** Use graphing software to graph $y = 2^x$ and $y = x + 1$ on the same number plane, and hence observe that the gradient of $y = 2^x$ at $(0, 1)$ is less than 1.
 - b** Similarly, graph $y = 3^x$ and $y = x + 1$ on the same number plane and hence observe that the gradient of $y = 3^x$ at $(0, 1)$ is greater than 1.
 - c** Choose a sequence of bases between 2 and 3 that make the line $y = x + 1$ more and more like a tangent to the curve at $(0, 1)$ — that is, the gradient at $(0, 1)$ becomes closer and closer to 1. In this way a reasonable approximation for e can be obtained.
- 7** Consider the five functions $y = \frac{1}{3}^x$, $y = \frac{1}{2}^x$, $y = 1^x$, $y = 2^x$, and $y = 3^x$.
- a** Explain why $\frac{1}{3}^x = 3^{-x}$, and $\frac{1}{2}^x = 2^{-x}$.
 - b** Sketch these graphs, or locate their graphs in this section or in Section 8F.
 - c** From the graphs, identify where each of these is increasing, and decreasing.
 - d** What is happening to these rates of increase, or decrease, as x increases?
 - e** What is happening to the five rates of increase as the base is increasing from $\frac{1}{3}$ to 3?

CHALLENGE

- 8** [Technology] At the start of this section, it was shown by first-principles differentiation that the gradient of $y = 2^x$ at its y -intercept is given by $\lim_{h \rightarrow 0} \frac{2^h - 1}{h}$. The investigation in Question 1 showed that the limit is about 0.7. Use a calculator or a spreadsheet to evaluate $\frac{2^h - 1}{h}$, correct to five decimal places, at least for the following values of h :
- a** 1
 - b** 0.1
 - c** 0.01
 - d** 0.001
 - e** 0.0001
 - f** 0.00001

10B Transformations of exponential functions

Learning intentions

- Use the calculator to evaluate expressions involving e .
- Apply transformations to the graph of $y = e^x$.

Transformations of exponential functions with bases other than e were discussed in Chapter 8. This short exercise first practises using the calculator to evaluate powers of e , and is then concerned with transformations of $y = e^x$.

Using the calculator to approximate powers of e

A calculator will provide approximate values of e^x correct to about ten significant figures. Most calculators label the function (e^x) and locate it above the button labelled $(\ln)$. Pressing (shift) first may be required.

Example 1 Approximating powers of e

Use your calculator to find, correct to four significant figures:

a e^2

b e^{-4}

c $e^{\frac{1}{3}}$

Solution

Using the function labelled (e^x) on the calculator:

a $e^2 \div 7.389$

b $e^{-4} \div 0.01832$

c $e^{\frac{1}{3}} \div 1.396$

The next three examples must first be put into index form before using the calculator. In particular, e itself must be written as e^1 , so that an approximation for the number e can be found using the function (e^x) with input $x = 1$.

Example 2 Approximating powers of e

Write each as a power of e , then approximate it correct to five decimal places.

a e

b $\frac{1}{e}$

c $\sqrt{e}$

Solution

a $e = e^1$
 $\div 2.71828$

b $\frac{1}{e} = e^{-1}$
 $\div 0.36788$

c $\sqrt{e} = e^{\frac{1}{2}}$
 $\div 1.64872$

The graphs of e^x and e^{-x}

The graphs of $y = e^x$ and $y = e^{-x}$ are fundamental to this whole course. Because x has been replaced by $-x$ in the second equation, the two graphs $y = e^x$ and $y = e^{-x}$ are reflections of each other in the y -axis.

For $y = e^x$:

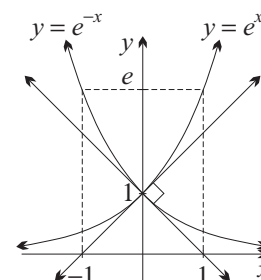
x	-2	-1	0	1	2
y	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2

For $y = e^{-x}$:

x	-2	-1	0	1	2
y	e^2	e	1	$\frac{1}{e}$	$\frac{1}{e^2}$

The two curves cross at $(0, 1)$.

By the definition of e , the gradient of $y = e^x$ at its y -intercept $(0, 1)$ is 1.



Hence by reflection, the gradient of $y = e^{-x}$ at its y -intercept $(0, 1)$ is -1 .

Thus the curves are perpendicular at their point of intersection.

Note: The function $y = e^{-x}$ is as important as $y = e^x$ in applications, or even more important. It describes a great many physical situations where a quantity ‘dies away exponentially’, such as the dying away of the sound of a plucked string.

Transformations of $y = e^x$

The usual methods of shifting and reflecting graphs can be applied to $y = e^x$. When the graph is shifted vertically, the horizontal asymptote at $y = 0$ will be shifted also. A small table of approximate values can be a very useful check.

All our standard transformations of $y = e^x$ have domain all real x , but the range may change.



Example 3 Transformations of $y = e^x$

Use transformations of the graph of $y = e^x$, confirmed by a table of values, to generate a sketch of each function. Show and state the y -intercept and the horizontal asymptote, and state the range, in both notations.

a $y = e^x + 3$

b $y = e^{x-2}$

c $y = e^{2x}$

Solution

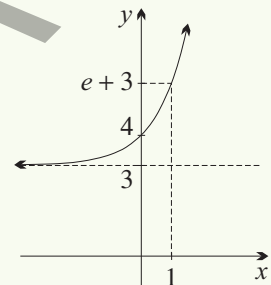
a Graph $y = e^x + 3$ by shifting $y = e^x$ up 3 units.

x	-1	0	1
y	$e^{-1} + 3$	4	$e + 3$
approximation	3.37	4	5.72

y -intercept: $(0, 4)$

asymptote: $y = 3$

range: $y > 3$, or we can write $(3, \infty)$



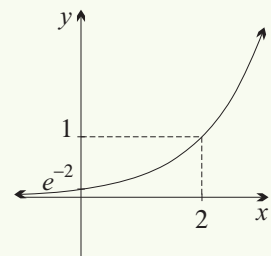
b Graph $y = e^{x-2}$ by shifting $y = e^x$ to the right by 2 units.

x	0	1	2	3
y	e^{-2}	e^{-1}	1	e
approximation	0.14	0.37	1	2.72

y -intercept: $(0, e^{-2})$

asymptote: $y = 0$ (the x -axis)

range: $y > 0$, or we can write $(0, \infty)$



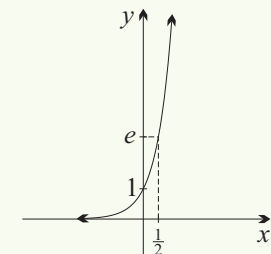
c Graph $y = e^{2x}$ by dilating $y = e^x$ horizontally with factor $\frac{1}{2}$.

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y	e^{-1}	1	e
approximation	0.37	1	2.72

y -intercept: $(0, 1)$

asymptote: $y = 0$ (the x -axis)

range: $y > 0$, or we can write $(0, \infty)$



Exercise 10B

FOUNDATION

Technology: The transformations of the graph of the exponential function $y = e^x$ throughout this exercise can be confirmed and further developed using graphing software, after which experimentation with further graphs can easily be done.

- Use the function (e^x) on your calculator to approximate correct to four decimal places.

a e^2	b e^{10}	c e^0	d e^1	e e^{-1}
f e^{-2}	g $e^{\frac{1}{2}}$	h $e^{-\frac{1}{2}}$	i $e^{-0.001}$	j e^{-6}
- Write each expression as a power of e . Then approximate correct to four significant figures.

a e	b $\frac{1}{e}$	c $\frac{1}{e^4}$	d $\sqrt{e}$	e $\sqrt[3]{e}$	f $\frac{1}{\sqrt{e}}$
- Write each expression as a power of e . Then approximate correct to four significant figures.

a $5e^2$	b $\frac{1}{64}e^6$	c $7\sqrt{e}$	d $\frac{3}{5}\sqrt{e}$	e $\frac{4}{e}$	f $\frac{5}{7e^4}$
- Sketch the graph of $y = e^x$, then use your knowledge of transformations to graph these functions, showing the horizontal asymptote. For each function, state the transformation, give the equation of the asymptote, and state the range in both notations.

a $y = e^x + 1$	b $y = e^x + 2$	c $y = e^x - 1$
d $y = e^x - 2$	e $y = 2e^x$	f $y = e^{\frac{1}{3}x}$
- Copy and complete the following tables of values for the functions $y = e^x$ and $y = e^{-x}$, giving your answers correct to two decimal places.

x	-2	-1	0	1	2
e^x					

x	-2	-1	0	1	2
e^{-x}					
 - Sketch both graphs on one number plane, and draw the tangents at each y-intercept.
 - What transformation exchanges these graphs?
 - We saw that the tangent to $y = e^x$ at its y-intercept has gradient 1. What is the gradient of $y = e^{-x}$ at its y-intercept? Explain why the two tangents are perpendicular.
 - Hence use point-gradient form to find the equations of the tangents to $y = e^x$ and $y = e^{-x}$ at the point $A(0, 1)$ where they intersect.
- Sketch the graph of $y = e^{-x}$. State the horizontal asymptote and the range.
 - Then use your knowledge of transformations to graph each function below. State the transformation of $y = e^{-x}$, give the equation of the asymptote, and state the range.

i $y = e^{-x} + 1$	ii $y = e^{-x} + 2$	iii $y = e^{-x} - 1$	iv $y = e^{-x} - 2$

DEVELOPMENT

Technology: The graph transformations in this exercise can be confirmed and further developed using graphing software.

- What transformation maps $y = e^x$ to $y = e^{x-1}$?
 - Sketch $y = e^{x-1}$.
 - Similarly sketch these functions:

i $y = e^{x-3}$	ii $y = e^{x+1}$	iii $y = e^{x+2}$

- 8 Let A , B and C be the points on $y = e^x$ with x -coordinates 0, 1 and 2.
- a Find the y -coordinates of A , B and C , and sketch the situation.
- b Find the gradient of the chord AB , find its equation, then show that its x -intercept is $-\frac{1}{e-1}$.
- c Similarly show that the x -intercept of the chord BC is $1 - \frac{1}{e-1}$.
- d Let P and Q be the points on $y = e^x$ with x -coordinates a and $a+1$. Show similarly that the x -intercept of the chord PQ is $a - \frac{1}{e-1}$.
- 9 Use the graph of $y = e^x$ and your knowledge of transformations to graph these functions. Show the horizontal asymptote and state the range in each case, using both notations.
- a $y = -e^x$ b $y = -e^{-x}$ c $y = 1 - e^x$ d $y = 3 - e^x$
- e $y = 1 - e^{-x}$ f $y = e^{-3x} - 2$ g $y = 2 - \frac{1}{2}e^{-\frac{1}{2}x}$ h $y = e^{-|x|}$

CHALLENGE

- 10 **Technology:** A spreadsheet will allow this investigation to be extended. The function e^x may be approximated by adding up a few terms of the infinite power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4} + \dots$$

Use this power series to approximate each power of e , correct to two decimal places. Then compare your answers with those given by your calculator.

- a e b e^{-1} c e^2 d e^{-2} e e^0

When x is large, many more terms are needed to approximate e^x correct to say two decimal places. Why is this so? Use your spreadsheet to investigate this.

Note: There are far quicker and more sophisticated ways to generate powers of e on a calculator or computer. But the Taylor-MacLauren series given above is simple and well-known, and dates from the early 18th century, when calculus was developing.

10C The logarithmic function base e

Learning intentions

- Work with logarithms base e
- Sketch $y = \log_e x$ and its transformations

We have seen that $e \doteq 2.7183$ is the most natural base to use for exponential functions in calculus. Work with the exponential function $y = e^x$ necessarily involves work with the logarithmic function $\log_e x$, because it is the inverse function of $y = e^x$. Thus e is also the most natural base to use for logarithms in calculus.

This section introduces $y = \log_e x$ and its graph, describing its relationship with the graph of $y = e^x$, and the transformations of its graph.

The logarithmic function

Because e is the natural base for logarithms, logarithms base e are called *natural logarithms*, and the function $y = \log_e x$ is called the logarithmic function to distinguish it from other logarithmic functions such as $\log_2 x$ and $\log_{10} x$ that have other bases.

5 The logarithmic function

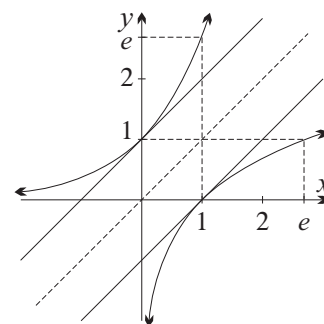
- Logarithms base e are called *natural logarithms*.
- The logarithmic function with base e ,

$$y = \log_e x$$

is called the logarithmic function to distinguish it from all other logarithmic functions $y = \log_a x$.

Below are tables of values and the graphs of the inverse functions $y = e^x$ and $y = \log_e x$. You can see that they are inverse functions because the rows are swapped. These are two of the most important graphs in the course.

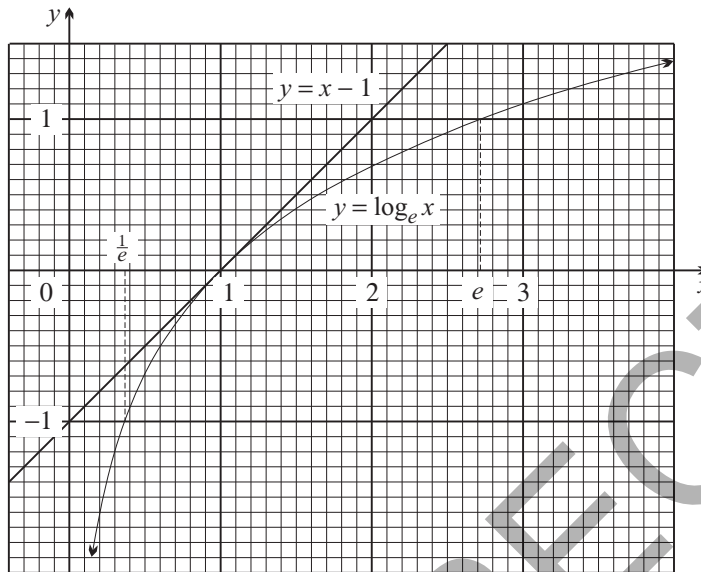
x	-2	-1	0	1	2
e^x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
x	$\frac{1}{e^2}$	$\frac{1}{e}$	1	e	e^2
$\log_e x$	-2	-1	0	1	2



We know already that the tangent to $y = e^x$ at its y -intercept $(0, 1)$ has gradient 1. When this graph is reflected in the line $y = x$, this tangent is reflected to a tangent to $y = \log_e x$ at its x -intercept $(1, 0)$. Both these tangents therefore have gradient 1.

Properties of the graph of $y = \log_e x$

The graph of $y = \log_e x$ is drawn below. The tangent has been drawn at the x -intercept $(1, 0)$ to show that the gradient of the curve there is exactly 1.



The graph of $y = \log_e x$ and its properties must be thoroughly known. Its properties correspond to the properties listed earlier of its inverse function $y = e^x$.

- The domain is $x > 0$, or we can write domain $= (0, \infty)$.
The range is all real y , or we can write range $= (-\infty, \infty)$.
- The y -axis $x = 0$ is a vertical asymptote to the curve.
- As $x \rightarrow 0^+$, $y \rightarrow -\infty$. (Read this as, 'As x approaches 0 from above ...'.)
As $x \rightarrow \infty$, $y \rightarrow \infty$.
- The curve has gradient 1 at its x -intercept $(1, 0)$.
- The curve is always concave down.

The notations for the logarithmic function — $\log_e x$, $\log x$ and $\ln x$

In calculus, and in mathematics generally, $\log_e x$ is by far the most important logarithmic function. For this reason, some authors write it simply as $\log x$. The problem is that many calculators, and even some mathematical software, use an engineering convention, where ' $\log x$ ' means $\log_{10} x$.

The function $\log_e x$ is also written as $\ln x$, where the ' n ' stands for 'natural logarithms'. Alternatively, ' n ' also stands for 'Napierian logarithms', in honour of the Scottish mathematician John Napier (1550–1617), who first developed tables of logarithms for calculations (first published in 1614).

The most sensible way to deal with this notational confusion is:

- Never write ' $\log x$ ' without a base — always write either $\log_{10} x$ or $\log_e x$.
- Use either $\log_e x$ or $\ln x$ for logarithms base e .

Note: Be very careful of the different convention on calculators, where $\boxed{\log}$ means $\log_{10} x$, and the function labelled $\boxed{\ln}$ is used to find logarithms base e .

The function $\boxed{e^x}$ is usually located on the same button as $\boxed{\ln}$ because the two functions $y = e^x$ and $y = \log_e x$ are inverses of each other.

6 The notation for the logarithmic function

- In this text, logarithms base e are written as $\log_e x$, and less often as $\ln x$.
- On calculators, however:
 - ▷ $\ln$ is used to approximate $\log_e x$. It is the inverse function of e^x .
 - ▷ $\log$ is used to approximate $\log_{10} x$. It is the inverse function of 10^x .

**Example 4 Approximating logs base e**

a Use your calculator to find, correct to four significant figures:

i $\log_e 10$

ii $\log_e \frac{1}{10}$

iii $\log_e 100$

b How are the answers to parts (ii) and (iii) related to the answer to part (i)?

Solution

a Using the function labelled $\ln$ on the calculator,

i $\log_e 10 \div 2.303$

ii $\log_e \frac{1}{10} \div -2.303$

iii $\log_e 100 \div 4.605$

b Using the log laws,

$$\log_e \frac{1}{10} = \log_e 10^{-1} = -\log_e 10 \quad \text{and} \quad \log_e 10^2 = 2\log_e 10,$$

and these relationships are clear from the approximations above.

Combining the logarithmic and exponential functions

As with any base, when the logarithmic and exponential functions base e are applied successively to a number, the result is the original number.

7 The logarithmic and exponential functions are inverse functions

$$\log_e e^x = x \quad \text{and} \quad e^{\log_e x} = x.$$

These identities follow immediately from the definition of the logarithmic function as the inverse function of the exponential function.

**Example 5 Demonstrating inverse functions on the calculator**

Use the functions labelled $\ln$ and e^x on your calculator to demonstrate that:

a $\log_e e^{10} = 10$

b $e^{\log_e 10} = 10$

Solution

a $\log_e e^{10} = \log_e 22\,026.46\dots$
 $\div 10$

b $e^{\log_e 10} = e^{2.302585\dots}$
 $\div 10$

Transformations of the logarithmic graph

The usual methods of transforming graphs can be applied to $y = \log_e x$. When the graph is shifted sideways, the vertical asymptote at $x = 0$ will also be shifted.

A small table of approximate values can be a very useful check, particularly when a sequence of transformations is involved. Remember that $y = \log_e x$ has domain $x > 0$ and that the vertical asymptote is at $x = 0$.

All our standard transformations of $y = \log_e x$ have range all real y , but the domain may change.



Example 6 Transformations of the graph of $\log_e x$

Use transformations of the graph of $y = \log_e x$, confirmed by a table of values, to generate a sketch of each function. State the transformation(s) used, and write down the domain (both notations), the x -intercept, and the vertical asymptote.

a $y = \log_e(-x)$

b $y = \log_e x - 2$

c $y = \log_e(x + 3)$

d $y = -2 \log_e(3x)$

Solution

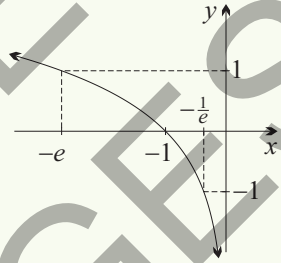
a Graph $y = \log_e(-x)$ by reflecting $y = \log_e x$ in the y -axis.

x	$-e$	-1	$-\frac{1}{e}$
y	1	0	-1

domain: $x < 0$ or $(-\infty, 0)$

x -intercept: $(-1, 0)$

asymptote: $x = 0$ (the y -axis)



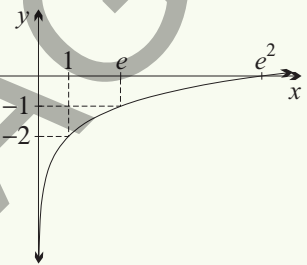
b Graph $y = \log_e x - 2$, which is $y + 2 = \log_e x$, by shifting $y = \log_e x$ down 2 units.

x	$\frac{1}{e}$	1	e	e^2
y	-3	-2	-1	0

domain: $x > 0$ or $(0, \infty)$

x -intercept: $(e^2, 0)$

asymptote: $x = 0$ (the y -axis)



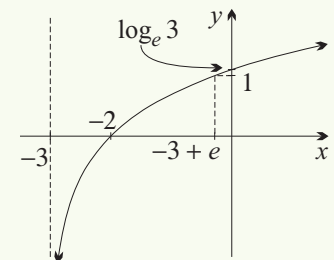
c Graph $y = \log_e(x + 3)$ by shifting $y = \log_e x$ left 3 units.

x	$\frac{1}{e} - 3$	-2	$e - 3$
y	-1	0	1

domain: $x > -3$ or $(-3, \infty)$

x -intercept: $(-2, 0)$

asymptote: $x = -3$



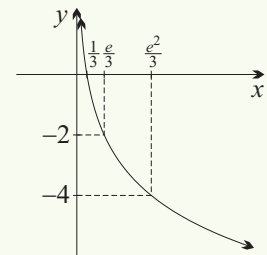
d Graph $y = -2 \log_e(3x)$ by dilating $y = \log_e x$ vertically factor -2 , then horizontally factor $\frac{1}{3}$.

x	$\frac{1}{3}$	$\frac{e}{3}$	$\frac{e^2}{3}$
y	0	-2	-4

domain: $x > 0$ or $(0, \infty)$

x -intercept: $(\frac{1}{3}, 0)$

asymptote: $x = 0$



Exercise 10C

FOUNDATION

Note: Remember that $\ln x$ means $\log_e x$, the natural logarithm. On the calculator, the key $(\ln)$ means $\log_e x$, and the key $(\log)$ unfortunately means $\log_{10} x$.

- 1 Use your calculator to approximate the following, correct to four decimal places where necessary. Read the note above and remember to use the $(\ln)$ key on the calculator.

$$\begin{array}{llll} \mathbf{a} \log_e 1 & \mathbf{b} \log_e 2 & \mathbf{c} \ln 3 & \mathbf{d} \ln 8 \\ \mathbf{e} \log_e \frac{1}{2} & \mathbf{f} \log_e \frac{1}{3} & \mathbf{g} \ln \frac{1}{8} & \mathbf{h} \ln \frac{1}{10} \end{array}$$

- 2 Use the functions labelled $(\ln)$ and (e^x) on your calculator to demonstrate that:

$$\begin{array}{lll} \mathbf{a} \log_e e^2 = 2 & \mathbf{b} \log_e e^3 = 3 & \mathbf{c} \ln e^1 = 1 \\ \mathbf{d} \ln e^{-2} = -2 & \mathbf{e} \log_e e^{-3} = -3 & \mathbf{f} \log_e e^{-1} = -1 \end{array}$$

- 3 Use the functions labelled $(\ln)$ and (e^x) on your calculator to demonstrate that:

$$\begin{array}{lll} \mathbf{a} e^{\log_e 2} = 2 & \mathbf{b} e^{\log_e 3} = 3 & \mathbf{c} e^{\log_e 1} = 1 \\ \mathbf{d} e^{\log_e 10} = 10 & \mathbf{e} e^{\ln \frac{1}{2}} = \frac{1}{2} & \mathbf{f} e^{\ln \frac{1}{10}} = \frac{1}{10} \end{array}$$

- 4 Solve each equation of x by first rewriting it as an index statement using the definition

$$x = \log_e a \quad \text{means} \quad e^x = a.$$

$$\begin{array}{lll} \mathbf{a} x = \ln 1 & \mathbf{b} x = \ln e & \mathbf{c} x = \log_e e^2 \\ \mathbf{d} x = \log_e \frac{1}{e} & \mathbf{e} x = \log_e \frac{1}{e^2} & \mathbf{f} x = \log_e \sqrt{e} \end{array}$$

- 5 Simplify each expression using the logarithm law $\log_e a^n = n \log_e a$, then evaluate it using the fact that $\log_e e = 1$.

$$\begin{array}{lllll} \mathbf{a} \log_e e^2 & \mathbf{b} \log_e e^5 & \mathbf{c} \log_e e^{200} & \mathbf{d} \log_e e^{-6} & \mathbf{e} \log_e \frac{1}{e^6} \\ \mathbf{f} \log_e e^{-1} & \mathbf{g} \log_e \frac{1}{e} & \mathbf{h} \log_e e^{\frac{1}{2}} & \mathbf{i} \ln \sqrt{e} & \mathbf{j} \ln \frac{1}{\sqrt{e}} \end{array}$$

- 6 Use the change-of-base formula to write these as logs base e , and hence approximate them correct to four significant figures:

$$\begin{array}{lll} \mathbf{a} \log_5 30 & \mathbf{b} \log_{\frac{1}{3}} 7 & \mathbf{c} \log_2 8 \end{array}$$

DEVELOPMENT

Technology: Again, the graph transformations in this exercise can be confirmed and further developed using graphing software.

- 7 Sketch the graph of $y = \log_e x$, then use your knowledge of transformations to graph these functions. In each case here the y -axis is a vertical asymptote and the domain is $x > 0$.

$$\begin{array}{llll} \mathbf{a} y = \log_e x + 1 & \mathbf{b} y = \log_e x + 2 & \mathbf{c} y = \ln x - 1 & \mathbf{d} y = \ln x - 2 \end{array}$$

- 8 **a** Copy and complete the following tables of values for the functions $y = \log_e x$ and $y = -\log_e x$, giving your answers correct to two decimal places.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_e x$					

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$-\log_e x$					

- b** Sketch both graphs on the same number plane, and draw the tangent to each at the x -intercept.
c It was shown at the beginning of this section that the tangent to $y = \log_e x$ at its y -intercept has gradient 1. Use this fact to find the gradient of $y = -\log_e x$ at its y -intercept, and hence explain why the two curves meet at right angles.

9 Use the log laws to simplify:

a $e \ln e$

b $\frac{1}{e} \ln \frac{1}{e}$

c $3 \ln e^2$

d $\log_e \sqrt{e}$

e $e \log_e e^3 - e \log_e e$

f $\log_e e + \log_e \frac{1}{e}$

10 Express as a single logarithm:

a $\log_e 3 + \log_e 2$

b $\log_e 100 - \log_e 25$

c $\log_e 2 - \log_e 3 + \log_e 6$

d $\ln 54 - \ln 10 + \ln 5$

11 Use the graph of $y = \log_e x$ and your knowledge of transformations to graph these functions. Show the vertical asymptote and state the domain in each case.

a $y = \ln(x - 1)$

b $y = \ln(x - 3)$

c $y = \log_e(x + 1)$

d $y = \log_e(x + 2)$

e $y = -\ln x$

f $y = \ln(-x)$

g $y = \frac{1}{2} \log_e x$

h $y = \log_e(\frac{1}{2}x)$

12 Sketch $y = -\log_e x$ and use your knowledge of transformations to graph these functions. Note that in each case the y -axis is a vertical asymptote and the domain is $x > 0$.

a $y = -\log_e x + 1$

b $y = -\log_e x + 2$

c $y = -\log_e x - 1$

d $y = -\log_e x - 2$

13 Use the log laws and the change-of-base formula to prove:

a $\log_e \frac{a}{b} = -\log_e \frac{b}{a}$.

b $\log_{\frac{1}{e}} x = -\log_e x$.

c $\log_{\frac{1}{e}} x^{-1} = \log_e x$.

14 [Parts (a) and (d) are examples from the Syllabus] Solve these equations using the identities for indices and logarithms, answering correct to four significant figures when necessary.

a $e^{x-2} = 12$

b $2^{4x-3} = 10$

c $3^{-2 \log_3 x} + \log_3 3^9 = \frac{6}{x}$

d $e^{2 \ln x} - \ln(e^x) - 4 = 0$

15 Consider the four functions $y = \log_{\frac{1}{3}} x$, $y = \log_{\frac{1}{2}} x$, $y = \log_2 x$, and $y = \log_3 x$.

a Explain why $\log_{\frac{1}{3}} x = -\log_3 x$, and $\log_{\frac{1}{2}} x = -\log_2 x$.

b Sketch these four graphs on the same coordinate plane.

CHALLENGE

16 Simplify:

a $\log_e e^e$

b $\log_e(\log_e e^e)$

c $\log_e(\log_e(\log_e e^e))$

17 Use a sequence of transformations to sketch each curve. State the domain (both notations).

a $y = -\log_e 2x - 1$

b $y = -2 \log_e(-\frac{1}{3}x + 1)$

c $y = -\log_e 4x - 3$

18 **Technology:** A spreadsheet will simplify this investigation and allow it to be extended. For $-1 < x \leq 1$, the function $\log_e(1 + x)$ may be approximated using the *power series*

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Use this power series to approximate each logarithm, correct to two decimal places. Then compare your answers with those given by your calculator.

a $\log_e 1 \frac{1}{2}$

b $\log_e \frac{5}{4}$

c $\log_e \frac{1}{2}$

d $\log_e \frac{1}{3}$

e $\log_e 1$

Chapter 10 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 10 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Use your calculator to approximate the following, correct to four significant figures:
a e **b** e^4 **c** e^{-2} **d** $e^{\frac{3}{2}}$
- Write each function as a single power of e :
a $e^{2x} \times e^{3x}$ **b** $e^{7x} \div e^x$ **c** $\frac{e^{2x}}{e^{6x}}$ **d** $(e^{3x})^3$
- Sketch the graph of each function on a separate number plane, and state its range:
a $y = e^x$ **b** $y = e^{-x}$ **c** $y = e^x + 1$ **d** $y = e^{-x} - 1$
- Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$, and find the x -intercept and y -intercept of this tangent.
- Use the appropriate button on your calculator to approximate the following, correct to four decimal places:
a $\log_{10} 27$ **b** $\log_{10} \frac{1}{2}$ **c** $\log_e 2$ **d** $\log_e 14$
- Write each equation below in logarithmic form. Then use the appropriate button on your calculator to approximate x correct to four decimal places.
a $10^x = 15$ **b** $10^x = 3$ **c** $e^x = 7$ **d** $e^x = \frac{1}{3}$
- Use the fact that $y = e^x$ and $y = \log_e x$ are inverse functions to simplify:
a $\log_e e^5$ **b** $\log_e e^{-\frac{1}{4}}$ **c** $e^{\ln 3}$ **d** $e^{\ln \frac{1}{5}}$
- Use the log laws to simplify:
a $e \log_e e$ **b** $\ln e^3$ **c** $\log_e \frac{1}{e}$ **d** $2e \log_e \sqrt{e}$
- Sketch graphs of these functions, clearly indicating the vertical asymptote in each case.
a $y = \log_e x$ **b** $y = \log_e(-x)$ **c** $y = \log_e(x - 2)$ **d** $y = \ln x + 1$

11

Radian measure of angles

Chapter introduction

Chapter 6 reviewed trigonometry, and developed it for all angles instead of only for acute angles, until the wave-like graphs of the sine and cosine functions could be displayed. These sine and cosine waves are fundamental in a vast variety of applications. They model such phenomena as sound waves, light and radio waves, vibrating strings, and tides. Many populations oscillate with the seasons, the economy tends to move in cycles, and in quantum mechanics, everything in the universe is some form of wave.

We will apply calculus to trigonometric functions next year, but before we can do that, we need to develop a new way to measure angles. We shall use *radians*, which are based on circles and π — one revolution is 2π radians — rather than the very ancient Babylonian system of *degrees*, where one revolution is the number of days in a year. This reliance on circles is hardly surprising, because Chapter 6 presented the general definition of the trigonometric functions based on circles.

With radian measure using π for the trig functions, and with Euler's number e as the appropriate base for exponential and log functions, we will be ready next year to extend calculus to both groups of functions. Most applications of calculus use functions that combine powers and these two groups of functions.

11A Radian measure of angle size

Learning intentions

- Define angle size in radians, and convert between degrees and radians.
- Find trig functions of angles given in radians, exactly and approximately.

The use of degrees to measure angles is based on astronomy. There are 360 days in the year — plus festival days — so 1° is the angle through which the Sun moves against the fixed stars each day (with apologies to Copernicus and Galileo).

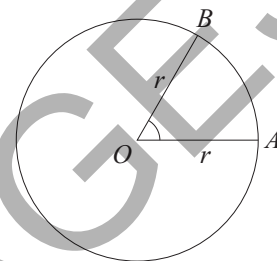
Mathematics is far too general a discipline to be tied to events in our solar system. We now develop a new system for measuring angles based on mathematics alone.

Radian measure of angle size

The size of an angle in radians is defined as the ratio of two lengths in a circle.

Given an angle with vertex O , construct a circle with centre O meeting the two arms of the angle at A and B .

The size of $\angle AOB$ in radians is the ratio of the arc length AB and the radius OA .



1 Radian measure

$$\angle AOB = \frac{\text{arc length } AB}{\text{radius } OA}$$

- This definition gives the same angle size, whatever the radius of the circle, because all circles are similar to one another.
- Angle size is the ratio of two lengths, so is a pure number with no dimension. Hence the unit ‘radians’ is not required when giving an angle size in radians.

This definition of angle size is very similar to the definitions of the six trigonometric functions, which are also ratios of lengths and so are also pure numbers.

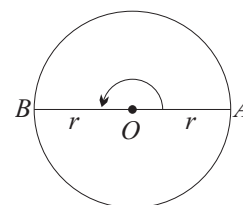
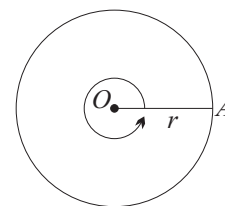
Some basic conversion between degrees and radians

- The arc subtended by a revolution is the whole circumference,

$$\begin{aligned} \text{so } 1 \text{ revolution} &= \frac{\text{circumference}}{\text{radius}} \\ &= \frac{2\pi r}{r} \\ &= 2\pi. \end{aligned}$$

- A straight angle subtends a semicircle,

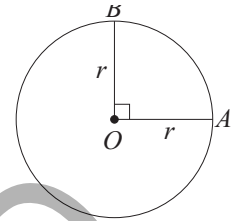
$$\begin{aligned} \text{so } 1 \text{ straight angle} &= \frac{\text{arc length of semicircle}}{\text{radius}} \\ &= \frac{\pi r}{r} \\ &= \pi. \end{aligned}$$



- A right angle subtends a quarter-circle,
so $1 \text{ right angle} = \frac{\text{arc length of quarter-circle}}{\text{radius}}$

$$= \frac{\frac{1}{2}\pi r}{r}$$

$$= \frac{\pi}{2}.$$



These three basic conversions should be memorised very securely.

2 Basic conversions between degrees and radian measure

$$360^\circ = 2\pi \qquad 180^\circ = \pi \qquad 90^\circ = \frac{\pi}{2}$$

Because $180^\circ = \pi$, an angle size in radians can be converted to an angle size in degrees by multiplying by $\frac{180^\circ}{\pi}$.

Conversely, degrees are converted to radians by multiplying by $\frac{\pi}{180}$.

3 Converting between degrees and radians

- To convert from radians to degrees,

$$\text{multiply by } \frac{180^\circ}{\pi}.$$

- To convert from degrees to radians,

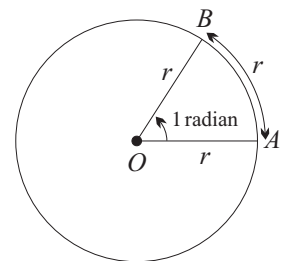
$$\text{multiply by } \frac{\pi}{180}.$$

- One radian and one degree:

$$1 \text{ radian} = \frac{180^\circ}{\pi} \div 57^\circ 18' \quad \text{and} \quad 1 \text{ degree} = \frac{\pi}{180} \div 0.0175.$$

One radian is by definition the angle subtended at the centre of a circle by an arc of length equal to the radius.

Notice that the sector OAB in the diagram to the right is almost an equilateral triangle, so 1 radian is about 60° . This makes sense of the value given above, that 1 radian is about 57° .



Example 1 Converting degrees to radians

Express these angle sizes in radians.

a 60°

b 270°

c 495°

d 37°

Solution

$$\begin{aligned} \text{a } 60^\circ &= 60 \times \frac{\pi}{180} \\ &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} \text{b } 270^\circ &= 270 \times \frac{\pi}{180} \\ &= \frac{3\pi}{2} \end{aligned}$$

$$\begin{aligned} \text{c } 495^\circ &= 495 \times \frac{\pi}{180} \\ &= \frac{11\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{d } 37^\circ &= 37 \times \frac{\pi}{180} \\ &= \frac{37\pi}{180} \end{aligned}$$

**Example 2** Converting radians to degrees

Express these angle sizes in degrees. Give exact answers, and then where appropriate give answers correct to the nearest degree.

a $\frac{\pi}{6}$

b 0.3

c $\frac{3\pi}{4}$

d 20

Solution

$$\begin{aligned}\mathbf{a} \quad \frac{\pi}{6} &= \frac{\pi}{6} \times \frac{180^\circ}{\pi} \\ &= 30^\circ\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad 0.3 &= \frac{3}{10} \times \frac{180^\circ}{\pi} \\ &= \frac{54^\circ}{\pi} \\ &\doteq 17^\circ\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \frac{3\pi}{4} &= \frac{3\pi}{4} \times \frac{180^\circ}{\pi} \\ &= 135^\circ\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad 20 &= 20 \times \frac{180^\circ}{\pi} \\ &= \frac{3600^\circ}{\pi} \\ &\doteq 1146^\circ\end{aligned}$$

Evaluating trigonometric functions of special angles

The trigonometric function of an angle is the same whether the angle size is given in degrees or radians. With angles whose related angle is one of the three special angles, it is a matter of recognising the special angles

$$\frac{\pi}{6} = 30^\circ \quad \text{and} \quad \frac{\pi}{4} = 45^\circ \quad \text{and} \quad \frac{\pi}{3} = 60^\circ.$$

**Example 3** Evaluating trig functions of special angles

Evaluate these trigonometric functions, using the special triangles sketched below.

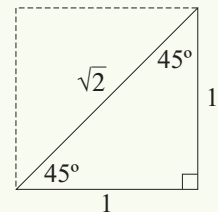
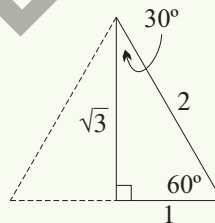
a $\sin \frac{\pi}{6}$

b $\operatorname{cosec} \frac{\pi}{4}$

Solution

$$\begin{aligned}\mathbf{a} \quad \sin \frac{\pi}{6} &= \sin 30^\circ \\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad \operatorname{cosec} \frac{\pi}{4} &= \frac{1}{\sin \frac{\pi}{4}} \\ &= \frac{1}{\sin 45^\circ} \\ &= \sqrt{2}\end{aligned}$$




Example 4 Angles whose the related angle is a special angle

Use special angles to evaluate these trigonometric functions.

a $\sin \frac{5\pi}{4}$

b $\sec \frac{11\pi}{6}$

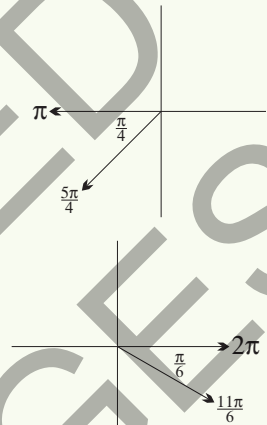
Solution

a Because $\frac{5\pi}{4}$ is in the third quadrant, with related angle $\frac{\pi}{4}$,

$$\begin{aligned}\sin \frac{5\pi}{4} &= -\sin \frac{\pi}{4} \\ &= -\sin 45^\circ \\ &= -\frac{1}{\sqrt{2}}.\end{aligned}$$

b Because $\frac{11\pi}{6}$ is in the fourth quadrant, with related angle $\frac{\pi}{6}$,

$$\begin{aligned}\sec \frac{11\pi}{6} &= +\sec \frac{\pi}{6} \\ &= \frac{1}{\cos \frac{\pi}{6}} \\ &= \frac{1}{\cos 30^\circ} \\ &= \frac{2}{\sqrt{3}}.\end{aligned}$$



Approximating trigonometric functions of other angles

For other angles, exact values of the trigonometric functions cannot normally be found. When calculator approximations are required, *it is vital to set the calculator to radians mode first.*

Your calculator has a key labelled **mode** or something similar to make the change — calculators set to the wrong mode routinely cause havoc at this point!

4 Setting the calculator to radians mode or degrees mode

From now on, always decide whether the calculator should be in radians mode or degrees mode before using any of the trigonometric functions.


Example 5 Using the calculator in radians mode

Evaluate correct to four decimal places:

a $\cos 1$

b $\cot 1.3$

Solution

Here the calculator must be set to radians mode.

a $\cos 1 \div 0.5403$

$$\begin{aligned}\mathbf{b} \quad \cot 1.3 &= \frac{1}{\tan 1.3} \\ &\div 0.2776\end{aligned}$$

Exercise 11A

FOUNDATION

Note: Be very careful throughout the remainder of this chapter whether your calculator is set in radians or degrees. The button used to make the change is usually labelled mode.

1 Express the following angles in radians.

a 90°

b 45°

c 30°

d 60°

e 120°

f 150°

g 135°

h 225°

i 360°

j 300°

k 270°

l 210°

2 Express the following angles in degrees.

a π

b 2π

c 4π

d $\frac{\pi}{2}$

e $\frac{\pi}{3}$

f $\frac{\pi}{4}$

g $\frac{2\pi}{3}$

h $\frac{5\pi}{6}$

i $\frac{3\pi}{4}$

j $\frac{3\pi}{2}$

k $\frac{4\pi}{3}$

l $\frac{7\pi}{4}$

m $\frac{11\pi}{6}$

3 Use your calculator in radian mode to evaluate, correct to two decimal places:

a $\sin 1$

b $\cos 2$

c $\tan 3$

d $\sin 0.7$

e $\cos 1.23$

f $\tan 5.678$

4 Use your calculator to express in radians, correct to three decimal places:

a 73°

b 14°

c 168°

d $21^\circ 36'$

e $95^\circ 17'$

f $211^\circ 12'$

5 Use your calculator to express in degrees and minutes, correct to the nearest minute:

a 2 radians

b 0.3 radians

c 1.44 radians

d 0.123 radians

e 3.1985 radians

f 5.64792 radians

DEVELOPMENT

6 Using the two special triangles, find the exact value of:

a $\sin \frac{\pi}{6}$

b $\sin \frac{\pi}{4}$

c $\cos \frac{\pi}{6}$

d $\tan \frac{\pi}{3}$

e $\tan \frac{\pi}{4}$

f $\cos \frac{\pi}{3}$

7 Using the two special triangles, find the exact value of:

a $\cot \frac{\pi}{4}$

b $\sec \frac{\pi}{4}$

c $\operatorname{cosec} \frac{\pi}{6}$

d $\cot \frac{\pi}{3}$

e $\operatorname{cosec} \frac{\pi}{3}$

f $\sec \frac{\pi}{6}$

8 Express in radians in terms of π :

a 20°

b 22.5°

c 36°

d 100°

e 112.5°

f 252°

9 Express in degrees:

a $\frac{\pi}{12}$

b $\frac{2\pi}{5}$

c $\frac{20\pi}{9}$

d $\frac{11\pi}{8}$

e $\frac{17\pi}{10}$

f $\frac{23\pi}{15}$

10 **a** Find the complement of $\frac{\pi}{6}$.

b Find the supplement of $\frac{\pi}{6}$.

11 Two angles of a triangle are $\frac{\pi}{3}$ and $\frac{2\pi}{9}$. Find, in radians, the third angle.

12 Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:

a $\sin \frac{2\pi}{3}$

b $\cos \frac{2\pi}{3}$

c $\cos \frac{5\pi}{6}$

d $\tan \frac{4\pi}{3}$

e $\tan \frac{3\pi}{4}$

f $\cos \frac{5\pi}{3}$

g $\sin \frac{5\pi}{4}$

h $\tan \frac{7\pi}{6}$

13 Using the two special triangles and your knowledge of angles of any magnitude, find the exact value of:

a $\sec \frac{2\pi}{3}$

b $\cot \frac{5\pi}{4}$

c $\operatorname{cosec} \frac{7\pi}{6}$

d $\cot \frac{5\pi}{6}$

e $\operatorname{cosec} \frac{5\pi}{3}$

f $\sec \frac{11\pi}{6}$

g $\cot \frac{4\pi}{3}$

h $\operatorname{cosec} \frac{7\pi}{4}$

14 Give each answer in degrees and then in radians.

a What angle do the hour and minute hands of a clock move through in 1 hour?

b Find the non-reflex angle between the hour and minute hands of a clock at:

i 2:00 pm

ii 2:15 pm

iii 2:30 pm

iv 2:45 pm

CHALLENGE

15 If $f(x) = \sin x$, $g(x) = \cos 2x$ and $h(x) = \tan 3x$, find, correct to three significant figures:

a $f(1) + g(1) + h(1)$

b $f(g(h(1)))$

16 Find, correct to three decimal places, the angle in radians through which:

a the second hand of a clock turns in 7 seconds,

b the hour hand of a clock turns between 6 am and 6:40 am.

11B Solving trigonometric equations

Learning intentions

- Solve trig equations in radians.

Once radians have been introduced into trigonometric functions, trigonometric equations can be solved in radians as well as in degrees.

Solving trigonometric equations in radians

Solving a trigonometric equation is done the same way whether the solution is to be given in radians or degrees.

5 Solving a trigonometric equation in radians

- First establish on a diagram which quadrants the angle can lie in.
- Then find the related angle — but use radian measure, not degrees.
- Use the diagram and the related angle to find all the answers, taking account of any restrictions on the angle.

Warning: When using the calculator's inverse trigonometric functions:

Never work with a negative number.

Always enter the absolute value of the number in order to find the related angle.

Alternatively, solve the equation first in degrees, then convert each answer to radians. But this can be clumsy, so only do it as a check until you are confident.



Example 6 Solving in radians trig equations with acute angles

Solve each trigonometric equation in radians, where the angle x is an acute angle.
Give answers in exact form, or correct to five significant figures.

a $\sin x = \frac{1}{2}$

b $\tan x = 3$

Solution

With acute angles, a quadrants diagram is not necessary.

a $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ (30° changed to radians)

b $\tan x = 3$

$x \doteq 1.1071$ (use the calculator)


Example 7 Solving in radians trig equations with general angles

Solve these trigonometric equations. Give answers in exact form if possible, otherwise correct to five significant figures.

a $\cos x = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$

b $\cos 3x = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$

c $\sin x = -\frac{1}{3}$, where $0 \leq x \leq 2\pi$

Solution

a $\cos x = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$.

Because $\cos x$ is negative, x is in quadrant 2 or 3.

The acute angle whose cosine is $+\frac{1}{2}$ is 60° , or $\frac{1\pi}{3}$.

Hence $x = \pi - \frac{1\pi}{3}$ or $\pi + \frac{1\pi}{3}$
 $= \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

b $\cos 3x = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$.

To find the domain of $3x$, multiply the condition by 3,

$\cos 3x = -\frac{1}{2}$, where $0 \leq 3x \leq 6\pi$.

Working with the diagram in part (a),

$3x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$ or $\frac{16\pi}{3}$

$x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}$ or $\frac{16\pi}{9}$

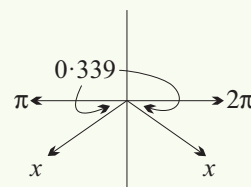
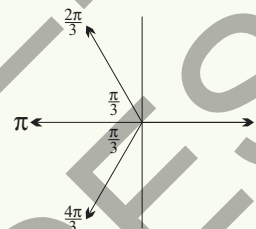
c $\sin x = -\frac{1}{3}$, where $0 \leq x \leq 2\pi$.

Because $\sin x$ is negative, x is in quadrant 3 or 4.

With the calculator in radians mode, enter $+\frac{1}{3}$, not $-\frac{1}{3}$, then the related angle $0.339\ 836\ \dots$ (store this in memory).

Hence $x = \pi + 0.339\ 836\ \dots$ or $2\pi - 0.339\ 836\ \dots$

$\div 3.4814$ or 5.9433 .


Solving trigonometric equations reducible to quadratics

Some trigonometric equations are quadratic equations in a trigonometric function. Some such equations can be solved directly, but it is often useful to make an algebraic substitution, as we did in Section 8E with exponential equations. Both methods are used in the next example.


Example 8 Solving in radians trig equations reducible to quadratics

Solve these trigonometric equations in radians, for $0 \leq x \leq 2\pi$.

a $\sin^2 x = \frac{3}{4}$

b $2\cos^2 x - \cos x - 1 = 0$

Solution

a This is easily done without any substitution.

$\sin^2 x = \frac{3}{4}$

$\sin x = \frac{\sqrt{3}}{2}$ or $\sin x = -\frac{\sqrt{3}}{2}$

The related angle is $\frac{1\pi}{3}$, and the angle can be in any of the four quadrants,

so $x = \frac{1\pi}{3}$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{5\pi}{3}$.

- b** Because of the factoring required, substitution is a better approach here.

$$2 \cos^2 x - \cos x - 1 = 0$$

Let $u = \cos x$.

Then $2u^2 - u - 1 = 0$, thus 'reducing' it to a quadratic equation,

$$2u^2 - 2u + u - 1 = 0$$

$$2u(u - 1) + (u - 1) = 0$$

$$(2u + 1)(u - 1) = 0$$

$$u = -\frac{1}{2} \text{ or } u = 1$$

$$\cos x = -\frac{1}{2} \text{ or } \cos x = 1.$$

When $\cos x = -\frac{1}{2}$, the related angle is $\frac{1\pi}{3}$, and the angle is quadrant 2 or 3,

so $x = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$.

When $\cos x = 1$, the solutions are boundary angles, so we use the graph.

From the graphs in Chapter 6, the answers are 0° and 360° .

Converting to radians, or using the graphs in radians in the later Section 11D,

$$x = 0 \text{ or } 2\pi.$$

Hence the solutions are $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π .

Exercise 11B

FOUNDATION

- 1** Find, in radians, the acute angle θ that satisfies:

a $\tan \theta = 1$

b $\sin \theta = \frac{1}{2}$

c $\cos \theta = \frac{1}{\sqrt{2}}$

d $\tan \theta = \frac{1}{\sqrt{3}}$

e $\sin \theta = \frac{\sqrt{3}}{2}$

f $\cos \theta = \frac{1}{2}$

- 2** Find, correct to three decimal places, the value of x between 0 and $\frac{\pi}{2}$ that satisfies each equation. (Your calculator needs to be in radians mode.)

a $\tan x = 3$

b $\sin x = 0.8$

c $\cos x = 0.4$

d $\sin x = 0.234$

e $\cos x = 0.987$

f $\tan x = 100$

- 3** Solve each equation for $0 \leq x \leq 2\pi$.

a $\sin x = \frac{1}{2}$

b $\cos x = -\frac{1}{2}$

c $\tan x = -1$

d $\sin x = 1$

e $2 \cos x = \sqrt{3}$

f $\sqrt{3} \tan x = 1$

g $\cos x + 1 = 0$

h $\sqrt{2} \sin x + 1 = 0$

DEVELOPMENT

- 4** Solve each equation for $0 \leq \theta \leq 2\pi$. Remember that a positive number has two square roots.

a $\sin^2 \theta = 1$

b $\tan^2 \theta = 1$

c $\cos^2 \theta = \frac{1}{4}$

d $\cos^2 \theta = \frac{3}{4}$

- 5** Consider the equation $\cos^2 \theta - \cos \theta = 0$, for $0 \leq \theta \leq 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \cos \theta$.

b Solve the quadratic equation for u .

c Hence find the values of θ that satisfy the original equation.

- 6** Consider the equation $\tan^2 \theta - \tan \theta - 2 = 0$, for $0 \leq \theta \leq 2\pi$.

a Write the equation as a quadratic equation in u by letting $u = \tan \theta$.

b Solve the quadratic equation for u .

c Hence find the values of θ that satisfy the original equation, giving the solutions correct to two decimal places where necessary.

- 7** Solve each equation for $0 \leq \theta \leq 2\pi$ by transforming it into a quadratic equation in u . Give your solutions correct to two decimal places where necessary.
- a** $\tan^2 \theta + \tan \theta = 0$ (Let $u = \tan \theta$.)
 - b** $2 \sin^2 \theta - \sin \theta = 0$ (Let $u = \sin \theta$.)
 - c** $\sin^2 \theta + \sin \theta - 2 = 0$ (Let $u = \sin \theta$.)
 - d** $\tan^2 \theta + \tan \theta - 6 = 0$ (Let $u = \tan \theta$.)
 - e** $2 \cos^2 \theta + \cos \theta - 1 = 0$
 - f** $2 \sin^2 \theta - \sin \theta - 1 = 0$
 - g** $3 \sin^2 \theta + 8 \sin \theta - 3 = 0$
 - h** $3 \cos^2 \theta - 8 \cos \theta - 3 = 0$

CHALLENGE

- 8** [Technology] Graphing programs provide an excellent way to see what is happening when an equation has many solutions. The equations in Question 2 are quite simple to graph, because $y = \text{LHS}$ is a single trigonometric function and $y = \text{RHS}$ is a horizontal line. Every point of intersection corresponds to a solution. Graph the LHS of each equation in question 2, and verify the solutions you obtained.
- 9** Use trigonometric identities to transform each equation so that it only involves one trigonometric function. Then solve it for $0 \leq x \leq 2\pi$, giving solutions correct to two decimal places where necessary.
- a** $2 \sin^2 x + \cos x = 2$
 - b** $\sec^2 x - 2 \tan x - 4 = 0$
 - c** $8 \cos^2 x = 2 \sin x + 7$
 - d** $6 \tan^2 x = 5 \sec x$
- 10** Solve each equation for $0 \leq \alpha \leq 2\pi$, giving solutions correct to two decimal places where necessary. Again, you will need trigonometric identities.
- a** $3 \sin \alpha = \operatorname{cosec} \alpha + 2$
 - b** $3 \tan \alpha - 2 \cot \alpha = 5$
- 11** Solve each equation for $0 \leq x \leq 2\pi$ by first dividing through by $\cos^2 x$. Give solutions correct to two decimal places where necessary.
- a** $\sin^2 x + \sin x \cos x = 0$
 - b** $\sin^2 x - 5 \sin x \cos x + 6 \cos^2 x = 0$

11C Arcs and sectors of circles

Learning intentions

- Develop and work with formulae for arc length and sector and segment area.

The lengths of arcs and the areas of sectors and segments can already be calculated using fractions of circles, but radian measure allows these formulae to be expressed in far more elegant forms.

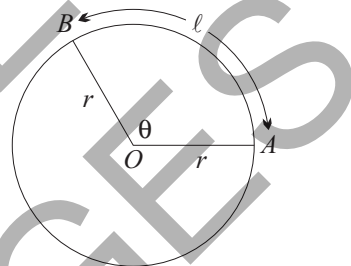
Arc length

In the diagram to the right, the arc AB has length ℓ and subtends an angle θ at the centre O of a circle with radius r .

The definition of angle size in radians is $\theta = \frac{\ell}{r}$.

Multiplying through by r , $\ell = r\theta$.

This is the standard formula for arc length:



6 Arc length

An arc subtending an angle θ at the centre of a circle of radius r has length

$$\ell = r\theta$$

Note: When the radius is 1, the arc length formula becomes $\ell = \theta$. Thus in a circle of radius 1, we can identify the arc length and the angle size in radians.

This is often taken as the definition of angle size in radians, but if this is done, it is important to remember that *an angle size in radians is still a pure number without units*, because it is the ratio of a length and a unit length.

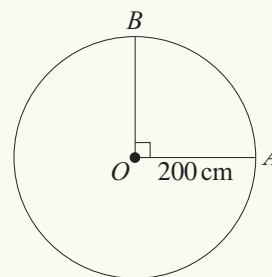


Example 9 Finding arc length

What is the length of an arc subtending a right angle at the centre of a circle of radius 200 cm?

Solution

$$\begin{aligned} \text{Arc length} &= r\theta \\ &= 200 \times \frac{\pi}{2} \quad (\text{a right angle has size } \frac{\pi}{2}) \\ &= 100\pi \text{ cm.} \end{aligned}$$




Example 10 Substituting into the formula for arc length

What is the radius of a circle in which an arc of length 5 metres subtends an angle of 120° at the centre?
Answer correct to the nearest centimetre.

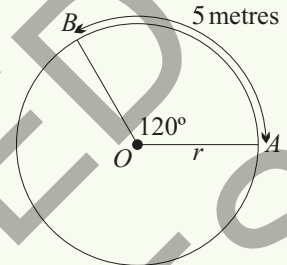
Solution

Substituting into $\ell = r\theta$,

$$5 = r \times \frac{2\pi}{3} \quad (120^\circ \text{ in radians is } \frac{2\pi}{3})$$

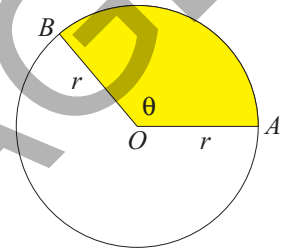
$$\boxed{\times \frac{3}{2\pi}} \quad r = \frac{15}{2\pi}$$

$$\div 2.39 \text{ metres.}$$


Area of a sector

In the diagram to the right, the shaded sector AOB is bounded by the arc AB and the radii OA and OB . Its area can be calculated as a fraction of the total area:

$$\begin{aligned} \text{area of sector} &= \frac{\theta}{2\pi} \times \text{area of circle} \\ &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 \theta. \end{aligned}$$


7 Area of a sector

An arc subtending an angle θ at the centre of a circle of radius r has area

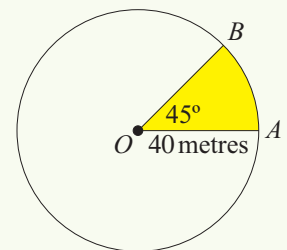
$$\text{Area} = \frac{1}{2} r^2 \theta$$


Example 11 Finding the area of a sector

What are the area and the perimeter of a sector subtending an angle of 45° at the centre of a circle of radius 40 metres?

Solution

$$\begin{aligned} \text{Area of sector} &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} \times 1600 \times \frac{\pi}{4} \quad (45^\circ \text{ in radians is } \frac{\pi}{4}) \\ &= 200\pi \text{ square metres.} \\ \text{Perimeter} &= \text{arc length} + 2r \\ &= \frac{\pi}{4} \times 40 + 2 \times 40 \\ &= 10\pi + 80 \text{ metres.} \end{aligned}$$




Example 12 Substituting into the formula for the area of a sector

A circular cake has radius 12 cm. What angle at the centre is subtended by a sector of area 100 cm^2 ? Answer correct to the nearest degree.

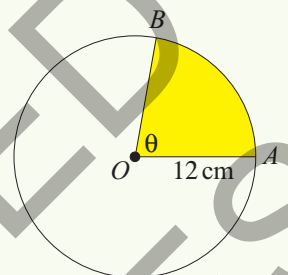
Solution

$$\text{Area of sector} = \frac{1}{2}r^2\theta.$$

$$100 = \frac{1}{2} \times 144 \times \theta$$

$$\boxed{\div 72} \quad \theta = \frac{100}{72} \times \frac{180^\circ}{\pi} \quad (\text{converting to degrees})$$

$$\doteq 80^\circ.$$


Proportionality of arc length, area of sector, and θ

We failed to remark in Section 11A that arc length is proportional to the angle θ at the centre. Yet we used that proportionality when we established that a right angle is $\frac{\pi}{2}$, a straight angle is π , and a revolution is 2π .

The proportionality follows from the symmetry of the circle — when you use radii to divide the circle into many little pieces, all with the same angle at the centre, these little pieces are all congruent. The statement of proportionality can be written in a way that is immediately obvious from the diagram:

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{\text{one revolution}}$$

Similarly, we used without comment the fact that the area of a sector is proportional to θ , which follows in the same way from the symmetry of the circle. Stating the proportionality in a way that is again obvious from the diagram:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{\text{one revolution}}$$

Combining the two second forms into a single identity:

8 Proportionality of arc length, area of sector, and θ

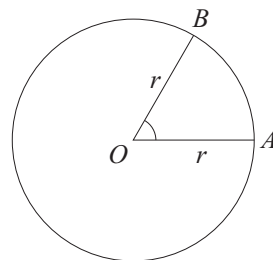
Suppose that an arc of a circle subtends an angle θ at the centre. Then:

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{\text{one revolution}}$$

Major and minor arcs and sectors

In the diagram to the right, the phrase ‘the arc AB ’ is ambiguous, because there are two arcs AB .

- The *minor arc* AB is the arc subtending the marked angle at the centre. This angle is less than a straight angle, and the minor arc is less than half the circumference
- The *major arc* AB is the *opposite arc* subtending the unmarked reflex angle at the centre. This angle is more than a straight angle, and the major arc is more than half the circumference.



The words ‘major’ and ‘minor’ are originally Latin words that simply mean ‘greater’ and ‘lesser’. They apply also to sectors in the obvious way — there are two *opposite sectors* AOB , the *minor sector* containing the marked angle, and the *major sector* containing the unmarked reflex angle.

The length of a major arc and the area of a major sector can be calculated with the usual arc length and sector formulae, but using the reflex angle that they subtend, as in the example below. Alternatively, they can be calculated by subtraction from the circumference or area of the whole circle.



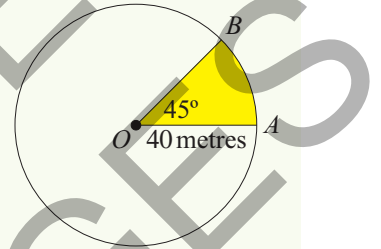
Example 13 Working with major and minor arcs and sections

Find the area and the perimeter of the major sector AOB in the diagram below.

Solution

The major arc and sector each subtend $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ at the centre, and are unshaded in the diagram.

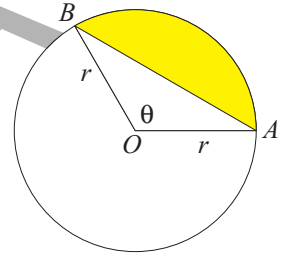
$$\begin{aligned}\text{Area of sector} &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2} \times 1600 \times \frac{7\pi}{4} \quad (315^\circ \text{ in radians is } \frac{7\pi}{4}) \\ &= 1400\pi \text{ square metres.} \\ \text{Perimeter} &= \text{arc length} + 2r \\ &= \frac{7\pi}{4} \times 40 + 2 \times 40 \\ &= 70\pi + 80 \text{ metres.}\end{aligned}$$



Area of a segment

In the diagram to the right, the chord AB divides the circle into two *segments*. The *minor segment* has been shaded, and the rest of the circle is the *major segment*.

Drawing the two radii OA and OB produces an isosceles triangle AOB , together with two opposite sectors. The areas of the two segments can now be found by adding or subtracting appropriate areas.



9 Area of a segment

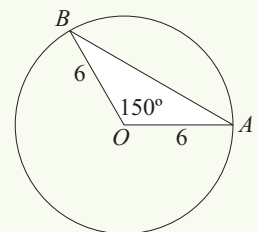
To find the area of a segment:

- Construct the radii from the ends of the chord, to form an isosceles triangle and two opposite sectors.
- Add or subtract the appropriate areas.



Example 14 Finding the area of segments

- Find the lengths of the minor and major arcs formed by two radii of a circle of radius 6 metres meeting at 150° .
- Find the areas of the minor and major sectors.
- Find the area of $\triangle AOB$.
- Find the areas of the major and minor segments.



Solution

The minor arc subtends 150° at the centre, which in radians is $\frac{5\pi}{6}$, and the major arc subtends 210° at the centre, which in radians is $\frac{7\pi}{6}$.

- a** Minor arc $= r\theta$ Major arc $= r\theta$
 $= 6 \times \frac{5\pi}{6}$ $= 6 \times \frac{7\pi}{6}$
 $= 5\pi$ metres. $= 7\pi$ metres.
- b** Minor sector $= \frac{1}{2}r^2\theta$ Major sector $= \frac{1}{2}r^2\theta$
 $= \frac{1}{2} \times 6^2 \times \frac{5\pi}{6}$ $= \frac{1}{2} \times 6^2 \times \frac{7\pi}{6}$
 $= 15\pi \text{ m}^2.$ $= 21\pi \text{ m}^2.$
- c** Area of $\triangle AOB = \frac{1}{2}r^2 \sin \theta$
 $= \frac{1}{2} \times 6^2 \times \sin 150^\circ$ (alternatively $\sin \frac{5\pi}{6}$)
 $= \frac{1}{2} \times 36 \times \frac{1}{2}$
 $= 9 \text{ m}^2.$
- d** The minor segment area is obtained by subtraction.
 Minor segment $=$ minor sector $- \triangle AOB$
 $= (15\pi - 9) \text{ m}^2.$
 The major segment area is obtained by addition.
 Major segment $=$ major sector $+ \triangle AOB$
 $= (21\pi + 9) \text{ m}^2.$

**Example 15** Substituting into the formulae for sector and segment area

Find, correct to the nearest mm, the radius of a circle in which:

- a** a sector, **b** a segment,
 of area 1 square metre subtends an angle of 90° at the centre of the circle.

Solution

Let the radius of the circle be r metres.

$$\begin{aligned} \text{Then area of sector } AOB &= \frac{1}{2}r^2 \times \frac{\pi}{2} \\ &= \frac{\pi}{4}r^2 \end{aligned}$$

and area of $\triangle AOB = \frac{1}{2}r^2$ (it is half a square).

- a** Substituting into the formula for the area of a sector,
 $\frac{\pi}{4} \times r^2 = 1$ (the area is 1 m^2)

$$\boxed{\times \frac{4}{\pi}}$$

$$r^2 = \frac{4}{\pi}$$

$$r \doteq 1.128 \text{ metres.}$$

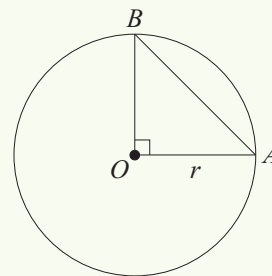
- b** Subtracting, area of segment $= \frac{\pi}{4}r^2 - \frac{1}{2}r^2$
 $= \frac{1}{4}r^2(\pi - 2).$

$$\text{Hence} \quad \frac{1}{4}r^2(\pi - 2) = 1 \quad (\text{the area is } 1 \text{ m}^2)$$

$$\boxed{\times \frac{4}{\pi - 2}}$$

$$r^2 = \frac{4}{\pi - 2}$$

$$r \doteq 1.872 \text{ metres.}$$



Exercise 11C

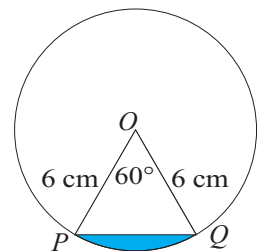
FOUNDATION

Note: Are you working with radians or degrees? Remember the button labelled mode.

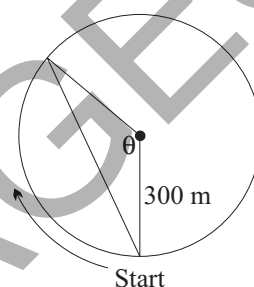
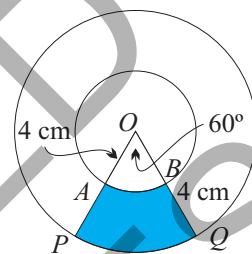
- 1 In the formula $\ell = r\theta$:
 - a Find ℓ , if $r = 18$ and $\theta = \frac{\pi}{6}$.
 - b Find ℓ , if $r = 10$ and $\theta = \frac{\pi}{4}$.
 - c Find r , if $\ell = 15$ and $\theta = 2$.
 - d Find r , if $\ell = 6\pi$ and $\theta = \frac{\pi}{4}$.
 - e Find θ , if $\ell = 2\pi$ and $r = 8$.
 - f Find θ , if $\ell = 3\pi$ and $r = 1.5$.
- 2 In the formula $A = \frac{1}{2}r^2\theta$:
 - a Find A , if $r = 4$ and $\theta = \frac{\pi}{4}$.
 - b Find A , if $r = 2$ and $\theta = \frac{2\pi}{3}$.
 - c Find θ , if $A = 16$ and $r = 4$.
 - d Find θ , if $A = 12\pi$ and $r = 6$.
 - e Find r , if $A = 54$ and $\theta = 3$.
 - f Find r , if $A = 40\pi$ and $\theta = \frac{1\pi}{5}$.
- 3 A circle has radius 6 cm. Find the length of an arc of this circle that subtends an angle at the centre of:
 - a 2 radians
 - b 0.5 radians
 - c $\frac{\pi}{3}$ radians
 - d $\frac{\pi}{4}$ radians
- 4 A circle has radius 8 cm. Find the area of a sector of this circle that subtends an angle at the centre of:
 - a 1 radian
 - b 3 radians
 - c $\frac{\pi}{4}$ radians
 - d $\frac{3\pi}{8}$ radians

DEVELOPMENT

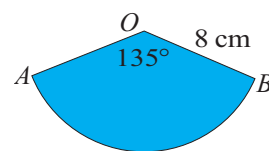
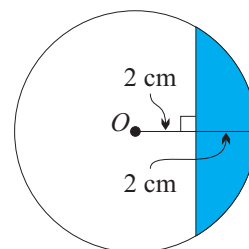
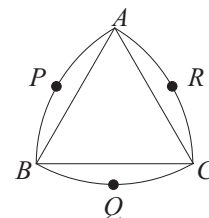
- 5 What is the radius of the circle in which an arc of length 10 cm subtends an angle of 2.5 radians at the centre?
- 6 If a sector of a circle of radius 4 cm has area 12 cm^2 , find the angle at the centre in radians.
- 7 A circle has radius 3.4 cm. Find, correct to the nearest millimetre, the length of an arc of this circle that subtends an angle at the centre of:
 - a 40°
 - b $73^\circ 38'$
 (Hint: Remember that θ must be in radians.)
- 8 Find, correct to the nearest square metre, the area of a sector of a circle of radius 100 metres if the angle at the centre is 100° .
- 9 A circle has radius 12 cm. Find, in exact form:
 - a the length of an arc that subtends an angle of 120° at the centre,
 - b the area of a sector in which the angle at the centre is 40° .
- 10 An arc of a circle of radius 7.2 cm is 10.6 cm in length. Find the angle subtended at the centre by this arc, correct to the nearest degree.
- 11 A sector of a circle has area 52 cm^2 and subtends an angle of $44^\circ 16'$. Find the radius in cm, correct to one decimal place.
- 12 Consider the diagram to the right.
 - a Find the exact area of sector OPQ .
 - b Find the exact area of $\triangle OPQ$.
 - c Hence find the exact area of the shaded minor segment.



- 13** A chord of a circle of radius 4 cm subtends an angle of 150° at the centre.
- Use the same method as the previous question to show that the area of the minor segment cut off by the chord is $\frac{4}{3}(5\pi - 3)\text{ cm}^2$.
 - By subtracting the area of the minor segment from the area of the circle, show that the area of the major segment cut off by the chord is $\frac{4}{3}(7\pi + 3)\text{ cm}^2$.
- 14** A circle has centre C and radius 5 cm, and an arc AB of this circle has length 6 cm. Find the area of the sector ACB .
- 15** The diagram to the right shows two concentric circles with common centre O .
- Find the exact perimeter of the region $APQB$.
 - Find the exact area of the region $APQB$.
- 16** An athlete runs at a steady 4 m/s around a circular track of radius 300 metres. She runs clockwise, starting at the southernmost point.
- How far has she run after 3 minutes?
 - What angle does this distance subtend at the centre?
 - How far, in a direct line across the field, is she from her start, correct to the nearest 0.01 m?
 - What is her true bearing from the centre then, correct to the nearest minute?

**CHALLENGE**

- 17** Triangle ABC is equilateral with side length 2 cm. Circular arcs AB , BC and CA have centres C , A and B respectively. Answer each part in exact form.
- Find the length of the arc AB .
 - Find the area of the sector $CAPBC$.
 - Find the length of the perimeter $APBQCRA$.
 - Find the area of $\triangle ABC$ and hence find the area enclosed by the perimeter $APBQCRA$.
- 18** Find the exact area of the shaded region of the circle shown in the diagram to the right.
- 19** A piece of paper is cut in the shape of a sector of a circle. The radius is 8 cm and the angle at the centre is 135° . The straight edges of the sector are placed together so that a cone is formed.
- Show that the base of the cone has radius 3 cm.
 - Show that the cone has perpendicular height $\sqrt{55}$ cm.
 - Hence find, in exact form, the volume of the cone.
 - Find the curved surface area of the cone.



11D Trigonometric graphs in radians

Learning intentions

- Graph the trig functions using radians.
- Identify period, amplitude and asymptotes (when appropriate).
- Identify symmetries of the trig graphs.

Now that angle size has been defined as a ratio, that is, as a pure number, the trigonometric functions can be drawn in their true shapes.

A note on the Syllabus: The graphs of cosecant, secant, and cotangent are not required in this course (but their algebraic properties are required). They are included here to complete the story, with all six interrelated graphs being viewed together. Reading their properties off their given graphs is a standard skill.

The graphs of the trig functions in radians

On the next page, the graphs of the six functions in radians have been drawn using the same scale on the x -axis and y -axis. This means that the gradient of the tangent at each point now equals the true value of the derivative there.

For example, place a ruler on the graph of $y = \sin x$ so that it makes a tangent to the curve at the origin. The ruler should lie along the line $y = x$, indicating that the tangent at the origin has gradient 1. In the language of calculus, this means that the derivative of $\sin x$ has value 1 when $x = 0$. This is where we will begin when the topic is taken up again in Year 12.

The amplitude of the sine and cosine functions

The *amplitude* of a wave is the maximum height of the wave above the mean position. Both $y = \sin x$ and $y = \cos x$ have a maximum value of 1, a minimum value of -1 and a mean value of 0. Thus both waves have amplitude 1.

10 The amplitudes of the sine and cosine functions

- $y = \sin x$ and $y = \cos x$ both have amplitude 1.

The other four trigonometric functions increase without bound near their asymptotes, so the idea of amplitude makes no sense. We can conveniently tie down the vertical scale of $y = \tan x$, however, by using the fact that $\tan \frac{\pi}{4} = 1$.

The periods of the trigonometric functions

The trigonometric functions are called *periodic functions* because each graph repeats itself exactly over and over again. The *period* of such a function is the length of the smallest repeating unit.

The graphs of $y = \sin x$ and $y = \cos x$ on the previous page are waves, with a pattern that repeats every revolution. Thus they both have period 2π .

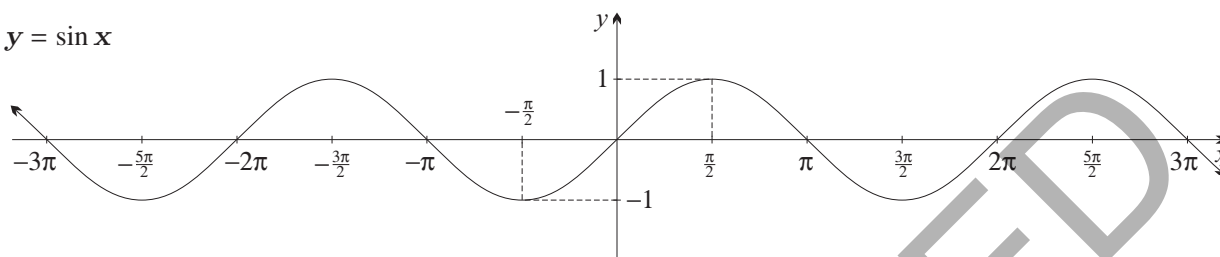
The graph of $y = \tan x$, on the other hand, has a pattern that repeats every half-revolution. Thus it has period π .

11 The periods of the sine, cosine and tangent functions

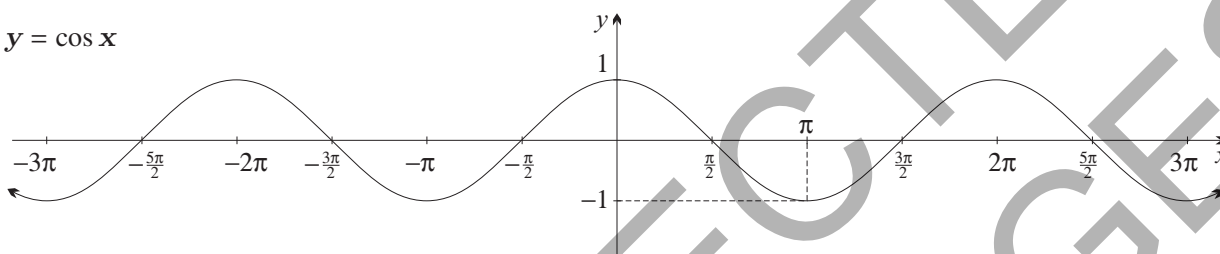
- $y = \sin x$ and $y = \cos x$ both have period 2π (that is, a full revolution).
- $y = \tan x$ has period π (that is, half a revolution).

The periods of the secant, cosecant and cotangent functions, and other properties and symmetries, are read off their graphs in the following investigation exercise.

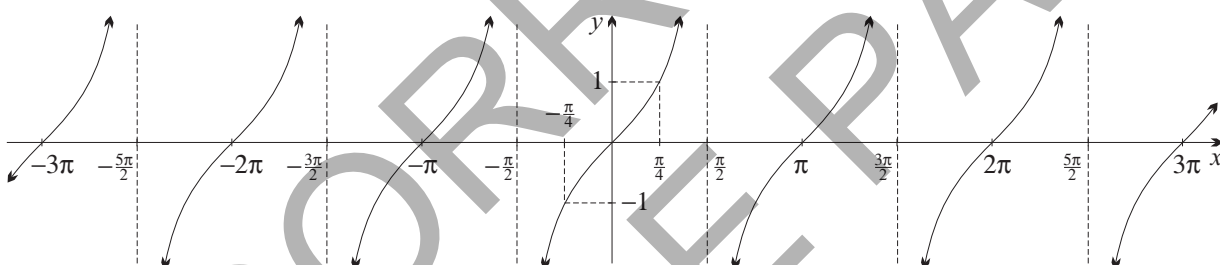
$$y = \sin x$$



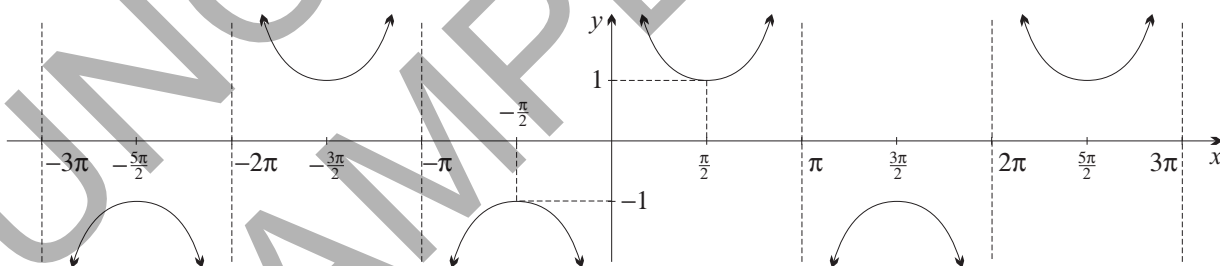
$$y = \cos x$$



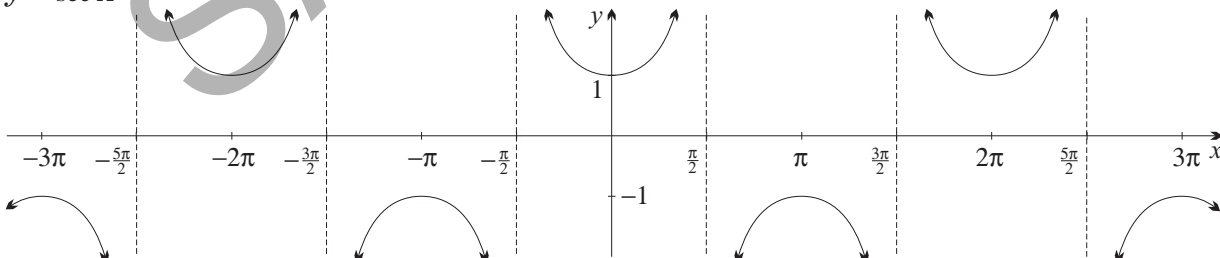
$$y = \tan x$$



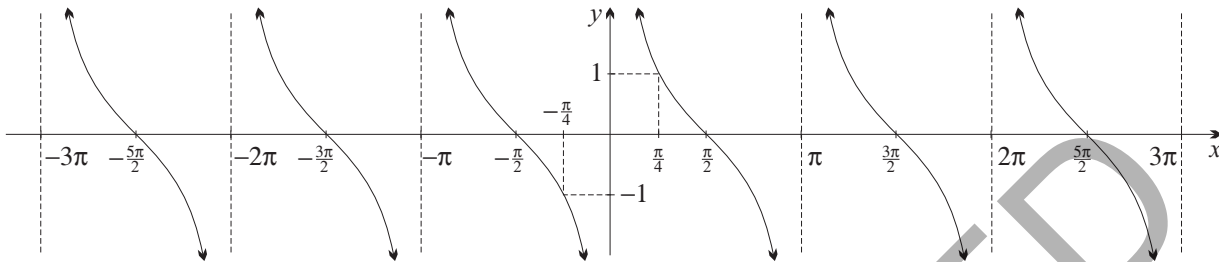
$$y = \operatorname{cosec} x$$



$$y = \sec x$$



$$y = \cot x$$



Oddness and evenness of the trigonometric functions

The graphs of $y = \sin x$ and $y = \tan x$ have point symmetry in the origin, as can easily be seen from their graphs on the previous page. This means that the functions $\sin x$ and $\tan x$ are odd functions. Algebraically, $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$.

The graph of $y = \cos x$, however, has line symmetry in the y -axis. This means that the function $\cos x$ is an even function. Algebraically, $\cos(-x) = \cos x$.

12 Oddness and evenness of the trigonometric functions

- The functions $\sin x$ and $\tan x$ are odd functions. Thus

$$\sin(-x) = -\sin x \quad \text{and} \quad \tan(-x) = -\tan x.$$
- The function $\cos x$ is an even function. Thus

$$\cos(-x) = \cos x.$$

See the following investigation exercise for the oddness and evenness of $\operatorname{cosec} x$, $\sec x$, and $\cot x$.

A preview of what is to come

Now that radian measure has been introduced, the way is clear in Year 12 to differentiate the trigonometric functions $\sin x$ and $\cos x$. The results are surprisingly simple:

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

These results are also reasonably obvious from the graphs.

- Look at the graph of $y = \sin x$ drawn on the previous page. Look at the places where the tangent is horizontal, where the tangent has its greatest positive slope, and where the tangent has its greatest negative slope. Then on a separate set of axes, begin a sketch of $y = \frac{d}{dx}(\sin x)$. This second sketch looks very much like $y = \cos x$.
- Do the same with the graph of $y = \cos x$. This time, the second sketch looks very much like $y = -\sin x$.

So differentiating a wave gives another wave shifted backwards by $\frac{\pi}{2}$, which is a quarter-revolution. But all that is next year's story.

Exercise 11D

FOUNDATION

This exercise is an investigation. Its purpose is familiarity with the six trigonometric functions, and in particular familiarity with their symmetries.

Except for question 1, refer all the time to the page at the start of this section where all six trigonometric graphs have been drawn in radians with the same scale on both axes.

An accurate drawing and an accurate gradient

- 1 **a** On graph paper, using a scale on both axes of 1 unit = 2 cm, draw an accurate graph of $y = \sin x$ for $-\pi \leq x \leq \pi$. Use your calculator to obtain a table of values, and in the interval from $x = -0.5$ to $x = 0.5$, plot points every 0.1 units. Also plot the important points at $x = \frac{\pi}{6}$, $x = \frac{\pi}{2}$, $x = \frac{5\pi}{6}$, $x = \pi$, and the opposites of these values.
- b** Place your ruler on the graph to confirm that the gradient of the tangent at the origin appears to be 1.

Reading the graphs backwards

Examine the graphs of the six trigonometric functions.

- 2 Use horizontal lines on the graph of $y = \sin x$ to find, if possible, at least six positive solutions of:

a $\sin x = 0$	b $\sin x = \frac{1}{2}$	c $\sin x = 1$	d $\sin x = 2$

Domains and ranges

Examine the graphs of the six trigonometric functions.

- 3 Write down the domain and the range of:

a $y = \sin x$	b $y = \cos x$	c $y = \tan x$
d $y = \operatorname{cosec} x$	e $y = \sec x$	f $y = \cot x$

Line symmetries

Examine the graphs of the six trigonometric functions.

- 4 **a** List all the vertical lines in which the graph of $y = \sin x$ is symmetric.
b Which other trigonometric graphs are symmetric in the same vertical lines, and why?
c In which vertical lines are the graphs of $y = \tan x$ and $y = \cot x$ symmetric?
- 5 **a** List all the vertical lines in which the graph of $y = \cos x$ is symmetric.
b Which of these lines is related to the fact that $\cos x$ is even, that is, $\cos(-x) = \cos x$?
c Which other trigonometric graphs are symmetric in the same vertical lines, and why?

Point symmetries

Examine the graphs of the six trigonometric functions, and recall that a graph has point symmetry about the point P if it has rotational symmetry of 180° about P .

- 6 **a** List all the points in which the graph of $y = \sin x$ has point symmetry.
b Which of these points is related to the fact that $\sin x$ is odd, that is, $\sin(-x) = -\sin x$?
c Which other graph has the same point symmetries as $y = \sin x$, and why?
- 7 **a** List all the points in which the graph of $y = \cos x$ has point symmetry.
b Which other graph has the same point symmetries as $y = \cos x$, and why?
- 8 **a** About which points do $y = \tan x$ and $y = \cot x$ have point symmetry?
b Relate these symmetries to the oddness or evenness of $y = \tan x$ and $y = \cot x$.

Translation symmetries

Examine the graphs of the six trigonometric functions.

- 9 a** What translation symmetry does $y = \sin x$ have? That is, what translations map the graph of $y = \sin x$ onto itself?
- b** What other trigonometric graphs have exactly these same translation symmetries?
- c** What translation symmetries do the other trigonometric graphs have?
- d** What periods do the six trigonometric functions have?

Transformations

Examine the graphs of the six trigonometric functions.

- 10 a** Reflection in some vertical lines transforms the graph of $y = \sin x$ onto the graph of $y = \cos x$. Identify all such vertical lines.
- b** For what other pair of trigonometric functions is the answer to the question in part (a) the same?
- c** Identify all vertical lines that reflect $y = \tan x$ onto $y = \cot x$.
- 11 a** Identify all translations that shift $y = \sin x$ onto the graph of $y = \cos x$.
- b** For what values of θ is it true that $\sin(x - \theta) = \cos x$?
- c** Identify all translations that shift $y = \tan x$ onto the graph of $y = \cot x$.
- 12** Identify any rotations about the origin that rotate the graph of one trigonometric function onto another.

Intersections

Examine the graphs of the six trigonometric functions.

- 13 a** Sketch $y = \sin x$ and $y = \cos x$ on one set of axes. Where do they intersect?
- b** Write down the corresponding trigonometric equation and solve it algebraically.
- 14 a** Sketch $y = \sin x$ and $y = \tan x$ on one set of axes. Where do they intersect?
- b** Write down the corresponding trigonometric equation and solve it algebraically.
- 15 a** Sketch $y = \cos x$ and $y = \tan x$ together. Very approximately, what is the smallest positive value of x where they intersect?
- 16 a** Describe the situation when $y = \sin x$ and $y = \operatorname{cosec} x$ are drawn together.
- b** What other pair of graphs have the same relationship?
- c** Name all the pairs of trigonometric functions whose graphs do not intersect when drawn together.

A harder question

- 17 a** Write down the trigonometric equation corresponding to the intersections of $y = \cos x$ and $y = \tan x$, and solve it algebraically.
- b** Write down the trigonometric equation corresponding to the intersections of $y = \tan x$ and $y = \sec x$, and show algebraically that it has no solutions.

Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- Express in radians in terms of π :
 a 180° b 20° c 240° d 315°
- Express in degrees:
 a $\frac{\pi}{6}$ b $\frac{3\pi}{5}$ c 3π d $\frac{5\pi}{3}$
- Find the exact value of:
 a $\sin \frac{1\pi}{3}$ b $\tan \frac{5\pi}{6}$ c $\operatorname{cosec} \frac{\pi}{4}$ d $\sec \frac{7\pi}{6}$
- Solve for $0 \leq x \leq 2\pi$:
 a $\cos x = \frac{1}{\sqrt{2}}$ b $\tan x = -\sqrt{3}$
- Solve each equation for $0 \leq \theta \leq 2\pi$ by reducing it to a quadratic equation in u . Give your solutions in terms of π , or approximated correct to two decimal places, as appropriate.
 a $2 \sin^2 \theta + \sin \theta = 0$ b $\cos^2 \theta - \cos \theta - 2 = 0$ c $2 \tan^2 \theta + 5 \tan \theta - 3 = 0$
- A circle has radius 12 cm. Find, in exact form:
 a the length of an arc that subtends an angle at the centre of 45° ,
 b the area of a sector in which the angle at the centre is 60° .
- A chord of a circle of radius 8 cm subtends an angle at the centre of 90° . Find, correct to three significant figures, the area of the minor segment cut off by the chord.
- Find, correct to the nearest minute, the angle subtended at the centre of a circle of radius 5 cm by an arc of length 13 cm.
- Find, correct to three significant figures, the radius of the circle in which a sector of area 20 cm^2 subtends an angle of 50° at the centre.

- 10** **a** Which of the six trigonometric graphs have amplitudes, and what are they?
b What are the periods of the six trigonometric graphs?
c Which of the six trigonometric functions are odd, and which are even?
- 11** **a** Give the smallest positive value of θ for which $\cos(x - \theta) = \sin x$.
b Give the smallest positive value of θ for which $\sin(x - \theta) = \cos x$.
c What is the smallest positive value of x for which $\sin x = \cos x$?

12

Probability

Chapter introduction

Probability arises when one performs an experiment that has various possible outcomes, but there is insufficient information to predict which of these outcomes will occur. The classic examples of this are tossing a coin, throwing a die, and drawing a card from a pack. Probability, however, is involved in almost every experiment done in science, and is fundamental to understanding statistics.

This chapter reviews some basic ideas of probability and develops a more systematic approach to solving probability problems. It concludes with the new topic of conditional probability.

12A Sets and Venn diagrams

Learning intentions

- Use the basic language and notation of set theory, with Venn diagrams.
- Relate the words ‘and’, ‘or’, and ‘not’ to intersection, union, and complement.

A probability experiment has many possible outcomes. We shall be using the language of set theory to handle and keep track of this *set of possible outcomes* while calculating probabilities. A brief review of sets and Venn diagrams is needed in preparation for their application to situations involving probability. The three key ideas needed in probability are the intersection of sets, the union of sets, and the complement of a set.

Logic is very close to the surface when we talk about sets and Venn diagrams. The three ideas of intersection, union and complement mentioned above correspond very precisely to the words ‘and’, ‘or’, and ‘not’.

Listing sets and describing sets

A *set* is a collection of things called *elements* or *members*. When a set is specified, it needs to be made absolutely clear what things are its members. This can be done by *listing* the members inside curly brackets. For example,

$$S = \{1, 3, 5, 7, 9\},$$

which is read as ‘ S is the set whose members are 1, 3, 5, 7, and 9’.

A set can also be specified by *writing a description* of the members inside curly brackets. For example,

$$T = \{\text{odd whole numbers less than 10}\},$$

which is read as ‘ T is the set of odd whole numbers less than 10’.

Equal sets

Two sets are called *equal* if they have exactly the same members. Hence the sets S and T in the previous paragraph are equal — written as $S = T$. The order in which the members are written doesn’t matter at all, and neither does repetition. For example,

$$\{1, 3, 5, 7, 9\} = \{3, 9, 7, 5, 1\} = \{5, 9, 1, 3, 7\} = \{1, 3, 1, 5, 1, 7, 9\}.$$

The size of a set

A set may be *finite*, such as the set above of odd whole numbers less than 10, or *infinite*, such as the set of all whole numbers.

If a set S is finite, then the symbol $|S|$ is used to mean the number of members of S . For example:

$$\text{If } A = \{5, 6, 7, 8, 9, 10\}, \text{ then } |A| = 6.$$

$$\text{If } B = \{\text{letters in the alphabet}\}, \text{ then } |B| = 26.$$

$$\text{If } C = \{12\}, \text{ then } |C| = 1.$$

$$\text{If } D = \{\text{odd numbers between 1.5 and 2.5}\}, \text{ then } |D| = 0.$$

The number of elements in a set S can also be written as $n(S)$.

The empty set

The last set D above is called *the empty set*, because it has no members at all. The usual symbol for the empty set is $\emptyset$. There is only one empty set, because any two empty sets have exactly the same members (that is, none at all), and so are equal.

1 Set terminology and notation

- A *set* is a collection of *elements*. For example,
 $S = \{1, 3, 5, 7, 9\}$ and $T = \{\text{odd integers from 0 to 10}\}$.
- Two sets are called *equal* if they have exactly the same members.
- The number of elements in a finite set S is written as $|S|$.
- The *empty set* is written as $\emptyset$ — there is only one empty set.

The number of elements in a set S can also be written as $n(S)$.

Intersection and union, and disjoint sets

There are two obvious ways of combining two sets A and B .

- The *intersection* $A \cap B$ of A and B is the set of elements in A and in B .
- The *union* $A \cup B$ of A and B is the set of elements in A or in B .

For example, if $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 6\}$, then

$$A \cap B = \{1, 3\}$$

$$A \cup B = \{0, 1, 2, 3, 6\}$$

- Two sets A and B are called *disjoint* if they have no elements in common, that is, if $A \cap B = \emptyset$.

For example, if $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$, then

$$A \cap B = \emptyset,$$

so A and B are disjoint.

‘And’ means intersection, ‘or’ means union

The important mathematical words ‘and’ and ‘or’ can be interpreted in terms of union and intersection:

$$A \cap B = \{\text{elements that are in } A \text{ and in } B\}$$

$$A \cup B = \{\text{elements that are in } A \text{ or in } B\}$$

Note: The word ‘or’ in mathematics always means ‘and/or’. Correspondingly, all the elements of $A \cap B$ are members of $A \cup B$.

2 The intersection and union of sets

Intersection of sets (corresponding to ‘and’):

- The *intersection* of two sets A and B is the set of elements in A and in B :

$$A \cap B = \text{elements in } A \text{ and in } B,$$

- The sets A and B are called *disjoint* when $A \cap B = \emptyset$.

Union of sets (corresponding to ‘or’):

- The *union* of A and B is the set of elements in A or in B :

$$A \cup B = \text{elements in } A \text{ or in } B.$$

Warning: The word ‘or’ in mathematics always means ‘and/or’.

Subsets of a set

A set A is called a *subset* of a set B if every member of A is a member of B . This relation is written as $A \subseteq B$. For example:

$$\{2, 3, 4\} \subseteq \{1, 2, 3, 4, 5\}$$

$$\{\text{vowels}\} \subseteq \{\text{letters in the alphabet}\}.$$

Because of the way subsets have been defined, every set is a subset of itself. Also, the empty set is a subset of every set. For example:

$$\{1, 3, 5\} \subseteq \{1, 3, 5\} \quad \text{and} \quad \emptyset \subseteq \{1, 3, 5\}.$$

If $A \subseteq S$ and $A \neq S$, A is called a *proper subset* of S , written $A \subset S$. This means every element of A is in S , but at least one member of S is not in A . For example:

$$\{2, 3, 4\} \subset \{1, 2, 3, 4, 5\} \quad \text{and} \quad \emptyset \subset \{\text{letters in alphabet}\}.$$

The universal set and the complement of a set

A *universal set* is the set of everything under discussion in a particular situation. For example, if $A = \{1, 3, 5, 7, 9\}$, then possible universal sets could be the set of all whole numbers, or the set of all integers, or even the set of all real numbers.

Once a universal set E is fixed, then the *complement* $\bar{A}$ of any set A is the set of all members of that universal set that are not in A . For example:

$$\begin{aligned} \text{If } A &= \{1, 3, 5, 7, 9\} \text{ and } E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \\ \text{then } \bar{A} &= \{2, 4, 6, 8, 10\}. \end{aligned}$$

Every member of a set A is either in A or in $\bar{A}$, but never in both A and $\bar{A}$:

$$A \cap \bar{A} = \emptyset, \quad \text{the empty set, and}$$

$$A \cup \bar{A} = E, \quad \text{the universal set.}$$

The complement of A is sometimes written as A' or A^c .

'Not' means complement

As mentioned in the introduction to this section, the important mathematical word 'not' can be interpreted in terms of the complementary set:

$$\bar{A} = \{\text{members of } E \text{ that are not members of } A\}.$$

3 Subsets, the universal set, and complements

- A set A is a *subset* of a set S , written as $A \subseteq S$, if every element of A is also an element of S .
- If $A \subseteq S$ and $A \neq S$, then A is a *proper subset* of S , written as $A \subset S$.
- If A and B are sets, then $A \cap B \subseteq A \cup B$.
- A *universal set* is a conveniently chosen set that contains all the elements relevant to the situation.
- Let A be a subset of a universal set E . The *complement* of A is the set of all elements of E that are not in A ,

$$\bar{A} = \{\text{elements of } E \text{ that are not in } A\}.$$

- A and $\bar{A}$ are disjoint sets whose union is the universal set:

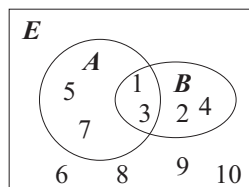
$$A \cap \bar{A} = \emptyset \quad \text{and} \quad A \cup \bar{A} = E$$

The complement of A is sometimes written as A' or A^c .

Venn diagrams

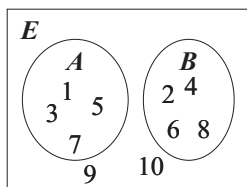
A *Venn diagram* is a diagram used to represent the relationship between sets. For example, the four diagrams below represent four different possible relationships between two sets A and B . In each case, the universal set is

$$E = \{1, 2, 3, \dots, 10\}.$$



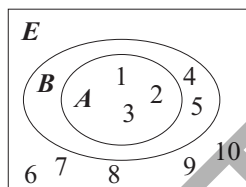
$$A = \{1, 3, 5, 7\}$$

$$B = \{1, 2, 3, 4\}$$



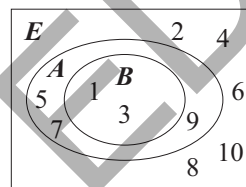
$$A = \{1, 3, 5, 7\}$$

$$B = \{2, 4, 6, 8\}$$



$$A = \{1, 2, 3\}$$

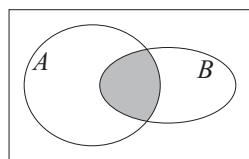
$$B = \{1, 2, 3, 4, 5\}$$



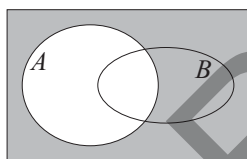
$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{1, 3\}$$

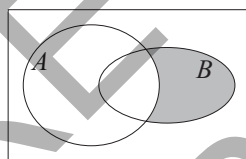
Sets can also be visualised by shading regions of the Venn diagram, as in the following examples:



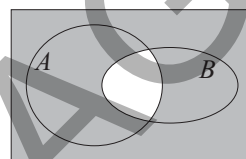
$$A \cap B$$



$$\bar{A}$$



$$\bar{A} \cap B$$



$$\overline{A \cap B}$$

The counting rule for finite sets

To calculate the size of the union $A \cup B$ of two finite sets, adding the sizes of A and of B will not do, because the members of the intersection $A \cap B$ would be counted twice. Hence $|A \cap B|$ needs to be subtracted again, and the rule is

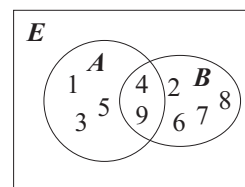
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

For example, the Venn diagram to the right shows the sets

$$A = \{1, 3, 4, 5, 9\} \quad \text{and} \quad B = \{2, 4, 6, 7, 8, 9\}.$$

From the diagram, $|A \cup B| = 9$, $|A| = 5$, $|B| = 6$ and $|A \cap B| = 2$, and the formula works because

$$9 = 5 + 6 - 2.$$



When two sets are disjoint, there is no overlap between A and B to cause any double counting. With $|A \cap B| = 0$, the counting rule becomes much simpler:

$$|A \cup B| = |A| + |B|.$$

4 The counting rule for finite sets

- Let A and B be finite sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

- Let A and B be disjoint finite sets. Then $|A \cap B| = 0$, so

$$|A \cup B| = |A| + |B|.$$

Problem solving using Venn diagrams

A Venn diagram is a very convenient way to sort out problems involving overlapping sets. But once there are more than a handful of elements, listing the elements is most inconvenient. Instead, we work only with the number of elements in each region, showing that number in brackets in the corresponding region.



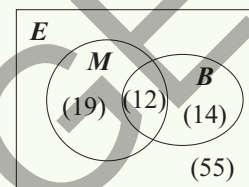
Example 1 Using a Venn diagram to visualise a problem

100 Sydneysiders were surveyed to find out how many of them had visited the cities of Melbourne and Brisbane. The survey showed that 31 people had visited Melbourne, 26 people had visited Brisbane and 12 people had visited both cities. Find how many people had visited:

- a** Melbourne or Brisbane,
- b** Brisbane but not Melbourne,
- c** only one of the two cities,
- d** neither city.

Solution

Let M be the set of people who have visited Melbourne, let B be the set of people who have visited Brisbane, and let E be the universal set of all people surveyed. Calculations should begin with the 12 people in the intersection of the two regions. Then the numbers shown in the other three regions of the Venn diagram can easily be found.



- a** Number visiting Melbourne or Brisbane $= 19 + 14 + 12$
 $= 45$
- b** Number visiting Brisbane but not Melbourne $= 14$
- c** Number visiting only one city $= 19 + 14$
 $= 33$
- d** Number visiting neither city $= 100 - 45$
 $= 55$

Exercise 12A

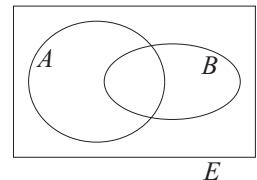
FOUNDATION

- 1** Write down the following sets by listing their members.
 - a** Odd positive integers less than 10.
 - b** The first six positive multiples of 6.
 - c** The numbers on a die.
 - d** The factors of 20.
- 2** Find $A \cup B$ and $A \cap B$ for each pair of sets.
 - a** $A = \{1, 3, 5\}$, $B = \{3, 5, 7\}$
 - b** $A = \{1, 3, 4, 8, 9\}$, $B = \{2, 4, 5, 6, 9, 10\}$
 - c** $A = \{h, o, b, a, r, t\}$, $B = \{b, i, c, h, e, n, o\}$
 - d** $A = \{j, a, c, k\}$, $B = \{e, m, m, a\}$
 - e** $A = \{\text{prime numbers less than } 10\}$, $B = \{\text{odd numbers less than } 10\}$

- 3** If $A = \{1, 4, 7, 8\}$ and $B = \{1, 2, 4, 5, 7\}$, state whether the following statements are true or false.
- a** $A = B$ **b** $|A| = 4$ **c** $|B| = 6$
d $A \subset B$ **e** $A \cup B = \{1, 2, 4, 5, 7, 8\}$ **f** $A \cap B = \{1, 4, 7\}$
- 4** If $A = \{1, 3, 5\}$, $B = \{3, 4\}$ and $E = \{1, 2, 3, 4, 5\}$, find:
- a** $|A|$ **b** $|B|$ **c** $A \cup B$ **d** $|A \cup B|$
e $A \cap B$ **f** $|A \cap B|$ **g** $\overline{A}$ **h** $\overline{B}$
- 5** If $A = \{\text{students who study Japanese}\}$ and $B = \{\text{students who study History}\}$, carefully describe each of the following sets.
- a** $A \cap B$ **b** $A \cup B$
- 6** If $A = \{\text{students with blue eyes}\}$ and $B = \{\text{students with blond hair}\}$, with universal set $E = \{\text{students at Clarence High School}\}$, carefully describe each of the following.
- a** $\overline{A}$ **b** $\overline{B}$ **c** $A \cup B$ **d** $A \cap B$
- 7** List all the subsets of each set.
- a** $\{a\}$ **b** $\{a, b\}$ **c** $\{a, b, c\}$ **d** $\emptyset$
- 8** State in each case whether or not $A \subset B$ (that is, whether A is a subset of B).
- a** $A = \{2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
b $A = \{3, 6, 9, 12\}$, $B = \{3, 5, 9, 11\}$
c $A = \{d, a, n, c, e\}$, $B = \{e, d, u, c, a, t, i, o, n\}$
d $A = \{a, m, y\}$, $B = \{s, a, r, a, h\}$
e $A = \emptyset$, $B = \{51, 52, 53, 54\}$

DEVELOPMENT

- 9** Let $A = \{1, 3, 7, 10\}$ and $B = \{4, 6, 7, 9\}$, and take the universal set to be the set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. List the members of:
- a** $\overline{A}$ **b** $\overline{B}$ **c** $A \cap B$
d $\overline{A \cap B}$ **e** $A \cup B$ **f** $\overline{A \cup B}$
- 10** Let $A = \{1, 3, 6, 8\}$ and $B = \{3, 4, 6, 7, 10\}$, and take the universal set to be the set $E = \{1, 2, 3, \dots, 10\}$. List the members of:
- a** $\overline{A}$ **b** $\overline{B}$ **c** $\overline{A \cup B}$ **d** $\overline{A \cap B}$
e $A \cup B$ **f** $\overline{A \cup B}$ **g** $A \cap B$ **h** $\overline{A \cap B}$
- 11** In each part, draw a copy of the diagram to the right and shade the corresponding region.
- a** $A \cap B$ **b** $\overline{A \cap B}$ **c** $A \cap \overline{B}$
d $A \cup B$ **e** $\overline{A \cup B}$ **f** $\emptyset$ ($\emptyset$ is the empty set)



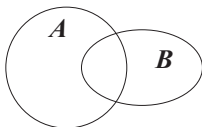
- 12** Answer true or false.

- a** If $A \subset B$ and $B \subset A$, then $A = B$.
b If $A \subset B$ and $B \subset C$, then $A \subset C$.

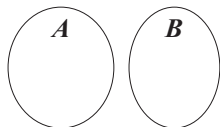
13 Copy and complete:

- a** If $P \subset Q$, then $P \cup Q = \dots$
b If $P \subset Q$, then $P \cap Q = \dots$

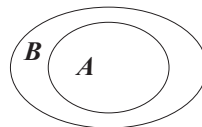
14 Select the Venn diagram that best shows the relationship between each pair of sets A and B :



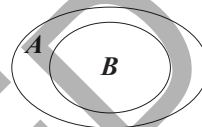
I



II



III



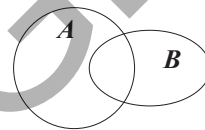
IV

- a** $A = \{\text{Tasmania}\}$, $B = \{\text{states of Australia}\}$
b $A = \{7, 1, 4, 8, 3, 5\}$, $B = \{2, 9, 0, 7\}$
c $A = \{l, e, a, r, n\}$, $B = \{s, t, u, d, y\}$
d $A = \{\text{politicians in Australia}\}$, $B = \{\text{politicians in NSW}\}$.

15 a Explain the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$ by making reference to the Venn diagram opposite.

- b** If $|A \cup B| = 17$, $|A| = 12$ and $|B| = 10$, find $|A \cap B|$.
c Show that the relationship in part (a) is satisfied when

$$A = \{3, 5, 6, 8, 9\} \quad \text{and} \quad B = \{2, 3, 5, 6, 7, 8\}.$$



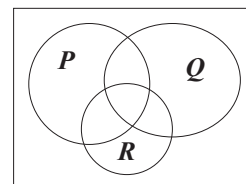
16 Use a Venn diagram to solve each problem.

- a** In a group of 20 people, there are 8 who play the piano, 5 who play the violin and 3 who play both. How many people play neither instrument?
b Each person in a group of 30 plays either tennis or golf. 17 play tennis, while 9 play both. How many play golf?
c In a class of 28 students, there are 19 who like geometry and 16 who like trigonometry. How many like both if there are 5 students who don't like either?

CHALLENGE

17 Shade each of the following regions on the given diagram (use a separate copy of the diagram for each part).

- a** $P \cap Q \cap R$
b $(P \cap R) \cup (Q \cap R)$
c $\overline{P} \cup \overline{Q} \cup \overline{R}$



18 A group of 80 people was surveyed about their approaches to keeping fit.

It was found that 20 jog, 22 swim and 18 go to the gym on a regular basis.

Further questioning found that 10 people both jog and swim, 11 people both jog and go to the gym, and 6 people both swim and go to the gym.

Finally, 43 people do none of these activities.

How many of the people do all three?

12B Probability and sample spaces

Learning intentions

- Understand ‘experiment’ in its mathematical sense, and the meanings of ‘random’.
- Interpret ‘equally likely possible outcomes’ in terms of lack of knowledge.
- Define probability using equally likely possible outcomes, and develop basic formulae.
- Solve simple problems using experimental probability.

Probability is an elusive idea, because we are talking about events that perhaps did not happen. Our first task is to develop a workable formula for probability that can serve as the foundation for the topic in this chapter. We will do this only when the possible results of an experiment can be divided into a finite number of equally likely possible outcomes.

The meaning of ‘experiment’ and ‘random experiment’

‘Experiment’ is probably the most important word in science — whenever scientists or doctors make observations, they are performing experiments.

But in probability theory, we use the word *experiment* to characterise a *procedure together with what is recorded*, thus excluding many exploratory experiments so common in science. We also require experiments to be *exactly repeatable*, which is seldom possible in science where there are so many variables to control.

Such an experiment may be *deterministic*, meaning that the outcome is always the same, or a *random experiment*, meaning that more than one outcome may occur.

Throwing four coins, and recording the number of heads, is a *random experiment* because there is more than one possible outcome. Throwing four coins into a bucket, and recording the number of coins in the bucket, is a *deterministic experiment* because 4 is the only possible outcome.

Equally likely possible outcomes

The idea of equally likely possible outcomes is well illustrated by the experiment of throwing a die and recording the number shown. (A *die*, plural *dice*, is a cube with its corners and edges rounded so that it rolls easily, and with the numbers 1–6 printed on its six sides.) There are six *possible outcomes*: 1, 2, 3, 4, 5, 6. This is a complete list of the possible outcomes, because each time the die is rolled, one, and only one, of these outcomes will occur.

Provided that we believe the die to be completely symmetric and not *biased* in any way, there is no reason for us to expect that any one outcome is more likely to occur than any of the other five. We may thus regard these six possible outcomes as *equally likely possible outcomes*.

With the results of the experiment now divided into six equally likely possible outcomes, we assign the probability $\frac{1}{6}$ to each of these six outcomes. Notice that these six probabilities are equal, and that they all add up to 1.

5 Equally likely possible outcomes

Suppose that the possible results of a random experiment can be divided into a finite number n of *equally likely possible outcomes*. This means that

- one and only one of these n outcomes will occur, and
- we have no reason to expect one outcome to be ‘more likely’ than another.

Then the probability $\frac{1}{n}$ is assigned to each of these equally likely possible outcomes.

‘Equally likely’ interpreted as lack of knowledge

The explanations above assumed that the terms ‘equally likely’ and ‘more likely’ already have a meaning in the mind of the reader. In the case of a thrown die, one could interpret the phrase ‘equally likely’ as a statement about the natural world — in this case that the die is perfectly symmetric.

A more convincing interpretation, however, is that we *lack entirely the knowledge to make any statement of preference for one outcome over another*. If we knew the future, rolling a die would not be a random experiment.

Two meanings of the word ‘random’

We defined a *random experiment* as an experiment with at least two possible outcomes — not necessarily equally likely.

Unfortunately, the word ‘random’ also has another meaning in mathematics and in common English. When we say that a person was chosen ‘at random’ from a hall, we mean that every person in the hall was equally likely to be chosen. When we say that a card is to be chosen ‘randomly’ from a pack, we mean that every card is equally likely to be chosen.

Be aware of the problem — but in practice, it is unlikely to cause problems.

6 Experiments, and the two meanings of ‘random’

- An *experiment* is a repeatable procedure together with what is recorded.
- A *random experiment* has two or more possible outcomes, not necessarily equally likely. A *deterministic experiment* has only one possible outcome.
- When, however, we say that a card is chosen *at random* from a pack, we mean that every card is *equally likely to be chosen*.

Warning: Be careful of this possible ambiguity.

The basic formula for the probability of an event

Suppose that I can win a game with a throw of at least 3 on a die.

‘Getting at least 3’ is then called the particular *event* under discussion.

The outcomes 3, 4, 5 and 6 are called *favourable* for this event

The other two possible outcomes 1 and 2 are called *unfavourable*.

The probability assigned to getting a score of at least 3 is then

$$\begin{aligned} P(\text{scoring at least 3}) &= \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}} \\ &= \frac{4}{6} \\ &= \frac{2}{3}. \end{aligned}$$

This is easily generalised:

7 The basic formula for probability

Suppose that an experiment has a finite number of equally likely possible outcomes.

- An *event* is a subset of the set of possible outcomes. These events are called *favourable* to the event, and the others are called *unfavourable*.
- The probability assigned to the event is

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$$

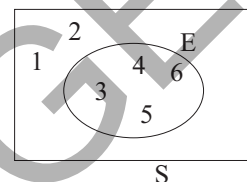
Warning: An *experiment* always includes what is recorded. For example, ‘tossing a coin and recording heads or tails’ is a different experiment from ‘tossing a coin and recording its distance from the wall’.

The sample space and the event space

Venn diagrams and the language of sets are a great help when explaining probability or solving probability problems.

The Venn diagram to the right shows the six equally likely possible outcomes when a die is thrown. The set of all these possible outcomes is

$$S = \{1, 2, 3, 4, 5, 6\}.$$



This set S is called a *uniform sample space*, and is represented by the outer rectangular box. The word ‘uniform’ indicates that the six possible outcomes are all equally likely, and the word ‘sample’ refers to the fact that running the experiment can be regarded as ‘sampling’. The event ‘scoring at least 3’ is identified with the set

$$E = \{3, 4, 5, 6\},$$

which is called the *event space*. It is represented by the ellipse, which is inside the rectangle because E is a subset of S .

Sample spaces and uniform sample spaces

In general, a sample space need not consist of equally likely possible outcomes.

8 The sample space and the event space

Sample space: A set of possible outcomes of an experiment is called a *sample space* if:

- one and only one of these outcomes will occur.

Uniform sample space: A finite sample space is called *uniform* if:

- all its possible outcomes are equally likely.

Event space: The set of all favourable outcomes is called the *event space*.

The assumptions of the chapter

First, the sample spaces in this chapter are assumed to be finite — this applies without further mention throughout the chapter.

Secondly, the discussions are restricted to situations where the results of an experiment can be reduced to a set of equally likely possible outcomes, that is, to a uniform sample space (assumed finite, as mentioned above).

Thus the basic probability formula can now be restated in set language as:

9 The basic formula for probability

Suppose that an event E is a subset of a uniform sample space S . Then

$$P(E) = \frac{|E|}{|S|}.$$

Probabilities involving playing cards

So many questions in probability involve a pack of playing cards that anyone studying probability needs to be familiar with them — acquire some cards and play some simple games with them. A pack of cards consists of 52 cards organised into four *suits*, each containing 13 cards. The four suits are:

two black suits: ♣ clubs ♠ spades

two red suits: ♦ diamonds ♥ hearts

Each of the four suits contains 13 cards:

A (ace), 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king).

An ace is also regarded as a 1. Three cards in each suit are called *court cards* (or *picture cards* or *face cards*) because they depict people in the royal court:

J (jack), Q (queen), K (king).

It is assumed that when a pack of cards is shuffled, the order is totally *random*, meaning that no one ordering of the cards is more likely to occur than any other.



Example 2 Using basic probability with playing cards

A card is drawn at random from a standard pack of 52 playing cards. Find the probability that the card is:

a the seven of hearts,	b any heart,
c any seven,	d any red card,
e any court card (that is, a jack, a queen, or a king),	f any green card,
g any red or black card.	

Solution

In each case, there are 52 equally likely possible outcomes.

a There is 1 seven of hearts, so $P(7♥) = \frac{1}{52}$.

b There are 13 hearts, so $P(\text{heart}) = \frac{13}{52}$
 $= \frac{1}{4}$.

c There are 4 sevens, so $P(\text{seven}) = \frac{4}{52}$
 $= \frac{1}{13}$.

d There are 26 red cards, so $P(\text{red card}) = \frac{26}{52}$
 $= \frac{1}{2}$.

e There are 12 court cards, so $P(\text{court card}) = \frac{12}{52}$
 $= \frac{3}{13}$.

f No card is green, so $P(\text{green card}) = \frac{0}{52}$
 $= 0$.

g All 52 cards are red or black, so $P(\text{red or black card}) = \frac{52}{52}$
 $= 1$.

Impossible and certain events

Parts (f) and (g) of Example 2 were intended to illustrate the probabilities of events that are impossible or certain.

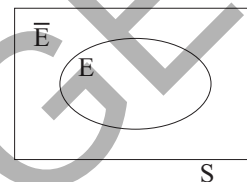
- Getting a green card is impossible because there are no green cards. Hence there are no favourable outcomes, and the probability is 0.
- Getting a red or black card is certain because all the cards are either red or black. Hence all possible outcomes are favourable, and the probability is 1.
- For the other five events, the probability lies between 0 and 1.

10 Impossible and certain events

- An impossible event has probability zero.
- A certain event has probability 1.
- The probability of any other event E lies in the open interval $0 < P(E) < 1$.

Complementary events and the word ‘not’

It is often easier to find the probability that an event does not occur than the probability that it does occur. The *complement* of an event E is the event ‘ E does not occur’. It is written as $\bar{E}$, or sometimes as E' or E^c .



Let S be a uniform sample space of the experiment. The complementary event $\bar{E}$ is represented by the region outside the circle in the Venn diagram to the right. Because $|\bar{E}| = |S| - |E|$, it follows that

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - P(E).$$

11 Complementary events and the word ‘not’

- The *complement* $\bar{E}$ of an event E is the event ‘ E does not occur’, normally read as ‘not E ’.
- $P(\bar{E}) = 1 - P(E)$.
- The symbols E' and E^c are also used for the complementary event.
- Always consider using complementary events in problems, particularly when the word ‘not’ occurs.

In Section 12A, the *complement* $\bar{E}$ of a set E was defined to be the set of things in S but not in E . The notation $\bar{E}$ for complementary event is quite deliberately the same notation as that for the complement of a set.

Example 3 Finding the probability that an event does not occur

A card is drawn at random from a pack of playing cards. Find the probability:

- that it is not a spade,
- that it is not a court card (jack, queen or king),
- that it is neither a red two nor a black six.

Solution

- Thirteen cards are spades, so

$$\begin{aligned} P(\text{spade}) &= \frac{13}{52} \\ &= \frac{1}{4}. \end{aligned}$$

$$\begin{aligned} \text{Hence, using complementary events, } P(\text{not a spade}) &= 1 - \frac{1}{4} \\ &= \frac{3}{4}. \end{aligned}$$

- b** Twelve cards are court cards, so $P(\text{court card}) = \frac{12}{52}$
 $= \frac{3}{13}$.
 Hence, using complementary events, $P(\text{court card}) = 1 - \frac{3}{13}$
 $= \frac{10}{13}$.
- c** There are two red 2s and two black 6s, so $P(\text{red 2 or black 6}) = \frac{4}{52}$
 $= \frac{1}{13}$.
 Hence, using complementary events, $P(\text{neither}) = 1 - \frac{1}{13}$
 $= \frac{12}{13}$.

Invalid arguments

Arguments offered in probability theory can be invalid for all sorts of subtle reasons, and it is common for a question to ask for comment on a given argument. It is most important in such a situation that any fallacy in the given argument be explained — it is not sufficient to offer only an alternative argument with a different conclusion.



Example 4 Explaining why an argument in probability is invalid

Comment on the validity of this argument.

‘Brisbane is one of 16 League teams, so the probability that Brisbane wins the premiership is $\frac{1}{16}$.’

Solution

[Identifying the fallacy] The division into 16 possible outcomes is correct (assuming that a tie for first place is impossible), but no reason has been offered as to why each team is equally likely to win, so the argument is invalid.

[Offering a replacement argument] If a team is selected at random from the 16 teams, then the probability that it is the premiership-winning team is $\frac{1}{16}$. Also, if someone knows nothing whatsoever about League, then for him the statement is correct. But these are different experiments.

Note: It is difficult to give a satisfactory account of this situation, indeed the idea of exact probabilities seems to have no meaning. Those with knowledge of the game would have some idea of ranking the 16 teams in order from most likely to win to least likely to win. If there is an organised system of betting, one may, or may not, agree to take this as an indication of the community’s collective wisdom on Brisbane’s chance of winning, and nominate it as ‘the probability’.

Experimental probability — relative frequency

When a drawing pin is thrown, there are two possible outcomes, point-up and point-down. But these outcomes are not equally likely, and there seems to be no way to analyse the results of the experiment into equally likely possible outcomes.

In the absence of a fancy argument from physics about rotating pins falling on a smooth surface, however, some estimate of probability can be gained by performing the experiment many times. We can then use the *relative frequencies* of the drawing pin landing pin-up and pin-down as estimates of the probabilities, as in Example 5 below (relative frequencies may have been introduced in earlier years). But an exact value is inaccessible by these methods, indeed the very idea of an exact value may well have no meaning.

The next example could raise difficult issues beyond this course, but the intention here is only to answer briefly in a common-sense manner.


Example 5 Using experimental probability

A drawing pin was thrown 400 times and fell point-up 362 times.

- What were the relative frequencies of the drawing pin falling point-up, and of falling point-down?
- What probability does this experiment suggest for the result ‘point-up’?
- A machine later repeated the experiment 1 000 000 times, and the pin fell point-up 916 203 times. Does this change your estimate in part (b)?

Solution

$$\begin{aligned}\text{a Relative frequency of point-up} &= \frac{362}{400} \\ &= 0.905\end{aligned}$$

$$\begin{aligned}\text{Relative frequency of point-down} &= \frac{38}{400} \\ &= 0.095\end{aligned}$$

- b** These results suggest that $P(\text{point-up}) \div 0.905$, but with only 400 trials, there would be little confidence in this result past the second decimal place, or even the first decimal place, because different *runs* of the same experiment would be expected to differ by small numbers. The safest conclusion would be that $P(\text{point-up}) \div 0.9$.
- c** The new results suggest that the estimate of the probability can now be refined to $P(\text{point-up}) \div 0.916$ — we can now be reasonably sure that the rounding to 0.9 in part (a) gave a value that was a little too low. (Did the machine throw the pin in a random manner, whatever that may mean?)

The next chapter will develop experimental probability further, using systematic tabulation of experimental data to estimate probabilities.

Exercise 12B
FOUNDATION

- A pupil has 3 tickets in the class raffle. If there were 60 tickets in the raffle and one ticket drawn, find the probability that the pupil:
 - wins,
 - does not win.
- A coin is tossed. Write down the probability that it shows:
 - a head,
 - a tail,
 - either a head or a tail,
 - neither a head nor a tail.
- If a die is rolled, find the probability that the uppermost face is:
 - a three,
 - an even number,
 - a number greater than four,
 - a multiple of three.
- A bag contains five red and seven blue marbles. If one marble is drawn from the bag at random, find the probability that it is:
 - red,
 - blue,
 - green.
- A bag contains eight red balls, seven yellow balls and three green balls. A ball is selected at random. Find the probability that it is:
 - red,
 - yellow or green,
 - not yellow.

- 6** In a bag there are four red capsicums, three green capsicums, six red apples and five green apples. One item is chosen at random. Find the probability that it is:
- | | |
|-----------------------|----------------------------|
| a green, | b red, |
| c an apple, | d a capsicum, |
| e a red apple, | f a green capsicum. |
- 7** A letter is chosen at random from the word TASMANIA. Find the probability that it is:
- a** an A,
b a vowel,
c a letter of the word HOBART.
- 8** A letter is randomly selected from the 26 letters in the English alphabet. Find the probability that the letter is:
- | | |
|----------------------------|--|
| a the letter S, | b a vowel, |
| c a consonant, | d the letter γ , |
| e either C, D or E, | f one of the letters of the word MATHS. |
- Note:** The letter Y is normally classified as a consonant.
- 9** A student has a 22% chance of being chosen as a prefect. What is the chance that he will not be chosen as a prefect?
- 10** When breeding labradors, the probability of breeding a black dog is $\frac{3}{7}$.
- a** What is the probability of breeding a dog that is not black?
b If you bred 56 dogs, how many would you expect not to be black?
- 11** A box containing a light bulb has a chance of $\frac{1}{15}$ of holding a defective bulb.
- a** If 120 boxes were checked, how many would you expect to hold defective bulbs?
b What is the probability that the box holds a bulb that works?
- 12** When a die is rolled, the theoretical probability of throwing a six is $\frac{1}{6}$, which is about 17%.
- a** How many sixes would you expect to get if a die is thrown 60 times?
b In an experiment, a certain die was thrown 60 times and a six turned up 18 times.
- What was the relative frequency of throwing a six?
 - What does this experiment suggest is the experimental probability of throwing a six?
 - Comment on whether the die is *biased* or *fair*.
- 13** Every day the principal goes early to the tuckshop to buy a sandwich for his lunch. He knows that the canteen makes 400 sandwiches a day — equal numbers of four types: bacon-lettuce-tomato; chicken; ham and cheese; and Vegemite. He always chooses a sandwich at random.
- a** What is his theoretical probability of obtaining a chicken sandwich?
b Over 20 days, the principal found he had bought 8 chicken sandwiches. What was his relative frequency, and therefore his experimental probability, of obtaining a chicken sandwich?
c Comment on why your answers for the theoretical and experimental probability might differ.

DEVELOPMENT

- 14** A number is selected at random from the integers 1, 2, 3, ..., 19, 20. Find the probability of choosing:
- | | | |
|---------------------------|------------------------------------|-----------------------------|
| a the number 4, | b a number greater than 15, | c an even number, |
| d an odd number, | e a prime number, | f a square number, |
| g a multiple of 4, | h the number e , | i a rational number. |
- 15** From a regular pack of 52 cards, one card is drawn at random. Find the probability that:
- | | | |
|------------------------------------|---|----------------------------------|
| a it is black, | b it is red, | c it is a king, |
| d it is the jack of hearts, | e it is a club, | f it is a picture card, |
| g it is a heart or a spade, | h it is a red five or a black seven, | i it is less than a four. |
- 16** A book has 150 pages. The book is randomly opened at a page. Find the probability that the page number is:
- | | | |
|---------------------------------|----------------------------|--------------------------------|
| a greater than 140, | b a multiple of 20, | c an odd number, |
| d a number less than 25, | e either 72 or 111, | f a three-digit number. |
- 17** An integer x , where $1 \leq x \leq 200$, is chosen at random. Determine the probability that it:
- | | | |
|------------------------------|-------------------------------|----------------------------------|
| a is divisible by 5, | b is a multiple of 13, | c has two digits, |
| d is a square number, | e is greater than 172, | f has three equal digits. |
- 18** A bag contains three times as many yellow marbles as blue marbles, and no other colours. If a marble is chosen at random, find the probability that it is:
- | | |
|------------------|----------------|
| a yellow, | b blue. |
|------------------|----------------|
- 19** Fifty tagged fish were released into a dam known to contain fish. Later a sample of 30 fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just prior to the sample of 30 being removed.

CHALLENGE

- 20** Comment on the following arguments. Identify precisely any fallacies in the arguments, and, if possible, give some indication of how to correct them.
- | |
|--|
| a 'On every day of the year it either rains or it doesn't. Therefore the chance that it will rain tomorrow is $\frac{1}{2}$.' |
| b 'When the Sydney Swans play Hawthorn, either Hawthorn wins, the Swans win, or the game is a draw. Therefore the probability that the next game between these two teams results in a draw is $\frac{1}{3}$.' |
| c 'When answering a multiple-choice test in which there are four possible answers to each question, the chance that Peter answers a question correctly is $\frac{1}{4}$.' |
| d 'A bag contains a number of red, white and black beads. If you choose one bead at random from the bag, the probability that it is black is $\frac{1}{3}$.' |
| e 'Four players play in a knockout tennis tournament resulting in a single winner. A man with no knowledge of the game or the players declares that one particular player will win his semi-final, but lose the final. The probability that he is correct is $\frac{1}{4}$.' |
- 21** A rectangular field is 60 metres long and 30 metres wide. A cow wanders randomly around the field. Find the probability that the cow is:
- | |
|--|
| a more than 10 metres from the edge of the field, |
| b not more than 10 metres from a corner of the field. |

12C Sample space graphs and tree diagrams

Learning intentions

- Use graphs and tree diagrams in multi-stage experiments.

Many experiments consist of several *stages*. For example, when a die is thrown twice, the two throws can be regarded as two separate stages of the one experiment. This section develops two approaches — graphing and tree diagrams — to investigating the sample space of a multi-stage experiment.

Graphing the sample space

The word ‘sample space’ is used in probability, rather than ‘sample set’, because the sample space of a multi-stage experiment takes on some of the characteristics of a space. In particular, the sample space of a two-stage experiment can be displayed on a two-dimensional graph, and the sample space of a three-stage experiment can be displayed in a three-dimensional graph.

The following example shows how a two-dimensional dot diagram can be used for calculations with the sample space of a die thrown twice.



Example 6 Using a diagram for a two-stage event

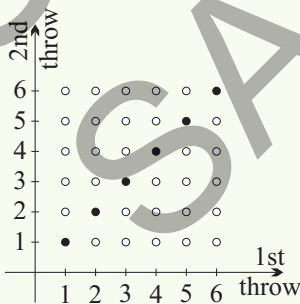
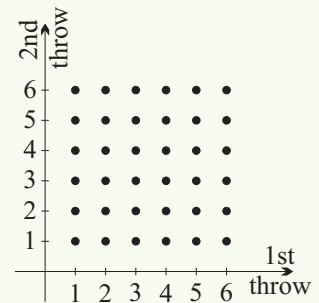
A die is thrown twice. Find the probability that:

a the pair is a double,	b at least one number is four,
c both numbers are greater than four,	d both numbers are even,
e the sum of the two numbers is six,	f the sum is at most four.

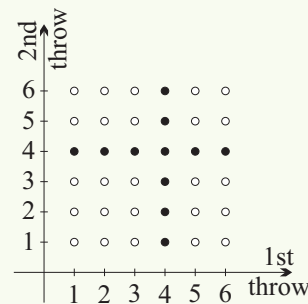
Solution

The horizontal axis in the diagram to the right represents the six possible outcomes of the first throw, and the vertical axis represents the six possible outcomes of the second throw. Each dot is an ordered pair, such as (3, 5), representing the first and second outcomes. The 36 dots therefore represent all 36 different possible outcomes of the two-stage experiment, all equally likely. This is the full sample space.

Parts (a)–(f) can now be answered by counting the dots representing the various event spaces.



- a** There are 6 doubles,
so $P(\text{double}) = \frac{6}{36}$ (count the dots)
 $= \frac{1}{6}$.



- b** 11 pairs contain a 4,
so $P(\text{at least one is a 4}) = \frac{11}{36}$.

- c** 4 pairs consist only of 5 or 6, so $P(\text{both greater than 4}) = \frac{4}{36}$
 $= \frac{1}{9}$.
- d** pairs have two even members, so $P(\text{both even}) = \frac{9}{36}$
 $= \frac{1}{4}$.
- e** pairs have sum 6, so $P(\text{sum is 6}) = \frac{5}{36}$.
- f** pairs have sum 2, 3 or 4, so $P(\text{sum at most 4}) = \frac{6}{36}$
 $= \frac{1}{6}$.

12 Graphing the sample space of a two-stage experiment

- The horizontal axis represents the first stage of the experiment.
- The vertical axis represents the second stage.

Complementary events in multi-stage experiments

Example 7 below continues the two-stage experiment in Example 6, but this time it involves working with complementary events.



Example 7 Using complementary events in two-stage experiment

A die is thrown twice.

- Find the probability of failing to throw a double six.
- Find the probability that the sum is not seven.

Solution

The same sample space can be used as in Example 6.

- a** The double six is one outcome amongst 36,

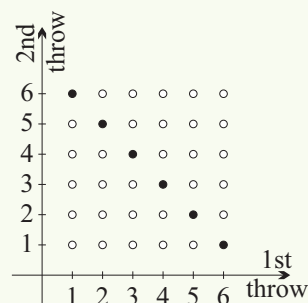
$$\text{so } P(\text{double six}) = \frac{1}{36}.$$

$$\text{Hence } P(\text{not throwing a double six}) = 1 - P(\text{double six}) \\ = \frac{35}{36}.$$

- b** The graph to the right shows that there are six outcomes giving a sum of seven,

$$\text{so } P(\text{sum is seven}) = \frac{6}{36} \\ = \frac{1}{6}.$$

$$\text{Hence } P(\text{sum is not seven}) = 1 - P(\text{sum is seven}) \\ = \frac{5}{6}.$$



Tree diagrams

Listing the sample space of a multi-stage experiment can be difficult, and the dot diagrams of the previous paragraph are hard to draw in more than two dimensions. Tree diagrams provide a very useful alternative way to display the sample space. Such diagrams have a column for each stage, plus an initial column labelled 'Start' and a final column listing the possible outcomes.



Example 8 Using a tree diagram in a three-stage experiment

A three-letter word is chosen in the following way. The first and last letters are chosen from the three vowels 'A', 'O' and 'U', with repetition not allowed, and the middle letter is chosen from 'L' and 'M'. List the sample space, then find the probability that:

- a** the word is 'ALO',
- b** the letter 'O' does not occur,
- c** 'M' and 'U' do not both occur,
- d** the letters are in alphabetical order.

Solution

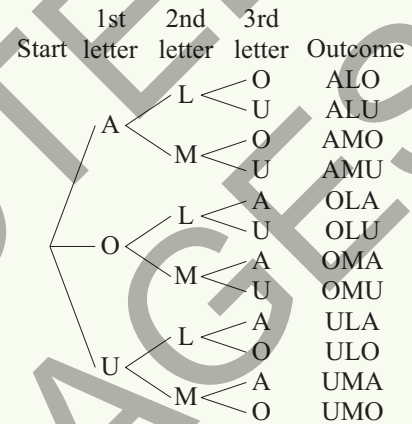
The tree diagram to the right lists all 12 equally likely possible outcomes. The two vowels must be different, because repetition was not allowed.

a $P(\text{'ALO'}) = \frac{1}{12}$

b $P(\text{no 'O'}) = \frac{4}{12}$
 $= \frac{1}{3}$

c $P(\text{not both 'M' and 'U'}) = \frac{8}{12}$
 $= \frac{2}{3}$

d $P(\text{alphabetical order}) = \frac{4}{12}$
 $= \frac{1}{3}$



13 A tree diagram of a multi-stage experiment

- The first column is headed 'Start'.
- The last column is headed 'Outcome', and lists all the paths through the tree.
- Each other column represents all the equally likely possible outcomes in the particular stage of the experiment.

The meaning of 'word'

In the example above, and throughout this chapter, 'word' simply means an arrangement of letters — the arrangement doesn't have to have any meaning or be a word in the dictionary. Thus 'word' simply becomes a convenient device for discussing arrangements of things in particular orders.

Invalid arguments

The next example illustrates another invalid argument in probability. As always, the solution first offers an explanation of the fallacy, before then offering an alternative argument with a different conclusion.



Example 9 Assessing invalid arguments in two-stage experiment

Comment on the validity of this argument.

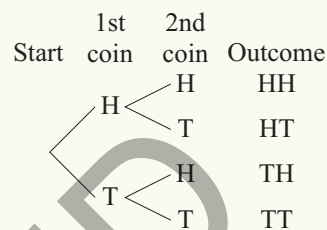
'When two coins are tossed together, there are three outcomes: two heads, two tails, and one of each. Hence the probability of getting one of each is $\frac{1}{3}$.'

Solution

[Identifying the fallacy] The division of the results into the three given outcomes is correct, but no reason is offered as to why these outcomes are equally likely.

[Supplying the correct argument] On the right is a tree diagram of the sample space. It divides the results of the experiment into four *equally likely* possible outcomes. Two of these outcomes, HT and TH, are favourable to the event 'one of each', so

$$P(\text{one of each}) = \frac{2}{4} = \frac{1}{2}.$$

**Exercise 12C****FOUNDATION**

- 1 A fair coin is tossed twice.
 - a Use a tree diagram to list the four possible outcomes.
 - b Hence find the probability that the two tosses result in:
 - i two heads,
 - ii a head and a tail,
 - iii a head on the first toss and a tail on the second.
- 2 A coin is tossed and a die is thrown.
 - a Use a tree diagram to list all 12 possible outcomes.
 - b Hence find the probability of obtaining:
 - i a head and an even number,
 - ii a tail and a number greater than 4,
 - iii a tail and a number less than 4,
 - iv a head and a prime number.
- 3 Two tiles are chosen at random, one after the other, from three Scrabble tiles T, O and 'E, and placed in order on the table.
 - a List the six possible outcomes in the sample space.
 - b Find the probability of choosing:
 - i two vowels,
 - ii a consonant then a vowel,
 - iii a T.
- 4 From a group of four students, Anna, Bill, Charlie and David, two are chosen at random, one after the other, to be on the Student Representative Council.
 - a List the 12 possible outcomes in the sample space.
 - b Hence find the probability that:
 - i Bill and David are chosen,
 - ii Anna is chosen,
 - iii Charlie is chosen but Bill is not,
 - iv neither Anna nor David is selected.
 - v Bill is chosen first.
 - vi Charles is not chosen second.
- 5 From the digits 2, 3, 8 and 9, a two-digit number is formed in which no digit is repeated.
 - a Use a tree diagram to list the possible outcomes.
 - b If the number was formed at random, find the probability that it is:
 - i the number 82,
 - ii a number greater than 39,
 - iii an even number,
 - iv a multiple of 3,
 - v a number ending in 2,
 - vi a perfect square.

- 6** A captain and vice-captain of a cricket team are to be chosen from Amanda, Belinda, Carol, Dianne and Emma.
- a** Use a tree diagram to list the possible pairings, noting that order is important.
- b** If the choices were made at random, find the probability that:
- i** Carol is captain and Emma is vice-captain,
 - ii** Belinda is either captain or vice-captain,
 - iii** Amanda is not selected for either position,
 - iv** Emma is vice-captain.

DEVELOPMENT

- 7** A coin is tossed three times. Draw a tree diagram to illustrate the possible outcomes. Then find the probability of obtaining:
- a** three heads, **b** a head and two tails, **c** at least two tails,
d at most one head, **e** more heads than tails, **f** a head on the second toss.
- 8** A green die and a red die are thrown simultaneously. List the set of 36 possible outcomes on a two-dimensional graph and hence find the probability of:
- a** obtaining a three on the green die, **b** obtaining a four on the red die,
c a double five, **d** a total score of seven,
e a total score greater than nine, **f** an even number on both dice,
g at least one two, **h** neither a one nor a four appearing,
i a five and a number greater than three, **j** the same number on both dice.
- 9** Suppose that the births of boys and girls are equally likely.
- a** In a family of two children, determine the probability that there are:
- i** two girls, **ii** no girls, **iii** one boy and one girl.
- b** In a family of three children, determine the probability that there are:
- i** three boys, **ii** two girls and one boy, **iii** more boys than girls.
- 10** A coin is tossed four times. Find the probability of obtaining:
- a** four heads, **b** exactly three tails, **c** at least two heads,
d at most one head, **e** two heads and two tails, **f** more tails than heads.
- 11** A hand of five cards contains a ten, jack, queen, king and ace. From the hand, two cards are drawn in succession, the first card not being replaced before the second card is drawn. Find the probability that:
- a** the ace is drawn, **b** the king is not drawn, **c** the queen is the second card drawn.

CHALLENGE

- 12** Three-digit numbers are formed from the digits 2, 5, 7 and 8, without repetition.
- a** Use a tree diagram to list all the possible outcomes. How many are there?
- b** Hence find the probability that the number is:
- i** greater than 528, **ii** divisible by 3, **iii** divisible by 13, **iv** prime.
- 13** If a coin is tossed n times, where $n > 1$, find the probability of obtaining:
- a** n heads, **b** at least one head and at least one tail.

12D Venn diagrams and the addition theorem

Learning intentions

- Use the addition rule and Venn diagrams.

In probability situations where an event is described using the logical words ‘and’, ‘or’, and ‘not’, Venn diagrams and the language of sets are a useful way to visualise the sample space and the event space.

A uniform sample space S is taken as the universal set because it includes all the equally likely possible outcomes. The event spaces are then subsets of S .

Mutually exclusive events and disjoint sets

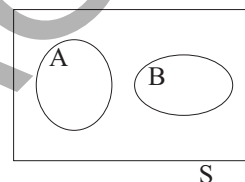
Two events are called *mutually exclusive* if they cannot both occur. For example:

If a die is thrown, the events ‘throwing a number less than three’ and ‘throwing a number greater than four’ cannot both occur, so are mutually exclusive.

If a card is drawn at random from a pack, the events ‘drawing a red card’ and ‘drawing a spade’ cannot both occur, so are mutually exclusive.

In the Venn diagram of such a situation, the two events A and B are represented as disjoint sets (*disjoint* means that their intersection is empty). The event ‘ A and B ’ is impossible, so has probability zero.

On the other hand, the event ‘ A or B ’ is represented on the Venn diagram by the union $A \cup B$ of the two sets. Because $|A \cup B| = |A| + |B|$ for disjoint sets, it follows that



$$\begin{aligned} P(A \text{ or } B) &= \frac{|A \cup B|}{|S|} \\ &= \frac{|A| + |B|}{|S|} \quad (\text{because } A \text{ and } B \text{ are disjoint}) \\ &= P(A) + P(B). \end{aligned}$$

14 Mutually exclusive events

Suppose that two mutually exclusive events A and B are subsets of the same uniform sample space S . Then the event ‘ A or B ’ is represented by $A \cup B$, and

$$P(A \text{ or } B) = P(A) + P(B).$$

The event ‘ A and B ’ cannot occur, and has probability zero.



Example 10 Recognising mutually exclusive events

If a die is thrown, find the probability that the result is less than three or greater than four.

Solution

The events ‘throwing a number less than three’ and ‘throwing a number greater than four’ are mutually exclusive,

so $P(\text{less than three or greater than four}) = P(\text{less than three}) + P(\text{greater than four})$

$$\begin{aligned} &= \frac{1}{3} + \frac{1}{3} \\ &= \frac{2}{3}. \end{aligned}$$


Example 11 **Recognising the union of mutually exclusive events**

If a card is drawn at random from a standard pack, find the probability that it is a red card or a spade.

Solution

‘Drawing a red card’ and ‘drawing a spade’ are mutually exclusive,

so $P(\text{a red card or a spade}) = P(\text{a red card}) + P(\text{a spade})$

$$= \frac{1}{2} + \frac{1}{4}$$

$$= \frac{3}{4}.$$


Example 12 **Mutually exclusive events in multi-stage experiment**

If three coins are tossed, find the probability of throwing an odd number of tails.

Solution

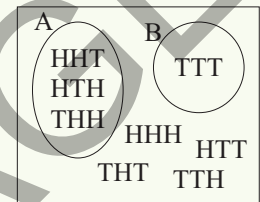
Let A be the event ‘one tail’ and B the event ‘three tails’. Then A and B are mutually exclusive, with

$$A = \{\text{HHT, HTH, THH}\} \quad \text{and} \quad B = \{\text{TTT}\}.$$

The full sample space has eight members altogether (a question in Exercise 12C lists them all), so

$$P(A \text{ or } B) = \frac{3}{8} + \frac{1}{8}$$

$$= \frac{1}{2}.$$


The events ‘A and B’ and ‘A or B’ — the addition rule

More generally, suppose that two events A and B are subsets of the same uniform sample space S , not necessarily mutually exclusive. The Venn diagram of the situation now represents the two events A and B as overlapping sets within the same universal set S .

The event ‘A and B’ is represented by the intersection $A \cap B$ of the two sets, and the event ‘A or B’ is represented by the union $A \cup B$.

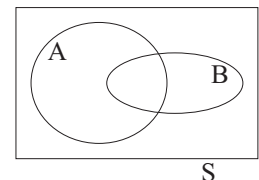
The general counting rule for sets is $|A \cup B| = |A| + |B| - |A \cap B|$, because the members of the intersection $A \cap B$ are counted in A , and are counted again in B , and so have to be subtracted. It follows then that

$$P(A \text{ or } B) = \frac{|A \cup B|}{|S|}$$

$$= \frac{|A| + |B| - |A \cap B|}{|S|}$$

$$= P(A) + P(B) - P(A \text{ and } B).$$

This rule is the *addition rule* of probability.



15 The events 'A or B' and 'A and B', and the addition rule

Suppose that two events A and B are subsets of the same uniform sample space S .

- The event 'A and B' is represented by the intersection $A \cap B$.
- The event 'A or B' is represented by the union $A \cup B$.

The addition rule:

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

The symbols $\cap$ and $\cup$, and the words 'and', 'or' and 'but'

As explained in the previous section, the word 'and' is closely linked with the intersection of event spaces. For this reason, the event A and B is also written as $A \cap B$, and the following two notations are used interchangeably:

$P(A \text{ and } B)$ means the same as $P(A \cap B)$.

Similarly, the word 'or' is closely linked with the union of event spaces — the word 'or' always means 'and/or' in logic and mathematics. Thus the event A or B is also written as $A \cup B$, and the following two notations are used interchangeably:

$P(A \text{ or } B)$ means the same as $P(A \cup B)$.

The word 'but' has the same logical meaning as 'and' — the difference in meaning between 'and' and 'but' is rhetorical, not logical. Using this notation, the addition rule now looks much closer to the counting rule for finite sets:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

**Example 13 Paying attention to 'and', 'or', and 'not'**

In a class of 30 girls, 13 play tennis and 23 play netball. If 7 girls play both sports, what is the probability that a girl chosen at random plays neither sport?

Solution

Let T be the event 'she plays tennis', and let N be the event 'she plays netball'.

Then $P(T) = \frac{13}{30}$

$$P(N) = \frac{23}{30}$$

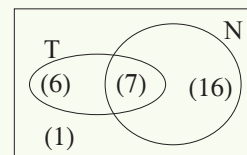
and $P(N \cap T) = \frac{7}{30}$ (meaning $P(N \text{ and } T)$).

Hence $P(N \cup T) = \frac{13}{30} + \frac{23}{30} - \frac{7}{30}$ (meaning $P(N \text{ or } T)$)

$$= \frac{29}{30}.$$

and $P(\text{neither sport}) = 1 - P(N \text{ or } T)$

$$= \frac{1}{30}.$$



Note: An alternative approach is shown in the diagram above. Starting with the 7 girls in the intersection, the numbers 6 and 16 can then be written into the respective regions 'tennis but not netball' and 'netball but not tennis'. These numbers add to 29, leaving only one girl playing neither tennis nor netball.

Complementary events and the addition rule

In the next example, the addition rule has to be applied in combination with the idea of complementary events. Some careful thinking is required when the words 'and' and 'or' are combined with 'not'.


Example 14 Using the addition rule with compound events

A card is drawn at random from a pack.

- a** Find the probability that it is not an ace, but also not a two.
- b** Find the probability that it is an even number, or a court card, or a red card (the court cards are jack, queen and king).

Warning: The word ‘or’ always means ‘and/or’ in logic and mathematics. Thus in part (b), there is no need to add ‘or any two of these, or all three of these’.

Solution

- a** The complementary event $\bar{E}$ is drawing a card that is an ace or a two.

$$\begin{aligned} \text{There are eight such cards, so } P(\text{ace or two}) &= \frac{8}{52} \\ &= \frac{2}{13}. \end{aligned}$$

$$\begin{aligned} \text{Hence } P(\text{not an ace, but also not a two}) &= 1 - \frac{2}{13} \\ &= \frac{11}{13}. \end{aligned}$$

(The words ‘and’ and ‘but’ have different emphasis, but they have the same logical meaning.)

- b** The complementary event $\bar{E}$ is drawing a card that is a black odd number less than ten. This complementary event has 10 members:

$$\bar{E} = \{A\clubsuit, 3\clubsuit, 5\clubsuit, 7\clubsuit, 9\clubsuit, A\spadesuit, 3\spadesuit, 5\spadesuit, 7\spadesuit, 9\spadesuit\}.$$

There are 52 possible cards to choose, so

$$\begin{aligned} P(\bar{E}) &= \frac{10}{52} \\ &= \frac{5}{26}. \end{aligned}$$

Hence, using the complementary event formula,

$$\begin{aligned} P(E) &= 1 - P(\bar{E}) \\ &= \frac{21}{26}. \end{aligned}$$

Exercise 12D
FOUNDATION

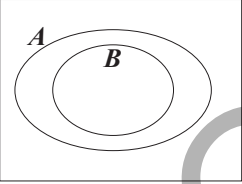
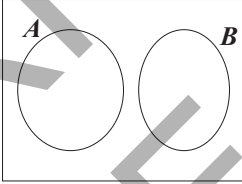
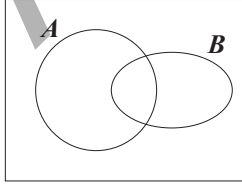
- 1** A die is rolled. If n denotes the number on the uppermost face, find:

a $P(n = 5)$ c $P(n = 4 \text{ or } n = 5)$ e $P(n \text{ is even or odd})$ g $P(n \text{ is even and divisible by three})$	b $P(n \neq 5)$ d $P(n = 4 \text{ and } n = 5)$ f $P(n \text{ is neither even nor odd})$ h $P(n \text{ is even or divisible by three})$
- 2** A card is selected from a regular pack of 52 cards. Find the probability that the card:

a is a jack, c is a jack or a ten, e is neither a jack nor a ten, g is a picture card, i is black or a picture card,	b is a ten, d is a jack and a ten, f is black, h is a black picture card, j is neither black nor a picture card.

- 3** Show that the following events A and B are disjoint by showing that $A \cap B = \emptyset$, and check that $P(A \text{ or } B) = P(A) + P(B)$.
- Two coins are thrown and $A = \{\text{two heads}\}$, $B = \{\text{one tail}\}$
 - A committee of two is formed by choosing at random from two men, Ricardo and Steve, and one woman, Tania. Suppose $A = \{\text{committee of two men}\}$, $B = \{\text{committee includes Tania}\}$.
- 4** A die is thrown. Let A be the event that an even number appears. Let B be the event that a number greater than 2 appears.
- Are A and B mutually exclusive?
 - Find:
 - $P(A)$
 - $P(B)$
 - $P(A \cap B)$
 - $P(A \cup B)$
- 5** Two dice are thrown. Let a and b denote the numbers rolled. Find:
- $P(a \text{ is odd})$
 - $P(b \text{ is odd})$
 - $P(a \text{ and } b \text{ are odd})$
 - $P(a \text{ or } b \text{ is odd})$
 - $P(\text{neither } a \text{ nor } b \text{ is odd})$
 - $P(a = 1)$
 - $P(b = a)$
 - $P(a = 1 \text{ and } b = a)$
 - $P(a = 1 \text{ or } b = a)$
 - $P(a \neq 1 \text{ and } a \neq b)$

DEVELOPMENT

- 6 a** 
 $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$.
 Find:
 - $P(\bar{A})$
 - $P(\bar{B})$
 - $P(A \text{ and } B)$
 - $P(A \text{ or } B)$
 - $P(\text{neither } A \text{ nor } B)$
- b** 
 $P(A) = \frac{2}{5}$ and $P(B) = \frac{1}{5}$.
 Find:
 - $P(\bar{A})$
 - $P(\bar{B})$
 - $P(A \text{ or } B)$
 - $P(A \text{ and } B)$
 - $P(\text{not both } A \text{ and } B)$
- c** 
 $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \text{ and } B) = \frac{1}{6}$.
 Find:
 - $P(\bar{A})$
 - $P(\bar{B})$
 - $P(A \text{ or } B)$
 - $P(\text{neither } A \text{ nor } B)$
 - $P(\text{not both } A \text{ and } B)$

- 7** Use the addition rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to answer the following questions.

- If $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{15}$, find $P(A \cup B)$.
- If $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{5}{6}$, find $P(A \cap B)$.
- If $P(A \cup B) = \frac{9}{10}$, $P(A \cap B) = \frac{1}{5}$ and $P(A) = \frac{1}{2}$, find $P(B)$.
- If A and B are mutually exclusive and $P(A) = \frac{1}{7}$ and $P(B) = \frac{4}{7}$, find $P(A \cup B)$.

- 8** An integer n is picked at random, where $1 \leq n \leq 20$. The events A , B , C and D are:

A : an even number is chosen,

B : a number greater than 15 is chosen,

C : a multiple of 3 is chosen,

D : a one-digit number is chosen.

- Are the events A and B mutually exclusive?
- Find $P(A)$, $P(B)$ and $P(A \text{ and } B)$, and hence evaluate $P(A \text{ or } B)$.

- b i** Are the events A and C mutually exclusive?
ii Find $P(A)$, $P(C)$ and $P(A \text{ and } C)$, and hence evaluate $P(A \text{ or } C)$.
- c i** Are the events B and D mutually exclusive?
ii Find $P(B)$, $P(D)$ and $P(B \text{ and } D)$, and hence evaluate $P(B \text{ or } D)$.
- 9** In a group of 50 students, there are 26 who study Latin and 15 who study Greek and 8 who study both languages. Draw a Venn diagram and find the probability that a student chosen at random:
- a** studies only Latin, **b** studies only Greek, **c** does not study either language.
- 10** During a game, all 21 members of an Australian Rules football team consume liquid. Some players drink only water, some players drink only GatoradeTM and some players drink both. There are 14 players who drink water and 17 players who drink GatoradeTM.
- a** How many drink both water and GatoradeTM?
b If one team member is selected at random, find the probability that:
- i** he drinks water but not GatoradeTM, **ii** he drinks GatoradeTM but not water.
- 11** Each student in a music class of 28 studies either the piano or the violin or both. It is known that 20 study the piano and 15 study the violin. Find the probability that a student selected at random studies both instruments.

CHALLENGE

- 12** List the 25 primes less than 100. A number is drawn at random from the integers from 1 to 100. Find the probability that:
- a** it is prime,
b it has remainder 1 after division by 4,
c it is prime and it has remainder 1 after division by 4,
d it is either prime or it has remainder 1 after division by 4.
- 13** A group of 60 students was invited to try out for three sports: rugby, soccer and cross country. Of these, 32 tried out for rugby, 29 tried out for soccer, 15 tried out for cross country, 11 tried out for rugby and soccer, 9 tried out for soccer and cross country, 8 tried out for rugby and cross country, and 5 tried out for all three sports. Draw a Venn diagram and find the probability that a student chosen at random:
- a** tried out for only one sport, **b** tried out for exactly two sports,
c tried out for at least two sports, **d** did not try out for a sport.

12E Multi-stage experiments and the product rule

Learning intentions

- Use the product rule in multi-stage experiments.
- Distinguish sampling with and without replacement.

This section deals with experiments that have a number of stages. The full sample space of such an experiment can quickly become too large to be conveniently listed, but instead a rule can be developed for multiplying together the probabilities associated with each stage.

Two-stage experiments — the product rule

Here is a simple question about a two-stage experiment:

‘Throw a die, then toss a coin. What is the probability of obtaining at least 2 on the die, followed by a head?’

Graphed to the right are the twelve possible outcomes of the experiment, all equally likely, with a box drawn around the five favourable outcomes. Thus

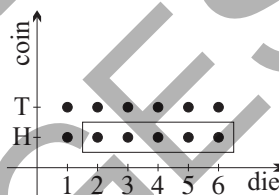
$$P(\text{at least 2 and a head}) = \frac{5}{12}.$$

Now consider the two stages separately. The first stage is throwing a die, and the desired outcome is $A =$ ‘getting at least 2’ — here there are six possible outcomes and five favourable outcomes, giving probability $\frac{5}{6}$. The second stage is tossing a coin, and the desired outcome is $B =$ ‘tossing a head’ — here there are two possible outcomes and one favourable outcome, giving probability $\frac{1}{2}$.

The full experiment then has $6 \times 2 = 12$ possible outcomes and there are $5 \times 1 = 5$ favourable outcomes. Hence

$$P(AB) = \frac{5 \times 1}{6 \times 2} = \frac{5}{6} \times \frac{1}{2} = P(A) \times P(B).$$

Thus the probability of the compound event ‘getting at least 2 and a head’ can be found by multiplying together the probabilities of the two stages. The argument here can easily be generalised to any two-stage experiment.



16 Two-stage experiments

If A and B are events in successive independent stages of an experiment, then

$$P(AB) = P(A) \times P(B),$$

where the word ‘independent’ means that the outcome of one stage does not affect the probabilities of the other stage.

Independent events

The word ‘independent’ needs further discussion. In the example above, the throwing of the die clearly does not affect the tossing of the coin, so the two events are independent.

Here is an obvious experiment where the two stages are not independent:

‘Choose someone at random from the NSW population. First note the elector’s gender. Then measure the person’s height.’


Example 15 Working with independent events

Two dice are thrown. What is the probability that the first throw is a double, and the second throw gives a sum of at least 4?

Solution

We saw in Section 12C that when two dice are thrown, there are 36 possible outcomes, graphed in the diagram to the right.

There are six doubles amongst the 36 possible outcomes,

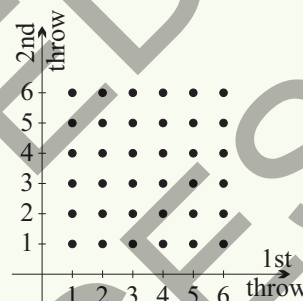
$$\begin{aligned}\text{so } P(\text{double}) &= \frac{6}{36} \\ &= \frac{1}{6}.\end{aligned}$$

All but the pairs (1, 1), (2, 1) and (1, 2) give a sum at least 4,

$$\begin{aligned}\text{so } P(\text{sum is at least 4}) &= \frac{33}{36} \\ &= \frac{11}{12}.\end{aligned}$$

Because the two stages are independent,

$$\begin{aligned}P(\text{double, sum at least 4}) &= \frac{1}{6} \times \frac{11}{12} \\ &= \frac{11}{72}.\end{aligned}$$



Multi-stage experiments — the product rule

The same arguments clearly apply to an experiment with any number of stages.

17 Multi-stage experiments

If $A_1, A_2, \dots, A_n$ are events in successive independent stages of a multi-stage experiment, then

$$P(A_1 A_2 \dots A_n) = P(A_1) \times P(A_2) \times \dots \times P(A_n).$$


Example 16 Using the product rule in a multi-stage experiment

A coin is tossed five times. Find the probability that:

a every toss is a head,

b no toss is a head,

c there is at least one head.

Solution

The five tosses are independent events.

$$\begin{aligned}\text{a } P(\text{HHHHH}) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{32}\end{aligned}$$

$$\begin{aligned}\text{b } P(\text{no heads}) &= P(\text{TTTTT}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{32}\end{aligned}$$

$$\begin{aligned}\text{c } P(\text{at least one head}) &= 1 - P(\text{no heads}) \\ &= 1 - \frac{1}{32} \\ &= \frac{31}{32}\end{aligned}$$

Sampling without replacement — an extension of the product rule

The product rule can be extended to the following question, where the two stages of the experiment are not independent.


Example 17 Using sampling without replacement

A box contains five discs numbered 1, 2, 3, 4 and 5.

Two numbers are drawn in succession, without replacement.

What is the probability that both are even?

Solution

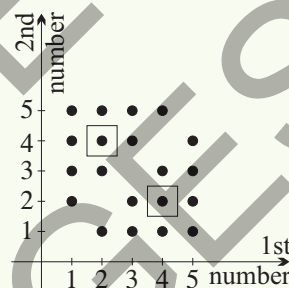
The probability that the first number is even is $\frac{2}{5}$. When this even number is removed, one even and three odd numbers remain, so the probability that the second number is also even is $\frac{1}{4}$.

$$\begin{aligned}\text{Hence } P(\text{both even}) &= \frac{2}{5} \times \frac{1}{4} \\ &= \frac{1}{10}.\end{aligned}$$

Note: The graph to the right allows the calculation to be checked by examining its full sample space.

Because doubles are not allowed (that is, there is no replacement), there are only 20 possible outcomes.

The two boxed outcomes are the only outcomes that consist of two even numbers, giving the same probability of $\frac{2}{20} = \frac{1}{10}$.


Listing the favourable outcomes

The product rule is often combined with a listing of the favourable outcomes. A tree diagram may help in producing that listing, although this is hardly necessary in the straightforward example below.


Example 18 Using listing with a multi-stage experiment

A coin is tossed four times. Find the probability that:

- a** the first three coins are heads, **b** the middle two coins are tails.

c Copy and complete this table:

number of heads	0	1	2	3	4
probability					

Solution

a $P(\text{the first three coins are heads}) = P(\text{HHHH}) + P(\text{HHHT})$
 (notice that the two events HHHH and HHHT are mutually exclusive)

$$= \frac{1}{16} + \frac{1}{16}$$

 (because each of these two probabilities is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$)

$$= \frac{1}{8}.$$

b $P(\text{middle two are tails}) = P(\text{HTTH}) + P(\text{HTTT}) + P(\text{TTTH}) + P(\text{TTTT})$

$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{1}{4}.$$

c $P(\text{no heads}) = P(\text{TTTT})$

$$= \frac{1}{16}$$

$$P(\text{one head}) = P(\text{HTTT}) + P(\text{THTT}) + P(\text{TTHT}) + P(\text{TTTH})$$

$$= \frac{4}{16}$$

$$P(\text{two heads}) = P(\text{HHTT}) + P(\text{HTHT}) + P(\text{THHT})$$

$$+ P(\text{HTTH}) + P(\text{THTH}) + P(\text{TTHH})$$

(these are all six possible orderings of H, H, T and T)

$$= \frac{6}{16}, \text{ and the last two results follow by symmetry.}$$

number of heads	0	1	2	3	4
probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Describing the experiment in a different way

Sometimes, the manner in which an experiment is described makes calculation difficult, but the experiment can be described in a different way so that the probabilities are the same but the calculations are much simpler.



Example 19 Retelling the story

Wes is sending Christmas cards to ten friends. He has two cards with angels, two with snow, two with reindeer, two with trumpets and two with Santa Claus.

What is the probability that Harry and Helmut get matching cards?

Solution

Describe the process in a different way as follows:

‘Wes decides that he will choose Harry’s card first and Helmut’s card second. Then he will choose the cards for his remaining eight friends.’

All that matters now is whether the card that Wes chooses for Helmut is the same as the card that he has already chosen for Harry. After he chooses Harry’s card, there are nine cards remaining, of which only one matches Harry’s card. Thus the probability that Helmut’s card matches is $\frac{1}{9}$.

Exercise 12E

FOUNDATION

1 Suppose that A , B , C and D are independent events, with $P(A) = \frac{1}{8}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$ and $P(D) = \frac{2}{7}$. Use the product rule to find:

- a** $P(AB)$ **b** $P(AD)$ **c** $P(BC)$ **d** $P(ABC)$ **e** $P(BCD)$ **f** $P(ABCD)$

2 A coin and a die are tossed. Use the product rule to find the probability of obtaining:

- a** a three and a head,
b a six and a tail,
c an even number and a tail,
d a number less than five and a head.

- 3** One set of cards contains the numbers 1, 2, 3, 4 and 5, and another set contains the letters A, B, C, D and E. One card is drawn at random from each set. Use the product rule to find the probability of drawing:
- a** 4 and B,
 - b** 2 or 5, then D,
 - c** 1, then A or B or C,
 - d** an odd number and C,
 - e** an even number and a vowel,
 - f** a number less than 3, and E,
 - g** the number 4, followed by a letter from the word MATHS.
- 4** Two marbles are picked at random, one from a bag containing three red and four blue marbles, and the other from a bag containing five red and two blue marbles. Find the probability of drawing:
- a** two red marbles,
 - b** two blue marbles,
 - c** a red marble from the first bag and a blue marble from the second.
- 5** A box contains five light bulbs, two of which are faulty. Two bulbs are selected, one at a time without replacement. Find the probability that:
- a** both bulbs are faulty,
 - b** neither bulb is faulty,
 - c** the first bulb is faulty and the second one is not,
 - d** the second bulb is faulty and the first one is not.

DEVELOPMENT

- 6** A die is rolled twice. Using the product rule, find the probability of throwing:
- a** a six and then a five,
 - b** a one and then an odd number,
 - c** a double six,
 - d** two numbers greater than four,
 - e** a number greater than four and then a number less than four.
- 7** A box contains twelve red and ten green discs. Three discs are selected, one at a time without replacement.
- a** What is the probability that the discs selected are red, green, red in that order?
 - b** What is the probability of this event if the disc is replaced after each draw?
- 8** **a** From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:
- i** a spade and then a heart,
 - ii** two clubs,
 - iii** a jack and then a queen,
 - iv** the king of diamonds and then the ace of clubs.
- b** Repeat the question if the first card is replaced before the second card is drawn.
- 9** A coin is weighted so that it is twice as likely to fall heads as it is tails.
- a** Write down the probabilities that the coin falls:
- i** heads,
 - ii** tails.
- b** If you toss the coin three times, find the probability of:
- i** three heads,
 - ii** three tails,
 - iii** head, tail, head in that order.
- 10** If a coin is tossed repeatedly, find the probability of obtaining at least one head in:
- a** two tosses,
 - b** five tosses,
 - c** ten tosses.

- 11** [Valid and invalid arguments] Identify any fallacies in the following arguments. If possible, give some indication of how to correct them.
- a** 'The probability that a Year 12 student chosen at random likes classical music is 50%, and the probability that a student plays a classical instrument is 20%. Therefore the probability that a student chosen at random likes classical music and plays a classical instrument is 10%.'
 - b** 'The probability of a die showing a prime is $\frac{1}{2}$, and the probability that it shows an odd number is $\frac{1}{2}$. Hence the probability that it shows an odd prime number is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.'
 - c** 'I choose a team at random from an eight-team competition. The probability that it wins any game is $\frac{1}{2}$, so the probability that it defeats all of the other seven teams is $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$.'
 - d** 'A normal coin is tossed and shows heads eight times. Nevertheless, the probability that it shows heads the next time is still $\frac{1}{2}$.'
- 12** A die is rolled twice. Using the product rule, find the probability of rolling:
- a** a double two,
 - b** any double,
 - c** a number greater than three, then an odd number,
 - d** a one and then a four,
 - e** a four and then a one,
 - f** a one and a four in any order,
 - g** an even number, then a five,
 - h** a five and then an even number,
 - i** an even number and a five in any order.
- 13** An archer fires three shots at a bullseye. He has a 90% chance of hitting the bullseye. Using H for hit and M for miss, list all eight possible outcomes. Then, assuming that successive shots are independent, use the product rule to find the probability that he will:
- a** hit the bullseye three times,
 - b** miss the bullseye three times,
 - c** hit the bullseye on the first shot only,
 - d** hit the bullseye exactly once,
 - e** miss the bullseye on the first shot only,
 - f** miss the bullseye exactly once.
- (Hint: Part (d) requires adding the probabilities of HMM, MHM and MMH, and part (f) requires a similar calculation.)
-
- CHALLENGE**
- 14** One layer of tinting material on a window cuts out $\frac{1}{5}$ of the sun's UV rays at a certain frequency.
- a** What fraction would be cut out by using two layers?
 - b** How many layers would be required to cut out at least $\frac{9}{10}$ of the sun's UV rays?
 - c** What relation does this question have to questions about probability?
- 15** In a lottery, the probability of the jackpot prize being won in any draw is $\frac{1}{60}$.
- a** What is the probability that the jackpot prize will be won in each of four consecutive draws?
 - b** How many consecutive draws need to be made for there to be a greater than 98% chance that at least one jackpot prize will have been won?
- 16** [This question is best done by retelling the story of the experiment, as explained in the notes above.] Nick has five different pairs of socks to last the working week, and they are scattered loose in his drawer. Each morning, he gets up before light and chooses two socks at random. Find the probability that he wears a matching pair:
- a** on the first morning,
 - b** on the last morning,
 - c** on the third morning,
 - d** the first two mornings,
 - e** every morning,
 - f** every morning but one.
- 17** John's bag has six socks, each black or white. He selects two socks at random. If the probability of selecting two white socks is $\frac{2}{3}$, find the probability of selecting two black socks.

12F Probability tree diagrams

Learning intentions

- Construct and use probability tree diagrams.

In more complicated problems, and particularly in unsymmetric situations, a probability tree diagram can be very useful in organising the various cases, in preparation for the application of the product rule.

Constructing a probability tree diagram

A *probability tree diagram* differs from the tree diagrams used in Section 12C for counting possible outcomes, in that the relevant probabilities are written on the branches and then multiplied together in accordance with the product rule. An example will demonstrate the method.

As before, these diagrams have one column labelled 'Start', a column for each stage, and a column listing the outcomes, but there is now an extra column labelled 'Probability' at the end.



Example 20 Using probability tree diagrams

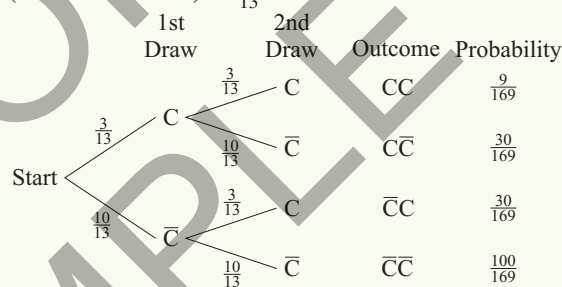
One card is drawn from each of two packs. Use a probability tree diagram to find the probability that:

- both cards are court cards (jack, queen or king),
- neither card is a court card,
- one card is a court card and the other is not.

Solution

In each pack of cards, there are 12 court cards out of 52 cards,

so $P(\text{court card}) = \frac{3}{13}$ and $P(\text{not a court card}) = \frac{10}{13}$.



Multiply the probabilities along each arm because they are successive stages.

Add the probabilities in the final column because they are mutually exclusive.

$$\begin{aligned} \text{a } P(\text{two court cards}) &= \frac{3}{13} \times \frac{3}{13} \\ &= \frac{9}{169} \end{aligned}$$

$$\begin{aligned} \text{b } P(\text{no court cards}) &= \frac{10}{13} \times \frac{10}{13} \\ &= \frac{100}{169} \end{aligned}$$

$$\begin{aligned} \text{c } P(\text{one court card}) &= \frac{3}{13} \times \frac{10}{13} + \frac{10}{13} \times \frac{3}{13} \\ &= \frac{30}{169} + \frac{30}{169} \\ &= \frac{60}{169} \end{aligned}$$

Note: The four probabilities in the last column of the tree diagram add exactly to 1, which is a useful check on the working. The three answers also add to 1.

In a *probability tree diagram*:

- Probabilities are written on the branches.
- There is a final column giving the probability of each outcome.



A bag contains six white marbles and four blue marbles. Three marbles are drawn in succession. At each draw, if the marble is white it is replaced, and if it is blue it is not replaced.

- a** no blue marbles,
- c** two blue marbles,

- b** one blue marble,
d three blue marbles.

With the ten marbles all in the bag,

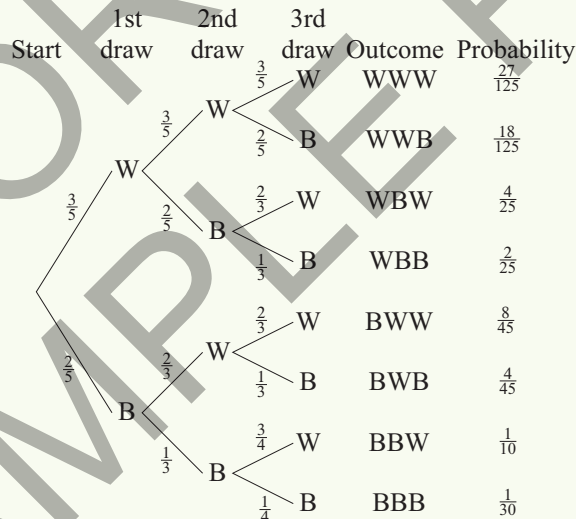
$$P(W) = \frac{6}{10} = \frac{3}{5} \quad \text{and} \quad P(B) = \frac{4}{10} = \frac{2}{5}.$$

If one blue marble is removed, there are six white and three blue marbles, so

$$P(W) = \frac{6}{9} = \frac{2}{3} \quad \text{and} \quad P(B) = \frac{3}{9} = \frac{1}{3}.$$

If two blue marbles are removed, there are six white and two blue marbles, so

$$P(W) = \frac{6}{8} = \frac{3}{4} \quad \text{and} \quad P(B) = \frac{2}{8} = \frac{1}{4}.$$



In each part, multiply the probabilities along each arm and then add the cases.

$$\mathbf{a} \quad P(\text{no blue marbles}) = \frac{27}{125}$$

$$\mathbf{b} \quad P(\text{one blue marble}) = \frac{18}{125} + \frac{4}{25} + \frac{8}{45}$$

$$= \frac{542}{1125}$$

$$\begin{aligned} \text{c } P(\text{two blue marbles}) &= \frac{2}{25} + \frac{4}{45} + \frac{1}{10} \\ &= \frac{121}{450} \end{aligned}$$

d $P(\text{three blue marbles}) = \frac{1}{30}$

Note: Again, as a check on the working, your calculator will show that the eight probabilities in the last column of the diagram add exactly to 1 and that the four answers above also add to 1.

Exercise 12F

FOUNDATION

- 1** A bag contains three black and four white discs. A disc is selected from the bag, its colour is noted, and it is then returned to the bag before a second disc is drawn.

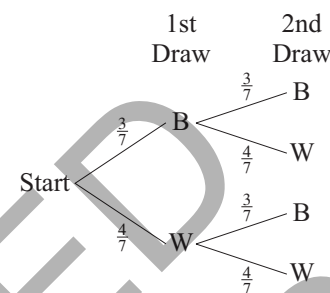
a By multiplying along the branches of the tree, find:

- i** $P(BB)$ **ii** $P(BW)$ **iii** $P(WB)$ **iv** $P(WW)$

b Hence, by adding, find the probability that:

- i** both discs drawn have the same colour,
ii the discs drawn have different colours.

c Draw your own tree diagram, and repeat part (b) if the first ball is not replaced before the second one is drawn.

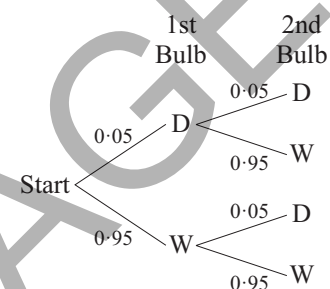


- 2** Two light bulbs are selected at random from a large batch of bulbs in which 5% are defective.

a By multiplying along the branches of the tree, find:

- i** $P(\text{both bulbs work})$,
ii $P(\text{the first works, but the second is defective})$,
iii $P(\text{the first is defective, but the second works})$,
iv $P(\text{both bulbs are defective})$.

b Hence find the probability that at least one bulb works.



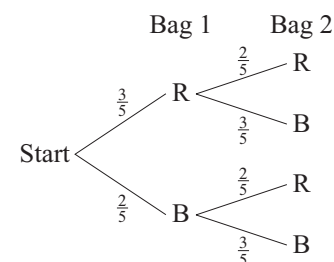
- 3** One bag contains three red and two blue balls and another bag contains two red and three blue balls. A ball is drawn at random from each bag.

a By multiplying along the branches of the tree, find:

- i** $P(RR)$ **ii** $P(RB)$ **iii** $P(BR)$ **iv** $P(BB)$

b Hence, by adding, find the probability that:

- i** the balls have the same colour,
ii the balls have different colours.



- 4** In group A there are three girls and seven boys, and in group B there are six girls and four boys. One person is chosen at random from each group. Draw a probability tree diagram.

a By multiplying along the branches of the tree, find:

- i** $P(GG)$ **ii** $P(GB)$ **iii** $P(BG)$ **iv** $P(BB)$

b Hence, by adding, find the probability that:

- i** two boys or two girls are chosen, **ii** one boy and one girl are chosen.

- 5** There is an 80% chance that Garry will pass his driving test and a 90% chance that Emma will pass hers. Draw a probability tree diagram, and find the chance that:

- a** Garry passes and Emma fails,
b Garry fails and Emma passes,
c only one of Garry and Emma passes,
d at least one fails.

DEVELOPMENT

- 6** The probability that a set of traffic lights will be green when you arrive at them is $\frac{3}{5}$.
A motorist drives through two sets of lights. Assuming that the two sets of traffic lights are not synchronised, find the probability that:
- a** both sets of lights will be green,
 - b** at least one set of lights will be green.
- 7** A factory assembles calculators. Each calculator requires a chip and a battery. It is known that 1% of chips and 4% of batteries are defective. Find the probability that a calculator selected at random will have at least one defective component.
- 8** The probability of a woman being alive at 80 years of age is 0.2, and the probability of her husband being alive at 80 years of age is 0.05. Find the probability that:
- a** they will both live to be 80 years of age,
 - b** only one of them will live to be 80.
- 9** Alex and Julia are playing in a tennis tournament. They will play each other twice, and each has an equal chance of winning the first game. If Alex wins the first game, his probability of winning the second game is increased to 0.55. If he loses the first game, his probability of winning the second game is reduced to 0.25. Find the probability that Alex wins exactly one game.
- 10** One bag contains four red and three blue discs, and another bag contains two red and five blue discs. A bag is chosen at random and then a disc is drawn from it. Find the probability that the disc is blue.
- 11** In a raffle in which there are 200 tickets, the first prize is drawn and then the second prize is drawn without replacing the winning ticket. If you buy 15 tickets, find the probability that you win:
- a** both prizes,
 - b** at least one prize.
- 12** A box contains 10 chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Hugo randomly selects and eats three chocolates from the box. Find the probability that:
- a** the first chocolate Hugo eats is caramel,
 - b** Hugo eats three mint chocolates,
 - c** Hugo eats exactly one caramel chocolate.
- 13** In an aviary there are four canaries, five cockatoos and three budgerigars. If two birds are selected at random, find the probability that:
- a** both are canaries,
 - b** neither is a canary,
 - c** one is a canary and one is a cockatoo,
 - d** at least one is a canary.
- 14** Max and Jack each throw a die. Find the probability that:
- a** they do not throw the same number,
 - b** the number thrown by Max is greater than the number thrown by Jack,
 - c** the numbers they throw differ by three.
- 15** In a large co-educational school, the population is 47% female and 53% male. Two students are selected from the school population at random. Find, correct to two decimal places, the probability that:
- a** both are male,
 - b** a boy and a girl are selected.

- 16** The numbers 1, 2, 3, 4 and 5 are each written on a card. The cards are shuffled and one card is drawn at random. The number is noted and the card is then returned to the pack. A second card is selected, and in this way a two-digit number is recorded. For example, a 2 on the first draw and a 3 on the second results in the number 23.
- a** What is the probability of the number 35 being recorded?
 - b** What is the probability of an odd number being recorded?
- 17** A twenty-sided die has the numbers from 1 to 20 on its faces.
- a** If it is rolled twice, what is the probability that the same number appears on the uppermost face each time?
 - b** If it is rolled three times, what is the probability that the number 15 appears on the uppermost face exactly twice?
- 18** An interviewer conducts a poll in Sydney and Melbourne on the popularity of the prime minister. In Sydney, 52% of the population approve of the prime minister, and in Melbourne her approval rating is 60%. If one of the two capital cities is selected at random and two electors are surveyed, find the probability that:
- a** both electors approve of the prime minister,
 - b** at least one elector approves of the prime minister.
- 19** In a bag there are four green, three blue and five red discs.
- a** Two discs are drawn at random, and the first disc is not replaced before the second disc is drawn. Find the probability of drawing:
 - i** two red discs,
 - ii** one red and one blue disc,
 - iii** at least one green disc,
 - iv** a blue disc on the first draw,
 - v** two discs of the same colour,
 - vi** two differently coloured discs.
 - b** Repeat part (a) if the first disc is replaced before the second disc is drawn.

CHALLENGE

- 20** In a game, two dice are rolled and the score given is the maximum of the two numbers on the uppermost faces. For example, if the dice show a three and a five, the score is a five.
- a** Find the probability that you score a one in a single throw of the two dice.
 - b** What is the probability of scoring three consecutive ones in three rolls of the dice?
 - c** Find the probability that you score a six in a single roll of the dice.
- 21** In each game of Sic Bo, three regular six-sided dice are thrown once.
- a** In a single game, what is the probability that all three dice show six?
 - b** What is the probability that exactly two of the dice show six?
 - c** What is the probability that exactly two of the dice show the same number?
 - d** What is the probability of rolling three different numbers on the dice?
- 22** A bag contains two green and two blue marbles. Marbles are drawn at random, one by one without replacement, until two green marbles have been drawn. What is the probability that exactly three draws will be required?

12G Conditional probability

Learning intentions

- Understand conditional probability and its formulae.
- Use the conditional probability formulae to solve problems.
- Relate conditional probability, independent events, and the product rule.
- Use two-way tables, and relate them to conditional probability.
- Summarise elementary probability in four formulae.

Sometimes our knowledge of the results of an experiment changes over time, and as we gain more information, our probabilities change. Conditional probability allows us to calculate these changing probabilities.

Reduced sample space and reduced event space

Drago will win a certain game if an odd number is thrown on a die. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, and the event space is $A = \{1, 3, 5\}$, so

$$\begin{aligned} P(\text{he wins}) &= \frac{3}{6} \\ &= \frac{1}{2}. \end{aligned}$$

Drago nervously turns away as the die is thrown, and someone calls out, ‘It’s a prime number!’ What, for him, is now the probability that he wins the game?

- First, the sample space has now been reduced to the set $B = \{2, 3, 5\}$ of prime numbers on the die. This is called the *reduced sample space*.
- Secondly, the event space has now been reduced to the set $\{3, 5\}$ of odd prime numbers on the die. This *reduced event space* is the intersection $A \cap B$ of the event space A and the reduced sample space B . Hence

$$\begin{aligned} P(\text{he wins, given that the number is prime}) &= \frac{|\text{reduced event space}|}{|\text{reduced sample space}|} \\ &= \frac{2}{3}. \end{aligned}$$

Alternatively, with B as the reduced sample space and $A \cap B$ as the reduced event space, we can write the working using a concise formula,

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{2}{3}, \end{aligned}$$

where $P(A|B)$ is the symbol for conditional probability, and means ‘the probability of A , given that B has occurred’.

(We are assuming, of course, that the person calling out was telling the truth.)

Conditional probability

It is only a matter of convenience whether we work with the reduced sample space and the reduced event space, or with formulae:

19 Conditional probability

Suppose that two events A and B are subsets of the same uniform sample space S . Then the *conditional probability* $P(A|B)$ of A , given that B has occurred, is obtained by removing all the elements not in B from the sample space S , and from the event space A . Thus there are two equivalent formulae,

$$P(A|B) = \frac{|\text{reduced event space}|}{|\text{reduced sample space}|} \quad \text{and} \quad P(A|B) = \frac{|A \cap B|}{|B|}.$$

Read $P(A|B)$ as ‘the probability of A , given that B has occurred’.

**Example 22** Work with conditional probability using reduced sample spaces

A card is drawn at random from a standard pack of 52 cards. Find the probability that it is a 9, 10, jack or queen if:

- a** Nothing more is known.
- b** It is known to be a court card (jack, queen or king)
- c** It is known to be an 8 or a 10.
- d** It is known not to be a 2, 3, 4 or 5.

Solution

Let the sample space S be the set of all 52 cards, and let

$$A = \{9\clubsuit, 9\spadesuit, 9\heartsuit, 9\diamondsuit, 10\clubsuit, 10\spadesuit, 10\heartsuit, 10\diamondsuit, J\clubsuit, J\spadesuit, J\heartsuit, J\diamondsuit, Q\clubsuit, Q\spadesuit, Q\heartsuit, Q\diamondsuit\}$$

- a** There is no reduction of the sample space or the event space,

$$\begin{aligned} \text{so } P(A) &= \frac{|A|}{|S|} \\ &= \frac{16}{52} \\ &= \frac{4}{13}. \end{aligned}$$

- b** The reduced sample space is

$$B = \{\text{court cards}\},$$

and the reduced event space is

$$A \cap B = \{\text{jacks and queens}\},$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{8}{12} \\ &= \frac{2}{3}. \end{aligned}$$

- c** The reduced sample space is

$$B = \{8\text{s and } 10\text{s}\},$$

and the reduced event space is

$$A \cap B = \{10\text{s}\},$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{4}{8} \\ &= \frac{1}{2}. \end{aligned}$$

- d** The reduced sample space is

$$B = \{\text{cards not } 2, 3, 4 \text{ or } 5\},$$

and the reduced event space is

$$A \cap B = A,$$

$$\begin{aligned} \text{so } P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{16}{36} \\ &= \frac{4}{9}. \end{aligned}$$

Another formula for conditional probability

Suppose as before that two events A and B are subsets of the same uniform sample space S . The previous formula gave $P(A|B)$ in terms of the sizes of the sets $A \cap B$ and B . By dividing through by $|S|$, we obtain a formula for $P(A|B)$ in terms of the probabilities of $A \cap B$ and B :

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{|A \cap B|}{|S|} \div \frac{|B|}{|S|} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

20 Another formula for conditional probability

For two events A and B ,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} .$$

The great advantage of this formula is that it can be used when nothing is known about the actual sizes of the sample space and the event space, as in the example below.



Example 23 Using conditional probability as a ratio of probabilities

In a certain population, 35% have blue eyes, 15% have blond hair, and 10% have blue eyes and blond hair. A person is chosen from this population at random.

- a** Find the probability that they have blond hair, given that they have blue eyes.
- b** Find the probability that they have blue eyes, given that they have blond hair.

Solution

From the given percentages, we know that

$$P(\text{blue eyes}) = 0.35, \quad P(\text{blond hair}) = 0.15, \quad P(\text{blue eyes and blond hair}) = 0.1.$$

$$\mathbf{a} \quad P(\text{blond hair, given blue eyes}) = \frac{P(\text{blond hair and blue eyes})}{P(\text{blue eyes})}$$

$$= \frac{0.1}{0.35}$$

$$\div 0.29$$

$$\mathbf{b} \quad P(\text{blue eyes, given blond hair}) = \frac{P(\text{blond hair and blue eyes})}{P(\text{blond hair})}$$

$$= \frac{0.1}{0.15}$$

$$\div 0.67$$

Independent events

We say that an event A is *independent* of an event B if knowing whether or not B has occurred does not affect our probability of A . That is,

$$A \text{ and } B \text{ are independent} \quad \text{means that} \quad P(A|B) = P(A) .$$

21 Independent events

- Two events A and B are called *independent* if the probability of A is not affected by knowing whether or not B has occurred:

$$P(A|B) = P(A)$$

- If A is independent of B , then B is independent of A .

The second dotpoint may seem trivial, but it needs proving.

Suppose that A is independent of B , meaning that $P(A|B) = P(A)$.

Then to prove that B is independent of A ,

$$\begin{aligned} P(B|A) &= \frac{P(B \cap A)}{P(A)} && \text{(using the formula for } P(B|A)) \\ &= \frac{P(A \cap B)}{P(B)} \times \frac{P(B)}{P(A)} && \text{(multiply top and bottom by } P(B)) \\ &= P(A|B) \times \frac{P(B)}{P(A)} && \text{(using the formula for } P(A|B)) \\ &= P(A) \times \frac{P(B)}{P(A)} && \text{(we are assuming that } P(A|B) = P(A)) \\ &= P(B), \text{ as required.} \end{aligned}$$

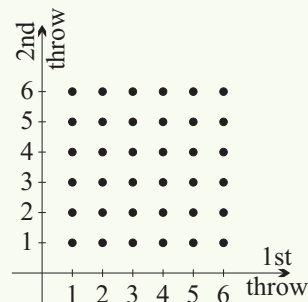
The consequence of this is that we can just say that ‘the two events A and B are independent’, which is what we would expect.

**Example 24** Conditional probability and independence

Two dice are thrown one after the other.

- Let A be the event ‘the first die is odd’.
- Let B be the event ‘the second die is 1, 2 or 3’.
- Let C be the event ‘the sum is five’.

Which of the three pairs of events are independent?

**Solution**

The sample space S has size 36, and

$$P(A) = \frac{18}{36} = \frac{1}{2}, \quad P(B) = \frac{18}{36} = \frac{1}{2}, \quad P(C) = \frac{4}{36} = \frac{1}{9}.$$

The last result follows because $5 = 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1$ are all the ways that a sum of five can be obtained. Taking intersections,

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4}, \quad P(B \cap C) = \frac{3}{36} = \frac{1}{12}, \quad P(C \cap A) = \frac{2}{36} = \frac{1}{18}.$$

Testing the definition of independence on the three pairs of events,

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} & P(B|C) &= \frac{P(B \cap C)}{P(C)} & P(C|A) &= \frac{P(C \cap A)}{P(A)} \\ &= \frac{1}{4} \div \frac{1}{2} & &= \frac{1}{12} \div \frac{1}{9} & &= \frac{1}{18} \div \frac{1}{2} \\ &= \frac{1}{2} & &= \frac{3}{4} & &= \frac{1}{9} \\ &= P(A) & &\neq P(B) & &= P(C) \end{aligned}$$

Hence A and B are independent, and C and A are independent, but B and C are not independent.

Note: The word ‘independent’ was introduced in Section 12E with multi-stage experiments. Its use in this section is a generalisation of its use in the multi-stage situation. For example, the events A and B above are independent in both senses, but the independence of C and A is a new idea.

The product rule

The formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ can be rearranged as

$$P(A \cap B) = P(A|B) \times P(B).$$

If the events are independent, then $P(A|B) = P(A)$, so

$$P(A \cap B) = P(A) \times P(B).$$

Conversely, if $P(A \cap B) = P(A) \times P(B)$, then $P(A|B) = P(A)$, meaning that the events are independent. Thus we have both a formula that applies to independent events, and another test for independence:

22 The product rule and independent events

- For any events A and B ,

$$P(A \text{ and } B) = P(A|B) \times P(B).$$

- If A and B are independent events, then

$$P(A \text{ and } B) = P(A) \times P(B).$$

- Conversely, if $P(A \text{ and } B) = P(A) \times P(B)$, then the events are independent.

Remember that $P(A \text{ and } B)$ and $P(A \cap B)$ mean exactly the same thing

All this is a generalisation of the product rule introduced with the multi-stage experiments and tree diagrams in Section 12E.



Example 25 Working with the product rule and independence

Calculate $P(A) \times P(B)$ and $P(B) \times P(C)$ and $P(C) \times P(A)$ in Example 24. Then use Box 22 above to confirm that these calculations give the same results for the independence of A and B , of B and C , and of C and A .

Solution

$$\begin{array}{lll} P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} & P(B) \times P(C) = \frac{1}{2} \times \frac{1}{9} & P(C) \times P(A) = \frac{1}{9} \times \frac{1}{2} \\ = \frac{1}{4} & = \frac{1}{18} & = \frac{1}{18} \\ = P(A \cap B) & \neq P(B \cap C) & = P(C \cap A) \end{array}$$

again showing that A and B are independent, that C and A are independent, but that B and C are not independent.



Example 26 Working with the product rule and independence

Example 23 gave some data about the incidence of blue eyes and blond hair in a population.

- Why does the product rule formula show immediately that ‘blue eyes’ and ‘blond hair’ are not independent?
- What would be the probability of having blue eyes and blond hair if the two characteristics were independent?

Solution

Box 22 above says that ‘blue eyes’ and ‘blond hair’ are independent if and only if

$$P(\text{blue eyes and blond hair}) = P(\text{blue eyes}) \times P(\text{blond hair}).$$

a Substituting the values from Example 23 into the formula gives:

$$\text{LHS} = 0.1 \quad \text{and} \quad \text{RHS} = 0.35 \times 0.15 = 0.0525$$

which are different, so ‘blue eyes’ and ‘blond hair’ are not independent.

b If they were independent, then $P(\text{blue eyes and blond hair}) = 0.0525$.

Using the sample space without any formulae

Sometimes, however, it is easier not to use any of the machinery in this section, but to go back to basics and deal directly with the sample space, with little or no notation.



Example 27 Using reduced samples with conditional probability

Yanick and Oskar can’t agree which movie to watch, so they play a game throwing two dice.

- If the sum is 7 or 8, Yanick chooses the movie.
- If the sum is 9, 10, 11 or 12, Oskar chooses the movie.
- If the sum is less than 7, they throw the dice again.

Who has the higher probability of choosing the movie?

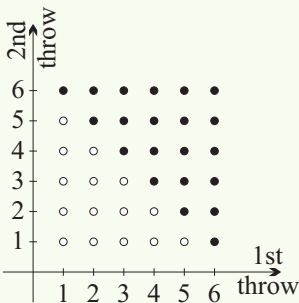
Solution

There are 36 possible throws in the original sample space. But we can ignore the 15 throws with sum less than 7, represented in the diagram by the open circles, because these throws are discarded if they occur. This leaves a sample space with only the 21 closed circles. There are $6 + 5 = 11$ throws with sum 7 or 8, so

$$P(\text{Yanick chooses}) = \frac{11}{21}$$

$$\text{and } P(\text{Oskar chooses}) = \frac{10}{21}.$$

Hence Yanick has the higher probability of choosing.



Two-way tables (contingency tables)

A *two-way table* or *contingency table* consists of two or more related frequency tables put together. In its simplest form, it has only four numbers in the table, yet it is surprisingly complicated to interpret.

A survey asked 200 adults whether they preferred black or red phone cases. Responses were recorded as Black or as Red, and the gender of the person was also recorded. The resulting *joint frequencies* are tabulated to the right.

	Black	Red
Men	38	12
Women	56	94

Let us ask the question, ‘Do men prefer black covers more than women do?’ If we glanced just at the joint frequencies 38 and 56 under ‘Black’, we might conclude that women prefer black covers more than men do. The data are deceptive, however, because far more women were questioned than men — for reasons that we have not been told. We need to take this into account.



Example 28 Interpreting a two-way table

Explain how to analyse the two-way table above to answer the question, ‘Do men prefer black covers more than women do?’

Solution

We can find the sums 50 and 150 of the two rows, and the sums 94 and 106 of the two columns. The grand total is 200, which checks the additions.

These five numbers are called *marginal frequencies*. The bias of the sample towards women is now clear.

	Black	Red	Sum
Men	38	12	50
Women	56	94	150
Sum	94	106	200

Each of the three rows, and each of the three columns, is a frequency table. The last row and the last column are called *marginal distributions*, and the inner two rows and columns are called *conditional distributions* (for reasons explained below).

Now we can answer the question. The proportion of men preferring black covers is $\frac{38}{50} = 76\%$, and the proportion of women preferring black covers is $\frac{56}{150} \div 37\%$, so the survey definitely says that men prefer black covers more than women do. (Because the survey was biased towards women, the proportion of people surveyed preferring black covers is $\frac{94}{200} = 47\%$, which is not the mean of 76% and 37%).

Similarly, the proportion of men preferring red covers is $\frac{12}{50} = 24\%$, and the proportion of women preferring red covers is $\frac{94}{150} \div 63\%$ (and the proportion of people surveyed preferring red covers is $\frac{106}{200} = 53\%$).

Conditional probability in two-way tables

Two-way tables are in this section because they are an example of conditional probability. The percentages in the example above can be regarded as estimates of the probabilities applying to the whole population. If we choose a person from the survey at random,

$$P(\text{person prefers black covers}) = \frac{94}{200} = 0.47.$$

To find the probability that the person prefers black covers given that the person is a man (or a woman), we use the *reduced sample space* of 50 men (or 150 women),

$$P(\text{prefers black} \mid \text{man}) = \frac{38}{50} = 0.76$$

$$P(\text{prefers black} \mid \text{woman}) = \frac{56}{150} \div 0.37.$$

We can also find conditional probabilities the other way around. For a person chosen at random from the survey,

$$P(\text{person is a man}) = \frac{50}{200} = 0.25$$

$$P(\text{man} \mid \text{prefers black}) = \frac{38}{94} \div 0.40$$

$$P(\text{man} \mid \text{prefers red}) = \frac{12}{106} \div 0.11.$$

23 Two-way tables and conditional probability

Two-way tables (or *contingency tables*) allow us to investigate whether two variables are related, and make estimates of conditional probability.

A final note about general sample spaces — a four-formula summary of probability

The theory developed in this chapter can be summarised in four results:

- $P(\text{not } A) = 1 - P(A)$
- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- $P(A \text{ and } B) = P(A) \times P(B)$ if and only if A and B are independent events.

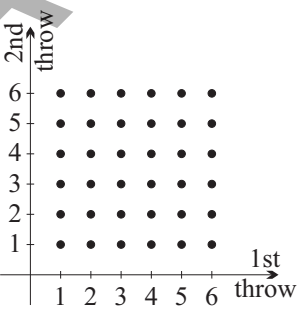
These four results were all developed using equally likely possible outcomes, which restricts them to experiments where there is a finite sample space that is uniform. The four results, however, hold in all theories of probability, either as axioms and definitions, or as results proven by different methods.

Exercise 12G

FOUNDATION

Note: Several questions in this exercise, and elsewhere, require the assumptions that a baby is equally likely to be a boy or a girl, that these are the only two possibilities, and that the sex of a child is independent of the sex of any previous children. These assumptions are simplifications of complex scientific and social considerations, and they do not describe the real situation.

- Two dice are thrown. The sample space of this experiment is shown in the dot diagram alongside. Following the throw, it is revealed that the first die shows an even number.
 - Copy the diagram and circle the reduced sample space.
 - Find the conditional probability of getting two sixes.
 - Find the conditional probability of getting at least one six.
 - Find the conditional probability that the sum of the two numbers is five.



- A poll is taken amongst 1000 people to determine their voting patterns in the last election.

	Coalition	Labor	Other	Total
Male	130	340	110	580
Female	210	190	20	420
Total	340	530	130	1000

- Determine the probability that a particular person in the group voted Coalition.
- What is the probability that a particular person voted Labor, given that they were female?
- What is the probability that a particular member of the group was male, if it is known that they voted Coalition?
- Robin voted neither Coalition nor Labor. What is the probability that Robin was female?

- 3** A student is investigating if there is any relationship between those who choose Extension 1 Mathematics (M) and those who choose Extension 1 English (E) at his school.

	M	$\bar{M}$	Total
E	29	27	
$\bar{E}$	95	42	
Total			

- a** Copy and complete the table by filling in the totals.
- b** What is the probability that a particular student chose:
- neither Extension 1 Mathematics nor Extension 1 English,
 - Extension 1 English, given that they chose Extension 1 Mathematics,
 - Extension 1 Mathematics, if it is known that they chose Extension 1 English,
 - Extension 1 Mathematics, given that they did not choose Extension 1 English.
- 4** Two cards are drawn in turn from a standard pack — the first card is replaced and the pack shuffled before the second card is drawn — and the suit of each card (S, H, D or C) is noted by the game master. A player wants to know the probability that both cards are hearts.
- a** What is the probability if nothing else is known?
- b** The first card is known to be a heart. List the reduced sample space. What is the conditional probability of two hearts?
- c** List the reduced sample space if at least one of the cards is known to be a heart. What is the probability of two hearts?
- d** List the reduced sample space if the first card is known to be red. What is the probability of two hearts?
- 5** In a certain game, the player tosses two coins and then throws a die. The sample space is shown in the table below.

	1	2	3	4	5	6
HH						
HT						
TH						
TT						

The rules of the game assign one point for each head, and zero points for each tail. This score is then added to the score on the die.

- a** Copy the table and fill in the point score for each outcome.
- b** Find the probability that a player gets more than 7 points.
- c** Suppose it is known that he has thrown two heads. Find the probability that a player gets more than 7 points.
- d** Find the probability that a player got an odd number of heads, given that their score was odd.
- 6** Use the formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ to answer the following questions.
- a** Find $P(A|B)$ if $P(A \cap B) = 0.5$ and $P(B) = 0.7$.
- b** Find $P(A|B)$ if $P(A \cap B) = 0.15$ and $P(B) = 0.4$.
- c** Find $P(E|F)$ if $P(E \cap F) = 0.8$ and $P(F) = 0.95$.

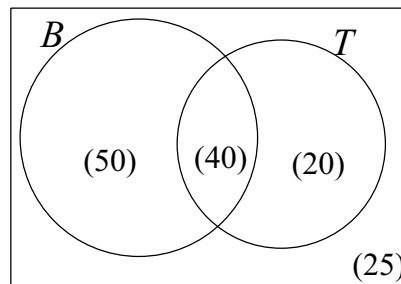
- 7** The two events A and B in the following experiments are known to be independent.
- $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cap B)$.
 - $P(A) = 0.3$ and $P(B) = 0.5$. Find $P(A \cap B)$.
 - $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A|B)$.
 - $P(A) = 0.7$ and $P(B) = 0.2$. Find $P(A|B)$.
- 8** Each of the following experiments involves two events, A and B . State in each case whether they are dependent or independent.
- $P(A|B) = 0.5$ and $P(A) = 0.4$ and $P(B) = 0.5$
 - $P(A|B) = 0.3$ and $P(A) = 0.3$ and $P(B) = 0.6$
 - $P(A|B) = \frac{3}{4}$ and $P(A) = \frac{2}{5}$ and $P(B) = \frac{3}{10}$
 - $P(A) = 0.3$ and $P(B) = 0.7$ and $P(A \cap B) = 0.21$
 - $P(A) = 0.2$ and $P(B) = 0.4$ and $P(A \cap B) = 0.8$
 - $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{3}$

DEVELOPMENT

- 9** **a** Draw a table showing the sample space if two dice are thrown in turn and their sum is recorded.
b Highlight the reduced sample space if the sum of the two dice is 5. Given that the sum of the two dice is 5, what is the probability that:
- the first die shows a 1,
 - at least one dice shows a 1,
 - at least one of the dice shows an odd number.
- 10** **a** For two events A and B it is known that $P(A \cup B) = 0.6$ and $P(A) = 0.4$ and $P(B) = 0.3$.
- Use the addition formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find $P(A \cap B)$.
 - Use the formula for conditional probability to find $P(A|B)$.
 - Find $P(B|A)$.
- b** Suppose that $P(V \cup W) = 0.7$ and $P(V) = 0.5$ and $P(W) = 0.35$. Find $P(V|W)$.
- c** Find $P(X|Y)$ if $P(X \cap Y) = 0.2$ and $P(X) = 0.3$ and $P(Y) = 0.4$.
- d** Find $P(A|B)$ if $P(A \cup B) = \frac{1}{3}$ and $P(A) = \frac{1}{5}$ and $P(B) = \frac{3}{10}$.

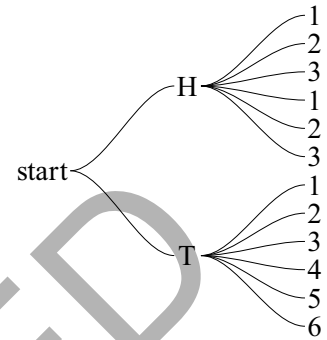
- 11** A student carries out a survey to determine how many students in his year group take the bus or train to school. The results are shown in the Venn diagram to the right.

Let B be the set of those who take the bus and T the set of those who take the train.



- Calculate the probability that a student takes the train.
- Find the probability $P(T|B)$ that a student chosen at random takes the train, given they take the bus. Use the reduced sample space B on the Venn Diagram, and show working. With reference to part (a), what can you conclude?
- Confirm your response to part (b) by showing that $P(T \cap B) = P(T)P(B)$.
- Show that $P(B|T) = P(B)$.
- Is it true that $P(B|T) = P(T|B)$?

- 12** A student is given the following experiment to perform:
A coin is flipped. If the result is heads, a three-sided die is tossed; if the result is tails a six-sided die is tossed. The result of the coin and die is recorded.
 The experiment is displayed in the tree diagram to the right. We have not used a weighted tree diagram, because the sample space is shown most clearly when each branch has an equal outcome. The student did not have a three-sided die, so instead he scratched out the numbers 4–6 on a regular six-sided die and replaced them with a second set of 1–3.



Complete these questions by identifying the reduced sample space in each situation.

- a** Find $P(\text{die shows } 1 \mid \text{coin shows H})$
b Find $P(\text{coin shows H} \mid \text{die shows } 1)$
- 13** A bag contains 3 blue and 2 green balls. Two selections are made without replacement and the colour of each ball drawn is noted.
a Draw a weighted tree diagram for the experiment.
b Find the conditional probability $P(\text{both are green} \mid \text{the second ball is green})$.
c Find $P(\text{at least one ball is green} \mid \text{at least one ball is blue})$.
- 14** Two dice are rolled. A three appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than seven.
- 15** A cricket team knows that in half of the games they played last season, they won the game and their star player Arnav was playing. Arnav consistently plays in 80% of the games. What is the probability that they will win this Saturday if Arnav is playing?
- 16** **a** A couple has two children, the older of which is a boy. What is the probability that they have two boys?
b A couple has two children and at least one is a boy. What is the probability that they have two boys?
- 17** A couple has three children, each being either a boy or a girl.
a List the sample space.
b Given that at least one is a boy, what is the probability that the oldest is male?
c Given that at least one of the first two children is a boy, what is the probability that the oldest is male?
- 18** A card is drawn from a standard pack. The dealer tells the players that it is a court card (jack, queen or king).
a What is the probability that it is a jack?
b What is the probability that it is either a jack or a red card?
c What is the probability that the next card drawn is a jack? Assume that the first card was not replaced.

- 19** Two dice are tossed in turn and the outcomes recorded. Let A be the probability that the first die is odd. Let S be the probability that the sum is odd. Let M be the probability that the product is odd.
- Use the definition of independence in Box 20 to find which of the three pairs of events are independent.
 - Confirm your conclusions using product-rule test for independence in Box 21.
- 20** The two events A and B in the following experiments are known to be independent.
- $P(A) = 0.4$ and $P(B) = 0.6$. Find $P(A \cup B)$.
 - The probability of event A occurring is 0.6 and the probability of event B occurring is 0.3. What is the probability that either A or B occurs?
- 21** A set of four cards contains two jacks, a queen and a king. Bob selects one card and then, without replacing it, selects another. Find the probability that:
- both Bob's cards are jacks,
 - at least one of Bob's cards is a jack,
 - given that one of Bob's cards is a jack, the other one is also a jack.
- 22** A small committee of two is formed from a group of 4 boys and 8 girls. If at least one of the members is a girl, what is the probability that both are girls?
- 23** Jack and Ben have been tracking their success rate of converting goals in their rugby games. Jack converts 70% of his goals and Ben converts 60%. At a recent home game, both get a kick, but only one converts his goal. What is the probability that it was Ben?
- 24** Susan picks two notes at random from four \$5, three \$10 and two \$20 notes. Given that at least one of the notes was \$10, what is the probability that Susan has picked up a total of \$20 or more?

CHALLENGE

- 25** Researchers are investigating a potential new test for a disease, because although the usual test is totally reliable, it is painful and expensive.
- The new test is intended to show a positive result when the disease is present. Unfortunately the test may show a *false positive*, meaning that the test result is positive even though the disease is not present. The test may also show a *false negative*, meaning that the test result is negative even though the disease is in fact present.
- The researchers tested a large sample of people who had symptoms that vaguely suggested that it was worth testing for the disease. They used the new test, and checked afterwards for the disease with the old totally reliable test, and came up with the following results:
- Of this sample group, 1% had the disease.
 - Of those who had the disease, 80% tested positive and 20% tested negative.
 - Of those who did not have the disease, 5% tested positive and 95% tested negative.
- What percentage of this group would test positive to this new test?
 - Use the conditional probability formula $P(A \cap B) = P(A|B)P(B)$ to find the probability that a person tests positive, but does not have the disease.
 - Find the probability that the new test gives a false positive. That is, find the probability the patient does not have the disease, given that the patient has tested positive.
 - Find the probability that the new test gives a false negative. That is, find the probability the patient has the disease, given that the patient has tested negative.
 - Comment on the usefulness of the new test.

- 26 a** Prove that for events A and B ,

$$P(A|B) = \frac{P(B|A)}{P(B)} \times P(A).$$

- b** Use this formula to produce an alternative solution to the probability of a false positive or negative in the previous question.
- 27** Prove the *symmetry of independence*, that is, if B is independent of A then A is independent of B .
- 28** [A notoriously confusing question] In a television game show, the host shows the contestant three doors, only one of which conceals the prize, and the game proceeds as follows. First, the contestant chooses a door. Secondly, the host opens one of the other two doors, showing the contestant that it is not the prize door. Thirdly, the host invites the contestant to change her choice, if she wishes. Analyse the game, and advise the contestant what to do.
- 29** A family has two children. Given that at least one of the children is a girl who was born on a Monday, what is the probability that both children are girls?

Chapter 12 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 12 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- If a die is rolled, find the probability that the uppermost face is:

a a two,	b an odd number,
c a number less than two,	d a prime number.
- A number is selected at random from the integers 1, 2, ..., 10. Find the probability of choosing:

a the number three,	b an even number,	c a square number,
d a negative number,	e a number less than 20,	f a multiple of three.
- From a regular pack of 52 cards, one card is drawn at random. Find the probability that the chosen card is:

a black,	b red,	c a queen,
d the ace of spades,	e a club or a diamond,	f not a seven.
- A student has a 63% chance of passing his driving test. What is the chance that he does not pass?
- A fair coin is tossed twice. Find the probability that the two tosses result in:

a two tails,	b a head followed by a tail,	c a head and a tail.
- Two dice are thrown. Find the probability of:

a a double three,	b a total score of five,
c a total greater than nine,	d at least one five,
e neither a four nor a five appearing,	f a two and a number greater than four,
g the same number on both dice,	h a two on at least one die.
- In a group of 60 tourists, 31 visited Queenstown, 33 visited Strahan and 14 visited both places. Draw a Venn diagram and find the probability that a tourist:

a visited Queenstown only,	b visited Strahan only,	c did not visit either place.
- A die is thrown. Let A be the event that an odd number appears. Let B be the event that a number less than five appears.

a Are A and B mutually exclusive?			
b Find:			
i $P(A)$	ii $P(B)$	iii $P(A \text{ and } B)$	iv $P(A \text{ or } B)$

- 9 The events A , B and C are independent, with $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$ and $P(C) = \frac{3}{5}$. Use the product rule to find:
- a** $P(AB)$ **b** $P(BC)$ **c** $P(AC)$ **d** $P(ABC)$
- 10 **a** From a standard pack of 52 cards, two cards are drawn at random without replacement. Find the probability of drawing:
- i** a club then a diamond, **ii** two hearts,
iii a seven then an ace, **iv** the queen of hearts then the eight of diamonds.
- b** Repeat part (a) if the first card is replaced before the second card is drawn.
- 11 There is a 70% chance that Harold will be chosen for the boys' debating team, and an 80% chance that Grace will be chosen for the girls' team. Draw a probability tree diagram and find the chance that:
- a** Harold is chosen, but Grace is not, **b** Grace is chosen, but Harold is not,
c only one of Harold and Grace is chosen, **d** neither Harold nor Grace is chosen.
- 12 In a park there are four Labradors, six German shepherds and five beagles. If two dogs are selected at random, find the probability that:
- a** both are beagles, **b** neither is a Labrador,
c at least one is a Labrador, **d** a beagle and German shepherd are chosen.
- 13 There are 500 tickets sold in a raffle. The winning ticket is drawn and then the ticket for second prize is drawn, without replacing the winning ticket. If you buy 20 tickets, find the probability that you win:
- a** both prizes, **b** at least one prize.
- 14 A restaurant wishes to streamline the rate at which it gets food out to its customers at their tables. It divides the evening service into two sittings — the first sitting (arrive 6:00 pm–leave 8:00 pm) and the second sitting (after 8:00 pm). Data are gathered over the course of one night on whether a customer orders an entrée or not.
- | | first sitting | second sitting | Total |
|--------------|---------------|----------------|-------|
| order entrée | 45 | 42 | |
| no entrée | 38 | 28 | |
| Total | | | |
- a** Copy and complete the table.
b How many people attended the restaurant that night?
c What is the probability that a customer ordered an entrée?
d What percentage of the customers attended the first sitting?
e The manager claims that someone attending the first session is more likely to order an entrée than someone attending the second sitting. Is this correct? Explain.
f It is discovered at the end of the evening that one customer absconded without paying her bill. Given that she ordered an entrée, what is the probability that she was in the first session?
g The next day the restaurant is expecting an exceptionally busy evening for the second sitting, with 90 customers placing a booking. Estimate how many entrées will be ordered.
h Can you suggest why there might be a difference in the ordering habits of those attending the restaurant?
i Why might this survey contain useful information for a restaurant?

- 15** In a certain experiment, the probability of events A and B are $P(A) = 0.6$ and $P(B) = 0.3$ respectively. In which of these cases are the events independent:
- a** if $P(A \cap B) = 0.18$ **b** if $P(A|B) = 0.3$ **c** if $P(A \cup B) = 0.72$
- 16** Two dice are rolled. A five appears on at least one of the dice. Find the probability that the sum of the uppermost faces is greater than nine.

13

Data and probability

Chapter introduction

Data bombard us constantly from every direction — prices of cars, daily temperatures and rainfall, birth weights of babies, marks in different subjects, how much milk people have for breakfast — and we struggle to make sense of it all.

The subject of *statistics* begins by making more precise the idea of an experiment and its associated random variable, and the dataset that is obtained when the experiment is repeated many times.

The next stage is *descriptive statistics*, which develops various systematic approaches to making sense of the data obtained. Tables, graphs, and summary statistics are used so that we can see the big picture.

Very quickly, however, statistics is combined with *probability theory*, inviting the language of *prediction* to be used. This may perhaps lead to a discussion of *causation*, which is so fundamental in science.

The previous chapter reviewed probability, but with clearer ideas and some additions. This chapter is about data, and again, it will clarify earlier ideas and make some advances. But the significant progress in this chapter is to use data to estimate probability far more carefully than the last chapter's brief comments about 'experimental probability'. This is in preparation for next year's treatment of discrete and continuous probability distributions, and particularly of the normal distribution, which is so important in any application of statistics.

13A Random variables and frequency tables

Learning intentions

- Clarify the language of experiment, dataset, and random variable.
- Use relative frequency as an estimate of the probabilities of the various values.
- Display data by plots, polygons, and histograms.
- Distinguish discrete, categorical, and continuous random variables.

This first section is mostly review, but the language is more precise than in earlier years. In particular, the concept of experiment is tightened, the resulting random variable is introduced, and distinctions are made amongst its various types.

The central idea is the organisation of data with the purpose of seeing the situation as a whole, and then the use of that data to estimate the probabilities of the random variable taking its various values when the experiment is performed.

A random experiment and its random variable

Statistical data are generated by random experiments, which are experiments with more than one possible outcome. As explained in the last chapter:

- Throwing four coins, and recording the number of heads, is a *random experiment* because there are five possible outcomes: 0, 1, 2, 3, and 4 heads.

Throwing four coins into an empty bucket, and counting the number of coins in the bucket, is a *deterministic experiment* because 4 is the only possible outcome — but we are not concerned with deterministic experiments here.

Denote by X the number of heads when the four coins are thrown. This variable X is a *random variable* because it is the result of a random experiment.

There are five possible outcomes 0, 1, 2, 3, and 4 of this experiment, and these five outcomes are called the *values* of the random variable X .

Frequency and relative frequency

The authors performed this experiment 1000 times, and recorded the *score* each time (actually, they set up a simulation on an Excel spreadsheet). The data that they collected are called collectively a *dataset*.

Here is their dataset, organised into a table, showing the five *values* (how many heads), the *frequency of each value*, and the *relative frequency of that value*:

Number of heads	0	1	2	3	4	Total
Frequency	66	226	395	260	53	1000
Relative frequency	0.066	0.226	0.395	0.260	0.053	1

The relative frequencies are simply the ratios of the frequencies over the total frequency. This table is the most straightforward way to organise the dataset so that its patterns can be seen and understood easily.

Relative frequency as an estimate of probability

We mentioned *experimental* (or empirical) probability briefly in the last chapter. It should now be clear that the five relative frequencies are *estimates* of the probabilities of throwing 0, 1, 2, 3, and 4 heads. For example, we can estimate:

$$P(3 \text{ heads}) = 0.260 \quad \text{and} \quad P(4 \text{ heads}) = 0.053.$$

Using notation based on the random variable X , we can extend this to:

$$\begin{aligned} P(X \geq 3) &= P(X = 3) + P(X = 4) & \text{and} & \quad P(X \leq 2) = 1 - P(X \geq 3) \\ &= 0.260 + 0.053 & & \quad = 1 - 0.313 \\ &= 0.313, & & \quad = 0.687. \end{aligned}$$

In this particular experiment, theoretical probabilities were calculated in Chapter 12, Example 18. Here is the comparison of the results:

Number of heads	0	1	2	3	4
Relative frequency	0.066	0.226	0.395	0.260	0.053
Theoretical probability	0.0625	0.25	0.375	0.25	0.0625

How close did you expect the estimated probabilities to be to the theoretical probabilities? That is a most important question, but it is very difficult, and beyond our course.

In most examples here, and in most applications of statistics, we cannot calculate theoretical probabilities, sometimes because the calculations are too difficult, but mostly because there is no theoretical model for the probabilities — the relative frequencies are the only way to find estimates of much-needed probabilities.

1 Organising data from a random experiment

Random experiment and random variable:

- A *random experiment* is an experiment with more than one possible outcome.
- A *random variable*, typically denoted by an upper-case letter such as X , is the outcome when a random experiment is run.
- The possible outcomes, typically denoted by x , are called the *values* of the random variable.

Scores and frequency:

- When an experiment is performed many times:
 - ▷ the outcome of each performance of the experiment is called a *score*, and
 - ▷ the collection of all the scores obtained is called a *dataset*.
- The dataset can conveniently be organised into a *frequency table* and a *relative frequency table*.
- The *frequency* of a value is the number of times it occurs.
- The *relative frequency* is the ratio of frequency over total number of scores.

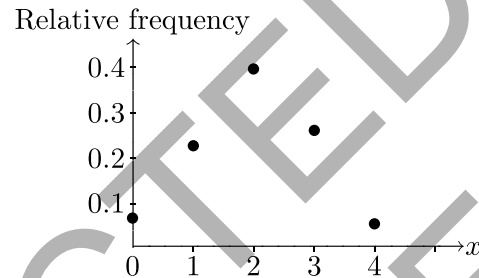
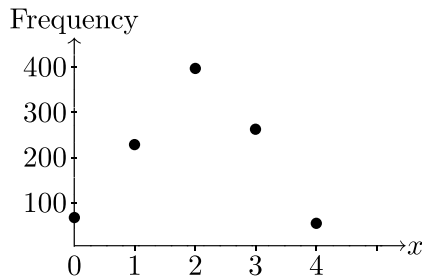
Relative frequency is an estimate of probability:

- The relative frequency of an outcome in a reasonably large dataset may be regarded as an estimate of the probability of the outcome.

Graphing the dataset using plots and polygons

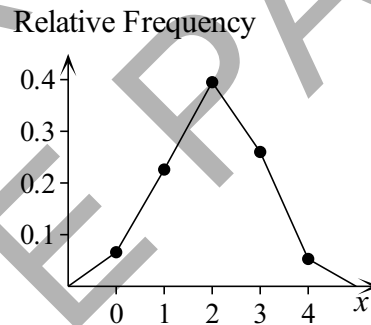
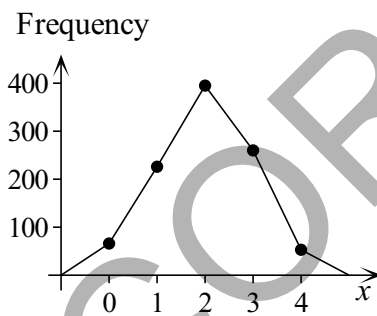
As always in this course, graphs make things clearer. Ever more inventive ways of displaying data are constantly being developed, but most displays are based somehow on standard plots and histograms, and they will be our main concern.

Here are plot graphs of the frequency table and relative frequency table of the coin-tossing above. Notice that we typically use x on the horizontal axis to denote the various values of the random variable X .



The graphs are identical apart from the vertical scale. The first emphasises the number of experiments performed, and the frequencies. The second emphasises the proportions, and thus provides estimates of the probabilities.

The plotted points are often joined up by line segments to form a graph known as a *polygon*. It is customary to start the polygon on the x -axis on both left and right — here this requires notionally adding -1 and 5 to the x -axis.

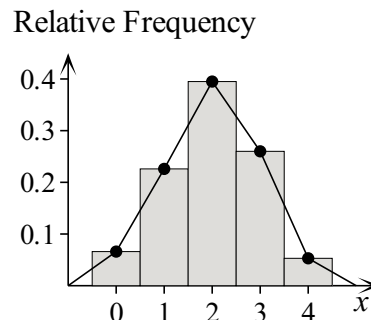
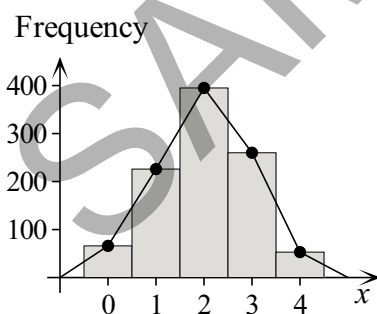


Graphing the dataset using histograms

Histograms have the same axes as the polygons, but use vertical columns.

- Each column is centred on the value.
- Each column is drawn touching its neighbours.

Here are frequency and relative frequency graphed in histograms:



- The polygon can be superposed on the histogram, as in the left-hand graph.
- Because each column is centred on the value, the line segments of the polygon join the midpoints of the top of each column.

- The polygon continues to the x -axis on both sides. It may be helpful to move the vertical axis left or right of zero on the x -axis. The origin has little meaning here.

By all means use spreadsheets and other graphing tools to draw these diagrams. Just make sure that you can also do it by hand — and be ready for different conventions.

The mode of a dataset

Any one of the diagrams above, or either of the tables, will identify the mode of the dataset, which is the score with the highest frequency — ‘mode’ means ‘fashion’. In the example above, the mode is 2, which has the highest frequency 395.

Often two or three or more values have the same maximum frequency. Such a dataset is called *bimodal* or *trimodal* or

2 Polygons and histograms

- The frequency or relative frequency of each score can be plotted as a point.
 - ▷ Joining the points by line segments creates a *polygon graph*.
 - ▷ It is customary to join the polygon to the x -axis on the left and right.
- Drawing columns centred on each value, and touching its neighbours, creates *histograms*.
 - ▷ The polygons can also be drawn superposed on the histograms.
 - ▷ The polygons’ line segments join the midpoints of the columns’ top edges.
- The *mode* is the score of highest frequency (use the table, or any diagram).
 - ▷ A dataset may possibly be *bimodal* or *trimodal* or

Discrete and categorical random variables

The tossing-four-coins experiment is an example of a *discrete random variable*, because its possible values are *numeric* (they are numbers), and they can be *listed* as 0, 1, 2, 3, 4 — this concept of ‘listing’ will be made more precise in Year 12.

But the values of a random variable don’t have to be numbers. They may be names, or objects, or any verbal description. The general word for this is ‘category’, and such a variable is called a *categorical random variable*.

We can still graph the dataset with a *column graph* that is similar to a histogram. It is customary to separate the columns of a column graph, because having the columns touching could be misleading.

Drawing polygons, however, does not make sense, unless the values are in some canonical order, such as days of the week



Example 1 Organising categorical data

A survey of 160 school students asked what was their usual mode of transport to school:

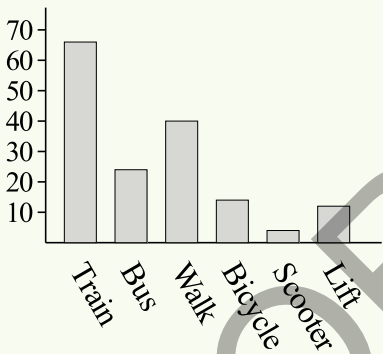
Transport	Train	Bus	Walk	Bicycle	Scooter	Lift	Total
Frequency	66	24	40	14	4	12	160

- a** Organise this into a table of relative frequencies (three decimal places).
- b** Draw a column graph of the frequencies, and identify the mode.
- c** Estimate the probability that a randomly chosen school student mostly:
 - i** catches a train or bus,
 - ii** has some mechanical assistance.
- d** Estimate what percentage of students come to school on four wheels.

Solution

Transport	Train	Bus	Walk	Bicycle	Scooter	Lift
Rel Freq	0.413	0.15	0.25	0.088	0.025	0.075

b Frequency



The mode is 'Train':
frequency 66, and relative frequency 0.413.

- c** Using the relative frequencies as estimates of the probabilities:
 - i** $P(\text{Train or bus}) \div 0.413 + 0.15$
 $= 0.563$
 - ii** $P(\text{Assistance}) = 1 - P(\text{Walk})$
 $= 1 - 0.25$
 $= 0.75$
- d** Probably only the 7.5% estimate from part (a) of the proportion who get a lift, but the data are unclear about the type of vehicle in which they get a lift, and about the type of scooter, and about the type of bus.

Continuous random variables

Our random variables so far have been discrete or categorical. It is far more common, however, that a sampling experiment involves a *continuous random variable*, such as the speed of a car, or the height of a person.

We cannot be precise here until Year 12, but for the purposes of this chapter, it is sufficient to say that the possible outcomes (values) of a continuous random variable are typically all the real numbers in an interval on the number line.

Outcomes are rarely repeated exactly in experiments with continuous data, and when that does happen, it is because our measurement of such a variable is only approximate — height or speed can only be measured to a few decimal places.

In order to deal with continuous random variables, we will need the idea of *grouping* of results, introduced in Section 13C, so that we can deal with:

speeds between 80 and 100 km/hour and heights ≤ 150 cm.

3 Three types of random variables, and their graphs

- A random variable is called *discrete* if its values are numeric — meaning that they are numbers — and they can be listed (more on ‘listing’ in Year 12).
 - ▷ A random variable whose values are names, or objects, or any verbal description, is called *categorical*.
 - ▷ The possible values of a *continuous random variables* are typically all real numbers in an interval on the number line (more precision in Year 12).
- A categorical dataset is graphed with frequency and relative frequency column graphs. These are not histograms, and it is customary to separate the columns.
 - ▷ The other datasets in the chapter normally use histograms and polygons. They all have the same procedures, detailed in Box 5 of the next section.

Exercise 13A

FOUNDATION

- 1 State whether each random variable is numeric or categorical. If it is numeric, state whether it is discrete or continuous. Comments may be appropriate.
 - a The favourite day of the week for a person chosen at random in Australia
 - b Height of Australian professional basketball players
 - c Age
 - d Political affiliation
 - e Colour of a counter drawn from a cup containing 5 red and 6 blue counters.
 - f Sex (male or female) of a child attending a particular primary school
 - g Sum of the numbers when two dice are thrown
 - h Shoe size
 - i Examination scores
- 2 Each experiment below returns a numerical result. Define a random variable X for each experiment and write down domain of possible values that X can take. This is also called the *sample space* for the experiment.
 - a A pupil is investigating the length of words chosen at random from the sentence *The quick brown fox jumped over the lazy dog*.
 - b Four die are tossed to investigate the number of sixes that are turned up.
 - c A reseacher is carrying out a census of a street in a city to determine household size.
 - d A digit is chosen at random from the 12-digit number 1.41421356237.

- 3** Cards in a set each have one of the numbers 1–5 written on them at random. The cards are shuffled. A card is selected and the number is recorded. This experiment is repeated; at each stage the card is returned to the pack and the cards are reshuffled. The frequency of each outcome is shown in the table below. Let the random variable X be the number written on a card selected at random.

x	0	1	2	3	4	5
f	10	20	30	25	5	10

- a** Record the relative frequencies in a new row in the table.
- b** Estimate:
- i** $P(X = 1)$
 - ii** $P(2 \leq X \leq 4)$
 - iii** $P(1 \leq X < 4)$
 - iv** $P(X = 6)$
 - v** $P(X \leq 2)$
 - vi** $P(X < 4)$
 - vii** $P(X \geq 1)$
 - viii** $P(X > 1)$
 - ix** $P(X \text{ is even})$
- 4** In a seaside town, researchers have asked random people how many days they have visited the local beach in the last week. Their responses are aggregated in the table below.

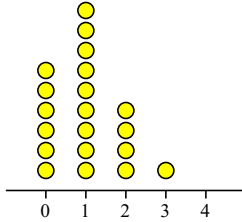
x	0	1	2	3	4	5	6	7
f	3	6	4	2	1	1	3	5
rf	0.12	0.24	0.16	0.08	0.04	0.04	0.12	0.20

- a** Express each of these probabilities in the form $P(X = x)$ or $P(X \leq x)$. The probability that a random person from the town visits the beach:
- i** four times a week;
 - ii** less than or equal to four times a week;
 - iii** no more than twice a week;
 - iv** less than four times a week.
 - v** more than twice a week.
 - vi** between 3 and 4 times a week (inclusive)
- b** Estimate each probability from part (a) using the relative frequencies from the table.
- 5** A die is tossed 50 times and the results recorded in the following table.

x	1	2	3	4	5	6
f	7	5	6	7	8	17
rf						

- a** Complete the table by filling in the row of relative frequencies.
- b** Is the die fair?
- c** What is the most likely outcome? What name is given to this outcome?
- d** Estimate the probability of throwing a 4.
- e** Estimate the probability of throwing a number bigger than 3.
- f** Estimate the number of sixes obtained if the dice is thrown 800 times.
- g** Find the probability that the sum of two dice rolls is 11.

- 6 The number of siblings for each member of a class is recorded in the dot plot below.



- Fill in a table showing the frequency and relative frequency as a decimal.
- What is the modal number of siblings?
- What is the percentage probability that a student chosen at random has no more than 2 siblings?
- What is the percentage probability that a family chosen at random from the families represented in the class has only a single child?

DEVELOPMENT

- 7 A teacher sets a simple pre-unit quiz before starting a new topic. The quiz is out of 10. The results are in the following table.

x	1	2	3	4	5	6	7	8	9
f	1	0	0	2	4	5	3	1	2

- Draw a simple histogram showing this data. Your histogram bars should centre around the scores 1, 2, 3, ..., 9 and there should be no gaps between the bars. Your graph should include a title and a label on each axis.
 - Join the centres of neighbouring bars to form a frequency polygon. By convention, the polygon is joined back to the horizontal axis at 0 and 10.
 - Comment on the shape of the distribution of the data.
 - If you have access to a spreadsheet program, you should be able to draw this graph as a column (bar) graph with the gaps between the bars set to 0. By adding a second graph to the same axes, you can superimpose the frequency polygon.
- 8 Clive is a frequent visitor to the zoo, and he is always interested to see how many lions are out of their dens and visible to the public. He records the number of lions he sees on each visit. The relative frequency of his results are shown in the following table.

x	1	2	3	4	5	6
rf	0.4	0	0.16	0	0.12	0.32

- Draw a simple relative frequency histogram showing this data. A vertical unit of 0.1 unit as 1 cm is recommended. Your histogram bars should centre around the scores 1, 2, 3, ..., 6 and there should be no gaps between the bars. Your graph should include a title and a label on each axis.
- Join the centres of neighbouring bars to form a relative frequency polygon. By convention, the polygon is joined back to the horizontal axis at 0 and 7.
- Can you suggest a story to explain any patterns you see in the data? It may need further investigation to check your explanation.
- If you have access to a spreadsheet program, you should be able to draw this graph as a column (bar) graph with the gaps between the bars set to 0. By adding a second graph to the same axes, you can superimpose the frequency polygon.

- 9** A school is investigating the number of times students buy lunch at the canteen in a week. They sample a group of students at random. Their experimental data is in the table below.

x	0	1	2	3	4	5
f	29	13	3	0	1	4

- Discuss what the school needs to consider when designing this experiment.
 - How many students are sampled?
 - What do you think X represents?
 - Add a relative frequency row.
 - What is the probability that a student chosen at random buys their lunch sometime during the week?
 - What is the probability that a students buys their lunch every school day?
 - The school population has 1200 students. Estimate how many lunches the canteen sells during the week.
- 10** Amy scoops up coins at random from a pile containing one 10 c coin and two 5 c coins.
- Name the 10 c coin T and the two 5 c coins F1 and F2, and write down as sets the seven possible non-empty scoops.
 - Let the random variable X be the value of the money that she picks up. What values can X take?
 - Amy repeats the experiment 20 times, replacing the coins after each selection. Complete the following table of her results, including the relative frequency.

x	5	10	15	20
f	4	8	6	2
rf				

- Assuming that she is equally likely to pick up any of these seven non-empty scoops, draw up a probability tree diagram for X and calculate the theoretical probability of each outcome, adding your calculations to the table in a row marked p .
 - Compare the theoretical probabilities with Amy's experimental outcomes.
- 11** An experiment with two possible outcomes is called a *Bernoulli* experiment. The outcome might be *Yes*, *No*; *True*, *False*; *In*, *Out*; *Success*, *Fail*; *Correct*, *Incorrect*, to give a few examples. If we take these two outcomes as 0 or 1, then a Bernoulli experiment generates a discrete random variable. A dart player records 1 if he hits the bulls eye, and 0 otherwise. His results are recorded in the following table.

x	0	1
f	40	7

- Add a relative frequency column.
- Explain what the frequency and relative frequency mean in this case.
- How many bull's eyes would he expect in 20 throws?

- 12** A teacher tells their class that the probability a pair of numbers chosen at random have no factors in common is $\frac{6}{\pi^2}$. A member of the computing and informatics club uses the GCD (*Greatest Common Divisor*) function in Excel to check if randomly selected pairs of numbers less than 100 have factors in common. Their results are displayed below.

x	0	1
f	31	19

- Discuss the design of their experiment.
 - Estimate the probability that a pair of numbers will have no factors in common. Does the experiment agree with the teacher's claimed theoretical probability?
 - Write your own spreadsheet to test this.
- 13** Most scientific calculators have a button to generate a random number. Your calculator may accept a function like RandInt#(1,6) or RandBetween(1,6), generating a number between 1 and 6 inclusive. Your calculator may have a function like Ran# generating a decimal between 0 and 1. Typing Ran# $\times$ 6+1 will generate a random dice roll; just discard any decimal portion remaining.
- Use your calculator to simulate rolling a dice 50 times. Record your results in a table with a tally column.
 - Calculate the frequency and relative frequency of each outcome.
 - Is the dice simulated by the calculator weighted towards any particular result?
 - If each outcome is equally likely, we call the distribution of probabilities *uniform*. Is the distribution you obtained uniform?
 - This simulation exercise can also be carried out in a spreadsheet.

CHALLENGE

- 14** **a** Simulate the sum of the outcomes when a pair of dice is rolled on your calculator. As you roll the dice, record your results in a dot plot. Roll a pair of die 50 times.
- b** Compare the results obtained by simulation in a spreadsheet. You can easily toss the pair of dice 500 times, for a more accurate simulation.
- 15** Take f as 1 if x is prime, otherwise its value is 0. Thus, f records the frequency of primes for x . The cumulative function $\pi(x)$ returns the number of primes less than or equal to x .
- a** Fill in the values in the table below.

x	1	2	3	4	5	6	7	8	9	10
f	0	1	1	0						
$\pi(x)$	0	1								

- b** A computer is used to calculate $\pi(n)$ for small values of x .

x	10	20	30	40	50	60	70	80	90	100
$\pi(x)$	4	8	10	12	15	17	19	22	24	25

Draw a polygon for $\pi(x)$.

- How many primes are there between 40 and 80? Check this result directly by writing down the primes.
- It is possible to write a spreadsheet in Excel to test for primes and to implement the results above. Vanilla Excel does not have a prime testing function, and you will likely have to use Excel's array feature to test the divisibility of x by all numbers less than $\sqrt{x}$. Use an internet search to investigate how to do this.

13B Cumulative frequency

Learning intentions

- Use cumulative frequency and cumulative relative frequency.
- Draw corresponding histograms and polygons.

Cumulative frequency is used when we want to know, for example, how many people eat 7 or fewer eggs per week, or have fewer than 10 installed apps on their phone, or were a hospital patient for more than 5 nights last year.

The median

The median is the middle score when the scores are arranged in ascending order. It divides the dataset into two equal halves. But some qualifications are needed.

4 The median

The median is the middle score, when the scores are arranged in ascending order.

- For an odd number of scores, take the middle score.
For example, with $43 = 21 + 1 + 21$ scores, the median is the 22nd score.
- For an even number of scores, take the mean of the two middle scores.
With $26 = 13 + 13$ scores, take the mean of the 13th and 14th scores.
- The median can be found from the cumulative frequency table.



Example 2 Finding the median with odd and even numbers of scores

Find the median of these datasets (which are written out in ascending order):

a 4 7 8 10 10 11 13 15 21 (9 = 4 + 1 + 4 scores, so take the 5th score)

b 4 7 8 10 10 11 11 13 13 21 (10 = 5 + 5 scores, take $\frac{1}{2}(5\text{th} + 6\text{th scores})$)

Solution

a The median of the 9 scores is the middle score, which is 10.

b The median of the 10 scores is the mean of the two middle scores, which are 10 and 11. Hence the median is $10\frac{1}{2}$.

Cumulative frequency tables

The *cumulative frequency* of some value x is the number of scores less than or equal to that value, that is, with $X \leq x$. A *cumulative frequency table* can be constructed quickly by repeated addition from an ordinary frequency table.

Similarly, a *cumulative relative frequency table* can be constructed from an ordinary relative frequency table, although it may be easier to do the divisions again rather than add up decimals. The more tedious the calculations, the more attractive spreadsheets become, provided that you have learnt first to do things by hand.

The histograms of both cumulative tables are drawn exactly as before, and because the cumulative frequencies increase from left to right, the heights will also increase from left to right. The polygons join the top right-hand corners of the columns, and are connected back down to the x -axis only on the left.

The terminology is becoming increasingly verbose. The authors will not object if the following abbreviations are made, *but only when working in this chapter*:

- f

frequency
- rf

relative frequency
- cf

cumulative frequency
- crf

cumulative relative frequency

Using tallies when organising the raw data

In the following example, tallies have been used to establish the frequencies from the raw data. Trying to count each score separately inevitably leads to errors. (Alternatively, the counting can be done on a spreadsheet.)



Example 3 Constructing cumulative frequency tables

At the start of Year 7, Cedar Heights High School selected 40 students at random from its new enrolment of the over 300 students, and gave them a spelling test marked out of 10. Here are the raw results:

4, 7, 2, 8, 7, 6, 3, 2, 8, 2, 9, 5, 8, 5, 8, 3, 6, 7, 5, 2,
10, 6, 7, 5, 6, 6, 9, 1, 5, 7, 8, 1, 6, 5, 7, 10, 6, 7, 8, 6.

- a Use a tally row to construct a frequency table for these results.
- b Add rows for the relative frequency, the cumulative frequency, and the relative cumulative frequency (rounded correct to at most three decimal places).
- c Find, with explanations, the median of the dataset.
- d If the test were given to a randomly chosen Year 7 student, find estimates of:

i $P(X \leq 3)$

ii $P(7 \leq X \leq 9)$

iii $P(X < \text{median})$
- e What conclusion could the school draw from this dataset?

Solution

a	Mark	0	1	2	3	4	5	6	7	8	9	10
	Tally											
	f	0	2	4	2	1	6	8	7	6	2	2
b	rf	0	0.05	0.1	0.05	0.025	0.15	0.2	0.175	0.15	0.05	0.05
	cf	0	2	6	8	9	15	23	30	36	38	40
	crf	0	0.05	0.15	0.2	0.225	0.375	0.575	0.75	0.9	0.95	1

- c The median is the mean of the 20th and 21st scores. From the cumulative frequency row, these scores are both 6, so the median is $\frac{1}{2}(6 + 6) = 6$.
- d Using the cumulative relative frequencies:

i $P(X \leq 3) = 0.2$

ii $P(7 \leq X \leq 9)$

$= P(X \leq 9) - P(X \leq 6)$

$= 0.95 - 0.575$

$= 0.375$

iii $P(X < \text{median})$

$= P(X \leq 5)$

$= 0.375$
- e The cumulative frequencies suggest that 8–9 students have poor spelling, or perhaps they had little experience in earlier years doing tests.

Cumulative frequency and relative cumulative frequency histograms and polygons

- The two cumulative histograms are constructed as for the non-cumulative histograms, with each column centred as before on the value.
- But the line segments of the polygon join the top right corners of the columns.
- The cumulative polygons extend to the x -axis only on the left — they meet the x -axis at the bottom left corner of the first column.

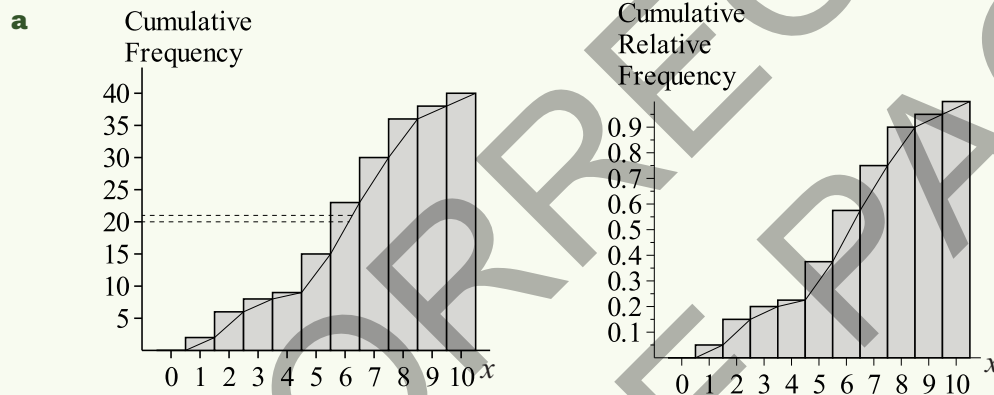


Example 4 Using the cumulative histograms and polygons

Continuing with the same example as in the previous example:

- Draw the cumulative frequency histogram and the cumulative relative frequency histogram, and superpose the polygons on them.
- Explain how to find the median from these graphs.

Solution



- On the cumulative frequency histogram, draw horizontal lines at 20 and 21. Both lines meet the histogram at the column corresponding to 6, so the median is $\frac{1}{2}(6 + 6) = 6$.

Summary of the procedures to draw histograms and polygons

Here is a summary of the procedures:

5 Procedures for drawing histograms and the superposed polygons

- For all the histograms:
 - ▷ The rectangles are centred on the values, and touch each other.
- For the frequency and relative frequency polygons:
 - ▷ The polygons' line segments join the centres of the columns' top edges.
 - ▷ The polygons extend left and right to the next class centre on the x -axis.
- For the cumulative frequency and cumulative relative frequency polygons:
 - ▷ The line segments of the polygon join the top right corners of the columns.
 - ▷ The polygons extend left to the bottom left corner of the first column.

These procedures are the same for all the histograms and polygons drawn in this chapter, and will not be repeated. They apply to the discrete data of this section, and to the grouped continuous and grouped discrete data of the final section

6 Summary of cumulative data

- For a dataset with random variable X and values x :
Cumulative frequency of x = number of scores with $X \leq x$.
- The cumulative frequency of the highest score is the total number of scores.

Note: The cumulative frequency polygon is sometimes called the *ogive*.

Exercise 13B

FOUNDATION

- 1 Find the median, mode and range of each dataset.

a 10 13 14 14 15 17 18

b 5 7 8 9 10 12 13 15 17

c 3 3 4 5 7 9 10 12 13 15

d 4 4 4 6 6 6 7 7 8 8 9 10

e 4 2 6 4 7 3 4 6 3 4 3 5 2 1 5 7 8

f 2 9 7 6 4 3 2 7 8 9 10 5 4 2 3 6 9 3

- 2 Construct a table of frequencies and cumulative frequencies for the following data. Find the number of scores, the mode(s) and the median.

a 2 2 2 3 3 4 4 4 4 5 5 6 6 8 8 9

b 1 4 7 6 3 8 9 3 4 6 6 8 4 3 6 1 2 2 5

c 9 3 3 5 4 9 12 10 6 12 10 14 7 7 3 10 12 7 4 8

d 5 4 7 3 3 6 7 7 7 6 7 6 2 3 7 7 7 7 3 3

- 3 The results of an experiment to investigate the value of a random variable X are listed in the following table.

x	0	1	2	3	4	5	6
f	3	2	4	2	5	3	2

- a** Copy the table and add a row listing cumulative frequencies and cumulative relative frequencies (to 3 decimal places).

- b** Estimate

i $P(X \leq 2)$

ii $P(X \leq 5)$

iii $P(2 < X \leq 5)$

- c** Estimate the median by finding x such that $P(X \leq x) = 0.5$.

- 4 The results of an experiment to investigate the value of a random variable X are listed in the following table.

x	0	1	2	3	4	5	6
f	2	4	6	12	8	5	1

- a** Copy the table and add a row listing cumulative frequencies and relative cumulative frequencies.

- b** Estimate

i $P(X \leq 1)$

ii $P(X \leq 5)$

iii $P(1 < X \leq 5)$

- c** Estimate

i $P(X \leq 3)$

ii $P(X > 3)$

5 Copy and complete the missing rows in these tables.

a

x	0	1	2	3	4	5	6
f							
cf	1	6	9	17	21	28	30

c (An experiment with 40 scores)

x	0	1	2	3	4
cf					
rf	0.15	0.225	0.05	0.225	0.35

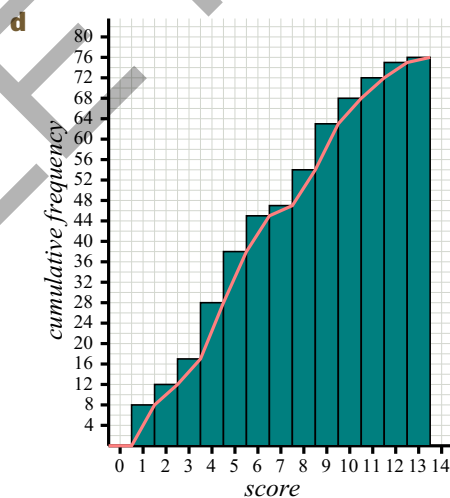
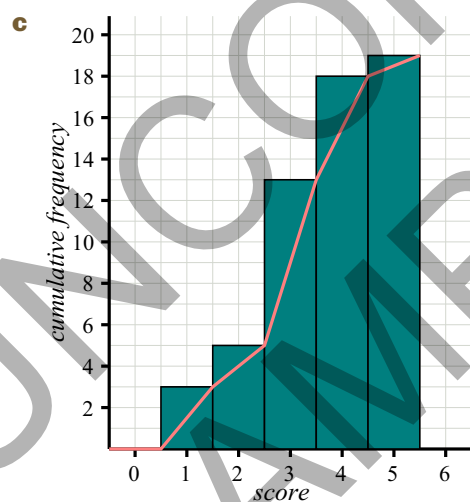
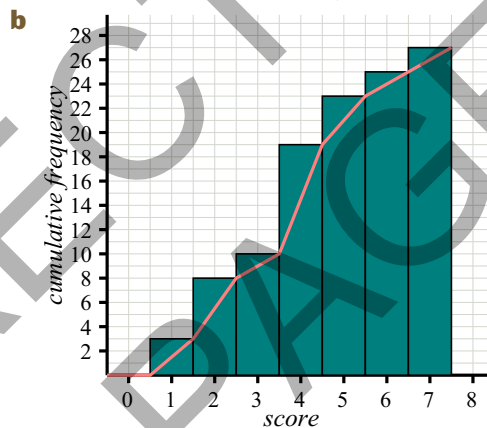
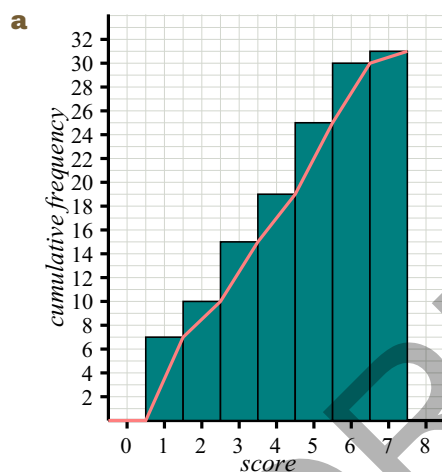
b

x	0	1	2	3	4	5	6
f							
cf	8	12	15	16	26	34	39

d

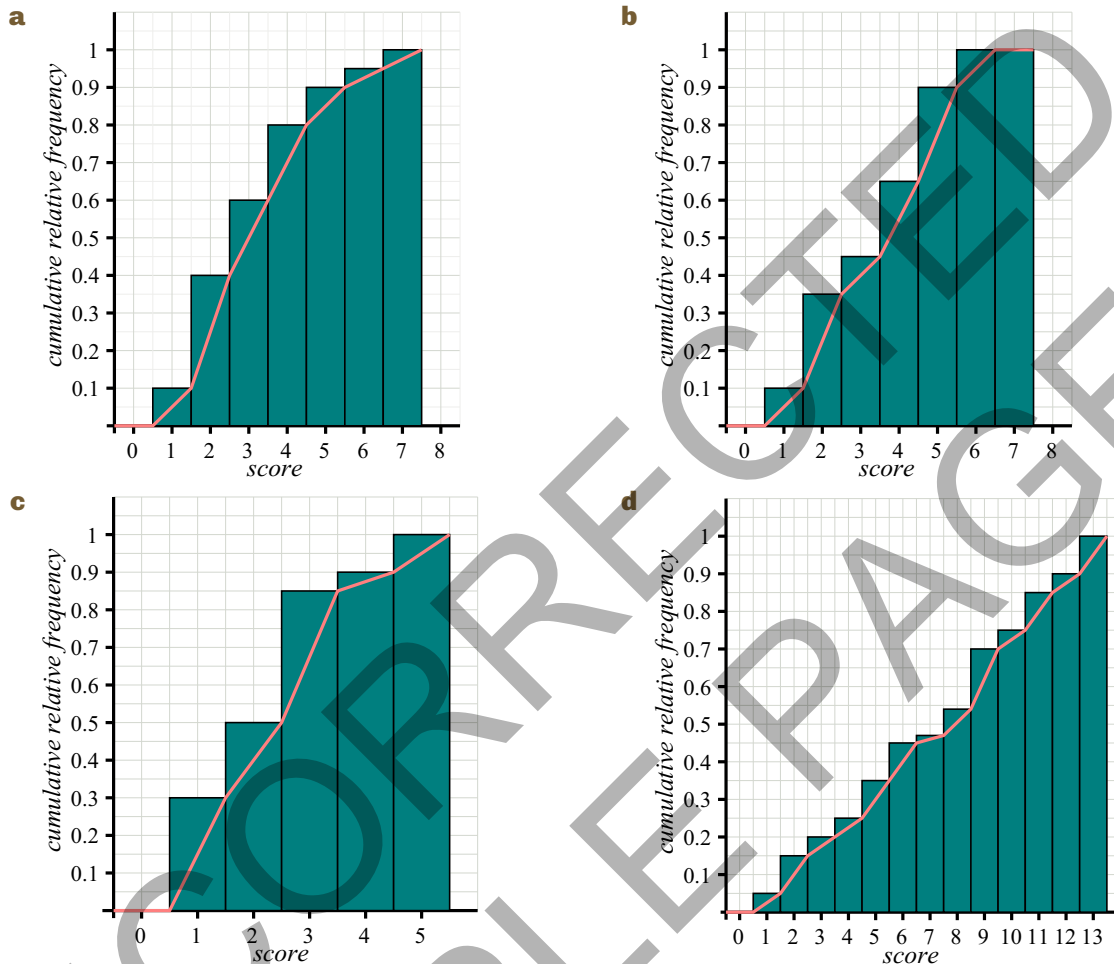
x	2	3	4	5	6
f					
cf					25
crf	0.2	0.52	0.60	0.88	1

6 Use the cumulative frequency polygon and cumulative frequency histogram to find the number of scores and the median for each experiment.



DEVELOPMENT

- 7 Use the cumulative relative frequency polygon and cumulative relative frequency histogram to estimate the median for each experiment.



- 8 The teacher mentioned in the previous exercise set a simple pre-unit quiz out of 10. The results are repeated again in the following table.

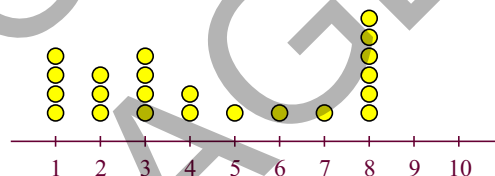
x	1	2	3	4	5	6	7	8	9
f	1	0	0	2	4	5	3	1	2

- Copy the table and add a cumulative frequency row.
- Draw a simple cumulative frequency histogram showing this data. Your histogram bars should centre around the scores 1, 2, 3, ... 9 and there should be no gaps between the bars. Your graph should include a title and a label on each axis.
- Join the **endpoints** each bar to its next neighbour to form a cumulative frequency polygon. By convention, the polygon is joined back to the horizontal axis at the start of the first bar.
- How many students got no more than 50%?
- How many students got less than 50%?
- The teacher is interested in what score 75% of the students failed to attain. What was the score?
- If you have access to a spreadsheet program, you should be able to draw this graph as a column (bar) graph with the gaps between the bars set to 0. By adding a second graph (a line graph joining the class centres) to the same axes, you can superimpose the frequency polygon.

- 9** As was mentioned in the previous exercise, Clive is a frequent visitor to the zoo. The random variable X is the number of lion sightings on each visit, and the relative frequency of his outcomes are shown in the following table.

x	1	2	3	4	5	6
rf	0.4	0	0.16	0	0.12	0.32

- Copy the table and add a cumulative relative frequency row.
 - Draw a simple cumulative relative frequency histogram showing this data. A vertical unit of 0.1 unit as 1 cm is recommended. Your histogram bars should centre around the scores 1, 2, 3, ... 9 and there should be no gaps between the bars. Your graph should include a title and a label on each axis.
 - Join the **endpoints** each bar to its next neighbour to form a cumulative relative frequency polygon. By convention, the polygon is joined back to the horizontal axis at the start of the first bar.
 - If you have access to a spreadsheet program, you should be able to draw this graph as a column (bar) graph with the gaps between the bars set to 0. By adding a second graph to the same axes, you can superimpose the frequency polygon.
- 10** A shop sells individual cupcakes and keeps a record of how many cupcakes each customer purchases. The results are shown in the dot plot to the right.



- Construct a table showing the cumulative frequency and cumulative relative frequency for the data.
 - Find the median sales of cupcakes.
 - Find the mode of the data.
 - The shop intends to pre-package cupcakes to streamline its sales for many customers. Discuss the advantage of selling the cupcakes in packages of
 - 3,
 - 4,
 - 8.
- 11** A school's results in Mathematics Advanced is found not to vary much from year to year. Last year's results are shown in the table below. The random variable X represents a student's band outcome.

x	1	2	3	4	5	6
f	0	0	2	4	10	6

- Copy the table and include a cumulative frequency and cumulative relative frequency row.
- What fraction of the students obtained a band 6 outcome?
- What is the probability that a student chosen at random from this year's cohort would get a band 6?
- Mike is sitting his HSC this year, and he is placed 10th in his cohort of 30 after his first assessment. Estimate his band based on his current results.
- How many places does Mike need to improve to achieve a band 6?
- Discuss issues with this method of estimation.

- 12** A basketball coach begins each training session by challenging his star player to shoot as many hoops as he can in two minutes. Over twenty-one sessions he records these results:

4 3 5 4 5 4 6 7 5 6 8 6 6 8 9 10 7 8 9 7 9

- a** Construct a dot plot of the data. (This can be used as an alternative to a tally.)
- b** Copy and fill in the following frequency table.

score x	3	4	5	6	7	8	9	10
frequency								
cumulative								

- c** What is the median number of hoops shot by the star player?
 - d** In his twenty-second session he shoots 11 hoops in the 2 minutes allowed. What is his new median score?
 - e** Is this frequency table a helpful way of displaying the scores?
- 13** An operator tracks the number of customers who pay the \$4 fee to take his amusement ride on each day of the week. His data is displayed in the following table.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
13	32	35	38	57	75	65

- a** Draw a column chart showing the data, with days of the week on the horizontal axis. Use a scale of 1 cm per 10 rides.
- b** Construct a table of cumulative frequencies, and draw a cumulative bar chart showing the number of rides sold up to and including that day of the week.

CHALLENGE

- 14** Sometime during the last year, a large restaurant began to hire out its 5 separate dining spaces for special group functions on Friday or Saturday night. The data collected has been used to generate the following cumulative relative frequency table. The random variable X is the number of rooms booked out on a night.

x	0	1	2	3	4	5
crf	0.15	0.275	0.4625	0.6625	0.85	1

Record keeping was poor, and the manager is not sure when they started hiring out the rooms and how many nights are represented in the data.

- a** Find the mode, that is the most common number of rooms booked out on a night.
- b** Find the number of nights that the modal number of rooms were booked.

13C Grouped data

Learning intentions

- Group data by intervals, and draw up the four types of frequency tables.
- Draw histograms and polygons for grouped data — continuous and discrete.

The main purpose of organising data into tables and graphs is to see the data as a whole. When a table has too many rows or columns, or a graph has too much detail, such an overview is much more difficult. The usual approach in such situations is to *group* the data. This reduces the number of rows or columns on the tables, and reduces the amount of clutter on the graphs.

Grouping continuous data

Here is some real data — the heights of 100 people of all ages, randomly chosen from a particular much larger community. The random variable here is continuous (assuming heights are real numbers), so two people are most unlikely to have exactly the same height, even though the data has been rounded to only a few significant figures. Hence almost all the scores should have frequency 1.

151.765	139.7	136.525	156.845	145.415	163.83	149.225	168.91	147.955	165.1
154.305	151.13	144.78	149.9	150.495	163.195	157.48	121.92	105.41	86.36
161.29	156.21	129.54	109.22	146.4	148.59	147.32	137.16	125.73	114.3
147.955	161.925	146.05	146.05	142.875	142.875	147.955	160.655	151.765	171.45
147.32	147.955	144.78	121.92	128.905	97.79	154.305	143.51	146.7	157.48
127	110.49	97.79	165.735	152.4	141.605	158.8	155.575	164.465	151.765
161.29	154.305	145.415	145.415	152.4	163.83	144.145	129.54	129.54	153.67
142.875	146.05	167.005	91.44	165.735	149.86	147.955	137.795	154.94	161.925
147.955	113.665	159.385	148.59	136.525	158.115	144.78	156.845	179.07	118.745
170.18	146.05	147.32	113.03	162.56	133.985	152.4	160.02	149.86	142.875

The data seem to be given correct to 0.005 cm, which seems less than one can reliably measure, and the trailing zeroes that we normally insert are missing — always question the credibility of raw data.

A frequency table here, with nearly all frequencies 1, would be useless. To make sense of this data, we need to *group* them, and the most obvious grouping is in 10 cm intervals because that results in 10 classes, which should show the big picture. Here is the table of frequencies (tallies omitted):

interval	80–90	90–100	100–110	110–120	120–130	130–140	140–150	150–160	160–170	170–180
centre	85	95	105	115	125	135	145	155	165	175
freq	1	3	2	5	8	6	34	22	16	3

The *class centre* on each row is the midpoint of the interval used in the grouping. This grouping makes the distribution of heights reasonably clear, and the resulting histograms will also be useful.

Grouping data is a form of rounding. It is useful because it allows us to see the big picture, but it always involves ignoring information, and the summary statistics (mode and median so far) for the grouped data may only be an approximation of the summary statistics of the raw data. Never discard the original data.

When the underlying random variable is continuous, the data are already grouped by the unavoidable rounding that all measurement involves. When those measurements involve several significant figures, as they do here, further grouping is usually required when displaying the data.

7 Grouping data

- Numeric data, whether discrete or continuous, may be *grouped* so that the resulting tables and graphs give a clearer overview of the data.
- Continuous data always needs grouping for frequency to have any meaning, unless the unavoidable rounding had already grouped the data sufficiently.
- Grouping normally involves *intervals of equal width* and *class centres*. The intervals are often called *bins*.
- Grouping involves ignoring information. This may or may not be an issue.
- *Data on a boundary should be counted in the lower interval.*

With a continuous variable, there may sometimes be data on a boundary, because data are always rounded. *Always count these scores as in the lower interval.* This follows the cumulative frequency convention in Section 13B, where we said that the cumulative frequency of a value x is the number of scores with $X \leq x$.

Relative and cumulative frequencies of grouped data

The relative frequencies, the cumulative frequencies, and the cumulative relative frequencies, of grouped data are calculated as before.



Example 5 Calculating relative and cumulative frequencies

- a** Using the grouped height data in the table above, add columns for relative frequency, cumulative frequency, and cumulative relative frequency.
- b** Give estimates of these probabilities when a person is chosen at random from the full community:
- i** $P(150 < X \leq 160)$
 - ii** $P(X > 140)$
 - iii** $P(X \leq 140)$
 - iv** $P(100 < X \leq 170)$

Solution

interval	80–90	90–100	100–110	110–120	120–130	130–140	140–150	150–160	160–170	170–180
centre	85	95	105	115	125	135	145	155	165	175
f	1	3	2	5	8	6	34	22	16	3
rf	0.01	0.03	0.02	0.05	0.08	0.06	0.34	0.22	0.16	0.03
cf	1	4	6	11	19	25	59	81	97	100
crf	0.01	0.04	0.06	0.11	0.19	0.25	0.59	0.81	0.97	1.00

- b**
- i** $P(150 < X \leq 160) = 0.22$
 - ii** $P(X \leq 140) = 0.25$
 - iii** $P(X > 140) = 1 - P(X \leq 140) = 0.75$
 - iv** $P(100 < X \leq 170) = P(X \leq 170) - P(X \leq 100) = 0.93$

Frequency histograms and polygons for grouped continuous data

Histograms and their procedures are particularly designed for grouped continuous data such as the dataset above. They were all described in detail in Box 5 of Section 13B.

An estimate of the median from the cumulative polygons

To find an estimate of the median (this will rarely be the exact median), draw a horizontal line halfway to the top of either the cumulative relative frequency polygon or the cumulative frequency polygon. The point where it meets the polygon gives, on the x -axis, an estimate of the median.



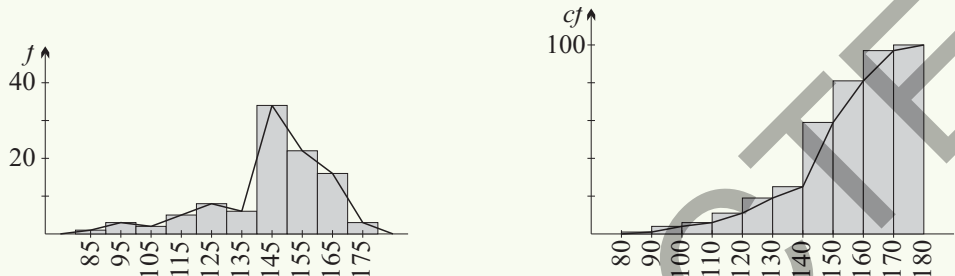
Example 6 Drawing and interpreting histograms and polygons for grouped data

Working still with the population data above:

- Draw the frequency histogram and the cumulative frequency histogram, with their superposed polygons.
- Estimate the median from the cumulative frequency histogram.
- Is it possible to estimate the proportion of children in the total population from these graphs?

Solution

a



- On the cumulative frequency polygon, draw horizontal lines at the middle cumulative frequencies 50 and 51. They meet the polygon at roughly 147 on the horizontal axis, so the estimate of the median is 147.
- That is not at all easy. First, one would have to define what a ‘child’ is. Secondly, height is not a determiner of being a child, because there are tall children of age 12–13, and some adults are rather short. Thirdly, men and women differ significantly in height. Hence any attempt would need detailed statistics relating height to age and sex, and complicated calculations to relate these statistics to the given heights. There is also an unexplained dip in the frequency of heights 130–140 cm, and there appear to be no babies in the dataset, so the randomness of the dataset is questionable.

Grouping discrete data

Histograms are designed for grouped data from a continuous random variable. Discrete data is also routinely grouped, but unfortunately, there are different conventions for the resulting tables and histograms. Don’t be surprised if your spreadsheet or software displays slightly different versions of these things.



Example 7 Preparing grouped discrete data

Section 13B prepared a frequency table for 40 spelling test marks in Year 7.

Mark	1	2	3	4	5	6	7	8	9	10
Frequency	2	4	2	1	6	8	7	6	2	2

Group the data by pairing the marks 1–2, 3–4, ..., and add the cumulative frequencies and the cumulative relative frequencies.

Solution

Interval	1–2	3–4	5–6	7–8	9–10
Class centre	$1\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	$7\frac{1}{2}$	$9\frac{1}{2}$
Frequency	6	3	14	13	4
Cumulative frequency	6	9	23	36	40
Cumulative relative frequency	0.15	0.225	0.575	0.9	1

Histograms and cumulative histograms for grouped discrete data

The procedures are the same as ever — see Box 5 of Section 13B. But a warning:

- Histograms and their procedures are designed for continuous data, and can be very approximate with discrete data, particularly with the polygons.

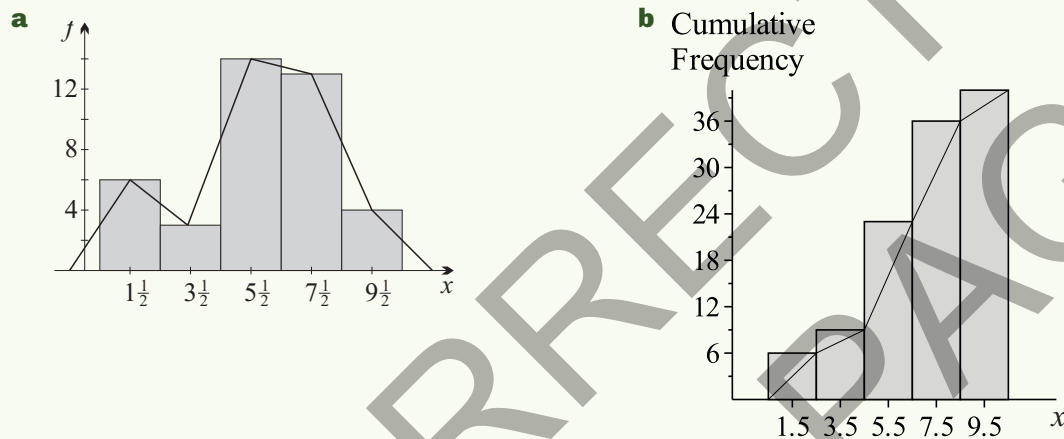


Example 8 Histogram and cumulative histogram for grouped discrete data

For the grouped data in the previous example, draw:

- the frequency histogram and polygon, and
- the cumulative frequency histogram and polygon.

Solution



Exercise 13C

FOUNDATION

- A horticulturalist has planted 40 dwarf apple trees and he is investigating the yield of fruit produced per tree when the trees are 3 years old. The yield, in kilograms, is given in the following data set.

18 24.5 30.2 27.8 50.4 34.4 53.8 47.5 64.4 57.2
 45.8 43.2 50.5 62.3 27.9 34.7 38.9 46.1 34.1 38.5
 43.6 38.4 39 46.7 48 52.3 57.1 43.4 46.4 38.9
 41.2 43.7 53.4 47 44.5 38.9 35 36.2 43.2 45.7

- Use a tally table to group the data set into classes of 5: $[15, 20)$, $[20, 25)$, $[25, 30)$, ...

lower class bound	15	20	25	30	35	40	45	50	55	60
tally										
frequency										

- Hence construct a frequency histogram for the grouped data.
- Add the frequency polygon to your graph.
- If spreadsheet software is available, use it to construct the histogram. You will need to set the bin width to 5. To ensure the first class begins at 15, you may need to choose the underflow bin to 20 (which means scores ≤ 20 form the first class).
 - In your spreadsheet, adjust the bin width and decide on the best class size to display the data.
- Add a cumulative frequency and cumulative relative frequency row to your table.
- Construct a cumulative relative frequency histogram for your grouped data.

- g** Use your cumulative relative frequency histogram to estimate the median of the grouped data. Do this by estimating, to the nearest kilogram, the weight of a fictitious tree at the 50% mark on your cumulative relative frequency polygon.
- h** **i** Identify which classes represent trees with yield in the lower 20%.
- ii** It is suggested that trees falling in the lower 20% should be removed. Discuss.

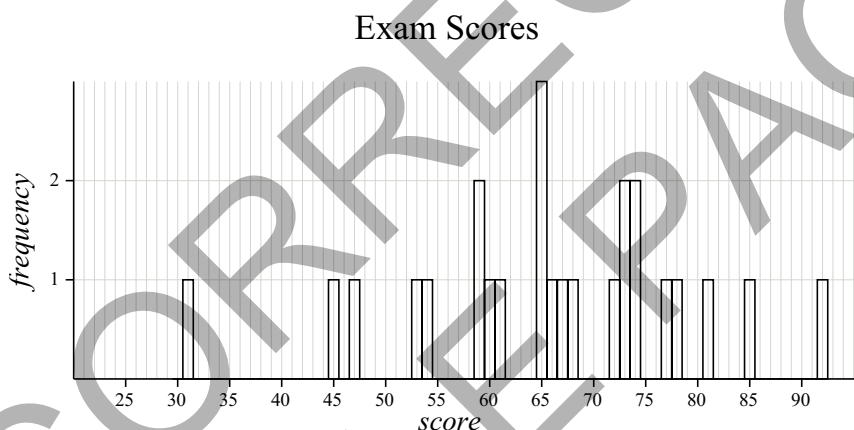
2 A teacher records the Examination results for their class:

47 85 65 73 31 66 45 65 74 81 92 74 77 65 54 53 67 78 72 61 60 59 68 59 73

a Copy the table below and fill in the missing values to group the data in bins of width 10 as shown.

x	[25, 35)	[35, 45)	[45, 55)	[55, 65)	[65, 75)	[75, 85)	[85, 95)
class centre							
Tally							
frequency							

- b** Construct a frequency histogram for the grouped data. Add a frequency polygon.
- c** The teacher also draws up a histogram with bin width 1, as below.



Discuss the advantages and disadvantages of the grouped data histogram you displayed, compared with the teacher's histogram.

d Draw up a table and construct a histogram with bins of width 5, that is $[30, 35)$, $[35, 40)$, ...

3 A business provides its employees with a mobile phone. Phones were purchased at the start of January and replaced over a three year period, due to faulty hardware, damage, planned upgrade or request for new features. The relative frequency for the replacement time period for a phone in months is shown in the following table. The data is grouped in 6 months bins. Thus, for example, 10% of phones are replaced between 12 and 18 months. The random variable X represents the lifespan of a random mobile phone in the company, before it is replaced.

x	[0, 6)	[6, 12)	[12, 18)	[18, 24)	[24, 30]	[30, 36]
rf	0.2	0.05	0.1	0.27	0.15	0.37

- a** X is a continuous random variable. Explain why $P(X = 6) = 0$ and write down the value of $P(X = x)$ for any x .
- b** Given $P(X < 6) = 0.2$, what is $P(X \leq 6)$?
- c** Estimate:
- i** the probability a phone is replaced in the first year, $P(X < 12)$;
 - ii** the probability a phone is replaced in the second year, $P(12 \leq X < 24)$;
 - iii** the probability a phone is replaced within the first 6 months of a year.

- 4 The business in the previous question has a similar scheme for laptops, but it replaces them over a 2 year period. The cumulative relative frequency for the replacement time period in months is shown in the following table. The data is grouped in 4 months bins. The random variable X represents the lifespan of a random laptop in the company, before it is replaced.

x	$[0, 4)$	$[4, 8)$	$[8, 12)$	$[12, 16)$	$[16, 20)$	$[20, 24]$
crf	0.12	0.22	0.37	0.57	0.79	1

- a What fraction of laptops are replaced in the first year?
 b Estimate the probability that a laptop lasts more than 8 months.
 c Estimate:
 i $P(4 < X < 20)$ ii $P(X > 16)$.

DEVELOPMENT

- 5 Xiommi recorded her time to get to work each day. Her results in minutes were:

22 30 23.5 27 25 21.5 39 30 32.5 33 35.5 37 42 22 23.5 27 29.5 23 34

- a Write the data out in order, and determine the median.
 b Group the data into classes by completing the following table.

class	20–24	24–28	28–32	32–36	36–40	40–44
class centre						
frequency						
cumulative						

- c Find the median of the grouped data. Does it agree with your answer to part (a)?
 d Draw a frequency histogram and polygon on the same chart.
 e Draw a cumulative frequency histogram and polygon on the same chart.
- 6 In a class experiment, students measured their heights. The results in centimetres were:

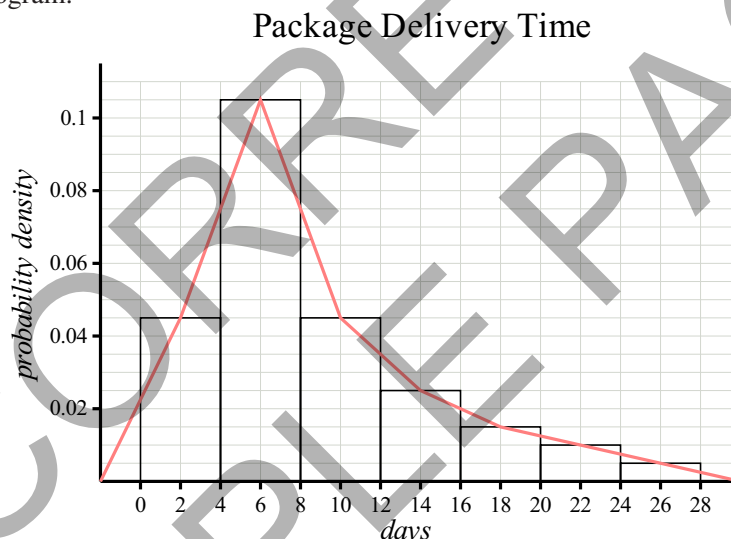
155 152 165 162 170 168 165 162 166 154 158 159
 163 166 164 164 159 157 163 154 166 158 159 163

- a Display the data in a frequency table.
 b Calculate the median of the dataset.
 c Why would it not be helpful to graph the data without first grouping it into classes?
 d Group the data into the intervals 150–154, 154–158, 158–162, 162–166, 166–170 and display your results in a grouped frequency table. Include any scores on a boundaries in the lower group, thus x is in the group 150–154 if $150 < x \leq 154$. (This is deliberately a different convention than we gave in the theory. Various conventions exist and you should always *read the question carefully*.)
 e Calculate the median of the dataset from this grouped frequency table.
 f Display your grouped data on a histogram with a frequency polygon joining the centres.
 g Construct a cumulative frequency histogram and cumulative frequency polygon — remember that the polygon starts at the bottom left-corner of the first rectangle and passes through the right-hand top corner of each rectangle.
 h Trace the line at frequency 12 (50%) until it meets the cumulative frequency polygon, and check whether this agrees with your answer for the median of the grouped data in part (e).
 i Construct a cumulative frequency histogram and cumulative frequency polygon of the *ungrouped* data.

- j** Compare your grouped and ungrouped cumulative histograms in parts (f) and (h). How similar are the graphs? Contrast the differences between the histogram of the grouped data in part (f) and what you would expect the histogram of the ungrouped data to look like (not drawn).
- k** Confirm that the line at frequency 12 meets your ungrouped polygon to give the same median as in part (g).
- l** If you have a spreadsheet program available, use it to construct a frequency histogram of the grouped data. The author found that to do this in Excel, he had to set the bin width to 4 and the underflow bin to 154 to force it to the desired starting bin. Excel follows the convention that data 154-158 is grouped as (154,158], as in this question. These distinctions become less important with continuous data and large data sets.

CHALLENGE

- 7** It is common for a histogram to plot *probability density* on the vertical axis, which is relative frequency divided by the bin width. The probability of an outcome in a class is the area of the rectangle (bin width $\times$ probability density). A delivery company tracks the time taken for packages to arrive at their destination. The data is shown on the following histogram.



- a** Find the probability that the package will arrive in between 4 and 8 days.
- b** Find the probability that the package will arrive in between 8 and 16 days. The polygon represents the smoothing out of the probability densities.
- c** What is the total area under the polygonal function?
- d** Estimate the probability that the package will take more than 18 days.

Chapter 13 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 13 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist



- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Chapter Review Exercise

- State whether each random variable is numeric or categorical. If it is numeric, state whether it is discrete or continuous.
 - In a street poll, people are asked to name their favourite movie.
 - The total rainfall in Sydney each year over the last 100 years.
 - Households in a suburb are asked how many pets they have.
 - Pupils are asked the time they spend playing video games per week.
- An internet popup is selling kitchen knives. In a recent promotion, they are trying to encourage customers to buy a second knife to keep as a spare, or to buy additional knives for friends and family. Let X be the random variable representing the number of knives a customer purchases. Data for one day in the promotional period is shown.

x	1	2	3	4	5	6
f	8	13	7	15	4	3

- Copy the table and add a row of relative frequencies.
- What is the mode? Explain in your own words what this means.
- What fraction of customers only buy one knife?
- What is the probability that a random customer will buy a single pair of knives?
- Estimate:
 - $P(X > 1)$
 - $P(1 \leq X \leq 3)$
 - $P(2 \leq X < 4)$
 - $P(X \text{ is even})$
- Find the probability that a customer bought a pair of knives, given they bought an even number of knives.
- Estimate the number of knives the company will sell over a seven day period.
- What assumptions did you have to make in answering the previous question? Were the assumptions reasonable?
- Construct a frequency histogram and polygon for the data.

- 3 A boy band is considering their career successes and decides to graph the percentage sales of their eight singles. The singles are ranked by date of production, from $x = 1$ to $x = 8$. The random variable X represents where a random single falls by order of production.

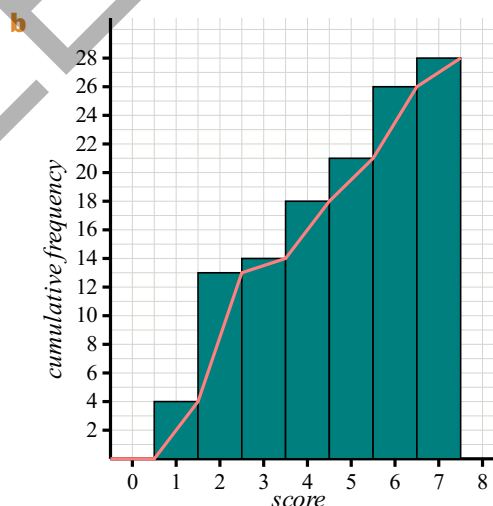
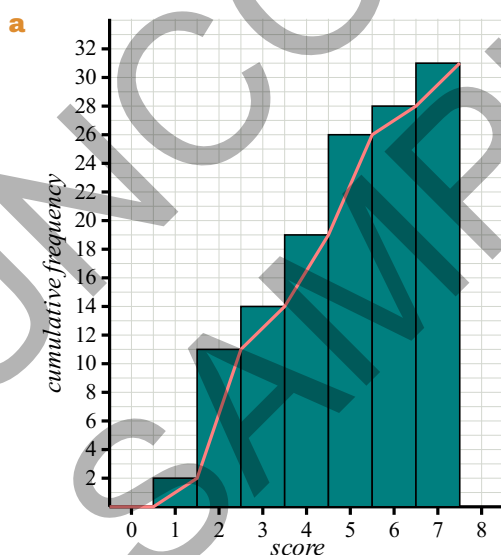
x	1	2	3	4	5	6	7	8
f	3	45	30	13	4	2	2	1

- a Is the data categorical or numerical?
- b Draw a relative frequency histogram of their sales. A scale of at least 4 cm as 0.1 unit on the vertical axis is recommended.
- c Is the data skewed? If so, is it skew left or right?
- 4 Find the median, mode and range of each dataset.
- a 2 5 6 6 6 7 8 9 10 12 13
- b 1 1 2 4 4 5 5 6 7 15
- c 14 10 12 7 7 8 15 16 17 10 5 9 2 8 7 12

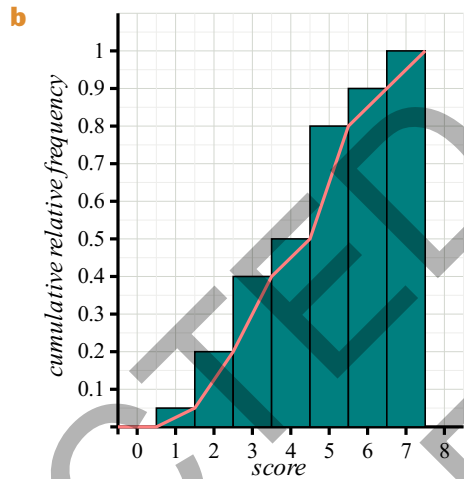
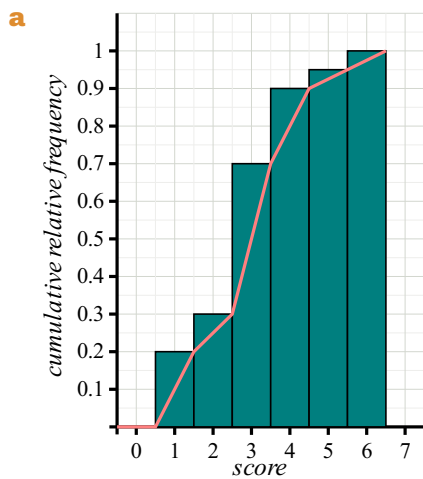
- 5 The results of an experiment to investigate the value of a random variable X are listed in the following table.

x	0	1	2	3	4	5	6
f	5	3	7	8	2	6	9

- a Copy the table and add a row listing cumulative frequencies and relative cumulative frequencies.
- b Estimate:
- i $P(X > 10)$
- ii $P(X \leq 5)$
- iii $P(1 < X \leq 3)$
- iv $P(X > 3)$
- 6 Use the cumulative frequency polygon and cumulative frequency histogram to find the number of scores and the median for each experiment.



- 7 Use the cumulative relative frequency polygon and cumulative relative frequency histogram to estimate the median for each experiment.



- 8 Data from an experiment is recorded in the following table.

<i>x</i>	1	2	3	4	5	6	7	8	9
<i>f</i>	2	4	1	5	3	6	2	1	3

- a** Copy the table and add a cumulative frequency row.
- b** Draw a cumulative frequency histogram showing this data. Add a cumulative frequency polygon.
- 9 A wildlife officer is investigating the length of adult crocodiles in a region of a national park. He records the following measurements.

2.8 3.2 3.25 5.2 3.4 6.1 4.2 2.85 3.3 3.4 2.9 2.95 2.6 4.8
3.9 3.6 4.6 3.45 3.4 3.1 3.3 4.6 4.75 3.1 3.2 2.7 5.7 4.2
3.3 4.3 2.6 5.6 3.3 3.2 3.4 3.1 4.5 5 3 3.8 4 3 3.5 5.4

- a** Complete the following table, grouping the crocodile lengths in bins of width 0.5 m.

<i>x</i>	[2.5, 3)	[3, 3.5)	[3.5, 4)	[4, 4.5)	[4.5, 5)	[5, 5.5)	[5.5, 6)	[6, 6.5)
tally								
<i>f</i>								
<i>cf</i>								

- b** Construct a frequency histogram and polygon for the grouped data.
- c** How many crocodiles were in the data set?
- d** Find the class interval containing the median crocodile length.
- e** The officer realises that the data actually includes two superimposed populations - female and male crocodiles. Each distribution has its own shape and its own mode and median. Suggest what you might be able to deduce from the histogram above about the two populations.