

CATHERINE SMITH | PAUL JOHANSEN | RAYMOND ROZEN | PAULINE HOLLAND

JACARANDA MATHS QUEST
SPECIALIST MATHEMATICS

12

FOR QUEENSLAND

**UNITS
3&4**

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JACARANDA MATHS QUEST
SPECIALIST
MATHEMATICS **12**
FOR QUEENSLAND

UNITS
3&4

CATHERINE SMITH | PAUL JOHANSEN | RAYMOND ROZEN | PAULINE HOLLAND

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ABOUT THIS RESOURCE

Jacaranda Maths Quest 12 Specialist Mathematics Units 3 & 4 for Queensland is expertly tailored to address comprehensively the intent and structure of the new syllabus. The *Jacaranda Maths Quest for Queensland* series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

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Chapter openers place mathematics in real-world contexts to drive engagement.

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Every chapter concludes with exam practice questions classified as Simple familiar, Complex familiar and Complex unfamiliar.

8 Rates of change and differential equations

8.1 Overview

The rate of change of one quantity with respect to another quantity is a measure of how the quantities are related or how a change in one quantity will affect a change in the other. As you have already learned, rates of change form the basis of the study of calculus. Differential equations build on the concepts already covered in calculus, differential and integral. Modelling the world around us using complex differential equations and sophisticated computers to solve these equations has led to an ever-increasing understanding of our world and the universe we inhabit. One example of the use of differential equations is in the science of meteorology. Super computers can analyse enormous amounts of weather data, but it is the use of differential equations to model long and short-term patterns that has enabled meteorologists to determine how changes in temperature and moisture in the atmosphere may lead to changes in the weather. This can mean the difference between life and death in communities now have advance warning of extreme weather conditions.



studyON
Unit 8 > Area 1 > Sequence 3 > Concept 1
Introduction to proof by mathematical induction Summary screen and practice questions

7.8 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacaranda.com.au.

Simple familiar

- For each of the following integrals, identify the domain of the integrand and whether the integrand is:
 $\int_0^1 (x + \frac{1}{x}) dx$, $\int_1^2 (x^2 + x^{-2}) dx$, $\int_0^1 \frac{x+2}{x^2+1} dx$, $\int_0^1 \sqrt{1-x^2} dx$
- Determine the area of the region bounded by the curve $y = x^2 - 8x$ and the x -axis.
- Determine the area of the region bounded by the curve $y = \frac{1}{x}$, the lines $x = 2$, $x = 3$ and the x -axis.
- Determine the area of the region bounded by the curve $y = x^2 - 4x$ and the line $y = 2x$.
- Determine the area of the region between the curve $y = \frac{1}{x}$ and the line $y = 2x$ from $x = 1$ to $x = 2$.
- Determine the volume of the solid formed by rotating the line $y = \frac{1}{2}$ about the x -axis from the origin to $x = 4$.
- Determine the volume of the solid formed by rotating the line $y = \frac{1}{2}$ about the x -axis from the origin to $x = 8$.
- Determine the volume of revolution when the region bounded by the curve $y = x^2 - 4x$ and the x -axis is rotated about the x -axis.
- Determine the volume of revolution when the region bounded by the curve $y = -5\ln(x+1) + 3$ and the coordinate axes is rotated about the y -axis.
- Determine the volume of revolution when the region bounded by the curve $y = \sqrt{1-x^2}$ and the coordinate axes is rotated about the y -axis.
- An exponential probability density function is given by $f(x) = \begin{cases} 2.2e^{-2.2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$
 - Determine $P(0 \leq T \leq 2)$ correct to four decimal places.
 - Determine $P(T \geq 2)$ correct to four decimal places.
- Complex familiar
- Determine the exact volume formed when the area bounded by the curve with the equation $y = x^2$, the x -axis and the line $x = 2$ is rotated about the y -axis.
 $\pi > 9.416$
- Consider the functions $f(x) = 5 \sin(\frac{\pi}{2}x)$ and $g(x) = \frac{1}{2}x$ where $x \geq 0$. Determine, correct to 3 decimal places, the area between functions f and g .
- The area of a quadrant of a circle is represented by $\int_0^{\sqrt{16-x^2}} \sqrt{16-x^2} dx$. Compare the areas obtained by using an exact calculus method and using Simpson's rule with 4 strips.
- Kately estimates that the average time between incoming phone calls is 2 days. If it has been one day without a call, determine the probability that there will be a call in the next 3 hours. Give your answer correct to four decimal places.

studyON
Unit 8 > Area 1 > Sequence 3 > Concept 1
Projectile motion Summary screen and practice questions

Exercise 4.6 Projectile motion

Technology active

- For the following questions give your answer to 2 decimal places, where a decimal answer is required.
- A soccer ball is kicked off the ground at an angle of 67.23° with an initial speed of 13 m/s. Calculate:
 - the time of flight
 - the range
 - the maximum height reached.
 - A ball is thrown so that its time of flight is $\frac{10\sqrt{3}}{g}$ and the ratio of the range to the maximum height is 2.

PRACTICE ASSESSMENT 1

Specialist Mathematics: Problem solving and modelling task

Unit 3: Mathematical induction, and further vectors, matrices and complex numbers

Topic 2: Vectors and matrices

Duration	Mode	Individual/group
4 weeks	Written report due to 10 pages (maximum 2000 words) including appropriate	Individual

- When expressed in component form, the vector $\mathbf{g} = [15, 32\sqrt{3}, 30]^\top$ is defined by which of the following:
 - $\mathbf{g} = 9\sqrt{3}\mathbf{i} - 6\sqrt{3}\mathbf{j} + 8\sqrt{3}\mathbf{k}$
 - $\mathbf{g} = 10.2\sqrt{3}\mathbf{i} - 7.1\sqrt{3}\mathbf{j} + 9.7\sqrt{3}\mathbf{k}$
 - $\mathbf{g} = 11.3\sqrt{3}\mathbf{i} - 6.0\sqrt{3}\mathbf{j} - 8.0\sqrt{3}\mathbf{k}$
 - $\mathbf{g} = 11.8\sqrt{3}\mathbf{i} - 5.9\sqrt{3}\mathbf{j} + 7.6\sqrt{3}\mathbf{k}$
- Determine the equivalent Cartesian equation for $x = 2z - 7$, $y = \frac{z}{2} + 7$.
 - $x = \frac{1}{2}z^2 + 2z - 7$
 - $x = \frac{1}{2}z^2 + 2z + \frac{1}{2}$
 - $x = \frac{1}{2}z^2 + 2z + \frac{1}{2}$
 - $x = \frac{1}{2}z^2 + 2z + \frac{1}{2}$
- A sphere contains the points $A(7, 13, 8)$ and $B(-1, 7, 2)$, that use the endpoints of its diameter. Calculate the Cartesian equation of the sphere:
 - $49 = (x+3)^2 + (y-9)^2 + (z-5)^2$
 - $80 = (x-3)^2 + (y-9)^2 + (z-5)^2$
 - $49 = (x-3)^2 + (y+9)^2 + (z-5)^2$
 - $80 = (x-3)^2 + (y+9)^2 + (z-5)^2$
- Calculate the vector equation for the line passing through the given points $A(5, -2, 1)$ and $B(6, 4, 7)$.
 - Determine the value for γ if the head of vector, $\mathbf{c} = 12\mathbf{i} + \gamma\mathbf{j} + 8\mathbf{k}$, touches the line $l: P(x, y, z) = (2, 1, 3) + t(1, 2, 3)$
 - $\gamma = 20$
 - $\gamma = 20.622$
 - $\gamma = 20.622$
- Calculate a vector perpendicular to both $\mathbf{a} = 5\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 9\mathbf{j} + 1\mathbf{k}$.
- Determine the volume of the parallelepiped containing the adjacent vectors:
 $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + 9\mathbf{j} - 8\mathbf{k}$ and $\mathbf{c} = 12\mathbf{i} - 9\mathbf{j} - 1\mathbf{k}$.
- Determine the equation of the plane containing the point $(-2, 9, 1)$ if the vector $\mathbf{a} = 5\mathbf{i} + 9\mathbf{j} + 8\mathbf{k}$ is normal to it.
- The path (measured in metres) of two objects in flight is given by the following vector functions of time:
 $\mathbf{r}_1(t) = (t^2 + 4)\mathbf{i} + (4t + 9)\mathbf{j} + (6t^2 + 1)\mathbf{k}$, $0 \leq t \leq 20$
 $\mathbf{r}_2(t) = (8t + 184)\mathbf{i} + (6.25t^2)\mathbf{j} + (3t + 109)\mathbf{k}$, $0 \leq t \leq 20$
Determine if the particles will cross paths at the same time, stating the time it occurs and the location at that point in time.
- The parametric equations, $x = 2 - t$, $y = 2 + t$, $z = 5 + 2t$, defines the path of particle.
Determine the point of intersection between the particle and the plane, $x + 2y - z = -14$.
- Complex familiar
- The vector product of two vectors $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ is $\mathbf{c} \times \mathbf{b} = 12\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}$. Calculate the values for p and q .
- Calculate the value of the determinant determined by the following points: $P(3, 8, 9)$, $Q(4, -8, -5)$ and $R(-6, 6, 2)$.

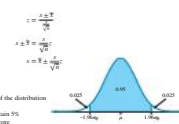
As you can see, 90% (9 out of 10) of the calculated confidence intervals contain the population mean.

This is the meaning of the 90% confidence interval.

Notice that the size of the confidence intervals varies between samples. The smaller the sample standard deviation, the smaller the confidence interval for the sample.

12.4.3 Verifying the formulae

We have learnt that when the sample size is large, that is $n > 30$, the distribution of T is normally distributed with a mean of $\mu_T = \mu$ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$. As a sample is normally selected in order to estimate the population mean and standard deviation, we have estimates for μ and σ on T and s respectively. We know that for normal distributions, $z = \frac{x - \mu}{\sigma/\sqrt{n}}$. This means that to find the upper and lower values of z ,



WORKED EXAMPLE 3

After surveying the 20 people in your class, you find that they plan to spend an average of \$4.33 on their lunch today. The standard deviation for your sample was \$0.23. Use these statistics to state a 90% confidence interval for the average amount that will be spent on lunch today.

THINK	WRITE
1. There are 20 people in the class. This is the sample size.	$n = 20$
2. \$4.33 is spent on lunch today is the sample mean.	$\bar{x} = 4.33$
3. The sample standard deviation is \$0.23.	$s = 0.23$
4. For a 90% confidence interval, $z = 1.645$.	$z = 1.645$

Chapter 10 — Modelling motion 1

Exercise 10.2 A body in equilibrium under concurrent forces

1. A body is in equilibrium under the forces shown in the diagram.

2. A body is in equilibrium under the forces shown in the diagram.

3. A body is in equilibrium under the forces shown in the diagram.

Each subtopic concludes with carefully graded Technology free and Technology active questions.

Two complete sets of practice assessments modelled on QCAA guidelines — a set for student revision and a quarantined set for teachers — are included. Exemplary responses and worked solutions are provided for teachers.

Chapter questions and activities are aligned with Marzano and Kenall's taxonomy of cognitive process — retrieval, comprehension, analysis and knowledge utilisation.

Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Free fully worked solutions are provided, enabling students to get help where they need it, whether at home or in the classroom — help at the point of learning is critical. Answers are provided at the end of each chapter in the print and online PDF.

Features of the Maths Quest series

Questions and topics are sequenced from lower to higher levels of complexity; ideas and concepts are logically developed and questions are carefully graded, allowing every student to achieve success.

An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.

- A spherical bubble is blown so that its radius is increasing at a constant rate of 2 millimetres per second. When its radius is 10 millimetres, calculate the rate at which:
 - its volume is increasing
 - its surface area is increasing
- A spherical basketball has a hole in it and its volume is decreasing at a rate of $6 \text{ cm}^3/\text{s}$. At what rate is the radius changing when the radius of the basketball is 6 cm?
- A metal ball is dissolving in an acid bath. Its radius is decreasing at a rate of 3 cm/s . At what rate is the ball's surface area changing when the radius of the ball is 2 cm?
- A football has its radius decreasing at a constant rate of 0.2 millimetres per week. Assume it remains spherical.
 - Show that the volume is decreasing at a rate that is proportional to its surface area.
 - If its initial radius is 30 millimetres, calculate how long it takes to disappear.
- A conical vase has a height of 40 cm and a radius of 8 cm. The axis of the vase is vertical and its vertex is downwards. Initially it is filled with water which leaks out through a small hole in the vertex at a rate of $6 \text{ cm}^3/\text{s}$. Calculate the rate at which the water level is falling when the height of the water is 16 cm.
- A cone is such that its radius is always equal to half its height. If the radius is decreasing at a rate of 2 cm/s, calculate the rate at which the volume of the cone is decreasing when the radius is 4 cm.
- The sides of an equilateral triangle are increasing at a rate of 2 centimetres per second. When the sides are $2\sqrt{3}$ centimetres, calculate the rate at which:
 - its area is increasing
 - its height is increasing
- Show that the formula for the volume, V , of a right circular cone with height h is given by

2.2 Introduction to vectors in three dimensions

Introduced in Unit 1, vectors have thus far only been considered in two-dimensional form. In this chapter, following a review of two-dimensional vectors, our study of vectors will be extended to three-dimensions.

2.2.1 Review of vectors in two-dimensions

Scalar and vector quantities

A scalar quantity is a real number only; it has magnitude, but no direction. Scalar quantities include temperature, mass and time.
A vector is a quantity that has magnitude and direction. Vectors include weight in a particular direction and displacement with a specific direction.
Vectors are directed line segments with a tail (start) and head (end). In the figure below, the head of the vector is at point B (indicated with an arrow), while the tail is at point A. In this instance, the vector could be expressed in a variety of forms.

Various forms of vector notation
 $\vec{AB} = \mathbf{a}$



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Exercise 4.6 Projectile motion

Technology active

For the following questions give your answer to 2 decimal places, where a decimal answer is required.

- A soccer ball is kicked off the ground at an angle of 67.2° with an initial speed of 13 m/s. Calculate:
 - the time of flight
 - the range
 - the maximum height reached

Country	Gold	Silver	Brass
United States of America	36	27	26
Russia	31	21	21
People's Republic of China	26	18	26
Russian Federation	29	19	17

A downloadable PDF of the entire print text

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In the Resources tab of every chapter there are two topic tests in downloadable, customisable Word format with worked solutions.

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Country	Gold	Silver	Brass
United States of America	36	27	26
Russia	31	21	21
People's Republic of China	26	18	26
Russian Federation	29	19	17

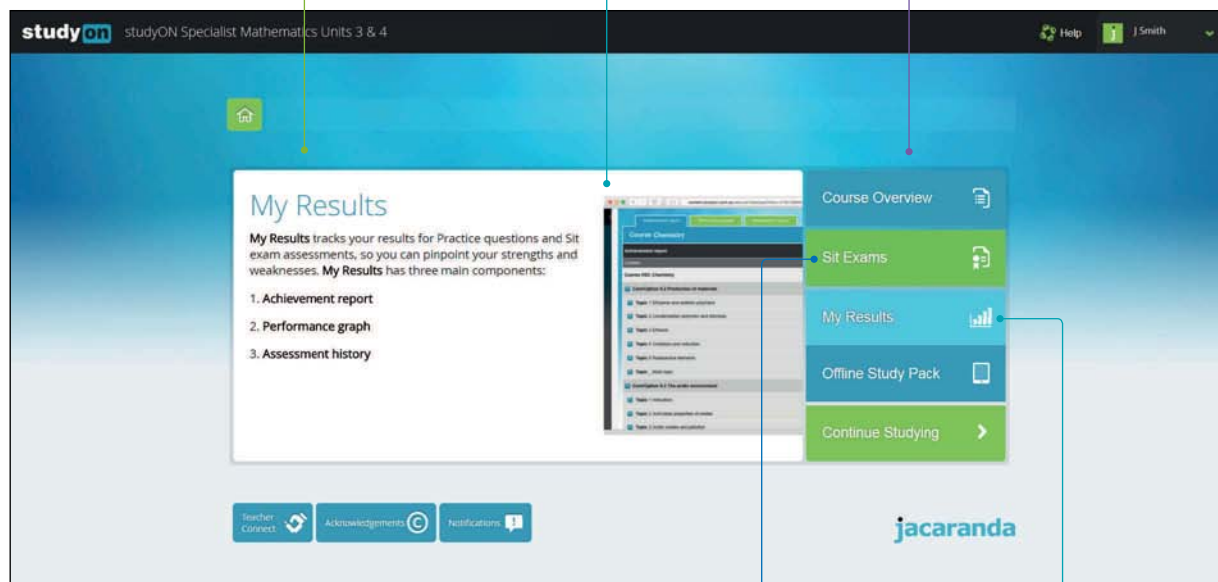
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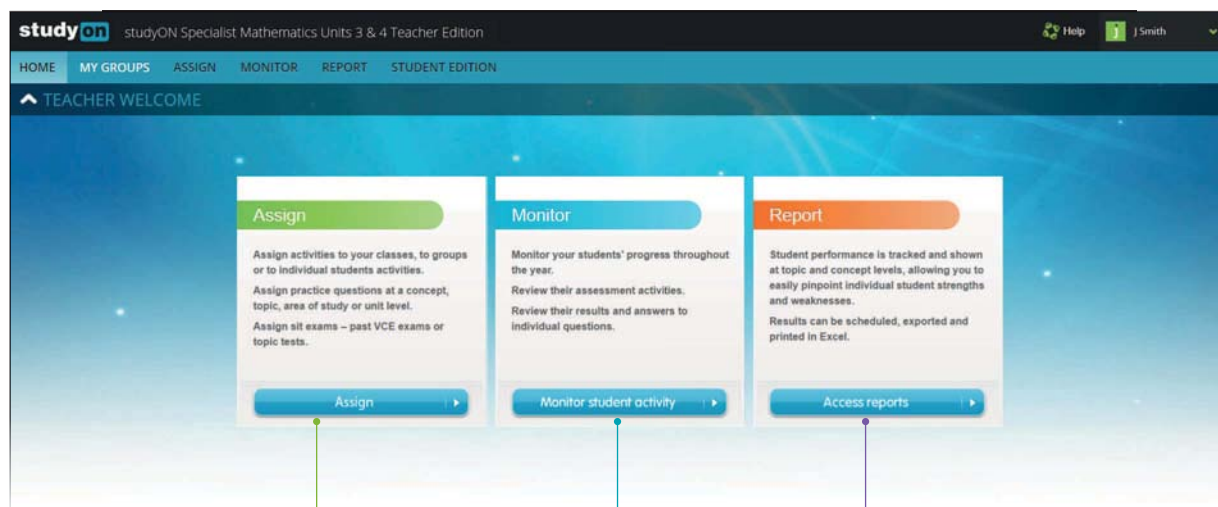


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Text

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1 Proof by mathematical induction

1.1 Overview

You were introduced to the concept of proofs in Unit 1 Chapter 3. In that chapter, we explored how to construct a logical argument based on definitions and axioms to prove that a statement or proposition is true. These concepts are extended here, through proof by mathematical induction, which is an indirect method of constructing a proof.

Euclid's proof that the number of primes is infinite contains suggestive use of proof by mathematical induction, but as a technique it was not until the 10th century when the Persian mathematician al-Karaji (953–1029) used a more overt style of induction-type proof to prove that $1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots + n)^2$. In 1838 the term mathematical induction was introduced, and a common understanding of the process was developed through the work of Augustus De Morgan (1806–1871).



LEARNING SEQUENCE

- 1.1 Overview
- 1.2 Introduction to proof by mathematical induction
- 1.3 Proof of divisibility
- 1.4 Further proof by induction
- 1.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Introduction to proof by mathematical induction

1.2.1 Proof by induction basics

Imagine a series of dominoes, placed just close enough so that if the first one topples, then it will cause the next one to fall and so on. This process is how a **proof by mathematical induction** works.

Notice that for the dominoes to topple, they firstly need to be close enough together *and* the first domino needs to fall. Without both of these steps happening, some (or all) of the dominoes will remain standing.

The first step in a proof by induction is to prove that the formula works for $n = 1$. This is the equivalent of knocking over the first domino and is known as the **initial statement**. This step is important because it is necessary to prove that the formula works at least once.

The second step is to assume that since that formula works once, the formula will also be true when $n = k$. It is then necessary to prove that the formula is also true for $n = k + 1$. This is the equivalent of having the dominoes close enough together that they will topple and is known as the **inductive step**.

If the formula is true for $n = k$ also true for $n = 1$ and if it is also true for $n = k + 1$, this means it is also true for $n = 2$. And as it is true for $n = 2$, it is also true for $n = 3$ and so on. Therefore, the formula is true for all natural numbers.



WORKED EXAMPLE 1

Use mathematical induction to prove that $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ for $n \geq 1$.

Note that this was first proven by a form of mathematical induction by the Persian mathematician al-Karaji (953–1029) in the 10th century.

THINK

1. Verify that the formula is true for $n = 1$.
2. There are some values of n that make the formula true. Write down the formula for $n = k$.
3. Knowing that $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$, add the next term $(k+1)$ to both sides.

WRITE

If $n = 1$,

$$\begin{aligned} \text{LHS} &= 1. & \text{RHS} &= \frac{1(1+1)}{2} \\ & & &= \frac{1(2)}{2} \\ & & &= 1 \end{aligned}$$

True for $n = 1$.

Assume that it is true for $n = k$.

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

When $n = k + 1$, then next term in the series is $k + 1$.

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \end{aligned}$$

4. Note that if $n = k + 1$, then

$$\frac{n(n+1)}{2} = \frac{(k+1)[(k+1)+1]}{2}$$

so we are trying to demonstrate that

$$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)[(k+1)+1]}{2}.$$

Additionally, $k + 1$ is a common factor and is part of the expression that we are working towards. Factorise and then simplify.

$$\begin{aligned} &= (k+1) \left(\frac{k}{2} + 1 \right) \\ &= (k+1) \left(\frac{k+2}{2} \right) \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

5. Write the expression so that it is clearly the same formula as when $n = k + 1$.
6. Write a statement to explain what you have demonstrated and concluded.

If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, by mathematical induction, the statement is true for $n \geq 1$.

WORKED EXAMPLE 2

Use mathematical induction to prove that $2 + 4 + 8 + \dots + 2^n = 2^{n+1} - 2$, $n \geq 1$.

THINK

1. Verify that the formula is true for $n = 1$.
2. There are some values of n that make the formula true. Write down the formula for $n = k$.
3. Knowing that $2 + 4 + 8 + \dots + 2^k = 2^{k+1} - 2$, add the next term (2^{k+1}) to both sides.
4. Note that if $n = k + 1$, then $2^{n+1} - 2 = 2^{k+1+1} - 2$ so we are trying to demonstrate that $2^{k+1} - 2 + 2^{k+1} = 2^{k+1+1} - 2$. $2^{k+1} + 2^{k+1}$ can be rewritten as $2 \times 2^{k+1}$.
5. Use index laws to write the expression so that it is clearly the same as the formula when $n = k + 1$.
6. Write a statement to explain what you have demonstrated and concluded.

WRITE

If $n = 1$,
 $\text{LHS} = 2^1 \quad \text{RHS} = 2^{1+1} - 2$
 $\quad \quad \quad = 2 \quad \quad \quad = 2$
 True for $n = 1$.
 Assume that it is true for $n = k$.
 $2 + 4 + 8 + \dots + 2^k = 2^{k+1} - 2$
 When $n = k + 1$, then next term in the series is 2^{k+1} .
 $2 + 4 + 8 + \dots + 2^k + 2^{k+1}$
 $\quad \quad \quad = 2^{k+1} - 2 + 2^{k+1}$
 $\quad \quad \quad = 2^{k+1} + 2^{k+1} - 2$
 $\quad \quad \quad = 2 \times 2^{k+1} - 2$
 $\quad \quad \quad = 2^{k+1+1} - 2$
 If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, by mathematical induction, the statement is true for $n \geq 1$.

1.2.2 Using $\sum$ notation

An alternative way to write the equation to be proved involves using the **summation sign**, $\sum$, the Greek letter Σ , pronounced 'sigma'). In Worked example 1, the expression $1 + 2 + 3 + \dots + n$ was used. An alternative way to write this is $\sum_{r=1}^n r$, which means to add up all the r values beginning with $r = 1$ and ending with $r = n$, that is $1 + 2 + 3 + \dots + n$.

In the next worked example, the expression $\sum_{r=1}^n 3r$ is used. When $r = 1$, $3r = 3$ and when $r = 2$, $3r = 6$.

Therefore, $\sum_{r=1}^n 3r$ can be rewritten as $3 + 6 + 9 + \dots + 3n$.

WORKED EXAMPLE 3

Use mathematical induction to prove that $\sum_{r=1}^n 3r = \frac{3n(1+n)}{2}$.

THINK

1. Verify that the formula is true for $n = 1$.
2. There are some values of n that make the formula true. Write down the formula for $n = k$.
3. Finding the sum of the first $k + 1$ terms is the same as finding the sum of the first k terms and then adding term $k + 1$.
In this instance, term $k + 1$ will be $3(k + 1)$ and we know that $\sum_{r=1}^k 3r = \frac{3k(1+k)}{2}$.
4. If the sum is true for the first $k + 1$ terms, then the sum will equal $\frac{3(k+1)[1+(k+1)]}{2}$.
Additionally, $k + 1$ and 3 are common factors and are part of the expression that we are working towards. Factorise and then simplify.
5. Write the expression so that it is clearly the same as the formula when $n = k + 1$.
6. Write a statement to explain what you have demonstrated and concluded.

WRITE

If $n = 1$,

$$\begin{aligned} \text{LHS} &= 3 \times 1 \\ &= 3 \end{aligned} \quad \begin{aligned} \text{RHS} &= \frac{3(1)(1+1)}{2} \\ &= \frac{3(2)}{2} \\ &= 3 \end{aligned}$$

True for $n = 1$.

Assume that it is true for $n = k$.

$$\begin{aligned} \sum_{r=1}^k 3r &= \frac{3k(1+k)}{2} \\ \sum_{r=1}^{k+1} 3r &= \sum_{r=1}^k 3r + 3(k+1) \\ &= \frac{3k(1+k)}{2} + 3(k+1) \\ &= 3(k+1) \left(\frac{k}{2} + 1 \right) \\ &= 3(k+1) \left(\frac{k+2}{2} \right) \\ &= \frac{3(k+1)[1+(k+1)]}{2} \end{aligned}$$

If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, by mathematical induction, the statement is true for $n \geq 1$.

Proof by mathematical induction procedure

1. Show the statement is true for $n = 1$
2. Assume the statement is true for $n = k$
3. Show the statement is true for $n = k + 1$
4. State your conclusion.

1.2.3 Review of set theory and logic symbols

Set notation	Meaning
R	The set of real numbers; numbers which can be represented on the number line, the set of which can be divided into irrational numbers and rational numbers (Q).
N	The set of natural numbers; $N = \{1, 2, 3, 4, \dots\}$
Z $Z = Z^+ \cup Z^- \cup \{0\}$	The set of integers; $Z = \{-3, -2, -1, 0, 1, 2, 3, \dots\}$ The set of integers consists of zero, the positive natural numbers and the negative integers.
Z^+ Z^-	The set of positive integers, the same as N The set of negative integers
Q $Q = \left\{ \frac{a}{b}, a, b \in R, b \neq 0 \right\}$	The set of rational numbers; numbers which can be written as the ratio of two integers, a, b where $b \neq 0$.
Logic symbols	Meaning
Proposition	A mathematical statement that is either true or false.
Proposition A or B	One or both of the propositions are true. Equivalent to $A \cup B$.
Proposition A and B	Both of the propositions are true. Equivalent to $A \cap B$.
$\neg A$	The negation of proposition A . The negation of a true statement is false, and the negation of a false statement is true. Equivalent to A' .
$\neg(A \text{ or } B) = \neg A \text{ and } \neg B$	Equivalent to $(A \cup B)'$
$\neg(A \text{ and } B) = \neg A \text{ or } \neg B$	Equivalent to $(A \cap B)'$
$P \Rightarrow Q$ $P \Leftrightarrow Q$	Implication: if P , then Q Equivalence: if P , then Q and if Q , then P
Quantifiers	
$\forall$	For all; all possible values for the variable are considered.
$\exists$	There exists; there is a value for the variable that would make the propositional function true.
R.T.P iff	Required to prove If and only if

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Introduction to proof by mathematical induction Summary screen and practice questions

Exercise 1.2 Introduction to proof by mathematical induction

Technology free

1. Consider the statement $2 + 4 + 6 + \dots + 2n = n(n + 1)$, $n \geq 1$.
 - a. Demonstrate that the statement is true for the first three possible values of n .
 - b. If $2 + 4 + 6 + \dots + 2k = k(k + 1)$, demonstrate that $2 + 4 + 6 + \dots + 2k + 2(k + 1) = (k + 1)(k + 2)$.
2. Consider the statement $\sum_{r=1}^n 2^{r-1} = 2^n - 1$.
 - a. Demonstrate that the statement is true for the first three possible values of n .
 - b. If $\sum_{r=1}^k 2^{r-1} = 2^k - 1$, demonstrate that $\sum_{r=1}^{k+1} 2^{r-1} = 2^{k+1} - 1$.
3. Consider the statement $3 + 5 + 7 + \dots (1 + 2n) = n(n + 2)$, $n \geq 1$.
 - a. Demonstrate that the statement is true for the first possible value of n .
 - b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
 - c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
 - d. Explain how you have demonstrated that $3 + 5 + 7 + \dots (1 + 2n) = n(n + 2)$.
4. Consider the statement $\sum_{r=1}^n (2r - 1) = n^2$.
 - a. Demonstrate that the statement is true for the first possible value of n .
 - b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
 - c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
 - d. Explain how you have demonstrated that $\sum_{r=1}^n (2r - 1) = n^2$.
5. **WE1** Use mathematical induction to prove that $5 + 6 + 7 + \dots + (n + 4) = \frac{n(n + 9)}{2}$, $n \geq 1$.
6. **WE2** Use mathematical induction to prove that $3 + 6 + 12 + \dots + 3 \times 2^{n-1} = 3(2^n - 1)$, $n \geq 1$.
7. **WE3** Use mathematical induction to prove that $\sum_{r=1}^n 2^r = 2^{n+1} - 2$.
8. Use mathematical induction to prove that $\sum_{r=1}^n 2r = n(n + 1)$.
9. Use mathematical induction to prove that $\sum_{r=1}^n \frac{1}{r(r + 1)} = 1 - \frac{1}{n + 1}$.
10. Use mathematical induction to prove that $1^2 + 3^2 + 5^2 + \dots + (2n - 1)^2 = \frac{1}{3}n(2n - 1)(2n + 1)$ for $n \geq 1$.
11. Use mathematical induction to prove that $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$, $n \geq 1$.
12. Use mathematical induction to prove that $\sum_{r=1}^n r^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

13. Use mathematical induction to prove that $\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$.

14. Use mathematical induction to prove that

$$a + (a + d) + (a + 2d) + \dots + (a + [n - 1]d) = \frac{n}{2}[2a + (n - 1)d], \text{ for } n \geq 1.$$

1.3 Proof of divisibility

1.3.1 Divisibility results

Proof by induction can also be used to prove that an expression is divisible by a certain number. An equation can be found by writing the expression as a multiple of that number. This means that if $n^3 - n$ is divisible by 3, then $n^3 - n = 3m$ (where m is an integer).

WORKED EXAMPLE 4

Use a proof by induction to demonstrate that $n^3 - n$ is divisible by 3, $n > 1$.

THINK

1. $n > 1$ means that the first value of n to check is $n = 2$. Verify that the formula is true for $n = 2$.
2. There are some values of n that make the claim true. If a number is divisible by 3, it can be written as $3m$, write this as a mathematical equation where $n = k$.
3. Consider the statement when $n = k + 1$.
4. Substitute $k^3 - k = 3m$ to demonstrate that the statement can be written as a multiple of 3.
5. Write a statement to explain that the statement is true for $n = k + 1$ if it is true for $n = k$.
6. Write the concluding statement.

WRITE

$$\text{If } n = 2, n^3 - n = 8 - 2 = 6$$

As 6 is divisible by 3, that claim is true for $n = 2$.

Assume that it is true for some $n = k$.

$$k^3 - k = 3m, \quad m \in \mathbb{Z}$$

$$\begin{aligned} (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3m + 3k^2 + 3k \\ &= 3(m + k^2 + k) \end{aligned}$$

The statement $k^3 - k = 3m$, $m \in \mathbb{Z}$ is divisible by 3.

Therefore if $k^3 - k$ is divisible by 3, then $(k+1)^3 - (k+1)$ is divisible by 3.

$n^3 - n$ is divisible by 3 if $n = 2$. It is true for $n = k + 1$ if it is true for $n = k$. Therefore, by mathematical induction, the statement is true $\forall n > 1$.

1.3.2 Using divisibility notation

When proving an expression is divisible by a certain number, an alternative way to write the equation to be proved involves using **| notation**. In Worked example 4, it was required to prove (R.T.P) that $n^3 - n$ is divisible by 3. An alternative way to write this is $3|(n^3 - n)$.

In the following worked example, the notation $5|(2^{4n+1} + 3)$ is used which is equivalent to $2^{4n+1} + 3$ is divisible by 5. In general terms, $a|b$ means b is divisible by a .

WORKED EXAMPLE 5

Use a proof by induction to demonstrate that $5|(2^{4n+1} + 3)$, $n \geq 1$.

THINK

1. Verify that the statement is true for $n = 1$.
2. There are some values of n that make the claim true. If a number is divisible by 5, it can be written as $5m$, write this as a mathematical equation where $n = k$.
3. Consider the statement when $n = k + 1$.
4. Rewrite using $2^{4k+4+1} = 2^{4k+1} \times 2^4$ and expand 2^4 . As we want $16(2^{4k+1} + 3) = 16 \times 2^{4k+1} + 16 \times 3$ it is necessary also subtract 16×3 to maintain the equality.
Substitute $2^{4k+1} + 3 = 5m$ to show that the claim is also true for $n = k + 1$.

Alternatively:

If $2^{4k+1} + 3 = 5m$ then we can substitute $2^{4k+1} = 5m - 3$ to show that the claim is also true for $n = k + 1$.

5. Write a statement to explain that the statement is true for $n = k + 1$ if it is true for $n = k$.
6. Write the concluding statement.

WRITE

$$\begin{aligned}\text{If } n = 1, \quad 2^{4n+1} + 3 &= 2^5 + 3 \\ &= 32 + 3 \\ &= 35\end{aligned}$$

As 35 is divisible by 5, the statement is true for $n = 1$.

Assume that it is true for some $n = k$.
 $2^{4k+1} + 3 = 5m$

$$\begin{aligned}2^{4(k+1)+1} + 3 &= 2^{4k+4+1} + 3 \\ 2^{4k+4+1} + 3 &= 2^{4k+1} \times 2^4 + 3 \\ &= 16 \times 2^{4k+1} + 3 \\ &= 16(2^{4k+1} + 3) - 3 \times 16 + 3 \\ &= 16(2^{4k+1} + 3) - 48 + 3 \\ &= 16 \times 5m - 45 \\ &= 5(16m - 9)\end{aligned}$$

Alternatively:

$$\begin{aligned}2^{4(k+1)+1} + 3 &= 2^{4k+4+1} + 3 \\ &= 2^{4k+1} \times 2^4 + 3 \\ &= (5m - 3) \times 16 + 3 \\ &= 16 \times 5m - 3 \times 16 + 3 \\ &= 16 \times 5m - 3 \times 15 \\ &= 5(16m - 9)\end{aligned}$$

This is divisible by 5, therefore $2^{4(k+1)+1} + 3$ is divisible by 5 if $2^{4k+1} + 3$ is divisible by 5.

$2^{4n+1} + 3$ is divisible by 5 if $n = 1$. It is true for $n = k + 1$ if it is true for $n = k$.

Therefore, the statement is true $\forall n \geq 1$.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 2

Proof of divisibility Summary screen and practice questions

Exercise 1.3 Proof of divisibility

Technology free

1. Consider the statement that $6^n + 4$ is divisible by 5, $n \in \mathbb{Z}^+ \cup \{0\}$.
 - a. Demonstrate that the statement is true for the first three possible values of n .
 - b. If $6^k + 4 = 5m$, $m \in \mathbb{Z}$, demonstrate that $6^{k+1} + 4$ is divisible by 5.
2. Consider the statement that $8^n - 1$ is divisible by 7, $n \in \mathbb{Z}^+$.
 - a. Demonstrate that the statement is true for the first three possible values of n .
 - b. If $8^k - 1 = 7m$, $m \in \mathbb{Z}$, demonstrate that $8^{k+1} - 1$ is divisible by 7.
3. Consider the statement: $3^{2n-1} + 1$ is divisible by 4, $n \in \mathbb{Z}^+$.
 - a. Demonstrate that the statement is true for the first possible value of n .
 - b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
 - c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
 - d. Explain how you have demonstrated that $3^{2n-1} + 1$ is divisible by 4, $n \in \mathbb{Z}^+$.
4. Consider the statement: $10^n - 4$ is divisible by 12, $n \geq 2$.
 - a. Demonstrate that the statement is true for the first possible value of n .
 - b. Assume that the statement is true for $n = k$. Document this assumption by writing a suitable mathematical equation.
 - c. Use the assumption from part **b** to demonstrate that if the statement is true for $n = k$, it will also be true for $n = k + 1$.
 - d. Explain how you have demonstrated that $10^n - 4$ is divisible by 12, $n \geq 2$.
5. **WE4** Use a proof by induction to demonstrate that $n^3 + 2n$ is divisible by 3, $n \geq 1$.
6. Use a proof by induction to demonstrate that $3^{4n} - 1$ is divisible by 80, $n \geq 1$.
7. Use a proof by induction to demonstrate that $5^n + 2 \times 11^n$ is divisible by 3, $n \geq 0$.
8. **WES** Use a proof by induction to demonstrate that $9 \mid (4^n + 15n - 1)$, $n \geq 1$.
9. Prove, using a proof by induction, that $9 \mid (4^n + 5^n)$, n is odd.
10. Prove, using a proof by induction, that $5 \mid (2^{3n} - 3^n)$, $n > 0$.
11. Prove, using a proof by induction, that $21 \mid (4^{n+1} + 5^{2n-1})$, $n > 0$.
12. Prove, using a proof by induction, that $11 \mid (10^n - (-1)^n)$, $n \geq 1$.
13. Prove, using a proof by induction, that $19 \mid \left(\frac{5}{4} \times 8^n + 3^{3n-1} \right)$, $n \in \mathbb{Z}^+$.
14. Prove by induction that $9 \mid [(3n + 1)7^n - 1]$, $n \in \mathbb{Z}^+$.

1.4 Further proof by induction

1.4.1 Using other notations

In Worked example 1, we proved that the statement $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$, $n \geq 1$ is true. We then

looked at how it is possible to write the statement using $\sum$ notation, namely $\sum_{r=1}^n r = \frac{n(n+1)}{2}$. It is not

strictly necessary to write $\sum_{r=1}^n, \sum_1^n$ can be written if it is clear from the context that the variable is r . Although each question and example completed so far has used the variable r , any variable can be used.

As the statement to be proved is a proposition, this can be abbreviated using the capital letter P followed by the variable to be considered in brackets.

WORKED EXAMPLE 6

Consider proposition $P(n)$: $\sum_1^n i = \frac{n(n+1)}{2}$.

- a. Identify $P(1)$.
- b. Identify $P(k)$.
- c. Identify term $k+1$ in $\sum_1^n i$.
- d. If $P(k)$ is true, demonstrate that $P(k+1)$ is true.

THINK

- a. $P(1)$ means that $n = 1$.
- b. $P(k)$ means that $n = k$.
- c. As i is variable in the summation, term $k+1$ will occur when $i = k+1$.
- d. Add term $k+1$ to both sides of $P(k)$ and demonstrate that the sum is equal to $\frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$.

WRITE

a. $P(1): \sum_1^1 i = \frac{1(2)}{2}$

$P(1): \sum_1^1 i = 1$

b. $P(k): \sum_1^k i = \frac{k(k+1)}{2}$

c. $i = k+1$

d. $P(k+1): \sum_1^{k+1} i = \sum_1^k i + (k+1)$
 $= \frac{k(k+1)}{2} + (k+1)$
 $= (k+1) \left(\frac{k}{2} + 1 \right)$
 $= (k+1) \left(\frac{k}{2} + \frac{2}{2} \right)$
 $= \frac{(k+1)(k+2)}{2}$

If $P(k)$ is true, then $P(k+1)$ is true.

1.4.2 Forming a proposition and proving it by induction

Sometimes it is necessary to first determine the proposition and then use proof by induction to demonstrate that the proposition is true.

WORKED EXAMPLE 7

In the Tower of Hanoi puzzle, disks from one tower need to be moved to another tower according to the following rules.



1. Only one disk can be moved at a time.
2. A disk cannot be placed on a smaller disk.

For a tower with n disks:

- a. Determine a rule for the minimum number of moves, M_n required to move disks from one tower to another tower.
- b. Use a proof by induction to prove your rule.

THINK

- a. 1. If there was one disk, it could be moved to a new tower with 1 move. Record this data in a table.
2. If there were two disks, number them d_1 and d_2 where $d_1 < d_2$.
Initially, d_1 would be on top and could be moved to tower 2.
 d_2 could be moved to tower 3.
 d_1 could be moved on top of d_2 and the problem is complete.
This took 3 moves. Record the data in the table.
3. Continue to explore the problem for an increasing number of disks and record the data in a table.
4. Observe that M_n is always 1 less than 2^n .
This means that the rule can be written as $M_n = 2^n - 1$.
- b. 1. Use a proof by induction to prove that the rule is correct. From the investigation, we know that the formula works for $n = 1$.
2. There are some values of n that make the formula true. Write down the formula for $n = k$.

WRITE

a.

n	M_n
1	1

n	M_n
1	1
2	3

n	M_n
1	1
2	3
3	7
4	15

n	$M_n = 2^n - 1$
1	$1 = 2^1 - 1$
2	$3 = 2^2 - 1$
3	$7 = 2^3 - 1$
4	$15 = 2^4 - 1$

- b. From the table, the formula is true for $n = 1$.

Assume this is true for $n = k$.

$$M_k = 2^k - 1$$

3. If it takes $2^k - 1$ moves to move k discs, determine how many moves it will take to move $k + 1$ discs.

4. If the rule is true for $n = k + 1$, simplify the expression to demonstrate that $M_{k+1} = 2^{k+1} - 1$. Write a statement to explain what you have demonstrated.

5. Write the concluding statement.

If all $k + 1$ disks are on the first tower, it will take $2^k - 1$ moves to move the top k disks onto the second tower. It will take 1 move to move the largest disk to the third tower and then $2^k - 1$ moves to move the other discs from tower 2 to tower 3 and complete the problem. Therefore,

$$\begin{aligned} M_{k+1} &= (2^k - 1) + 1 + (2^k - 1) \\ &= 2^k + 2^k - 1 \\ &= 2 \times 2^k - 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

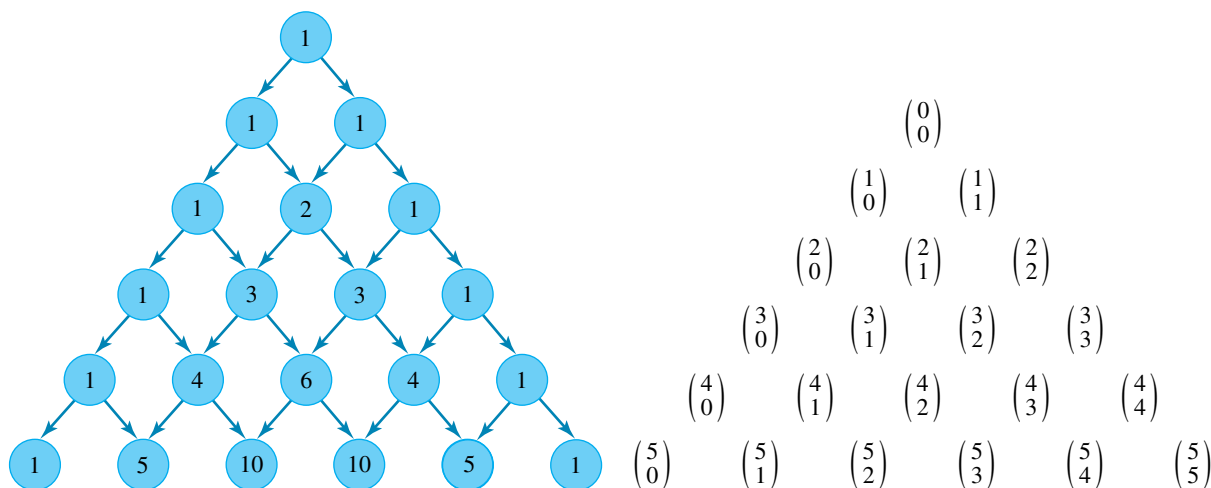
If the statement is true for $n = k$, it is also true for $n = k + 1$.

If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, by mathematical induction, the statement is true for $n \geq 1$.

The Tower of Hanoi puzzle was invented in 1883 by the French mathematician Edouard Lucas (1842 – 1891). In the original puzzle, there were 64 gold discs that needed to be moved.

1.4.3 Pascal's triangle

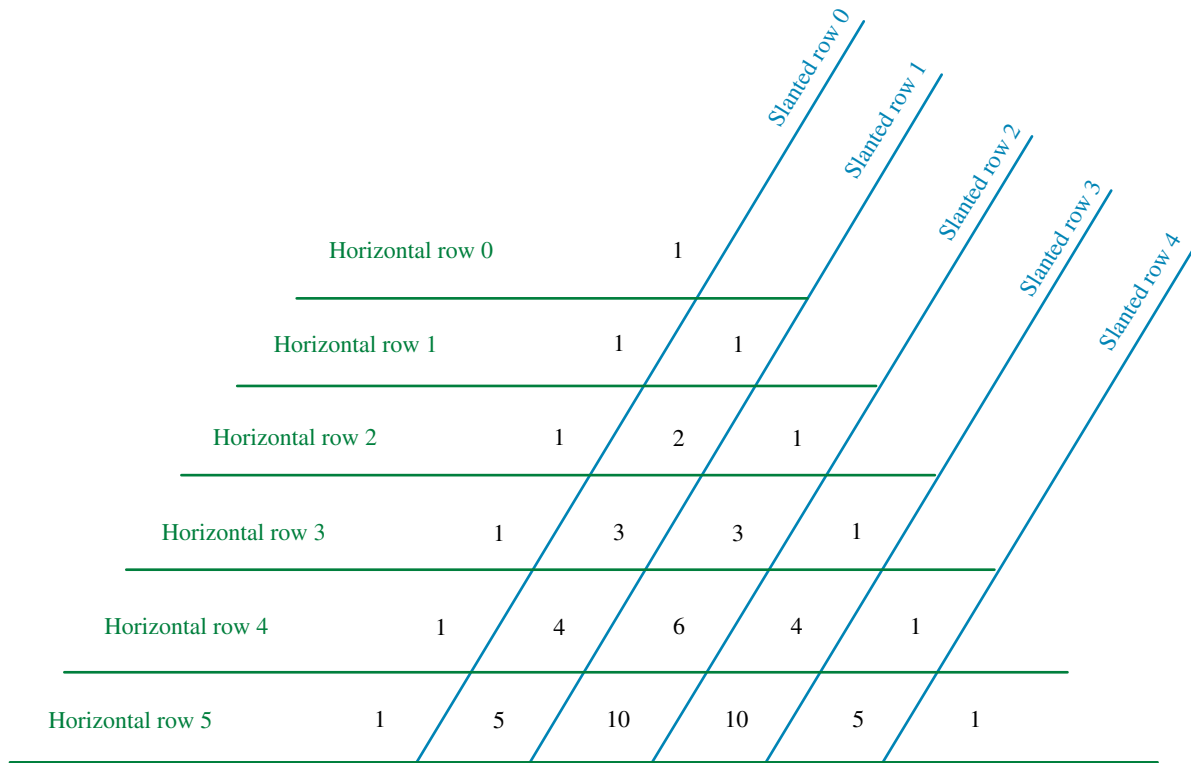
In Unit 1 you were introduced to Pascal's triangle – which links the mathematical fields of combinations, probability and the binomial theorem. Recall that each row of Pascal's triangle begins and ends with a 1 and every other element is generated by adding the pair of elements directly above it.



These elements can also be represented using combinatorial notation ${}^nC_r$ or $\binom{n}{r}$.

This leads to Pascal's identity: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ which will be proven below.

There are many patterns that can be found in Pascal's triangle. In order to talk about them, we need to be able to number the horizontal and slanted rows.



WORKED EXAMPLE 8

Prove, using mathematical induction, Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

THINK

1. Substitute $r = 1$ into the equation to demonstrate Pascal's identity is true for $r = 1$.

Recall the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2. There are some values of n that make the formula true. Write down the formula for $r = k$. Demonstrate this assumption using both combinatorial notation and factorial notation.

WRITE

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

$$\text{LHS: } \binom{n}{1} = \frac{n!}{1!(n-1)!}$$

$$= n$$

$$\text{RHS: } \binom{n-1}{0} + \binom{n-1}{1} = 1 + \frac{(n-1)!}{1!(n-2)!}$$

$$= 1 + (n-1)$$

$$= n$$

The identity is true for $r = 1$.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$

3. Knowing that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

substitute $r = k + 1$ and use factorial notation to write the identity.

$$\begin{aligned} \binom{n}{k+1} &= \binom{n-1}{k} + \binom{n-1}{k+1} \\ \frac{n!}{(k+1)!(n-k-1)!} &= \frac{(n-1)!}{k!(n-k-1)!} \\ &\quad + \frac{(n-1)!}{(k+1)!(n-k-2)!} \end{aligned}$$

4. If the identity is true for $r = k$, demonstrate that it must also be true for $r = k + 1$.

$$\begin{aligned} \text{LHS: } \binom{n}{k+1} &= \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{n!(n-k)}{(k+1)k!(n-k)(n-k-1)!} \\ &= \frac{(n-k)}{(k+1)} \frac{n!}{k!(n-k)!} \end{aligned}$$

Substitute

$$\begin{aligned} \frac{n!}{k!(n-k)!} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ \binom{n}{k+1} &= \frac{n-k}{k+1} \left(\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \right) \\ &= \frac{(n-k)(n-1)!}{(k+1)(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(k+1)k!(n-k-1)!} \\ &= \frac{(n-1)k(n-1)!}{(k+1)k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{(k+1)k!(n-k-1)(n-k-2)!} \\ &= \frac{k(n-1)!}{(k+1)k!(n-k-1)!} + \frac{(n-k)(n-1)!}{(n-k-1)(k+1)!(n-k-2)!} \\ &= \frac{k}{k+1} \frac{(n-1)!}{k!(n-k-1)!} + \frac{n-k}{n-k-1} \frac{(n-1)!}{(k+1)!(n-k-2)!} \\ \text{As } \frac{k}{k+1} &= \frac{k+1-1}{k+1} = 1 - \frac{1}{k+1} \text{ and } \frac{n-k}{n-k-1} = \frac{n-k-1+1}{n-k-1} = 1 + \frac{1}{n-k-1} \\ \binom{n}{k+1} &= \left(1 - \frac{1}{k+1} \right) \frac{(n-1)!}{k!(n-k-1)!} + \left(1 + \frac{1}{n-k-1} \right) \frac{(n-1)!}{(k+1)!(n-k-2)!} \\ &= \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k+1)k!(n-k-1)!} + \frac{(n-1)!}{(k+1)!(n-k-2)!} \\ &\quad + \frac{(n-1)!}{(n-k-1)(k+1)!(n-k-2)!} \\ &= \frac{(n-1)!}{k!(n-k-1)!} - \frac{(n-1)!}{(k+1)!(n-k-1)!} + \frac{(n-1)!}{(k+1)!(n-k-2)!} \\ &\quad + \frac{(n-1)!}{(k+1)!(n-k-1)!} \\ &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k+1)!(n-k-2)!} \\ &= \binom{n-1}{k} + \binom{n-1}{k+1} \end{aligned}$$

5. Write the concluding statement.

If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, by mathematical induction, the statement is true.

study on

Units 3 & 4

Area 1

Sequence 1

Concept 3

Further proof by induction Summary screen and practice questions

Exercise 1.4 Further proof by induction

Technology free

1. **WE6** Consider proposition $P(n): \sum_{i=1}^n 2^i = 2^{n+1} - 2$.

a. Identify $P(1)$.

b. Identify $P(k)$.

c. Identify term $k + 1$ in $\sum_{i=1}^n 2^i$.

d. If $P(k)$ is true, demonstrate that $P(k + 1)$ is true.

2. Consider proposition $P(n): \sum_{i=1}^n 2i = n(n + 1)$.

a. Identify $P(1)$.

b. Identify $P(k)$.

c. Identify term $k + 1$ in $\sum_{i=1}^n 2i$.

d. If $P(k)$ is true, demonstrate that $P(k + 1)$ is true.

3. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^n 3^i = \frac{3^{n+1} - 3}{2}$ is true.

4. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^n 2 \times 3^{i-1} = 3^n - 1$ is true.

5. Use a proof by induction to demonstrate that $P(n): \sum_{i=1}^n (3i - 1) = \frac{n(3n + 1)}{2}$

6. Consider the proposition $P(n): \sum_{r=2}^n (r - 1)r = \frac{(n - 1)n(n + 1)}{3}$.

a. Demonstrate that $P(2)$ is true.

b. Identify $P(k)$.

c. Identify $P(k + 1)$.

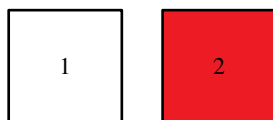
d. Use a proof by induction to prove $P(n)$.

7. **WE7** In a puzzle similar to the Tower of Hanoi, a series of squares can be either red, white or blue. The squares can change colour according to the following rules.

1. If a square is going to change colour, it must be the leftmost square of its current colour.
2. When a square changes colour, it must be the leftmost square of the new colour.
All squares begin as the same colour (for example red) and the puzzle ends when they are all the same colour, but not the original colour (for example blue).
For 2 red squares



3. Move 1: Only square 1 can change colour as it is the left-most red square. Let's make it white.

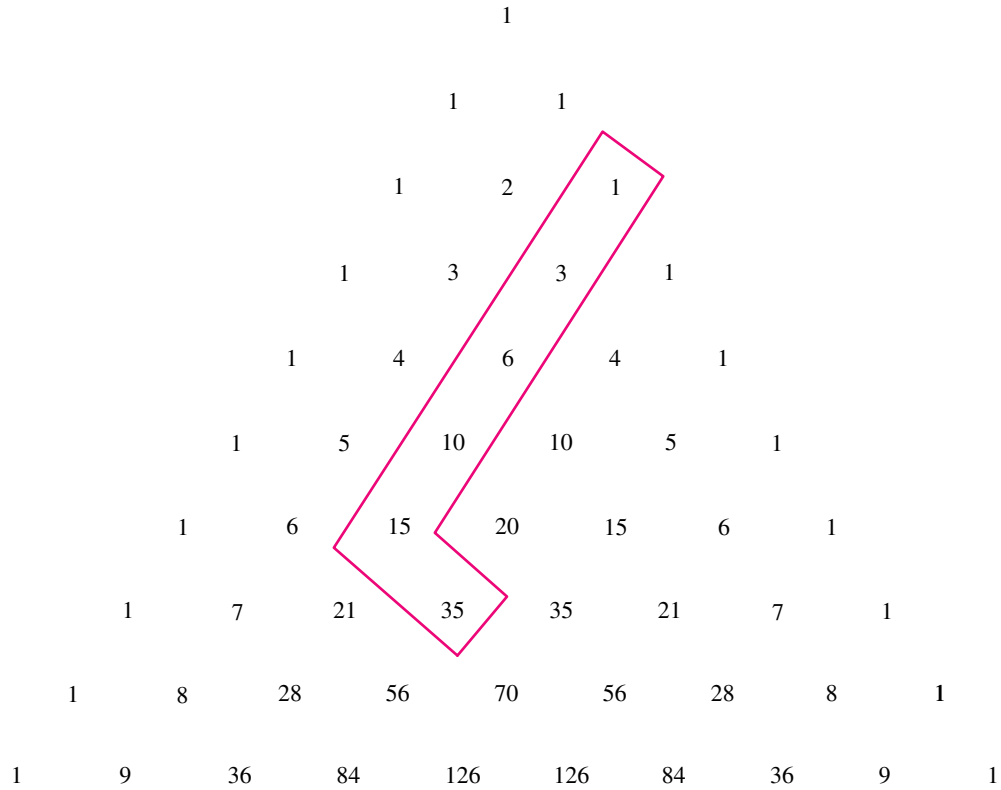


Move 2: Either square can change colour. As 1 is the left-most white square, it could become red or blue. As 2 is the left-most red square, it could become blue, but it cannot become white because square 1 is already white and to the left of square 2.

For a series of n squares:

- a. Determine a rule for the minimum number of moves, M_n required to change the colour of all squares according to the rules.
 - b. Use a proof by induction to prove your rule.
8. **WE8** Use mathematical induction to demonstrate that the sum of the entries in the n th horizontal row of Pascal's Triangle is 2^n . You may use Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ in your proof.
 9. Use mathematical induction to prove the binomial expansion $(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}a^0b^n$. You may use Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ in your proof.
 10. The hockey stick formula involves adding the first r terms of slanted row n in Pascal's triangle. An example of a possible 'hockey stick' is shown next page.
 - a. Determine a rule for the sum of the first r terms of slanted row 2 in Pascal's triangle.
 - b. Use a proof by induction to prove your rule. You may use Pascal's Identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ in your proof.
 11. The hockey stick formula can be generalised for adding the first r terms of slanted row n in Pascal's triangle.
 - a. Determine a rule for the sum of the first r terms of slanted row n in Pascal's triangle.
 - b. Use a proof by induction to prove your rule for a given slanted row n .
 12. Demonstrate al-Karaji's proof of $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$. Use this proof for the sum of n numbers to prove by induction that $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$.
 13. Use mathematical induction to prove that $\sum_{r=1}^n r^4 = \frac{6n^5 + 15n^4 + 10n^3 - n}{30}$.

14. The Fibonacci numbers 1, 1, 2, 3, 5, ... can be found using the recursive formula $F_{n+1} = F_n + F_{n-1}$ where $F_1 = F_2 = 1$. Use mathematical induction to prove that the Fibonacci numbers can be found using the formula $F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$.



1.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Use mathematical induction to prove that $\sum_{r=1}^n (r+1) = \frac{n(n+3)}{2}$.
2. Use mathematical induction to prove that $\sum_{r=1}^n 1.5^{r-1} = 2 \times 1.5^n - 2$.
3. Prove by induction that $\sum_{r=1}^n (3r-2) = \frac{n(3n-1)}{2}$.
4. Use a proof by induction to prove that $3 + 9 + 27 + \dots + 3^n = \frac{3^{n+1} - 3}{2}$.
5. Use a proof by induction to prove that $P(n): \sum_{r=1}^n \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$.
6. Prove by induction $P(n): (1+x)^n \geq 1+nx$.
7. Use proof by induction to demonstrate that $3^{2n} - 1$ is divisible by 8 for $n \geq 1$.
8. Prove by induction that $7 \mid (12^n + 2(5^{n-1})), n \geq 1$.
9. Use a proof by induction to demonstrate that $9 \mid (4^n + 6n - 1), n \geq 1$.
10. Prove by induction that $9 \mid (10^n - 1), n \geq 1$.
11. Prove by induction that $2^{n+2}3^n + 5n - 4$ is divisible by 25 for $n \in \mathbb{N}$.
12. Use a proof by induction to prove that $7^n - 2^n$ is divisible by 5 for $n \in \mathbb{Z}^+$.

Complex familiar

13. Use proof by induction to demonstrate that $19 \mid (5^{2n+1} + 3^{n+2} \times 2^{n-1}), n \geq 1$.
14. Use proof by induction to demonstrate that $\sum_{r=1}^n (r^2 + r) = \frac{n^3}{3} + n^2 + \frac{2n}{3}$.
15. Use proof by induction to demonstrate that $P(n): \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true.
16. Use proof by induction to demonstrate that $n^3 + 3n^2 + 2n$ is divisible by 6.

Complex unfamiliar

17. Prove by induction that $15 \mid (4^n + 5^n + 6^n)$ for odd n .
18. Use proof by induction to prove that $n^5 - n$ is divisible by 30 where $n \geq 2$.
19. Prove by induction that $11 \mid (5^{5n+1} + 4^{5n+2} + 3^{5n}), n \geq 0$.
20. Use induction to prove that $\sum_{i=1}^n (F_i)^2 = F_n F_{n+1}$ where F_i is the i th term in the Fibonacci sequence.

study on

Units 3 & 4 Sit exam

Answers

1 Proof by mathematical induction

Exercise 1.2 Introduction to proof by mathematical induction

1. a. $n = 1$. LHS = 2. RHS = $1(1 + 1) = 2$. LHS = RHS
 $n = 2$. LHS = $2 + 4 = 6$. RHS = $2(2 + 1) = 6$.
 LHS = RHS
 $n = 3$. LHS = $2 + 4 + 6 = 12$. RHS = $3(3 + 1) = 12$.
 LHS = RHS
 b. Sample responses can be found in the worked solutions in the online resources.
 2. a. $n = 1$. $\sum_{r=1}^n 2^{r-1} = 2^0 = 1$, $2^n - 1 = 2^1 - 1 = 1$. True
 $n = 2$. $\sum_{r=1}^n 2^{r-1} = 2^0 + 2^1 = 3$, $2^n - 1 = 2^2 - 1 = 3$. True
 $n = 3$. $\sum_{r=1}^n 2^{r-1} = 2^0 + 2^1 + 2^2 = 7$,
 $2^n - 1 = 2^3 - 1 = 7$. True
 b. Sample responses can be found in the worked solutions in the online resources.
 3. a. $n = 1$. LHS = 3. RHS = $n(n + 2) = 1(3) = 3$. True
 b. $3 + 5 + 7 + \dots (1 + 2k) = k(k + 2)$
 c. Sample responses can be found in the worked solutions in the online resources.
 d. If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, the statement is true for $n \geq 1$.
 4. a. $n = 1$. $\sum_{r=1}^n (2r - 1) = 2 - 1 = 1$. $n^2 = 1$. True
 b. $\sum_{r=1}^k (2r - 1) = k^2$
 c. Sample responses can be found in the worked solutions in the online resources.
 d. If the statement is true for $n = k$, it is also true for $n = k + 1$. The statement is true for $n = 1$. Therefore, the statement is true for $n \geq 1$.
- 5–14. Sample responses can be found in the worked solutions in the online resources.

Exercise 1.3 Proof of divisibility

1. a. $n = 0$, $6^n + 4 = 1 + 4 = 5$. Divisible by 5.
 $n = 1$, $6^n + 4 = 6 + 4 = 10$. Divisible by 5.
 $n = 2$, $6^n + 4 = 36 + 4 = 40$. Divisible by 5.
 b. Sample responses can be found in the worked solutions in the online resources.
2. a. $n = 1$, $8^n - 1 = 8 - 1 = 7$. Divisible by 7.
 $n = 2$, $8^n - 1 = 64 - 1 = 63$. Divisible by 7.
 $n = 3$, $8^n - 1 = 512 - 1 = 511 = 73 \times 7$. Divisible by 7.
 b. Sample responses can be found in the worked solutions in the online resources.
3. a. $n = 1$. $3^{2n-1} + 1 = 3 + 1 = 4$. Divisible by 4.
 b. $3^{2k-1} + 1 = 4m$, $m \in \mathbb{Z}$.
 c. Sample responses can be found in the worked solutions in the online resources.

- d. $3^{2n-1} + 1$ is divisible by 4 when $n = 1$. It is divisible by 4 for $n = k + 1$ if it is divisible by 4 for $n = k$. Therefore, it is divisible by 4 for $n \in \mathbb{Z}^+$.
 4. a. $n = 2$. $10^n - 4 = 100 - 4 = 96 = 12 \times 8$. Divisible by 12.
 b. $10^n - 4 = 12m$, $m \in \mathbb{Z}$.
 c. Sample responses can be found in the worked solutions in the online resources.
 d. $10^n - 4$ is divisible by 12 when $n = 2$. It is divisible by 12 for $n = k + 1$ if it is divisible by 12 for $n = k$. Therefore it is divisible by 12 for $n \geq 2$.
- 5–14. Sample responses can be found in the worked solutions in the online resources.

Exercise 1.4 Further proof by induction

1. a. $P(1): \sum_{i=1}^1 2^i = 2$
 b. $P(k): \sum_{i=1}^k 2^i = 2^{k+1} - 2$
 c. 2^{k+1}
 d. Sample responses can be found in the worked solutions in the online resources.
2. a. $P(1): \sum_{i=1}^1 2i = 2$
 b. $P(k): \sum_{i=1}^k 2i = k(k + 1)$
 c. $2(k + 1)$
 d. Sample responses can be found in the worked solutions in the online resources.
- 3–5. Sample responses can be found in the worked solutions in the online resources.
6. a. LHS: $\sum_{r=2}^2 (r - 1)r = (2 - 1) \times 2 = 2$
 RHS: $\frac{(2 - 1)2(2 + 1)}{3} = 2$. $P(2)$ is true.
 b. $P(k): \sum_{r=2}^k (r - 1)r = \frac{(k - 1)k(k + 1)}{3}$
 c. $P(k + 1): \sum_{r=2}^{k+1} (r - 1)r = \frac{k(k + 1)(k + 2)}{3}$
 d. Sample responses can be found in the worked solutions in the online resources.
7. a. $M_n = 2^n - 1$
 b. Sample responses can be found in the worked solutions in the online resources.
- 8, 9. Sample responses can be found in the worked solutions in the online resources.
10. a. $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \dots + \binom{2+r-1}{2} = \binom{2+r}{3}$
 b. Sample responses can be found in the worked solutions in the online resources.

11. a. $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \dots + \binom{n+r-1}{n} =$
 $\binom{n+r}{n+1}$

- b. Sample responses can be found in the worked solutions in the online resources.

12–14. Sample responses can be found in the worked solutions in the online resources.

1.5 Review: exam practice

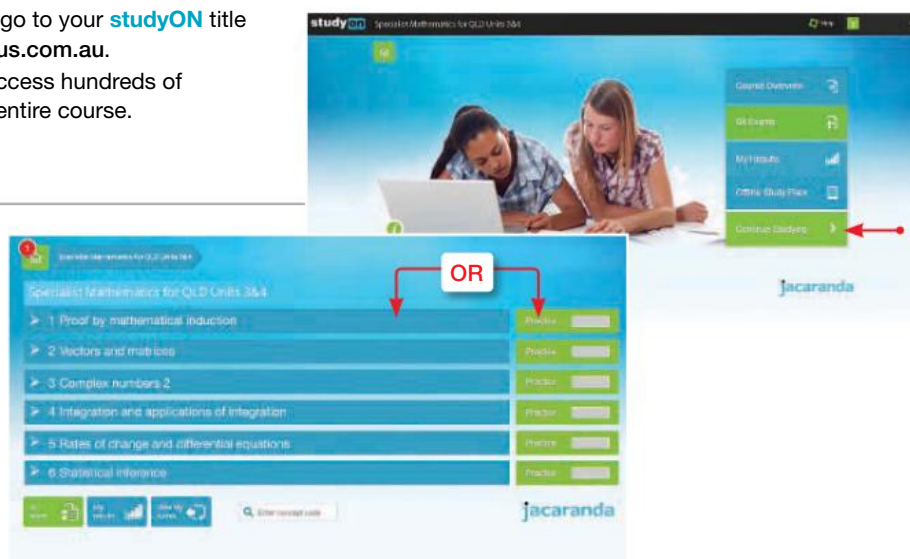
1–20. Sample responses can be found in the worked solutions in the online resources.

REVISION UNIT 3 Mathematical induction, and further vectors, matrices and complex numbers

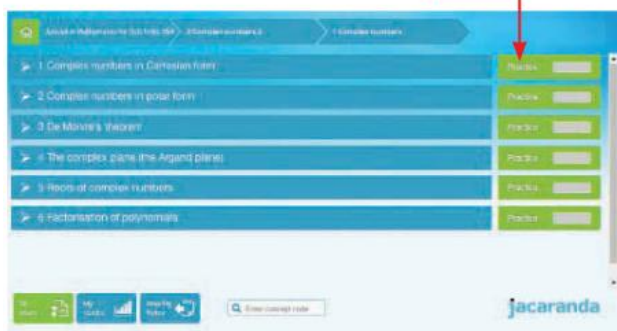
TOPIC 1 Proof by mathematical induction

- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



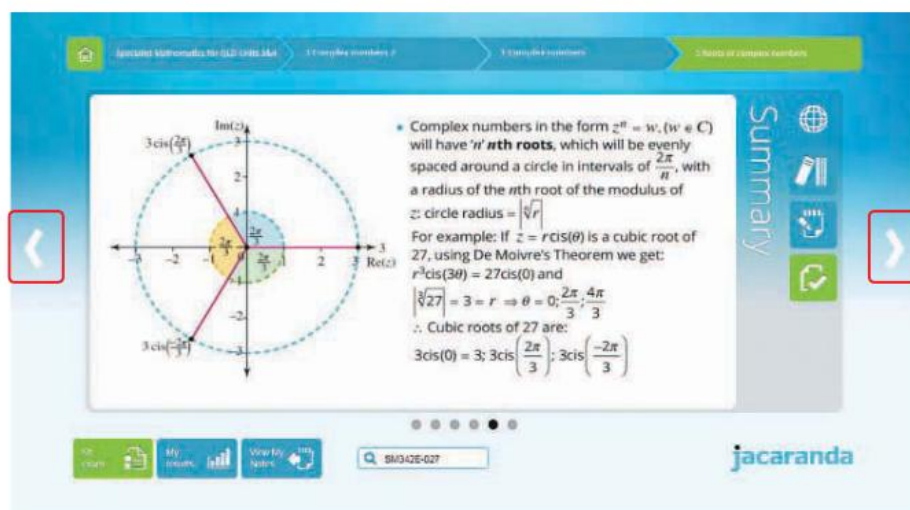
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



2 Vectors in three dimensions

2.1 Overview

Until this point, the study of vectors has been restrained to only two dimensions. By extending our knowledge to three dimensions, the list of applications is endless. The product of two vectors can extend from simply being a scalar quantity (dot product), to that of a vector product which is a vector. The vector product is valuable in determining areas and volumes of specific shapes, the turning force (torque) applied to an object at a specific point and is central in developing the equations of planes. By extending our study of vectors to three dimensions, it allows us to strengthen our understanding of Cartesian equations and expand our knowledge through the introduction to parametric equations.

Parametric equations link our conventional understanding of equations of two variable with a third, known as a parameter. The famous Brachistochrone problem incorporates three variables and was posed by the Mathematician James Bernoulli (1654–1705). Bernoulli was determined to identify the path (x, y) in which a particle could slide from two points (P and Q) in the least possible time (t). Challenging the likes of fellow scholars such as Newton and L'Hopital, Bernoulli found that if the particle followed a path in the shape of an inverted cycloid, it would reach its destination in the shortest possible time. Applications of equations involving the parameter t are inherent in modelling problems such as the vector equation of a straight line in three dimensions and determining points of collision between the paths of two or more particles.



LEARNING SEQUENCE

- 2.1 Overview
- 2.2 Introduction to vectors in three dimensions
- 2.3 Geometric proofs using vectors
- 2.4 Cartesian and parametric equations
- 2.5 The vector equation of a straight line
- 2.6 The vector product
- 2.7 Applications of vectors
- 2.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Introduction to vectors in three dimensions

Introduced in Unit 1, vectors have thus far only been considered in two-dimensional form. In this chapter, following a review of two-dimensional vectors, our study of vectors will be extended to three-dimensions.

2.2.1 Review of vectors in two-dimensions

Scalar and vector quantities

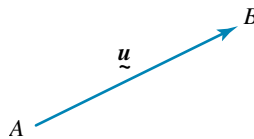
A **scalar** quantity is a real number only, it has magnitude, but no direction. Scalar quantities include temperature, mass and time.

A **vector** is a quantity that has **magnitude** and **direction**. Vectors include weight in a particular direction and displacement with a specific direction.

Vectors are directed line segments with a tail (start) and head (end). In the figure below, the head of the vector is at point B (indicated with an arrow), while the tail is at point A . In this instance, the vector could be expressed in a variety of forms.

Various forms of vector notation

$$\overrightarrow{AB} = \underline{u} = u$$



The *tilde* symbol ($\sim$) is commonly used because drawing bold letters is often difficult.

Equality of vectors

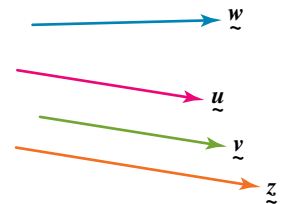
Two vectors are equal if *both* their magnitude and direction are equal.

In this figure, the following statements can be made:

$\underline{u} = \underline{v}$ (direction and magnitude equal)

$\underline{u} \neq \underline{w}$ (directions are not equal)

$\underline{u} \neq \underline{z}$ (magnitudes are not equal)



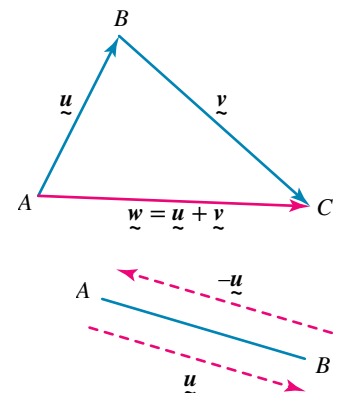
Addition and subtraction of vectors

The triangle rule of vectors can be applied in both the addition and subtraction of vectors.

To add two vectors, take the tail of one vector and join it to the head of another. The result of this addition is the vector from the tail of the first vector to the head of the second vector.

If $\underline{u}$ is the vector from A to B , then $-\underline{u}$ is the vector from B to A .

To subtract two vectors, add the negative of the second vector to the first vector. For example, $A - B = A + (-B)$.



Multiplying a vector by a scalar

Multiplying a vector by a *positive* number (scalar) affects only the *magnitude* of the vector, not the direction.

If the scalar is negative, then the direction is reversed.

Cartesian form of a vector in two-dimensions

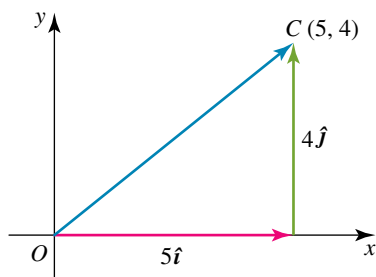
Thus far, points have been considered in 2-dimensions. A point, $P(x, y)$, is determined by its distances from two mutually perpendicular axes.

On the Cartesian plane, a **position vector** is the directed line segment joining the origin with the point P . It defines a point, P , by magnitude and direction relative to the origin, O , and can be written as $\overrightarrow{OP}$ or $\underline{u}$.

The position vector $\underline{u}$ is the sum of the two directional unit vectors, $\hat{i}$ and $\hat{j}$ where:

- $\hat{i}$ is a unit vector along the x -axis, with magnitude 1.
- $\hat{j}$ is a unit vector along the y -axis, with magnitude 1.

For example, if the point C has coordinates $(5, 4)$, then $\underline{u} = 5\hat{i} + 4\hat{j}$.



This is the **Cartesian form** of a vector, and is given by,

$$\underline{u} = \overrightarrow{OP} = x\hat{i} + y\hat{j}$$

The position vector, $\underline{u} = 5\hat{i} + 4\hat{j}$ can also be expressed as an ordered pair $(5, 4)$ or in column vector notation as, $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

The magnitude of a vector

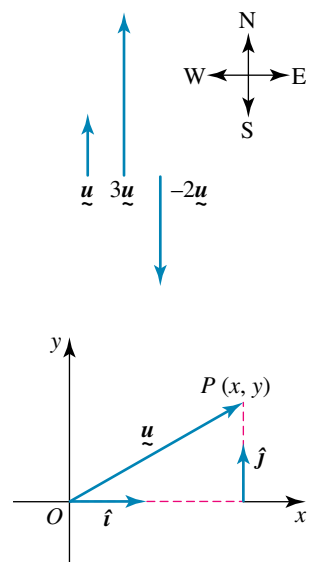
By using Pythagoras' theorem on a position vector, we can determine its length, or magnitude.

The magnitude of the vector $\underline{u} = x\hat{i} + y\hat{j}$ is also called the norm and is denoted as $|\underline{u}|$ or r .

$$|\underline{u}| = \sqrt{x^2 + y^2}$$

The magnitude of $\underline{u} = 5\hat{i} + 4\hat{j}$ can be calculated using the above formula

$$\begin{aligned} |\underline{u}| &= \sqrt{5^2 + 4^2} \\ &= \sqrt{41} \end{aligned}$$



The unit vector

The magnitude of a vector is also used to determine the **unit vector**. If $\underline{u}$ is the vector, then the unit vector, denoted by $\hat{\underline{u}}$, is given by,

$$\hat{\underline{u}} = \frac{\underline{u}}{|\underline{u}|}$$

For example, the unit vector for $\underline{u} = 5\hat{i} + 4\hat{j}$ is:

$$\begin{aligned}\hat{\underline{u}} &= \frac{\underline{u}}{|\underline{u}|} \\ &= \frac{5\hat{i} + 4\hat{j}}{\sqrt{41}} \\ &= \frac{5}{\sqrt{41}}\hat{i} + \frac{4}{\sqrt{41}}\hat{j} \quad \text{or} \quad \frac{1}{\sqrt{41}}(5\hat{i} + 4\hat{j})\end{aligned}$$

Polar form

In **polar form**, the vector $\underline{u} = x\hat{i} + y\hat{j}$, can be expressed in terms of its magnitude $|\underline{u}|$ and direction θ . The polar form of a vector $\underline{u}$ is $[r, \theta]$, where:

- r is the magnitude of $\underline{u}$
- θ is the anti-clockwise angle formed from the positive x -axis (polar axis)
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

The position vector $\underline{u} = 5\hat{i} + 4\hat{j}$ can be expressed in polar form as shown.

$$\underline{u} = 5\hat{i} + 4\hat{j}$$

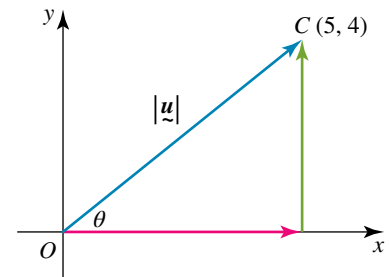
$$|\underline{u}| = \sqrt{41} \text{ (previously calculated)}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right)$$

$$\theta \approx 0.6747 \text{ radians}$$

$$\theta \approx 38.66^\circ$$

$$\therefore \underline{u} = [\sqrt{51}, 38.66^\circ]$$



Resolving vectors into components

If the two-dimensional vector $\underline{u}$ makes an angle of θ with the positive x -axis and it has a magnitude of r , then we can find its x - and y -components using the formulas:

$$\begin{aligned}x &= |r| \cos \theta \\ y &= |r| \sin \theta\end{aligned}$$

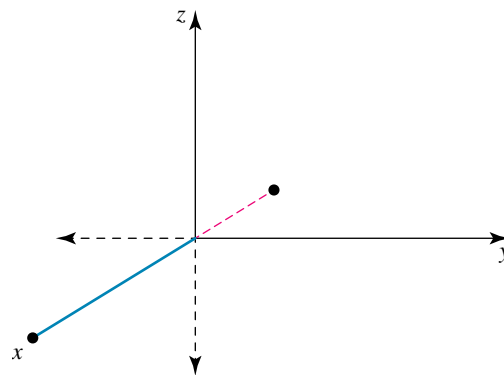
For example, the vector in polar form $\underline{v} = [5, 233.13^\circ]$, can be converted to component (Cartesian) form:

$$\begin{aligned}\underline{v} &= 5 \cos(233.13^\circ)\hat{i} + 5 \sin(233.13^\circ)\hat{j} \\ &= -3\hat{i} - 4\hat{j}\end{aligned}$$

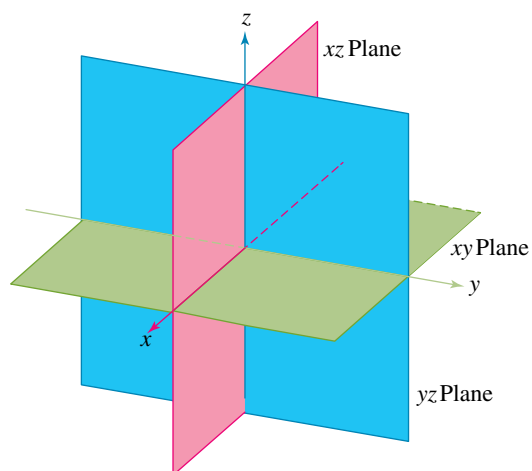
2.2.2 Introduction to vectors in three-dimensions

Cartesian coordinates for three-dimensional planes

In the three-dimensional coordinate system, the z -axis exists perpendicular to both the x - and y -axes at the origin. The three-dimensions are shown with the x -axis coming out of the page and the y - and z -axes lying flat on the page.

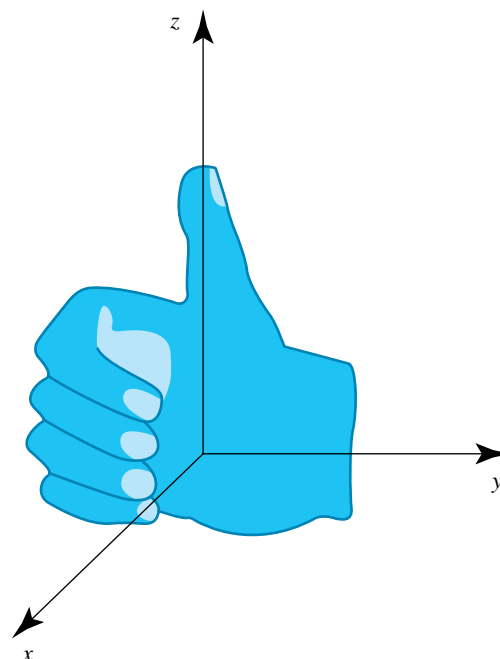


The space consists of 8 regions called octants that are divided by three perpendicular planes. Points in the first octant have positive values for x , y and z .

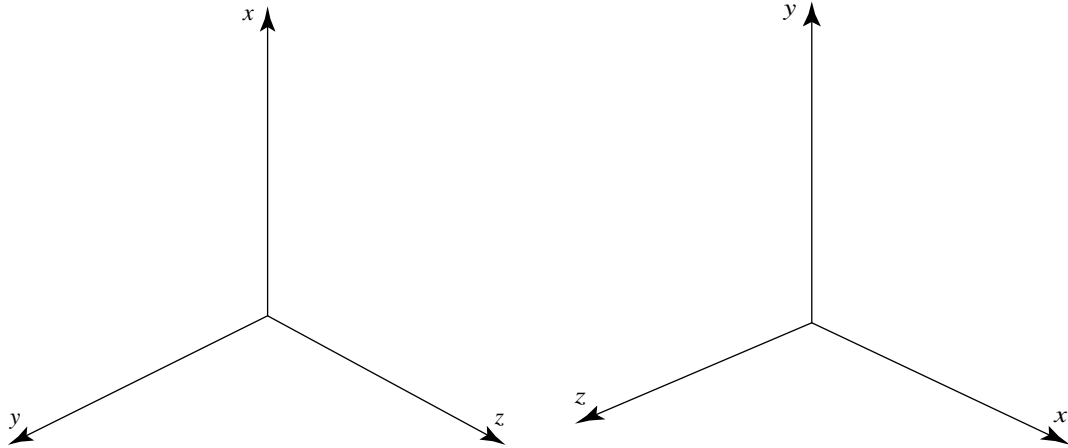


The orientation of the axes in the diagram indicates that it is a **right-handed coordinate system**. Systems can be either right- or left-handed in orientation, however the right-handed is the preferred option.

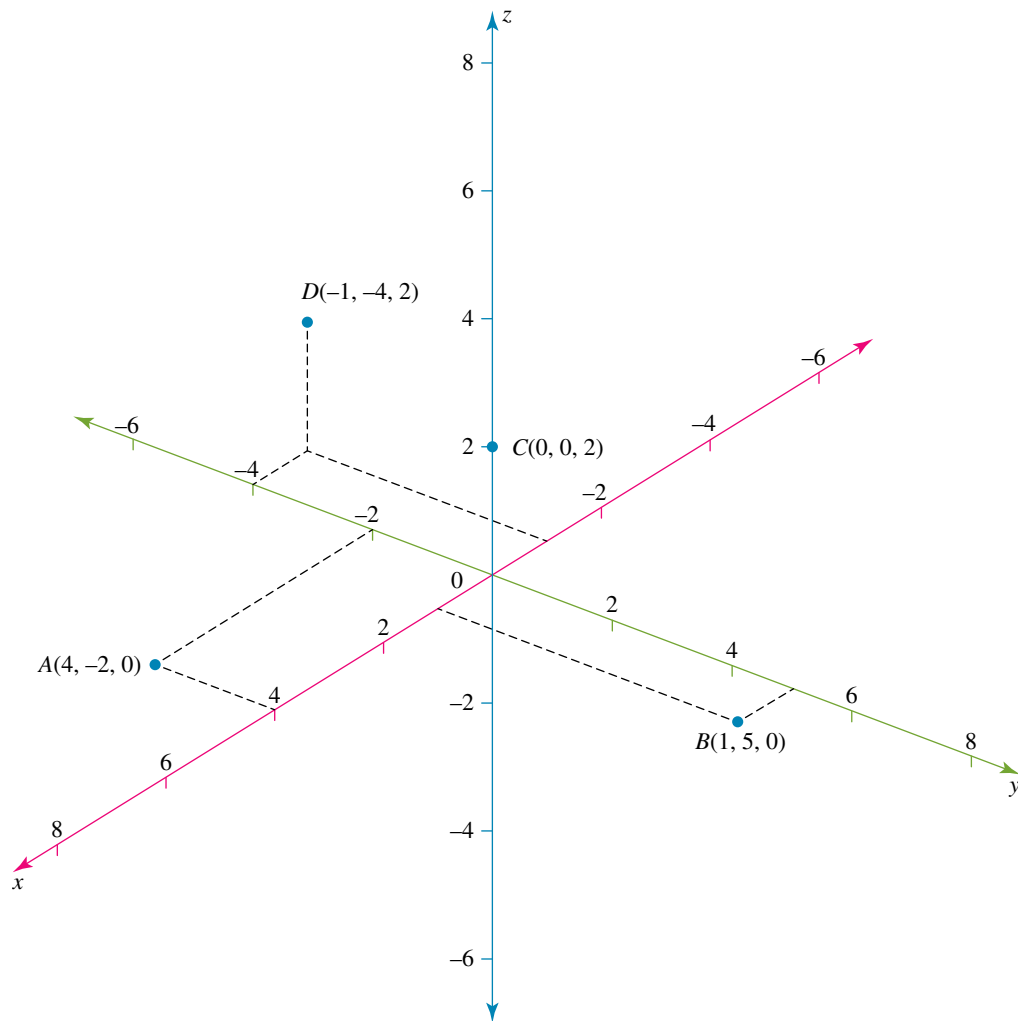
As the fingers on your right-hand curl from the x -axis to the y -axis, your thumb points in the direction of the positive z -axis.



The following arrangements are also examples of right-handed systems.



In three-dimensional space the coordinates (x, y, z) define a point $P(x, y, z)$, with each point made up of three components. Points $A(4, -2, 0)$, $B(1, 5, 0)$, $C(0, 0, 2)$ and $D(-1, -4, 2)$ are shown in the diagram.



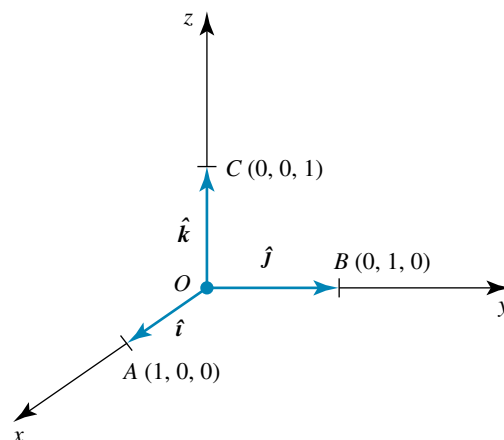
The location of a point relative to the origin can be expressed as the position vector $\overrightarrow{OP}$, in terms of the unit vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$. These are known as the **standard unit vectors**:

Let $\hat{i}$ be a unit vector in the positive direction, parallel to the x -axis, with magnitude 1.

Let $\hat{j}$ be a unit vector in the positive direction, parallel to the y -axis, with magnitude 1.

Let $\hat{k}$ be a unit vector in the positive direction, parallel to the z -axis, with magnitude 1.

Points A , B and C , can be expressed in **component form** in terms of the standard unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.



$$\begin{aligned}\hat{i} &= (1, 0, 0) & \therefore \overrightarrow{OA} &= \hat{i} \\ \hat{j} &= (0, 1, 0) & \therefore \overrightarrow{OB} &= \hat{j} \\ \hat{k} &= (0, 0, 1) & \therefore \overrightarrow{OC} &= \hat{k}\end{aligned}$$

In three-dimensions, a position vector beginning at the origin $O(0, 0, 0)$ and terminating at the point $P(x, y, z)$ is defined as,

$$\begin{aligned}\overrightarrow{OP} &= x\hat{i} + y\hat{j} + z\hat{k} \\ \underline{u} &= x\hat{i} + y\hat{j} + z\hat{k}\end{aligned}$$

A vector expressed in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$ is said to be in component form (or Cartesian form), and the process is called the **resolution of vectors**.

The position vector $\overrightarrow{OP}$ can be determined using a vector sum.

$$\begin{aligned}\therefore \overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP} \\ &= x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\end{aligned}$$

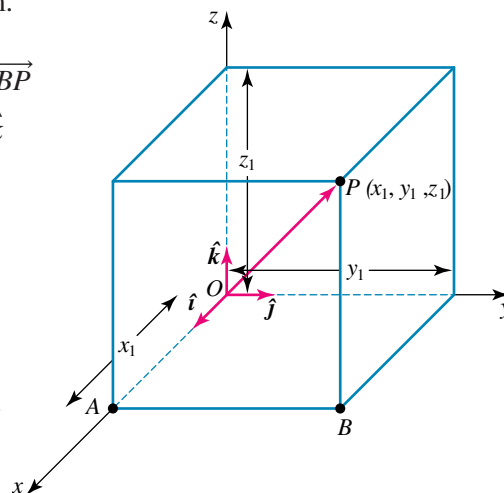
Magnitude and unit vectors in three-dimensions.

The magnitude of the three-dimensional vector $\underline{u} = x\hat{i} + y\hat{j} + z\hat{k}$ is defined by the formula,

$$|\underline{u}| = \sqrt{x^2 + y^2 + z^2}$$

If $\underline{u}$ is the vector, then the unit vector denoted by $\hat{u}$, is given by,

$$\hat{u} = \frac{\underline{u}}{|\underline{u}|}$$



WORKED EXAMPLE 1

If the point P has the coordinates $(-7, 3, -4)$, determine:

- The vector $\overrightarrow{OP}$
- The magnitude of $\overrightarrow{OP}$
- The unit vector parallel to $\overrightarrow{OP}$

THINK

- The $\hat{i}, \hat{j}, \hat{k}$ components of the vector $\overrightarrow{OP}$ correspond to the x -, y - and z -coordinates respectively.
- Recall that the magnitude of the vector $\underline{u} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by $|\underline{u}| = \sqrt{x^2 + y^2 + z^2}$.
1. Let $\overrightarrow{OP} = \underline{p}$, the unit vector parallel to $\overrightarrow{OP}$ is notated as $\hat{p}$. Recall the unit vector formula $\hat{p} = \frac{\underline{p}}{|\underline{p}|}$.
2. The unit vector is often expressed as a scalar multiple of $\underline{p}$ as shown here.

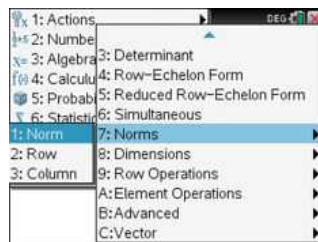
WRITE

- $P(-7, 3, -4)$ has coordinates $x = -7, y = 3, z = -4$
 $\overrightarrow{OP} = -7\hat{i} + 3\hat{j} - 4\hat{k}$
- $|\overrightarrow{OP}| = \sqrt{(-7)^2 + (3)^2 + (-4)^2}$
 $= \sqrt{74}$
- $\hat{p} = \frac{\underline{p}}{|\underline{p}|}$
 $= \frac{-7\hat{i} + 3\hat{j} - 4\hat{k}}{\sqrt{74}}$
 $= \frac{1}{\sqrt{74}}(-7\hat{i} + 3\hat{j} - 4\hat{k})$

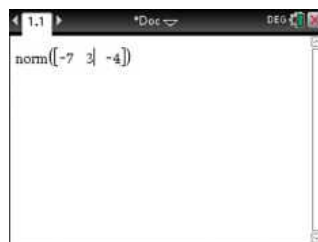
TI | THINK

1. On a Calculator page, select MENU 7: Matrix Vector 7: Norms 1: Norm

WRITE



2. Complete the entry line $([-7, 3, -4])$



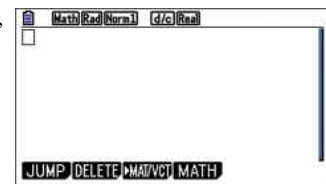
3. Press the ENTER button. The answer appears on the screen.



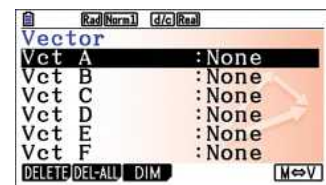
CASIO | THINK

1. On a Run-Matrix screen, select MAT/VCT

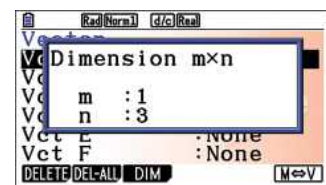
WRITE



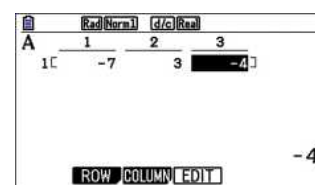
2. To switch between the vector and matrix screens, press the $M \leftrightarrow V$ button. Press the EXE button to define the Vct A.



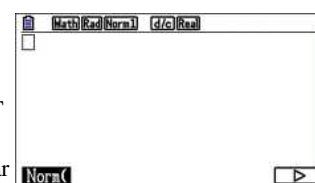
3. Complete the entry line as
1
3
Press the EXE button.



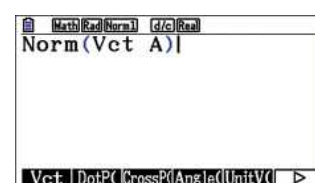
- Complete the entry line as
-7 3 -4
Press the EXIT button.



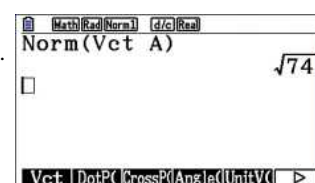
- Press the EXIT button to return to the Run-Matrix screen. Select OPTN MAT/VCT $\triangleright$ (x3 times)
The screen should appear as shown.



- Select, Norm $\triangleright$ (x3 times) Vct ALPHA X, θ , T)



- Press the EXE button to complete the calculation. The answer appears on the screen.



Note that the TI calculator can only display the magnitude in decimal form.

The vector between two points in three-dimensions

In the diagram opposite, $\underline{a} = \overrightarrow{OA}$ is the position vector of point A and $\underline{b} = \overrightarrow{OB}$ is the position vector of point B. The vector describing the location of A relative to B denoted by $\overrightarrow{BA}$, is found using vector addition as $-\underline{b} + \underline{a}$ or $\underline{a} - \underline{b}$.

The vector describing the location of B relative to A denoted by $\overrightarrow{AB}$, shown, is $\underline{b} - \underline{a}$.

If A and B are points defined by the position vectors $\underline{a}$ and $\underline{b}$ respectively, then

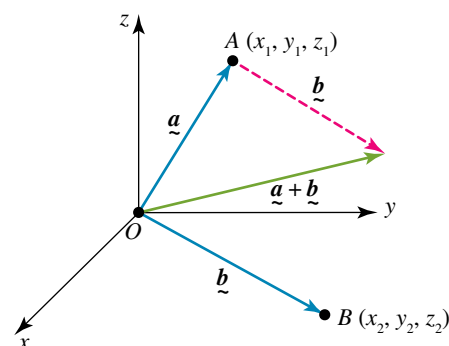
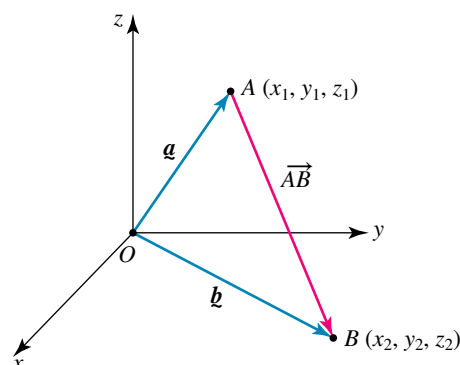
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \underline{b} - \underline{a}\end{aligned}$$

Addition and subtraction of three-dimensional vectors in component form

Given, $\underline{a} = (x_1, y_1, z_1)$ and $\underline{b} = (x_2, y_2, z_2)$

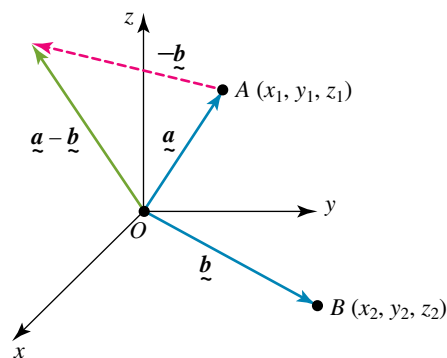
Addition of vectors in component form is:

$$\begin{aligned}\underline{a} + \underline{b} &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}\end{aligned}$$



Subtraction of vectors in component form is:

$$\begin{aligned}\underline{a} - \underline{b} &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) - (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}\end{aligned}$$



WORKED EXAMPLE 2

Three points, A , B and C have the coordinates $(-3, 7, -1)$, $(2, -2, 11)$ and $(8, -7, 12)$ respectively. Calculate:

- The vector $\overrightarrow{AB}$
- The vector $\overrightarrow{BC}$
- The vector sum $\underline{u} = 3\overrightarrow{BC} - 4\overrightarrow{AB}$

THINK

1. Define the vector $\overrightarrow{OA}$.
2. Define the vector $\overrightarrow{OB}$.
3. $\overrightarrow{AB}$ is determined by subtracting $\overrightarrow{OA}$ from $\overrightarrow{OB}$.

- Define the vector $\overrightarrow{OC}$.
 $\overrightarrow{BC}$ is determined by subtracting $\overrightarrow{OB}$ from $\overrightarrow{OC}$.

1. The vectors $\overrightarrow{BC}$ and $\overrightarrow{AB}$ have been defined. Substitute these into the vector sum $\underline{u} = 3\overrightarrow{BC} - 4\overrightarrow{AB}$.
2. Simplify the vector sum by multiplying the respective vectors by the scalar values. Define $\underline{u}$ by applying the rules for adding and subtracting vectors as components.

WRITE

- $$\begin{aligned}\overrightarrow{OA} &= -3\hat{i} + 7\hat{j} - \hat{k} \\ \overrightarrow{OB} &= 2\hat{i} - 2\hat{j} + 11\hat{k} \\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (2\hat{i} - 2\hat{j} + 11\hat{k}) - (-3\hat{i} + 7\hat{j} - \hat{k}) \\ &= (2 + 3)\hat{i} + (-2 - 7)\hat{j} + (11 + 1)\hat{k} \\ &= 5\hat{i} - 9\hat{j} + 12\hat{k}\end{aligned}$$
- $$\begin{aligned}\overrightarrow{OC} &= 8\hat{i} - 7\hat{j} + 12\hat{k} \\ \overrightarrow{BC} &= \overrightarrow{OC} - \overrightarrow{OB} \\ &= (8\hat{i} - 7\hat{j} + 12\hat{k}) - (2\hat{i} - 2\hat{j} + 11\hat{k}) \\ &= (8 - 2)\hat{i} + (-7 + 2)\hat{j} + (12 - 11)\hat{k} \\ &= 6\hat{i} - 5\hat{j} + \hat{k}\end{aligned}$$
- $$\begin{aligned}\underline{u} &= 3\overrightarrow{BC} - 4\overrightarrow{AB} \\ &= 3(6\hat{i} - 5\hat{j} + \hat{k}) - 4(5\hat{i} - 9\hat{j} + 12\hat{k}) \\ &= 18\hat{i} - 15\hat{j} + 3\hat{k} - 20\hat{i} + 36\hat{j} - 48\hat{k} \\ &= -2\hat{i} + 21\hat{j} - 45\hat{k}\end{aligned}$$

Scalar multiples and parallel vectors

When a vector is multiplied by a scalar, λ , each coefficient is multiplied by the scalar.

$$\begin{aligned}\lambda \underline{a} &= \lambda (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\ &= \lambda x_1\hat{i} + \lambda y_1\hat{j} + \lambda z_1\hat{k}\end{aligned}$$

Two vectors can be defined as parallel if one is a scalar multiple of the other. That is, $\underline{b}$ is parallel to $\underline{a}$ if $\underline{b} = \lambda \underline{a}$.

Equality of vectors.

Consider two vectors, $\underline{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\underline{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$. The vectors are equal if and only if, $x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$.

The midpoint of a vector in three-dimensions

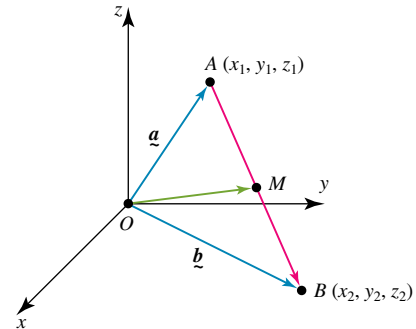
Consider the two vectors,

$$\underline{a} = \overrightarrow{OA} \text{ and } \underline{b} = \overrightarrow{OB}$$

The midpoint, M , of the line segment AB , can be represented by the vector,

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}. \text{ Alternatively, it can be defined as,}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}.$$



WORKED EXAMPLE 3

The location of two points are defined by $P(4, 8, -2)$ and $Q(12, -3, 6)$.

Calculate the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PQ .

THINK

1. Define the vector $\overrightarrow{OP}$.
2. Define the vector $\overrightarrow{OQ}$.
3. Recall the rule that defines a midpoint as a vector.
4. Calculate $\overrightarrow{OM}$ by substituting $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ into the formula.
5. Conclude with a statement.

WRITE

$$\overrightarrow{OP} = 4\hat{i} + 8\hat{j} - 2\hat{k}$$

$$\overrightarrow{OQ} = 12\hat{i} - 3\hat{j} + 6\hat{k}$$

$$\overrightarrow{OM} = \frac{\overrightarrow{OP} + \overrightarrow{OQ}}{2}$$

$$= \frac{(4\hat{i} + 8\hat{j} - 2\hat{k}) + (12\hat{i} - 3\hat{j} + 6\hat{k})}{2}$$

$$= \frac{16\hat{i} + 5\hat{j} + 4\hat{k}}{2}$$

$$= 8\hat{i} + \frac{5}{2}\hat{j} + 2\hat{k}$$

The midpoint between the points P and Q is represented by the position vector

$$\overrightarrow{OM} = 8\hat{i} + \frac{5}{2}\hat{j} + 2\hat{k}.$$

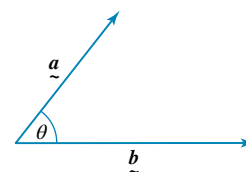
2.2.3 The dot product

Review of the dot product

When a vector is added or subtracted to another vector, or multiplied by a scalar quantity, the result is a vector. However, when vectors are multiplied this can either be in the form of the **dot (scalar) product** or **vector (cross) product**. The dot product results in a scalar quantity, whereas the vector product results in a vector.

The dot or scalar product is also known as the **inner product** and can be calculated using two different methods. The first method is used if the magnitude of two vectors and the angle, θ , between them are known.

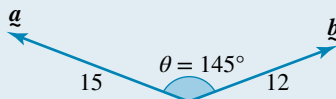
$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$$



Where θ is the angle between the tail of the vectors $\underline{a}$ and $\underline{b}$.

WORKED EXAMPLE 4

Examine the diagram and determine $\underline{a} \cdot \underline{b}$.



THINK

1. The magnitude of the two vectors and the angle between is given.
2. Recall the dot product formula and substitute in relevant values.

WRITE

$$|\underline{a}| = 15, |\underline{b}| = 12, \theta = 145^\circ$$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}| \cos \theta \\ &= 15 \times 12 \times \cos(145^\circ) \\ &= -147.45 \end{aligned}$$

The dot (scalar) product of vectors expressed in component form

The second method of calculating the dot (scalar) product is used with the vectors in component form. Consider the standard unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. The magnitude of each vector is 1 and can be expressed as $|\hat{i}| = 1$, $|\hat{j}| = 1$ and $|\hat{k}| = 1$. Begin by examining $\hat{i} \cdot \hat{i}$ in detail.

- The angle between $\hat{i}$ and $\hat{i}$ is 0° .
- $\cos(0^\circ) = 1$

By definition $\hat{i} \cdot \hat{i}$ is

$$\begin{aligned} \hat{i} \cdot \hat{i} &= |\hat{i}||\hat{i}| \cos(0^\circ) \\ &= |\hat{i}|^2 \quad (\text{where } |\hat{i}| = 1) \\ &= 1 \end{aligned}$$

Extrapolating from $\hat{i}$ to $\hat{j}$ and $\hat{k}$:

$$\hat{i} \cdot \hat{i} = 1, \hat{j} \cdot \hat{j} = 1 \text{ and } \hat{k} \cdot \hat{k} = 1$$

The unit vectors are mutually perpendicular to each other, that is the angle between $\hat{i}$, $\hat{j}$ and $\hat{k}$ are all 90° . Given that $\cos(90^\circ) = 0$ and the dot product is commutative, it follows that $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0$, $\hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$ and $\hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.

This allows us to calculate the dot product for any two vectors, $\underline{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\underline{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, where x, y and z are constants.

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \cdot (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) \\ &= x_1x_2(\hat{i} \cdot \hat{i}) + y_1y_2(\hat{j} \cdot \hat{j}) + z_1z_2(\hat{k} \cdot \hat{k}) \\ &= x_1x_2 + y_1y_2 + z_1z_2 \end{aligned}$$

We only need to multiply the corresponding x -, y - and z - components of two vectors to find their dot product.

WORKED EXAMPLE 5

Given the vectors, $\underline{a} = 5\hat{i} + 2\hat{j} - 8\hat{k}$ and $\underline{b} = -2\hat{i} + \hat{j} - 3\hat{k}$, determine $\underline{a} \cdot \underline{b}$.

THINK

1. Substitute for the given vectors.
2. Add the products of the corresponding x -, y -, z -components.
3. State the final result.

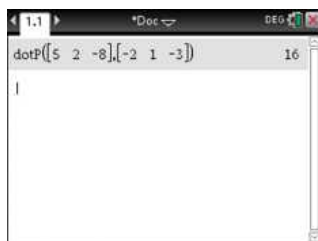
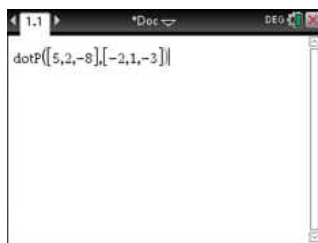
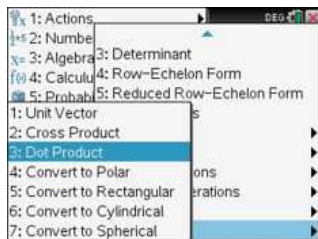
WRITE

$$\begin{aligned}\underline{a} \cdot \underline{b} &= (5\hat{i} + 2\hat{j} - 8\hat{k}) \cdot (-2\hat{i} + \hat{j} - 3\hat{k}) \\ \underline{a} \cdot \underline{b} &= (5 \times -2) + (2 \times 1) + (-8 \times -3) \\ &= -10 + 2 + 24 \\ &= 16 \\ \underline{a} \cdot \underline{b} &= 16\end{aligned}$$

TI | THINK

1. On a Calculator page, select MENU
7: Matrix Vector
C: Vector
3: Dot Product
Press the ENTER button.
2. Complete the entry line
 $([5, 2, -8], [-2, 1, -3])$
3. Press the ENTER button.
The answer appears on the screen.

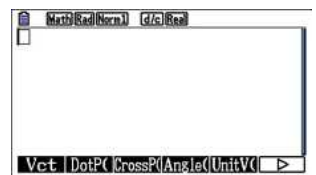
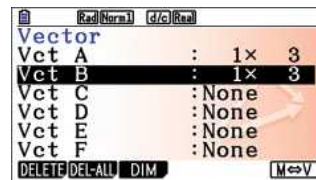
WRITE



CASIO | THINK

1. On a Run-Matrix screen, select MAT/VCT
Define Vectors A and B as shown in Worked example 1b.
2. On a Run-Matrix screen, select OPTN MAT/VCT
▷ (x3 times) DotP
3. Complete the entry line as
Vct ALPHA X, θ , T,
Vct ALPHA Log)
4. Press the EXE button to perform the calculation.
The answer appears on the screen.

WRITE



Determining the angle between two vectors

Given two formulas for calculating the dot product:

$$\begin{aligned}\underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}| \cos \theta \\ \underline{a} \cdot \underline{b} &= x_1x_2 + y_1y_2 + z_1z_2\end{aligned}$$

they can be combined to give:

$$x_1x_2 + y_1y_2 + z_1z_2 = |\underline{a}||\underline{b}| \cos \theta$$

Rearranging this formula gives the angle between two vectors.

The angle between two vectors

$$\cos \theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\underline{a}||\underline{b}|} \text{ or } \cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

The dot product of perpendicular vectors

It was shown that the dot product of perpendicular unit vectors is equal to zero.

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0, \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0 \text{ and } \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

This is true for any two, non-zero vectors, $\underline{a}$ and $\underline{b}$. If the two vectors $\underline{a}$ and $\underline{b}$ are perpendicular (**orthogonal**) the angle between them is 90° , and the equation $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$ becomes,

$$\begin{aligned} \underline{a} \cdot \underline{b} &= |\underline{a}||\underline{b}| \cos(90^\circ) \\ &= |\underline{a}||\underline{b}| \times 0 \quad (\text{Recall } \cos(90^\circ) = 0) \\ &= 0 \end{aligned}$$

If $\underline{a} \cdot \underline{b} = 0$, then the vectors $\underline{a}$ and $\underline{b}$ are perpendicular.

Properties of the dot product

- The dot product is a scalar. It is the result of multiplying scalar quantities.
- If the magnitude of two vectors and the angle, θ , between them are known it is calculated by: $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$. If the vectors are in component form, it is calculated by:
 $\underline{a} \cdot \underline{b} = x_1x_2 + y_1y_2 + z_1z_2$
- The order of multiplication is unimportant (*commutative* property): $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$
- The dot product is *distributive*: $\underline{c} \cdot (\underline{a} + \underline{b}) = \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{b}$
- Since the angle between the same vector and itself is 0° : $\underline{a} \cdot \underline{a} = |\underline{a}|^2$
- If $\underline{a} \cdot \underline{b} = 0$, then the vectors $\underline{a}$ and $\underline{b}$ are perpendicular.

WORKED EXAMPLE 6

The vectors given by, $\underline{a} = 6\hat{i} + y\hat{j} - \hat{k}$ and $\underline{b} = -3\hat{i} + 4\hat{j} + 2\hat{k}$, are perpendicular. Calculate the value of y .

THINK

1. Write $\underline{a} \cdot \underline{b}$ as the dot product of the two vectors in component form.
2. Add the products of the corresponding x -, y -, z -components.
3. State the dot product relationship of perpendicular vectors.
4. Substitute the initial solution for $\underline{a} \cdot \underline{b}$ [1] into [2] to solve for y .
5. The solution is stated.

WRITE

$$\underline{a} \cdot \underline{b} = (6\hat{i} + y\hat{j} - \hat{k}) \cdot (-3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (6 \times -3) + (y \times 4) + (-1 \times 2) \\ &= -18 + 4y - 2 \\ &= 4y - 20 \end{aligned} \quad [1]$$

$$\begin{aligned} \underline{a} \text{ is perpendicular to } \underline{b} \\ \therefore \underline{a} \cdot \underline{b} &= 0 \end{aligned} \quad [2]$$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= 0 \\ 4y - 20 &= 0 \\ y &= \frac{20}{4} \end{aligned}$$

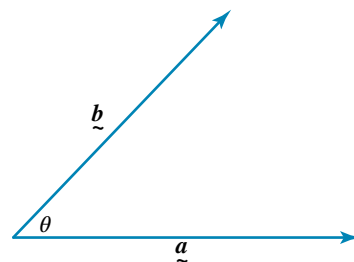
$$\therefore y = 5$$

2.2.4 Review of vector projections (vector resolute)

We have resolved vectors parallel and perpendicular to the x - and y -axes. Now we will examine the more generalised process of resolving one vector parallel and perpendicular to another vector.

Consider the two vectors $\underline{a}$ and $\underline{b}$, shown. The angle between them, as for a dot product, is given by θ . It can be shown that $\underline{b}$ is made up of a **projection** acting in the direction of $\underline{a}$ and another projection acting perpendicular to $\underline{a}$.

The first step is to find the projection in the direction of $\underline{a}$.



The scalar projection (scalar resolute)

- The length OA is the **scalar resolute** of $\underline{b}$ on $\underline{a}$.
- It effectively indicates 'how much' of $\underline{b}$ is in the direction of $\underline{a}$.

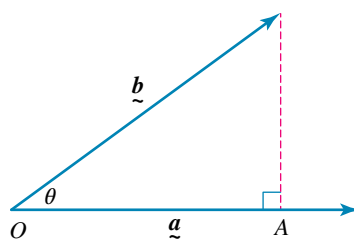
Using trigonometry $OA = |\underline{b}| \cos \theta$, and the dot product

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta:$$

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$$

$$\underline{a} \cdot \underline{b} = |\underline{a}|OA$$

$$\begin{aligned} \therefore OA &= \frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}, \text{ where } \hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|} \\ &= \hat{\underline{a}} \cdot \underline{b} \end{aligned}$$

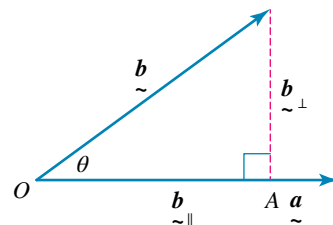


The scalar resolute of $\underline{b}$ on $\underline{a}$ is given by $\hat{\underline{a}} \cdot \underline{b}$ or $\frac{\underline{a} \cdot \underline{b}}{|\underline{a}|}$, where $\hat{\underline{a}}$ is the unit vector in the direction of $\underline{a}$.

The scalar resolute of $\underline{b}$ on $\underline{a}$ is not equal to the scalar resolute of $\underline{a}$ on $\underline{b}$.

Vector projection (vector resolute)

Consider now the vector joining O to A . Its magnitude is just the scalar resolute $\hat{\underline{a}} \cdot \underline{b}$, while its direction is the same as $\underline{a}$, that is $\underline{a}$. This quantity is called the vector resolute of $\underline{b}$ parallel to $\underline{a}$ and is denoted by the symbol $\underline{b}_{\parallel}$.



The vector resolute of $\underline{b}$ parallel to $\underline{a}$ is given by

$$\underline{b}_{\parallel} = (\hat{\underline{a}} \cdot \underline{b})\hat{\underline{a}}$$

Consider the geometry of the above figure. The original vector $\underline{b}$ is the sum of $\underline{b}_{\perp}$ and $\underline{b}_{\parallel}$. The vector resolute of $\underline{b}$ perpendicular to $\underline{a}$ can be calculated simply as shown below.

$$\underline{b} = \underline{b}_{\parallel} + \underline{b}_{\perp} \quad (\text{by addition of vectors})$$

$$\underline{b}_{\perp} = \underline{b} - \underline{b}_{\parallel} \quad (\text{by rearranging the vector equation})$$

The vector resolute of $\underline{b}$ perpendicular to $\underline{a}$ is given by

$$\underline{b}_{\perp} = \underline{b} - (\hat{\underline{a}} \cdot \underline{b})\hat{\underline{a}}$$

WORKED EXAMPLE 7

Let $\underline{a} = 8\hat{i} - 6\hat{j} + 5\hat{k}$ and $\underline{b} = -6\hat{i} + 2\hat{j} - \hat{k}$. Determine:

- the scalar resolute of $\underline{b}$ on $\underline{a}$
- the vector resolute of $\underline{b}$ parallel to $\underline{a}$, namely $\underline{b}_{\parallel}$
- the vector resolute of $\underline{b}$ perpendicular to $\underline{a}$, namely $\underline{b}_{\perp}$

THINK

1. Calculate the magnitude of $\underline{a}$.
2. Calculate the unit vector in the direction of $\underline{a}$, $\hat{a}$, by dividing $\underline{a}$ by $|\underline{a}|$.
3. Determine the scalar resolute of $\underline{b}$ on $\underline{a}$, by substituting into the formula $\hat{a} \cdot \underline{b}$.
4. Simplify $\hat{a} \cdot \underline{b}$ using dot product laws for unit vectors.

WRITE

$$\begin{aligned} \text{a. } |\underline{a}| &= \sqrt{(8)^2 + (-6)^2 + (5)^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

$$\begin{aligned} \hat{a} &= \frac{\underline{a}}{|\underline{a}|} \\ &= \frac{(8\hat{i} - 6\hat{j} + 5\hat{k})}{5\sqrt{5}} \end{aligned}$$

$$\hat{a} \cdot \underline{b} = \left(\frac{8\hat{i}}{5\sqrt{5}} - \frac{6\hat{j}}{5\sqrt{5}} + \frac{5\hat{k}}{5\sqrt{5}} \right) \cdot (-6\hat{i} + 2\hat{j} - \hat{k})$$

$$\begin{aligned} \hat{a} \cdot \underline{b} &= \frac{-48}{5\sqrt{5}} + \frac{-12}{5\sqrt{5}} + \frac{-5}{5\sqrt{5}} \\ &= -\frac{65}{5\sqrt{5}} \\ &= -\frac{13}{\sqrt{5}} \end{aligned}$$

1. Determine $\underline{b}_{\parallel}$ by recalling the formula $\underline{b}_{\parallel} = (\hat{a} \cdot \underline{b})\hat{a}$

$$\begin{aligned} \text{b. } \underline{b}_{\parallel} &= (\hat{a} \cdot \underline{b})\hat{a} \\ &= \left(-\frac{13}{\sqrt{5}} \right) \left(\frac{8\hat{i}}{5\sqrt{5}} - \frac{6\hat{j}}{5\sqrt{5}} + \frac{\hat{k}}{\sqrt{5}} \right) \\ &= -\frac{104}{25}\hat{i} + \frac{78}{25}\hat{j} - \frac{13}{5}\hat{k} \end{aligned}$$

2. Simplify.

1. Determine $\underline{b}_{\perp}$ by recalling the formula $\underline{b}_{\perp} = \underline{b} - \underline{b}_{\parallel}$

$$\text{c. } \underline{b}_{\perp} = \underline{b} - (\hat{a} \cdot \underline{b})\hat{a}$$

2. Substitute in the vectors, $\underline{b}_{\perp}$ and $\underline{b}_{\parallel}$.

$$\underline{b}_{\perp} = (-6\hat{i} + 2\hat{j} - \hat{k}) - \left(-\frac{104}{25}\hat{i} + \frac{78}{25}\hat{j} - \frac{13}{5}\hat{k} \right)$$

3. Simplify.

$$\underline{b}_{\perp} = -\frac{46}{25}\hat{i} - \frac{28}{25}\hat{j} + \frac{8}{5}\hat{k}$$

2.2.5 Three dimensional vectors in polar form

In two-dimensions, a vector in polar form was defined by $\underline{u} = [r, \theta]$, where r is the magnitude of $\underline{u}$ and θ is the anti-clockwise angle formed from the positive x -axis.

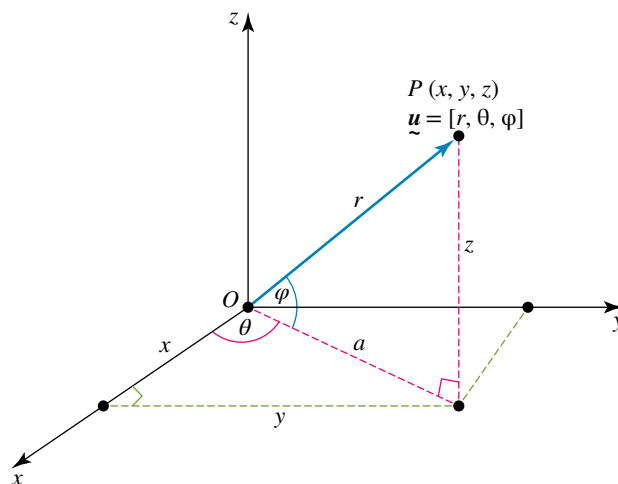
In three dimensions, the point $P(x, y, z)$ can be defined by the position vector, $\overrightarrow{OP} = \underline{u}$.

The vector, $\underline{u}$, can then be represented in polar form $\underline{u} = [r, \theta, \varphi]$ where,

- r is the magnitude of $\underline{u}$, where $r \geq 0$ and is given by $r = \sqrt{x^2 + y^2 + z^2}$.
- θ (**azimuth angle**) is the anti-clockwise angle formed from the positive x -axis to the projection of $\underline{u}$ onto the xy -plane, where $0^\circ \leq \theta \leq 360^\circ$.
- φ (**altitude angle**) is the angle between the line segment $\overrightarrow{OP}$ and the xy -plane, where $-90^\circ \leq \varphi \leq 90^\circ$.

The relationship between Cartesian (x, y, z) and polar $[r, \theta, \varphi]$ coordinates can be examined by referring to the diagram to the right. The findings will be useful to convert between one form to the other.

Two trigonometric ratios can be formed by considering the triangle defining the altitude angle, φ .

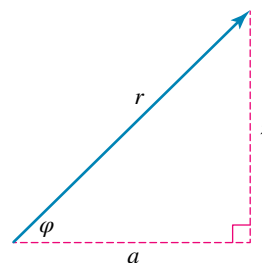


$$\sin \varphi = \frac{z}{r}$$

$$\therefore z = r \sin \varphi \quad [1]$$

$$\cos \varphi = \frac{a}{r}$$

$$\therefore a = r \cos \varphi \quad [2]$$



The second triangle forms a further three trigonometric ratios in terms of θ, a, x and y :

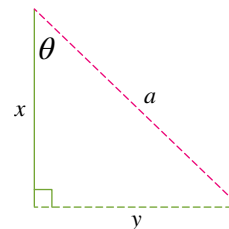
$$\sin \theta = \frac{y}{a}$$

$$\therefore y = a \sin \theta \quad [3]$$

$$\cos \theta = \frac{x}{a}$$

$$\therefore x = a \cos \theta \quad [4]$$

$$\tan \theta = \frac{y}{x} \quad [5]$$



x and y can be expressed in terms r and the two angles, through the process of substitution.

Substituting [2] into [4] and substituting [2] into [3] gives

$$x = r \cos \varphi \cos \theta$$

$$y = r \cos \varphi \sin \theta$$

These formulae are useful when working with vectors in three-dimensions. A summary of the necessary rules to remember are detailed below.

When considering the three-dimensional vector, $\underline{u} = [r, \theta, \varphi] = (x\hat{i} + y\hat{j} + z\hat{k})$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$x = r \cos \varphi \cos \theta$$

$$y = r \cos \varphi \sin \theta$$

$$z = r \sin \varphi$$

$$\sin \varphi = \frac{z}{r}$$

$$\tan \theta = \frac{y}{x}$$

Vectors expressed in the polar form, $\underline{u} = [r, \theta, \varphi]$, are often related to the spherical coordinate system which may be used to identify points on the Earth's surface.

WORKED EXAMPLE 8

- Express $\underline{u} = [10, 30^\circ, 45^\circ]$ in component form, $(x\hat{i} + y\hat{j} + z\hat{k})$. Give your answer as an exact value.
- Determine the altitude angle, to three decimal places, between the vector, $\underline{y} = -12\hat{i} + 9\hat{j} - 8\hat{k}$ and the xy -plane.
- Convert $\underline{y} = -12\hat{i} + 9\hat{j} - 8\hat{k}$ to polar form, giving your answer to three decimal places.

THINK

1. State the magnitude, r , and the two angles (θ, φ) .
 2. Recall the formula $x = r \cos \varphi \cos \theta$ to resolve the x -component. Substitute values $(r, \varphi$ and $\theta)$ and evaluate.
 3. Recall the formula $y = r \cos \varphi \sin \theta$ to resolve the y -component. Substitute values $(r, \varphi$ and $\theta)$ and evaluate.
 4. Recall the formula $z = r \sin \varphi$ to resolve the z -component. Substitute values $(r$ and $\varphi)$ and evaluate.
 5. Conclude by representing the vector in component form.
1. State the coefficient (x, y, z) of each component of the vector.
 2. Calculate the magnitude of $\underline{y}$. In polar form, $|\underline{y}| = r$.

WRITE

$$\begin{aligned}
 \text{a. } \underline{u} &= [10, 30^\circ, 45^\circ] \\
 r &= 10, \theta = 30^\circ, \varphi = 45^\circ \\
 x &= r \cos \varphi \cos \theta \\
 &= 10 \cos(45^\circ) \cos(30^\circ) \\
 &= 10 \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{3}}{2} \right) \\
 &= \frac{5\sqrt{6}}{2} \\
 y &= r \cos \varphi \sin \theta \\
 &= 10 \cos(45^\circ) \sin(30^\circ) \\
 &= 10 \left(\frac{\sqrt{2}}{2} \right) \left(\frac{1}{2} \right) \\
 &= \frac{5\sqrt{2}}{2} \\
 z &= r \sin \varphi \\
 &= 10 \sin(45^\circ) \\
 &= 10 \left(\frac{\sqrt{2}}{2} \right) \\
 &= 5\sqrt{2} \\
 \underline{u} &= x\hat{i} + y\hat{j} + z\hat{k} \\
 &= \frac{5\sqrt{6}}{2}\hat{i} + \frac{5\sqrt{2}}{2}\hat{j} + 5\sqrt{2}\hat{k}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } \underline{y} &= -12\hat{i} + 9\hat{j} - 8\hat{k} \\
 \therefore x &= -12, y = 9, z = -8 \\
 r &= \sqrt{(-12)^2 + (9)^2 + (-8)^2} \\
 &= 17
 \end{aligned}$$



3. Calculate the altitude angle by recalling the formula $\sin \varphi = \frac{z}{r}$. Substitute values for z and r .
Solve for φ remembering $-90^\circ \leq \varphi \leq 90^\circ$.

$$\sin \varphi = \frac{z}{r}$$

$$\sin \varphi = \frac{-8}{17}$$

$$\varphi = \sin^{-1} \left(\frac{-8}{17} \right)$$

$$\varphi \approx -28.072^\circ$$

The angle between $\underline{y}$ and the xy - plane is -28.072°

- c. 1. Recall the formula to calculate the azimuth angle, $\tan \theta = \frac{y}{x}$.
Substitute values for x and y .

c. $\tan \theta = \frac{y}{x} = \frac{9}{-12}$

$$\theta = \tan^{-1} \left(\frac{9}{-12} \right)$$

$$\theta \approx -36.870^\circ$$

2. Determine in which quadrant θ lies.
As the x -value is negative and the y -value is positive, the angle lies in the second quadrant.
Calculate the 2nd quadrant solution by subtracting the magnitude of the angle from 180° .

The angle is in the second quadrant as $x < 0$ and $y > 0$

$$\theta = 180^\circ - 36.870^\circ$$

$$\theta = 143.130^\circ$$

3. State the polar equivalent for $\underline{y}$.

$$\therefore \underline{y} = -12\hat{i} + 9\hat{j} - 8\hat{k} \text{ in polar form is } \underline{y} = [17, 143.130^\circ, -28.072^\circ]$$

study on

Units 3 & 4 > Area 2 > Sequence 1 > Concept 1

Introduction to vectors in three dimension Summary screen and practice questions

Exercise 2.2 Introduction to vectors in three dimensions

Technology free

- State the coordinates of a point whose position vector is defined as:
 - $\overrightarrow{OA} = 5\hat{i} - 4\hat{j}$
 - $\overrightarrow{OB} = -2\hat{i} + 6\hat{k}$
 - $\overrightarrow{OC} = \frac{5}{2}\hat{i} + 11\hat{j} - 12\hat{k}$
 - $\overrightarrow{OD} = -4\hat{i} - \hat{j} + 8\hat{k}$
- Determine the position vector of each of the following points relative to the origin, O :
 - $A(5, 3, 2)$
 - $B(2, 0, -6)$
 - $C\left(-\frac{2}{5}, \frac{2}{3}, -\frac{22}{17}\right)$
 - $D(-11, 1, -4)$
- State if the following are either vector or scalar quantities.
 - Velocity
 - Volume
 - Force applied in a downward direction
 - Time
 - Speed

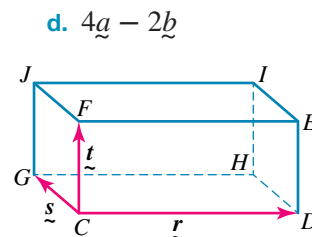
4. Refer to the vectors, $\underline{a} = 4\hat{i} - 2\hat{j} + 7\hat{k}$ and $\underline{b} = -3\hat{i} + 5\hat{j} - \hat{k}$, to calculate:

- a. $-\underline{a}$ b. $2\underline{b}$ c. $\underline{a} + \underline{b}$

5. A rectangular prism (box) $CDEFGHIJ$ can be defined by three vectors, $\underline{r}$, $\underline{s}$ and $\underline{t}$, as shown.

Express in terms of $\underline{r}$, $\underline{s}$ and $\underline{t}$

- a. the vector joining C to E
b. the vector joining C to I
c. the vector joining D to J .



- d. $4\underline{a} - 2\underline{b}$

Technology active

6. An aeroplane travels 600 km east, then 900 km south. Calculate how far the aeroplane is from its starting point and determine the true bearing of the resultant displacement.

7. **WE1** If the point P has the coordinate $(8, 4, 1)$, determine:

- a. the vector $\overrightarrow{OP}$ b. the magnitude of $\overrightarrow{OP}$
c. the unit vector parallel to $\overrightarrow{OP}$.

8. If the point P has coordinates $(6, 2, -8)$, determine:

- a. the vector $\overrightarrow{OP} = \underline{p}$
b. the magnitude of the vector, $|\underline{p}|$
c. the unit vector parallel to $\underline{p}$
d. a vector 7 units long in the direction of $\underline{p}$.

9. **WE2** Three points, A , B and C have the coordinates $(3, 4, 2)$, $(8, 5, 10)$ and $(9, 7, 11)$ respectively. Calculate:

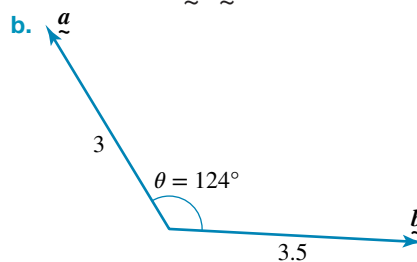
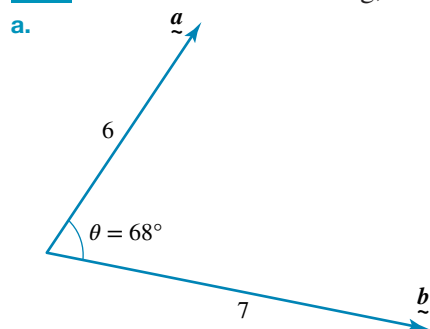
- a. the vector $\overrightarrow{AB}$ b. the vector $\overrightarrow{BC}$
c. the vector sum $\underline{u} = 3\overrightarrow{BC} - 4\overrightarrow{AB}$.

10. **WE3** The location of two points are defined by, $P(1, 5, -6)$ and $Q(9, 7, 4)$, respectively. Calculate the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PQ .

11. Four points are given by $P(-2, 5, 0)$, $Q(0, 8, 4)$, $R(3, -6, 12)$ and $S(6, 17, 16)$. Refer to these points to determine:

- a. $\overrightarrow{OP}$
b. $\overrightarrow{PQ}$
c. $\underline{c} = 2\overrightarrow{PQ} - 3\overrightarrow{QR}$
d. the position vector $\overrightarrow{OM}$, where M is the midpoint of the line segment PS
e. if $\overrightarrow{PQ}$ and $\overrightarrow{QS}$ are parallel.

12. **WE4** For each of the following, examine the diagram and determine $\underline{a} \cdot \underline{b}$.



13. **WE5** Given the vectors $\underline{a} = 2\hat{i} - 5\hat{j} + \hat{k}$ and $\underline{b} = -3\hat{i} + 7\hat{j} - 4\hat{k}$, determine $\underline{a} \cdot \underline{b}$.

14. **WE6** The vectors given by, $\underline{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\underline{b} = 4\hat{i} + y\hat{j} + 3\hat{k}$, are perpendicular. Determine the value of y .

15. Let $\underline{a} = \hat{i} - 2\hat{j} + 8\hat{k}$, $\underline{b} = 3\hat{i} + 6\hat{j} - 9\hat{k}$ and $\underline{c} = 5\hat{i} + 4\hat{j} - 2\hat{k}$. Demonstrate, using these vectors, the property:

$$\underline{c} \cdot (\underline{a} - \underline{b}) = \underline{c} \cdot \underline{a} - \underline{c} \cdot \underline{b}$$

16. Calculate the angle between the two vectors, $\underline{a} = -5\hat{i} + 6\hat{j} + 2\hat{k}$ and $\underline{b} = 7\hat{i} + 11\hat{j} - 2\hat{k}$.
17. **WE7** Let $\underline{a} = 20\hat{i} - 18\hat{j} + 8\hat{k}$ and $\underline{b} = -5\hat{i} + 4\hat{j} - 2\hat{k}$. Determine:
- the scalar resolute of $\underline{b}$ on $\underline{a}$
 - the vector resolute of $\underline{b}$ parallel to $\underline{a}$, namely $\underline{b}_{\parallel}$
 - the vector resolute of $\underline{b}$ perpendicular to $\underline{a}$, namely $\underline{b}_{\perp}$.
18. **WE8a** For each of the following, express $\underline{u} = [r, \theta, \varphi]$ in component form, $x\hat{i} + y\hat{j} + z\hat{k}$.
- $\underline{u} = [6, 61^\circ, 32^\circ]$
 - $\underline{u} = [20, 153^\circ, -64^\circ]$
 - $\underline{u} = [3, 120^\circ, -40^\circ]$
 - $\underline{u} = [15, 325^\circ, 36^\circ]$
19. a. **WE8b** Determine the altitude angle between the vector, $\underline{u} = x\hat{i} + y\hat{j} + z\hat{k}$ and the xy -plane in each case below.
- b. **WE8c** Convert $\underline{u} = x\hat{i} + y\hat{j} + z\hat{k}$ to polar form in each case below.
- $\underline{u} = 2\hat{i} + 5\hat{j} - 3\hat{k}$
 - $\underline{u} = 3\hat{i} - 4\hat{j} + 2\hat{k}$
 - $\underline{u} = 5\hat{i} + 6\hat{j} + 7\hat{k}$
 - $\underline{u} = -4\hat{i} + 5\hat{j} - 10\hat{k}$
20. A boat travels 500 km in a direction S40°E and then 333 km in a southerly direction. Calculate the resultant displacement from the starting position giving both magnitude and direction as a true bearing.
21. Let $\underline{a} = 6\hat{i} + 2\hat{j} - \hat{k}$. Determine a vector parallel to $\underline{a}$ such that their dot product is -82 .
22. An injured mountain climber is located at a position relative to a camp given by the vector $\underline{b} = 150\hat{i} + 200\hat{j} + 100\hat{k}$. A rescue helicopter heads off from the camp in a direction parallel to the vector $\underline{a} = 200\hat{i} + 250\hat{j} + 125\hat{k}$. All measurements are in metres.
- Determine how far, in metres correct to two decimal places, the rescue helicopter is from the camp when closest to the mountain climber.
 - Calculate the minimum distance, in metres correct to two decimal places, between the helicopter and the mountain climber.



2.3 Geometric proofs using vectors

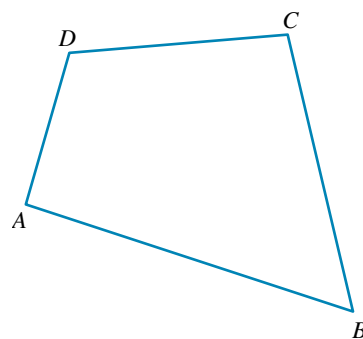
2.3.1 Geometrical shapes

Geometry is a branch of mathematics that explores the relationship between lines, shapes, solids and angles. It is often referred to as deductive geometry because problems are solved by referring to mathematical statements considered to be true, known as axioms or postulates. The most famous of which are Euclid's postulates, explored in Unit 1, Topic 3 (Chapter 4 of the Jacaranda Maths Quest 11 Specialist Mathematics Units 1 & 2 for Queensland). Euclid's first postulate stated that a straight line can be drawn between any two points.

Straight lines define the edges of many shapes and solids and can be expressed as directed line segments. When considered in this way, vectors can be used to prove and solve many geometric problems. Consider the common geometric shapes and their vector properties below.

Quadrilateral – General

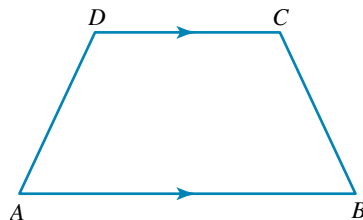
A plane four-sided figure with no two sides necessarily parallel nor equal in length.



Trapezium

A plane four-sided figure with one pair of sides parallel, but not equal.

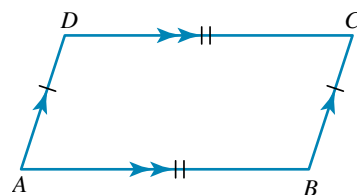
In trapezium $ABCD$, since AB is parallel to DC , $\overrightarrow{AB} = \lambda \overrightarrow{DC}$; where λ is a scalar.



Parallelogram

A plane four-sided figure with two sets of sides parallel and equal in length.

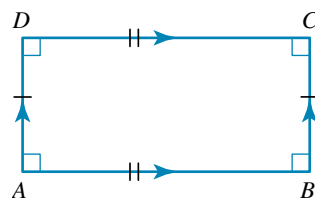
In parallelogram $ABCD$, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$. Also, $|\overrightarrow{AB}| = |\overrightarrow{DC}|$ and $|\overrightarrow{AD}| = |\overrightarrow{BC}|$.



Rectangle

A parallelogram with all angles being 90° .

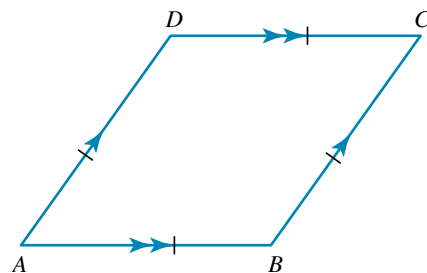
In rectangle $ABCD$, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$ and thus $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{BC} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{CD} \cdot \overrightarrow{DA} = 0$ and $\overrightarrow{DA} \cdot \overrightarrow{AB} = 0$, since all these sides are perpendicular.



Rhombus

A parallelogram with all sides equal in length.

In rhombus $ABCD$, $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$. $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$, since all sides are equal in length.



Square

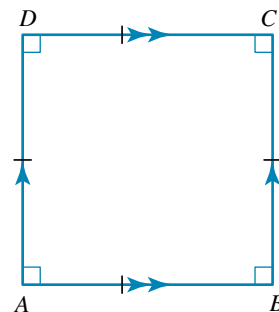
A rhombus with all angles 90° .

In square $ABCD$,

$$\overrightarrow{AB} = \overrightarrow{DC}, \overrightarrow{AD} = \overrightarrow{BC} \text{ and}$$

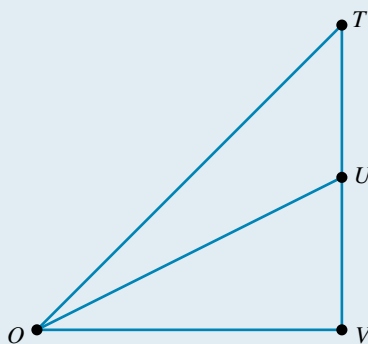
$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0, \overrightarrow{BC} \cdot \overrightarrow{CD} = 0, \overrightarrow{CD} \cdot \overrightarrow{DA} = 0, \overrightarrow{DA} \cdot \overrightarrow{AB} = 0$, since all these sides are perpendicular.

$$\text{Also, } |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{DC}| = |\overrightarrow{AD}|$$



WORKED EXAMPLE 9

If O is the origin and U is the midpoint of the line segment TV , prove that $\overrightarrow{OU} = \frac{1}{2}(\overrightarrow{OT} + \overrightarrow{OV})$.



THINK

1. Define the vector $\overrightarrow{OV}$ as the sum of the other two other vectors.
2. Rearrange [1] to make $\overrightarrow{TV}$ as the subject.
3. Define the vector $\overrightarrow{OU}$ as the sum of $\overrightarrow{OT}$ and $\overrightarrow{TU}$.
As U is the midpoint of $\overrightarrow{TV}$, $\overrightarrow{TU} = \frac{1}{2}\overrightarrow{TV}$.
4. Substitute [2] into [3] and simplify the right-hand side.
Expand $\frac{1}{2}(\overrightarrow{OV} - \overrightarrow{OT})$.
Collect like vectors $\overrightarrow{OT}$.
Factorise to demonstrate the LHS = RHS.
5. Write a concluding statement.

WRITE

$$\text{Let, } \overrightarrow{OV} = \overrightarrow{OT} + \overrightarrow{TV} \quad [1]$$

$$\overrightarrow{TV} = \overrightarrow{OV} - \overrightarrow{OT} \quad [2]$$

$$\overrightarrow{OU} = \overrightarrow{OT} + \frac{1}{2}\overrightarrow{TV} \quad [3]$$

$$\overrightarrow{OU} = \overrightarrow{OT} + \frac{1}{2}(\overrightarrow{OV} - \overrightarrow{OT})$$

$$\overrightarrow{OU} = \overrightarrow{OT} + \frac{1}{2}\overrightarrow{OV} - \frac{1}{2}\overrightarrow{OT}$$

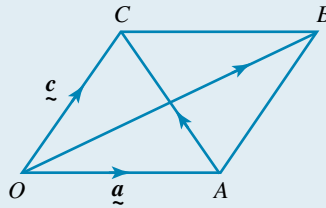
$$= \frac{1}{2}\overrightarrow{OT} + \frac{1}{2}\overrightarrow{OV}$$

$$= \frac{1}{2}(\overrightarrow{OT} + \overrightarrow{OV})$$

$$\therefore \overrightarrow{OU} = \frac{1}{2}(\overrightarrow{OT} + \overrightarrow{OV})$$

WORKED EXAMPLE 10

Prove that if the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle.



THINK

1. Define the vectors $\underline{a}$ and $\underline{c}$ in terms of vectors joining the points A, B, C, D . State equivalent vectors using the properties of a parallelogram.
2. Express the diagonals in terms of the vectors, $\underline{a}$ and $\underline{c}$.
3. Assume the diagonals are equal in length.
4. Express the diagonals in terms of $\underline{a}$ and $\underline{c}$ and square both sides to apply the dot product property, $|\underline{a}|^2 = \underline{a} \cdot \underline{a}$.
5. Expand both sides. Remember that $\underline{a} \cdot \underline{a} = |\underline{a}|^2$ and $\underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}$.
6. Rewrite $|\underline{OB}|^2 = |\underline{AC}|^2$ in terms of the dot product calculations and collect like terms.
7. State the conclusion.

WRITE

Let O, A, B, C be a parallelogram.

$$\underline{a} = \overrightarrow{OA} = \overrightarrow{CB}$$

$$\underline{c} = \overrightarrow{OC} = \overrightarrow{AB}$$

$$\begin{aligned} \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} & \overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \underline{c} - \underline{a} & &= \underline{a} + \underline{c} \end{aligned}$$

$$|\overrightarrow{OB}| = |\overrightarrow{AC}|$$

$$\begin{aligned} |\overrightarrow{OB}|^2 &= |\overrightarrow{AC}|^2 \\ (\underline{a} + \underline{c}) \cdot (\underline{a} + \underline{c}) &= (\underline{c} - \underline{a}) \cdot (\underline{c} - \underline{a}) \end{aligned}$$

$$\begin{aligned} \text{LHS} &= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{c} \\ &= |\underline{a}|^2 + 2\underline{a} \cdot \underline{c} + |\underline{c}|^2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \underline{c} \cdot \underline{c} + \underline{c} \cdot -\underline{a} + -\underline{a} \cdot \underline{c} + -\underline{a} \cdot -\underline{a} \\ &= |\underline{c}|^2 - 2\underline{a} \cdot \underline{c} + |\underline{a}|^2 \end{aligned}$$

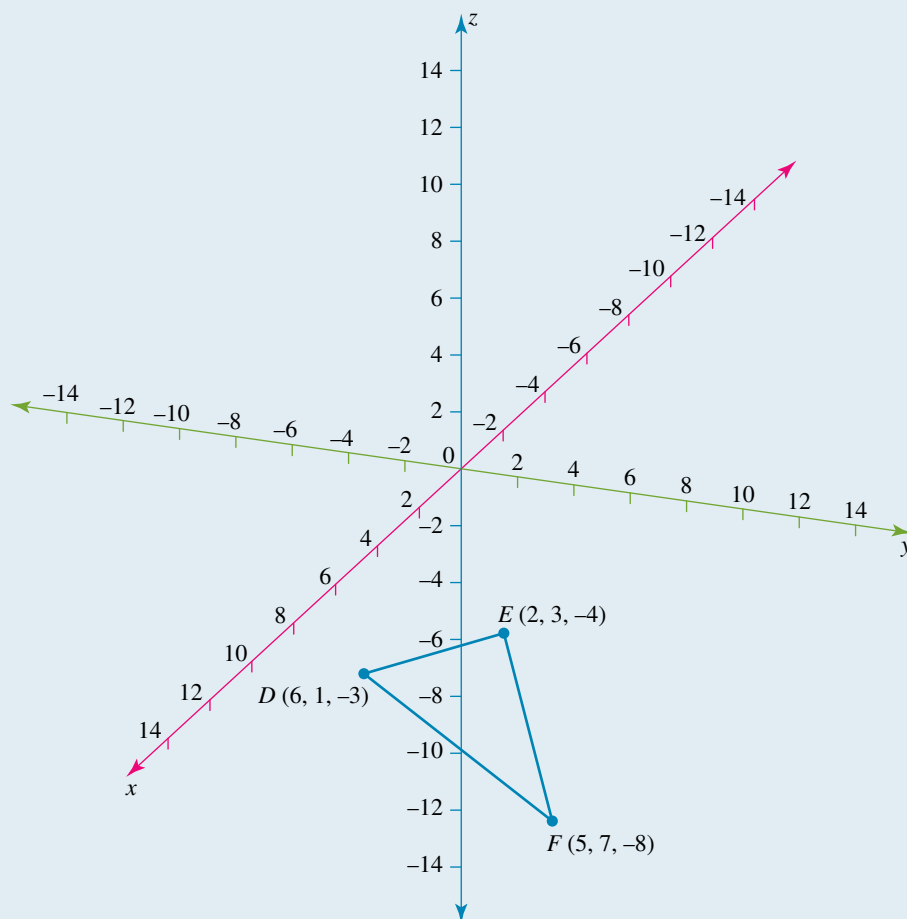
$$\begin{aligned} |\underline{a}|^2 + 2\underline{a} \cdot \underline{c} + |\underline{c}|^2 &= |\underline{c}|^2 - 2\underline{a} \cdot \underline{c} + |\underline{a}|^2 \\ 2\underline{a} \cdot \underline{c} + 2\underline{a} \cdot \underline{c} &= |\underline{c}|^2 + |\underline{a}|^2 - |\underline{a}|^2 - |\underline{c}|^2 \\ 4\underline{a} \cdot \underline{c} &= 0 \\ \underline{a} \cdot \underline{c} &= 0 \end{aligned}$$

Since $\underline{a} \cdot \underline{c} = 0$ this implies that $\angle COA = 90^\circ$.
 $\therefore \overrightarrow{OA}$ is $\perp$ to $\overrightarrow{OC}$, $OACB$ is a rectangle.

This can extend to simple applications of plane shapes in three dimensions.

WORKED EXAMPLE 11

Given the points $D(6, 1, -3)$, $E(2, 3, -4)$ and $F(5, 7, -8)$, prove that DEF forms a right-angled triangle at E , and hence calculate the area of the triangle as an exact value.



THINK

1. Define each of the vertices as position vectors.
2. The line segments that form the triangle can be written as vectors in terms of $\overrightarrow{OD}$, $\overrightarrow{OE}$, $\overrightarrow{OF}$.
3. If $\overrightarrow{DE}$ is perpendicular to $\overrightarrow{EF}$, then $\overrightarrow{DE} \cdot \overrightarrow{EF} = 0$.
4. As the dot product is 0, the angle between the vectors is 90° .

WRITE

$$D(6, 1, -3) \quad \therefore \overrightarrow{OD} = 6\hat{i} + \hat{j} - 3\hat{k}$$

$$E(2, 3, -4) \quad \therefore \overrightarrow{OE} = 2\hat{i} + 3\hat{j} - 4\hat{k}$$

$$F(5, 7, -8) \quad \therefore \overrightarrow{OF} = 5\hat{i} + 7\hat{j} - 8\hat{k}$$

$$\begin{aligned} \overrightarrow{DE} &= \overrightarrow{OE} - \overrightarrow{OD} \\ &= (2\hat{i} + 3\hat{j} - 4\hat{k}) - (6\hat{i} + \hat{j} - 3\hat{k}) \\ &= -4\hat{i} + 2\hat{j} - \hat{k} \end{aligned}$$

$$\begin{aligned} \overrightarrow{EF} &= \overrightarrow{OF} - \overrightarrow{OE} \\ &= (5\hat{i} + 7\hat{j} - 8\hat{k}) - (2\hat{i} + 3\hat{j} - 4\hat{k}) \\ &= 3\hat{i} + 4\hat{j} - 4\hat{k} \end{aligned}$$

$$\begin{aligned} \overrightarrow{DE} \cdot \overrightarrow{EF} &= (-4\hat{i} + 2\hat{j} - \hat{k}) \cdot (3\hat{i} + 4\hat{j} - 4\hat{k}) \\ &= -12 + 8 + 4 \\ &= 0 \end{aligned}$$

Therefore $\overrightarrow{DE}$ is $\perp$ to $\overrightarrow{EF}$, hence the triangle is right-angled.

5. Recall the area of a triangle can be calculated by the rule, $A = \frac{1}{2}bh$.

In this case $b = |\overrightarrow{DE}|$ and $h = |\overrightarrow{EF}|$

$$|\overrightarrow{DE}| = \sqrt{(-4)^2 + (2)^2 + (-1)^2} \\ = \sqrt{21}$$

$$|\overrightarrow{EF}| = \sqrt{(3)^2 + (4)^2 + (-4)^2} \\ = \sqrt{41}$$

6. Substitute the magnitude of the two vectors into the area rule and state the result.

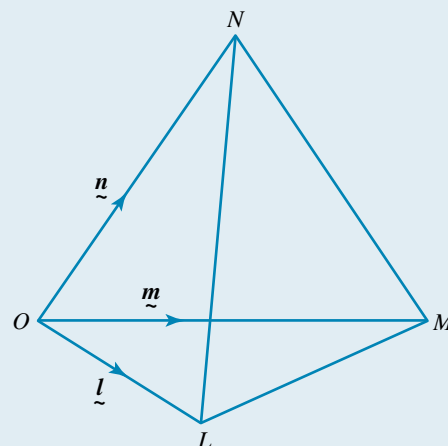
$$A = \frac{1}{2}|\overrightarrow{DE}||\overrightarrow{EF}| \\ = \frac{1}{2}\sqrt{21} \times \sqrt{41} \\ = \frac{1}{2}\sqrt{861} \text{ units}^2$$

WORKED EXAMPLE 12

A tetrahedron is defined by the vertices, O , L , M and N .

Given that, $\overrightarrow{OL} = \underline{l}$, $\overrightarrow{OM} = \underline{m}$ and $\overrightarrow{ON} = \underline{n}$

- Express $\overrightarrow{LM}$ in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$
- Express $\overrightarrow{MN}$ in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$
- Express $\overrightarrow{LN}$ in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$
- Develop an expression for the line joining the midpoints of OL to LN .



THINK

1. Choose a path from L to M that travels along edges defined as vectors $\underline{l}$, $\underline{m}$ or $\underline{n}$.
2. Recall that $\overrightarrow{LO} = -\overrightarrow{OL}$ (negative of a vector).
3. Express the RHS in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$.
1. Choose a path from M to N that travels along edges defined as vectors $\underline{l}$, $\underline{m}$ or $\underline{n}$.
2. Recall that $\overrightarrow{MO} = -\overrightarrow{OM}$ (negative of a vector).
3. Express the RHS in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$.
1. Choose a path from L to N that travels along edges defined as vectors $\underline{l}$, $\underline{m}$ or $\underline{n}$.
2. Recall that $\overrightarrow{LO} = -\overrightarrow{OL}$ (negative of a vector).
3. Express the RHS in terms of $\underline{l}$, $\underline{m}$ or $\underline{n}$.

WRITE

a. $\overrightarrow{LM} = \overrightarrow{LO} + \overrightarrow{OM}$

$$\overrightarrow{LM} = \overrightarrow{OM} - \overrightarrow{OL}$$

$$\overrightarrow{LM} = \underline{m} - \underline{l}$$

$$\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON}$$

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$

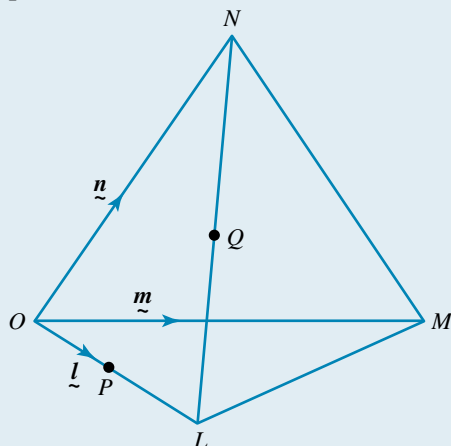
$$\overrightarrow{MN} = \underline{n} - \underline{m}$$

$$\overrightarrow{LN} = \overrightarrow{LO} + \overrightarrow{ON}$$

$$\overrightarrow{LN} = \overrightarrow{ON} - \overrightarrow{OL}$$

$$\overrightarrow{LN} = \underline{n} - \underline{l}$$

- d. 1. Define the midpoints of $\overrightarrow{OL}$ and $\overrightarrow{LN}$ as P and Q respectively. Draw a sketch to assist visualising the problem.



Let P be the midpoint of OL .
Let Q be the midpoint of LN .

2. $\overrightarrow{OP}$ is parallel to $\vec{l}$ ($\overrightarrow{OL}$) and is half its magnitude.

$$\overrightarrow{OP} = \frac{1}{2} \overrightarrow{OL} = \frac{1}{2} \vec{l}$$

3. $\overrightarrow{OQ}$ is the vector sum of $\overrightarrow{OL}$ and one half of $\overrightarrow{LN}$.

$$\overrightarrow{OQ} = \overrightarrow{OL} + \frac{1}{2} \overrightarrow{LN}$$

$\overrightarrow{OL}$ can be expressed as $\vec{l}$.

$$= \vec{l} + \frac{1}{2} (\vec{n} - \vec{l})$$

$\overrightarrow{LN}$ was defined as $\vec{n} - \vec{l}$.

$$= \vec{l} + \frac{1}{2} \vec{n} - \frac{1}{2} \vec{l}$$

Expand and simplify to then collect like vectors.

$$= \frac{1}{2} \vec{l} + \frac{1}{2} \vec{n}$$

4. $\overrightarrow{PQ}$ can be determined by subtracting $\overrightarrow{OP}$ from $\overrightarrow{OQ}$.

The vector joining P to Q is

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \left(\frac{1}{2} \vec{l} + \frac{1}{2} \vec{n} \right) - \left(\frac{1}{2} \vec{l} \right)$$

$$= \frac{1}{2} \vec{n}$$

study on

Units 3 & 4 > Area 2 > Sequence 1 > Concept 2

Geometric proofs using vectors Summary screen and practice questions

Exercise 2.3 Geometric proofs using vectors

Technology free

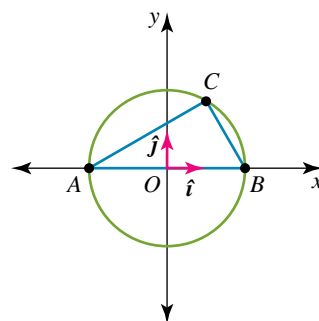
- WE9** $OABC$ is a rhombus. The points P, Q, R and S are the midpoints of the sides OA, AB, BC and OC respectively. Prove that $PQRS$ is a rectangle.
- WE10** Prove that if the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.
- WE11** Given the points $A(-3, 5, 4)$, $B(2, 3, 5)$ and $C(4, 6, 1)$, prove that ABC forms a right-angled triangle at B , and hence calculate the area of the triangle.

4. Given the points $A(4, 7, 3)$, $B(8, 7, 1)$ and $C(6, 5, 2)$:
- Show that ABC forms an isosceles triangle.
 - Let M be the midpoint of AB , and show that MC is perpendicular to AB .

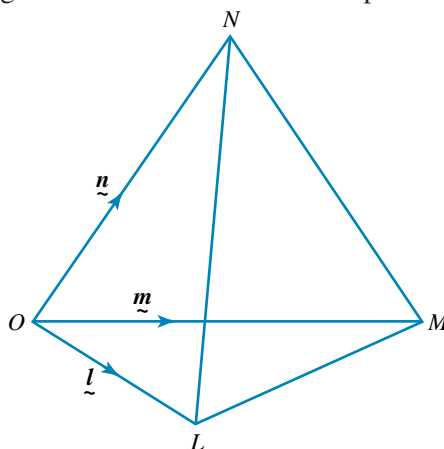
5. Prove Pythagoras' theorem.

6. a. Prove that the angle inscribed in a semicircle is a right angle.

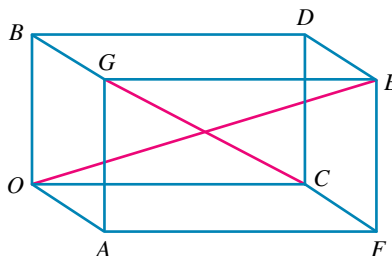
b. The diagram shows a circle of radius r with centre at the origin O on the x - and y -axes. The points A and B lie on the diameter of the circle and are on the x -axis; their coordinates are $(-r, 0)$ and $(r, 0)$ respectively. The point C has coordinates (a, b) and lies on the circle, where a , b and r are all positive real constants. Show that CA is perpendicular to CB .



7. Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
8. $OABC$ is a square. The points P , Q , R and S are the midpoints of the sides OA , AB , BC and OC respectively. Prove that $PQRS$ is a square.
9. Prove that the line segments joining the midpoints of consecutive sides of a rectangle form a rhombus.
10. AB and CD are two diameters of a circle with centre O . Letting $\overrightarrow{OA} = \underline{a}$ and $\overrightarrow{OC} = \underline{c}$, prove that $ACBD$ is a rectangle.
11. A median is formed inside a triangle when the midpoint of a side is joined to its opposite vertex. Prove that medians inside a triangle sum to zero.
12. Prove that the vector $\underline{\hat{a}} + \underline{\hat{b}}$ bisects the angle between the vectors $\underline{a}$ and $\underline{b}$.
13. Four points, $ABCD$, form a quadrilateral. Prove that the midpoints of the sides of the quadrilateral form a parallelogram.
14. A parallelogram is defined by the vectors $\underline{a}$ and $\underline{b}$. Prove the parallelogram law that states,
 $|\underline{a} + \underline{b}|^2 + |\underline{a} - \underline{b}|^2 = 2|\underline{a}|^2 + 2|\underline{b}|^2$.
15. **WE12** A tetrahedron is defined by the vertices, O , L , M and N . Given that, $\overrightarrow{OL} = \underline{l}$, $\overrightarrow{OM} = \underline{m}$ and $\overrightarrow{ON} = \underline{n}$, show that the line segment formed between the midpoints of $\overrightarrow{LN}$ to $\overrightarrow{MN}$ is parallel to the $\overrightarrow{LM}$.



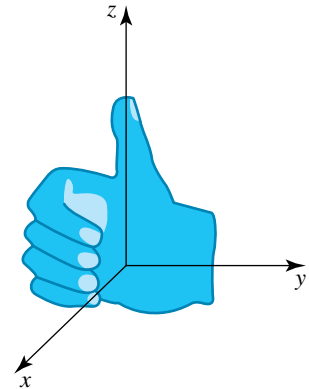
16. A storage container is formed in the shape of a rectangular prism. Two support beams are being installed on the inside of the container that would join O to E and G to C . Prove that the diagonals bisect one another, if and only if, the dimensions of the container are a Pythagorean triad.



2.4 Cartesian and parametric equations

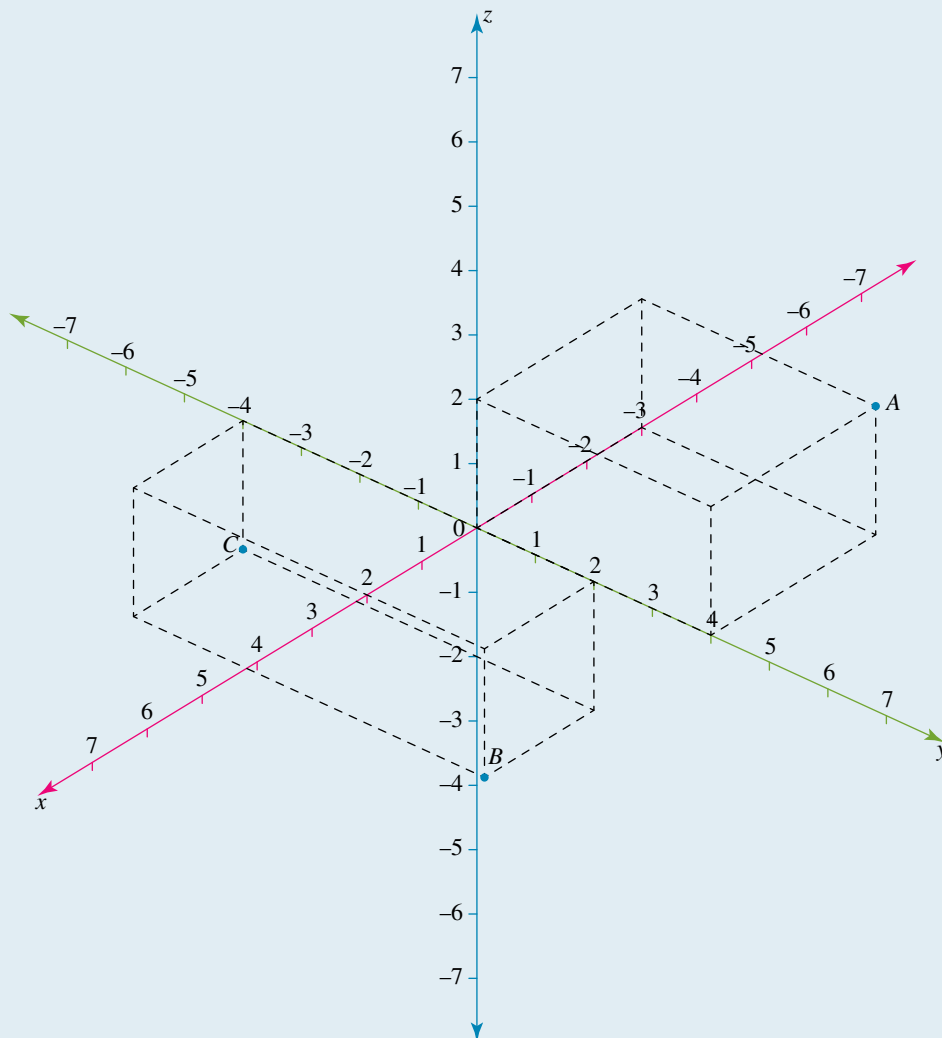
2.4.1 Cartesian coordinates for three-dimensional space

The three-dimensional coordinate system was introduced at the beginning of this chapter. The position of a point in three-dimensional space can be defined in terms of the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ and the point P can be written in the form of an ordered triple, $P(x, y, z)$. Three-dimensional axes are commonly drawn as a right-handed system - as the fingers on your right-hand curl from the x -axis to the y -axis, your thumb points in the direction of the positive z -axis.



WORKED EXAMPLE 13

Refer to the three-dimensional space below to state the coordinates of A , B and C .



THINK

1. Point A is 3 units in the negative direction of the x -axis, 4 units in the positive direction of the y -axis and 2 units in the positive direction of the z -axis.
2. Point B is 2 units in the positive direction of the x -axis, 2 units in the positive direction of the y -axis and 2 units in the negative direction of the z -axis.
3. Point C does not have an x -axis component, but is 4 units in the negative direction of the y -axis and 2 units in the negative direction of the z -axis.

WRITE

Point A is defined as, $A(-3, 4, 2)$

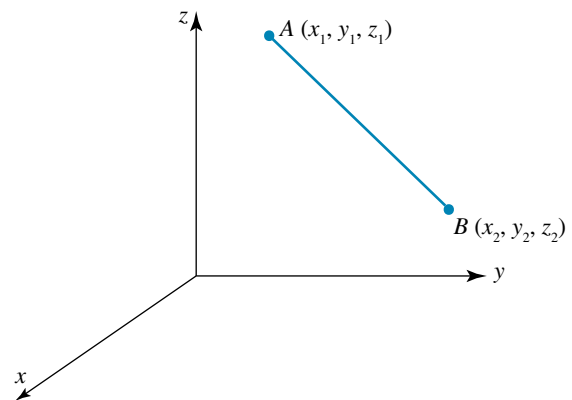
Point B is defined as, $B(2, 2, -2)$

Point C is defined as, $C(0, -4, -2)$

2.4.2 Distances between points in three-dimensions

Many of the formulae previously studied can be applied to three-dimensional Cartesian systems.

The distance between two points in three-dimensional space, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, can be calculated using Pythagoras's theorem twice to produce the formula for the distance between two points.



The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Like midpoint calculations in two-dimensional space, the midpoint M_{AB} of a line segment defined by the Cartesian coordinates $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ can be found by taking the average of each of the coordinates.

The midpoint of the line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

WORKED EXAMPLE 14

A line segment is formed between two points, $A(-2, 5, 8)$ and $B(-10, 11, -4)$.

- a. Calculate the exact distance between A and B .
- b. Determine the midpoint of the line segment AB .



THINK

- a. 1. Recall the distance formula for two points in three dimensional space.
2. Substitute the x, y, z coordinates into the formula for points A and B .
3. Express the distance in simplified surd form.
- b. 1. Recall the midpoint formula for three dimensions.
2. Substitute the x, y, z coordinates into the formula for points A and B .
3. Express the midpoint, M_{AB} as coordinates (x, y, z) .

WRITE

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(-10 - -2)^2 + (11 - 5)^2 + (-4 - 8)^2}$$

$$d = \sqrt{64 + 36 + 144}$$

$$d = \sqrt{244}$$

$$d = 2\sqrt{61} \text{ units}$$

$$M_{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

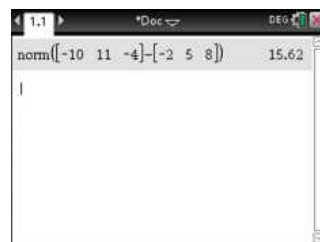
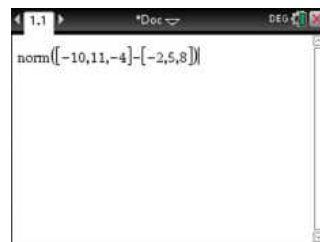
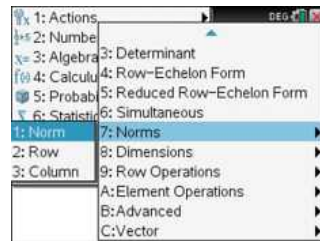
$$= \left(\frac{-2 + -10}{2}, \frac{5 + 11}{2}, \frac{8 + -4}{2} \right)$$

$$M_{AB} = (-6, 8, 2)$$

TI | THINK

- a. 1. On a Calculator page, select MENU
7: Matrix Vector
7: Norms
1: Norm
Press the ENTER button.
2. Complete the entry line
($[-10, 11, -4] - [-2, 5, 8]$)
3. Press the ENTER button.
The answer appears on the screen.

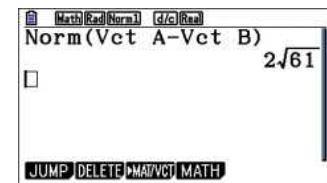
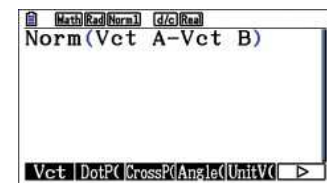
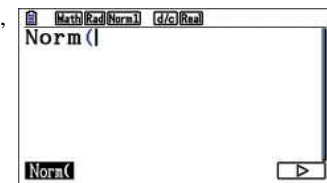
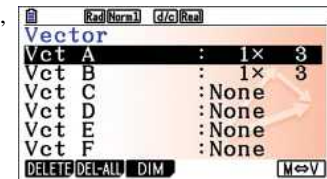
WRITE



CASIO | THINK

- a. 1. On a Run-Matrix screen, select MAT/VCT
Define Vectors A and B as shown in Worked example 1b.
2. On a Run-Matrix screen, select OPTN
MAT/VCT
▷ (x3 times)
Norm(
3. Complete the entry line as
Vct ALPHA
 X, θ, T -Vct
ALPHA
Log)
4. Press the EXE button.
The answer appears on the screen

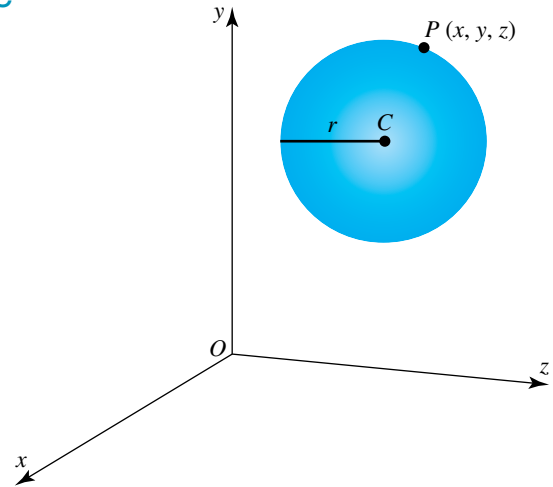
WRITE



2.4.3 The Cartesian equation of a sphere

The **sphere** can be defined, in formal terms, as the locus of points $P(x, y, z)$ that is equidistant from a given point C , where C is, $C(x_c, y_c, z_c)$. More simply, a sphere is simply the set of all points $P(x, y, z)$ that are an equal distance from the centre $C(x_c, y_c, z_c)$.

The equation of a sphere, with radius r and centre $C(x_c, y_c, z_c)$, can be determined by applying the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ where the distance d is the radius of the sphere, r .



If $P(x, y, z)$ is a point on the sphere, the equation for a sphere is

$$r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$$

WORKED EXAMPLE 15

- Determine the Cartesian equation for the sphere with centre $C(-3, 7, 2)$ and radius $r = 9$.
- Determine the Cartesian equation for the sphere that has $A(6, 12, 11)$ and $B(-2, 4, -1)$ as the endpoints of its diameter.
- Determine the centre and radius of the sphere with the Cartesian equation $x^2 + y^2 + z^2 - 12x + 4y + 8z + 20 = 0$

THINK

- Identify the coordinates of the centre of the circle. Express these values in terms of the variables x_c, y_c, z_c .
 - Square the given radius so that it can be expressed as r^2 .
 - Recall the general form for the equation of a sphere and substitute in the values.
- State the given points, which define the endpoints of a diameter.
 - The midpoint, M_{AB} , defines the center of the circle. Recall the rule for the midpoint and substitute the coordinates for points A and B .

WRITE

$$\begin{aligned} \text{a. } C(x_c, y_c, z_c) &= C(-3, 7, 2) \\ \therefore x_c &= -3, y_c = 7, z_c = 2 \end{aligned}$$

$$\begin{aligned} \text{Radius, } r &= 9 \\ \therefore r^2 &= 9^2 = 81 \\ r^2 &= (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 \\ 81 &= (x + 3)^2 + (y - 7)^2 + (z - 2)^2 \end{aligned}$$

$$\text{b. } A(6, 12, 11) \text{ and } B(-2, 4, -1)$$

$$\begin{aligned} M_{AB} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right) \\ &= \left(\frac{6 + (-2)}{2}, \frac{12 + 4}{2}, \frac{11 + (-1)}{2} \right) \\ &= (2, 8, 5) \\ \therefore \text{The centre is given by } C(2, 8, 5) \end{aligned}$$

3. The centre of the circle defines the value for the associated variables x_c, y_c, z_c .

$$C(x_c, y_c, z_c) = C(2, 8, 5)$$

$$x_c = 2, y_c = 8, z_c = 5$$

4. The radius of a sphere, r , is the distance from the centre to any point on its surface. The radius can then be defined by calculating the distance from the centre $C(2, 8, 5)$ to the point $A(6, 12, 11)$.

Calculate the distance from the centre to one of the endpoints of the diameter.

$$r = \sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2}$$

$$r = \sqrt{(6 - 2)^2 + (12 - 8)^2 + (11 - 5)^2}$$

$$r = \sqrt{16 + 16 + 36}$$

$$r = \sqrt{68}$$

$$r^2 = 68 \text{ units}$$

5. Recall the general form for the equation of a sphere and substitute in the values.

The Cartesian equation of the sphere is

$$r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$$

$$68 = (x - 2)^2 + (y - 8)^2 + (z - 5)^2$$

- c. 1. Complete the square on each of the x -, y - and z - components to rearrange the equation into the general form of a sphere $r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$. Remember to add any values required to complete the square to both side of the equation.

$$c. x^2 + y^2 + z^2 - 12x + 4y + 8z + 20 = 0$$

$$x^2 - 12x + y^2 + 4y + z^2 + 8z + 20 = 0$$

$$x^2 - 12x + y^2 + 4y + z^2 + 8z = -20$$

$$x^2 - 12x + \left(\frac{-12}{2}\right)^2 + y^2 + 4y + \left(\frac{4}{2}\right)^2 + z^2 + 8z + \left(\frac{8}{2}\right)^2$$

$$= -20 + \left(\frac{-12}{2}\right)^2 + \left(\frac{4}{2}\right)^2 + \left(\frac{8}{2}\right)^2$$

$$x^2 - 12x + 36 + y^2 + 4y + 4 + z^2 + 8z + 16$$

$$= -20 + 36 + 4 + 16$$

$$(x - 6)^2 + (y + 2)^2 + (z + 4)^2 = 36$$

$$(x - 6)^2 + (y - (-2))^2 + (z - (-4))^2 = 36$$

$$\therefore C(6, -2, -4)$$

2. The centre of the circle can be recognised from the equation generated.

3. To determine the radius, recall the general form of the sphere $r^2 = (x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2$.

The centre is $(6, -2, -4)$ with a radius of 6.

$$r^2 = 36$$

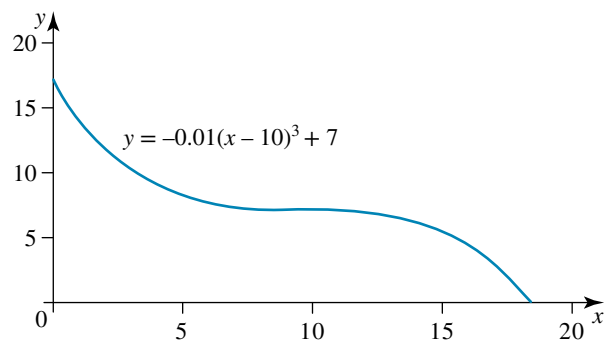
$$r = \sqrt{36}$$

$$= 6$$

2.4.4 Parametric equations in two-dimensions

A function defines a dependent variable, y , in terms of an independent variable x . These types of equations are in the form $y = f(x)$ and have graphs that are intersected only once by a vertical line. That is, for every x -value there is a unique y -value.

Consider the Cartesian equation, $y = -0.01(x - 10)^3 + 7$ shown graphically in the diagram. In this case, the function could model the position of a particle travelling along a water slide. The function can then define the point (x, y) , where the position of the particle is as a function of x .



The table exhibits this relationship as ordered pairs.

x	0	5	10	15
$y = f(x)$	17	8.25	7	5.75

While the path of the particle can be described by the Cartesian equation, $y = -0.01(x - 10)^3 + 7$, it would also be useful to know *when* the particle reaches a point location. This is achieved by introducing *time* as a third variable. Here, time is referred to as the **parameter**.

Written as a **parametric equation**, the location of the particle on the slide can be defined in terms of the parameter, that is:

$$x = t, y = -0.01(t - 10)^3 + 7$$

For each number t , there are corresponding values of x and y (and hence corresponding positions), shown in the table below.

t	0	1	5	10	18
$x = t$	0	1	5	10	18
$y = -0.01(t - 10)^3 + 7$	17	14.29	8.25	7	1.88
(x, y)	(0, 17)	(1, 14.29)	(5, 8.25)	(10, 7)	(18, 1.88)

Any letter can be used to represent the parameter. While the letter t is commonly used, it does not have to be defined as time. Where equations of curves are being examined, the angle θ is often the parameter.

If $f(t)$ and $g(t)$ are two continuous functions defined by the parameter t , then the equations $x = f(t)$ and $y = g(t)$ are referred to as parametric equations. The collection of points (x, y) obtained as t varies is called a **plane curve**.

Simple algebraic techniques can be used to convert a set of parametric equations into Cartesian form (and vice versa). The following examples will focus on parametric equations in two dimensions.

WORKED EXAMPLE 16

Determine the equivalent Cartesian equation for the following parametric equations.

a. $x = 2t, y = t + 3$

b. $x = t - 3, y = 2t^2$

c. $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

THINK

- a. 1. Define each parametric equation as equation [1] and [2].
2. To convert to a Cartesian equation, the parameter t needs to be eliminated. Solve for t in equation [1].
3. Substitute the solution for t into [2]. That is, t is being replaced with x .
The Cartesian equation represents the graph of a straight line.

WRITE

a. $x = 2t$ [1]
 $y = t + 3$ [2]
 $x = 2t$
 $t = \frac{x}{2}$
 $y = t + 3$ [2]
 $y = \frac{x}{2} + 3$

- b. 1.** Define each parametric equation as equation [1] and [2].
- 2.** The parameter t needs to be eliminated. Solve for t in [1].
- 3.** Substitute the solution for t into [2]. That is, t is being replaced with x .
The Cartesian equation represents the graph of a parabola.

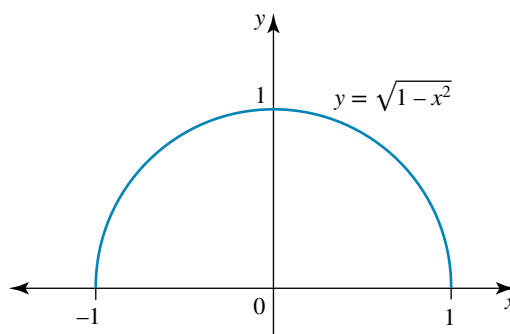
$$\begin{aligned}\text{b. } x &= t - 3 & [1] \\ y &= 2t^2 & [2] \\ x &= t - 3 \\ t &= x + 3 \\ y &= 2t^2 & [2] \\ y &= 2(x + 3)^2 \\ y &= 2(x^2 + 6x + 9) \\ y &= 2x^2 + 12x + 18\end{aligned}$$

- c. 1.** Define each parametric equation as equation [1] and [2].
- 2.** As the parameter t is not easily eliminated, a different approach is needed.
Try squaring both sides of equation [1] and [2].
- 3.** Add the equations, [1] + [2].
- 4.** Rewrite both fractions as the square of the numerators and denominators. Add the fractions as they have like denominators.
- 5.** The numerator can be factorised as $t^4 + 2t^2 + 1 = (t^2 + 1)^2$ using the substitution $\alpha = t^2$.
- 6.** Cancel like terms and state the relation. In this case it is a circle of radius 1.

$$\begin{aligned}\text{c. } x &= \frac{2t}{1+t^2} & [1] \\ y &= \frac{1-t^2}{1+t^2} & [2] \\ \text{Squaring [1] gives, } x^2 &= \left(\frac{2t}{1+t^2}\right)^2 \\ \text{Squaring [2] gives, } y^2 &= \left(\frac{1-t^2}{1+t^2}\right)^2 \\ x^2 + y^2 &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 \\ &= \frac{(2t)^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} \\ &= \frac{4t^2 + 1 - 2t^2 + t^4}{(1+t^2)^2} \\ &= \frac{t^4 + 2t^2 + 1}{(1+t^2)^2} \\ &= \frac{(t^2 + 1)^2}{(t^2 + 1)^2} \\ &= 1 \\ \therefore x^2 + y^2 &= 1\end{aligned}$$

Parametric equations are useful when describing the curve of complex graphs and relations. In Worked example 16c, the parametric equation described a circle. If the equation of a circle was written in the form $y = f(x)$, it would be $y = \sqrt{1 - x^2}$. When graphed, it produces the top half of a circle. The bottom half of the circle could be expressed as $y = -\sqrt{1 - x^2}$.

Curves such as circles and ellipses can be expressed in parametric form when the angle θ or A is the parameter. When parametric equations involve trigonometric functions, we often need to make use of standard trigonometric identities to eliminate the parameter. Recalling the Pythagorean identities, addition identities, double-angle identities and factorisation identities will help calculate and solve Cartesian equations of parametric equations.



1. The Pythagorean identities

$$\sin^2(A) + \cos^2(A) = 1$$

$$\sec^2(A) = 1 + \tan^2(A)$$

$$\operatorname{cosec}^2(A) = 1 + \cot^2(A)$$

3. The double-angle identities

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = 1 - 2 \sin^2(A)$$

$$\tan(2A) = \frac{2 \tan(A)}{1 - \tan^2(A)}$$

5. Multiple-angle identities

$$\sin(3A) = 3 \sin(A) - 4 \sin^3(A)$$

$$\cos(3A) = 4 \cos^3(A) - 3 \cos(A)$$

$$\sin(4A) = 4 \sin(A) \cos(A) - 8 \sin^3(A) \cos(A)$$

$$\cos(4A) = 8 \cos^4(A) - 8 \cos^2(A) + 1$$

7. Half-angle identities

$$\sin(A) = 2 \sin\left(\frac{A}{2}\right) \cos\left(\frac{A}{2}\right)$$

$$\cos(A) = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right)$$

$$= 2 \cos^2\left(\frac{A}{2}\right) - 1$$

$$= 1 - 2 \sin^2\left(\frac{A}{2}\right)$$

2. The addition identities

$$\sin(A \pm B) = \sin(A) \cos(B) \pm \cos(A) \sin(B)$$

$$\cos(A \pm B) = \cos(A) \cos(B) \mp \sin(A) \sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A) \tan(B)}$$

4. Angle sum and angle difference identities

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

$$\sin(A - B) = \sin(A) \cos(B) - \cos(A) \sin(B)$$

$$\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$$

$$\cos(A - B) = \cos(A) \cos(B) + \sin(A) \sin(B)$$

6. Product identities

$$\sin(A) \sin(B) = \frac{1}{2} (\cos(A - B) - \cos(A + B))$$

$$\cos(A) \cos(B) = \frac{1}{2} (\cos(A - B) + \cos(A + B))$$

$$\sin(A) \cos(B) = \frac{1}{2} (\sin(A + B) + \sin(A - B))$$

$$\cos(A) \sin(B) = \frac{1}{2} (\sin(A + B) - \sin(A - B))$$

WORKED EXAMPLE 17

Calculate the Cartesian equation for the following parametric equations:

a. $x = \cos \theta$, $y = \sin \theta$

b. $x = 3 \cos \theta$, $y = 5 \sin \theta$

c. $x = 2 \sec \theta$, $y = 3 \tan \theta$

d. $x = \sin \theta$, $y = \cos 2\theta$

THINK

- a. 1. Define each parametric equation as equation [1] and [2].
2. Square equations [1] and [2] to allow the Pythagorean identities to be used, and define new equations.
3. Add equations [3] and [4].

WRITE

a. $x = \cos \theta$	[1]
$y = \sin \theta$	[2]
$x^2 = \cos^2 \theta$	[3]
$y^2 = \sin^2 \theta$	[4]
$x^2 + y^2 = \cos^2 \theta + \sin^2 \theta$	[3] + [4]

4. Substitute the Pythagorean identity $\sin^2(A) + \cos^2(A) = 1$. The result is the Cartesian equation of a circle.

$$x^2 + y^2 = 1$$

- b. 1. Define each parametric equation as equation [1] and [2].

$$\begin{aligned} \text{b. } x &= 3 \cos \theta & [1] \\ y &= 5 \sin \theta & [2] \end{aligned}$$

2. Begin by squaring equations [1] and [2]. Rewrite the resulting equation in terms of either $\cos^2 \theta$ or $\sin^2 \theta$.

$$\begin{aligned} x^2 &= 9 \cos^2 \theta \\ \frac{x^2}{9} &= \cos^2 \theta & [3] \end{aligned}$$

$$y^2 = 25 \sin^2 \theta$$

3. Add equations [3] and [4].

$$\frac{y^2}{25} = \sin^2 \theta \quad [4]$$

$$\frac{x^2}{9} + \frac{y^2}{25} = \cos^2 \theta + \sin^2 \theta \quad [3] + [4]$$

4. Substitute the Pythagorean identity $\sin^2(A) + \cos^2(A) = 1$. The result is the Cartesian equation of an ellipse.

$$\frac{x^2}{9} + \frac{y^2}{25} = 1$$

- c. 1. Define each parametric equation as equation [1] and [2].

$$\begin{aligned} \text{c. } x &= 2 \sec \theta & [1] \\ y &= 3 \tan \theta & [2] \end{aligned}$$

2. Square equation [1] and rearrange the result making $\sec^2 \theta$ the subject.

$$x^2 = 4 \sec^2 \theta \therefore \sec^2 \theta = \frac{x^2}{4} \quad [3]$$

3. Square equation [2] and rearrange the result making $\tan^2 \theta$ the subject.

$$y^2 = 9 \tan^2 \theta \therefore \tan^2 \theta = \frac{y^2}{9} \quad [4]$$

4. Subtract equation [4] from [3],

$$\sec^2 \theta - \tan^2 \theta = \frac{x^2}{4} - \frac{y^2}{9} \quad [3] - [4]$$

5. Substitute the Pythagorean identity $\sec^2(A) - \tan^2(A) = 1$. The result is the Cartesian equation of a hyperbola.

$$\begin{aligned} \sec^2 \theta - \tan^2 \theta &= 1 \\ \frac{x^2}{4} - \frac{y^2}{9} &= 1 \end{aligned}$$

- d. 1. Define each parametric equation as equation [1] and [2].

$$\begin{aligned} \text{d. } x &= \sin \theta & [1] \\ y &= \cos 2\theta & [2] \end{aligned}$$

2. Begin by squaring equation [1].

$$x^2 = \sin^2 \theta \quad [3]$$

3. Recall the double-angle identity $\cos(2A) = 1 - 2\sin^2(A)$. Substitute into equation [2] then [3] to form the Cartesian equation.

$$\begin{aligned} \cos 2\theta &= 1 - 2\sin^2 \theta \\ y &= 1 - 2x^2 \end{aligned}$$

2.4.5 Parametric equations as vector functions

Recall the vector location, by definition, of a point $P(x, y)$, is given by the position vector $\overrightarrow{OP} = x\hat{i} + y\hat{j}$. Now consider a vector where the x - and y - components are not constants but vary dependent on a parameter, t . If we write $\underline{r}(t)$ to represent $\overrightarrow{OP}$, according to the parameter t , then the position of any particle can be defined by a parametric equation.

The position of any particle in two-dimensional space whose location varies according to a parameter, t can be defined by

$$\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$$

where $x(t)$ and $y(t)$ are the respective horizontal and vertical components of the position vector at any real value of the parameter t .

Equations written in this form are defined as a **vector function**. That is, the vector location of a particle is a function of t . In many instances, the parameter t , is time.

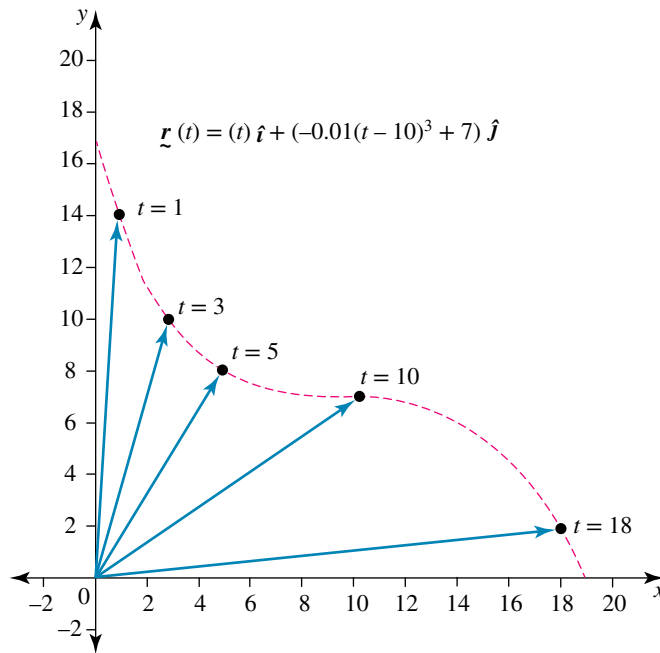
Returning to the example of the path of a particle on a water slide defined by the parametric equations

$$x = t, y = -0.01(t - 10)^3 + 7.$$

The parametric equation can then be used to define the path of the same particle as a vector function.

$$\underline{r}(t) = (t)\hat{i} + (-0.01(t - 10)^3 + 7)\hat{j}$$

A sketch of the particle's path can be depicted using vectors. For each new value of t , a new position vector, $\underline{r}(t)$, is formed.



It is common to define the domain for a vector function. In the case above, the curve was graphed for the interval, $0 \leq t \leq 18.88$.

WORKED EXAMPLE 18

Let a particle's position (metres), as a function of time (seconds), be given by

$$\underline{r}(t) = (t)\hat{i} + (-t^2 + 6t)\hat{j}, \text{ for } 0 \leq t \leq 4.$$

- Determine the particle's location at $t = 1$ second.
- Generate a table of values to model the position, $\underline{r}(t) = x\hat{i} + y\hat{j}$, of the particle for integer values of t .

THINK

- a. Substitute $t = 1$ into the equation $\underline{r}(t) = (t)\hat{i} + (-t^2 + 6t)\hat{j}$ to determine the particle's position at $t = 1$.
- b. The process used to determine $\underline{r}(1)$ is repeated for all valid integer values of t for the stated domain $0 \leq t \leq 4$. Summarise the results in table form.

WRITE

- a. For $\underline{r}(t)$, let $t = 1$.
- $$\underline{r}(t) = (t)\hat{i} + (-t^2 + 6t)\hat{j}$$
- $$\underline{r}(1) = (1)\hat{i} + (-(1)^2 + 6 \times 1)\hat{j}$$
- $$= \hat{i} + 5\hat{j}$$

b.

t	$x(t)$	$y(t)$	$\underline{r}(t)$
0	0	0	$0\hat{i} + 0\hat{j}$
1	1	5	$\hat{i} + 5\hat{j}$
2	2	8	$2\hat{i} + 8\hat{j}$
3	3	9	$3\hat{i} + 9\hat{j}$
4	4	8	$4\hat{i} + 8\hat{j}$

WORKED EXAMPLE 19

The position of a particle is given by the vector function $\underline{r}(t) = (2\sqrt{t})\hat{i} + (5t - 2)\hat{j}$, for $t \geq 0$.

- a. Express the position of the particle as a Cartesian equation.
- b. Calculate the domain and range of the equation, then graph the path of the particle on a Cartesian plane.
- c. At a certain point on the particles path, its location is $\underline{r}(t) = x\hat{i} + 8\hat{j}$. Use the vector function to calculate the value for x .

THINK

- a. 1. Define each parametric equation as equation [1] and [2].
2. The parameter t needs to be eliminated. Solve for t in the equation [1].
3. Substitute [3] into [2]. Simplify and state the Cartesian equation.

WRITE

$$x = 2\sqrt{t} \quad [1]$$

$$y = 5t - 2 \quad [2]$$

$$x = 2\sqrt{t}$$

$$\frac{x}{2} = \sqrt{t}$$

$$t = \left(\frac{x}{2}\right)^2$$

$$t = \frac{x^2}{4} \quad [3]$$

Substitute [3] into [2].

$$t = \frac{x^2}{4} \quad [3]$$

$$y = 5t - 2 \quad [2]$$

$$y = 5\left(\frac{x^2}{4}\right) - 2$$

$$y = \frac{5x^2}{4} - 2$$

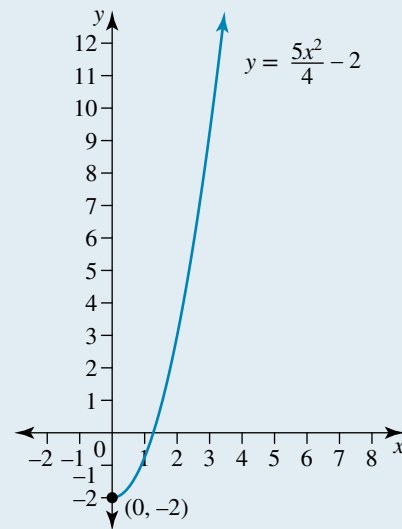
- b. 1. As $t \geq 0$ calculate the domain and range by substituting $t = 0$ into $x = 2\sqrt{t}$ and $y = 5t - 2$.

- b. When $t = 0$

$$\begin{aligned}x &= 2\sqrt{t} \\&= 2\sqrt{0} \\&= 0 \\y &= 5t - 2 \\&= 5 \times 0 - 2 \\&= -2\end{aligned}$$

As $t \geq 0$, it follows from the parametric equations that $x \geq 0$ and $y \geq -2$.

2. Construct the graph of the Cartesian equation using suitable technology.



- c. 1. Equate the two position vectors.
2. Equate the y -component of $r(t) = x\hat{i} + 8\hat{j}$ with the known parametric equation $y = (5t - 2)\hat{j}$
3. Solve for t .

4. Determine the value of x by evaluating the parametric equation for when $t = 2$

5. State the answer.

c. $2\sqrt{t}\hat{i} + (5t - 2)\hat{j} = x\hat{i} + 8\hat{j}$
 $(5t - 2)\hat{j} = 8\hat{j}$

$$\begin{aligned}5t - 2 &= 8 \\t &= \frac{8 + 2}{5}\end{aligned}$$

$$t = 2$$

$x\hat{i}$ can be calculated by substituting

$$\begin{aligned}t = 2 \text{ in } x &= 2\sqrt{t} \\x &= 2\sqrt{2} \\x &= 2\sqrt{2}\end{aligned}$$

Parametric equations as vector functions in three dimensions

The use of parameters can be extended to three-dimensional spaces. The vector function, $\underline{r}(t)$, would contain the parametric equations,

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

The position of any particle in three-dimensional space whose location varies according to a parameter, t , would consist of three components

$$\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

where $x(t)$, $y(t)$, $z(t)$ are the respective components of the position vector at any real value of the parameter t .

The procedures studied in the two-dimensional examples, are applicable in three dimensions, as shown in the following worked example.

WORKED EXAMPLE 20

The position (metres) of a bird is given by $\underline{r}(t) = (2t)\hat{i} + (0.01t^3)\hat{j} + (0.5t^2)\hat{k}$ for $t \geq 0$ seconds.

- Determine the position of the bird at $t = 0$ and $t = 15$ seconds.
- Calculate the straight-line distance of the bird from the origin at $t = 15$ seconds, to two decimal places.

THINK

- Substitute $t = 0$ into the equation $\underline{r}(t) = (2t)\hat{i} + (0.01t^3)\hat{j} + (0.5t^2)\hat{k}$ to determine the bird's position at $t = 0$.
 - Substitute $t = 15$ into the equation $\underline{r}(t) = (2t)\hat{i} + (0.01t^3)\hat{j} + (0.5t^2)\hat{k}$ to determine the bird's position at $t = 15$.
- The distance of the bird from the origin at $t = 15$ is the magnitude of the position vector at $t = 15$.
Recall the magnitude formula for three-dimensional vectors
 $|\underline{u}| = \sqrt{x^2 + y^2 + z^2}$.
 - Conclude with a statement.

WRITE

- Let $t = 0$.

$$\underline{r}(0) = (2 \times 0)\hat{i} + (0.01(0)^3)\hat{j} + (0.5(0)^2)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$
Hence, at $t = 0$ the bird's position is the origin.
Let $t = 15$.

$$\underline{r}(15) = (2 \times 15)\hat{i} + (0.01(15)^3)\hat{j} + (0.5(15)^2)\hat{k}$$

$$= 30\hat{i} + 33.75\hat{j} + 112.5\hat{k}$$
Hence at $t = 15$, the bird's position is $30\hat{i} + 33.75\hat{j} + 112.5\hat{k}$.
- $$|\underline{r}(15)| = \sqrt{x^2 + y^2 + z^2}$$

$$= \sqrt{(30)^2 + (33.75)^2 + (112.5)^2}$$

$$= 121.22$$

After 15 seconds of flight the bird is 121.22 metres from the origin.

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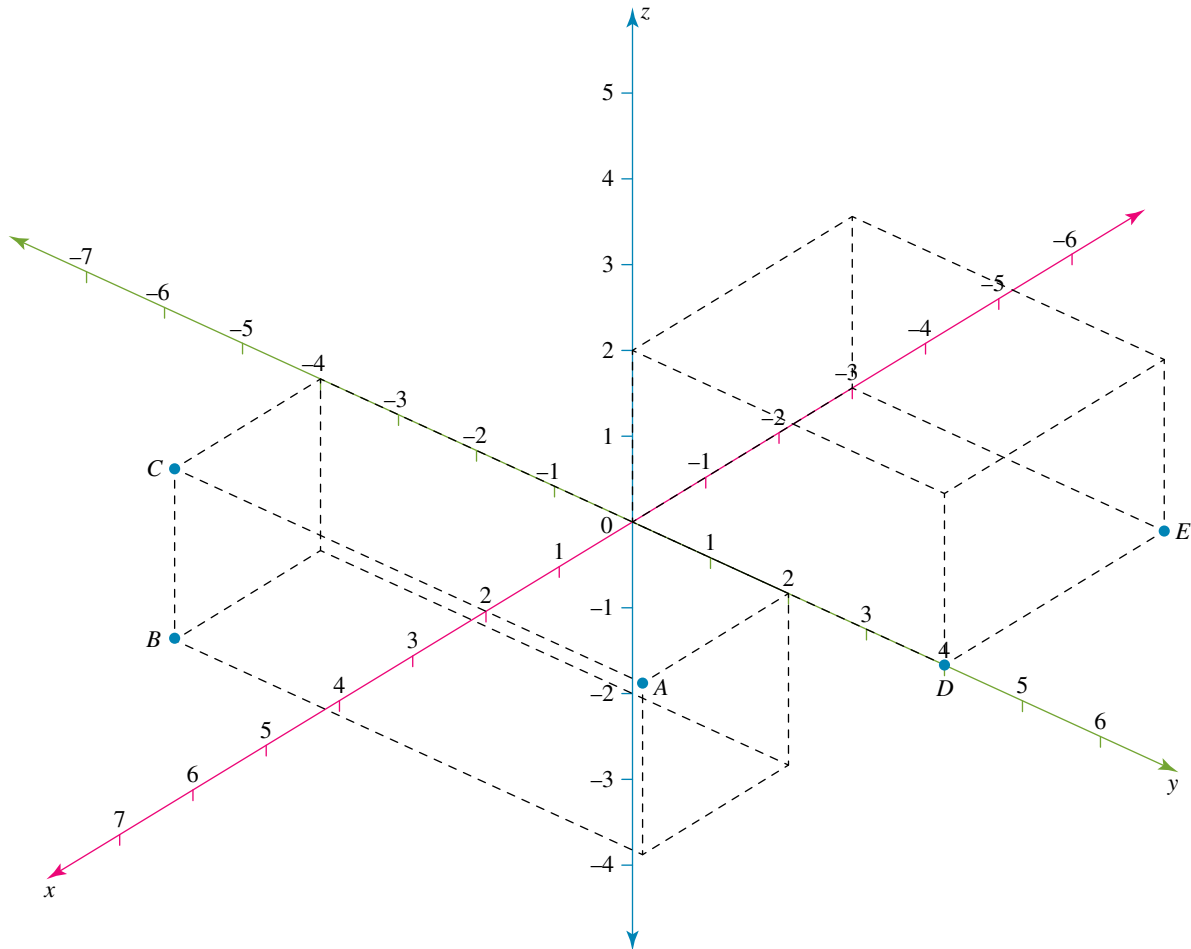
Units 3 & 4 > Area 2 > Sequence 1 > Concept 3

Cartesian and parametric equations Summary screen and practice questions

Exercise 2.4 Cartesian and parametric equations

Technology free

1. **WE13** Refer to the three-dimensional space below to state the coordinates of the following points.
 a. Point A b. Point B c. Point C d. Point D e. Point E



2. **WE14a** A line segment is formed between two points, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Calculate the exact distance between A and B.
 a. $A(-5, 8, 12)$ and $B(7, 15, 21)$ b. $A(7, -2, 9)$ and $B(3, 1, -2)$
 c. $A(5, 2, 1)$ and $B(-3, 7, -11)$ d. $A(1, -2, 7)$ and $B(-9, 3, -2)$
3. **WE14b** A line segment is formed between two points, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Determine the midpoint of the line segment AB.
 a. $A(5, 2, -7)$ and $B(9, 12, 5)$ b. $A(-10, 2, -2)$ and $B(4, 6, -12)$
 c. $A(7, -2, 10)$ and $B(1, -10, -2)$ d. $A(2, 2, 0)$ and $B(9, -6, -9)$
4. **WE15a** Determine the Cartesian equation for the sphere with centre $C(x_c, y_c, z_c)$ and radius r .
 a. $r = 6$ and $C(2, 8, -3)$ b. $r = 3$ and $C(-3, 1, -9)$
 c. $r = 7$ and $C(1, 0, 6)$ d. $r = \sqrt{11}$ and $C(-9, -2, 7)$
5. **WE15b** Determine the Cartesian equation for the sphere that has $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as the endpoints of its diameter.
 a. $A(7, 13, 8)$ and $B(-1, 3, 2)$ b. $A(-5, -3, -1)$ and $B(3, 1, 11)$
 c. $A(2, -4, 5)$ and $B(-8, 0, -25)$ d. $A(1, -4, 8)$ and $B(9, -8, -2)$

6. Express the following equations in the expanded form of a sphere, $x^2 + y^2 + z^2 + ax + by + cz + d = 0$.
- $(x - 5)^2 + (y - 7)^2 + (z - 1)^2 = 144$
 - $(x + 1)^2 + (y - 3)^2 + (z + 2)^2 = 81$
 - $(x + 3)^2 + (y - 1)^2 + z^2 = 15$
 - $(x - 1)^2 + (y - 4)^2 + (z + 3)^2 = 56$
7. **WE15c** Each of the following equations have been expressed in the expanded form of a sphere. Determine the centre and radius of the following spheres.
- $x^2 - 10x + y^2 + 4y + z^2 + 12z - 16 = 0$
 - $x^2 + y^2 + z^2 + 2x + 8y - 16z = 0$
 - $x^2 + y^2 + z^2 + 20x - 12y + 24z + 180 = 0$
 - $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$
8. **WE16** Determine the equivalent Cartesian equation for the following parametric equations.
- $x = 3t + 2, y = 2t + 5$
 - $x = t - 5, y = -t^2 + t$
 - $x = 2t - 1, y = 4t^2 + 7$
 - $x = t^3, y = \frac{t^2}{4}$
 - $x = 4 + \frac{1}{t}, y = 4 - t$
 - $x = \frac{2}{\sqrt{t+1}}, y = \frac{t}{t+1}$
9. **WE17** Calculate the Cartesian equation for the following parametric equations.
- $x = \sin \theta, y = \cos \theta$
 - $x = \cos 3\theta, y = \sin 3\theta$
 - $x = 4 \cos \theta, y = 7 \sin \theta$
 - $x = \operatorname{cosec} \theta, y = \cot \theta$
10. Determine the equivalent Cartesian equation for the following vector functions.
- $\underline{r}(t) = 5t \hat{i} + \frac{t}{3} \hat{j}$
 - $\underline{r}(t) = 4t \hat{i} + (t^2 - t) \hat{j}$
 - $\underline{r}(t) = 3t \hat{i} + (t^3 + 2) \hat{j}$
 - $\underline{r}(t) = (2 \cos t) \hat{i} + (2 \sin t) \hat{j}$
11. **WE18** A particle's position (metres), as a function of time (seconds), is given by $\underline{r}(t) = (t) \hat{i} + (-0.5t^2 + 4t) \hat{j}$, for $0 \leq t \leq 4$.
- Determine the particle's location at $t = 1$ second.
 - Generate a table of integer values to model the position, $\underline{r}(t) = x \hat{i} + y \hat{j}$, of the particle for integer values of t .

Technology active

12. Let a particle's position (metres), as a function of time (seconds), is given by $\underline{r}(t) = (2t) \hat{i} + (t^2 - 8t + 16) \hat{j}$, for $0 \leq t \leq 5$.
- Determine the particle's location at $t = 3$ s.
 - Generate a table of integer values to model the position of the object for integer values of t .
 - Calculate a Cartesian equation and determine the domain and range.
 - Graph the position of the particle on a Cartesian plane.
13. **WE19** The position of a particle is given by the vector function $\underline{r}(t) = \left(\sqrt[3]{\frac{t}{4}}\right) \hat{i} + (2t + 7) \hat{j}$, for $t \geq 0$.
- Express the position of the particle as a Cartesian equation.
 - Calculate the domain and range of the equation, then graph the path of the particle on a Cartesian plane.
 - At a certain point on the particle's path, its location is $\underline{r}(t) = x \hat{i} + 71 \hat{j}$. Use the vector function to calculate the value for x .

14. The position of an object is defined by the vector function, $\underline{r}(t) = (3 - 5t)\hat{i} + (2 + 3t)\hat{j} + (t + 1)\hat{k}$, for $t \in \mathbb{R}$.
- Generate a table of values for the position of the object at $t = 1$, $t = 3$ and $t = 5$.
 - Sketch the graph of the vector function to describe any interesting features.
15. The position of two particles are defined by the following equations,

$$\underline{r}_a(t) = 5 \sin(t)\hat{i} + 5 \cos(t)\hat{j} + 10\hat{k}$$

$$\underline{r}_b(t) = 6 \sin(t)\hat{i} + 6 \cos(t)\hat{j} - 2\hat{k}$$

- Use appropriate technology to sketch the position of both particles.
 - Compare the two graphs to describe any similarities or differences.
16. **WE20** The position (metres) of a bird is given by $\underline{r}(t) = (3t)\hat{i} + (0.02t^3)\hat{j} + (0.4t^2)\hat{k}$ for $t \geq 0$ seconds.
- Determine the position of the bird at $t = 0$ and $t = 12$ seconds.
 - Calculate the straight-line distance of the bird from the origin at $t = 12$ seconds.
17. A paragliding enthusiast begins their flight from a cliff face. The path of the paraglider is given by $\underline{r}(t) = (850 - 15t)\hat{i} + (950 - 0.4t^2)\hat{j} + (350 - 0.01t^3)\hat{k}$ for $0 \leq t \leq b$. The pilot aims to finish the flight somewhere near the airport, defined as the point $O(0, 0, 0)$.
- Calculate the initial position of the paraglider.
 - Determine the domain by calculating the largest possible value for b . State any assumptions made in developing a solution.
 - Determine the ground level distance the pilot is from the airport when the paraglider lands.
 - Analyse the flight path of the paraglider and determine if it at any point it ascended in altitude.
- 
18. For each of the following vector equations, use technology to sketch the equation of the path.
- $x = \cos(4t), y = \sin(6t)$ for $t \geq 0$
 - $x = \sin(5t), y = \cos(t)$ for $t \geq 0$
 - $x = -\cos(2t), y = \sin(3t)$ for $t \geq 0$
19. For each of the following vector equations, use technology to sketch the equation of the path.
- The Spiral: $\underline{r}(t) = t \cos(t)\hat{i} + t \sin(t)\hat{j}$, for $0 \leq t \leq 6\pi$
 - The Cycloid: $\underline{r}(t) = 4(t - \sin(t))\hat{i} + 4(1 - \cos(t))\hat{j}$, for $-2\pi \leq t \leq 4\pi$
 - The Witch of Agnesi: $\underline{r}(t) = 2 \cot(t)\hat{i} + 2 \sin^2(t)\hat{j}$, for $-\pi \leq t \leq \pi$
 - Lissajous curve: $\underline{r}(t) = 3 \cos(t)\hat{i} + 2 \sin(2t)\hat{j}$, for $-\pi \leq t \leq \pi$
20. The path of particle is defined by the vector function, $\underline{r}(t) = \frac{3t}{1+t^3}\hat{i} + \frac{3t^2}{1+t^3}\hat{j}$, for $-10 \leq t \leq 10$.
Show that the Cartesian equation is given by $x^3 + y^3 - 3xy = 0$

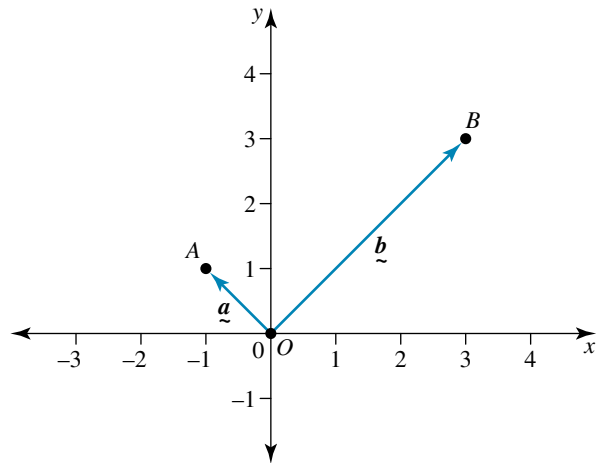
2.5 The vector equation of a straight line

In your studies thus far, the equation of a line has been considered only in terms of a scalar equation in the form $y = mx + c$. Equations in this form describe the straight-line relationship existing between two variables (x, y). We will now explore how position vectors can be used to define a straight line in both two and three dimensions.

2.5.1 Lines in two dimensions

Consider the two-dimensional plane containing the points $A(-1, 1)$ and $B(3, 3)$. The location of these points can be expressed as position vectors:

- $\overrightarrow{OA} = \underline{a} = -1\hat{i} + 1\hat{j}$
- $\overrightarrow{OB} = \underline{b} = 3\hat{i} + 3\hat{j}$

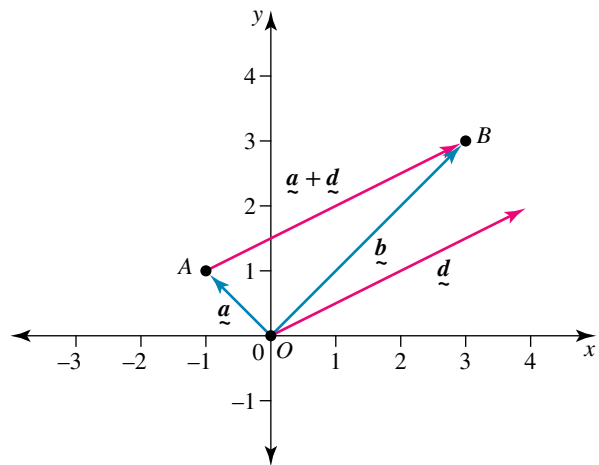


Similarly, the location of the vector connecting the two points is defined as

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \underline{d} = 4\hat{i} + 2\hat{j}$$

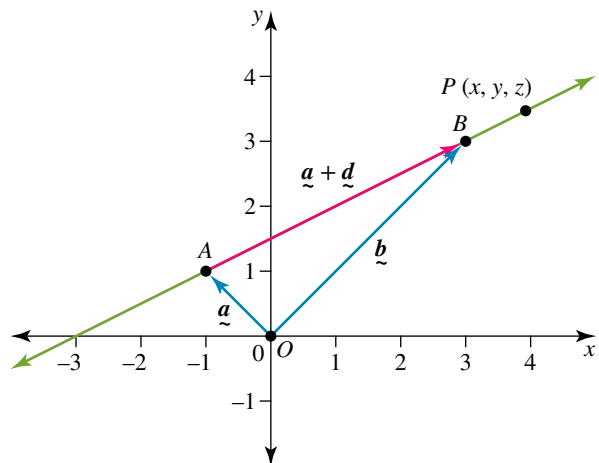
The **line segment** joining points A and B can now be described with the two vectors, $\underline{a}$ and $\underline{d}$.

- The 'starting point' is the head of position vector $\underline{a} = -1\hat{i} + 1\hat{j}$
- The line segment is formed by travelling (adding) in the direction of $\underline{d} = 4\hat{i} + 2\hat{j}$.



Vectors can also be used to define a line in both two and three dimensions. In this example a line can be formed by extending the line segment connecting A and B .

The position of any point $P(x, y, z)$ on this line can be calculated by adding scalar multiples of $\underline{d}$ to the initial position vector $\underline{a}$.



2.5.2 The vector equation of a straight line in two dimensions

The **vector equation** of a line is given by $\underline{r} = \underline{a} + k\underline{d}$.

Where:

- $\underline{r}$ is the position vector of any point on the line
- $\underline{a}$ is the position vector of a known point on the line
- $\underline{d}$ is the vector travelling in the same direction (parallel) to the line
- the varying (scalar) parameter is k , where $k \in \mathbb{R}$.

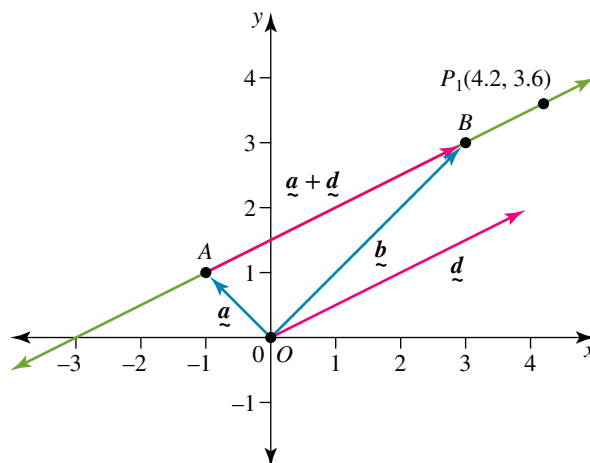
In changing the parameter k , the vector $\underline{r}$ can have its head at any point on the line. Consider again the two-dimensional plane containing the points $A(-1, 1)$ and $B(3, 3)$.

- The line between points A and B is defined by the vector equation, $\underline{r} = (-1\hat{i} + 1\hat{j}) + k(4\hat{i} + 2\hat{j})$.
- Two points on the line will be located by varying the parameter k .

Scenario 1: $k = 1.3$

$$\begin{aligned}\underline{r} &= (-1\hat{i} + 1\hat{j}) + 1.3(4\hat{i} + 2\hat{j}) \\ &= -1\hat{i} + 1\hat{j} + 5.2\hat{i} + 2.6\hat{j} \\ &= 4.2\hat{i} + 3.6\hat{j}\end{aligned}$$

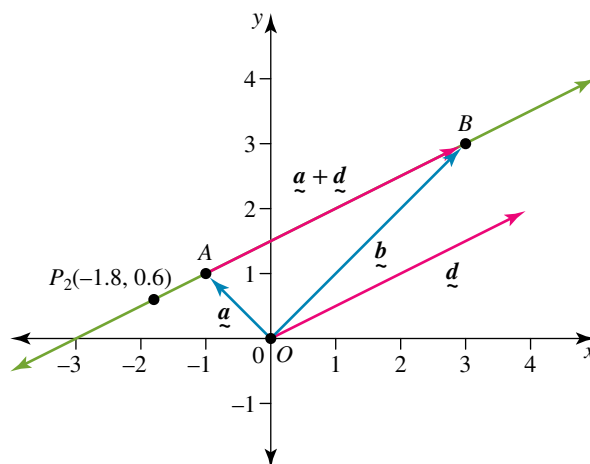
$$\therefore P_1(x, y) = (4.2, 3.6)$$



Scenario 2: $k = -0.2$

$$\begin{aligned}\underline{r} &= (-1\hat{i} + 1\hat{j}) - 0.2(4\hat{i} + 2\hat{j}) \\ &= -1\hat{i} + 1\hat{j} - 0.8\hat{i} - 0.4\hat{j} \\ &= -1.8\hat{i} + 0.6\hat{j}\end{aligned}$$

$$\therefore P_2(x, y) = (-1.8, 0.6)$$



WORKED EXAMPLE 21

- Determine a vector equation, in the two-dimensional space (x, y) , for the line passing through the points $A(-1, 7)$ and $B(4, -2)$.
- Calculate the vector location of the point $\underline{r}$ on the line if the parameter $k = 2$.

THINK

1. Define the position vectors $\underline{a}$ and $\underline{b}$.
2. Calculate the direction vector, $\underline{d}$, between points A and B . This vector is parallel to the line.
3. Recall the formula for the vector equation of a line and substitute in the expressions for $\underline{d}$ and $\underline{a}$. The vector equation of the line has been determined.

WRITE

$$\begin{aligned}\text{a. } \overrightarrow{OA} &= \underline{a} = -1\hat{i} + 7\hat{j} \\ \overrightarrow{OB} &= \underline{b} = 4\hat{i} - 2\hat{j} \\ \underline{d} &= \underline{b} - \underline{a} \\ \underline{d} &= (4\hat{i} - 2\hat{j}) - (-1\hat{i} + 7\hat{j}) \\ \underline{d} &= 5\hat{i} - 9\hat{j} \\ \underline{r} &= \underline{a} + k\underline{d} \\ \underline{r} &= (-\hat{i} + 7\hat{j}) + k(5\hat{i} - 9\hat{j})\end{aligned}$$

- b. The location of $\underline{r}$ can be calculated by substituting $k = 2$ into the vector equation. The solution is expressed as a position vector.

$$\begin{aligned}\text{b. Let } k &= 2 \\ \underline{r} &= (-\hat{i} + 7\hat{j}) + 2(5\hat{i} - 9\hat{j}) \\ &= (-\hat{i} + 7\hat{j}) + (10\hat{i} - 18\hat{j}) \\ &= 9\hat{i} - 11\hat{j}\end{aligned}$$

2.5.3 Lines in three dimensions

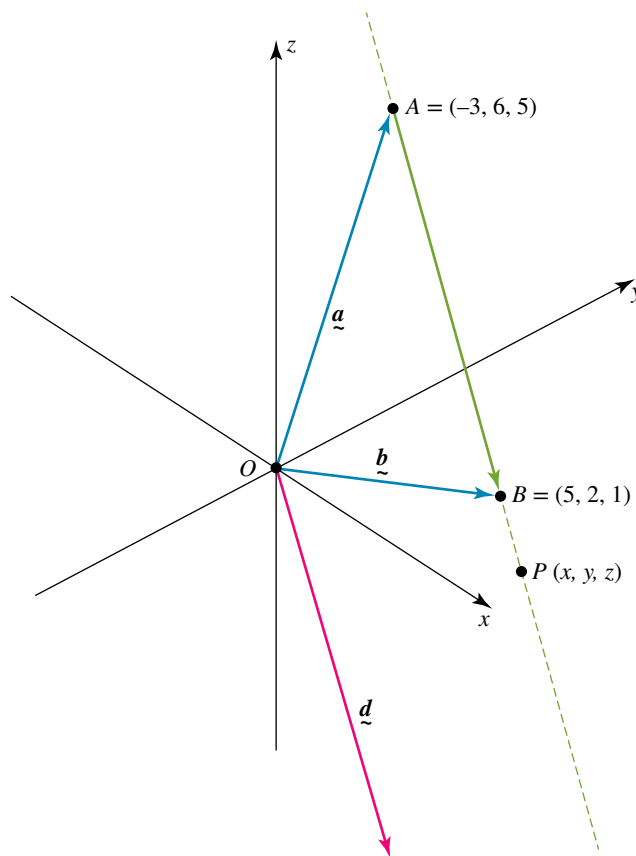
Vector equations can be developed in three-dimensional space providing we have:

- the position vector $\underline{a} = (x_1\hat{i}, y_1\hat{j}, z_1\hat{k})$ that corresponds to a point on the line and,
- a second vector $\underline{d} = (x_2\hat{i}, y_2\hat{j}, z_2\hat{k})$ that is a vector in the direction of the line. Alternatively,
- the equation of a line can be defined if two points (A and B) on the same line are known, as shown in the diagram.

As for two-dimensional space, the vector equation $\underline{r} = \underline{a} + k\underline{d}$ will define the location of any point $P(x, y, z)$ in a three-dimensional space, in most instances.

Consider any point $P(x, y, z)$ which lies along a line and can be located in terms of $\underline{a}$ and $\underline{d}$, where:

- $\underline{d}$ is the position vector between the points A and B .
- The line containing points A and B and runs in the same direction as $\underline{d}$.
- The position of any point $P(x, y, z)$ can be determined by starting at $\underline{a}$ and then travelling along the line in scalar (k) products of $\underline{d}$.
- Hence, the formula is given by $\underline{r} = \underline{a} + k\underline{d}$.



WORKED EXAMPLE 22

- Determine a vector equation, in the three-dimensional space (x, y, z) , for the line passing through the points $A(-3, 6, 5)$ and $B(5, 2, 1)$.
- Verify if the point $C(65, -28, -29)$ exists on the line.
- The position vector $\underline{e} = -27\hat{i} + 18\hat{j} + z\hat{k}$ terminates on the line formed between points A and B . Calculate the value of z .

THINK

- a. 1. Define the position vectors $\underline{a}$ and $\underline{b}$.

WRITE

$$\begin{aligned}\text{a. } \overrightarrow{OA} &= \underline{a} = -3\hat{i} + 6\hat{j} + 5\hat{k} \\ \overrightarrow{OB} &= \underline{b} = 5\hat{i} + 2\hat{j} + 1\hat{k}\end{aligned}$$

2. Calculate the direction vector, $\underline{d}$, between points A and B . This vector is parallel to the line.
3. Recall the formula for the vector equation of a line and substitute in the expressions for $\underline{d}$ and $\underline{a}$. The vector equation of the line has been determined.
- b. 1. In most instances it is possible to verify a solution by substituting values into an equation. As k is unknown this is not possible. The first step is to equate one of the components $\underline{r}(\hat{i}, \hat{j}, \hat{k})$ with the point $C(65, -28, -29)$.
2. Solve for k .
3. Substitute $k = 8.5$ into the vector equation $\underline{r}$ to determine if the point $C(65, -28, -29)$ lies on the line.
- c. 1. The vector equation of the line was developed in part a.
Determine the value of k by equating either the $\hat{i}$ or $\hat{j}$ components from the vectors $\underline{e}$ and $\underline{r}$.
2. Substitute $k = -3$ into the vector equation $\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$ to determine the position vector $\underline{e}$.
3. Equate the vectors $\underline{r}$ and $\underline{e}$.
4. Solve for z by equating the $\hat{k}$ components.
- $$\underline{d} = \underline{b} - \underline{a}$$
- $$\underline{d} = (5\hat{i} + 2\hat{j} + 1\hat{k}) - (-3\hat{i} + 6\hat{j} + 5\hat{k})$$
- $$\underline{d} = 8\hat{i} - 4\hat{j} - 4\hat{k}$$
- $$\underline{r} = \underline{a} + k\underline{d}$$
- $$\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$$
- b. $\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$
 $C(65, -28, -29)$
 Equate the $\hat{i}$ components to calculate k .
 $65 = -3 + 8k$
- $$65 = -3 + 8k$$
- $$\frac{65 + 3}{8} = k$$
- $$k = 8.5$$
- $$\underline{r}(8.5) = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + 8.5(8\hat{i} - 4\hat{j} - 4\hat{k})$$
- $$= (-3\hat{i} + 6\hat{j} + 5\hat{k}) + (68\hat{i} - 34\hat{j} - 34\hat{k})$$
- $$= 65\hat{i} - 28\hat{j} - 29\hat{k}$$
- $\therefore C(65, -28, -29)$ does lie on the line.
- c. $\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$
 $\underline{e} = -27\hat{i} + 18\hat{j} + z\hat{k}$
 Equate the $\hat{i}$ components to calculate k .
 $-27 = -3 + 8k$
 $\frac{-27 + 3}{-8} = k$
 $k = -3$
- Let $k = -3$
- $$\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + -3(8\hat{i} - 4\hat{j} - 4\hat{k})$$
- $$= -27\hat{i} + 18\hat{j} + 17\hat{k}$$
- Hence the position vector $\underline{e}$ is
- $$\underline{e} = -27\hat{i} + 18\hat{j} + 17\hat{k}$$
- $$-27\hat{i} + 18\hat{j} + 17\hat{k} = -27\hat{i} + 18\hat{j} + z\hat{k}$$
- $$z\hat{k} = 17\hat{k}$$
- $$z = 17$$

2.5.4 Vector equations in parametric and Cartesian form

We have seen how lines can be expressed as vector equations in the form $\underline{r} = \underline{a} + k\underline{d}$. These equations may also be expressed in either parametric or Cartesian form. To demonstrate this, the vector equation from Worked example 22 will be used.

Parametric form

- The vector equation was $\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$
- Points on the line are in the form $P(x, y, z)$
- Therefore, any point on the line is given by, $(x, y, z) = (-3 + 8k, 6 - 4k, 5 - 4k)$

The vector equation can be rewritten in parametric form by equating the components:

$$x = -3 + 8k, \quad y = 6 - 4k, \quad z = 5 - 4k$$

The parametric equation of a line in three-dimensional space is given by the general formula

$$x = a_1 + d_1k, \quad y = a_2 + d_2k, \quad z = a_3 + d_3k$$

Cartesian form

The position of the point $P(x, y, z)$ is defined by the parameter k . Consequently, each of the equations that exist in the parametric form of a line, can be rewritten in terms of k .

$$\begin{aligned} x &= -3 + 8k & y &= 6 - 4k & z &= 5 - 4k \\ \frac{x + 3}{8} &= k & \frac{y - 6}{-4} &= k & \frac{z - 5}{-4} &= k \\ k &= \frac{x + 3}{8} & k &= \frac{y - 6}{-4} & k &= \frac{z - 5}{-4} \end{aligned}$$

The line connecting points A and B can now be written in Cartesian form as the value for k is equivalent in the three equations above. The Cartesian equation is given by,

$$\begin{aligned} \frac{x + 3}{8} &= \frac{y - 6}{-4} = \frac{z - 5}{-4} = k \\ \frac{x + 3}{8} &= \frac{y - 6}{-4} = \frac{z - 5}{-4} \end{aligned}$$

The Cartesian equation of a line is likewise referred to as the **symmetric equation of a line**.

The Cartesian equation of line in three-dimensional space is given by the general formula

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

In this form, the line passes through the point $(x, y, z) = (a_1, a_2, a_3)$, and is parallel to the vector $\underline{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$.

The relationship between the three forms of a line in vector space is summarised below using the vector equation $\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$ as an example.

Vector equation	Parametric form	Cartesian equation
$\underline{r} = (-3\hat{i} + 6\hat{j} + 5\hat{k}) + k(8\hat{i} - 4\hat{j} - 4\hat{k})$ $\underline{a} = -3\hat{i} + 6\hat{j} + 5\hat{k}$ $\underline{d} = 8\hat{i} - 4\hat{j} - 4\hat{k}$	$x = -3 + 8k$ $y = 6 - 4k$ $z = 5 - 4k$ $(a_1, a_2, a_3) = (-3, 6, 5)$ $(d_1, d_2, d_3) = (8, -4, -4)$	$\frac{x+3}{8} = \frac{y-6}{-4} = \frac{z-5}{-4}$ $(a_1, a_2, a_3) = (-3, 6, 5)$ $(d_1, d_2, d_3) = (8, -4, -4)$

All forms are appropriate when an equation is required to model a line in three-dimensional vector geometry. The Cartesian equation is often preferred if you are required to calculate the point on the line that intersects a plane. The intersection points occur when either x, y or z equal zero. Upon making the relevant substitution, the location of the point $P(x, y, z)$ can then be calculated.

WORKED EXAMPLE 23

The vector equation of a line is given by $\underline{r} = (8\hat{i} + 2\hat{j} - 3\hat{k}) + k(3\hat{i} - 4\hat{j} + 2\hat{k})$.

- Convert the vector equation into parametric form.
- Express the parametric form of the line as a Cartesian equation.

THINK

- Write an equation linking the general location of the point $P(x, y, z)$ to the components of the vector equation.
 - Equate the components to write three equations representing each of the three dimensions.
- Rewrite the equation for the x -component in terms of k .
 - Rewrite the equation for the y -component in terms of k .
 - Rewrite the equation for the z -component in terms of k .
 - Equate the three equations as they are equivalent values of k .
 - Express the final answer as a Cartesian equation in terms of only x, y and z .

WRITE

a. $(x, y, z) = (8 + 3k, 2 - 4k, -3 + 2k)$

$$x = 8 + 3k$$

$$y = 2 - 4k$$

$$z = -3 + 2k$$

b. $x = 8 + 3k$

$$k = \frac{x-8}{3}$$

$$y = 2 - 4k$$

$$k = \frac{y-2}{-4}$$

$$z = -3 + 2k$$

$$k = \frac{z+3}{2}$$

$$\frac{x-8}{3} = \frac{y-2}{-4} = \frac{z+3}{2} = k$$

$$\frac{x-8}{3} = \frac{y-2}{-4} = \frac{z+3}{2}$$

Exercise 2.5 The vector equation of a straight line

Technology free

- Calculate the two-dimensional vector equation for the line passing through each of the following pairs of points (x, y) .
 - $A(2, 1)$ and $B(6, 7)$
 - $A(0, 6)$ and $B(9, -12)$
 - $A(-4, 6)$ and $B(-1, 12)$
 - $A(-2, -4)$ and $B(5, 3)$
 - $A(2, 8)$ and $B(7, -1)$
 - $A(-7, -5)$ and $B(1, 15)$
- WE21** a. Determine a vector equation, in the two-dimensional space (x, y) , for the line passing through the points $A(-2, 8)$ and $B(5, -3)$.
 b. Calculate the vector location of the point $\underline{r}$ on the line if the parameter $k = 3$.
- Determine the location of the point $P(x, y)$ on the following lines.
 - $\underline{r} = (4\hat{i} + 2\hat{j}) + k(3\hat{i} - 5\hat{j})$ when $k = 2$
 - $\underline{r} = (-5\hat{i} + 6\hat{j}) + k(-2\hat{i} + \hat{j})$ when $k = -3$
- Calculate the three-dimensional vector equation for the line passing through each of the following pairs of points (x, y, z) .
 - $A(-2, 2, 1)$ and $B(0, 6, 7)$
 - $A(4, -1, 11)$ and $B(-13, 9, -12)$
 - $A(5, 4, 11)$ and $B(-1, 1, 8)$
 - $A(6, -1, 2)$ and $B(5, 3, -9)$
 - $A(-3, -2, 1)$ and $B(5, 7, -13)$
 - $A(1, 5, 8)$ and $B(7, 0, -1)$
- WE22** a. Determine a vector equation, in the three-dimensional space (x, y, z) , for the line passing through the points $A(-4, 7, 6)$ and $B(6, 3, 2)$.
 b. Verify if the point $C(96, -33, -34)$ exists on the line.
 c. The position vector $\underline{e} = -34\hat{i} + 19\hat{j} + z\hat{k}$ terminates on the line formed between points A and B . Calculate the value of z .
- Calculate the vector location of the point $P(\hat{i}, \hat{j}, \hat{k})$ on the following lines.
 - $\underline{r} = (-3\hat{i} + \hat{j} - 5\hat{k}) + k(6\hat{i} - 5\hat{j} - 2\hat{k})$ when $k = 5$
 - $\underline{r} = (4.5\hat{i} + \hat{j} - 8\hat{k}) + k(2\hat{i} + \hat{j} + 3\hat{k})$ when $k = -2$
- Verify, through suitable calculations, if the following points lie on each of the lines below.

	Vector equation of the line	Point (x, y, z)
a.	$\underline{r} = (-3\hat{i} + 2\hat{j}) + k(5\hat{i} + 9\hat{j})$	$P(12, 29, 0)$
b.	$\underline{r} = (-7\hat{i} + 4\hat{j} - 8\hat{k}) + k(\hat{i} - 8\hat{j} + 3\hat{k})$	$P(-3, -28, 2)$
c.	$\underline{r} = (8\hat{i} + 6\hat{j} - 5\hat{k}) + k(2\hat{i} - 2\hat{j} - 9\hat{k})$	$P(4, 10, 13)$

- Determine the vector equation of the line through the point $P(6, 5, 1)$ that is parallel to the vector $\underline{u} = 6\hat{i} - 2\hat{j} + 3\hat{k}$.
- Calculate the vector equation of a line that is parallel to the z -axis and through the point $P(1, 2, 4)$.
- A line in 3D vector space is defined by the equation $\underline{r} = (4\hat{i} + \hat{j} - 5\hat{k}) + k(-2\hat{i} + 3\hat{j} + 5\hat{k})$. The head of the vector $\overrightarrow{OT} = (x\hat{i} + 10\hat{j} + z\hat{k})$ touches the line. Calculate the values for x and z .

11. **WE23a** The vector equation of a line is given by $\underline{r} = \underline{a} + k\underline{d}$. Convert the following vector equations into parametric form.
- a. $\underline{r} = (-\hat{i} + 3\hat{j} + 7\hat{k}) + k(2\hat{i} + 5\hat{j} - 3\hat{k})$ b. $\underline{r} = (-7\hat{i} + 8\hat{j} + 3\hat{k}) + k(19\hat{i} + 11\hat{j} + k)$
- c. $\underline{r} = (5\hat{i} + \hat{j} - 3\hat{k}) + k(2\hat{i} + 2\hat{j} - 6\hat{k})$
12. **WE23b** Express the following parametric equations of lines as Cartesian equations.
- a. $x = 1 + 5k, y = 2 + 5k, z = 3 + 5k$ b. $x = -2 + 6k, y = 4 - 9k, z = 8 - 2k$
- c. $x = 11 + 12k, y = 3 + 21k, z = 12 + 13k$
13. Convert the following Cartesian equations into the form $\underline{r} = \underline{a} + k\underline{d}$.
- a. $\frac{x-2}{3} = \frac{y-5}{2} = \frac{z+8}{5}$ b. $\frac{x+7}{-2} = \frac{y+1}{6} = \frac{z-2}{3}$
14. Determine the point at which the following lines intersect the specified plane.

Cartesian equation	Plane
a. $\frac{x-2}{3} = \frac{y-5}{2} = \frac{z+8}{5}$	xy-plane
b. $\frac{x+7}{-2} = \frac{y+1}{6} = \frac{z-2}{3}$	xz-plane

Technology active

15. Calculate the distance between the point $Q(-2, 1, 3)$ and the line defined by the equations, $x = 4 - t, y = t, z = 2 + 2t$. Give your answer to two decimal places.
16. The position of an aircraft is defined by the vector equation $\underline{r} = (2\hat{i} - \hat{k}) + k(\hat{i} + \hat{j} + \hat{k})$.

A second aircraft is flying in a straight-line path and its location was recorded on two occasions, defined by the vectors $\underline{a} = 5\hat{i} + 8\hat{j} + 11\hat{k}$ and $\underline{b} = 9\hat{i} + 17\hat{j} + 23\hat{k}$.

In the interest of public safety, determine if it is possible for the two aircraft to cross paths.

Justify your solution with suitable calculations and state the assumptions considered to be true when developing the response.



17. Two lines are formed between the points D, E, F and G . The first line joins $D(-1, 2, 1)$ and $E(3, -3, 5)$. The second line passes through $F(2, -1, 4)$ and $G(12, -2, 9)$. Calculate the acute angle between the two lines.

2.6 The vector product

2.6.1 Multiplying vectors to produce vectors

Up until this point, the product of two vectors has only been calculated as a scalar quantity. The dot product (or scalar or inner product) for two vectors $\underline{a}$ and $\underline{b}$ is defined by:

- $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta$: when the magnitudes of, and angle between $\underline{a}$ and $\underline{b}$ are known.
- $\underline{a} \cdot \underline{b} = a_1b_1 + a_2b_2 + a_3b_3$: when $\underline{a}$ and $\underline{b}$ are expressed in component form.

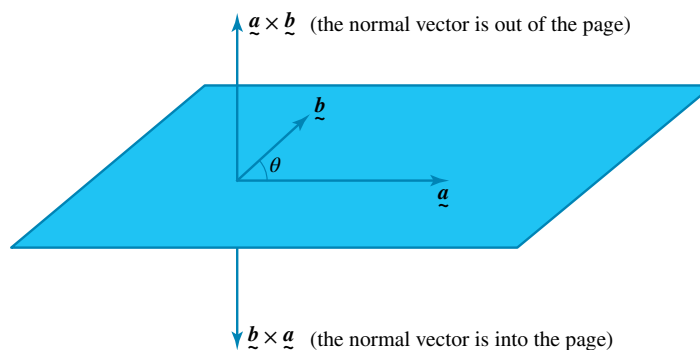
Applications of the dot product include:

- determining if vectors are perpendicular
- calculating the angle between two vectors
- calculating projections and components of vectors.

Is it possible to multiply two vectors so that their product isn't restricted to a scalar quantity? The vector product of two vectors $\underline{a}$ and $\underline{b}$, unlike the scalar product, is a vector. For this reason, it is called the **vector product**, and has applications in fields as diverse as geology, physics, mechanics and engineering. Furthermore, it is extremely useful in describing relationships between lines and planes in three dimensions.

We have seen how two vectors, $\underline{a}$ and $\underline{b}$, can be used to calculate and define the vector equation of a line between two points. The same two vectors, $\underline{a}$ and $\underline{b}$, provided that they are not parallel, define a **plane**. A plane is a flat surface extending indefinitely in all directions.

The orientation of the plane can be described by a vector perpendicular (orthogonal) to the plane defined by vectors $\underline{a}$ and $\underline{b}$. This perpendicular vector is called a normal vector, (or simply the **normal**) and is the cross product of the two vectors which define the plane.



The vector product is defined as

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin(\theta) \underline{\hat{n}}$$

where θ is the angle between $\underline{a}$ and $\underline{b}$,

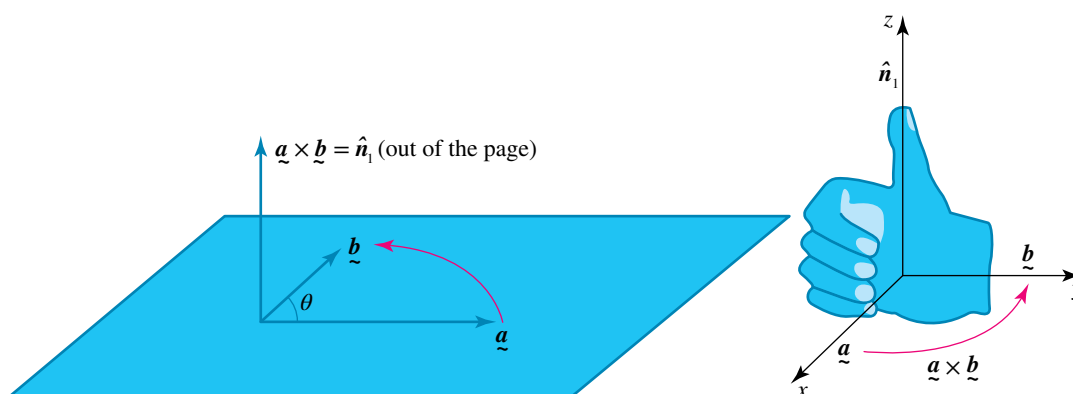
$\underline{\hat{n}}$ is the unit vector perpendicular to both $\underline{a}$ and $\underline{b}$ in the right-hand sense.

Note: Vector product is commonly called **cross product**.

The right-hand sense is applicable in a right-handed coordinate system, as previously described in Section 2.2. The vector product produces a vector perpendicular to the vectors $\underline{a}$ and $\underline{b}$, travelling in a direction that is either out of the page or into the page. This is explained below.

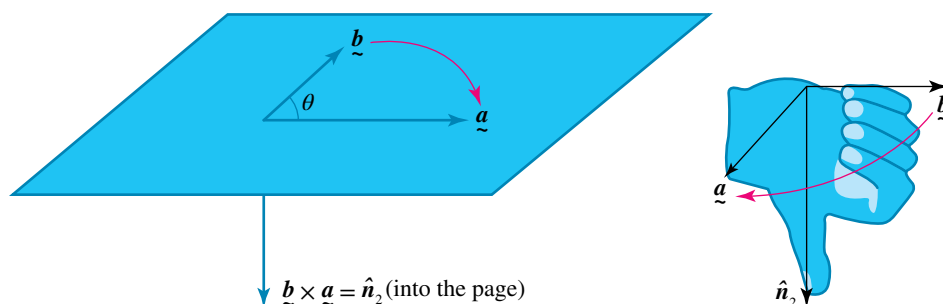
Normal Vector $\underline{\hat{n}}_1$ - out of the page

As the fingers on your right-hand curl from $\underline{a}$ and $\underline{b}$, your thumb points in the direction of the normal vector $\underline{\hat{n}}_1 = \underline{a} \times \underline{b}$. In this case, it is pointing out of the page.



Normal Vector $\hat{n}_2$ - into the page

In order to curl your right hand from $\vec{b}$ to $\vec{a}$, you would have to rotate your wrist so your thumb pointed down. The normal vector, $\hat{n}_2 = \vec{b} \times \vec{a}$, in this instance points into the page.



Remember the direction of the normal vector by thinking of it as turning on a water tap.

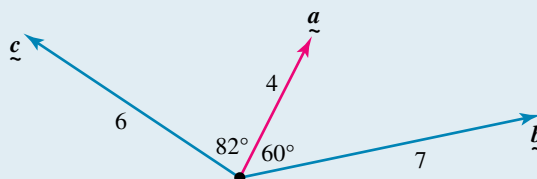
- Turning a tap in a clockwise direction moves it inward (into the page).
- Rotating the tap head anti-clockwise moves it in an upward direction (out of the page).

WORKED EXAMPLE 24

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ exist in a horizontal plane. Calculate the following vector products and in each case state the direction of $\hat{n}$.

a. $\vec{a} \times \vec{b}$

b. $\vec{b} \times \vec{c}$



THINK

1. Recall the vector product formula.
 2. Substitute the values for $|\vec{a}| = 4$, $|\vec{b}| = 7$ and $\theta = 60^\circ$.
 3. Calculate the vector product, including the direction. Recognise that $\vec{a}$ turns to $\vec{b}$ in a clockwise direction, indicating the direction of $\hat{n}$.
1. Recall the vector product formula.
 2. Substitute the values for $|\vec{b}| = 7$, $|\vec{c}| = 6$ and $\theta = 142^\circ$.
 3. Calculate the vector product. Recognise that $\vec{b}$ rotates anti-clockwise to $\vec{c}$, indicating the direction of $\hat{n}$.

WRITE

$$\begin{aligned}
 \text{a. } \vec{a} \times \vec{b} &= |\vec{a}||\vec{b}| \sin \theta \hat{n} \\
 &= 4 \times 7 \times \sin(60^\circ) \hat{n} \\
 &= 28 \times \frac{\sqrt{3}}{2} \hat{n} \\
 &= 14\sqrt{3} \hat{n} \text{ into the page}
 \end{aligned}$$

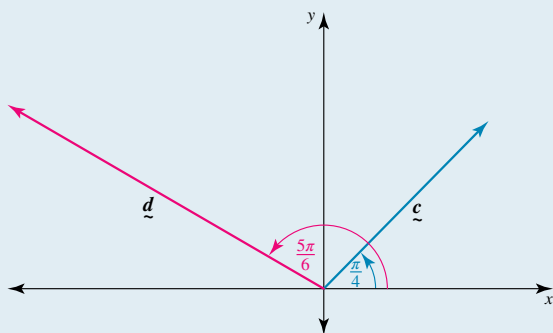
$$\begin{aligned}
 \text{b. } \vec{b} \times \vec{c} &= |\vec{b}||\vec{c}| \sin \theta \hat{n} \\
 &= 7 \times 6 \times \sin(142^\circ) \hat{n} \\
 &= 25.86 \hat{n} \text{ out of the page}
 \end{aligned}$$

WORKED EXAMPLE 25

The vectors $\underline{c} = \left[2\sqrt{2}, \frac{\pi}{4} \right]$ and $\underline{d} = \left[6, \frac{5\pi}{6} \right]$ exist in a horizontal plane. Calculate the exact value of the vector product, $\underline{c} \times \underline{d}$.

THINK

1. Recognise that the vectors are in polar form, $\underline{v} = [|\underline{v}|, \theta]$ and state their magnitude.
2. Sketch the vectors to get a clearer perspective of their orientation.



3. Recall the rule for vector product, $\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{n}$. The direction of the normal vector can be determined by applying the right hand rule. In this case, $\underline{c} \times \underline{d}$ is turning in an anticlockwise direction, which is out of the page.

WRITE

The magnitudes of the vectors are,

$$|\underline{c}| = 2\sqrt{2} \text{ and } |\underline{d}| = 6.$$

Both vectors are in a positive direction from the x -axis. The angle, θ , between the vectors is,

$$\theta = \frac{5\pi}{6} - \frac{\pi}{4} = \frac{7\pi}{12}$$

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{n}$$

$$\underline{c} \times \underline{d} = 2\sqrt{2} \times 6 \times \sin \left(\frac{7\pi}{12} \right) \hat{n}$$

$$= 12\sqrt{2} \sin \left(\frac{7\pi}{12} \right) \hat{n}$$

out of the page.

2.6.2 Applications: the area of parallelograms and triangles

The vector product can be used to calculate the area of particular shapes. The area of a parallelogram is given by the rule, Area = base $\times$ height. Consider the relationship between the area of a parallelogram and the vector product.

The height of the parallelogram can be expressed using basic trigonometric ratios.

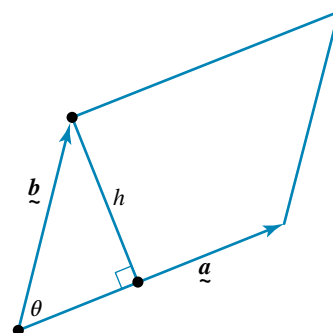
$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin \theta = \frac{h}{|\underline{b}|}$$

$$h = |\underline{b}| \sin \theta$$

The area of the parallelogram is then given by.

$$\begin{aligned} \text{Area} &= \text{base} \times \text{height} \\ &= |\underline{a}| \times |\underline{b}| \sin \theta \\ &= |\underline{a}| |\underline{b}| \sin \theta \end{aligned}$$



Recalling the formula for the vector product, $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n}$, we can recognise that the area of a parallelogram, defined by the vectors $\underline{a}$ and $\underline{b}$, is equal to the magnitude of the vector product of $\underline{a}$ and $\underline{b}$. That is, the area is given by:

$$\text{Area}_{\text{parallelogram}} = |\underline{a} \times \underline{b}|.$$

The vector product can be used to calculate the area of a triangle via a similar procedure.

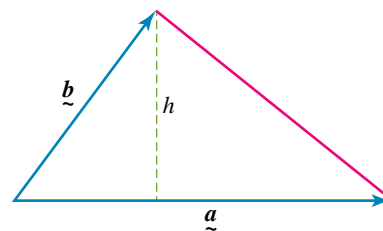
Recall that the area of any triangle is given by, $\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$.

Given that $h = |\underline{b}| \sin \theta$, the area can be calculated

$$\text{Area} = \frac{1}{2} \times |\underline{a}| \times |\underline{b}| \sin \theta$$

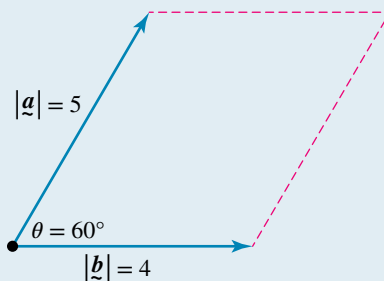
Thus, the area of a triangle, defined by the vectors $\underline{a}$ and $\underline{b}$, is equal to,

$$\begin{aligned} \text{Area}_{\text{triangle}} &= \frac{1}{2} |\underline{a} \times \underline{b}| \\ &= \frac{1}{2} |\underline{a}| \times |\underline{b}| \sin \theta \\ &= \frac{1}{2} |\underline{a}||\underline{b}| \sin \theta \end{aligned}$$

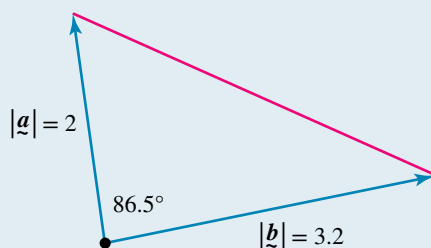


WORKED EXAMPLE 26

- a. Use the vector formula, $\text{Area} = |\underline{a}||\underline{b}|\sin \theta$ to determine the area of the parallelogram formed by the vectors with adjacent sides, $\underline{a}$ and $\underline{b}$. Give your answer to 2 decimal places.



- b. Use the formula $\text{Area} = \frac{1}{2} |\underline{a}||\underline{b}|\sin \theta$ to calculate the area of the triangle. Give your answer to 2 decimal places.



THINK

- a. 1. Recall and state the formula for the area of a parallelogram.

$$\text{Area} = |\underline{a}||\underline{b}| \sin \theta$$

2. Substitute the values for $|\underline{a}| = 5$, $|\underline{b}| = 4$ and $\theta = 60^\circ$.

3. Calculate the area and simplify the answer to two decimal places.

- b. 1. Recall and state the formula for the area of a triangle. $\text{Area} = \frac{1}{2}|\underline{a}||\underline{b}| \sin \theta$

2. Substitute the values for $|\underline{a}| = 2$, $|\underline{b}| = 3.2$ and $\theta = 86.5^\circ$.

3. Calculate the area and simplify the solution to two decimal places.

WRITE

a. $\text{Area} = |\underline{a}||\underline{b}| \sin \theta$

$$= 5 \times 4 \times \sin 60^\circ$$

$$= 20 \times \frac{\sqrt{3}}{2}$$

$$= 10\sqrt{3}$$

$$\approx 17.32 \text{ square units}$$

b. $\text{Area} = \frac{1}{2}|\underline{a}||\underline{b}| \sin \theta$

$$= \frac{1}{2} \times 2 \times 3.2 \times \sin 86.5^\circ$$

$$\approx 3.19 \text{ square units}$$

2.6.3 The vector product in $\hat{i} - \hat{j} - \hat{k}$ form

Recall that the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the mutually perpendicular unit vectors that can be used to describe three-dimensional space. The x -, y - and z - directions cannot be arranged in any order, even though they may be mutually perpendicular. The form is a right-hand triad. This means that the direction of z must be that obtained by anticlockwise rotation of x to y .

Consider some special vector products. Remember that $\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n}$ where $\hat{n}$ is a unit vector perpendicular to $\underline{a}$ and $\underline{b}$ in the right-hand sense.

$$\hat{i} \times \hat{i}$$

$$\underline{a} = \hat{i} \text{ and } \underline{b} = \hat{i}$$

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n}$$

$$\hat{i} \times \hat{i} = 1 \times 1 \sin (0^\circ) \hat{n}$$

$$\hat{i} \times \hat{i} = \underline{0}$$

$$\therefore \hat{i} \times \hat{i} = \underline{0}$$

$$\hat{j} \times \hat{k}$$

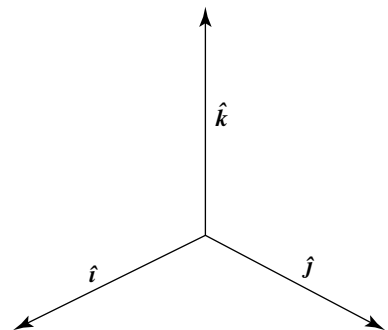
$$\underline{a} = \hat{j} \text{ and } \underline{b} = \hat{k}$$

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n}$$

$$\hat{j} \times \hat{k} = 1 \times 1 \sin (90^\circ) \hat{n}$$

$$\hat{j} \times \hat{k} = 1\hat{n} \text{ in the direction of } +\hat{n}$$

$$\therefore \hat{j} \times \hat{k} = \hat{i}$$



$$\hat{i} \times \hat{k}$$

$$\underline{a} = \hat{i} \text{ and } \underline{b} = \hat{k}$$

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin \theta \hat{n}$$

$$\hat{i} \times \hat{k} = 1 \times 1 \sin (90^\circ) \hat{n}$$

$$\hat{i} \times \hat{k} = 1\hat{n} \text{ in the direction of } -\hat{j}$$

$$\therefore \hat{i} \times \hat{k} = -\hat{j}$$

Continuing in this way, we can show that the cross product for the basic unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are:

$$\hat{i} \times \hat{i} = \underline{0} \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k} \quad \hat{j} \times \hat{j} = \underline{0} \quad \hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j} \quad \hat{k} \times \hat{j} = -\hat{i} \quad \hat{k} \times \hat{k} = \underline{0}$$

A useful way of summarising these results is shown in the figure.
A product following the direction indicated (clockwise) produces a positive value.

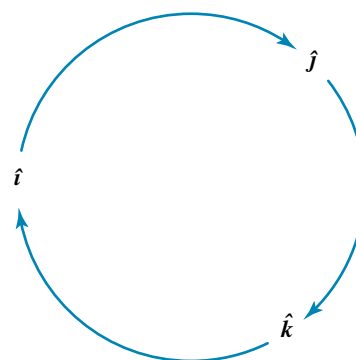
For example,

$$\hat{i} \times \hat{j} = \hat{k}$$

A product in the anticlockwise direction produces a negative value.

For example,

$$\hat{k} \times \hat{j} = -\hat{i}$$



To evaluate the vector product of two vectors that are expressed in component form we can use the Distributive Law. This is illustrated in the following worked example.

WORKED EXAMPLE 27

If $\underline{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$, $\underline{b} = -2\hat{i} + 3\hat{j} - \hat{k}$ and $\underline{c} = -3\hat{i} + 4\hat{j} - 7\hat{k}$, calculate the following vector products using the distributive law of vectors.

- $\underline{a} \times \underline{b}$
- $\underline{c} \times \underline{b}$
- Justify that the vector product from **b** is orthogonal to vectors $\underline{c}$ and $\underline{b}$ respectively.

THINK

- Express $\underline{a}$ and $\underline{b}$ as a product in component form.
- Calculate the product $\underline{a} \times \underline{b}$ by applying the Distributive Law for vectors.
- Simplify the calculation by collecting like vector components.
- Express $\underline{c}$ and $\underline{b}$ as a product in component form.
- Calculate the product $\underline{c} \times \underline{b}$ applying the Distributive Law for vectors.
- Simplify the calculation by collecting like vector components.
- Let $\underline{c} \times \underline{b} = \underline{n}$. Vectors, $\underline{c}$ and $\underline{n}$, are perpendicular if $\underline{c} \cdot \underline{n} = 0$.
- Calculate $\underline{c} \cdot \underline{n}$ by recalling the formula $\underline{c} \cdot \underline{n} = x_1x_2 + y_1y_2 + z_1z_2$. Multiply the corresponding components.
- Vectors, $\underline{b}$ and $\underline{n}$, are perpendicular if $\underline{b} \cdot \underline{n} = 0$. Calculate $\underline{b} \cdot \underline{n}$ by recalling the formula $\underline{c} \cdot \underline{n} = x_1x_2 + y_1y_2 + z_1z_2$. Multiply the corresponding components.

WRITE

- $$\begin{aligned}\underline{a} \times \underline{b} &= (3\hat{i} + 2\hat{j} - 4\hat{k}) \times (-2\hat{i} + 3\hat{j} - \hat{k}) \\ &= 3\hat{i} \times -2\hat{i} + 3\hat{i} \times 3\hat{j} + 3\hat{i} \times -\hat{k} \\ &\quad + 2\hat{j} \times -2\hat{i} + 2\hat{j} \times 3\hat{j} + 2\hat{j} \times -\hat{k} \\ &\quad + -4\hat{k} \times -2\hat{i} + -4\hat{k} \times 3\hat{j} + -4\hat{k} \times -\hat{k} \\ &= 9\hat{k} + 3\hat{j} + 4\hat{k} - 2\hat{i} + 8\hat{j} + 12\hat{i} \\ &= 10\hat{i} + 11\hat{j} + 13\hat{k}\end{aligned}$$
- $$\begin{aligned}\underline{c} \times \underline{b} &= (-3\hat{i} + 4\hat{j} - 7\hat{k}) \times (-2\hat{i} + 3\hat{j} - \hat{k}) \\ &= -3\hat{i} \times -2\hat{i} + -3\hat{i} \times 3\hat{j} + -3\hat{i} \times -\hat{k} \\ &\quad + 4\hat{j} \times -2\hat{i} + 4\hat{j} \times 3\hat{j} + 4\hat{j} \times -\hat{k} \\ &\quad + -7\hat{k} \times -2\hat{i} + -7\hat{k} \times 3\hat{j} + -7\hat{k} \times -\hat{k} \\ &= -9\hat{k} - 3\hat{j} + 8\hat{k} - 4\hat{i} + 14\hat{j} + 21\hat{i} \\ &= 17\hat{i} + 11\hat{j} - \hat{k}\end{aligned}$$
- Let $\underline{c} \times \underline{b} = \underline{n}$
 $\underline{c}$ and $\underline{n}$ are orthogonal if $\underline{c} \cdot \underline{n} = 0$

$$\begin{aligned}\underline{c} \cdot \underline{n} &= (-3\hat{i} + 4\hat{j} - 7\hat{k}) \cdot (17\hat{i} + 11\hat{j} - \hat{k}) \\ &= -51 + 44 + 7 \\ &= 0\end{aligned}$$

 $\underline{b}$ and $\underline{n}$ are orthogonal if $\underline{b} \cdot \underline{n} = 0$

$$\begin{aligned}\underline{b} \cdot \underline{n} &= (-2\hat{i} + 3\hat{j} - \hat{k}) \cdot (17\hat{i} + 11\hat{j} - \hat{k}) \\ &= -34 + 33 + 1 \\ &= 0\end{aligned}$$

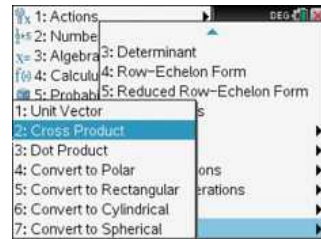
4. Write a justifying statement.

$\therefore$ The vector product $\underline{c} \times \underline{b} = \underline{n}$ is orthogonal to both vectors $\underline{c}$ and $\underline{b}$.

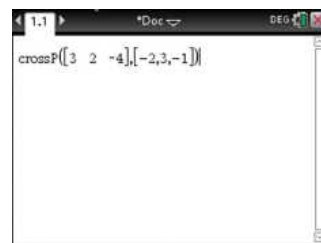
TI| THINK

- a.1. On a Calculator page, select
MENU
7: Matrix Vector
C: Vector
2: Cross Product
Press the ENTER button.

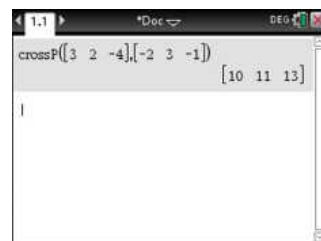
WRITE



2. Complete the entry line
($\begin{bmatrix} -3, 2, -4 \end{bmatrix},$
 $\begin{bmatrix} -2, 3, -1 \end{bmatrix}$)



3. Press the ENTER button. The answer appears on the screen.



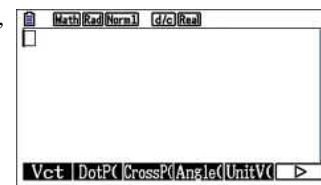
CASIO| THINK

- a.1. On a Run-Matrix screen, select MAT/VCT
Define Vectors
A and B as shown in
Worked Example 1b.

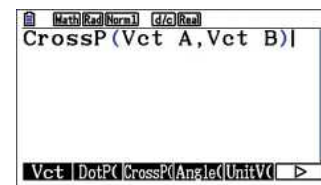
WRITE



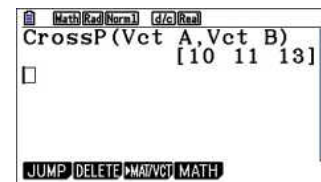
2. On a Run-Matrix screen, select OPTN MAT/VCT
▷ (x2 times)CrossP(



3. Complete the entry line as
Vct ALPHA
X, θ , T, Vct ALPHA
Log)



4. Press the EXE button.
The answer appears on
the screen



2.6.4 Calculating the vector product using determinants

The Distributive Law for vectors is a valid method for calculating the vector product when the vectors are in component form. The process is rather arduous, and the same result can be calculated more efficiently using determinants. First, we shall review the definitions of a determinant in both two and three dimensions.

For a 2×2 matrix:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then, } \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix:

$$\text{If } A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then, } \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The elements of a determinant are generally numerical figures. It is possible to replace the first row of the matrix with the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. When calculating the determinant in this case, the sub-determinants (minors) become scalar multiples of the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d & e & f \\ g & h & i \end{vmatrix} = \hat{i} \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \hat{j} \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \hat{k} \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

If the vectors exist in the form, $\underline{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$ and $\underline{b} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$, then the vector product can be defined in terms of determinants as,

$$\begin{aligned} \underline{a} \times \underline{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1) \end{aligned}$$

Hopefully it is now clearer as to why the vector product can also be called the cross product.

WORKED EXAMPLE 28

- Calculate $\underline{a} \times \underline{b}$ using determinants, given $\underline{a} = -\hat{i} + 4\hat{j} + 6\hat{k}$ and $\underline{b} = 4\hat{i} + 2\hat{j} - 5\hat{k}$.
- Calculate $\underline{b} \times \underline{a}$ to investigate if the vector product is commutative.

THINK

- Express the vector product, $\underline{a} \times \underline{b}$ as the determinant of a 3×3 matrix.

- Calculate the determinant for the matrix.
Remember to write out the expansion with the signed minors (cofactors).

- Complete the expansion being careful with + and - signs.

- Express the vector product, $\underline{b} \times \underline{a}$ as the determinant of a 3×3 matrix.

- Calculate the determinant for the matrix. Remember to write out the expansion with the signed minors (cofactors).

- Complete the expansion being careful with + and - signs.

WRITE

$$\begin{aligned} \text{a. } \underline{a} \times \underline{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & 6 \\ 4 & 2 & -5 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} 4 & 6 \\ 2 & -5 \end{vmatrix} - \hat{j} \begin{vmatrix} -1 & 6 \\ 4 & -5 \end{vmatrix} + \hat{k} \begin{vmatrix} -1 & 4 \\ 4 & 2 \end{vmatrix} \\ &= \hat{i}(4 \times -5 - 6 \times 2) - \hat{j}(-1 \times -5 - 6 \times 4) \\ &\quad + \hat{k}(-1 \times 2 - 4 \times 4) \\ &= \hat{i}(-20 - 12) - \hat{j}(5 - 24) + \hat{k}(-2 - 16) \\ &= -32\hat{i} + 19\hat{j} - 18\hat{k} \end{aligned}$$

$$\begin{aligned} \text{b. } \underline{b} \times \underline{a} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -5 \\ -1 & 4 & 6 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix} - \hat{j} \begin{vmatrix} 4 & -5 \\ -1 & 6 \end{vmatrix} + \hat{k} \begin{vmatrix} 4 & 2 \\ -1 & 4 \end{vmatrix} \\ &= \hat{i}(2 \times 6 - -5 \times 4) - \hat{j}(4 \times 6 - -5 \times -1) \\ &\quad + \hat{k}(4 \times 4 - 2 \times -1) \\ &= \hat{i}(12 + 20) - \hat{j}(24 - 5) + \hat{k}(16 + 2) \\ &= 32\hat{i} - 19\hat{j} + 18\hat{k} \end{aligned}$$

4. Recall that an operation is commutative if $a \circ b = b \circ a$. Conclude with a statement in response to the investigation.

The cross product is not commutative because $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$

2.6.5 Properties of the vector product

Although the vector product does not form a commutative operation with vectors, it is interesting to note that $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$. This property is known as being anti-commutative. The following six theorems summarise some of the properties of the vector product.

Six properties of the cross product

1. $\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$
2. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$
3. $(\underline{a} + \underline{b}) \times \underline{c} = \underline{a} \times \underline{c} + \underline{b} \times \underline{c}$
4. $\underline{a} \cdot (\underline{b} \times \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c}$
5. $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$
6. $(k\underline{a}) \times \underline{b} = k(\underline{a} \times \underline{b}) = \underline{a} \times (k\underline{b})$

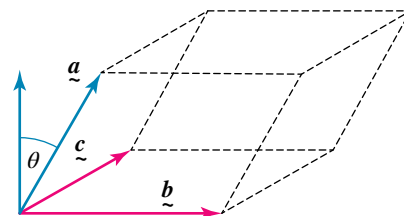
where $\underline{a}, \underline{b}$ and $\underline{c}$ are vectors, and k is a scalar.

2.6.6 The scalar triple product

The fourth theorem, $\underline{a} \cdot (\underline{b} \times \underline{c})$, is referred to as the **scalar triple product**. As the name suggests, the product is a scalar quantity, and is sometimes referred to as the **triple product**.

The triple product has numerous applications, one of which is related to the volume of a parallelepiped. The volume of a parallelepiped whose adjacent sides are the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ is,

$$\text{Volume} = |\underline{a} \cdot (\underline{b} \times \underline{c})|$$



In three steps, this formula can be proved by combining the theories of both the dot and vector products.

1. The area of the base is given by the vector product $A = |\underline{b} \times \underline{c}|$.
2. If θ is the angle between $\underline{a}$ and $\underline{b} \times \underline{c}$, the height of the parallelepiped will be $h = |\underline{a}| \cos \theta$.
3. The vector formula for the volume of a parallelepiped is then developed by combining the equations, $A = |\underline{b} \times \underline{c}|$ and $h = |\underline{a}| \cos \theta$,

$$\begin{aligned} \text{Volume} &= A \times h \\ &= |\underline{a}| \cos \theta \times |\underline{b} \times \underline{c}| \\ &= |\underline{a} \cdot (\underline{b} \times \underline{c})| \end{aligned}$$

Determinants can be used to simplify the triple product calculation, $\underline{a} \cdot (\underline{b} \times \underline{c})$.

If $\underline{a} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$, $\underline{b} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$ and $\underline{c} = (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$, the scalar triple product can be calculated by

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

WORKED EXAMPLE 29

Determine the volume of the parallelepiped containing the adjacent vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$.

$$\underline{a} = \hat{i} + 2\hat{j} + \hat{k}, \quad \underline{b} = 5\hat{i} + 2\hat{j} - 4\hat{k} \quad \text{and} \quad \underline{c} = 3\hat{i} - 4\hat{j} - \hat{k}.$$

THINK

1. State the formula for the volume, which is the scalar triple product.
2. Express the scalar triple product as the determinate of a 3×3 matrix. Calculate the determinant of the matrix.
3. Remember to write out the expansion with the signed minors (cofactors).
4. Complete the expansion being careful with + and - signs.

WRITE

$$\text{Volume} = |\underline{a} \cdot (\underline{b} \times \underline{c})|$$

$$V = \begin{vmatrix} 1 & 2 & 1 \\ 5 & 2 & -4 \\ 3 & -4 & -1 \end{vmatrix}$$

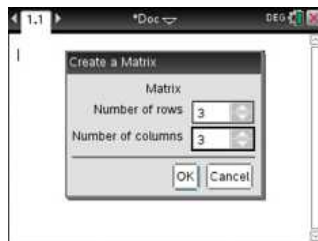
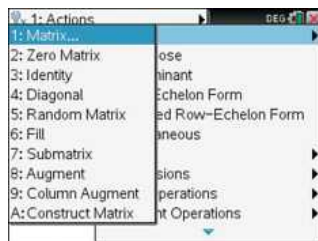
$$V = 1 \begin{vmatrix} 2 & -4 \\ -4 & -1 \end{vmatrix} - 2 \begin{vmatrix} 5 & -4 \\ 3 & -1 \end{vmatrix} + 1 \begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix}$$

$$\begin{aligned} V &= |-18 - 14 - 26| \\ &= |-58| \\ &= 58 \end{aligned}$$

TI| THINK

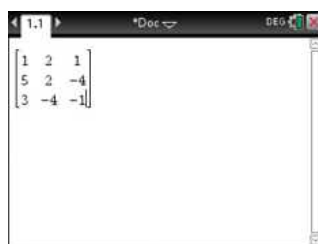
1. On a Calculator page, select MENU
7: Matrix & Vector
1: Create
1: Matrix
Press the ENTER button.
2. Complete the entry line as 3 3 Press the OK button.

WRITE



3. Complete the entry line as

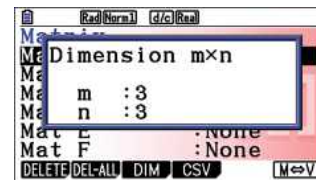
$$\begin{bmatrix} 1 & 2 & 1 \\ 5 & 2 & -4 \\ 3 & -4 & -1 \end{bmatrix}$$
Press the ENTER button when finished.



CASIO| THINK

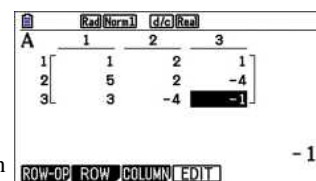
1. On a Run-Matrix screen, select
MAT/VCT
To switch between the vector and matrix screens, press the M $\leftrightarrow$ V button.
2. Press the EXE button.
Complete the entry line as
3 3 Press the EXE button.

WRITE



3. Complete the entry line as

$$\begin{bmatrix} 1 & 2 & 1 \\ 5 & 2 & -4 \\ 3 & -4 & -1 \end{bmatrix}$$
Press the EXIT button when finished.

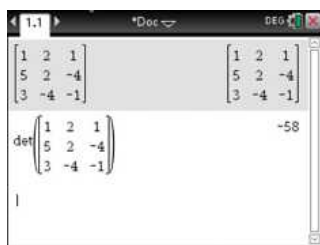


4. On a Calculator page, select MENU
7: Matrix & Vector
3: Determinant
CTRL (-)

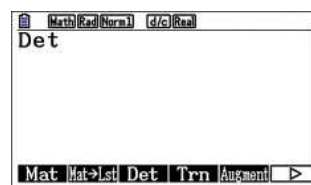


5. Press the ENTER button.
The answer appears on the screen.

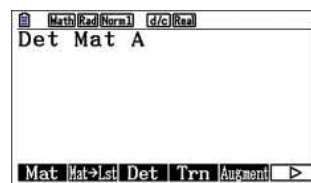
Note: The absolute value of determinant is the volume of the parallelepiped.



4. On a Run-Matrix screen, select OPTN
MAT/VCT
DET



5. Complete the entry line as
SHIFT
2
ALPHA
X, θ , T



6. Press the EXE button.
The answer appears on the screen.
Note: The absolute value of determinant is the volume of the parallelepiped.



study on

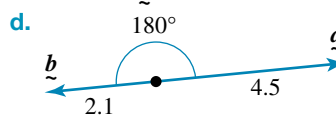
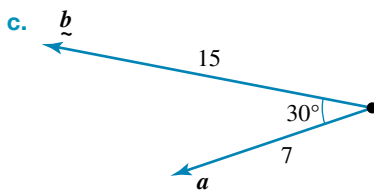
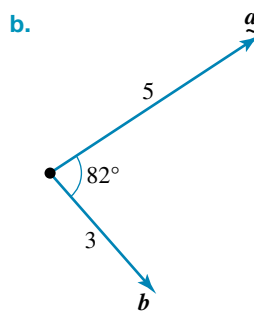
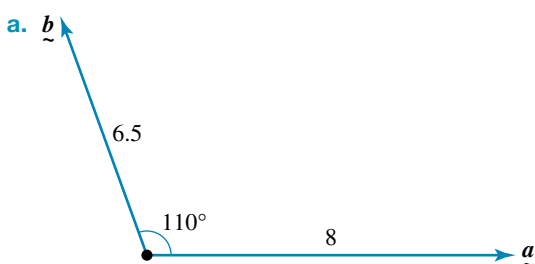
Units 3 & 4 > Area 2 > Sequence 1 > Concept 5

The vector product Summary screen and practice questions

Exercise 2.6 The vector product

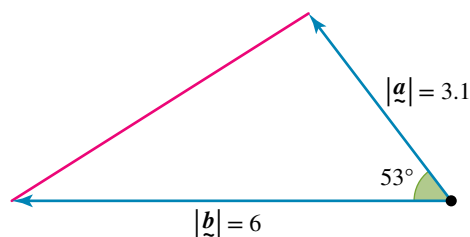
Technology active

1. **WE24** The vectors $\underline{a}$ and $\underline{b}$ exist in a horizontal plane. Calculate $\underline{a} \times \underline{b}$, and state the direction of $\hat{n}$.

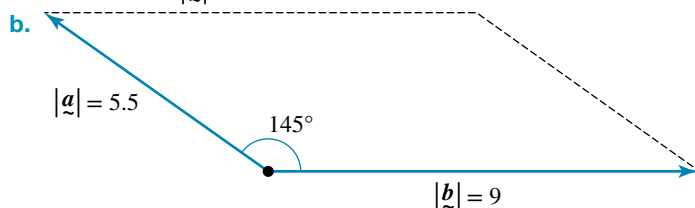
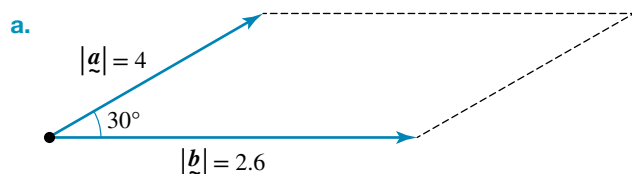


2. **WE25** The vectors $\underline{a}$ and $\underline{b}$ exist in a horizontal plane. Calculate the vector product, $\underline{a} \times \underline{b}$, for the following pairs of vectors.
- a. $\underline{a} = [12, 25^\circ]$ and $\underline{b} = [10, 75^\circ]$
 - b. $\underline{a} = [21, 182^\circ]$ and $\underline{b} = [16, 267^\circ]$
 - c. $\underline{a} = [4, 173^\circ]$ and $\underline{b} = [7, 22^\circ]$
 - d. $\underline{a} = [5, 25^\circ]$ and $\underline{b} = [7, 345^\circ]$

3. **WE26b** Use the formula $\text{Area} = \frac{1}{2}|\underline{a}||\underline{b}| \sin \theta$ to calculate the area of the following triangles.



4. **WE26a** Use the vector formula, $\text{Area} = |\underline{a}||\underline{b}| \sin \theta$ to determine the area of the parallelograms formed by the vectors with adjacent sides, $\underline{a}$ and $\underline{b}$.



5. The table below summaries the nine possibilities when the vector product is calculated using the unit vectors $\hat{i}$, $\hat{j}$ or $\hat{k}$. Complete the table by using your knowledge of the Distributive Law for vectors.

		Second vector $\underline{b}$		
		$\hat{i}$	$\hat{j}$	$\hat{k}$
First Vector $\underline{a}$	$\times$			
	$\hat{i}$			$-\hat{j}$
	$\hat{j}$		$\underline{0}$	
	$\hat{k}$			

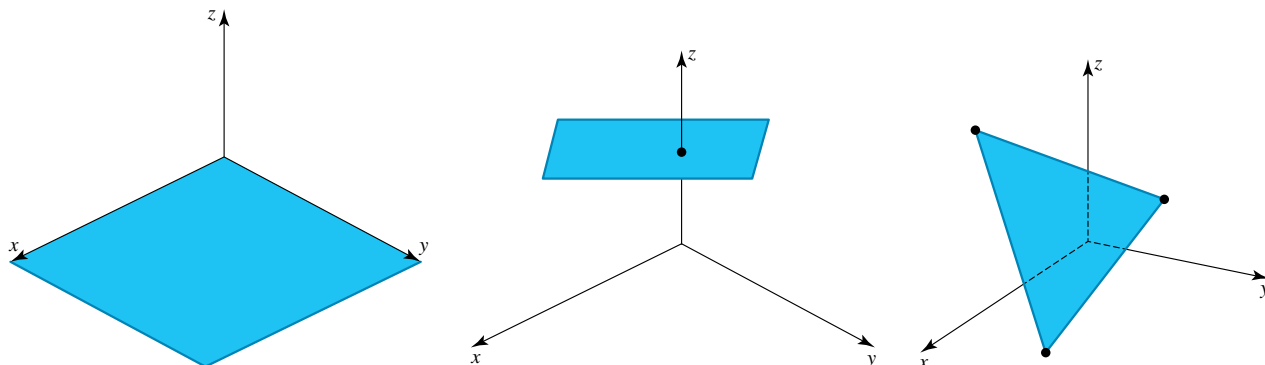
6. Use the vector product of vectors to calculate a vector perpendicular to both $\underline{a}$ and $\underline{b}$.
- a. $\underline{a} = 2\hat{i} + 3\hat{j} - 7\hat{k}$ and $\underline{b} = -\hat{i} + 7\hat{j} - 2\hat{k}$ b. $\underline{a} = -5\hat{i} - \hat{j} - 3\hat{k}$ and $\underline{b} = 11\hat{i} - 4\hat{j} - 2\hat{k}$
- c. $\underline{a} = \hat{i} + 12\hat{j} + 8\hat{k}$ and $\underline{b} = -9\hat{i} + 3\hat{j} + 7\hat{k}$ d. $\underline{a} = 3.5\hat{i} + 1.5\hat{j} - 2\hat{k}$ and $\underline{b} = 0.5\hat{i} - 2\hat{j}$
7. **WE27** If $\underline{a} = 4\hat{i} + 3\hat{j} - 2\hat{k}$, $\underline{b} = -3\hat{i} + 5\hat{j} - \hat{k}$ and $\underline{c} = -7\hat{i} + \hat{j} - 2\hat{k}$, calculate the following vector products using the Distributive Law for vectors
- a. $\underline{a} \times \underline{b}$ b. $\underline{c} \times \underline{b}$
- c. Justify that the vector product from b, is orthogonal to vectors $\underline{c}$ and $\underline{b}$ respectively.
8. The two vectors $\underline{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\underline{b} = a\hat{i} - 2\hat{j} + 4\hat{k}$ exist in the same three-dimensional space, and $\underline{a} \cdot \underline{b} = 28$. Calculate $\underline{a} \times \underline{b}$ to determine if it is possible to find a unit vector orthogonal to $\underline{a}$ and $\underline{b}$.
9. **WE28** a. Calculate $\underline{a} \times \underline{b}$ using determinants, given $\underline{a} = -3\hat{i} + 7\hat{j} - 5\hat{k}$, $\underline{b} = 8\hat{i} + \hat{j} - 2\hat{k}$.
b. Calculate $\underline{b} \times \underline{a}$ to investigate if the vector product is commutative.
10. Use determinants to calculate the following vector products, $\underline{a} \times \underline{b}$.
- a. $\underline{a} = 5\hat{i} - 2\hat{j} - 9\hat{k}$ and $\underline{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ b. $\underline{a} = 3\hat{j} - 8\hat{k}$ and $\underline{b} = -4\hat{i} + 6\hat{j} + 7\hat{k}$
- c. $\underline{a} = 3\hat{i} - 2\hat{j} - 4\hat{k}$ and $\underline{b} = 7\hat{i} + 2\hat{j} - \hat{k}$ d. $\underline{a} = -7\hat{i} + \hat{j} - 6\hat{k}$ and $\underline{b} = 6\hat{i} - 12\hat{j} + 13\hat{k}$

11. **WE29** Determine the volume of the parallelepiped containing the adjacent vectors, $\underline{a}$, $\underline{b}$ and $\underline{c}$.
- $\underline{a} = 5\hat{i} + 3\hat{j} + 2\hat{k}$, $\underline{b} = 7\hat{i} + 3\hat{j} - 2\hat{k}$ and $\underline{c} = 8\hat{i} - 7\hat{j} - 3\hat{k}$
 - $\underline{a} = \hat{i} - 2\hat{j} + 8\hat{k}$, $\underline{b} = -3\hat{i} - 2\hat{j} + 9\hat{k}$ and $\underline{c} = -5\hat{i} + 6\hat{j} - 2\hat{k}$
 - $\underline{a} = 8\hat{i} + 3\hat{j} - 7\hat{k}$, $\underline{b} = -4\hat{i} + 3\hat{j} + \hat{k}$ and $\underline{c} = 7\hat{i} - 11\hat{j} + 6\hat{k}$
 - $\underline{a} = 7\hat{i} + 5\hat{j} + 6\hat{k}$, $\underline{b} = -2\hat{i} - 4\hat{j} - 6\hat{k}$ and $\underline{c} = 2\hat{i} - 4\hat{j} + 8\hat{k}$
12. The vertices of a triangle are defined by the vectors, $\underline{a} = 7\hat{i} + 4\hat{j} + 8\hat{k}$, $\underline{b} = \hat{i} + 14\hat{j} - 8\hat{k}$ and $\underline{c} = 8\hat{i} + 3\hat{j} + 2\hat{k}$. Calculate the area using the rule $A = \frac{1}{2}|\underline{a} \times \underline{b}|$.
13. Use the vector product to prove the property $\underline{a} \times \underline{a} = \underline{0}$, given that $\underline{a} = (x, y, z)$.
14. Calculate the scalar triple product for the vectors, $\underline{a} = 3\hat{i} + 3\hat{j} - 6\hat{k}$, $\underline{b} = 2\hat{i} + 4\hat{j}$ and $\underline{c} = 3\hat{i} + 9\hat{j} + 6\hat{k}$, to determine if they are coplanar (exist on the same plane).
15. The vector product of two vectors $\underline{a} = 3\hat{i} + 5\hat{j} - 3\hat{k}$ and $\underline{b} = 4\hat{i} + b_y\hat{j} + b_z\hat{k}$ is $\underline{a} \times \underline{b} = -7\hat{i} - 15\hat{j} - 32\hat{k}$. Calculate the values of b_y and b_z .
16. Calculate each of the following scalar triple products.
- $\hat{j} \cdot \hat{k} \times \hat{k}$
 - $\hat{i} \cdot \hat{j} \times \hat{k}$
 - $\hat{i} \cdot \hat{k} \times \hat{j}$
 - $\hat{k} \cdot \hat{j} \times \hat{i}$
17. Use the vector product to calculate the angle between the vectors $\underline{a} = 3\hat{i} + 4\hat{j} - 2\hat{k}$ and $\underline{b} = 2\hat{i} + \hat{j} + 2\hat{k}$.
18. A tetrahedron is a pyramid consisting of four triangular faces. The volume of a tetrahedron is given by the rule, $V = \frac{1}{6} |\underline{a} \cdot (\underline{b} \times \underline{c})|$. Calculate the volume of a tetrahedron with vertices $A(6, 2, 0)$, $B(7, 5, 0)$, $C(3, 8, 1)$ and $D(13, 7, 2)$.
19. Verify through suitable calculations, that the quadrilateral with vertices defined by $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ is a parallelogram. Hence, calculate its area using determinants.
- $$\begin{aligned}\underline{a} &= 6\hat{i} + 3\hat{j} + \hat{k} \\ \underline{b} &= 3\hat{i} + 7\hat{j} + 4\hat{k} \\ \underline{c} &= 3\hat{i} + 5\hat{j} + 9\hat{k} \\ \underline{d} &= 6\hat{i} + \hat{j} + 6\hat{k}\end{aligned}$$

2.7 Applications of vectors

2.7.1 The vector and Cartesian equation of a plane in three dimensions

A two-dimensional line can be represented by the scalar equation $y = mx + c$. In three-dimensional space, the line between two points is modelled by the vector equation of a line, $\underline{r} = \underline{a} + k\underline{d}$. In this section we will model a plane in three dimensions through the application of vectors. A plane can be described as a 'flat' 2-dimensional surface extending indefinitely. To help you visualise a plane, three graphical examples are shown below.



Determining the equation of a plane using a point and the normal

The equation of a plane written in Cartesian form, is expressed as $ax + by + cz + d = 0$. To calculate the equation of a plane, two important features are required:

1. A point $P(x, y, z)$ that exists on the plane.
2. A vector that is perpendicular to the plane.

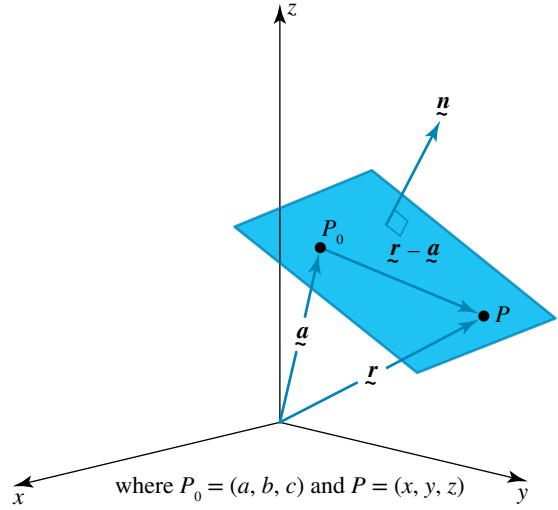
Remember this vector is called the normal.

Consider the diagram shown. The locations of points P and P_0 can be defined by position vectors.

That is, $P(x, y, z) = \underline{r}$ and $P_0(a, b, c) = \underline{a}$. The vector connecting the two points can then be defined as $\overrightarrow{P_0P} = \underline{r} - \underline{a}$.

As we are asserting that $\underline{r} - \underline{a}$ lies on the plane it will always be perpendicular to the normal. That is,

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$



This is defined as the **vector equation of a plane**.

The vector equation of a plane is given by

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

This can be written as

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

To translate this into Cartesian form, the vector, $\underline{r} - \underline{a}$ is to be expressed in terms of its Cartesian coordinates.

$$\begin{aligned} \overrightarrow{P_0P} &= \underline{r} - \underline{a} \\ \underline{r} - \underline{a} &= (x\hat{i} + y\hat{j} + z\hat{k}) - (a\hat{i} + b\hat{j} + c\hat{k}) \\ &= (x - a)\hat{i} + (y - b)\hat{j} + (z - c)\hat{k} \end{aligned}$$

In component form, the normal vector is $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$. The dot product calculation results in:

$$\begin{aligned} (\underline{r} - \underline{a}) \cdot \underline{n} &= 0 \\ [(x - a)\hat{i} + (y - b)\hat{j} + (z - c)\hat{k}] \cdot [n_1\hat{i} + n_2\hat{j} + n_3\hat{k}] &= 0 \\ n_1(x - a) + n_2(y - b) + n_3(z - c) &= 0 \end{aligned}$$

This is the equation of a plane containing the point, $P_0(a, b, c)$, that is perpendicular to $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$.

The equation can then be transformed into Cartesian form, $ax + by + cz + d = 0$, by expanding out and collecting like terms. It is interesting to note that the coefficients in this form are the components of the normal vector.

The equation of a plane containing the points $P(x, y, z)$ and $P_0(a, b, c)$ that is perpendicular to the normal $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ is given by

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

WORKED EXAMPLE 30

Calculate the equation of the plane perpendicular to $\underline{n} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, containing the point $P_0(2, 4, -1)$ using a dot product calculation.

THINK

1. Define the vectors $\underline{r}$, $\underline{a}$ and $\underline{n}$.
2. Determine the vector $\underline{r} - \underline{a}$ by applying the formula
$$\overrightarrow{P_0P} = \underline{r} - \underline{a}$$
$$= (x - a)\hat{i} + (y - b)\hat{j} + (z - c)\hat{k}$$
3. Recall the formal dot product definition for a plane and substitute in the vectors.
4. Complete the dot product calculations.
5. Expand and simplify.

WRITE

$$\begin{aligned}\underline{r} &= (x, y, z), \underline{a} = (2, 4, -1) \text{ and} \\ \underline{n} &= 2\hat{i} + 3\hat{j} + 4\hat{k} \\ (\underline{r} - \underline{a}) &= (x - 2)\hat{i} + (y - 4)\hat{j} + (z + 1)\hat{k} \\ (\underline{r} - \underline{a}) \cdot \underline{n} &= 0 \\ ((x - 2)\hat{i} + (y - 4)\hat{j} + (z + 1)\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) &= 0 \\ 2(x - 2) + 3(y - 4) + 4(z + 1) &= 0 \\ 2x - 4 + 3y - 12 + 4z + 4 &= 0 \\ 2x + 3y + 4z - 12 &= 0\end{aligned}$$

The procedure completed in Worked example 30 was very detailed. The same result can be achieved in fewer steps by recalling the formula, $n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$. This is demonstrated in the next example and is the preferred option unless specified otherwise.

WORKED EXAMPLE 31

Determine the equation of the plane containing the point $(-5, 3, 6)$ if the vector $\underline{n} = 2\hat{i} + 4\hat{j} - 7\hat{k}$ is normal to it.

THINK

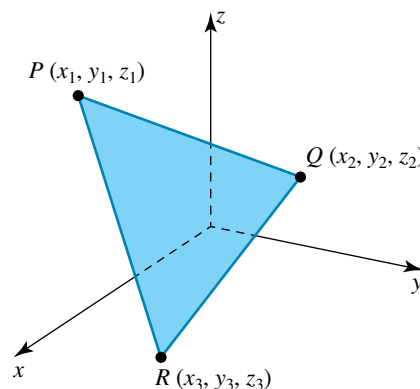
1. Recall the general formula for the equation of a plane.
2. Substitute the components of point (a, b, c) and $\underline{n}$.
Expand and collect like terms.
3. State your answer.

WRITE

$$\begin{aligned}n_1(x - a) + n_2(y - b) + n_3(z - c) &= 0 \\ 2(x + 5) + 4(y - 3) - 7(z - 6) &= 0 \\ 2x + 10 + 4y - 12 - 7z + 42 &= 0 \\ 2x + 4y - 7z + 40 &= 0 \\ \text{The equation of the plane is} \\ 2x + 4y - 7z + 40 &= 0\end{aligned}$$

Determining the equation of a plane using three points

In both worked examples above, the equation was calculated based on a known point on the plane and a vector normal to it. Consider the plane shown, defined by the three non-collinear points. A normal vector can be calculated by applying our knowledge of the vector product. The following example demonstrates this procedure.



WORKED EXAMPLE 32

- Calculate the Cartesian equation of the plane determined by the points $P(-2, 1, 3)$, $Q(3, 5, 6)$ and $R(-5, 3, 5)$
- Determine the axis intercepts for the plane.

THINK

- Calculate and define two vectors relating to the points P, Q and R .
 - Find a vector perpendicular to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. This can be found by calculating $\overrightarrow{PQ} \times \overrightarrow{PR}$. This is calculated as the determinant.
 - State the normal vector.
 - Recall the general formula for the equation of a plane.
 - Substitute the components of point $P(-2, 1, 3)$ and the normal vector into the formula.
 - State your answer.
- Calculate the x -intercept $(x, 0, 0)$ when $y = 0, z = 0$.
 - Calculate the y -intercept $(0, y, 0)$ when $x = 0, z = 0$.
 - Calculate the z -intercept $(0, 0, z)$ when $x = 0, y = 0$.
 - State the three intercepts.

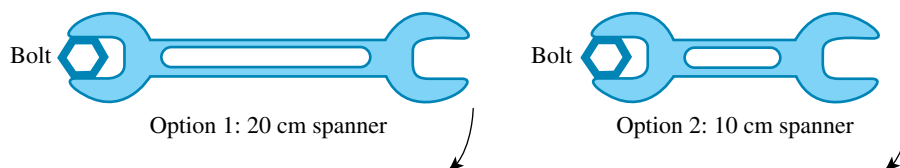
WRITE

$$\begin{aligned}
 \text{a. } \overrightarrow{PQ} &= (3 - (-2))\hat{i} + (5 - 1)\hat{j} + (6 - 3)\hat{k} \\
 &= 5\hat{i} + 4\hat{j} + 3\hat{k} \\
 \overrightarrow{PR} &= (-5 - (-2))\hat{i} + (3 - 1)\hat{j} + (5 - 3)\hat{k} \\
 &= -3\hat{i} + 2\hat{j} + 2\hat{k} \\
 \underline{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\
 \underline{n} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 4 & 3 \\ -3 & 2 & 2 \end{vmatrix} \\
 \underline{n} &= 2\hat{i} - 19\hat{j} + 22\hat{k} \\
 n_1(x - a) + n_2(y - b) + n_3(z - c) &= 0 \\
 2(x + 2) - 19(y - 1) + 22(z - 3) &= 0 \\
 2x + 4 - 19y + 19 + 22z - 66 &= 0 \\
 2x - 19y + 22z - 43 &= 0 \\
 \therefore 2x - 19y + 22z - 43 &= 0 \\
 \text{b. Let } y = 0 \text{ and } z = 0, \\
 2x - 19(0) + 22(0) &= 43 \\
 x &= \frac{43}{2} \\
 \text{Let } x = 0 \text{ and } z = 0, \\
 2(0) - 19y + 22(0) &= 43 \\
 y &= -\frac{43}{19} \\
 \text{Let } x = 0 \text{ and } y = 0, \\
 2(0) - 19(0) + 22z &= 43 \\
 z &= \frac{43}{22} \\
 \text{The plane intersects the axes at} \\
 \left(\frac{43}{2}, 0, 0\right), \left(0, -\frac{43}{19}, 0\right) \text{ and } \left(0, 0, \frac{43}{22}\right)
 \end{aligned}$$

If the equation of a plane is written in the form $ax + by + cz = d$, the intercepts can be calculated by the following formulae, $x = \frac{d}{a}$, $y = \frac{d}{b}$, $z = \frac{d}{c}$.

2.7.2 Torque

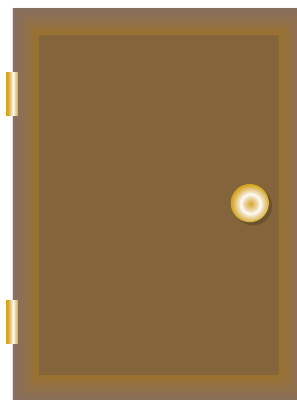
The vector product has further applications in both engineering and physics. Consider tightening a bolt on an item that you hope will never loosen. In order to turn the bolt inward, the spanner needs to be rotated clockwise. Two options are presented below.



Which spanner do you think would be more effective if both were applied with the same force? The turning force depends on how far the force is applied from the point of application, and so the first option of the longer spanner would be the best choice.

This principle is observed in many other applications such as:

1. The location of the handle on a hinged door. If the handle was placed too close to the hinges, a person would find it difficult to open the door. In most instances the door handle is on the opposite side to the hinges thereby increasing the distance from the point of application.
2. When a young child rides a bicycle with 'coaster brakes' they often try to stop by applying a force at the peddle. The length of the crank arm, along with the applied force, determines the magnitude of turning force applied at the crank.



This turning force is referred to as **torque**, and is defined by:

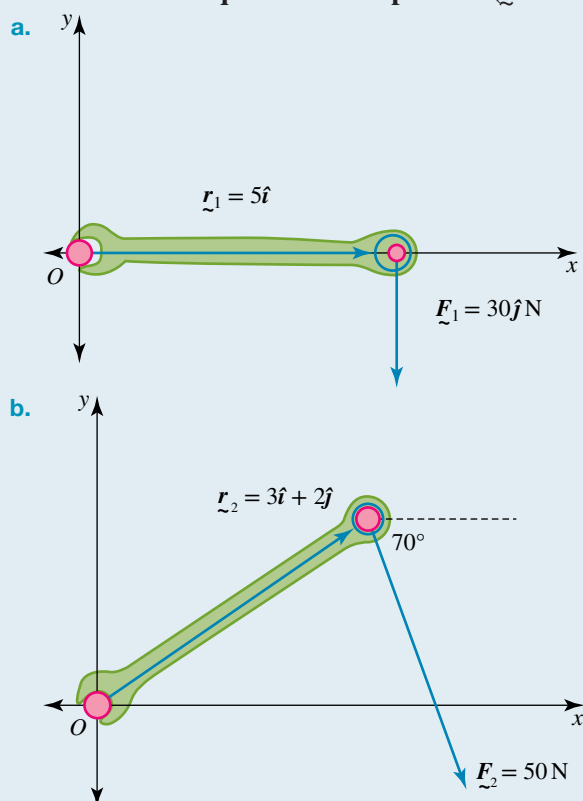
$$\text{Torque} = |\underline{r} \times \underline{F}|$$

where $\underline{F}$ is the force and $\underline{r}$ is the vector from the point of application of the force to the point, or axis, about which the turning is considered.

As this equation suggests, the size of the torque depends not only on the force and the distance the force is from the point of application, but also on the angle at which the force is applied. The torque is a maximum when the angle between $\vec{F}$ and $\vec{r}$ is 90° . If $\vec{F}$ is measured in Newtons (N) and $\vec{r}$ is measured in metres (m) then torque is measured in Newton metres (Nm).

WORKED EXAMPLE 33

The diagrams depict two different spanners being acted on by a force, $\vec{F}_n$. For each scenario, calculate the torque about the point O ($\vec{r}$ is in metres).



THINK

- a. 1. Recall the torque formula. Substitute in the vectors, $\vec{r}$ and $\vec{F}$.
2. Calculate the vector product using the determinant method.
3. The vector product is in component form. Determine the magnitude of $\vec{r} \times \vec{F}$.

WRITE

a. $\vec{r} = \overrightarrow{OP} = 5\hat{i}, \vec{F} = -30\hat{j}$

$$\begin{aligned}\text{Torque} &= |\vec{r} \times \vec{F}| \\ &= |(5\hat{i}) \times (-30\hat{j})|\end{aligned}$$

$$\begin{aligned}\vec{r} \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 0 \\ 0 & -30 & 0 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} 0 & 0 \\ -30 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 5 & 0 \\ 0 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 5 & 0 \\ 0 & -30 \end{vmatrix} \\ &= -150\hat{k}\end{aligned}$$

$$\begin{aligned}|\vec{r} \times \vec{F}| &= |-150| \\ &= 150 \text{ Nm} \\ \text{The torque is } 150 \text{ Nm.}\end{aligned}$$

- b. 1. Force needs to be expressed as a vector in component form.
 $\underline{F} = |\underline{F}|(\cos \theta \hat{i} + \sin \theta \hat{j})$

2. State the torque formula. Substitute in the vectors, $\underline{r}$ and $\underline{F}$.

3. Calculate the vector product using the determinant method.

4. The vector product is in component form. Determine the magnitude of $\underline{r} \times \underline{F}$.

$$\begin{aligned} \text{b. } \underline{r} &= \overrightarrow{OP} = 3\hat{i} + 2\hat{j} \\ \underline{F} &= 50 \cos(70^\circ)\hat{i} - 50 \sin(70^\circ)\hat{j} \\ &= 17.10\hat{i} - 46.98\hat{j} \end{aligned}$$

$$\begin{aligned} \text{Torque} &= |\underline{r} \times \underline{F}| \\ &= |(3\hat{i} + 2\hat{j}) \times (17.10\hat{i} - 46.98\hat{j})| \end{aligned}$$

$$\begin{aligned} |\underline{r} \times \underline{F}| &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 17.10 & -46.98 & 0 \end{vmatrix} \\ &= \hat{i} \begin{vmatrix} 2 & 0 \\ -46.98 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ 17.10 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 2 \\ 17.10 & -46.98 \end{vmatrix} \\ &= -175.14\hat{k} \end{aligned}$$

$$\begin{aligned} |\underline{r} \times \underline{F}| &= |-175.14| \\ &= 175.14 \text{ Nm} \end{aligned}$$

The torque is 175.14 Nm.

2.7.3 Applications of vector functions of time

Vector equations of lines, vector functions and parametric equations are convenient methods to describe the position of a particle in two or three dimensions according to a given parameter, t . As shown in many of the previous sections of work, time is a common parameter. The following examples explore various applications of vector functions of time in both abstract and real life instances.

WORKED EXAMPLE 34

The paths of two model sail boats are defined by the following vector functions of time, where time is measured in minutes and distance in metres.

$$\text{Boat A: } \underline{r}_A(t) = (t + 13)\hat{i} + 6t\hat{j}$$

$$\text{Boat B: } \underline{r}_B(t) = (3t - 5)\hat{i} + (5 + t)\hat{j}$$

- Calculate the distance between the two boats at $t = 8$ minutes.
- Calculate if, at any given time t (seconds), they are in the same location. That is, determine if there is any possibility of a collision.



THINK

- Evaluate the position of boat A by letting $t = 8$, expressed as $\underline{r}_A(8)$.
- Evaluate the position of boat B by letting $t = 8$, expressed as $\underline{r}_B(8)$.

WRITE

$$\begin{aligned} \text{a. } \underline{r}_A(8) &= (8 + 13)\hat{i} + 6 \times 8\hat{j} \\ &= 21\hat{i} + 48\hat{j} \end{aligned}$$

$$\begin{aligned} \underline{r}_B(8) &= (3 \times 8 - 5)\hat{i} + (5 + 8)\hat{j} \\ &= 19\hat{i} + 13\hat{j} \end{aligned}$$

3. The vector joining the two boats at $t = 8$ min, $\underline{r}_B(8) - \underline{r}_A(8)$, defines the directed line segment between the two boats. The magnitude of the resultant vector is then calculated.
This represents the straight line distance between boat A and B.

$$\begin{aligned} |\underline{r}_B(8) - \underline{r}_A(8)| &= |(19\hat{i} + 13\hat{j}) - (21\hat{i} + 48\hat{j})| \\ &= |-2\hat{i} - 35\hat{j}| \\ &= \sqrt{(-2)^2 + (-35)^2} \\ &= \sqrt{1229} \end{aligned}$$

The boats are approximately 35.06 metres apart.

- b. 1. The boats will collide if they reach the same point at the same time.
Equate the two vector functions.
2. Equate the $\hat{i}$ and $\hat{j}$ components and solve for t in both cases.

b. $\underline{r}_A = \underline{r}_B$
 $(t + 13)\hat{i} + 6t\hat{j} = (3t - 5)\hat{i} + (5 + t)\hat{j}$

$\hat{i}$ component:	$\hat{j}$ component:
$t + 13 = 3t - 5$	$6t = 5 + t$
$18 = 2t$	$5t = 5$
$2t = 18$	$t = 1$
$t = 9$	

There is no common time for when the two functions are in the same location.

The boats do not collide.

3. State your observations.

4. State the answer.

WORKED EXAMPLE 35

The location of two particles can be defined by the following vector functions of time.

$$\begin{aligned} \underline{r}_A(t) &= (144 - t^2)\hat{i} + (2t^2)\hat{j} + (0.25(t - 2)^3)\hat{k} \\ \underline{r}_B(t) &= (t^2 - 5t - 6)\hat{i} + (240 - 4t)\hat{j} + ((t - 4)^2 + 92)\hat{k} \end{aligned}$$

Calculate if, at any given time t , $t \geq 0$ seconds, they are in the same location.

THINK

1. The particles will be in the same location if they reach the same point at the same time.
Equate the two vector functions.

WRITE

$$\begin{aligned} \underline{r}_A &= \underline{r}_B \\ (144 - t^2)\hat{i} + 2t^2\hat{j} + (0.25(t - 2)^3)\hat{k} &= (t^2 - 5t - 6)\hat{i} \\ &+ (240 - 4t)\hat{j} + ((t - 4)^2 + 92)\hat{k} \end{aligned}$$

2. Equate the $\hat{i}$ and $\hat{j}$ components and collect like terms.

$\hat{i}$ component:	$\hat{j}$ component:
$144 - t^2 = t^2 - 5t - 6$	$2t^2 = 240 - 4t$
$0 = 2t^2 - 5t - 150$	$2t^2 + 4t - 240 = 0$
	$t^2 + 2t - 120 = 0$

3. Factorise both quadratic equations and apply the Null factor Law to solve for t .

$0 = (2t + 15)(t - 10)$	$(t - 10)(t + 12) = 0$
$(2t + 15) = 0$ or $(t - 10) = 0$	$(t - 10) = 0$ or $(t + 12) = 0$
$t = -7.5$ or $t = 10$	$t = 10$ or $t = -12$

4. State your observations.	As time cannot be negative, we can reject the solutions of $t = -7.5$ and $t = -12$. There is a common time in which the two particles are in the same location. This occurs when $t = 10$ seconds.
5. Determine the position vector of both particles at $t = 10$.	$\underline{r}_A(10) = (144 - (10)^2)\hat{i} + 2(10)^2\hat{j} + (0.25(10 - 2)^3)\hat{k}$ $= 44\hat{i} + 200\hat{j} + 128\hat{k}$ $\underline{r}_B(10) = (10^2 - 5(10) - 6)\hat{i} + (240 - 4(10))\hat{j} + ((10 - 4)^2 + 92)\hat{k}$ $= 44\hat{i} + 200\hat{j} + 128\hat{k}$
6. Summarise findings.	The two particles will be in the same location when $t = 10$ seconds. The position vector at this location is $44\hat{i} + 200\hat{j} + 128\hat{k}$.

WORKED EXAMPLE 36

The parametric equation, $x = 4 - t$, $y = 1 + 3t$, $z = 4 + 2t$, defines the path of particle. Determine the point of intersection between the particle and the plane, $2x + 4y - 2z = 1$.

THINK

- Define the three parametric equations as [1] to [3].
- The point of intersection must satisfy both the equation of the plane and line. Substitute [1], [2] and [3] into the equation of the plane. Continue by solving for t .
- Evaluate the parametric equation, for $t = -0.5$ to calculate the point of intersection.
- State the solution as the point of intersection.

WRITE

$$\begin{aligned}
 x &= 4 - t & [1] \\
 y &= 1 + 3t & [2] \\
 z &= 4 + 2t & [3] \\
 2(4 - t) + 4(1 + 3t) - 2(4 + 2t) &= 1 \\
 8 - 2t + 4 + 12t - 8 - 4t &= 1 \\
 6t + 4 &= 1 \\
 t &= \frac{1 - 4}{6} \\
 t &= -0.5 \\
 x &= 4 - (-0.5) \\
 &= 4.5 \\
 y &= 1 + 3(-0.5) \\
 &= -0.5 \\
 z &= 4 + 2(-0.5) \\
 &= 3 \\
 \text{The particle intersects the plane at} & \\
 (4.5, -0.5, 3) &
 \end{aligned}$$

study on

Units 3 & 4 > Area 2 > Sequence 1 > Concept 6

Applications of vectors Summary screen and practice questions

Exercise 2.7 Applications of vectors

Technology active

- WE30&31** Calculate the equation of the plane perpendicular to $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$, containing the point $P_0 = (x_0, y_0, z_0)$.
 - $P_0 = (3, 6, -2)$ and $\underline{n} = 5\hat{i} - 2\hat{j} + \hat{k}$
 - $P_0 = (-2, 1, 4)$ and $\underline{n} = -3\hat{i} + 5\hat{j} + 4\hat{k}$
 - $P_0 = (-1, 0, 3)$ and $\underline{n} = -7\hat{i} + 4\hat{j} - 6\hat{k}$
 - $P_0 = (4, -1, -4)$ and $\underline{n} = -4\hat{i} - \hat{j} + 3\hat{k}$
- Determine a unit vector, $\hat{n}$, to each of the following planes.
 - $4x - 2y - z = 12$
 - $-x + 2y - 12 = 2z$
 - $4 = -6x + 7y - 3z$
- WE32**
 - Calculate the Cartesian equation of the plane determined by the points below.
 - Determine the axis intercepts for each plane.
 - $P(1, 4, 2)$, $Q(2, 3, 5)$ and $R(1, 9, 8)$
 - $P(-2, 5, 3)$, $Q(3, -1, 2)$ and $R(-7, 1, 0)$
 - $P(-1, 4, 8)$, $Q(1, 8, -3)$ and $R(12, 9, -5)$
 - $P(-6, 7, 1)$, $Q(-9, 2, 4)$ and $R(3, -3, 8)$
- For each of the following, determine if the point $P(x, y, z)$, lies on the given plane.
 - $P(2, 9, -1)$, plane: $4x - 2y + z = -11$
 - $P(-1, -6, 4)$, plane: $3x - z + 1 = 10$
 - $P(1, 4, -7)$, plane: $-x + 3y + 5z = -24$
- Two planes are parallel if both normal vectors are parallel. Similarly, if two planes are perpendicular, then so too are their normal vectors.
 - Examine the two planes and determine if they are parallel. $P_1: 4x - 2y + 4z = 15$ and $P_2: 2x - y + 2z = -23$
 - Determine the equation of the plane that is parallel to $11x - 2y + 6z = 12$ and contains the point $P(1, -8, 2)$.
 - Calculate the value of c if the two planes, $P_1: 8x - 6y - 2z = 1$ and $P_2: 2x - y + cz = -6$, are perpendicular.
- The angle between two planes is calculated using the normal vectors $\underline{n}_1$ and $\underline{n}_2$. The angle can be determined by the formula, $\cos \theta = \frac{|\underline{n}_1 \cdot \underline{n}_2|}{|\underline{n}_1| \times |\underline{n}_2|}$. Calculate the angle between the two planes, $4x - y + 2z = 0$ and $2x + 3y - z = -1$ to three decimal places.
- Explain why the equation of a plane cannot be formed by the points $P(-2, 6, -18)$, $Q(-12, 8, -68)$ and $R(13, 3, 57)$.
- Calculate the equation of the plane that is perpendicular to the plane $8x + 2y + z = 0$, passing through the points $P(2, 3, 4)$ and $Q(5, 7, 8)$.
 - Justify that the intersection of the two planes form a right-angle.
- Triangular shaped shade sails are often used in parks and schools. The vertices of a shade sail can be located by the points $L(3, -5, 3)$, $M(-4, -1, 4)$ and $N(1, 2, 5)$, where all units are in metres.
 - Develop an equation to model the plane created by the erected triangular shade cloth. State the assumptions considered true when developing models and solutions.
 - Calculate the area of the shade cloth to two decimal places.
- A plane is perpendicular to the vector $\underline{n}$ and contains the point P . Prove that the shortest distance, D , from the plane to a point Q is given by $D = \frac{|\underline{n} \cdot \overrightarrow{PQ}|}{|\underline{n}|}$.



- b. Apply the formula, $D = \frac{|\underline{n} \cdot \overrightarrow{PQ}|}{|\underline{n}|}$, to calculate the shortest distance from $Q(6, 5, 4)$ to the plane, $x + 5y - 2z = 8$. (Hint, $P(x, y, z)$ is any point on the plane)
- c. If a plane is in the form, $ax + by + cz = d$, the shortest distance to a point $Q = (x_1, y_1, z_1)$ can be calculated by the equation, $D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$. Use this alternative approach to verify the solution obtained in Question 10b.

11. A line is formed when two non-parallel planes intersect. A vector in the direction of the line of intersection can be calculated by the vector product, $\underline{n}_1 \times \underline{n}_2$. This implies that the vector product of the two normals are parallel to the line of intersection. Calculate the parametric equation for the line of intersection of the planes, $8x - 9y - 2z = 4$ and $-6x + 2y - z = 13$.

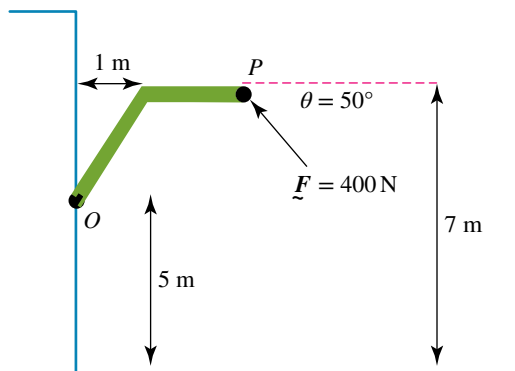
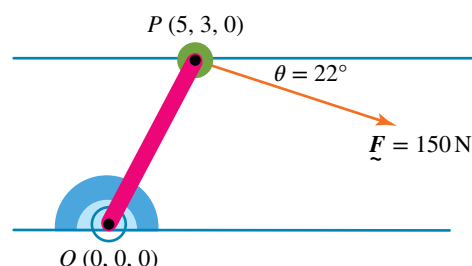
12. **WE33** A force, $\underline{F}$, acts on an object at a point, P . In each of the following cases, calculate the torque about O . The coordinate location of points is measured in metres.

- a. $P(10, 0, 0)$ and $\underline{F} = -20\hat{k}$ N
- b. $P(6, 1, -2)$ and $\underline{F} = (4\hat{i} - 7\hat{j} + \hat{k})$ N
- c. $P(2, -5, 6)$ and $\underline{F} = (-5\hat{i} + 2\hat{j} - 3\hat{k})$ N

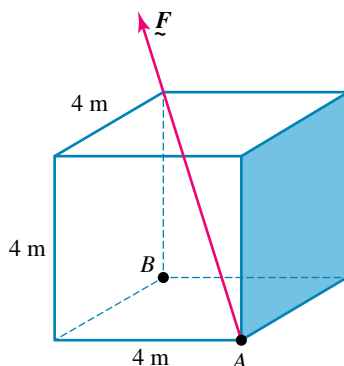
13. To rotate the level arm shown on the right, the turning force must be greater than 400 Nm. Calculate the torque about the point O , given the force, $\underline{F} = (x\hat{i} + y\hat{j} + z\hat{k})$, acting on the object at P .

14. A metal bracket is fixed to the wall at a rotatable point O . The force, $\underline{F}$, is applied at P . The magnitude of the torque at point O is 1586.7 Nm.

Calculate the vector location of point P .



15. A force of 400 N is applied on the cube at point A, as shown. Calculate the torque of the force about B to two decimal places.



16. **WE34** The paths of two model sail boats are defined by the following vector functions of time, where time is measured in minutes and distance in metres.

$$\text{Boat A: } \underline{r}_A(t) = (9t - 7)\hat{i} + 11t\hat{j}$$

$$\text{Boat B: } \underline{r}_B(t) = (2t + 14)\hat{i} + (9 + t)\hat{j}$$

- a. Calculate the distance between the two boats at $t = 5$ minutes.
b. Calculate if, at any given time t (minutes), they are in the same location. That is, determine if there is any possibility of a collision.
17. **WE35** The flight path, measured in metres, of two helicopters is given by the vector functions of time (minutes) below.

$$\underline{r}_A(t) = (169 + 0.5t^2)\hat{i} + (10t + 120)\hat{j} + (0.45t^3 + 2)\hat{k}$$

$$\underline{r}_B(t) = (4t + 179)\hat{i} + (240 - 0.2t^2)\hat{j} + (2t^2 + 20)\hat{k}$$

Determine if the helicopters will ever cross paths.
State any assumptions used in developing models and solutions.



18. Two particles are moving in a circular motion in a three-dimensional space. The position of the first particle is defined by

$$x_a = 6 \sin t, \quad y_a = 4 \cos t, \quad z_a = 5 \quad 0 \leq t \leq 10\pi$$

The position of the second particle is defined by

$$x_b = 2 \cos t - 6, \quad y_b = 2 + 2 \sin t, \quad z_b = 5 \quad 0 \leq t \leq 10\pi$$

The position of both particles are functions of time (t) given in seconds. Calculate the time and position of when the two particles first intersect.

19. **WE36** The parametric equation, $x = 2 + 4t, y = -4 - 2t, z = 3 + 3t$, defines the path of particle. Determine the point of intersection between the particle and the plane, $-4x + 3y + 7z = 21$.
20. The vector equation of the line, $\underline{r} = (-3\hat{i} + 5\hat{j} - 2\hat{k}) + k(2\hat{i} + \hat{j} - 6\hat{k})$, defines the path of particle. Determine the point of intersection between the particle and the plane, $7x + 4y + 8z = 1$.

2.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

- MC** Consider the points $P(-2, 5, 1)$, $Q(5, 8, 4)$ and $R(9, -6, 12)$. The resultant vector of the vector sum, $2\overrightarrow{OP} + \overrightarrow{QP} - 4\overrightarrow{QR}$, is:

A. $-43\hat{i} + 56\hat{j} - 30\hat{k}$	B. $-27\hat{i} + 63\hat{j} - 33\hat{k}$
C. $-10\hat{i} + 57\hat{j} - 27\hat{k}$	D. $-42\hat{i} + 61\hat{j} - 35\hat{k}$
- MC** Determine the angle between the given vectors, $\underline{a} = 2\hat{i} - 5\hat{j} + 8\hat{k}$ and $\underline{b} = 9\hat{i} + 7\hat{j} - 8\hat{k}$.

A. $\theta = 129.24^\circ$	B. $\theta = 127.81^\circ$
C. $\theta = 127.09^\circ$	D. $\theta = 128.56^\circ$

3. **MC** When expressed in component form, the vector $\underline{u} = [15, 325^\circ, 36^\circ]$ is defined by which of the following?
- A. $\underline{u} = 9.94\hat{i} - 6.96\hat{j} + 8.82\hat{k}$ B. $\underline{u} = 10.25\hat{i} - 7.14\hat{j} + 9.72\hat{k}$
 C. $\underline{u} = 11.36\hat{i} - 6.07\hat{j} - 8.01\hat{k}$ D. $\underline{u} = 11.87\hat{i} - 5.99\hat{j} + 7.67\hat{k}$
4. **MC** Determine the equivalent Cartesian equation for $x = 2t - 7, y = \frac{t^2}{7} + t$
- A. $y = \frac{1}{4}x^2 + 2x - \frac{21}{2}$
 B. $y = \frac{1}{16}x^2 + x + \frac{15}{2}$
 C. $y = \frac{1}{26}x^2 + 2x + \frac{21}{8}$
 D. $y = \frac{1}{28}x^2 + x + \frac{21}{4}$
5. **MC** A sphere contains the points, $A(7, 13, 8)$ and $B(-1, 3, 2)$, that are the endpoints of its diameter. Calculate the Cartesian equation of the sphere.
- A. $45 = (x + 3)^2 + (y - 8)^2 + (z - 5)^2$ B. $50 = (x - 3)^2 + (y - 8)^2 + (z - 5)^2$
 C. $52 = (x - 3)^2 + (y + 8)^2 + (z - 6)^2$ D. $49 = (x - 2)^2 + (y - 7)^2 + (z - 6)^2$
6. a. Calculate the vector equation for the line passing through the given points, $A(5, -2, 13)$ and $B(6, 4, 7)$.
 b. Determine the value for $y \times z$ if the head of vector, $\underline{c} = 12\hat{i} + y\hat{j} + z\hat{k}$, touches the line at $P(x, y, z)$.
7. **MC** Calculate $\underline{a} \times \underline{b}$ for the given vectors, $\underline{a} = [19, 7^\circ]$ and $\underline{b} = [24, 145^\circ]$.
- A. $\underline{a} \times \underline{b} = 305.12\hat{n}$
 B. $\underline{a} \times \underline{b} = 299.82\hat{n}$
 C. $\underline{a} \times \underline{b} = -311.89\hat{n}$
 D. $\underline{a} \times \underline{b} = 309.74\hat{n}$
8. Calculate a vector perpendicular to both $\underline{a} = 5\hat{i} + 6\hat{j} - 4\hat{k}$ and $\underline{b} = 4\hat{i} - 9\hat{j} + 7\hat{k}$.
9. Determine the volume of the parallelepiped containing the adjacent vectors, $\underline{a} = 4\hat{i} + 4\hat{j} + 2\hat{k}, \underline{b} = 8\hat{i} + 6\hat{j} - 3\hat{k}$ and $\underline{c} = 12\hat{i} - 5\hat{j} - 7\hat{k}$.
10. Determine the equation of the plane containing the point $(-2, 9, 1)$ if the vector $\underline{n} = 3\hat{i} + 5\hat{j} + 8\hat{k}$ is normal to it.
11. The path (measured in metres) of two objects in flight is given by the following vector functions of time (minutes):

$$\underline{r}_a(t) = (t^2 + 4)\hat{i} + (4t + 9)\hat{j} + (0.5t^2 + 1)\hat{k}, \quad 0 \leq t \leq 20$$

$$\underline{r}_b(t) = (8t + 184)\hat{i} + (0.25t^2)\hat{j} + (3t + 109)\hat{k}, \quad 0 \leq t \leq 20$$

Determine if the particles will cross paths at the same time, stating the time it occurs and the location at that point in time.

12. The parametric equation, $x = 2 - t, y = 2 + t, z = 5 + 2t$, defines the path of particle. Determine the point of intersection between the particle and the plane, $x + 2y - 3z = -14$.

Complex familiar

13. The vector product of two vectors $\underline{a} = -2\hat{i} + 11\hat{j} - \hat{k}$ and $\underline{b} = -3\hat{i} + b_y\hat{j} + b_z\hat{k}$ is $\underline{a} \times \underline{b} = 12\hat{i} + 5\hat{j} + 31\hat{k}$. Calculate the values of b_y and b_z .
14. Calculate the equation of the plane determined by the following points: $P(3, 8, 9), Q(4, -8, -5)$ and $R(-5, 6, 2)$.

15. From base camp, a group of explorers can observe a famous ruin that can be located by the position vector, $\underline{r} = 150\hat{i} + 160\hat{j} + 650\hat{k}$, where all units are in metres.

Convert the position vector to polar form.

Interpret the both the magnitude and altitude angle to assess if the path from camp to the ruin would be reasonable.

16. Given $\underline{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\underline{b} = 8\hat{i} + 3\hat{k}$ and $\underline{c} = 6\hat{i} + 2\hat{j} + 6\hat{k}$, prove that the vector product is not associative. That is, $\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$.



Complex unfamiliar

17. A golfer is trying to hit the golf ball as far as possible from ground level. The path of the golf ball, according to time t (seconds), can be modelled by the parametric equations,

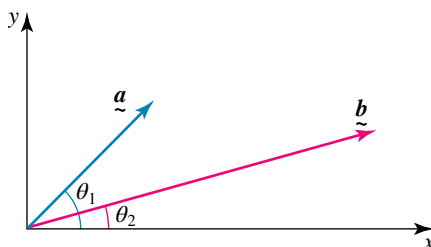
$$x = (v_0 \cos \theta) t, y = (v_0 \sin \theta) t - \frac{1}{2}gt^2 \quad (\text{where } g = 9.8)$$

When the golfer strikes the ball, $\theta = 30^\circ$ and $v_0 = 66 \text{ ms}^{-1}$.

Calculate the maximum height the ball reaches and when it strikes the ground. State any assumptions considered to be true when developing models and/or solutions. Justify the reasonableness of the results by considering the location of the golf ball when it returns to ground level.



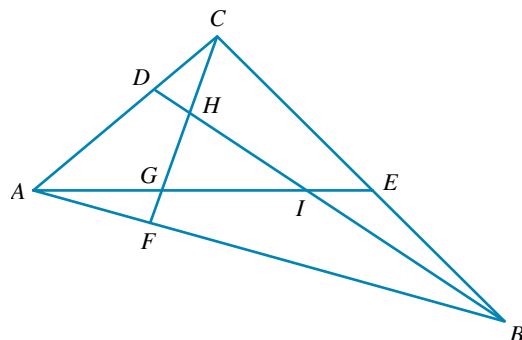
18. Consider the two vectors, $\underline{a}$ and $\underline{b}$, in the diagram below.



Verify the identity $\sin(\theta_a - \theta_b) = \sin \theta_a \cos \theta_b - \cos \theta_a \sin \theta_b$. The proof should be constructed by calculating the vector product, $\underline{a} \times \underline{b}$.

19. Calculate the equation of the plane containing the point $P(-3, 2, 5)$, and containing the line of intersection of the planes $x - 2y + 4z = 3$ and $-3x + 5y - 4z = -5$. Technology can be used to determine a particular point on the line formed by the intersection of the planes.

20. In the triangle ABC , D divides AC in the ratio $2:1$, E divides CB in the ratio $2:1$ and F divides BA in the ratio $2:1$. Prove that the area of the triangle GHI is $\frac{1}{7}$ times the area of triangle ABC .



study on

Units 3 & 4 Sit exam

Answers

2 Vectors in three dimensions

Exercise 2.2 Introduction to vectors in three dimensions

- $A(5, -4)$
 - $C\left(\frac{5}{2}, 11, -12\right)$
- $5\hat{i} + 3\hat{j} + 2\hat{k}$
 - $-\frac{2}{5}\hat{i} + \frac{2}{3}\hat{j} - \frac{22}{17}\hat{k}$
- Vector
 - Vector
 - Scalar
- $-4\hat{i} + 2\hat{j} - 7\hat{k}$
 - $\hat{i} + 3\hat{j} + 6\hat{k}$
- $\overrightarrow{CE} = \underline{r} + \underline{t}$
 - $\overrightarrow{DJ} = -\underline{r} + \underline{s} + \underline{t}$
- The aeroplane is 1081.67 km at a bearing $146.31^\circ T$ from its starting location.
- $8\hat{i} + 4\hat{j} + \hat{k}$
 - $\frac{1}{9}(8\hat{i} + 4\hat{j} + \hat{k})$
- $6\hat{i} + 2\hat{j} - 8\hat{k}$
 - $\frac{1}{\sqrt{26}}(3\hat{i} + \hat{j} - 4\hat{k})$
- $5\hat{i} + \hat{j} + 8\hat{k}$
 - $-17\hat{i} + 2\hat{j} - 29\hat{k}$
- $5\hat{i} + 6\hat{j} - \hat{k}$
 - $-2\hat{i} + 5\hat{j}$
 - $2\hat{i} + 3\hat{j} + 4\hat{k}$
 - $-5\hat{i} + 48\hat{j} - 16\hat{k}$
 - $2\hat{i} + 11\hat{j} + 8\hat{k}, \therefore M(2, 11, 8)$
 - $3\overrightarrow{PQ} = \overrightarrow{QS}$ Therefore, the vectors are parallel.
- 15.73
 - 5.87
- 45
- $y = -18$
- $-76 = -76$
 $\therefore \underline{c} \cdot (\underline{a} - \underline{b}) = \underline{c} \cdot \underline{a} - \underline{c} \cdot \underline{b}$
- $\theta = 75.29^\circ$
- $\frac{-94}{\sqrt{197}}$
 - $-\frac{940}{197}\hat{i} + \frac{846}{197}\hat{j} - \frac{376}{197}\hat{k}$
 - $-\frac{45}{197}\hat{i} - \frac{58}{197}\hat{j} - \frac{18}{197}\hat{k}$
- $2.47\hat{i} + 4.45\hat{j} + 3.18\hat{k}$
 - $-1.15\hat{i} + 1.99\hat{j} - 1.93\hat{k}$
- $\varphi = -29.12^\circ$
 - $\left[\sqrt{38}, 68.20^\circ, -29.12^\circ\right]$
 - $\varphi = 21.80^\circ$
 - $\left[\sqrt{29}, 306.87^\circ, 21.80^\circ\right]$
 - $\varphi = 41.87^\circ$
 - $\left[\sqrt{110}, 50.19^\circ, 41.87^\circ\right]$
 - $\varphi = -57.37^\circ$
 - $\left[\sqrt{141}, 128.66^\circ, -57.37^\circ\right]$

- The resultant displacement of the boat from its starting position is 939.692 km, in the direction, $160^\circ T$.
- $\lambda = -2; \lambda \underline{a} = -12\hat{i} - 4\hat{j} + 2\hat{k}$
- The helicopter has travelled 269.14 metres when it reaches the closest point. The distance to the mountain climber at that point is 8.13 metres.

Exercise 2.3 Geometric proofs using vectors

- Sample responses can be found in the worked solutions in the online resources.
- $\frac{1}{2}\sqrt{870}$ units squared
- Sample responses can be found in the worked solutions in the online resources.

Exercise 2.4 Cartesian and parametric equations

- $A(2, 2, 0)$
 - $C(2, -4, 0)$
 - $E(-3, 4, 0)$
- $\sqrt{274}$
 - $\sqrt{146}$
 - $\sqrt{233}$
 - $\sqrt{206}$
- $(7, 7, -1)$
 - $(-3, 4, -7)$
 - $(4, -6, 4)$
 - $\left(\frac{11}{2}, -2, \frac{-9}{2}\right)$
- $36 = (x - 2)^2 + (y - 8)^2 + (z + 3)^2$
 - $9 = (x + 3)^2 + (y - 1)^2 + (z + 9)^2$
 - $49 = (x - 1)^2 + (y)^2 + (z - 6)^2$
 - $11 = (x + 9)^2 + (y + 2)^2 + (z - 7)^2$
- $50 = (x - 3)^2 + (y - 8)^2 + (z - 5)^2$
 - $56 = (x + 1)^2 + (y + 1)^2 + (z - 5)^2$
 - $254 = (x + 3)^2 + (y + 2)^2 + (z + 10)^2$
 - $45 = (x - 5)^2 + (y + 6)^2 + (z - 3)^2$
- $x^2 + y^2 + z^2 - 10x - 14y - 2z - 69 = 0$
 - $x^2 + y^2 + z^2 + 2x - 6y + 4z - 67 = 0$
 - $x^2 + y^2 + z^2 + 6x - 2y - 5 = 0$
 - $x^2 + y^2 + z^2 - 2x - 8y + 6z - 30 = 0$
- $C(5, -2, -6), r = 9$
 - $C(-1, -4, 8), r = 9$
 - $C(-10, 6, -12), r = 10$
 - $C(-1, -2, -3), r = 5$
- $y = \frac{2}{3}x + \frac{11}{3}$
 - $y = -x^2 - 9x - 20$
 - $y = x^2 + 2x + 8$
 - $y = \frac{\left(\sqrt[3]{x}\right)^2}{4}$ or $y = \frac{1}{4}x^{\frac{2}{3}}$
 - $y = \frac{4x - 17}{x - 4}$
 - $y = \frac{4 - x^2}{4} = -0.25x^2 + 1$

9. a. $x^2 + y^2 = 1$
 b. $x^2 + y^2 = 1$
 c. $\frac{x^2}{16} + \frac{y^2}{49} = 1$
 d. $x^2 - y^2 = 1$

10. a. $y = \frac{x}{15}$
 b. $y = \frac{x^2}{16} - \frac{x}{4}$
 c. $y = \frac{x^3}{27} + 2$
 d. $x^2 + y^2 = 4$

11. a. $\hat{i} + 3.5\hat{j}$

b.

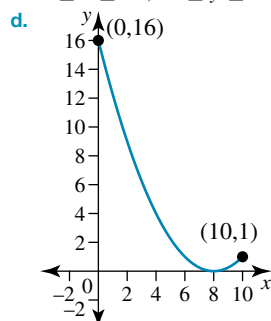
t	$x(t)$	$y(t)$	$\underline{r}(t)$
0	0	0	$0\hat{i} + 0\hat{j}$
1	1	3.5	$\hat{i} + 3.5\hat{j}$
2	2	6	$2\hat{i} + 6\hat{j}$
3	3	7.5	$3\hat{i} + 7.5\hat{j}$
4	4	8	$4\hat{i} + 8\hat{j}$

12. a. $6\hat{i} + \hat{j}$

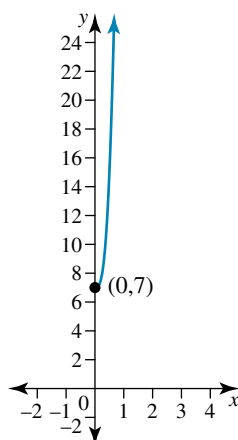
b.

t	$x(t)$	$y(t)$	$\underline{r}(t)$
0	0	0	$0\hat{i} + 16\hat{j}$
1	2	9	$2\hat{i} + 9\hat{j}$
2	4	4	$4\hat{i} + 4\hat{j}$
3	6	1	$6\hat{i} + 1\hat{j}$
4	8	0	$8\hat{i}$
5	10	1	$10\hat{i} + \hat{j}$

- c. $y = \frac{x^2}{4} - 4x + 16$
 $0 \leq x \leq 10, 0 \leq y \leq 16$



13. a. $y = 8x^3 + 7$
 b. $0 \leq x \leq +\infty$ and $7 \leq y \leq +\infty$



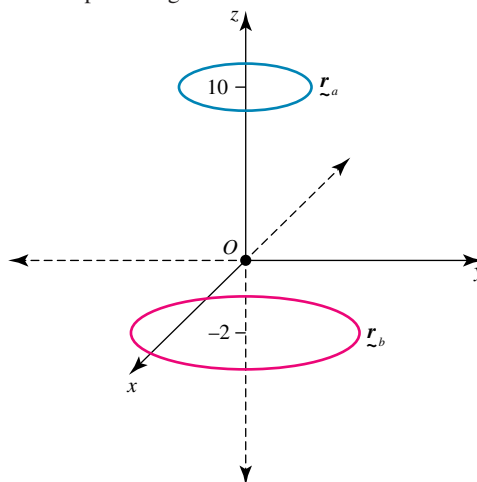
- c. $2\hat{i} + 71\hat{j}$

14. $\underline{r}(t) = (3 - 5t)\hat{i} + (2 + 3t)\hat{j} + (t + 1)\hat{k}$, for $t \in \mathbb{R}$

t	$x(t)$	$y(t)$	$z(t)$	$\underline{r}(t)$
1	-2	5	2	$-2\hat{i} + 5\hat{j} + 2\hat{k}$
3	-12	11	4	$-12\hat{i} + 11\hat{j} + 4\hat{k}$
5	-22	17	6	$-22\hat{i} + 17\hat{j} + 6\hat{k}$

- b. The path of the particle could be described as a vector equation of a line.

15. a. Sketch the plot using 3D software.



- b. $\underline{r}_a(t)$ is a circle with radius, $r = 5$ shifted 10 units up
 $\underline{r}_b(t)$ is a circle with radius, $r = 6$ the circle has been shifted 2 units downward

16. a. $\underline{r}(0) = 0\hat{i} + 0\hat{j} + 0\hat{k}$
 $\underline{r}(12) = 36\hat{i} + 34.56\hat{j} + 5.6\hat{k}$

- b. 50.22 units

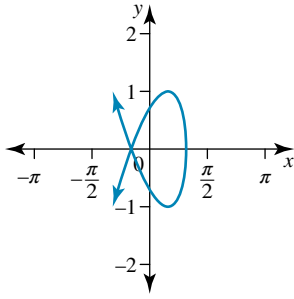
17. a. $850\hat{i} + 950\hat{i} + 350\hat{k}$

- b. The largest value for $b = 32.71$

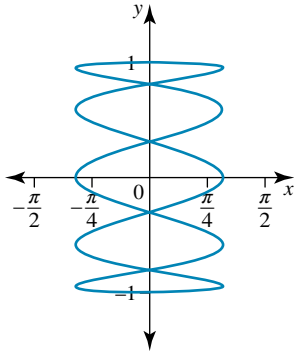
- c. 633.75 m

- d. No, it never ascends in flight path, it is always descending.

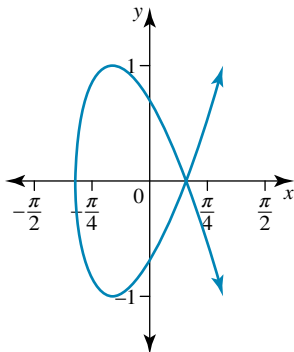
18. a. $x = \cos(4t), y = \sin(6t)$ for $t \geq 0$



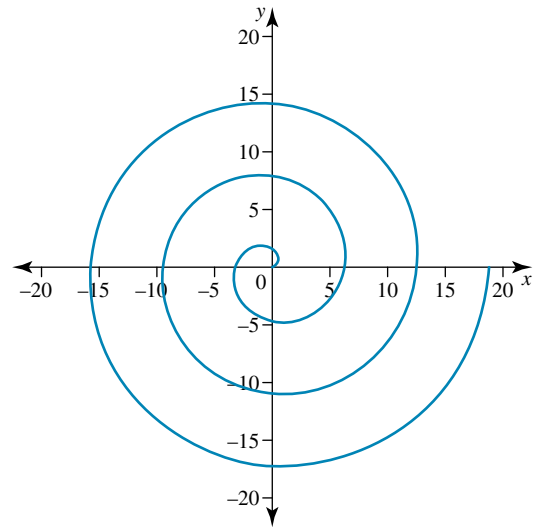
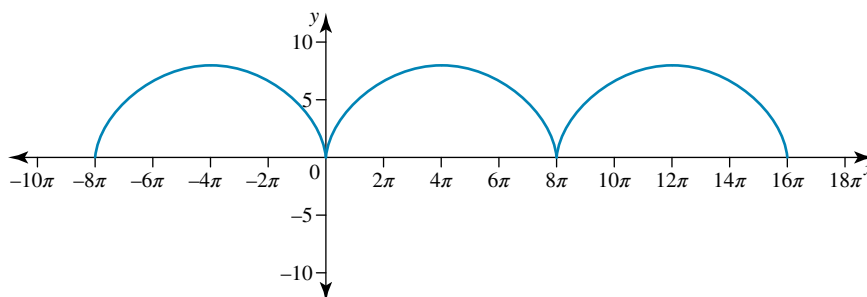
- b. $x = \sin 5t$
 $y = \cos t$ for $t \geq 0$



- c. $x = -\cos 2t$,
 $y = \sin 3t$ for $t \geq 0$



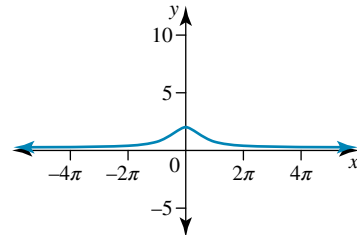
19. a. $\vec{r}(t) = t \cos(t) \hat{i} + t \sin(t) \hat{j}$,
for $0 \leq t \leq 6\pi$



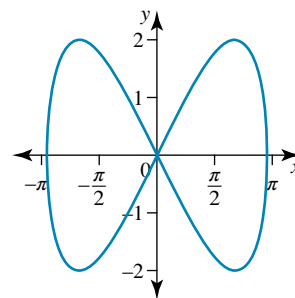
- b. $\vec{r}(t) = 4(t - \sin(t)) \hat{i} + 4(1 - \cos(t)) \hat{j}$,
for $-2\pi \leq t \leq 4\pi$

*See image at foot of the page

- c. $\vec{r}(t) = 2 \cot(t) \hat{i} + 2 \sin^2(t) \hat{j}$,
for $-\pi \leq t \leq \pi$



- d. $\vec{r}(t) = 3 \cos(t) \hat{i} + 2 \sin(2t) \hat{j}$,
for $-\pi \leq t \leq \pi$



20. Sample responses can be found in the worked solutions in the online resources.

Exercise 2.5 The vector equation of a straight line

- $\underline{r} = (2\hat{i} + 1\hat{j}) + k(4\hat{i} + 6\hat{j})$
 - $\underline{r} = (6\hat{j}) + k(9\hat{i} - 18\hat{j})$
 - $\underline{r} = (-4\hat{i} + 6\hat{j}) + k(3\hat{i} + 6\hat{j})$
 - $\underline{r} = (-2\hat{i} - 4\hat{j}) + k(7\hat{i} + 7\hat{j})$
 - $\underline{r} = (2\hat{i} + 8\hat{j}) + k(5\hat{i} - 9\hat{j})$
 - $\underline{r} = (-7\hat{i} - 5\hat{j}) + k(8\hat{i} + 20\hat{j})$
- $\underline{r} = (-2\hat{i} + 8\hat{j}) + k(7\hat{i} - 11\hat{j})$
 - $19\hat{i} - 25\hat{j}$
- $P(10, -8)$
 - $P(1, 3)$
- $\underline{r} = (-2\hat{i} + 2\hat{j} + \hat{k}) + k(2\hat{i} + 4\hat{j} + 6\hat{k})$
 - $\underline{r} = (4\hat{i} - 1\hat{j} + 11\hat{k}) + k(-17\hat{i} + 10\hat{j} - 23\hat{k})$
 - $\underline{r} = (5\hat{i} + 4\hat{j} + 11\hat{k}) + k(-6\hat{i} - 3\hat{j} - 3\hat{k})$
 - $\underline{r} = (6\hat{i} - 1\hat{j} + 2\hat{k}) + k(-\hat{i} + 4\hat{j} - 11\hat{k})$
 - $\underline{r} = (-3\hat{i} - 2\hat{j} + \hat{k}) + k(8\hat{i} + 9\hat{j} - 14\hat{k})$
 - $\underline{r} = (\hat{i} + 5\hat{j} + 8\hat{k}) + k(6\hat{i} - 5\hat{j} - 9\hat{k})$
- $\underline{r} = (-4\hat{i} + 7\hat{j} + 6\hat{k}) + k(10\hat{i} - 4\hat{j} - 4\hat{k})$
 - $C(96, -33, -34)$ does lie on the line.
 - $z = 18$
- $27\hat{i} - 24\hat{j} - 15\hat{k}$
 - $0.5\hat{i} - 1\hat{j} - 14\hat{k}$
- $P(12, 29, 0)$ does lie on the line.
 - $P(-3, -28, 2)$ does not exist on the line.
 - $P(4, 10, 13)$ does exist on the line
- $\underline{r} = (6\hat{i} + 5\hat{j} + \hat{k}) + k(6\hat{i} - 2\hat{j} + 3\hat{k})$
- $\underline{r} = (\hat{i} + 2\hat{j} + 4\hat{k}) + k(\hat{k})$
- $x = -2$ and $z = 10$
- $x = -1 + 2k, y = 3 + 5k, z = 7 - 3k$
 - $x = -7 + 19k, y = 8 + 11k, z = 3 + k$
 - $x = 5 + 2k, y = 1 + 2k, z = -3 - 6k$
- $x - 1 = y - 2 = z - 3$
 - $\frac{x+2}{6} = \frac{4-y}{9} = \frac{8-z}{2}$
 - $\frac{x-11}{12} = \frac{y-3}{21} = \frac{z-12}{13}$
- $\underline{r} = (2\hat{i} + 5\hat{j} - 8\hat{k}) + k(3\hat{i} + 2\hat{j} + 5\hat{k})$
 - $\underline{r} = (-7\hat{i} - \hat{j} + 2\hat{k}) + k(-2\hat{i} + 6\hat{j} + 3\hat{k})$
- $\left(\frac{34}{5}, \frac{41}{5}, 0\right)$
 - $\left(-\frac{44}{6}, 0, \frac{15}{6}\right)$
- $D = \frac{\sqrt{147}}{\sqrt{6}} \approx 4.95$ units
- The aircraft do not cross paths.
- $\theta = 39.92^\circ$

Exercise 2.6 The vector product

- $48.86\hat{n}$ out of the page
 - $14.854\hat{n}$ into the page
 - $52.5\hat{n}$ into the page

d. 0

- $91.93\hat{n}$
 - $334.72\hat{n}$
 - $13.57\hat{n}$
 - $22.50\hat{n}$

- 12.04 units^2
 - 7.43 units^2
- 5.2 units^2
 - 28.39 units^2

5.

$\times$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

- $43\hat{i} + 11\hat{j} + 17\hat{k}$
 - $-10\hat{i} - 43\hat{j} + 31\hat{k}$
 - $60\hat{i} - 79\hat{j} + 111\hat{k}$
 - $-4\hat{i} - \hat{j} - \frac{31}{4}\hat{k}$
- $\underline{a} \times \underline{b} = 7\hat{i} + 10\hat{j} + 29\hat{k}$
 - $\underline{b} \times \underline{a} = 9\hat{i} - \hat{j} - 32\hat{k}$
 - Sample responses can be found in the worked solutions in the online resources.
- A unit vector normal to vectors $\underline{a}$ and $\underline{b}$ is not possible. The two vectors do not define a plane as they are scalar multiples of each other.
- $-9\hat{i} - 46\hat{j} - 59\hat{k}$
 - $\underline{a} \times \underline{b} \neq \underline{b} \times \underline{a}$. Not commutative.
- $19\hat{i} - 38\hat{j} + 19\hat{k}$
 - $69\hat{i} + 32\hat{j} + 12\hat{k}$
 - $10\hat{i} - 25\hat{j} + 20\hat{k}$
 - $-59\hat{i} + 55\hat{j} + 78\hat{k}$
- 246 units^3
 - 164 units^3
- $6\sqrt{59} \text{ units}^2$
- Sample responses can be found in the Worked solutions in the online resources.
- The vectors are coplanar as the scalar triple product equals zero.
- $b_z = 1$ and $b_y = -4$
- 0
 - 1
 - 1
 - 1
- 68.199°
- 2.61 units^3
- 30.61 units^2

Exercise 2.7 Applications of vectors

- $5x - 2y + z = 1$
 - $-3x + 5y + 4z = 27$
 - $-7x + 4y - 6z = -11$
 - $-4x - y + 3z = -27$
- $\frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} - \hat{k})$
 - $\frac{1}{3}(-\hat{i} + 2\hat{j} - 2\hat{k})$

- c. $\frac{1}{\sqrt{94}}(6\hat{i} - 7\hat{j} + 3\hat{k})$
3. i. a. $21x + 6y - 5z = 35$
 b. $x = \frac{5}{3}, y = \frac{35}{6}, z = -7$
 ii. a. $14x + 20y - 50z = -78$
 b. $x = \frac{-39}{7}, y = \frac{-39}{10}, z = \frac{-39}{-25}$
 iii. a. $3x - 117y - 42z = -807$
 b. $x = -269, y = \frac{269}{39}, z = \frac{269}{14}$
 iv. a. $-5x + 48y + 75z = 441$
 b. $x = -269, y = \frac{147}{16}, z = \frac{147}{25}$
4. a. $P(2, 9, -1)$ exists on the plane.
 b. $P(-1, -6, -4)$ doesn't exist on plane.
 c. $P(1, 4, -7)$ exists on the plane.
5. a. $\underline{n}_1 = 2\underline{n}_2 \therefore$ The planes are parallel
 b. $11x - 2y + 6z = 39$
 c. 11
6. 79.923°
7. Points are collinear. A plane cannot be formed.
8. a. $4x - 29y + 26z = 25$
 b. $P_1 \perp P_2$
9. a. $x + 12y - 41z = -180$
 b. 21.37m^2
10. a. $|D| = \frac{\underline{n} \cdot \overrightarrow{PQ}}{|\underline{n}|}$. Sample responses can be found in the Worked Solutions in the online resources.
 b. 2.74 units
 c. Sample responses can be found in the Worked Solutions in the online resources.
11. $x = -\frac{11}{10} + 13k, y = 20k, z = -\frac{32}{5} - 38k$
12. a. 200 Nm
 b. 49.81 Nm
 c. 32.03 Nm
13. $698.19\text{ Nm} > 400\text{ Nm}$, hence sufficient force.
14. $\overrightarrow{OP} = 3.5\hat{i} + 2\hat{j}$
15. 1306.39 Nm
16. a. The boats are approximately 43.32 metres apart.

b. $\underline{r}_A(3) \neq \underline{r}_B(3)$. No collision.

17. $\underline{r}(10) = 219\hat{i} + 220\hat{j} + 452\hat{k}$

18. $t = \frac{3\pi}{2}$ seconds

19. $(-78, 36, -57)$

20. $(-4.2, 4.4, 1.6)$

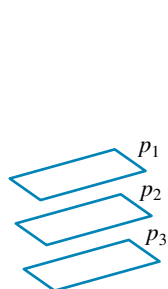
2.8 Review: exam practice

- B
- C
- A
- D
- B
- a. $\underline{r} = (5\hat{i} - 2\hat{j} + 13\hat{k}) + k(\hat{i} + 6\hat{j} - 6\hat{k})$
 b. $y \times z = -1160$
- A
- $6\hat{i} - 51\hat{j} - 69\hat{k}$
- 372 units³
- $3x + 5y + 8z = 47$
- $\underline{r}_b(18) = \underline{r}_a(18) = 328\hat{i} + 81\hat{j} + 163\hat{k}$
- $P(1, 3, 7)$
- $b_z = 1$ and $b_y = 1$
- $84x + 119y - 130z = 34$
- The straight-line distance is 686 metres. The altitude angle is 71.36° . This may be considered very steep and hence unreasonable to climb.
- Sample responses can be found in the worked solutions in the online resources.
- Horizontal distance is 384.94 metres. The maximum height is 55.56 metres which occurred at $t = 3.367$ seconds. The result is unreasonable in terms of general driving distances. It is assumed that no other forces, aside from gravity, act on the ball during its flight, that the ground is level and no spin/sideways movement occurs.
- Sample responses can be found in the worked solutions in the online resources.
- $34x - 58y + 56z - 62 = 0$
- $\frac{1}{7}$ the area of $\triangle ABC$. Sample responses can be found in the worked solutions in the online resources.

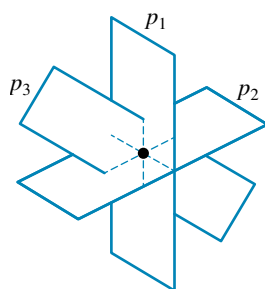
3 Solving systems of linear equations and the application of matrices

3.1 Overview

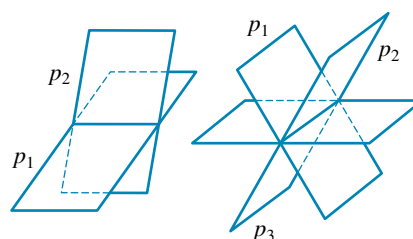
Systems of linear equations, and their applications, are closely related to the study of algebra, geometry, vectors and matrices. Often referred to as linear algebra, this branch of mathematics simplifies higher order systems of equations by translating the information into matrices. The equation of a plane was studied in the previous chapter using both a vector and an analytical geometry approach. If you were to consider three planes simultaneously, you would be modelling three equations in a system containing three variables (x, y, z). Systems in three dimensions may have zero, one or infinitely many solutions. The diagrams represent generalised geometric interpretations of each solution type.



Zero-solutions



One unique solution



Infinitely many-solutions

Matrices are used to find solutions across a wide range of disciplines including ranking players at competitive tournaments through dominance matrices, network flow analysis, code breaking (cryptology), polynomial curve fitting, input/output models in economics using Leontief matrices and the use of Leslie matrices in ecological studies.

Leslie matrices are primarily used in ecology as a powerful predictor of population growth. Combining the principles of both matrix multiplication and powers of matrices, estimates can be made on future population numbers based on the species reproductive and survival rates.

LEARNING SEQUENCE

- 3.1 Overview
- 3.2 Solving linear equations using matrix algebra
- 3.3 Solving a system of linear equations using Gaussian elimination
- 3.4 The three cases for solutions of systems of linear equations
- 3.5 Using technology for matrix calculations
- 3.6 Dominance and Leslie matrices
- 3.7 Applications of matrices
- 3.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 Solving linear equations using matrix algebra

3.2.1 System of linear equations

Linear algebra is fundamental in the study of engineering, business, economics, network analysis, electrical circuits and vectors. This branch of mathematics explores **linear equations**, systems of linear equations and the various techniques that can be applied to solve them.

Recall from previous units that a linear equation in two-dimensions, x and y is given by,

$$ax + by = c, \text{ where } a, b \text{ and } c \in R$$

A linear equation in three-dimensions contains three variables x , y and z . It is commonly expressed in the form,

$$ax + by + cz = d, \text{ where } a, b, c \text{ and } d \in R$$

Linear equations contain variables to the first power. The variables are not products of other variables, nor do they contain roots of variables or are connected to logarithmic, exponential or trigonometric functions.

In general, a linear equation with n variables, is of the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots + a_nx_n = b$$

Where $a_1, a_2, \dots, a_n$ and $b \in R$ and are the coefficients of $x_1, x_2, \dots, x_n$.

Other examples of linear and non-linear equations include:

<i>Linear equations</i>	<i>Non-linear equations</i>
$4x - 11y = 12$	$2yz + 4xy - z^2 = 12$
$2x - y + 7z = 14$	$\cos^2(x) + \ln(y) + z = 100$
$7x - \frac{121}{7}y + 3z = \pi$	$4\sqrt[5]{x} - \frac{2}{y} = -50$

A finite list of linear equations is referred to as a **system of linear equations**.

The general form of a system of m linear equations in n variables is given by

$$\begin{array}{cccccccc}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n & = & b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n & = & b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n & = & b_3 \\
 \vdots & & \vdots \\
 a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n & = & b_m
 \end{array}$$

Where $a_1, a_2, \dots, a_n$ are the coefficients of the variables $x_1, x_2, \dots, x_n$ and b is a constant term.

For example, the following system consists of two linear equations in two variables, x and y . As the number of equations $m = 2$ and the number of variables $n = 2$ are equal, it is referred to as a square system.

$$\begin{array}{l}
 55x + 38y = 207 \\
 17x - 6y = -7
 \end{array}$$

In three dimensions, an example of a square system is shown.

$$\begin{array}{l}
 4x + 2y - 7z = -27 \\
 5x - 5y + z = 0 \\
 -2x + 3y - 4z = -16
 \end{array}$$

In a system of linear equations, the variables are called the unknowns. Consider the following system of linear equations, where x and y represent the price of two different items.

$$\begin{aligned}120x + 10y &= 70 \\ 38x + 20y &= 39\end{aligned}$$

The **unique solution** for this set of simultaneous equations is $x = \frac{1}{2}, y = 1$. These scalar values represent a sequence of numbers that satisfy every equation in the system as shown below.

$\begin{aligned}\text{LHS} &= 120x + 10y \\ &= 120 \times \frac{1}{2} + 10 \times 1 \\ &= 70 \\ &= \text{RHS}\end{aligned}$	$\begin{aligned}\text{LHS} &= 38x + 20y \\ &= 38 \times \frac{1}{2} + 20 \times 1 \\ &= 39 \\ &= \text{RHS}\end{aligned}$

If there is only one possible solution, such as the example above, then it is known as a unique solution. The point of intersection of two lines is an example of a unique solution. The three possible solution types for a system of linear equations will be examined in detail later in this chapter. Until that point we will focus on systems with one unique solution.

Systems of linear equations have either zero, one or an infinite number of solutions. The set of all possible solutions is called the solution set.

WORKED EXAMPLE 1

Identify which of the following equations are non-linear:

- a. $-3x - 2y + 6z = 1$ b. $3x - 4yz = 12$ c. $7x + 9y + z = -2$ d. $12x^2 + y - z = 14$

THINK

- a. This equation contains variables raised to the first power.
- b. This equation contains a term that is product of two variables, yz .
- c. This equation contains variables raised to the first power.
- d. This equation contains a variable, x , that is not raised to the first power.

WRITE

- a. $-3x - 2y + 6z = 1$ is linear.
- b. $3x - 4yz = 12$ is non-linear.
- c. $7x + 9y + z = -2$ is linear.
- d. $12x^2 + y - z = 14$ is non-linear

WORKED EXAMPLE 2

Given the following systems of linear equations:

a. $-4x + y - 6z = -7$ $x + 5y - 8z = 13$	b. $x + 3y - 4z = -21$ $9z - 12z + 2z = 41$ $-6x + y - 2z = -16$

- i. State the number of m equations and n variables in the system.
- ii. Justify whether the system is square or not.
- iii. Determine if $x = 1, y = -2, z = 4$ is a valid solution for all equations in the system.

THINK

- a. i. The system contains two equations, each containing three variables.
- ii. Square systems exist if $m = n$.
- iii. Assess the validity of the solution $x = 1, y = -2, z = 4$ by substituting the corresponding values for x, y and z into each equation.
The solution satisfies each equation if the LHS equals the RHS.
- b. i. The system contains three equations, each containing three variables.
- ii. Square systems exist if $m = n$.
- iii. Assess the validity of the solution $x = 1, y = -2, z = 4$.
Substitute the corresponding values for x, y and z into each equation.
The solutions satisfies each equations if the LHS equals the RHS.

WRITE

- a. $m \text{ equations} = 2$
 $n \text{ variables} = 3$
 $m \neq n$, therefore it is not a square system.
- $$-4x + y - 6z = -7$$
- $$\begin{aligned} \text{LHS} &= -4 \times 1 + -2 - 6 \times 4 \\ &= -4 - 2 - 24 \\ &= -30 \\ &\neq \text{RHS} \end{aligned}$$
- $$x + 5y - 8z = 13$$
- $$\begin{aligned} \text{LHS} &= 1 + 5 \times -2 - 8 \times 4 \\ &= 1 - 10 - 32 \\ &= -41 \\ &\neq \text{RHS} \end{aligned}$$
- $(1, -2, 4)$ is not a valid solution as it doesn't satisfy all equations in the system.
- b. $m \text{ equations} = 3$
 $n \text{ variables} = 3$
 $m = n$, therefore it can be categorised as a square system.
- $$x + 3y - 4z = -21$$
- $$\begin{aligned} \text{LHS} &= 1 + 3 \times -2 - 4 \times 4 \\ &= 1 - 6 - 16 \\ &= -21 \\ &= \text{RHS} \end{aligned}$$
- $$9x - 12y + 2z = 41$$
- $$\begin{aligned} \text{LHS} &= 9 \times 1 - 12 \times -2 + 2 \times 4 \\ &= 9 + 24 + 8 \\ &= 41 \\ &= \text{RHS} \end{aligned}$$
- $$-6x + y - 2z = -16$$
- $$\begin{aligned} \text{LHS} &= 6 \times 1 + -2 - 2 \times 4 \\ &= 6 - 2 - 8 \\ &= -4 \\ &= \text{RHS} \end{aligned}$$
- $(1, -2, 4)$ is a valid solution as it satisfies all equations in the system.

3.2.2 Solving systems of linear equations using inverse matrices

Review of inverse matrices and determinants

For every square matrix, the determinant, can be calculated.

For the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = \Delta = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

The determinant can then be used to calculate the inverse of a matrix.

For the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the inverse, A^{-1} is given by $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

When a matrix is multiplied by its inverse, the identity matrix is obtained. That is $AA^{-1} = I$, where, for a 2×2 matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Remember that an inverse does not exist for a **singular matrix** because the determinant equals zero. A non-singular matrix is often referred to as an **invertible matrix** because an inverse does exist.

Solving systems of linear equations using inverse matrices

In this section, the solution of a system of linear equations will be calculated by means of matrix algebra and inverses. Consider the system of equations $ax + by = e$ and $cx + dy = f$. The coefficients, variables and scalar quantities can be translated into matrix form.

Coefficient matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Unknown/variable matrix (column vector) is $\begin{bmatrix} x \\ y \end{bmatrix}$

Value/scalar matrix (column vector) is $\begin{bmatrix} e \\ f \end{bmatrix}$

If we let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} e \\ f \end{bmatrix}$, the system of equations $ax + by = e$ and $cx + dy = f$ can be written in the form of a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$AX = B$$

To solve matrix equations for X in this form, inverse matrices must be used. Recall that matrix division is not possible. Begin by pre-multiplying both sides by A^{-1} , remembering that the order is important.

$$A^{-1}AX = A^{-1}B$$

Given that $A^{-1}A = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the matrix equation has a unique solution in the form,

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

If A is a non-singular matrix, in a system of m linear equations with n variables, the unique solution is given by $X = A^{-1}B$.

WORKED EXAMPLE 3

Solve the following system of linear equations by using an inverse matrix.

$$4x + y = 12$$

$$3x + 2y = 14$$

THINK

1. Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.
2. Express matrices A , B and X in the form of a matrix equation.
3. Multiply both sides from the left by A^{-1} to create a general matrix solution for X .
4. Determine the inverse matrix, A^{-1} , using algebraic techniques.
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
5. Apply the rules for matrix multiplication. The product is a 2×1 matrix. Apply the rules for scalar multiplication.
6. Apply the rules for the equality of matrices to then state the solution.

WRITE

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$$

Assuming A is invertible,

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\begin{aligned} A^{-1} &= \frac{1}{4 \times 2 - 1 \times 3} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \end{aligned}$$

$$X = A^{-1}B$$

$$\begin{aligned} X &= \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \times \begin{bmatrix} 12 \\ 14 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 24 + -14 \\ -36 + 56 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\therefore$ The solution is $x = 2, y = 4$
 $(x, y) = (2, 4)$

Inverse matrices are not limited to 2×2 systems. For any square system, the inverse matrix can be found using technology. Once the inverse matrix has been calculated, the general formula $X = A^{-1}B$ can then be applied to calculate the solution.

WORKED EXAMPLE 4

Solve the following system of linear equations by using an inverse matrix that has been calculated using technology.

$$3x - 5y + z = 25$$

$$-2x - 7y - 10z = -23$$

$$-6x + 2y - 9z = -54$$



THINK

- Convert the system of linear equations to coefficient (A), variable (X) and value (B) matrices.
- Express matrices A , B and X in the form of a matrix equation.
- Multiply both sides from the left by A^{-1} . A general matrix solution for X now exists.
- Complete the matrix operations using appropriate technology.
- Apply the rules for the equality of matrices to then state the solution.

WRITE

$$A = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$$

Assuming A is invertible,

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1} \begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$\therefore$ The solution is $x = 2$, $y = -3$ and $z = 4$.
 $(x, y, z) = (2, -3, 4)$

TI | THINK

- On a Calculator page, select

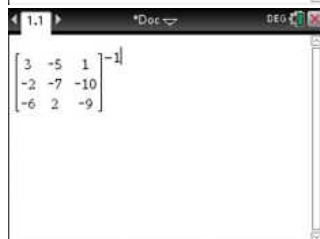
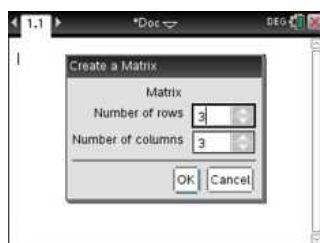


- Complete the entry line as
3
3
Press the OK button.

- Complete the entry line as

$$\begin{bmatrix} 3 & -5 & 1 \\ -2 & -7 & -10 \\ -6 & 2 & -9 \end{bmatrix}^{-1}$$

WRITE



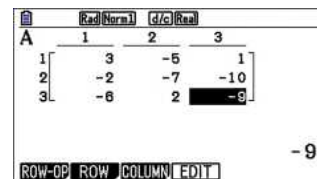
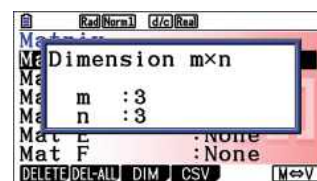
CASIO | THINK

- On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the $M \leftrightarrow V$ button.

- Press the EXE button.
Complete the entry line as
3
3
Press the EXE button.

- Complete the entry line as
3 -5 1
-2 -7 -10
-6 2 -9
Press the EXIT button when finished.

WRITE

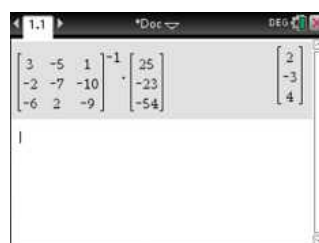
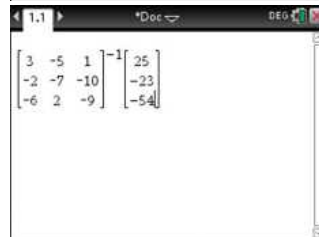
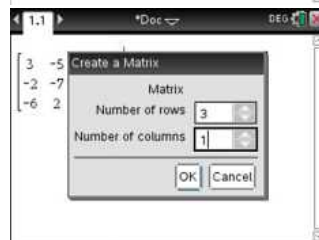
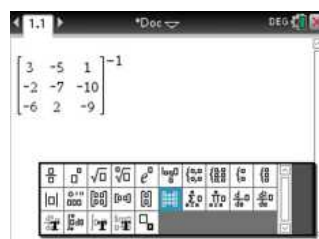


4. Press the  button,
followed by the  button.
button.

5. Complete the entry line as
3
1
Press the OK button.

6. Complete the entry line as
 $\begin{bmatrix} 25 \\ -23 \\ -54 \end{bmatrix}$

7. Press the ENTER button.
The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.

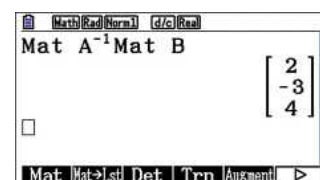
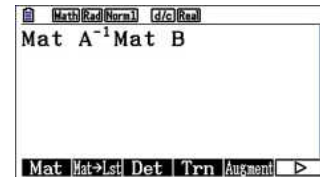
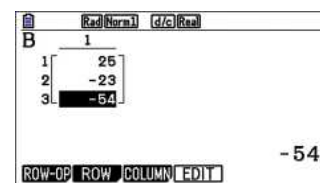
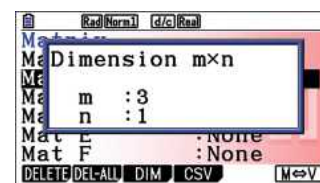


4. Highlight the Mat B row and press the EXE button. Complete the entry line as
3
1
Press the EXE button

5. Complete the entry line as
25
-23
-54
Press the EXIT button twice when finished.

6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
ALPHA
X,θ,T
^
-1
×
Mat
ALPHA
Log

7. Press the EXE button.
The answer appears on the screen.
Note: This can also be calculated using a row-reduction method. This is shown in a later example.



3.2.3 Matrix algebra

When solving matrix equations up until this point, both sides of the equation were multiplied by A^{-1} to yield the solution $X = A^{-1}B$. This process is an example of how a matrix equation can be solved through the application of matrix algebra. **Matrix algebra** shares many similarities with the algebra used to solve elementary linear equations, however not all.

For instance, the algebraic properties of a matrix equation and a simple linear equation are equivalent in the following examples.

1. The use of an additive inverse to solve linear system.

Matrix algebra	Algebra
$X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix}$ $X + \begin{bmatrix} 2 & -2 \\ 3 & -7 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ -3 & 7 \end{bmatrix}$ $X = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$	$x + 2 = 3$ $x + 2 - 2 = 3 - 2$ $x = 1$

2. The use of a scalar multiple to solve a linear system.

Matrix algebra	Algebra
$3X = \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $\frac{1}{3} \times 3X = \frac{1}{3} \times \begin{bmatrix} 12 & -6 \\ 9 & -3 \end{bmatrix}$ $X = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$	$3x = 12$ $\frac{1}{3} \times 3x = \frac{1}{3} \times 12$ $x = 4$

In contrast, the method for factorising is not consistent in both systems of algebra. Consider three matrices of equal dimension C , D and X .

The matrix expression, $XC + XD$, can be factorised as it follows the left distributive law of matrix algebra.

$$XC + XD = X(C + D)$$

A common misconception occurs when attempting to factorise a matrix expression such as $XC + DX$.

$$XC + DX \neq X(C + D)$$

In this case X is common to both terms, however it is incorrect to assume that $XC = CX$ and $DX = XD$. It is also important to note that the common factor, X , does not appear on the same side of both terms.

If the left distributive law of matrices was applied to $X(C + D)$ the resultant expression would be $XC + XD$.

The application of matrix algebra is vital when solving systems of linear equations. This is especially true when attempting to solve more complex equations such as:

$$AX + 2BX - C = D \text{ or } 3A(X + I) = C - D.$$

Summary of matrix algebra laws

$A + B = B + A$	Commutative law for addition
$(A + B) + C = A + (B + C)$	Associative law for addition
$A + 0 = A$	Additive identity
$A + (-A) = 0$	Additive inverse
$(AB)C = A(BC)$	Associative law for multiplication
$AI = A = IA$	Multiplicative identity
$AA^{-1} = A^{-1}A = I$	Multiplicative inverse
$A(B + C) = AB + AC$	Left distributive law
$(B + C)A = BA + CA$	Right distributive law

The following worked examples will apply the above laws to solve a range of matrix equations.

WORKED EXAMPLE 5

Determine a general solution for the variable matrix X in the following matrix equations.

a. $XA - B = C$

b. $AX + BX = C$

THINK

- a. 1. Apply the additive inverse law to isolate XA by adding B to both sides of the equation.
2. Apply the multiplicative inverse law by multiplying both sides by A^{-1} to the right.
3. Simplify $AA^{-1} = I$
4. State the general solution for X .
- b. 1. The right distributive law can be applied to factorise the LHS as X appears on the right of both terms.
2. Apply the multiplicative inverse law by multiplying both sides by $(A + B)^{-1}$ to the left.
3. Simplify $(A + B)(A + B)^{-1} = I$
4. State the general solution for X .

WRITE

- a. $XA - B = C$
 $XA - B + B = C + B$
 $XA = C + B$
 Assuming A is invertible and $C + B$ exists,
 $XAA^{-1} = (C + B)A^{-1}$
 $XI = (C + B)A^{-1}$
 $\therefore X = (C + B)A^{-1}$
- b. $AX + BX = C$
 $(A + B)X = C$
 Assuming $(A + B)$ is non-singular and $A + B$ exists,
 $(A + B)^{-1}(A + B)X = (A + B)^{-1}C$
 $IX = (A + B)^{-1}C$
 $\therefore X = (A + B)^{-1}C$

WORKED EXAMPLE 6

Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$ to determine a unique solution for each of the following linear systems.

a. $B + XA = C$

b. $AX - BX = C$

THINK

- a. 1. Apply the additive inverse law to isolate XA by adding $-B$ to both sides of the equation.
2. Simplify both sides.
3. Apply the multiplicative inverse law by multiplying both sides by A^{-1} to the right.
4. A general solution for X has been determined.
5. Recall the inverse matrix formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ and}$$

apply it to determine

$$A^{-1} = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}^{-1}.$$

WRITE

- a. $B + XA = C$
 $B + (-B) + XA = C + (-B)$
 $XA = C - B$
 Assuming A is non-singular.
 $XAA^{-1} = (C - B)A^{-1}$
 $XI = (C - B)A^{-1}$
 $X = (C - B)A^{-1}$

$$A^{-1} = \frac{1}{2 \times 3 - (-6) \times (-4)} \begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-18} \begin{bmatrix} 3 & 6 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

6. Substitute for the given matrices C, B and A^{-1} into $X = (C - B)A^{-1}$ and then simplify the RHS by applying the rule for subtraction of matrices to simplify $(C - B)$.

7. Apply the rules for matrix multiplication.

8. State the solution.

- b. 1. The LHS can be factorised because X appears on the right of both terms.
 2. Apply the multiplicative inverse law by multiplying both sides by $(A - B)^{-1}$ to the left.
 3. Simplify $(A - B)^{-1}(A - B) = I$ by applying the multiplicative inverse law.
 4. Simplify $(A - B)$ by applying the matrix rule for subtraction.

5. Determine $(A - B)^{-1}$ by applying the rule for the inverse of a 2×2 matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

6. Substitute for the given matrices into $X = (C - B)A^{-1}$.

Continue by applying the rules for matrix multiplication.

7. State the solution.

$$X = \left(\begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix} \right) \times \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

$$X = \begin{bmatrix} 4 & -10 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} -\frac{1}{6} & -\frac{1}{3} \\ -\frac{2}{9} & -\frac{1}{9} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{14}{9} & -\frac{2}{9} \\ \frac{5}{18} & \frac{8}{9} \end{bmatrix}$$

b. $AX - BX = C$

$$(A - B)X = C$$

Assuming $(A - B)$ is invertible.

$$(A - B)^{-1}(A - B)X = (A - B)^{-1}C$$

$$IX = (A - B)^{-1}C$$

$$X = (A - B)^{-1}C$$

$$A - B = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -9 \\ -9 & -1 \end{bmatrix}$$

$$(A - B)^{-1} = \frac{1}{3 \times -1 - -9 \times -9} \begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \frac{1}{-84} \begin{bmatrix} -1 & 9 \\ 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix}$$

$$X = (A - B)^{-1}C$$

$$= \begin{bmatrix} \frac{1}{84} & -\frac{3}{28} \\ -\frac{3}{28} & -\frac{1}{28} \end{bmatrix} \times \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{5}{28} & -\frac{13}{21} \\ -\frac{11}{28} & \frac{4}{7} \end{bmatrix}$$

Exercise 3.2 Solving linear equations using matrix algebra

Technology free

- WE1** Identify which of the following equations are non-linear.
 - $4x - 5 \sin(y) + z = 1$
 - $8x + 9y + 11z = 98$
 - $-8x + y = 7$
 - $4x^2 - 8yz = 12$
- WE2** Given the following systems of linear equations.
 - $$\begin{aligned} 6x + 3y - 5z &= 23 \\ -2x + y + 3z &= -5 \end{aligned}$$
 - $$\begin{aligned} x + 3y - 4z &= 23 \\ 9x - 12y + 2z &= -62 \\ -6x + y - 2z &= 23 \end{aligned}$$
 - State the number of m equations and n variables in the system.
 - Justify whether the system is square or not.
 - Determine if $x = -2, y = 3, z = -4$ is a valid solution set for all equations in the system.
- WE3** Solve the following system of linear equations by using an inverse matrix.
 - $$\begin{aligned} 5x + 2y &= 26 \\ 6x + 3y &= 30 \end{aligned}$$
 - $$\begin{aligned} -4x + 9y &= 15 \\ x - 3y &= -6 \end{aligned}$$
 - $$\begin{aligned} 5x - 4y &= 14 \\ 9x + 10y &= 94 \end{aligned}$$
 - $$\begin{aligned} -7x + 8y &= 92 \\ 6x - 3y &= -48 \end{aligned}$$

Technology active

- WE4** Solve the following system of linear equations by using an inverse matrix that has been calculated using technology.
 - $$\begin{aligned} 2x + 2y - 8z &= 38 \\ -5x - y - 7z &= 40 \\ 8x - 7y - 5z &= 17 \end{aligned}$$
 - $$\begin{aligned} x + 7y - 4z &= 34 \\ -7x - 7y - 9z &= -129 \\ -7x - 5z &= -60 \end{aligned}$$
 - $$\begin{aligned} -x + 8y - 4z &= -38 \\ -x - 5y + 4z &= 17 \\ x + 3y - 5z &= -13 \end{aligned}$$
 - $$\begin{aligned} 3x + 3y + 2z &= -31 \\ -5x - 4y - z &= 42 \\ 3x + 3y - 4z &= -19 \end{aligned}$$
- The points awarded at a school athletics carnival was determined by tallying the number of team members finishing in either 1st, 2nd, 3rd or 4th place in the events.

Team	1 st	2 nd	3 rd	4 th	Total Points
Blue	5	6	2	1	105
Green	3	2	8	2	90
Yellow	8	2	2	2	110
Red	10	2	3	4	141



Determine the points awarded for each of the respective placings using an inverse matrix.

6. Every month a food stall operates at the local market. Four food items are sold, shown in the table below. Create a matrix equation and then use technology to calculate the individual cost per food item.

	Number of items sold				
Month	Lolly bag	Pizza	Meat Pie	Hamburger	Total sales
January	12	61	12	11	\$565
February	17	36	17	13	\$432
March	23	25	23	8	\$361
April	38	51	9	9	\$494

7. **MC** Identify which of the following matrix operations exhibit the right distributive law.
A. $A + O = A$ **B.** $P(Q + R) = PQ + PR$
C. $(T + U)V = TV + UV$ **D.** $C^{-1}C = CC^{-1} = I$
8. **MC** Determine which of the following statements is false. It is assumed that all the matrices are square and of the same dimension.
A. $AI = IA$ **B.** $CX - XD = X(C - D)$
C. $TX - XT = -XT + TX$ **D.** $XM + XN = X(M + N)$
9. Construct an expression for X from the matrix equation $EX + X + F = G$.
10. **WE5** Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).
a. $AXB = C$ **b.** $XA = B$ **c.** $AXB = I$ **d.** $AX + X = C$
11. Determine a general solution for the variable matrix X for the following matrix equations. (It is assumed that all the matrices are square matrices and non-singular).
a. $AX + BX + C = D$ **b.** $2X + 3AX = B$ **c.** $AX + 3BX = C$ **d.** $7A(X - I) = B + C$
12. **WE6** Use $A = \begin{bmatrix} 2 & -6 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.
a. $XA = B$ **b.** $B + XA = C$ **c.** $AX - BX = C$ **d.** $CXD = 2I$
13. Use $A = \begin{bmatrix} 3 & -1 \\ 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -8 \\ -2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 7 & 5 \\ -1 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -5 \\ 3 & 4 \end{bmatrix}$, to determine a unique solution for each of the following linear systems.
a. $CX = B$ **b.** $ACX = 4I$ **c.** $CX - 5DX = B$ **d.** $3A(X + I) = C - D$
14. Assuming that A is a square invertible matrix, use matrix algebra to justify the mathematical statement below.

$$(A + I)(A - I) = A^2 - I$$

15. Solve the following system of nonlinear equations for the unknown angles α and β , where $0 \leq \alpha \leq \frac{\pi}{2}$ and $0 \leq \beta \leq \frac{\pi}{2}$.

$$2 \sin \alpha + 3 \cos \beta = \frac{3\sqrt{2} + 2}{2}$$

$$\sin \alpha - \cos \beta = \frac{1 - \sqrt{2}}{2}$$

3.3 Solving a system of linear equations using Gaussian elimination

3.3.1 Gaussian elimination

A variety of methods can be used to solve a system of simultaneous equations, such as those applied in the previous section. In elementary mathematics, simultaneous equations are often solved using the substitution or elimination method. Both techniques, and especially the later, are applicable when solving matrix equations using the **Gaussian elimination** method.

Consider a set of two simultaneous linear equations in two variables, x and y .

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

As we saw earlier, this system can be translated into the following matrices,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e \\ f \end{bmatrix} \text{ and } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

where A is the coefficient matrix, B is the value matrix and X is the variable or unknown matrix.

We can combine matrices A and B as $\left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array} \right]$ where this is referred to as an augmented matrix, denoted by $[A|B]$.

Consider the system below.

System	Coefficient matrix	Value matrix	Augmented matrix
$5x - 2y = 8$ $3x + 4y = 10$	$\begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix}$	$\begin{bmatrix} 8 \\ 10 \end{bmatrix}$	$\left[\begin{array}{cc c} 5 & -2 & 8 \\ 3 & 4 & 10 \end{array} \right]$

The purpose of Gaussian elimination is to solve a system of linear equations using elementary row operation procedures which correspond to the operations used when solving equations by elimination.

Row operation		Example notation
1.	Multiply a row (equation) by a constant.	If the third row was multiplied by -2 it would be denoted as $-2 \times R_3$
2.	Add and subtract one row (equation) to or from another.	If row 2 was multiplied by 4 and the result is then added to row 3 it would be denoted as $R_3 + 4R_2$
3.	Swap rows (equations).	If we interchange rows one and two it is denoted as $R_1 \leftrightarrow R_2$

The objective of these procedures is to produce a vector with ones in the leading diagonal and zeros under the leading diagonal. This is known as **row-echelon form**. The following matrices are in row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & 10 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 3 \end{array} \right]$$

Once row-echelon form is achieved, the solution can be found by using an algebraic method referred to as back substitution. The following worked example demonstrates this approach.

WORKED EXAMPLE 7

Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{aligned} 4x + 8y &= 24 \\ -3x + y &= 17 \end{aligned}$$

THINK

1. Translate the linear equations to matrix form.

2. Express A and B in augmented matrix form.

3. Begin the process of elementary row operations to reduce the augmented matrix to row-echelon form. Divide the first row by four.

4. Continue to produce row-echelon form so $a_{21} = 0$. Add $3R_1$ to R_2 .

5. Multiply R_2 by $\frac{1}{7}$ so $a_{22} = 1$. The second row has now reached the desired form.

6. State the solution for second row.

7. Rewrite the first row of the augmented matrix in terms of x and y . Substitute $y = 5$ into row one to then solve for x .

8. State the unique solution for the system of linear equations.

WRITE

$$A = \begin{bmatrix} 4 & 8 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 24 \\ 17 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{cc|c} 4 & 8 & 24 \\ -3 & 1 & 17 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ -3 & 1 & 17 \end{array} \right] \frac{1}{4}R_1$$

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 7 & 35 \end{array} \right] R_2 + 3R_1 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 5 \end{array} \right] \frac{1}{7}R_2$$

$$\therefore y = 5.$$

Use back-substitution to find a solution.

$$x + 2y = 6$$

$$x + 2(5) = 6$$

$$x + 10 = 6$$

$$x + 10 - 10 = 6 - 10$$

$$x = -4$$

The unique solution for the system of linear equations is $(x, y) = (-4, 5)$.

In the previous worked example the coefficient matrix was transformed into row-echelon form. To determine the solution, the process of back substitution was applied.

Another approach to determine a solution using Gaussian elimination is the **reduced row-echelon form**. In this method, the solution set (x, y) is attained by continuing elementary row operations until all elements above the leading diagonal are zero. In this form, the augmented matrix will have I on the left-hand side and the solution will be on the right-hand side.

$$[A|B] \rightarrow [I|X]$$

The solution from the previous worked example was $x = -4, y = 5$. The reduced row echelon matrix for this system would be $\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 5 \end{array} \right]$. The following matrices are in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & | & -8 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & 11 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & 0 & | & -5 \\ 0 & 0 & 0 & 1 & | & 10 \end{bmatrix}$$

$$x = -8, y = -3, z = 11 \quad w = -2, x = 4, y = -5, z = 10$$

WORKED EXAMPLE 8

Solve the following system of linear equations using Gaussian elimination and back substitution.

$$\begin{aligned} 5x + 2y - 2z &= 2 \\ 2x + 5y + 3z &= -8 \\ x + 3y - 4z &= -11 \end{aligned}$$

THINK

1. Translate the linear equations to matrix form.
2. Express A and B in augmented matrix form.
3. The coefficient for x in row 3 is 1. Interchange rows 1 and 3.
4. Multiples of row 1 can be added/subtracted to row 2 and 3 so that $a_{21} = 0$ and $a_{31} = 0$.
5. Subtract $13 \times R_2$ from R_3 so that $a_{32} = 0$.
6. Complete two operations. First multiply R_2 by -1 so $a_{22} = 1$. Follow this by multiplying R_2 by $-\frac{1}{125}$ so $a_{33} = 1$.
7. State the solution for the third row.
8. Rewrite the second row of the augmented matrix in terms of x and y . Substitute in $z = 1$ to then solve for y .

WRITE

$$A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & 3 \\ 1 & 3 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ -8 \\ -11 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 5 & 2 & -2 & 2 \\ 2 & 5 & 3 & -8 \\ 1 & 3 & -4 & -11 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 2 & 5 & 3 & -8 \\ 5 & 2 & -2 & 2 \end{array} \right] \quad R_1 \leftrightarrow R_3$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & -13 & 18 & 57 \end{array} \right] \quad \begin{array}{l} R_2 - 2R_1 \rightarrow R_2 \\ R_3 - 5R_1 \rightarrow R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & -1 & 11 & 14 \\ 0 & 0 & -125 & -125 \end{array} \right] \quad R_3 - 13 \times R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -4 & -11 \\ 0 & 1 & -11 & -14 \\ 0 & 0 & 1 & 1 \end{array} \right] \quad \begin{array}{l} -1 \times R_2 \rightarrow R_2 \\ -\frac{1}{125} \times R_3 \rightarrow R_3 \end{array}$$

$$\therefore z = 1$$

Use back-substitution to find a solution.

$$y - 11z = -14 \quad (\text{From } R_2)$$

$$y - 11(1) = -14$$

$$y - 11 + 11 = -14 + 11$$

$$y = -3$$



9. Rewrite the first row of the augmented matrix in terms of x , y and z .
Substitute $y = -3$ and $z = 1$ to solve for x .

$$\begin{aligned}x + 3y - 4z &= -11 && \text{(From } R_1\text{)} \\x + 3(-3) - 4(1) &= -11 \\x - 9 - 4 &= -11 \\x - 13 &= -11 \\x &= 2\end{aligned}$$

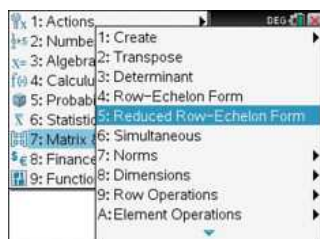
10. State the unique solution for the system of linear equations.

The unique solution for the system of linear equations is $x = 2$, $y = -3$ and $z = 1$.
 $\therefore (x, y, z) = (2, -3, 1)$

TI | THINK

1. On a Calculator page, select MENU
7: Matrix and Vector
5: Reduced Row-Echelon Form

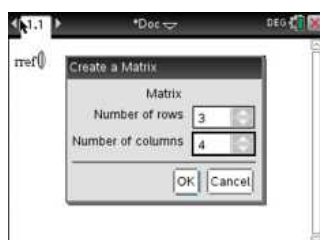
WRITE



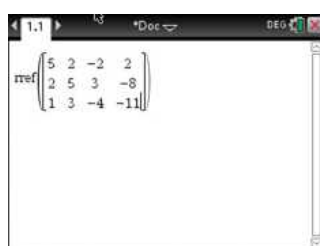
2. On a calculator screen, select



3. Complete the entry line as
3
4
Press the OK button.



4. Complete the entry line as
5 2 -2 2
2 5 3 -8
1 3 -4 -11



5. Press the ENTER button. The answer appears on the screen.



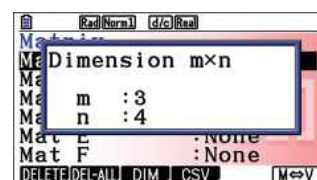
CASIO | THINK

1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the $M \leftrightarrow V$ button.

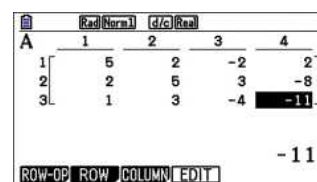
WRITE



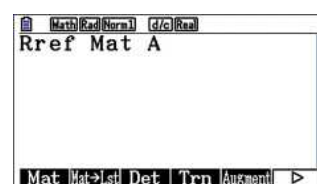
2. Press the EXE button. Complete the entry line as
3
4
Press the EXE button.



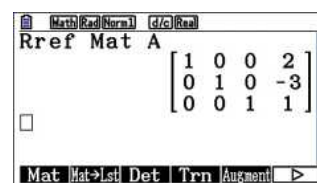
3. Complete the entry line as
5 2 -2 2
2 5 3 -8
1 3 -4 -11
Press the EXIT button twice when finished.



4. On a Run-Matrix screen, select OPTN,
MAT/VCT
>
Rref
> (x3)
Mat
SHIFT
X,θ,T



5. Press the EXE button. The answer appears on the screen.



The row-echelon form and reduced row-echelon form of a matrix give rise to the various types of matrices.

A **triangular matrix** is a type of square matrix in which either all the elements above or all the elements below the leading diagonal are zero. Triangular matrices come in two types:

1. An **upper triangular matrix** is a type of square matrix in which all the elements below the leading diagonal are zero.
2. A **lower triangular matrix** is a type of square matrix in which all the elements above the leading diagonal are zero.

Examples of triangular matrices are shown below:

$$\begin{bmatrix} 1 & 6 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Upper triangular matrix} \quad \begin{bmatrix} -1 & 0 & 0 \\ 6 & 9 & 0 \\ -2 & -1 & 2 \end{bmatrix} \quad \text{Lower triangular matrix}$$

Row-echelon form is a type of an upper triangular matrix.

A **diagonal matrix** is a square matrix where all elements off the leading diagonal are zero.

For example,

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Reduced row-echelon form can be classified as a diagonal matrix where the elements on the leading diagonal are all 1s.

study on

Units 3 & 4 > Area 2 > Sequence 2 > Concept 2

Solving a system of linear equations using Gaussian elimination Summary screen and practice questions

Exercise 3.3 Solving a system of linear equations using Gaussian elimination

Technology free

1. **MC** A matrix in row echelon form is defined by which description?
 - A. Contains ones in the leading diagonal and ones under the leading diagonal.
 - B. Contains ones in the leading diagonal and zeros under the leading diagonal. In addition, a row consisting of only zeros may exist as the bottom row of the matrix.
 - C. Contains ones in the leading diagonal and zeros in every other cell.
 - D. Contains ones in every cell.
2. **MC** Identify which of the following matrices is reduced row echelon form.

A. $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -7 \end{array} \right]$

B. $\left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 0 & 5 \end{array} \right]$

C. $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 8 \end{array} \right]$

D. $\left[\begin{array}{cc|c} 1 & 9 & 4 \\ 0 & 1 & 7 \end{array} \right]$
3. **MC** The augmented matrix $\left[\begin{array}{ccc|c} 2 & 7 & -4 & 5 \\ 1 & 2 & 3 & 8 \\ 4 & 8 & -12 & -20 \end{array} \right]$ changes to $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 8 \\ 2 & 7 & -4 & 5 \\ 1 & 2 & -3 & -5 \end{array} \right]$. Identify which row operations were completed for that change to occur.

A. $R_2 - R_1, R_3 \leftrightarrow R_2$

B. $R_2 \leftrightarrow R_1, R_3 \times \frac{1}{4}$

C. $R_2 \leftrightarrow R_3, R_3 \times 4$

D. $R_2 \leftrightarrow R_3, R_3 \div 4$

4. **WE7** Solve the following system of linear equations using Gaussian elimination and back substitution.
- a. $6x - 4y = 10$
 $-2x + 7y = 8$
- b. $x + 3y = -2$
 $3x - 9y = -24$
- c. $9x + 8y = 34$
 $-3x + 2y = -2$
- d. $-5x - 2y = 19$
 $4x + y = -8$

5. **WE8** Solve the following system of linear equations using Gaussian elimination and back substitution.

- a. $6x + 6y + z = -12$
 $x + 4y - z = -8$
 $3x + 3y - 5z = -6$
- b. $-2x + 6y = -20$
 $-x - 6y - 3z = 26$
 $-3x - 4z = 6$
- c. $6x - 6y + 2z = -8$
 $-5x + 7y - 2z = 16$
 $-6x + 7y - 5z = 7$
- d. $-4x + 2y + 2z = 38$
 $4x - y + 2z = -17$
 $-2x + y + 5z = 35$
- e. $5x + 3y - z = -23$
 $-x + 4y + 6z = -15$
 $-2x + 4y + 6z = -12$

6. Solve the following set of simultaneous equations, containing four variables, using Gaussian elimination and back substitution.

$$\begin{aligned} 2w + 8x - 4y + 2z &= -46 \\ -10w + x - 8y - 6z &= -25 \\ 4w - 5x + 2y - 5z &= 56 \\ -8w - 16x + 4y + 3z &= 33 \end{aligned}$$

7. The row echelon matrix of a system of linear equations is $\left[\begin{array}{ccc|c} 1 & 4 & -2 & 30 \\ 0 & 1 & 3 & 110 \\ 0 & 0 & 1 & 30 \end{array} \right]$.

Assuming the independent variables are x , y and z , use back substitution to calculate the unique solution for the system.

Technology active

8. One group of students purchase 4 regular coffees and 3 large coffees for \$33. A second group purchase 2 regular and 5 large coffees for \$34. Use Gaussian elimination to calculate the cost of both coffee sizes.



9. The cost of a movie ticket varies according to the classification of student, adult or pensioner. The table below summarises the cost for three different families to attend the cinema.

Number of students	Number of adults	Number of pensioners	Total cost (\$)
6	2	1	82
4	1	2	58
2	2	2	70



Use Gaussian elimination to determine the total cost for a family containing two students, one adult and two pensioners.

10. Convert the augmented matrix, $\left[\begin{array}{ccc|c} 5 & 2 & 4 & 32 \\ -4 & 7 & 8 & -15 \\ -3 & 11 & 1 & 9 \end{array} \right]$, into a system of linear equations.

11. A quadratic in the form, $y(x) = a_0 + a_1x + a_2x^2$, contains the points $A(2, 1)$, $B(3, 7)$ and $C(4, 17)$. If n points from a curve are known it is possible to develop a polynomial to the $n - 1$ degree through a process known as interpolating a polynomial. Points A , B and C can be translated into an augmented matrix given by,

$$\left[\begin{array}{ccc|c} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{array} \right] \quad \text{where} \quad \begin{array}{l} A(2, 1) = (x_1, y_1) \\ B(3, 7) = (x_2, y_2) \\ C(4, 17) = (x_3, y_3) \end{array}$$

The values for a_0 , a_1 and a_2 are determined when the augmented matrix has reached reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a_0 \\ 0 & 1 & 0 & a_1 \\ 0 & 0 & 1 & a_2 \end{array} \right]$$

Develop an augmented matrix and perform elementary row operations to determine the values for a_0 , a_1 and a_2 .

12. In the previous questions we used augmented matrices in the form $[A|B]$ to find solutions. This method can be used to determine the inverse of matrix A . To achieve this, matrix A is augmented with the identity matrix I , $[A|I]$. The inverse is found by completing elementary row reductions until you produce $[I|A^{-1}]$. Follow this method to determine the inverse of matrix A where:

$$A = \begin{bmatrix} 3 & 12 & -6 \\ 2 & 4 & -6 \\ 1 & 3 & -4 \end{bmatrix}.$$

13. The initial stages in finding a solution to a system of linear equations was attempted below. Identify the location of the five errors in the solution below.

$$\begin{array}{l} x - 5y + z = -6 \\ 3x + y - 2z = 4 \\ -5x + 4z - y = -4 \end{array} \quad \begin{array}{l} \left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & 4 & -1 & -4 \end{array} \right] \\ \left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & 22 \\ 0 & -21 & 4 & -34 \end{array} \right] \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 5R_1 \rightarrow R_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 1 & -\frac{5}{16} & \frac{11}{8} \\ 0 & 0 & -1 & 44 \end{array} \right] \begin{array}{l} R_2 \div 16 \rightarrow R_2 \\ 5R_3 - 21R_1 \rightarrow R_3 \end{array} \end{array}$$

3.4 The three cases for solutions of systems of linear equations

The study of linear equations until this point has been focussed on the numerous methods that can be used to solve a system. You may have observed that all systems were common in that each had exactly one solution. That is, each system had a unique solution. However, the solution set is not limited to only one solution type.

Systems of linear equations have either:

- A unique (one) solution
- No solution
- Infinitely many solutions

A system containing at least one solution is defined as **consistent**. An **inconsistent** system has no solution.

3.4.1 Geometric interpretation of solutions of lines in two-dimensions

Recall the general form of a system in two dimensions. These systems are given by

$$a_1x + b_1y = c_1 \quad [1]$$

$$a_2x + b_2y = c_2 \quad [2].$$

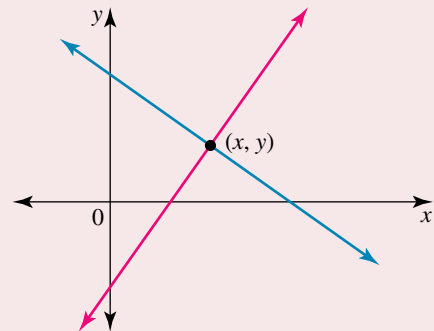
Systems in this form were previously expressed in the form of a matrix equation, $AX = B$. The unknown matrix X was solved using various techniques, with the most functional method being the inverse matrix approach. This was possible if A was non-singular ($\Delta \neq 0$). The coefficient matrix, A , will be the point of reference when investigating the three possible solution types below.

A unique solution (Non-parallel, distinct lines)

- A is *non-singular* ($\det A \neq 0$)
- The system is *consistent*.
- The two lines are *not parallel*.
- The two lines intersect at a one point that represents a *unique solution*.

Properties of the coefficients

- $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$

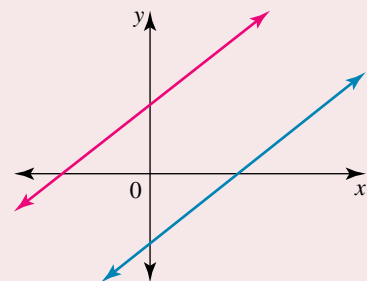


No solution (Parallel but distinct lines)

- A is *singular* ($\det A = 0$)
- The system is *inconsistent*.
- The two lines are *parallel* and *distinct* (the lines will have different y-intercepts).
- The two lines do not intersect and there is *no solution*.

Properties of the coefficients

- $\frac{a_2}{a_1} = \frac{b_2}{b_1} \neq \frac{c_2}{c_1}$

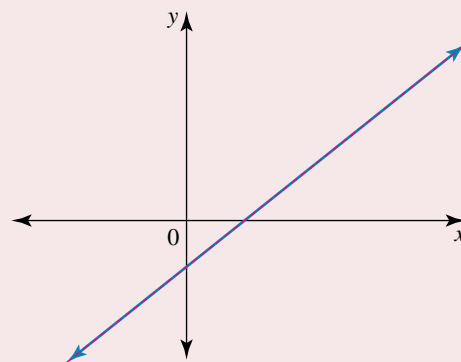


Infinitely many solutions (Parallel and Coincident lines)

- A is singular ($\det A = 0$)
- The system is *consistent*.
- The two lines are *parallel* and *not distinct* (the lines will have the same y -intercept).
- The two lines are *coincident* meaning they overlap and there is an *infinite number of solutions*.

Properties of the coefficients

- $\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1}$



WORKED EXAMPLE 9

Solve the following system of linear equations by using matrices and interpret the solution geometrically.

$$\begin{aligned} 2x + 7y &= 16 \\ -4x - 14y &= 20 \end{aligned}$$

THINK

1. Define the matrices A , X and B .
2. Translate into a matrix equation.
3. Write as a general equation and solve for X .
4. Calculate the determinant using $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ to determine if A is singular.
5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.
6. Rearrange both equations into the form $y = mx + c$,

WRITE

$$A = \begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 7 \\ -4 & -14 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\det(A) = \begin{vmatrix} 2 & 7 \\ -4 & -14 \end{vmatrix} = 2 \times -14 - 7 \times -4 = 0$$

$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.

$$2x + 7y = 16$$

$$7y = 16 - 2x$$

$$\rightarrow y_1 = -\frac{2}{7}x + \frac{16}{7}$$

$$-4x - 14y = 20$$

$$14y = -4x - 20$$

$$\rightarrow y_2 = -\frac{2}{7}x - \frac{10}{7}$$

7. The lines are parallel because the gradients are equal.

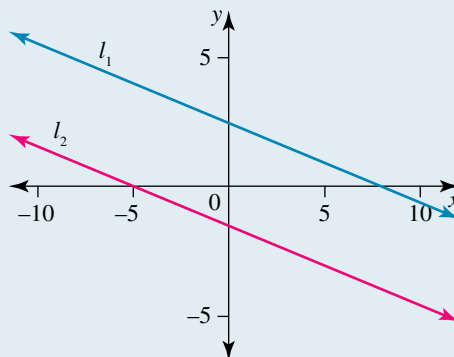
The two lines are distinct because the y-intercepts are different.

8. Sketch the graphs. Note that the two lines are parallel with distinct y-intercepts, which means they will never intersect and hence have no solution.

$$\text{Line [1]: } m_1 = \frac{-2}{7} \text{ y-intercept} = \frac{16}{7}$$

$$\text{Line [2]: } m_2 = -\frac{2}{7} \text{ y-intercept} = -\frac{10}{7}$$

The lines are parallel, yet distinct as they have different y-intercepts.



9. State the answer.

There is no solution as the lines are parallel and distinct.

WORKED EXAMPLE 10

- a. Solve the following system of linear equations by using matrices and interpret the solution geometrically.
b. Describe the solution set in parametric form.

$$\begin{aligned} 3x - 4y &= -2 \\ 9x - 12y &= -6 \end{aligned}$$

THINK

- a. 1. Define the matrices A , X and B .
2. Translate the matrices into a matrix equation.
3. Write as a general equation and solve for X .

4. Recall the determinant rule

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and calculate $|A|$ to determine if A is singular.

WRITE

a. $A = \begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

$$\begin{bmatrix} 3 & -4 \\ 9 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

$$\det(A) = \begin{vmatrix} 3 & -4 \\ 9 & -12 \end{vmatrix} = 3 \times -12 - -4 \times 9 = 0$$

5. As $\det(A) = 0$, matrix A is singular, hence A^{-1} does not exist.
6. Rearrange both equations into the form $y = mx + c$.

7. The lines are parallel because the gradients are equal.
The two lines are coincident because the y -intercepts are the same.

8. Sketch the graphs. Note that the two lines overlap, meaning there are an infinite number of solutions.

$\det(A) = 0$, so A is singular. The system either has no solution or infinitely many solutions.

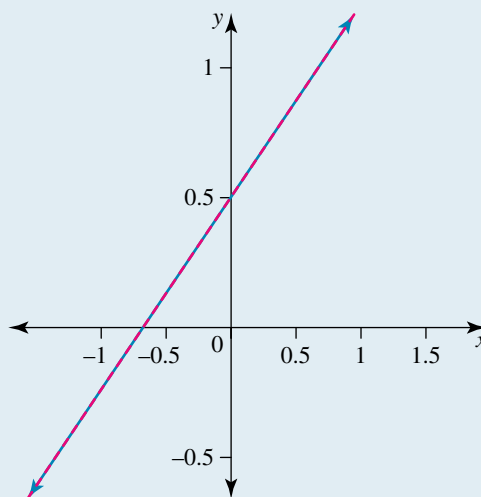
$$\begin{aligned} 3x - 4y &= -2 \\ 4y &= 3x + 2 \\ \rightarrow y_1 &= \frac{3}{4}x + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 9x - 12y &= -6 \\ 12y &= 9x + 6 \\ \rightarrow y_2 &= \frac{3}{4}x + \frac{1}{2} \end{aligned}$$

Line [1]: $m_1 = \frac{3}{4}$, $y_1\text{-int} = \frac{1}{2}$

Line [2]: $m_2 = \frac{3}{4}$, $y_2\text{-int} = \frac{1}{2}$

The two lines are coincident because they are identical.



- b. 1. The two lines are identical. Rewrite either equation in parametric form. Begin by making x the subject of the equation.
2. Express y in terms of t .
3. State the answer.

b. $3x - 4y = -2$
 $3x = 4y - 2$
 $x = \frac{4y - 2}{3}$

Let $y = t$ so that $x = \frac{4t - 2}{3}$

As a coordinate: $\left(\frac{4t - 2}{3}, t\right)$

There are an infinite number of solutions of the form $(x, y) = \left(\frac{4t - 2}{3}, t\right)$.

3.4.2 Geometric interpretation of solutions of lines in three-dimensions

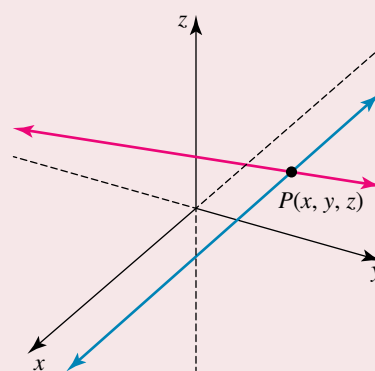
The vector equation of a line in three-dimensions was studied in Chapter 2. Recall that a line can be expressed in three different forms.

Vector equation form	Parametric form	Cartesian form
$\underline{r} = (-5\hat{i} + 2\hat{j} + 7\hat{k}) + k(3\hat{i} - 4\hat{j} - 9\hat{k})$ $\underline{a} = -5\hat{i} + 2\hat{j} + 7\hat{k}$ $\underline{d} = 3\hat{i} - 4\hat{j} - 9\hat{k}$ where $k \in \mathbb{R}$	$x = -5 + 3k$ $y = 2 - 4k$ $z = 7 - 9k$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$	$\frac{x+5}{3} = \frac{y-2}{-4} = \frac{z-7}{-9}$ $(a_1, a_2, a_3) = (-5, 2, 7)$ $(d_1, d_2, d_3) = (3, -4, -9)$

When examining two lines (l_1 and l_2) in three dimensions, three solution types are possible.

Consistent system – one unique solution

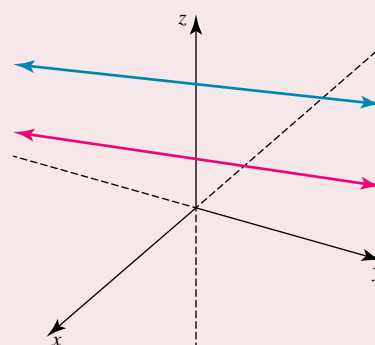
- The system is *consistent*.
- The two lines are *not parallel*.
- The two lines intersect at a one point that represents a *unique solution*.



Inconsistent system – two cases

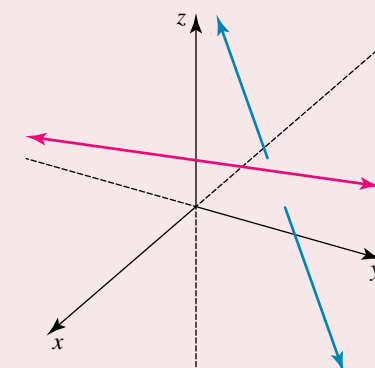
Case 1: Parallel but distinct lines

- The system is *inconsistent*.
- The two lines are *parallel* and *distinct*.
- The two lines do not intersect and there is *no solution*.



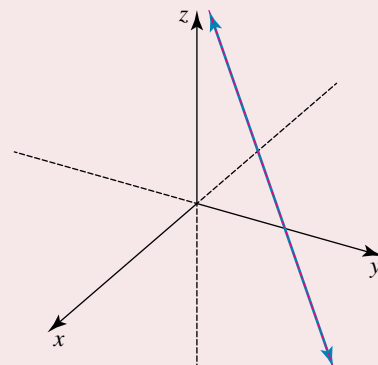
Case 2: Not parallel but distinct lines

- The system is *inconsistent*.
- The two lines are referred to as skew lines and do not intersect, hence there is *no solution*.



Consistent system – infinitely many solutions (parallel and coincident lines)

- The system is *consistent*.
- The two lines are *parallel* and *not distinct*.
- The two lines are *coincident* meaning they overlap and there is an *infinite number of solutions*.



WORKED EXAMPLE 11

- a. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2 .
b. Describe the geometric nature of the solution.

$$l_1: (5\hat{i} - 3\hat{j} - 7\hat{k}) + k(-4\hat{i} + \hat{j} + 5\hat{k})$$

$$l_2: (-5\hat{i} + 2\hat{j} - 9\hat{k}) + k(6\hat{i} - 4\hat{j} + 7\hat{k})$$

THINK

- a. 1. Analyse the direction vectors to determine if the lines are parallel.
They are not equal, nor is one a scalar multiple of the other.
2. Rewrite the equations of both lines in parametric form.
3. Both lines are in terms of the same parameter, k , where $k \in R$. Equate the x component of l_1 and l_2 to determine a possible value for k .
4. Determine the position on both lines when the parameter is set to $k = 1$ by substituting $k = 1$ into the parametric equations.

WRITE

a. $\underline{d}_1 = -4\hat{i} + \hat{j} + 5\hat{k}$

$$\underline{d}_2 = 6\hat{i} - 4\hat{j} + 7\hat{k}$$

$$\underline{d}_1 \neq m \times \underline{d}_2, \text{ where } m \in R$$

The two lines are not parallel.

$$l_1: x = 5 - 4k$$

$$y = -3 + k$$

$$z = -7 + 5k$$

$$l_2: x = -5 + 6k$$

$$y = 2 - 4k$$

$$z = -9 + 7k$$

Solve for k by equating the x component of l_1 and l_2 :

$$5 - 4k = -5 + 6k$$

$$10 = 10k$$

$$k = 1$$

Let $k = 1$, for lines, l_1 and l_2 :

$$l_1: x = 5 - 4 \times 1 = 1$$

$$y = -3 + 1 = -2$$

$$z = -7 + 5 \times 1 = -2$$

$$(x_1, y_1, z_1) = (1, -2, -2)$$

$$l_2: x = -5 + 6 \times 1 = 1$$

$$y = 2 - 4 \times 1 = -2$$

$$z = -9 + 7 \times 1 = -2$$

$$(x_2, y_2, z_2) = (1, -2, -2)$$

5. State the solution.

$$(x_1, y_1, z_1) = (x_2, y_2, z_2).$$

A unique solution exists at

$$(x, y, z) = (1, -2, -2).$$

b. Describe the geometric nature of the solution.

b. The two distinct lines intersect at one unique point, $(1, -2, -2)$.

In the previous worked example, both vector equations are in terms of the parameter, k , where $k \in R$. When determining a point of intersection, it was simply found by equating the components, (x, y, z) . In some of the more complex systems you may find that two lines are defined by different parameters, k_1 and k_2 .

WORKED EXAMPLE 12

A system of linear equations in three variables contains the lines l_1 and l_2 .

- Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.
- Describe the geometric nature of the system by referring to any possible solution/s.

$$l_1: \frac{x+6}{2} = \frac{y+2}{5} = \frac{z+4}{-3} = k_1, \quad l_2: \frac{x+3}{4} = \frac{y+6}{10} = \frac{z}{-6} = k_2$$

THINK

- Analyse the direction vectors to determine if the lines are parallel.

Both lines are in Cartesian form,

$$\frac{x-a_1}{d_1} = \frac{x-a_2}{d_2} = \frac{x-a_3}{d_3} = k. \text{ The direction of}$$

the line can be expressed as a vector given by

$$\underline{d}_1 = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}.$$

- As the lines are parallel, they may overlap (coincident) or be distinct. Rewrite the equations of both lines in parametric form.

- Determine a point on the first line by setting the parameter to any given value. Let $k_1 = 0$.

WRITE

$$\text{a. } \underline{d}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$$

$$\underline{d}_2 = 4\hat{i} + 10\hat{j} - 6\hat{k}$$

$$\underline{d}_1 = \frac{1}{2} \times \underline{d}_2 \rightarrow \therefore \underline{d}_1 \parallel \underline{d}_2$$

The two lines are parallel.

$$l_1: x = -6 + 2k_1$$

$$y = -2 + 5k_1$$

$$z = -4 - 3k_1$$

$$l_2: x = -3 + 4k_2$$

$$y = -6 + 10k_2$$

$$z = -6k_2$$

For l_1 , let $k_1 = 0$

$$l_1: x = -6 + 2 \times 0 = -6$$

$$y = -2 + 5 \times 0 = -2$$

$$z = -4 - 3 \times 0 = -4$$

The following point exists on

$$l_1: (x_1, y_1, z_1) = (-6, -2, -4)$$

4. If the lines coincident, then the same point found on l_1 will satisfy the second line, l_2 . Verify whether $(-6, -2, -4)$ lies on the second line.

Substitute $(-6, -2, -4)$ into the parametric form of l_2 .

$$x = -3 + 4k_2$$

$$-6 = -3 + 4k_2$$

$$k_2 = -\frac{3}{4}$$

$$y = -6 + 10k_2$$

$$-2 = -6 + 10k_2$$

$$k_2 = \frac{2}{5}$$

$$z = -6k_2$$

$$-4 = -6k_2$$

$$k_2 = \frac{2}{3}$$

The values for k_2 are not equal which indicates the point $(-6, -2, -4)$ does not lie on l_2 .

A solution does not exist for the simultaneous system.

5. State the solution.

- b. Describe the geometric nature of the solution.

- b. The two lines are parallel and distinct which means they will never intersect.

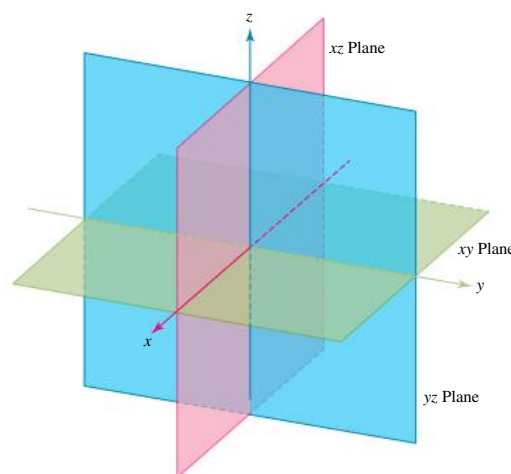
3.4.3 Geometric interpretation of solutions of planes in three-dimensions

Coordinate geometry in three dimensions was introduced in Chapter 2. These systems are defined by three variables to specify the location of a point $P(x, y, z)$. A right-handed system, consisting of three orthogonal planes is shown. Each of these planes could be written in Cartesian form, $ax + by + cz + d = 0$. When considering the equations of these planes simultaneously you are investigating the solution for a 3×3 linear system. In the system shown in the figure, the unique solution exists as the intersection of the three planes at the origin, $P(0, 0, 0)$.

Systems comprising of three planes can have either one, zero or infinitely many solutions.

Review of properties of a plane

To begin this study, it important to refresh your memory on the properties of a plane and the associated terminology.



The equation of the plane passing through (a, b, c) and perpendicular to $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ is given by:

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

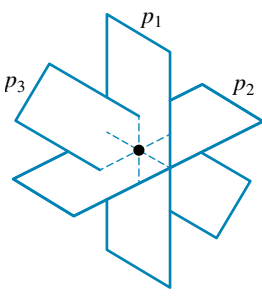
or

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

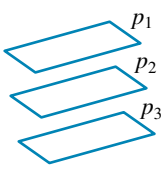
- To determine if two or more planes are parallel you must compare their normal vector $\underline{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$. They are parallel if they are equal or a scalar multiple of the other.
- Coplanar objects are those lying in the same plane.
- Three vectors are coplanar if the scalar triple product, $\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3)$, is zero.
- The scalar triple product can be found, with or without technology, by calculating the determinant.

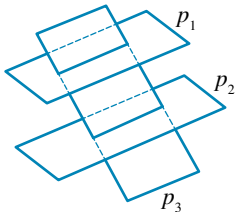
$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

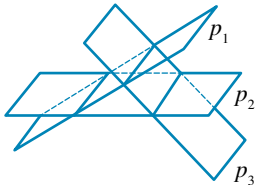
Consistent system – one unique solution

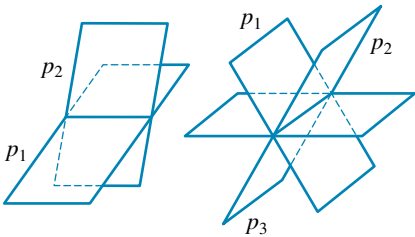
Geometric interpretation	Planes intersect at one unique point, $P(x_1, y_1, z_1)$	
$AX = B$	A is invertible.	
Not coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) \neq 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$	

Inconsistent system – zero solutions

Geometric interpretation	The normal vector of three planes are parallel yet distinct.	
$AX = B$	A is not invertible.	
Coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & a_{12} & a_{13} & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \end{array} \right]$	

Inconsistent system – zero solutions		
Geometric interpretation	Two planes are distinct with parallel normals. The third plane is not parallel.	
$AX = B$	A is not invertible.	
Coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array} \right]$ The final row is a false statement, so this system has no solution (they intersect as lines in parallel pairs).	

Inconsistent system – zero solutions		
Geometric interpretation	The normals are not parallel.	
$AX = B$	A is not invertible.	
Coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array} \right]$ The final row is a false statement, so this system has no solution. The planes intersect as pairs in a line.	

Consistent system – infinitely many solutions		
Geometric interpretation	The normals are coplanar and the planes intersect as a line. Infinitely many solutions can be defined parametrically.	
$AX = B$	A is not invertible.	
Coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & 0 \end{array} \right]$ The third equation is eliminated, hence the free variable is expressed in terms of t, $z = t$.	

(Continued)

(Continued)

Consistent system – infinitely many solutions		
Geometric interpretation	The three planes are coincident. There are infinitely many solutions.	
$AX = B$	A is not invertible.	
Coplanar	$\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$	
Augmented matrix in triangular form	$\left[\begin{array}{ccc c} 1 & a_{12} & a_{13} & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$	

WORKED EXAMPLE 13

The following descriptions are the geometric and/or algebraic properties of a system of linear equations in three variables. Make an inference on the type of solution set each system would produce based on the given information.

- The three planes are not parallel, $\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$ and the final row of the triangular matrix is a false statement.
- The three normals are not parallel. The coefficient matrix in the equation $AX = B$ is non-singular.
- When solving the system using Gaussian elimination the augmented matrix reaches the

triangular form $\left[\begin{array}{ccc|c} 1 & 0 & a_{13} & a \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 0 & c \end{array} \right]$. The planes intersect as pairs in the form of parallel lines.

THINK

- The three planes are not parallel, hence the system could be consistent or inconsistent. The normals are coplanar therefore it can't be unique. The final row of the triangular matrix is false, so the planes may intersect in pairs of lines.
- As the coefficient matrix is invertible, and the planes are not parallel the only solution type is unique.
- The final row of the augmented triangular matrix is false so the system is inconsistent. As the planes intersect as pairs in the form of parallel lines there is no unique solution.

WRITE

- It is an inconsistent system with zero solutions.
- It is a consistent system containing one unique solution.
- It is an inconsistent system containing no solution.

WORKED EXAMPLE 14

Consider the following system of linear equations.

$$\begin{aligned} 3x + 2y + 5z &= 27 \\ x + 5y - 4z &= -2 \\ -4x + 3y - 6z &= -10 \end{aligned}$$

- Determine the number of possible solutions by examining the normal vectors algebraically.
- Convert the equations to an augmented matrix and then use technology to solve the system.
- Describe how the solution can be interpreted geometrically.

THINK

1. Determine if the normal vectors are equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.

2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.

The determinant may be calculated using technology. (See Chapter 2, Worked example 29).

$$\det \left(\begin{bmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{bmatrix} \right)$$

1. Convert the linear system of equations into augmented matrix form where:
 A = Coefficient matrix.
 B = Value matrix.

2. The augmented matrix can be reduced into triangular form using technology (see Worked example 8).

$$\text{rref} \left(\begin{bmatrix} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{bmatrix} \right)$$

Note: The same could be achieved using Gaussian elimination.

3. State the type of solution and describe it geometrically.

WRITE

$$\mathbf{n}_1 = 3\hat{i} + 2\hat{j} + 5\hat{k}$$

$$\mathbf{n}_2 = \hat{i} + 5\hat{j} - 4\hat{k}$$

$$\mathbf{n}_3 = -4\hat{i} + 3\hat{j} - 6\hat{k}$$

The normals are not parallel, hence the planes are not parallel.

$$\begin{aligned} \mathbf{n}_1 \cdot \mathbf{n}_2 \times \mathbf{n}_3 &= (3\hat{i} + 2\hat{j} + 5\hat{k}) \cdot (\hat{i} + 5\hat{j} - 4\hat{k}) \times (-4\hat{i} + 3\hat{j} - 6\hat{k}) \\ &= \begin{vmatrix} 3 & 2 & 5 \\ 1 & 5 & -4 \\ -4 & 3 & -6 \end{vmatrix} \\ &= 3 \begin{vmatrix} 5 & -4 \\ 3 & -6 \end{vmatrix} - 2 \begin{vmatrix} 1 & -4 \\ -4 & -6 \end{vmatrix} + 5 \begin{vmatrix} 1 & 5 \\ -4 & 3 \end{vmatrix} \\ &= 105 \end{aligned}$$

As $\Delta \neq 0$, the planes are not coplanar and intersect at a unique point.

$$\mathbf{b.} \quad [A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array} \right]$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

The three planes intersect at $(x, y, z) = (-2, 4, 5)$.

3. The system is consistent and has one unique solution. Geometrically, the three planes intersect at a unique point.

WORKED EXAMPLE 15

Consider the following system of linear equations to answer the following.

$$-x - 5y - 8z = -22$$

$$6x + 9y + 6z = 69$$

$$-2x - y + 2z = -17$$

- Determine the number of possible solutions by examining the normal vectors algebraically.
- Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
- Solve the system of linear equations, expressing the solution in parametric form.
- Describe how the solution can be interpreted geometrically.

THINK

1. Examine the normal vectors to determine if they are equivalent or a scalar multiple of another normal vector. State if any of the planes are parallel.
2. If a unique solution exists, the normals will not be coplanar. Recall the scalar triple product formula. This can be found by calculating the determinant of the 3×3 matrix.

The determinant may be calculated using technology (see Chapter 2, Worked example 29).

$$\det \begin{pmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{pmatrix}$$

1. Convert the linear system of equations into augmented matrix form $[A|B]$ where,
 A = Coefficient matrix.
 B = Constant matrix.

WRITE

$$\underline{n}_1 = -\hat{i} - 5\hat{j} - 8\hat{k}$$

$$\underline{n}_2 = 6\hat{i} + 9\hat{j} + 6\hat{k}$$

$$\underline{n}_3 = -2\hat{i} - \hat{j} + 2\hat{k}$$

The normal vectors are not parallel, hence the planes are not parallel.

$$\begin{aligned} \underline{n}_1 \cdot \underline{n}_2 \times \underline{n}_3 &= (-\hat{i} - 5\hat{j} - 8\hat{k}) \cdot (6\hat{i} + 9\hat{j} + 6\hat{k}) \times (-2\hat{i} - \hat{j} + 2\hat{k}) \\ &= \begin{vmatrix} -1 & -5 & -8 \\ 6 & 9 & 6 \\ -2 & -1 & 2 \end{vmatrix} \\ &= -1 \begin{vmatrix} 9 & 6 \\ -1 & 2 \end{vmatrix} - (-5) \begin{vmatrix} 6 & 6 \\ -2 & 2 \end{vmatrix} + (-8) \begin{vmatrix} 6 & 9 \\ -2 & -1 \end{vmatrix} \\ &= 0 \end{aligned}$$

As $\Delta = 0$ the three planes are coplanar and it has already been noted that they are not parallel.

Geometrically, this could be the planes intersecting in a line or not at all.

$$[A|B] = \left[\begin{array}{ccc|c} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{array} \right]$$

2. The augmented matrix can be reduced into triangular form using technology (see Worked example 8).

$$\text{rref} \left(\begin{bmatrix} -1 & -5 & -8 & -22 \\ 6 & 9 & 6 & 69 \\ -2 & -1 & 2 & -17 \end{bmatrix} \right)$$

- c. 1. The third row of the triangular matrix is equivalent to, $0x + 0y + 0z = 0$. This informs us that there are infinitely many solutions.
2. Express the first two rows of the triangular matrix as two equations in terms of x, y and z . Let the free variable $z = t$.
3. Rewrite equation [2] by substituting t for z and then make y the subject.
4. Rewrite equation [1] by substituting t for z and then make x the subject.
5. State the solution in terms of the parameter, t .

- d. State the type of solution and describe it geometrically.

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

- c. The system has infinitely many solutions because the planes intersect in a line.

$$x - 2z = 7 \quad [1]$$

$$y + 2z = 3 \quad [2]$$

Let $z = t$ as it is the free variable.

$$y + 2t = 3$$

$$y = 3 - 2t \quad [3]$$

$$x - 2t = 7$$

$$x = 7 + 2t$$

The intersection line is given by the parametric equations:

$$x = 7 + 2t$$

$$y = 3 - 2t$$

$$z = t$$

- d. This is a consistent system with the solution representing the intersection of three planes in the form of a line.

Note: The solution could easily be converted to two other forms.

- Vector equation: $\underline{r} = (7\hat{i} + 3\hat{j} + 0\hat{k}) + t(2\hat{i} - 2\hat{j} + \hat{k})$

- Cartesian form: $\frac{x-7}{2} = \frac{y-3}{-2} = \frac{z}{1}$

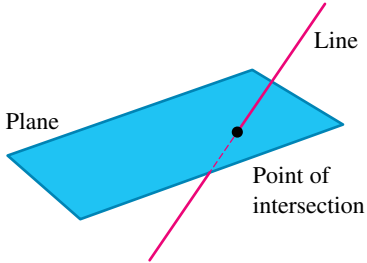
3.4.4 The intersection of lines and planes

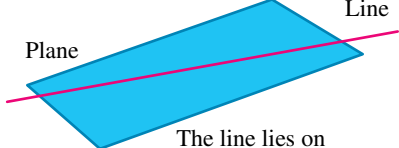
When considering the intersection of a plane and line in three-dimensions, there are three possible solution types.

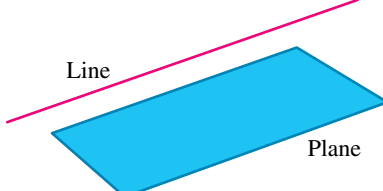
- **One (unique) solution** where the plane and line intersect at a single point $P(x, y, z)$.
- **Infinitely many solutions** in which the line lies on the plane.
- **No solution** as the line and plane are parallel yet distinct.

To examine these solutions geometrically, a vector approach will be applied in the first instance. The plane and line are defined as follows.

- **Line** (l_1): $\underline{r} = \underline{a} + k\underline{d}$ where $\underline{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is the position vector of a known point on the line and $\underline{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$ is the direction vector *parallel* to the line.
- **Plane** (p_1): $ax + by + cz + d = 0$, where $\underline{n} = a\hat{i} + b\hat{j} + c\hat{k}$ is *perpendicular* to the plane.

Consistent system – one unique solution		
Examine the direction and normal vectors.	$\underline{d} \cdot \underline{n} \neq 0$. Therefore, the line and plane are <i>not parallel</i> .	 Plane Line Point of intersection
Geometric interpretation	The line and plane intersect at a unique point, $P(x, y, z)$ so <i>one solution</i> exists.	

Consistent system – infinitely many solutions		
Examine the direction and normal vectors.	$\underline{d} \cdot \underline{n} = 0$ Therefore, the line and plane are <i>parallel</i> .	 Plane Line The line lies on the plane
Infinitely many or no solutions?	Find a point $P(x, y, z)$ that lies on the line l_1 . This point <i>will satisfy</i> the equation of the plane p_1 .	
Geometric interpretation	The line lies on the plane and <i>infinitely many solutions</i> exist.	

Inconsistent system – zero solutions		
Examine the direction and normal vectors.	$\underline{d} \cdot \underline{n} = 0$ Therefore, the line and plane are <i>parallel</i> . (i.e. the line is above or below the plane)	 Line Plane
Infinitely many or no solutions?	Find a point $P(x, y, z)$ that lies on the line l_1 . This point <i>will not satisfy</i> the equation of the plane p_1 .	
Geometric interpretation	The line is parallel yet distinct to the plane. <i>No solution</i> exists.	

WORKED EXAMPLE 16

Consider a three-dimensional system containing the line $l_1: \underline{r} = (2\hat{i} + 3\hat{j} - 3\hat{k}) + k(-\hat{i} + 2\hat{j} - 3\hat{k})$ and plane given by $p_1: 2x + 7y - 4z = 61$.

- Examine the line and plane to identify if they intersect.
- If they do intersect, determine the solution, otherwise state the system has zero solutions.

THINK

a. 1. Examine the normal $\underline{n}$ and direction $\underline{d}$ vectors. Determine if they are perpendicular by calculating the dot product.

2. State if the line and plane intersect.

b. 1. Convert the vector equation of the line into parametric equations.

2. Substitute the parametric equations into the equation of the plane.

3. Solve for k .

4. Determine the point of intersection by substituting $k = 1$ for the parametric equations.

5. State the solution.

WRITE

a. $\underline{n} = 2\hat{i} + 7\hat{j} - 4\hat{k}, \underline{d} = -\hat{i} + 2\hat{j} - 3\hat{k}$

$$\begin{aligned}\underline{n} \cdot \underline{d} &= (2\hat{i} + 7\hat{j} - 4\hat{k}) \cdot (-\hat{i} + 2\hat{j} - 3\hat{k}) \\ &= 2 \times -1 + 7 \times 2 + -4 \times -3 \\ &= 24\end{aligned}$$

$$\underline{n} \cdot \underline{d} \neq 0$$

As $\underline{n} \cdot \underline{d} \neq 0$ the normal and direction vectors are not perpendicular. The plane and line are not parallel, hence a single point of intersection must exist.

b. $l_1: \underline{r} = (2\hat{i} + 3\hat{j} - 3\hat{k}) + k(-\hat{i} + 2\hat{j} - 3\hat{k})$

$$l_1: \begin{cases} x = 2 - k \\ y = 3 + 2k \\ z = -3 - 3k \end{cases}$$

$$2(2 - k) + 7(3 + 2k) - 4(-3 - 3k) = 61$$

$$2(2 - k) + 7(3 + 2k) - 4(-3 - 3k) = 61$$

$$4 - 2k + 21 + 14k + 12 + 12k = 61$$

$$24k + 37 = 61$$

$$k = \frac{61 - 37}{24}$$

$$k = 1$$

$$l_1, k = 1: \begin{cases} x = 2 - 1 = 1 \\ y = 3 + 2 \times 1 = 5 \\ z = -3 - 3 \times 1 = -6 \end{cases}$$

The line and plane intersect at a single (unique) point $(1, 5, -6)$.

When determining the intersection of a line and plane the vector approach is not the only method. In Worked example 16, the equation of the line was converted to parametric equations and subsequently substituted into the equation of the plane. A single value for k was found, indicating that a unique solution existed. The following worked example will examine systems containing infinitely many or no solutions.

WORKED EXAMPLE 17

For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.

a. $l_1: \begin{cases} x = 2 + 4k \\ y = -3 + 2k \\ z = 3 - 2k \end{cases}, \quad p_1: 3x + y + 7z = 24$

b. $l_2: \begin{cases} x = 3 + 5k \\ y = -2 + 3k \\ z = 3 - k \end{cases}, \quad p_2: -2x + 4y + 2z = 1$



THINK

- a. 1. Substitute the parametric equations into the equation of the plane.
 2. Solve for k .
 3. Interpret the solution.
- b. 1. Substitute the parametric equations into the equation of the plane.
 2. Solve for k .
 3. Interpret the solution.

WRITE

a. $3(2 + 4k) + 1(-3 + 2k) + 7(3 - 2k) = 18$

$$\begin{aligned} 6 + 12k - 3 + 2k + 21 - 14k &= 24 \\ 24 + 0k &= 24 \\ 0k &= 0 \end{aligned}$$

For any value of k the equation is true. The line lies on the plane meaning there are infinitely many solutions.

b. $-2(3 + 5k) + 4(-2 + 3k) + 2(3 - k) = 1$

$$\begin{aligned} -6 - 10k - 8 + 12k + 6 - 2k &= 1 \\ 0k - 8 &= 1 \\ 0k &= 9 \end{aligned}$$

There are no possible values for k that make this equation true. The line is parallel yet distinct from the plane meaning there are zero solutions.

study on

Units 3 & 4

Area 2

Sequence 2

Concept 3

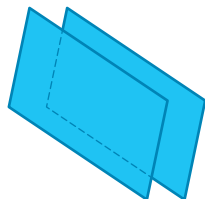
The three cases for solutions of systems of linear equations Summary screen and practice questions

Exercise 3.4 The three cases for solutions of systems of linear equations

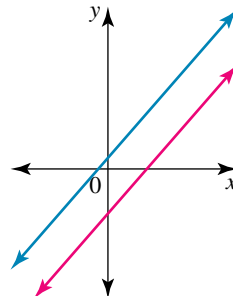
Technology free

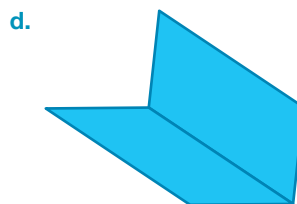
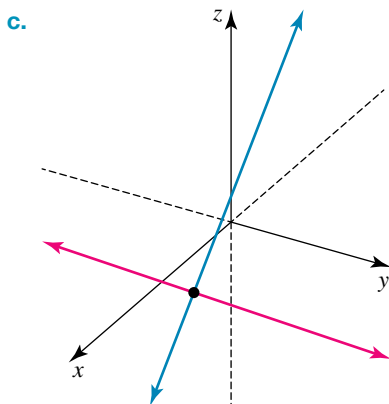
1. **MC** A unique solution may exist in a square system of two dimensions if the lines exhibit which of the following geometric features?
- A. Parallel and distinct
 B. Coincidental
 C. Not parallel
 D. Homogenous
2. Refer to the following graphs and state how many solutions exist.

a.



b.





3. **WE9** Solve the following system of linear equations by using matrices and interpret the solution geometrically.
- a. $8x + 2y = 12$
 $4x - 7y = -10$
- b. $-2x + y = -2$
 $6x - 3y = 7$
- c. $-3x + y = -14$
 $-2x - 5y = 2$
- d. $4x - 8y = 4$
 $6x - 12y = -3$
4. **WE10** a. Solve the following systems of linear equations by using matrices and interpret the solution geometrically.
b. Describe the solution set in parametric form.
- i. $10x - 6y = 12$
 $5x - 3y = 6$
- ii. $x - 2y = -4$
 $-4x + 8y = 16$
5. The following systems contain two lines in the form, $l_1: a_1x + b_1y = c_1$ and $l_2: a_2x + b_2y = c_2$. Determine the nature of the solution set by interpreting the values for a_n, b_n and c_n algebraically.
- a. $x + 2y = 15$
 $3x - y = 10$
- b. $3x + 7y = 9$
 $6x + 14y = 18$
- c. $-16x + 6y = -14$
 $-4x + \frac{3}{2}y = -4$
- d. $-x + 4y = -3$
 $-5x + 20y = -12$
6. In each case below, the number of solutions has been defined for the system. Determine the restrictions on the values for a, b and c that would make the statement on the solution type true.
- a. $\begin{cases} 5x - 2y = a \\ 10x - 4y = 24 \end{cases}$ No solution
- b. $\begin{cases} 8x - 2y = -12 \\ -4x + by = c \end{cases}$ Infinitely many solutions
7. **WE11** a. Determine the unique solution for the following systems of linear equations defined by the lines l_1 and l_2 .
b. Describe the geometric nature of the solution.
- i. $l_1: (\hat{i} - \hat{j} - 4\hat{k}) + k(-5\hat{i} - 4\hat{j} - 5\hat{k})$
 $l_2: (-15\hat{i} + 7\hat{j} - 40\hat{k}) + k(-\hat{i} - 6\hat{j} + 4\hat{k})$
- ii. $l_1: (-\hat{i} + 2\hat{j} + 2\hat{k}) + k(-\hat{i} - 4\hat{j} - 4\hat{k})$
 $l_2: (-9\hat{i} - 5\hat{j} - 9\hat{k}) + k(7\hat{i} + 3\hat{j} + 7\hat{k})$
- iii. $l_1: (4\hat{i} + 7\hat{j} + 3\hat{k}) + k(6\hat{i} + \hat{j} + 5\hat{k})$
 $l_2: (60\hat{i} - 14\hat{j} + 3\hat{k}) + k(-2\hat{i} + 4\hat{j} + 5\hat{k})$
8. a. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2 .
b. Describe the geometric nature of the solution.
- i. $l_1: \begin{cases} x = 3 - 3k \\ y = 3 - 4k \\ z = 4 + 4k \end{cases}$ $l_2: \begin{cases} x = -33 + k \\ y = -15 - 2k \\ z = -14 + 6k \end{cases}$
- ii. $l_1: k = \frac{x+1}{4} = \frac{y-5}{-3} = \frac{z+8}{4}$ $l_2: \begin{cases} x = 53 - 5k \\ y = -25 + 2k \\ z = -14 + 5k \end{cases}$

9. **WE12** A system of linear equations in three variables contains the lines l_1 and l_2 .
- Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.
 - Describe the geometric nature of the system by referring to any possible solution/s.

i. $l_1: \frac{x+4}{-6} = \frac{y-5}{1} = \frac{z+4}{8} = k_1, \quad l_2: \frac{x+1}{-12} = \frac{y-2}{2} = \frac{z+7}{16} = k_2$

ii. $l_1: \frac{x-4}{-5} = \frac{y+4}{-5} = \frac{z-1}{6} = k_1, \quad l_2: \frac{x+18}{-2} = \frac{y+17}{7} = \frac{z-27}{2} = k_2$

iii. $l_1: \frac{x-6}{-8} = \frac{y-3}{5} = \frac{z-1}{-6} = k_1, \quad l_2: \frac{x+34}{4} = \frac{y-62}{6} = \frac{z+13}{7} = k_2$

10. A system of linear equations in three variables contains the lines l_1 and l_2 .
- Determine if a solution exists for the simultaneous equations. If so, calculate the point of intersection.
 - Describe the geometric nature of the system by referring to any possible solution/s.

i. $l_1: \frac{x+6}{-6} = \frac{y+4}{-4} = \frac{z+7}{-2} = k_1$

$l_2: x = -1 - 30k_2, y = -7 - 20k_2, z = -10 - 10k_2$

ii. $l_1: \begin{cases} x = 2 + k_1 \\ y = 6 - 3k_1 \\ z = 2 + 5k_1 \end{cases} \quad l_2: \begin{cases} x = 1 - 4k_2 \\ y = -9 + 3k_2 \\ z = 39 + k_2 \end{cases}$

11. **WE13** The following descriptions relate to the geometric and/or algebraic properties of a system of linear equations in three variables. Make an inference on the type of solution set each system would produce based on given information.

- a. The three planes are coincident.

b. The diagonal augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array} \right]$.

- c. The three planes are not parallel, $\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) = 0$ and the final row of the triangular matrix is a false statement. The three planes meet along a line defined by the vector equation,

$\vec{r} = (4\hat{i} - 6\hat{j} - 11\hat{k}) + k(-3\hat{i} + 13\hat{j} + 2\hat{k})$.

Technology active

12. **WE14** Consider the following system of linear equations to answer the following.

a. $5x + 8y + z = -5$
 $-7x - 4y - 5z = -11$
 $-3x - y - 6z = 4$

b. $-7x + 7y + 42z = 9$
 $5x - 2y + 8z = -11$
 $7x - 4y - 4z = 2$

c. $3x + 6y - 2z = -25$
 $8x - 4y + 6z = -58$
 $-5x - 5y - 3z = 43$

d. $-6x - 6y - z = -64$
 $x - y + 2z = 4$
 $x + 7y - 7z = 24$

e. $6x + 3y - 24z = 8$
 $-3x - y + 6z = -2$
 $4x + 8z = 10$

f. $-4x - 5y - 4z = 5$
 $4x + 5y + 4z = -7$
 $7x - 6y + 7z = 4$

- Determine the number of possible solutions by examining the normal vectors algebraically.
 - Convert the equations to an augmented matrix and then use technology to solve the system.
 - Describe how the solution can be interpreted geometrically.
13. In each of the following cases, determine the equation of the line formed by the intersection between the two planes. Give all possible answers in parametric form.

a. $p_1: 8x - 2y + 3z = 1$
 $p_2: -8x + 6y + z = -5$

b. $p_1: -3x - 2y + z = 1$
 $p_2: 7x + y + 5z = 6$

c. $p_1: \frac{2}{3}x - \frac{7}{6}y + \frac{1}{2}z = 2$
 $p_2: -4x + 7y + -3z = -11$

14. Investigate the number of possible solutions by examining the normal vectors algebraically. Describe how the systems could be interpreted geometrically.

a. $2x + 3y - 4z = 8$
 $2x + 3y - 4z = -5$
 $2x + 3y - 4z = -4$

c. $4x + 7y - 4z = 5$
 $16x + 28y - 16z = -3$
 $7x - z = -1$

b. $-6x + 8y + 5z = 11$
 $-12x + 16y + 10z = -5$
 $24x - 32y - 20z = 0$

d. $-7x + 4y - z = 5$
 $-42x + 24y - 6z = -1$
 $-3x - 2y + 5z = 7$

15. The equations below are of three coincidental planes. Determine the real values for d_2 and a_3 .

$$p_1: 8x + 12y - 4z = 16$$

$$p_2: 4x + 6y - 2z = d_2$$

$$p_3: a_3x + 18y - 6z = 24$$

16. a. Determine the line of intersection between the following planes. Express the solution as a vector equation.

$$p_1: -4x - 2y + 5z = -18 \quad p_2: 5x + 3y - \frac{11}{2}z = 26$$

- b. Calculate $\underline{n}_1 \times \underline{n}_2$. How does this relate to the solution obtained in a.?

17. **WE15** Consider the following systems of linear equations to answer the following.

a. $3x + 2y - 16z = 22$
 $-4x - 4y + 20z = -28$
 $x + 3y - 3z = 5$

b. $6x + 5y - 4z = 42$
 $-3x + 4y + 28z = 135$
 $x - 4y - 20z = -109$

c. $-3x + y + \frac{11}{2}z = 40$
 $5x + 4y + 5z = 58$
 $3x + 5y + \frac{19}{2}z = 92$

d. $4x + 2y - 18z = 58$
 $8x + 8y - 32z = 120$
 $x + y - 4z = 15$

- i. Determine the number of possible solutions by examining the normal vectors algebraically.
ii. Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
iii. Solve the system of linear equations, expressing the solution in parametric form.
iv. Describe how the solution can be interpreted geometrically.
18. Determine if the solution set is valid for the following systems.

a. $-5x + 7y + 44z = 79$
 $9x + 8y - 38z = 2$
 $4x - 2y - 28z = -38$ Solution: $\begin{cases} x = -6 + 6t \\ y = 7 - 2t \\ z = t \end{cases}$

b. $-7x + 3y + 22z = 136$
 $2x + 2y - 4z = -24$
 $2x - y - 10z = -63$ Solution: $\begin{cases} x = -25 + 4t \\ y = 13 - 2t \\ z = t \end{cases}$

19. **WE16** a. Consider a three-dimensional system containing the line

$$l_1: \underline{r} = (2\hat{i} + 3\hat{j} + \hat{k}) + k(4\hat{i} + 6\hat{j} - 2\hat{k}) \text{ and plane given by } p_1: 5x - 2y + 4z = 8.$$

- i. Examine the line and plane to identify if they intersect.
ii. Determine the solution if they do intersect, otherwise state the system has zero solutions.

- b. Consider a three-dimensional system containing the line $l_1: \underline{r} = (2\hat{i} + 7\hat{j} - 9\hat{k}) + k(3\hat{i} + \hat{j} - \hat{k})$ and plane given by $p_1: 2x - 3y + 3z = 4$.
- Examine the line and plane to identify if they intersect.
 - Determine the solution if they do intersect, otherwise state the system has zero solutions.
- c. Consider a three-dimensional system containing the line $l_1: \underline{r} = (\hat{i} - 2\hat{j} + 5\hat{k}) + k(2\hat{i} + 4\hat{j} - 6\hat{k})$ and plane given by $p_1: -3x - 2y + z + 34 = 0$.
- Examine the line and plane to identify if they intersect.
 - Determine the solution if they do intersect, otherwise state the system has zero solutions.
20. **WE17** For each of the following systems, calculate if the plane and line intersect. If so, determine how many solutions exist.
- a. $l_1: \begin{cases} x = 2 + 4k \\ y = -1 - 2k \\ z = 7 - 3k \end{cases}, p_1: 5x - 3y + 2z - 47 = 0$
- b. $l_1: \begin{cases} x = 5 - 2k \\ y = 4 - 9k \\ z = 1 - 3k \end{cases}, p_1: 3x - 2y + 4z = 37$
- c. $l_1: \begin{cases} x = 1 + 3k \\ y = -3 + 2k \\ z = -2 - k \end{cases}, p_1: 5x - 7y + z - 24 = 0$
21. An object is travelling a straight-line path according to the vector equation, $l_1: (-2\hat{i} - 3\hat{j} - 5\hat{k}) + k(-6\hat{i} - 4\hat{j} - \hat{k})$. The sea floor is a flat plane modelled by the equation, $24x - 16y - 31z = -16$. Determine if the object will ever intersect the sea floor and calculate a solution if possible.



3.5 Using technology for matrix calculations

3.5.1 Matrix operations using a graphic calculator

Until this point, systems of linear equations and associated operations have been completed algebraically. Augmented matrices can be reduced to row-echelon form by either:

- Completing elementary row operations algebraically.
- Using technology (for example: graphics calculator, MS Excel, Geogebra, Wolfram, etc.)

In Worked example 14, the unknown matrix X , was calculated through technology, where,



$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 5 & 27 \\ 1 & 5 & -4 & -2 \\ -4 & 3 & -6 & -10 \end{array} \right]$$

The solution can be calculated using either a TI or Casio graphics calculator as shown below.

	TI	CASIO
Input a matrix.		
If the matrix is invertible, a reduced row-echelon matrix is produced using the <i>rref</i> command.		
The matrix in reduced row-echelon form.		
Unique solution	$(x, y, z) = (-2, 4, 5)$	$(x, y, z) = (-2, 4, 5)$

Most of the basic matrix operations can be completed using technology. It is important that you are competent in this area when answering technology active questions. The aim of the worked examples that follow is to build your proficiencies in the use of technology. Begin each matrix investigation by first defining the associated matrices.

WORKED EXAMPLE 18

Refer to the matrices below and use technology to calculate the following matrix operations.

$$A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$$

a. $2A - B$

b. $\det A$

TI | THINK

- a. 1. On a Calculator page, select



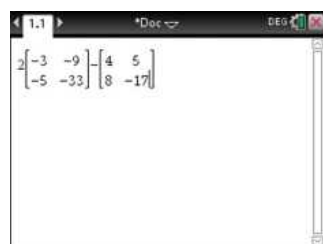
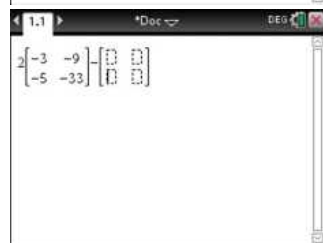
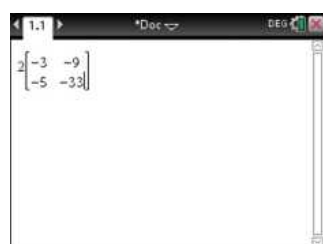
2. Complete the entry line as
-3
-9
-5
-33

3. Complete the entry line as
-
Then select
2



4. Press the button, followed by the button.
Complete the entry line as
4
5
8
-17

WRITE



CASIO | THINK

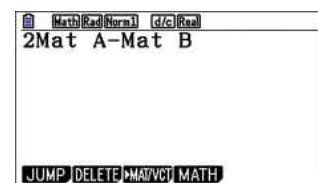
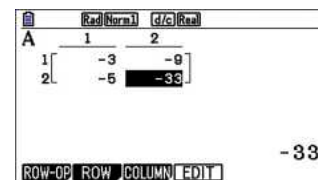
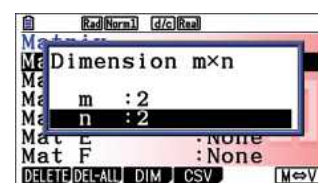
- a. 1. On a Run-Matrix screen, select MAT/VCT
To switch between the vector and matrix screens, press the $M \leftrightarrow V$ button.

2. Press the EXE button.
Complete the entry line as
2
2
Press the EXE button.

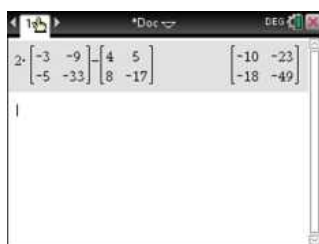
3. Complete the entry line as
-3
-9
-5
-33
Press the EXIT button when finished.
Repeat steps 1-3 to define Matrix B.

4. On a Run-Matrix screen, select
2
SHIFT+2 (Mat)
ALPHA
X,θ,T
-
SHIFT+2 (Mat)
ALPHA
Log
Press the EXE button.

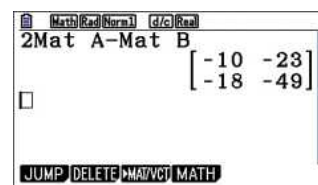
WRITE



5. Press the ENTER button.
The answer appears on the screen.



5. The answer appears on the screen.



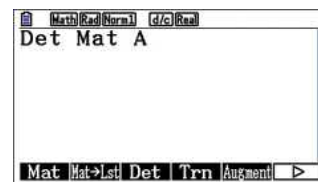
- b. 1. On a Calculator page, select
MENU
7: Matrix & Vector
3: Determinant
2



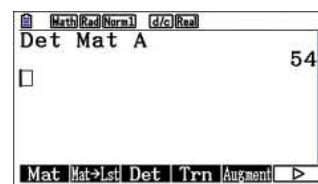
2. Complete the entry line as
-3
-9
-9
-33
Press the ENTER button.
The answer appears on the screen.



- b. 1. On a Run-Matrix screen, select
OPTN
MAT/VCT
DET
SHIFT+2 (Mat)
ALPHA
X,θ,T



2. Press the EXE button.
The answer appears on the screen.



3.5.2 Powers of matrices

A logical extension of matrix multiplication is to calculate the power of a matrix, where

$$\begin{aligned} A^1 &= A \\ A^2 &= A \times A \\ A^3 &= A^2 \times A \end{aligned}$$

In general form, $A^n = A^{n-1}A$, where n is a positive integer. But what dimensions can matrix A have?

Matrix powers – an investigation

We can investigate powers of matrices by completing the following steps. Remember to use pronumerals for the elements of A , not constant values.

1. a. Let matrix A be any 3×2 matrix.
b. Find A^2 .
c. What do you notice?
2. a. Let matrix A be any 2×2 matrix.
b. Find A^2 .
3. a. Let matrix A be any 2×3 matrix.
b. Find A^2 .
c. What do you notice?
4. a. What general conclusion can you make concerning the order of a matrix that is to be raised to a power?
b. Justify your conclusion by referring to the dimensions of matrices involved in a product.

From the above investigations we can conclude that $A \times A$ must be conformable; that is, the number of columns of the first factor in the product should be the same as the number of rows in the second factor.

That is, A must be a *square* matrix where $n \times n$ is multiplied by $n \times n$ to get A^2 .

Hence powers of matrices are only defined for square matrices.

WORKED EXAMPLE 19

If $A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$ use technology to determine:

a. A^4

b. $(A \times B)^2$

TI | THINK

- a. 1. On a Calculator page, select

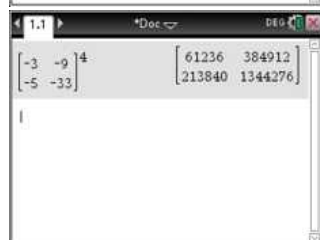
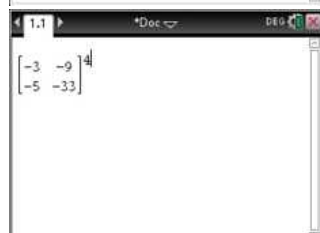
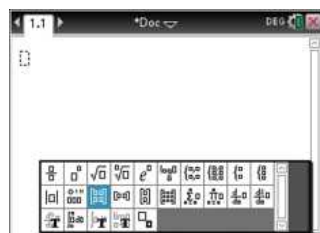


2. Complete the entry line as

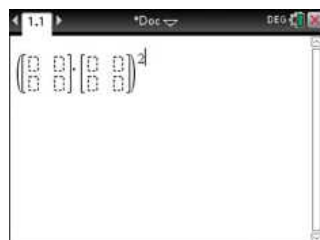
$$\begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}^4$$

3. Press the ENTER button.
The answer appears on the screen.

WRITE



- b. 1. On a Calculator page, select



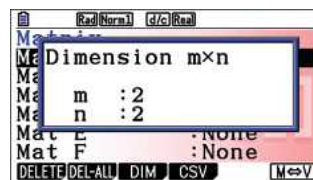
CASIO | THINK

- a. 1. On a Run-Matrix screen, select
MAT/VCT
To switch between the
vector
and matrix screens, press
the $M \leftrightarrow V$ button.

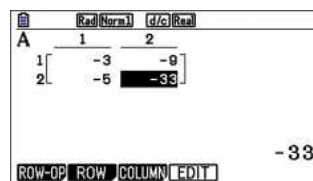
WRITE



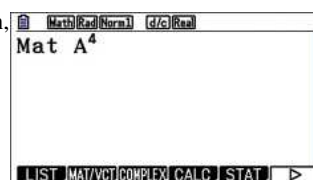
2. Press the EXE button.
Complete the entry line as
2
2
2
Press the EXE button.



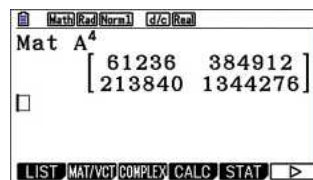
3. Complete the entry
line as
-3 -9
-5 -33
Press the EXIT button
twice
when finished.



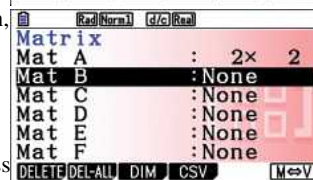
4. On a Run-Matrix screen, select
SHIFT
2
ALPHA
X,θ,T
^
4



5. Press the EXE button.
The answer appears on the screen.



- b. 1. On a Run-Matrix screen, select
MAT/VCT
To switch between the
vector
and matrix screens, press
the $M \leftrightarrow V$ button.



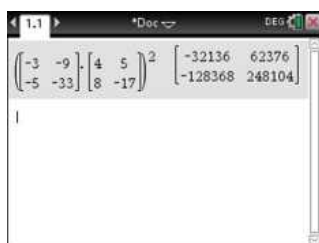
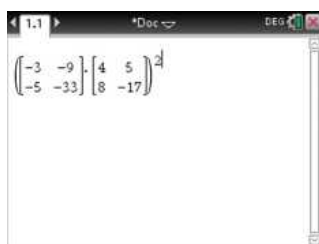
2. Complete the entry

$$\begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}$$

Press the EXIT button twice when finished.

3. Press ENTER.

The answer appears on the screen.



2. Highlight Mat B and press the EXE button.

Complete the entry

line as

2

2

Press the EXE button.

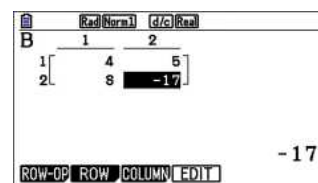
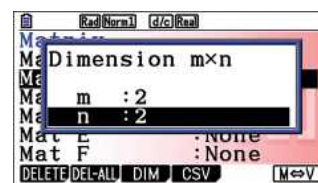
3. Complete the entry

line as

4 5

8 -17

Press the EXIT button twice when finished.



4. On a Run-Matrix screen,

select

(

SHIFT

2

ALPHA

X,θ,T

×

SHIFT

2

ALPHA

log

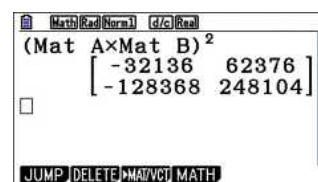
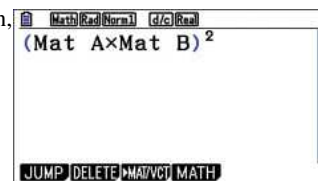
)

^

2

5. Press the EXE button.

The answer appears on the screen.



study on

Units 3 & 4 Area 2 Sequence 2 Concept 4

Using technology for matrix calculations Summary screen and practice questions

Exercise 3.5 Using technology for matrix calculations

Technology active

1. **WE18** Refer to the matrices below to calculate the following matrix operations.

$$A = \begin{bmatrix} -3 & -9 \\ -5 & -33 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 8 & -17 \end{bmatrix}, C = \begin{bmatrix} 9 \\ -9 \end{bmatrix}, D = \begin{bmatrix} 2 & -6 & 4 \\ 1 & -4 & 10 \\ -2 & 6 & -6 \end{bmatrix}, E = \begin{bmatrix} -42 \\ -49 \\ 48 \end{bmatrix}$$

a. $11A - 8B$

b. $\frac{1}{5}(AB)$

c. $\det(B)$

d. $|D|$

e. A^{-1}

f. D^{-1}

g. $X = D^{-1}E$

h. $X = B^{-1}C$

2. Consider $A = \begin{bmatrix} 2 & -1 & 1 \\ -3 & 5 & -4 \\ 3 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 & 4 \\ 6 & -1 & 1 \\ 0 & -3 & 5 \end{bmatrix}$, when examining if the following statements are true or false.

a. $A^3 \times B^2 = B^2 \times A^3$

b. $A^3 B^3 = (AB)^3$

3. **WE19** If $A = \begin{bmatrix} 6 & -2 \\ 1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} -10 \\ -2 \end{bmatrix}$, $D = \begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 3 \\ 5 & -1 & 2 \end{bmatrix}$, $E = \begin{bmatrix} 10 \\ 9 \\ 24 \end{bmatrix}$,

determine:

a. A^3

b. $A^2 \times (B^{-1})^3$

c. $2D^3$

d. X if, $\begin{bmatrix} 122 & 121 & 120 \\ 119 & 118 & 117 \\ 116 & 115 & 114 \end{bmatrix} = D^2 - 5X$

4. Determine if the following systems are either consistent or inconsistent by completing the 'ref' or 'rref' operation on a graphics calculator.

a. $A|B = \left[\begin{array}{ccc|c} 10 & -8 & -2 & 3 \\ 14 & -19 & -5 & -12 \\ 15 & -12 & -3 & 7 \end{array} \right]$

b. $A|B = \left[\begin{array}{cc|c} -8 & -7 & 12 \\ 16 & 14 & -20 \end{array} \right]$

c. $A|B = \left[\begin{array}{cccc|c} 3 & 4 & 7 & 1 & -96 \\ 2 & 6 & -1 & 0 & 26 \\ -4 & 7 & 2 & 6 & 72 \\ 4 & 3 & -2 & -1 & 8 \end{array} \right]$

d. $A|B = \left[\begin{array}{ccc|c} 5 & -3 & 1 & 19 \\ 6 & -9 & 8 & 23 \\ 12 & -3 & -4 & 41 \end{array} \right]$

5. One week's production at a café, in matrix form is given by,

$$A = \begin{bmatrix} \text{Coffee} & \text{Tea} & \text{Juice} \\ 78 & 34 & 14 \\ 17 & 11 & 10 \end{bmatrix} \begin{array}{l} \text{Units of material} \\ \text{Units of labour} \end{array}$$

Assume that in this week's production, the cost for each of material is \$0.35 and the cost of each unit of labour is \$0.58. Calculate the total cost to manufacture coffee, tea and juice.

6. Consider the two matrices $A = \begin{bmatrix} -8 & 2 \\ -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 \\ 4 & -9 \end{bmatrix}$. Calculate the following.

a. $(A - B)(A + B)$

b. $(A + B)^2$

7. A nilpotent matrix exists if $A^2 = 0$. Nilpotent matrices are singular. Determine if the following matrices are nilpotent.

a. $A = \begin{bmatrix} 8 & -32 \\ 2 & -8 \end{bmatrix}$

b. $\begin{bmatrix} 9 & -81 \\ 1 & -9 \end{bmatrix}$

8. An idempotent matrix exists if $A^2 = A$. Idempotent matrices are singular. Determine if the following matrices are idempotent.

a. $A = \begin{bmatrix} 6 & -3 \\ 10 & -5 \end{bmatrix}$

b. $B = \begin{bmatrix} -6 & 6 \\ -7 & 7 \end{bmatrix}$

9. If $A = \begin{bmatrix} -2 & -4 \\ 7 & 0 \end{bmatrix}$, then validate the mathematical statement $A^2 + 2A = -28I$.

10. Demonstrate, using numerical methods, that $A^2B = A(AB)$ given, $A = \begin{bmatrix} 5 & -3 & -2 \\ 4 & 0 & -6 \\ 2 & 7 & 5 \end{bmatrix}$ and

$$B = \begin{bmatrix} 7 & 4 & 1 \\ 9 & 8 & 1 \\ 2 & 9 & 2 \end{bmatrix}.$$

11. The sum of three numbers is 25. When double the first number and twice the second number are added to the third number the sum is 43. The last number is one less than the first. Construct a suitable matrix equation to calculate the values for each number.

Technology free

12. a. Construct a suitable matrix equation to determine suitable values for c given,

$$\begin{bmatrix} 5 & 5 & c \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ c \end{bmatrix} = -100$$

- b. Validate the solution from a.

3.6 Dominance and Leslie matrices

3.6.1 Dominance matrices

Have you ever wondered how tennis players are seeded or ranked? It obviously has something to do with their performance against past opponents where points are earned at every tournament based on how far a player advances in the event. In simple games such as a knock-out competition, one loss and you are out of the competition. Only the winners continue to play. Dominance matrices are often used to determine player rankings.

The following example will explain how matrices are used to establish the seedings or rankings of players in round-robin situations where each player plays every other player, thereby creating a more just system of ranking.

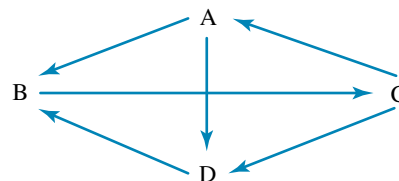
Consider 4 players Anna, Belle, Carla and Diane (A, B, C, D), who on past performances have shown that A defeats D and B, D defeats B, C defeats A and D, and B defeats C.

This situation can be represented on a **digraph** — a network diagram that has arrows on the edges, where $A \rightarrow B$ indicates that A defeats B.

The information from the digraph can be converted into matrix form (a *dominance matrix*) as below:

$$M = \begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

where 1s are used to indicate ‘defeats’ and 0s to indicate otherwise. Obviously A can’t defeat A so a ‘0’ is used along the leading diagonal.



Notice also that:

1. there are as many 1s as there are paths
2. corresponding elements occur on either side of the leading diagonal. That is, if A defeats B(1), then a 0 will be stored in the B defeats A element on the opposite side of the leading diagonal.

When the elements of each of the rows are added they yield a **dominance vector**, showing how many players each has defeated.

$$V = \begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

This result can be readily checked from the original digraph by counting the number of arrows *out of* each node.

Note, from now on the row/column labels will be omitted.

It can be seen from this information that A and C are ranked equally, and B and D are ranked equally. So we still need to distinguish between A and C, and B and D to establish the ranking.

We assume in most ranking situations that if A defeats B and B defeats C then A will defeat C. This relationship is described as being *transitive*, where if variable $a < b$ and $b < c$, then $a < c$.

In our example, A defeats B who defeats C, and A defeats D who defeats B. This means that A has second-order influence over C and B but not D. (A doesn't defeat anyone who defeats D). The matrix M^2 can be used to investigate second-order influence.

$$\begin{aligned} M^2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

Notice that the leading diagonal is still 0. It is impossible for a player to have second-order influence over themselves. Row 1 represents the second-order influence of player A over the other players. The element 2 in row 3 occurs because C defeats 2 players (A and D) who defeat B.

We can find the second-order dominance vector, $V_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, but how much importance should it be given?

If we assign equal importance, we calculate the sum of the first-order influence and the second-order influence.

$$\begin{aligned} M + M^2 &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{aligned}$$

This gives a dominance vector $\begin{bmatrix} 4 \\ 3 \\ 5 \\ 2 \end{bmatrix} = V_1 + V_2$ and allows us to rank the competitors in the order

C, A, B, D.

If we wanted to investigate third-order influence, we could calculate $M^3 (= M^2M)$.

$$M^2M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ giving } V_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

At this stage, notice that the leading diagonal is no longer 0. If there were more players in the tournament, we could continue finding powers of M , but with 4 players, we stop at M^3 . In general, if there are n players, we stop at M^{n-1} .

In most scenarios, it is probably unfair to assign equal importance to first-, second- and third-order influence. We can allocate arbitrary constants to weight the influence; that is, $M + xM^2 + yM^3$. The resulting dominance vector can be found by calculating $V_1 + xV_2 + yV_3$.

If we choose $x = 0.5$ and $y = 0.25$, the dominance vector would be

$$V_1 + 0.5V_2 + 0.25V_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 2 \\ 3 \\ 1 \end{bmatrix} + 0.25 \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.75 \\ 2.75 \\ 4.25 \\ 2 \end{bmatrix}$$

This would rank the four players as C, A, B, D.

When you compare the final seeding with the initial information, we can see that A and C both won 2 games, but the wins by A were against the lower ranked B and D. Players B and D both won 1 game but B managed to defeat the higher placed C. This justifies the seeding as produced.

WORKED EXAMPLE 20

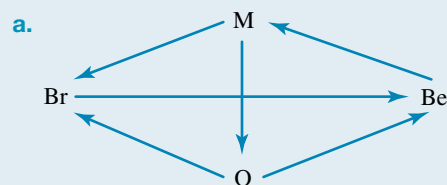
We want to seed 4 badminton players, Oliver, Benjamin, Brooke and Millicent. In past matches, Millicent defeated Oliver and Brooke, both Oliver and Brooke defeated Benjamin, Benjamin defeated Millicent, and Oliver defeated Brooke.

- Construct a digraph to represent this information.
- By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.

THINK

- A diagram represents the matches played in the past. $M \rightarrow O$ indicates that M defeats O.

WRITE



b. 1. The information from the digraph is translated to a dominance matrix, M . The 1s indicate 'defeats'.

2. M^2 helps us investigate the second order influence.

3. Dominance vectors are calculated by adding the elements of each row from the matrices. The resulting dominance vector is the sum of V_1 and V_2 .

4. State the ranked order of players according to the resulting dominance vector, $V_1 + V_2$.

b.

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{Oliver} \\ \text{Benjamin} \\ \text{Brooke} \\ \text{Millicent} \end{matrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

$$V_1 + V_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 5 \end{bmatrix} \begin{matrix} \text{O} \\ \text{Be} \\ \text{Br} \\ \text{M} \end{matrix}$$

The players would be ranked Millicent, Oliver, Benjamin, Brooke.

3.6.2 Leslie matrices

There are many factors that affect the size of populations of animal species. The first of these is the survival rate; that is, the proportion of animals that survive until the next year (or whatever time period is being used). Clearly, a more accurate picture of the survival rate can be obtained if we use survival rates specific to the different age groups. For example, a saltwater crocodile may produce many offspring in a year but not many of the babies survive their first year. However, once they reach 1 year of age it is much more likely that they will get to 2 years of age. A second factor influencing the population of a species is its reproductive rates or how many babies are produced per year. Again, reproductive rates that are specific to particular age groups will yield a more accurate analysis.

To see how matrices can be used to model animal populations, consider the following example.

A wildlife officer is interested in the number of a certain species of rat in a habitat and has compiled the following information on this species.

We have a population breakdown by age:

Age (x) (at the end of the year)	Number (N) (of females)
1	55
2	32
3	25



It is assumed that the rats do not live longer than three years and that there are the same number of males and females.

Fecundity (reproductive) rates, F_x , (the number of female offspring per female per year) have been calculated:

Age (x)	Fecundity (F_x)
1	0.6
2	2.4
3	0.5

The survival rates, S_x , of each age group have also been estimated:

Age (x)	Survival rate (S_x)
1	0.5
2	0.6

Given this information and an initial starting population, it is possible to predict the future population. Note that the 1-year-old rats that survive the year become 2-year-old rats in the next year.

As each age group of females produce babies, the number of 1-year-old rats in the next year can be calculated as:

$$\begin{aligned} \text{Number of 1-year-old rats} = & \text{number of 1-year-olds} \times F_1 \\ & \text{number of 2-year-olds} \times F_2 \\ & \text{number of 3-year-olds} \times F_3 \end{aligned}$$

The number of 2-year-old rats in the next year = $0.5 \times$ number of 1-year-olds.

The number of 3-year-old rats in the next year = $0.6 \times$ number of 2-year-olds.

It is useful to formulate the relationships within a population of rats in any year, and between the populations of rats over several years, using matrices. The population of rats at the end of year 1 can be written:

$$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

A matrix called the **Leslie matrix** can be used to estimate the next year's population given the current population. The Leslie matrix is given by:

$$L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$$

The estimated rat population at the end of the second year is N_2 , where $N_2 = L \times N_1$.

A Leslie matrix has fecundity rates across the top row, survival rates along the sub-diagonal and 0s elsewhere.

$$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix} \text{ where } F_i \text{ is the fecundity rate of the } i \text{ th age group and } S_i \text{ is the survival rate of the}$$

i th age group.

We can also calculate the number of females after any year using the Leslie matrix and the number of females after year 1.

Consider the number of females after years 2, 3, 4, . . . , r .

$$\text{After year 2: } N_2 = L \times N_1$$

$$\text{After year 3: } N_3 = L \times N_2 = L \times L \times N_1 = L^2 \times N_1$$

$$\text{After year 4: } N_4 = L \times N_3 = L \times L^2 \times N_1 = L^3 \times N_1$$

$$\text{After year } r: N_r = L^{r-1} \times N_1$$

In general terms, $N_r = L \times N_{r-1}$ and $N_r = L^{r-1} \times N_1$.

WORKED EXAMPLE 21

Using the Leslie matrix, $L = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}$, and the female rat population in year 1,

$$N_1 = \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}.$$

- a. State the meaning of the number 2.4 in the matrix.
- b. Calculate the female rat population at the start of year 2.
- c. Calculate the total rat population after 6 years, assuming equal numbers of male and female rats.

THINK

- a. The first row of the Leslie matrix gives fecundity rates. The second column refers to 2-year-olds.
- b. 1. The population in the next year is found by multiplying the Leslie matrix by the population this year.
2. Perform the matrix multiplications.

3. Write the answer.

WRITE

- a. The number 2.4 refers to the number of female rats born on average to a 2-year-old rat.

$$\text{b. } N_2 = L \times N_1$$

$$\begin{aligned} N_2 &= \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix} \\ &= \begin{bmatrix} 122.3 \\ 27.5 \\ 19.2 \end{bmatrix} \end{aligned}$$

The total female population is $122.3 + 27.5 + 19.2 = 169$.

- c. 1. Calculate the population after 6 years using the formula $N_r = L^{r-1} \times N_1$.

2. Perform the matrix multiplications.

3. Add the female populations.

4. Calculate the total rat population assuming that there are equal numbers of males and females.

5. Answer the question.

$$c. N_6 = L^5 \times N_1$$

$$N_6 = \begin{bmatrix} 0.6 & 2.4 & 0.5 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0 \end{bmatrix}^5 \begin{bmatrix} 55 \\ 32 \\ 25 \end{bmatrix}$$

$$= \begin{bmatrix} 521.9 \\ 171.9 \\ 73.3 \end{bmatrix}$$

The total female population is
 $521.9 + 171.9 + 73.3 = 767.1$.

Number of males and females $= 2 \times 767.1$
 $= 1534.2$

The total rat population after 6 years is approximately 1534.

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Dominance and Leslie matrices Summary screen and practice questions

Exercise 3.6 Dominance and Leslie matrices

Technology free

1. **WE20** The top four chess players in a tournament are Breanna, Kayley, Teagan and Cameron. In past matches, Cameron defeated Breanna and Teagan, both Breanna and Teagan defeated Kayley, Kayley defeated Cameron, and Breanna defeated Teagan.
 - a. Construct a digraph to represent this information.
 - b. By giving equal importance to first- and second-order influence, use dominance matrices to rank the players.
2. Three friends have noticed that when they played chess, Mair defeated Ann and Janine, and Ann defeated Janine. Use dominance matrices to rank these players.
3. A round-robin netball match was arranged for house competitions where Barnes lost to all but Cunningham, Cunningham lost to Leslie but defeated Hamilton. No teams went undefeated.
 - a. If it is decided to give equal importance to first- and second-order influence, use dominance matrices to rank the students' houses.
 - b. If house points are allocated as 20, 15, 10, 5 for the overall ranking, how many points did each house receive?



4. Five schools are debating in a round-robin tournament — the following table shows the results.

	Clifton	Warwick	Goondiwindi	Stanthorpe	Ipswich
Clifton	—	L	W	W	L
Warwick		—	W	W	L
Goondiwindi			—	L	W
Stanthorpe				—	W

Clifton lost to Warwick and Ipswich and so on.

- Construct a dominance matrix of this information.
 - A total of 15 points is divided in the ratio 5: 4: 3: 2: 1 and awarded according to the ranking of the schools at the end of the tournament. If this division of points is allocated according to the figures produced by the dominance matrix sum $M + 0.8M^2 + 0.5M^3$, list the number of points each school wins.
5. Suppose that only 15% of baby saltwater crocodiles live to 1 year of age and that 60% of 1-year-olds survive to 2 years of age. In one season, 40 crocodile eggs hatch. How many of these survive to:
- 1 year of age?
 - 2 years of age?

Technology active

6. **WE21** The change in the rat population in a certain habitat is described by the following Leslie matrix:

$$L = \begin{bmatrix} 0.7 & 1.4 & 0.7 \\ 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

The initial female rat population is $N_1 = \begin{bmatrix} 33 \\ 18 \\ 8 \end{bmatrix}$.

- State the meaning of the number 1.4 in the matrix.
 - State the meaning of the number 0.5 in the matrix.
 - Calculate the female rat population for year 3.
 - Calculate the total rat population after 9 years, assuming equal numbers of male and female rats.
7. The numbers of a particular species of bird have been monitored over three years and the following Leslie matrix has been developed to model the population. The birds are aged 1, 2 or 3 years.

$$L = \begin{bmatrix} 0 & 3.1 & 2.2 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

- Determine how many new birds do 1-year-old birds produce on average.
- Determine how many new birds do 2-year-old birds produce on average.
- Calculate what proportion of 1-year-old birds survive the first year.
- If the female population in year 1 is $N_1 = \begin{bmatrix} 245 \\ 101 \\ 32 \end{bmatrix}$, calculate the population of female birds in year 3.

8. Consider the following data associated with sheep flocks.

Age (years)	Birth rate	Survival rate	Number (of females) initially
0	0	0.90	450
1	0.50	0.85	2200
2	0.45	0.50	860
3	0.40	0	55

- a. Construct a Leslie matrix for the sheep flock.
b. Use the Leslie matrix to predict the sheep population in 5 years time.
9. On a farm, the following data are recorded for the sheep flock in a particular year.

Age	Number of females at start of year	Number of females at the end of the year	Number of females born to this group
0	330	280	0
1	880	870	350
2	150	0	12

Assume that there are no female sheep aged over 2 years.

- a. Calculate the survival rate of:
i. the 0-year-old group
ii. the 1-year-old group
iii. the 2-year-old group.
- b. Calculate the birth rate for:
i. the 0-year-old group
ii. the 1-year-old group
iii. the 2-year-old group.
- c. Construct a Leslie matrix for the sheep flock.
- d. Use the matrix to predict the sheep population in 3 years time.
10. The Leslie matrix (L) below shows the fecundity rate of a female mouse population while N_1 shows the population in the first year.



$$L = \begin{bmatrix} 0.8 & 2.2 & 0.6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}, N_1 = \begin{bmatrix} 65 \\ 28 \\ 31 \end{bmatrix}.$$

Calculate the total number of rats in the ninth year. It is assumed that there is equal number of male and female rats.

3.7 Applications of matrices

3.7.1 Markov chains and eigenvectors

Andrei Markov was a Russian mathematician whose name is given to a technique that calculates the probability associated with the state of a system as it makes various transitions. It answers questions such as, ‘What is the probability that James will be late to work today given that he was late yesterday?’ or ‘What is the likelihood that the next note in a piece of music will be an A, given that the previous note was a G’? or, ‘What can be said about the long-term behaviour of the piece of music?’.

In Palmdale only two brands of soft drink are sold — Blue or Red. Suppose, at the start of the first month, 50% of the population drank Blue and 50% of the population drank Red. The people of Palmdale do not always stick to the same drink. At the end of the month some of them change to the other drink. Suppose at the end of each month 40% of people who drink Blue change to Red and 70% of those who drink Red change to Blue.

As time goes by the proportions are as follows.



	Start	End of month 1	End of month 2
Blue	50%	$50\% \times 60\% + 70\% \times 50\% = 65\%$	$65\% \times 60\% + 70\% \times 35\% = 63.5\%$
Red	50%	$50\% \times 30\% + 40\% \times 50\% = 35\%$	$35\% \times 30\% + 40\% \times 65\% = 36.5\%$

What is the long-term behaviour of the way in which the market share is distributed? Even the calculation of the behaviour for three months is complicated. The proportion of market share each month is called the **state** of the system. The sequence of states from month to month forms a **Markov chain**.

A **transition matrix**, T , can be used to simplify the calculations. As its name suggests, the matrix assists in the calculation of the transition in drinking habits from one month to the next.

$$\begin{array}{c}
 \text{From} \quad \text{From Red to Blue} \\
 \text{Blue Red} \quad \swarrow \\
 \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{array}{l} \text{Blue} \\ \text{Red} \end{array} \text{ To} \\
 T = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}
 \end{array}$$

Note that the columns of a transition matrix always sum to 1. At any time, the state is represented by the column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$, where x is the proportion of Blue and y is the proportion of Red. Thus, if the initial state, S_0 , is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, then the state after:

$$1 \text{ month, } S_1, \text{ is: } \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$2 \text{ months, } S_2, \text{ is: } \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.635 \\ 0.365 \end{bmatrix}.$$

Often we are interested in the long-term behaviour of the market or the **steady state**, as it is often called.

The steady state will be a value of $\begin{bmatrix} x \\ y \end{bmatrix}$ such that:

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}.$$

This means the market share remains the same from one month to the next.

$$\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$$

In this case

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

gives $x = 0.6x + 0.7y$ and $y = 0.4x + 0.3y$.

Recalling that x and y are proportions that add to 1.

$$x + y = 1$$

Solving the three equations simultaneously gives:

$$\begin{aligned} x &= 0.6364 \\ y &= 0.3636. \end{aligned}$$

That is, the steady state or long-term behaviour of the market is that Blue has 63.64% of the market and Red has 36.36%.

Markov chains and long term behaviour

1. A transition matrix, T , is used to calculate the likelihood of certain states given the states at a previous stage.
2. The sequence of states, S_i , is called a Markov chain.
3. The values of T are arranged as follows:

$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} \text{A} & \text{B} \end{array} \\ \begin{bmatrix} & \\ & \end{bmatrix} & \begin{array}{cc} \text{A} & \text{B} \\ \text{To} & \end{array} \end{array}$$

4. If a system can be in one of r states, the $r \times 1$ column matrix (or vector) p gives the likelihood of each of the states.
5. If a system is initially in a state p after n stages the system is in a state given by $T^n p$.
6. The sum of columns in a transition matrix is 1.
7. The sum of probabilities in the state matrix is 1.
8. The long-term behaviour of the system can be found by considering $T^{50} p$, first checking that T^{51} is the same as T^{50} .

WORKED EXAMPLE 22

The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & x \end{bmatrix}$.

a. Calculate the value of x .



- b. If the state of the system is $\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$, calculate the state of the system 2 stages later, using technology.

THINK

- Form an equation by adding the elements of the third column. The sum of each column must be 1.
1. The state of the system will be $p = T^2 p_0$ where p_0 is the initial state of the system. (Use a graphics calculator to make the calculations, see instructions below.)
2. Write the answer.

WRITE

$$\begin{aligned} a. \quad 0.5 + 0.4 + x &= 1 \\ x &= 0.1 \end{aligned}$$

$$b. \quad p = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2 \begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$

$$\text{The steady state will be } p = \begin{bmatrix} 0.3125 \\ 0.3625 \\ 0.3250 \end{bmatrix}.$$

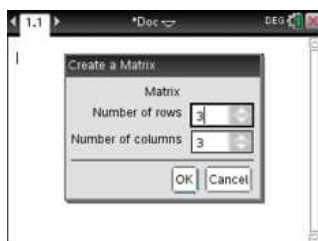
TI | THINK

- On a Calculator page, select

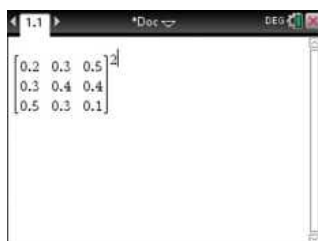


- Complete the entry line as
3
3
Press the OK button.

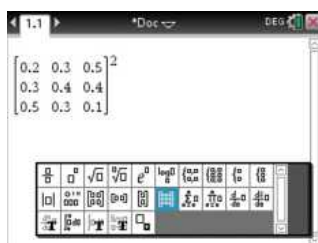
WRITE



- Complete the entry line as
as
 $\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.4 & 0.4 \\ 0.5 & 0.3 & 0.1 \end{bmatrix}^2$



- Press the  button,
followed by the  button.



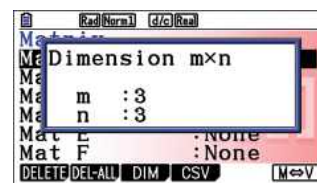
CASIO | THINK

- On a Run-Matrix screen, select
MAT/VCT
To switch between the
vector and matrix screens,
press the M $\leftrightarrow$ V button.

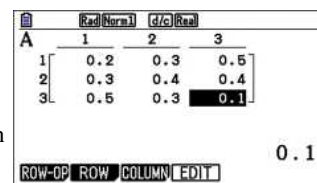
WRITE



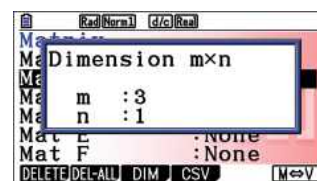
- Press the EXE button.
Complete the entry line as
3
3
Press the EXE button.



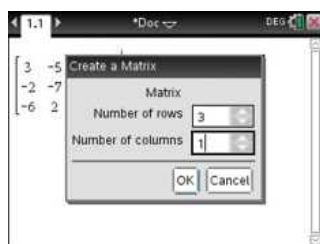
- Complete the entry line as
0.2 0.3 0.5
0.3 0.4 0.4
0.5 0.3 0.1
Press the EXIT button when
finished.



- Highlight the Mat B row
and press the EXE button.
Complete the entry line as
3
1
Press the EXE button.

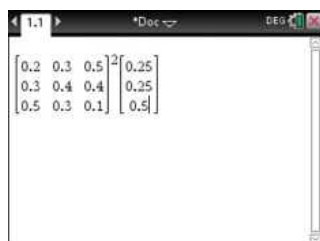


5. Complete the entry line as
3
-2
-6
1
Press the OK button.

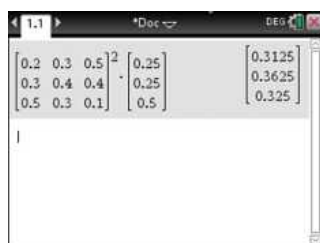


6. Complete the entry line as

$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$



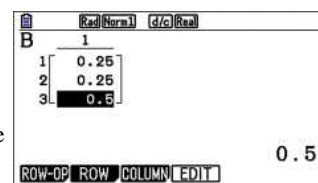
7. Press the ENTER button.
The answer appears on the screen.



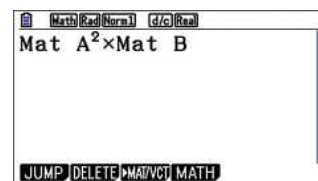
5. Complete the entry line as

$$\begin{bmatrix} 0.25 \\ 0.25 \\ 0.50 \end{bmatrix}$$

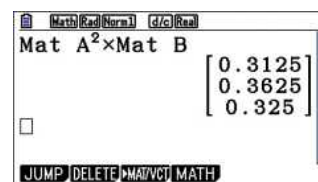
Press the EXIT button twice when finished.



6. On a Run-Matrix screen, select
OPTN,
MAT/VCT
Mat
ALPHA
X,θ,T
^
2
×
Mat
ALPHA
Log



7. Press the EXE button.
The answer appears on the screen.
example.



WORKED EXAMPLE 23

The Nee Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 60%. If it is not raining on a particular day, the chance that it will rain on the following day is 50%.

- Construct a transition matrix to represent this situation.
- Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.
- In the long term, determine what proportion of days it rains in the Nee Islands.

THINK

- Label the columns and rows with the possible states and enter the appropriate values for the transition matrix.
- b. 1.** Define the initial state; that is, define the matrix for the state on Tuesday.

WRITE

- From

Rain	Dry	
$\begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$	Rain	To
	Dry	

$\therefore T = \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}$

- The state on Tuesday (rain) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



2. Find the state on Friday by calculating $T^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where T is the transition matrix. The matrix T is raised to the power of 3 as Friday is three days after Tuesday.

3. Answer the question.

- c. 1. Find the long-term term behaviour by considering a large (say $n = 50$) power of the transition matrix, T .
2. Check that T^{51} is the same. They are the same (to 4 decimal places), which indicates that a steady-state solution is found.
3. Choose a starting value, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and calculate the long-term proportions. *Note: Any starting value could have been used.*

4. Answer the question.

The state on Friday, $\begin{bmatrix} x \\ y \end{bmatrix}$, is given by:

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.556 & 0.555 \\ 0.444 & 0.445 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.556 \\ 0.444 \end{bmatrix}. \end{aligned}$$

The likelihood of rain on Friday is 0.556.

$$\begin{aligned} \text{c. Consider } T^{50} &= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^{50} \\ &= \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix} \\ \text{and } T^{51} &= \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}^{51} \\ &= \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix} \end{aligned}$$

If the starting value is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the long-term

$$\begin{aligned} \text{behaviour is given by } T^{50} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5556 & 0.5556 \\ 0.4444 & 0.4444 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5556 \\ 0.4444 \end{bmatrix}. \end{aligned}$$

So, the long-term probability of rain is 0.5556 and dry is 0.4444.

3.7.2 Eigenvalues

For any $n \times n$ matrix A , if there is a $n \times 1$ column matrix, X , such that $AX = \lambda X$, then X is called an **eigenvector** of A and λ is called an **eigenvalue** of the matrix, A . That is, for matrix A , the eigenvalue λ acts as a scalar on the eigenvector X . It is important to remember that eigenvalues are numbers whereas eigenvectors are vectors.

In the vector equation $AX = \lambda X$, both λ and X are unknowns. To solve this, we need to rewrite the equation as a linear system by expressing λX as λIX .

$$\begin{aligned} AX &= \lambda X \quad [1] \\ AX - \lambda X &= 0 \end{aligned}$$

$AX - \lambda IX = 0$ where I is the identity matrix.

$$(A - \lambda I)X = 0 \quad [2]$$

In order for the linear system to have a non-trivial (nonzero) solution, $A - \lambda I$ must be singular. That is, $\det(A - \lambda I) = 0$. It follows from equation [2] that:

$$\det(A - \lambda I) = 0. \quad [3]$$

The eigenvalues of A are the solutions to the equation $\det(A - \lambda I) = 0$.

If A is a 2×2 matrix, equation [3] is a quadratic equation in λ .

$$\begin{aligned} \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) &= \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \\ &= ad - a\lambda - bc - d\lambda + \lambda^2 \\ &= \lambda^2 - \lambda(a + d) + ad - bc \end{aligned}$$

If A is a 3×3 matrix then equation [3] is an equation in λ of degree 3.

$$\begin{aligned} \det \left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right) &= \det \begin{bmatrix} a - \lambda & b & c \\ d & e - \lambda & f \\ g & h & i - \lambda \end{bmatrix} \\ &= -\lambda^3 + \lambda^2(a + e + i) + \lambda(fh - ei + cg + bd - ea - ai) \\ &\quad + cdh - ecg + bfg - bdi + eai - afh \end{aligned}$$

In general, any $n \times n$ matrix will have n eigenvalues. Additionally, the sum of the diagonal entries of the square matrix is equal to the sum of the eigenvalues (check this in Worked example 24).

Eigenvectors of a matrix, A , are important because they represent objects that are in a sense unchanged by a transformation represented by the matrix, A . In physics, economics and biology, eigenvectors are important features in mathematical modelling. In the previous section on Markov chains, the values for a steady state were to be found by solving the equation $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}$. If we substitute $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and use eigenvector notation, we obtain $X = TX$.

Thus, the steady state, X , is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

Eigenvectors and eigenvalues

For a matrix, A , if there is a column matrix (vector), X , such that:

$$AX = \lambda X$$

then X is called an eigenvector of A and λ is called an eigenvalue of the matrix, A . The steady state, $X = TX$, is an eigenvector corresponding to an eigenvalue, $\lambda = 1$.

WORKED EXAMPLE 24

If $A = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$, determine:

- eigenvalues associated with the matrix
- eigenvectors associated with the matrix.



THINK

a. 1. To find the eigenvalues, λ , substitute matrix values into the expression $A - \lambda I$.

2. Substitute for $A - \lambda I$ in $\det(A - \lambda I) = 0$.

3. Find the determinant and solve for λ .

4. Write the eigenvalues.

b. 1. Recall the relationship linking the eigenvectors with the eigenvalues.

2. Substitute $\lambda = 5$ and $\lambda = -2$.

3. Use $\lambda = 5$ first. Substitute values for matrix A and assign unknowns to matrix X .

4. Substitute for A and X .

Consider the first row of the product.
(You will obtain the same result if the second row is used.)

5. As you cannot determine independent values for a and b , state the relationship between them.

6. Repeat this procedure for $\lambda = -2$. Again, substitute for A and X . Consider the first row of the product only.

7. State the eigenvector for $\lambda = -2$.

8. Answer the question.

WRITE

$$\begin{aligned} \text{a. } A - \lambda I &= \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1 - \lambda & 6 \\ 2 & 2 - \lambda \end{bmatrix} \end{aligned}$$

$$\text{So } \begin{vmatrix} 1 - \lambda & 6 \\ 2 & 2 - \lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(2 - \lambda) - 12 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda - 5)(\lambda + 2) = 0$$

Using the Null Factor Law $\lambda = 5$ and $\lambda = -2$

Thus, the eigenvalues are 5 and -2 .

$$\text{b. } AX = \lambda X$$

$$AX = 5X \text{ or } AX = -2X$$

$$\text{Let } X = \begin{bmatrix} a \\ b \end{bmatrix} \text{ then with } \lambda = 5,$$

$$AX = 5X.$$

$$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 5 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a + 6b = 5a$$

$$6b = 4a$$

$$b = \frac{4a}{6}$$

$$= \frac{2a}{3}$$

$$X \text{ is any vector of the form } \begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a + 6b = -2a$$

$$6b = -3a$$

$$b = \frac{-3a}{6}$$

$$= -\frac{a}{2}$$

$$X \text{ is any vector of the form } \begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}.$$

The eigenvectors for the given matrix are any

$$\text{vectors of the form } \begin{bmatrix} a \\ \frac{2a}{3} \end{bmatrix} \text{ or } \begin{bmatrix} a \\ -\frac{a}{2} \end{bmatrix}.$$

3.7.3 Leontief Matrices

Wassily Leontief was born in St Petersburg in 1906. Although he trained in Russia and Germany his main contribution to economic theory took place while he was a professor of economics at Harvard University in the United States. He received the Nobel Prize for economics in 1973 and died in 1999. Leontief developed input–output analysis, a technique greatly valued by economists involved in planning and development.



Consider an industry or economy that produces energy (E), raw materials (R) and manufactured goods (M).

There is an inter-relationship between these three activities. To produce energy you require raw materials and manufactured goods. Likewise, to produce raw materials requires energy and manufactured goods.

The Leontief model looks at an economy such as this and relates input and output. It answers questions such as: ‘what inputs (how much of each of E , R and M) are needed to produce a certain output?’.

Matrices are used to record the relationship between the industries.

Suppose the relationship can be quantified and represented by a consumption matrix, C .

$$C = \begin{array}{ccc|c} E & R & M & \\ \hline 0.2 & 0.3 & 0.3 & E \\ 0.4 & 0.1 & 0.3 & R \\ 0.3 & 0.4 & 0.1 & M \end{array}$$

This matrix should be interpreted as follows. The number 0.4 in the second row, first column means that every unit of E produced requires 0.4 units of R .

Output and input can be represented using 3×1 matrices, p and q . For example, $p = \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix}$ represents an output of 3 units of E , 4 units of R and 7 units of M .

The input, q , required to give an output, p , can be obtained by using the equation:

$$q = Cp.$$

WORKED EXAMPLE 25

Consider the consumption matrix

$$C = \begin{array}{ccc|c} E & R & M & \\ \hline 0.2 & 0.3 & 0.3 & E \\ 0.4 & 0.1 & 0.3 & R \\ 0.3 & 0.4 & 0.1 & M \end{array}$$



- State the meaning of the value 0.4 in the third row of the matrix.
- Calculate the inputs needed to produce 1 unit of E , 0 units of M and 0 units of R .
- Calculate the inputs needed to produce 2 units of E , 2 units of R and 3 units of M .

THINK

- Consider the row and column for element **0.4** in the third row. The row is linked to M and the column is linked to R .
 1. Write the output as a 3×1 matrix.
 2. Recall the equation relating input to output, $q = Cp$.
 3. Multiply the consumption matrix given, C , by the output matrix p .
 4. State the answer as a sentence.
1. Write the output as a 3×1 matrix.
 2. Recall the equation relating input to output, $q = Cp$.
 3. Multiply the consumption matrix given, C , by the output matrix p .
 4. State the answer as a sentence.

WRITE

- This entry means it takes 0.4 units of M to make 1 unit of R .
- Let $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ because the amount of $E = 1$ with the quantities of R and M being 0.
 $q = Cp$

$$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \end{bmatrix}$$

The input q required is 0.2 units of E , 0.4 units of R and 0.3 units of M .
- $q = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$
 $q = Cp$

$$q = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$$

The input needed is 1.9 units of E , 1.9 units of R and 1.7 units of M .

In the previous worked example, you will notice that the products of the process $p = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ exceeded the inputs $q = \begin{bmatrix} 1.9 \\ 1.9 \\ 1.7 \end{bmatrix}$. This surplus, $p - q = \begin{bmatrix} 0.1 \\ 0.1 \\ 1.3 \end{bmatrix}$, can be used to meet external demand, d .

In a well-balanced economy, it is possible to choose a production, p , such that it matches a given demand, d , and $p - q = d$. Note that, as $q = Cp$, this equation is equivalent to $p - Cp = d$. That is, production less consumption inputs required is equal to demand.

This gives:

$$\begin{aligned}p - Cp &= d \\(I - C)p &= d \\p &= (I - C)^{-1} d\end{aligned}$$

where I is the identity matrix.

You may think that, provided $(I - C)^{-1}$ exists, the equation $p = (I - C)^{-1} d$ ensures that we can always find production levels, p , such that the demand will always be met. This is not always the case, for although the values can be computed in theory, in practice some of the elements of p may turn out to be negative. This would not make sense and thus the process would fail for that value of d .

WORKED EXAMPLE 26

Given the consumption matrix $C = \begin{bmatrix} E & R & M \\ 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \begin{matrix} E \\ R \\ M \end{matrix}$

- a. Determine the production, p , correct to 2 decimal places, that will match the demand $d = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.
- b. Demonstrate that it is always possible to find values for production that will exactly match a given demand, d .

THINK

- a. 1. Recall the production equation $p = (I - C)^{-1} d$ to determine production, given demand.
2. Substitute values for C and d .
3. Determine p using a graphics calculator.
4. Write the answer.

WRITE

a. $p = (I - C)^{-1} d$.

$$p = \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.4 & 0.1 \end{bmatrix} \right]^{-1} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$$

$$p = \begin{bmatrix} 19.6875 \\ 17.1875 \\ 18.6458\bar{3} \end{bmatrix}$$

When the demand is:

$$E = 5, R = 2 \text{ and } M = 4$$

the production that exactly matches it is:

$$E = 19.69, R = 17.19 \text{ and } M = 18.65.$$

- b. 1. Use the equation, $p = (I - C)^{-1} d$ to find p .
2. Recognise that the values of production need to be positive.

b. $(I - C)^{-1} = \begin{bmatrix} 2.40 & 1.35 & 1.25 \\ 1.56 & 2.19 & 1.25 \\ 1.49 & 1.42 & 2.08 \end{bmatrix}$

As long as $(I - C)^{-1}$ exists and has positive entries, it will be always possible to find values of p to match the demand, d .

The next worked example uses a two-industry economy and illustrates how we can form a consumption matrix from information about the economy.

WORKED EXAMPLE 27

Consider the following two industry input–output models. To produce one tonne of coal requires 0.2 tonnes of coal and 0.3 tonnes of steel. To produce one tonne of steel requires 0.4 tonnes of coal and 0.2 tonnes of steel.

- Construct the consumption matrix for this economy.
- Determine the production required to meet the external demand of 80 tonnes of coal and 20 tonnes of steel.

THINK

- Construct the consumption matrix using C and S across the top and down the side, where C is coal and S is steel.
1. Recall the formula relating production, p , to demand, d and substitute the consumption matrix calculated in **a.** and the external demand given, d .

- Write the answer in words.

WRITE

$$\begin{array}{cc} & \begin{array}{cc} C & S \end{array} \\ \begin{array}{c} C \\ S \end{array} & = \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \begin{array}{c} C \\ S \end{array} \end{array}$$

$$\begin{aligned} \text{b. } p &= (I - C)^{-1} d \\ &= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \right]^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix} \\ &= \begin{bmatrix} 0.8 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 80 \\ 20 \end{bmatrix} \\ &= \begin{bmatrix} 138.5 \\ 76.9 \end{bmatrix} \end{aligned}$$

The production that meets the demand is 138.5 tonnes of coal and 76.9 tonnes of steel.

3.7.4 Cryptology

Cryptology is the science of rendering a communication meaningless to anyone intercepting the message. It is an area of mathematics that has grown rapidly, and is used on a daily basis through secure messaging apps and internet banking.

Enciphering is the coding of a message (plain text) into a form that is meaningless to a third party. **Deciphering** is the reverse process; taking the code and translating it back into the original plain text message.

Consider the simplest of ciphers — a mono-alphabetic substitution; that is, the substitution of one letter for another. A cipher such as this is easy to crack if one has enough text to work with. A disadvantage of substitution ciphers is that they preserve the frequencies of individual letters, making it relatively easy to break the code by statistical methods. The percentage frequency of each letter in English correct to one decimal place is:

A	B	C	D	E	F	G	H	I	J	K	L	M
8.2	1.5	2.8	4.3	12.7	2.2	2.0	6.1	7.0	0.2	0.8	4.0	2.4
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
6.7	7.5	1.9	0.1	6.0	6.3	9.1	2.8	1.0	2.3	0.1	2.0	0.1

Naturally, coded texts vary in content, and the frequency of letters in a code will not follow these general proportions exactly. However, they do provide a starting point for someone attempting to decipher a text. The appearance of apostrophes and frequent letter associations in common words such as ‘the’ may also provide clues.

Enciphering

In the discussion to follow, we assume that each plain-text and cipher-text letter, except Z, is assigned the numerical value that specifies its position in the standard alphabet.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	0

In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by a procedure that involves modulo arithmetic. Modulo arithmetic plays an important role in cryptology.

Let us consider this first.

Recall that we say:

$$45 \equiv 3 \text{ modulo } 7, \text{ because the remainder when 45 is divided by 7 is 3.}$$

This statement is usually abbreviated to:

$$45 \equiv 3 \pmod{7}.$$

When operating in mod 7 there are only 7 distinct numbers 0, 1, 2, 3, 4, 5 and 6. Every other number can be linked to one of these remainders.

As there are 26 letters in the English alphabet, modulo 26 is frequently used in cryptology.

For example, $87 = 9 \pmod{26}$. This means that the remainder when 87 is divided by 26 is 9. Another way of looking at this is $87 = 26 \times 3 + 9$.

For negative numbers, we still look for a positive remainder. For example, $-38 = 14 \pmod{26}$. That is, $-38 = 26 \times -2 + 14$.

For modulo 26, $Z_{26} = \{0, 1, 2, \dots, 25\}$.

If a is a number in Z_m , then a number, a^{-1} , in Z_m is called a *reciprocal* or *multiplicative inverse* of a modulo m if $aa^{-1} = a^{-1}a = 1 \pmod{m}$.

Not every element of Z_m will have a multiplicative inverse.

The easiest way to search for a numbers inverse is to use the multiplication table and search along its row for the number 1.

- For instance, if we choose the number 2 in mod 26, and then searched its row for the number 1, we will not find it.
- The reciprocal of 3 in mod 26 is 9 because the product of the two numbers is 1.

The results for multiplication modulo 26 are listed in the table below.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	2	4	6	8	10	12	14	16	18	20	22	24	0	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	1	4	7	10	13	16	19	22	25	2	5	8	11	14	17	20	23
4	4	8	12	16	20	24	2	6	10	14	18	22	0	4	8	12	16	20	24	2	6	10	14	18	22
5	5	10	15	20	25	4	9	14	19	24	3	8	13	18	23	2	7	12	17	22	1	6	11	16	21
6	6	12	18	24	4	10	16	22	2	8	14	20	0	6	12	18	24	4	10	16	22	2	8	14	20
7	7	14	21	2	9	16	23	4	11	18	25	6	13	20	1	8	15	22	3	10	17	24	5	12	19
8	8	16	24	6	14	22	4	12	20	2	10	18	0	8	16	24	6	14	22	4	12	20	2	10	18
9	9	18	1	10	19	2	11	20	3	12	21	4	13	22	5	14	23	6	15	24	7	16	25	8	17
10	10	20	4	14	24	8	18	2	12	22	6	16	0	10	20	4	14	24	8	18	2	12	22	6	16
11	11	22	7	18	3	14	25	10	21	6	17	2	13	24	9	20	5	16	1	12	23	8	19	4	15
12	12	24	10	22	8	20	6	18	4	16	2	14	0	12	24	10	22	8	20	6	18	4	16	2	14
13	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13	0	13
14	14	2	16	4	18	6	20	8	22	10	24	12	0	14	2	16	4	18	6	20	8	22	10	24	12
15	15	4	19	8	23	12	1	16	5	20	9	24	13	2	17	6	21	10	25	14	3	18	7	22	11
16	16	6	22	12	2	18	8	24	14	4	20	10	0	16	6	22	12	2	18	8	24	14	4	20	10
17	17	8	25	16	7	24	15	6	23	14	5	22	13	4	21	12	3	20	11	2	19	10	1	18	9
18	18	10	2	20	12	4	22	14	6	24	16	8	0	18	10	2	20	12	4	22	14	6	24	16	8
19	19	12	5	24	17	10	3	22	15	8	1	20	13	6	25	18	11	4	23	16	9	2	21	14	7
20	20	14	8	2	22	16	10	4	24	18	12	6	0	20	14	8	2	22	16	10	4	24	18	12	6
21	21	16	11	6	1	22	17	12	7	2	23	18	13	8	3	24	19	14	9	4	25	20	15	10	5
22	22	18	14	10	6	2	24	20	16	12	8	4	0	22	18	14	10	6	2	24	20	16	12	8	4
23	23	20	17	14	11	8	5	2	25	22	19	16	13	10	7	4	1	24	21	18	15	12	9	6	3
24	24	22	20	18	16	14	12	10	8	6	4	2	0	24	22	20	18	16	14	12	10	8	6	4	2
25	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1

From the table we can see that 9 is the multiplicative inverse of 3 (mod 26) as $9 \times 3 = 1 \pmod{26}$.
That is, $3^{-1} \pmod{26} = 9$.

WORKED EXAMPLE 28

Simplify each of the following.

a. $200 \bmod 26$

b. $5^{-1} \bmod 26$

c. $-9 \bmod 26$

THINK

- Since $\bmod 26$ refers to the remainder when dividing by 26, divide 200 by 26 and find the remainder.
- Use the table to find the multiplicative inverse of 5. 5^{-1} means $5^{-1} \times 5 = 1$.
- Divide -9 by 26 and find the positive remainder. Use $-9 = -1 \times 26 + 17$ rather than $-9 = 0 \times 26 - 9$.

WRITE

- $200 \div 26 = 7$ remainder 18.
 $200 \bmod 26 = 18$.
- From the table, $5 \times 21 = 1 \pmod{26}$.
Therefore $5^{-1} \bmod 26 = 21$.
- $-9 \div 26 = -1$ remainder 17.
Therefore $-9 \bmod 26 = 17$.

Now we can use this to transform plain text to cipher text. In the simplest ciphers, successive *pairs* of plain text are transformed into cipher text by the following procedure

Step 1 Choose a 2×2 matrix with integer entries $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This matrix will be used to perform the encoding.

Step 2 Group successive plain-text letters into pairs, adding an arbitrary 'dummy' letter to fill out the pair if the plain text has an odd number of letters, and replace each plain-text letter by its last numerical value.

Step 3 Successively convert each plain-text pair, p_1p_2 , into a column matrix: $P = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, and form the product AP . We will call P a *plain-text* matrix and AP the corresponding *cipher-text* matrix.

Step 4 Convert each cipher-text matrix into its alphabetic equivalent using modulo 26 if the numbers are greater than 26.

WORKED EXAMPLE 29

Use the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ to encipher the message: THE GAME IS UP.

THINK

- Group the letters in pairs, doubling the last if there is an odd number.
- Convert each letter to a number from 0 to 25. Use the table on page 171.
- Premultiply the first pair of numbers by the given cipher matrix.
- Convert each number in the resulting matrix to module 26; that is, find the remainder when each number is divided by 26. In this case, $48 = 22 \pmod{26}$ and $36 = 10 \pmod{26}$.

WRITE

TH EG AM EI SU PP

20 8 5 7 1 13 5 9 19 21 16 16

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 36 \end{bmatrix} \\ = \begin{bmatrix} 22 \\ 10 \end{bmatrix} \pmod{26}$$



5. Repeat steps 3 and 4 for the remaining pairs.

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} &= \begin{bmatrix} 17 \\ 19 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \end{bmatrix} \pmod{26} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 13 \end{bmatrix} &= \begin{bmatrix} 15 \\ 27 \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \end{bmatrix} \pmod{26} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} &= \begin{bmatrix} 19 \\ 23 \end{bmatrix} = \begin{bmatrix} 19 \\ 23 \end{bmatrix} \pmod{26} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 19 \\ 21 \end{bmatrix} &= \begin{bmatrix} 59 \\ 61 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \pmod{26} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 16 \\ 16 \end{bmatrix} &= \begin{bmatrix} 48 \\ 48 \end{bmatrix} = \begin{bmatrix} 22 \\ 22 \end{bmatrix} \pmod{26} \end{aligned}$$

So 20 8 5 7 1 13 5 9 19 21 16 16 becomes
22 10 17 19 15 1 19 23 7 9 22 22

6. Convert each number to a letter of the alphabet using the table under Enciphering.

The enciphered message is
V J Q S O A S W G I V V.

Because the plain text was grouped in pairs and enciphered by a 2×2 matrix, the cipher in the previous worked example is referred to as a *2-cipher*. It is possible to group the plain text in triples and encipher by a 3×3 matrix with integer entries; this is called a *3-cipher*. In general, for an *n-cipher*, plain text is grouped into sets of n letters and enciphered by an $n \times n$ matrix with integer entries.

Deciphering

To decipher a message that has been encrypted using matrix A , we use the inverse of A , A^{-1} . That is, for the enciphering 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the deciphering 2×2 matrix, A^{-1} is given by

$$A^{-1} = (ad - bc)^{-1} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \pmod{26}$$

where $(ad - bc)^{-1}$ is the multiplicative inverse of $(ad - bc)$ in modulo 26.

Note that not all numbers have an inverse in mod 26. The inverse of 3 is 9 because $9 \times 3 = 27 = 1 \pmod{26}$; but there is no inverse for 2 or 4 or 13. (See the table on page 174.)

Note also that the inverse of -3 is different from the inverse of 3. The inverse of 3 is 9 but $-3 \times 9 = -27 = 25 \pmod{26}$. The inverse of -3 is 17 as $-3 \times 17 = -51 = 1 \pmod{26}$.

WORKED EXAMPLE 30

Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

DPOOOHZZVOQOVYYVQQ

THINK

1. To find the inverse matrix, A^{-1} , first calculate the determinant of matrix A .

WRITE

$$\begin{aligned} \det A &= 2 \times 2 - 1 \times 1 \\ &= 3 \end{aligned}$$

2. Find the multiplicative inverse (mod 26) of $\det A$ using the relationship that $(\det A)^{-1} \times (\det A) = 1$ for modulo 26. (Use the table.)
3. Find A^{-1} .

From the table, the multiplicative inverse of 3 is 9.

$$9 \times 3 = 27 = 1 \pmod{26}$$

$$\begin{aligned} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} &= 9 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 18 & -9 \\ -9 & 18 \end{bmatrix} \\ &= \begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix} \end{aligned}$$

4. Convert the elements in the resulting matrix to modulo 26.
5. Convert the cipher text to a string of numbers. (Use the table.) Group the numbers in pairs.
6. Multiply the first pair of numbers by the deciphering matrix $\begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix}$.
7. Convert the elements in the resulting matrix to modulo 26.
8. Repeat steps 6 and 7 for all pairs of numbers. List the decoded numbers.
9. Convert the decoded numbers to text.

D	P	O	O	O	H	Z	Z	V
4	16	15	15	15	8	0	0	22
O	Q	O	V	Y	Y	V	Q	Q
15	17	25	22	25	25	22	17	17

$$\begin{aligned} \begin{bmatrix} 18 & 17 \\ 17 & 18 \end{bmatrix} \begin{bmatrix} 4 \\ 16 \end{bmatrix} &= \begin{bmatrix} 344 \\ 356 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 18 \end{bmatrix} \pmod{26} \end{aligned}$$

4 16 15 15 15 8 0 0 22 15 17 25 22 25 25 22 17 17
 gives
 6 18 5 5 16 9 0 0 1 20 15 13 15 18 18 15 23 23
 This gives FREEPIZZATOMORROWW and
 can be interpreted as: Free Pizza Tomorrow

studyon

Units 3 & 4 > Area 2 > Sequence 2 > Concept 6

Applications of matrices Summary screen and practice questions

Exercise 3.7 Applications of matrices

Technology active

1. **WE22** The transition matrix for a system is $\begin{bmatrix} 0.3 & 0.2 & 0.50 \\ 0.3 & 0.2 & 0.25 \\ 0.4 & x & 0.25 \end{bmatrix}$.
 - a. Calculate the value of x .

- b. If the current state of the system is $\begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$, calculate the state of the system at each of the following stages:
- i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.
- c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:
- i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.
- d. If the current state of the system is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, calculate the state of the system at each of the following stages:
- i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.
2. The transition matrix for a system is $\begin{bmatrix} 0.2 & 0.2 & c \\ 0.3 & b & 0.25 \\ a & 0.1 & 0.20 \end{bmatrix}$.
- a. Calculate the values of a, b and c .
- b. If the current state of the system is $\begin{bmatrix} 0.1 \\ 0.2 \\ 0.7 \end{bmatrix}$, calculate the state of the system after each of the following stages:
- i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.
- c. If the current state of the system is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, calculate the state of the system for each of the following stages:
- i. 1 stage later ii. 2 stages later iii. 3 stages later iv. 4 stages later.
3. **WE23** The Robinson Islands are very wet. If it is raining on a particular day, the chance that it will rain the next day is 70%. If it is not raining on a particular day, the chance that it will rain on the following day is 52%.
- a. Construct a transition matrix to represent this situation.
- b. Using the transition matrix, calculate the probability that, if it is raining on Tuesday, it will also be raining on Friday of the same week.
- c. Using technology, determine the long term proportion of days it rains in the Nee Islands.
4. **WE24** If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$, determine the:
- a. eigenvalues associated with the matrix.
- b. eigenvectors associated with the matrix.
5. If $A = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$, determine the:
- a. eigenvalues associated with the matrix.
- b. eigenvectors associated with the matrix.
6. If matrix T is a 2×2 transition matrix, then the columns of T will each add to give 1. Show that $\lambda = 1$ is always an eigenvalue for such a transition matrix.

7. a. Show that the eigenvalues of a matrix, $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ are found by solving the following equation for λ :

$$\lambda^2 - 5\lambda - 2 = 0$$

- b. Show that the matrix A is a solution to the following equation.

$$A^2 - 5A - 2I = 0$$

8. **WE25** Given the consumption matrix $C = \begin{matrix} & \begin{matrix} G & H & T \end{matrix} \\ \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} & \begin{matrix} G \\ H \\ T \end{matrix} \end{matrix}$,

- a. state the meaning of the value 0.6 in the third row of the matrix
b. calculate the inputs needed to produce: 1 unit of H
c. calculate the inputs needed to produce 12 units of G , 15 units of H and 30 units of T .

9. Given the consumption matrix $C = \begin{matrix} & \begin{matrix} G & H & T \end{matrix} \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} & \begin{matrix} G \\ H \\ T \end{matrix} \end{matrix}$, calculate the inputs needed to produce:

- a. 3 units of G .
b. 8 units of G , 25 units of H and 10 units of T .

10. **WE26** Consider the consumption matrix:

$$C = \begin{matrix} & \begin{matrix} G & H & T \end{matrix} \\ \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.3 \\ 0.6 & 0.1 & 0.1 \end{bmatrix} & \begin{matrix} G \\ H \\ T \end{matrix} \end{matrix}$$

- a. Calculate the production needed to match the demand $d = \begin{bmatrix} 30 \\ 45 \\ 25 \end{bmatrix}$.
b. Show that a suitable production can be found to match any demand.

11. Consider the following three-industry economy:

The external demands for A , B and C are 20, 30 and 10 respectively. The exchange matrix is:

$$M = \begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.1 & 0.1 & 0.2 \\ 0.5 & 0.6 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{matrix}$$

- a. What is the meaning of the entry 0.5 in the matrix?
b. Calculate the required output for each of the three industries.
12. **WE27** A two-industry economy consists of coal and steel. To produce one tonne of coal requires 0.3 tonnes of coal and 0.4 tonnes of steel. To produce one tonne of steel requires 0.45 tonnes of coal and 0.2 tonnes of steel.
- a. Construct the consumption matrix.
b. What production is required to meet the demand of 45 tonnes of coal and 25 tonnes of steel?



13. Consider the following hypothetical three-industry economy.

To manufacture 1 unit of A requires	0.2 units of A	0.3 units of B	0.1 units of C
To manufacture 1 unit of B requires	0.1 units of A	0.2 units of B	0.2 units of C
To manufacture 1 unit of C requires	0.2 units of A	0.1 units of B	0.3 units of C

- a. Construct the consumption matrix for this economy.
b. What inputs are required to produce 10 units of A , 20 units of B and 60 units of C ?
c. How much of each should be produced to meet the external demand of 45 units of A , 30 units of B and 40 units of C ?
14. **WE28** Simplify each of the following.
a. $300 \bmod 26$ b. $7^{-1} \bmod 26$ c. $-14 \bmod 26$ d. $200 \bmod 26$ e. $8^{-1} \bmod 26$
f. $-5 \bmod 26$ g. $431 \bmod 26$ h. $15^{-1} \bmod 26$ i. $-15 \bmod 26$
15. **WE29** Encode the message: LEAVE IMMEDIATELY using the enciphering matrix $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$.
16. Encode the message: THE IDES OF MARCH using the enciphering matrix $\begin{bmatrix} 2 & 5 \\ 1 & 5 \end{bmatrix}$.
17. The matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ has been used to encrypt the following message:
O L Y P X F R K R M E L.
a. What is $A^{-1} \bmod 26$?
b. Use this matrix to decipher the message.
18. **WE30** Decode this cipher text that has been encoded using the matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$
I V W L A E D C O U E D V W K R Y R.
19. Encode the message: ALAN TURING IS A LEGEND using a 3×3 enciphering matrix $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$.

3.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple Familiar

1. Given the following systems of linear equations,
a. $7x + 2y - 4z = 24$ b. $2x - y + 3z = 25$
 $x - 8y + 2z = 22$ $3x + 5y - 4z = -3$
 $8x + 2y - 5z = 26$
i. state the number of m equations and n variables in the system
ii. justify whether the system is square or not
iii. determine if $x = 6, y = -1, z = 4$ is a valid solution set all equations in the system.
2. **MC** Identify which of the following matrices is row echelon form.
A. $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -7 \end{bmatrix}$ B. $\begin{bmatrix} 0 & -1 & -2 \\ 1 & 2 & 5 \end{bmatrix}$ C. $\begin{bmatrix} 4 & 3 & 3 \\ 0 & 2 & 8 \end{bmatrix}$ D. $\begin{bmatrix} 1 & 9 & 4 \\ 0 & 1 & 7 \end{bmatrix}$
3. State if the following are *true* or *false*.
a. A consistent system could have an infinite number of solutions.
b. A linear system in three unknowns can have no solution.
c. A line is formed when three planes intersect. This system is inconsistent.

- d. Two parallel lines in three dimensions are distinct. This system is inconsistent.
- e. A 3×3 linear system is consistent because $\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) \neq 0$.
4. **MC** A system consists of the following equations.

$$\begin{aligned}a_1x_1 + b_1y_1 + c_1z_1 &= d_1 \\a_2x_2 + b_2y_2 + c_2z_2 &= d_2 \\a_3x_3 + b_3y_3 + c_3z_3 &= d_3\end{aligned}$$

The solution for this system is $x = -2 + 4t, y = 3 + 8t, z = 4 - t$ where $t \in \mathbb{R}$. In geometric terms, the system and solution are best described by which of the following:

- A. It is an inconsistent system.
- B. It is a consistent system containing three planes. The planes intersect with the solution forming a line. Infinitely many solutions exist.
- C. It is a consistent system containing three planes. The planes intersect at the unique position defined by $(x, y, z) = (-2, 3, 4)$
- D. It is an inconsistent system containing three planes. Two of the planes are parallel and distinct.
5. Consider $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -4 \\ -3 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, when calculating if the following statements

are true or false.

a. $A^2(A + B) = A^3 + AB$

b. $(A^3 - B^3)A^3 = A^3A^3 - A^3B^3$

6. In a round robin tennis competition, Tim recorded two losses to Nelly and Peter. Nelly's only loss was to Peter. Peter lost to Earnest. Use dominance matrices and the formula $M + 0.5M^2$ to rank the players.
7. Brooke Cove is a very monsoonal area with prevalent heavy rainfall. If it is raining on a particular day, the chance that it will rain the next day is 89%. If it is not raining on a particular day, the chance that it will rain on the following day is 71%.



- a. Construct a transition matrix to represent this situation.
- b. Using the transition matrix, calculate the probability that, if it is raining on Wednesday, it will also be raining on Saturday of the same week.
- c. In the long term, determine on what proportion of days does it rain in the Brooke Cove.
8. Solve the following system of linear equations by using an inverse matrix.

$$\begin{aligned}7x + 2y + 5z &= 17 \\3x + 8y + z &= 9 \\2x - 7y + z &= -15\end{aligned}$$

9. Solve the following system of linear equations using Gaussian elimination and back substitution.
- a. $6x - 12y = -18$
 $-2x + 7y = 12$
- b. $-2x + 6y + z = -14$
 $-x + 6y - 3z = -4$
 $-30y - 4z = -30$

10. Determine the unique solution for the system of linear equations defined by the lines l_1 and l_2 . Describe the geometric nature of the solution.

$$\begin{aligned}l_1: & (-3\hat{i} + 3\hat{j} - 7\hat{k}) + k(6\hat{i} + 3\hat{j} + 6\hat{k}) \\l_2: & (6\hat{i} + 9\hat{j} - 6\hat{k}) + k(-3\hat{i} - 3\hat{j} + 5\hat{k})\end{aligned}$$

11. Consider the following system,

$$-6x + 2y - 5z = -22$$

$$8x - 3y - 4z = 49$$

$$-3x - 7y - 4z = -48$$

- Determine the number of possible solutions by examining the normal vectors algebraically.
 - Convert the equations to an augmented matrix and then use technology to solve the system.
 - Describe how the solution can be interpreted geometrically.
12. Consider the following system of linear equations,

$$2x + 4y + 12z = 22$$

$$8x + 3y - 30z = 23$$

$$8x - 4y - 72z = -12$$

- Determine the number of possible solutions by examining the normal vectors algebraically.
- Convert the equations to an augmented matrix and then use technology to reduce it to upper triangular form.
- Solve the system of linear equations, expressing the solution in parametric form.
- Describe how the solution can be interpreted geometrically.

Complex familiar

13. Determine a general solution for the variable matrix X in the following matrix equations. (It is assumed that square matrices are non-singular).
- a. $2X - 3B = C$ b. $BX - 2X + C = 0$ c. $A(X + I) + 2C = B$ d. $XA + B = XC - 2D$
14. The sum of three numbers is 10. When triple the first number is added to the sum of the second and third number the result is 26. The second number is 6 less than the third. Develop a suitable matrix equation and apply Gaussian elimination techniques to determine a unique solution.
15. Determine if the following systems are either consistent or inconsistent by completing the 'ref' or 'rref' operation on a graphics calculator.

a. $[A|B] = \left[\begin{array}{ccc|c} 36 & 24 & 16 & 84 \\ -9 & -6 & -4 & -21 \\ 6 & 4 & \frac{8}{3} & 14 \end{array} \right]$

b. $[A|B] = \left[\begin{array}{cc|c} -3 & 6 & 12 \\ 6 & -12 & -40 \end{array} \right]$

16. The numbers of a particular species of lizard, all aged 1, 2 or 3 years, have been monitored over many years and the following Leslie matrix has been developed to model the population.

$$L = \begin{bmatrix} 0 & 5.30 & 3.50 \\ 0.20 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$



- On average, how many new lizards do 1-year-old lizards produce?
- On average, how many new lizards do 2-year-old lizards produce?
- What proportion of 1-year-old lizards survive the first year?
- If the initial female population is $N_1 = \begin{bmatrix} 35 \\ 46 \\ 22 \end{bmatrix}$, calculate the female population in 3 years time.

Complex Unfamiliar

17. Determine the solution/s to the following system where $x, y, z \in R$.

$$7x^3 + 2y^2 + 3z = -35$$

$$2x^3 + 5y^2 - 5z = 24$$

$$3x^3 + y^2 - 4z = -19$$

18. The linear equations below form a consistent system with infinitely many solutions.

$$p_1: -4x - 2y - 4z = -12$$

$$p_2: -x + y + 5z = 9$$

$$p_3: 2x + b_3y + 4z = d_3$$

Calculate the unknowns values, b_3 and d_3 .

19. A consistent linear system is given by:

$$3x + 3y + 6z = 3a$$

$$10x + 10z = 10b$$

$$8x + 4y + 12z = 4c$$

Use Gaussian elimination to determine the relationship between a, b and c that must exist if the initial statement about the system type is true.

20. A study was carried out on a dingo population that inhabited a remote sand island. The population has the following characteristics.

- Half the dingoes survive the first year.
- Of the surviving dingoes, half survive the second year.
- In the first year of life, these dingoes produce no offspring.
- On average, 6 dingoes are produced in the second year and 8 in the third year.
- The current population is given by the initial state matrix,

$$V_1 = \begin{bmatrix} 60 \\ 40 \\ 36 \end{bmatrix} \quad \begin{array}{l} 0 < t \leq 1 \\ 1 < t \leq 2 \\ 2 < t \leq 3 \end{array}$$

- Determine the dingo population in 5 and 10 years' time.
- Compare the populations at both 5 and 10 years to show that a common ratio doesn't exist in terms of the number of dingoes per age group.
- It is possible that a population, consisting of three age groups, will continuously abide by a fixed ratio. For example,

$$X = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \quad \begin{array}{l} 0 < t \leq 1 \\ 1 < t \leq 2 \\ 2 < t \leq 3 \end{array}$$

Determine an alternative for V_1 so that the ratio between age groups remains constant.

studyon

Units 3 & 4 Sit exam

Answers

3 Solving systems of linear equations and the application of matrices

Exercise 3.2 Solving linear equations using matrix algebra

- a. and d. are non-linear.
- a. i. $m = 2, n = 3$
ii. Not a square system as $m \neq n$
iii. Not a valid solution.
- b. i. $m = 3, n = 3$
ii. It is a square system as $m = n$
iii. The solution set satisfies all equations in the system.
- a. $x = 6, y = -2$
b. $x = 3, y = 3$
c. $x = 6, y = 4$
d. $x = -4, y = 8$
- a. $x = -1, y = 0, z = -5$
b. $x = 5, y = 7, z = 5$
c. $x = 6, y = -3, z = 2$
d. $x = -4, y = -5, z = -2$
- 1st place is awarded 10 points
2nd place is awarded 7 points
3rd place is awarded 5 points
4th place is awarded 3 points
- The cost per individual item are:
Lolly bag \$1
Pizza \$8
Meat pie \$5
Hamburger \$6
- C
- B
- $X = (E + I)^{-1}(G - F)$
- a. $X = (A^{-1}C)B^{-1}$
b. $X = BA^{-1}$
c. $X = A^{-1}IB^{-1}$
d. $X = (A + I)^{-1}C$
- a. $X = (A + B)^{-1}(D - C)$
b. $X = (I_2 + 3A)^{-1}B$
c. $X = (A + 3B)^{-1}C$
d. $X = (7A)^{-1}(B + C - 7AI)$

$$12. \text{ a. } X = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{31}{18} & -\frac{19}{9} \end{bmatrix} \quad \text{b. } X = \begin{bmatrix} -\frac{1}{18} & -\frac{7}{9} \\ \frac{4}{9} & \frac{2}{9} \end{bmatrix}$$

$$\text{c. } X = \begin{bmatrix} -\frac{43}{84} & -\frac{53}{84} \\ \frac{11}{28} & -\frac{9}{28} \end{bmatrix} \quad \text{d. } X = \begin{bmatrix} \frac{64}{203} & \frac{78}{203} \\ -\frac{2}{7} & -\frac{2}{7} \end{bmatrix}$$

$$13. \text{ a. } X = \begin{bmatrix} -\frac{5}{8} & -\frac{19}{16} \\ \frac{7}{8} & \frac{1}{16} \end{bmatrix}$$

$$\text{b. } X = \begin{bmatrix} -\frac{7}{20} & \frac{3}{10} \\ \frac{13}{20} & -\frac{11}{30} \end{bmatrix}$$

$$\text{c. } X = \begin{bmatrix} \frac{30}{217} & \frac{11}{31} \\ -\frac{2}{217} & -\frac{9}{31} \end{bmatrix}$$

$$\text{d. } X = \begin{bmatrix} -\frac{31}{45} & \frac{23}{45} \\ \frac{16}{15} & -\frac{14}{5} \end{bmatrix}$$

14. True, LHS = RHS

$$15. \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{4}$$

Exercise 3.3 Solving a system of linear equations using Gaussian elimination

- B
- C
- B
- a. $x = 3, y = 2$
b. $x = -5, y = 1$
c. $x = 2, y = 2$
d. $x = 1, y = -12$
- a. $x = 0, y = -2, z = 0$
b. $x = -2, y = -4, z = 0$
c. $x = 3, y = 5, z = 2$
d. $x = -5, y = 5, z = 4$
e. $x = -3, y = -3, z = -1$
- $w = 2, x = -3, y = 4, z = -5$
- $x = 10, y = 20, z = 30$
- Large = \$5, Regular = \$4.50
- $A = 22, S = 5, P = 8$
Two students, 1 adult and 2 pensioners will cost
Cost = \$48
- $5x + 2y + 4z = 32$
 $-4x + 7y + 8z = -15$
 $-3x + 11y + z = 9$

$$11. y(x) = 2x^2 - 4x + 1 \text{ or } y(x) = 1 - 4x + 2x^2$$

$$12. A^{-1} = \begin{bmatrix} \frac{1}{9} & \frac{5}{3} & -\frac{8}{3} \\ \frac{1}{9} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{6} & -\frac{2}{3} \end{bmatrix}$$

$$13. \left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 3 & 1 & -2 & 4 \\ -5 & 4 & -1 & -4 \end{array} \right] \quad \begin{array}{l} R_2 - 3R_1 \rightarrow R_2 \\ R_3 + 5R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 16 & -5 & 22 \\ 0 & -21 & 4 & -34 \end{array} \right] \quad \begin{array}{l} R_2 \div 16 \rightarrow R_2 \\ 5R_3 - 21R_1 \rightarrow R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & -6 \\ 0 & 1 & \frac{-5}{16} & \frac{11}{8} \\ 0 & 0 & -1 & 44 \end{array} \right]$$

18. a. The parametric equation satisfies all equations in the system. It is a valid solution set.
 b. The parametric equation satisfies all equations in the system. It is a valid solution set.
19. a. Infinitely many solutions
 b. Zero solutions
 c. Unique solution at $P(5, 6, -7)$
20. a. Unique solution at $P(6, -3, 4)$
 b. No solution (line is parallel yet distinct from the plane).
 c. Infinitely many solutions (the line lies on the plane).
21. $(x, y, z) = \left(-\frac{1124}{49}, -\frac{831}{49}, -\frac{416}{49}\right)$

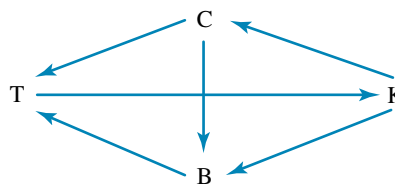
Exercise 3.5 Using technology for matrix calculations

1. a. $\begin{bmatrix} -65 & -139 \\ -119 & -227 \end{bmatrix}$
 b. $\begin{bmatrix} -\frac{84}{5} & \frac{138}{5} \\ -\frac{284}{5} & \frac{536}{5} \end{bmatrix}$
 c. -108
 d. 4
 e. $\begin{bmatrix} -\frac{11}{18} & \frac{1}{6} \\ \frac{5}{54} & -\frac{1}{18} \end{bmatrix}$
 f. $\begin{bmatrix} -\frac{9}{7} & -3 & -11 \\ -\frac{7}{2} & -1 & -4 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$
 g. $\begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix}$
 h. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. a. $\text{LHS} \neq \text{RHS}$
 $\therefore$ Statement is false.
 b. $\text{LHS} \neq \text{RHS}$
 $\therefore$ The statement is false.
3. a. $\begin{bmatrix} 178 & -250 \\ 125 & 303 \end{bmatrix}$
 b. $\begin{bmatrix} 1114 & 586 \\ 343 & 343 \\ 599 & 281 \\ -343 & 343 \end{bmatrix}$
 c. $\begin{bmatrix} 74 & -62 & 30 \\ 30 & -152 & 84 \\ 210 & -48 & 26 \end{bmatrix}$
 d. $\begin{bmatrix} -\frac{108}{5} & -24 & -\frac{121}{5} \\ -\frac{104}{5} & -21 & -\frac{123}{5} \\ -\frac{91}{5} & -\frac{123}{5} & -\frac{108}{5} \end{bmatrix}$
4. a. It is an inconsistent system.
 b. It is an inconsistent system.
 c. It is a consistent system.
 d. It is a consistent system.
5. Total cost for coffee is \$37.16.
 Total cost for tea is \$18.28.
 Total cost for juice is \$10.70.

6. a. $\begin{bmatrix} 71 & 4 \\ 63 & -28 \end{bmatrix}$
 b. $\begin{bmatrix} 45 & 44 \\ -11 & 12 \end{bmatrix}$
7. a. A is nilpotent.
 b. A is nilpotent.
8. a. A is idempotent.
 b. A is idempotent.
9. Statement is true, $\text{LHS} = \text{RHS}$
10. Statement is true, $\text{LHS} = \text{RHS}$
11. The unknown numbers are 8, 10 and 7.
12. a. $c = -\frac{20}{3}$
 b. Statement is true, $\text{LHS} = \text{RHS}$

Exercise 3.6 Dominance and Leslie matrices

1. a.



- b. The players would be ranked Cameron, Breanna, Kayley, Teagan.
2. Players can be ranked Mair, Ann, Janine
3. a. The houses are ranked Hamilton, Leslie, Cunningham, Barnes.
 b. Hamilton received 20 points.
 Leslie received 15 points.
 Cunningham received 10 points.
 Barnes received 5 points.

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} C \\ W \\ G \\ S \\ I \end{matrix}$$

4. a.
 b. Warwick 5 points, Ipswich 4 points, Stanthorpe 3 points, Clifton 2 points, Goondiwindi 1 point
5. a. 6
 b. 3.6
6. a. 1.4 females per year
 b. 50% live to age 3
 c. 91 female rats
 d. 674 (337 females)
7. a. 0 new birds
 b. 3.1 new birds
 c. 0.2 survive
 d. 338 females
8. a. $L = \begin{bmatrix} 0 & 0.50 & 0.45 & 0.4 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.85 & 0 & 0 \\ 0 & 0 & 0.50 & 0 \end{bmatrix}$
 b. 7460 sheep (3730 females)
9. a. i. 0.85
 ii. 0.99
 iii. 0
 b. i. 0
 ii. 0.4
 iii. 0.08
- c. $\begin{bmatrix} 0 & 0.40 & 0.08 \\ 0.85 & 0 & 0 \\ 0 & 0.99 & 0 \end{bmatrix}$
- d. 769 female sheep, 1538 total population.
10. $N_9 = \begin{bmatrix} 2705 \\ 870 \\ 223 \end{bmatrix}$
 7596 (3798 female).

Exercise 3.7 Applications of matrices

- $x = 0.6$
 - $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.392 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.4 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.35 \\ 0.25 \\ 0.40 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.355 \\ 0.255 \\ 0.390 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.353 \\ 0.255 \\ 0.393 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.40 \\ 0.25 \\ 0.35 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.345 \\ 0.258 \\ 0.398 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.354 \\ 0.254 \\ 0.392 \end{bmatrix}$
 - $a = 0.5, b = 0.7, c = 0.55$
 - $P = \begin{bmatrix} 0.455 \\ 0.345 \\ 0.210 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.274 \\ 0.428 \\ 0.299 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.305 \\ 0.456 \\ 0.239 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.284 \\ 0.470 \\ 0.246 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.375 \\ 0.395 \\ 0.230 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.281 \\ 0.447 \\ 0.273 \end{bmatrix}$
 - $P = \begin{bmatrix} 0.296 \\ 0.465 \\ 0.240 \end{bmatrix}$
 - $\begin{bmatrix} 0.7 & 0.52 \\ 0.3 & 0.48 \end{bmatrix}$
 - 0.636 or 63.6%
 - The long-term probability of rain is 0.63415 and dry is 0.36585
 - $\lambda = 7, \lambda = -2$
 - $\begin{bmatrix} a \\ -\frac{4a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ a \end{bmatrix}$
 - $\lambda = -0.2, \lambda = 1$
 - $\begin{bmatrix} a \\ \frac{7a}{5} \end{bmatrix}$ and $\begin{bmatrix} a \\ -a \end{bmatrix}$
 - $\lambda = 1$
 - Sample responses can be found in the worked solutions in the online resources.
 - Sample responses can be found in the worked solutions in the online resources.
 - This entry means it takes 0.6 units of T to make 1 unit of G .
 - The input q is 0.5 units of G , 0.4 units of H and 0.1 units of T .
 - The input q is 17.1 units of G , 17.4 units of H and 11.7 units of T .
 - The input needed is 0.6 units of G , 0.6 units of H and 1.2 units of T .
 - The input needed is 7.6 units of G , 14.6 units of H and 6.7 units of T .

- 10.** **a.** The production that exactly matches the demand is 109 units of G , 171 units of H and 119 units of T .
b. It will always be possible to find values of P to match the demand d .
- 11.** **a.** To produce 1 unit of A requires 0.5 units of C .
b. The required output is 37 units of A , 50 units of B and 59 units of C .
- 12.** **a.** $c = \begin{bmatrix} 0.3 & 0.45 \\ 0.4 & 0.20 \end{bmatrix} \begin{matrix} C \\ S \end{matrix}$
b. The production that meets the demand is 124 tonnes of coal and 93 tonnes of steel.
- 13.** **a.** $C = \begin{bmatrix} 0.2 & 0.1 & 0.2 \\ 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$
b. The input q is 16 units of A , 13 units of B and 23 units of C .
c. The production that meets the demand is 90 units of A , 83 units of B and 94 units of C .
- 14.** **a.** 14 **b.** 15 **c.** 12
d. 18 **e.** Not possible **f.** 21
g. 15 **h.** 7 **i.** 11
- 15.** The enciphered message is
M K P G K X M Z V Y U N C S U G
- 16.** The enciphered message is BHCXGCIPIYSNMTQ
- 17.** **a.** $A^{-1} \text{ mod } 26 = \begin{bmatrix} 8 & 11 \\ 11 & 4 \end{bmatrix}$
b. R E L A X B E H A P P Y or RELAX BE HAPPY
- 18.** YOU SHOULD SEE ENIGMA.
- 19.** CZOAWJRXKKRDOODONEZBPP.

3.8 Review: exam practice

1. a. i. $m = 2, n = 3$
ii. The system is not square because $m \neq n$
iii. The solution is valid for all equations in the system.
- b. i. $m = 3, n = 3$
ii. The system is square because $m = n$
iii. The solution is valid for all equations in the system.
2. D
3. a. True b. True c. False
d. True e. True
4. B
5. a. False: $LHS \neq RHS$ b. False: $LHS \neq RHS$
6. The players are ranked Peter, Nelly, Earnest and Tim.
7. a. $\begin{bmatrix} 0.89 & 0.71 \\ 0.11 & 0.29 \end{bmatrix}$
b. 0.8666
c. The long-term probability of rain is 0.86585 and dry is 0.13415.
8. The unique solution is the point $(x, y, z) = (-6, 2, 11)$.
9. a. The unique solution is the point $(x, y) = (1, 2)$.
b. The unique solution is the point $(x, y, z) = (10, 1, 0)$.
10. A unique solution exists at $(3, 6, -1)$.
The two distinct lines intersect at one unique point.
11. a. The planes are not coplanar. This implies that the planes intersect at a unique point or intersect as pairs of lines.
b. The system is consistent and has one unique solution.
The three planes intersect at $(7, 5, -2)$.
c. The planes intersect at one unique point, $(7, 5, -2)$.

12. a. The three planes are coplanar, and not parallel.
Geometrically, this could be the planes intersecting in a line or not at all.

b. $[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -6 & 1 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right]$

The system has infinitely many solutions because the planes intersect in a line.

c. $x = 1 + 6t$
 $y = 5 - 6t$
 $z = t$

- d. This is a consistent system with the solution representing the intersection of three planes in the form of a line.

13. a. $X = \frac{1}{2}(C + 3B)$

b. $X = -(B - 2)^{-1}C$

c. $X = [A^{-1}(B - 2C)] - I$

d. $X = (-2D - B)(A - C)^{-1}$

14. $(x, y, z) = (8, -2, 4)$

15. a. The system is consistent (infinite number of solutions).

- b. It is an inconsistent system (parallel yet distinct lines).

16. a. 0

b. 5.3

c. 0.2

d. $\begin{bmatrix} 77.4 \\ 64.2 \\ 1.8 \end{bmatrix}$; 143 female lizards

17. $(x, y, z) = (-2, \pm 3, 1)$

18. The line is defined by the parametric equations:

$$x = -1 + t, y = 8 - 4t, z = t$$

$$p_3: 2x + \frac{3}{2}y + 4z = 10$$

$$\therefore b_3 = \frac{3}{2}, d_3 = 10$$

19. The relationship is $0 = a + b - c$

20. a. $X_5 = \begin{bmatrix} 2076 \\ 852 \\ 85 \end{bmatrix}$; Population = 3013

$$X_{10} = \begin{bmatrix} 83\,856 \\ 20\,246 \\ 5382 \end{bmatrix}; \text{Population} = 109\,484$$

b. $\begin{bmatrix} 24t \\ 10t \\ t \end{bmatrix} \neq \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$

The ratios are not equivalent at 5 and 10 years

c. $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$

This is in a ratio of 16: 4: 1. If the initial dingo

population conforms to $X = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix}$, the population

each year that follows will remain in the same ratio.

4 Vector calculus

4.1 Overview

The motion of a particle has been studied since ancient times and has intrigued both mathematicians and physicists. The motion of particles can take many forms, for example, a speck of dust sitting on the outer edge of a rotating vinyl record or a ball travelling toward the ground in a straight line. In this chapter we will examine the motion of particles in both two and three dimensions and how differentiation and integration techniques can be used to analyse the motion. Consider the image below of the young athlete throwing a javelin. The initial speed and the angle at which the javelin is released will both have an effect on the distance the javelin will travel. These variables will be investigated in the study of projectile motion in this chapter.



LEARNING SEQUENCE

- 4.1. Overview
- 4.2. Position vectors as functions of time: circles, ellipses and hyperbolas
- 4.3. Differentiation of vectors
- 4.4. Integration of vectors
- 4.5. Straight line motion with constant and variable acceleration
- 4.6. Projectile motion
- 4.7. Circular motion
- 4.8. Review: exam practice

Fully worked solutions are available for this chapter in the resources section of your eBookPLUS at www.jacplus.com.au.

4.2 Position vectors as functions of time: circles, ellipses and hyperbolas

4.2.1 Position vectors as functions

Parametric equations

The **parametric equations** of a particle as it moves in two dimensions were examined in Chapter 2, Section 2.4. The path traced out by the particle is defined in terms of a third variable, t , which is called a parameter. In the two-dimensional case the path is described by two parametric equations, as both the x - and y -coordinates depend upon the parameter, t .

$$x = x(t) \quad [1]$$

$$y = y(t) \quad [2]$$

The position vector of the particle is given by $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions. This is also called a vector function of the scalar real variable t , where t is called the parameter, and often represents time.

If we can eliminate the parameter from the two parametric equations and obtain an equation of the form $y = f(x)$, this is called an **explicit relationship** or the equation of the path. It may not be possible to obtain an explicit relationship, but often an **implicit relationship** of the form $f(x, y) = 0$ can be formed. The relationship between x and y is called the **Cartesian equation of the path**. Careful consideration must be given to the possible values of t , which then specify the domain (the x -values) and the range (the y -values) of the equation of the path.

Closest approach

Given the position vector of a particle, $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors in the x and y directions, it is possible to determine the position or coordinates of the particle at a given value of t . It is also possible to calculate the value of t and the coordinates when the particle is closest to the origin.

WORKED EXAMPLE 1

A particle moves so that its vector equation is given by $\underline{r}(t) = (3t - 4)\hat{i} + (4t - 3)\hat{j}$ for $t \geq 0$. Determine:

- the distance of the particle from the origin when $t = 2$
- the distance of the particle from the origin at any time t
- the closest distance of the particle from the origin.

THINK

- To determine the position of the particle at $t = 2$, substitute the value of t .
- To calculate the distance of the particle from the origin, determine the magnitude of the vector by applying the formula $|\underline{u}| = \sqrt{x^2 + y^2}$
- State the distance from the origin at this time.

WRITE

$$\begin{aligned} \text{a. } \underline{r}(2) &= (6 - 4)\hat{i} + (8 - 3)\hat{j} \\ &= 2\hat{i} + 5\hat{j} \\ \text{At } t = 2 \text{ the particle is at } (2, 5). \\ |\underline{r}(2)| &= \sqrt{2^2 + 5^2} \\ |\underline{r}(2)| &= \sqrt{29} \text{ units} \end{aligned}$$

- b. 1.** Determine the magnitude of the vector at time t by applying the formula
 $|r| = \sqrt{x^2 + y^2}$
- 2.** Expand and simplify to state the distance in terms of t .
- c. 1.** For the closest (or minimum distance), we need to use calculus.
- 2.** Use the chain rule.
- 3.** Solve for the value of t .
- 4.** Determine the position vector at this value.
- 5.** Determine the magnitude of the vector at this time, which represents the closest distance of the particle from the origin.
- 6.** State the final result.

b. $\underline{r}(t) = (3t - 4)\hat{i} + (4t - 3)\hat{j}$
 $|\underline{r}(t)| = \sqrt{(3t - 4)^2 + (4t - 3)^2}$
 $|\underline{r}(t)| = \sqrt{9t^2 - 24t + 16 + 16t^2 - 24t + 9}$
 $= \sqrt{25t^2 - 48t + 25}$

c. The minimum distance occurs when
 $\frac{d}{dt} (|\underline{r}(t)|) = 0.$
 $|\underline{r}(t)| = (25t^2 - 48t + 25)^{\frac{1}{2}}$
 $\frac{d}{dt} (|\underline{r}(t)|) = \frac{\frac{1}{2} \times (50t - 48)}{\sqrt{25t^2 - 48t + 25}} = 0$
 $50t - 48 = 0$
 $t = \frac{24}{25}$
 Substitute $t = \frac{24}{25}$:
 $\underline{r}\left(\frac{24}{25}\right) = \left(3 \times \frac{24}{25} - 4\right)\hat{i} + \left(4 \times \frac{24}{25} - 3\right)\hat{j}$
 $= -\frac{28}{25}\hat{i} + \frac{21}{25}\hat{j}$
 $|\underline{r}(t)|_{\min} = \sqrt{\left(-\frac{28}{25}\right)^2 + \left(\frac{21}{25}\right)^2}$
 $|\underline{r}(t)|_{\min} = \frac{7}{5}$

Collision problems

There are a number of problems that can be formulated around the motion of two moving particles on different curves in two dimensions.

1. Do the particles collide? For two particles to collide, they must be at exactly the same coordinates at exactly the same time.
2. Do the paths of the particles cross without colliding? This will happen when they are at the same coordinates but at different times.
3. What is the distance between the particles at a particular time? To determine this, the magnitude of the difference between their respective position vectors must be found.



WORKED EXAMPLE 2

Two particles move so that their position vectors are given by $\underline{r}_A(t) = (3t - 8)\hat{i} + (t^2 - 18t + 87)\hat{j}$ and $\underline{r}_B(t) = (20 - t)\hat{i} + (2t - 4)\hat{j}$ for $t \geq 0$. Determine:

- when and where the particles collide
- where their paths cross
- the distance between the particles when $t = 10$.

THINK

1. Equate the $\hat{i}$ components for each particle.

2. Solve this equation for t .

3. Equate the $\hat{j}$ components for each particle.
4. Solve this equation for t .

5. Evaluate the position vectors at the common time, $t = 7$.

6. State the result for when the particles collide.
1. The particles paths cross at different values of t . Introduce a different parameter, s .
2. Equate the $\hat{i}$ components for each particle.
3. Solve this equation for t .
4. Equate the $\hat{j}$ components for each particle.
5. Solve this equation for s by substituting $t = 28 - 3s$.

6. Note that when $s = 7$, the particles collide, so use $s = 5$.
7. Determine the time when the paths cross by solving for t when $s = 5$.
8. Evaluate the position vectors at the required time, $t = 13$.
9. State the required result.

WRITE

- $$\underline{r}_A(t) = (3t - 8)\hat{i} + (t^2 - 18t + 87)\hat{j}$$

$$\underline{r}_B(t) = (20 - t)\hat{i} + (2t - 4)\hat{j}$$

$$3t - 8 = 20 - t$$

$$4t = 28$$

$$t = 7$$

$$t^2 - 18t + 87 = 2t - 4$$

$$t^2 - 20t + 91 = 0$$

$$(t - 7)(t - 13) = 0$$

$$t - 7 = 0 \quad \text{or} \quad t - 13 = 0$$

$$\therefore t = 7, 13$$

The common solution is when $t = 7$.

$$\underline{r}_A(7) = 13\hat{i} + 10\hat{j}$$

$$\underline{r}_B(7) = 13\hat{i} + 10\hat{j}$$

The particles collide when $t = 7$ at the point $(13, 10)$.
- Let $\underline{r}_A(s) = (3s - 8)\hat{i} + (s^2 - 18s + 87)\hat{j}$ and $\underline{r}_B(t) = (20 - t)\hat{i} + (2t - 4)\hat{j}$.

$$3s - 8 = 20 - t$$

$$t = 28 - 3s$$

$$s^2 - 18s + 87 = 2t - 4$$

Substitute $t = 28 - 3s$:

$$s^2 - 18s + 87 = 2(28 - 3s) - 4$$

$$s^2 - 18s + 87 = 56 - 6s - 4$$

$$s^2 - 12s + 35 = 0$$

$$(s - 5)(s - 7) = 0$$

Using the Null Factor Law $s = 5, 7$

$$\underline{r}_A(5) = (15 - 8)\hat{i} + (25 - 90 + 87)\hat{j}$$

$$= 7\hat{i} + 22\hat{j}$$

When $s = 5$, $t = 28 - 15$
 $= 13$

$$\underline{r}_B(13) = (20 - 13)\hat{i} + (26 - 4)\hat{j}$$

$$= 7\hat{i} + 22\hat{j}$$

The paths cross at the point $(7, 22)$.

- c. 1. Evaluate both the position vectors at the required time, $t = 10$.

2. Calculate the difference between these two vectors.
3. The distance between the particles is the magnitude of the difference between these vectors.
4. State the required distance.

- c. Substitute $t = 10$:

$$\underline{r}_A(10) = (30 - 8)\hat{i} + (100 - 180 + 87)\hat{j} \\ = 22\hat{i} + 7\hat{j}$$

$$\underline{r}_B(10) = (20 - 10)\hat{i} + (20 - 4)\hat{j} \\ = 10\hat{i} + 16\hat{j}$$

$$\underline{r}_A(10) - \underline{r}_B(10) = 22\hat{i} + 7\hat{j} - (10\hat{i} + 16\hat{j}) \\ = 12\hat{i} - 9\hat{j}$$

$$|\underline{r}_A(10) - \underline{r}_B(10)| = |12\hat{i} - 9\hat{j}| \\ = \sqrt{12^2 + (-9)^2} \\ = \sqrt{225}$$

$$|\underline{r}_A(10) - \underline{r}_B(10)| = 15 \text{ units}$$

4.2.2 Parametric and Cartesian equations of circles, ellipses and hyperbolas

When eliminating the parameter a number of different methods may be required. These could include direct substitution or trigonometric identities.

WORKED EXAMPLE 3

Given the vector equation, $\underline{r}(t) = (2 + 3 \cos(t))\hat{i} + (3 \sin(t) - 1)\hat{j}$, for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK

1. Define the two parametric equations as [1] and [2]
2. Express the trigonometric ratios $\cos(t)$ and $\sin(t)$ in terms of x and y respectively.
3. Square equations [3] and [4] to utilise a Pythagorean identity and define results as [5] and [6].
4. Add equations [5] and [6].
5. Apply the Pythagorean identity, $\cos^2(t) + \sin^2(t) = 1$. The result is the Cartesian equation of a circle.

WRITE

$$x = 2 + 3 \cos(t) \quad [1]$$

$$y = 3 \sin(t) - 1 \quad [2]$$

$$\frac{x-2}{3} = \cos(t) \quad [3]$$

$$\frac{y+1}{3} = \sin(t) \quad [4]$$

$$\frac{(x-2)^2}{9} = \cos^2(t) \quad [5]$$

$$\frac{(y+1)^2}{9} = \sin^2(t) \quad [6]$$

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{9} = \cos^2(t) + \sin^2(t)$$

$$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{9} = 1$$

$$\therefore (x-2)^2 + (y+1)^2 = 9$$

This is a circle of radius 3, with centre at $(2, -1)$

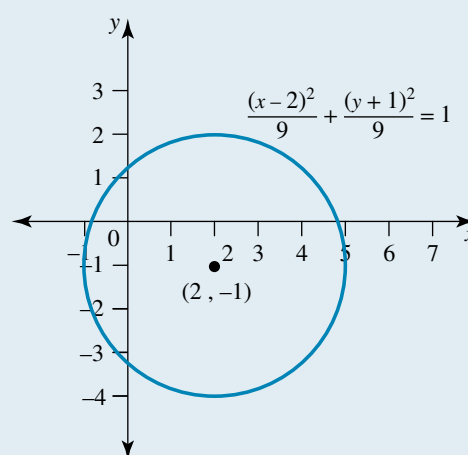
6. Determine the domain.

Since $-1 \leq \cos(t) \leq 1$, it follows from the parametric equation $x(t) = 2 + 3 \cos(t)$ that the domain is $-1 \leq x \leq 5$, that is $[-1, 5]$.

7. Determine the range.

Since $-1 \leq \sin(t) \leq 1$, it follows from the parametric equation $y(t) = 3 \sin(t) - 1$ that the range is $-4 \leq y \leq 2$, that is $[-4, 2]$.

8. The graph is the whole circle. The exact coordinates of the x - and y -intercepts are not required in this case.



WORKED EXAMPLE 4

Given the vector equation, $\underline{r}(t) = (6 \cos(t))\hat{i} + (5 \sin(t))\hat{j}$, for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK

- Define the two parametric equations as [1] and [2]
- Square equations [1] and [2] and define results as [3] and [4].
- Add equations [3] and [4].
- Apply the Pythagorean identity, $\cos^2(t) + \sin^2(t) = 1$. The result is the Cartesian equation of an ellipse
- Determine the domain.

WRITE

$$x = 6 \cos(t) \quad [1]$$

$$y = 5 \sin(t) \quad [2]$$

$$x^2 = 36 \cos^2(t)$$

$$\frac{x^2}{36} = \cos^2(t) \quad [3]$$

$$y^2 = 25 \sin^2(t)$$

$$\frac{y^2}{25} = \sin^2(t) \quad [4]$$

$$\frac{x^2}{36} + \frac{y^2}{25} = \cos^2(t) + \sin^2(t) \quad [3] + [4]$$

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

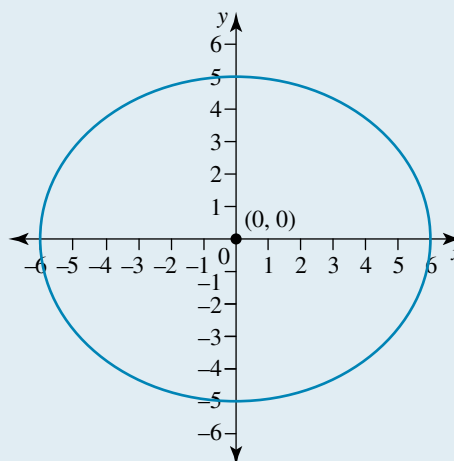
This is an ellipse, with centre at $(0, 0)$ and semi-major and semi-minor axes 6 and 5.

Since $-1 \leq \cos(t) \leq 1$, it follows from the parametric equation $x(t) = 6 \cos(t)$ that the domain is $-6 \leq x \leq 6$, that is $[-6, 6]$.

6. Determine the range.

7. The graph is the whole ellipse. The exact ordinates of the x - and y -intercepts are not required in this case.

Since $-1 \leq \sin(t) \leq 1$, it follows from the parametric equation $y(t) = 5 \sin(t)$ that the range is $-5 \leq y \leq 5$, that is $[-5, 5]$.



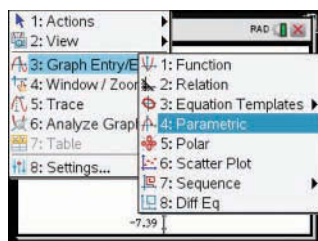
TI | THINK

1. On a Home page, select Graph

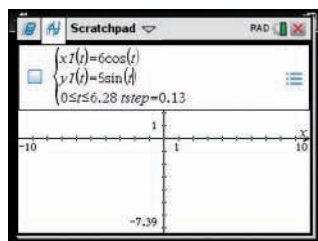
WRITE



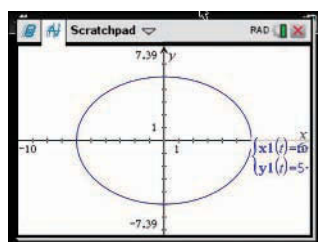
2. Parametric equations can be graphed by selecting, MENU
3: Graph Entry/Edit
4: Parametric



3. Complete the entry line as
 $6 \cos(t)$
 $5 \sin(t)$
Note: t is entered by pressing the T button on the alpha keyboard.



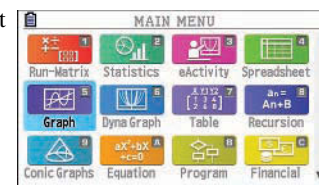
4. Press the ENTER button. The answer appears on the screen.



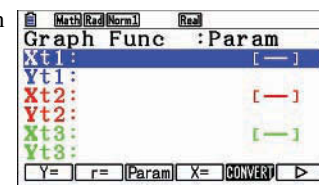
CASIO | THINK

1. On a Menu screen, select GRAPH

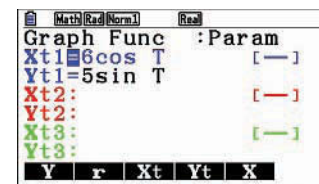
WRITE



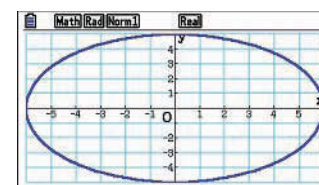
2. Parametric equations can be graphed by selecting, TYPE Param



3. Complete the entry line as
 $6 \cos T$
 $5 \sin T$
Note: T is entered by pressing the X, θ, T button.



4. Press the EXE button. The answer appears on the screen.



WORKED EXAMPLE 5

Given the vector equation $\underline{r}(t) = 3 \sec(2t)\hat{i} + 4 \tan(2t)\hat{j}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.

THINK

1. Define two parametric equations as [1] and [2].
2. Express the trigonometric ratios $\sec(2t)$ and $\tan(2t)$ in terms of x and y respectively.
3. Eliminate the parameter using an appropriate trigonometric identity to determine the Cartesian equation of the path. In this case the expression is given as an implicit equation.
4. Determine the domain.
5. Determine the range.
6. The graph is the whole hyperbola.

WRITE

$$x = 3 \sec(2t) \quad [1]$$

$$y = 4 \tan(2t) \quad [2]$$

$$[1] \Rightarrow \sec(2t) = \frac{x}{3}$$

$$[2] \Rightarrow \tan(2t) = \frac{y}{4}$$

Since $\sec^2(2t) - \tan^2(2t) = 1$, it follows that

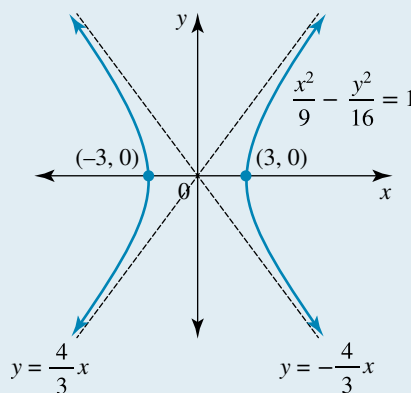
$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

This is a hyperbola with centre at the origin. A hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has asymptotes when $y = \pm \frac{b}{a}x$.

Therefore the asymptotes of $\frac{x^2}{9} - \frac{y^2}{16} = 1$ are $y = \pm \frac{4x}{3}$.

It follows from the parametric equation $x(t) = 3 \sec(2t)$ that the domain is $(-\infty, -3] \cup [3, \infty)$.

It follows from the parametric equation $y(t) = 4 \tan(2t)$ that the range is $\mathbb{R}$.



The parametric representation of a curve is not necessarily unique. This can be seen by comparing Worked examples 5 and 6.

WORKED EXAMPLE 6

Show that the parametric equations $x(t) = \frac{3}{2} \left(t + \frac{1}{t} \right)$ and $y(t) = 2 \left(t - \frac{1}{t} \right)$ where $t \in \mathbb{R} \setminus \{0\}$

represent the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

THINK

1. Define the two parametric equations as [1] and [2].

2. Express the equations in a form to eliminate the parameter.

3. Square both equations [3] and [4].

4. Subtract the equations [5] – [6] to eliminate the parameter.

5. Divide by 4.

WRITE

$$x = \frac{3}{2} \left(t + \frac{1}{t} \right) \quad [1]$$

$$y = 2 \left(t - \frac{1}{t} \right) \quad [2]$$

$$\frac{2x}{3} = t + \frac{1}{t} \quad [3]$$

$$\frac{y}{2} = t - \frac{1}{t} \quad [4]$$

$$\frac{4x^2}{9} = t^2 + 2 + \frac{1}{t^2} \quad [5]$$

$$\frac{y^2}{4} = t^2 - 2 + \frac{1}{t^2} \quad [6]$$

[5] – [6]:

$$\frac{4x^2}{9} - \frac{y^2}{4} = t^2 + 2 + \frac{1}{t^2} - \left(t^2 - 2 + \frac{1}{t^2} \right)$$

$$\frac{4x^2}{9} - \frac{y^2}{4} = 4$$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

This gives the hyperbola as required.

studyon

Units 3 & 4 > Area 2 > Sequence 3 > Concepts 1&2

Position vectors as functions of time Summary screen and practice questions

Circles, ellipses and hyperbolas Summary screen and practice questions

Exercise 4.2 Position vectors as functions of time: circles, ellipses and hyperbolas

Technology free

- WE1** A particle moves so that its vector equation is given by $\underline{r}(t) = (t - 2)\hat{i} + (3t - 1)\hat{j}$ for $t \geq 0$. Determine:
 - the distance of the particle from the origin when $t = 5$
 - the distance of the particle from the origin at any time t
 - the closest distance of the particle from the origin.
- A particle moves so that its vector equation is given by $\underline{r}(t) = \sqrt{t}\hat{i} + (2t + 3)\hat{j}$ for $t \geq 0$. Determine:
 - the distance of the particle from the origin when $t = 4$
 - the value t when the distance of the particle from the origin is $15\sqrt{2}$.
- WE2** Two particles move so that their position vectors are given by $\underline{r}_A(t) = (2t + 6)\hat{i} + (t^2 - 6t + 45)\hat{j}$ and $\underline{r}_B(t) = (t + 11)\hat{i} + (7t + 5)\hat{j}$ for $t \geq 0$. Determine:
 - when and where the particles collide
 - where their paths cross
 - the distance between the particles when $t = 10$.

4. Two particles move so that their position vectors are given by $\underline{r}_A(t) = (-t^2 + 12t - 22)\hat{i} + (19 - 3t)\hat{j}$ and $\underline{r}_B(t) = (18 - 2t)\hat{i} + (t + 3)\hat{j}$ for $t \geq 0$. Determine:
- when and where the particles collide
 - where their paths cross
 - the distance between the particles when $t = 5$.
5. a. A particle moves so that its vector equation is given by $\underline{r}(t) = (2t - 1)\hat{i} + (t - 3)\hat{j}$ for $t \geq 0$. Determine:
- the distance of the particle from the origin when $t = 4$
 - the distance of the particle from the origin at any time t
 - the closest distance of the particle from the origin.
- b. A particle moves so that its vector equation is given by $\underline{r}(t) = (4t - 3)\hat{i} + (3t + 4)\hat{j}$ for $t \geq 0$. Determine:
- the distance of the particle from the origin when $t = 2$
 - the distance of the particle from the origin at any time t
 - the closest distance of the particle from the origin.
6. a. A boat moves so that its vector equation is given by $\underline{r}(t) = (2t - 3)\hat{i} + 2\sqrt{t}\hat{j}$ for $t \geq 0$, where distance is measured in kilometres. Determine:
- the distance of the boat from the origin when $t = 4$
 - the closest distance of the boat from the origin
 - the times when the boat is 3 kilometres from the origin.
- b. A particle moves so that its vector equation is given by $\underline{r}(t) = (2t - 7)\hat{i} + (2t + 2)\hat{j}$ for $t \geq 0$. Determine:
- the closest distance of the particle from the origin
 - the time when the particle is $9\sqrt{5}$ units from the origin.
7. a. A particle moves so that its vector equation is given by $\underline{r}(t) = (at + b)\hat{i} + (ct^2 + d)\hat{j}$ for $t \geq 0$. If $\underline{r}(2) = 5\hat{i} + 7\hat{j}$ and $\underline{r}(4) = 13\hat{i} + 19\hat{j}$, calculate the values of a, b, c and d .
- b. A particle moves so that its vector equation is given by $\underline{r}(t) = (at + b)\hat{i} + (ct^2 + dt)\hat{j}$ for $t \geq 0$. If $\underline{r}(4) = 13\hat{i} + 4\hat{j}$ and $\underline{r}(6) = 17\hat{i} + 18\hat{j}$, calculate the values of a, b, c and d .
8. Two particles move so that their position vectors are given by $\underline{r}_A(t) = (3t - 43)\hat{i} + (-t^2 + 26t - 160)\hat{j}$ and $\underline{r}_B(t) = (17 - t)\hat{i} + (2t - 25)\hat{j}$ for $t \geq 0$. Determine:
- when and where the particles collide
 - where their paths cross
 - the distance between the particles when $t = 12$.
9. Two particles move so that their position vectors are given by $\underline{r}_A(t) = (t^2 - 6)\hat{i} + (2t + 2)\hat{j}$ and $\underline{r}_B(t) = (7t - 16)\hat{i} + \frac{1}{5}(17t - t^2)\hat{j}$ for $t \geq 0$. Determine:
- when and where the particles collide
 - the distance between the particles when $t = 10$.
10. Two particles move so that their position vectors are given by $\underline{r}_A(t) = (-t^2 + 12t + 53)\hat{i} + (3t + 38)\hat{j}$ and $\underline{r}_B(t) = (2t + 29)\hat{i} + (86 - t)\hat{j}$ for $t \geq 0$. Determine:
- when and where the particles collide
 - where their paths cross
 - the distance between the particles when $t = 20$.
11. a. A particle moves so that its vector equation is given by $\underline{r}(t) = t\hat{i} + \frac{1}{t}\hat{j}$ for $t > 0$. Determine the closest distance of the particle from the origin.
- b. A particle moves so that its vector equation is given by $\underline{r}(t) = e^{-t}\hat{i} + e^t\hat{j}$ for $t \in \mathbb{R}$. Determine the closest distance of the particle from the origin.
12. **WE3** Given the vector equation $\underline{r}(t) = 3\cos(t)\hat{i} + 4\sin(t)\hat{j}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.



13. Given the vector equation $\underline{r}(t) = (5 - 2 \cos(t))\hat{i} + (3 \sin(t) - 4)\hat{j}$, for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.
14. **WE4** Given the vector equation $\underline{r}(t) = 5 \sec(2t)\hat{i} + 3 \tan(2t)\hat{j}$ for $t \geq 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.
15. **WE5** Given the vector equation $\underline{r}(t) = 4 \cot\left(\frac{t}{2}\right)\hat{i} + 3 \operatorname{cosec}\left(\frac{t}{2}\right)\hat{j}$ for $t > 0$, determine and sketch the Cartesian equation of the path, and state the domain and range.
16. **WE6** Show that the parametric equations $x = \frac{5}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{3}{2}\left(t - \frac{1}{t}\right)$ where $t \in \mathbb{R} \setminus \{0\}$ represent the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$.
17. Show that the parametric equations $x = \frac{6t}{1+t^2}$ and $y = \frac{3(1-t^2)}{1+t^2}$ represent the circle $x^2 + y^2 = 9$.

Technology active

For questions 18–23, determine and sketch the Cartesian equation of the path for each of the given vector equations, and state the domain and range.

18. a. $\underline{r}(t) = 2t\hat{i} + 4t^2\hat{j}$ for $t \geq 0$ b. $\underline{r}(t) = (t-1)\hat{i} + 3t\hat{j}$ for $t \geq 0$
19. a. $\underline{r}(t) = 2t\hat{i} + \frac{1}{t}\hat{j}$ for $t > 0$
 b. $\underline{r}(t) = 2t\hat{i} + (t^2 - 4t)\hat{j}$ for $t \geq 0$
 c. $\underline{r}(t) = \left(t + \frac{1}{t}\right)\hat{i} + \left(t - \frac{1}{t}\right)\hat{j}$ for $t > 0$
20. a. $\underline{r}(t) = e^{-2t}\hat{i} + e^{2t}\hat{j}$ for $t \geq 0$
 b. $\underline{r}(t) = e^{-t}\hat{i} + (2 + e^{2t})\hat{j}$ for $t \geq 0$
 c. $\underline{r}(t) = e^t\hat{i} + (2 + e^{2t})\hat{j}$ for $t \geq 0$
21. a. $\underline{r}(t) = 3 \cos(t)\hat{i} + 3 \sin(t)\hat{j}$ for $t \geq 0$
 b. $\underline{r}(t) = 4 \cos(t)\hat{i} + 3 \sin(t)\hat{j}$ for $t \geq 0$
 c. $\underline{r}(t) = 4 \sec(t)\hat{i} + 3 \tan(t)\hat{j}$ for $t \geq 0$
22. a. $\underline{r}(t) = (1 + 3 \cos(t))\hat{i} + (3 \sin(t) - 2)\hat{j}$ for $t \geq 0$
 b. $\underline{r}(t) = (4 + 3 \cos(t))\hat{i} + (2 \sin(t) - 3)\hat{j}$ for $t \geq 0$
 c. $\underline{r}(t) = (2 - 3 \sec(t))\hat{i} + (5 \tan(t) - 4)\hat{j}$ for $t \geq 0$
23. a. $\underline{r}(t) = \cos^2(t)\hat{i} + \sin^2(t)\hat{j}$ for $t \geq 0$
 b. $\underline{r}(t) = \cos^3(t)\hat{i} + \sin^3(t)\hat{j}$ for $t \geq 0$
 c. $\underline{r}(t) = \cos^4(t)\hat{i} + \sin^4(t)\hat{j}$ for $t \geq 0$
24. If a and b are positive real numbers, show that the following vector equations give the same Cartesian equation.
 a. $\underline{r}(t) = a \cos(t)\hat{i} + a \sin(t)\hat{j}$ and $\underline{r}(t) = \left(\frac{2at}{1+t^2}\right)\hat{i} + \left(\frac{a(1-t^2)}{1+t^2}\right)\hat{j}$
 b. $\underline{r}(t) = a \cos(t)\hat{i} + b \sin(t)\hat{j}$ and $\underline{r}(t) = \left(\frac{2at}{1+t^2}\right)\hat{i} + \left(\frac{b(1-t^2)}{1+t^2}\right)\hat{j}$
25. For each of the following vector equations, sketch the equation of the path, using technology.
 a. $\underline{r}(t) = \cos(2t)\hat{i} + \sin(4t)\hat{j}$ for $t \geq 0$
 b. $\underline{r}(t) = \cos(2t)\hat{i} + \sin(6t)\hat{j}$ for $t \geq 0$
26. For each of the following vector equations, sketch the equation of the path, using technology.
 a. The cycloid $\underline{r}(t) = 2(t - \sin(t))\hat{i} + 2(1 - \cos(t))\hat{j}$ for $t \geq 0$
 b. The cardioid $\underline{r}(t) = 2 \cos(t)(1 + \cos(t))\hat{i} + 2 \sin(t)(1 + \cos(t))\hat{j}$ for $t \geq 0$

4.3 Differentiation of vectors

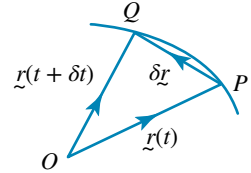
4.3.1 Vector functions

The function $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is called a vector function. It represents the position vector of a particle at time t in two dimensions. As t changes, both the x - and y -coordinates change; thus, the particle is moving along a curve. The equation of the curve is called the Cartesian equation of the path.

The derivative of a vector function

Consider an origin, O , and let P be the position of the particle at time t , so that $\overrightarrow{OP} = \underline{r}(t)$.

Suppose that Q is a neighbouring point close to P at time $t + \delta t$, so that $\overrightarrow{OQ} = \underline{r}(t + \delta t)$.



$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ \delta \underline{r} &= \underline{r}(t + \delta t) - \underline{r}(t)\end{aligned}$$

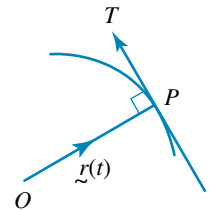
Consider $\frac{\delta \underline{r}}{\delta t} = \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t}$, where $\delta t \neq 0$. Because $\delta \underline{r}$ is a vector and δt is a scalar, $\frac{\delta \underline{r}}{\delta t}$ is a vector parallel to $\delta \underline{r}$ or the vector $\overrightarrow{PQ}$. As $\delta t \rightarrow 0$, provided the limit exists, we define $\frac{d\underline{r}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\underline{r}(t + \delta t) - \underline{r}(t)}{\delta t}$

$$\begin{aligned}\frac{d\underline{r}}{dt} &= \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t)\hat{i} + y(t + \delta t)\hat{j} - (x(t)\hat{i} + y(t)\hat{j})}{\delta t} \\ \frac{d\underline{r}}{dt} &= \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t) - x(t)}{\delta t} \hat{i} + \lim_{\delta t \rightarrow 0} \frac{y(t + \delta t) - y(t)}{\delta t} \hat{j} \\ \frac{d\underline{r}}{dt} &= \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}\end{aligned}$$

The vector $\frac{d\underline{r}}{dt} = \dot{\underline{r}}$, where the dot indicates the derivative with respect to t , is a vector parallel to the tangent T to the curve at the point P .

The unit tangent vector at $t = a$ is denoted by $\hat{\underline{s}} = \frac{\dot{\underline{r}}(a)}{|\dot{\underline{r}}(a)|}$ and is defined as the unit vector in the direction of $\underline{r}$ at $t = a$.

We do not need to use first principles to find the derivatives of vectors.



4.3.2 Rules for differentiating vectors

In the following sections it is assumed that the derivatives exist for the functions given.

Derivative of a constant vector

If $\underline{c}$ is a constant vector, that is a vector which does not change and is independent of t , then $\frac{d\underline{c}}{dt} = \underline{0}$.

Note that $\frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \underline{0}$.

Derivative of a sum or difference of vectors

The sum or difference of two vectors can be differentiated as the sum or difference of the individual derivatives. That is,

$$\frac{d}{dt}(\underline{a} + \underline{b}) = \frac{d\underline{a}}{dt} + \frac{d\underline{b}}{dt} \quad \text{and} \quad \frac{d}{dt}(\underline{a} - \underline{b}) = \frac{d\underline{a}}{dt} - \frac{d\underline{b}}{dt}.$$

Using these rules, if $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, then $\frac{d\underline{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$. Simply put, to differentiate a vector we merely differentiate each component using the rules for differentiation.

WORKED EXAMPLE 7

Calculate a unit tangent vector to $\underline{r}(t) = e^{3t}\hat{i} + \sin(2t)\hat{j}$ at the point where $t = 0$.

THINK

1. Differentiate the vector by differentiating the individual components of the vector.
2. State the derivative vector.
3. Evaluate the derivative vector at the required value, $t = 0$.
4. Find the magnitude of the derivative vector by recalling the formula
5. State the required result, which is a unit vector by substituting into the formula $\hat{\underline{r}} = \frac{\underline{\dot{r}}(a)}{|\underline{\dot{r}}(a)|}$.

WRITE

$$\underline{r}(t) = e^{3t}\hat{i} + \sin(2t)\hat{j}$$

$$\frac{d\underline{r}}{dt} = \frac{d}{dt}(e^{3t})\hat{i} + \frac{d}{dt}(\sin(2t))\hat{j}$$

$$\frac{d\underline{r}}{dt} = 3e^{3t}\hat{i} + 2\cos(2t)\hat{j}$$

$$\left. \frac{d\underline{r}}{dt} \right|_{t=0} = 3\hat{i} + 2\hat{j}$$

$$\begin{aligned} \left| \frac{d\underline{r}(0)}{dt} \right| &= \sqrt{3^2 + 2^2} \\ &= \sqrt{13} \end{aligned}$$

$$\hat{\underline{r}} = \frac{1}{\sqrt{13}}(3\hat{i} + 2\hat{j})$$

4.3.3 Vectors describing motion

Velocity vector

Because $\underline{r}(t)$ represents the position vector, $\underline{v}(t) = \frac{d\underline{r}}{dt} = \underline{\dot{r}}(t)$ represents the velocity vector. Note the single dot above $\underline{r}$ indicates the derivative with respect to time. Furthermore, if $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, then $\underline{\dot{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$.

Note the alternative notations that may be used, for example $\underline{v}(t) = \frac{d\underline{r}}{dt} = \underline{\dot{r}}(t)$.

Speed and momentum

The speed of a moving particle is the magnitude of the velocity vector. The speed at time t is given by $|\underline{\dot{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$.

Momentum is the quantity of motion that an object has, and it is dependent upon both the mass of the object and the speed of the object. As all objects have mass, all moving objects have momentum. If the particle has a mass of m , then the momentum, $\underline{p}$, acting on the particle is given by $\underline{p} = m\dot{\underline{r}}(t)$.

Acceleration vector

Since $\underline{v}(t) = \frac{d\underline{r}}{dt} = \dot{\underline{r}}(t)$ represents the velocity vector, differentiating again with respect to t gives the acceleration vector. The acceleration vector is given by $\underline{a}(t) = \frac{d}{dt}(\dot{\underline{r}}(t)) = \ddot{\underline{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$. Note that the two dots above the variables indicate the second derivative with respect to time.

Vectors describing motion

Position vector: $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

Velocity vector: $\underline{v}(t) = \frac{d\underline{r}}{dt}$
 $= \dot{\underline{r}}(t)$
 $= \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$

Speed: $|\dot{\underline{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$

Momentum: $\underline{p} = m\dot{\underline{r}}(t)$

Acceleration vector: $\underline{a}(t) = \frac{d^2\underline{r}}{dt^2}$ or
 $\underline{a}(t) = \frac{d\underline{v}}{dt}$
 $= \frac{d}{dt}(\dot{\underline{r}}(t))$
 $= \ddot{\underline{r}}(t)$
 $= \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$

WORKED EXAMPLE 8

A particle spirals outwards so that its position vector is given by $\underline{r}(t) = t \cos(t)\hat{i} + t \sin(t)\hat{j}$ for $t \geq 0$. Calculate:

- the velocity vector.
- the speed of the particle at time t and hence find the speed when $t = \frac{3\pi}{4}$.
- the acceleration vector.

THINK

1. State the parametric equations.
2. Differentiate x with respect to t . The dot notation is used for the derivative with respect to t . Recall and apply the product rule.

WRITE

a. $\underline{r}(t) = t \cos(t)\hat{i} + t \sin(t)\hat{j}$
 Then $x(t) = t \cos(t)$ and $y(t) = t \sin(t)$.
 $\frac{dx}{dt} = \dot{x} = \frac{d}{dt}(t \cos(t))$
 $= \cos(t) \frac{d}{dt}(t) + t \frac{d}{dt}(\cos(t))$
 $= \cos(t) - t \sin(t)$

3. Differentiate y with respect to t .
Recall and apply the product rule.

$$\begin{aligned}\frac{dy}{dt} &= \dot{y} = \frac{d}{dt}(t \sin(t)) \\ &= \sin(t) \frac{d}{dt}(t) + t \frac{d}{dt}(\sin(t)) \\ &= \sin(t) + t \cos(t)\end{aligned}$$

4. State the velocity vector.

$$\dot{\mathbf{r}}(t) = (\cos(t) - t \sin(t))\hat{i} + (\sin(t) + t \cos(t))\hat{j}$$

- b. 1. Calculate the speed of the particle at time t , by applying the speed formula, $|\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$.
Substitute for the derivatives and expand.

$$\begin{aligned}\text{b. } |\dot{\mathbf{r}}(t)| &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{x}^2 &= (\cos(t) - t \sin(t))^2 \\ &= \cos^2(t) - 2t \cos(t) \sin(t) + t^2 \sin^2(t) \\ \dot{y}^2 &= (\sin(t) + t \cos(t))^2 \\ &= \sin^2(t) + 2t \sin(t) \cos(t) + t^2 \cos^2(t) \\ \dot{x}^2 + \dot{y}^2 &= \cos^2(t) - 2t \cos(t) \sin(t) + t^2 \sin^2(t) + \sin^2(t) \\ &\quad + 2t \sin(t) \cos(t) + t^2 \cos^2(t) \\ &= \cos^2(t) + \sin^2(t) + t^2 \sin^2(t) + t^2 \cos^2(t) \\ &= 1 + t^2\end{aligned}$$

3. State the speed at time t .

$$\begin{aligned}|\dot{\mathbf{r}}(t)| &= \sqrt{1 + t^2} \\ \left| \dot{\mathbf{r}}\left(\frac{3\pi}{4}\right) \right| &= \sqrt{1 + \left(\frac{3\pi}{4}\right)^2} \\ &= \sqrt{\frac{16 + 9\pi^2}{16}}\end{aligned}$$

4. Find the speed at the required time,
 $t = \frac{3\pi}{4}$.

5. State the speed in simplified form.

$$\left| \dot{\mathbf{r}}\left(\frac{3\pi}{4}\right) \right| = \frac{1}{4} \sqrt{16 + 9\pi^2}$$

- c. 1. Recall the acceleration vector formula $\mathbf{a}(t) = \frac{d}{dt}(\dot{\mathbf{r}}(t)) = \ddot{\mathbf{r}}(t)$
 $= \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$ and apply first to the $\hat{i}$ component.

$$\begin{aligned}\text{c. } \frac{d^2x}{dt^2} &= \ddot{x} = \frac{d}{dt}(\cos(t) - t \sin(t)) \\ &= -\sin(t) - \frac{d}{dt}(t \sin(t)) \\ &= -\sin(t) - \sin(t) - t \cos(t) \\ &= -2 \sin(t) - t \cos(t)\end{aligned}$$

2. Determine the $\hat{j}$ component of the acceleration vector.

$$\begin{aligned}\frac{d^2y}{dt^2} &= \ddot{y} = \frac{d}{dt}(\sin(t) + t \cos(t)) \\ &= \cos(t) + \frac{d}{dt}(t \cos(t)) \\ &= \cos(t) + \cos(t) - t \sin(t) \\ &= 2 \cos(t) - t \sin(t)\end{aligned}$$

3. State the acceleration vector in terms of t .

$$\begin{aligned}\ddot{\mathbf{r}}(t) &= x(t)\hat{i} + y(t)\hat{j} \\ \ddot{\mathbf{r}}(t) &= -(2 \sin(t) + t \cos(t))\hat{i} + (2 \cos(t) - t \sin(t))\hat{j}\end{aligned}$$

4.3.4 Extension to three dimensions

If $\mathbf{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is the position vector of a particle moving in three dimensions, then the velocity vector is given by $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k}$. The speed is given by

$$|\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \text{ and the acceleration vector is given by } \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \ddot{\mathbf{r}}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}.$$

WORKED EXAMPLE 9

A particle has a position vector given by $\underline{r}(t) = t^3\hat{i} + 6\sin(3t)\hat{j} + 12e^{-\frac{t}{2}}\hat{k}$ for $t \geq 0$. Determine:

- the velocity vector
- the acceleration vector.

THINK

1. Differentiate the position vector to determine the velocity vector.
2. State the derivative or velocity vector.
1. Differentiate the velocity vector to determine the acceleration vector.
2. State the acceleration vector.

WRITE

$$\underline{r}(t) = t^3\hat{i} + 6\sin(3t)\hat{j} + 12e^{-\frac{t}{2}}\hat{k}$$

$$\dot{\underline{r}}(t) = \frac{d}{dt}(t^3)\hat{i} + \frac{d}{dt}(6\sin(3t))\hat{j} + \frac{d}{dt}\left(12e^{-\frac{t}{2}}\right)\hat{k}$$

$$\dot{\underline{r}}(t) = 3t^2\hat{i} + 18\cos(3t)\hat{j} - 6e^{-\frac{t}{2}}\hat{k}$$

$$\underline{\ddot{r}}(t) = \frac{d}{dt}(3t^2)\hat{i} + \frac{d}{dt}(18\cos(3t))\hat{j} - \frac{d}{dt}\left(6e^{-\frac{t}{2}}\right)\hat{k}$$

$$\underline{\ddot{r}}(t) = 6t\hat{i} - 54\sin(3t)\hat{j} + 3e^{-\frac{t}{2}}\hat{k}$$

4.3.5 The gradient of the curve

The Cartesian equation of a curve can be determined as either an explicit relationship, $y = f(x)$, or an implicit relationship, $f(x, y) = c$. Therefore, the gradient of the curve can be determined using either explicit or implicit differentiation. Alternatively, the gradient of the curve can be found using parametric

differentiation, since $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$. Techniques such as these have been studied in earlier chapters. It may be necessary to solve equations to calculate the maximum or minimum speeds of a particle, the maximum or minimum values of the acceleration, or the force acting on a particle, given its mass.

WORKED EXAMPLE 10

A particle moves so that its position vector is given by $\underline{r}(t) = (3 - 2\cos(2t))\hat{i} + (2 + 3\sin(2t))\hat{j}$ for $0 \leq t \leq \pi$.

- Determine the coordinates where the tangent of the curve is parallel to the x -axis.
- Determine and sketch the Cartesian equation of the path, stating its domain and range.
- Determine the maximum and minimum values of the speed.

THINK

1. State the parametric equations.
2. Differentiate x with respect to t . The dot notation can be used for the derivative with respect to t .
3. Differentiate y with respect to t .
4. Apply the chain rule to find $\frac{dy}{dx}$.

WRITE

$$\underline{r}(t) = (3 - 2\cos(2t))\hat{i} + (2 + 3\sin(2t))\hat{j}$$

$$x(t) = 3 - 2\cos(2t) \text{ and } y(t) = 2 + 3\sin(2t).$$

$$x(t) = 3 - 2\cos(2t)$$

$$\frac{dx}{dt} = \dot{x} = 4\sin(2t)$$

$$y(t) = 2 + 3\sin(2t)$$

$$\frac{dy}{dt} = \dot{y} = 6\cos(2t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\dot{y}}{\dot{x}}$$

5. Substitute for the derivatives.

$$\begin{aligned}\frac{dy}{dx} &= \frac{6 \cos(2t)}{4 \sin(2t)} \\ &= \frac{3 \cos(2t)}{2 \sin(2t)}\end{aligned}$$

6. The tangent is parallel to the x -axis when the gradient is zero.

$$\frac{dy}{dx} = 0 \Rightarrow \cos(2t) = 0, \text{ but } \sin(2t) \neq 0.$$

7. Solve for the values of t .

$$\text{Since } \cos(2t) = 0, 0 \leq t \leq \pi$$

$$2t = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$t = \frac{\pi}{4}, \frac{3\pi}{4}$$

8. Calculate the coordinates when the tangent is parallel to the x -axis by substituting the values of t into the parametric equations.

$$\text{When } t = \frac{\pi}{4},$$

$$x\left(\frac{\pi}{4}\right) = 3 - 2 \cos\left(\frac{\pi}{2}\right) = 3 \text{ and}$$

$$y\left(\frac{\pi}{4}\right) = 2 + 3 \sin\left(\frac{\pi}{2}\right) = 5.$$

At $(3, 5)$, the gradient is zero.

$$\text{When } t = \frac{3\pi}{4},$$

$$x\left(\frac{3\pi}{4}\right) = 3 - 2 \cos\left(\frac{3\pi}{2}\right) = 3 \text{ and}$$

$$y\left(\frac{3\pi}{4}\right) = 2 + 3 \sin\left(\frac{3\pi}{2}\right) = -1.$$

At $(3, -1)$, the gradient is zero.

b. 1. To determine the Cartesian equation of the path, eliminate the parameter from the parametric equations.

$$\text{b. } x = 3 - 2 \cos(2t) \Rightarrow \cos(2t) = \frac{x-3}{-2}$$

$$y = 2 + 3 \sin(2t) \Rightarrow \sin(2t) = \frac{y-2}{3}$$

$$\frac{(x-3)^2}{4} + \frac{(y-2)^2}{9} = 1$$

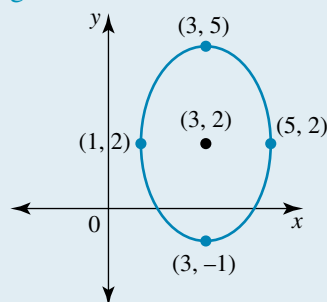
2. State the Cartesian equation of the path by substituting the x - and y -terms into the Pythagorean identity $\cos^2(2t) + \sin^2(2t) = 1$.

3. State the curve and its domain and range.

The curve is an ellipse with centre at $(3, 2)$ and semi-major and minor axes of 2 and 3. The domain is 3 ± 2 or $[1, 5]$ and the range is 2 ± 3 or $[-1, 5]$.

4. Sketch the curve.

Note that at the points $(3, 5)$ and $(3, -1)$ the gradient is zero.



- c. 1. Determine the speed at time t by applying the formula $|\dot{\underline{r}}(t)| = \sqrt{\dot{x}^2 + \dot{y}^2}$ and substituting for the derivatives.

2. Substitute the Pythagorean identity $\sin^2(2t) = 1 - \cos^2(2t)$ to express the speed in terms of $\cos(2t)$ only.

3. State the speed at time t .

4. Determine the maximum value of the speed.

5. State the maximum value of the speed.

6. Determine the minimum value of the speed.

7. State the minimum value of the speed.

$$\begin{aligned} |\dot{\underline{r}}(t)| &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ &= \sqrt{(4 \sin(2t))^2 + (6 \cos(2t))^2} \\ &= \sqrt{16 \sin^2(2t) + 36 \cos^2(2t)} \\ |\dot{\underline{r}}(t)| &= \sqrt{16(1 - \cos^2(2t)) + 36 \cos^2(2t)} \\ &= \sqrt{16 + 20 \cos^2(2t)} \\ &= \sqrt{4(4 + 5 \cos^2(2t))} \\ |\dot{\underline{r}}(t)| &= 2\sqrt{4 + 5 \cos^2(2t)} \end{aligned}$$

The maximum value of the speed occurs when $\cos^2(2t) = 1$.

$$\begin{aligned} |\dot{\underline{r}}(t)|_{\max} &= \sqrt{4(4 + 5 \cos^2(2t))} \\ &= 2\sqrt{4 + 5} \end{aligned}$$

$$|\dot{\underline{r}}(t)|_{\max} = 6$$

The minimum value of the speed occurs when $\cos^2(2t) = 0$.

$$|\dot{\underline{r}}(t)|_{\min} = 2\sqrt{4}$$

$$|\dot{\underline{r}}(t)|_{\min} = 4$$

4.3.6 Applications of vector calculus

Given the vector equation of a moving particle, its motion can be analysed.

WORKED EXAMPLE 11

The position vector, $\underline{r}(t)$, of a golf ball at a time t seconds is given by $\underline{r}(t) = 15t\hat{i} + (20t - 4.9t^2)\hat{j}$ for $t \geq 0$, where the distance is in metres, $\hat{i}$ is a unit vector horizontally forward and $\hat{j}$ is a unit vector vertically upwards above ground level.

- Determine when the golf ball hits the ground.
- Determine where the golf ball hits the ground.
- Determine the initial speed and angle of projection.
- Calculate the maximum height reached.
- Show that the golf ball travels in a parabolic path.



THINK

1. The time when the golf ball is at ground level is when the $\hat{j}$ component is zero.

2. We can ignore the initial time, $t = 0$, when the ball is at ground level.
State when the golf ball hits the ground.

WRITE

$$\text{a. } y(t) = 20t - 4.9t^2 = 0$$

$$t(20 - 4.9t) = 0$$

$$t = 0 \text{ or } 20 - 4.9t = 0$$

$$t = \frac{20}{4.9}$$

$$t \approx 4.08$$

At $t = 0$ and at $t = 4.08$ seconds the ball is at the ground level.

The golf ball hits the ground after 4.08 seconds.

- b. 1. The distance, or range, R , travelled when the golf ball hits the ground is the value of the $\hat{i}$ component at this time. Substitute $t = 4.08$ into the $\hat{i}$ component.

2. State where the golf ball hits the ground.

- c. 1. To determine the velocity vector, differentiate the position vector.
2. Calculate the initial velocity vector, that is, when $t = 0$.
3. Determine the initial speed by calculating the magnitude of the initial velocity vector.

4. The initial angle of projection is the angle the initial velocity vector makes with the $\hat{i}$ axis.

5. State the required results.

- d. 1. The golf ball will rise until the vertical component of its velocity is zero. Set the j component of the velocity vector to 0 to calculate the time this occurs.

2. The maximum height reached, H , is the value of the $\hat{j}$ component of the position vector at this time, $t = 2.04$.

3. State the maximum height reached.

- e. 1. Write the parametric equations.

2. Substitute the value of t into the equation for y .

3. Simplify and form common denominators.

4. State the result.

$$\begin{aligned} \text{b. } R &= x(4.08) \\ &= 15 \times 4.08 \\ &= 61.2 \end{aligned}$$

The golf ball hits the ground at a distance of 61.2 metres from the initial point.

$$\begin{aligned} \text{c. } \underline{r}(t) &= 15\hat{i} + (20t - 4.9t^2)\hat{j} \\ \underline{\dot{r}}(t) &= 15\hat{i} + (20 - 9.8t)\hat{j} \\ \underline{\dot{r}}(0) &= 15\hat{i} + 20\hat{j} \end{aligned}$$

The initial speed is the magnitude of the initial velocity vector.

$$|\underline{\dot{r}}(0)| = \sqrt{15^2 + 20^2} = 25 \text{ m/s}$$

The angle the initial velocity vector makes with the $\hat{i}$ axis is

$$\tan^{-1}\left(\frac{20}{15}\right) = 53.13^\circ.$$

The golf ball is struck with an initial speed of 25 m/s at an angle of 53.13° .

$$\text{d. } y(t) = 20 - 9.8t = 0$$

$$\begin{aligned} t &= \frac{20}{9.8} \\ &= 2.04 \end{aligned}$$

$$\begin{aligned} H = y(2.04) &= 20 \times 2.04 - 4.9 \times 2.04^2 \\ &= 20.41 \end{aligned}$$

The golf ball reaches a maximum height of 20.41 metres.

$$\begin{aligned} \text{e. } x &= 15t \Rightarrow t = \frac{x}{15} \\ y &= 20t - 4.9t^2 \\ y &= 20\left(\frac{x}{15}\right) - 4.9\left(\frac{x}{15}\right)^2 \end{aligned}$$

$$y = -\frac{x}{2250}(49x - 3000)$$

The parametric equation is of the form of a parabola, $y = ax(x - b)$ with $a < 0$. Therefore, the golf ball travels in a parabolic path.

study on

Units 3 & 4 > Area 2 > Sequence 3 > Concept 3

Differentiation of vectors Summary screen and practice questions

Exercise 4.3 Differentiation of vectors

Technology free

- WE7** Calculate a unit tangent vector to $\underline{r}(t) = (e^{2t} + e^{-2t})\hat{i} + (e^{2t} - e^{-2t})\hat{j}$ at the point where $t = 0$.
- Calculate a unit tangent vector to $\underline{r}(t) = \cos(2t)\hat{i} + \sin(2t)\hat{j}$ at the point where $t = \frac{\pi}{6}$.
- WE8** A particle moves so that its position vector is given by $\underline{r}(t) = te^{-2t}\hat{i} + te^{2t}\hat{j}$ for $t \geq 0$. Calculate:
 - the velocity vector.
 - the speed of the particle at time t and hence find its speed when $t = \frac{1}{2}$.
 - the acceleration vector.
- A particle of mass 4 kg moves so that its position vector is given by $\underline{r}(t) = \cos^2(t)\hat{i} + \sin^2(t)\hat{j}$ for $t \geq 0$, where t is measured in seconds and the distance is in metres. Calculate the magnitude of the momentum acting on the particle at time $t = \frac{3\pi}{8}$.
- WE9** A particle moves so that its position vector is given by $\underline{r}(t) = 2t^4\hat{i} + 4\cos(2t)\hat{j} + 6e^{-2t}\hat{k}$ for $t \geq 0$. Determine:
 - the velocity vector.
 - the acceleration vector.
- A particle moves so that its position vector is given by $\underline{r}(t) = 8\cos\left(\frac{\pi t}{4}\right)\hat{i} + 8\sin\left(\frac{\pi t}{4}\right)\hat{j} + 4e^{-2t}\hat{k}$ for $t \geq 0$. Calculate the magnitude of the acceleration vector when $t = 1$.

Technology active

- WE10** A particle moves so that its position vector is given by $\underline{r}(t) = (4 + 3\cos(2t))\hat{i} + (3 - 2\sin(2t))\hat{j}$ for $0 \leq t \leq \pi$.
 - Determine the coordinates where the gradient of the curve is parallel to the x -axis.
 - Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - Determine the maximum and minimum values of the speed.
- A particle moves so that its position vector is given by $\underline{r}(t) = 3\sec(t)\hat{i} + 2\tan(t)\hat{j}$ for $0 \leq t \leq 2\pi$.
 - Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - Determine the coordinates where the gradient is $\frac{4}{3}$.
- WE11** The position vector $\underline{r}(t)$ of a soccer ball at a time $t \geq 0$ seconds is given by $\underline{r}(t) = 5t\hat{i} + (12t - 4.9t^2)\hat{j}$, where the distance is in metres, $\hat{i}$ is a unit vector horizontally forward and $\hat{j}$ is a unit vector vertically upwards above ground level.
 - Determine when the soccer ball hits the ground.
 - Determine where the soccer ball hits the ground.
 - Determine the initial speed and angle of projection.
 - Calculate the maximum height reached.
 - Show that the soccer ball travels in a parabolic path.
- A girl throws a tennis ball. The position vector, $\underline{r}(t)$, of the tennis ball at a time $t \geq 0$ seconds is given by $\underline{r}(t) = 24t\hat{i} + (2 + 7t - 4.9t^2)\hat{j}$, where the distance is in metres, $\hat{i}$ is a unit vector horizontally forward and $\hat{j}$ is a unit vector vertically upwards above ground level.
 - How long before the tennis ball hits the ground?
 - Determine where the tennis ball hits the ground.
 - Determine the initial speed and angle of projection.
 - Calculate the maximum height reached.
 - Show that the tennis ball travels in a parabolic path.



11. Determine a unit tangent vector to each of the following at the point indicated.
- $\underline{r}(t) = 2t\hat{i} + 4t^2\hat{j}$, $t \geq 0$ at $t = 1$
 - $\underline{r}(t) = 2t\hat{i} + 8t^3\hat{j}$, $t \geq 0$ at $t = 1$
 - $\underline{r}(t) = 3t^2\hat{i} + (t^2 - 4t)\hat{j}$, $t \geq 0$ at $t = 3$
 - $\underline{r}(t) = \left(t + \frac{1}{t}\right)\hat{i} + \left(t - \frac{1}{t}\right)\hat{j}$, $t \geq 0$ at $t = 2$
12. Determine a unit tangent vector to each of the following at the point indicated.
- $\underline{r}(t) = e^{-2t}\hat{i} + e^{2t}\hat{j}$, $t \geq 0$ at $t = 0$
 - $\underline{r}(t) = (t^2 + t)\hat{i} + (t^2 - t)\hat{j}$, $t \geq 0$ at $t = 1$
 - $\underline{r}(t) = \cos^2(t)\hat{i} + \cos(2t)\hat{j}$, $t \geq 0$ at $t = \frac{\pi}{3}$
 - $\underline{r}(t) = \sin(t)\hat{i} + \sin(2t)\hat{j}$, $t \geq 0$ at $t = \frac{\pi}{3}$
13. For each of the following position vectors, determine the velocity vector and the acceleration vector.
- $\underline{r}(t) = (t^2 + 9)\hat{i} + \left(\frac{1}{1+t}\right)\hat{j}$
 - $\underline{r}(t) = \ln(3t)\hat{i} + (5t^2 + 4t)\hat{j}$
 - $\underline{r}(t) = 8e^{-\frac{t}{2}}\hat{i} + 4e^{2t}\hat{j}$
 - $\underline{r}(t) = \ln(3t + 4)\hat{i} + 4\cos(3t)\hat{j}$
14. For each of the following position vectors, determine the velocity vector and the acceleration vector.
- $\underline{r}(t) = e^{-2t}\hat{i} + (t^4 - 2t^2)\hat{j} + (5t^2 - 3)\hat{k}$
 - $\underline{r}(t) = 3\cos(2t)\hat{i} - 4\sin(2t)\hat{j} + (12t - 5t^2)\hat{k}$
 - $\underline{r}(t) = t^2\sin(2t)\hat{i} + te^{-2t}\hat{j} + 10t\hat{k}$
 - $\underline{r}(t) = \cos^2(3t)\hat{i} + t^3\cos(3t)\hat{j} + (12 + t^2)\hat{k}$
15. a. A particle moves so that its position vector is given by $\underline{r}(t) = 3\cos(2t)\hat{i} + 3\sin(2t)\hat{j}$, $t \geq 0$.
- Determine the Cartesian equation of the path.
 - Show that the speed is constant.
 - Show that the acceleration is directed inwards.
 - Show that the velocity vector is perpendicular to the acceleration vector.
- b. A particle moves so that its position vector is given by $\underline{r}(t) = 4\cos(3t)\hat{i} + 2\sqrt{2}\sin(3t)(\hat{j} - \hat{k})$, $t \geq 0$.
- Show that the speed is constant.
 - Show that the acceleration vector is perpendicular to the position vector.
16. a. A particle of mass 3 kg moves so that its position vector is given by $\underline{r}(t) = (2 + 4\sin(2t))\hat{i} + (4\cos(2t) - 3)\hat{j}$, $t \geq 0$, where the distance is in metres.
- Determine and sketch the Cartesian equation of the path.
 - Show that the velocity vector is perpendicular to the acceleration vector.
 - Determine the magnitude of the momentum acting on the particle.
 - Determine the magnitude of the resultant force acting on the body after $\frac{\pi}{8}$ seconds.
 - Calculate the gradient of the curve when $t = \frac{\pi}{8}$.
- b. A particle of mass 2 kg moves so that its position vector is given by $\underline{r}(t) = (5 + 2\cos(2t))\hat{i} + (3 + 4\sin(2t))\hat{j}$, $0 \leq t \leq 2\pi$, where the distance is in metres.
- Determine and sketch the Cartesian equation of the path.
 - Determine the magnitude of the momentum acting on the particle after $\frac{\pi}{12}$ seconds.

- iii. Determine the magnitude of the resultant force acting on the body after $\frac{\pi}{12}$ seconds.
 - iv. Determine the maximum and minimum values of the speed.
 - v. Calculate the values of t for which the gradient of the curve is $-2\sqrt{3}$.
17. A particle moves so that its position vector is given by $\underline{r}(t) = a \cos(nt)\hat{i} + a \sin(nt)\hat{j}$ for $t \geq 0$, where a and n are positive constants.
- a. Determine the Cartesian equation of the path.
 - b. Show that the speed is constant.
 - c. Show that the acceleration is directed inwards.
 - d. Show that the velocity vector is perpendicular to the position vector.
 - e. Describe the motion.
18. a. A particle moves so that its position vector is given by $\underline{r}(t) = 2 \sec(t)\hat{i} + 3 \tan(t)\hat{j}$ for $0 \leq t \leq 2\pi$.
- i. Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - ii. Calculate the gradient at the point where $t = \frac{\pi}{4}$.
- b. A particle moves so that its position vector is given by $\underline{r}(t) = 2 \sec^2(t)\hat{i} + 3 \tan^2(t)\hat{j}$ for $0 \leq t \leq 2\pi$.
- i. Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - ii. Calculate the gradient at any point.
 - iii. Calculate the minimum value of the speed.
19. a. A particle moves so that its position vector is given by $\underline{r}(t) = 3 \operatorname{cosec}(t)\hat{i} + 4 \cot(t)\hat{j}$ for $0 \leq t \leq 2\pi$.
- i. Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - ii. Calculate the gradient when $t = \frac{\pi}{3}$.
 - iii. Calculate the minimum value of the speed.
- b. A particle moves so that its position vector is given by $\underline{r}(t) = (1 + 2 \operatorname{cosec}(t))\hat{i} + (4 - 3 \cot(t))\hat{j}$ for $0 \leq t \leq 2\pi$.
- i. Determine and sketch the Cartesian equation of the path, stating its domain and range.
 - ii. Determine the values of t for which the gradient of the curve is 3.
20. A girl throws a tennis ball for her dog. The position vector $\underline{r}(t)$ of the tennis ball at a time $t \geq 0$ seconds is given by $\underline{r}(t) = 10t\hat{i} + (10t - 4.9t^2)\hat{j}$ where the distance is in metres, $\hat{i}$ is a unit vector horizontally forward and $\hat{j}$ is a unit vector vertically upwards above ground level.
- a. Determine the time taken to reach the ground.
 - b. Determine the horizontal distance covered.
 - c. Determine the initial speed and angle of projection.
 - d. Calculate the maximum height reached.
 - e. Show that the tennis ball travels in a parabolic path.



4.4 Integration of vectors

4.4.1 The constant vector

When integrating a function, remember to include the constant of integration, which is a scalar. When integrating a vector function, the constant of integration is a vector. This follows since if $\underline{c}$ is a constant

vector, then $\frac{d}{dt}(\underline{c}) = \underline{0}$.

4.4.2 Rules for integrating vectors

When differentiating a vector, we differentiate its components, so when we integrate a vector, we integrate each component using the rules for finding antiderivatives. If $\underline{q}(t) = x(t)\hat{i} + y(t)\hat{j}$ is a vector function, then we

define $\int \underline{q}(t) dt = \int x(t) d\hat{i} + \int y(t) d\hat{j} + \underline{c}$, where $\underline{c}$ is a constant vector.

Note that in the two-dimensional case, $\underline{c} = c_1\hat{i} + c_2\hat{j}$, where c_1 and c_2 are real numbers.

Velocity vector to position vector

Differentiating the position vector with respect to time gives the velocity vector, therefore integrating the velocity vector with respect to time, gives the position vector. Given the velocity vector

$\underline{v}(t) = \dot{\underline{x}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$, the position vector is $\underline{x}(t) = \int \dot{x}(t)d\hat{i} + \int \dot{y}(t)d\hat{j} = x(t)\hat{i} + y(t)\hat{j} + \underline{c}$. Note that an initial condition must be given for us to be able to determine the constant vector of integration.

WORKED EXAMPLE 12

The velocity vector of a particle is given by $\dot{\underline{x}}(t) = 2\hat{i} + 6t\hat{j}$ for $t \geq 0$. If $\underline{x}(1) = 3\hat{i} + \hat{j}$, determine the position vector at any time t .

THINK

1. Integrate the velocity vector to obtain the position vector using the given rules.
2. Perform the integration. Do not forget to add in a constant vector.
3. Substitute to find the value of the constant vector.
4. Solve for the constant vector.
5. Substitute this value of the constant vector into the position vector.
6. Simplify the position vector to give the final result.

WRITE

$$\begin{aligned}\dot{\underline{x}}(t) &= 2\hat{i} + 6t\hat{j} \\ \underline{x}(t) &= \int 2dt\hat{i} + \int 6tdt\hat{j} \\ \underline{x}(t) &= 2t\hat{i} + 3t^2\hat{j} + \underline{c} \\ \text{Substitute } t = 1 \text{ and use the given condition.} \\ \underline{x}(1) &= 2\hat{i} + 3\hat{j} + \underline{c} = 3\hat{i} + \hat{j} \\ 2\hat{i} + 3\hat{j} + \underline{c} &= 3\hat{i} + \hat{j} \\ \underline{c} &= 3\hat{i} + \hat{j} - (2\hat{i} + 3\hat{j}) \\ \underline{c} &= \hat{i} - 2\hat{j} \\ \underline{x}(t) &= 2t\hat{i} + 3t^2\hat{j} + \underline{c} \\ &= (2t\hat{i} + 3t^2\hat{j}) + (\hat{i} - 2\hat{j}) \\ \underline{x}(t) &= (2t + 1)\hat{i} + (3t^2 - 2)\hat{j}\end{aligned}$$

Acceleration vector to position vector

Differentiating the velocity vector with respect to time gives the acceleration vector, therefore integrating the acceleration vector with respect to time, gives the velocity vector. Given the acceleration vector

$\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = \ddot{\underline{x}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$, the velocity vector is

$\underline{v}(t) = \dot{\underline{x}}(t) = \int \ddot{x}(t)d\hat{i} + \int \ddot{y}(t)d\hat{j} = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \underline{c}_1$, where $\underline{c}_1$ is a constant vector. By integrating again, as above, we can find the position vector.

Note that two sets of information must be given to determine the two constant vectors of integration. This process is a generalisation of the techniques used in earlier studies of integration.

Acceleration, velocity and position vectors

Given the acceleration vector $\underline{a}(t) = \frac{d\underline{v}(t)}{dt} = \underline{\ddot{x}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$ we can obtain the:

Velocity vector: $\underline{v}(t) = \underline{\dot{x}}(t) = \int \ddot{x}(t)d\hat{i} + \int \ddot{y}(t)d\hat{j} = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \underline{c}_1$

Position vector: $\underline{r}(t) = \int \dot{x}(t)d\hat{i} + \int \dot{y}(t)d\hat{j} = x(t)\hat{i} + y(t)\hat{j} + \underline{c}_2$

WORKED EXAMPLE 13

The acceleration vector of a particle is given by $\underline{\ddot{x}}(t) = 6t\hat{i}$, where $t \geq 0$. Given that $\underline{\dot{x}}(2) = 6\hat{i} - 3\hat{j}$ and $\underline{r}(2) = 4\hat{i} - 2\hat{j}$, determine the position vector at time t .

THINK

1. Integrate the acceleration vector to obtain the velocity vector using the given rules.
2. Perform the integration. Do not forget to add in the first constant vector.
3. To determine the first constant vector substitute $t = 2$ into the integration and equate with the first given condition, $\underline{\dot{x}}(2) = 6\hat{i} - 3\hat{j}$.
4. Solve for the first constant vector.
5. Substitute back for the first constant vector, $\underline{\dot{x}}(t) = 3t^2\hat{i} + \underline{c}_1$.
6. Simplify to give the velocity vector.
7. To obtain the position vector, integrate the velocity vector using the given rules.
8. Perform the integration. Do not forget to add in a second constant vector.
9. To find the second constant vector substitute $t = 2$ into the integration and equate with the second given condition, $\underline{r}(2) = 4\hat{i} - 2\hat{j}$.
10. Solve for the second constant vector.
11. Substitute back for the second constant vector, $\underline{r}(t) = (t^3 - 6t)\hat{i} - 3t\hat{j} + \underline{c}_2$.
12. Simplify the position vector to give the final result.

WRITE

$$\begin{aligned}\underline{\ddot{x}}(t) &= 6t\hat{i} \\ \underline{\dot{x}}(t) &= \int 6td\hat{i} \\ \underline{\dot{x}}(t) &= 3t^2\hat{i} + \underline{c}_1 \\ \underline{\dot{x}}(2) &= 3(2)^2\hat{i} + \underline{c}_1 \\ &= 3(2)^2\hat{i} + \underline{c}_1 \\ &= 12\hat{i} + \underline{c}_1 \\ 12\hat{i} + \underline{c}_1 &= 6\hat{i} - 3\hat{j} \\ \underline{c}_1 &= (6\hat{i} - 3\hat{j}) - 12\hat{i} \\ &= -6\hat{i} - 3\hat{j} \\ \underline{\dot{x}}(t) &= 3t^2\hat{i} + (-6\hat{i} - 3\hat{j}) \\ \underline{\dot{x}}(t) &= (3t^2 - 6)\hat{i} - 3\hat{j} \\ \underline{r}(t) &= \int (3t^2 - 6)d\hat{i} - \int 3t d\hat{j} \\ \underline{r}(t) &= (t^3 - 6t)\hat{i} - 3t\hat{j} + \underline{c}_2 \\ \text{Substitute } t = 2 \text{ and use the second} \\ \text{given condition.} \\ \underline{r}(2) &= (8 - 12)\hat{i} - 6\hat{j} + \underline{c}_2 = 4\hat{i} - 2\hat{j} \\ -4\hat{i} - 6\hat{j} + \underline{c}_2 &= 4\hat{i} - 2\hat{j} \\ \underline{c}_2 &= (4\hat{i} - 2\hat{j}) - (-4\hat{i} - 6\hat{j}) \\ &= 8\hat{i} + 4\hat{j} \\ \underline{r}(t) &= (t^3 - 6t)\hat{i} - 3t\hat{j} + (8\hat{i} + 4\hat{j}) \\ \underline{r}(t) &= (t^3 - 6t + 8)\hat{i} + (4 - 3t)\hat{j}\end{aligned}$$

4.4.3 Determining the Cartesian equation

If the position vector is known then the parametric equations of the path of the particle can be determined. Eliminating the parameter, we can determine the Cartesian equation of the curve along which the particle moves.

WORKED EXAMPLE 14

The acceleration vector of a moving particle is given by $\cos\left(\frac{t}{2}\right)\hat{i} - \sin\left(\frac{t}{2}\right)\hat{j}$ for $0 \leq t \leq 4\pi$, where t is the time. The initial velocity is $2\hat{j}$ and the initial position is $\hat{i} - 3\hat{j}$. Determine the Cartesian equation of the path.

THINK

- To obtain the velocity vector, state and integrate the acceleration vector using the given rules.
- Perform the integration. Do not forget to add in the first constant vector.
Recall that $\int \cos(kx) dx = \frac{1}{k} \sin(kx)$ and $\int \sin(kx) dx = -\frac{1}{k} \cos(kx)$.
- Initially means when $t = 0$. Substitute $t = 0$ and equate with the first given condition to solve for the first constant vector.
- Substitute back for the first constant vector and integrate the velocity vector to obtain the position vector, using the given rules.
- Perform the integration. Do not forget to add in a second constant vector.
- Initially means when $t = 0$. Substitute $t = 0$ and equate with the second given condition to solve for the second constant vector.
- Substitute back for the second constant vector and state the position vector,
 $\underline{r}(t) = -4 \cos\left(\frac{t}{2}\right)\hat{i} + 4 \sin\left(\frac{t}{2}\right)\hat{j} + \underline{c}_2$.
- State the parametric equations.
- Eliminate the parameter and label the equations.

WRITE

$$\begin{aligned}\underline{\ddot{r}}(t) &= \cos\left(\frac{t}{2}\right)\hat{i} - \sin\left(\frac{t}{2}\right)\hat{j} \\ \underline{\dot{r}}(t) &= \int \cos\left(\frac{t}{2}\right) d\hat{t} + \int \sin\left(\frac{t}{2}\right) d\hat{j} \\ \text{Since } \int \cos(kx) dx &= \frac{1}{k} \sin(kx) \text{ and } \int \sin(kx) dx = -\frac{1}{k} \cos(kx) \text{ with } k = \frac{1}{2}, \\ \underline{\dot{r}}(t) &= 2 \sin\left(\frac{t}{2}\right)\hat{i} + 2 \cos\left(\frac{t}{2}\right)\hat{j} + \underline{c}_1 \\ \underline{\dot{r}}(0) &= 2 \sin(0)\hat{i} + 2 \cos(0)\hat{j} + \underline{c}_1 \\ &= 2 \times 0\hat{i} + 2\hat{j} + \underline{c}_1 \\ 2\hat{j} + \underline{c}_1 &= 2\hat{j} \\ \underline{c}_1 &= \underline{0} \\ \underline{\dot{r}}(t) &= 2 \sin\left(\frac{t}{2}\right)\hat{i} + 2 \cos\left(\frac{t}{2}\right)\hat{j} \\ \underline{r}(t) &= \int 2 \sin\left(\frac{t}{2}\right) d\hat{t} + \int 2 \cos\left(\frac{t}{2}\right) d\hat{j} \\ \underline{r}(t) &= -4 \cos\left(\frac{t}{2}\right)\hat{i} + 4 \sin\left(\frac{t}{2}\right)\hat{j} + \underline{c}_2 \\ \underline{r}(0) &= -4 \cos(0)\hat{i} + 4 \sin(0)\hat{j} + \underline{c}_2 \\ &= -4 \times 1\hat{i} + 4 \times 0\hat{j} + \underline{c}_2 \\ -4\hat{i} + \underline{c}_2 &= \hat{i} - 3\hat{j} \\ \underline{c}_2 &= 5\hat{i} - 3\hat{j} \\ \underline{r}(t) &= -4 \cos\left(\frac{t}{2}\right)\hat{i} + 4 \sin\left(\frac{t}{2}\right)\hat{j} + (5\hat{i} - 3\hat{j}) \\ \underline{r}(t) &= \left(5 - 4 \cos\left(\frac{t}{2}\right)\right)\hat{i} + \left(4 \sin\left(\frac{t}{2}\right) - 3\right)\hat{j} \\ x &= 5 - 4 \cos\left(\frac{t}{2}\right), y = 4 \sin\left(\frac{t}{2}\right) - 3 \\ \cos\left(\frac{t}{2}\right) &= \frac{5-x}{4}, \sin\left(\frac{t}{2}\right) = \frac{y+3}{4}\end{aligned}$$

10. Square equations and to utilise the Pythagorean identity $\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$.

$$\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$$

$$\left(\frac{5-x}{4}\right)^2 + \left(\frac{y+3}{4}\right)^2 = 1$$

11. State the Cartesian equation in the implicit form.

$$(x-5)^2 + (y+3)^2 = 16$$

This is a circle with centre at (5, -3) and radius 4.

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Integration of vectors Summary screen and practice questions

Exercise 4.4 Integration of vectors

Technology free

- WE12** The velocity vector of a particle is given by $\dot{\mathbf{r}}(t) = (4t - 4)\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ for $t \geq 0$. If $\mathbf{r}(1) = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$, determine the position vector at time t .
- The initial position of a particle is given by $3\hat{\mathbf{i}} + \hat{\mathbf{j}}$. If the velocity vector of the particle is given by $\dot{\mathbf{r}}(t) = 6\sin(3t)\hat{\mathbf{i}} + 4e^{-2t}\hat{\mathbf{j}}$, determine the position vector.
- WE13** The acceleration vector of a particle is given by $\ddot{\mathbf{r}}(t) = -12t^2\hat{\mathbf{j}}$, where $t \geq 0$ is the time. Given that $\dot{\mathbf{r}}(2) = -2\hat{\mathbf{i}} - 16\hat{\mathbf{j}}$ and $\mathbf{r}(2) = \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$, determine the position vector at time t .
- A particle is moving such that $\mathbf{r}(1) = -2\hat{\mathbf{i}} + 7\hat{\mathbf{j}}$ and $\dot{\mathbf{r}}(1) = 6\hat{\mathbf{i}} + 10\hat{\mathbf{j}}$. The acceleration vector of the particle is given by $\ddot{\mathbf{r}}(t) = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, where $t \geq 0$ is the time. Determine the position vector at time t .
- WE14** The acceleration vector of a moving particle is given by $-45\cos(3t)\hat{\mathbf{i}} + 45\sin(3t)\hat{\mathbf{j}}$, where $t \geq 0$ is the time. The initial velocity is $-15\hat{\mathbf{j}}$ and the initial position is $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}$. Determine the Cartesian equation of the path.
- A particle is moving such that its initial position is $2\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and its initial velocity is $10\hat{\mathbf{j}}$. The acceleration vector of the particle is given by $12\cos(2t)\hat{\mathbf{i}} - 20\sin(2t)\hat{\mathbf{j}}$, where $0 \leq t \leq \pi$ and t is the time. Determine the Cartesian equation of the path.
- The velocity vector of a particle is given by $\dot{\mathbf{r}}(t) = e^{-\frac{t}{3}}\hat{\mathbf{i}} + 4t^3\hat{\mathbf{j}}$, where $t \geq 0$ is the time. If initially the particle is at the origin, determine the position vector.
 - The velocity vector of a particle is given by $\dot{\mathbf{r}}(t) = 2t\hat{\mathbf{i}} + 6\sin(2t)\hat{\mathbf{j}}$, where $t \geq 0$ is the time. If initially the particle is at the origin, determine the position vector.
 - The velocity vector of a particle is given by $\dot{\mathbf{r}}(t) = \frac{1}{\sqrt{16-t^2}}\hat{\mathbf{i}} - \frac{t}{\sqrt{t^2+9}}\hat{\mathbf{j}}$, where $t \geq 0$ is the time. If $\mathbf{r}(0) = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, determine the position vector.
 - The velocity vector of a particle is given by $\dot{\mathbf{r}}(t) = \frac{2}{2t+1}\hat{\mathbf{i}} + \frac{72}{(3t+2)^2}\hat{\mathbf{j}}$, where $t \geq 0$ is the time. If $\mathbf{r}(0) = 5\hat{\mathbf{i}} + \hat{\mathbf{j}}$, determine the position vector at time t .
- The acceleration vector of a particle is given by $\ddot{\mathbf{r}}(t) = 6\hat{\mathbf{i}}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\mathbf{r}}(0) = 4\hat{\mathbf{j}}$ and the initial position is $\mathbf{r}(0) = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, determine the position vector at time t .
 - The acceleration vector of a particle is given by $\ddot{\mathbf{r}}(t) = 4\hat{\mathbf{j}}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\mathbf{r}}(0) = 2\hat{\mathbf{i}}$ and the initial position is $\mathbf{r}(0) = 4\hat{\mathbf{i}} + \hat{\mathbf{j}}$, determine the position vector at time t .

- c. A moving particle starts at position $\underline{r}(0) = 3\hat{i} - 2\hat{j}$ with an initial velocity of zero. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = 8\hat{i} + 6\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
- d. A moving particle starts at position $\underline{r}(0) = 3\hat{i} + 4\hat{j}$ with an initial velocity $\dot{\underline{r}}(0) = 8\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = 4\hat{i} + 2\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
9. a. The acceleration vector of a particle is given by $\ddot{\underline{r}}(t) = (6t - 8)\hat{i}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underline{r}}(0) = 4\hat{j}$ and the initial position is $\underline{r}(0) = 5\hat{i} - 3\hat{j}$, determine the position vector at time t .
- b. The acceleration vector of a particle is given by $\ddot{\underline{r}}(t) = (12t - 6)\hat{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underline{r}}(0) = 8\hat{i}$ and the initial position is $\underline{r}(0) = 7\hat{i} - 4\hat{j}$, determine the position vector at time t .
- c. A particle is moving such that $\underline{r}(1) = 9\hat{i} - 5\hat{j}$ and $\dot{\underline{r}}(1) = 6\hat{i} - 2\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = 2\hat{i} + 2\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
- d. A particle is moving such that $\underline{r}(1) = 8\hat{i} - 8\hat{j}$ and $\dot{\underline{r}}(1) = 2\hat{i} - 10\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = -2\hat{i} + 6\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
10. a. A particle is moving such that $\underline{r}(1) = 2\hat{i}$ and $\dot{\underline{r}}(1) = 4\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = \frac{2}{t^3}\hat{i} - \frac{4}{t^3}\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
- b. A particle is moving such that $\underline{r}(4) = 8\hat{i} + \hat{j}$ and $\dot{\underline{r}}(4) = \hat{i} - 2\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = \frac{-1}{\sqrt{t^3}}\hat{i} + 2\hat{j}$, where $t \geq 0$ is the time. Determine the position vector at time t .
- c. The acceleration vector of a particle is given by $\ddot{\underline{r}}(t) = \frac{-9}{(3t+1)^2}\hat{i} + \frac{32}{(2t+1)^3}\hat{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underline{r}}(0) = 3\hat{i} - 8\hat{j}$ and the initial position is $\underline{r}(0) = 4\hat{i} + 3\hat{j}$, determine the position vector at time t .
- d. The acceleration vector of a particle is given by $\ddot{\underline{r}}(t) = \frac{-9}{(3t+1)^2}\hat{i} - \frac{24}{(2t+1)^4}\hat{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $\dot{\underline{r}}(0) = 2\hat{i} - \hat{j}$ and the initial position is $\underline{r}(0) = 6\hat{i} + 8\hat{j}$, determine the position vector at time t .
11. a. When a ball is thrown, its acceleration vector is given by $\ddot{\underline{r}}(t) = -10\hat{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $15\hat{i} + 20\hat{j}$ and the initial position is $2\hat{j}$, determine the Cartesian equation of the path.
- b. When a ball is thrown, its acceleration vector is given by $\ddot{\underline{r}}(t) = -9.8\hat{j}$, where $t \geq 0$ is the time. If initially the velocity vector is $5\hat{i} + 10\hat{j}$ and the initial position is $\hat{j}$, determine the Cartesian equation of the path.
12. a. The acceleration vector of a moving particle is given by $\ddot{\underline{r}}(t) = 4e^{-2t}\hat{i} + 2e^{2t}\hat{j}$, where $t \in \mathbb{R}$. If $\dot{\underline{r}}(0) = -2\hat{i} + 2\hat{j}$ and $\underline{r}(0) = 5\hat{i} - 2\hat{j}$, determine the Cartesian equation of the path.
- b. The acceleration vector of a moving particle is given by $\ddot{\underline{r}}(t) = -e^{-\frac{t}{2}}\hat{i} + 2e^{\frac{t}{2}}\hat{j}$, where $t \in \mathbb{R}$. If $\dot{\underline{r}}(0) = 2\hat{i} + 4\hat{j}$ and $\underline{r}(0) = -2\hat{i} + 3\hat{j}$, determine the Cartesian equation of the path.
13. a. A particle is moving such that $\underline{r}(0) = \hat{i} + 5\hat{j}$ and $\dot{\underline{r}}(0) = 4\hat{j}$. The acceleration vector of the particle is given by $\ddot{\underline{r}}(t) = 8\cos(2t)\hat{i} - 8\sin(2t)\hat{j}$, where t is the time and $0 \leq t \leq 2\pi$. Determine the Cartesian equation of the path.
- b. The acceleration vector of a moving particle is given by $-\cos\left(\frac{t}{3}\right)\hat{i} - \sin\left(\frac{t}{3}\right)\hat{j}$, where $0 \leq t \leq 6\pi$. If $\dot{\underline{r}}(0) = 3\hat{j}$ and $\underline{r}(0) = 2\hat{i} + 5\hat{j}$, determine the Cartesian equation of the path.
14. a. A particle is moving such that its initial position is $3\hat{i} + 5\hat{j}$ and its initial velocity $-6\hat{j}$. The acceleration vector of the particle is given by $\underline{a}(t) = 9\cos(3t)\hat{i} + 18\sin(3t)\hat{j}$, where t is the time and $0 \leq t \leq 2\pi$. Determine the Cartesian equation of the path.

- b. The acceleration vector of a moving particle is given by $-3 \cos\left(\frac{t}{2}\right)\hat{i} + \sin\left(\frac{t}{2}\right)\hat{j}$, where $0 \leq t \leq 4\pi$. If $\underline{v}(0) = -2\hat{j}$ and $\underline{r}(0) = 5\hat{i} + 3\hat{j}$, determine the Cartesian equation of the path.
15. a. The acceleration vector of a moving particle is given by $-2 \cos(t)\hat{i} - 8 \cos(2t)\hat{j}$, where $t \geq 0$. If initially the velocity vector is zero and the initial position is $2\hat{j}$, determine the Cartesian equation of the path, stating the domain and range.
- b. A moving particle is such that its initial position is $\hat{i} + 3\hat{j}$ and its initial velocity $4\hat{j}$. The acceleration vector of the particle is given by $-8 \sin(2t)\hat{i} - 96 \cos(4t)\hat{j}$, where t is the time and $0 \leq t \leq \pi$. Determine the Cartesian equation of the path, stating the domain and range.
16. a. Particle A has an acceleration of $2\hat{i} + 4\hat{j}$, an initial velocity of $-6\hat{i} + 5\hat{j}$ and an initial position of $13\hat{i} - 17\hat{j}$. Particle B has an acceleration of $6\hat{i} + 8\hat{j}$, an initial velocity of $-8\hat{i} - 20\hat{j}$ and an initial position of $\hat{i} + 40\hat{j}$. Show that the two particles collide and calculate the time and point of collision.
- b. Car A has an acceleration of $2\hat{i} - 2\hat{j}$ at time t , and after 1 second its velocity is $-5\hat{i} + 6\hat{j}$ and its position is $-3\hat{i} + 2\hat{j}$. Car B has an acceleration of $2\hat{i} - 6\hat{j}$ at time t , and after 1 second its velocity is $-2\hat{i} + 6\hat{j}$ and its position is $-15\hat{i} + 34\hat{j}$. Show that the two cars collide and calculate the time and point of collision.



Technology active

17. a. The acceleration vector of a particle is given by $-n^2r \cos(nt)\hat{i} - n^2r \sin(nt)\hat{j}$, at time t where a , b , and r are all real constants. If $\underline{v}(0) = nr\hat{j}$ and $\underline{r}(0) = (a + r)\hat{i} + b\hat{j}$, determine the Cartesian equation of the path.
- b. A particle is moving such that its initial position is $(h + a)\hat{i} + k\hat{j}$ and its initial velocity is $bn\hat{j}$, where a , b , n , h and k , are all real constants. The acceleration vector of the particle is given by $-n^2a \cos(nt)\hat{i} - n^2b \sin(3t)\hat{j}$, where t is the time. Determine the Cartesian equation of the path.
18. a. The acceleration vector of a moving particle is given by $(4 \cos(2t) - 2 \cos(t))\hat{i} + (4 \sin(2t) - 2 \sin(t))\hat{j}$, where $0 \leq t \leq 2\pi$, and $\underline{v}(\pi) = -4\hat{j}$ and $\underline{r}(\pi) = -3\hat{j}$. Determine the position vector and sketch the equation of the path.
- b. A particle is moving such that $\underline{r}\left(\frac{\pi}{2}\right) = \hat{i} + 4\hat{j}$ and $\underline{v}\left(\frac{\pi}{2}\right) = -4\hat{i} - 4\hat{j}$. The acceleration vector of the particle is given by $-(16 \cos(4t) + 4 \cos(t))\hat{i} + (16 \sin(4t) - 4 \sin(t))\hat{j}$. Determine the position vector and sketch the equation of the path.
- c. The acceleration vector of a moving particle is given by $-8 \cos(2t)\hat{i} - 108 \sin(6t)\hat{j}$, where $0 \leq t \leq 2\pi$ and $\underline{r}\left(\frac{\pi}{4}\right) = -4\hat{i}$ and $\underline{v}\left(\frac{\pi}{4}\right) = -3\hat{j}$. Determine the position vector and sketch the equation of the path.

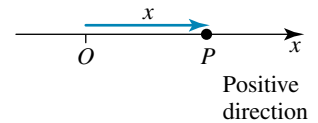
4.5 Straight line motion with constant and variable acceleration

4.5.1 Rectilinear motion

Kinematics is the name given to the study of the motion of bodies, objects or particles. This section will deal with motion in a straight line only — **rectilinear motion** — and all objects considered will be treated as points for mathematical convenience; that is, the objects do not rotate or change shape. First, some basic concepts are explained.

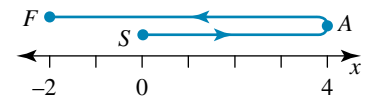
Position

The position of a particle moving in a straight line measures its location from a fixed point of reference, usually the origin O on the line. Positions to the right of O are usually taken as positive. The point P in this diagram has a position coordinate x .



Displacement

The displacement of a moving particle is its change in position relative to a fixed point. Displacement gives both the distance and direction that a particle is from a fixed point.



For example, a particle which moves from S to F via A is shown here on a displacement–time line (or position–time line).

The distance travelled from S to F is 10 units. The displacement of F from S is -2 .

Velocity

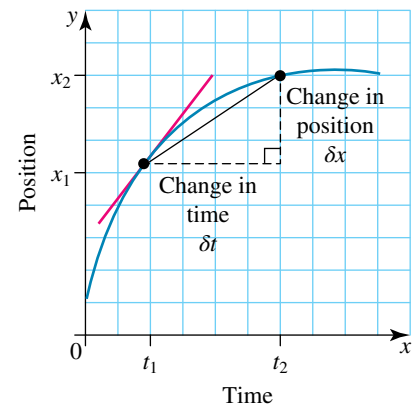
The average velocity of a particle is the rate of change of its position with respect to time. This can be illustrated on a position–time graph. For this curve the position x at time t is shown. The curve $x(t)$ is shown.

$$\begin{aligned}\text{Average velocity} &= \frac{\text{change in position}}{\text{change in time}} \\ &= \frac{x_2 - x_1}{t_2 - t_1} \\ &= \frac{\delta x}{\delta t}\end{aligned}$$

The instantaneous velocity, $v(t)$, at time t is defined as the limiting value of the average velocity as δt approaches 0.

That is, velocity is the gradient of a position–time graph

$$\text{or } v(t) = \lim_{\delta t \rightarrow 0} \frac{\delta x}{\delta t} \text{ or } v(t) = \frac{dx}{dt}$$

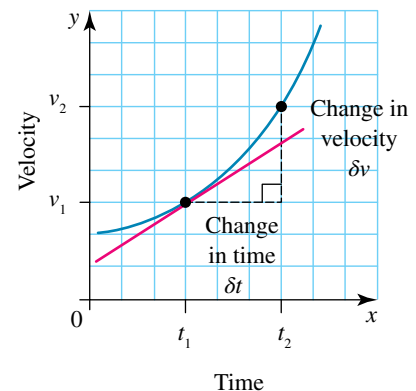


Acceleration

The average acceleration of a particle a during a time interval $t_2 - t_1$ is the rate of change of its velocity with respect to time.

$$\begin{aligned}\text{Average acceleration} &= \frac{v_2 - v_1}{t_2 - t_1} \\ &= \frac{\delta v}{\delta t}\end{aligned}$$

The instantaneous acceleration, a , at time t is the limiting value of the average acceleration as t approaches 0.



That is, acceleration is the gradient of a velocity–time graph at a particular time; or

$$a = \lim_{\delta t \rightarrow 0} \frac{\delta v}{\delta t} \\ = \frac{dv}{dt}$$

4.5.2 Motion with variable acceleration

We can apply these definitions of velocity and acceleration to equations defining the path of a particle as a function of time. This allows the calculation of a particle's velocity and acceleration at any given time.

WORKED EXAMPLE 15

The position of a particle moving in a straight line is given by:

$$x(t) = 2t^3 + t \ln(t) - 4, t > 0$$

Determine:

a. the velocity at any time t

b. the acceleration at any time t .

THINK

a. 1. Recall velocity is the gradient of a position–time graph, that is $v(t) = \frac{dx}{dt}$. Differentiate x with respect to t , using the product rule on $t \ln(t)$.

2. Simplify.

b. Recall acceleration is the gradient of a velocity–time graph, that is $a = \frac{dv}{dt}$. Differentiate v with respect to t .

WRITE

a. $x = 2t^3 + t \ln(t) - 4$

$$v = \frac{dx}{dt} \\ = 6t^2 + \left(t \times \frac{1}{t}\right) + \ln(t) \times 1 \\ = 6t^2 + \ln(t) + 1$$

b. $a = \frac{dv}{dt} \\ = 12t + \frac{1}{t}$

Using anti-differentiation

If the acceleration, a , of a particle is known in terms of time, t , then the velocity, $v(t)$, can be found by anti-differentiation. Thus,

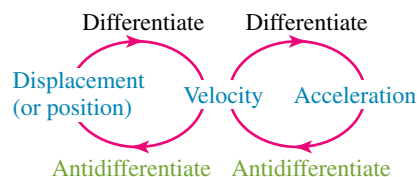
$$v(t) = \int a(t)dt + c_1$$

where c_1 is a constant due to integration; generally it is the initial velocity, $v(0)$.

Similarly, the position, $x(t)$, can be found by anti-differentiating velocity with respect to time:

$$x(t) = \int v(t)dt + c_2$$

where c_2 is a constant due to integration; generally it is the initial position, $x(0)$.



Determining the constant

The constant of anti-differentiation can be determined if initial conditions are given.

$$\text{Velocity: } v(t) = \int a(t) dt + c_1$$

$$\text{Displacement: } x(t) = \int v(t) dt + c_2$$

WORKED EXAMPLE 16

The velocity of an object which is initially 3 metres to the left of O is given by:

$$v(t) = 3t^2 - 2t - 5 \text{ m/s}$$

Determine:

- a. the displacement from O at any time t
- b. the acceleration at any time t
- c. when the object is at rest
- d. the distance travelled in the first second
- e. the acceleration when the velocity is 0.

THINK

- a. 1. To determine the displacement, x , anti-differentiate with respect to t .
- 2. To calculate c , substitute the initial condition $t = 0$ and $x = -3$ into the equation.
- 3. Solve the equation for c .
- 4. State the rule for x .
- b. To determine the acceleration, $\dot{v}(t)$, differentiate the velocity, that is, $\dot{v}(t) = \frac{d}{dt}[v(t)]$.
- c. 1. The object is at rest when $v(t) = 0$.
- 2. Factorise the LHS.
- 3. Solve for t by applying the Null Factor Law.
- 4. State the answer (t cannot be negative).
(The answer can be verified by graphing the velocity function on a graphics calculator.)
- d. 1. The velocity is always negative during the first second and so the distance travelled will equal the magnitude of the displacement during the first second. The object is always moving to the left during the first second.

WRITE

- a. $v(t) = 3t^2 - 2t - 5$
$$x(t) = \int (3t^2 - 2t - 5) dt$$
$$= t^3 - t^2 - 5t + c$$

When $t = 0, x = -3$.

$$-3 = 0^3 - 0^2 - 5(0) + c$$
$$-3 = c$$
$$x(t) = t^3 - t^2 - 5t - 3$$
- b. $a(t) = \dot{v}(t) = \frac{d}{dt}(3t^2 - 2t - 5)$
$$a(t) = \dot{v}(t) = 6t - 2$$
- c. $v(t) = 3t^2 - 2t - 5 = 0$
$$(3t - 5)(t + 1) = 0$$
$$t = \frac{5}{3} \text{ or } t = -1,$$

The domain is $t \geq 0$, so only the first solution is valid.
The object is at rest after $1\frac{2}{3}$ seconds.
- d. At $t = 0$
$$v(0) = 3(0)^2 - 2(0) - 5$$
$$= 3 \times 0^2 - 2 \times 0 - 5$$
$$= -5 \text{ m/s}$$
$$t = 1$$
$$v(1) = 3(1)^2 - 2(1) - 5$$
$$= 3 \times 1^2 - 2 \times 1 - 5$$
$$= -4 \text{ m/s}$$

Also, $v \neq 0$ during the first second.
Thus, the distance travelled will equal the magnitude of the displacement during the first second.

2. The displacement is given by by
 $x(t) = t^3 - t^2 - 5t - 3$ as per step a.1.
 The distance is thus $x(1) - x(0)$.

$$\begin{aligned} x(t) &= t^3 - t^2 - 5t - 3 \\ \text{Distance travelled} &= |x(1) - x(0)| \\ &= |(1^3 - 1^2 - 5 - 3) - (-3)| \\ &= |-8 + 3| \\ &= |-5| \\ &= 5 \end{aligned}$$

The object moves 5 m to the left during the first second.

- e. 1. Find when the velocity is 0 by solving $v(t) = 0$ as in part c.
2. Solve by applying the Null factor Law. Accept only positive solutions.
3. Substitute the solution into the equation for acceleration.
4. Evaluate for a .
5. State the answer.

$$\begin{aligned} \text{e. } v(t) &= 3t^2 - 2t - 5 = 0 \\ (3t - 5)(t + 1) &= 0 \\ t &= \frac{5}{3} \text{ or } t = -1, \text{ but the domain is } \\ t &\geq 0, \text{ so only } t = \frac{5}{3} \text{ is valid.} \\ a &= \frac{dv}{dt} \\ v(t) &= 3t^2 - 2t - 5 \\ a(t) &= 6t - 2 \\ \text{When } t &= \frac{5}{3}, a = 6 \times \frac{5}{3} - 2 \\ a &= 10 - 2 \\ &= 8 \\ \text{When the velocity is } 0, \text{ the} \\ \text{acceleration is } 8 \text{ m/s}^2. \end{aligned}$$

WORKED EXAMPLE 17

The acceleration of a particle moving in a straight line is given by: $\frac{dv}{dt} = 5e^t - 6t + 4 \text{ cm/s}^2$, where v is the velocity at any time t .

If the particle starts at the origin with a velocity of -1 cm/s , calculate:

- a. the velocity at any time t
 b. the displacement $x(t)$ from the origin at any time t
 c. the displacement from the origin after 1 second.

THINK

- a. 1. To calculate v , anti-differentiate the acceleration, $\frac{dv}{dt}$.
2. To determine the constant substitute the initial condition $v = -1$ when $t = 0$.
3. Solve for the constant of anti-differentiation.
4. State the velocity.

WRITE

$$\begin{aligned} \text{a. } \frac{dv}{dt} &= 5e^t - 6t + 4 \\ v(t) &= \int (5e^t - 6t + 4)dt + c \\ &= 5e^t - 3t^2 + 4t + c \\ \text{When } t &= 0, v = -1. \\ -1 &= 5e^0 - 3(0)^2 + 4(0) + c \\ -1 &= 5 + c \\ c &= -6 \\ \text{Therefore, } v(t) &= 5e^t - 3t^2 + 4t - 6. \end{aligned}$$

- b.**
1. The displacement is determined by anti-differentiating v with respect to t .
 2. Substitute the initial condition $x = 0$ when $t = 0$.
 3. Solve for the constant of anti-differentiation.
 4. State the displacement.
- c.**
1. Substitute $t = 1$ into the rule for x .
 2. Evaluate.
 3. Give an approximate answer correct to a useful number of decimal places.
(The answer can be verified by graphing the displacement function on a graphics calculator.)

b. $x(t) = \int (5e^t - 3t^2 + 4t - 6) dt$
 $= 5e^t - t^3 + 2t^2 - 6t + c$
 When $t = 0, x = 0$.
 $0 = 5e^0 - (0)^3 + 2(0)^2 - 6(0) + c$
 $0 = 5 + c$
 $c = -5$
 Therefore, $x(t) = 5e^t - t^3 + 2t^2 - 6t - 5$.

c. $x(1) = 5e^1 - (1)^3 + 2(1)^2 - 6(1) - 5$
 $= 5e - 1 + 2 - 6 - 5$
 $= 5e - 10$
 Therefore, the displacement from O after 1 second is approximately 3.59 cm.

4.5.3 Motion under constant acceleration

Having investigated the link between position, velocity and acceleration, we shall now review the situation where acceleration is constant. These quantities are vectors, but as motion is in a straight line, we are going to use the sign of the quantity to indicate direction.

Suppose a particle is travelling in a straight line with constant acceleration a . The equation of motion can be found using the quadratic expression $x = lt^2 + mt + n$

The derivative of this function results in $v = 2lt + m$ and $a = 2l$ (a constant value)

Unless otherwise stated, assume that the object starts at 0.

Motion under constant acceleration

Under constant acceleration the equation of motion is determined by $x = lt^2 + mt + n$.

The velocity is the derivative of the equation of motion: $v = 2lt + m$

The acceleration is the derivative of the equation of velocity: $a = 2l$

WORKED EXAMPLE 18

A particle moves from rest with a constant acceleration, reaching a speed of 20 m/s in 8 seconds. Calculate:

a. the acceleration

b. the distance travelled by the particle.

THINK

1. List the given information and what has to be found.
2. As acceleration is constant; the position will be a quadratic.

WRITE

- a. $v_0 = 0$ m/s, $v_8 = 20$ m/s, $t = 8$ s, $x = 0$ m
Require a .
 $x = lt^2 + mt + n$



3. Derive the equation for velocity and substitute known values to solve for l and m . State the equation for v .

$$v = 2lt + m$$

Initially, $t = 0$ and $v = 0$; therefore $m = 0$

$$\text{When } t = 8, 20 = 2l \times 8$$

$$l = 1.25$$

$$v = 2.5t$$

$$a = 2.5$$

The acceleration is 2.5 m/s^2 .

4. Differentiate v to find a .

- b. 1. Substitute the values for l, m, n into the equation for x . Assume that the initial position is the origin.
2. As we are finding distance, check if the object changes direction in the time period.
3. Solve for $t = 8 \text{ s}$. State the solution.

- b. Assume that the particle starts at the origin.

Therefore $n = 0$

$$x = 1.25t^2$$

The equation $x = 1.25t^2$ is increasing and therefore does not change direction. The distance covered will be the same as position.

$$t = 8 \text{ s}, x = 1.25 \times 8^2 = 80$$

The distance covered is 80 m .

THINK

Alternative method 1 – developing the equations

- a. Use the definition of acceleration formula.

$$\begin{aligned} \text{average acceleration} &= \frac{\text{change in velocity}}{\text{change in time}} \\ &= \frac{20 - 0}{8 - 0} \\ &= \frac{20}{8} \\ &= 2.5 \text{ m/s}^2 \end{aligned}$$

- b. 1. To determine the velocity, v , integrate with respect to t .

$$a(t) = 2.5$$

$$\begin{aligned} v(t) &= \int 2.5 \, dt \\ &= 2.5t + c \end{aligned}$$

When $t = 0$ and $v = 0$:

$$0 = 2.5 \times 0 + c$$

$$0 = c$$

$$v(t) = 2.5t$$

$$\begin{aligned} x(t) &= \int v(t) \, dt \\ &= \int 2.5t \, dt \\ &= 1.25t^2 + c \end{aligned}$$

When $t = 0$ and $x = 0$:

$$0 = 1.25 \times 0^2 + c$$

$$0 = c$$

$$x(t) = 1.25t^2$$

$$\begin{aligned} x(8) &= 1.25(8)^2 \\ &= 80 \end{aligned}$$

The distance covered is 80 m .

3. State the rule for v .

4. The determine the position, x , integrate with respect to t .

5. The particle is initially at rest. To calculate c , substitute $t = 0$ and $x = 0$ into the equation and solve for c .

6. State the rule for x .

7. Calculate the distance travelled by the particle after 8 seconds by making the substitution $t = 8$.

8. State the answer.

Alternative method 2 – constant acceleration formulas

- b. Once the acceleration has been determined, the distance travelled by the particle can be calculated using the constant acceleration formula

$$s = ut + \frac{1}{2}at^2 \text{ where } u = 0, t = 8$$

and $a = 2.5$.

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 0 \times 8 + \frac{1}{2} \times 2.5 \times 8^2 \\ &= 80 \end{aligned}$$

WORKED EXAMPLE 19

A ball is thrown directly upward from a platform 16 metres above the ground with an initial velocity of 24.5 m/s. Determine:

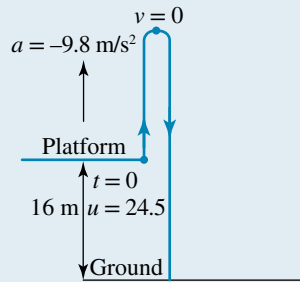
- the time taken to reach the maximum height
- the maximum height above the ground that the ball reaches
- the total time the ball is in the air
- the time taken for the ball to fall to the ground from the maximum height
- the speed of the ball when it hits the ground
- the speed of the ball when it returns to the level of the platform.

THINK

- a. 1. Sketch a diagram of the motion of the ball. Consider *up* to be positive and *down* to be negative.

WRITE

- a. $v_0 = 24.5$ and $v_t = 0$



- At its maximum height, the velocity of the ball is 0 m/s.
- List what is given and what has to be found (acceleration is due to gravity).
- As acceleration is constant; the position will be a quadratic.
- Derive the equations for velocity and acceleration. Substitute known values to solve for l and m .
State the equation for v .

$v_t = 0$ at the ball's maximum height.

$$v_0 = 24.5 \text{ m/s}, a = -9.8 \text{ m/s}^2, v_t = 0$$

Require x_t

$$x = lt^2 + mt + n$$

$$v = 2lt + m$$

Initially, $t = 0$ and $v = 24.5$; therefore

$$m = 24.5$$

$$a = 2l$$

$$-9.8 = 2l$$

$$l = -4.9$$

$$v = -9.8t + 24.5$$

$$0 = -9.8t + 24.5$$

$$t = 2.5$$

- Find when $v = 0$.
Solve for t .

7. State the solution.

- b. 1. Let the ground position be $x = 0$.
State the equation for x . Find the height when $t = 2.5$ s.

2. State the solution.

- c. 1. When the ball lands, $x = 0$. Solve for t . Use the quadratic formula or graphics calculator to solve.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

2. State the solution.

- d. 1. The ball took 2.5 s to reach its maximum height, and 5.585 s to land.

2. State the solution.

- e. 1. It takes 5.585 s for the ball to reach the ground. Determine the velocity at $t = 5.585$ s.

2. State the solution (speed doesn't have direction).

- f. 1. Calculate the time when the position is 16 m by substituting $x = 16$ into position equation.

2. Calculate the velocity after 5 s by using the velocity equation.
State the solution (speed doesn't have direction).

It takes 2.5 s for the ball to reach its maximum height.

- b. Initially, $x = 16$, therefore $n = 16$

$$x = -4.9t^2 + 24.5t + 16$$

$$t = 2.5, x = -4.9 \times 2.5^2 + 24.5 \times 2.5 + 16 = 46.625$$

The maximum height above the ground is 46.625 m.

- c. When the ball lands, $x = 0$.

$$0 = -4.9t^2 + 24.5t + 16$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-24.5 \pm \sqrt{(24.5)^2 - 4 \times -4.9 \times 16}}{-9.8}$$

$$t = 5.585, -0.585$$

$$t = 5.585 \text{ or } t = -0.585$$

Ignore the negative solution.

The ball takes 5.585 s to reach the ground.

- d. Time to fall from the maximum height

$$= 5.585 - 2.5$$

$$= 3.085$$

It takes 3.085 s to fall from the maximum height.

- e. $v = -9.8 \times 5.585 + 24.5$

$$= -30.23$$

The speed when it hits the ground is 30.23 m/s.

- f. $x = 16$

$$16 = -4.9t^2 + 24.5t + 16$$

$$0 = -4.9t^2 + 24.5t$$

$$= -4.9t(t - 5)$$

$$t = 0 \text{ or } t = 5$$

The ball will return to the platform level after 5 s.

$$t = 5, v = -9.8 \times 5 + 24.5$$

$$= -24.5$$

The speed when it passes the platform will be 24.5 m/s.

THINK

Alternative method – developing the equations

- a. 1. The path of the ball is considered as a vertical straight line with the origin O at ground level. Vertically up is taken as the positive direction.

WRITE

$$a = \frac{dv}{dt} = -9.8$$

2. To determine the velocity, v , integrate with respect to t .

$$\begin{aligned} v(t) &= \int \frac{dv}{dt} dt \\ &= \int -9.8 dt \\ &= -9.8t + c \end{aligned}$$

3. The particle has an initially velocity of 24.5 m/s. To calculate c , substitute $t = 0$ and $v = 24.5$ into the equation and solve for c .

$$\begin{aligned} \text{When } t = 0 \text{ and } v = 24.5: \\ 24.5 &= -9.8 \times 0 + c \\ 24.5 &= c \end{aligned}$$

4. State the rule for v .

$$v(t) = -9.8t + 24.5$$

5. At maximum height, the velocity of the ball is 0 m/s.

$$0 = -9.8t + 24.5$$

$$t = \frac{24.5}{9.8}$$

$$t = 2.5 \text{ s}$$

6. State the solution.

It takes 2.5 s for the ball to reach its maximum height.

- b. 1. To determine the position, x , integrate $v(t)$ with respect to t .

$$\begin{aligned} x(t) &= \int (-9.8t + 24.5) dt \\ &= -4.9t^2 + 24.5t + c \end{aligned}$$

2. The ball is initially released from a height of 16 m above the ground. To calculate c , substitute $t = 0$ and $x = 16$ into the equation and solve for c .

$$\begin{aligned} \text{When } t = 0 \text{ and } x = 16: \\ 16 &= -4.9 \times 0^2 + 24.5 \times 0 + c \\ 16 &= c \end{aligned}$$

3. State the rule for x .

$$x(t) = -4.9t^2 + 24.5t + 16$$

4. The maximum height is reached when the ball is instantaneously at rest, i.e. when $v = -9.8t + 24.5 = 0$, which has previously been calculated to occur when $t = 2.5$.

Find x when $t = 2.5$.

$$\begin{aligned} x(2.5) &= -4.9(2.5)^2 + 24.5(2.5) + 16 \\ &= 46.625 \end{aligned}$$

5. State the solution.

The maximum height above the ground is 46.625 m.

- c. f. Now that the position and velocity functions have been determined, the other parts of this question can now be answered as previously shown.

study on

Units 3 & 4

Area 2

Sequence 3

Concept 5

Straight line motion with constant and variable acceleration Summary screen and practice questions

Exercise 4.5 Straight line motion with constant and variable acceleration

Technology active

1. **WE15** For each of the following, determine **i.** the velocity and **ii.** the acceleration at any time t .

a. $x = t^3 + 2t^2 - 5t$

b. $x = 4t - \sqrt{t+2}$

c. $x = e^t + 2e^{-t}$

d. $x = t^2 - \ln(t+3)$

e. $x = \sin^{-1} \frac{t}{3}$

f. $x = t(5-t)(t+2)$

2. The displacement from O of a particle travelling in a straight line is: $x = t^3 - 6t^2 + 11t - 6$, where x is in cm and t is in seconds.
- At what time does it pass through the origin?
 - Determine the velocity at any time.
 - When is the velocity equal to 0?
 - Calculate the minimum velocity.
 - Calculate the distance travelled in the first second.
3. An object travelling in a straight line has its displacement given by:

$$x = 2 \cos(3t - 1) + 3$$

Determine:

- the minimum and maximum displacement
 - when the velocity is first equal to 0
 - how long after it is first at rest is it next at rest
 - the acceleration in terms of the displacement.
4. **WE16** The velocity of an object which is initially 2 m right of O is given by:

$$v(t) = 6t^2 - 2t - 8 \text{ m/s}$$

Determine:

- the displacement $x(t)$ from the origin at any time t
 - the acceleration $\ddot{x}(t)$ at any time t
 - when the object is at rest
 - the distance travelled in the first second
 - the acceleration when the velocity is 0.
5. The velocity of an object moving in a straight line, which is initially positioned at $x = -1$, is given by:

$$\frac{dx}{dt} = 2t^2 - 9t + 4 \text{ cm/s}$$

Determine:

- the displacement at time t
 - the acceleration at time t
 - when the object is at rest
 - the distance travelled during $t = 3$ to $t = 5$
 - the average speed during the time interval $t = 3$ to $t = 5$
 - the average velocity during the time interval $t = 3$ to $t = 5$.
6. The acceleration of a body moving in a straight line is given by:

$$a(t) = 6 - 12t \text{ m/s}^2$$

Initially, the body is at rest at $x = 4$. Determine:

- the velocity at any time
 - the position at any time
 - the time and position when the velocity is 0 m/s
 - the distance travelled in the first 3 seconds.
7. **WE17** The acceleration of a particle travelling in a straight line is given by:

$$\frac{dv}{dt} = 4e^{2t} - 2t + 3 \text{ cm/s}^2$$

If the particle starts from the origin with a velocity of 2 cm/s, then determine:

- the velocity at any time t
- the displacement from the origin at any time t
- the displacement from the origin after 3 seconds (correct to 2 decimal places).

8. **MC** A particle travelling in a straight line has an initial velocity of -1 m/s and initial displacement of -2 m. The acceleration of the particle is given by:

$$\ddot{x} = 6t - 2 \text{ m/s}^2$$

- a. **MC** The particle is stationary when t is equal to:

A. $\frac{1}{3}$ B. 2 s C. 3 s D. 1 s

- b. **MC** The particle is at the origin when t is equal to:

A. 0 s B. 2 s C. -1 s D. 3 s

9. **MC** If the velocity of a body moving in a straight line is: $x = t^2 + 2t - 3$ cm/s then the distance, in cm, travelled in the first 2 seconds is equal to:

A. $1\frac{2}{3}$ B. $2\frac{1}{3}$ C. 4 D. 1

10. A body is travelling in a straight line with a velocity given by:

$$\frac{dx}{dt} = \frac{t}{t^2 + 1} \text{ m/s}$$

If the body starts at the origin, determine:

- a. the initial velocity b. the displacement at any time
c. the acceleration at any time d. the maximum velocity
e. the minimum acceleration.

11. The acceleration of a stone falling from rest from a height of 100 m is given by:

$$\ddot{x} = -10e^{-t} \text{ m/s}^2$$

- a. Determine the velocity at any time t .
b. Sketch a velocity versus time graph.
c. Estimate the time taken, to the nearest second, to reach the ground.

12. The velocity of a particle is given by:

$$\frac{dx}{dt} = \frac{4}{1 + t^2} \text{ m/s}$$

where the initial displacement is 1 m. Determine:

- a. the displacement at any time t b. the maximum velocity and when it occurs
c. the acceleration at any time t d. the acceleration when $t = 1$.

13. The acceleration of a particle is:

$$a(t) = 4 \sin(2t) \text{ where } t = 0, v = -2 \text{ and } x = 0$$

Determine:

- a. the velocity at any time t b. the displacement at any time t
c. the acceleration in terms of the displacement x d. the acceleration in terms of the velocity v .

14. A particle moves from rest with constant acceleration of 2 m/s^2 . Answer the following questions on the particle, which has travelled 25 metres.

- a. **MC** The velocity at this time is equal to:

A. 12.5 m/s B. 10 m/s C. 25 m/s D. 5 m/s

- b. **MC** The time taken for the particle to reach a speed of 20 m/s is:

A. 5 s B. 40 s C. 15 s D. 10 s

- c. **MC** The distance travelled when it reaches 20 m/s is:

A. 100 m B. 20 m C. 1600 m D. 5 m

15. **WE18** A particle moving from rest with constant acceleration reaches a speed of 20 m/s in 5 seconds. Calculate:
- the acceleration
 - the distance travelled.
16. A train decelerates uniformly for 20 seconds over a distance of 320 metres before coming to a stop. Calculate:
- the acceleration
 - the initial speed.
17. A car accelerates from rest at 12 m/s^2 over a distance of 400 metres. Calculate:
- the time taken to travel the 400 metres
 - the final velocity of the car.
18. An object accelerates at 9 m/s^2 over a distance of 40 metres until it reaches a speed of 60 m/s. Calculate:
- the initial velocity
 - the time taken.
19. An object is thrown from the ground at 39.2 m/s. Calculate:
- the maximum height reached by the object
 - the total time that the object is in flight.
20. **WE19** A stone is dropped from a tower that is 50 metres above ground level. Determine:
- how long it takes the stone to reach the ground
 - the speed of the stone on impact.
21. A stone is dropped down a deep well and it takes 4 seconds for it to hit the water at the bottom.
- How deep is the well?
 - What is the speed of the stone on impact with the water?

4.6 Projectile motion

The motion of a particle when acted upon by gravity and air resistance is called projectile motion.

In the modelling approach to projectile motion in this course, certain assumptions are made. The first is to assume that the projectile, no matter how big, is treated as a point particle. Further assumptions are to ignore air resistance, assume that the Earth is flat, ignore the rotation of the Earth and ignore the variations in gravity due to height. For projectiles moving close to the Earth's surface at heights of no more than approximately two hundred metres, these assumptions are generally valid.



4.6.1 General theory of a projectile

Consider a projectile fired in a vertical two-dimensional plane from the origin, O , with an initial speed of $V \text{ m/s}$ at an angle of α degrees to the horizontal. T is the time of flight, H is the maximum height reached by the projectile during its motion, and R is the range on the horizontal plane, that is, the horizontal distance travelled.

As the motion is in two dimensions, $x = x(t)$ is the horizontal displacement and $y = y(t)$ is the vertical displacement at time t seconds, where $0 \leq t \leq T$.

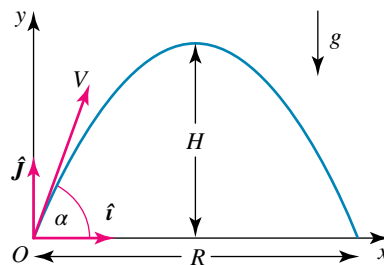
Taking $\hat{i}$ as a unit vector of 1 metre in the positive x direction and $\hat{j}$ as a unit vector of 1 metre in the positive upwards y direction,

$\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is the position vector,

$\underline{\dot{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j}$ is the velocity vector and

$\underline{\ddot{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j}$ is the acceleration vector.

The only force acting on the projectile is the weight force, which acts downwards, so that $\underline{g} = -g\hat{j}$, where $g = 9.8 \text{ m/s}^2$.



Using vectors:

$\ddot{\mathbf{r}}(t) = -g\hat{\mathbf{j}}$ and $\dot{\mathbf{r}}(t) = -gt\hat{\mathbf{j}} + \mathbf{c}_1$ (integrating the acceleration vector to get the velocity vector). But $\dot{\mathbf{r}}(0) = V\cos(\alpha)\hat{\mathbf{i}} + V\sin(\alpha)\hat{\mathbf{j}}$ (the initial velocity vector).

Therefore, $\mathbf{c}_1 = V\cos(\alpha)\hat{\mathbf{i}} + V\sin(\alpha)\hat{\mathbf{j}}$.

$\dot{\mathbf{r}}(t) = V\cos(\alpha)\hat{\mathbf{i}} + (V\sin(\alpha) - gt)\hat{\mathbf{j}}$ (substituting for the first constant vector)

$\mathbf{r}(t) = Vt\cos(\alpha)\hat{\mathbf{i}} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\hat{\mathbf{j}} + \mathbf{c}_2$ (integrating the velocity vector to get the position vector)

As the projectile is fired from the origin, O , $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{c}_2 = \mathbf{0}$; thus,

$$\mathbf{r}(t) = Vt\cos(\alpha)\hat{\mathbf{i}} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

Note that if the projectile is not initially fired from the origin but from some initial height h above ground level, then $\mathbf{c}_2 = h\hat{\mathbf{j}}$.

Then the parametric equations of the projectile are

$$x(t) = Vt\cos(\alpha) \text{ and } y(t) = Vt\sin(\alpha) - \frac{1}{2}gt^2 + h.$$

A projectile can be described by the equations:

$$\text{Cartesian equation: } \mathbf{r}(t) = Vt\cos(\alpha)\hat{\mathbf{i}} + \left(Vt\sin(\alpha) - \frac{1}{2}gt^2\right)\hat{\mathbf{j}}$$

$$\text{Parametric equations: } x(t) = Vt\cos(\alpha) \text{ and } y(t) = Vt\sin(\alpha) - \frac{1}{2}gt^2 + h.$$

Time of flight

The time of flight is the time that the projectile takes to go up and come down again, or the time at which it returns to ground level. To find the time of flight, solve $y = 0$ for t .

$$Vt\sin(\alpha) - \frac{1}{2}gt^2 = 0$$

$$t\left(V\sin(\alpha) - \frac{1}{2}gt\right) = 0$$

$$t = 0 \quad \text{or} \quad \left(V\sin(\alpha) - \frac{1}{2}gt\right) = 0$$

$$\frac{1}{2}gt = V\sin(\alpha)$$

$$t = \frac{2V\sin(\alpha)}{g}$$

The result $t = 0$ represents the time when the projectile was fired, so $T = \frac{2V\sin(\alpha)}{g}$ represents the time of flight.

Time of flight:

$$T = \frac{2V\sin(\alpha)}{g}$$

The range

The range is the horizontal distance travelled in time T .

$$R = x(T) = VT \cos(\alpha)$$

Substitute $T = \frac{2V \sin(\alpha)}{g}$:

$$R = V \cos(\alpha) \left(\frac{2V \sin(\alpha)}{g} \right)$$

Expand:

$$R = \frac{V^2 2 \sin(\alpha) \cos(\alpha)}{g}$$

Applying the double-angle formula $\sin(2A) = 2 \sin(A) \cos(A)$,

$$R = \frac{V^2 \sin(2\alpha)}{g}$$

Note: for maximum range, $\sin(2\alpha) = 1$, so $2\alpha = 90^\circ$ or $\alpha = 45^\circ$. This applies only for a projectile fired from ground level.

Range of flight:

$$R = \frac{V^2 \sin(2\alpha)}{g}$$

Maximum height

The maximum height occurs when the particle is no longer rising. This occurs when the vertical component of the velocity is zero, that is, $\dot{y}(t) = 0$.

Solving $\dot{y} = V \sin(\alpha) - gt = 0$ gives $t = \frac{V \sin(\alpha)}{g} = \frac{T}{2}$, which is half the time of flight.

This applies only for a projectile fired from ground level.

Substituting for $\frac{T}{2}$ into the y component gives

$$H = y\left(\frac{T}{2}\right) = V \sin(\alpha) \left(\frac{V \sin(\alpha)}{g} \right) - \frac{g}{2} \left(\frac{V \sin(\alpha)}{g} \right)^2$$

$$H = \frac{V^2 \sin^2(\alpha)}{g} - \frac{V^2 \sin^2(\alpha)}{2g}$$

$$H = \frac{V^2 \sin^2(\alpha)}{2g}$$

Maximum height:

$$H = \frac{V^2 \sin^2(\alpha)}{2g}$$

WORKED EXAMPLE 20

A golf ball is hit off the ground at an angle of 53.13° to the horizontal with an initial speed of 25 m/s. Calculate:

- the time of flight
- the range
- the maximum height reached.

THINK

1. State the value of the parameters.
2. Apply the time of flight formula to the value of the parameters.
3. State the time of flight.
1. Apply the range of flight formula to the value of the parameters.
2. State the range.
1. Apply the maximum height formula to the value of the parameters.
2. State the maximum height reached.

WRITE

$$\begin{aligned} \text{a. } V &= 25 \text{ and } \alpha = 53.13^\circ \\ T &= \frac{2V \sin(\alpha)}{g} \\ &= \frac{2 \times 25 \sin(53.13^\circ)}{9.8} \\ T &= 4.08 \text{ seconds.} \\ \text{b. } R &= \frac{V^2 \sin(2\alpha)}{g} \\ &= \frac{25^2 \sin(2 \times 53.13^\circ)}{9.8} \\ R &= 61.22 \text{ metres.} \\ \text{c. } H &= \frac{V^2 \sin^2(\alpha)}{2g} \\ &= \frac{25^2 \sin^2(53.13^\circ)}{2 \times 9.8} \\ H &= 20.41 \text{ metres.} \end{aligned}$$

Improving the range

The formulas for the time of flight $T = \frac{2V \sin(\alpha)}{g}$

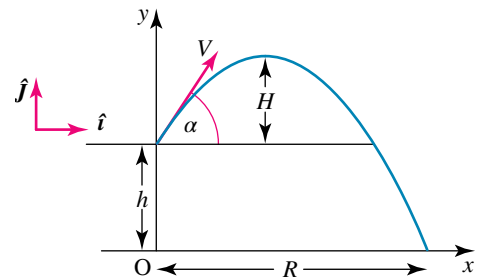
and the range $R = \frac{V^2 \sin(2\alpha)}{g}$ derived above are only true when the projectile is released from the origin, O. In many situations, the projectile is released from a height h above ground level.

In these situations, we need to solve the appropriate parametric equations $x(t) = Vt \cos(\alpha)$ and $y(t) = h + Vt \sin(\alpha) - \frac{1}{2}gt^2$ to find the time of flight and the range. The result for the

maximum height, $H = \frac{V^2 \sin^2(\alpha)}{2g}$, is still valid, but it gives

only the height above h .

It can also be shown that the path of the projectile is a parabola. An interesting question in these situations is how we determine the maximum range on the horizontal plane (ground level) for a given speed of projection.



WORKED EXAMPLE 21

A boy throws a ball from the top of a hill 2 metres above ground level with an initial speed of 10 m/s at an angle of 40° to the horizontal.

- Calculate the time of flight and the horizontal distance travelled.
- Calculate the maximum height reached.
- Determine the speed and angle at which the ball lands.
- Show that the ball travels in a parabolic path.

THINK

1. State the value of the parameters.

2. Start with the acceleration vector and the initial velocity vector to find the velocity vector at time t .

3. State the position vector at time t .

4. The time the ball hits the ground when the vertical component is zero. That is, when $y = 2 + 10t \sin(40^\circ) - 4.9t^2 = 0$. Use technology to solve the quadratic.

5. Determine the horizontal distance travelled by substituting $t = 1.57$ into the x -component of the position vector determined in step 3.

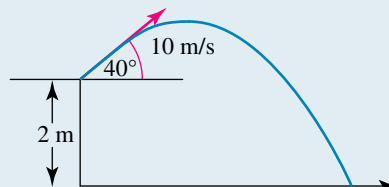
1. The ball will rise and reach maximum height when the vertical component of its velocity is zero.

2. State the maximum height reached by substituting into the y -component of the position vector determined in step a. 3.

1. Use the time when the ball lands to find its velocity vector at this time.
2. Find the speed at which the ball lands.

WRITE

- $V = 10 \text{ m/s}$ and $\alpha = 40^\circ$



$$\ddot{\mathbf{r}}(t) = -9.8\hat{j}$$

$$\dot{\mathbf{r}}(0) = 10 \cos(40^\circ)\hat{i} + 10 \sin(40^\circ)\hat{j}$$

$$\dot{\mathbf{r}}(t) = 10 \cos(40^\circ)\hat{i} + (10 \sin(40^\circ) - 9.8t)\hat{j}$$

Since the initial position is 2 metres above the ground, $\mathbf{r}(0) = 2\hat{j}$.

$$\mathbf{r}(t) = 10t \cos(40^\circ)\hat{i} + (2 + 10t \sin(40^\circ) - 4.9t^2)\hat{j}$$

$$y = 2 + 10t \sin(40^\circ) - 4.9t^2 = 0$$

$$t = -0.26, 1.57$$

Since $t \geq 0$, the time of flight is 1.57 seconds.

$$x(1.57) = 10 \times 1.57 \cos(40^\circ)$$

$$= 12.04$$

The horizontal distance travelled is 12.04 metres.

- Solve $\dot{y} = 10 \sin(40^\circ) - 9.8t = 0$ for t .

$$t = \frac{10 \sin(40^\circ)}{9.8}$$

$$= 0.656$$

Note that this is not half the time of flight.

$$y(0.656) = 2 + 10 \times 0.656 \sin(40^\circ) - 4.9 \times 0.656^2$$

The maximum height reached is 4.11 metres.

- $\dot{\mathbf{r}}(1.57) = 10 \cos(40^\circ)\hat{i} + (10 \sin(40^\circ) - 9.8 \times 1.57)\hat{j}$
 $= 7.66\hat{i} - 8.96\hat{j}$

$$|\dot{\mathbf{r}}(1.57)| = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$= \sqrt{7.66^2 + (-8.96)^2}$$

The ball lands with a speed of 11.8 m/s.

3. Determine the angle that the velocity vector makes with the ground, when the ball lands.

Let ψ be the angle with the ground when the ball lands.

$$\tan(\psi) = \frac{\dot{y}}{\dot{x}}$$

$$\begin{aligned}\psi &= \tan^{-1} \left(\frac{8.96}{7.66} \right) \\ &= 49.5^\circ\end{aligned}$$

- d. 1. State the parametric equations and eliminate the parameter t .

$$\begin{aligned}\text{d. } x &= 10t \cos(40^\circ), y = 2 + 10t \sin(40^\circ) - 4.9t^2 \\ t &= \frac{x}{10 \cos(40^\circ)}\end{aligned}$$

$$y = 2 + \frac{10x \sin(40^\circ)}{10 \cos(40^\circ)} - 4.9 \left(\frac{x}{10 \cos(40^\circ)} \right)^2$$

2. The Cartesian equation is parabolic as it is of the form $y = ax^2 + bx + c$.

$$y = 2 + x \tan(40^\circ) - \frac{49x^2}{1000} \sec^2(40^\circ)$$

The equation of the path

It can be shown that the path of the projectile is a parabola. Transposing $x = Vt \cos(\alpha)$ for t gives

$$t = \frac{x}{V \cos(\alpha)}. \text{ Substituting this into } y(t) = Vt \sin(\alpha) - \frac{1}{2}gt^2 \text{ gives}$$

$$y = V \sin(\alpha) \left(\frac{x}{V \cos(\alpha)} \right) - \frac{g}{2} \left(\frac{x}{V \cos(\alpha)} \right)^2.$$

Simplifying this gives

$$y = x \tan(\alpha) - \frac{gx^2}{2V^2 \cos^2(\alpha)} \text{ or } y = x \tan(\alpha) - \frac{gx^2 \sec^2(\alpha)}{2V^2}.$$

This is of the form $y = ax + bx^2$, so the path of the projectile is a parabola.

Finding the angle of projection to the horizontal

Utilising the trigonometric identity $\sec^2(\alpha) = 1 + \tan^2(\alpha)$, the Cartesian equation of a projectile can be

written in the form $y = x \tan(\alpha) - \frac{gx^2}{2V^2}(1 + \tan^2 \alpha)$. Alternatively, this can be rearranged into a quadratic in

$$\tan(\alpha) \text{ as } \frac{gx^2}{2V^2} \tan^2(\alpha) - x \tan(\alpha) + \left(y + \frac{gx^2}{2V^2} \right) = 0.$$

This equation is useful for finding the angle of projection, α , if we are given the coordinates of a point (x, y) through which the particle passes and the initial speed of projection, V . Because this equation is a quadratic in $\tan(\alpha)$, it is probable that we can obtain two values for α .

Angle of projection

To determine the angle of projection to the horizontal:

$$\frac{gx^2}{2V^2} \tan^2(\alpha) - x \tan(\alpha) + \left(y + \frac{gx^2}{2V^2} \right) = 0$$

WORKED EXAMPLE 22

A basketballer shoots for goal from the three-point line. He throws the ball with an initial speed of 15 m/s and the ball leaves his hands at a height of 2.1 metres above the ground. Calculate the possible angles of projection if he is to score a goal.

Data: Distance from goal to three-point line: 6.25 m

Height of ring: 3.05 m

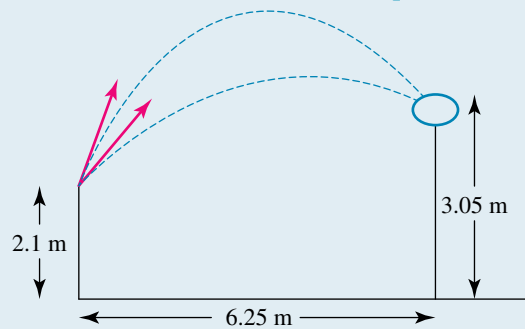


THINK

1. State the value of the parameters.
Drawing a diagram will help.

WRITE

$V = 15$ but α is unknown and required to be found.



2. State where the projectile will pass through.
3. Substitute in the appropriate values to obtain a quadratic in $\tan(\alpha)$.

The basketball must pass through the point where $x = 6.25$ and $y = 3.05 - 2.1 = 0.95$.

Substitute $V = 15$, $x = 6.25$ and $y = 0.95$:

$$\frac{gx^2}{2V^2} \tan^2(\alpha) - x \tan(\alpha) + \left(y + \frac{gx^2}{2V^2} \right) = 0$$

$$\frac{9.8 \times 6.25^2}{2 \times 15^2} \tan^2(\alpha) - 6.25 \tan(\alpha) + \left(0.95 + \frac{9.8 \times 6.25^2}{2 \times 15^2} \right) = 0$$

4. Simplify the quadratic to be solved.
5. Apply the quadratic formula to solve. First find the discriminant of the quadratic equation.

$$0.8507 \tan^2(\alpha) - 6.25 \tan(\alpha) + 1.8007 = 0$$

$$a = 0.8507, b = -6.25, c = 1.8007$$

$$\Delta = b^2 - 4ac$$

$$= (-6.25)^2 - 4 \times 0.8507 \times 1.8007$$

$$= 32.935$$

$$\sqrt{\Delta} = 5.739$$

6. Use the quadratic formula to solve for $\tan(\alpha)$.

$$\tan(\alpha) = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$= \frac{6.25 \pm 5.739}{2 \times 0.8507}$$

$$= 7.0465, 0.3004$$

7. Find the values of α .
8. State the final result.

$$\alpha = \tan^{-1}(7.0465), \tan^{-1}(0.3004)$$

There are two possible angles of projection:
 $\alpha = 81.92^\circ$ or 16.92° .

4.6.2 Proofs involving projectile motion

Often projectile motion problems involve parameters rather than specific given values. In these types of problems, we are required to prove or show that a certain equation is valid. To do this, we can use the general equations and mathematically manipulate these to show the desired result. Note that we can only give results in terms of the given parameters.

WORKED EXAMPLE 23

A ball is projected at an angle of α from a point on a horizontal plane. If the ball reaches a maximum height of H , show that the time interval between the instants when the ball is at heights

of $H \sin^2(\alpha)$ is $2\cos(\alpha)\sqrt{\frac{2H}{g}}$.

THINK

1. Draw a diagram.

2. Recall the formula for the maximum height,

$$H = \frac{V^2 \sin(\alpha)}{2g}$$

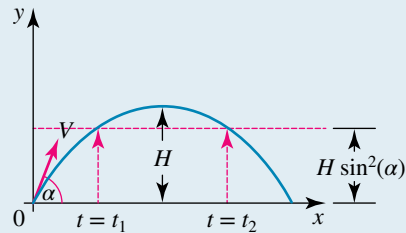
3. Since the initial speed of projection is unknown, express V in terms of the given values.

4. Use the given equation for y .

5. We need to find the values of t at the given heights of $H \sin^2(\alpha)$.

6. Find the discriminant of the quadratic.

WRITE



$$H = \frac{V^2 \sin(\alpha)}{2g}$$

$$V^2 = \frac{2gH}{\sin^2(\alpha)}$$

$$y = Vt \sin(\alpha) - \frac{1}{2}gt^2$$

$$\text{Let } y = H \sin^2(\alpha).$$

$$H \sin^2(\alpha) = Vt \sin(\alpha) - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 - Vt \sin(\alpha) + H \sin^2(\alpha) = 0$$

$$a = \frac{1}{2}g, b = -V \sin(\alpha), c = H \sin^2(\alpha)$$

$$\Delta = b^2 - 4ac$$

$$= V^2 \sin^2(\alpha) - 4 \times \frac{1}{2}g \times H \sin^2(\alpha)$$

$$= V^2 \sin^2(\alpha) - 2gH \sin^2(\alpha)$$



7. Use previous results to eliminate the unknown, V .

Substitute $V^2 \sin^2(\alpha) = 2gH$ into the discriminant:

$$\begin{aligned}\Delta &= 2gH - 2gH \sin^2(\alpha) \\ &= 2gH(1 - \sin^2(\alpha)) \\ &= 2gH \cos^2(\alpha)\end{aligned}$$

8. Use the quadratic formula to determine the two times.

$$\begin{aligned}t &= \frac{V \sin(\alpha) \pm \sqrt{\Delta}}{g}, \text{ so} \\ t_1 &= \frac{V \sin(\alpha) - \sqrt{\Delta}}{g} \text{ and} \\ t_2 &= \frac{V \sin(\alpha) + \sqrt{\Delta}}{g}, \text{ where } t_2 > t_1.\end{aligned}$$

9. Determine the time interval and simplify.

$$\begin{aligned}t_2 - t_1 &= \frac{2\sqrt{\Delta}}{g} \\ &= \frac{2\sqrt{2gH \cos^2(\alpha)}}{g} \\ &= \frac{2 \cos(\alpha)}{g} \sqrt{2gH}\end{aligned}$$

10. State the required result.

$$t_2 - t_1 = 2 \cos(\alpha) \sqrt{\frac{2H}{g}}$$

4.6.3 Incorporating air resistance and three-dimensional motion

In reality, projectiles will move in a three-dimensional framework rather than a two-dimensional plane. Also, with the effects of air resistance being included, the path is not necessarily a parabola.

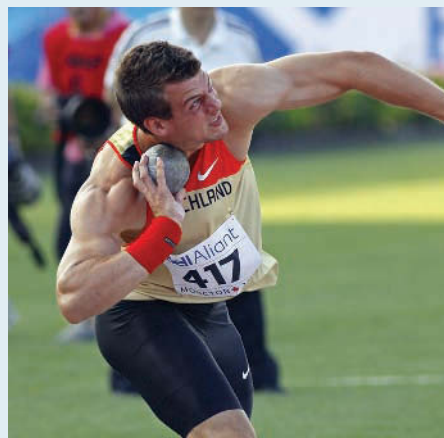
WORKED EXAMPLE 24

A shot is thrown by a shot-put competitor on level ground. At a time t in seconds measured from the point of release of the shot, the position vector $\underline{r}(t)$ of the shot is given by

$$\underline{r}(t) = 7t\hat{i} + \left(5t + 6\left(e^{-\frac{t}{2}} - 1\right)\right)\hat{j} + (2 + 12t - 5t^2)\hat{k}$$

where $\hat{i}$ is a unit vector in the east direction, $\hat{j}$ is a unit vector in the north direction and $\hat{k}$ is a unit vector vertically up. The origin, O , of the coordinate system is at ground level and displacements are measured in metres. Let P be the point where the shot hits the ground.

- Calculate the time taken for the shot to hit the ground.
- Determine how far from O the shot hits the ground.
- Calculate the initial speed of projection.
- Calculate the speed and angle at which the shot hits the ground.



THINK

- a. 1. Determine when the shot hits the ground.
 2. Solve for the values of t using the quadratic formula.

- b. 1. Find the position vector where the shot hits the ground at $t = 2.56$ seconds.
 2. Determine the distance where the shot hits the ground by calculating the magnitude of the position vector at $t = 2.56$ seconds.

- c. 1. Determine the velocity vector.
 2. Determine the initial velocity vector, at $t = 0$.
 3. Find the initial speed of projection by calculating the magnitude of the velocity vector.

- d. 1. Determine the velocity vector when the shot hits the ground, that is at $t = 2.56$ seconds.
 2. Find the magnitude of the velocity vector.
 3. Determine the angle at which the shot hits the ground.

WRITE

- a. The shot hits the ground when the $\hat{k}$ component is zero, that is, when $2 + 12t - 5t^2 = 0$.

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (12)^2 - 4 \times -5 \times 2 \\ &= 184 \\ \sqrt{\Delta} &= 2\sqrt{46} \\ t &= \frac{-b \pm \sqrt{\Delta}}{2a}\end{aligned}$$

$$= \frac{-12 \pm 2\sqrt{46}}{2 \times -5}$$

$$= -0.1565, 2.5565$$

$$t = -0.1565 \text{ or } t = 2.5565.$$

Since $t \geq 0$, the shot hits the ground at 2.56 seconds.

$$\begin{aligned}\mathbf{r}(2.56) &= 7 \times 2.56\hat{i} + \left(5 \times 2.56 + 6 \left(e^{-\frac{2.56}{2}} - 1\right)\right)\hat{j} \\ &\quad + (2 + 12 \times 2.56 - 5 \times 2.56^2)\hat{k} \\ \mathbf{r}(2.56) &= 17.896\hat{i} + 8.453\hat{j}\end{aligned}$$

$$\begin{aligned}|\mathbf{r}(2.56)| &= \sqrt{17.895^2 + 8.453^2} \\ &= 19.79 \text{ metres}\end{aligned}$$

$$\begin{aligned}\mathbf{\dot{r}}(t) &= 7\hat{i} + \left(5 - 3e^{-\frac{t}{2}}\right)\hat{j} + (12 - 10t)\hat{k} \\ \mathbf{\dot{r}}(0) &= 7\hat{i} + (5 - 3e^0)\hat{j} + 12\hat{k} \\ &= 7\hat{i} + 2\hat{j} + 12\hat{k}\end{aligned}$$

$$\begin{aligned}|\mathbf{\dot{r}}(0)| &= \sqrt{7^2 + 2^2 + 12^2} \\ &= \sqrt{197}\end{aligned}$$

The initial speed of projection is 14.04 m/s.

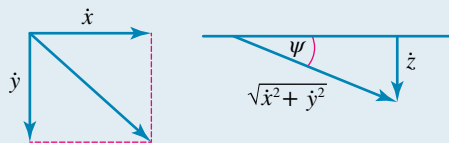
$$\begin{aligned}\mathbf{\dot{r}}(2.56) &= 7\hat{i} + \left(5 - 3e^{-\frac{2.56}{2}}\right)\hat{j} + (12 - 10 \times 2.56)\hat{k} \\ &= 7\hat{i} + 4.16\hat{j} - 13.56\hat{k}\end{aligned}$$

$$\begin{aligned}|\mathbf{\dot{r}}(2.56)| &= \sqrt{7^2 + 4.16^2 + (-13.56)^2} \\ &= \sqrt{250.34} \\ &\approx 15.82\end{aligned}$$

The required angle, ψ , is the angle between the downwards component and the combined east and north components.

$$\tan(\psi) = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$





4. Find the angle.

$$\tan(\psi) = \frac{13.56}{\sqrt{7^2 + 4.16^2}}$$

$$\psi = \tan^{-1}(1.665)$$

5. State the final result.

The shot hits the ground with a speed of 15.82 m/s at an angle of 59.015°.

study on

Units 3 & 4 > Area 2 > Sequence 3 > Concept 6

Projectile motion Summary screen and practice questions

Exercise 4.6 Projectile motion

Technology active

For the following questions give your answer to 2 decimal places, where a decimal answer is required.

- WE20** A soccer ball is kicked off the ground at an angle of $67^\circ 23'$ with an initial speed of 13 m/s. Calculate:
 - the time of flight
 - the range
 - the maximum height reached.
- A ball is thrown so that its time of flight is $\frac{10\sqrt{3}}{g}$ and the ratio of the range to the maximum height reached is $\frac{4\sqrt{3}}{3}$. Calculate the initial speed and angle of projection.

- WE21** A cricketer smashes a cricket ball at an angle of 35° to the horizontal from a point 0.5 m above the ground with an initial speed of 30 m/s.
 - Calculate the time of flight and the horizontal distance travelled.
 - Calculate the maximum height reached.
 - Determine the speed and angle at which the ball lands.
 - Show that the ball travels in a parabolic path.
- A cricket outfielder attempts to throw the cricket ball back towards the stumps. He throws the ball from a height of 1.8 m above the ground with an initial speed of 20 m/s at an angle of 20° to the horizontal.
 - Calculate the time of flight and the horizontal distance travelled.
 - Calculate the maximum height reached.
 - Determine the speed and angle at which the ball lands.
 - Show that the ball travels in a parabolic path.



5. **WE22** A tennis player hits the ball 4 feet above the baseline of a tennis court with a speed of 100 feet per second. If the ball travels in a vertical plane towards the centre of the net and just grazes the top of the net, calculate the angles at which it could have been hit. Data: Tennis court dimensions 78×27 feet
Net height at the centre 3 feet
Use $g = 32 \text{ ft/s}^2$.



6. A gun is mounted on top of a cliff face 122.5 metres high and fires a shell at a speed of 49 m/s in order to hit a target which is located at a horizontal distance of 346.482 metres from the base of the cliff. Determine:
- the angle of projection
 - the time of flight
 - the speed and angle at which the shell hits the target.
7. **WE23** A particle is projected at a given angle α from a point on a horizontal plane. If the particle reaches a maximum height of H , show that the time interval between the instants when the particle is at heights of $\frac{H}{2}$ is $2\sqrt{\frac{H}{g}}$.
8. A particle is projected with an initial speed of $3g$ m/s. The particle reaches heights of g metres at times t_1 and t_2 . Given that $t_2 - t_1 = 1$, show that the range of the particle is given by $\frac{9g\sqrt{3}}{2}$.
9. **WE24** A football is kicked by a footballer. At a time t in seconds measured from the point of impact, the position vector $\underline{r}(t)$ of the tip of the football is given by



$$\underline{r}(t) = 60(1 - e^{-\frac{t}{2}})\hat{i} + 2t\hat{j} + (1 + 12t - 4.9t^2)\hat{k}$$

where $\hat{i}$ is a unit vector horizontally forward, $\hat{j}$ is a unit vector to the right and $\hat{k}$ is a unit vector vertically up.

The origin, O , of the coordinate system is at ground level and all displacements are measured in metres.

- Calculate how far from the origin the football hits the ground.
 - Calculate the speed and angle at which the football hits the ground.
10. A javelin is thrown by an athlete on level ground. The time t is in seconds, measured from the release of the javelin. The position vector $\underline{r}(t)$ of the tip of the javelin is given by
- $$\underline{r}(t) = 20t\hat{i} + \left(2\pi t - 3\sin\left(\frac{\pi t}{6}\right)\right)\hat{j} + (1.8 + 14.4t - 5t^2)\hat{k}$$
- where $\hat{i}$ is a unit vector in the east direction, $\hat{j}$ is a unit vector in the north direction and $\hat{k}$ is a unit vector vertically up. The origin, O , of the coordinate system is at ground level, and all displacements are measured in metres.
- Calculate how long will it take for the javelin to strike the ground.
 - Determine how far from O the tip of the javelin hits the ground.
 - Calculate the speed and angle at which the javelin's tip strikes the ground.
11. **a.** A cricket ball is hit by a batsman off his toes with an initial speed of 20 m/s at an angle of 30° with the horizontal. Determine:
- the time of flight
 - the range on the horizontal plane
 - the greatest height reached.

- b. A golf ball is hit at an angle of 20° and its range is 150 metres. Determine:
- the initial speed of projection
 - the greatest height reached
 - the time of flight.
12. a. A rock is thrown horizontally from the top of a cliff face and strikes the ground 30 metres from the base of the cliff after 1 second.
- Determine the height of the cliff.
 - Determine the initial speed of projection.
 - Determine the speed and angle at which the rock strikes the ground.
 - Show that the rock travels in a parabolic path.
- b. A motorcycle is driven at 150 km/h horizontally off the top of a cliff face and strikes the ground after 2 seconds. Determine:
- the height of the cliff
 - how far from the base of the cliff the motorcycle strikes the ground
 - the speed and angle at which it strikes the ground.
13. a. An object is projected from the top of a building 100 metres high at an angle of 45° with a speed of 10 m/s. Determine:
- the time of flight to reach ground level
 - how far from the edge of the building the object strikes the ground
 - the greatest height the object reaches above the building
 - the speed and angle at which the object strikes the ground.
- b. A stone is thrown with a speed of 15 m/s at an angle of 20° from a cliff face and strikes the ground after 3.053 seconds. Determine:
- the height of the cliff
 - how far from the edge of the cliff the stone strikes the ground
 - the greatest height the stone reaches above the ground
 - the speed and angle at which the stone strikes the ground.
14. a. A rugby player uses a place kick to kick the ball off the ground. To score a goal, the ball must pass through a point 40 metres out and 8 metres vertically above the point of release. If he kicks the ball at a speed of 30 m/s, calculate the possible angles of projection.
- b. A catapult throws a stone with a speed of 7 m/s from a point at the top of a cliff face 40 metres high to hit a ship which is at a horizontal distance of 20 metres from the base of the cliff. Calculate the possible angles of projection.
15. a. A shell is fired with an initial speed of 147 m/s for maximum range. There is a target 2 km away on the same horizontal level.
- How far above the target does the shell pass?
 - How far beyond the target does the shell strike the ground?
 - What is the minimum speed of projection required to just reach the target?
- b. A projectile falls a metres short of its target when fired at an angle of α , and a metres beyond the target when fired at the same muzzle velocity but at an angle of β . If θ is the angle required for a direct hit on the target, show that $\sin(2\theta) = \frac{1}{2}(\sin(2\alpha) + \sin(2\beta))$.



16. a. A gun is mounted on top of a cliff face 122.5 metres high and fires a shell at a speed of 49 m/s in order to hit a target that is located at a horizontal distance of 346.482 metres from the base of the cliff. Calculate:
- the angle of projection
 - the time of flight
 - the speed and angle at which the shell hits the target.
- b. A baseball is initially hit from a distance of 1.5 metres above the ground at an angle of 35° . The ball reaches a maximum height of 8 metres above the ground. Calculate:



- the initial speed of projection
 - the distance of the outfielder from the hitter if the outfielder just catches the ball at ground level
 - the speed and angle at which the baseball strikes the fielder's hands.
17. a. A projectile is fired with an initial speed of $\sqrt{2ga}$ to hit a target at a horizontal distance of a from the point of projection and a vertical distance of $\frac{a}{2}$ above it. Show that there are two possible angles of projection, that the ratio of the two times to hit the target is $\sqrt{5}$, and that the ratio of the maximum heights reached in each case is $\frac{9}{5}$.
- b. A projectile is fired with an initial speed of $\sqrt{2ga}$ at an angle of α to hit a target at a horizontal distance of a from the point of projection and a vertical distance of b .
- Show that $a \tan^2(\alpha) - 4a \tan(\alpha) + (4b + a) = 0$.
 - Show that it is impossible to hit the target if $4b > 3a$.
 - Show that if $4b = 3a$, then $\alpha = \tan^{-1}(2)$.
18. a. A particle is projected to just clear two walls. The walls are both of height 6 metres and are at distances of 5 metres and 10 metres from the point of projection. Show that if α is the angle of projection, then $\tan(\alpha) = \frac{9}{5}$.
- b. A particle is projected to just clear two walls. The walls are both h metres high and are at distances of a metres and b metres from the point of projection. Show that the angle of projection is $\tan^{-1} \left(\frac{h(a+b)}{ab} \right)$.
- c. A particle is projected to just clear two walls. The first wall is h_1 metres high and at a distance of a metres from the point of projection; the second wall is h_2 metres high and b metres from the point of projection, where $b > a > 0$. Show that the angle of projection is $\tan^{-1} \left(\frac{b^2 h_1 - a^2 h_2}{ab(b-a)} \right)$.
19. A soccer ball is kicked off the ground. Its position vector is given by

$$\underline{r}(t) = 6t\hat{i} + 28t\hat{j} + \frac{12\sqrt{2}}{5} \sin\left(\frac{\pi t}{2}\right)\hat{k}, \text{ where } \hat{i} \text{ and } \hat{j}$$

are unit vectors in the horizontal plane at right angles to each other, and $\hat{k}$ is a unit vector in the vertical direction. Displacements are measured in metres, and t is the time in seconds after the ball is kicked.

- Calculate the initial speed at which the soccer ball is kicked.
- At what angle from the ground was the soccer ball kicked?
- Determine the maximum height reached by soccer ball.



- d. After the soccer ball has reached its maximum height and is on its downwards trajectory, a player jumps and heads the soccer ball when it is 2.4 metres above the ground.
- What time has elapsed from the instant that the soccer ball is kicked until the player heads the ball?
 - How far, measured along the ground, was the player from where the ball was kicked?
 - If the soccer ball has a mass of 430 grams, calculate the magnitude of the momentum in kg m/s of the ball at the instant when it strikes the player's head. Give your answer correct to 2 decimal places.
20. a. An object is thrown horizontally with a speed of V m/s from the top of a cliff face h metres high and strikes the ground at a distance of R metres from the base of the cliff after a time of T seconds. Show that:
- $T = \sqrt{\frac{2h}{g}}$
 - $R = VT$
 - the speed at which the object hits the ground is $\sqrt{V^2 + 2gh}$
 - the angle at which the object strikes the ground is $\tan^{-1}\left(\frac{2h}{R}\right)$
 - the object travels in a parabolic path, $y = h\left(1 - \frac{x^2}{R^2}\right)$.
- b. A stone is projected horizontally from the top of a cliff of height H metres with a speed of U m/s. At the same instant another object is fired from the base of the cliff with a speed of V m/s at an angle of α . Given that the two objects collide after a time of T seconds, show that

$$\alpha = \cos^{-1}\left(\frac{U}{V}\right) \text{ and } V^2 = U^2 + \frac{H^2}{T^2}.$$

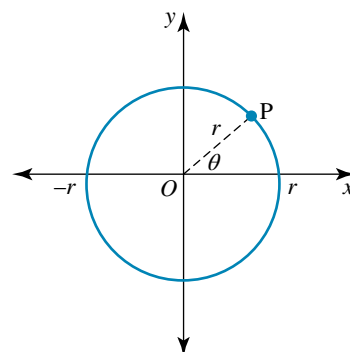
21. A projectile is fired at an angle of α with an initial speed of V . It reaches a maximum height of H and has a horizontal range of R . Show that:
- $R \tan(\alpha) = \frac{1}{2}gT^2$
 - $\tan(\alpha) = \frac{4H}{R}$
 - $T^2 = \frac{8H}{g}$
 - $V = \sqrt{2g\left(H + \frac{R^2}{16H}\right)}$
 - the equation of the path is $y = \frac{4Hx}{R^2}(R - x)$ for $0 \leq x \leq R$.

4.7 Circular motion

4.7.1 Uniform circular motion

Parametric equations in the form $x = (r \cos(t))\hat{i}$ and $y = (r \sin(t))\hat{j}$ were studied in the first section of this chapter when describing the position of particle. The Cartesian form of this equation is given by $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$, which is the general form of a circle.

A particle P travelling at a constant speed along a circular path of radius r is defined as being in **uniform circular motion**. Consider the diagram on the right, which exhibits the Cartesian equation of the path defined as $x^2 + y^2 = r^2$, where $r > 0$.



Position of a particle on a circle

The position vector of particle, P , on a circle of radius, r , can be defined as

$$\begin{aligned}\underline{r} &= \overrightarrow{OP} \\ &= x\hat{i} + y\hat{j} \\ &= (r \cos \theta)\hat{i} + (r \sin \theta)\hat{j}\end{aligned}$$

Where θ is the angle r makes with the positive x -axis. The position of the particle varies according to time, t , hence $\theta = f(t)$. Note that when $\theta = 0$, $t = 0$.

WORKED EXAMPLE 25

Given the vector equation, $\underline{r}(t) = \cos 6t\hat{i} + \sin 6t\hat{j}$, for $t \geq 0$, determine:

a. the velocity vector, $\underline{v}(t)$

b. the acceleration vector, $\underline{a}(t)$.

THINK

a. 1. To obtain the velocity vector, differentiate the position vector $\underline{r}(t)$.

2. State velocity vector.

b. 1. Differentiate the velocity vector to obtain the acceleration vector.

2. State acceleration vector.

WRITE

$$\begin{aligned}\text{a. } \underline{v}(t) &= \dot{\underline{r}}(t) \\ &= \frac{d}{dt}(\cos(6t))\hat{i} + \frac{d}{dt}(\sin(6t))\hat{j} \\ &= -6 \sin(6t)\hat{i} + 6 \cos(6t)\hat{j} \\ \therefore \underline{v}(t) &= -6 \sin(6t)\hat{i} + 6 \cos(6t)\hat{j}\end{aligned}$$

$$\begin{aligned}\text{b. } \underline{a}(t) &= \ddot{\underline{r}}(t) \\ &= \frac{d}{dt}(-6 \sin(6t))\hat{i} + \frac{d}{dt}(6 \cos(6t))\hat{j} \\ &= -36 \cos(6t)\hat{i} - 36 \sin(6t)\hat{j} \\ \therefore \underline{a}(t) &= -36(\cos(6t)\hat{i} + \sin(6t)\hat{j})\end{aligned}$$

4.7.2 Angular velocity

The rate at which the position vector $\overrightarrow{OP}$ of a rotating particle changes with respect to time is given by the derivative $\frac{d\theta}{dt}$. This is referred to as the **angular velocity** of the particle and is often denoted by the

lowercase Greek letter omega, ω , so that $\frac{d\theta}{dt} = \omega$. Assuming the particle moves at a constant speed around the circle, then the angular velocity will be constant. By integrating the equation, $\int \frac{d\theta}{dt} = \omega$, with respect to t , we obtain $\theta = \omega t$ (as ω is constant and $\theta = 0$ when $t = 0$).

Angular velocity

The position of a particle, P , moving with angular velocity ω can be described by the vector,

$$\underline{r}(t) = r \cos(\omega t)\hat{i} + r \sin(\omega t)\hat{j}$$

Where the unit of angular velocity is radians per second.

4.7.3 Velocity

In the Worked example 25, the three equations of motion were,

- Position: $\underline{r}(t) = (\cos(6t))\hat{i} + (\sin(6t))\hat{j}$, for $t \geq 0$
- Velocity: $\underline{v}(t) = -6 \sin(6t)\hat{i} + 6 \cos(6t)\hat{j}$, for $t \geq 0$
- Acceleration: $\underline{a}(t) = -36(\cos(6t)\hat{i} + \sin(6t)\hat{j})$, for $t \geq 0$

Can any generalisations be made about these equations?

Velocity vector

If particle P is travelling in a circular path of radius r at a constant speed, the following statements can be made:

1. The velocity vector, $\underline{v}(t)$, can be defined as: $\underline{v}(t) = -r\omega \sin(\omega t)\hat{i} + r\omega \cos(\omega t)\hat{j}$
2. $\underline{v}(t)$ is perpendicular to $\underline{r}(t)$
3. The magnitude of velocity (speed) of the particle is defined as $|\underline{v}(t)| = v = r\omega$

The proofs to these three statements exist below.

1. The velocity vector

$$\begin{aligned}\underline{v}(t) &= \frac{d\underline{r}}{dt} \\ &= \frac{d}{dt} ((r \cos(\omega t))\hat{i} + (r \sin(\omega t))\hat{j}) \\ &= (-r\omega \sin(\omega t))\hat{i} + (r\omega \cos(\omega t))\hat{j} \\ &= (-r\omega \sin(\theta))\hat{i} + (r\omega \cos(\theta))\hat{j} \quad \text{where } \theta = \omega t\end{aligned}$$

2. $\underline{v}(t)$ is perpendicular to $\underline{r}(t)$

If the velocity and position vectors are perpendicular, the dot product will equal zero.

$$\begin{aligned}\underline{r} \cdot \underline{v} &= ((r \cos(\omega t))\hat{i} + (r \sin(\omega t))\hat{j}) \cdot ((-r\omega \sin(\omega t))\hat{i} + (r\omega \cos(\omega t))\hat{j}) \\ &= (r \cos(\omega t))(-r\omega \sin(\omega t)) + (r \sin(\omega t))(r\omega \cos(\omega t)) \\ &= -r^2\omega \cos(\omega t) \sin(\omega t) + r^2\omega \cos(\omega t) \sin(\omega t) \\ &= 0 \\ \therefore \underline{r}(t) &\perp \underline{v}(t)\end{aligned}$$

3. The magnitude of velocity (speed) of the particle is defined as $|\underline{v}(t)| = v = r\omega$

$$\begin{aligned}v &= |\underline{v}(t)| \\ &= \sqrt{(-r\omega \sin(\omega t))^2 + (r\omega \cos(\omega t))^2} \\ &= \sqrt{r^2\omega^2 (\sin^2(\omega t) + \cos^2(\omega t))} \\ &= \sqrt{r^2\omega^2 \times 1} \\ &= r\omega\end{aligned}$$

WORKED EXAMPLE 26

A particle of mass 6 kg rotates with a constant speed of 10 m/s in a circular path of radius 5 metres. Determine:

- a. the angular velocity of the particle
- b. the position vector of the particle after 7 seconds if $\theta = 0$ when $t = 0$
- c. the velocity and acceleration vectors after 7 seconds.

THINK

- a. Recall the formula for speed. Rearrange the equation to make ω the subject. Substitute in values for the speed (v) and radius (r) to calculate the angular velocity.
- b. 1. Substitute in values for ω and r to develop the position vector $\underline{r}(t) = (r \cos(\omega t))\hat{i} + (r \sin(\omega t))\hat{j}$
2. Determine the position of the particle at 7 seconds by letting $t = 7$.
- c. 1. Calculate the velocity vector by differentiating the position vector.
2. Determine the velocity vector of the particle at 7 seconds by letting $t = 7$.
3. Calculate the acceleration vector by differentiating the velocity vector.
4. Determine the acceleration vector of the particle at 7 seconds by letting $t = 7$.

WRITE

a. $v = r\omega$
 $\omega = \frac{v}{r}$
 $= \frac{10}{5}$
 $= 2 \text{ rad/s}$

b. At $t = 0$, $\theta = 0$.
 $\underline{r}(t) = (r \cos(\omega t))\hat{i} + (r \sin(\omega t))\hat{j}$
 $= (5 \cos(2t))\hat{i} + (5 \sin(2t))\hat{j}$
 $\underline{r}(7) = (5 \cos(2 \times 7))\hat{i} + (5 \sin(2 \times 7))\hat{j}$
 $\approx 0.684\hat{i} + 4.953\hat{j}$

c. $\underline{\dot{r}}(t) = \frac{d}{dt}((5 \cos(2t))\hat{i} + (5 \sin(2t))\hat{j})$
 $= (-10 \sin(2t))\hat{i} + (10 \cos(2t))\hat{j}$
 $\underline{\dot{r}}(7) = (-10 \sin(2 \times 7))\hat{i} + (10 \cos(2 \times 7))\hat{j}$
 $\approx -9.906\hat{i} + 1.367\hat{j}$

$\underline{\ddot{r}}(t) = \frac{d}{dt}((-10 \sin(2t))\hat{i} + (10 \cos(2t))\hat{j})$
 $= (-20 \cos(2t))\hat{i} + (-20 \sin(2t))\hat{j}$
 $\underline{\ddot{r}}(7) = (-20 \cos(2 \times 7))\hat{i} + (-20 \sin(2 \times 7))\hat{j}$
 $\approx -2.735\hat{i} - 19.812\hat{j}$

WORKED EXAMPLE 27

A particle moves in a circular path of radius 6 metres, completing 120 revolutions per minute. Calculate:

- a. the angular velocity in radians per second
 b. the speed of the particle in metres per second.

THINK

- a. Convert one minute to seconds. Calculate the angular velocity given one revolution is 2π radians.
- b. Recall the formula for speed (v) and substitute values for r and ω .

WRITE

a. $\omega = 120 \text{ revolutions per minute (rpm)}$
 $= \frac{120 \times 2\pi}{60}$
 $= 4\pi \text{ rad/s}$

b. $v = r\omega$
 $= 6 \times 4\pi$
 $= 24\pi \text{ m/s } (v \approx 75.398 \text{ m/s})$

Period of circular motion

The period of the motion is the time taken, T , to complete one revolution and is given by

$$T = \frac{2\pi}{\omega}. \text{ Where } \omega \text{ is the angular velocity.}$$

4.7.4 Acceleration

Acceleration of circular motion

If P is travelling in a circular path of radius r at a constant speed, then the following statements can be made about the acceleration vector, $\underline{a}(t)$:

1. The acceleration vector, $\underline{a}(t)$, can be defined as: $\underline{a}(t) = -\omega^2 \underline{r}(t)$
2. The magnitude of acceleration is given by, $|\underline{a}(t)| = \frac{v^2}{r}$

The proofs to these statements exist below.

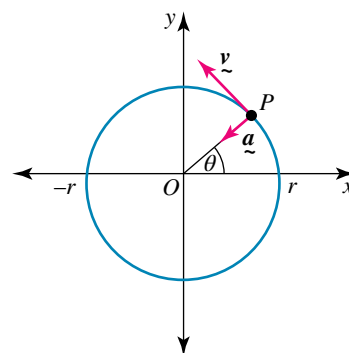
$$1. \underline{a}(t) = -\omega^2 \underline{r}(t)$$

$$\begin{aligned}\underline{a}(t) &= \frac{d\underline{v}}{dt} \\ &= \frac{d}{dt} ((-r\omega \sin(\omega t))\hat{i} + (r\omega \cos(\omega t))\hat{j}) \\ &= -r\omega^2 \cos(\omega t)\hat{i} - r\omega^2 \sin(\omega t)\hat{j} \\ &= -\omega^2 (r \cos(\omega t)\hat{i} + r \sin(\omega t)\hat{j}) \\ &= -\omega^2 \underline{r}(t)\end{aligned}$$

Consider the diagram. The acceleration vector is heading to the centre of its circular path (referred to as centripetal acceleration).

$$2. \text{ The magnitude of acceleration is given by, } |\underline{a}(t)| = \frac{v^2}{r}$$

$$\begin{aligned}a &= |\underline{a}(t)| \\ &= \sqrt{(-\omega^2 r \cos(\omega t))^2 + (-\omega^2 r \sin(\omega t))^2} \\ &= \sqrt{\omega^4 r^2 \cos^2(\omega t) + \omega^4 r^2 \sin^2(\omega t)} \\ &= \sqrt{\omega^4 r^2 (\cos^2(\omega t) + \sin^2(\omega t))} \\ &= \sqrt{\omega^4 r^2} \\ &= \omega^2 r \\ &= \frac{v^2}{r^2} r \quad \left(\text{given } \omega = \frac{v}{r} \right) \\ &= \frac{v^2}{r}\end{aligned}$$



WORKED EXAMPLE 28

A particle is travelling in a circular path with a radius of 2 metres at a constant speed. The period of the motion is 4 seconds. Calculate:

- a. the angular velocity
- b. the speed of the particle
- c. the magnitude of the acceleration after 6 seconds.

THINK

- a. The period of motion is given as 4 seconds. Determine the angular velocity, ω , by recalling the formula for the time T for a revolution.

- b. Recall the equation for the magnitude of velocity (speed). Substitute in values for r and ω to calculate v .

- c. Recall either formula for the magnitude of acceleration $|a| = \omega^2 r$ or $a = \frac{v^2}{r}$. Calculate the magnitude by substituting in values for ω and r .

WRITE

$$\text{a. } T = \frac{2\pi}{\omega}$$

$$\omega = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{4}$$

$$\omega = \frac{\pi}{2} \text{ rad/s}$$

$$\begin{aligned} \text{b. } v &= r\omega \\ &= 2 \times \frac{\pi}{2} \\ &= \pi \text{ m/s} \end{aligned}$$

$$\begin{aligned} \text{c. } a &= \omega^2 r & \left(a = \frac{v^2}{r} \text{ is also applicable} \right) \\ a &= \left(\frac{\pi}{2} \right)^2 \times 2 \\ a &= \frac{\pi^2}{2} \text{ m/s}^2 \end{aligned}$$

studyon

Units 3 & 4 > Area 2 > Sequence 3 > Concept 7

Circular motion Summary screen and practice questions

Exercise 4.7 Circular motion

Technology free

- WE25** Given the vector equation, $\underline{r}(t) = (\cos(7t))\hat{i} + (\sin(7t))\hat{j}$ for $t \geq 0$, determine:
 - the velocity vector, $\underline{v}(t)$
 - the acceleration vector, $\underline{a}(t)$.
- Given the vector equation, $\underline{r}(t) = (2 \cos(3t))\hat{i} + (2 \sin(3t))\hat{j}$ for $t \geq 0$, determine:
 - the velocity vector, $\underline{v}(t)$
 - the acceleration vector, $\underline{a}(t)$.
- A particle rotates with constant speed in a circular path according to the position vector, $\underline{r}(t) = (5 \cos(2t))\hat{i} + (5 \sin(2t))\hat{j}$, for $t \geq 0$.
 - Determine the radius of the circular path.
 - Calculate the speed, v , at which the particle moves.
- WE26** A particle of mass 8 kg rotates with a constant speed of 12 m/s in a circular path of radius 8 metres. Determine the following:
 - the angular velocity of the particle
 - the position vector of the particle after 4 seconds if $\theta = 0$ when $t = 0$
 - the velocity and acceleration vectors after 4 seconds.

Technology active

5. An object travels in a circle of radius 2 metres at a constant speed of 4 m/s. Calculate the following:
 - a. the angular velocity of the particle
 - b. the position vector of the particle after 3 seconds if $\theta = 0$ when $t = 0$
 - c. the velocity and acceleration vectors after 3 seconds.
6. A car travels around a circular track at a constant speed of 24 m/s. The radius of the circle is 80 metres. Determine the following:
 - a. the angular velocity of the particle
 - b. the position vector of the particle after 1 second if $\theta = 0$ when $t = 0$
 - c. the velocity and acceleration vectors after 1 second
 - d. the magnitude of the acceleration after 1 second
 - e. the period of revolution.
7. A toy train travels in a circular path of radius 3 metres at a constant speed. The train completes 2 revolutions each second. Calculate:
 - a. the speed of the object
 - b. the magnitude of the acceleration of the train.
8. **WE27** A particle moves in a circular path of radius 24 metres, completing 360 revolutions per minute. Calculate:
 - a. the angular velocity in radians per second
 - b. the speed of the particle in metres per second.
9. A particle connected to a string 4 metres long is moving in a circular path at a speed of 35 metres per second. Calculate the angular velocity of the particle.
10. A satellite travels in a circular orbit with a period of 13 hours. The radius of orbit is 29 000 km. Determine:
 - a. the angular velocity
 - b. the satellite's speed.
11. **WE28** A particle is travelling in a circular path with a radius of 7 metres at a constant speed. The period of the motion is 124 seconds. Calculate:
 - a. the angular velocity
 - b. the speed of the particle
 - c. the magnitude of the acceleration after 3 seconds.
12. A particle is travelling in a circular path with a radius of 8 metres at a constant speed. The period of the motion is 5 seconds. Calculate:
 - a. the angular velocity
 - b. the speed of the particle
 - c. the magnitude of the acceleration after 9 seconds.
13. The diameter of the wheel of a unicycle is 1.2 metres and is rotating at 60 rpm. A small stone is lodged in the tread on the outer edge of the wheel. Calculate:
 - a. the period
 - b. the angular velocity
 - c. the speed of the stone on the outer edge
 - d. the magnitude of the acceleration of the stone on the outside edge of the wheel.
14. Kristen is spinning her 1.2 metre diameter umbrella horizontally at a rate of 2 revolutions in 4 seconds. If she holds the rim 1.1 m above the ground, calculate the diameter of the horizontal circle where the droplets meet the ground. (The drops leave the horizontal rim of the umbrella tangentially).



4.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** A stone is attached to string and is swung around in a circle at a constant speed of 4π m/s. The radius of the circular path is 16π metres. The magnitude of the acceleration of the stone is:

- A. $\frac{\pi}{3}$ m/s²
B. $\frac{\pi}{12}$ m/s²
C. π m/s²
D. 16 m/s²

2. **MC** Consider the vector equation, $\underline{r}(t) = (1 + 6 \cos(t))\hat{i} + (6 \sin(t) - 5)\hat{j}$, for $t \geq 0$. The Cartesian equation is defined by:

- A. $\frac{(x-1)^2}{36} + \frac{(y+5)^2}{36} = 1$
B. $\frac{(x+1)^2}{6} + \frac{(y+5)^2}{6} = 1$
C. $\frac{(x-1)^2}{36} + \frac{(y-5)^2}{36} = 1$
D. $\frac{(x+1)^2}{36} + \frac{(y+5)^2}{36} = 1$

3. **MC** A particle moves in such a way that its velocity vector at a time t is given by $4e^{\frac{t}{2}}\hat{i} - 2\sin\left(\frac{t}{2}\right)\hat{j}$.

Initially the position vector of the particle is $3\hat{i} - 3\hat{j}$. The position vector of the particle at a time t is given by:

- A. $\left(8e^{\frac{t}{2}} - 5\right)\hat{i} - \left(4\cos\left(\frac{t}{2}\right) + 3\right)\hat{j}$
B. $\left(8e^{\frac{t}{2}} - 5\right)\hat{i} - \left(2 + \cos\left(\frac{t}{2}\right)\right)\hat{j}$
C. $\left(8e^{\frac{t}{2}} - 5\right)\hat{i} + \left(4\cos\left(\frac{t}{2}\right) - 7\right)\hat{j}$
D. $\left(1 + 2e^{\frac{t}{2}}\right)\hat{i} - \left(\cos\left(\frac{t}{2}\right) - 4\right)\hat{j}$

4. **MC** A particle moves so that its position vector at a time t , is given by $\underline{r}(t) = 2t\hat{i} + 2\sqrt{t}\hat{j}$ for $t \geq 0$. The distance of the particle from the origin after 4 seconds is equal to:

- A. $4\sqrt{5}$
B. $\frac{3}{2}$
C. $\frac{5}{2}$
D. 8

5. **MC** A golf ball is hit off the ground, its position vector, at a time t seconds after being hit, is given by

$$\underline{r}(t) = 15\sqrt{2}t\hat{i} + \left(15\sqrt{2}t - \frac{1}{2}gt^2\right)\hat{k} \text{ for } t \geq 0, \text{ where } \hat{i} \text{ is a unit vector in metres horizontally forward,}$$

and $\hat{k}$ is a unit vector in metres vertically upwards. Students, when analysing the motion of the golf ball, stated some propositions:

- Alicia stated that the golf ball is hit with an initial velocity of 30 m/s for maximum range.
- Betty stated that the golf ball hits the ground again after a time of 4.33 seconds.
- Colin stated that the golf ball reaches a maximum height of 22.96 metres.
- David stated that the golf ball first hits the ground at a distance of 91.84 metres from where it was hit.
- Edward stated that the golf ball travels in a parabolic path.

Then:

- A. only Alicia, Betty and Edward are correct
B. only Alicia, Betty and Colin are correct
C. only Alicia, Betty, Colin and David are correct
D. all of Alicia, Betty, Colin, David and Edward are correct.

6. Two particles, P and Q , have position vectors $\underline{p} = 3 \cos(2t)\hat{i} + 4 \sin(2t)\hat{j}$ and $\underline{q} = 4 \sin(nt)\hat{i} + 3 \cos(nt)\hat{j}$ respectively at a time, t seconds, $t \geq 0$.
 - a. Determine the Cartesian equation of the paths for both P and Q .
 - b. Determine the coordinates where the paths of P and Q cross.
7. A particle moves so that at time t its position vector is given by $\underline{r}(t) = 8 \cos(t)\hat{i} + 6 \sin(t)\hat{j}$, for $0 \leq t \leq 2\pi$. The particle moves along a curve C . Determine:
 - a. the coordinates of the point A when $t = \frac{\pi}{4}$
 - b. the coordinates of the point B when $t = \frac{5\pi}{6}$
 - c. the Cartesian equation of the curve C .
8. Two particles move so that their position vectors are given by $\underline{r}_A(t) = (2t + 5)\hat{i} + (t^2 - 6t + 48)\hat{j}$ and $\underline{r}_B(t) = (t + 11)\hat{i} + (7t + 6)\hat{j}$ for $t \geq 0$. Calculate:
 - a. when and where the particles collide
 - b. the distance between the particles when $t = 9$.
9. A particle moves along a curve defined by $\underline{r}(t) = (2 + 3 \sec(t))\hat{i} + (4 \tan(t) - 1)\hat{j}$ for $t \geq 0$. Determine the Cartesian equation of the path, and sketch the graph, stating its domain and range.
10. The acceleration of a body moving in a straight line is given by: $a(t) = 8 - 14t \text{ m/s}^2$. Initially, the body is at rest at $x = 5$. Calculate:
 - a. the velocity at any time
 - b. the position at any time
 - c. the time and position when the velocity is 0 m/s.
11. A particle moves so that its position vector is given by $\underline{r}(t) = 3t^4\hat{i} + 2t^2\hat{j} + e^{2t}\hat{k}$. Determine:
 - a. the velocity vector
 - b. the acceleration vector.
12. The acceleration of a particle is given by $\underline{\ddot{r}}(t) = (6t - 4)\hat{i}$, where $t \geq 0$ is the time. If initially the velocity vector is $\underline{\dot{r}}(t) = 5\hat{j}$ and the initial position is $\underline{r}(0) = 4\hat{i} - 2\hat{j} + \hat{k}$, calculate the position vector at time t .

Complex familiar

13. A particle of mass 13 kg rotates with a constant speed of 36 m/s in a circular path of radius 12 metres. Determine the following:
 - a. the angular velocity of the particle
 - b. the position vector of the particle after 4 seconds if $\theta = 0$ when $t = 0$
 - c. the acceleration vector after 4 seconds.
14. A ball is dropped from a tower that is 60 metres above the ground level.
 - a. Calculate how long it takes the stone to reach the ground.
 - b. Determine the speed of the stone on impact.
15. A girl throws a ball from the top of a hill 3 metres above ground level with an initial speed of 12 m/s at an angle of 42° to the horizontal.
 - a. Calculate the time of flight and the horizontal distance.
 - b. Calculate the maximum height reached.
 - c. Determine the speed and angle at which the ball lands.
16. A particle rotates with constant speed in a circular path according to the position vector, $\underline{r}(t) = (10 \cos(5t))\hat{i} + (10 \sin(5t))\hat{j}$, for $t \geq 0$.
 - a. Determine the radius of the circular path.
 - b. Calculate the speed, v , at which the particle moves.

Complex unfamiliar

17. a. When a projectile is fired at an angle of α with an initial speed of V from a cliff of height h for maximum range R , provided that $V < \sqrt{gR}$, show that:

i. $\tan(\alpha) = \frac{V^2}{gR}$

ii. $h = \frac{gR^2 - V^4}{2V^2g}$

iii. $R = \frac{V}{g} \sqrt{V^2 + 2gH}$

iv. $T = \frac{\sqrt{2(V^2 + gh)}}{g}$.

- b. A shot-putter can throw the shot with a release speed of 20 m/s. The shot leaves her hand 2 metres above the ground. Calculate the maximum range, the angle of projection and the time of flight.
18. A curve given by the Cartesian equation $y = \frac{abx}{x^2 + a^2}$, where a and b are positive constants, is called the serpentine curve.

- a. Show that $\frac{dy}{dx} = \frac{ab(a-x)(a+x)}{(x^2 + a^2)^2}$ and that the graph of the serpentine curve has turning points at

$$\left(a, \frac{b}{2}\right) \text{ and } \left(-a, -\frac{b}{2}\right).$$

- b. Show that $\frac{d^2y}{dx^2} = \frac{2abx(x^2 - 3a^2)}{(x^2 + a^2)^3}$ and hence show that the points of inflection on the serpentine curve are given by $\left(\pm\sqrt{3}a, \pm\frac{\sqrt{3}b}{4}\right)$.

- c. Show that the vector equation of the serpentine curve is given by $\underline{r}(t) = a \cot(t)\hat{i} + \frac{b}{2} \sin(2t)\hat{j}$ for $t \geq 0$.

- d. Sketch the graph of the serpentine curve.

19. a. The position vector of a moving particle is given by $\underline{r}(t) = 2 \sin(t)\hat{i} + (2 \sin(t) \tan(t))\hat{j}$, for $t > 0$.

Show that the particle moves along the curve $y = \frac{x^2}{\sqrt{4 - x^2}}$.

- b. A curve called the Witch of Agnesi is defined by the parametric equations $x = at$ and $y = \frac{a}{1 + t^2}$.

Show that Cartesian equation is given by $y = \frac{a^3}{a^2 + x^2}$.

20. a. A particle moves so that its vector equation is given by $\underline{r}(t) = (5t - 2)\hat{i} + (12t - 2)\hat{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.
- b. A particle moves so that its vector equation is given by $\underline{r}(t) = (at + b)\hat{i} + (ct + d)\hat{j}$ for $t \geq 0$. Find the closest distance of the particle from the origin.

study on

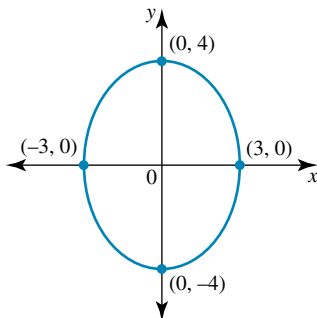
Units 3 & 4 Sit exam

Answers

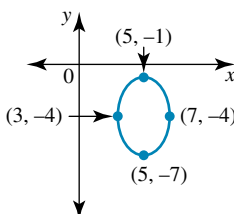
4 Vector calculus

Exercise 4.2 Position vectors as functions of time: circles, ellipses and hyperbolas

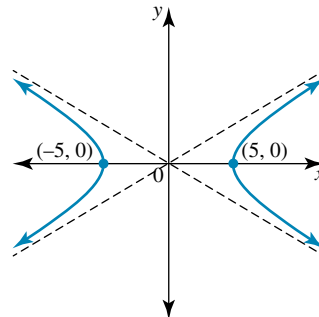
1. a. $\sqrt{205}$ b. $\sqrt{10t^2 - 10t + 5}$
c. $\frac{\sqrt{10}}{4}$
2. a. $5\sqrt{2}$ b. 9
3. a. $t = 5, (16, 40)$
b. $(36, 180)$
c. $5\sqrt{5}$
4. a. $t = 4, (10, 7)$ b. $(-2, 13)$
c. $\sqrt{41}$
5. a. i. $5\sqrt{2}$ ii. $\sqrt{5t^2 - 10t + 10}$
iii. $\sqrt{5}$
b. i. $5\sqrt{5}$ ii. $5\sqrt{t^2 + 1}$
iii. 5
6. a. i. $\sqrt{41}$ ii. $\sqrt{5}$ iii. 0, 2
b. i. $\frac{9\sqrt{2}}{2}$ ii. 8
7. a. $a = 4, b = -3, c = 1, d = 3$
b. $a = 2, b = 5, c = 1, d = -3$
8. a. $t = 15, (2, 5)$ b. $(8, -7)$ c. 15
9. a. $t = 2, (-2, 6); t = 5, (19, 12)$
b. $8\sqrt{26}$
10. a. $t = 12, (53, 74)$ b. $(89, 56)$
c. $80\sqrt{5}$
11. a. $\sqrt{2}$ b. $\sqrt{2}$
12. Ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$; domain $[-3, 3]$, range $[-4, 4]$



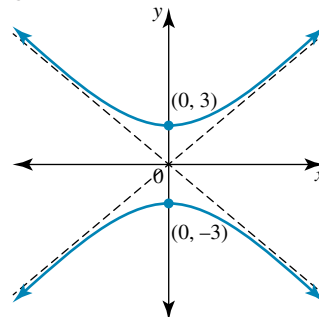
13. Ellipse $\frac{(x-5)^2}{4} + \frac{(y+4)^2}{9} = 1$; domain $[3, 7]$, range $[-7, -1]$



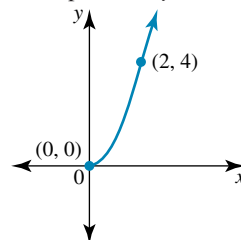
14. Hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$, asymptotes $y = \pm \frac{3x}{5}$; domain $(-\infty, -5] \cup [5, \infty)$, range R



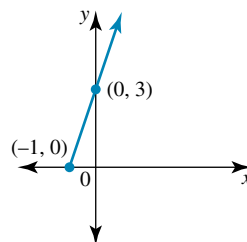
15. Hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$, asymptotes $y = \pm \frac{3x}{4}$; domain R , range $(-\infty, -3] \cup [3, \infty)$



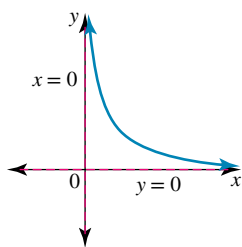
16. Sample responses can be found in the worked solutions in the online resources.
17. Sample responses can be found in the worked solutions in the online resources.
18. a. Part of a parabola, $y = x^2$; domain $[0, \infty)$, range $[0, \infty)$



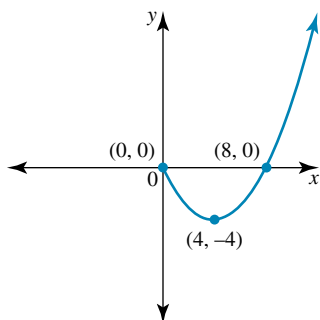
- b. Part of a straight line, $y = 3x + 3$; domain $[-1, \infty)$, range $[0, \infty)$



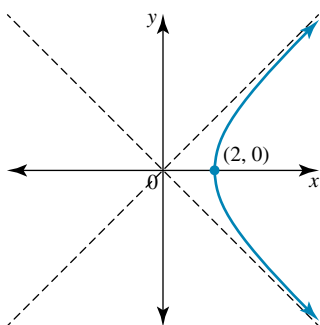
19. a. $y = \frac{2}{x}$; domain $[0, \infty)$, range $[0, \infty)$



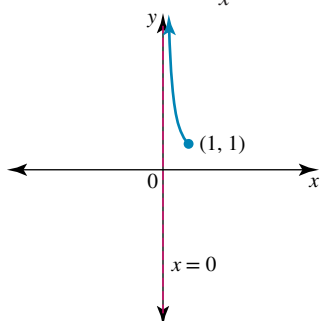
- b. Part of a parabola, $y = \frac{1}{4}(x^2 - 8x)$; domain $[0, \infty)$, range $[-4, \infty)$



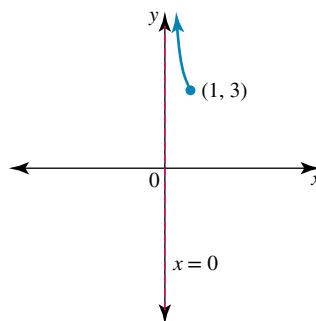
- c. Part of a hyperbola, $y = \sqrt{x^2 - 4}$; domain $[2, \infty)$, range R



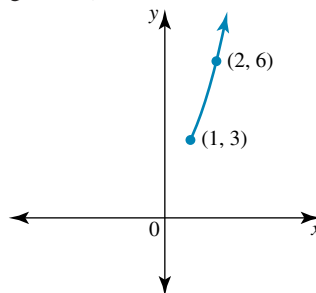
20. a. Part of a hyperbola, $y = \frac{1}{x}$; domain $(0, 1]$, range $[1, \infty)$



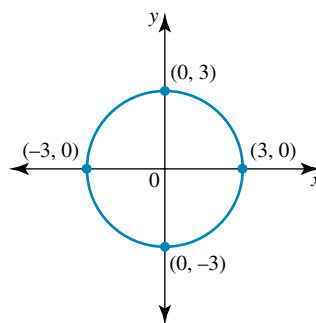
- b. Part of a truncus, $y = 2 + \frac{1}{x^2}$; domain $(0, 1]$, range $[3, \infty)$



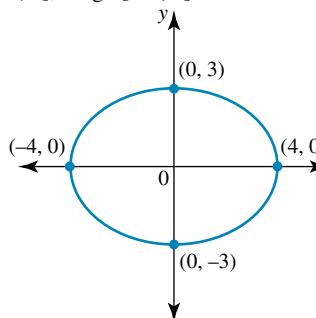
- c. Part of a parabola, $y = 2 + x^2$; domain $[1, \infty)$, range $[3, \infty)$



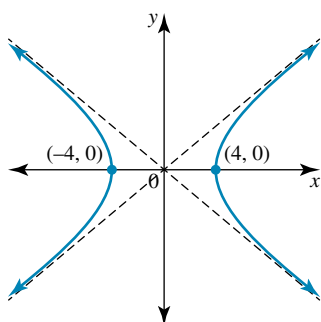
21. a. Circle with centre at the origin, radius 3, $x^2 + y^2 = 9$; domain $[-3, 3]$, range $[-3, 3]$



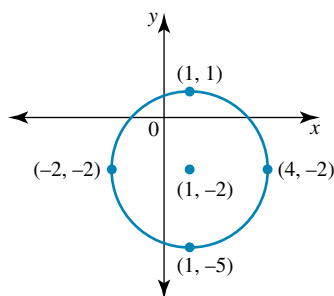
- b. Ellipse with centre at the origin, $\frac{x^2}{16} + \frac{y^2}{9} = 1$; domain $[-4, 4]$, range $[-3, 3]$



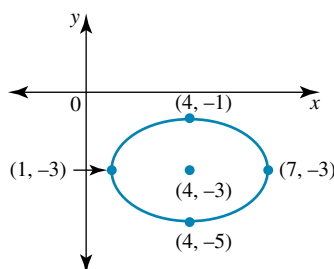
- c. Hyperbola with centre at the origin, $\frac{x^2}{16} - \frac{y^2}{9} = 1$, asymptotes $y = \pm \frac{3x}{4}$; domain $(-\infty, -4] \cup [4, \infty)$, range R



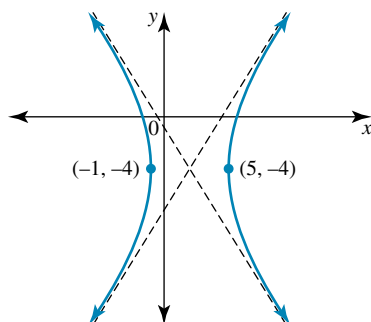
22. a. Circle with centre at $(1, -2)$, radius 3,
 $(x - 1)^2 + (y + 2)^2 = 9$; domain $[-2, 4]$, range $[-5, 1]$



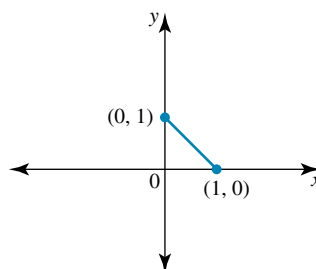
- b. Ellipse with centre at $(4, -3)$, $\frac{(x - 4)^2}{9} + \frac{(y + 3)^2}{4} = 1$;
 domain $[1, 7]$, range $[-5, -1]$



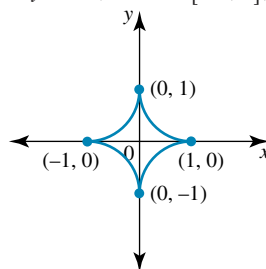
- c. Hyperbola with centre at $(2, -4)$,
 $\frac{(x - 2)^2}{9} - \frac{(y + 4)^2}{25} = 1$, asymptotes $y = \frac{5x}{3} - \frac{22}{3}$,
 $y = -\frac{5x}{3} - \frac{22}{3}$; domain $(-\infty, -1] \cup [5, \infty)$, range R



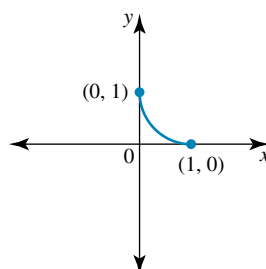
23. a. Part of a straight line, $y = 1 - x$; domain $[0, 1]$,
 range $[0, 1]$



- b. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$; domain $[-1, 1]$, range $[-1, 1]$

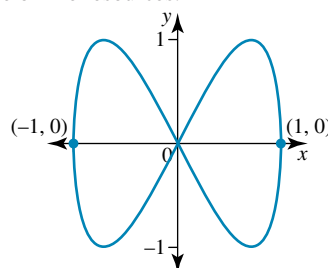


- c. $\sqrt{y} + \sqrt{x} = 1$; domain $[0, 1]$, range $[0, 1]$

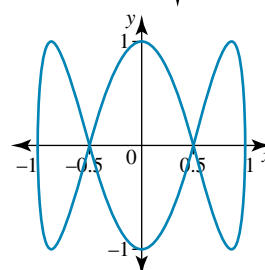


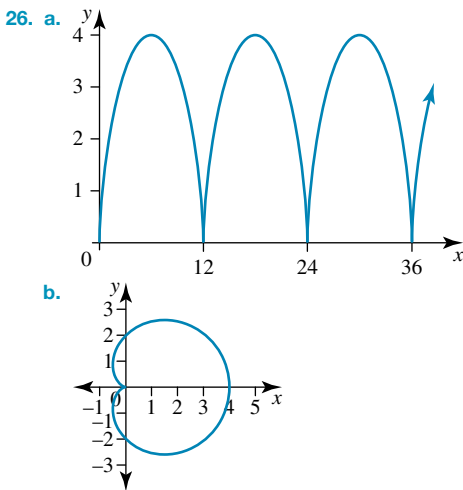
24. Sample responses can be found in the worked solutions in the online resources.

25. a.



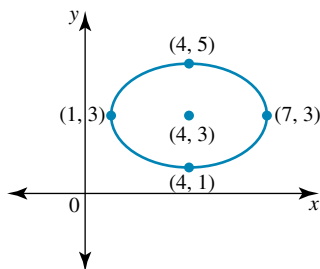
- b.



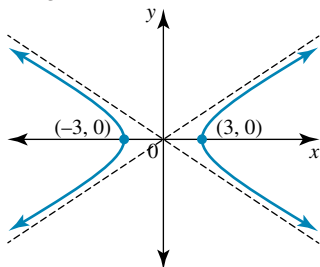


Exercise 4.3 Differentiation of vectors

1. $\hat{j}$
2. $\frac{1}{2}(-\sqrt{3}\hat{i} + \hat{j})$
3. a. $e^{-2t}(1-2t)\hat{i} + e^{2t}(1+2t)\hat{j}$
b. $2e$
c. $4(t-1)e^{-2t}\hat{i} + 4(t+1)e^{2t}\hat{j}$
4. 4 kg m/s
5. a. $8t^3\hat{i} - 8\sin(2t)\hat{j} - 12e^{-2t}\hat{k}$
b. $24t^2\hat{i} - 16\cos(2t)\hat{j} + 24e^{-2t}\hat{k}$
6. $\frac{1}{2}\sqrt{\pi^4 + 1024e^{-4}}$
7. a. (4, 1), (4, 5)
b. $\frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} = 1$; domain [1, 7], range [1, 5]



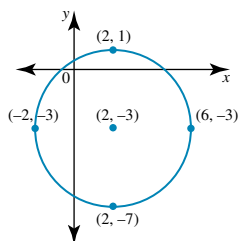
- c. $|\dot{x}(t)|_{\max} = 4$, $|\dot{x}(t)|_{\min} = 3$
8. a. $\frac{x^2}{9} - \frac{y^2}{4} = 1$; domain $|x| \geq 3$, range R , asymptotes $y = \pm \frac{2x}{3}$



- b. $\left(2\sqrt{3}, \frac{2\sqrt{3}}{3}\right), \left(-2\sqrt{3}, -\frac{2\sqrt{3}}{3}\right)$

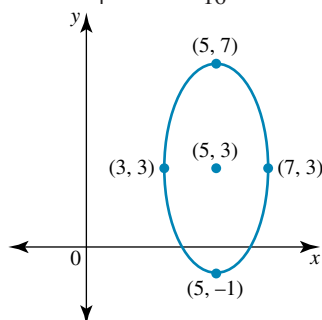
9. a. 2.45 s
c. 13 m/s, 67.38°
e. $y = -\frac{x}{250}(49x - 600)$
10. a. 1.67 s
c. 25 m/s, 16.26°
e. $y = 2 + \frac{7x}{24} - \frac{49x^2}{5760}$
11. a. $\frac{1}{\sqrt{17}}(\hat{i} + 4\hat{j})$
b. $\frac{1}{\sqrt{145}}(\hat{i} + 12\hat{j})$
c. $\frac{1}{\sqrt{82}}(9\hat{i} + \hat{j})$
d. $\frac{1}{\sqrt{34}}(3\hat{i} + 5\hat{j})$
12. a. $\frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$
b. $\frac{1}{\sqrt{10}}(3\hat{i} + \hat{j})$
c. $-\frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$
d. $\frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$
13. a. $\dot{x}(t) = 2\hat{i} - \frac{1}{(1+t)^2}\hat{j}$, $\ddot{x}(t) = \frac{2}{(1+t)^3}\hat{j}$
b. $\dot{x}(t) = \frac{1}{t}\hat{i} + (10t + 4)\hat{j}$, $\ddot{x}(t) = \frac{-1}{t^2}\hat{i} + 10\hat{j}$
c. $\dot{x}(t) = -4e^{-\frac{t}{2}}\hat{i} + 8e^{2t}\hat{j}$, $\ddot{x}(t) = 2e^{-\frac{t}{2}}\hat{i} + 16e^{2t}\hat{j}$
d. $\dot{x}(t) = \frac{3}{3t+4}\hat{i} - 12\sin(3t)\hat{j}$,
 $\ddot{x}(t) = \frac{-9}{(3t+4)^2}\hat{i} - 36\cos(3t)\hat{j}$
14. a. $\dot{x}(t) = -2e^{-2t}\hat{i} + (4t^3 - 4t)\hat{j} + 10t\hat{k}$,
 $\ddot{x}(t) = 4e^{-2t}\hat{i} + (12t^2 - 4)\hat{j} + 10\hat{k}$
b. $\dot{x}(t) = -6\sin(2t)\hat{i} - 8\cos(2t)\hat{j} + (12 - 10t)\hat{k}$,
 $\ddot{x}(t) = -12\cos(2t)\hat{i} + 16\sin(2t)\hat{j} - 10\hat{k}$
c. $\dot{x}(t) = (2t\sin(2t) + 2t^2\cos(2t))\hat{i}$
 $+ e^{-2t}(1 - 2t)\hat{j} + 10\hat{k}$,
 $\ddot{x}(t) = (8t\cos(2t) + (2 - 4t^2)\sin(2t))\hat{i}$
 $+ 4e^{-2t}(t - 1)\hat{j}$
d. $\dot{x}(t) = -3\sin(6t)\hat{i} + 3t^2\cos(3t)\hat{j} - t\sin(3t)\hat{k} + 2t\hat{k}$,
 $\ddot{x}(t) = -18\cos(6t)\hat{i} + ((6t - 9t^3)\cos(3t) - 18t^2\sin(3t))\hat{j} + 2\hat{k}$
15. a. i. Circle, $x^2 + y^2 = 9$
ii. The speed is 6.
iii. $\ddot{x}(t) = -4\dot{x}(t)$
iv. Sample responses can be found in the worked solutions in the online resources.
b. i. The speed is 12.
ii. Sample responses can be found in the worked solutions in the online resources.

16. a. i. Circle, $(x - 2)^2 + (y + 3)^2 = 16$



- ii. Sample responses can be found in the worked solutions in the online resources.
 iii. 24 kg m/s
 iv. 48 N
 v. -1

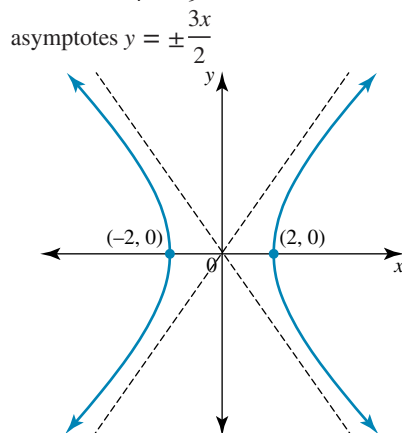
- b. i. Ellipse, $\frac{(x - 5)^2}{4} + \frac{(y - 3)^2}{16} = 1$



- ii. $4\sqrt{13}$ kg m/s
 iii. $8\sqrt{7}$ N
 iv. $|\dot{r}(t)|_{\max} = 8$, $|\dot{r}(t)|_{\min} = 4$
 v. $\frac{\pi}{12}$, $\frac{7\pi}{12}$, $\frac{13\pi}{12}$, $\frac{19\pi}{12}$

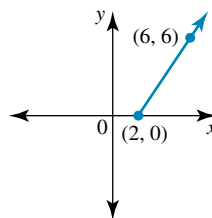
17. a. Circle, $x^2 + y^2 = a^2$
 b - d. Sample responses can be found in the worked solutions in the online resources.
 e. The motion is circular.

18. a. i. Hyperbola, $\frac{x^2}{4} - \frac{y^2}{9} = 1$; domain $|x| \geq 2$, range R ,



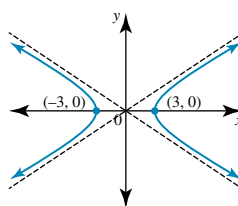
- ii. $\frac{3\sqrt{2}}{2}$

- b. i. Straight line, $\frac{x}{2} - \frac{y}{3} = 1$; domain $[2, \infty)$, range $[0, \infty)$



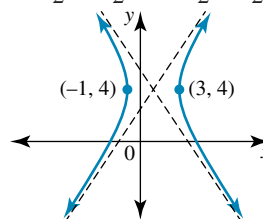
- ii. $\frac{3}{2}$
 iii. 3

19. a. i. Hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$; domain $|x| \geq 3$, range R ;
 asymptotes $y = \pm \frac{4x}{3}$



- ii. $\frac{8}{3}$
 iii. 4

- b. i. Hyperbola, $\frac{(x - 1)^2}{4} - \frac{(y - 4)^2}{9} = 1$; domain $(-\infty, -1] \cup [3, \infty)$, range R , asymptotes
 $y = -\frac{3x}{2} + \frac{11}{2}$, $y = \frac{3x}{2} + \frac{5}{2}$



- ii. $\frac{2\pi}{3}$, $\frac{4\pi}{3}$

20. a. 2.04 s
 b. 20.4 m
 c. 14.14 m/s, 45°
 d. 5.1 m
 e. $y = -\frac{x}{1000}(49x - 1000)$

Exercise 4.4 Integration of vectors

- $(2t^2 - 4t + 5)\hat{i} + (4 - 3t)\hat{j}$
- $(5 - 2\cos(3t))\hat{i} + (3 - 2e^{-2t})\hat{j}$
- $(5 - 2t)\hat{i} + (16t - t^4 - 10)\hat{j}$
- $(3t^2 - 5)\hat{i} + (t^2 + 8t - 2)\hat{j}$
- $(x + 2)^2 + (y - 4)^2 = 25$; circle with centre $(-2, 4)$, radius 5
- $\frac{(x - 5)^2}{9} + \frac{(y + 2)^2}{25} = 1$; ellipse with centre $(5, -2)$, semi-major and minor axes 3, 5

7. a. $3(1 - e^{-\frac{t}{3}})\hat{i} + t^4\hat{j}$
 b. $t^2\hat{i} + 3(1 - \cos(2t))\hat{j}$
 c. $\left(3 + \sin^{-1}\left(\frac{t}{4}\right)\right)\hat{i} + (5 - \sqrt{t^2 + 9})\hat{j}$
 d. $(5 + \ln(2t + 1))\hat{i} + \left(4 - \frac{12}{(3t + 2)^2}\right)\hat{j}$
8. a. $(3t^2 + 2)\hat{i} + (4t - 3)\hat{j}$
 b. $(2t + 4)\hat{i} + (2t^2 + 1)\hat{j}$
 c. $(4t^2 + 3)\hat{i} + (3t^2 - 2)\hat{j}$
 d. $(2t^2 + 3)\hat{i} + (t^2 + 8t + 4)\hat{j}$
9. a. $(t^3 - 4t^2 + 5)\hat{i} + (4t - 3)\hat{j}$
 b. $(8t + 7)\hat{i} + (2t^3 - 3t^2 - 4)\hat{j}$
 c. $(t^2 + 4t + 4)\hat{i} + (t^2 - 4t - 12)\hat{j}$
 d. $(5 + 4t - t^2)\hat{i} + (3t^2 - 16t + 5)\hat{j}$
10. a. $\left(t + \frac{1}{t}\right)\hat{i} + 2\left(t - \frac{1}{t}\right)\hat{j}$
 b. $4\sqrt{t}\hat{i} + (t^2 - 10t + 25)\hat{j}$
 c. $(\ln(3t + 1) + 4)\hat{i} + \left(\frac{4}{2t + 1} - 1\right)\hat{j}$
 d. $(\ln(3t + 1) + 3t + 2)\hat{i} + \left(4t - \frac{1}{(2t + 1)^2}\right)\hat{j}$

11. a. $y = -\frac{x^2}{45} + \frac{4x}{3} + 2, x \geq 0$

b. $y = -\frac{49x^2}{250} + 2x + 1, x \geq 0$

12. a. $y = \frac{1}{x - 4} + 3$

b. $y = \frac{32}{2 - x} - 5$

13. a. $(x - 3)^2 + (y - 5)^2 = 4$

b. $(x + 7)^2 + (y - 5)^2 = 81$

14. a. $(x - 4)^2 + \frac{(y - 5)^2}{4} = 1$

b. $\frac{(x + 7)^2}{144} + \frac{(y - 3)^2}{16} = 1$

15. a. $y = x^2 - 4, [-2, 2], [-4, 0]$

b. $y = -3x(x - 2), [-1, 3], [-9, 3]$

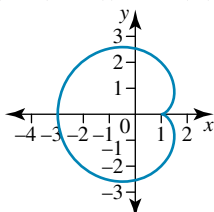
16. a. $t = 3, (4, 16)$

b. $t = 5, (-7, 10)$

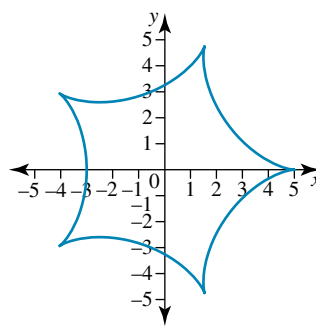
17. a. $(x - a)^2 + (y - b)^2 = r^2$

b. $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$

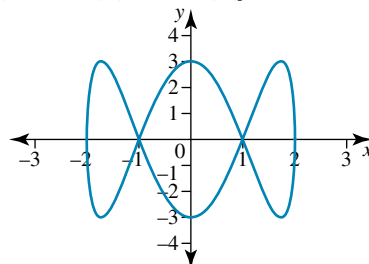
18. a. $\underline{r}(t) = (2\cos(t) - \cos(2t))\hat{i} + (2\sin(t) - \sin(2t))\hat{j}$



b. $\underline{r}(t) = (4\cos(t) + \cos(4t))\hat{i} + (4\sin(t) - \sin(4t))\hat{j}$



c. $\underline{r}(t) = 2\cos(2t)\hat{i} + 3\sin(6t)\hat{j}$



Exercise 4.5 Straight line motion with constant and variable acceleration

1. a. i. $3t^2 + 4t - 5$

ii. $6t + 4$

b. i. $4 - \frac{1}{2\sqrt{t+2}}$

ii. $\frac{1}{4(t+2)^{\frac{3}{2}}}$

c. i. $e^t - 2e^{-t}$

ii. $e^t + 2e^{-t}$

d. i. $2t - \frac{1}{t+3}$

ii. $2 + \frac{1}{(t+3)^2}$

e. i. $\frac{1}{\sqrt{9-t^2}}$

ii. $\frac{t}{(9-t^2)^{\frac{3}{2}}}$

f. i. $10 + 6t - 3t^2$

ii. $6 - 6t$

2. a. $t = 1, 2$ and 3

b. $v = 3t^2 - 12t + 11$

c. $t = \frac{6 - \sqrt{3}}{3}$ and $\frac{6 + \sqrt{3}}{3}$

d. -1 cm/s

e. 6 cm

3. a. Min, $x = 1$, max, $x = 5$

b. $t = \frac{1}{3}$

c. $t = \frac{\pi}{3}$

d. $a = -9x + 27$

4. a. $x = 2t^3 - t^2 - 8t + 2$

b. $x = 12t - 2$

c. $t = \frac{4}{3}$ s

d. 7 m

e. 14 m/s²

5. a. $x = \frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t - 1$

b. $\ddot{x} = 4t - 9$

c. $t = \frac{1}{2}, t = 4$

d. 7 cm

e. 3.5 cm/s

f. $\frac{2}{3}$ cm/s

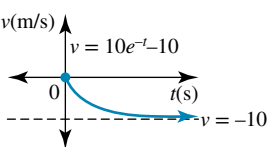
6. a. $v = 6t - 6t^2$

b. $x = 3t^2 - 2t^3 + 4$

c. $t = 0, x = 4$ and $t = 1, x = 5$

d. 29 m

- b.** 39.2 m/s



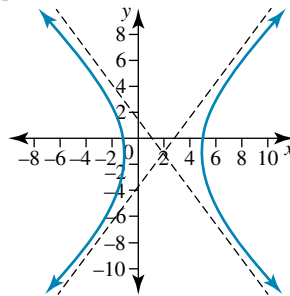
- b. $\underline{r}(3) \approx 1.920\hat{i} - 0.559\hat{j}$
 c. $\dot{\underline{r}}(3) \approx 1.920\hat{i} - 0.559\hat{j}$
 $\ddot{\underline{r}}(3) \approx -7.681\hat{i} + 2.235\hat{j}$
6. a. $\omega = \frac{3}{10} \text{ rad/s}$
 b. $\underline{r}(1) \approx 76.427\hat{i} + 23.642\hat{j}$
 c. $\dot{\underline{r}}(1) \approx -7.092\hat{i} + 22.928\hat{j}$
 $\ddot{\underline{r}}(1) \approx -6.878\hat{i} - 2.128\hat{j}$
 d. $|\ddot{\underline{r}}(1)| \approx 7.200 \text{ m/s}^2$
 e. $T = \frac{20\pi}{3} \text{ s}$
7. a. $v = 12\pi \text{ m/s}$
 b. $|\ddot{\underline{r}}(t)| = |\underline{a}(t)| = 48\pi^2 \text{ m/s}^2$
8. a. $\omega = 12\pi \text{ rad/s}$
 b. $v = 288\pi \text{ m/s}$
9. $\omega = \frac{35}{4} \text{ rad/s}$
10. a. $\omega = \frac{2\pi}{13} \text{ radians/hour}$
 b. $v \approx 14\,016.336 \text{ km/hour}$
11. a. $\omega = \frac{\pi}{62} \text{ rad/s}$
 b. $v = \frac{7\pi}{62} \text{ m/s} \approx 0.355 \text{ m/s}$
 c. $|\underline{a}| = \frac{7\pi^2}{3844} \text{ m/s}^2$ ($a \approx 0.018 \text{ m/s}^2$)
12. a. $\omega = \frac{2\pi}{5} \text{ rad/s}$
 b. $v = \frac{16\pi}{5} \text{ m/s}$ ($v \approx 10.053 \text{ m/s}$)
 c. $|\underline{a}| = \frac{32\pi^2}{25} \text{ m/s}^2$ ($a \approx 12.633 \text{ m/s}^2$)
13. a. $\therefore T = 1 \text{ s}$
 b. $\omega = 2\pi \text{ rad/s}$
 c. $v = 1.2\pi \text{ m/s}$
 d. $|\underline{a}| = 2.4\pi^2 \text{ m/s}^2$
14. The diameter of the circle: $D = 2.986 \text{ m}$

4.8 Review: exam practice

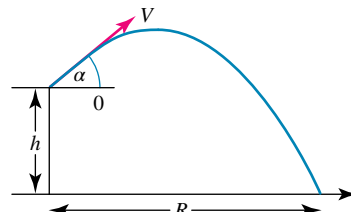
- C
 - A
 - C
 - A
 - D
6. a. $p: \frac{x^2}{9} + \frac{y^2}{16} = 1, t \geq 0$
 $q: \frac{x^2}{16} + \frac{y^2}{9} = 1, t \geq 0$
- b. $\left(\frac{12}{5}, \frac{12}{5}\right), \left(\frac{-12}{5}, \frac{12}{5}\right), \left(\frac{-12}{5}, \frac{-12}{5}\right), \left(\frac{12}{5}, \frac{-12}{5}\right)$
7. a. $\underline{r}\left(\frac{\pi}{4}\right) = 4\sqrt{2}\hat{j} + 3\sqrt{2}\hat{j}$
 b. $\underline{r}\left(\frac{5\pi}{6}\right) = -4\sqrt{3}\hat{j} + 3\hat{j}$
 c. $\frac{x^2}{64} + \frac{y^2}{36} = 1$
8. a. The particles collide at $17\hat{i} + 48\hat{j}$ after 6 seconds of flight.

b. $|\underline{r}_B(9) - \underline{r}_A(9)| = \sqrt{333} \text{ m}$

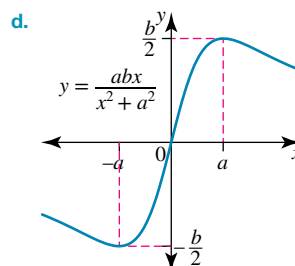
9. hyperbola, centre $(2, -1)$ dom $(-\infty, -1] \cup [5, \infty)$ ran R



10. a. $v(t) = 8t - 7t^2$
 b. $x(t) = -\frac{7}{3}t^3 + 4t^2 + 5$
 c. $x(0) = 5 \text{ m}, x\left(\frac{8}{7}\right) = \frac{991}{147} \text{ m}$
11. a. $\underline{v}(t) = 12t^3\hat{i} + 4\hat{j} + 2e^{2t}\hat{k}$
 b. $\underline{a}(t) = 36t^2\hat{i} + 4\hat{j} + 4e^{2t}\hat{k}$
12. $\underline{r}(t) = (t^3 - 2t^2 + 4)\hat{i} + (5t - 2)\hat{j} + \hat{k}$
13. a. $\omega = 3 \text{ rad/s}$
 b. $\underline{r}(4) = 10.126\hat{i} - 6.439\hat{j}$
 c. $\ddot{\underline{r}}(4) = -91.136\hat{i} + 57.950\hat{j}$
14. a. $t = 3.50 \text{ s}$ b. 34.30 m/s
15. a. The ball travels a horizontal distance of 17.251 metres.
 The total time of flight is 2.033 seconds.
 b. The maximum height reached is 6.673 metres.
 c. The ball makes a 53.340° angle with the ground and lands with a speed of 14.242 m/s.
16. a. $r = 10$ b. $v = 50 \text{ m/s}$
- 17.



- a. Sample responses can be found in the worked solutions in the online resources.
- b. $R = 42.77 \text{ m}$
 $\alpha = 43.7^\circ$
 $T = 2.96 \text{ s}$
18. a-c. Sample responses can be found in the worked solutions in the online resources.



19. Sample responses can be found in the worked solutions in the online resources.
20. a. $|\underline{r}(t)|_{\min} = \frac{14}{13}$
 b. $\underline{r}\left(\frac{-(ab + cd)}{a^2 + c^2}\right) = \frac{ad - bc}{\sqrt{a^2 + c^2}}$

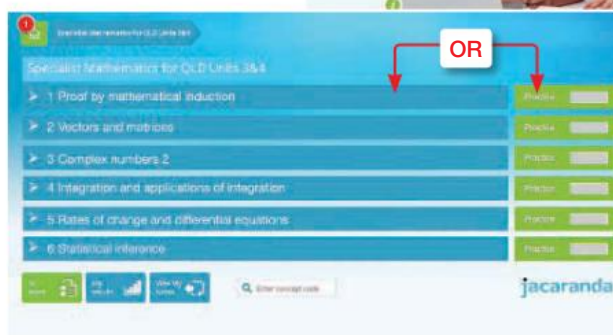
REVISION UNIT 3 Mathematical induction, and further vectors, matrices and complex numbers

TOPIC 2 Vectors and matrices

- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



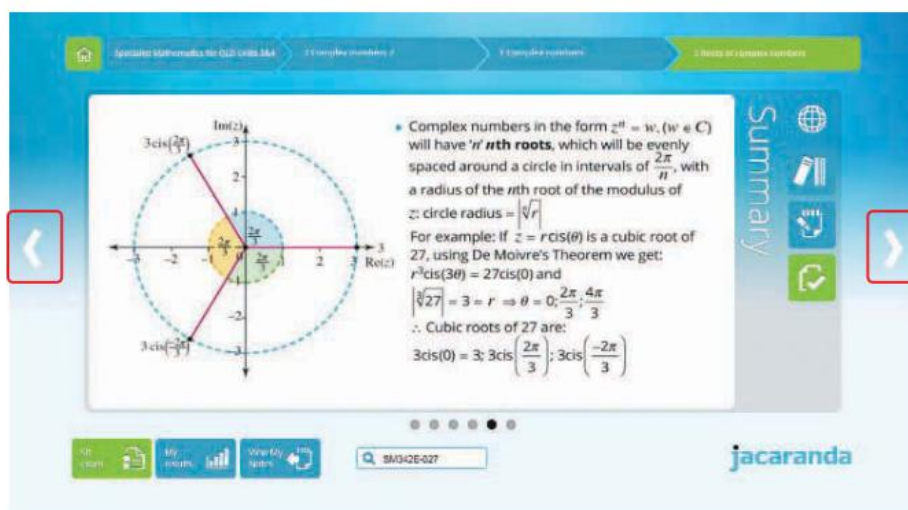
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



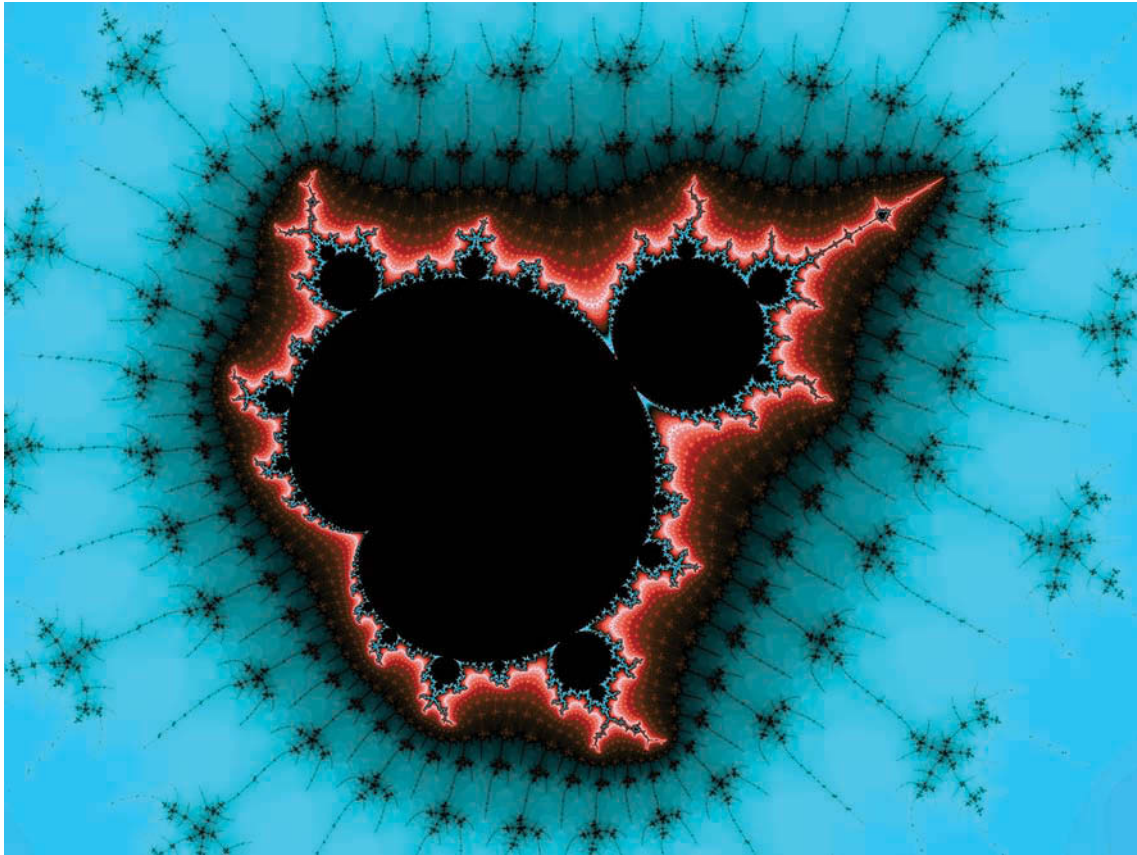
- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** in the sequence and practice questions at the concept level for a more granular set of questions.



5 Complex numbers

5.1 Overview

Complex numbers have many real-world applications, however, they are most commonly used in engineering and physics. For example, complex numbers can be used to analyse variations in voltage and current in an AC circuit. In mathematics, complex numbers are used to solve equations that were once thought to be ‘impossible’. With the introduction of the imaginary number i , the complex number system was created, producing a new field of mathematics and visually striking objects such as the Mandelbrot set.



LEARNING SEQUENCE

- 5.1 Overview
- 5.2 Complex numbers in Cartesian form
- 5.3 Complex numbers in polar form
- 5.4 De Moivre's theorem
- 5.5 The complex plane (the Argand plane)
- 5.6 Roots of complex numbers
- 5.7 Factorisation of polynomials
- 5.8 Review: exam practice

Fully worked solutions are available for this chapter in the resources section of your eBookPLUS at www.jacplus.com.au.

5.2 Complex numbers in Cartesian form

5.2.1 The definition of a complex number

The **complex number system** is an extension of the real number system. Complex numbers are numbers that include the number i , known as the imaginary unit.

The imaginary number

The imaginary unit i is defined as $i^2 = -1$

This means that solutions to equations such as $x^2 = -1$ can now be found in terms of i .

The general form of a complex number is represented by z and defined as $z = a + bi$, where a and $b \in R$, and $z \in C$, where C is used to denote the set of complex numbers (in the same way that R denotes the set of real numbers). Note that $z = a + bi$ is one single number but is composed of two parts: a real part and an imaginary part. The real part is written as $\text{Re}(z) = a$ and the imaginary part is written as $\text{Im}(z) = b$.

A complex number in the form $z = a + bi$, where both a and b are real numbers, is called the Cartesian form or rectangular form or standard form of a complex number. Throughout this chapter, it is assumed that all equations are solved over C .

WORKED EXAMPLE 1

Evaluate each of the following.

a. $\text{Re}(6 - 12i)$

b. $\text{Im}(3 - i + i^3)$

THINK

- a. The real part of the complex number $6 - 12i$ is 6.
- b. 1. Simplify $3 - i + i^3$ first by recalling that $i^2 = -1$.
2. Look for the coefficient of i to determine the imaginary part of any complex number.

WRITE

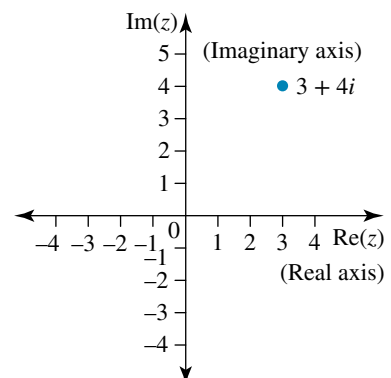
a. $\text{Re}(6 - 12i) = 6$

b. $\begin{aligned}\text{Im}(3 - i + i^3) &= \text{Im}(3 - i + i^2 \times i) \\ &= \text{Im}(3 - i - i) \\ &= \text{Im}(3 - 2i) \\ \text{Im}(3 - 2i) &= -2\end{aligned}$

5.2.2 Review of basic operations on complex numbers in Cartesian form

Real numbers are represented on a number line. This representation helps us to understand certain properties of these numbers. We see that 4 is bigger than 3 and that $\sqrt{2}$ lies between 1 and 2. Complex numbers are represented in two dimensions using the Argand diagram, and this image helps us to visualise the complex number. For example, the complex number $3 + 4i$ can be represented on the **Argand plane** or Argand diagram shown right.

You will recall that complex numbers can be added, subtracted, multiplied and divided. However, subtraction and division are not treated as distinct operations. Subtraction of a number is achieved by adding the negative of that number and division is achieved by multiplying by the inverse of the number.



Review of complex number arithmetic

If $z = a + bi$ and $w = c + di$ then

Addition: $z + w = a + bi + c + di = (a + c) + (b + d)i$

Subtraction: $z - w = z + (-w) = (a - c) + (b - d)i$

Scalar multiplication: $k \times z = k(a + bi) = ka + kbi$, where $k \in R$

Multiplication: $z \times w = (a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$

WORKED EXAMPLE 2

If $p = 5 - 3i$ and $q = -2 + 7i$, evaluate each of the following without using a calculator.

a. $2p - q$

b. $p \times q$

c. $(p + q)^2$

THINK

1. Substitute for p and q .
2. Multiply each part of p by 2.
3. Simplify the real and imaginary parts.
2. Substitute for p and q .
Use the Distributive Law to expand the brackets.
Simplify the real and imaginary parts recalling that $i^2 = -1$.
1. Substitute for p and q .
2. Simplify the real and imaginary parts.
3. Expand by recalling $(a + b)^2 = a^2 + 2ab + b^2$.
4. Simplify $9 + 24i + 16i^2$ by recalling that $i^2 = -1$.

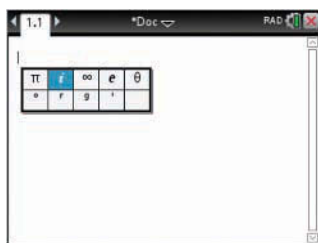
WRITE

$$\begin{aligned} \text{a. } 2p - q &= 2(5 - 3i) - (-2 + 7i) \\ &= 10 - 6i + 2 - 7i \\ &= 12 - 13i \\ \text{b. } p \times q &= (5 - 3i) \times (-2 + 7i) \\ &= -10 + 35i + 6i - 21i^2 \\ &= -10 + 41i - 21 \times -1 \\ &= 11 + 41i \\ \text{c. } (p + q)^2 &= ((5 - 3i) + (-2 + 7i))^2 \\ &= (3 + 4i)^2 \\ &= 9 + 24i + 16i^2 \\ &= 9 + 24i - 16 \\ &= -7 + 24i \end{aligned}$$

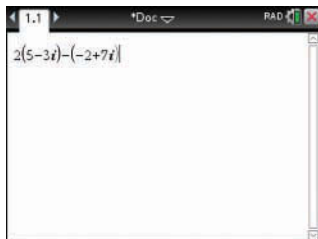
TI | THINK

- a. 1. Note: i can be entered by pressing the π $\triangleright$ button to open a menu.

WRITE



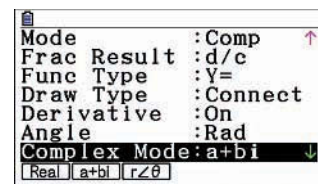
2. On a Calculator page, complete the entry line $2(5 - 3i) - (-2 + 7i)$



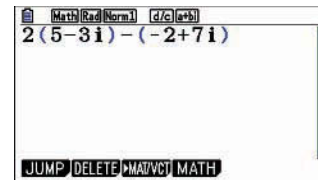
CASIO | THINK

- a. 1. On a Run-Matrix screen, select
SHIFT
MENU
Complex Mode
 $a + bi$
Once this has been set, press the EXIT button.

WRITE



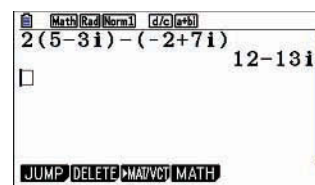
2. On a Run-Matrix screen, complete the entry line $2(5 - 3i) - (-2 + 7i)$
Note: i can be entered by selecting SHIFT followed by 0.



3. Press the ENTER button.
The answer appears on the screen.



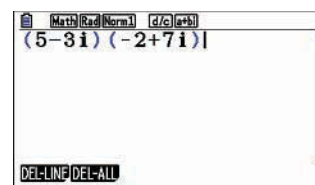
3. Press the EXE button.
The answer appears on the screen.



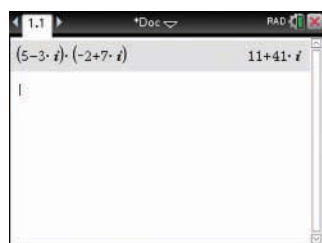
- b. 1. On a Calculator page, complete the entry line $(5 - 3i)(-2 + 7i)$



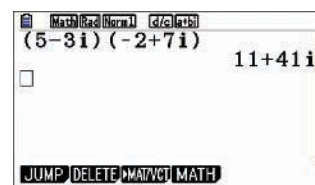
- b. 1. On a Run-Matrix screen, complete the entry line $(5 - 3i)(-2 + 7i)$
Note: i can be entered by selecting SHIFT followed by 0.



2. Press the ENTER button.
The answer appears on the screen.



2. Press the EXE button.
The answer appears on the screen.



Equality of complex numbers

Two complex numbers are equal if and only if their real parts and their imaginary parts are both equal. For example, if $5 + yi = x - 3i$, then from equating the real part, we find that $x = 5$, and from equating the imaginary part, we find that $y = -3$.

Powers of i

As $i^2 = -1$, it follows that

$$i^3 = i \times i^2 = i \times -1 = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

$$\text{and } i^5 = i \times i^4 = i \times 1 = i.$$

A pattern can be seen for repetitions of the powers of i . Any even power of i will give ± 1 , while any odd power of i will give $\pm i$.

Multiplication by i

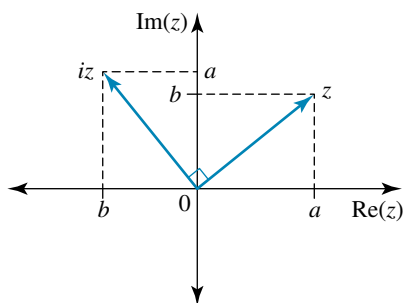
If $z = a + bi$, then iz is given by

$$iz = i(a + bi)$$

$$= ia + i^2b$$

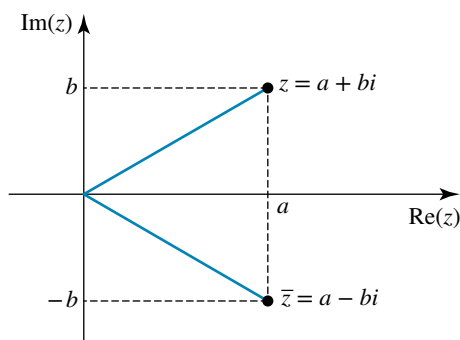
$$= -b + ai$$

The complex number iz is a rotation of z by 90° anticlockwise.



5.2.3 The Complex conjugate of a complex number

The **complex conjugate** $\bar{z}$, is a reflection of the point $a + bi$ about the $\text{Re}(z)$ axis. The geometric interpretation is shown in the diagram below.



The complex conjugate

If $z = a + bi$, the complex conjugate of z is denoted by $\bar{z}$ and is defined as $\bar{z} = a - bi$.

The complex conjugate can be determined by changing the sign of the imaginary component.

WORKED EXAMPLE 3

Determine the complex conjugate of each of the following complex numbers.

a. $v = 12 - 6i$

b. $w = -3 + \sqrt{6}i$

THINK

- Change the sign of the imaginary component.
- Change the sign of the imaginary component.

WRITE

$$\bar{v} = 12 + 6i$$

$$\bar{w} = -3 - \sqrt{6}i$$

5.2.4 Inverses and division of complex numbers

The multiplicative inverse z^{-1} of a complex number exist such that $z \times z^{-1} = 1$. Given a non-zero complex number z , the inverse can be calculated by applying your knowledge of the complex conjugate.

The multiplicative inverse of a complex number

$$\text{If } z = a + bi \text{ with } z \neq 0, \text{ then } z^{-1} = \frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{a - bi}{a^2 + b^2}.$$

WORKED EXAMPLE 4

Determine z^{-1} given $z = -8 + 2i$.

THINK

1. Write z^{-1} as a rational expression: $z^{-1} = \frac{1}{z}$.
2. Multiply the numerator and denominator by the complex conjugate of $-8 + 2i$.
3. Express in standard form, $a + bi$.

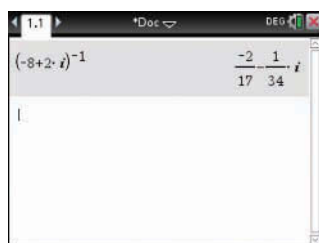
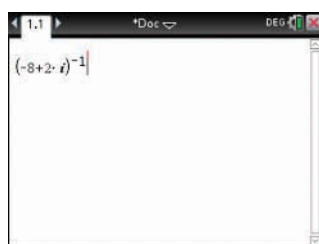
WRITE

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{-8 + 2i} \\ &= \frac{1}{-8 + 2i} \times \frac{-8 - 2i}{-8 - 2i} \\ &= \frac{-8 - 2i}{64 - 4i^2} \\ &= \frac{-8 - 2i}{68} \\ &= -\frac{2}{17} - \frac{1}{34}i \end{aligned}$$

TI | THINK

1. On a Calculator page, complete the entry line $(-8 + 2i)^{-1}$
2. Press the ENTER button. The answer appears on the screen.

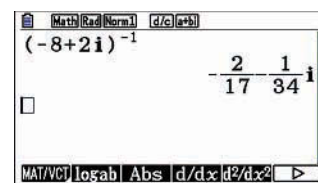
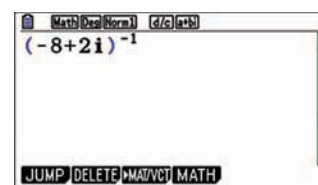
WRITE



CASIO | THINK

1. On a Run-Matrix screen, complete the entry line $(-8 + 2i)^{-1}$
Note: i can be entered by selecting SHIFT followed by 0.
2. Press the EXE button. The answer appears on the screen.

WRITE



Complex conjugates are helpful when dividing complex numbers. The following investigation will consider the two complex numbers $z_1 = a + bi$ and $z_2 = c + di$. To perform the operation $\frac{z_1}{z_2}$ we must begin by multiplying the numerator and denominator by $\bar{z}_2$.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + bi}{c + di} \\ &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{(ac + bd)}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2} \end{aligned}$$

Multiply the fraction by $\frac{\bar{z}_2}{\bar{z}_2}$

Simplify the numerator and denominator by applying the distributive law of multiplication.

Collect real and imaginary components

Expressed in the form $a + bi$

Division of complex numbers

For division of complex numbers z_1 and z_2

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} \quad \text{where } z_2 \neq 0$$

WORKED EXAMPLE 5

Express $\frac{1-3i}{4+i}$ in the form $a+bi$.

THINK

1. Multiply both the numerator and denominator by the complex conjugate of $4+i$.
2. Expand and simplify the expressions obtained in the numerator and denominator.
3. Express in standard form, $a+bi$.

WRITE

$$\begin{aligned} \frac{1-3i}{4+i} &\times \frac{4-i}{4-i} \\ &= \frac{4-i-12i+3i^2}{4^2-4i+4i-i^2} \\ &= \frac{1-13i}{17} \\ &= \frac{1}{17} - \frac{13}{17}i \end{aligned}$$

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Units 3 & 4

Area 3

Sequence 1

Concept 1

Complex numbers in Cartesian form Summary screen and practice questions

Exercise 5.2 Complex numbers in Cartesian form

Technology Free

1. Using the imaginary number i , write expressions for:
 - a. $\sqrt{-25}$
 - b. $\sqrt{-144}$
 - c. $\sqrt{-\frac{4}{25}}$
 - d. $\sqrt{-23}$
2. Express each of the following in the form $a+bi$, where a and b are real numbers.
 - a. $i^3 + i^2$
 - b. $2i^{12} + 3i^{15}$
 - c. $2i^5 - 3i^6 + 4i^7$
 - d. $i(i^4 + i^5)$
3. **WE1** Evaluate each of the following.
 - a. $\text{Re}(8-9i)$
 - b. $\text{Im}(12-i^2+2i^5)$
 - c. $\text{Im}(17-i)$
 - d. $\text{Im}(7-3i^2+i^4-2i)$
4.
 - a. Draw an Argand diagram and mark on it the complex numbers: $u = 2+i$, $v = 1+2i$, and $w = -2i$.
 - b. On the same diagram mark the points: iu , iv and iw .
 - c. Describe, geometrically, the relationship between the complex number z and iz on the Argand diagram.
5.
 - a. Draw an Argand diagram and mark on it the complex numbers: $u = 2+i$, $v = 1+2i$, and $w = -2i$.
 - b. On the same diagram mark the points: $-u$, $-v$ and $-w$.
 - c. Describe, geometrically, the relationship between the complex numbers z and $-z$ on the Argand diagram.

6. a. Draw an Argand diagram and mark on it the numbers: $u = 2 + i$, $v = 1 + 2i$, and $w = -2i$.
 b. On the same diagram mark the points: $-iu$, $-iv$ and $-iw$.
 c. Describe, geometrically, the relationship between the complex numbers z and $-iz$ on the Argand diagram.
7. **WE2** If $p = 6 - 2i$ and $q = -3 + 3i$, evaluate each of the following without using a calculator.
 a. $2p - q$ b. $p \times q$ c. $(p + q)^2$
8. If $s = 4 + i$, $t = 5 - 4i$ and $u = -7 + 8i$, evaluate each of the following without using a calculator.
 a. $s - u$ b. $3(t + u)$ c. $s \times t$ d. $s(tu)$
9. **WE3** Determine the complex conjugate of each of the following complex numbers.
 a. $e = 14 + 9i$ b. $f = -2 + \sqrt{7}i$ c. $g = 9 - 2\sqrt{7}i$ d. $h = -1 - i$
10. If $v = 2 + 3i$ and $w = 5 - i$, demonstrate that $\overline{v - w} = \overline{v} - \overline{w}$.
11. **WE4** Determine z^{-1} given z in each of the following cases.
 a. $z = 1 + i$ b. $z = 2 - 6i$ c. $z = -3 + 2i$ d. $z = -4 - 4i$
12. **WE5** Express $\frac{3 - i}{2 + 5i}$ in the form $a + bi$.
13. If $w = 3 + 3i$ and $v = 8 - 2i$, then:
 a. **MC** $\text{Im}(\overline{w}v)$ is equal to:
 A. -30 B. -32 C. -34 D. 30
 b. **MC** The expression $\overline{vw}$ is equal to:
 A. $30 + 18i$ B. $30 - 18i$ C. $-30 - 18i$ D. $-30 + 12i$
14. Let $z_1 = a + bi$ and $z_2 = c + di$. Prove the identity, $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$.
15. Let $z_1 = a + bi$ and $z_2 = c + di$. Prove the identity, $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$.
16. Suppose $p = (a_1, b_1)$, $q = (a_2, b_2)$ and $r = (a_3, b_3)$.
 Prove the distributive law: $p(q + r) = pq + pr$.
17. Calculate the values of x and y that satisfy each of the following.
 a. $(3 + 2i)(x + yi) = 37 - 10i$ b. $(5 + 4i)(x + yi) = -32 - i$

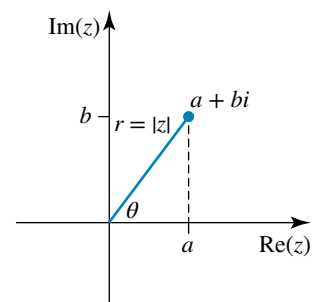
5.3 Complex numbers in polar form

5.3.1 Review of complex numbers in polar form

In the preceding section, we expressed complex numbers in Cartesian form. Another way in which complex numbers can be represented is polar form. This form has two parts: the modulus and the argument.

To demonstrate this, consider the complex number $z = a + bi$ as pictured in the Argand diagram on the right.

The length, magnitude or modulus of the complex number is the distance of the complex number from the origin. This distance is represented by r or $|z|$. Note that this distance is always a positive real number.



The modulus of a complex number

For $z = a + bi$, the modulus of z is defined by

$$r = |z| = \sqrt{a^2 + b^2}$$

The angle θ that the line segment makes with the positive real axis is called the **argument of z** , and is denoted by $\arg(z)$.

The argument of a complex number

For $z = a + bi$, the argument of z is defined by

$$\arg(z) = \theta = \tan^{-1} \left(\frac{b}{a} \right)$$

Usually this angle is given in radians, as multiples of π , although it can also be given in degrees. Angles measured anticlockwise from the positive real axis are positive angles, and angles measured clockwise from the positive real axis are negative angles. Because any integer multiple of 2π radians (or 360°) can be added or subtracted to any angle to get the same result on the Argand plane, all complex numbers have more than one representation. That is, there is no unique angle to represent a complex number.

To overcome this problem, we often refer to the **principal value** of the argument of z . The principal value is defined to be the angle $\text{Arg}(z)$ where $\text{Arg}(z) \in (-\pi, \pi]$. (Note the use of the capital letter A to denote the principal value.)

Principal value of the argument

For a non-zero complex number z , the principal value of the argument of z is defined by

$$-\pi < \text{Arg}(z) \leq \pi$$

Polar form

In the diagram on the previous page, the trigonometric properties of a right-angled triangle can be used to deduce that $a = r \cos(\theta)$ and $b = r \sin(\theta)$. Therefore, a complex number $z = a + bi$ can be written as $z = r \cos(\theta) + r \sin(\theta)i$. This latter form is known as polar form of a complex number. It is sometimes called the **modulus-argument** form of a complex number.

Polar form of a complex number

To express a complex number in polar form:

$$z = r \cos(\theta) + r \sin(\theta)i$$

$$z = r(\cos(\theta) + i \sin(\theta))$$

$$z = r \text{cis}(\theta)$$

where 'cis' is shorthand for ' $\cos + i \sin$ '.

Note that the polar form of a complex number can also be written as $z = |z| \text{cis}(\theta)$, since $r = |z|$.

WORKED EXAMPLE 6

- Express $2 - 2\sqrt{3}i$ in polar form.
- Express $6 \text{cis} \left(\frac{\pi}{4} \right)$ in Cartesian form.



THINK

- a. 1. To express a complex number in polar form, r and θ are required for $r \operatorname{cis}(\theta)$.
Recall that $r = \sqrt{a^2 + b^2}$.
2. Calculate θ in radians. Recall
 $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for $a + bi$.
3. Express $2 - 2\sqrt{3}i$ in polar form, that is in the form $r \operatorname{cis}(\theta)$.
- b. 1. Identify r and θ .
2. To convert into Cartesian form, use the relationship $z = r \cos(\theta) + r \sin(\theta)i$.

WRITE

a. $r = \sqrt{(2)^2 + (-2\sqrt{3})^2}$
 $= \sqrt{4 + 12}$
 $= 4$

$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right)$
 $= -\frac{\pi}{3}$

$2 - 2\sqrt{3}i = 4 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

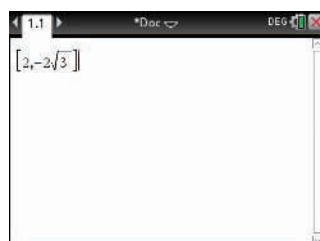
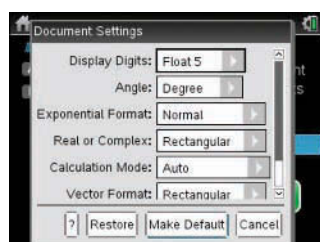
b. For $6 \operatorname{cis}\left(\frac{\pi}{4}\right)$; $r = 6$ and $\theta = \frac{\pi}{4}$

$6 \operatorname{cis}\left(\frac{\pi}{4}\right) = 6 \cos\left(\frac{\pi}{4}\right) + 6 \sin\left(\frac{\pi}{4}\right)i$
 $= 6 \times \frac{\sqrt{2}}{2} + 6i \times \frac{\sqrt{2}}{2}$
 $= 3\sqrt{2} + 3\sqrt{2}i \text{ or } 3\sqrt{2}(1 + i)$

TI | THINK

- a. 1. On Home page, select
5 Settings
2: Document Settings
2. Set the calculator to the appropriate format by completing the entry lines
Angle: Degree
Real or Complex: Rectangular
3. On a Calculator page, complete the entry line
 $[2, -2\sqrt{3}]$

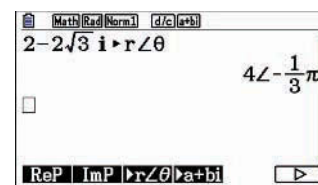
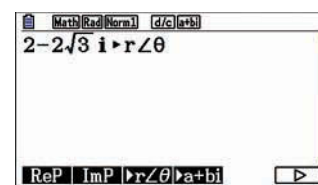
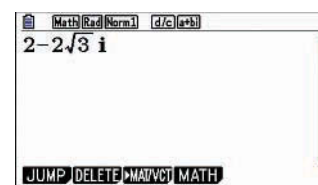
WRITE



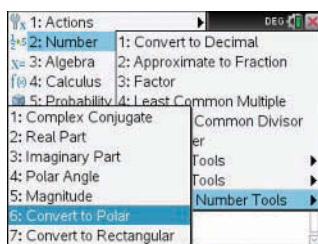
CASIO | THINK

- a. 1. On a Run-Matrix screen, complete the entry line
 $2 - 2\sqrt{3}i$
2. To convert the complex number into polar form, select
OPTN
COMPLEX
 $\triangleright r\angle\theta$
3. Press the EXE button.
The answer appears on the screen.

WRITE



4. To convert the complex number to Polar form, select MENU
2: Number
9: Complex Number
Tools
6: Convert to Polar



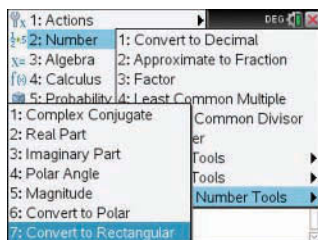
5. Press the ENTER button.
The answer appears on the screen.
Note: The answer states the angle in degrees
 $\left(-60^\circ = -\frac{\pi}{3}\right)$.



- b. 1. On a Calculator page, complete the entry line $(6 \angle 45)$
Note: The $\angle$ symbol can be entered by pressing CTRL



2. To convert a complex number from Polar to Cartesian form, select MENU
2: Number
9: Complex Number
Tools
7: Convert to Rectangular



3. Press ENTER.
The answer appears on the screen.



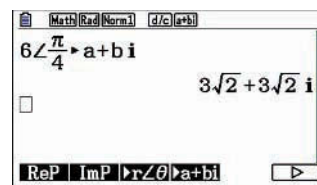
- b. 1. On a Run-Matrix screen, complete the entry line $6 \angle \frac{\pi}{4}$



2. To convert the complex number from Polar to Cartesian form, select OPTN
COMPLEX
 $\triangleright a + bi$



3. Press the EXE button.
The answer appears on the screen.



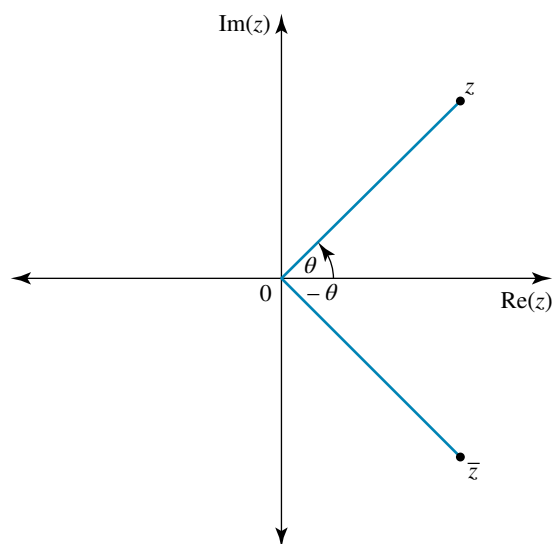
Note that the TI calculator will only return an approximate answer to $3\sqrt{2}$.



5.3.2 The complex conjugate in polar form

We have seen previously that the complex conjugate is a reflection of the point z in the horizontal (real) axis. As polar form uses angles as a means of determining the location of a complex number, it can be seen in the diagram on the right that the complex conjugate of a complex number positioned at an angle of θ from the positive real axis would be located at an angle of $-\theta$ from the positive real axis.

If the polar form of a complex number z is $r \operatorname{cis}(\theta)$, then it follows that the complex conjugate in polar form is given by $r \operatorname{cis}(-\theta)$.



$$\begin{aligned}\bar{z} &= r \cos(-\theta) + r \sin(-\theta)i \\ &= r \cos(\theta) - r \sin(\theta)i \\ &= r [\cos(\theta) - i \sin(\theta)] \\ &= r \operatorname{cis}(-\theta)\end{aligned}$$

The trigonometric identities, $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$ were used to simplify the above calculation. These will also be useful when proving modulus and argument identities in polar form.

The complex conjugate in polar form

If $z = r \operatorname{cis}(\theta)$, then $\bar{z} = r \operatorname{cis}(-\theta)$

5.3.3 Basic operations on complex numbers in polar form

When performing operations in polar form, there is no convenient method for adding and subtracting. For addition or subtraction, the complex numbers need to be converted to Cartesian form first. Multiplying and dividing complex numbers are much easier when they are expressed in polar form. Consider the two complex numbers z_1 and z_2 given by,

$$z_1 = r_1 (\cos(\theta_1) + i \sin(\theta_1)) \text{ and } z_2 = r_2 (\cos(\theta_2) + i \sin(\theta_2)).$$

To determine the product $z_1 \times z_2$:

$$\begin{aligned}z_1 \times z_2 &= r_1 (\cos(\theta_1) + i \sin(\theta_1)) \times r_2 (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1) + i \sin(\theta_1)) (\cos(\theta_2) + i \sin(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1) \cos(\theta_2) + \cos(\theta_1) i \sin(\theta_2) + i \sin(\theta_1) \cos(\theta_2) + i^2 \sin(\theta_1) \sin(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1) \cos(\theta_2) + \cos(\theta_1) i \sin(\theta_2) + i \sin(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) \\ &= r_1 r_2 [\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) + i (\cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2))]\end{aligned}$$

Using the following trigonometric identities,

$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1) \cos(\theta_2) \pm \cos(\theta_1) \sin(\theta_2) \quad [1]$$

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1) \cos(\theta_2) \mp \sin(\theta_1) \sin(\theta_2) \quad [2]$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \quad [3]$$

we have

$$\begin{aligned}z_1 \times z_2 &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \\ &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)\end{aligned}$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1 (\cos(\theta_1) + i \sin(\theta_1))}{r_2 (\cos(\theta_2) + i \sin(\theta_2))} \\ &= \frac{r_1 (\cos(\theta_1) + i \sin(\theta_1))}{r_2 (\cos(\theta_2) + i \sin(\theta_2))} \times \frac{(\cos(\theta_2) - i \sin(\theta_2))}{(\cos(\theta_2) - i \sin(\theta_2))} \\ &= \frac{r_1 (\cos(\theta_1) \times \cos(\theta_2) + \cos(\theta_1) \times -i \sin(\theta_2) + i \sin(\theta_1) \times \cos(\theta_2) + i \sin(\theta_1) \times -i \sin(\theta_2))}{r_2 (\cos(\theta_2) \times \cos(\theta_2) + \cos(\theta_2) \times -i \sin(\theta_2) + i \sin(\theta_2) \times \cos(\theta_2) + i \sin(\theta_2) \times -i \sin(\theta_2))} \\ &= \frac{r_1 (\cos(\theta_1) \cos(\theta_2) - i \cos(\theta_1) \sin(\theta_2) + i \cos(\theta_2) \sin(\theta_1) + \sin(\theta_1) \sin(\theta_2))}{r_2 (\cos^2(\theta_2) - \cancel{i \cos(\theta_2) \sin(\theta_2)} + \cancel{i \cos(\theta_2) \sin(\theta_2)} + \sin^2(\theta_2))} \\ &= \frac{r_1 (\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2) + i (\cos(\theta_2) \sin(\theta_1) - \cos(\theta_1) \sin(\theta_2)))}{r_2 (\cos^2(\theta_2) + \sin^2(\theta_2))} \end{aligned}$$
$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1 (\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2))}{r_2} \\ &= \frac{r_1}{r_2} \text{cis} (\theta_1 - \theta_2)\end{aligned}$$

Multiplication and division in polar form

$$z_1 \times z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \text{ and } \frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

MMAA: **m**ultiply **m**oduli, **a**dd **a**rguments
DMSA: **d**ivide **m**oduli, **s**ubtract **a**rguments

If $z_1 = 4 \operatorname{cis} \left(\frac{2\pi}{3} \right)$ and $z_2 = 5 \operatorname{cis} \left(\frac{\pi}{4} \right)$, determine

a. $z_1 \times z_2$

a. 1. Simplify using the multiplication rule
 $z_1 \times z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2).$

$$\begin{aligned}\text{a. } z_1 \times z_2 &= 4 \operatorname{cis} \left(\frac{2\pi}{3} \right) \times 5 \operatorname{cis} \left(\frac{\pi}{4} \right) \\ &= 4 \times 5 \operatorname{cis} \left(\frac{2\pi}{3} + \frac{\pi}{4} \right)\end{aligned}$$

2. The argument lies between $-\pi < \theta \leq \pi$ so it does not need to be simplified.

Express the answer in the required format.

- b. 1. Simplify using the rule

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2).$$

2. The argument lies between $-\pi < \theta \leq \pi$ so it does not need to be simplified.

Express the answer in the required format.

$$= 20 \operatorname{cis}\left(\frac{11\pi}{12}\right)$$

$$\text{b. } \frac{z_1}{z_2} = \frac{4 \operatorname{cis}\left(\frac{2\pi}{3}\right)}{5 \operatorname{cis}\left(\frac{\pi}{4}\right)}$$

$$= \frac{4}{5} \operatorname{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{4}{5} \operatorname{cis}\left(\frac{5\pi}{12}\right)$$

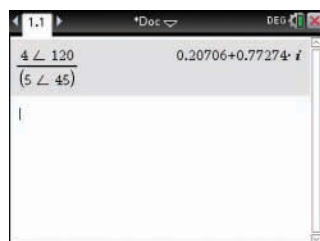
TI | THINK

- b. 1. On a Calculator page, complete the entry line $(4\angle 120) / (5\angle 45)$



2. Press the ENTER button.

The answer appears on the screen in Cartesian form.



3. To convert the answer to Polar form, select CTRL

(-)

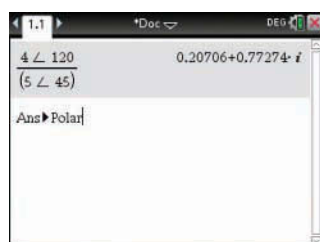
MENU

2: Number

9: Complex Number

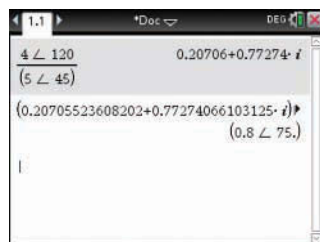
Tools

6: Convert to Polar



4. Press the ENTER button.

The answer appears on the screen.

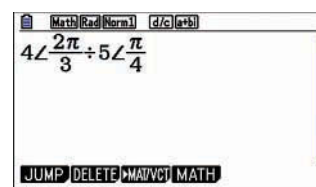


CASIO | THINK

- b. 1. On a Run-Matrix screen, complete the entry line

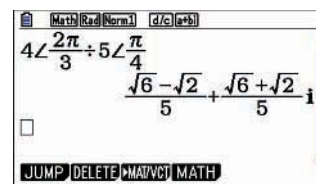
$$4\angle \frac{2\pi}{3} \div 5\angle \frac{\pi}{4}$$

WRITE

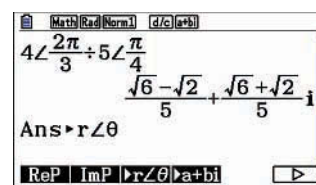


2. Press the EXE button.

The answer appears on the screen in Cartesian form.

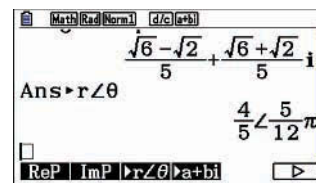


3. To convert the answer to polar, select SHIFT(-) OPTN COMPLEX $\triangleright r\angle\theta$



4. Press the EXE button.

The answer appears on the screen in Polar form.



5.3.4 Proving identities involving the modulus and argument

The product of a complex number and its conjugate, $z \times \bar{z}$, is a real number. Consider the following where $a, b \in \mathbb{R}$.

$$\begin{aligned} z\bar{z} &= (a + bi) \times (a - bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \\ &= |z|^2 \end{aligned}$$

This is an example of using a Cartesian approach to prove the identity $z\bar{z} = |z|^2$.

Below is a list of some common complex number identities which can be proved using the polar form of a complex number.

- i. $z\bar{z} = |z|^2$
- ii. $|z_1 z_2| = |z_1| |z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- iii. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$, $z_2 \neq 0$
- iv. $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$
- v. $\left| \frac{1}{z^n} \right| = \frac{1}{|z|^n}$ and $\arg\left(\frac{1}{z^n}\right) = -n \arg(z)$, $z \neq 0$
- vi. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- vii. $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$
- viii. $z + \bar{z} = 2\operatorname{Re}(z)$
- ix. $z - \bar{z} = 2i\operatorname{Im}(z)$

WORKED EXAMPLE 8

a. Prove the identity $z\bar{z} = |z|^2$ using polar arithmetic.

b. Verify the identity $z - \bar{z} = 2i \operatorname{Im}(z)$ using $z = 2 \operatorname{cis}\left(\frac{\pi}{6}\right)$.

THINK

- a. 1.** Define the complex number z and its conjugate $\bar{z}$ in polar form.
- 2.** Begin with the LHS. Substitute the expressions for z and $\bar{z}$.
- 3.** Recall and apply the rule $z_1 \times z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$ to expand $z\bar{z}$.
- 4.** Simplify the modulus and argument components.
- 5.** Recall that $\operatorname{cis}(\theta)$ is short for $\cos(\theta) + i \sin(\theta)$. Use exact values to evaluate $\cos(0)$ and $\sin(0)$.
- 6.** The LHS has been simplified to $|z|^2$.
- 7.** Finalise the proof with a suitable statement.

WRITE

a. Let $z = |z| \operatorname{cis}(\theta)$ and $\bar{z} = |z| \operatorname{cis}(-\theta)$

$$\begin{aligned} \text{LHS} &= z\bar{z} \\ &= (|z| \operatorname{cis}(\theta)) \times (|z| \operatorname{cis}(-\theta)) \\ &= |z| \times |z| \operatorname{cis}(\theta + (-\theta)) \\ &= |z|^2 \operatorname{cis}(0) \\ &= |z|^2 (\cos(0) + i \sin(0)) \\ &= |z|^2 (1 + i \times 0) \\ &= |z|^2 \\ &= \text{RHS} \end{aligned}$$

Since the LHS = RHS then $z\bar{z} = |z|^2$.

- b. 1. Define the complex number z and its conjugate $\bar{z}$ in polar form.

2. Begin with the LHS. Substitute the expressions for z and $\bar{z}$.

Since there is no rule to follow for subtracting complex numbers in polar form we must convert them into Cartesian form first. Recall that $\text{cis}(\theta)$ is short for $\cos(\theta) + i \times \sin(\theta)$.

3. Use exact values and simplify where possible.

4. In this case, polar arithmetic needs to be applied to simplify the RHS. Make a substitution for $z = 2 \text{cis}\left(\frac{\pi}{6}\right)$.

5. Recall that $\text{cis}(\theta)$ is short for $\cos(\theta) + i \times \sin(\theta)$.

6. Extract the imaginary component of z .

7. Simplify the RHS.

8. Finalise the proof with a suitable statement.

b. Let $z = 2 \text{cis}\left(\frac{\pi}{6}\right)$ and $\bar{z} = 2 \text{cis}\left(-\frac{\pi}{6}\right)$

$$\text{LHS} = z - \bar{z}$$

$$= 2 \text{cis}\left(\frac{\pi}{6}\right) - 2 \text{cis}\left(-\frac{\pi}{6}\right)$$

$$= 2 \cos\left(\frac{\pi}{6}\right) + 2 \sin\left(\frac{\pi}{6}\right) i -$$

$$\left(2 \cos\left(-\frac{\pi}{6}\right) + 2 \sin\left(-\frac{\pi}{6}\right) i\right)$$

$$= 2 \times \frac{\sqrt{3}}{2} + 2 \times \frac{1}{2} \times i - 2 \times \frac{\sqrt{3}}{2} - 2 \times \frac{-1}{2} i$$

$$= \cancel{\sqrt{3}} + i - \cancel{\sqrt{3}} + i$$

$$= 2i$$

$$\text{RHS} = 2i \text{Im}(z)$$

$$= 2i \text{Im}\left(2 \text{cis}\left(\frac{\pi}{6}\right)\right)$$

$$= 2i \text{Im}\left(2 \cos\left(\frac{\pi}{6}\right) + 2 \sin\left(\frac{\pi}{6}\right) i\right)$$

$$= 2i \times 2 \left(\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 2i \times 2 \times \frac{1}{2}$$

$$= 2i$$

$$= \text{LHS}$$

Since the LHS = RHS then

$$z - \bar{z} = 2i \text{Im}(z) \text{ for } z = 2 \text{cis}\left(\frac{\pi}{6}\right).$$

study on

Units 3 & 4 > Area 3 > Sequence 1 > Concept 2

Complex numbers in polar form Summary screen and practice questions

Exercise 5.3 Complex numbers in polar form

Technology Free

1. Determine the modulus of each of the following.

a. $z = 3 + i$

b. $z = 12 + 5i$

c. $z = 2 + \sqrt{5}i$

2. Calculate the Argument of z for each of the following in the interval $(-\pi, \pi]$, giving an exact answer.

a. $z = 8 + 8i$

b. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

c. $z = -\sqrt{3} + i$

3. a. **WE6** Express $2\sqrt{3} + 2i$ in polar form.
 b. Express $15 \operatorname{cis} \left(\frac{2\pi}{3} \right)$ in Cartesian form.
 c. Express $-\frac{1}{\sqrt{3}} - \frac{1}{3}i$ in polar form.
 d. Express $4 \operatorname{cis} \left(-\frac{\pi}{4} \right)$ in Cartesian form.
4. Simplify $3 \left(\cos \left(\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right) + 4 \operatorname{cis} \left(\frac{5\pi}{6} \right)$, giving your answer in Cartesian form.
5. Express each of the following using a principal argument.
 a. $\operatorname{cis} \left(\frac{7\pi}{4} \right)$ b. $\operatorname{cis} \left(\frac{-12\pi}{11} \right)$ c. $\operatorname{cis} \left(\frac{7\pi}{5} \right)$
6. **WE7** If $z_1 = 2 \operatorname{cis} \left(\frac{\pi}{6} \right)$ and $z_2 = 6 \operatorname{cis} \left(\frac{3\pi}{4} \right)$, determine:
 a. $z_1 \times z_2$ b. $\frac{z_2}{z_1}$
7. Express each of the following in the form $r \operatorname{cis} (\theta)$ where $\theta \in (-\pi, \pi]$.
 a. $3 \operatorname{cis} \left(\frac{\pi}{4} \right) \times 4 \operatorname{cis} \left(\frac{\pi}{2} \right)$ b. $15 \operatorname{cis} \left(\frac{5\pi}{6} \right) \div 5 \operatorname{cis} \left(\frac{\pi}{3} \right)$
 c. $\sqrt{5} \operatorname{cis} \left(\frac{-7\pi}{12} \right) \times 4 \operatorname{cis} \left(\frac{5\pi}{12} \right)$ d. $6 \operatorname{cis} \left(\frac{3\pi}{4} \right) \div 2 \operatorname{cis} \left(\frac{\pi}{8} \right)$
 e. $\sqrt{7} \operatorname{cis} \left(-\frac{3\pi}{4} \right) \times \sqrt{7} \operatorname{cis} \left(-\frac{4\pi}{5} \right)$ f. $14 \operatorname{cis} \left(\frac{-7\pi}{12} \right) \div 7 \operatorname{cis} \left(\frac{5\pi}{6} \right)$
8. Given $u = 1 + \sqrt{3}i$ and $v = 4 \operatorname{cis} \left(\frac{\pi}{6} \right)$, determine each of the following, expressing your answers in exact polar form where possible.
 a. uv b. $2u - 3v$ c. $\frac{v}{u}$
9. **WE8** Verify the identity $z - \bar{z} = 2i \operatorname{Im}(z)$ using $z = 3 \operatorname{cis} \left(\frac{\pi}{3} \right)$.
10. Use polar arithmetic to prove the following identities.
 a. $|z_1 z_2| = |z_1| |z_2|$ b. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
11. Prove the following identities using a polar approach.
 a. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$ b. $\arg \left(\frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2)$
12. Prove the following identities using polar arithmetic.
 a. $|z^n| = |z|^n$ b. $\arg(z^n) = n \arg(z)$
13. Given $z = r \operatorname{cis} (\theta)$, prove the following identities.
 a. $\left| \frac{1}{z^n} \right| = \frac{1}{|z|^n}, z \neq 0$ b. $\arg \left(\frac{1}{z^n} \right) = -n \arg(z), z \neq 0$
14. Verify the following identities if $z_1 = 4 \operatorname{cis} \left(\frac{\pi}{3} \right)$ and $z_2 = 7 \operatorname{cis} \left(\frac{\pi}{6} \right)$.
 a. $|z_1 z_2| = |z_1| |z_2|$ b. $z_1 + \bar{z}_1 = 2\operatorname{Re}(z_1)$
 c. $\arg \left(\frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2)$ d. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
15. Prove the identity, $\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}$ using polar arithmetic.

16. Prove the identity, $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$ by considering all complex numbers in polar form.



17. Use polar arithmetic to prove the identity, $z + \bar{z} = 2\text{Re}(z)$.
18. Given $z = |z|\text{cis}(\theta)$, prove the identity, $z - \bar{z} = 2i\text{Im}(z)$.
19. a. Let $u = \frac{1}{4}(\sqrt{3} - i)$ and $v = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$.
- Determine uv , working with both numbers in Cartesian form and giving your answer in Cartesian form.
 - Determine uv , working with both numbers in polar form and giving your answer in polar form.
 - Hence, deduce the exact value of $\sin\left(\frac{\pi}{12}\right)$.
 - Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{\pi}{12}\right)$.
- b. Let $u = \sqrt{2}(1 - i)$ and $v = 2\text{cis}\left(\frac{2\pi}{3}\right)$.
- Determine uv , working with both numbers in Cartesian form and giving your answer in Cartesian form.
 - Determine uv , working with both numbers in polar form and giving your answer in polar form.
 - Hence, deduce the exact value of $\sin\left(\frac{5\pi}{12}\right)$.
 - Using the formula $\sin(x - y)$, verify your exact value for $\sin\left(\frac{5\pi}{12}\right)$.

5.4 De Moivre's theorem

In the previous section you would have observed that multiplying and dividing complex numbers is a simpler process when the complex numbers are in polar form.

The product and quotient of complex numbers can be determined using

$$z_1 \times z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2) \text{ and } \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2) \text{ where } r_1 = |z_1| \text{ and } r_2 = |z_2|.$$

To calculate powers of z it is necessary to complete repeated multiplication.

Consider

$$z^n = z \times z \times z \times \dots \times z \quad (n \text{ times})$$

Now consider the repeated multiplication of the complex number $z = r \operatorname{cis}(\theta)$.

$$z^1 = r^1 \operatorname{cis}(\theta)$$

$$z^2 = r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) = r \times r \operatorname{cis}(\theta + \theta) = r^2 \operatorname{cis}(2\theta)$$

$$z^3 = r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) = r^3 \operatorname{cis}(3\theta)$$

$$z^4 = r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) \times r \operatorname{cis}(\theta) = r^4 \operatorname{cis}(4\theta)$$

This can be generalized in a form known as **De Moivre's theorem**.

De Moivre's theorem

De Moivre's theorem states that $(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$.

This theorem can be proved using mathematical induction demonstrated in the following worked example.

WORKED EXAMPLE 9

Prove De Moivre's theorem, $(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta)$, for positive integer values of n through the process of mathematical induction.

THINK

1. State the proposition.
2. State what is required to prove (RTP).
3. Test for when $n = 1$.
4. Assume the theorem is true for $n = k$.
5. Rewrite De Moivre's theorem for when $n = k + 1$.
6. Approach the proof by considering the LHS.
7. Simplify by grouping the real and imaginary components.

WRITE

Let $P(n)$ be the proposition that

$$(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta), n \in \mathbb{N}, n \geq 1.$$

RTP: Both $P(1)$ is true and $P(k+1)$ is true given $P(k)$ is true.

$$n = 1; \text{ LHS} = (r \operatorname{cis}(\theta))^1 \\ = r \operatorname{cis}(\theta)$$

$$\text{RHS} = r^1 \operatorname{cis}(1 \times \theta) \\ = r \operatorname{cis}(\theta)$$

$$\text{LHS} = \text{RHS} \Rightarrow P(1) \text{ is true.}$$

Assume $P(k)$ is true $k \in \mathbb{N}, k \geq 1$

$$(r \operatorname{cis}(\theta))^k = r^k \operatorname{cis}(k\theta)$$

$$[r(\cos \theta + i \sin \theta)]^k = r^k (\cos k\theta + i \sin k\theta)$$

$$\text{RTP: } (r \operatorname{cis}(\theta))^{k+1} = r^{k+1} \operatorname{cis}((k+1)\theta)$$

LHS:

$$\begin{aligned} (r \operatorname{cis}(\theta))^{k+1} &= (r \operatorname{cis}(\theta))^k \times (r \operatorname{cis}(\theta))^1 \\ &= r^k \operatorname{cis}(k\theta) \times r^1 \operatorname{cis}(\theta)^1 \\ &= r^{k+1} (\cos(k\theta) + i \sin(k\theta)) (\cos(\theta) + i \sin(\theta)) \\ &= r^{k+1} \begin{pmatrix} \cos(k\theta) \cos(\theta) + \cos(k\theta) i \sin(\theta) \\ + i \sin(k\theta) \cos(\theta) + i^2 \sin(k\theta) \sin(\theta) \end{pmatrix} \\ &= r^{k+1} \begin{pmatrix} (\cos(k\theta) \cos(\theta) - \sin(k\theta) \sin(\theta)) \\ + i \begin{pmatrix} \cos(k\theta) \sin(\theta) \\ + \sin(k\theta) \cos(\theta) \end{pmatrix} \end{pmatrix} \end{aligned}$$

8. Recall the trigonometric identities used in Section 5.3.3. The HCF is θ so simplify by factorising.

$$\begin{aligned} &= r^{k+1} (\cos(k\theta + \theta) + i \sin(k\theta + \theta)) \\ &= r^{k+1} (\cos((k+1)\theta) + i \sin((k+1)\theta)) \\ &= r^{k+1} \operatorname{cis}((k+1)\theta) \\ &= \text{RHS of } P(k+1) \end{aligned}$$

That is $P(k+1)$ is true whenever $P(k)$ is true.

9. Write the answer.

By the principle of mathematical induction it is true for all values of n . That is, $(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta)$ is true for all values of $n \in N, n \geq 1$.

De Moivre's theorem is also applicable for negative integer values of n . A proof will not be conducted here, however an opportunity to do so exists in Exercise 5.4.

WORKED EXAMPLE 10

Simplify the following using De Moivre's theorem.

a. $\left(3 \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5$ b. $\left(8 \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^{-3}$

THINK

- a. 1. Use De Moivre's theorem for calculating the result of raising a complex number to a power.
2. Evaluate 3^5 and convert the angle into a principal value.

$$\frac{5\pi}{4} = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}.$$

- b. 1. Use De Moivre's theorem for calculating the result of raising a complex number to a power.
2. Use index laws to help simplify 8^{-3} and simplify the argument.
3. Express the angle as a principal value.

4. Write $r \operatorname{cis}(\theta)$ in expanded form to simplify further.

WRITE

$$\begin{aligned} \text{a. } \left(3 \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5 &= 3^5 \operatorname{cis}\left(5 \times \frac{\pi}{4}\right) \\ &= 243 \operatorname{cis}\left(-\frac{3\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \text{b. } \left(8 \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^{-3} &= 8^{-3} \operatorname{cis}\left(-3 \times \frac{2\pi}{3}\right) \\ &= \frac{1}{8^3} \operatorname{cis}\left(-\cancel{3} \times \frac{2\pi}{\cancel{3}}\right) \\ &= \frac{1}{512} \operatorname{cis}(-2\pi) \\ &= \frac{1}{512} \operatorname{cis}(0) \\ &= \frac{1}{512} \cos(0) + i \times \frac{1}{512} \sin(0) \\ &= \frac{1}{512} \end{aligned}$$

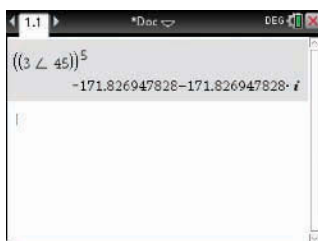
TI | THINK

- a. 1. On a Calculator page, complete the entry line $(3\angle 45)^5$

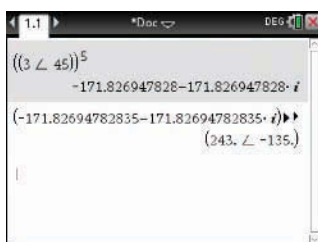
WRITE



2. Press the ENTER button.
The answer appears on the screen in Cartesian form.



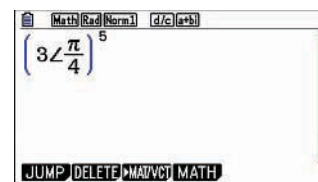
3. Convert the answer to Polar form.
The answer appears on the screen.



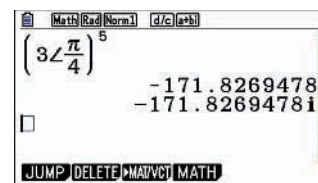
CASIO | THINK

- a. 1. On a Run-Matrix screen, complete the entry line $\left(3\angle\frac{\pi}{4}\right)^5$

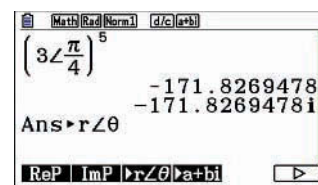
WRITE



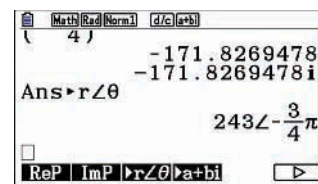
2. Press the EXE button.
The answer appears on the screen in Cartesian form.



3. To convert the answer to polar form, select
SHIFT(−)
OPTN
COMPLEX
▷ $r\angle\theta$



4. Press the EXE button.
The answer appears on the screen.



study on

Units 3&4 > Area 3 > Sequence 1 > Concept 3

De Moivre's theorem Summary screen and practice questions

Exercise 5.4 De Moivre's theorem

Technology Free

- WE9** Prove De Moivre's theorem, $(r \operatorname{cis}(\theta))^n = r^n \operatorname{cis}(n\theta)$, for negative integer values of n through the process of mathematical induction.
- WE10** Simplify the following using De Moivre's theorem.
 - $\left(2 \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5$
 - $\left(3 \operatorname{cis}\left(-\frac{3\pi}{4}\right)\right)^6$
 - $\left(3 \operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-4}$
 - $\left(\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^{-6}$
- Use De Moivre's theorem to simplify each of the following.
 - $\left(2 \operatorname{cis}\left(\frac{3\pi}{4}\right)\right)^5 \times \left(3 \operatorname{cis}\left(\frac{\pi}{8}\right)\right)^6$
 - $\left(4 \operatorname{cis}\left(\frac{\pi}{4}\right)\right)^5 \times \left(2 \operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^{-3}$

4. Use De Moivre's theorem to simplify each of the following.

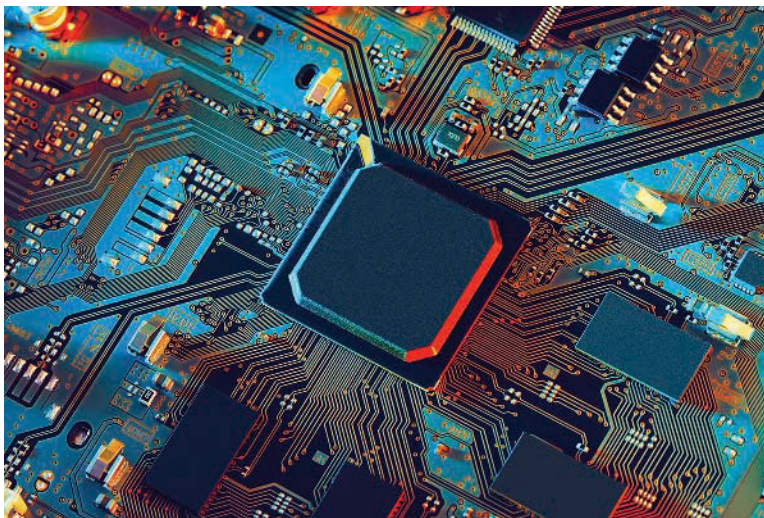
a. $\frac{\left(3 \operatorname{cis}\left(\frac{\pi}{3}\right)\right)^4}{\left(\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^3}$ b. $\frac{\left(4 \operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^7}{\left(2 \operatorname{cis}\left(-\frac{3\pi}{5}\right)\right)^{-6}}$

5. Use De Moivre's theorem to express the following in the form $a + bi$.

a. $(2 - 2i)^4$ b. $(\sqrt{2} + \sqrt{6}i)^{-6}$

6. Determine the integer values for a and b that satisfy the following equation

$$(1 - \sqrt{3}i)^7 = a + b\sqrt{3}i.$$



7. If $4\left(\left(\frac{4+4i}{1-i}\right)^4 - \left(\frac{1}{1+i}\right)^4\right) = a + bi$, determine the value of a and b .

8. Determine all values of n such that $(-\sqrt{3} + 1)^n + (-\sqrt{3} - i)^n = 0$.

9. a. If $z = 2 + 2i$, determine each of the following.

i. z^8 ii. $\operatorname{Arg}(z^8)$

- b. If $z = -3\sqrt{3} + 3i$, calculate each of the following.

i. z^6 ii. $\operatorname{Arg}(z^6)$

10. If $z = \operatorname{cis}(\theta)$, show that

a. $|z + 1| = 2 \cos\left(\frac{\theta}{2}\right)$ b. $\operatorname{Arg}(1 + z) = \frac{\theta}{2}$

11. a. By considering the real and imaginary parts of the equation $(\operatorname{cis}(\theta))^2 = \operatorname{cis}(2\theta)$, show that

i. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

ii. $\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$

- b. By considering the real and imaginary parts of the equation $(\operatorname{cis}(\theta))^3 = \operatorname{cis}(3\theta)$, show that

i. $\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta)$

ii. $\sin(3\theta) = 3 \sin(\theta) - 4 \sin^3(\theta)$

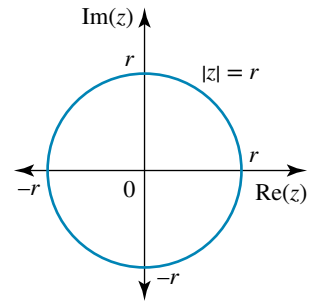
5.5 The complex plane (the Argand plane)

In previous sections, complex numbers have been used to represent points on the Argand plane. If we consider z as a complex variable, we can sketch subsets or regions on the Argand plane.

In this section, $z = a + bi$ will be expressed as $z = x + yi$.

5.5.1 Circles

The equation $|z| = r$ where $z = x + yi$ can be rewritten as $\sqrt{x^2 + y^2} = r$ (by definition of the modulus of z). Squaring both sides yields $x^2 + y^2 = r^2$. This represents a circle with centre $(0, 0)$ and radius r . Geometrically, $|z| = r$ represents the set of points, or what is called the locus of points, on the Argand plane that are at r units from the origin.



WORKED EXAMPLE 11

Sketch and describe the region of the complex plane defined by $\{z: |z + 2 - 3i| = 4\}$.

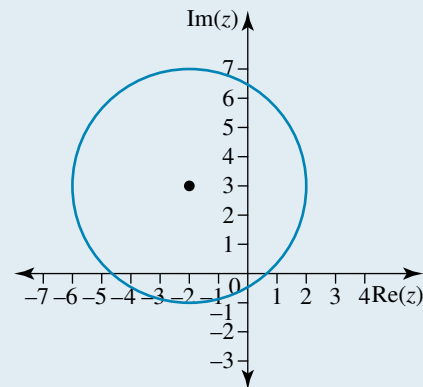
THINK

1. Consider the equation.
Substitute $z = x + yi$ into the given statement.
2. Group the real and imaginary parts.
3. Recall that if $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$.
4. Square both sides.
5. Sketch and identify the graph on the Argand plane.

WRITE

$$\begin{aligned} |z + 2 - 3i| &= 4 \\ |x + yi + 2 - 3i| &= 4 \\ |(x + 2) + i(y - 3)| &= 4 \\ \sqrt{(x + 2)^2 + (y - 3)^2} &= 4 \end{aligned}$$

$(x + 2)^2 + (y - 3)^2 = 16$
The equation represents a circle with centre $(-2, 3)$ and radius 4.



5.5.2 Lines

If $z = x + yi$, then $\text{Re}(z) = x$ and $\text{Im}(z) = y$. The equation $a \text{Re}(z) + b \text{Im}(z) = c$ where a, b and $c \in \mathbb{R}$ represents the line $ax + by = c$.

WORKED EXAMPLE 12

Sketch and describe the region of the complex plane defined by $\{z: 2\operatorname{Re}(z) - 3\operatorname{Im}(z) = 6\}$.

THINK

1. Consider the equation.
2. Determine the axial intercepts.
3. Identify and sketch the equation.

WRITE

$$2\operatorname{Re}(z) - 3\operatorname{Im}(z) = 6,$$

As $z = x + yi$, then $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$.

This is a straight line with the Cartesian equation $2x - 3y = 6$.

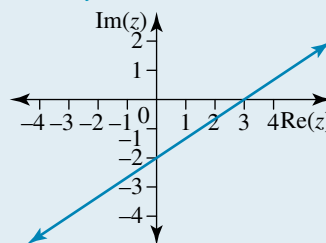
When $y = 0$, $2x = 6 \Rightarrow x = 3$.

$(3, 0)$ is the intercept with the real axis.

When $x = 0$, $-3y = 6 \Rightarrow y = -2$.

$(0, -2)$ is the intercept with the imaginary axis.

The equation represents the line $2x - 3y = 6$.



Lines in the complex plane can also be represented as a set of points that are equidistant from two other fixed points. The equations of a line in the complex plane can thus have multiple representations.

WORKED EXAMPLE 13

Sketch and describe the region of the complex plane defined by $\{z: |z - 2i| = |z + 2|\}$.

THINK

1. Consider the equation as a set of points.
Substitute $z = x + yi$ into the given statement.
2. Group the real and imaginary parts together.
3. Recall that if $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$.
4. Square both sides, expand, and cancel like terms.
5. Identify the required line.
6. Identify the line geometrically.

WRITE

$$|z - 2i| = |z + 2|$$

$$|x + yi - 2i| = |x + yi + 2|$$

$$|x + (y - 2)i| = |(x + 2) + yi|$$

$$\sqrt{x^2 + (y - 2)^2} = \sqrt{(x + 2)^2 + y^2}$$

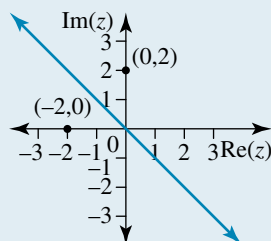
$$x^2 + y^2 - 4y + 4 = x^2 + 4x + 4 + y^2$$

$$-4y = 4x$$

$$y = -x$$

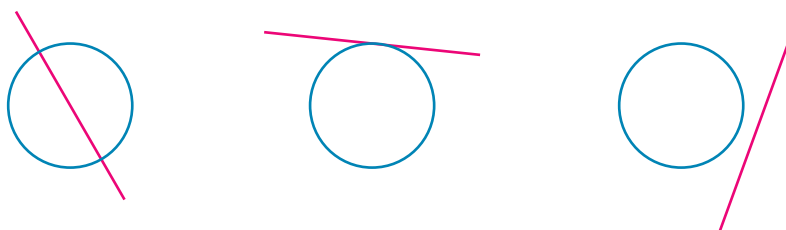
The line is the set of points that is equidistant from the two points $(0, 2)$ and $(-2, 0)$.

7. Sketch the required line.



5.5.3 Intersection of lines and circles

The coordinates of the points of intersection between a straight line and a circle can be found algebraically by solving the system of equations. If there are two solutions to the equations, the line intersects the circle at two points. If there is one solution to the equation, the line and the circle touch at one point, and the line is a tangent to the circle at the point of contact. If there are no solutions to the equations, the line does not intersect the circle.



WORKED EXAMPLE 14

- a. Two sets of points in the complex plane are defined by $S = \{z: |z| = 5\}$ and $T = \{z: 2\operatorname{Re}(z) - \operatorname{Im}(z) = 10\}$. Determine the coordinates of the points of intersection between S and T .
- b. Two sets of points in the complex plane are defined by $S = \{z: |z| = 3\}$ and $T = \{z: 2\operatorname{Re}(z) - \operatorname{Im}(z) = k\}$. Determine the values of k for which the line through T is a tangent to the circle S .

THINK

- a. 1. Find the Cartesian equation of S .
Substitute $z = x + yi$ into the statement.
2. Recall that if $z = x + yi$ then $|z| = \sqrt{x^2 + y^2}$.
3. Square both sides and identify the boundary of S .
4. Find the Cartesian equation of T .
5. Solve equations [1] and [2] for x and y by substitution.

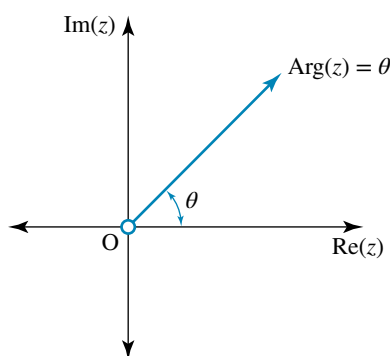
WRITE

- a. $|z| = 5$
 $|x + yi| = 5$
 $\sqrt{x^2 + y^2} = 5$
 $x^2 + y^2 = 25$ [1]
 S is a circle with centre $(0, 0)$ and radius 5.
- Substitute $z = x + yi$:
 $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$
 $2\operatorname{Re}(z) - \operatorname{Im}(z) = 10$
 $2x - y = 10$ [2]
 T is a straight line.
- $y = 2x - 10$ [2]
 $x^2 + (2x - 10)^2 = 25$ [1]

6. Expand and simplify. 7. Solve for x. 8. Find the corresponding y-values. 9. State the coordinates of the two points of intersection. b. 1. Find the Cartesian equation of S. Substitute $z = x + yi$ into the statement. 2. Recall that if $z = x + yi$ then $z = \sqrt{x^2 + y^2}$. 3. Square both sides and identify the boundary of S. 4. Find the Cartesian equation of T. 5. Solve equations [1] and [2] for x and y by substitution. 6. Expand and simplify. 7. If the line through T is a tangent to the circle S, there will be only one solution for x. For a quadratic equation, one solution can be found by determining when the discriminant $= 0$. 8. Solve the discriminant equal to zero for k. 9. State the value of k for which the line through T is a tangent to the circle S.	$x^2 + 4x^2 - 40x + 100 = 25$ $5x^2 - 40x + 75 = 0$ $5(x^2 - 8x + 15) = 0$ $x^2 - 8x + 15 = 0$ $(x - 5)(x - 3) = 0$ $x = 5 \text{ or } x = 3$ From $y = 2x - 10$ [2], when $x = 5 \Rightarrow y = 0$ and when $x = 3 \Rightarrow y = -4$. The points of intersection are $(5, 0)$ and $(3, -4)$. b. $z = 3$ $x + yi = 3$ $\sqrt{x^2 + y^2} = 3$ $x^2 + y^2 = 9$ [1] S is a circle with centre $(0, 0)$ and radius 3. Substitute $z = x + yi$: $\operatorname{Re}(z) = x$ and $\operatorname{Im}(z) = y$ $2\operatorname{Re}(z) - \operatorname{Im}(z) = 2x - y = k$ [2] T is a straight line. $y = 2x - k$ [2] $x^2 + (2x - k)^2 = 9$ [1] $x^2 + 4x^2 - 4kx + k^2 = 9$ $5x^2 - 4kx + k^2 - 9 = 0$ The discriminant $\Delta = b^2 - 4ac = 0$, where $a = 5, b = -4k$ and $c = k^2 - 9$. $\Delta = (-4k)^2 - 4 \times 5 \times (k^2 - 9)$ $= 16k^2 - 20(k^2 - 9)$ $= -4k^2 + 180$ $= 4(45 - k^2)$ $45 - k^2 = 0$ $k = \pm\sqrt{45}$ $k = \pm 3\sqrt{5}$

5.5.4 Rays

$\operatorname{Arg}(z) = \theta$ represents the set of all points on the half-line or ray that has one end at the origin and makes an angle of θ with the positive real axis. Note that the endpoint, in this case the origin, is not included in the set. We indicate this by placing a small open circle at this point.



WORKED EXAMPLE 15

Determine the Cartesian equation and sketch the graph defined by $\left\{ z: \text{Arg}(z - 1 + i) = -\frac{\pi}{4} \right\}$.

THINK

- Find the Cartesian equation of the ray.
Substitute $z = x + yi$ into the statement.
- Group the real and imaginary parts.
- Use the definition of the argument. $\text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right)$.
- Simplify by taking the tangent of both sides then use your knowledge of exact values of $\tan$ to evaluate $\tan\left(-\frac{\pi}{4}\right)$.
- State the Cartesian equation of the ray.
- Rewrite the original statement in the form $\text{Arg}(z - \alpha)$ to identify the point from which the ray starts.
- Determine the angle the ray makes.
- Describe the ray.

WRITE

$$\text{Arg}(z - 1 + i) = -\frac{\pi}{4}$$

$$\text{Arg}(x + yi - 1 + i) = -\frac{\pi}{4}$$

$$\text{Arg}((x - 1) + (y + 1)i) = -\frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{y + 1}{x - 1}\right) = -\frac{\pi}{4} \text{ for } x > 1$$

$$\frac{y + 1}{x - 1} = \tan\left(-\frac{\pi}{4}\right)$$

$$\frac{y + 1}{x - 1} = -1 \text{ for } x > 1$$

$$y + 1 = -(x - 1) \text{ for } x > 1$$

$$y = -x \text{ for } x > 1.$$

$$\text{Arg}(z - 1 + i) = -\frac{\pi}{4} \text{ becomes}$$

$$\text{Arg}(z - (1 - i)) = -\frac{\pi}{4}$$

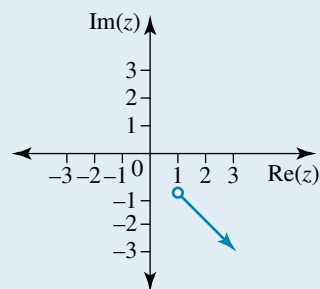
The ray starts from the point $(1, -1)$.

The ray makes an angle of $-\frac{\pi}{4}$ with the positive real axis.

The point $(1, -1)$ is not included.

Alternatively, consider the ray from the origin making an angle of $-\frac{\pi}{4}$ with the positive real axis to have been translated one unit to the right parallel to the real axis, and one unit down parallel to the imaginary axis.

9. Sketch the required ray.



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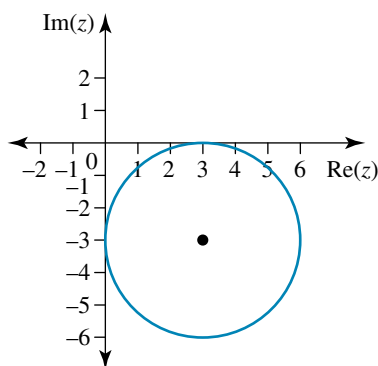
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The complex plane (the Argand plane) Summary screen and practice questions

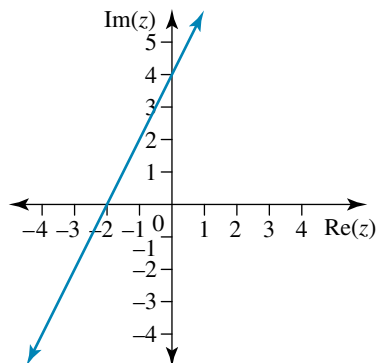
Exercise 5.5 The complex plane (the Argand plane)

Technology Free

- WE11** Sketch and describe the region of the complex plane defined by $\{z: |z - 3 + 2i| = 4\}$.
- The region of the complex plane shown below can be described by $\{z: |z - (a + bi)| = r\}$. Find the values of a , b and r .

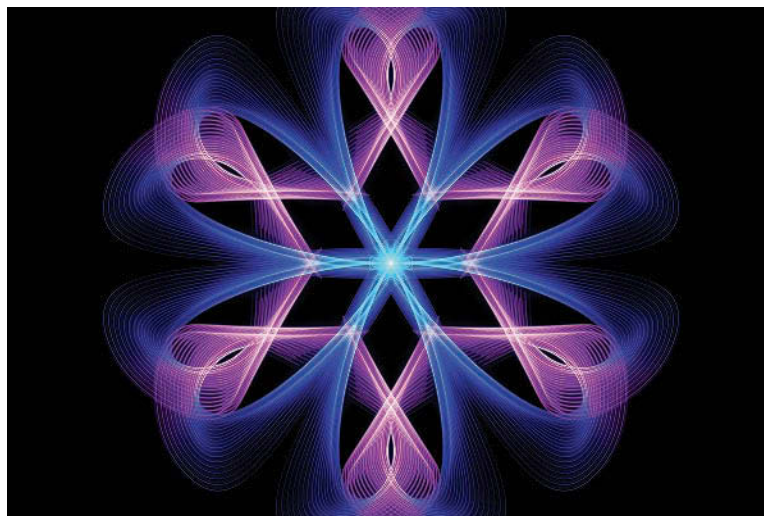


- WE12** Sketch and describe the region of the complex plane defined by $\{z: 4\text{Re}(z) + 3\text{Im}(z) = 12\}$.
- The region of the complex plane shown can be described by $\{z: a\text{Re}(z) + b\text{Im}(z) = 8\}$. Determine the values of a and b .



- WE13** Sketch and describe the region of the complex plane defined by $\{z: |z + 3i| = |z - 3|\}$.

6. Sketch and describe the region of the complex plane defined by $\{z: |z - i| = |z + 3i|\}$.
7. a. **WE14** Two sets of points in the complex plane are defined by $S = \{z: |z| = 3\}$ and $T = \{z: 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 12\}$. Determine the coordinates of the points of intersection between S and T .
- b. Two sets of points in the complex plane are defined by $S = \{z: |z| = 4\}$ and $T = \{z: 4\operatorname{Re}(z) - 2\operatorname{Im}(z) = k\}$. Determine the values of k for which the line through T is a tangent to the circle S .
8. a. Two sets of points in the complex plane are defined by $S = \{z: |z| = \sqrt{29}\}$ and $T = \{z: 3\operatorname{Re}(z) - \operatorname{Im}(z) = 1\}$. Determine the coordinates of the points of intersection between S and T .
- b. Two sets of points in the complex plane are defined by $S = \{z: |z| = 5\}$ and $T = \{z: 2\operatorname{Re}(z) - 3\operatorname{Im}(z) = k\}$. Determine the values of k for which the line through T is a tangent to the circle S .
9. **WE15** Determine the Cartesian equation and sketch the graph defined by $\left\{z: \operatorname{Arg}(z - 2) = \frac{\pi}{6}\right\}$.
10. Determine the Cartesian equation and sketch the graph defined by $\left\{z: \operatorname{Arg}(z + 3i) = -\frac{\pi}{2}\right\}$.
11. For each of the following, sketch and determine the Cartesian equation of the set, and describe the region.
- a. $\{z: |z| = 3\}$ b. $\{z: |z| = 2\}$ c. $\{z: |z + 2 - 3i| = 2\}$ d. $\{z: |z - 3 + i| = 3\}$
12. Illustrate each of the following and describe the subset of the complex plane.
- a. $\{z: \operatorname{Im}(z) = 2\}$ b. $\{z: \operatorname{Re}(z) + 2\operatorname{Im}(z) = 4\}$
c. $\{z: 3\operatorname{Re}(z) + 2\operatorname{Im}(z) = 6\}$ d. $\{z: 2\operatorname{Re}(z) - \operatorname{Im}(z) = 6\}$
13. Sketch and describe each of the following sets, clearly indicating which boundaries are included.
- a. $\{z: |z - 2| = |z - 4|\}$ b. $\{z: |z + 4i| = |z - 4|\}$
c. $\{z: |z + 4| = |z - 2i|\}$ d. $\{z: |z + 2 - 3i| = |z - 2 + 3i|\}$
14. a. Show that the complex equation $\left\{z: \operatorname{Im}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a straight line and determine its equation.
- b. Show that the complex equation $\left\{z: \operatorname{Re}\left(\frac{z - 2i}{z - 3}\right) = 0\right\}$ represents a circle and determine its centre and radius.
15. a. Determine the Cartesian equation of $\{z: |z - 3| = 2|z + 3i|\}$.
- b. Determine the locus of the set of points in the complex plane given by $\{z: |z + 3| = 2|z + 6i|\}$.
- c. Let $S = \{z: |z - 6| = 2|z - 3i|\}$ and $T = \{z: |z - (a + bi)| = r\}$. Given that $S = T$, calculate the values of a , b and r .
- d. Let $\{z: |z + 3| = 2|z - 3i|\}$ and $T = \{z: |z - (a + bi)| = r\}$. Given that $S = T$, calculate the values of a , b and r .



16. Four sets of points in the complex plane are defined by $R = \{z: (z - 3 + 4i)(\bar{z} - 3 - 4i) = 25\}$, $S = \{z: |z - 3 + 4i| = 5\}$, $T = \{z: 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 25\}$ and $U = \{z: |z| = |z - 6 + 8i|\}$.
- Determine the Cartesian equations of T and U and show that $T = U$.
 - Determine the Cartesian equations of S and R and show that $S = R$.
 - Sketch S and T on one Argand plane and determine $u: S = R$ where $u \in C$.
17. a. Two sets of points in the complex plane are defined by $S = \{z: |z| = 3\}$ and $T = \{z: 3\operatorname{Re}(z) + 4\operatorname{Im}(z) = 15\}$. Show that the line T is a tangent to the circle S and determine the coordinates of the point of contact.
- Two sets of points in the complex plane are defined by $S = \{z: |z| = r\}$ and $T = \{z: 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = 10\}$. Given that the line T is a tangent to the circle S , determine the value of r .
 - Two sets of points in the complex plane are defined by $S = \{z: |z| = 6\}$ and $T = \{z: 3\operatorname{Im}(z) - 4\operatorname{Re}(z) = 8\}$. Determine the coordinates of the points of intersection between S and T .
 - Two sets of points in the complex plane are defined by $S = \{z: |z| = 6\}$ and $T = \{z: 3\operatorname{Re}(z) - 4\operatorname{Im}(z) = k\}$. Determine the values of k for which the line through T is a tangent to the circle S .
18. Sketch and describe the following subsets of the complex plane.
- $\{z: \operatorname{Arg}(z) = \frac{\pi}{6}\}$
 - $\{z: \operatorname{Arg}(z + i) = \frac{\pi}{4}\}$
 - $\{z: \operatorname{Arg}(z - 2) = \frac{3\pi}{4}\}$
 - $\{z: \operatorname{Arg}(z + 2 - i) = -\frac{\pi}{2}\}$
19. a. Let $S = \{z: |z| = 2\}$ and $T = \{z: \operatorname{Arg}(z) = \frac{\pi}{4}\}$. Sketch the sets S and T on the same Argand diagram and calculate $z: S = T$.
- Let $S = \{z: |z| = 3\}$ and $T = \{z: \operatorname{Arg}(z) = -\frac{\pi}{4}\}$. Sketch the sets S and T on the same Argand diagram and calculate $z: S = T$.
 - Sets of points in the complex plane are defined by $S = \{z: |z + 3 + i| = 5\}$ and $R = \{z: \operatorname{Arg}(z + 3) = -\frac{3\pi}{4}\}$
 - Determine the Cartesian equation of S .
 - Determine the Cartesian equation of R .
 - If $u \in C$, calculate u where $S = R$.
20. a. Show that the complex equation $|z - a|^2 - |z - bi|^2 = a^2 + b^2$, where a and b are real and $b \neq 0$, represents a line.
- Show that the complex equation $|z - a|^2 + |z - bi|^2 = a^2 + b^2$, where a and b are real, represents a circle, and determine its centre and radius.
 - Show that the complex equation $3z\bar{z} + 6z + 6\bar{z} + 2 = 0$ represents a circle, and determine its centre and radius.
 - Consider the complex equation $az\bar{z} + bz + b\bar{z} + c = 0$ where a , b and c are real.
 - If $b^2 > ac$ and $a \neq 0$, what does the equation represent?
 - If $a = 0$ and $b \neq 0$, what does the equation represent?
 - Show that the complex equation $z\bar{z} + (3 + 2i)z + (3 - 2i)\bar{z} + 4 = 0$ represents a circle, and determine its centre and radius.
 - Consider the complex equation $az\bar{z} + \bar{b}z + b\bar{z} + c = 0$ where a and c are real and $b = \alpha + \beta i$ is complex. Show that the equation represents a circle provided $b\bar{b} > ac$ and $a \neq 0$, and determine the circle's centre and radius.
21. a. Show that the complex equation $\left\{z: \operatorname{Im}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where a and b are real represents a straight line if $ab \neq 0$.
- Show that the complex equation $\left\{z: \operatorname{Re}\left(\frac{z - ai}{z - b}\right) = 0\right\}$ where a and b are real represents a circle if $ab \neq 0$. State the circle's centre and radius.

5.6 Roots of complex numbers

5.6.1 Determine and examine the n^{th} roots of complex numbers.

In Section 5.4, we considered raising complex numbers in polar form to integer powers.

Given $z = r \operatorname{cis}(\theta)$, integer powers of z were calculated using De Moivre's theorem where $z^n = r^n \operatorname{cis}(n\theta)$. The focus in this section is to find the solutions (or **roots**) to the equation $z^n = w$, where $w \in \mathbb{C}$. That is, given z^n , we are interested in calculating z , defined as an n^{th} root of z^n .

Consider the equation $z^n = w$.

Let $z = r_1 \operatorname{cis}(\phi)$ and $w = r_2 \operatorname{cis}(\theta)$

$$\therefore (r_1 \operatorname{cis}(\phi))^n = r_2 \operatorname{cis}(\theta)$$

Applying De Moivre's theorem,

$$r_1^n (\cos(n\phi) + i \sin(n\phi)) = r_2 (\cos(\theta) + i \sin(\theta))$$

Now equate the moduli and arguments from the left- and right-hand sides.

Modulus:	Argument:
$r_1^n = r_2$ $\therefore r_1 = \sqrt[n]{r_2}$ $= (r_2)^{\frac{1}{n}}$	$\cos(n\phi) + i \sin(n\phi) = \cos(\theta) + i \sin(\theta)$ $\operatorname{cis}(n\phi) = \operatorname{cis}(\theta)$ $n\phi = \theta + 2k\pi \quad \text{where } k \in \mathbb{Z}$ $\phi = \frac{\theta + 2k\pi}{n} \quad \text{where } k \in \mathbb{Z}$

The investigation above illustrates that if $z^n = w$ where $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$, the equation has n distinct solutions, in which z is termed an n^{th} root of w .

Because the solutions start repeating, we keep using k until n solutions are found. The roots are generally denoted as $z_1, z_2, \dots, z_n$ where z_1 represents the first root. The first root is commonly known as the principal n^{th} root of the complex number.

WORKED EXAMPLE 16

Use De Moivre's theorem to solve $z^2 = 4 \operatorname{cis}\left(\frac{\pi}{3}\right)$ for z .

THINK

1. Use De Moivre's theorem.
2. Simplify the modulus and argument.
3. Let $k = 0$ to calculate the first root.

WRITE

$$z = \sqrt[2]{4} \operatorname{cis}\left(\frac{\frac{\pi}{3} + 2k\pi}{2}\right)$$

$$z = 2 \operatorname{cis}\left(\frac{\pi}{6} + k\pi\right)$$

$$\begin{aligned} \text{Let } k = 0, z_1 &= 2 \operatorname{cis}\left(\frac{\pi}{6} + 0 \times \pi\right) \\ &= 2 \operatorname{cis}\left(\frac{\pi}{6}\right) \end{aligned}$$



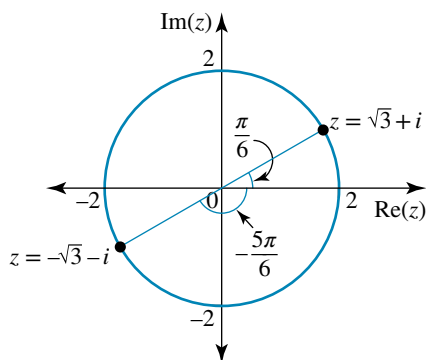
4. Let $k = 1$ to calculate the second root.

$$\begin{aligned}\text{Let } k = 1, z_2 &= 2 \operatorname{cis} \left(\frac{\pi}{6} + 1 \times \pi \right) \\ &= 2 \operatorname{cis} \left(\frac{7\pi}{6} \right) \\ &= 2 \operatorname{cis} \left(-\frac{5\pi}{6} \right)\end{aligned}$$

5. State both answers.

$$z_1 = 2 \operatorname{cis} \left(\frac{\pi}{6} \right), z_2 = 2 \operatorname{cis} \left(-\frac{5\pi}{6} \right)$$

In cartesian form the roots are given by $z_1 = \sqrt{3} + i$ and $z_2 = -\sqrt{3} - i$. Plotting the solutions on the Argand plane, it can be seen that the roots to the equation lie on a circle with a radius of 2. The two solutions are separated around the circle by 180° . Although there are two roots, they are not complex conjugates of one another.



WORKED EXAMPLE 17

- a. If $z^3 = 4 - 4\sqrt{3}i$, solve the equation for z using a polar method.
b. Plot the solutions on an Argand diagram and comment on their location.

THINK

- a. 1. Express z in polar form
($z = r \operatorname{cis}(\theta)$).

WRITE

$$\begin{aligned}\text{a. } |z| &= \sqrt{(4)^2 + (-4\sqrt{3})^2} \\ |z| &= 8 \\ \theta &= \tan^{-1} \left(\frac{-4\sqrt{3}}{4} \right) \\ \theta &= \frac{-\pi}{3} \\ \therefore z &= 4 - 4\sqrt{3}i = 8 \operatorname{cis} \left(\frac{-\pi}{3} \right) \\ z^3 &= 8 \operatorname{cis} \left(-\frac{\pi}{3} \right) \\ z &= \sqrt[3]{8} \operatorname{cis} \left(\frac{-\frac{\pi}{3} + 2k\pi}{3} \right)\end{aligned}$$

2. Use De Moivre's theorem.

3. Simplify the modulus and argument.

$$z = 2 \operatorname{cis} \left(\frac{-\pi}{9} + \frac{2k\pi}{3} \right)$$

4. Let $k = 0$ to calculate the first root. Convert z_1 to standard form.

$$\begin{aligned} \text{Let } k = 0, z_1 &= 2 \operatorname{cis} \left(\frac{-\pi}{9} + \frac{2 \times 0 \times \pi}{3} \right) \\ &= 2 \operatorname{cis} \left(\frac{-\pi}{9} \right) \\ &\approx 1.879 - 0.684i \end{aligned}$$

5. Let $k = 1$ to calculate the second root. Convert z_2 to standard form.

$$\begin{aligned} \text{Let } k = 1, z_2 &= 2 \operatorname{cis} \left(\frac{-\pi}{9} + \frac{2 \times 1 \times \pi}{3} \right) \\ &= 2 \operatorname{cis} \left(\frac{5\pi}{9} \right) \\ &\approx -0.347 + 1.970i \end{aligned}$$

6. Let $k = 2$ to calculate the third root. Convert z_3 to standard form.

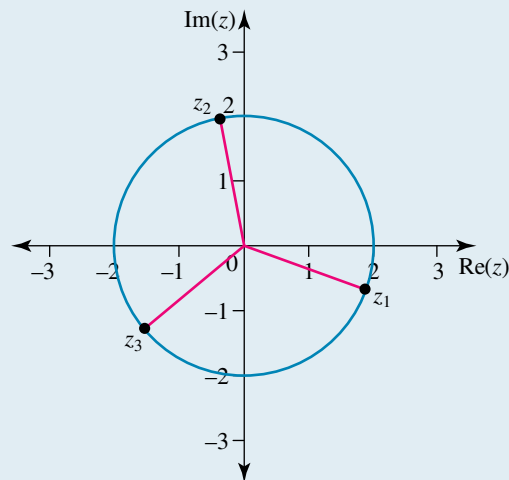
$$\begin{aligned} \text{Let } k = 2, z_3 &= 2 \operatorname{cis} \left(\frac{-\pi}{9} + \frac{2 \times 2 \times \pi}{3} \right) \\ &= 2 \operatorname{cis} \left(\frac{-7\pi}{9} \right) \\ &\approx -1.532 - 1.286i \end{aligned}$$

State all answers.

$$z = 2 \operatorname{cis} \left(-\frac{\pi}{9} \right), 2 \operatorname{cis} \left(\frac{5\pi}{9} \right) \text{ and } 2 \operatorname{cis} \left(-\frac{7\pi}{9} \right)$$

b. Use the Cartesian form to plot the solutions on an Argand diagram.

b.



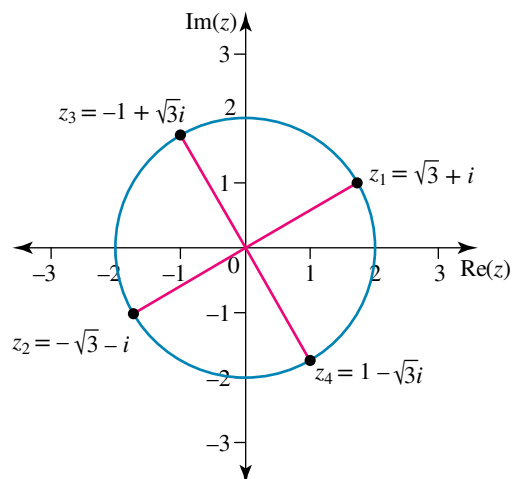
The three roots are equally spaced around a circle of radius 2.

The angle between each solution is $\frac{2\pi}{3}$ (120°).

In the previous worked example there were 3 solutions that were evenly spaced by $\frac{2\pi}{n} = \frac{2\pi}{3}$. Consider the problem below in which the fourth roots of a complex number are determined.

$$z^4 = -8 + 8\sqrt{3}i$$

$$\text{Solutions } \begin{cases} z_1 = \sqrt{3} + i \\ z_2 = -\sqrt{3} - i \\ z_3 = -1 + \sqrt{3}i \\ z_4 = 1 - \sqrt{3}i \end{cases}$$



In the above case there are 4 solutions that were evenly spaced by $\frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$. In general, equations in the form $z^n = w$ where $z, w \in C$ and $n \in N$, have the following properties.

Solutions of $z^n = w$

Solutions to $z^n = w$ where $w \in C$ are in the form $z = \sqrt[n]{r} \operatorname{cis} \left(\frac{\theta + 2k\pi}{n} \right)$, $k = 0, 1, 2, \dots, n-1$.

The solutions are equidistant from the origin and lie on a circle of radius $\sqrt[n]{r}$.

The solutions are evenly spaced around the circle with the arguments differing by $\frac{2\pi}{n}$.

The following example demonstrates an alternative method for solving equations of the form $z^n = w$ where $w \in C$.

WORKED EXAMPLE 18

Solve $z^2 = 3 - 4i$ using a Cartesian approach.

THINK

- Express z in Cartesian form and substitute this into the equation.
- Expand the brackets using $(a + b)^2 = a^2 + 2ab + b^2$ and use $i^2 = -1$ to simplify the equation.
- Equate the coefficients of the real parts and the imaginary parts.
- Rearrange $2ab = -4$ for b .
- Substitute this new equation into $a^2 - b^2 = 3$ and simplify.
- Multiply through by a^2 .

WRITE

$$\begin{aligned} z &= a + bi \\ (a + bi)^2 &= 3 - 4i \\ a^2 + 2abi + b^2i^2 &= 3 - 4i \\ a^2 - b^2 + 2abi &= 3 - 4i \\ a^2 - b^2 = 3 \text{ and } 2ab &= -4 \\ 2ab = -4 \Rightarrow b &= -\frac{2}{a} \\ a^2 - \left(-\frac{2}{a}\right)^2 &= 3 \\ a^2 - \frac{4}{a^2} &= 3 \\ a^4 - 4 &= 3a^2 \\ a^4 - 3a^2 - 4 &= 0 \end{aligned}$$

7. Factorise this quartic by identifying the hidden quadratic equation.
8. Use the null factor theorem to calculate the answers for a . Recall that for any complex number $z = a + bi$ that $a, b \in R$, this means we can ignore any solutions to $(a^2 + 1) = 0$.
9. Determine the corresponding values of b for each values of a .
10. State the solutions to the equation.

$$\begin{aligned}
 (a^2)^2 - 3(a^2) - 4 &= 0 \\
 (a^2 - 4)(a^2 + 1) &= 0 \\
 a^2 - 4 = 0 \text{ or } a^2 + 1 &= 0 \\
 (a - 2)(a + 2) &= 0 \text{ since } a, b \in R \\
 \therefore a = 2 \text{ or } a = -2
 \end{aligned}$$

When $a = 2, b = -1$ and when $a = -2, b = 1$.

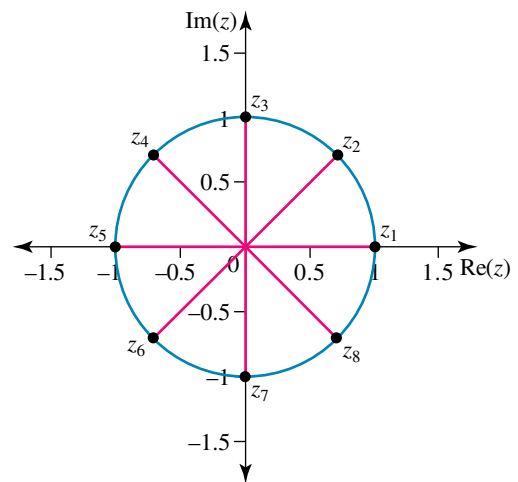
The solutions to the equation $z^2 = 3 - 4i$ are $z = 2 - i$ and $z = -2 + i$.

5.6.2 Determine and examine the n^{th} roots of unity

Since $1 = 1 + 0i = \text{cis}(0)$, De Moivre's theorem is often used to solve equations in the form $z^n = 1$. The solutions to this form of an equation are known as the n^{th} roots of 1 (unity). When interpreting the solutions geometrically, the roots represent the n vertices of a regular polygon of n sides inscribed inside a circle of radius one. This circle is referred to as the unit circle defined by the equation $|z| = 1$.

Using our knowledge of solving equations of the form $z^n = w$ where $w \in C$, we can deduce that the solutions to the equation $z^8 = 1$ where $z \in C$ will lie on a circle of radius 1 and be evenly spaced around the circle by $\frac{\pi}{4}$.

The eight 8th roots of 1 can be interpreted geometrically using an Argand diagram.



The solution set is $z \in \left\{ 1, -1, i, -i, \sqrt{2} \left(\frac{1}{2} - \frac{1}{2}i \right), \sqrt{2} \left(\frac{1}{2} + \frac{1}{2}i \right), \sqrt{2} \left(-\frac{1}{2} + \frac{1}{2}i \right), \sqrt{2} \left(-\frac{1}{2} - \frac{1}{2}i \right) \right\}$. If you were to examine the non-real roots you will find that they occur as conjugate

pairs. In this case, n is even so there are two real roots $\{1, -1\}$. If n is odd, the only real root will be 1 and the other roots will be imaginary occurring as conjugate pairs.

If $z^n = 1$, where $z \in C$ and $n \in N$, then the n^{th} roots of 1 (unity) are given by

$$z = \text{cis} \left(\frac{2k\pi}{n} \right), k = 0, 1, 2, \dots, n-1.$$

WORKED EXAMPLE 19

- a. Determine all solutions to the equation, $z^3 = 1$, over C . Express the solutions in polar form.
- b. Represent these solutions using a polar grid.

THINK

- a. 1. Express 1 in polar form.
2. Use De Moivre's theorem.
3. Simplify the modulus and argument.
4. Let $k = 0$ to calculate the first root.
5. Let $k = 1$ to calculate the second root.
6. Let $k = 2$ to calculate the third root.

- b. Plot the three roots on a polar grid. The circle has a centre $(0, 0)$ and radius, $r = 1$.
The three solutions are spaced by $\frac{2\pi}{3}$.

WRITE

a. $1 = \text{cis}(0)$

$$z^3 = \text{cis}(0)$$

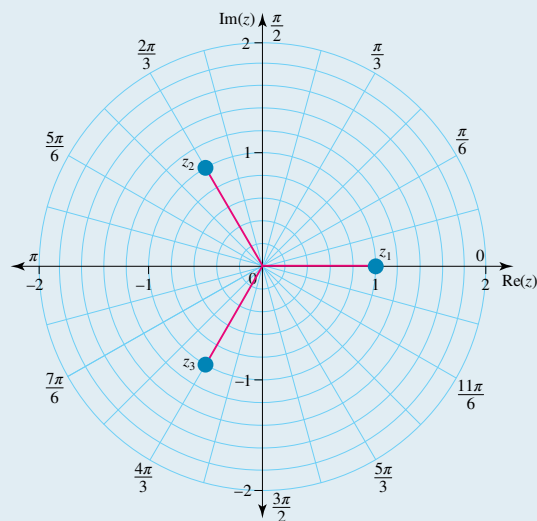
$$z = \sqrt[3]{1} \text{cis}\left(\frac{0 + 2k\pi}{3}\right)$$

$$z = 1 \text{cis}\left(\frac{2k\pi}{3}\right)$$

$$\begin{aligned} \text{Let } k = 0, z_1 &= \text{cis}\left(\frac{2 \times 0 \times \pi}{3}\right) \\ &= \text{cis}(0) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{Let } k = 1, z_2 &= \text{cis}\left(\frac{2 \times 1 \times \pi}{3}\right) \\ &= \text{cis}\left(\frac{2\pi}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{Let } k = 2, z_3 &= \text{cis}\left(\frac{2 \times 2 \times \pi}{3}\right) \\ &= 2 \text{cis}\left(\frac{-2\pi}{3}\right) \end{aligned}$$



Note: If you were to construct lines between the three solutions, an equilateral triangle would be formed.

5.6.3 Solving $z^n = a$ where $a \in R$

Equations of the form $z^n = a$ where $a \in R$, can be solved in the same way in which equations of the form $z^n = w$ where $w \in C$ were solved. That is, if

$$z^n = a, \quad a \in R \text{ and } n \in N,$$

$$\text{then } z = |a|^{\frac{1}{n}} \text{cis}\left(\frac{\theta + 2k\pi}{n}\right), \quad k = 0, 1, 2, \dots, n-1.$$

WORKED EXAMPLE 20

Determine the solutions to $\{z: z^3 = -27\}$, stating the solution set in Cartesian form.

THINK

- Express -27 in polar form.
- Use De Moivre's theorem.
- Let $k = 0$ to calculate the first root.
- Let $k = 1$ to calculate the second root.
- Let $k = 2$ to calculate the third root.
- State the solution set in Cartesian form. Note that the imaginary roots occur as a conjugate pair.

WRITE

$$\begin{aligned} -27 &= |-27| \operatorname{cis}(\pi) \\ &= 27 \operatorname{cis}(\pi) \end{aligned}$$

$$z^3 = 27 \operatorname{cis}(\pi)$$

$$z = \sqrt[3]{27} \operatorname{cis}\left(\frac{\pi + 2k\pi}{3}\right)$$

$$z = 3 \operatorname{cis}\left(\frac{\pi + 2k\pi}{3}\right)$$

Let $k = 0$,

$$\begin{aligned} z_1 &= 3 \operatorname{cis}\left(\frac{\pi + 2 \times 0 \times \pi}{3}\right) \\ &= 3 \operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

Let $k = 1$,

$$\begin{aligned} z_2 &= 3 \operatorname{cis}\left(\frac{\pi + 2 \times 1 \times \pi}{3}\right) \\ &= 3 \operatorname{cis}(\pi) \end{aligned}$$

Let $k = 2$,

$$\begin{aligned} z_1 &= 3 \operatorname{cis}\left(\frac{\pi + 2 \times 2 \times \pi}{3}\right) \\ &= 3 \operatorname{cis}\left(\frac{5\pi}{3}\right) \\ &= 3 \operatorname{cis}\left(-\frac{\pi}{3}\right) \end{aligned}$$

$$z_1 = \frac{3}{2} + \frac{3\sqrt{3}}{2}i, z_2 = -3, z_3 = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

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Units 3 & 4 > Area 3 > Sequence 1 > Concept 5

Roots of complex numbers Summary screen and practice questions

Exercise 5.6 Roots of complex numbers

Technology Free

- WE16** Use De Moivre's theorem to solve $z^2 = 5 \operatorname{cis}\left(\frac{\pi}{6}\right)$ for z .
- Use De Moivre's theorem to solve $z^3 = 27 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ for z .

- ## 5.7 Factorisation of polynomials

A polynomial in z is an expression of the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0$, where $n \in \mathbb{N}$ is the degree (highest power) of $P(z)$ and a_n (with $a_n \neq 0$) are the coefficients.

For example, $P(z) = 3z^4 - 5z^2 + 6$ is a polynomial of degree 4 over R and $P(z) = 2iz^3 + 3z^2 - 8i$ is a polynomial of degree 3 over C .

Factor theorem: If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

The fundamental theorem of algebra

That is, if $P_n(z)$ is a polynomial of degree n over C , then there exists a $z_1 \in C$ such that $P_n(z_1) = 0$. This important result can be used to show that a polynomial of degree n , with $n \in N$, has n solutions.

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Firstly, the fundamental theorem of algebra guarantees that there is a $z_1 \in C$ such that $P_n(z_1) = 0$. The factor theorem states that if $P_n(z_1) = 0$ for some z_1 then $(z - z_1)$ is a factor of $P_n(z)$ so that $P_n(z) = (z - z_1)P_{n-1}(z)$, where $P_{n-1}(z)$ is a polynomial of degree $n - 1$.

Now by applying the fundamental theorem of algebra to $P_{n-1}(z)$ there is a $z_2 \in C$ such that $P_{n-1}(z_2) = 0$ and the factor theorem ensures that $P_{n-1}(z) = (z - z_2)P_{n-2}(z)$.

Hence $P_n(z) = (z - z_1)(z - z_2)P_{n-2}(z)$. By applying this method to each successive polynomial we can state the following:

Implications of the fundamental theorem of algebra

A polynomial of degree n will always have exactly n linear factors. That is;

$$P_n(z) = (z - z_1)(z - z_2)(z - z_3) \dots (z - z_n)P_0(z)$$

where $z_1, z_2, z_3, \dots, z_n \in C$ and $P_0(z)$ is a constant.

Note: Although n solutions are obtained, the fundamental theorem of algebra does not prescribe that they are necessarily distinct.

5.7.2 Solving quadratic Equations

Recall the quadratic equation $az^2 + bz + c = 0$. If the coefficients a , b and c are all real, then the roots depend upon the discriminant, $\Delta = b^2 - 4ac$.

If $\Delta > 0$, then there are two distinct real roots.

If $\Delta = 0$, then there is one real root.

If $\Delta < 0$, then there is one pair of complex conjugate roots.

Relationship between the roots and coefficients

Given a quadratic equation with real coefficients, if the discriminant is negative, then the roots occur in complex conjugate pairs. A relationship can be formed between the roots and the coefficients.

Given a quadratic $az^2 + bz + c = 0$, if $a \neq 0$, then

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0.$$

Let the roots be α and β , so the factors are

$$(z - \alpha)(z - \beta).$$

Expanding gives

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0 \text{ or}$$

$$z^2 - (\text{sum of the roots}) + (\text{product of the roots}) = 0$$

so that

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

This provides a relationship between the roots and coefficients.

Rather than solving a quadratic equation, consider the reverse problem of forming a quadratic equation with real coefficients, given one of the roots.

A polynomial equation $P(z) = 0$ can be solved by first factorising $P(z)$ so that:

$$\text{if } P(z) = (z - z_1)(z - z_2) \dots (z - z_n) = 0$$

then $z = z_1, z_2, \dots, z_n$.

WORKED EXAMPLE 21

Solve $z^2 + 6z + 13 = 0$ over C using the completing the square method.

THINK

1. Use the 'complete the square' method.
2. Express the RHS as the difference of two squares by converting 4 into complex form ($i^2 = -1$).
Write the expression in brackets as a quadratic.
3. Apply the null factor theorem to state the solutions.

WRITE

$$\begin{aligned}0 &= z^2 + 6z + 13 \\0 &= z^2 + 6z + 9 - 9 + 13 \\0 &= (z + 3)^2 + 4 \\0 &= (z + 3)^2 - 4i^2 \\0 &= (z + 3)^2 - (2i)^2 \\0 &= (z + 3 + 2i)(z + 3 - 2i) \\(z + 3 + 2i) &= 0 \text{ or } (z + 3 - 2i) = 0 \\ \text{The roots occur as a pair complex} \\ \text{conjugates: } z &= -3 - 2i, -3 + 2i.\end{aligned}$$

WORKED EXAMPLE 22

Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(-11 + 2i) = 0$.

THINK

1. State the given root.
2. The conjugate is also a root.
3. State the linear factors.
4. Expand the linear factors and state the quadratic equation.

WRITE

$$\begin{aligned}\text{Let } \alpha &= -11 + 2i \\ \text{Let } \beta &= -11 - 2i \\ P(z) &= (z - \alpha)(z - \beta) \\ &= (z + 11 - 2i)(z + 11 + 2i) \\ P(z) &= (z + 11)^2 - (2i)^2 \\ &= z^2 + 22z + 121 - 4i^2 \\ &= z^2 + 22z + 125\end{aligned}$$

A quadratic equation in the form $P(z) = az^2 + bz + c = 0$ can also be solved by using the quadratic formula as an alternative method. Given a, b and $c \in R$ the equation can be

solved using $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.



5.7.3 Solving cubic equations

A cubic polynomial equation of the form $az^3 + bz^2 + cz + d = 0$ with $z \in C$, but with all the coefficients real, will have three linear factors. These may be repeated, but the cubic must have at least one real factor. When solving $az^3 + bz^2 + cz + d = 0$, the roots can be three real roots, not necessarily all distinct, or they can be one real root and one pair of complex conjugate roots.

WORKED EXAMPLE 23

Determine the roots of $z^3 + 6z^2 + 61z + 106 = 0$.

THINK

1. Determine the real root using trial and error.
2. Apply the factor theorem.
3. We can find the quadratic factor using algebraic methods.
4. Compare the two expressions for $P(z)$ to calculate c .
5. Compare the two expressions for $P(z)$ and use equation [1] to calculate b .
6. State $P(z)$ as the product of a linear and quadratic factor.
7. Factorise the quadratic equation by completing the square. Apply the null factor theorem.
8. State the three roots.

WRITE

$$P(-1) = (-1)^3 + 6(-1)^2 + 61(-1) + 106 = 50$$

$$P(-2) = (-2)^3 + 6(-2)^2 + 61(-2) + 106 = 0$$

Therefore $(z + 2)$ is a factor.

$$\begin{aligned} P(z) &= z^3 + 6z^2 + 61z + 106 \\ &= (z + 2)(z^2 + bz + c) \\ &= z^3 + bz^2 + cz + 2z^2 + 2bz + 2c \end{aligned}$$

$$\begin{aligned} 106 &= 2c & [1] \\ c &= 53 \end{aligned}$$

$$61z = 2bz + cz \quad [2]$$

$$\begin{aligned} 61z &= 2bz + 53z \\ b &= \frac{61z - 53z}{2z} \end{aligned}$$

$$b = 4$$

$$P(z) = z^3 + 6z^2 + 61z + 106$$

$$= (z + 2)(z^2 + 4z + 53)$$

$$0 = (z + 2)(z^2 + 4z + 4 - 4 + 53)$$

$$0 = (z + 2)((z + 2)^2 + 49)$$

$$0 = (z + 2)((z + 2)^2 - (7i)^2)$$

$$0 = (z + 2)(z + 2 + 7i)(z + 2 - 7i)$$

The roots are one real and one pair of complex conjugates:

$$z = -2, -2 + 7i, -2 - 7i$$

The conjugate root theorem

The preceding results are true not only for quadratic and cubic equations, but for any n^{th} degree polynomial. In general, provided that all the coefficients of the polynomial are real, if the roots are complex, they must occur in conjugate pairs.

Note that if one of the coefficients is a complex number, then the roots do not occur in conjugate pairs.

WORKED EXAMPLE 24

If $P(z) = z^3 + bz^2 + cz - 87 = 0$ where b and c are real, and $P(-2 + 5i) = 0$, determine the values of b and c , and state all the roots of $P(z) = 0$.

THINK

1. State the given root.

WRITE

$$\text{Let } \alpha = -2 + 5i$$



- The conjugate is also a root.
- Express $P(z)$ as the product of three linear factors.
- Expand the complex conjugate factors to form a quadratic factor.
- Compare the two expressions for $P(z)$. Expand the linear and quadratic factors.
- Compare the two expressions for $P(z)$ and equate the constant terms to find χ .
- Compare the two expressions for $P(z)$ and equate coefficients to find b and c .
- State the values for b and c along with the three zeros of $P(z)$ over $\mathbb{C}$.

$$\text{Let } \beta = -2 - 5i$$

$$\begin{aligned} P(z) &= (z - \alpha)(z - \beta)(z - \chi) \\ &= (z + 2 + 5i)(z + 2 - 5i)(z - \chi) \end{aligned}$$

$$\begin{aligned} P(z) &= ((z + 2)^2 - (5i)^2)(z - \chi) \\ &= (z^2 + 4z + 4 - 25i^2)(z - \chi) \\ &= (z^2 + 4z + 29)(z - \chi) \end{aligned}$$

$$\begin{aligned} z^3 + bz^2 + cz - 87 &= (z - \chi)(z^2 + 4z + 29) \\ &= z^3 + 4z^2 + 29z - \chi z^2 - 4\chi z - 29\chi \\ &= z^3 + (4 - \chi)z^2 + (29 - 4\chi)z - 29\chi \end{aligned}$$

$$-87 = -\chi \times 29 \quad [1]$$

$$\chi = 3$$

$$bz^2 = (4 - \chi)z^2 \quad \text{and} \quad cz = (29 - 4\chi)z$$

$$\begin{aligned} bz^2 &= (4 - 3)z^2 & \text{and} & \quad cz = (29 - 4 \times 3)z \\ b &= 1 & \text{and} & \quad c = 17 \end{aligned}$$

$$P(z) = (z - 3)(z + 2 + 5i)(z + 2 - 5i) = 0$$

$$\therefore b = 1 \text{ and } c = 17$$

The roots are one real root and one pair of complex conjugates: $z = 3, -2 - 5i, -2 + 5i$.

5.7.4 Solving quartic equations

A quartic of the form $P(z) = az^4 + bz^3 + cz^2 + dz + e$ with all real coefficients can have four linear factors. The roots can be either all real roots; two real roots and one pair of complex conjugate roots; or two pairs of complex conjugate roots.

WORKED EXAMPLE 25

Solve for z if $z^4 + 10z^2 - 11 = 0$.

THINK

- Use appropriate substitution to reduce the quartic equation to a quadratic equation.
- Factorise the new expression.
- Substitute z^2 for a .
- Express each factor as the difference of two squares by recalling $i^2 = -1$.
- Express $P(z)$ as the product of four linear factors.
- Apply the null factor theorem and state the roots.

WRITE

$$\text{Let } a = z^2, \text{ then } a^2 = z^4$$

$$z^4 + 10z^2 - 11 = 0$$

$$a^2 + 10a - 11 = 0$$

$$(a + 11)(a - 1) = 0$$

$$(z^2 + 11)(z^2 - 1) = 0$$

$$\left(z^2 - (\sqrt{11}i)^2\right)(z^2 - (1)^2) = 0$$

$$(z + \sqrt{11}i)(z - \sqrt{11}i)(z + 1)(z - 1) = 0$$

$$z = 1, -1, \sqrt{11}i, -\sqrt{11}i$$

In the previous worked examples, algebraic methods were used to determine unknown coefficients for quadratic factors. Polynomial long division is an alternative method and is demonstrated in the following worked example.

WORKED EXAMPLE 26

Given that $P(4) = 0$ where $P(z) = z^4 - 2z^3 - 6z^2 + 32z - 160$, use polynomial long division to calculate all the roots for $P(z)$.

THINK

1. State the given real root.
2. Since one root is real determine another real root by testing $p(-4)$.
3. Express $P(z)$ as the product of two quadratic factors.
4. Complete polynomial long division to calculate the unknown quadratic factor.
5. Express $P(z)$ as the product of two quadratic factors by using the result from the long division operation.
6. Factorise the quadratic equation by completing the square.
7. Express $P(z) = 0$ as the product of four linear factors.
8. Apply the null factor law to state the four solutions for $P(z) = 0$.

WRITE

Let $z_1 = 4$

$$P(-4) = (-4)^4 - 2(-4)^3 - 6(-4)^2 + 32(-4) - 160 = 0$$

Thus $z_2 = -4$ is another root.

$$\begin{aligned} P(z) &= (z - z_1)(z - z_2)(z - z_3)(z - z_4) \\ &= (z - 4)(z + 4)(z - z_3)(z - z_4) \\ &= (z^2 - 16)(az^2 + bz + c) \end{aligned}$$

$$\begin{array}{r} z^2 - 16 \overline{) z^4 - 2z^3 - 6z^2 + 32z - 160} \\ \underline{-(z^4 \quad \downarrow \quad -16z^2 \quad \downarrow \quad \downarrow)} \\ -2z^3 + 10z^2 + 32z - 160 \\ \underline{-(-2z^3 \quad \downarrow \quad +32z \quad \downarrow)} \\ 10z^2 - 160 \\ \underline{-(10z^2 \quad -160)} \\ 0 \end{array}$$

$$P(z) = (z^2 - 16)(z^2 - 2z + 10)$$

$$\begin{aligned} P(z) &= (z^2 - 16)(z^2 - 2z + 1 - 1 + 10) \\ &= (z^2 - 16)((z - 1)^2 + 9) \\ &= (z^2 - 16)((z - 1)^2 - (3i)^2) \\ 0 &= (z + 4)(z - 4)(z - 1 + 3i)(z - 1 - 3i) \end{aligned}$$

The solutions occur as a pair of real roots and a pair of complex conjugate roots.

$$z = -4, 4, 1 - 3i, 1 + 3i.$$

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Units 3 & 4 > Area 3 > Sequence 1 > Concept 6

Factorisation of polynomials Summary screen and practice questions

Exercise 5.7 Factorisation of polynomials

Technology free

- WE21** Solve $z^2 - 14z + 74 = 0$ over C using the completing the square method.
- Solve $4z^2 + 4z + 82 = 0$ over C using the completing the square method.
- Solve $z(z + 4) = -29$ for z using the quadratic formula.
- WE22** Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(-1 + 13i) = 0$.
- Determine the equation of the quadratic $P(z)$ with real coefficients given that $P(6 - 5i) = 0$.
- WE23** Determine the roots of $z^3 + 9z^2 + 24z - 34 = 0$.
- Solve the equation $z^3 + 13z^2 + 97z + 85 = 0$ over C .
- Determine the roots of $z^3 + 4z^2 - 2z - 20 = 0$.
- WE24** If $P(z) = z^3 + bz^2 + cz + 29 = 0$ where b and c are real, and $P(5 + 2i) = 0$, determine the values of b and c , and state all the roots of $P(z) = 0$.
- If $P(z) = z^3 + bz^2 + cz - 195 = 0$ where b and c are real, and $P(7 + 4i) = 0$, determine the values of b and c , and state all the roots of $P(z) = 0$.
- If $P(z) = z^3 + bz^2 + cz + 51 = 0$ where b and c are real, and $P(-1 - 4i) = 0$, determine the values of b and c , and state all the roots of $P(z) = 0$.
- Develop an equation with integer coefficients that have the following roots.
a. $z_1 = -9, z_2 = 7 - 8i, z_3 = 7 + 8i$ **b.** $z_1 = 11, z_2 = \alpha, z_3 = 4 - 7i, z_4 = \beta$
- WE25** Solve for z if $z^4 + 5z^2 - 36 = 0$.
- Solve for z if $z^4 - 24z^2 - 1225 = 0$.
- Solve for z if $z^4 + 62z^2 - 128 = 0$.
- WE26** Given that $P(3) = 0$ where $P(z) = z^4 + 10z^3 + 25z^2 - 90z - 306$, use polynomial long division to calculate all the roots for $P(z)$.
- If $z = 4$ is a zero of $P(z) = z^3 + 8z^2 + 69z - 468$, use polynomial long division to calculate all the roots for $P(z)$.
- Given $P(z) = z^4 - 2z^3 - 6z^2 + 32z - 160 = (z - 1 - 3i)(z - z_2)(z - z_3)(z - z_4)$, use polynomial long division to calculate all the roots for $P(z)$.

Technology active

- Given $P(z) = z^4 + bz^3 + 18z^2 + 32z + 32$ and $P(4i) = 0$, calculate the value of the real constant b and determine all the roots.
- A quartic is given by the equation, $P(z) = (az^2 + bz + c)(z^2 - 10z + 41) + d$, where $d \in R$. If $P(z) = z^4 - 8z^3 + 23z^2 + 62z + 81$ in expanded form, determine the real value d .

5.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

- MC** The values for a and b that satisfy the equation $(12 - 5i)(a + bi) = 56 + 33i$ are:
A. $a = -3, b = 4$ **B.** $a = 3, b = -4$ **C.** $a = 3, b = 5$ **D.** $a = 3, b = 4$
- MC** The Cartesian equation for $\{z : |z - 5i| = |z + 5|\}$ is best described by:
A. $y = -x$ **B.** $y = -4x$ **C.** $y = 4x$ **D.** $y = x$
- MC** A cubic polynomial with real coefficients is given by $P(z) = (z - 4)(z - 4 + 3i)(z - z_3)$. The expression for z_3 is defined as:
A. 4 **B.** -4 **C.** $4 + 3i$ **D.** $-4 + 3i$
- Determine z^{-1} given $z = 4 - 8i$.
- Use De Moivre's theorem to simplify $\left(4 \operatorname{cis} \left(-\frac{\pi}{3}\right)\right)^5 \div \left(2 \operatorname{cis} \left(\frac{\pi}{4}\right)\right)^4$.

6. Determine the Cartesian equation and sketch the graph of $\{z : |z + 1 - 2i| = 4\}$.
7. Express $2 \operatorname{cis} \left(\frac{-\pi}{4} \right) \times (5 - 5i)$ in the form $z = a + bi$.
8. Use De Moivre's theorem to solve $z^3 = 125 \operatorname{cis} \left(\frac{-\pi}{4} \right)$ for z .
9. If $z = a + bi$, determine the solutions to the equation $\frac{z-1}{z+1} = z + 3$.
10.
 - a. If $z^4 = \sqrt{3} + i$, determine the complex number z using a polar method.
 - b. Plot the solutions on an Argand diagram and comment on their location.
11.
 - a. Determine all solutions to the equation, $z^6 = 1$, over C . Express the solutions in polar form.
 - b. Represent these solutions using a polar grid.
12. Solve the equation $z^4 + 55z^2 - 576 = 0$ over C .

Complex familiar

13. Prove the following identities using polar arithmetic.
 - a. $z\bar{z} = |z|^2$
 - b. $|z^n| = |z|^n$
14. Use De Moivre's theorem to determine z^9 , given $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Verify the solution by evaluating $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^9$ using a graphics calculator.
15. If $P(z) = z^3 + bz^2 + cz + 29 = 0$ where b and c are real, and $P(5 + 2i) = 0$, determine the values of b and c , and state all the roots of $P(z) = 0$.
16. Given $P(z) = z^4 + 8z^3 + 30z^2 + 56z + 65 = (z + 3 + 2i)(z - z_2)(z - z_3)(z - z_4)$, use polynomial division to calculate all the roots for $P(z)$.

Complex unfamiliar

17. Determine a polynomial of degree 5 with integer coefficients, that has the following roots. A graphics calculator can be used to expand factors.

$$z_1 = 6i, z_2 = 2 - 4i, z_3 = 3$$

18. Prove the triangle inequality given by $|z + v| \leq |z| + |v|$.
19. If $z = \operatorname{cis}(\theta)$, prove that $\frac{1+z}{1-z} = i \cot\left(\frac{\theta}{2}\right)$.
20. Show that the complex equation $|z - c| = 3|z - \bar{c}|$ represents a circle and determine its centre and radius.

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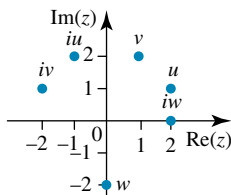
Units 3 & 4 Sit exam

Answers

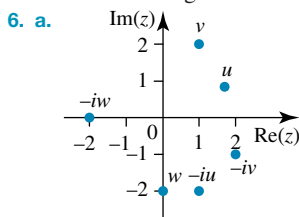
5 Complex numbers

Exercise 5.2 Complex numbers in Cartesian form

- a. $5i$
c. $\frac{2}{5}i$
- a. $-1 - i$
c. $3 - 2i$
- a. 8
c. -1
- a. $12i$
d. $\sqrt{23}i$
b. $2 - 3i$
d. $-1 + i$
b. 2
d. -2



- b. As shown in Q4a
c. Multiplying a complex number z by i rotates z 90° anticlockwise about the origin.
- a.
b. As shown in Q5a
c. Multiplying a complex number z by -1 rotates z by 180° about the origin.



- b. As shown in Q6a
c. Multiplying a complex number z by $-i$ rotates z 90° clockwise about the origin.
- a. $15 - 7i$
b. $-12 + 24i$
c. $8 + 6i$
- a. $11 - 7i$
b. $-6 + 12i$
c. $24 - 11i$
- a. $-80 + 269i$
b. $\bar{f} = -2 - \sqrt{7}i$
c. $\bar{g} = 9 + 2\sqrt{7}i$
d. $h = -1 + i$
- a. $\bar{e} = 14 - 9i$
b. $\bar{f} = -2 - \sqrt{7}i$
c. $\bar{g} = 9 + 2\sqrt{7}i$
d. $h = -1 + i$
- Sample responses can be found in the worked solutions in the online resources.
- a. $\frac{1}{2} - \frac{1}{2}i$
b. $\frac{1}{20} + \frac{3}{20}i$
c. $-\frac{3}{13} - \frac{2}{13}i$
d. $-\frac{1}{8} + \frac{1}{8}i$
- $\frac{1}{29} - \frac{17}{29}i$
- a. A
b. B
- Sample responses can be found in the worked solutions in the online resources.
- a. $x = 7, y = -8$
b. $x = -4, y = 3$

Exercise 5.3 Complex numbers in polar form

- a. $\sqrt{10}$
b. 13
c. 3
- a. $\frac{\pi}{4}$
b. $\frac{\pi}{3}$
c. $\frac{5\pi}{6}$
- a. $z = 4 \operatorname{cis} \left(\frac{\pi}{6} \right)$
b. $\frac{15}{2} (-1 + \sqrt{3}i)$
c. $z = \frac{2}{3} \operatorname{cis} \left(\frac{-5\pi}{6} \right)$
d. $2\sqrt{2}(1 - i)$
- $\frac{3 - 4\sqrt{3}}{2} + \frac{3\sqrt{3} + 4}{2}i$
- a. $\operatorname{cis} \left(\frac{-\pi}{4} \right)$
b. $\operatorname{cis} \left(\frac{10\pi}{11} \right)$
c. $\operatorname{cis} \left(-\frac{3\pi}{5} \right)$
- a. $12 \operatorname{cis} \left(\frac{11\pi}{12} \right)$
b. $3 \operatorname{cis} \left(\frac{7\pi}{12} \right)$
- a. $12 \operatorname{cis} \left(\frac{3\pi}{4} \right)$
b. $3 \operatorname{cis} \left(\frac{\pi}{2} \right)$
c. $4\sqrt{5} \operatorname{cis} \left(\frac{-\pi}{6} \right)$
d. $3 \operatorname{cis} \left(\frac{5\pi}{8} \right)$
e. $7 \operatorname{cis} \left(\frac{9\pi}{20} \right)$
f. $2 \operatorname{cis} \left(\frac{7\pi}{12} \right)$
- a. $8 \operatorname{cis} \left(\frac{\pi}{2} \right)$
b. $8.7671 \operatorname{cis} (-2.8481)$
c. $2 \operatorname{cis} \left(-\frac{\pi}{6} \right)$

9–18. Sample responses can be found in the worked solutions in the online resources.

- a. i. $\frac{1}{4} (\sqrt{3} + 1) + \frac{1}{4} (\sqrt{3} - 1)i$
ii. $\frac{\sqrt{2}}{2} \operatorname{cis} \left(\frac{\pi}{12} \right)$
iii. $\sin \left(\frac{\pi}{12} \right) = \frac{1}{4} (\sqrt{6} - \sqrt{2})$
iv. Sample responses can be found in the worked solutions in the online resources.
- i. $(\sqrt{6} - \sqrt{2}) + (\sqrt{6} + \sqrt{2})i$
ii. $4 \operatorname{cis} \left(\frac{5\pi}{12} \right)$
iii. $\sin \left(\frac{5\pi}{12} \right) = \frac{1}{4} (\sqrt{6} + \sqrt{2})$
iv. Sample responses can be found in the worked solutions in the online resources.

Exercise 5.4 De Moivre's theorem

- Sample responses can be found in the worked solutions in the online resources.
- a. $32 \operatorname{cis} \left(-\frac{3\pi}{4} \right)$
b. $729 \operatorname{cis} \left(-\frac{\pi}{2} \right)$
c. $\frac{1}{81} \operatorname{cis} \left(\frac{-2\pi}{3} \right)$
d. 1
- a. $23\,328 \operatorname{cis} \left(\frac{\pi}{2} \right)$
b. $128 \operatorname{cis} \left(\frac{3\pi}{4} \right)$
- a. $81 \operatorname{cis} \left(\frac{\pi}{3} \right)$
b. $1\,048\,576 \operatorname{cis} \left(-\frac{14\pi}{15} \right)$
- a. -64
b. $\frac{1}{512}$
- a = 64, b = -64

7. $a = 1025, b = 0$

8. $n = \frac{3(2k+1)}{5}$

9. a. i. 4096

ii. 0

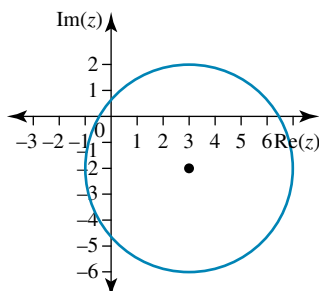
b. i. $-46\,656$

ii. π

10, 11. Sample responses can be found in the worked solutions in the online resources.

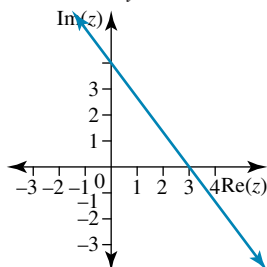
Exercise 5.5 The complex plane (the Argand plane)

1. Circle $(x-3)^2 + (y+2)^2 = 16$, with centre $(3, -2)$ and radius 4



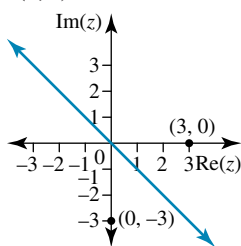
2. $a = 3, b = -3, r = 3$

3. The line $4x + 3y = 12$

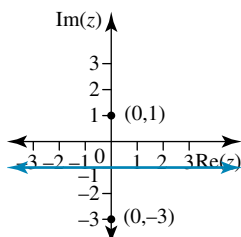


4. $a = -4, b = 2$

5. Line $y = -x$; the set of points equidistant from $(0, -3)$ and $(3, 0)$



6. Line $y = -1$; the set of points equidistant from $(0, 1)$ and $(0, -3)$

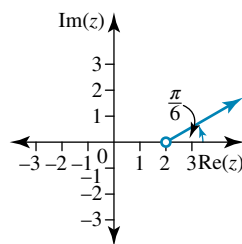


7. a. $(0, 3), \left(\frac{72}{25}, \frac{21}{25}\right)$

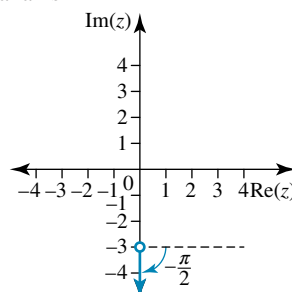
b. $\pm 8\sqrt{5}$

8. a. $(2, 5), \left(-\frac{7}{5}, -\frac{26}{5}\right)$ b. $\pm 5\sqrt{13}$

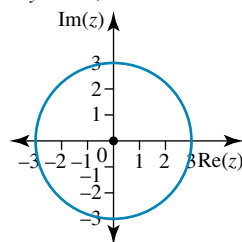
9. A ray from $(2, 0)$ making an angle of $\frac{\pi}{6}$ or 30° with the real axis



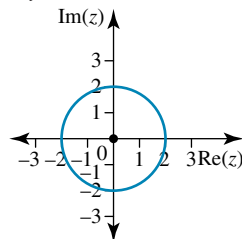
10. A ray from $(0, -3)$ making an angle of $-\frac{\pi}{2}$ or 90° with the real axis



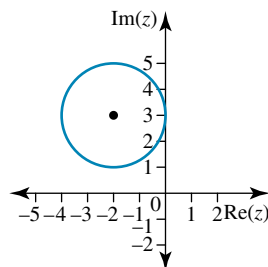
11. a. $x^2 + y^2 = 9$; circle with centre $(0, 0)$, radius 3



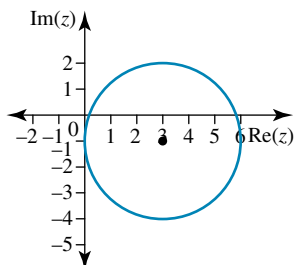
- b. $x^2 + y^2 = 4$; circle with centre $(0, 0)$, radius 2



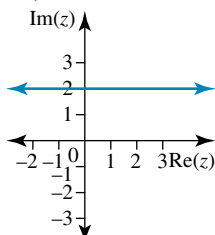
- c. $(x+2)^2 + (y-3)^2 = 4$; circle with centre $(-2, 3)$, radius 2



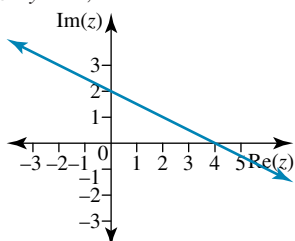
- d. $(x - 3)^2 + (y + 1)^2 = 9$; circle with centre $(3, -1)$, radius 3



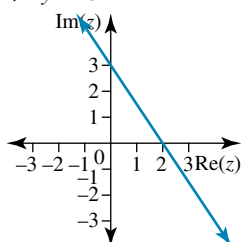
12. a. $y = 2$; line



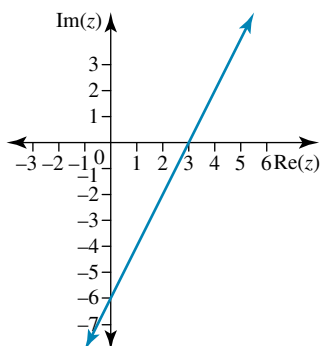
- b. $x + 2y = 4$; line



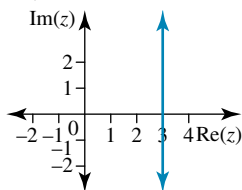
- c. $3x + 2y = 6$



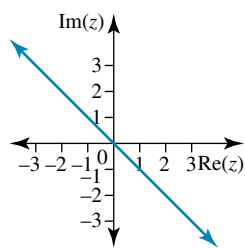
- d. $y = 2x - 6$



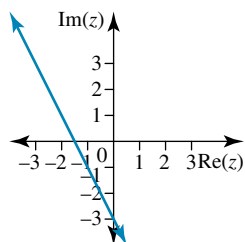
13. a. $x = 3$; line



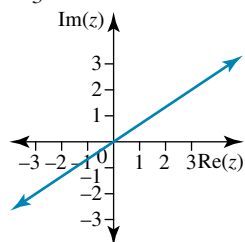
- b. $y = -x$



- c. $y = -2x - 3$



- d. $y = \frac{2x}{3}$



14. a. $y = -\frac{2x}{3} + 2$

- b. $\left(x - \frac{3}{2}\right)^2 + (y - 1)^2 = \frac{13}{4}$; circle with centre $\left(\frac{3}{2}, 1\right)$, radius $\frac{\sqrt{13}}{2}$

15. a. $(x + 1)^2 + (y + 4)^2 = 8$; circle with centre $(-1, -4)$, radius $2\sqrt{2}$

- b. $(x - 1)^2 + (y + 8)^2 = 20$; circle with centre $(1, -8)$, radius $2\sqrt{5}$

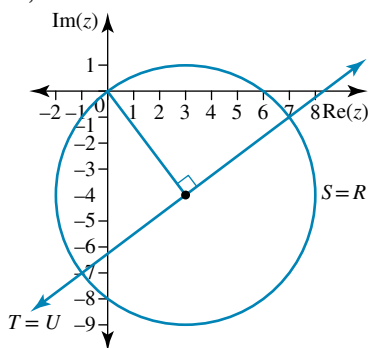
- c. $a = -2, b = 4, r = 2\sqrt{5}$

- d. $a = 1, b = 4, r = 2\sqrt{2}$

16. a. $3x - 4y = 25$; line

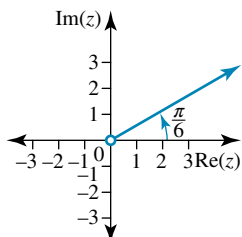
- b. $(x - 3)^2 + (y + 4)^2 = 25$; circle with centre $(3, -4)$, radius 5

- c. $7 - i, -1 - 7i$

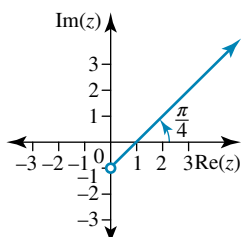


17. a. $\left(\frac{9}{5}, \frac{12}{5}\right)$ b. 2
c. $(-2, 0), \left(-\frac{14}{25}, \frac{48}{25}\right)$ d. ± 30

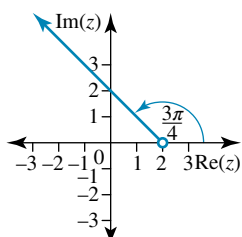
18. a. $y = \frac{x}{\sqrt{3}}$ for $x > 0$; a ray from $(0, 0)$ making an angle of 30° with the real axis



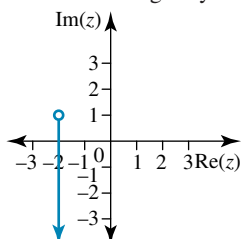
- b. $y = x - 1$ for $x > 0$; a ray from $(0, -1)$ making an angle of 45° with the real axis



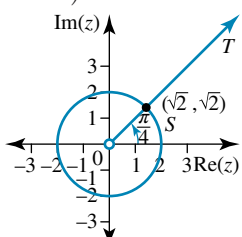
- c. $y = 2 - x$ for $x < 2$; a ray from $(2, 0)$ making an angle of 135° with the real axis



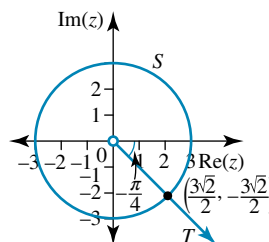
- d. $x = -2$ for $y < 1$; a ray from $(-2, 1)$ going down parallel to the imaginary axis.



19. a. $(\sqrt{2}, \sqrt{2})$



- b. $\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$



- c. i. S is the circle with centre $(-3, -1)$ and radius 5: $(x + 3)^2 + (y + 1)^2 = 25$.
ii. T is $y = x + 3$ for $x < -3$, the ray from $(-3, 0)$ making an angle of -135° with the real axis.
iii. $u = -7 - 4i$

20. a. $y = \frac{a}{b}x + b$; line

- b. $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}$; circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$, radius $\frac{\sqrt{a^2 + b^2}}{2}$

- c. $(x + 2)^2 + y^2 = \frac{10}{3}$; circle with centre $(-2, 0)$, radius $\frac{\sqrt{30}}{3}$

- d. i. $\left(x + \frac{b}{a}\right)^2 + y^2 = \frac{b^2 - ac}{a^2}$; circle with centre $\left(-\frac{b}{a}, 0\right)$, radius $\frac{\sqrt{b^2 - ac}}{a}$

- ii. If $a = 0$ and $b \neq 0$, the equation is $x = -\frac{c}{2b}$, a line.

- e. $(x + 3)^2 + (y - 2)^2 = 9$; circle with centre $(-3, 2)$, radius 3

- f. $\left(x + \frac{\alpha}{a}\right)^2 + \left(y + \frac{\beta}{a}\right)^2 = \frac{b\bar{b} - ac}{a^2}$; circle with centre $\left(-\frac{\alpha}{a}, -\frac{\beta}{a}\right)$, radius $\frac{\sqrt{b\bar{b} - ac}}{a}$

21. a. $y = -\frac{a}{b}x + a$, $ab \neq 0$; line

- b. $\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4}$; circle with centre $\left(\frac{b}{2}, \frac{a}{2}\right)$, radius $\frac{\sqrt{a^2 + b^2}}{2}$

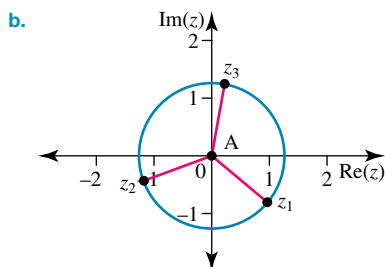
Exercise 5.6 Roots of complex numbers

1. $z = \sqrt{5} \operatorname{cis} \left(\frac{\pi}{12}\right), \sqrt{5} \operatorname{cis} \left(-\frac{11\pi}{12}\right)$

2. $z = 3 \operatorname{cis} \left(\frac{2\pi}{9}\right), 3 \operatorname{cis} \left(\frac{8\pi}{9}\right), 3 \operatorname{cis} \left(-\frac{4\pi}{9}\right)$

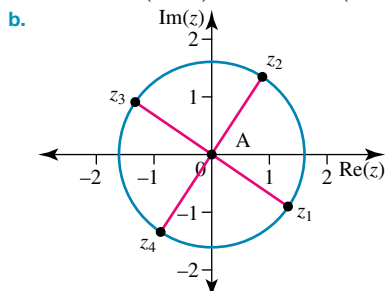
3. $z = 2 \operatorname{cis} \left(-\frac{3\pi}{16}\right), 2 \operatorname{cis} \left(\frac{5\pi}{16}\right), 2 \operatorname{cis} \left(\frac{13\pi}{16}\right), 2 \operatorname{cis} \left(-\frac{11\pi}{16}\right)$

4. a. $z = \sqrt[3]{2} \operatorname{cis} \left(-\frac{2\pi}{9}\right), \sqrt[3]{2} \operatorname{cis} \left(\frac{4\pi}{9}\right), \sqrt[3]{2} \operatorname{cis} \left(-\frac{8\pi}{9}\right)$



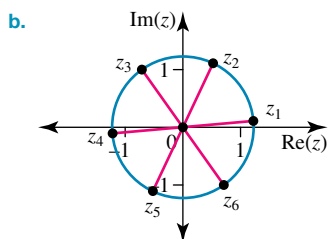
The three roots are equally spaced around a circle of radius $\sqrt[3]{2}$. The angle between each solution is $\frac{2\pi}{3}$.

5. a. $z = \sqrt[4]{5\sqrt{2}} \operatorname{cis}\left(-\frac{3\pi}{16}\right), \sqrt[4]{5\sqrt{2}} \operatorname{cis}\left(\frac{5\pi}{16}\right), \sqrt[4]{5\sqrt{2}} \operatorname{cis}\left(\frac{13\pi}{16}\right), \sqrt[4]{5\sqrt{2}} \operatorname{cis}\left(-\frac{11\pi}{16}\right)$



The three roots are equally spaced around a circle of radius $\sqrt[4]{5\sqrt{2}} \approx 1.631$. The angle between each solution is $\frac{\pi}{2}$.

6. a. $z = \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(\frac{\pi}{36}\right), \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(\frac{13\pi}{36}\right), \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(\frac{25\pi}{36}\right), \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(-\frac{35\pi}{36}\right), \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(-\frac{23\pi}{36}\right), \sqrt[6]{2\sqrt{3}} \operatorname{cis}\left(-\frac{11\pi}{36}\right)$



The three roots are equally spaced around a circle of radius $\sqrt[6]{2\sqrt{3}} \approx 1.230$. The angle between each solution is $\frac{\pi}{3}$.

7. a. $z = \operatorname{cis}\left(\frac{\pi}{6}\right), \operatorname{cis}\left(\frac{5\pi}{6}\right), \operatorname{cis}\left(-\frac{\pi}{2}\right)$

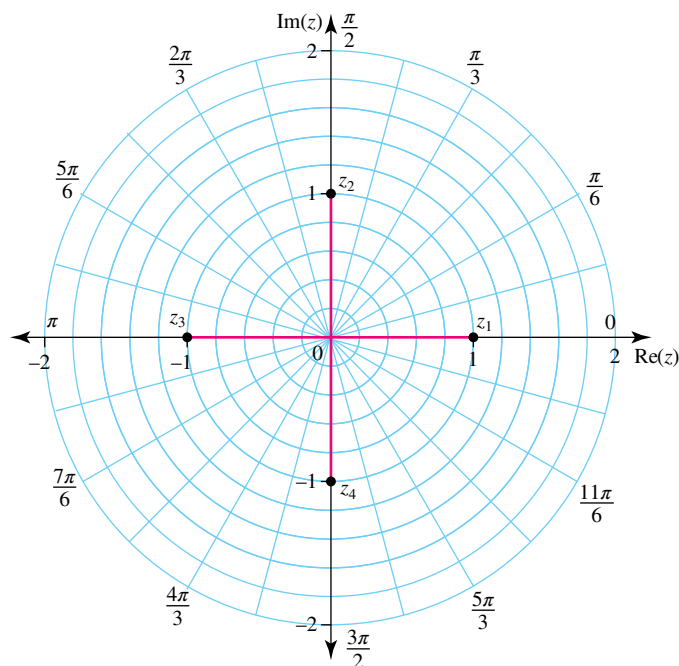
b. $z = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{8}\right), 2\sqrt{2} \operatorname{cis}\left(\frac{5\pi}{8}\right), 2\sqrt{2} \operatorname{cis}\left(-\frac{7\pi}{8}\right), 2\sqrt{2} \operatorname{cis}\left(-\frac{3\pi}{8}\right)$

8. $\sqrt{3} + i, -\sqrt{3} - i$

9. Sample responses can be found in the worked solutions in the online resources.

10. a. $z = 1, \operatorname{cis}\left(\frac{\pi}{2}\right), \operatorname{cis}(\pi), \operatorname{cis}\left(\frac{3\pi}{2}\right)$

b.

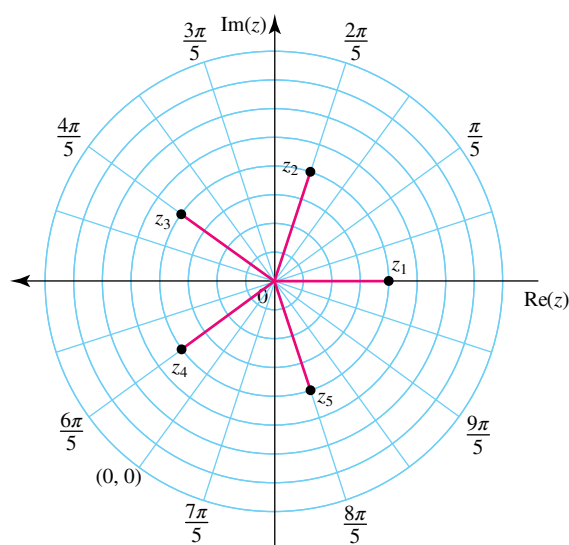


11. Sample responses can be found in the worked solutions in the online resources.

12. a. $z =$

$1, \operatorname{cis}\left(\frac{\pi}{5}\right), \operatorname{cis}\left(\frac{2\pi}{5}\right), \operatorname{cis}\left(\frac{-4\pi}{5}\right), \operatorname{cis}\left(\frac{-2\pi}{5}\right)$

b.



13. The solutions would be equally spaced by $\frac{2\pi}{5}$ radians and lie on a circle of radius $10^{\frac{1}{5}}$. The modulus of each solution would change from $|z| = 1$ to $|z| = 10^{\frac{1}{5}}$.

14. a. $z = -1 - \sqrt{3}i, z = -1 + \sqrt{3}i, z = 2$
 b. $z = 4i, z = -4i, z = -4, z = 4$
 15. See Figure in Below *

Exercise 5.7 Factorisation of polynomials

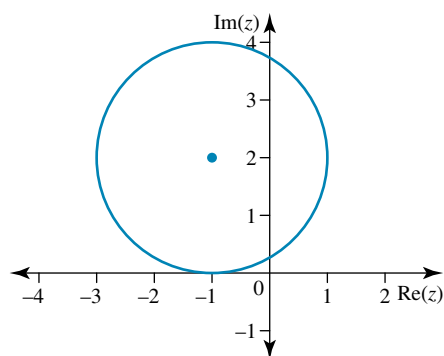
1. $z = 7 + 5i, 7 - 5i$
2. $z = -\frac{1}{2} + \frac{9}{2}i, -\frac{1}{2} - \frac{9}{2}i$
3. $z = -2 + 5i, -2 - 5i$
4. $P(z) = z^2 + 2z + 170$
5. $P(z) = z^2 - 12z + 61$
6. $z = 1, -5 - 3i, -5 + 3i$
7. $z = -1, -6 - 7i, -6 + 7i$
8. $z = -1, -3 - i, -3 + i$
9. $b = -9, c = 19z = -1, 5 + 2i, 5 - 2i$
10. $b = -17, c = 107z = 3, 7 + 4i, 7 - 4i$
11. $b = 5, c = 23z = -3, -1 + 4i, -1 - 4i$
12. a. $P(z) = z^3 - 5z^2 - 13z + 1017$
 b. $P(z) = z^4 - 8z^3 - 56z^2 + 968z + 7865$
13. $z = 2, -2, 3i, -3i$
14. $z = 7, -7, 5i, -5i$
15. $z = 8, -8, \sqrt{2}i, -\sqrt{2}i$
16. $z = -3, 3, -5 - 3i, -5 + 3i$
17. $z = 4, -6 - 9i, -6 + 9i$
18. $z = 4, -4, 1 + 3i, 1 - 3i$
19. $b = 2, z = \pm 4i, z = -1 \pm i$
20. $d = -1$

4. $z^{-1} = \frac{1}{20} + \frac{i}{10}$

5. $64 \operatorname{cis} \left(\frac{-2\pi}{3} \right)$

6. c, $(-1, 2), r = 2$.

$$(x + 1)^2 + (y - 2)^2 = 4$$



7. $-10\sqrt{2}i$

8. $z = 5 \operatorname{cis} \left(\frac{-\pi}{12} \right), 5 \operatorname{cis} \left(\frac{7\pi}{12} \right), 5 \operatorname{cis} \left(\frac{-9\pi}{12} \right)$

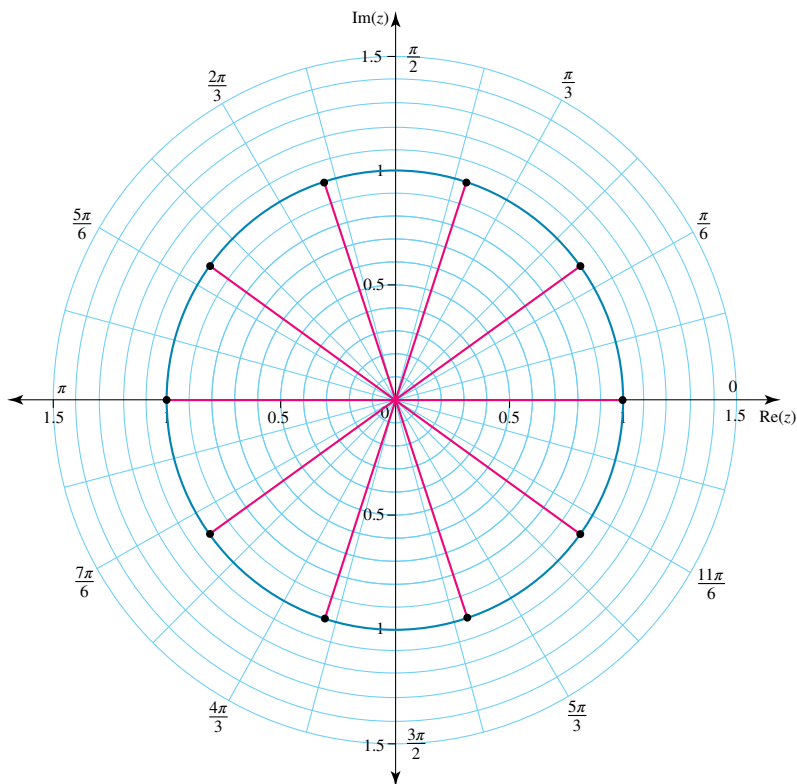
9. $z = -\frac{3}{2} - \frac{\sqrt{7}}{2}i, -\frac{3}{2} + \frac{\sqrt{7}}{2}i$

10. a. $z = \sqrt[3]{2} \operatorname{cis} \left(\frac{\pi}{18} \right), \sqrt[3]{2} \operatorname{cis} \left(\frac{13\pi}{18} \right), \sqrt[3]{2} \operatorname{cis} \left(-\frac{11\pi}{18} \right)$

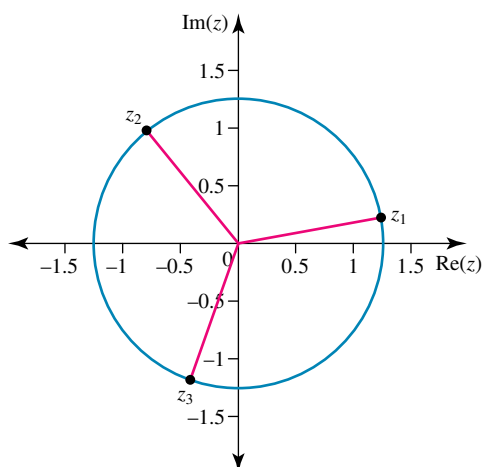
5.8 Review: exam practice

1. D
2. A
3. C

*15.



b.



The three roots are evenly spaced by $\frac{2\pi}{3}$ radians. The modulus of each root is $2^{\frac{1}{3}}$

11. a. $z = \text{cis}(0), \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right),$
 $\text{cis}(\pi), \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\left(-\frac{\pi}{3}\right)$

b. *See figure below

12. $z = 8i, -8i, 3, -3$

13. Sample responses can be found in the worked solutions in the online resources.

14. $z^9 = 1$

15. $z = 5 + 2i, 5 - 2i, -1$

16. $z = -3 + 2i, -3 - 2i, -1 + 2i, -1 - 2i$

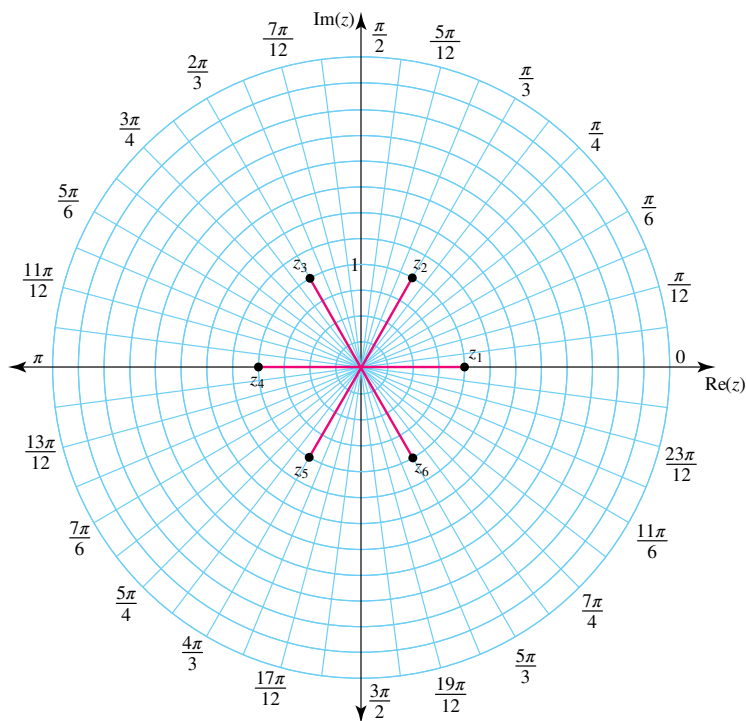
17. $P(z) = z^5 - 7z^4 + 68z^3 - 312z^2 + 1152z - 2160$

18, 19. Sample responses can be found in the worked solutions in the online resources.

20. Circle; $\frac{9b^2}{16} = (x - a)^2 + \left(y + \frac{5b}{4}\right)^2$.

Centre = $\left(a, -\frac{5b}{4}\right), r = \frac{3b}{4}$

*11. b.

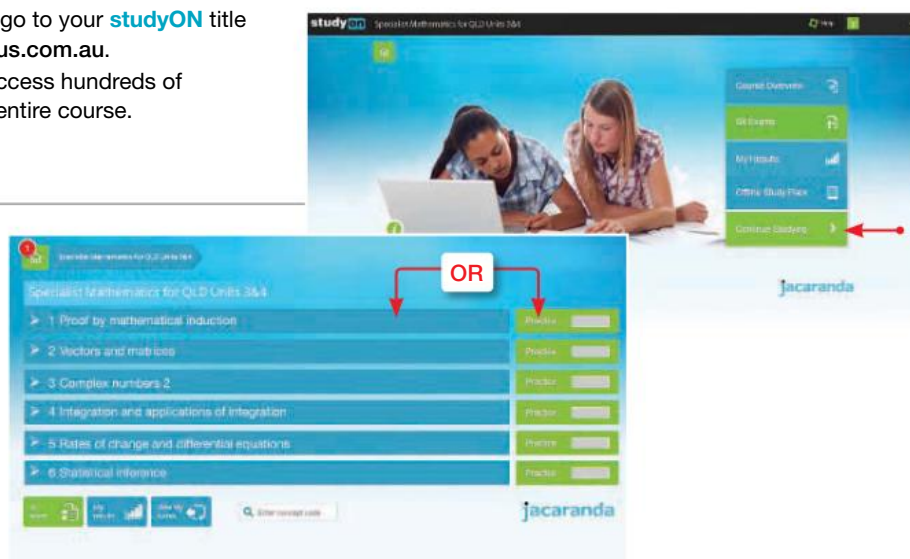


REVISION UNIT 3 Mathematical induction, and further vectors, matrices and complex numbers

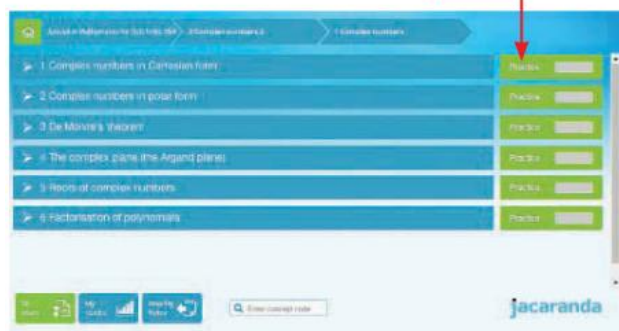
TOPIC 3 Complex numbers 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



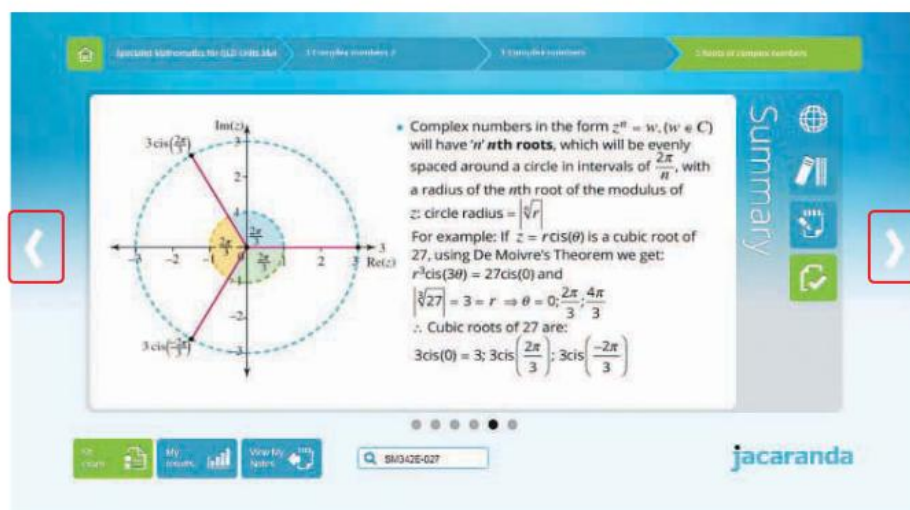
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 1

Specialist Mathematics: Problem solving and modelling task

Unit 3: Mathematical induction, and further vectors, matrices and complex numbers

Topic 2: Vectors and matrices

Conditions

Duration	Mode	Individual/group
4 weeks	Written report, up to 10 pages (maximum 2000 words) excluding appendix	Individual
Resources available		
The use of technology is required, for example: • calculator • spreadsheets • internet • other mathematical software.		
Milestones		
Week 1		
Week 2		
Week 3		
Week 4 (assessment submission)		
Criterion	Marks allocated	Result
Formulate *Assessment objectives 1, 2, 5	4	
Solve *Assessment objectives 1, 6	7	
Evaluate and verify *Assessment objectives 4, 5	5	
Communicate *Assessment objective 3	4	
Total	20	
Scaffolding	The approach to problem-solving and modelling on the following flowchart must be used.	

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Context

In competitions consisting of a number of rounds, there needs to be a method to determine the rank order of the teams (or individual competitors) at the end of the competition. The rules of the competition will normally specify how to determine the rank order.



Task

Your task is to decide if the use of dominance matrices would be a fairer method to rank order teams than the one currently used. Choose a context (particular sporting competition or other form of competition) as the basis of your investigation. Check with your teacher that your context is suitable.



To complete this task, you must:

- present your findings as an investigative report based on the approach to problem-solving and mathematical modelling outlined in the Specialist Mathematics syllabus and on the flow chart on the following page of this instrument
- respond with a range of understanding and skills, such as using mathematical language, appropriate calculations, tables of data, graphs and diagrams
- provide a response that highlights the real-life application of mathematics
- respond using a written report format that can be read and interpreted independently of the instrument task sheet
- develop a unique response
- use both analytic procedures and technology.

Approach to problem-solving and modelling

Formulate

Design a plan to investigate whether dominance matrices would be a fairer method to determine the rank order than the current one. Translate the problem into a mathematically purposeful representation by first determining the applicable mathematical and/or statistical principles, concepts, techniques and technology that are required to make progress with the problem. Identify and document appropriate assumptions, variables and observations, based on the logic of a proposed solution and model.

Solve

Select and apply mathematical and statistical procedures, concepts and techniques previously learnt to determine if the current system or dominance matrices is a fairer system of determining a rank order. Synthesise and refine your model and generate and test hypotheses with secondary data and information. This process may require returning to the initial observations and assumptions and reconsidering and modifying them to ensure the problem has been solved or can actually be solved. The use of technology must be demonstrated in the process of solving the problem.

Is it solved?

Evaluate and verify

Once a possible solution has been achieved, consider the reasonableness of the solution and the utility of the model in terms of the problem. Evaluate your results and make a judgment about the solution to the problem in relation to whether the current system or dominance matrices is the fairer system.

This will involve exploring the strengths and limitations of the solution and/or model. Where necessary, this will require going back through the process to further refine your model to provide a valid solution.

Is the solution verified?

Communicate

Clearly and fully communicate your development of solutions and models to determine whether the current system or dominance matrices is fairer.

Communicate your findings systematically and concisely using mathematical, statistical and everyday language, including through a suitable introduction, body and conclusion.

Draw conclusions by discussing the key results and the strengths and limitations of the solution and/or model. Offer further explanation, justification, and/or recommendations, framed in the context of the initial problem.

PRACTICE ASSESSMENT 2

Specialist Mathematics: Unit 3 examination

Unit 3: Mathematical induction, and further vectors, matrices and complex numbers

Topic 1: Proof by mathematical induction

Topic 2: Vectors and matrices

Topic 3: Complex numbers 2

Conditions

Technique	Response Type	Duration	Reading
Technology active	Short response	120 minutes	5 minutes
Resources		Instructions	
• QCAA formula sheet • Notes not permitted • Non-CAS graphics calculator permitted		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to two decimal places.	
Criterion		Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6		60	

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Part A: Simple familiar — total marks: 36

Question 1 (6 marks)

Use Gaussian elimination to solve the system of equations.

$$\begin{aligned}x + y + z &= 6 \\x + 2z &= 5 \\x - y - 2z &= -1\end{aligned}$$

Use substitution to verify that your solution is correct.

Question 2 (4 marks)

A Leslie matrix has been developed to model a lizard population in their first, second and third years of life.

$$L = \begin{bmatrix} 0 & 6.8 & 4.9 \\ 0.3 & 0 & 0 \\ 0 & 0.35 & 0 \end{bmatrix}$$

- Identify the average number of new lizards produced by lizards in their first year of life.
- Identify the average number of new lizards produced by lizards in their second year of life.
- Identify the expected survival rate of lizards in their second year.
- If there are initially 35, 48 and 23 female lizards in their first, second and third years respectively, calculate, with the aid of technology, the total expected female population in 3-years-time.

Question 3 (8 marks)

Sketch the following on the complex plane:

a. $\left\{z: \arg(z) = -\frac{\pi}{4}\right\}$

b. $\{z: \operatorname{Re}(z) = 2\}$

c. $\{z: |z + 2 - 3i| = 2\}$

d. $\left\{z: \arg(z + i) = \frac{\pi}{6}\right\}$

[illegible]

Question 4 (5 marks)

Solve $z^2 + 6z + 13 = 0$, $z \in \mathbb{C}$. Verify your solutions algebraically.

This image shows a single sheet of white paper with horizontal blue or grey ruling lines. The lines are evenly spaced and run across the width of the page. There are approximately 20 lines visible. The paper has a slight shadow on its right side, suggesting it's resting on a surface.

Question 5 (7 marks)

Solve $z^4 = -8\sqrt{2} + 8\sqrt{2}i$. Leave your answers in the form $z = r \operatorname{cis} \theta$, $-180^\circ < \theta \leq 180^\circ$. Use a sketch to verify the reasonableness of your solution.

Question 6 (6 marks)

Consider an object moving along a path so that $\tilde{r}(t) = 2 \sin(t) \hat{i} + 3 \cos(t) \hat{j}$

- Determine the Cartesian equation that describes the path of the object.
- Determine $\underline{v}(t) = \dot{\underline{r}}(t)$
- Determine the position and direction of motion when $t = 0$.

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Part B: Complex familiar — total marks: 12

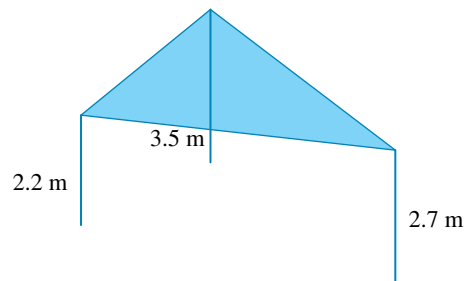
Question 7 (5 marks)

Prove, by mathematical induction, the following proposition for $n \in \mathbb{N}$:

$$P(n) = \sum_{r=1}^n \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

Question 8 (7 marks)

A triangular shade cloth is to be suspended by three poles of different heights as shown. The bases of the poles are positioned 10 m away from each other. Determine the area of the shade cloth needed. Your solution must include appropriate justification of procedures and results.



Question 9 (9 marks)

Tom decides that if the velocity is increased to 20 m/s at an angle of 60° , the supplies will arrive faster. By also considering where the supplies would land, evaluate the reasonableness of Tom's claim.

[illegible]

If $P(z) = z^4 - (1 + 3i)z^3 + 3(i - 1)z^2 + (7 + i)z - 4 - i$ and $P(z)$ can also be expressed as $P(z) = (z - 1)((z - a)^3 + b)$ for $z, a \in \mathbb{C}, b \in \mathbb{Z}$, determine the values of a and b .

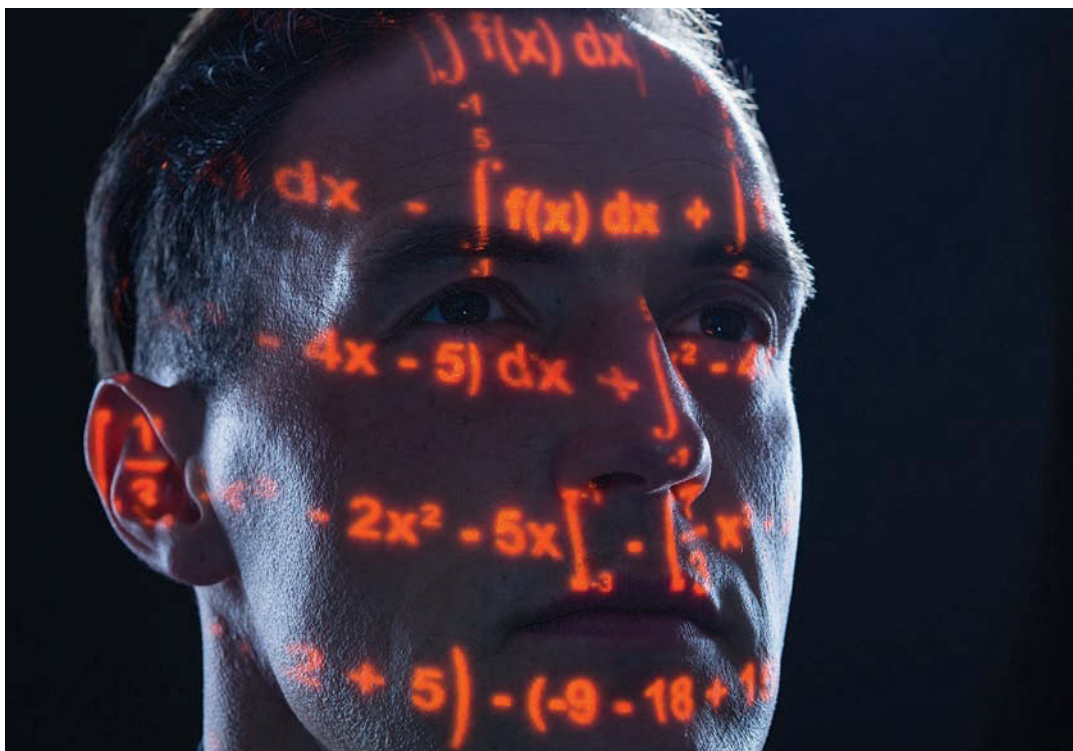
[illegible]

6 Integration techniques

6.1 Overview

Gottfried Wilhelm von Leibniz (1643 – 1727) and Isaac Newton (1646 – 1716) are credited with independently developing the basics of calculus, identifying the inverse relationship between differentiation and integration.

You began your study of integration in Mathematical Methods, learning how to integrate a variety of functions. In this chapter, you will explore methods of integration that allow you to integrate combinations of functions.



LEARNING SEQUENCE

- 6.1 Overview
- 6.2 Integration by linear substitution
- 6.3 Integration by non-linear substitutions
- 6.4 Integration using the trigonometric identities
- 6.5 Integration of inverse trigonometric functions
- 6.6 Integration by parts
- 6.7 Integration involving partial fractions
- 6.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.2 Integration by linear substitution

When differentiating a function in the form $f(g(x))$, it is possible to make the substitution $u = g(x)$ and use the chain rule to determine the derivative. When determining an anti-derivative, it is also possible to make an appropriate substitution in order to rewrite the integrand as a function that has a known integral. To begin, we will explore functions when the substitution $u = ax + b$ is appropriate.

6.2.1 Determining integrals in the form $\int (ax + b)^n dx$

Determining integrals in the form $\int (ax + b)^n dx, n \in \mathbb{Z}^+$

Integrals in this form could be found by expanding. However, as n becomes larger, this becomes impractical.

WORKED EXAMPLE 1

Determine $\int (2x - 5)^4 dx$.

THINK

1. Although $(2x - 5)^4$ could be expanded, this would result in a sum with 5 terms. Let $u = 2x - 5$ and use differentiation to determine dx .

WRITE

Let $u = 2x - 5$

$$\frac{du}{dx} = 2$$

$$du = 2 dx$$

$$dx = \frac{du}{2}$$

2. Rewrite the integral, substituting for $2x - 5$ and dx .

$$\begin{aligned}\int (2x - 5)^4 dx &= \int u^4 \frac{du}{2} \\&= \int \frac{1}{2} u^4 du \\&= \frac{1}{2} \left(\frac{u^5}{5} \right) + c \\&= \frac{1}{10} u^5 + c \\&= \frac{1}{10} (2x - 5)^5 + c\end{aligned}$$

3. As the integrand is now expressed as a function of u only, it can be integrated.

4. As the original integral was in terms of x , substitute $u = 2x - 5$.

Determining integrals in the form $\int (ax + b)^n dx, n \in \mathbb{Q}, n \neq -1$

In this instance, it is not possible to use expansion to rewrite the integral, so substitution must be used. It may be necessary to rewrite the integrand so that it is in the form $(ax + b)^n$.

WORKED EXAMPLE 2

The gradient of a curve is given by $\frac{1}{\sqrt{4x + 9}}$. Determine the particular curve which passes through the origin.

THINK

1. If the original curve is y , then the gradient is $\frac{dy}{dx}$.
2. Integrate both sides to obtain an expression for y .
3. Use index laws to express the integrand as a function to a power. Note $\sqrt{u} = u^{\frac{1}{2}}$.
4. The integrand can be rewritten using $u = 4x + 9$. Differentiate to determine du .

WRITE

$$\frac{dy}{dx} = \frac{1}{\sqrt{4x+9}}$$

$$y = \int \frac{1}{\sqrt{4x+9}} dx$$

$$y = \int (4x+9)^{-\frac{1}{2}} dx$$

$$\text{Let } u = 4x + 9$$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

$$dx = \frac{du}{4}$$

5. Rewrite the integral, substituting for $4x + 9$ and dx .

$$\begin{aligned} \int (4x+9)^{-\frac{1}{2}} dx &= \int u^{-\frac{1}{2}} \frac{du}{4} \\ &= \int \frac{1}{4} u^{-\frac{1}{2}} du \\ &= \frac{1}{4} \left(2u^{\frac{1}{2}} \right) + c \\ &= \frac{1}{2} u^{\frac{1}{2}} + c \end{aligned}$$

7. As the original integral was in terms of x , substitute $u = 4x + 9$.

$$y = \frac{1}{2} \sqrt{4x+9} + c$$

8. As the curve passes through the origin, c can be determined by substituting $x = 0$ and $y = 0$.

$$0 = \frac{1}{2} \sqrt{9} + c$$

$$c = -\frac{3}{2}$$

9. Substitute for c and state the equation for y .

$$y = \frac{1}{2} \sqrt{4x+9} - \frac{3}{2}$$

$$y = \frac{\sqrt{4x+9} - 3}{2}$$

Determining integrals in the form $\int (ax+b)^n dx, n \neq -1$

In Mathematical Methods, you learnt that $\int \frac{1}{x} dx = \ln(x), x > 0$. This is because if $y = \ln(x)$, then

$$x = e^y, x > 0$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$= \frac{1}{x}$$

Therefore $\int \frac{1}{x} dx = \ln(x) + c, x > 0$.

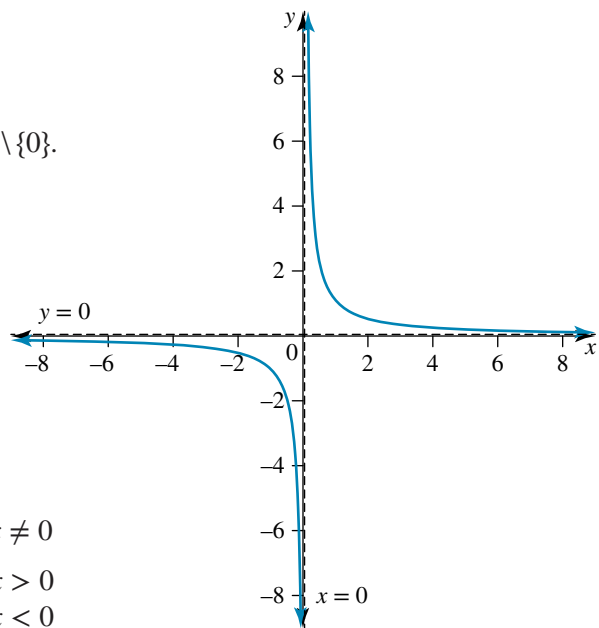
However, as $\frac{1}{x}$ is defined for $R \setminus \{0\}$, $\int \frac{1}{x} dx$ exists for $R \setminus \{0\}$.

Consider the function $y = \ln|x|, x \neq 0$.

We can use the chain rule to differentiate this function

if $u = |x|, x \neq 0$. Notice that $u > 0$ so we know

that $\frac{d}{du} \ln(u) = \frac{1}{u}$.



$$\begin{aligned} y &= \ln(u), u > 0 \\ \frac{dy}{du} &= \frac{1}{u} \\ u &= |x|, \quad x \neq 0 \\ &= \begin{cases} x & , \quad x > 0 \\ -x & , \quad x < 0 \end{cases} \\ \frac{du}{dx} &= \begin{cases} 1 & , \quad x > 0 \\ -1 & , \quad x < 0 \end{cases} \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \begin{cases} \frac{1}{u} \times 1 & , \quad x > 0 \\ \frac{1}{u} \times -1 & , \quad x < 0 \end{cases} \end{aligned}$$

Substituting $u = |x|$

$$\begin{aligned} &= \begin{cases} x & , \quad x > 0 \\ -x & , \quad x < 0 \end{cases} \\ \frac{dy}{dx} &= \begin{cases} \frac{1}{|x|} & , \quad x > 0 \\ -\frac{1}{|x|} & , \quad x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{x} & , \quad x > 0 \\ -\frac{1}{-x} & , \quad x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{x} & , \quad x > 0 \\ \frac{1}{x} & , \quad x < 0 \end{cases} \\ &= \frac{1}{x}, \quad x \neq 0 \end{aligned}$$

The anti-derivative of x^n where $n = -1$

$$\text{As } \frac{d}{dx} \ln |x| = \frac{1}{x}, x \neq 0, \quad \int \frac{1}{x} dx = \ln |x| + c, x \neq 0.$$

This result must be used in Specialist Mathematics.

WORKED EXAMPLE 3

Determine the anti-derivative of $\frac{1}{5x+4}$ for $x \neq -\frac{4}{5}$.

THINK

- Express the question using integral notation.
- The integrand can be rewritten using $u = 5x + 4$. Differentiate to determine du .
- Rewrite the integral, substituting for $5x + 4$ and dx .
- Use $\int \frac{1}{x} dx = \ln |x|, x \neq 0$ to find the integral.
- As the original integral was in terms of x , substitute $u = 5x + 4$.

WRITE

$$\int \frac{1}{5x+4} dx$$

$$\text{Let } u = 5x + 4$$

$$\frac{du}{dx} = 5$$

$$du = 5dx$$

$$dx = \frac{du}{5}$$

$$\begin{aligned} \int \frac{1}{5x+4} dx &= \int \frac{1}{u} \frac{du}{5} \\ &= \frac{1}{5} \int \frac{1}{u} du \\ &= \frac{1}{5} \ln |u| + c \\ &= \frac{1}{5} \ln |5x+4| + c \end{aligned}$$

The last three examples can be summarised as follows

Integrals of the form $\int (ax+b)^n dx, n \in \mathbb{Q}$

$$\int (ax+b)^n dx = \begin{cases} \frac{1}{a(n+1)} (ax+b)^{n+1} + c & n \neq -1 \\ \frac{1}{a} \ln |ax+b| + c & n = -1 \end{cases}$$

6.2.2 Evaluating definite integrals using a linear substitution

When using substitution to evaluate definite integrals, there is no need to return to an expression in terms of x provided that the terminals are expressed in terms of u .

WORKED EXAMPLE 4

Calculate $\int_0^1 \frac{4}{(3x+2)^2} dx$.

THINK

1. Use index laws to express the integrand as a function to a power.
2. The integrand can be rewritten using $u = 3x + 2$. Differentiate to determine du .
3. Rewrite the integral, substituting for $3x + 2$ and dx . Note that the integral is occurring between $x = 0$ and $x = 1$.
4. Determine values for u for $x = 0$ and $x = 1$.
5. As the integrand is now expressed as a function of u only, it can be integrated.
6. Evaluate the definite integral.
7. State the final result.

WRITE

$$\int_0^1 4(3x+2)^{-2} dx$$

$$\text{Let } u = 3x + 2$$

$$\frac{du}{dx} = 3$$

$$du = 3dx$$

$$dx = \frac{du}{3}$$

$$\begin{aligned} \int_{x=0}^{x=1} 4(3x+2)^{-2} dx &= \int_{x=0}^{x=1} 4u^{-2} \frac{du}{3} \\ &= \frac{4}{3} \int_{x=0}^{x=1} u^{-2} du \end{aligned}$$

$$x = 0: u = 3(0) + 2 = 2$$

$$x = 1: u = 3(1) + 2 = 5$$

$$\frac{4}{3} \int_{u=2}^{u=5} u^{-2} du$$

$$= \frac{4}{3} \left[\frac{u^{-1}}{-1} \right]_2^5$$

$$= \frac{4}{3} \left[\frac{-1}{u} \right]_2^5$$

$$= \frac{4}{3} \left[\frac{-1}{5} - \frac{-1}{2} \right]$$

$$= \frac{4}{3} \left(\frac{-2}{10} + \frac{5}{10} \right)$$

$$= \frac{4}{10}$$

$$= \frac{2}{5}$$

$$\int_0^1 \frac{4}{(3x+2)^2} dx = \frac{2}{5}$$

6.2.3 Other linear substitutions

Integrals of the form $\int f(x) [g(x)]^n dx$ where $g(x)$ is a linear function can be solved using the substitution $u = g(x)$. Because $g(x)$ is linear, it is possible to find u as a function of x and therefore find $f(x)$ as a function of u .

WORKED EXAMPLE 5

Determine the following:

a. $\int x(2x-5)^4 dx$

b. $\int \frac{6x-5}{4x^2-12x+9} dx$ for $x \neq \frac{3}{2}$.

THINK

- a. 1. The integrand can be rewritten using $u = 2x - 5$. Differentiate to determine du .
2. Rewrite the integral, substituting for $2x - 5$ and dx .
3. The integrand is not a function of u only. Rearrange u to find an expression for x .
4. Substitute for x in the integral.
5. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to expand the brackets before integrating.
6. The final solution will need to be written in terms of x . If we substitute at this point, we will get $\frac{1}{4} \left[\frac{(2x-5)^6}{6} + (2x-5)^5 \right] + c$ which will need to be factorised. It is possibly easier to factorise first and then substitute for u .
7. As the original integral was in terms of x , substitute $u = 2x - 5$ and simplify.

- b. 1. Factorise the denominator.

WRITE

- a. Let $u = 2x + 5$

$$\frac{du}{dx} = 2$$

$$du = 2 dx$$

$$dx = \frac{du}{2}$$

$$\begin{aligned} \int x(2x-5)^4 dx &= \int x u^4 \frac{du}{2} \\ &= \frac{1}{2} \int x u^4 du \end{aligned}$$

$$u = 2x - 5$$

$$2x = u + 5$$

$$x = \frac{u+5}{2}$$

$$\begin{aligned} \frac{1}{2} \int x u^4 du &= \frac{1}{2} \int \frac{(u+5)}{2} u^4 du \\ &= \frac{1}{4} \int (u+5) u^4 du \\ &= \frac{1}{4} \int (u^5 + 5u^4) du \\ &= \frac{1}{4} \left(\frac{u^6}{6} + \frac{5u^5}{5} \right) + c \\ &= \frac{1}{4} \left(\frac{u^6}{6} + u^5 \right) + c \\ &= \frac{1}{4} \frac{u^5}{6} (u+6) + c \\ &= \frac{1}{24} u^5 (u+6) + c \\ &= \frac{1}{24} (2x-5)^5 (2x-5+6) + c \\ &= \frac{1}{24} (2x-5)^5 (2x+1) + c \end{aligned}$$

b. $\int \frac{6x-5}{4x^2-12x+9} dx = \int \frac{6x-5}{(2x-3)^2} dx$



2. The integrand can be rewritten using $u = 2x - 3$.
Differentiate to determine du .

Let $u = 2x - 3$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$dx = \frac{du}{2}$$

3. Rewrite the integral, substituting for $2x - 3$ and dx .

$$\begin{aligned}\int \frac{6x-5}{(2x-3)^2} dx &= \int \frac{6x-5}{u^2} \frac{du}{2} \\ &= \frac{1}{2} \int \frac{6x-5}{u^2} du\end{aligned}$$

4. The integrand is not a function of u only. Rearrange u to find an expression for $6x$.

$$u = 2x - 3$$

$$2x = u + 3$$

$$6x = 3u + 9$$

5. Substitute for x in the integral.

$$\begin{aligned}\frac{1}{2} \int \frac{6x-5}{u^2} du &= \frac{1}{2} \int \frac{3u+9-5}{u^2} du \\ &= \frac{1}{2} \int \frac{3u+4}{u^2} du\end{aligned}$$

6. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to write the integrand as the sum of two terms before integrating.

$$= \frac{1}{2} \int \left(\frac{3u}{u^2} + \frac{4}{u^2} \right) du$$

$$= \frac{1}{2} \int \left(\frac{3}{u} + 4u^{-2} \right) du$$

$$= \frac{3}{2} \int \frac{1}{u} du + \frac{4}{2} \int u^{-2} du$$

$$= \frac{3}{2} \ln |u| + \frac{4}{2} \times \frac{u^{-1}}{-1} + c$$

$$= \frac{3}{2} \ln |u| - \frac{2}{u} + c$$

7. Integrate noting that $\int \frac{1}{x} dx = \ln |x|, x \neq 0$.

8. As the original integral was in terms of x , substitute $u = 2x - 3$.

$$= \frac{3}{2} \ln |2x-3| - \frac{2}{2x-3} + c$$

6.2.4 Definite integrals using other linear substitutions

WORKED EXAMPLE 6

Calculate $\int_0^8 \frac{x}{\sqrt{2x+9}} dx$.

THINK

1. Use index laws to express the integrand as a function of a power.

WRITE

$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \int_0^8 x(2x+9)^{-\frac{1}{2}} dx$$

2. The integrand can be rewritten using $u = 2x + 9$. Differentiate to determine du .

$$\begin{aligned}\text{Let } u &= 2x + 9 \\ \frac{du}{dx} &= 2 \\ du &= 2dx \\ dx &= \frac{du}{2}\end{aligned}$$

3. Rewrite the integral, substituting for $2x + 9$ and dx . Note that the integral is occurring between $x = 0$ and $x = 8$.

$$\begin{aligned}\int_{x=0}^{x=8} x(2x+9)^{-\frac{1}{2}} dx &= \int_{x=0}^{x=8} x u^{-\frac{1}{2}} \frac{du}{2} \\ &= \frac{1}{2} \int_{x=0}^{x=8} x u^{-\frac{1}{2}} du\end{aligned}$$

4. The integrand is not a function of u only. Rearrange u to find an expression for x . Determine values for u when $x = 0$ and $x = 8$.

$$\begin{aligned}u &= 2x + 9 \\ 2x &= u - 9 \\ x &= \frac{u - 9}{2}\end{aligned}$$

5. Substitute for x in the integral and the new terminals.

$$\begin{aligned}x = 0: u &= 9 \\ x = 8: u &= 2 \times 8 + 9 = 25 \\ &= \frac{1}{2} \int_{u=9}^{u=25} \frac{(u-9)}{2} u^{-\frac{1}{2}} du \\ &= \frac{1}{4} \int_9^{25} (u-9) u^{-\frac{1}{2}} du\end{aligned}$$

6. As the integrand is now expressed as a function of u only, it can be integrated. It will be necessary to expand the brackets before integrating.

$$\begin{aligned}&= \frac{1}{4} \int_9^{25} \left(u^{\frac{1}{2}} - 9u^{-\frac{1}{2}} \right) du \\ &= \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} - 9 \times 2u^{\frac{1}{2}} \right]_9^{25} \\ &= \frac{1}{4} \left[\frac{2}{3} \sqrt{u^3} - 18\sqrt{u} \right]_9^{25} \\ &= \frac{1}{4} \left[\left(\frac{2}{3} \sqrt{25^3} - 18\sqrt{25} \right) - \left(\frac{2}{3} \sqrt{9^3} - 18\sqrt{9} \right) \right] \\ &= \frac{1}{4} \left[\left(\frac{2}{3} \times 5^3 - 18 \times 5 \right) - \left(\frac{2}{3} \times 3^3 - 18 \times 3 \right) \right] \\ &= \frac{22}{3}\end{aligned}$$

7. Evaluate the definite integral.

8. State the final result.

$$\int_0^8 \frac{x}{\sqrt{2x+9}} dx = \frac{22}{3}$$

When integrating functions of the form $f(g(x))$ when $g(x)$ is a linear function, the substitution $u = g(x)$ can be used. If determining an indefinite integral, it is necessary to write the final solution in terms of x .

If determining a definite integral by substitution, it is not necessary to rewrite the integral in terms of x provided that the terminals are values for u .

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 1

Integration by linear substitution Summary screen and practice questions

Exercise 6.2 Integration by linear substitution

Technology free

1. **WE1** Determine:

a. $\int (5x - 9)^6 dx$

b. $\int (3x + 4)^7 dx$

c. $\int (3x - 5)^6 dx$

d. $\int (1 - 2x)^5 dx$

2. **WE2** The gradient of a curve is given by $\frac{1}{\sqrt{16x + 25}}$. Determine the particular curve which passes through the origin.

3. Given that $f'(x) = \frac{1}{(3x - 7)^2}$ and $f(2) = 3$, determine the value of $f(1)$.

4. **WE3** Determine the anti-derivative of:

a. $\frac{1}{3x - 5}$ for $x \neq \frac{5}{3}$

b. $\frac{1}{7 - 2x}$ for $x \neq \frac{7}{2}$

c. $\frac{2}{3x - 1}$ for $x \neq \frac{1}{3}$

d. $\frac{3}{1 - 2x}$ for $x \neq \frac{1}{2}$

5. Determine the anti-derivative of the following:

a. $(3x + 5)^6$

b. $\frac{1}{(3x + 5)^2}$

c. $\frac{1}{(3x + 5)^3}$

d. $\frac{1}{\sqrt[3]{3x + 5}}$

6. Determine each of the following:

a. $\int (6x + 7)^8 dx$

b. $\int \frac{1}{\sqrt{6x + 7}} dx$

c. $\int \frac{1}{6x + 7} dx$ for $x \neq -\frac{7}{6}$

d. $\int \frac{1}{(6x + 7)^2} dx$

7. **WE4** Calculate the following:

a. $\int_{-1}^0 \frac{9}{(2x + 3)^3} dx$

b. $\int_0^4 \frac{6}{\sqrt{3x + 4}} dx$

c. $\int_0^{13} (2x + 1)^{\frac{1}{3}} dx$

d. $\int_2^3 \frac{1}{(2x - 3)^2} dx$

8. **WE5** Determine:

a. $\int x(5x - 9)^5 dx$

b. $\int \frac{2x - 1}{9x^2 - 24x + 16} dx$ for $x \neq \frac{4}{3}$

c. $\int \frac{x}{(2x + 7)^3} dx$

d. $\int \frac{x}{\sqrt{6x + 5}} dx$

9. For each of the following, determine $f(x)$.

a. $f'(x) = x(3x + 5)^6$

b. $f'(x) = \frac{x}{(3x + 5)^2}$ for $x \neq -\frac{5}{3}$

c. $f'(x) = \frac{x}{(3x + 5)^3}$

d. $f'(x) = \frac{x}{\sqrt[3]{3x + 5}}$

10. Determine each of the following:

a. $\int x(6x + 7)^8 dx$

b. $\int \frac{x}{\sqrt{6x + 7}} dx$

c. $\int \frac{x}{6x + 7} dx$

d. $\int \frac{x}{(6x + 7)^2} dx$ for $x \neq -\frac{7}{6}$

11. a. Given that $\frac{dx}{dt} = \frac{1}{(2 - 5t)^2}$ and $x(0) = 0$, determine an equation for x in terms of t .

b. A certain curve has its gradient given by $\frac{1}{\sqrt{3 - 2x}}$ for $x < \frac{3}{2}$. If the point $\left(-\frac{1}{2}, -2\right)$ lies on the curve, determine the equation of the curve.

- c. Given that $f'(x) = \frac{3}{3-2x}$ and that $f(0) = 0$, calculate $f(1)$.
- d. A certain curve has a gradient equal $\frac{x}{\sqrt{2x+9}}$, determine the equation of the particular curve which passes through the origin.
12. **WE6** Calculate:
- a. $\int_1^2 x\sqrt{x-1} dx$ b. $\int_0^3 \frac{x^2}{\sqrt{x+1}} dx$ c. $\int_0^5 \frac{x}{\sqrt{3x+1}} dx$ d. $\int_0^1 \frac{15x}{(3x+2)^2} dx$
13. Determine each of the following.
- a. $\int_1^2 (3x-4)^5 dx$ b. $\int_1^2 x(3x-4)^5 dx$ c. $\int_0^{13} \frac{1}{\sqrt[3]{2t+1}} dt$ d. $\int_0^{13} \frac{t}{\sqrt[3]{2t+1}} dt$
14. If a and b are non-zero real constants, determine the following:
- a. $\int \sqrt{ax+b} dx$ b. $\int x\sqrt{ax+b} dx$
- c. $\int \frac{1}{ax+b} dx$ for $x \neq \frac{-b}{a}$ d. $\int \frac{x}{ax+b} dx$ for $x \neq \frac{-b}{a}$
15. If a and b are non-zero real constants, determine the following:
- a. $\int \frac{1}{\sqrt{ax+b}} dx$ b. $\int \frac{x}{\sqrt{ax+b}} dx$
- c. $\int \frac{1}{(ax+b)^2} dx$ d. $\int \frac{x}{(ax+b)^2} dx$ for $x \neq \frac{-b}{a}$
16. If a, b, c , and d are non-zero real constants, determine the following:
- a. $\int \frac{cx+d}{ax+b} dx$ for $x \neq \frac{-b}{a}$
- b. $\int \frac{cx+d}{(ax+b)^2} dx$ for $x \neq \frac{-b}{a}$
- c. $\int \frac{cx^2+d}{(ax+b)^2} dx$ for $x \neq \frac{-b}{a}$
- d. $\int \frac{cx^2+d}{ax+b} dx$ for $x \neq \frac{-b}{a}$

6.3 Integration by non-linear substitutions

We have looked at how it is possible to integrate functions of the form $\int f(x) [g(x)]^n dx$ where $g(x)$ is a linear function by making the substitution $u = g(x)$.

It is sometimes also possible to use a non-linear substitution. If the function is in the form $f(g(x)) g'(x)$, then the substitution $u = g(x)$ can be used.

6.3.1 Indefinite integration using non-linear substitutions

This method relies on the derivative, or constant multiples of the derivative, being present and recognisable. It is important that after making the substitution $u = g(x)$ that the integrand is rewritten entirely in terms of u .

WORKED EXAMPLE 7

Determine $\int \frac{3x}{(x^2 + 9)^2} dx$.

THINK

1. The function is in the form $f(g(x)) g'(x)$ where $g(x) = x^2 + 9$. Let $u = x^2 + 9$ and differentiate to determine dx .
2. Rewrite the integral substituting for $x^2 + 9$ and dx .
3. Simplify the integral noting that $\frac{3x}{2x} = \frac{3}{2}$.
4. The integrand is written in terms of u only, so it can be integrated.
Rewrite the integral using power notation and integrate.
5. Substitute for u so that the final answer is in terms of x only.

WRITE

Let $u = x^2 + 9$

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$dx = \frac{du}{2x}$$

$$\begin{aligned} \int \frac{3x}{(x^2 + 9)^2} dx &= \int \frac{3x}{u^2} \frac{du}{2x} \\ &= \int \frac{3x}{2x} \frac{1}{u^2} du \\ &= \frac{3}{2} \int \frac{1}{u^2} du \\ &= \frac{3}{2} \int u^{-2} du \\ &= \frac{3}{2} \times \frac{u^{-1}}{-1} + c \\ &= -\frac{3}{2u} + c \\ &= -\frac{3}{2(x^2 + 9)} + c \end{aligned}$$

Using the logarithm function

This uses the standard integral $\int \frac{1}{x} dx = \ln |x| + c$ where $x \neq 0$.

WORKED EXAMPLE 8

Determine $\int \frac{x-3}{x^2-6x+13} dx$.

THINK

1. The function is in the form $f(g(x)) g'(x)$ where $g(x) = x^2 - 6x + 13$. Let $u = x^2 - 6x + 13$ and differentiate to determine dx .
2. Rewrite the integral substituting for $x^2 - 6x + 13$ and dx .

WRITE

Let $u = x^2 - 6x + 13$

$$\frac{du}{dx} = 2x - 6$$

$$du = (2x - 6) dx$$

$$dx = \frac{du}{2(x-3)}$$

$$\int \frac{x-3}{x^2-6x+13} dx = \int \frac{x-3}{u} \frac{du}{2(x-3)}$$

3. Simplify the integral noting that $\frac{x-3}{2(x-3)} = \frac{1}{2}$

$$= \int \frac{x-3}{2(x-3)} \frac{du}{u}$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + c$$

4. The integrand is written in terms of u only, so it can be integrated.

$$\text{Integrate using } \int \frac{1}{x} dx = \ln |x| + c$$

5. Substitute for u so that the final answer is in terms of x only.

$$= \frac{1}{2} \ln |x^2 - 6x + 13| + c$$

6. The function $x^2 - 6x + 13 > 0$ for all x meaning the absolute value sign is not needed.

$$= \frac{1}{2} \ln (x^2 - 6x + 13) + c$$

Note that since $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$, it follows that $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$. This pattern can be used when integrating. The previous worked example could have been solved as follows:

$$\text{If } f(x) = x^2 - 6x + 13, f'(x) = 2x - 6$$

$$\begin{aligned} \int \frac{x-3}{x^2-6x+13} dx &= \frac{1}{2} \int \frac{2x-6}{x^2-6x+13} dx \\ &= \frac{1}{2} \ln |x^2-6x+13| + c \\ &= \frac{1}{2} \ln (x^2-6x+13) + c \quad \text{since } x^2-6x+13 > 0. \end{aligned}$$

Using trigonometric functions

It is also possible to use the standard trigonometric integrals.

Anti-derivatives of trigonometric functions

$$\int \sin(x) dx = -\cos(x) + c$$

$$\int \cos(x) dx = \sin(x) + c$$

$$\int \sec^2(x) dx = \tan(x) + c$$

WORKED EXAMPLE 9

Determine $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$.

THINK

1. The function is in the form $f(g(x)) g'(x)$ where $g(x) = \frac{1}{x}$. Let $u = \frac{1}{x}$ and differentiate to determine dx .

WRITE

$$\begin{aligned} \text{Let } u &= \frac{1}{x} \\ &= x^{-1} \\ \frac{du}{dx} &= -x^{-2} \\ &= -\frac{1}{x^2} \end{aligned}$$

2. Rewrite the integral substituting for $\frac{1}{x}$ and dx .

3. Simplify the integral noting that $\frac{-x^2}{x^2} = -1$

4. The integrand is written in terms of u only, so it can be integrated. Integrate using

$$\int \cos(x) dx = \sin(x) + c.$$

5. Substitute for u so that the final answer is in terms of x only.

$$x^2 du = -dx$$

$$dx = -x^2 du$$

$$\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx = \int \frac{1}{x^2} \cos(u) (-x^2 du)$$

$$= \int \frac{-x^2}{x^2} \cos(u) du$$

$$= - \int \cos(u) du$$

$$= -\sin(u) + c$$

$$= -\sin\left(\frac{1}{x}\right) + c$$

WORKED EXAMPLE 10

Determine $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx$.

THINK

1. The function is in the form $f(g(x)) g'(x)$ where $g(x) = \sqrt{x}$. Let $u = \sqrt{x}$ and differentiate to determine dx .

2. Rewrite the integral substituting for $\sqrt{x}$ and dx .

3. Simplify the integral noting that $\frac{2\sqrt{x}}{\sqrt{x}} = 2$.

4. The integrand is written in terms of u only, so it can be integrated.

$$\text{Integrate using } \int \sin(x) dx = -\cos(x) + c.$$

5. Substitute for u so that the final answer is in terms of x only.

WRITE

$$\text{Let } u = \sqrt{x} \\ = x^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$dx = 2\sqrt{x} du$$

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = \int \frac{\sin(u)}{\sqrt{x}} 2\sqrt{x} du$$

$$= \int \frac{2\sqrt{x}}{\sqrt{x}} \sin(u) du$$

$$= 2 \int \sin(u) du$$

$$= -2 \cos(u) + c$$

$$= -2 \cos(\sqrt{x}) + c$$

Note: When rewriting the integral in terms of u , shown above in step 2, we notice that $\sqrt{x}$ (or u) also appears in the denominator. This could lead to trying to evaluate $\int \frac{\sin(u)}{u} dx$ or even $\int \frac{\sin(u)}{u} 2\sqrt{x} du$, both of which cannot be determined since the variable in the integrand does not match the variable of integration. This example highlights the importance of taking care when using a substitution method to integrate.

Using other functions

Using the chain rule, $\frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)}$. This means that

The anti-derivative of $g'(x)e^{g(x)}$

$$\int g'(x)e^{g(x)} dx = e^{g(x)} + c$$

WORKED EXAMPLE 11

Determine $\int \sin(2x) e^{\cos(2x)} dx$.

THINK

1. The function is in the form $f(g(x)) g'(x)$ where $g(x) = \cos(2x)$. Let $u = \cos(2x)$ and differentiate to determine dx .

2. Rewrite the integral substituting for $\cos(2x)$ and dx .

3. Simplify the integral noting that $\frac{\sin(2x)}{-2\sin(2x)} = -\frac{1}{2}$.

4. The integrand is written in terms of u only, so it can be integrated.

Integrate using $\int e^x dx = e^x + c$.

5. Substitute for u so that the final answer is in terms of x only.

WRITE

Let $u = \cos(2x)$

$$\frac{du}{dx} = -2\sin(2x)$$

$$du = -2\sin(2x) dx$$

$$dx = \frac{du}{-2\sin(2x)}$$

$$\begin{aligned} \int \sin(2x) e^{\cos(2x)} dx &= \int \sin(2x) e^u \frac{du}{-2\sin(2x)} \\ &= \int \frac{\sin(2x)}{-2\sin(2x)} e^u du \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{\cos(2x)} + c \end{aligned}$$

6.3.2 Definite integration using non-linear substitutions

When using substitution to evaluate definite integrals there is no need to return to an expression in terms of x providing the terminals are expressed in terms of u .

WORKED EXAMPLE 12

Calculate $\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2 + 4}} dt$.

THINK

- The function is in the form $f(g(t)) g'(t)$ where $g(t) = t^2 + 4$. Let $u = t^2 + 4$ and differentiate to determine dt .
- Rewrite the integral substituting for $t^2 + 4$ and dt . Note that the integral is occurring between $t = 0$ and $t = \sqrt{5}$.
- Simplify the integral noting that $\frac{t}{2t} = \frac{1}{2}$. Determine values for u when $t = 0$ and $t = \sqrt{5}$.
- The integrand is written in terms of u only, so it can be integrated. Rewrite the integral using power notation and integrate.
- Evaluate the definite integral.
- State the final result.

WRITE

Let $u = t^2 + 4$

$$\frac{du}{dt} = 2t$$

$$du = 2t dt$$

$$dt = \frac{du}{2t}$$

$$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2 + 4}} dt = \int_{t=0}^{t=\sqrt{5}} \frac{t}{\sqrt{u}} \frac{du}{2t}$$

$$t = 0: u = 0^2 + 4 = 4$$

$$t = \sqrt{5}: u = (\sqrt{5})^2 + 4 = 9$$

$$\begin{aligned} \int_{t=0}^{t=\sqrt{5}} \frac{t}{\sqrt{u}} \frac{du}{2t} &= \int_{u=4}^{u=9} \frac{t}{2t} \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} \int_4^9 \frac{1}{\sqrt{u}} du \\ &= \frac{1}{2} \int_4^9 u^{-\frac{1}{2}} du \\ &= \frac{1}{2} \left[2u^{\frac{1}{2}} \right]_4^9 \\ &= \frac{1}{2} \left[2\sqrt{u} \right]_4^9 \\ &= \frac{1}{2} (2\sqrt{9} - 2\sqrt{4}) \\ &= \frac{1}{2} (6 - 4) \\ &= 1 \end{aligned}$$

$$\int_0^{\sqrt{5}} \frac{t}{\sqrt{t^2 + 4}} dt = 1$$

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 2

Integration by non-linear substitutions Summary screen and practice questions

Exercise 6.3 Integration by non-linear substitutions

Technology free

1. **WE7** Determine:

a. $\int \frac{8x}{(x^2 + 16)^3} dx$

c. $\int 2x(x^2 + 3)^4 dx$

b. $\int \frac{5x}{\sqrt{2x^2 + 3}} dx$

d. $\int 2x(6 - x^2)^{-3} dx$

2. **WE8** Determine:

a. $\int \frac{x+2}{x^2 + 4x + 29} dx$

c. $\int \frac{x+1}{x^2 + 2x + 9} dx$

b. $\int \frac{x^2}{x^3 + 9} dx$ for $x \neq \sqrt[3]{-9}$

d. $\int \frac{3x}{x^2 + 9} dx$

3. Determine the integral of the following:

a. $x(x^2 + 4)^5$

b. $\frac{x}{x^2 + 4}$

c. $\frac{x}{(x^2 + 9)^2}$

d. $\frac{x}{\sqrt{x^2 + 9}}$

4. Determine the following:

a. $\int \frac{x^2}{(x^3 + 27)^3} dx$

c. $\int \frac{x^2}{x^3 + 8} dx$ for $x \neq -2$

b. $\int \frac{x^2}{\sqrt{x^3 + 27}} dx$

d. $\int x^2(x^3 + 8)^3 dx$

5. Determine $f(x)$ given:

a. $f'(x) = (x - 2)(x^2 - 4x + 13)^3$

c. $f'(x) = \frac{4 - x}{\sqrt{x^2 - 8x + 25}}$

b. $f'(x) = \frac{x - 2}{(x^2 - 4x + 13)^2}$

d. $f'(x) = \frac{4 - x}{x^2 - 8x + 25}$

6. Determine the anti-derivative of the following:

a. $3x^2(x^3 - 5)^2$

b. $2(x + 2)(x^2 + 4x)^{-3}$

c. $(2x + 5)\sqrt{x^2 + 5x}$

d. $\frac{2x - 3}{(x^2 - 3x)^4}$

7. Determine the integral of the following functions:

a. $6x^2(x^3 - 2)^5$

c. $x^2(x^3 - 1)^7$

b. $x(4 - x^2)^3$

d. $(x + 3)(x^2 + 6x - 2)^4$

8. Given the following functions, determine the anti-derivative:

a. $(x + 1)(x^2 + 2x + 3)^{-4}$

c. $\frac{2x - 5}{(x^2 - 5x + 2)^6}$

b. $\frac{4x + 6}{\sqrt{x^2 + 3x}}$

d. $(x^2 - 1)\sqrt{4 - 3x + x^3}$

9. Determine:

a. $\int x(x^2 + 1)^{\frac{5}{2}} dx$

c. $\int (2x + 1)\sqrt{x + x^2 - 3} dx$

b. $\int x\sqrt{1 - x^2} dx$

d. $\int \frac{3x^2 + 4x}{\sqrt{x^3 + 2x^2}} dx$

10. **WE9** Determine:

a. $\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$

b. $\int x \cos(x^2) dx$

c. $\int (3x^2 + 5) \cos(x^3 + 5x) dx$

d. $\int (2x + 3) \sin(x^2 + 3x - 2) dx$

11. **WE10** Determine:

a. $\int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$

b. $\int \frac{\ln(6x)}{4x} dx$

c. $\int \frac{\ln(3x)}{2x} dx$

d. $\int \frac{(4x - 2) \ln(x^2 - x)}{x^2 - x} dx$

12. **WE11** Determine:

a. $\int \cos(3x) e^{\sin(3x)} dx$

b. $\int \sec^2(2x) e^{\tan(2x)} dx$

c. $\int (6x - 3) e^{x^2 - x + 3} dx$

d. $\int x^2 e^{x^3 + 2} dx$

13. Determine the following:

a. $\int \frac{e^{2x}}{4e^{2x} + 5} dx$

b. $\int \frac{e^{-3x}}{(2e^{-3x} - 5)^2} dx$

c. $\int \frac{e^{-2x}}{(3e^{-2x} + 4)^3} dx$

d. $\int \frac{2e^{2x} + 1}{(e^{2x} + x)^2} dx$

14. Determine the following.

a. $\int \frac{1}{x} \sin(\ln(4x)) dx$

b. $\int \frac{1}{x} \cos(\ln(3x)) dx$

c. $\int \frac{e^{2x}}{e^x + 2} dx$

d. $\int \frac{e^{3x}}{e^x - 1} dx$ for $x \neq 0$.

15. a. A certain curve has a gradient given by $x \sin(x^2)$. Determine the equation of the particular curve which passes through the origin.

b. Calculate the value of y when $x = \frac{3}{\pi}$ given the following:

$$\frac{dy}{dx} = \frac{1}{x^2} \sec^2\left(\frac{1}{x}\right)$$

when $x = \frac{4}{\pi}$, $y = 0$

c. Given that $f'(x) = \frac{5 - x}{x^2 - 10x + 29}$ and $f(0) = 0$, calculate $f(1)$.

d. If a curve has a gradient $\sin(2x) e^{\cos(2x)}$ and passes through $\left(\frac{\pi}{4}, 0\right)$, calculate the value of y when $x = 0$.

16. **WE12** Calculate:

a. $\int_0^{2\sqrt{2}} \frac{x}{\sqrt{2x^2 + 9}} dx$

b. $\int_0^1 \frac{x}{(3x^2 + 5)^2} dx$

c. $\int_{-2}^0 4x e^{x^2} dx$

d. $\int_0^1 (4x + 7) \sqrt{2x^2 + 7x} dx$

17. Determine the value of the following definite integrals:

a. $\int_3^5 \frac{3t}{\sqrt{t^2 - 9}} dt$

b. $\int_0^2 \frac{x}{4x^2 + 9} dx$

c. $\int_1^2 \frac{x}{(2x^2 + 1)^2} dx$

d. $\int_{-1}^1 \frac{t}{\sqrt{2t^2 + 3}} dt$

18. Calculate the following:

a. $\int_4^5 \frac{3 - x}{x^2 - 6x + 34} dx$

b. $\int_2^3 \frac{2 - x}{(x^2 - 4x + 5)^2} dx$

c. $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}} dt$

d. $\int_0^{\frac{\pi}{8}} \sec^2(2\theta) e^{\tan(2\theta)} d\theta$

6.4 Integration using the trigonometric identities

6.4.1 Review of trigonometric identities

Your formula sheet contains the following double-angle identities:

$$\begin{aligned}\sin(2A) &= 2 \sin(A) \cos(A) \\ \cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= 1 - 2 \sin^2(A) \\ &= 2 \cos^2(A) - 1\end{aligned}$$

The functions being used will often be written in terms of a multiple of x . If we substitute $A = ax$ then the formulas become:

$$\begin{aligned}\sin(2ax) &= 2 \sin(ax) \cos(ax) \\ \sin(ax) \cos(ax) &= \frac{1}{2} \sin(2ax) \\ \hline \cos(2ax) &= 1 - 2 \sin^2(ax) \\ 2 \sin^2(ax) &= 1 - \cos(2ax) \\ \sin^2(ax) &= \frac{1}{2} (1 - \cos(2ax)) \\ \hline \cos(2ax) &= 2 \cos^2(ax) - 1 \\ 2 \cos^2(ax) &= \cos(2ax) + 1 \\ \cos^2(ax) &= \frac{1}{2} (\cos(2ax) + 1)\end{aligned}$$

The Pythagorean identities can also be rewritten.

$$\begin{aligned}\sin^2(A) + \cos^2(A) &= 1 & \tan^2(A) + 1 &= \sec^2(A) & \cot^2(A) + 1 &= \operatorname{cosec}^2(A) \\ \sin^2(ax) + \cos^2(ax) &= 1 & \tan^2(ax) + 1 &= \sec^2(ax) & \cot^2(ax) + 1 &= \operatorname{cosec}^2(ax)\end{aligned}$$

In summary, the following substitutions may be useful:

Trigonometric identities

$$\begin{aligned}\sin(ax) \cos(ax) &= \frac{1}{2} \sin(2ax) \\ \sin^2(ax) &= \frac{1}{2} (1 - \cos(2ax)) \\ \cos^2(ax) &= \frac{1}{2} (1 + \cos(2ax)) \\ \sin^2(ax) + \cos^2(ax) &= 1 \\ \tan^2(ax) + 1 &= \sec^2(ax) \\ \cot^2(ax) + 1 &= \operatorname{cosec}^2(ax)\end{aligned}$$

Note: Through choosing different trigonometric substitutions, your answers to integration questions may appear to be different to those in the answers or those generated by sites like Wolfram Alpha. Using the trigonometry identities that you studied in Unit 2 Topic 2 it can be demonstrated that solutions are identical or only different by a constant term.

6.4.2 Integrating even powers of $\sin(ax)$ or $\cos(ax)$

Integrating $\sin^2(ax)$ or $\cos^2(ax)$

When integrating multiples of $\sin^2(ax)$ or $\cos^2(ax)$, the substitutions $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$ or

$\cos^2(ax) = \frac{1}{2}(\cos(2ax) + 1)$ can be used.

WORKED EXAMPLE 13

Determine the anti-derivative of the following:

a. $\cos^2\left(\frac{x}{4}\right)$

b. $2\sin^2\left(\frac{x}{2}\right)$

THINK

a. 1. $\cos^2\left(\frac{x}{4}\right)$ can be rewritten using

$$\cos^2(ax) = \frac{1}{2}(\cos(2ax) + 1) \text{ where } a = \frac{1}{4}.$$

2. Express the question using integral notation.

3. Rewrite the integrand using the substitution.

4. Integrate noting that $\frac{d}{dx} \sin\left(\frac{x}{2}\right) = \frac{1}{2} \cos\left(\frac{x}{2}\right)$ so

$$\int \frac{1}{2} \cos\left(\frac{x}{2}\right) dx = \sin\left(\frac{x}{2}\right) + c.$$

b. 1. $\sin^2\left(\frac{x}{2}\right)$ can be rewritten using

$$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax)) \text{ where } a = \frac{1}{2}.$$

2. Express the question using integral notation.

3. Rewrite the integrand using the substitution.

4. Integrate noting that $\int \cos(x) dx = \sin(x) + c$.

WRITE

a. Use $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

$$\begin{aligned} \cos^2\left(\frac{x}{4}\right) &= \frac{1}{2} \left(1 + \cos\left(\frac{2x}{4}\right)\right) \\ &= \frac{1}{2} + \frac{1}{2} \cos\left(\frac{x}{2}\right) \end{aligned}$$

$$\begin{aligned} \int \cos^2\left(\frac{x}{4}\right) dx &= \int \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{x}{2}\right)\right) dx \\ &= \frac{x}{2} + \sin\left(\frac{x}{2}\right) + c \end{aligned}$$

b. Use $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$

$$\begin{aligned} 2\sin^2\left(\frac{x}{2}\right) &= 2 \left[\frac{1}{2} \left(1 - \cos\left(\frac{2x}{2}\right)\right)\right] \\ &= 1 - \cos(x) \end{aligned}$$

$$\begin{aligned} \int 2\sin^2\left(\frac{x}{2}\right) dx &= \int (1 - \cos(x)) dx \\ &= x - \sin(x) + c \end{aligned}$$

Integrating $\sin^{2n}(ax)$ or $\cos^{2n}(ax)$

The method used for $\sin^2(ax)$ and $\cos^2(ax)$ can be used whenever there is an even power through repeated application of the double-angle identities.

WORKED EXAMPLE 14

Determine $\int \cos^4(2x) dx$.

THINK

1. Rewrite $\cos^4(2x)$ as a power of $\cos^2(2x)$ so that the double-angle identity can be used.
2. $\cos^2(2x)$ can be rewritten using the substitution $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$ where $a = 2$.
3. Expand the integrand.

4. $\cos^2(4x)$ can be rewritten using the substitution $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$ where $a = 4$.

5. Express the question using integral notation and rewrite the integrand.

6. Integrate noting that $\frac{d}{dx} \sin(8x) = 8 \cos(8x)$ so

$$\int \cos(8x) dx = \frac{1}{8} \sin(8x) + c \text{ and}$$

$$\frac{d}{dx} \sin(4x) = 4 \cos(4x) \text{ so}$$

$$\int \cos(4x) dx = \frac{1}{4} \sin(4x) + c.$$

WRITE

$$\cos^4(2x) = [\cos^2(2x)]^2$$

$$= \left[\frac{1}{2} (\cos(4x) + 1) \right]^2$$

$$= \frac{1}{4} (1 + \cos(4x))^2$$

$$= \frac{1}{4} (\cos^2(4x) + 2 \cos(4x) + 1)$$

$$= \frac{1}{4} \left[\frac{1}{2} (\cos(8x) + 1) + 2 \cos(4x) + 1 \right]$$

$$= \frac{1}{4} \left[\frac{1}{2} \cos(8x) + \frac{1}{2} + 2 \cos(4x) + 1 \right]$$

$$= \frac{1}{4} \left[\frac{1}{2} \cos(8x) + 2 \cos(4x) + \frac{3}{2} \right]$$

$$\int \cos^4(2x) dx = \frac{1}{4} \int \left(\frac{1}{2} \cos(8x) + 2 \cos(4x) + \frac{3}{2} \right) dx$$

$$= \frac{1}{4} \left[\frac{1}{2} \times \frac{1}{8} \sin(8x) + 2 \times \frac{1}{4} \sin(4x) + \frac{3x}{2} \right] + c$$

$$= \frac{1}{64} \sin(8x) + \frac{1}{8} \sin(4x) + \frac{3x}{8} + c$$

6.4.3 Integrating $\sin^n(ax) \cos^n(ax)$, $n \geq 1$

When the powers of $\sin(ax)$ and $\cos(ax)$ are equal, then the substitution $\sin(ax) \cos(ax) = \frac{1}{2} \sin(2ax)$ can be useful. It sometimes necessary to then use the substitution $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$.

WORKED EXAMPLE 15

Determine the following indefinite integrals:

a. $\int \sin(x) \cos(x) \, dx$

b. $\int 8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) \, dx$

THINK

a. 1. $\sin(x) \cos(x)$ can be rewritten using
 $\sin(ax) \cos(ax) = \frac{1}{2} \sin(2ax)$ where $a = 1$.

2. Rewrite the integrand using the substitution.

3. Integrate noting that
 $\frac{d}{dx} \cos(2x) = -2 \sin(2x)$ so
 $\int \sin(2x) \, dx = -\frac{1}{2} \cos(2x) + c$.

b. 1. Rewrite $8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right)$ as a power
of $\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)$.

2. $\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)$ can be rewritten using
 $\sin(ax) \cos(ax) = \frac{1}{2} \sin(2ax)$ where $a = \frac{1}{2}$.

Make the substitution and simplify.

3. $\sin^2(x)$ can be rewritten using
 $\sin^2(ax) = \frac{1}{2} (1 - \cos(2ax))$ where $a = 1$.

4. Rewrite the integrand using the substitution.

5. Integrate noting that $\frac{d}{dx} \sin(2x) = 2 \cos(2x)$
so $\int \cos(2x) \, dx = \frac{1}{2} \sin(2x) + c$.

WRITE

a. Use $\sin(ax) \cos(ax) = \frac{1}{2} \sin(2ax)$

$$\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

$$\begin{aligned} \int \sin(x) \cos(x) \, dx &= \int \frac{1}{2} \sin(2x) \, dx \\ &= \frac{1}{2} \int \sin(2x) \, dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left(-\frac{1}{2} \cos(2x) \right) + c \\ &= -\frac{1}{4} \cos(2x) + c \end{aligned}$$

b. $8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) = 8 \left(\sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \right)^2$

$$= 8 \left(\frac{1}{2} \sin\left(\frac{2x}{2}\right) \right)^2$$

$$= 8 \times \frac{1}{4} \sin^2(x)$$

$$= 2 \sin^2(x)$$

$$= 2 \left[\frac{1}{2} (1 - \cos(2x)) \right]$$

$$= 1 - \cos(2x)$$

$$\int 8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) \, dx = \int (1 - \cos(2x)) \, dx$$

$$= x - \frac{1}{2} \sin(2x) + c$$

6.4.4 Integrating $\sin(ax) \cos^n(ax)$ or $\cos(ax) \sin^n(ax)$, $n > 1$

When the integrand involves $\sin(ax) \cos^n(ax)$ or $\cos(ax) \sin^n(ax)$, it is in the form $f(g(x)) g'(x)$ where $g(x) = \cos(ax)$ and $g(x) = \sin(ax)$, respectively. This means that a non-linear substitution of $u = g(x)$ can be used to find the integral.

WORKED EXAMPLE 16

Determine $\int \sin(3x) \cos^4(3x) dx$.

THINK

1. The integrand is in the form $f(g(x)) g'(x)$ where $g(x) = \cos(3x)$.
Let $u = \cos(3x)$ and use differentiation to determine dx .

2. Rewrite the integral, substituting for $\cos(3x)$ and dx .

3. Simplify the integrand, noting that $\frac{\sin(3x)}{\sin(3x)} = 1$ and $-\frac{1}{3}$ can be factored in front of the integral sign.

4. As the integrand is now expressed as a function of u only, it can be integrated.

5. As the original integral was written in terms of x , substitute $u = \cos(3x)$.

WRITE

Let $u = \cos(3x)$

$$\frac{du}{dx} = -3 \sin(3x)$$

$$du = -3 \sin(3x) dx$$

$$dx = \frac{du}{-3 \sin(3x)}$$

$$\begin{aligned} \int \sin(3x) \cos^4(3x) dx \\ = \int \sin(3x) u^4 \frac{du}{-3 \sin(3x)} \end{aligned}$$

$$= \int \frac{\sin(3x)}{-3 \sin(3x)} u^4 du$$

$$= -\frac{1}{3} \int u^4 du$$

$$= -\frac{1}{3} \left(\frac{u^5}{5} \right) + c$$

$$= -\frac{u^5}{15} + c$$

$$= -\frac{1}{15} \cos^5(3x) + c$$

6.4.5 Integrating odd powers of $\sin(ax)$ or $\cos(ax)$

When integrating $\sin^n(ax)$ or $\cos^n(ax)$ when n is odd, or $\sin^m(ax) \cos^n(ax)$ when either m or n is odd (or they both are), then the Pythagorean Identity $\sin^2(ax) + \cos^2(ax) = 1$ is used to rewrite the integrand so that it is in the form $\sin(ax) \cos^k(ax)$ or $\cos(ax) \sin^k(ax)$. It can then be integrated like the previous example.

WORKED EXAMPLE 17

Determine the anti-derivatives of the following:

a. $\cos^3(x)$

b. $\cos^4(2x) \sin^3(2x)$

THINK

- a. 1. Notice that $\cos^3(x) = \cos(x) \cos^2(x)$. The Pythagorean identity can be used to rewrite $\cos^2(x)$ as $1 - \sin^2(x)$.

2. Express the question using integral notation.

WRITE

a. $\cos^2(x) = 1 - \sin^2(x)$

$$\cos^3(x) = \cos(x) \cos^2(x)$$

$$= \cos(x) (1 - \sin^2(x))$$

$$\int \cos^3(x) dx = \int \cos(x) (1 - \sin^2(x)) dx$$



3. The integrand is in the form $f(g(x)) g'(x)$ where $g(x) = \sin(x)$.
Let $u = \sin(x)$ and use differentiation to determine dx .

4. Rewrite the integral, substituting for $\sin(x)$ and dx .

5. Simplify the integrand, noting that $\frac{\cos(x)}{\cos(x)} = 1$.

6. As the integrand is now expressed as a function of u only, it can be integrated.

7. As the original integral was written in terms of x , substitute $u = \sin(x)$.

- b. 1. There is an odd power for $\sin(2x)$. Notice $\sin^3(2x) = \sin(2x) \sin^2(2x)$. The Pythagorean Identity can be used to rewrite $\sin^2(2x)$ as $1 - \cos^2(2x)$.

2. Express the question using integral notation.

3. The integrand is in the form $f(g(x)) g'(x)$ where $g(x) = \cos(2x)$.
Let $u = \cos(2x)$ and use differentiation to determine dx .

4. Rewrite the integral, substituting for $\cos(2x)$ and dx .

5. Simplify the integrand, noting that $\frac{\sin(2x)}{\sin(2x)} = 1$ and $-\frac{1}{2}$ can be factored in front of the integral sign.

$$\text{Let } u = \sin(x)$$

$$\frac{du}{dx} = \cos(x)$$

$$du = \cos(x) dx$$

$$dx = \frac{du}{\cos(x)}$$

$$\int \cos(x) (1 - \sin^2(x)) dx$$

$$= \int \cos(x) (1 - u^2) \frac{du}{\cos(x)}$$

$$= \int \frac{\cos(x)}{\cos(x)} (1 - u^2) du$$

$$= \int (1 - u^2) du$$

$$= u - \frac{u^3}{3} + c$$

$$= \sin(x) - \frac{1}{3} \sin^3(x) + c$$

$$\text{b. } \sin^2(2x) = 1 - \cos^2(2x)$$

$$\cos^4(2x) \sin^3(2x)$$

$$= \cos^4(2x) \sin(2x) \sin^2(2x)$$

$$= \cos^4(2x) \sin(2x) (1 - \cos^2(2x))$$

$$= \sin(2x) (\cos^4(2x) - \cos^6(2x))$$

$$\int \cos^4(2x) \sin^3(2x) dx$$

$$= \int \sin(2x) (\cos^4(2x) - \cos^6(2x)) dx$$

$$\text{Let } u = \cos(2x)$$

$$\frac{du}{dx} = -2 \sin(2x)$$

$$du = -2 \sin(2x) dx$$

$$dx = \frac{du}{-2 \sin(2x)}$$

$$\int \sin(2x) (\cos^4(2x) - \cos^6(2x)) dx$$

$$= \int \sin(2x) (u^4 - u^6) \frac{du}{-2 \sin(2x)}$$

$$= \int \frac{\sin(2x)}{-2 \sin(2x)} (u^4 - u^6) du$$

$$= -\frac{1}{2} \int (u^4 - u^6) du$$

6. As the integrand is now expressed as a function of u only, it can be integrated.

$$= -\frac{1}{2} \left(\frac{u^5}{5} - \frac{u^7}{7} \right) + c$$

$$= -\frac{u^5}{10} + \frac{u^7}{14} + c$$

$$= \frac{u^7}{14} - \frac{u^5}{10} + c$$

7. As the original integral was written in terms of x , substitute $u = \cos(2x)$.

$$= \frac{1}{14} \cos^7(2x) - \frac{1}{10} \cos^5(2x) + c$$

Integrals involving odd powers

If there is an odd power of either $\sin(ax)$, $\cos(ax)$ (or both):

1. Rewrite the function as either $\sin^{2n+1}(ax) = \sin(ax) \sin^{2n}(ax)$ or $\cos^{2n+1}(ax) = \cos(ax) \cos^{2n}(ax)$. If both functions have odd powers, only rewrite one of them.
2. Use the Pythagorean identity to rewrite the even power. The integrand will now be in the form $\sin(ax) \cos^k(ax)$ or $\cos(ax) \sin^k(ax)$.
3. Use an appropriate substitution ($u = \cos(ax)$ or $u = \sin(ax)$) to evaluate.

6.4.6 Integrating products of $\sin(ax)$ and $\cos(bx)$

In the products investigated so far, the integrand has involved powers of $\sin(ax)$ and $\cos(ax)$. In order to be able to write the integrand in the form $f(g(x)) g'(x)$ it is necessary for the angles to be the same size. Therefore a different approach must be used when the angles are not the same size. In this case, it may be possible to use the double-angle identities or the product identities. The following worked example is completed both ways.

WORKED EXAMPLE 18

Determine $\int \cos(x) \sin(2x) dx$.

THINK

1. **Method 1** The integrand involves x and $2x$. Rewrite $\sin(2x)$ using $\sin(2A) = 2 \sin(A) \cos(A)$ where $A = x$
2. The integrand is in the form $f(g(x)) g'(x)$ where $g(x) = \cos(x)$. Let $u = \cos(x)$ and use differentiation to determine dx .
3. Rewrite the integral, substituting for $\cos(x)$ and dx .

WRITE

$$\sin(2A) = 2 \sin(A) \cos(A)$$

$$\begin{aligned} \int \cos(x) \sin(2x) dx &= \int \cos(x) 2 \sin(x) \cos(x) dx \\ &= \int 2 \cos^2(x) \sin(x) dx \end{aligned}$$

$$\text{Let } u = \cos(x)$$

$$\frac{du}{dx} = -\sin(x)$$

$$du = -\sin(x) dx$$

$$dx = \frac{du}{-\sin(x)}$$

$$\begin{aligned} &\int 2 \cos^2(x) \sin(x) dx \\ &= \int 2u^2 \sin(x) \frac{du}{-\sin(x)} \end{aligned}$$

4. Simplify the integrand, noting that $\frac{\sin(x)}{\sin(x)} = 1$ and -2 can be factored in front of the integral sign.

$$= \int \frac{2 \sin(x)}{-\sin(x)} u^2 du$$

$$= -2 \int u^2 du$$

$$= -2 \frac{u^3}{3} + c$$

$$= -\frac{2u^3}{3} + c$$

5. As the integrand is now expressed as a function of u only, it can be integrated.

$$= -\frac{2}{3} \cos^3(x) + c$$

6. As the original integral was written in terms of x , substitute $u = \cos(x)$.

1. **Method 2** The integrand involves x and $2x$. Rewrite $\cos(x) \sin(2x)$ using $\cos(A) \sin(B) = \frac{1}{2} (\sin(A+B) - \sin(A-B))$ where $A = x$ and $B = 2x$.

$$\cos(A) \sin(B) = \frac{1}{2} (\sin(A+B) - \sin(A-B))$$

$$\cos(x) \sin(2x) = \frac{1}{2} (\sin(x+2x) - \sin(x-2x))$$

$$= \frac{1}{2} (\sin(3x) - \sin(-x))$$

$$= \frac{1}{2} (\sin(3x) + \sin(x))$$

2. Note that $\sin(-x) = -\sin(x)$.

3. We know how to integrate each of those terms, so rewrite the integral.

$$\int \cos(x) \sin(2x) dx = \frac{1}{2} \int (\sin(3x) + \sin(x)) dx$$

4. Integrate noting that $\frac{d}{dx} \cos(x) = -\sin(x)$ so

$$= \frac{1}{2} \left(-\frac{1}{3} \cos(3x) - \cos(x) \right) + c$$

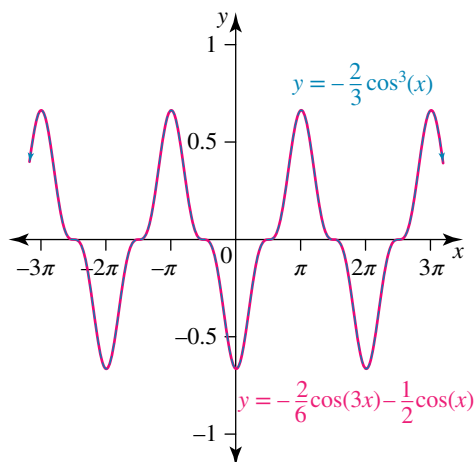
$$\int \sin(x) dx = -\cos(x) + c \text{ and}$$

$$\frac{d}{dx} \cos(3x) = -3 \sin(3x) \text{ so}$$

$$\int \sin(3x) dx = -\frac{1}{3} \cos(3x) + c.$$

$$= -\frac{1}{6} \cos(3x) - \frac{1}{2} \cos(x) + c$$

Note: It appears that the different methods have resulted in different results. However, it is possible to use the trigonometric identities to verify that the solutions are identical. It is also possible to graph both solutions to verify that they are the same. In both instances, the solutions may differ by a constant value, and both be valid solutions (the “+c” is used because there are a family of solutions that will all differentiate to give the original function, the only difference between the solutions is the constant value).



6.4.7 Integrating powers of $\tan(ax)$ or $\cot(ax)$

We know that $\frac{d}{dx} \tan(x) = \sec^2(x)$. This means that the Pythagorean Identity $\tan^2(ax) + 1 = \sec^2(ax)$ can be used to rewrite powers of $\tan(ax)$ into a function that we can integrate.

The quotient rule can be used to determine $\frac{d}{dx} \cot(x)$.

$$\begin{aligned}
 \frac{d}{dx} (\cot(x)) &= \frac{d}{dx} \left(\frac{\cos(x)}{\sin(x)} \right) \\
 &= \frac{d}{dx} \left(\frac{u}{v} \right) \\
 &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\
 &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
 &= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \quad \begin{array}{l} u = \cos(x) \\ \frac{du}{dx} = -\sin(x) \end{array} \\
 &= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} \\
 &= \frac{-1}{\sin^2(x)} \\
 &= -\operatorname{cosec}^2(x)
 \end{aligned}$$

Therefore,

The derivative of $\cot(x)$

$$\frac{d}{dx} \cot(x) = -\operatorname{cosec}^2(x)$$

The Pythagorean Identity $\cot^2(ax) + 1 = \operatorname{cosec}^2(ax)$ can be used to rewrite powers of $\cot(ax)$ into a function we can integrate.

WORKED EXAMPLE 19

Determine the anti-derivative of each of the following integrals:

a. $\tan^2(x)$

b. $3\cot^2(3x) \operatorname{cosec}^2(3x)$

THINK

- a. 1. We know that $\frac{d}{dx} \tan(x) = \sec^2(x)$ and $\tan^2(ax) + 1 = \sec^2(ax)$. Use the identity to rewrite $\tan^2(x)$ where $a = 1$.
2. Express the question using integral notation.
3. Rewrite the integrand using the substitution.

WRITE

a. $\tan^2(x) = \sec^2(x) - 1$

$$\begin{aligned}
 &\int \tan^2(x) \, dx \\
 &= \int (\sec^2(x) - 1) \, dx
 \end{aligned}$$

4. Integrate using $\int \sec^2(x) dx = \tan(x) + c$.

$$= \tan(x) - x + c$$

- b. 1. As $\frac{d}{dx} \cot(3x) = -3 \operatorname{cosec}^2(3x)$, the integrand is of the form $f(g(x)) g'(x)$ where $g(x) = \cot(3x)$. Let $u = \cot(3x)$ and use differentiation to determine dx .

b. Let $u = \cot(3x)$

$$\frac{du}{dx} = -3 \operatorname{cosec}^2(3x)$$

$$du = -3 \operatorname{cosec}^2(3x) dx$$

$$dx = \frac{du}{-3 \operatorname{cosec}^2(3x)}$$

2. Express the question using integral notation.

$$\int 3 \cot^2(3x) \operatorname{cosec}^2(3x) dx$$

3. Rewrite the integral, substituting for $\cot(3x)$ and dx .

$$\begin{aligned} & \int 3 \cot^2(3x) \operatorname{cosec}^2(3x) dx \\ &= \int 3 u^2 \operatorname{cosec}^2(3x) \frac{du}{-3 \operatorname{cosec}^2(3x)} \end{aligned}$$

4. Simplify the integrand, noting that

$$\frac{3 \operatorname{cosec}^2(3x)}{-3 \operatorname{cosec}^2(3x)} = -1.$$

$$= \int \frac{3 \operatorname{cosec}^2(3x)}{-3 \operatorname{cosec}^2(3x)} u^2 du$$

$$= \int -u^2 du$$

5. As the integrand is now expressed as a function of u only, it can be integrated.

$$= -\frac{u^3}{3} + c$$

6. As the original integral was written in terms of x , substitute $u = \cot(3x)$.

$$= -\frac{1}{3} \cot^3(3x) + c$$

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 3

Integration using the trigonometric identities Summary screen and practice questions

Exercise 6.4 Integration using the trigonometric identities

Technology free

1. Determine the following:

a. $\frac{d}{dx} \cos(2x)$

b. $\frac{d}{dx} \sin(4x)$

c. $\frac{d}{dx} \tan(3x)$

d. $\frac{d}{dx} \cot(5x)$

2. **WE13** Determine the anti-derivative of the following:

a. $\cos^2(x)$

b. $\sin^2(2x)$

c. $2 \cos^2(4x)$

d. $4 \sin^2(3x)$

e. $\cos^2(5x)$

f. $\sin^2(6x)$

3. Determine the following integrals:

a. $\int \cos^2\left(\frac{x}{2}\right) dx$

b. $\int \sin^2\left(\frac{x}{3}\right) dx$

c. $\int 3 \cos^2\left(\frac{x}{6}\right) dx$

d. $\int 2 \sin^2\left(\frac{x}{4}\right) dx$

e. $\int \cos^2\left(\frac{2x}{3}\right) dx$

f. $\int \sin^2\left(\frac{3x}{2}\right) dx$

4. **WE14** Determine the anti-derivative of the following:

a. $\cos^4(3x)$ b. $\cos^4\left(\frac{x}{2}\right)$ c. $\sin^4(x)$ d. $\sin^4(2x)$

5. **WE15a** Determine the following indefinite integrals:

a. $\int 2 \sin(x) \cos(x) dx$ b. $\int 4 \sin(2x) \cos(2x) dx$
c. $\int \sin(3x) \cos(3x) dx$ d. $\int -2 \sin(4x) \cos(4x) dx$

6. **WE15b** Determine the following indefinite integrals.

a. $\int \sin^2(x) \cos^2(x) dx$ b. $\int \sin^2(2x) \cos^2(2x) dx$ c. $\int 2 \sin^2(4x) \cos^2(4x) dx$
d. $\int 2 \sin^2(3x) \cos^2(3x) dx$ e. $\int 6 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) dx$ f. $\int 4 \sin^2\left(\frac{x}{3}\right) \cos^2\left(\frac{x}{3}\right) dx$

7. **WE16** Determine the following integrals:

a. $\int \sin(x) \cos^4(x) dx$ b. $\int \sin(2x) \cos^3(2x) dx$ c. $\int \sin\left(\frac{x}{2}\right) \cos^5\left(\frac{x}{2}\right) dx$
d. $\int \cos(3x) \sin^4(3x) dx$ e. $\int \cos\left(\frac{x}{5}\right) \sin^6\left(\frac{x}{5}\right) dx$ f. $\int \cos\left(\frac{2x}{3}\right) \sin^7\left(\frac{2x}{3}\right) dx$

8. **WE17a** Determine the anti-derivatives of the following:

a. $\sin^3(x)$ b. $\cos^3(2x)$ c. $6 \sin^3(4x)$
d. $4 \cos^3(3x)$ e. $\sin^3(7x)$ f. $\cos^3(6x)$

9. Determine the following:

a. $\int 3 \sin^3\left(\frac{x}{2}\right) dx$ b. $\int 2 \cos^3\left(\frac{x}{3}\right) dx$ c. $\int \sin^3\left(\frac{3x}{2}\right) dx$
d. $\int \cos^3\left(\frac{5x}{2}\right) dx$ e. $\int \sin^3\left(\frac{3x}{4}\right) dx$ f. $\int \cos^3\left(\frac{4x}{3}\right) dx$

10. **WE17b** Determine the anti-derivatives of the following:

a. $\cos^2(x) \sin^3(x)$ b. $\sin^2(x) \cos^3(x)$ c. $\cos^2(2x) \sin^3(2x)$
d. $\sin^2(3x) \cos^3(3x)$ e. $\cos^2\left(\frac{x}{2}\right) \sin^3\left(\frac{x}{2}\right)$ f. $\sin^2\left(\frac{3x}{2}\right) \cos^3\left(\frac{3x}{2}\right)$

11. Determine the following.

a. $\int 4 \cos^2\left(\frac{x}{3}\right) \sin^3\left(\frac{x}{3}\right) dx$ b. $\int -6 \sin^2\left(\frac{5x}{4}\right) \cos^3\left(\frac{5x}{4}\right) dx$
c. $\int \sin^3(x) \cos^4(x) dx$ d. $\int \cos^3(2x) \sin^4(2x) dx$
e. $\int 2 \sin^3(2x) \cos^5(2x) dx$ f. $\int -2 \cos^3(3x) \sin^6(3x) dx$

12. **WE18** Determine the following.

a. $\int \sin(x) \cos(2x) dx$ b. $\int \cos(2x) \cos(4x) dx$ c. $\int \sin(3x) \cos(6x) dx$
d. $\int \cos(4x) \cos(8x) dx$ e. $\int \sin\left(\frac{x}{2}\right) \cos(x) dx$ f. $\int \cos\left(\frac{x}{3}\right) \cos\left(\frac{2x}{3}\right) dx$

13. **WE19** Determine the anti-derivative of each of the following integrals:

a. $1 + \tan^2(2x)$ b. $1 + \tan^2\left(\frac{x}{3}\right)$ c. $\tan^2(x) \sec^2(x)$
d. $\tan^3(x) \sec^2(x)$ e. $4 \tan^5(2x) \sec^2(2x)$ f. $8 \tan^4\left(\frac{x}{2}\right) \sec^2\left(\frac{x}{2}\right)$

14. Determine the following integrals:

a. $\int \tan^2(x) \sec^4(x) dx$

b. $\int 6 \cot^2(2x) \operatorname{cosec}^4(2x) dx$

c. $\int 2 \tan^2\left(\frac{x}{2}\right) \sec^4\left(\frac{x}{2}\right) dx$

d. $\int 3 \cot^3(3x) \operatorname{cosec}^4(3x) dx$

e. $\int \tan^4\left(\frac{x}{5}\right) \sec^4\left(\frac{x}{5}\right) dx$

f. $\int 12 \cot^5(6x) \operatorname{cosec}^6(6x) dx$

15. If $f'(x) = 6 \sin(x) \cos^2(x)$ and $f\left(\frac{\pi}{3}\right) = 0$, determine $f(x)$.

16. If $f'(x) = 4 \sin^2(2x) \cos^2(2x)$ and $f\left(\frac{\pi}{4}\right) = \pi$, determine $f(x)$.

17. Determine $g(x)$ if $g'(x) = \sin^3\left(\frac{x}{2}\right) \cos^4\left(\frac{x}{2}\right)$ and $g(0) = -\frac{4}{35}$.

18. Determine the following integrals:

a. $\int \sin(x) \cos^n(x) dx$

b. $\int \cos(x) \sin^n(x) dx$

c. $\int \sec^2(x) \tan^n(x) dx$ for
 $n \in \mathbb{Z}$ with $n > 1$

d. $\int \sin^3(x) \cos^n(x) dx$

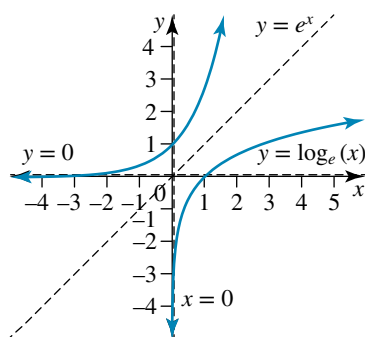
e. $\int \cos^3(x) \sin^n(x) dx$

f. $\int \sec^4(x) \tan^n(x) dx$ for
 $n \in \mathbb{Z}$ with $n > 1$

6.5 Integration of inverse trigonometric functions

6.5.1 Inverse functions

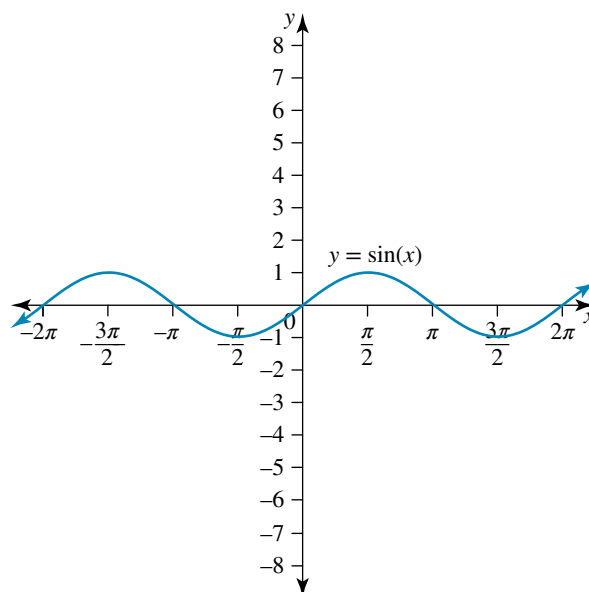
In Mathematical Methods you explored the inverse relationship between $y = a^x$ and $x = \log_a(y)$. If you graph $y = e^x$ and $y = \ln(x)$ you can see that they are reflections of each other in the line $y = x$. As both functions are one-to-one functions, the inverse is also a function.



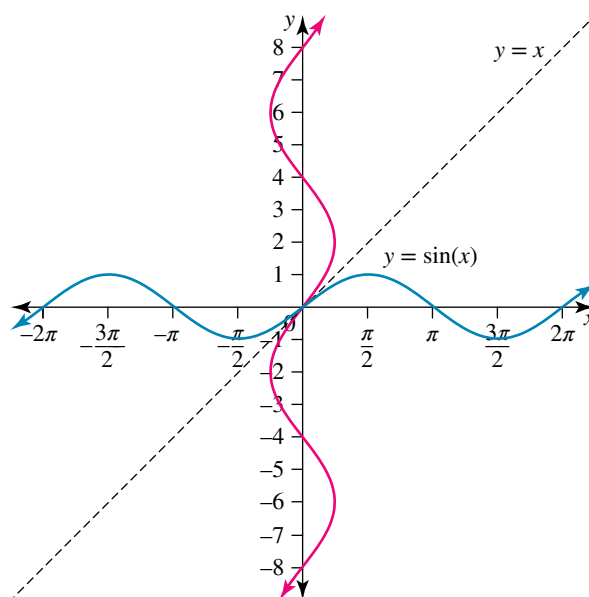
All circular functions are periodic functions. They are many-to-one functions meaning that if the domain of the circular function is not restricted to make it a one-to-one function, then the inverse of a trigonometric function is not a function.

The inverse sine function

The sine function $y = \sin(x)$, is a many-to-one function.

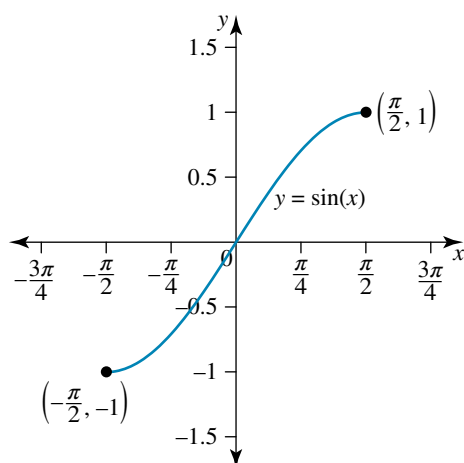


If it is reflected in the line $y = x$, the result is a one-to-many relation.

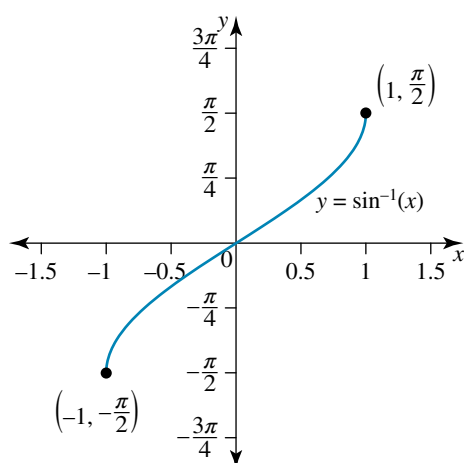


If we want the inverse to be a function, it is necessary to restrict the domain of the sine function.

Because the sine function is cyclic, there are an infinite number of possible domain restrictions, such as $\left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. The restriction $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is normally chosen. This will make the domain of the sine function $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range $[-1, 1]$.

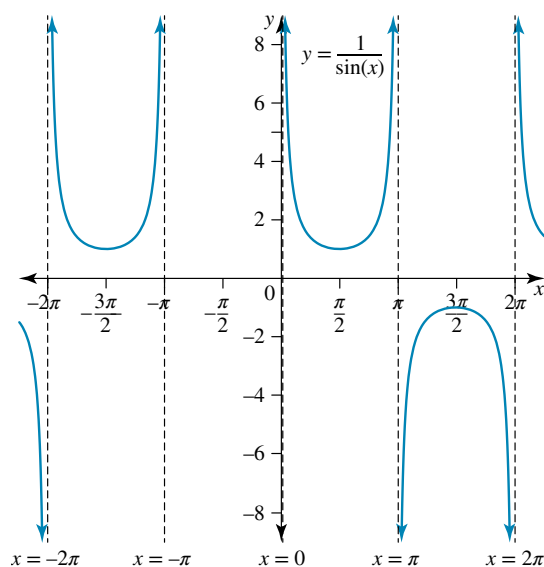


And the inverse function will be



The domain of this inverse function is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Note this is different to graphing the reciprocal function $y = \frac{1}{\sin(x)}$. The graph of $y = \frac{1}{\sin(x)}$ (which you studied in Unit 2 Topic 2) looks like:



The inverse function can be written as $y = \sin^{-1}(x)$ or $y = \arcsin(x)$. Be careful not to confuse $y = \sin^{-1}(x)$ and $y = \frac{1}{\sin(x)}$.

Although there are an infinite number of solutions to $\sin(x) = \frac{1}{2}$, there is only one solution to $x = \sin^{-1}\left(\frac{1}{2}\right)$ because $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 20

Calculate the following:

a. $\sin^{-1}(2)$

c. $\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

THINK

- a. 1. Write an equivalent statement.
2. The range of $\sin(x)$ is $[-1, 1]$ so there is no solution to $\sin(x) = 2$.

- b. 1. Write an equivalent statement.

2. This relationship is from a special triangle.

3. $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so there will be only one solution to

$$x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right).$$

- c. 1. Let $x = \sin^{-1}\left(\frac{1}{3}\right)$ and write an equivalent statement.

2. Substitute $x = \sin^{-1}\left(\frac{1}{3}\right)$ into $\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

3. We know that $\sin(x) = \frac{1}{3}$.

Note: as $\sin(x)$ and $\sin^{-1}(x)$ are inverse functions, this question could have solved by realising that $\sin(\sin^{-1}(x)) = x$ if $x \in [-1, 1]$

b. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

d. $\sin^{-1}\left(\sin\left(\frac{13\pi}{12}\right)\right)$

WRITE

- a. Let $x = \sin^{-1}(2)$
 $\sin(x) = 2$
 As $2 \notin [-1, 1]$ there is no solution to $\sin(x) = 2$.

b. Let $x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
 $\sin(x) = \frac{\sqrt{3}}{2}$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so the only solution is
 $x = \frac{\pi}{3}.$

c. Let $x = \sin^{-1}\left(\frac{1}{3}\right)$

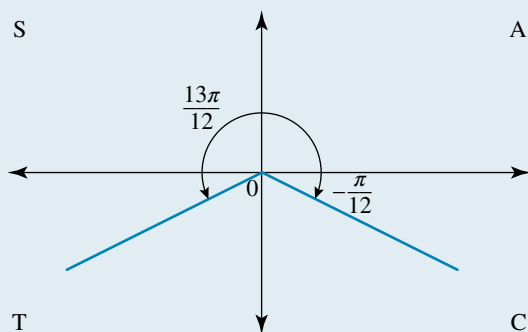
$$\sin(x) = \frac{1}{3}$$

$$\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \sin(x)$$

$$= \frac{1}{3}$$

d. 1. Write an equivalent statement.

2. As $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $x \neq \frac{13\pi}{12}$.
 $\frac{13\pi}{12}$ is in quadrant 3.



d. Let $x = \sin^{-1} \left(\sin \left(\frac{13\pi}{12} \right) \right)$

$$\sin(x) = \sin \left(\frac{13\pi}{12} \right)$$

$\frac{13\pi}{12}$ is in quadrant 3.

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\begin{aligned} \sin \left(\frac{13\pi}{12} \right) &= -\sin \left(\frac{\pi}{12} \right) \\ &= \sin \left(-\frac{\pi}{12} \right) \end{aligned}$$

So the statement

$$\sin(x) = \sin \left(\frac{13\pi}{12} \right) \text{ becomes,}$$

$$\sin(x) = \sin \left(-\frac{\pi}{12} \right).$$

$$\text{Therefore } x = -\frac{\pi}{12}.$$

Properties of the inverse sine function

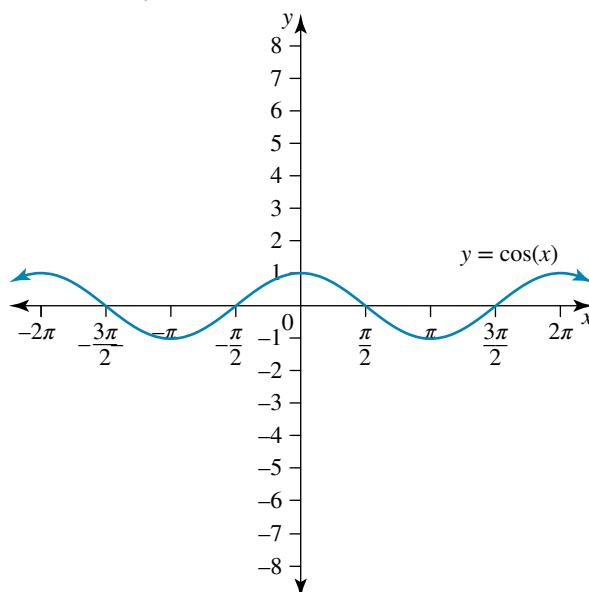
For the inverse sine function $f: [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $f(x) = \sin^{-1}(x)$,

$$\sin(\sin^{-1}(x)) = x, \quad x \in [-1, 1]$$

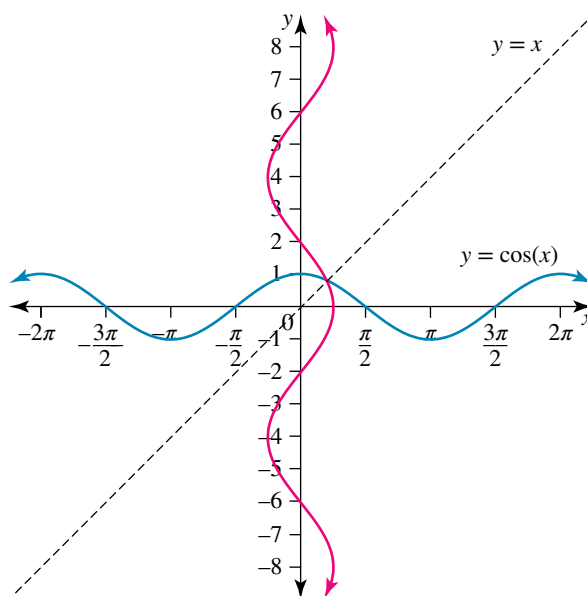
$$\sin^{-1}(\sin(x)) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

The inverse cosine function

The cosine function $y = \cos(x)$, is a many-to-one function.

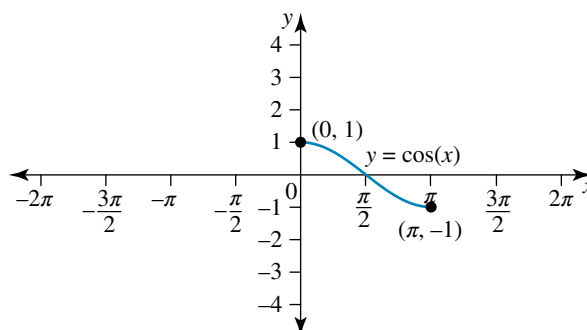


If it is reflected in the line $y = x$, the result is a one-to-many relation.

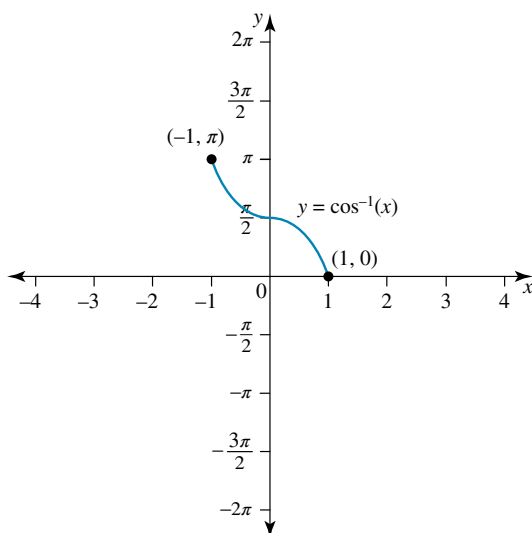


If we want the inverse to be a function, it is necessary to restrict the domain of the cosine function.

Because the cosine function is cyclic, there are an infinite number of possible domain restrictions, such as $[-\pi, 0]$, $[0, \pi]$ or $[\pi, 2\pi]$. The restriction $[0, \pi]$ is normally chosen. This will make the domain of the cosine function $[0, \pi]$ and the range $[-1, 1]$.



And the inverse function will be:



The domain of this inverse function is $[-1, 1]$ and the range is $[0, \pi]$.

Note this is different to graphing the reciprocal function $y = \frac{1}{\cos(x)}$ (which you studied in Unit 2 Topic 2).

The inverse function can be written as $y = \cos^{-1}(x)$ or $y = \arccos(x)$.

Although there are an infinite number of solutions to $\cos(x) = \frac{\sqrt{2}}{2}$, there is only one solution to $x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ because $x \in [0, \pi]$, so $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 21

Calculate the following:

a. $\cos^{-1}\left(\frac{3}{2}\right)$

b. $\cos^{-1}\left(\cos\left(-\frac{\pi}{5}\right)\right)$

THINK

a. 1. Write an equivalent statement

2. The range of $\cos(x)$ is $[-1, 1]$ so there is no solution to $\cos^{-1}\left(\frac{3}{2}\right)$.

b. 1. Write an equivalent statement.

2. As $x \in [0, \pi]$ $x \neq -\frac{\pi}{5}$.
 $-\frac{\pi}{5}$ is in quadrant 4.

WRITE

a. Let $x = \cos^{-1}\left(\frac{3}{2}\right)$

$$\cos(x) = \frac{3}{2}$$

As $\frac{3}{2} \notin [-1, 1]$ there is no solution to $\cos(x) = \frac{3}{2}$.

b. Let $x = \cos^{-1}\left(\cos\left(-\frac{\pi}{5}\right)\right)$

$$\cos(x) = \cos\left(-\frac{\pi}{5}\right)$$

$-\frac{\pi}{5}$ is in quadrant 4.

$$x \in [0, \pi]$$

$$\cos\left(-\frac{\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right)$$

Therefore $x = \frac{\pi}{5}$.

Properties of the inverse cosine function

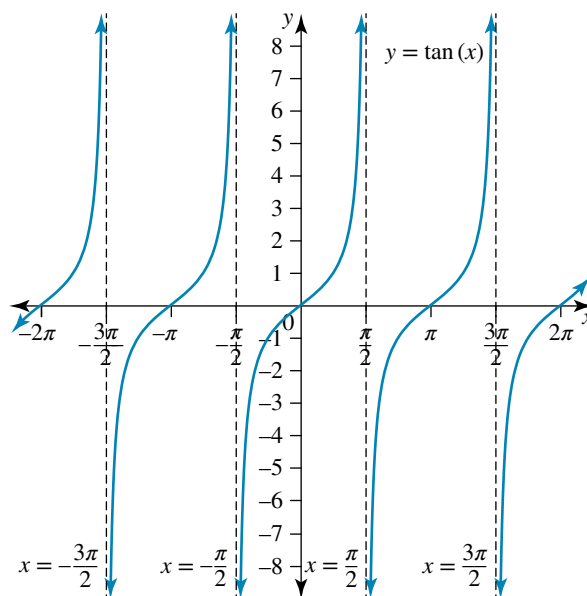
For the inverse cosine function $f: [-1, 1] \rightarrow [0, \pi]$, $f(x) = \cos^{-1}(x)$,

$$\cos(\cos^{-1}(x)) = x, \quad x \in [-1, 1]$$

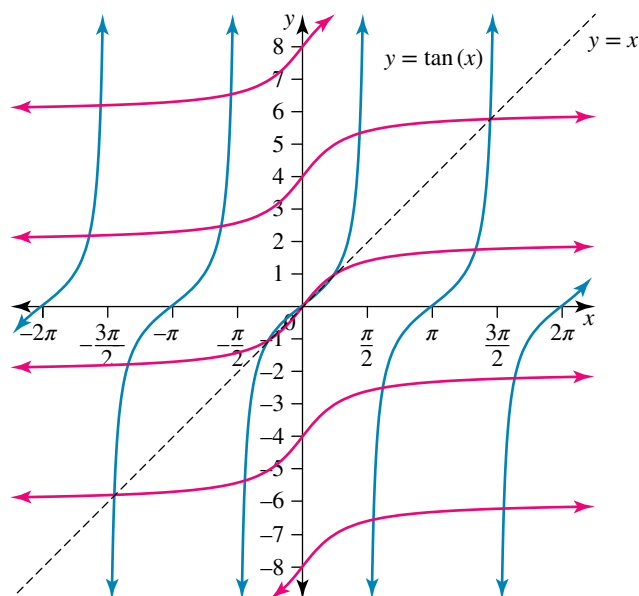
$$\cos^{-1}(\cos(x)) = x, \quad x \in [0, \pi]$$

The inverse tangent function

The tangent function $y = \tan(x)$, is a many-to-one function.

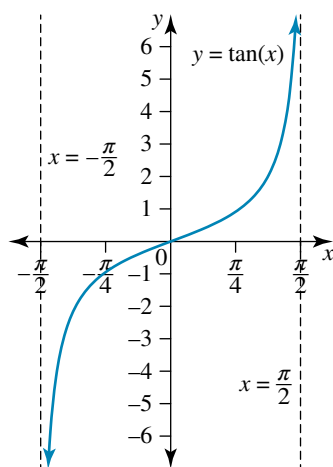


If it is reflected in the line $y = x$, the result is a one-to-many relation.

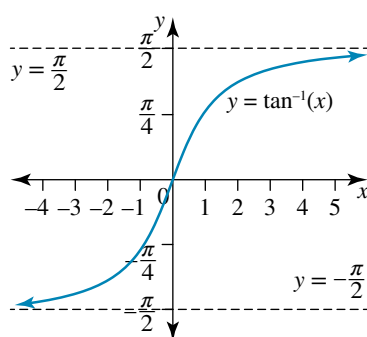


If we want the inverse to be a function, it is necessary to restrict the domain of the tangent function.

Because the tangent function is cyclic, there are an infinite number of possible domain restrictions, such as $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. The restriction $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is normally chosen. This will make the domain of the tangent function $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the range R .



And the inverse function will be:



The domain of this inverse function is $\mathbb{R}$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The inverse function can be written as $y = \tan^{-1}(x)$ or $y = \arctan(x)$. Be careful not to confuse $y = \tan^{-1}(x)$ and $y = \frac{1}{\tan(x)}$.

Although there are an infinite number of solutions to $\tan(x) = 1$, there is only one solution to $x = \tan^{-1}(1)$ because $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so $\tan^{-1}(1) = \frac{\pi}{4}$. Note this is also the solution that your calculator returns.

WORKED EXAMPLE 22

Calculate the following:

- a. $\tan(\tan^{-1}(2))$ b. $\tan^{-1}\left(\tan\left(\frac{13\pi}{12}\right)\right)$

THINK

a. 1. Let $x = \tan^{-1}(2)$ and write an equivalent statement.

2. Substitute $x = \tan^{-1}(2)$ in $\tan(\tan^{-1}(2))$.

WRITE

a. Let $x = \tan^{-1}(2)$

$$\tan(x) = 2$$

$$\tan(\tan^{-1}(2)) = \tan(x)$$

3. We know that $\tan(x) = 2$.

$$= 2$$

Note: as $\tan(x)$ and $\tan^{-1}(x)$ are inverse functions, this question could have solved by realising that $\tan(\tan^{-1}(x)) = x$ if $x \in R$.

b. 1. Write an equivalent statement.

$$\text{b. Let } x = \tan^{-1}\left(\tan\left(\frac{13\pi}{12}\right)\right)$$

$$\tan(x) = \tan\left(\frac{13\pi}{12}\right)$$

$\frac{13\pi}{12}$ is in quadrant 3.

$$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\tan\left(\frac{13\pi}{12}\right) = \tan\left(\frac{\pi}{12}\right)$$

$$\text{Therefore } x = \frac{\pi}{12}.$$

2. As $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) x \neq \frac{13\pi}{12}$.

$\frac{13\pi}{12}$ is in quadrant 3.

Properties of the inverse tangent function

For the inverse cosine function $f: R \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $f(x) = \tan^{-1}(x)$,

$$\tan(\tan^{-1}(x)) = x, \quad x \in R$$

$$\tan^{-1}(\tan(x)) = x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

6.5.5 Derivatives of the inverse trigonometric functions

It is possible to determine the derivative of the inverse trigonometric functions. In order to determine these derivatives, it is necessary to use the Pythagorean Identities. Your formula sheet lists the formulas as

Derivatives involving inverse trigonometric functions

$$\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad x \in (-a, a)$$

$$\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}, \quad x \in (-a, a)$$

$$\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}, \quad x \in R$$

The inverse sine function

Let $y = \sin^{-1}(x)$. We know that $x \in [-1, 1]$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Rearranging: $x = \sin(y)$

Differentiating: $\frac{dx}{dy} = \cos(y)$

Taking reciprocals: $\frac{dy}{dx} = \frac{1}{\cos(y)}$

The Pythagorean Identity states that $\cos^2(y) + \sin^2(y) = 1$

$$\cos^2(y) = 1 - \sin^2(y)$$

$$\begin{aligned}\text{As } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \cos(y) \in [0, 1], \text{ as } x = \sin(y), \cos(y) &= \sqrt{1 - \sin^2(y)} \\ &= \sqrt{1 - x^2}\end{aligned}$$

$$\begin{aligned}\text{Therefore } \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ &= \frac{1}{\sqrt{1 - x^2}}\end{aligned}$$

We have demonstrated that $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$

We can use the chain rule to determine $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right)$

$$\begin{aligned}\text{Let } y &= \sin^{-1}\left(\frac{x}{a}\right) \text{ and } u = \frac{x}{a} \\ \Rightarrow y &= \sin^{-1}(u) \text{ with } \frac{du}{dx} = \frac{1}{a}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{\sqrt{1 - u^2}} \times \frac{1}{a}\end{aligned}$$

Replacing $u = \frac{x}{a}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{a\sqrt{1 - \left(\frac{x}{a}\right)^2}} \\ &= \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a\sqrt{\frac{a^2}{a^2} - \frac{x^2}{a^2}}} \\
&= \frac{1}{a\sqrt{\frac{a^2 - x^2}{a^2}}} \\
&= \frac{1}{a \times \frac{1}{a}\sqrt{a^2 - x^2}} \\
&= \frac{1}{\sqrt{a^2 - x^2}}
\end{aligned}$$

Therefore $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}.$

WORKED EXAMPLE 23

Determine $\frac{d}{dx} \sin^{-1}(3x).$

THINK

1. We know that $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$ so

$$\frac{x}{a} = 3x$$

$$a = \frac{1}{3}$$

2. Use the formula to write $\frac{d}{dx} \sin^{-1}(3x).$

3. Simplify. Note that $\sqrt{\frac{1}{9}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}.$

WRITE

$$\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad a = \frac{1}{3}$$

$$\begin{aligned}
\frac{d}{dx} \sin^{-1}(3x) &= \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} \\
&= \frac{1}{\sqrt{\frac{1}{9} - x^2}} \\
&= \frac{1}{\sqrt{\frac{1}{9} - \frac{9x^2}{9}}} \\
&= \frac{1}{\sqrt{\frac{1}{9}(1 - 9x^2)}} \\
&= \frac{1}{\frac{1}{3}\sqrt{1 - 9x^2}} \\
&= \frac{3}{\sqrt{1 - 9x^2}}
\end{aligned}$$

The inverse cosine function

Let $y = \cos^{-1}(x)$. We know that $x \in [-1, 1]$ and $y \in [0, \pi]$

Rearranging: $x = \cos(y)$

Differentiating: $\frac{dx}{dy} = -\sin(y)$

Taking reciprocals: $\frac{dy}{dx} = \frac{-1}{\sin(y)}$

The Pythagorean Identity states that $\cos^2(y) + \sin^2(y) = 1$

$$\sin^2(y) = 1 - \cos^2(y)$$

As $y \in [0, \pi]$, $\sin(y) \in [0, 1]$, as $x = \cos(y)$, $\sin(y) = \sqrt{1 - \cos^2(y)}$
 $= \sqrt{1 - x^2}$

Therefore $\frac{dy}{dx} = \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1 - x^2}}$

We have demonstrated that $\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}$.

It is possible to use the chain rule to demonstrate that $\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$ using a similar process shown earlier for the inverse sine function. You may wish to try this yourself.

WORKED EXAMPLE 24

Determine the derivative of $\cos^{-1}\left(\frac{2x}{3}\right)$.

THINK

1. We know that $\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$ so

$$\frac{x}{a} = \frac{2x}{3}, a = \frac{3}{2}$$

2. Use the formula to write $\frac{d}{dx} \cos^{-1}\left(\frac{2x}{3}\right)$.

3. Simplify. Note that $\sqrt{\frac{1}{4}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$.

WRITE

$$\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}, a = \frac{3}{2}$$

$$\begin{aligned} \frac{d}{dx} \cos^{-1}\left(\frac{2x}{3}\right) &= \frac{-1}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} \\ &= \frac{-1}{\sqrt{\frac{9}{4} - x^2}} \\ &= \frac{-1}{\sqrt{\frac{9}{4} - \frac{4x^2}{4}}} \\ &= \frac{-1}{\sqrt{\frac{1}{4}(9 - 4x^2)}} \\ &= \frac{-1}{\frac{1}{2}\sqrt{9 - 4x^2}} \\ &= \frac{-2}{\sqrt{9 - 4x^2}} \end{aligned}$$

The inverse tangent function

Let $y = \tan^{-1}(x)$. We know that $x \in \mathbb{R}$ and $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Rearranging: $x = \tan(y)$

Differentiating: $\frac{dx}{dy} = \sec^2(y)$

Taking reciprocals: $\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

The Pythagorean Identity states that $\sec^2(y) = 1 + \tan^2(y)$

As $x = \tan(y)$, $\sec^2(y) = 1 + x^2$

Therefore $\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1 + x^2}$

We have demonstrated that $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}$

It is possible to use the chain rule to demonstrate that $\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$ using a similar process shown earlier for the inverse sine function. You may wish to try this yourself.

WORKED EXAMPLE 25

If $y = \tan^{-1}(2x)$, determine $\frac{dy}{dx}$.

THINK

1. We know that $\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$ so $\frac{x}{a} = 2x$
 $a = \frac{1}{2}$

2. Use the formula to find $\frac{dy}{dx}$.

3. Simplify. Note that in this instance you can multiply by $\frac{4}{4}$ to assist with simplifying the fraction in the denominator.

WRITE

$$\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}, a = \frac{1}{2}$$

$$y = \tan(2x)$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2 + x^2}$$

$$\begin{aligned} &= \frac{\frac{1}{2}}{\frac{1}{4} + x^2} \\ &= \frac{\frac{1}{2}}{\frac{1}{4} + x^2} \times \frac{4}{4} \\ &= \frac{2}{1 + 4x^2} \end{aligned}$$

6.5.6 Integration using the inverse trigonometric functions

We have established that

$$\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx} \cos^{-1} \left(\frac{x}{a} \right) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx} \tan^{-1} \left(\frac{x}{a} \right) = \frac{a}{a^2 + x^2}.$$

From these results, the following can be stated:

Anti-derivatives involving inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c, \quad x \in (-a, a)$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1} \left(\frac{x}{a} \right) + c, \quad x \in (-a, a)$$

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c, \quad x \in R$$

Integrals involving the inverse sine function

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$ can be used to determine integrals. Notice that the coefficient of x^2 is 1, so it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 26

Determine the following integrals:

a. $\int \frac{12}{\sqrt{36 - x^2}} dx$

b. $\int \frac{4}{\sqrt{49 - 36x^2}} dx$

THINK

a. Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$
with $a = 6$.

b. 1. As we have $36x^2$, the substitution $u = 6x$ could be used so that the denominator becomes $\sqrt{49 - u^2}$. This means that the function is in the form $f(g(x)) g'(x)$ where $g(x) = 6x$. Let $u = 6x$ and differentiate to determine dx .

WRITE

a. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c, \quad a = 6$
 $\int \frac{12}{\sqrt{36 - x^2}} dx = 12 \int \frac{1}{\sqrt{6^2 - x^2}} dx$
 $= 12 \sin^{-1} \left(\frac{x}{6} \right) + c$

b. Let $u = 6x$
 $\frac{du}{dx} = 6$
 $du = 6 dx$
 $dx = \frac{du}{6}$

2. Rewrite the integral, substituting for $6x$ and dx .

$$\begin{aligned}\int \frac{4}{\sqrt{49-36x^2}} dx &= \int \frac{4}{\sqrt{49-(6x)^2}} dx \\ &= \int \frac{4}{\sqrt{49-(u)^2}} \frac{du}{6}\end{aligned}$$

3. Use $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ with $a = 7$.

$$\begin{aligned}&= \frac{4}{6} \int \frac{1}{\sqrt{7^2-u^2}} du \\ &= \frac{2}{3} \sin^{-1}\left(\frac{u}{7}\right) + c\end{aligned}$$

4. Substitute for u so that the final answer is in terms of x only.

$$= \frac{2}{3} \sin^{-1}\left(\frac{6x}{7}\right) + c$$

Integrals involving the inverse cosine function

$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ can also be used to determine integrals. Notice that the coefficient of x^2 is 1, so, like using the previous formula, it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 27

The gradient of a curve that passes through the origin is given by $\frac{-4}{\sqrt{81-25x^2}}$. Determine the equation of the curve.

THINK

1. If the original curve is y , then the gradient is $\frac{dy}{dx}$.
2. Integrate both sides to obtain an expression for y .
3. As we have $25x^2$, the substitution $u = 5x$ could be used so that the denominator becomes $\sqrt{81-u^2}$. This means that the function is in the form $f(g(x)) g'(x)$ where $g(x) = 5x$. Let $u = 5x$ and differentiate to determine dx .

4. Rewrite the integral, substituting for $5x$ and dx .

5. Use $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$ with $a = 9$.

WRITE

$$\begin{aligned}\frac{dy}{dx} &= \frac{-4}{\sqrt{81-25x^2}} \\ y &= \int \frac{-4}{\sqrt{81-25x^2}} dx\end{aligned}$$

Let $u = 5x$

$$\frac{du}{dx} = 5$$

$$du = 5 dx$$

$$dx = \frac{du}{5}$$

$$\begin{aligned}\int \frac{-4}{\sqrt{81-25x^2}} dx &= \int \frac{-4}{\sqrt{81-(5x)^2}} dx \\ &= \int \frac{-4}{\sqrt{81-(u)^2}} \frac{du}{5}\end{aligned}$$

$$\begin{aligned}&= \frac{4}{5} \int \frac{-1}{\sqrt{9^2-u^2}} du \\ &= \frac{4}{5} \cos^{-1}\left(\frac{u}{9}\right) + c\end{aligned}$$

6. Substitute for u so that the final answer is in terms of x only.

7. As the curve passes through the origin, c can be determined by substituting $x = 0$ and $y = 0$.

8. Substitute for c and state the equation for y .

$$y = \frac{4}{5} \cos^{-1} \left(\frac{5x}{9} \right) + c$$

$$0 = \frac{4}{5} \cos^{-1} (0) + c$$

$$0 = \frac{4}{5} \times \frac{\pi}{2} + c$$

$$c = -\frac{2\pi}{5}$$

$$y = \frac{4}{5} \cos^{-1} \left(\frac{5x}{9} \right) - \frac{2\pi}{5}$$

Integrals involving the inverse tangent function

$\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c$ can also be used to determine integrals. Notice that the coefficient of x^2 is 1, so, like using the previous formulas, it might be necessary to use an appropriate substitution in order to use the formula.

WORKED EXAMPLE 28

Determine the anti-derivatives of the following:

a. $\frac{8}{4 + x^2}$

b. $\frac{4}{16 + 9x^2}$

THINK

a. 1. Express the question using integral notation.

2. Use $\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c$ with $a = 2$.

b. 1. Express the question using integral notation

2. As we have $36x^2$, the substitution $u = 6x$ could be used so that the denominator becomes $49 + u^2$. This means that the function is in the form $f(g(x)) g'(x)$ where $g(x) = 6x$. Let $u = 6x$ and differentiate to determine dx .

3. Rewrite the integral, substituting for $6x$ and dx .

WRITE

a. $\int \frac{8}{4 + x^2} dx$

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c, \quad a = 2$$

$$\begin{aligned} \int \frac{8}{4 + x^2} dx &= \int \frac{4 \times 2}{2^2 + x^2} dx \\ &= 4 \tan^{-1} \left(\frac{x}{2} \right) + c \end{aligned}$$

b. $\int \frac{4}{16 + 9x^2} dx$

Let $u = 3x$

$$\frac{du}{dx} = 3$$

$$du = 3 dx$$

$$dx = \frac{du}{3}$$

$$\begin{aligned} \int \frac{4}{16 + 9x^2} dx &= \int \frac{4}{16 + (3x)^2} dx \\ &= \int \frac{4}{16 + (u)^2} \frac{du}{3} \end{aligned}$$

4. Use $\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c$ with $a = 4$.

$$= \frac{1}{3} \int \frac{4}{4^2 + u^2} du$$

$$= \frac{1}{3} \tan^{-1} \left(\frac{u}{4} \right) + c$$

5. Substitute for u so that the final answer is in terms of x only.

$$= \frac{1}{3} \tan^{-1} \left(\frac{3x}{4} \right) + c$$

Integrals involving completing the square

A quadratic expression, $ax^2 + bx + c$ can be expressed, by completing the square, in the form, $(px + q)^2 + k$.

Recognition of this can then be used to integrate expressions of the form $\frac{1}{ax^2 + bx + c}$ with $a > 0$ and

$\Delta = b^2 - 4ac < 0$, which will then involve the inverse tangent function.

Integrating expressions of the form $\frac{1}{\sqrt{ax^2 + bx + c}}$ with $a < 0$ and $\Delta = b^2 - 4ac > 0$ will involve the inverse sine or cosine functions.

WORKED EXAMPLE 29

Determine the following integrals:

a. $\int \frac{1}{9x^2 + 12x + 29} dx$

b. $\int \frac{1}{\sqrt{21 - 12x - 9x^2}} dx$

THINK

- a. 1. Use completing the square to express the denominator in the form $(px + q)^2 + k$.
2. Write the denominator as the sum of two squares.
3. This means that the function is in the form $f(g(x)) g'(x)$ where $g(x) = 3x + 2$. Let $u = 3x + 2$ and differentiate to determine dx .
4. Rewrite the integral, substituting for $3x + 2$ and dx .

WRITE

$$\begin{aligned} \text{a. } & 9x^2 + 12x + 29 \\ &= (3x)^2 + 2(3x)2 + 2^2 + 25 \\ &= (3x + 2)^2 + 25 \\ & \int \frac{1}{9x^2 + 12x + 29} dx = \int \frac{1}{(3x + 2)^2 + 25} dx \end{aligned}$$

$$\text{Let } u = 3x + 2$$

$$\frac{du}{dx} = 3$$

$$du = 3 dx$$

$$dx = \frac{du}{3}$$

$$\int \frac{1}{(3x+2)^2 + 25} dx = \int \frac{1}{(u)^2 + 25} \frac{du}{3}$$



5. Use $\int \frac{a}{a^2 + x^2} dx = \tan^{-1} \left(\frac{x}{a} \right) + c$ with $a = 5$. Note that we need the numerator to be 5 to be able to use the formula.

6. Substitute for u so that the final answer is in terms of x only.

- b. 1. Use completing the square to express the denominator in the form $(px + q)^2 + k$.
2. Write the denominator as the difference of two squares within the square root.
3. This means that the function is in the form $f(g(x)) g'(x)$ where $g(x) = 3x + 2$. Let $u = 3x + 2$ and differentiate to determine dx .

4. Rewrite the integral, substituting for $3x + 2$ and dx .

5. Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$ with $a = 5$.

6. Substitute for u so that the final answer is in terms of x only.

$$\begin{aligned} &= \frac{1}{3} \int \frac{1}{5^2 + u^2} du \\ &= \frac{1}{3} \times \frac{1}{5} \int \frac{5}{5^2 + u^2} du \\ &= \frac{1}{15} \tan^{-1} \left(\frac{u}{5} \right) + c \\ &= \frac{1}{15} \tan^{-1} \left(\frac{3x + 2}{5} \right) + c \end{aligned}$$

$$\begin{aligned} \text{b. } 21 - 12x - 9x^2 &= 21 - (9x^2 + 12x) \\ &= 21 - ((3x)^2 + 2(3x)2 + 2^2) + 4 \\ &= 25 - (3x + 2)^2 \\ \int \frac{1}{\sqrt{21 - 12x - 9x^2}} dx &= \int \frac{1}{\sqrt{25 - (3x + 2)^2}} dx \end{aligned}$$

$$\text{Let } u = 3x + 2$$

$$\frac{du}{dx} = 3$$

$$du = 3 dx$$

$$dx = \frac{du}{3}$$

$$\begin{aligned} \int \frac{1}{\sqrt{25 - (3x+2)^2}} dx &= \int \frac{1}{\sqrt{25 - (u)^2}} \frac{du}{3} \\ &= \frac{1}{3} \int \frac{1}{\sqrt{5^2 - u^2}} du \\ &= \frac{1}{3} \sin^{-1} \left(\frac{u}{5} \right) + c \\ &= \frac{1}{3} \sin^{-1} \left(\frac{3x + 2}{5} \right) + c \end{aligned}$$

Integrals involving substitutions and inverse trigonometric functions

Using the properties of indefinite integrals, $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$ we can break up complicated integrals, into two manageable integrals.

WORKED EXAMPLE 30

Determine $\int \frac{4x + 5}{\sqrt{25 - 16x^2}} dx$.

THINK

1. $\int \frac{a}{\sqrt{25-16x^2}} dx$ can be integrated using the inverse sine function.
 $\int \frac{bx}{\sqrt{25-16x^2}} dx$ can be integrated using a non-linear substitution. Therefore to integrate $\int \frac{4x+5}{\sqrt{25-16x^2}} dx$ express the integral as the sum of two integrals.
2. Consider the first integral. The integrand is in the form $f(g(x)) g'(x)$ where $g(x) = 25 - 16x^2$. Let $u = 25 - 16x^2$ and differentiate to determine dx .
3. Rewrite the integral, substituting for $25 - 16x^2$ and dx .
4. Simplify the integral noting that $\frac{4x}{32x} = \frac{1}{8}$.
5. Write the integrand using power notation and complete the integration.
6. Substitute for u so that the final answer is in terms of x only.
7. Consider the second integral. This can be integrated using $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ if the substitution $u = 4x$ is used. Let $u = 4x$ and differentiate to determine dx .
8. Rewrite the integral, substituting for $4x$ and dx .

WRITE

$$\begin{aligned}\text{Let } I &= \int \frac{4x+5}{\sqrt{25-16x^2}} dx \\ &= \int \frac{4x}{\sqrt{25-16x^2}} dx + \int \frac{5}{\sqrt{25-16x^2}} dx\end{aligned}$$

$$\text{Let } u = 25 - 16x^2$$

$$\frac{du}{dx} = -32x$$

$$du = -32x dx$$

$$dx = -\frac{du}{32x}$$

$$\int \frac{4x}{\sqrt{25-16x^2}} dx = \int \frac{4x}{\sqrt{u}} \left(-\frac{du}{32x}\right)$$

$$= \int -\frac{4x}{32x} \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{8} \int \frac{1}{\sqrt{u}} du$$

$$= -\frac{1}{8} \int u^{-\frac{1}{2}} du$$

$$= -\frac{1}{8} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= -\frac{1}{4} u^{\frac{1}{2}} + c$$

$$= -\frac{1}{4} \sqrt{u} + c$$

$$\int \frac{4x}{\sqrt{25-16x^2}} dx = -\frac{1}{4} \sqrt{25-16x^2} + c$$

$$\text{Let } u = 4x$$

$$\frac{du}{dx} = 4$$

$$du = 4 dx$$

$$dx = \frac{du}{4}$$

$$\begin{aligned}\int \frac{5}{\sqrt{25-16x^2}} dx &= \int \frac{5}{\sqrt{25-(4x)^2}} dx \\ &= \int \frac{5}{\sqrt{25-(u)^2}} \frac{du}{4}\end{aligned}$$

9. Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$ with $a = 5$.
$$= \frac{5}{4} \int \frac{1}{\sqrt{5^2 - u^2}} du$$

$$= \frac{5}{4} \sin^{-1} \left(\frac{u}{5} \right) + c$$
10. Substitute for u so that the final answer is in terms of x only.
$$\int \frac{5}{\sqrt{25 - 16x^2}} dx = \frac{5}{4} \sin^{-1} \left(\frac{4x}{5} \right) + c$$
11. As we have now determined $\int \frac{4x}{\sqrt{25 - 16x^2}} dx$ and
$$I = \int \frac{4x}{\sqrt{25 - 16x^2}} dx + \int \frac{5}{\sqrt{25 - 16x^2}} dx$$

$$= -\frac{1}{4} \sqrt{25 - 16x^2} + \frac{5}{4} \sin^{-1} \left(\frac{4x}{5} \right) + c$$

$$= \frac{5}{4} \sin^{-1} \left(\frac{4x}{5} \right) - \frac{1}{4} \sqrt{25 - 16x^2} + c$$

 $\int \frac{5}{\sqrt{25 - 16x^2}} dx$ we can now state the original integral. It is only necessary to include one arbitrary constant.

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Integration of inverse trigonometric functions Summary screen and practice questions

Exercise 6.5 Integration of inverse trigonometric functions

Technology free

1. **WE20** Calculate the following:

a. $\sin^{-1}(1.1)$

b. $\sin^{-1} \left(\sin \left(\frac{5\pi}{3} \right) \right)$

c. $\sin(\sin^{-1}(0.9))$

d. $\sin^{-1} \left(-\frac{6}{5} \right)$

e. $\sin^{-1} \left(\sin \left(\frac{7\pi}{6} \right) \right)$

f. $\sin \left(\sin^{-1} \left(\frac{1}{3} \right) \right)$

2. **WE21** Calculate the following:

a. $\cos^{-1}(1.2)$

b. $\cos^{-1} \left(\cos \left(\frac{7\pi}{6} \right) \right)$

c. $\cos \left(\cos^{-1} \left(\frac{\pi}{6} \right) \right)$

d. $\cos^{-1} \left(\frac{4}{3} \right)$

e. $\cos^{-1} \left(\cos \left(\frac{11\pi}{3} \right) \right)$

f. $\cos \left(\cos^{-1} \left(\frac{1}{4} \right) \right)$

3. **WE22** Calculate the following:

a. $\tan^{-1}(\sqrt{3})$

b. $\tan^{-1} \left(-\frac{\sqrt{3}}{3} \right)$

c. $\tan^{-1} \left(\tan \left(\frac{7\pi}{6} \right) \right)$

d. $\tan \left(\tan^{-1} \left(\frac{5}{4} \right) \right)$

e. $\tan^{-1} \left(\tan \left(\frac{5\pi}{3} \right) \right)$

f. $\tan(\tan^{-1}(1.1))$

4. **WE23** Determine:

a. $\frac{d}{dx} \sin^{-1} \left(\frac{x}{4} \right)$

b. $\frac{d}{dx} \sin^{-1}(5x)$

5. **WE24** Determine the derivative of:

a. $\cos^{-1} \left(\frac{3x}{4} \right)$

b. $\cos^{-1} \left(\frac{2x-3}{5} \right)$

6. **WE25** Determine $\frac{dy}{dx}$ for each of the following:
- a. $y = \tan^{-1}(4x)$ b. $y = \tan^{-1}\left(\frac{2x-3}{5}\right)$
7. Given $f(x) = \sin^{-1}(4x)$, determine $f'\left(\frac{1}{8}\right)$.
8. Determine the gradient of the curve $y = \cos^{-1}\left(\frac{3x-4}{5}\right)$ at the point where $x = \frac{4}{3}$.
9. **WE26** Determine the following integrals:
- a. $\int \frac{1}{\sqrt{100-x^2}} dx$ b. $\int \frac{2}{\sqrt{4-x^2}} dx$ c. $\int \frac{12}{\sqrt{64-9x^2}} dx$ d. $\int \frac{1}{\sqrt{36-25x^2}} dx$
10. **WE27** The gradient of a curve that passes through the origin is given by $\frac{-2}{\sqrt{25-16x^2}}$. Determine the equation of the curve.
11. On a certain curve the gradient is given by $\frac{-2}{\sqrt{4-x^2}}$. Determine the equation of the curve that passes through the point $(2, 3)$.
12. **WE28** Determine the anti-derivatives of the following:
- a. $\frac{10}{25+x^2}$ b. $\frac{1}{100+x^2}$ c. $\frac{12}{64+9x^2}$ d. $\frac{1}{36+25x^2}$
13. Determine the derivative of each of the following:
- a. $\sin^{-1}\left(\frac{\sqrt{x}}{3}\right)$ b. $\sin^{-1}\left(\frac{3}{4x}\right)$ c. $\cos^{-1}\left(\frac{e^{2x}}{4}\right)$
- d. $\cos^{-1}\left(\frac{4}{3x}\right)$ e. $\tan^{-1}\left(\frac{x^2}{3}\right)$ f. $\tan^{-1}\left(\frac{6}{5x}\right)$
14. **WE29** Determine the following integrals:
- a. $\int \frac{1}{4x^2-12x+25} dx$ b. $\int \frac{1}{\sqrt{7+12x-4x^2}} dx$ c. $\int \frac{1}{25x^2-20x+13} dx$ d. $\int \frac{1}{\sqrt{12+20x-25x^2}} dx$
15. Determine the anti-derivative of the following:
- a. $\frac{2}{\sqrt{5-4x-x^2}}$ b. $\frac{2}{x^2+4x+13}$ c. $\frac{6}{\sqrt{24-30x-9x^2}}$ d. $\frac{6}{74+30x+9x^2}$
16. Determine $f(x)$ for each of the following:
- a. $f'(x) = \frac{x}{\sqrt{36-25x^2}}$ b. $f'(x) = \frac{5x}{64+25x^2}$
- c. $f'(x) = \frac{10x}{\sqrt{49-25x^2}}$ d. $f'(x) = \frac{x}{49+36x^2}$
17. **WE30** Determine:
- a. $\int \frac{5-3x}{\sqrt{29-9x^2}} dx$ b. $\int \frac{3x+5}{9x^2+25} dx$ c. $\int \frac{1-4x}{\sqrt{1-16x^2}} dx$ d. $\int \frac{8+x}{1+16x^2} dx$
18. Determine the anti-derivative of the following:
- a. $\frac{3x-4}{\sqrt{9-16x^2}}$ b. $\frac{3+4x}{9+16x^2}$ c. $\frac{5-2x}{\sqrt{5-2x^2}}$ d. $\frac{5-2x}{25+2x^2}$
19. If a, b, p and q are positive real constants, determine the following:
- a. $\int \frac{1}{\sqrt{p^2-q^2x^2}} dx$ b. $\int \frac{1}{p^2+q^2x^2} dx$ c. $\int \frac{ax+b}{\sqrt{p^2-q^2x^2}} dx$ d. $\int \frac{ax+b}{p^2+q^2x^2} dx$

20. Determine each of the following:

a. $\int_3^5 \frac{x-4}{x^2-6x+13} dx$ b. $\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{2x+1}{4x^2-12x+25} dx$ c. $\int_1^4 \frac{3x+2}{\sqrt{8+2x-x^2}} dx$ d. $\int_{-2}^1 \frac{2x-3}{\sqrt{12-8x-4x^2}} dx$

6.6 Integration by parts

6.6.1 Introduction to integration by parts

The product rule states that $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$.

Integrating both sides with respect to x gives $uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$.

Rearranging yields $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$.

Integration by parts

The formula for the technique of integration by parts is

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Or using different notation: If $h(x) = f(x) g(x)$ then the product rule says the $h'(x) = f'(x) g(x) + f(x) g'(x)$.

Integrating both sides with respect to x gives $f(x) g(x) = \int f'(x) g(x) dx + \int f(x) g'(x) dx$

Rearranging yields $\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$.

Using either formula is known as integration by parts. The formula for integration by parts is found on your formula sheet.

When integrating a product, integration by parts may be a suitable method.

When identifying u (or $f(x)$), notice that the function you need to integrate involves $\frac{du}{dx}$ (or $f'(x)$). This means that a good choice for u (or $f(x)$) is often a polynomial, as repeated differentiation of a polynomial will eventually result in a constant.

When identifying $\frac{dv}{dx}$ (or $g'(x)$) it is necessary to be able to determine v (or $g(x)$).

WORKED EXAMPLE 31

Determine $\int x \sin(x) dx$.

THINK

- As the integrand is a product, try to integrate by parts. We need to be able to express $\int x \sin(x) dx$ as

$$\int u \frac{dv}{dx} dx.$$

WRITE

$$\int x \sin(x) dx = \int u \frac{dv}{dx} dx$$

2. Let $u = x$ and $\frac{dv}{dx} = \sin(x)$.

3. Differentiate u to find $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to find v .

4. Use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ and substitute for u, v and $\frac{dv}{dx}$.

Integrate using $\int \cos(x) dx = \sin(x) + c$.

5. State the answer.

$$u = x, \frac{dv}{dx} = \sin(x)$$

$$\frac{du}{dx} = 1, v = \int \sin(x) dx = -\cos(x)$$

$$\int x \sin(x) dx = uv - \int v \frac{du}{dx} dx$$

$$= x \times -\cos(x) - \int -\cos(x) \times 1 dx$$

$$= -x \cos(x) + \int \cos(x) dx$$

$$= -x \cos(x) + \sin(x) + c$$

$$\int x \sin(x) dx = -x \cos(x) + \sin(x) + c$$

Sometimes integration by parts needs to be repeated until the $\int v \frac{du}{dx} dx$ (or $\int f'(x) g(x) dx$) is a function that you are able to integrate. The next example will use the $f(x)$ notation.

WORKED EXAMPLE 32

Determine $\int x^2 e^x dx$.

THINK

1. As the integrand is a product, try to integrate by parts. We need to be able to express $\int x^2 e^x dx$ as $\int f(x) g'(x) dx$.

2. As x^2 is a polynomial, let $f(x) = x^2$. This means that $g'(x) = e^x$.

3. Differentiate $g(x)$ to find $g'(x)$ and integrate $f'(x)$ to find $f(x)$.

4. Use the integration by parts formula $\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$ and substitute for $f(x)$, $g(x)$ and $g'(x)$.

5. Integration by parts is needed in order to find $\int 2x e^x dx$ where $f(x) = 2x$ and $g'(x) = e^x$.

WRITE

$$\text{Let } I = \int x^2 e^x dx$$

$$= \int f(x) g'(x) dx$$

$$f(x) = x^2, g'(x) = e^x$$

$$f'(x) = 2x, g(x) = \int e^x dx = e^x$$

$$I = f(x) g(x) - \int f'(x) g(x) dx$$

$$= x^2 \times e^x - \int (2x \times e^x) dx$$

$$= x^2 e^x - \int (2x e^x) dx$$

$$f(x) = 2x, g'(x) = e^x$$

$$f'(x) = 2, g(x) = e^x$$

$$\int (2x e^x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

$$= 2x \times e^x - \int (2 \times e^x) dx$$

$$= 2x e^x - 2e^x + c$$

6. Substitute for $\int 2x e^x dx$ into the original integral.

$$\begin{aligned} I &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - (2x e^x - 2e^x) + c \\ &= x^2 e^x - 2x e^x + 2e^x + c \end{aligned}$$

Sometimes it isn't obvious that integration by parts is the required method. We know that $\frac{d}{dx} \ln(x) = \frac{1}{x}$, but we do not know what $\int \ln(x) dx$ is. By letting $u = \ln(x)$ it is possible to use integration by parts to find the integral. This can also be used to anti-differentiate $\sin^{-1}(ax)$, $\cos^{-1}(ax)$ and $\tan^{-1}(ax)$.

WORKED EXAMPLE 33

Use integration by parts to determine $\int \ln(x) dx$.

THINK

1. The integrand does not appear to be a product however, we can write $\ln(x)$ as $\ln(x) \times 1$.
2. State the formula for integration by parts. We need to be able to express $\int (\ln(x) \times 1) dx$ as $\int u \frac{dv}{dx} dx$.
3. Let $u = \ln(x)$ and $\frac{dv}{dx} = 1$.
4. Differentiate u to find $\frac{du}{dx}$ and integrate $\frac{dv}{dx}$ to find v .
5. Use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ and substitute for u, v and $\frac{dv}{dx}$. Simplify the integrand, noting that $x \times \frac{1}{x} = 1$ and then integrate.

WRITE

$$\int \ln(x) dx = \int (\ln(x) \times 1) dx$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$u = \ln(x), \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{x}, v = x$$

$$\begin{aligned} \int \ln(x) dx &= \int u \frac{dv}{dx} dx \\ &= uv - \int v \frac{du}{dx} dx \\ &= \ln(x) \times x - \int \left(x \times \frac{1}{x} \right) dx \\ &= x \ln(x) - \int 1 dx \\ &= x \ln(x) - x + c \end{aligned}$$

Sometimes we need to integrate a product of cyclic functions. A function is cyclic if repeated differentiation will eventually bring you back to the original function (or a multiple of the original function).

WORKED EXAMPLE 34

Determine $\int e^x \sin(x) dx$

THINK

1. As the integrand is a product, try to integrate by parts. Let $I = \int e^x \sin(x) dx$.
2. As both factors are cyclic, it does not matter which function we choose to be $f(x)$ and which one we choose to be $g'(x)$.
3. Differentiate $f(x)$ to find $f'(x)$ and integrate $g'(x)$ to find $g(x)$.
4. Use $\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$ and substitute for $f(x)$, $g(x)$ and $g'(x)$. Label the result as equation [1] so that we can come back to it later.
5. Use integration by parts again to find $\int e^x \cos(x) dx$ where $f(x) = \cos(x)$ and $g'(x) = e^x$.
6. Notice that we are back to the original integral $I = \int e^x \sin(x) dx$. Label this as equation [2] so we can use it again.
7. Substitute equation [2] into equation [1].
8. We now have I on both sides of the equation. Rearrange the equation to make I the subject.
9. State your answer, remembering to include " $+ c$ ".

WRITE

$$I = \int e^x \sin(x) dx = \int f(x) g'(x) dx$$

$$f(x) = \sin(x), g'(x) = e^x$$

$$f'(x) = \cos(x), g(x) = e^x$$

$$\begin{aligned} I &= f(x) g(x) - \int f'(x) g(x) dx \\ &= \sin(x) e^x - \int (\cos(x) e^x) dx \\ &= e^x \sin(x) - \int (e^x \cos(x)) dx \quad [1] \end{aligned}$$

$$\begin{aligned} f(x) &= \cos(x), g'(x) = e^x \\ f'(x) &= -\sin(x), g(x) = e^x \\ \int e^x \cos(x) dx &= \int f(x) g'(x) dx \\ &= f(x) g(x) - \int f'(x) g(x) dx \\ &= \cos(x) e^x - \int (-\sin(x)) e^x dx \\ &= e^x \cos(x) + \int e^x \sin(x) dx \\ \int e^x \cos(x) dx &= e^x \cos(x) + I \quad [2] \end{aligned}$$

$$\begin{aligned} I &= e^x \sin(x) - (e^x \cos(x) + I) \\ &= e^x \sin(x) - e^x \cos(x) - I \\ 2I &= e^x \sin(x) - e^x \cos(x) \end{aligned}$$

$$\begin{aligned} I &= \frac{1}{2} (e^x \sin(x) - e^x \cos(x)) \\ &= \frac{e^x}{2} (\sin(x) - \cos(x)) \end{aligned}$$

$$\int e^x \sin(x) dx = \frac{e^x}{2} (\sin(x) - \cos(x)) + c$$

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Units 3 & 4

Area 4

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Concept 5

Integration by parts Summary screen and practice questions

Exercise 6.6 Integration by parts

Technology free

1. Use integration by parts to determine the following:

a. $\int x \cos(x) dx$ (Let $u = x$)

b. $\int x e^x dx$ (Let $u = x$)

c. $\int x \ln(x) dx$ (Let $u = \ln(x)$)

d. $\int x \sin(2x) dx$ (Let $u = x$)

2. **WE31** Determine:

a. $\int x \sin(4x) dx$

b. $\int x \cos(3x) dx$

c. $\int x e^{-2x} dx$

d. $\int x \cos\left(\frac{x}{2}\right) dx$

3. **WE32** Determine:

a. $\int x^2 \cos(x) dx$

b. $\int x^2 \sin(x) dx$

4. Determine the anti-derivative of the following:

a. $x^2 e^{2x}$

b. $x^2 (x + 1)^5$

5. **WE33** Use integration by parts to determine:

a. $\int \ln(2x) dx$

b. $\int \sin^{-1}(x) dx$

6. Determine the anti-derivative of each of the following:

a. $(x - 1) \sin(x)$

b. $(x + 2) e^x$

7. Determine the following integrals:

a. $\int x^3 \ln(x) dx$

b. $\int (x + 1) \ln(x) dx$

8. Determine:

a. $\int x^2 \sin^{-1}(x) dx$

b. $\int (x + 1)^2 e^x dx$

9. Use integration by parts to determine:

a. $\int x^3 e^x dx$

b. $\int \frac{x}{e^x} dx$

10. **WE34** Determine:

a. $\int e^x \cos(x) dx$

b. $\int e^{2x} \cos(x) dx$

11. Determine $f(x)$ given:

a. $f'(x) = e^x \sin(2x)$

b. $f'(x) = e^{2x} \cos(3x)$

12. Determine $\int x^n \ln(ax) dx, n \neq -1$.

13. Use integration by parts to determine $\int \frac{\ln(x)}{x} dx$.

14. Use integration by parts to determine $\int \sin(\ln(x)) dx$.

6.7 Integration involving partial fractions

A rational function is a ratio of two functions, both of which are polynomials. For example, $\frac{mx + k}{ax^2 + bx + c}$ is a rational function; it has a linear function in the numerator and a quadratic function in the denominator. In the preceding section, we integrated a certain expression of this form when $a > 0$ and $\Delta = b^2 - 4ac < 0$,

meaning that the quadratic function was expressed as the sum of two squares. In this section, we examine cases when $a \neq 0$ and $\Delta = b^2 - 4ac > 0$. This means that the quadratic function in the denominator can now be factorised into linear factors. Integrating expressions of this kind does not involve a new integration technique, just an algebraic method of expressing the integrand into its partial fractions decomposition.

6.7.1 Integration by partial fractions

Using one of the properties of indefinite integrals, $\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$, we can break up complicated integrals into two manageable integrals. Through integration, a rational expression is rearranged into the sum of rational expressions that we can integrate.

We have already observed that $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$.

To integrate rational expressions that are not in this format, the rational expression is rewritten as partial fractions.

The rational expression must be written as a proper fraction first. That is, the degree of the numerator polynomial is less than the degree of the denominator polynomial.

If the denominator's linear factors are unique (each factor is different), the partial fractions are written as follows (where the original rational expression is a proper fraction).

Partial fractions with distinct linear factors

$$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

The rational expression $\frac{2x^2 - 5x - 4}{(x+1)(x-2)x}$ would be written as $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x}$. This means that $\frac{Ax(x-2) + Bx(x+1) + C(x+1)(x-2)}{(x+1)(x-2)x} = \frac{2x^2 - 5x - 4}{(x+1)(x-2)x}$. To determine the values of A, B and C , equate the numerators and solve for A, B and C . There are two main methods that can be used to determine the values for A, B and C . These will be demonstrated in the next worked example.

WORKED EXAMPLE 35

If $Ax(x-2) + Bx(x+1) + C(x+1)(x-2) = 2x^2 - 5x - 4$, calculate values for A, B and C .

THINK

1. Method 1

The values for A, B and C can be found by choosing values for x that make the factors equal 0. Let $x = 0, x = -1$ and $x = 2$.

WRITE

$x = 0$:

$$C(1)(-2) = -4$$

$$C = 2$$

$x = -1$:

$$A(-1)(-1-2) = 2(-1)^2 - 5(-1) - 4$$

$$3A = 3$$

$$A = 1$$

$x = 2$:

$$B(2)(2+1) = 2(2^2) - 5(2) - 4$$

$$6B = -6$$

$$B = -1$$

1. Method 2

Expand

$Ax(x-2) + Bx(x+1) + C(x+1)(x-2)$ so that the coefficients of x^2 , x and the constant term can be equated.

$$2. (A+B+C)x^2 + (-2A+B-C)x - 2C = 2x^2 - 5x - 4$$

Equate the coefficients.

3. Use equation [3] to solve for C and substitute this value into equations [1] and [2].

4. Equations [4] and [5] are in terms of A and B only, so use elimination to solve.

5. State the solutions.

$$\begin{aligned} Ax(x-2) + Bx(x+1) + C(x+1)(x-2) \\ = A(x^2 - 2x) + B(x^2 + x) + C(x^2 - x - 2) \\ = (A+B+C)x^2 + (-2A+B-C)x - 2C \end{aligned}$$

$$A+B+C=2 \quad [1]$$

$$-2A+B-C=-5 \quad [2]$$

$$-2C=-4 \quad [3]$$

$$[3]: -2C = -4$$

$$C = 2$$

$$[1]: A+B+2=2$$

$$A+B=0 \quad [4]$$

$$[2]: -2A+B-2=-5$$

$$-2A+B=-3 \quad [5]$$

$$[4] - [5]: 3A = 3$$

$$A = 1$$

$$[4]: B = -1$$

$$A = 1, B = -1, C = 2$$

$$\text{This means that } \frac{2x^2 - 5x - 4}{(x+1)(x-2)x} = \frac{1}{x+1} + \frac{-1}{x-2} + \frac{2}{x} = \frac{1}{x+1} - \frac{1}{x-2} + \frac{2}{x}$$

WORKED EXAMPLE 36

For $\frac{x+7}{(x+2)(x-3)}$,

a. identify the partial fractions

b. determine the anti-derivative.

THINK

a. 1. The degree of the numerator is 1. The degree of the denominator is 2. The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers.

2. Add the fractions together by finding a common denominator.

3. As the denominator are equal, equate the numerators.

4. Solve for A and B by choosing values for x that make the factors equal 0. Let $x = 3$ and $x = -2$.

WRITE

$$\text{a. Let } \frac{x+7}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$\frac{x+7}{(x+2)(x-3)} = \frac{A(x-3) + B(x+2)}{(x+2)(x-3)}$$

$$x+7 = A(x-3) + B(x+2)$$

$$x = 3:$$

$$3+7 = A(3-3) + B(3+2)$$

$$10 = 5B$$

$$B = 2$$

$$x = -2:$$

$$-2+7 = A(-2-3) + B(-2+2)$$

$$5 = -5A$$

$$A = -1$$

5. Rewrite the rational expression as a sum of the partial fractions.

b. 1. Express the question using integral notation.

2. Rewrite the integral as partial fractions.

3. Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ to determine the integral.

$$\frac{x+7}{(x+2)(x-3)} = \frac{-1}{x+2} + \frac{2}{x-3}$$

$$\begin{aligned} \text{b. } \int \frac{x+7}{(x+2)(x-3)} dx &= \int \left(\frac{-1}{x+2} + \frac{2}{x-3} \right) dx \\ &= - \int \frac{1}{x+2} dx + 2 \int \frac{1}{x-3} dx \\ &= -\ln |x+2| + 2 \ln |x-3| + c \\ &= 2 \ln |x-3| - \ln |x+2| + c \end{aligned}$$

Note: the answer can be simplified using the logarithm laws to be $\ln \left(\frac{(x-3)^2}{|x+2|} \right) + c$.

If the denominator is not expressed in factorised form, it is necessary to factorise the denominator first.

WORKED EXAMPLE 37

For $\frac{2x-3}{x^2-3x-4}$,

a. identify the partial fractions

b. determine the anti-derivative.

THINK

- a. 1. The degree of the numerator is 1. The degree of the denominator is 2. The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers. Factorise the denominator to identify the denominators of the partial fractions.
2. Add the fractions together by finding a common denominator.
3. As the denominators are equal, equate the numerators.
4. Solve for A and B by choosing values for x that make the factors equal 0. Let $x = 4$ and $x = -1$.

5. Rewrite the rational expression as a sum of the partial fractions.

b. 1. Express the question using integral notation.

WRITE

$$\begin{aligned} \text{a. } \frac{2x-3}{x^2-3x-4} &= \frac{2x-3}{(x-4)(x+1)} \\ &= \frac{A}{x-4} + \frac{B}{x+1} \end{aligned}$$

$$\frac{2x-3}{x^2-3x-4} = \frac{A(x+1) + B(x-4)}{(x-4)(x+1)}$$

$$2x-3 = A(x+1) + B(x-4)$$

$$x = 4:$$

$$2(4) - 3 = A(4+1) + B(4-4)$$

$$5 = 5A$$

$$A = 1$$

$$x = -1:$$

$$2(-1) - 3 = A(-1+1) + B(-1-4)$$

$$-5 = -5B$$

$$B = 1$$

$$\frac{2x-3}{x^2-3x-4} = \frac{1}{x-4} + \frac{1}{x+1}$$

$$\text{b. } \int \frac{2x-3}{x^2-3x-4} dx$$

2. Rewrite the integral as partial fractions.

$$= \int \left(\frac{1}{x-4} + \frac{1}{x+1} \right) dx$$

3. Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ to determine the integral.

$$= \ln |x-4| + \ln |x+1| + c$$

4. Note that logarithm laws can be used to simplify the answer.

$$= \ln |(x-4)(x+1)| + c$$

Note: the original question was in the form $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ where $f(x) = x^2 - 3x - 4$ and $f'(x) = 2x - 3$, so the question could have been integrated by recognising that pattern.

Repeated factors

If the denominator's linear factors are repeated, the partial fractions are written as follows (where the original rational expression is a proper fraction).

Partial fractions with repeated linear factors

$$\frac{f(x)}{(ax+b)^2} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

WORKED EXAMPLE 38

Determine the anti-derivative of $\frac{8x+10}{(x+2)^2}$.

THINK

- The degree of the numerator is 1.
The degree of the denominator is 2.
The rational expression is a proper fraction. It can be written as the sum of 2 rational numbers. The factor $(x+2)$ is repeated.
- Add the fractions together by finding a common denominator.
- As the denominators are equal, equate the numerators.
- It is possible to solve for B by letting $x = -2$ and then equate the coefficients to solve for A .
Alternatively, expand the right hand side and equate the coefficients.
- Rewrite the rational expression as the sum of the partial fractions.
- Express the question using integral notation.

WRITE

$$\text{Let } \frac{8x+10}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\frac{8x+10}{(x+2)^2} = \frac{A(x+2)}{(x+2)^2} + \frac{B}{(x+2)^2}$$

$$8x+10 = A(x+2) + B$$

$$8x+10 = Ax + 2A + B$$

$$\text{Consider coefficient of } x : A = 8$$

$$\text{Consider constant: } 10 = 2A + B$$

$$10 = 2 \times 8 + B$$

$$B = -6$$

$$\frac{8x+10}{(x+2)^2} = \frac{8}{x+2} - \frac{6}{(x+2)^2}$$

$$\int \frac{8x+10}{(x+2)^2} dx$$

7. Rewrite the integral as partial fractions.

8. Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ to integrate $\int \frac{8}{x+2} dx$. To find $\int \frac{6}{(x+2)^2} dx$ rewrite the integrand to include negative powers.

9. Simplify by writing with positive powers.

$$\begin{aligned} &= \int \left(\frac{8}{x+2} - \frac{6}{(x+2)^2} \right) dx \\ &= 8 \int \frac{1}{x+2} dx - 6 \int (x+2)^{-2} dx \\ &= 8 \ln |x+2| - \frac{6(x+2)^{-1}}{-1} + c \\ &= 8 \ln |x+2| + \frac{6}{x+2} + c \end{aligned}$$

6.7.2 Non-proper rational functions

If the degree of the numerator is not smaller than the degree of the denominator, the rational function is not proper. It is necessary to use division to rewrite the function.

For example, if the rational function is the function of two quadratic functions, then it is in the form

$$\frac{rx^2 + sx + t}{ax^2 + bx + c}, r \neq 0, a \neq 0. \text{ Through division, this can be rewritten as } \frac{rx^2 + sx + t}{ax^2 + bx + c} = q + \frac{mx + k}{ax^2 + bx + c}$$

where $q = \frac{r}{a}$. The proper rational expression $\frac{mx + k}{ax^2 + bx + c}$ can be rewritten as partial fractions so that it can be integrated.

WORKED EXAMPLE 39

Determine $\int \frac{2x^2 + 5x + 3}{x^2 + 3x - 4} dx$.

THINK

1. The degree of the numerator is 2. The degree of the denominator is 2. This is not a proper fraction, so use division to obtain a polynomial and a proper rational fraction.

WRITE

$$x^2 + 3x - 4 \overline{) \begin{array}{r} 2x^2 + 5x + 3 \\ -(2x^2 + 6x - 8) \\ \hline -x + 11 \end{array}}$$

2. $\frac{11-x}{x^2 + 3x - 4}$ can be rewritten as partial fractions.

3. Add the fractions together by finding a common denominator.

4. As the denominators are equal, equate the numerators.

$$\begin{aligned} \therefore \frac{2x^2 + 5x + 3}{x^2 + 3x - 4} &= 2 + \frac{11-x}{x^2 + 3x - 4} \\ \frac{11-x}{x^2 + 3x - 4} &= \frac{11-x}{(x+4)(x-1)} \\ &= \frac{A}{x+4} + \frac{B}{x-1} \\ \frac{11-x}{(x+4)(x-1)} &= \frac{A(x-1) + B(x+4)}{(x+4)(x-1)} \\ 11-x &= A(x-1) + B(x+4) \end{aligned}$$

5. Solve for A and B by choosing values for x that make the factors equal 0. Let $x = -4$ and $x = 1$.

$$x = -4:$$

$$11 - (-4) = A(-4 - 1) + B(-4 + 4)$$

$$15 = -5A$$

$$A = -3$$

$$x = 1:$$

$$11 - 1 = A(1 - 1) + B(1 + 4)$$

$$10 = 5B$$

$$B = 2$$

6. Rewrite $\frac{11-x}{x^2+3x-4}$ as the sum of partial fractions.

$$\frac{11-x}{x^2+3x-4} = -\frac{3}{x+4} + \frac{2}{x-1}$$

7. Rewrite the integral.

$$\begin{aligned} \int \frac{2x^2+5x+3}{x^2+3x-4} dx &= \int \left(2 + \frac{11-x}{x^2+3x-4} \right) dx \\ &= \int \left(2 - \frac{3}{x+4} + \frac{2}{x-1} \right) dx \end{aligned}$$

8. Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ to determine the integral.

$$\begin{aligned} &= \int 2 dx - 3 \int \frac{1}{x+4} dx + 2 \int \frac{1}{x-1} dx \\ &= 2x - 3 \ln |x+4| + 2 \ln |x-1| + c \end{aligned}$$

Note that logarithm laws can be used to simplify this answer if desired.

6.7.3 Rational functions involving non-linear factors

If the denominator does not factorise into linear factors, then the partial fractions are written so that the degree of the numerator is one degree less than the degree of the denominator. This means that if the denominator is a quadratic (with no real factors), then the numerator is written in the form $Ax + B$. The following partial fraction representation highlights this concept.

Partial fractions with an irreducible quadratic factor

$$\frac{px^2 + qx + r}{(x + \alpha)(x^2 + \beta^2)} = \frac{Ax + B}{x^2 + \beta^2} + \frac{C}{x + \alpha}$$

WORKED EXAMPLE 40

Determine $\int \frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} dx$.

THINK

- Recall the factor theorem which states that if $P(x)$ is a polynomial and $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$. Use the factor theorem to factorise the polynomial in the denominator.
- Factorise the denominator.

WRITE

$$f(x) = x^3 + 2x^2 + 4x + 8$$

$$f(1) = 1 + 2 + 4 + 8 \neq 0$$

$$f(-2) = -8 + 8 - 8 + 8 = 0$$

$$(x + 2) \text{ is a factor.}$$

$$f(x) = x^2(x + 2) + 4(x + 2)$$

$$= (x + 2)(x^2 + 4)$$

3. As the degree of the numerator is less than the degree of the denominator, write it as partial fractions. As $x^2 + 4$ is a quadratic, its numerator will be $Ax + B$.

4. Add the fractions together by finding a common denominator.
 5. As the denominator are equal, equate the numerators.
 6. C can be determined by letting $x = -2$.

7. Equate the co-efficients to identify A and B .

8. Rewrite the rational expression as the sum of the partial fractions.
 9. Rewrite the integral using partial fractions.

10. As $\frac{d}{dx}(x^2 + 4) = 2x$, $\frac{2x+3}{x^2+4}$ needs to be rewritten as $\frac{2x}{x^2+4} + \frac{3}{x^2+4}$.
 11. Use $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ to integrate $\frac{2x}{x^2+4}$ and $\frac{1}{x+2}$. Use $\int \frac{a}{a^2+x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$ to integrate $\frac{3}{x^2+4}$.

$$\frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x + 2}$$

$$= \frac{(Ax + B)(x + 2)}{(x + 2)(x^2 + 4)} + \frac{C(x^2 + 4)}{(x + 2)(x^2 + 4)}$$

$$x^2 + 7x + 2 = (Ax + B)(x + 2) + C(x^2 + 4)$$

$$x = -2:$$

$$4 - 14 + 2 = C(4 + 4)$$

$$-8 = 8C$$

$$C = -1$$

$$x^2 + 7x + 2 = (Ax + B)(x + 2) - 1(x^2 + 4)$$

$$x^2 + 7x + 2 = Ax^2 + 2Ax + Bx + 2B - x^2 - 4$$

$$x^2 + 7x + 2 = (A - 1)x^2 + (2A + B)x + (2B - 4)$$

$$x^2: A - 1 = 1$$

$$A = 2$$

$$\text{constant: } 2 = 2B - 4$$

$$B = 3$$

$$\frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} = \frac{2x + 3}{x^2 + 4} - \frac{1}{x + 2}$$

$$\int \frac{x^2 + 7x + 2}{x^3 + 2x^2 + 4x + 8} dx = \int \left(\frac{2x + 3}{x^2 + 4} - \frac{1}{x + 2} \right) dx$$

$$= \int \left(\frac{2x}{x^2 + 4} + \frac{3}{x^2 + 4} - \frac{1}{x + 2} \right) dx$$

$$= \int \frac{2x}{x^2 + 4} dx - \int \frac{1}{x + 2} dx + \frac{3}{2} \int \frac{2}{x^2 + 2^2} dx$$

$$= \ln(x^2 + 4) - \ln|x + 2| + \frac{3}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 6

Integration involving partial fractions Summary screen and practice questions

Exercise 6.7 Integration involving partial fractions

Technology free

1. **WE35** Calculate values for A , B and C .
 - a. $Ax + B(x - 1) = 3x - 2$
 - b. $A(x + 2) + B(x - 3) = x - 8$
 - c. $A(x - 4) + B = 3x - 2$
 - d. $A(3x + 1) + B(x - 2) = 5x + 4$
 - e. $A(x + 2)(x - 3) + Bx(x - 3) + Cx(x + 2) = 3x^2 - x + 6$
 - f. $Ax(x + 2) + Bx(x + 3) + C(x + 2)(x + 3) = x^2 - 3x - 12$
2. **WE36** For $\frac{1}{(x + 1)(x + 2)}$
 - a. identify the partial fractions
 - b. determine the anti-derivative.
3. For each of the following, determine the integral by first identifying the partial fractions.
 - a. $\int \frac{12}{(x - 2)(x + 2)} dx$
 - b. $\int \frac{6x}{(x + 3)(x - 1)} dx$
 - c. $\int \frac{3x}{(x - 2)(x + 1)} dx$
 - d. $\int \frac{12 - 2x}{(1 - x)(3 - x)} dx$
4. Determine the following anti-derivatives:
 - a. $\frac{x + 20}{(x - 4)(x + 4)}$
 - b. $\frac{x + 4}{x(x - 2)}$
 - c. $\frac{7x - 4}{(x - 2)(x + 3)}$
 - d. $\frac{8x - 10}{(2x + 1)(x - 3)}$
5. Determine $f(x)$ for each of the following:
 - a. $f'(x) = \frac{9x - 11}{(3x - 2)(x + 1)}$
 - b. $f'(x) = \frac{11 - 3x}{(2 - x)(x + 3)}$
 - c. $f'(x) = \frac{4 - x}{x(x + 2)}$
 - d. $f'(x) = \frac{9x + 8}{(x - 3)(x + 4)}$
6. **WE37** For $\frac{5x + 10}{24 - 2x - x^2}$
 - a. identify the partial fractions
 - b. determine the anti-derivative.
7. For each of the following, determine the integral by first identifying the partial fractions.
 - a. $\int \frac{3x + 10}{x^2 + 2x} dx$
 - b. $\int \frac{5x - 4}{x^2 - x - 2} dx$
 - c. $\int \frac{x + 3}{x^2 + 3x + 2} dx$
 - d. $\int \frac{6x - 1}{x^2 - 5x - 6} dx$
8. Determine the following anti-derivatives:
 - a. $\frac{5x - 7}{x^2 - 4x + 3}$
 - b. $\frac{x + 16}{x^2 + 7x + 6}$
 - c. $\frac{7x + 9}{x^2 - 9}$
 - d. $\frac{7x + 1}{x^2 - 1}$
9. Determine $f(x)$ for each of the following:
 - a. $f'(x) = \frac{5x}{2x^2 - 3x - 2}$
 - b. $f'(x) = \frac{16 - 2x}{3x^2 + 7x - 6}$
 - c. $f'(x) = \frac{x + 4}{2x^2 - 5x + 2}$
 - d. $f'(x) = \frac{4}{4 - x^2}$
10. **WE38** Determine the anti-derivative of:
 - a. $\frac{4x + 5}{(x + 2)^2}$
 - b. $\frac{5x - 26}{(x - 5)^2}$
 - c. $\frac{2x + 1}{x^2 + 6x + 9}$
 - d. $\frac{6x - 5}{4x^2 - 12x + 9}$
11. **WE39** Determine:
 - a. $\int \frac{x^2 - 1}{x^2 + 3x} dx$
 - b. $\int \frac{x^2 + 2x + 4}{x^2 - 4x} dx$
 - c. $\int \frac{x^2 - x}{(x + 3)(x + 1)} dx$
 - d. $\int \frac{x^2 + x + 4}{x^2 - 2x - 3} dx$

12. Determine the anti-derivative of:

a. $\frac{x^2 + 3x - 2}{x^2 - 4}$

b. $\frac{x^2 - x + 2}{x^2 + 2x + 1}$

c. $\frac{2x^2 - 9x + 7}{x^2 - 6x + 9}$

d. $\frac{x - 1}{x + 5}$

13. Determine $g(x)$ for each of the following:

a. $g'(x) = \frac{x^3 + 4x^2 - x}{(x + 2)(x + 1)}$

b. $g'(x) = \frac{2x^3 + x^2 - 5}{x^2 - 1}$

c. $g'(x) = \frac{x^3 + 4x - 13}{x^2 - 4x - 5}$

d. $g'(x) = \frac{x^3 + x^2 - 4x}{x^2 - 4x + 4}$

14. **WE40** Determine:

a. $\int \frac{4x - 2}{x^2 + 9} dx$

b. $\int \frac{x + 1}{x^2 + 4} dx$

c. $\int \frac{19 - 3x}{x^3 - 2x^2 + 9x - 18} dx$

d. $\int \frac{25}{x^3 + 3x^2 + 16x + 48} dx$

15. Determine the following integrals:

a. $\int \frac{5x^2 + 2x + 17}{(x - 1)(x + 2)(x - 3)} dx$

b. $\int \frac{x^2 + 18x + 5}{(x + 1)(x - 2)(x + 3)} dx$

c. $\int \frac{x^2 + 8x + 9}{(x - 1)(x + 2)^2} dx$

d. $\int \frac{x^2 + 5x + 1}{(x^2 + 1)(2 - x)} dx$

16. If a, b, p and q are non-zero real constant, then determine each of the following:

a. $\int \frac{1}{b^2x^2 - a^2} dx$

b. $\int \frac{x}{b^2x^2 - a^2} dx$

c. $\int \frac{x}{(ax - b)^2} dx$

d. $\int \frac{1}{(px + a)(qx + b)} dx$

6.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Use the substitution $u = 2x - 1$ to determine the following integrals:

a. $\int (2x - 1)^6 dx$

b. $\int \frac{1}{(2x - 1)^2} dx$

c. $\int \frac{1}{2x - 1} dx$

d. $\int \frac{1}{\sqrt[3]{2x - 1}} dx$

2. Use the substitution $u = x^2 + 2$ to determine the following integrals:

a. $\int \frac{x}{x^2 + 2} dx$

b. $\int x \cos(x^2 + 2) dx$

c. $\int 3x e^{x^2 + 2} dx$

d. $\int_0^2 \frac{4x}{(x^2 + 2)} dx$

3. Use an appropriate trigonometric identity to determine the following:

a. $\int \cos(2x) \sin(2x) dx$

b. $\int (\cos^2(2x) + \sin^2(2x)) dx$

c. $\int \cos^3(2x) \sin(2x) dx$

d. $\int \cos(2x) \sin^3(2x) dx$

4. Determine each of the following:

a. $\int \tan(3x) dx$

b. $\int \cot(3x) dx$

c. $\int \tan(3x) \sec^2(3x) dx$

d. $\int \tan^2(3x) \sec^2(3x) dx$

5. Calculate the following:

a. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $\tan^{-1}(-\sqrt{3})$

c. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

d. $\tan(\tan^{-1}(1.5))$

6. Calculate the following:

a. $\sin\left(\sin^{-1}\left(\frac{3}{2}\right)\right)$ b. $\sec\left(\sec^{-1}(\sqrt{2})\right)$ c. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$ d. $\sin^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

7. Determine the derivative of the following:

a. $\sin^{-1}\left(\frac{x}{2}\right)$ b. $\tan^{-1}(3x)$ c. $\cos^{-1}(x^2)$ d. $\tan^{-1}(\sqrt{x})$

8. Use the substitution $u = \sin^{-1}\left(\frac{x}{a}\right)$ to determine $\int \frac{1}{\sqrt{a^2 - x^2}} dx$.

9. Use the substitution $u = \tan^{-1}\left(\frac{x}{a}\right)$ to determine $\int \frac{a}{a^2 + x^2} dx$.

10. Determine the anti-derivative of the following:

a. $\frac{1}{\sqrt{16 - x^2}}$ b. $\frac{1}{\sqrt{1 - 16x^2}}$ c. $\frac{10}{\sqrt{49 - 25x^2}}$ d. $\frac{-2}{\sqrt{1 - 4x^2}}$

11. Use integration by parts to determine the following:

a. $\int x \sin(x) dx$ b. $\int 2x \cos(3x) dx$ c. $\int x^2 \sin(x) dx$ d. $\int x^2 \ln(x) dx$

12. Use partial fractions to determine the anti-derivative of the following:

a. $\frac{5x - 13}{(x - 3)(x - 2)}$ b. $\frac{x + 4}{(x + 1)^2}$ c. $\frac{x + 1}{x^2(x - 1)}$ d. $\frac{x^2 + 8}{x^2 - 5x + 6}$

Complex familiar

13. Use completing the square to rewrite the denominator and then determine the integral of $\frac{x + 1}{x^2 + x + 1}$.

14. Use integration by parts to determine the anti-derivative of $\frac{\cos(3x)}{e^x}$.

15. Use integration by parts to determine $\int \tan^{-1}(x) dx$.

16. The gradient of a curve is given by $\frac{1}{\sqrt{4 - x^2}}$. Determine the equation of the curve that passes through the point $(\sqrt{3}, \pi)$.

Complex unfamiliar

17. Determine $\int x (\ln(x))^2 dx$.

18. Determine $\int x \tan^{-1}(x) dx$.

19. Use the substitution $u = \sqrt{x}$ to determine $\int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx$.

20. Determine $\int \frac{\cos(x)}{\sin^2(x) + \sin(x) - 6} dx$.

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Answers

6 Integration techniques

Exercise 6.2 Integration by linear substitution

1. a. $\frac{1}{35} (5x - 9)^7 + c$ b. $\frac{1}{24} (3x + 4)^8 + c$
 c. $\frac{1}{21} (3x - 5)^7 + c$ d. $-\frac{1}{12} (1 - 2x)^6 + c$
2. $y = \frac{1}{8} (\sqrt{16x + 25} - 5)$
3. $\frac{11}{4}$
4. a. $\frac{1}{3} \ln |3x - 5| + c$ b. $-\frac{1}{2} \ln |7 - 2x| + c$
 c. $\frac{2}{3} \ln |3x - 1| + c$ d. $-\frac{3}{2} \ln |1 - 2x| + c$
5. a. $\frac{1}{21} (3x + 5)^7 + c$ b. $-\frac{1}{3(3x + 5)} + c$
 c. $-\frac{1}{6(3x + 5)^2} + c$ d. $\frac{1}{2} \sqrt[3]{(3x + 5)^2} + c$
6. a. $\frac{1}{54} (6x + 7)^9 + c$ b. $\frac{1}{3} \sqrt{6x + 7} + c$
 c. $\frac{1}{6} \ln |6x + 7| + c$ d. $-\frac{1}{6(6x + 7)} + c$
7. a. 2 b. 8 c. 30 d. $\frac{1}{3}$
8. a. $\frac{1}{175} (5x - 9)^7 + \frac{3}{50} (5x - 9)^6 + c =$
 $\frac{1}{350} (10x + 3) (5x - 9)^6 + c$
 b. $\frac{2}{9} \ln |3x - 4| - \frac{5}{9(3x - 4)} + c$
 c. $\frac{-(4x + 7)}{8(2x + 7)^2} + c$
 d. $\frac{1}{27} (3x - 5) \sqrt{6x + 5} + c$
9. a. $\frac{1}{72} (3x + 5)^8 - \frac{5}{63} (3x + 5)^7 + c =$
 $\frac{1}{504} (21x - 5) (3x + 5)^7 + c$
 b. $\frac{1}{9} \ln |3x + 5| + \frac{5}{9(3x + 5)} + c$
 c. $\frac{-(6x + 5)}{18(3x + 5)^2} + c$
 d. $\frac{1}{10} (2x - 5) (3x + 5)^{\frac{2}{3}} + c$
10. a. $\frac{1}{360} (6x + 7)^{10} - \frac{7}{324} (6x + 7)^9 + c =$
 $\frac{1}{3240} (54x - 7) (6x + 7)^9 + c$
 b. $\frac{1}{27} (3x - 7) \sqrt{6x + 7} + c$
 c. $\frac{x}{6} - \frac{7}{36} \ln |6x + 7| + c$
 d. $\frac{7}{36(6x + 7)} + \frac{1}{36} \ln |6x + 7| + c$

11. a. $x = \frac{t}{2(2 - 5t)}$
 b. $f(x) = -\sqrt{3 - 2x}$
 c. $\frac{3}{2} \ln(3)$
 d. $y = \frac{1}{3} (x - 9) \sqrt{2x + 9} + 9$
12. a. $\frac{16}{15}$ b. $\frac{76}{15}$
 c. 4 d. $\frac{5}{3} \ln\left(\frac{5}{2}\right) - 1$
13. a. $\frac{7}{2}$ b. $\frac{47}{7}$ c. 6 d. $\frac{333}{10}$
14. a. $\frac{2}{3a} (ax + b)^{\frac{3}{2}} + c$
 b. $\frac{2}{15a^2} (3ax - 2b) (ax + b)^{\frac{3}{2}} + c$
 c. $\frac{1}{a} \ln |ax + b| + c$
 d. $\frac{x}{a} - \frac{b}{a^2} \ln |ax + b| + c$
15. a. $\frac{2}{a} \sqrt{ax + b} + c$
 b. $\frac{2}{3a^2} (ax - 2b) \sqrt{ax + b} + c$
 c. $\frac{-1}{a(ax + b)} + c$
 d. $\frac{b}{a^2(ax + b)} + \frac{1}{a^2} \ln |ax + b| + c$
16. a. $\frac{ad - bc}{a^2} \ln |ax + b| + \frac{cx}{a} + k$
 b. $\frac{c}{a^2} \ln |ax + b| + \frac{bc - ad}{a^2(ax + b)} + k$
 c. $\frac{cx}{a^2} - \frac{2bc}{a^3} \ln |ax + b| - \frac{a^2d + cb^2}{a^3(ax + b)} + k$
 d. $\frac{a^2d + cb^2}{a^3} \ln |ax + b| + \frac{cx^2}{2a} - \frac{bcx}{a^2} + k$

Exercise 6.3 Integration by non-linear substitutions

1. a. $\frac{-2}{(x^2 + 16)^2} + c$ b. $\frac{5}{2} \sqrt{2x^2 + 3} + c$
 c. $\frac{1}{5} (x^2 + 3)^5 + c$ d. $\frac{1}{2(6 - x^2)^2} + c$
2. a. $\frac{1}{2} \ln(x^2 + 4x + 29) + c$ b. $\frac{1}{3} \ln |x^3 + 9| + c$
 c. $\frac{1}{2} \ln(x^2 + 2x + 9) + c$ d. $\frac{3}{2} \ln(x^2 + 9) + c$
3. a. $\frac{1}{12} (x^2 + 4)^6 + c$ b. $\frac{1}{2} \ln(x^2 + 4) + c$
 c. $\frac{-1}{2(x^2 + 9)} + c$ d. $\sqrt{x^2 + 9} + c$
4. a. $\frac{-1}{6(x^3 + 27)^2} + c$ b. $\frac{2}{3} \sqrt{x^3 + 27} + c$
 c. $\frac{1}{3} \ln |x^3 + 8| + c$ d. $\frac{1}{12} (x^3 + 8)^4 + c$

5. a. $\frac{1}{8}(x^2 - 4x + 13)^4 + c$
 c. $-\sqrt{x^2 - 8x + 25} + c$
6. a. $\frac{1}{3}(x^3 - 5)^3 + c$
 c. $\frac{2}{3}(x^2 + 5x)^{\frac{3}{2}} + c$
7. a. $\frac{1}{3}(x^3 - 2)^6 + c$
 c. $\frac{1}{24}(x^3 - 1)^8 + c$
8. a. $-\frac{1}{6}(x^2 + 2x + 3)^{-3} + c$
 c. $-\frac{1}{5}(x^2 - 5x + 2)^{-5} + c$
9. a. $\frac{1}{7}(x^2 + 1)^{\frac{7}{2}} + c$
 c. $\frac{2}{3}\sqrt{x + x^2 - 3}^3 + c$
10. a. $\cos\left(\frac{1}{x}\right) + c$
 c. $\sin(x^3 + 5x) + c$
11. a. $2\sin(\sqrt{x}) + c$
 c. $\frac{1}{4}(\ln(3x))^2 + c$
12. a. $\frac{1}{3}e^{\sin(3x)} + c$
 c. $3e^{x^2 - x + 3} + c$
13. a. $\frac{1}{8}\ln(4e^{2x} + 5) + c$
 c. $\frac{1}{12(3e^{-2x} + 4)^2} + c$
14. a. $-\cos(\ln(4x)) + c$
 b. $\sin(\ln(3x)) + c$
 c. $e^x - 2\ln(e^x + 2) + c$
 d. $\frac{1}{2}e^{2x} + e^x + \ln|e^x - 1| + c$
15. a. $f(x) = \frac{1}{2}(1 - \cos(x^2))$
 b. $1 - \sqrt{3}$
 c. $\frac{1}{2}\ln\left(\frac{29}{20}\right)$
 d. $\frac{1}{2}(1 - e)$
16. a. 1 b. $\frac{1}{80}$ c. $2 - 2e^4$ d. 18
17. a. 12 b. $\frac{1}{4}\ln\left(\frac{5}{3}\right)$ c. $\frac{1}{18}$ d. 0
18. a. $\frac{1}{2}\ln\left(\frac{26}{29}\right)$ b. $-\frac{1}{4}$ c. $2e(e - 1)$ d. $\frac{1}{2}(e - 1)$

Exercise 6.4 Integration using the trigonometric identities

1. a. $-2\sin(2x)$
 c. $3\sec^2(3x)$
2. a. $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$
 c. $x + \frac{1}{8}\sin(8x) + c$
 e. $\frac{1}{2}x + \frac{1}{20}\sin(10x) + c$
3. a. $\frac{1}{2}x + \frac{1}{2}\sin(x) + c$
 c. $\frac{3}{2}x + \frac{9}{2}\sin\left(\frac{x}{3}\right) + c$
 e. $\frac{1}{2}x + \frac{3}{8}\sin\left(\frac{4x}{3}\right) + c$
4. a. $\frac{3}{8}x + \frac{1}{12}\sin(6x) + \frac{1}{96}\sin(12x) + c$
 b. $\frac{3}{8}x + \frac{1}{2}\sin(x) + \frac{1}{16}\sin(2x) + c$
 c. $\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c$
 d. $\frac{3}{8}x - \frac{1}{8}\sin(4x) + \frac{1}{64}\sin(8x) + c$
5. a. $\sin^2(x) + c$ or $-\frac{1}{2}\cos(2x) + c$
 b. $-\frac{1}{2}\cos(4x) + c$ or $\sin^2(2x) + c$
 c. $-\frac{1}{12}\cos(6x) + c$ or $\frac{1}{6}\sin^2(3x) + c$
 d. $\frac{1}{8}\cos(8x) + c$ or $-\frac{1}{4}\sin^2(4x) + c$
6. a. $\frac{1}{8}x - \frac{1}{32}\sin(4x) + c$
 c. $\frac{1}{4}x - \frac{1}{64}\sin(16x) + c$
 e. $\frac{3}{4}x - \frac{3}{8}\sin(2x) + c$
7. a. $-\frac{1}{5}\cos^5(x) + c$
 c. $-\frac{1}{3}\cos^6\left(\frac{x}{2}\right) + c$
 e. $\frac{5}{7}\sin^7\left(\frac{x}{5}\right) + c$
8. a. $\frac{1}{3}\cos^3(x) - \cos(x) + c$
 b. $\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + c$
 c. $\frac{1}{2}\cos^3(4x) - \frac{3}{2}\cos(4x) + c$
 d. $\frac{4}{3}\sin(3x) - \frac{4}{9}\sin^3(3x) + c$
 e. $\frac{1}{21}\cos^3(7x) - \frac{1}{7}\cos(7x) + c$
 f. $\frac{1}{6}\sin(6x) - \frac{1}{18}\sin^3(6x) + c$
- b. $4\cos(4x)$
 d. $-5\operatorname{cosec}^2(5x)$
 b. $\frac{1}{2}x - \frac{1}{8}\sin(4x) + c$
 d. $2x - \frac{1}{3}\sin(6x) + c$
 f. $\frac{1}{2}x - \frac{1}{24}\sin(12x) + c$
 b. $\frac{1}{2}x - \frac{3}{4}\sin\left(\frac{2x}{3}\right) + c$
 d. $x - 2\sin\left(\frac{x}{2}\right) + c$
 f. $\frac{1}{2}x - \frac{1}{6}\sin(3x) + c$
- b. $\frac{1}{8}x - \frac{1}{64}\sin(8x) + c$
 d. $\frac{1}{4}x - \frac{1}{48}\sin(12x) + c$
 f. $\frac{1}{2}x - \frac{3}{8}\sin\left(\frac{4x}{3}\right) + c$
 b. $-\frac{1}{8}\cos^4(2x) + c$
 d. $\frac{1}{15}\sin^5(3x) + c$
 f. $\frac{3}{16}\sin^8\left(\frac{2x}{3}\right) + c$

9. a. $2 \cos^3\left(\frac{x}{2}\right) - 6 \cos\left(\frac{x}{2}\right) + c$
 b. $6 \sin\left(\frac{x}{3}\right) - 2 \sin^3\left(\frac{x}{3}\right) + c$
 c. $\frac{2}{9} \cos^3\left(\frac{3x}{2}\right) - \frac{2}{3} \cos\left(\frac{3x}{2}\right) + c$
 d. $\frac{2}{5} \sin\left(\frac{5x}{2}\right) - \frac{2}{15} \sin^3\left(\frac{5x}{2}\right) + c$
 e. $\frac{4}{9} \cos^3\left(\frac{3x}{4}\right) - \frac{4}{3} \cos\left(\frac{3x}{4}\right) + c$
 f. $\frac{3}{4} \sin\left(\frac{4x}{3}\right) - \frac{1}{4} \sin^3\left(\frac{4x}{3}\right) + c$
10. a. $\frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + c$
 b. $\frac{1}{3} \sin^3(x) - \frac{1}{5} \sin^5(x) + c$
 c. $\frac{1}{10} \cos^5(2x) - \frac{1}{6} \cos^3(2x) + c$
 d. $\frac{1}{9} \sin^3(3x) - \frac{1}{15} \sin^5(3x) + c$
 e. $\frac{2}{5} \cos^5\left(\frac{x}{2}\right) - \frac{2}{3} \cos^3\left(\frac{x}{2}\right) + c$
 f. $\frac{2}{9} \sin^3\left(\frac{3x}{2}\right) - \frac{2}{15} \sin^5\left(\frac{3x}{2}\right) + c$
11. a. $\frac{12}{5} \cos^5\left(\frac{x}{3}\right) - 4 \cos^3\left(\frac{x}{3}\right) + c$
 b. $\frac{24}{25} \sin^5\left(\frac{5x}{4}\right) - \frac{8}{5} \sin^3\left(\frac{5x}{4}\right) + c$
 c. $\frac{1}{7} \cos^7(x) - \frac{1}{5} \cos^5(x) + c$
 d. $\frac{1}{10} \sin^5(2x) - \frac{1}{14} \sin^7(2x) + c$
 e. $\frac{1}{8} \cos^8(2x) - \frac{1}{6} \cos^6(2x) + c$
 f. $\frac{2}{27} \sin^9(3x) - \frac{2}{21} \sin^7(3x) + c$
12. a. $\cos(x) - \frac{2}{3} \cos^3(x) + c$ or $-\frac{1}{6} \cos(3x) + \frac{1}{2} \cos(x) + c$
 b. $\frac{1}{2} \sin(2x) - \frac{1}{3} \sin^3(2x) + c$ or $\frac{1}{4} \sin(2x) + \frac{1}{12} \sin(6x) + c$
 c. $\frac{1}{3} \cos(3x) - \frac{2}{9} \cos^3(3x) + c$ or $-\frac{1}{18} \cos(9x) + \frac{1}{6} \cos(3x) + c$
 d. $\frac{1}{4} \sin(4x) - \frac{1}{6} \sin^3(4x) + c$ or $\frac{1}{8} \sin(4x) + \frac{1}{24} \sin(12x) + c$
 e. $2 \cos\left(\frac{x}{2}\right) - \frac{4}{3} \cos^3\left(\frac{x}{2}\right) + c$ or $-\frac{1}{3} \cos\left(\frac{3x}{2}\right) - \cos\left(\frac{x}{2}\right) + c$
 f. $3 \sin\left(\frac{x}{3}\right) - 2 \sin^3\left(\frac{x}{3}\right) + c$ or $-\frac{1}{2} \cos(x) + \frac{3}{2} \cos\left(\frac{x}{3}\right) + c$
13. a. $\frac{1}{2} \tan(2x) + c$
 b. $3 \tan\left(\frac{x}{3}\right) + c$
 c. $\frac{1}{3} \tan^3(x) + c$
 d. $\frac{1}{4} \tan^4(x) + c$
 e. $\frac{1}{3} \tan^6(2x) + c$
 f. $\frac{16}{5} \tan^5\left(\frac{x}{2}\right) + c$
14. a. $\frac{1}{3} \tan^3(x) + \frac{1}{5} \tan^5(x) + c$
 b. $-\cot^3(2x) - \frac{3}{5} \cot^5(2x) + c$
 c. $\frac{4}{3} \tan^3\left(\frac{x}{2}\right) + \frac{4}{5} \tan^5\left(\frac{x}{2}\right) + c$
 d. $-\frac{1}{4} \cot^4(3x) - \frac{1}{6} \cot^6(3x) + c$
 e. $\tan^5\left(\frac{x}{5}\right) + \frac{5}{7} \tan^7\left(\frac{x}{5}\right) + c$
 f. $-\frac{1}{3} \cot^6(6x) - \frac{1}{2} \cot^8(6x) - \frac{1}{5} \cot^{10}(6x) + c$
15. $f(x) = -2 \cos^3(x) + \frac{1}{4}$
16. $f(x) = \frac{x}{2} - \frac{1}{16} \sin(8x) + \frac{7\pi}{8}$
17. $g(x) = \frac{2}{7} \cos^7\left(\frac{x}{2}\right) - \frac{2}{5} \cos^5\left(\frac{x}{2}\right)$
18. a. $\frac{-1}{n+1} \cos^{n+1}(x) + c$
 b. $\frac{1}{n+1} \sin^{n+1}(x) + c$
 c. $\frac{1}{n+1} \tan^{n+1}(x) + c$
 d. $\frac{1}{n+3} \cos^{n+3}(x) - \frac{1}{n+1} \cos^{n+1}(x) + c$
 e. $\frac{1}{n+1} \sin^{n+1}(x) - \frac{1}{n+3} \sin^{n+3}(x) + c$
 f. $\frac{1}{n+3} \tan^{n+3}(x) + \frac{1}{n+1} \tan^{n+1}(x) + c$

Exercise 6.5 Integration of inverse trigonometric functions

1. a. Does not exist
 b. $-\frac{\pi}{3}$
 c. 0.9
 d. Does not exist
 e. $-\frac{\pi}{6}$
 f. $\frac{1}{3}$
2. a. Does not exist
 b. $\frac{5\pi}{6}$
 c. $\frac{\pi}{6}$
 d. Does not exist
 e. $\frac{\pi}{3}$
 f. $\frac{1}{4}$
3. a. $\frac{\pi}{3}$
 b. $-\frac{\pi}{6}$
 c. $\frac{\pi}{6}$
 d. $\frac{5}{4}$
 e. $-\frac{\pi}{3}$
 f. 1.1
4. a. $\frac{1}{\sqrt{16-x^2}}$
 b. $\frac{5}{\sqrt{1-25x^2}}$

$$5. \text{ a. } \frac{-3}{\sqrt{16-9x^2}} \quad \text{b. } \frac{-1}{\sqrt{(x+1)(4-x)}}$$

$$6. \text{ a. } \frac{4}{1+16x^2} \quad \text{b. } \frac{5}{2x^2-6x+17}$$

$$7. \frac{8\sqrt{3}}{3}$$

$$8. -\frac{3}{5}$$

$$9. \text{ a. } \sin^{-1}\left(\frac{x}{10}\right) + c$$

$$\text{b. } 2 \sin^{-1}\left(\frac{x}{2}\right) + c$$

$$\text{c. } 4 \sin^{-1}\left(\frac{3x}{8}\right) + c$$

$$\text{d. } \frac{1}{5} \sin^{-1}\left(\frac{5x}{6}\right) + c$$

$$10. y = \frac{1}{4} \left(2 \cos^{-1}\left(\frac{4x}{5}\right) - \pi \right)$$

$$11. y = 2 \cos^{-1}\left(\frac{x}{2}\right) + 3$$

$$12. \text{ a. } 2 \tan^{-1}\left(\frac{x}{5}\right) + c$$

$$\text{b. } \frac{1}{10} \tan^{-1}\left(\frac{x}{10}\right) + c$$

$$\text{c. } \frac{1}{2} \tan^{-1}\left(\frac{3x}{8}\right) + c$$

$$\text{d. } \frac{1}{30} \tan^{-1}\left(\frac{5x}{6}\right) + c$$

$$13. \text{ a. } \frac{1}{2\sqrt{x(9-x)}}$$

$$\text{b. } \frac{-3}{|x|\sqrt{16x^2-9}}$$

$$\text{c. } \frac{-2e^{2x}}{\sqrt{16-e^{4x}}}$$

$$\text{d. } \frac{4}{|x|\sqrt{9x^2-16}}$$

$$\text{e. } \frac{6x}{x^4+9}$$

$$\text{f. } \frac{-30}{25x^2+36}$$

$$14. \text{ a. } \frac{1}{8} \tan^{-1}\left(\frac{2x-3}{4}\right) + c$$

$$\text{b. } \frac{1}{2} \sin^{-1}\left(\frac{2x-3}{4}\right) + c$$

$$\text{c. } \frac{1}{15} \tan^{-1}\left(\frac{5x-2}{3}\right) + c$$

$$\text{d. } \frac{1}{5} \sin^{-1}\left(\frac{5x-2}{4}\right) + c$$

$$15. \text{ a. } 2 \sin^{-1}\left(\frac{x+2}{3}\right) + c$$

$$\text{b. } \frac{2}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + c$$

$$\text{c. } 2 \sin^{-1}\left(\frac{3x+5}{7}\right) + c$$

$$\text{d. } \frac{2}{7} \tan^{-1}\left(\frac{3x+5}{7}\right) + c$$

$$16. \text{ a. } -\frac{1}{25} \sqrt{36-25x^2} + c$$

$$\text{b. } \frac{1}{10} \ln(64+25x^2) + c$$

$$\text{c. } -\frac{2}{5} \sqrt{49-25x^2} + c$$

$$\text{d. } \frac{1}{72} \ln(36x^2+49) + c$$

$$17. \text{ a. } \frac{5}{3} \sin^{-1}\left(\frac{3x}{5}\right) + \frac{1}{3} \sqrt{25-9x^2} + c$$

$$\text{b. } \frac{1}{6} \ln(9x^2+25) + \frac{1}{3} \tan^{-1}\left(\frac{3x}{5}\right) + c$$

$$\text{c. } \frac{1}{4} \left(\sin^{-1}(4x) + \sqrt{1-16x^2} \right) + c$$

$$\text{d. } 2 \tan^{-1}(4x) + \frac{1}{32} \ln(1+16x^2) + c$$

$$18. \text{ a. } \cos^{-1}\left(\frac{4x}{3}\right) - \frac{3}{16} \sqrt{9-16x^2} + c$$

$$\text{b. } \frac{1}{4} \tan^{-1}\left(\frac{4x}{3}\right) + \frac{1}{8} \ln(16x^2+9) + c$$

$$\text{c. } \sqrt{5-2x^2} + \frac{5\sqrt{2}}{2} \sin^{-1}\left(\frac{\sqrt{10}x}{5}\right) + c$$

$$\text{d. } \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\sqrt{2}x}{5}\right) - \frac{1}{2} \ln(2x^2+25) + c$$

$$19. \text{ a. } \frac{1}{q} \sin^{-1}\left(\frac{qx}{p}\right) + c$$

$$\text{b. } \frac{1}{q} \tan^{-1}\left(\frac{qx}{p}\right) + c$$

$$\text{c. } \frac{b}{q} \sin^{-1}\left(\frac{qx}{p}\right) - \frac{a}{q^2} \sqrt{p^2-q^2x^2} + c$$

$$\text{d. } \frac{b}{q} \tan^{-1}\left(\frac{qx}{p}\right) + \frac{a}{2q^2} \ln(q^2x^2+p^2) + c$$

$$20. \text{ a. } \frac{1}{2} \ln(2) - \frac{\pi}{8}$$

$$\text{b. } \frac{1}{4} \ln(2) + \frac{\pi}{8}$$

$$\text{c. } 9 + \frac{5\pi}{2}$$

$$\text{d. } \sqrt{3} - \frac{5\pi}{3}$$

Exercise 6.6 Integration by parts

$$1. \text{ a. } \cos(x) + x \sin(x) + c$$

$$\text{b. } (x-1)e^x + c$$

$$\text{c. } \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + c$$

$$\text{d. } \frac{1}{4} \sin(2x) - \frac{1}{2}x \cos(2x) + c$$

$$2. \text{ a. } -\frac{1}{4}x \cos(4x) + \frac{1}{16} \sin(4x) + c$$

$$\text{b. } \frac{1}{3}x \sin(3x) + \frac{1}{9} \cos(3x) + c$$

$$\text{c. } -\frac{1}{4}(2x+1)e^{-2x} + c$$

$$\text{d. } 2x \sin\left(\frac{x}{2}\right) + 4 \cos\left(\frac{x}{2}\right) + c$$

$$3. \text{ a. } x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$

$$\text{b. } -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + c$$

$$4. \text{ a. } \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + c$$

$$\text{b. } \frac{1}{6}x^2(x+1)^6 - \frac{1}{21}x(x+1)^7 + \frac{1}{168}(x+1)^8 + c$$

$$5. \text{ a. } x \ln(2x) - x + c$$

$$\text{b. } x \sin^{-1}(x) + \sqrt{1-x^2} + c$$

$$6. \text{ a. } -x \cos(x) + \sin(x) + \cos(x) + c$$

$$\text{b. } (x+1)e^x + c$$

$$7. \text{ a. } \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + c$$

$$\text{b. } \frac{1}{2}(x+1)^2 \ln(x) - \frac{1}{4}x^2 - x - \frac{1}{2} \ln(x) + c$$

$$8. \text{ a. } \frac{1}{3}x^3 \sin^{-1}(x) + \frac{1}{9}x^2 \sqrt{1-x^2} + \frac{2}{9} \sqrt{1-x^2} + c$$

$$\text{b. } (x^2+1)e^x + c$$

$$9. \text{ a. } e^x(x^3-3x^2+6x-6) + c$$

$$\text{b. } -e^{-x}(x+1) + c$$

$$10. \text{ a. } \frac{e^x}{2} (\cos(x) + \sin(x)) + c$$

$$\text{b. } \frac{e^{2x}}{5} (2 \cos(x) + \sin(x)) + c$$

$$11. \text{ a. } f(x) = \frac{e^x}{5} (\sin(2x) - 2 \cos(2x)) + c$$

$$\text{b. } f(x) = 13e^{2x} (2 \cos(3x) + 3 \sin(3x)) + c$$

12. $\frac{x^{n+1}}{n+1} \left(\ln(ax) - \frac{1}{n+1} \right) + c$
 13. $\frac{1}{2} [\ln(x)]^2 + c, x > 0$
 14. $\frac{x}{2} [\sin(\ln(x)) - \cos(\ln(x))] + c$

Exercise 6.7 Integration involving partial fractions

1. a. $A = 1, B = 2$
 b. $A = -1, B = 2$
 c. $A = 3, B = 10$
 d. $A = 2, B = -1$
 e. $A = -1, B = 2, C = 2$
 f. $A = 2, B = 1, C = -2$
2. a. $\frac{1}{x+1} - \frac{1}{x+2}$ b. $\ln \left| \frac{x+1}{x+2} \right| + c$
3. a. $\frac{3}{x-2} - \frac{3}{x+2}, 3 \ln \left| \frac{x-2}{x+2} \right| + c$
 b. $\frac{9}{2(x+3)} + \frac{3}{2(x-1)}, \frac{9}{2} \ln |x+3| + \frac{3}{2} \ln |x-1| + c$
 c. $\frac{2}{x-2} + \frac{1}{x+1}, 2 \ln |(x-2)| + \ln |x+1| + c$
 d. $\frac{5}{1-x} - \frac{3}{3-x}, 3 \ln |3-x| - 5 \ln |1-x| + c$
4. a. $3 \ln |x-4| - 2 \ln |x+4| + c$
 b. $3 \ln |x-2| - 2 \ln |x| + c$
 c. $2 \ln |x-2| + 5 \ln |x+3| + c$
 d. $2 \ln |(2x+1)(x-3)| + c$
5. a. $f(x) = 4 \ln |x+1| - \ln |3x-2| + c$
 b. $f(x) = 4 \ln |x+3| - \ln |2-x| + c$
 c. $f(x) = 2 \ln |x| - 3 \ln |x+2| + c$
 d. $f(x) = 5 \ln |x-3| + 4 \ln |x+4| + c$
6. a. $\frac{-2}{x+6} + \frac{3}{4-x}$
 b. $-2 \ln |x+6| - 3 \ln |4-x| + c$
7. a. $5 \ln |x| - 2 \ln |x+2| + c$
 b. $2 \ln |x-2| + 3 \ln |x+1| + c$
 c. $2 \ln |x+1| - \ln |x+2| + c$
 d. $5 \ln |x-6| + \ln |x+1| + c$
8. a. $4 \ln |x-3| + \ln |x-1| + c$
 b. $3 \ln |x+1| - 2 \ln |x+6| + c$
 c. $2 \ln |x+3| + 5 \ln |x-3| + c$
 d. $3 \ln |x+1| + 4 \ln |x-1| + c$
9. a. $\frac{1}{2} \ln |2x+1| + 2 \ln |x-2| + c$
 b. $\frac{4}{3} \ln |3x-2| - 2 \ln |x+3| + c$
 c. $2 \ln |x-2| - \frac{3}{2} \ln |2x-1| + c$
 d. $\ln \left| \frac{2+x}{2-x} \right| + c$
10. a. $4 \ln |x+2| + \frac{3}{x+2} + c$
 b. $5 \ln |x-5| + \frac{1}{x-5} + c$
 c. $\frac{5}{x+3} + 2 \ln |x+3| + c$
- d. $\frac{3}{2} \ln |2x-3| - \frac{2}{2x-3} + c$
 11. a. $x - \frac{1}{3} \ln |x| - \frac{8}{3} \ln |x+3| + c$
 b. $x + 7 \ln |x-4| - \ln |x| + c$
 c. $x + \ln |x+1| - 6 \ln |x+3| + c$
 d. $x + 4 \ln |x-3| - \ln |x+1| + c$
12. a. $x + \ln |x+2| + 2 \ln |x-2| + c$
 b. $x - 3 \ln |x+1| - \frac{4}{x+1} + c$
 c. $2x + 3 \ln |x-3| + \frac{2}{x-3} + c$
 d. $x - 6 \ln |x+5| + c$
13. a. $\frac{1}{2} x^2 + x - 10 \ln |x+2| + 4 \ln |x+1| + c$
 b. $x^2 + x + 3 \ln |x+1| - \ln |x-1| + c$
 c. $\frac{1}{2} x^2 + 4x + 22 \ln |x-5| + 3 \ln |x+1| + c$
 d. $\frac{1}{2} x^2 + 5x - \frac{4}{x-2} + 12 \ln |x-2| + c$
14. a. $2 \ln(x^2+9) - \frac{2}{3} \tan^{-1} \left(\frac{x}{3} \right) + c$
 b. $\frac{1}{2} \ln(x^2+4) + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + c$
 c. $\ln |x-2| - \frac{1}{2} \ln(x^2+9) - \frac{5}{3} \tan^{-1} \left(\frac{x}{3} \right) + c$
 d. $\ln |x+3| - \frac{1}{2} \ln(x^2+16) + \frac{3}{4} \tan^{-1} \left(\frac{x}{4} \right) + c$
15. a. $-4 \ln |x-1| + \frac{33}{15} \ln |x+2| + \frac{34}{5} \ln |x-3| + c$
 b. $2 \ln |x+1| + 3 \ln |x-2| - 4 \ln |x+3| + c$
 c. $2 \ln |x-1| - \ln |x+2| - \frac{1}{x+2} + c$
 d. $\ln(x^2+1) - \tan^{-1}(x) - 3 \ln |2-x| + c$
16. a. $\frac{1}{2ab} \ln \left| \frac{bx-a}{bx+a} \right| + c$
 b. $\frac{1}{2b^2} \ln |b^2x^2 - a^2| + c$
 c. $\frac{1}{a^2} \ln |ax-b| - \frac{b}{a^2(ax-b)} + c$
 d. $\frac{1}{aq-bp} \ln \left| \frac{qx+b}{px+a} \right| + c$

6.8 Review: exam practice

1. a. $\frac{1}{14} (2x-1)^7 + c$ b. $-\frac{1}{2(2x-1)} + c$
 c. $\frac{1}{2} \ln |2x-1| + c$ d. $\frac{3}{4} \sqrt{(2x-1)^2} + c$
2. a. $\frac{1}{2} \ln(x^2+2) + c$ b. $\frac{1}{2} \sin(x^2+2) + c$
 c. $\frac{3}{2} e^{x^2+2} + c$ d. $2 \ln(3)$
3. a. $-\frac{1}{8} \cos(4x) + c$ b. $x + c$
 c. $-\frac{1}{8} \cos^4(2x) + c$ d. $\frac{1}{8} \sin^4(2x) + c$

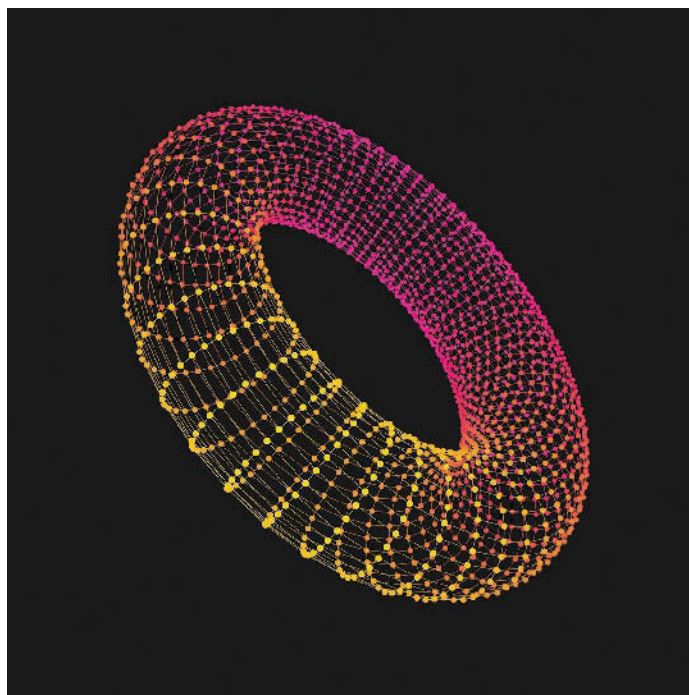
4. a. $-\frac{1}{3} \ln |\cos(3x)| + c$
 c. $\frac{1}{6} \tan^2(3x) + c$
5. a. $\frac{5\pi}{6}$ b. $-\frac{\pi}{3}$
6. a. Does not exist
 c. $\frac{7\pi}{6}$
7. a. $\frac{1}{\sqrt{4-x^2}}$
 c. $-\frac{2x}{\sqrt{1-x^4}}$
8. $\sin^{-1}\left(\frac{x}{a}\right) + c$
9. $\tan^{-1}\left(\frac{x}{a}\right) + c$
10. a. $\sin^{-1}\left(\frac{x}{4}\right) + c$
 c. $2 \sin^{-1}\left(\frac{5x}{7}\right) + c$
11. a. $-x \cos(x) + \sin(x) + c$
 b. $\frac{2}{3} x \sin(3x) + \frac{2}{9} \cos(3x) + c$
- b. $\frac{1}{3} \ln |\sin(3x)| + c$
 d. $\frac{1}{9} \tan^3(3x) + c$
- c. $\frac{\sqrt{2}}{2}$ d. 1.5
- b. $\frac{\sqrt{2}}{2}$
 d. $-\frac{\pi}{3}$
- b. $\frac{3}{1+9x^2}$
- d. $\frac{1}{2\sqrt{x}(1+x)}$
- b. $\frac{1}{4} \sin^{-1}(4x) + c$
 d. $\cos^{-1}(2x) + c$
- c. $-x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + c$
 d. $\frac{1}{3} x^3 \ln(x) - \frac{1}{9} x^3 + c$
12. a. $3 \ln|x-2| + 2 \ln|x-3| + c$
 b. $\ln|x+1| - \frac{3}{x+1} + c$
 c. $-2 \ln|x| + \frac{1}{x} + 2 \ln|x-1| + c$
 d. $x - 12 \ln|x-2| + 17 \ln|x-3| + c$
13. $\frac{1}{2} \ln(x^2 + x + 1) + \frac{\sqrt{3}}{3} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$
14. $\frac{3 \sin(3x) - \cos(3x)}{10e^x} + c$
15. $x \tan^{-1}(x) - \frac{1}{2} \ln(1+x^2) + c$
16. $y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{2\pi}{3}$
17. $\frac{1}{2} x^2 (\ln(x))^2 - \frac{1}{2} x^2 \ln(x) + \frac{1}{4} x^2 + c$
18. $\frac{1}{2} x^2 \tan^{-1}(x) + \frac{1}{2} \tan^{-1}(x) - \frac{1}{2} x + c$
19. $-x + 4\sqrt{x} - 4 \ln(1 + \sqrt{x}) + c$
20. $\frac{1}{5} \ln\left(\frac{2 - \sin(x)}{\sin(x) + 3}\right) + c$

7 Applications of integral calculus

7.1 Overview

In chapter 6 you learnt some techniques of integration. These techniques can be used to calculate the area between functions and the axes and also calculate the area between two different functions. In some cases, it is not possible to calculate the integral by algebraic techniques. In this instance, a method to approximate the area is required. Integration can also be used to determine the volume of solid.

In Mathematical Methods, you learnt to use the trapezoidal rule to approximate areas under curves, in this chapter, you learn about using Simpson's rule to determine the approximation. This method can also be used to estimate the area of irregular shapes.



LEARNING SEQUENCE

- 7.1 Overview
- 7.2 Area between a function and the axes
- 7.3 Area between functions
- 7.4 Volumes of solids of revolution
- 7.5 Volumes of revolution
- 7.6 Approximation using Simpson's rule
- 7.7 Exponential probability density function
- 7.8 Review: exam practice

Fully worked solutions are available for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.2 Area between a function and the axes

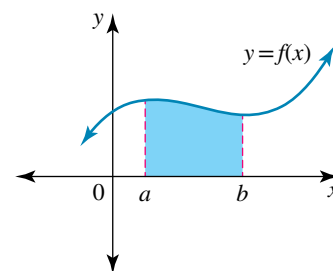
7.2.1 The domain of the integrand

Basic integration techniques and evaluating areas bounded by functions and the x -axis have been covered in the Mathematical Methods course.

The Fundamental Theorem of Calculus states that if $f(x)$ is continuous over the closed interval $[a, b]$ and $\int f(x) dx = F(x) + c$,

then $\int_a^b f(x) dx = F(b) - F(a)$. This result will also give the area

bounded by the function $y = f(x)$, the x -axis and the lines $x = a$ and $x = b$, if $f(x) \geq 0$ over $[a, b]$.



WORKED EXAMPLE 1

For each of the following integrals, identify the domain of the integrand and whether the integrand exists.

a. $\int_{-2}^2 \frac{1}{\sqrt{9-x^2}} dx$

b. $\int_0^4 \frac{2}{(x-1)(x+3)} dx$

THINK

a. 1. For the integrand to exist, $\sqrt{9-x^2}$ must be greater than 0.

2. Solve the inequation for x .

3. Since the domain of the function $\frac{1}{\sqrt{9-x^2}}$ is $x \in (-3, 3)$ and is continuous over this domain the integral will exist for all values of x between the terminals -2 and 2 .

b. 1. The integrand does not exist for $x = -3$ and 1 , as these values make the denominator equal to zero.

2. State the domain of the function.

3. The integral does not exist for all values of x between the terminals 0 and 4 (as 1 lies in the interval).

WRITE

a. The integrand exists if $\sqrt{9-x^2} > 0$.
That is, $9-x^2 > 0$.

$$x^2 < 9$$

$$-3 < x < 3$$

The domain is $(-3, 3)$.

The integral exists.

b. $x \neq -3, 1$

$$\mathbb{R} \setminus \{-3, 1\}$$

The integral does not exist.

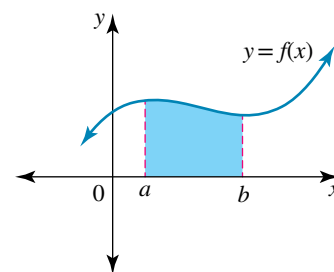
When evaluating a definite integral, the result can be either a positive or negative number depending on whether the area sits above or below the x -axis. As the definite integral is used to calculate area, this means that the area is a signed area, that is it may be positive or negative. When stating the area of a region, the result is always positive.

7.2.2 Areas above the x-axis

If $f(x) \geq 0$ for the region $a \leq x \leq b$, then the region is

above the x -axis and $\int_a^b f(x) dx > 0$.

This means that the area of the region can be found using $A = \int_a^b f(x) dx$.

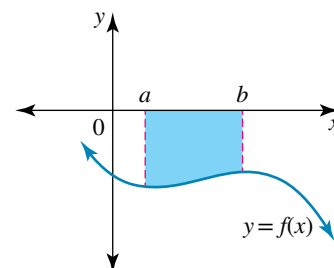


7.2.3 Areas below the x-axis

If $f(x) < 0$ for the region $a \leq x \leq b$, then the region is below the x -axis and

$\int_a^b f(x) dx < 0$. This means that the area of the region can be found using:

$$\begin{aligned} A &= \left| \int_a^b f(x) dx \right| \\ &= - \int_a^b f(x) dx \\ &= \int_b^a f(x) dx \end{aligned}$$



Notice how a and b have swapped positions.

WORKED EXAMPLE 2

Determine the area of the region bounded by the curve $y = x^2 - 4x + 3$ and the x -axis.

THINK

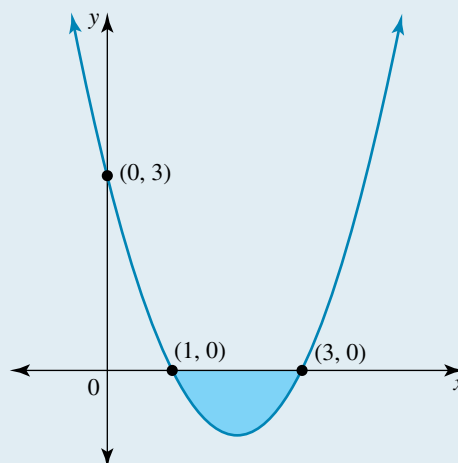
- Factorise the quadratic to determine the x -intercepts. The y -intercept can be easily found by substituting $x = 0$ into the equation.
- Sketch the graph, shading the required area.

WRITE

$$y = x^2 - 4x + 3$$

$$y = (x - 3)(x - 1)$$

The graph crosses the x -axis at $x = 1$ and $x = 3$ and crosses the y -axis at $y = 3$.



3. The required area is below the x -axis and will evaluate to a negative number. The area must be given by the absolute value or the negative of this definite integral.

$$A = \left| \int_1^3 (x^2 - 4x + 3) dx \right|$$

$$= - \int_1^3 (x^2 - 4x + 3) dx$$

4. Anti-differentiate the integrand.

$$A = - \left[\frac{1}{3}x^3 - 2x^2 + 3x \right]_1^3$$

5. Evaluate the definite integral.

$$A = - \left[\left(\frac{1}{3} \times 3^3 - 2 \times 3^2 + 3 \times 3 \right) - \left(\frac{1}{3} \times 1^3 - 2 \times 1^2 + 3 \times 1 \right) \right]$$

$$= \frac{4}{3}$$

6. State the value of the required area.

The area is $\frac{4}{3}$ square units.

7.2.4 Areas above and below the x -axis

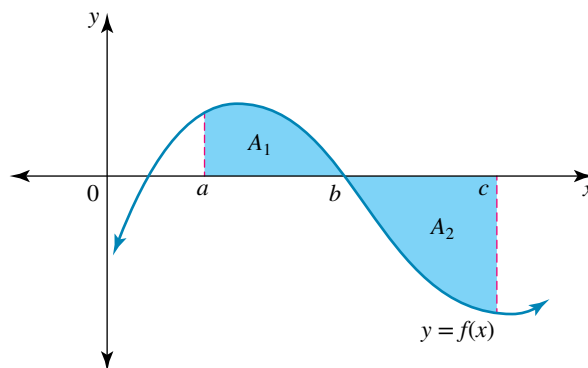
For regions that are combinations of areas above and below the x -axis, each area needs to be calculated by separate integrals.

For example, from the diagram $A_1 = \int_a^b f(x) dx$

and $A_2 = - \int_b^c f(x) dx$. This means that the total area can be found using $\text{Area} = A_1 + A_2$

$$= \int_a^b f(x) dx - \int_b^c f(x) dx.$$

Note that this is not the same as finding $\int_a^c f(x) dx$.



WORKED EXAMPLE 3

Determine the area bounded by the curve $y = x^3 - 9x$ and the x -axis.

THINK

1. Factorise the cubic to determine the x -intercepts.

WRITE

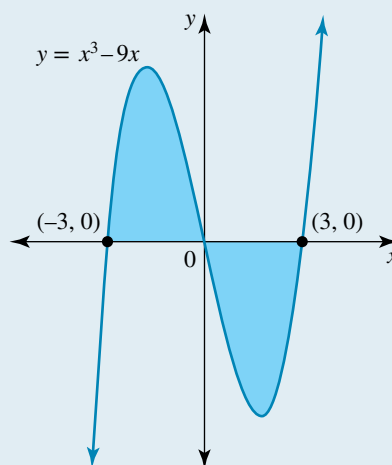
$$y = x^3 - 9x$$

$$y = x(x^2 - 9)$$

$$y = x(x + 3)(x - 3)$$

The graph crosses the x -axis at $x = 0$ and $x = \pm 3$.

2. Sketch the graph, shading the required area.



3. The required area is given by a definite integral.

If we calculate $A = \int_{-3}^3 (x^3 - 9x)dx$ it comes to zero, as the positive and negative area have cancelled out.

4. Anti-differentiate the integrand.

5. Evaluate the definite integral. Note that A_2 gives a negative answer since the area is below the x -axis. When calculating the total area, we take the absolute value of any areas below the x -axis to ensure they are positive. The total area required is the sum of A_1 and $|A_2|$ or $2 \times A_1$ due to the symmetry of the two areas.

6. State the value of the required area.

$$\text{Total area} = \int_{-3}^0 (x^3 - 9x)dx + \left| \int_0^3 (x^3 - 9x)dx \right|$$

$$\text{Let } A_1 = \int_{-3}^0 (x^3 - 9x)dx \text{ and}$$

$$A_2 = \int_0^3 (x^3 - 9x)dx, \text{ so that } A_1 > 0 \text{ and}$$

$$A_2 < 0, \text{ but } A_1 = |A_2| \text{ by symmetry.}$$

$$A_1 = \int_{-3}^0 (x^3 - 9x)dx$$

$$A_1 = \left[\frac{1}{4}x^4 - \frac{9}{2}x^2 \right]_{-3}^0$$

$$A_1 = \left[0 - \left(\frac{1}{4} \times (-3)^4 - \frac{9}{2} \times (-3)^2 \right) \right]$$

$$A_1 = \frac{81}{4}$$

$$= 20\frac{1}{4}$$

$$A_2 = \int_0^3 (x^3 - 9x)dx = -\frac{81}{4} = -20\frac{1}{4}$$

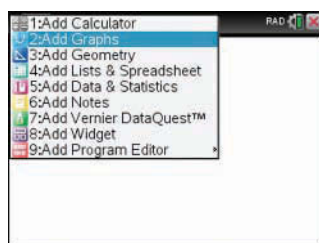
$$A_1 + |A_2| = 2 \times \frac{81}{4} = \frac{81}{2} = 40\frac{1}{2}$$

The area is $40\frac{1}{2}$ square units.

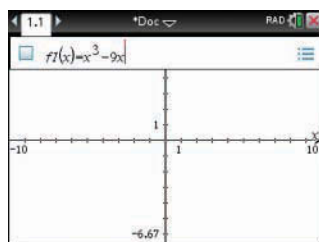
TI | THINK

- On a Calculator page, press MENU then select
2: Add Graphs
Press MENU then select 9: Settings
Change Display Digits to Float 12 and then click on Make Default.

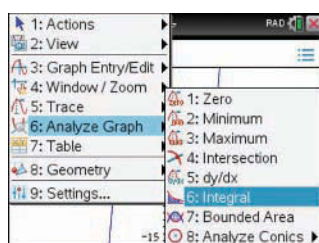
WRITE



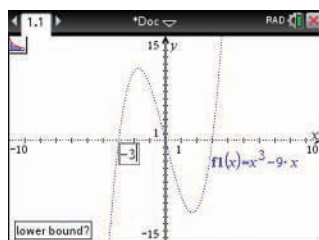
- Complete the entry line as $x^3 - 9x$ then press the ENTER button.



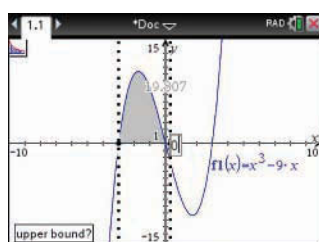
- Press MENU then select
6: Analyze Graph
6: Integral



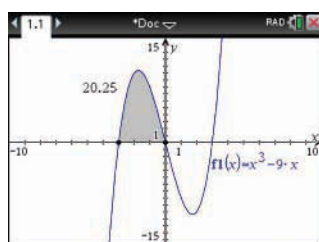
- Select the lower bound, $x = -3$, by entering the number -3.



- Select the upper bound, $x = 0$, by entering the number 0.



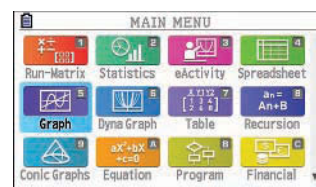
- The answer appears on the screen.
Note: Due to symmetry, the final answer can be determined as $2 \times 20.25 = 40.5$. Alternatively, the process can be repeated for the region, $0 \leq x \leq 3$.



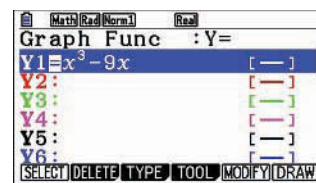
CASIO | THINK

- On a Main Menu screen, select GRAPH

WRITE



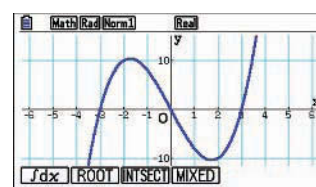
- Complete the entry line as $x^3 - 9x$ then press the EXE or DRAW button.



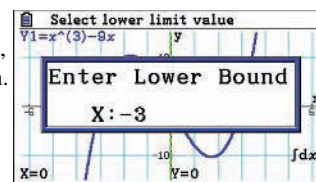
- To calculate the definite integral, select SHIFT

$$\int dx$$

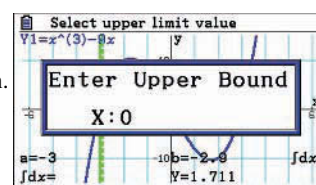
$$\int dx \text{ (a second time)}$$



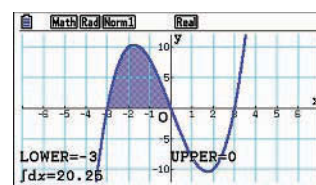
- Select the lower bound, $x = -3$, by first pressing -3, followed by the EXE button.



- Select the upper bound, $x = 0$, by first pressing 0, followed by the EXE button.



- The answer appears on the screen.
Note: Due to symmetry, the final answer can be determined as $2 \times 20.25 = 40.5$. Alternatively, the process can be repeated for the region, $0 \leq x \leq 3$.



When we are determining areas under functions that are not easy to sketch, the area can be calculated provided that the x -intercepts can be determined. If there are no x -intercepts in the region $[a, b]$, then the area can be found using $\left| \int_a^b f(x) dx \right|$. If an x -intercept occurs at c where $a < c < b$, then the area can be found using $\left| \int_a^c f(x) dx \right| + \left| \int_c^b f(x) dx \right|$.

WORKED EXAMPLE 4

Determine the area between the graph of $y = \frac{x}{x^2 - 1}$, the x -axis, $x = 2$ and $x = 3$.

THINK

1. It is necessary to identify if the x -intercept occurs in the region $2 \leq x \leq 3$.
2. As the x -intercept is not in the region $2 \leq x \leq 3$, the area can be found by calculating $\int_2^3 \frac{x}{x^2 - 1} dx$.
3. Check the domain of the function to see if the integral exists.
4. Use $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$ to determine the integral.
5. Evaluate the definite integral.
6. State the value of the required area.

WRITE

$$\frac{x}{x^2 - 1} = 0$$

$$x = 0$$

The x -intercept is not in the region $2 \leq x \leq 3$.

$$\text{Area} = \int_2^3 \frac{x}{x^2 - 1} dx$$

$$x^2 - 1 \neq 0$$

$$x \neq \pm 1$$

$$\text{dom}(y) = \mathbb{R} \setminus \{-1, 1\}$$

The integral exists.

$$= \frac{1}{2} \int_2^3 \frac{2x}{x^2 - 1} dx$$

$$= \frac{1}{2} [\ln |x^2 - 1|]_2^3$$

$$= \frac{1}{2} (\ln(9 - 1) - \ln(4 - 1))$$

$$= \frac{1}{2} (\ln(8) - \ln(3))$$

$$= \frac{1}{2} \ln\left(\frac{8}{3}\right)$$

$$\text{The area is } \frac{1}{2} \ln\left(\frac{8}{3}\right) \text{ or}$$

approximately 0.49 square units.

studyon

Units 3 & 4

Area 4

Sequence 2

Concept 1

Area between a function and the axes Summary screen and practice questions

Exercise 7.2 Area between a function and the axes

Technology free

1. **WE1** For each of the following integrals, identify the domain of the integrand and whether the integrand exists.

a. $\int_1^2 \frac{1}{9-x^2} dx$

b. $\int_0^1 \frac{-1}{\sqrt{4-x^2}} dx$

c. $\int_4^5 \frac{dx}{\sqrt{16-x^2}}$

d. $\int_{-1}^2 \frac{dx}{1+x^2}$

e. $\int_1^4 \frac{2}{x} dx$

f. $\int_1^2 \frac{dx}{x(x+1)}$

2. Evaluate the integrals in question 1 provided that the integrand, $f(x)$, exists for all values within the domain of the integral.
3. For each of the following integrals, identify the domain of the integrand and whether the integrand exists.

a. $\int_{-1}^1 \frac{4x+10}{x^2+5x+6} dx$

b. $\int_0^2 \frac{1}{(x-1)^2} dx$

c. $\int_1^3 \frac{3x+2}{x^2-8x+12} dx$

d. $\int_0^{\frac{\sqrt{3}}{2}} \frac{dx}{4x^2+9}$

e. $\int_{-1}^0 \frac{dx}{\sqrt{1-9x^2}}$

f. $\int_{\frac{1}{2}}^2 (2x-1)^{\frac{3}{2}} dx$

4. Evaluate the integrals in question 3 provided that the integrand, $f(x)$, exists for all values within the domain of the integral.

5. **WE2** Determine the area bounded by the following curves and the x -axis.

a. $y = x^2 - 5x + 6$

b. $y = 1 - x^2$

c. $y = 8 - x^2$

d. $y = 12 - 3x^2$

e. $y = x^2 - 2x - 15$

f. $y = 3x^2 - 10x - 8$

6. **WE3** Determine the area bounded by the following curves and the x -axis.

a. $y = x^3 - 4x$

b. $f(x) = x(x-2)(x-3)$

c. $h(x) = (x-1)(x+2)(x+5)$

d. $g(x) = x^3 - 4x^2$

7. a. **WE4** Determine the area between the graph of

$y = \frac{x}{x^2+2}$, the x -axis, $x = 1$ and $x = 2$.

- b. Determine the area between the graph of $y = \frac{x}{x^2+2}$, the x -axis, $x = -1$ and $x = 1$.

- c. Determine the area between the graph of $y = 3 - e^{2x}$, the x -axis, $x = -2$ and $x = 0$.

- d. Determine the area between the graph of $y = 3 - e^{2x}$, the x -axis, $x = -1$ and $x = 1$.

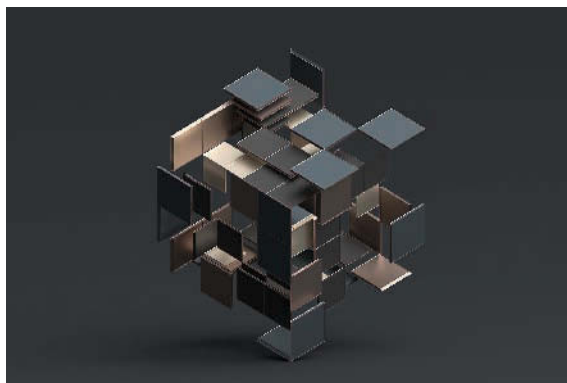
8. For each of the following functions, identify the maximal domain of the function and determine the area described.

- a. The area between the function $y = \sqrt{2x+1}$, the coordinate axes and the line $x = 4$.

- b. The area between the function $y = \frac{1}{3x+5}$, the coordinate axes and $x = 3$.

- c. The area between the function $y = \frac{1}{(2x+3)^2}$, the x -axis, $x = 1$ and $x = 2$.

- d. The area between the function $y = \frac{x}{\sqrt{16-3x}}$, the coordinate axes and $x = 5$.



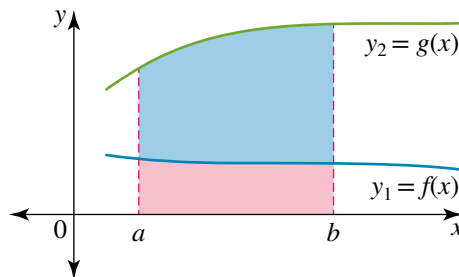
9. a. The graph of $y = \frac{x}{x^2 - c}$ has vertical asymptotes at $x = \pm 2$. Identify the value of c and determine the area bounded by the curve, the x -axis and the lines $x = 3$ and $x = 5$.
- b. Determine the area bounded by $y = x \cos(x^2)$, the x -axis, $x = 0$ and $x = \frac{\sqrt{\pi}}{2}$.
- c. Determine the area bounded by $y = xe^{-x^2}$, the x -axis, $x = 0$ and $x = 2$.
- d. Determine the area bounded by $y = \frac{x}{\sqrt{x^2 + 4}}$, the coordinate axes and $x = 2\sqrt{3}$.
10. a. Determine the area bounded by the curve $y = \frac{5}{x^2 + x - 6}$, the coordinate axes and $x = -1$.
- b. Determine the area bounded by the curve $y = \frac{2x - 3}{4 + 3x - x^2}$, the x -axis, $x = 2$ and $x = 3$.
- c. Determine the area bounded by $y = \frac{21}{40 - 11x - 2x^2}$, the coordinate axes and $x = -5$.
- d. Determine the area bounded by $y = \frac{x^3 - 9x + 9}{x^2 - 9}$, the x -axis, $x = 4$ and $x = 6$.
11. Determine the area between the curve $y^2 = 4 - x$ and the y -axis.
12. Determine the area of the loop with the equation $y^2 = x^2(4 - x)$.
13. If $a > 0$, determine the area bounded by the curve $y^2 = a - x$ and the y -axis.
14. If $a > 0$, determine the area of the loop $y^2 = x^2(a - x)$.

7.3 Area between functions

When trying to determine the area between two functions, $f(x)$ and $g(x)$, over an interval $a \leq x \leq b$ the method depends on whether the functions intersect or do not intersect in the interval.

7.3.1 Area between two functions that do not intersect in the required interval

If $y_1 = f(x)$ and $y_2 = g(x)$ are two continuous functions that do not intersect between $x = a$ and $x = b$, then the area between the curves can be found by subtracting the lower function from the higher function.



If we call the area between $y_2 = g(x)$, $x = a$ and $x = b$, A_1 , then $A_1 = \int_a^b g(x) dx$.

If we call the pink area between $y_1 = f(x)$, $x = a$ and $x = b$, A_2 , then $A_2 = \int_a^b f(x) dx$.

The required blue area can be found using $A = A_1 - A_2$

$$= \int_a^b g(x)dx - \int_a^b f(x)dx.$$

By the properties of definite integrals,

$$A = A_1 - A_2$$

$$= \int_a^b (g(x) - f(x)) dx$$

$$= \int_a^b (y_2 - y_1) dx$$

Note: it is only necessary to evaluate one definite integral, the difference between y_1 and y_2 .

WORKED EXAMPLE 5

Determine the area between the parabola $y = x^2 - 2x - 15$ and the straight line $y = 2x - 3$.

THINK

- Factorise the quadratic to determine the x -intercepts. The y -intercept can be found by substituting $x = 0$ into the equation.
- Determine the x -values of the points of intersection between the parabola and the straight line.
- Sketch the graph of the parabola and the straight line on one set of axes, shading the required area.

WRITE

$$y = x^2 - 2x - 15$$

$$y = (x - 5)(x + 3)$$

The parabola crosses the x -axis at $x = 5$ and $x = -3$ and crosses the y -axis at $y = -15$.

The straight line crosses the x -axis at $x = \frac{3}{2}$ and crosses the y -axis at $y = -3$.

Let $y_1 = x^2 - 2x - 15$ and $y_2 = 2x - 3$.

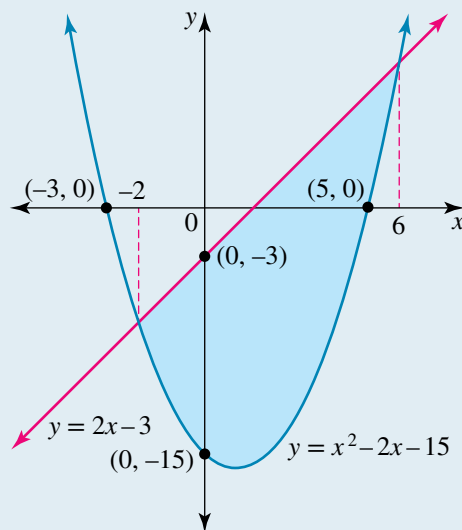
To find the points of intersection, solve $y_1 = y_2$.

$$x^2 - 2x - 15 = 2x - 3$$

$$x^2 - 4x - 12 = 0$$

$$(x - 6)(x + 2) = 0$$

$$x = 6, -2$$



4. The line $y_2 = 2x - 3$ is always above the parabola $y_1 = x^2 - 2x - 15$ in the region bounded by the line and the curve. This means that the area can be determined

by calculating $\int_{-2}^6 (y_2 - y_1) dx$.

5. Anti-differentiate the integrand.

6. Evaluate the definite integral.

7. State the value of the required area between the parabola and the straight line.

$$A = \int_a^b (y_2 - y_1) dx \text{ with } a = -2, b = 6,$$

$$y_1 = x^2 - 2x - 15 \text{ and } y_2 = 2x - 3.$$

$$y_2 - y_1 = -x^2 + 4x + 12$$

$$A = \int_{-2}^6 (-x^2 + 4x + 12) dx$$

$$A = \left[-\frac{1}{3}x^3 + 2x^2 + 12x \right]_{-2}^6$$

$$A = \left[\left(-\frac{1}{3} \times 6^3 + 2 \times 6^2 + 12 \times 6 \right) - \left(-\frac{1}{3} \times (-2)^3 + 2 \times (-2)^2 + 12 \times (-2) \right) \right]$$

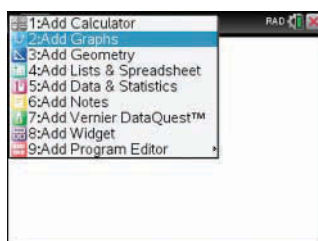
$$A = 85\frac{1}{3}$$

The area between the straight line and the parabola is $85\frac{1}{3}$ square units.

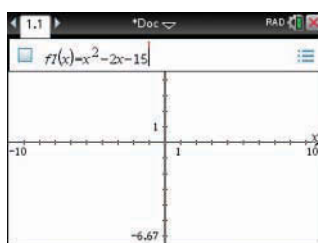
TI | THINK

1. On a Calculator page, press MENU then select 2: Add Graphs

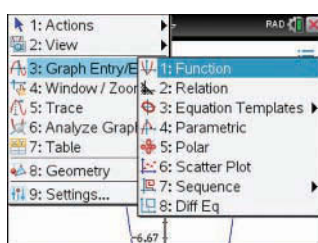
WRITE



2. Complete the entry line as $x^2 - 2x - 15$ then press the ENTER button.



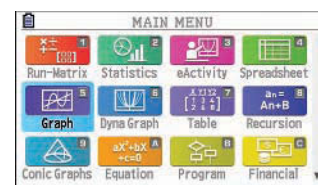
3. Press MENU then select 3: Graph Entry/Edit 1: Function



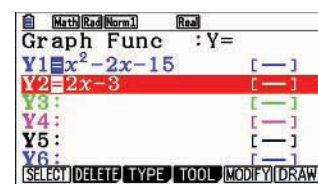
CASIO | THINK

1. On a Main Menu screen, select GRAPH

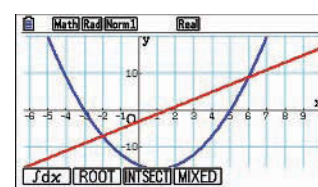
WRITE



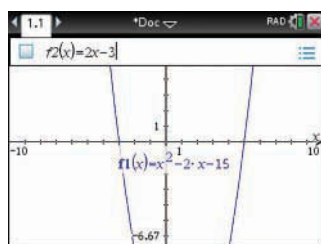
2. Complete the entry line in Y1 and Y2 as $x^2 - 2x - 15$ $2x - 3$ then press the EXE or DRAW button.



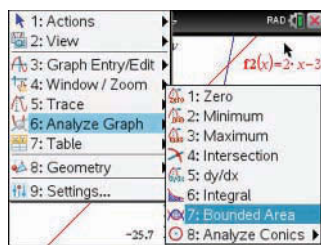
3. To calculate the area bounded by the two functions select SHIFT F5
 $\int dx$
 INTSECT



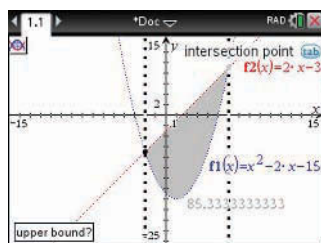
4. Complete the entry line as $2x - 3$ then press the ENTER button.



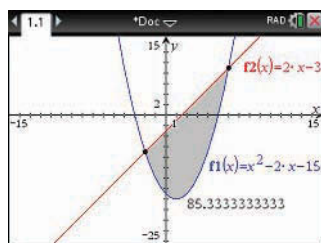
5. Press MENU then select
6: Analyze Graph
7: Bounded Area



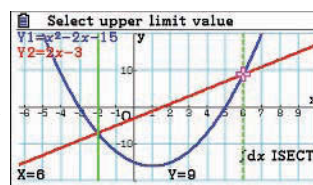
6. Select the two intersection points using the cursor.
Press the ENTER button on both intersection points.



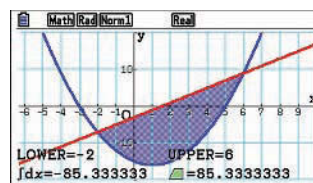
7. The answer appears on the screen as textbox next to the shaded area.



4. Begin by pressing the EXE button to select the lower intersection point.
Press the right cursor button to select the upper intersection point.
Select this intersection point by pressing the EXE button.

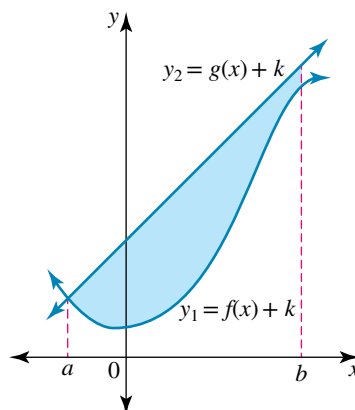
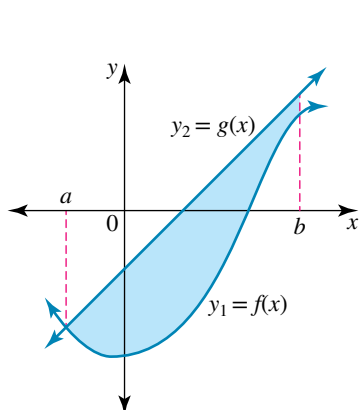


5. The answer appears on the screen.



Note: When finding areas between curves, it does not matter if some of the area is above or below the x -axis.

We can translate both curves up k units parallel to the y -axis so that the area between the curves lies entirely above the x -axis as shown.



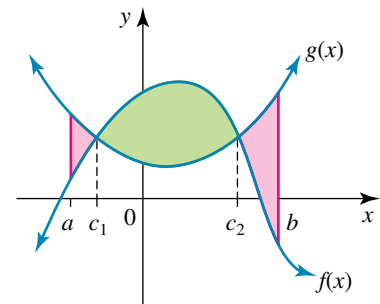
$$\begin{aligned}
 A &= \int_a^b (g(x) + k) dx - \int_a^b (f(x) + k) dx \\
 &= \int_a^b (g(x) - f(x)) dx \\
 &= \int_a^b (y_2 - y_1) dx
 \end{aligned}$$

Provided that $y_2 \geq y_1$ for $a \leq x \leq b$, it does not matter if some or all of the area is above or below the x -axis, as the required area between the curves will be a positive number.

7.3.2 Area between two functions that intersect in the required interval

When determining the area between functions, the lower function is subtracted from the higher function. This means that if the functions intersect in the interval, then the lower and higher functions will change, so it is necessary to break the interval into sub-intervals.

For the graph shown, the area is found by calculating

$$\int_a^{c_1} (g(x) - f(x)) dx + \int_{c_1}^{c_2} (f(x) - g(x)) dx + \int_{c_2}^b (g(x) - f(x)) dx$$


WORKED EXAMPLE 6

Determine the area between $f(x) = \frac{4}{x}$ and $g(x) = x$ from $x = 1$ to $x = 3$.

THINK

1. State the functions.
2. Determine the x -coordinates of the point of intersection.
3. Identify if the functions intersect in the interval $1 \leq x \leq 3$.
4. Sketch $f(x)$ and $g(x)$ on the same set of axes. Shade the area between the functions, $x = 1$ and $x = 3$. Indicate that the functions intersect at $x = 2$ in the region $1 \leq x \leq 3$

WRITE

$$f(x) = \frac{4}{x}$$

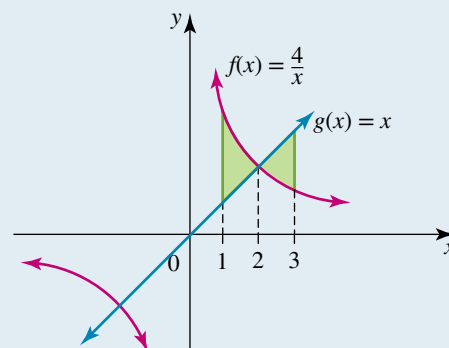
$$g(x) = x$$

$$\frac{4}{x} = x$$

$$x^2 = 4$$

$$x = \pm 2$$

$x = 2$ occurs in the interval $1 \leq x \leq 3$



5. Within the interval $1 \leq x \leq 2$, $f(x)$ is higher than $g(x)$.
 Within the interval $2 \leq x \leq 3$, $g(x)$ is higher than $f(x)$.
 State the area as the sum of two integrals for the sub-intervals.

6. Anti-differentiate the integrands.

7. Evaluate the definite integrals.

8. State the value of the required area.

$$\begin{aligned}
 A &= \int_1^2 (f(x) - g(x)) dx + \int_2^3 (g(x) - f(x)) dx \\
 &= \int_1^2 \left(\frac{4}{x} - x \right) dx + \int_2^3 \left(x - \frac{4}{x} \right) dx \\
 &= \left[4 \ln |x| - \frac{1}{2} x^2 \right]_1^2 + \left[\frac{1}{2} x^2 - 4 \ln |x| \right]_2^3 \\
 &= \left(4 \ln(2) - \frac{1}{2} (2)^2 \right) - \left(4 \ln(1) - \frac{1}{2} (1)^2 \right) + \\
 &\quad \left(\frac{1}{2} (3)^2 - 4 \ln(3) \right) - \left(\frac{1}{2} (2)^2 - 4 \ln(2) \right) \\
 &= 4 \ln(2) - 2 - 4 \ln(1) + \frac{1}{2} + \frac{9}{2} - \\
 &\quad 4 \ln(3) - 2 + 4 \ln(2) \\
 &= 4 \ln \left(\frac{2 \times 2}{3} \right) + \frac{10}{2} - 4 \\
 &= 4 \ln \left(\frac{4}{3} \right) + 1 \\
 &\text{The area is } 4 \ln \left(\frac{4}{3} \right) + 1 \text{ or} \\
 &\text{approximately 2.15 square units.}
 \end{aligned}$$

Therefore, when calculating areas between two functions over an interval, it must be determined whether the functions intersect within that interval. If they do, the area is broken into sub-intervals.

As with areas under curves, the graph should be sketched to identify the higher and lower function.

If the graph is not sketched, then the absolute value of the integral is used to ensure that the resulting area has a positive value.

7.3.3 Using Technology

Graphics calculators could be used to either assist with the graphing or to verify the area obtained.

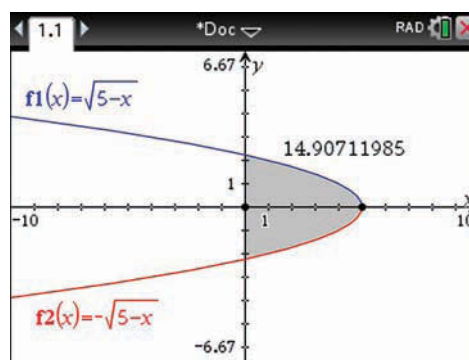
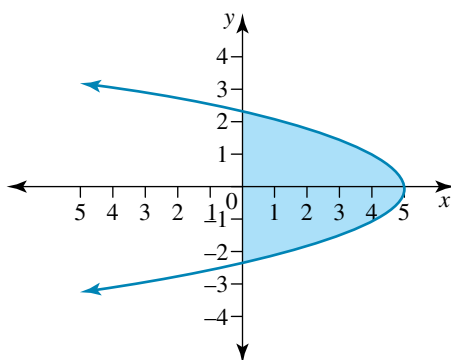
You should already be able to use your calculator to sketch graphs, determine the roots of graphs and obtain the points of intersection between graphs. These skills can also be useful when solving integration questions.

When performing integral calculations, your calculator can determine the area between a function and the x -axis over a particular domain. This can be seen in the technology section of Worked example 3.

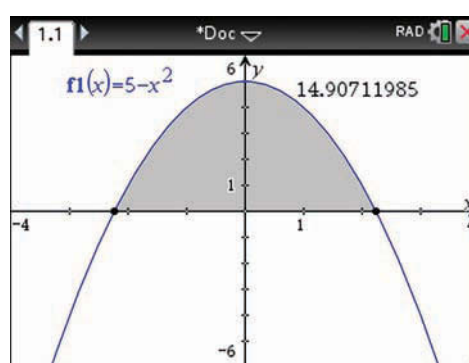
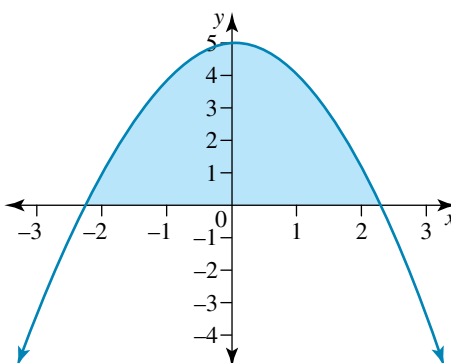
Your calculator can also determine the area bounded by a function and the x -axis (without you needing to calculate the roots of the function first) and also determine the area bounded by two functions. This is due to the inbuilt features of the calculators which can automatically determine points of interest such as roots and points of intersection. This can be seen in the technology section of Worked example 5.

One area that requires further investigation is to note that these graphics calculators are only able to calculate definite integrals in the form $\int_{x=a}^{x=b} f(x) dx$. This means that if you wish to calculate the area

between $x = 5 - y^2$ and the y -axis, it is necessary to rewrite the curve as $y^2 = 5 - x$ so $y = \pm\sqrt{5 - x}$. Both functions would need to be graphed and the area between them calculated. As you can see, the area is approximately 14.91 square units.



Alternatively, the same area could be obtained by determining $\int_{-\sqrt{5}}^{\sqrt{5}} (5 - y^2) dy$. To use the calculator to do this, it is necessary to use the inverse relations (effectively swapping the x and y -axes). In this case, instead of graphing $x = 5 - y^2$ as two functions $y = \pm\sqrt{5 - x}$, determine the inverse relation and graph $y = 5 - x^2$. In this instance, the area can be found by determining the area between the function and the x -axis. As demonstrated, the result is an area of 14.91 square units.



The next worked example will be solved using technology.

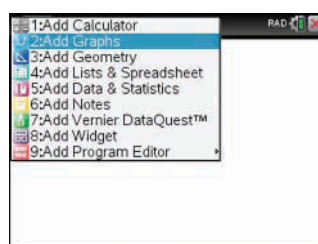
WORKED EXAMPLE 7

Use technology to determine the area bounded by the curves $x = y^2 - 2$ and $y = x^3$.

TI | THINK

- On a Calculator page, press MENU then select 2: Add Graphs

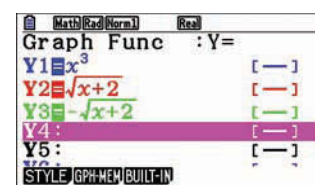
WRITE



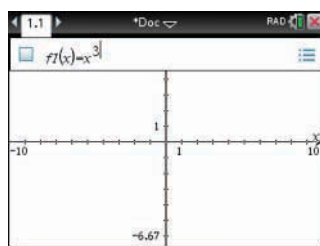
CASIO | THINK

- On a GRAPH screen, complete the entry line in Y1, Y2 and Y3 as x^3 , $\sqrt{x+2}$, $-\sqrt{x+2}$ then press the EXE or DRAW button.

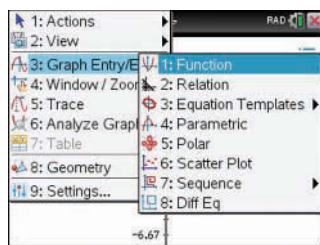
WRITE



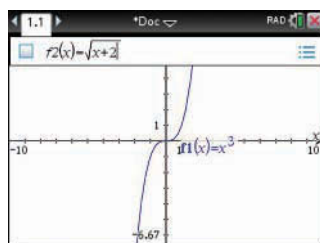
2. Complete the entry line as $f1(x) = x^3$ then press the ENTER button.



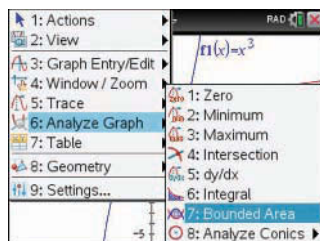
3. Press MENU then select 3: Graph Entry/Edit 1: Function



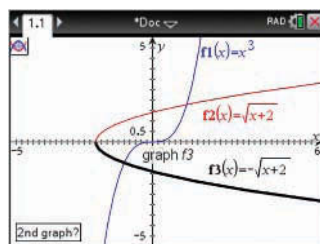
4. Complete the entry line as $\sqrt{x+2}$ then press the ENTER button. Repeat steps 3 and 4 to include $-\sqrt{x+2}$



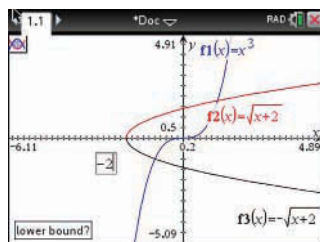
5. Press MENU then select 6: Analyze Graph 7: Bounded Area



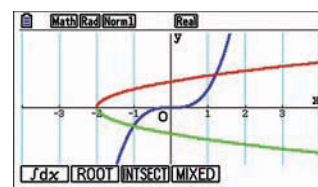
6. Use the directional cursor to select $f2(x)$ and $f3(x)$.



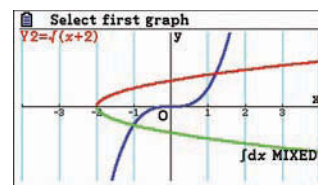
7. Enter the lower boundary by completing the entry line -2 Continue by pressing the ENTER button.



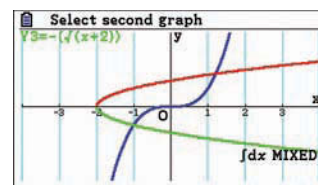
2. To calculate the area bound by Y2 and Y3, select SHIFT F5 $\int dx$ MIXED



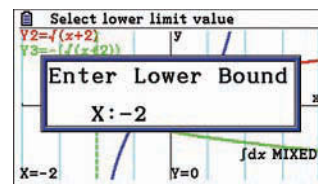
3. Use the directional cursor to select the first graph, Y2. Continue by pressing the EXE button.



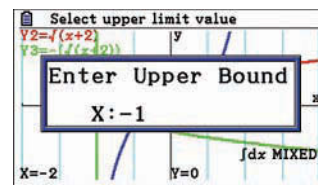
4. Use the directional cursor to select the second graph, Y3. Continue by pressing the EXE button.



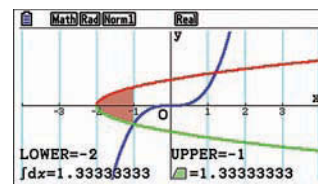
5. Enter the lower boundary by completing the entry line -2 Continue by pressing the EXE button.



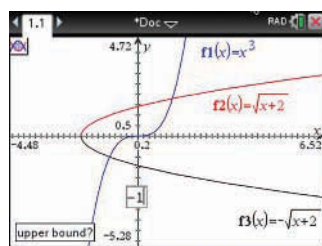
6. Enter the lower boundary by completing the entry line -1 Continue by pressing the EXE button.



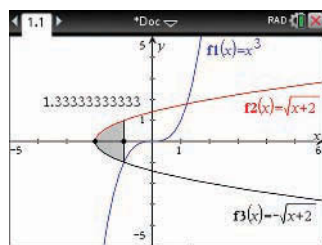
7. The answer appears on the screen.



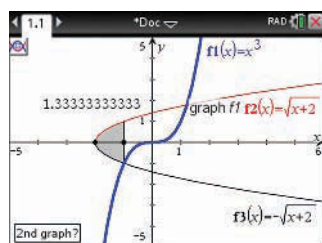
8. Enter the upper boundary by completing the entry line -1
Continue by pressing the ENTER button.



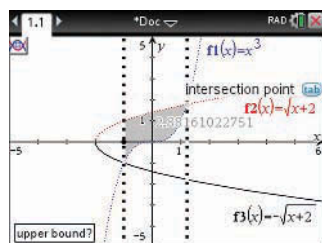
9. The answer appears on the screen.



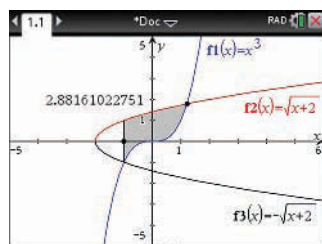
10. Press MENU then select 6: Analyze Graph 7: Bounded Area Use the directional cursor to select $f_2(x)$ $f_1(x)$ and.



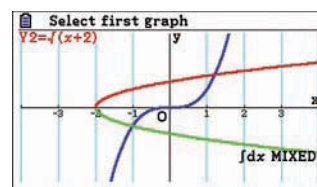
11. Enter the lower boundary by completing the entry line -1
Continue by pressing the ENTER button. The upper boundary can be selected by using the cursor to locate the intersection point.



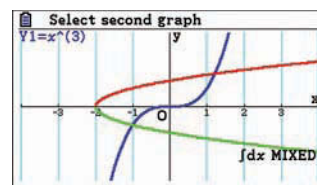
12. The answer appears on the screen.



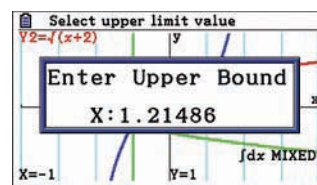
8. To calculate the area bound by Y2 and Y1, use the directional cursor to select the first graph, Y2. Continue by pressing the EXE button.



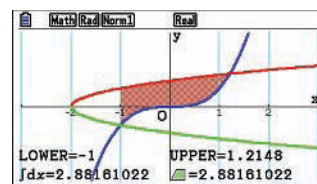
9. Use the directional cursor to select the second graph, Y1. Continue by pressing the EXE button.



10. Enter the lower and upper boundaries by completing the entry line -1
EXE
1.21486
EXE



11. The answer appears on the screen.



Determine the total area and state the solution.

Total area = $1.3333333 + 2.8816102 = 4.21$ square units (correct to two decimal places).

When it is necessary to identify points of intersection, but it is difficult or impossible to do so algebraically, technology can be used. Technology can also be used to determine integral calculations in similar circumstances.

study on

Units 3 & 4

Area 4

Sequence 2

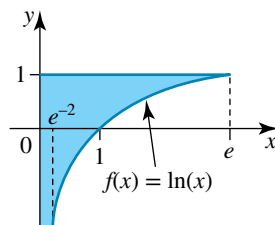
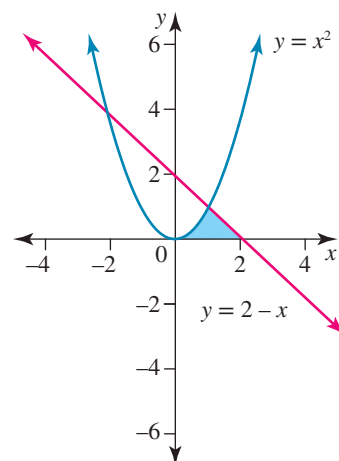
Concept 2

Area between functions Summary screen and practice questions

Exercise 7.3 Area between functions

Technology free

- WE5** Determine the area between the parabola $y = x^2 - 3x - 18$ and the straight line $y = 4x - 10$.
 - Determine the area between the parabola $y = x^2 - 2x - 8$ and the straight line $y = 1 - 2x$.
 - Determine the area between the parabola $y = x^2 - 2x - 35$ and the straight line $y = 4x - 8$.
 - Determine the area between the parabola $y = x^2 + 5x - 14$ and the straight line $y = 2x + 4$.
- WE6** Determine the area between $f(x) = \frac{9}{x}$ and $g(x) = x$ from $x = 1$ to $x = 4$.
- Determine the area between the functions.
 - $y = x^2$ and $y = x$
 - $y = x^3$ and $y = x$
 - $y = x^4$ and $y = x$
 - $y = x^5$ and $y = x$
- Determine the area between the pair of curves below, over the given interval.
 - $y = \sin(x)$, $y = \cos(x)$, $0 \leq x \leq \pi$
 - $y = (x - 1)^2$, $y = (x + 1)^2$, $-1 \leq x \leq 1$
 - $y = \frac{1}{x}$, $y = 4x$, $\frac{1}{4} \leq x \leq 1$
 - $y = e^x$, $y = -e^x$, $-2 \leq x \leq 1$
- Determine the following areas.
 - The area between the line $2y + x - 5 = 0$ and the hyperbola $y = \frac{2}{x}$.
 - The area between the line $9y + 3x - 10 = 0$ and the hyperbola $y = \frac{1}{3x}$.
- Determine the area of the region bounded by the curve $y = 4\sqrt{x - 1}$, the curve $y = 4\sqrt{3 - x}$ and the x -axis.
- Calculate the area of the shaded region.
- Determine the area bounded by the curves $y = 5\sin^2(2x)$, $y = 5\cos^2(2x)$ and the lines $x = 0$ and $x = \frac{\pi}{4}$.
- We know that the area of a circle with radius r is πr^2 . Use the equation $x^2 + y^2 = r^2$ to justify the formula for area of a circle.
- The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab . Use integration to justify this formula.
- A cross-section of a platform is shown (all measurements are in metres). Determine the cross-sectional area of the platform.



Technology active

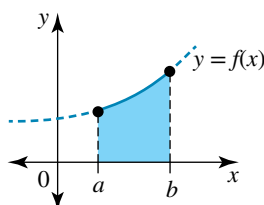
For questions 12–14 give all answers correct to two decimal places.

- WE7** Use technology to determine the area bounded by the curves $x = y^2 - 3$ and $y = x^3$.
- Use technology to determine the area of the region bounded by the curves $y = 4\ln(3 - x)$, $y = 3\sqrt{x + 3}$ and the x -axis.
 - If the line $x = c$ divides the region found in part **a.** into two regions with equal area, determine the value of c .
- Use technology to determine the area of the region bounded by the curve $y = \ln(x)$ the line $y = 6 - x$ and the x -axis.
 - If the line $y = c$ divides the region found in part **a.** into two regions with equal area, determine the value of c .

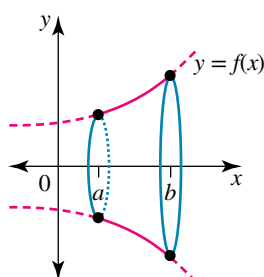
7.4 Volumes of solids of revolution

7.4.1 Rotations about the x-axis

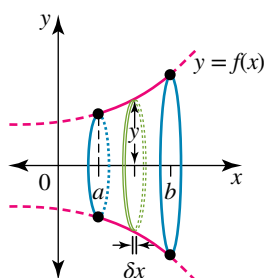
Suppose that a curve $y = f(x)$ is continuous on the closed interval $a \leq x \leq b$. The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ can be rotated 360° about x-axis resulting in a **solid of revolution**.



The solid generated is symmetrical about the x-axis and any vertical cross-section is a circle with a radius equal to the value of y at that point. For example, the radius of the circle at $x = a$ is $f(a)$.



A thin vertical slice could be considered to be a cylinder, with at radius y and a height δx . The volume of the cylinder is $\pi y^2 \delta x$.



To determine the volume of the solid of revolution, the volumes of all of the cylinders between $x = a$ and $x = b$ are added together and the limit as $\delta x \rightarrow 0$ determined.

$$\begin{aligned} V &= \lim_{\delta x \rightarrow 0} \sum_{x=a}^{x=b} \pi y^2 \delta x \\ &= \int_a^b \pi y^2 dx \\ &= \pi \int_a^b [f(x)]^2 dx \end{aligned}$$

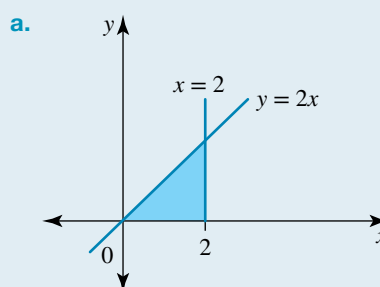
WORKED EXAMPLE 8

- Sketch the graph of $y = 2x$ and identify the region bounded by the graph, the x -axis and the line $x = 2$.
- Calculate the volume of the solid of revolution when the region is rotated about the x -axis.

THINK

- Sketch the lines $y = 2x$ and $x = 2$ and shade the area bounded by $y = 2x$, $x = 2$ and the x -axis.

WRITE



- The volume generated is bounded by $x = 0$ and $x = 2$. The formula for volume of a solid formed by rotation about the x -axis is $\pi \int_a^b [f(x)]^2 dx$.

b. $V = \pi \int_0^2 [f(x)]^2 dx$

- The integrand needs to be expressed in terms of x .
Substitute $f(x) = 2x$.

$$= \pi \int_0^2 (2x)^2 dx$$

- Simplify the integrand.

$$= 4\pi \int_0^2 x^2 dx$$

- Determine the anti-derivative.

$$= 4\pi \left[\frac{1}{3}x^3 \right]_0^2$$

$$= 4\pi \left[\frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 \right]$$

- Evaluate the integral.

$$= 4\pi \left[\frac{8}{3} - 0 \right]$$

$$= \frac{32\pi}{3}$$

- State the volume.

The volume is $\frac{32\pi}{3}$ cubic units.

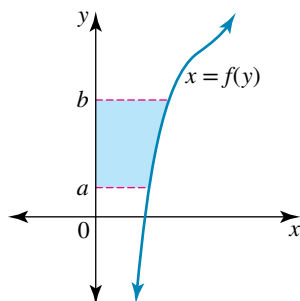
In the worked example above, the solid generated is a cone. The height of the cone is 2 units and the radius of the base is 4 units. Using the formula for volume of a cone:

$$\begin{aligned}
 V &= \frac{1}{3}\pi r^2 h \\
 &= \frac{1}{3}\pi (4)^2 (2) \\
 &= \frac{32\pi}{3} \text{ cubic units}
 \end{aligned}$$

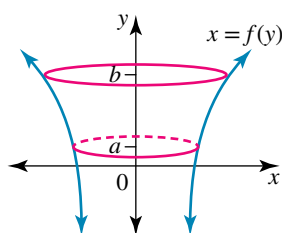
This is the same result as determined in the worked example.

7.4.2 Rotations about the y-axis

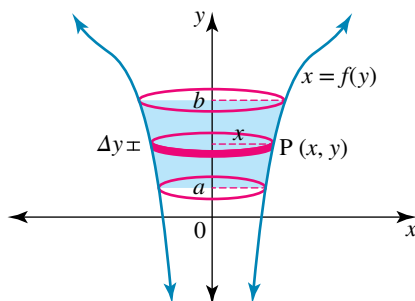
Curves can also be rotated about the y-axis. In this instance, the region is bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and the y-axis.



The solid generated is symmetrical about the y-axis and any horizontal cross-section is a circle with a radius equal to the value of x at that point. For example, the radius of the circle at $y = a$ is $f(a)$.



A thin horizontal slice could be considered to be a cylinder, with radius x and a height δy . The volume of the cylinder is $\pi x^2 \delta y$.



To determine the volume of the solid of revolution, the volumes of all of the cylinders between $y = a$ and $y = b$ are added together and the limit as $\delta y \rightarrow 0$ determined.

$$\begin{aligned} V &= \lim_{\delta y \rightarrow 0} \sum_{y=a}^{y=b} \pi x^2 \delta y \\ &= \int_a^b \pi x^2 dy \\ &= \pi \int_a^b [f(y)]^2 dy \end{aligned}$$

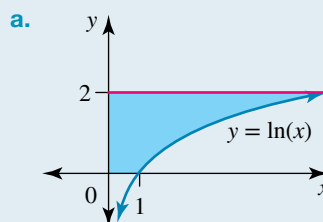
WORKED EXAMPLE 9

- Sketch the region bounded by the curve $y = \ln(x)$, the x -axis, the y -axis and the line $y = 2$.
- Calculate the volume of the solid of revolution formed when the region is rotated about the y -axis.

THINK

- Sketch the curve $y = \ln(x)$ and the line $y = 2$. Shade the area bounded by $y = \ln(x)$, the x -axis, the y -axis and the line $y = 2$.

WRITE



- The volume generated is bounded by $y = 0$ and $y = 2$. The formula for volume of a solid formed by

rotation about the y -axis is $\pi \int_a^b [f(y)]^2 dy$.

- The integrand needs to be expressed in terms of y . Rearrange $y = \ln(x)$ to make x the subject.

- Substitute $f(y) = e^y$ into the formula for volume.

- Simplify the integrand.

- Determine the anti-derivative.

- Evaluate the integral.

- State the volume.

$$\text{b. } V = \pi \int_0^2 [f(y)]^2 dy$$

$$y = \ln(x)$$

$$x = e^y$$

$$f(y) = e^y$$

$$V = \pi \int_0^2 (e^y)^2 dy$$

$$= \pi \int_0^2 e^{2y} dy$$

$$= \pi \left[\frac{1}{2} e^{2y} \right]_0^2$$

$$= \pi \left[\frac{1}{2} e^4 - \frac{1}{2} e^0 \right]$$

$$= \frac{\pi}{2} (e^4 - 1)$$

The volume is $\frac{\pi}{2} (e^4 - 1) \approx 84.19$ cubic units.

Volumes of solids of revolution

To determine the volume of the solid of revolution formed by rotating $y = f(x)$ about the x -axis from $x = a$ to $x = b$, evaluate the integral $V = \pi \int_a^b y^2 dx = \pi \int_a^b [f(x)]^2 dx$.

To determine the volume of the solid of revolution formed by rotating $x = f(y)$ about the y -axis from $y = a$ to $y = b$, evaluate the integral $V = \pi \int_a^b x^2 dy = \pi \int_a^b [f(y)]^2 dy$.

7.4.3 Applications

The volume of some common geometric shapes can now be found using calculus, by rotation lines or curves about the x - or y -axis and using the techniques described.

WORKED EXAMPLE 10

A drinking glass has a base diameter of 5 cm, a top diameter of 7 cm and a height of 11 cm. Determine the capacity of the glass to the nearest mL.



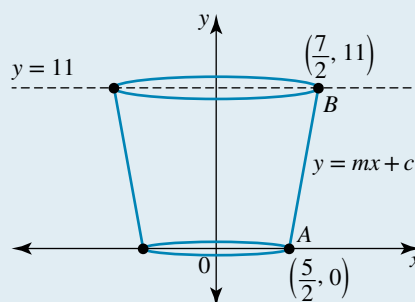
THINK

- Sketch the graph and identify the area to rotate.
Write the coordinates of the points A and B .

- Determine the gradient of the line segment AB joining the points $A\left(\frac{5}{2}, 0\right)$ and $B\left(\frac{7}{2}, 11\right)$.
- Use $y - y_1 = m(x - x_1)$ to determine the equation of the line segment AB .

- The glass is formed by rotating AB about the y -axis. The formula for rotation about the y -axis is $V = \pi \int_a^b x^2 dy$. We are interested in the region between $y = 0$ and $y = 11$.

WRITE



$$m = \frac{11 - 0}{\frac{7}{2} - \frac{5}{2}} = 11$$

$$y - 0 = 11 \left(x - \frac{5}{2} \right)$$

$$y = 11x - \frac{55}{2}$$

$$V = \pi \int_a^b [f(y)]^2 dy$$

$$= \pi \int_0^{11} x^2 dy$$

5. The integrand needs to be expressed in terms of y .

Rearrange $y = 11x - \frac{55}{2}$ to make x the subject.

$$y = 11x - \frac{55}{2}$$

$$y + \frac{55}{2} = 11x$$

$$x = \frac{1}{11}y + \frac{5}{2}$$

$$f(y) = \frac{1}{11}y + \frac{5}{2}$$

6. Substitute the equation for x into the formula for volume so that the integrand is expressed in terms of y only.

$$V = \pi \int_0^{11} [f(y)]^2 dy$$

$$= \pi \int_0^{11} \left(\frac{1}{11}y + \frac{5}{2} \right)^2 dy$$

$$= \pi \left[\frac{1}{3} \left(\frac{1}{11}y + \frac{5}{2} \right)^3 \times 11 \right]_0^{11}$$

$$= \frac{11\pi}{3} \left[\left(\frac{1}{11}y + \frac{5}{2} \right)^3 \right]_0^{11}$$

$$= \frac{11\pi}{3} \left[\left(1 + \frac{5}{2} \right)^3 - \left(0 + \frac{5}{2} \right)^3 \right]$$

$$= \frac{11\pi}{3} \left[\left(\frac{7}{2} \right)^3 - \left(\frac{5}{2} \right)^3 \right]$$

$$= \frac{11\pi}{3} \times \frac{109}{4}$$

$$= \frac{1199\pi}{12}$$

$$\approx 314$$

9. State the volume in cm^3 . Use $1 \text{ cm}^3 = 1 \text{ mL}$ to state the volume of the glass to the nearest mL.

The volume is 314 cm^3 .

This means that the glass can hold 314 mL.

study on

Units 3 & 4 > Area 4 > Sequence 2 > Concept 3

Volumes of solids of revolution Summary screen and practice questions

Exercise 7.4 Volumes of solids of revolution

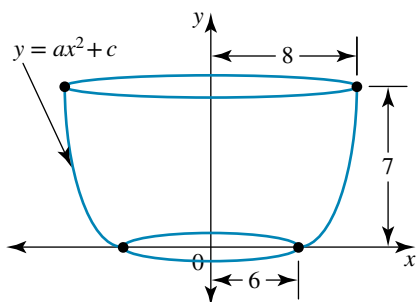
Technology free

- WE8** Sketch the graph of $y = \sqrt{x}$ and identify the region bounded by the graph, the x -axis and the line $x = 4$.
- Calculate the volume of the solid of revolution when the region is rotated about the x -axis.

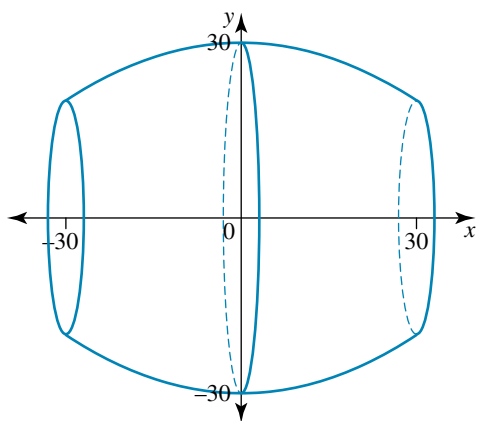
2. The area bounded by the curve $y = 4 - x^2$ and the coordinate axis is rotated about the x -axis to form a solid of revolution. Determine the volume of the solid formed.
3. a. **WE9** Sketch the region bounded by the curve $y = \sqrt{x}$, the y -axis, the origin and the line $y = 2$.
b. Calculate the volume of the solid of revolution formed when the region is rotated about the y -axis.
4. The area bounded by the curve $y = 4 - x^2$ and the coordinate axes is rotated about the y -axis to form a solid of revolution. Calculate the volume of the solid.
5. Consider the line $y = 3x$.
a. Determine the volume of the solid of revolution formed by rotating the section of the line between the origin and $x = 5$ about the x -axis.
b. Determine the volume of the solid of revolution formed by rotating the section of the line between the origin and $y = 5$ about the y -axis.
6. A cone is formed by rotating the line segment of $2x + 3y = 6$ cut off by the coordinate axes about:
a. the x -axis
b. the y -axis.
Determine the volume of the solid formed in each case.
7. If the region bounded by the curve $y = \sec(2x)$, the co-ordinate axes and $x = \frac{\pi}{8}$ is rotated 360° about the x -axis, determine the volume of the solid formed.
8. If the region bounded by the curve $y = 2e^{\frac{x}{2}}$, the coordinate axes and $x = 2$ is rotated 360° about the x -axis, determine the volume of the solid formed.
9. **WE10** A plastic bucket has a base diameter of 20 cm, a top diameter of 26 cm and a height of 24 cm. Determine the capacity of the bucket to the nearest litre.



10. A soup bowl has a base radius of 6 cm, a top radius of 8 cm, and a height of 7 cm. The edge of the bowl is a parabola of the form $y = ax^2 + c$. Determine the capacity of the soup bowl to the nearest mL.



11. a. If the region bounded by the curve $y = 3 \sin(2x)$, the origin, the x -axis and the first intercept the curve makes with the x -axis is rotated 360° about the x -axis, determine the volume of the solid formed.
- b. If the region bounded by the curve $y = 4 \cos(3x)$, the origin, the x -axis and the first intercept the curve makes with the x -axis is rotated 360° about the x -axis, determine the volume of the solid formed.
12. Consider the curve $y = 3x^2 + 4$.
- a. Determine the volume of the solid formed by rotating the area between the curve, the coordinate axes and the line $x = 2$ about the x -axis.
- b. Determine the volume of the solid formed by rotating the area between the curve, $y = 4$ and $y = 10$ about the y -axis.
13. Consider the area between the curve $y = x^2 - 9$ and the x -axis.
- a. If the area described is rotated 360° about the x -axis, determine the volume of the solid formed.
- b. If the area described is rotated 360° about the y -axis, determine the volume of the solid formed.
14. a. Determine the volume of the solid formed by rotating the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ about the x -axis.
- b. Determine the volume of the solid formed by rotating the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ about the y -axis.
- c. Determine the volume of the solid formed by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x -axis.
- d. Determine the volume of the solid formed by rotating the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y -axis.
- e. An egg can be regarded as an ellipsoid (formed by rotating an ellipse). If the egg has a total length of 57 mm and its diameter at the centre is 44 mm, determine an equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that can be rotated to form the required ellipsoid.
- f. Use the equation from e to determine the volume of the egg in cubic millimetres.
15. The diagram below shows a wine barrel. The barrel has a total length of 60 cm, a total height of 60 cm at the middle and a total height of 40 cm at the ends.
- a. If the upper arc can be represented by a parabola, demonstrate that its equation is given by $y = -\frac{x^2}{90} + 30$ for $-30 \leq x \leq 30$.
- b. If the arc is rotated about the x -axis, determine the capacity of the wine barrel to the nearest litre.



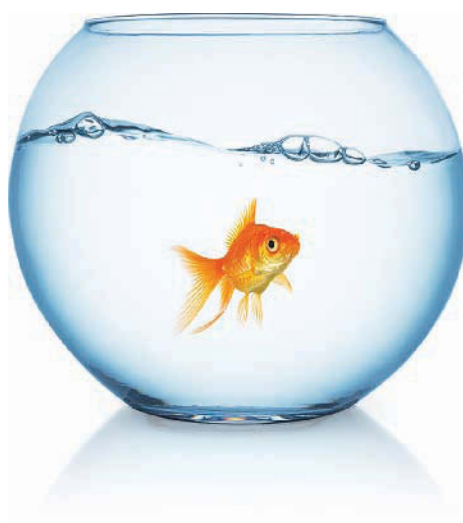
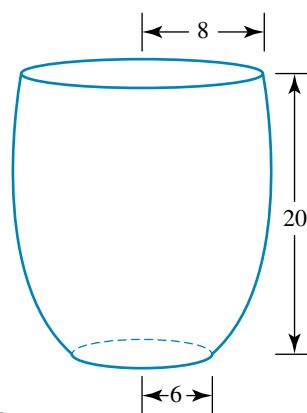
16. A vase has a base radius of 6 cm, its top radius is 8 cm and its height is 20 cm. Determine the volume of the vase in cubic centimetres if the side is modelled by:

- a. a straight line of the form $y = ax + b$
- b. a parabola of the form $y = ax^2 + c$
- c. a cubic $y = ax^3 + c$
- d. a quartic of the form $y = ax^4 + c$.

17. a. The region bounded by the curve $\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$, the x -axis, $x = 0$ and $x = 2$ is rotated about the x -axis. Determine the volume of the solid formed.

- b. The region bounded by the curve $\sqrt{x} + \sqrt{y} = 2$ and the coordinate axes is rotated about the x -axis. Determine the volume of the solid formed.

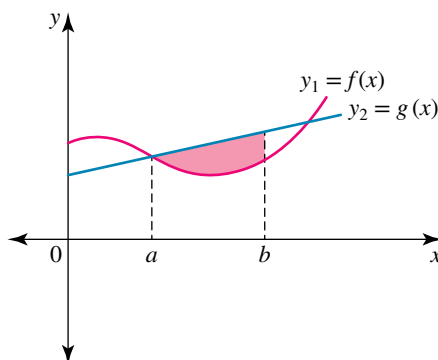
18. A fish bowl consists of a portion of a sphere of radius 20 cm. The bowl is filled with water so that the radius of the water at the top is 16 cm and the base of the bowl has a radius of 12 cm. If the total height of the water in the bowl is 28 cm, determine the volume of water in the bowl.



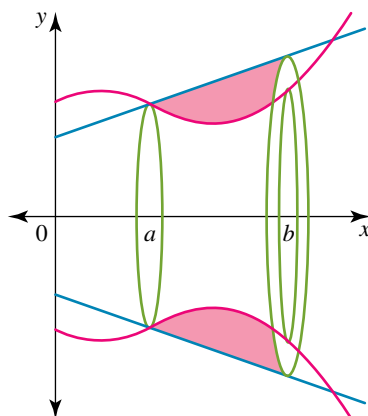
7.5 Volumes of revolution

7.5.1 Rotations about the x -axis

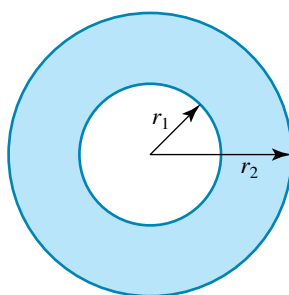
Suppose that there are two curves $y_1 = f(x)$ and $y_2 = g(x)$ that are continuous over the closed interval $a \leq x \leq b$ and over the interval, $g(x) \geq f(x)$.



If the area bounded by the functions and the lines $x = a$ and $x = b$ is rotated about the x -axis, the resulting object will be hollow in the middle.

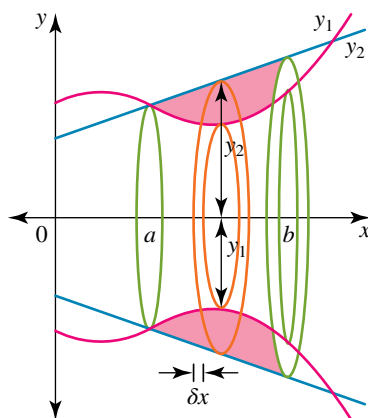


A cross-section of the shape would look like a circular washer. The inner radius, r_1 , would be equal to $y_1 = f(x)$ and the outer radius, r_2 , would be equal to $y_2 = g(x)$.



The area of this cross-section can be found by determining $\pi(r_2^2 - r_1^2)$.

The volume of the thin vertical slice of height δx would have a volume of $\pi(y_2^2 - y_1^2)\delta x$.



The volume of the solid of revolution can be found by adding the volume of the thin circular washers between $x = a$ and $x = b$ and finding the limit as $\delta x \rightarrow 0$.

$$\begin{aligned} V &= \sum_{x=a}^{x=b} \pi (y_2^2 - y_1^2) \delta x \\ &= \int_a^b \pi (y_2^2 - y_1^2) dx \\ &= \pi \int_a^b ([g(x)]^2 - [f(x)]^2) dx \end{aligned}$$

WORKED EXAMPLE 11

Determine the volume formed when the area bounded by the curve $y = 4 - x^2$ and the line $y = 3$ is rotated about:

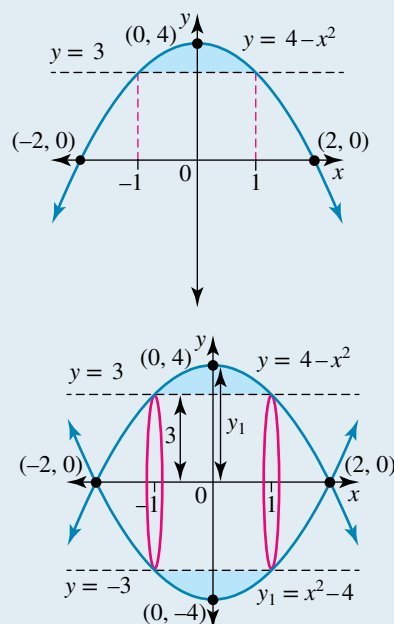
- a. the x -axis b. the y -axis.

THINK

- a. 1. Determine the point of intersection between the curve and the line.
2. Sketch the graph and identify the area to rotate about the x -axis

WRITE

- a. Let $y_1 = 4 - x^2$ and $y_2 = 3$
- $$y_1 = y_2$$
- $$4 - x^2 = 3$$
- $$x^2 = 1$$
- $$x = \pm 1$$



3. The volume can be found using
- $$V = \pi \int_a^b ([g(x)]^2 - [f(x)]^2) dx.$$
- $a = -1$ and $b = 1$.

$$V = \pi \int_{-1}^1 ([g(x)]^2 - [f(x)]^2) dx$$

4. The integrand needs to be expressed in terms of x . In the shaded area, $4 - x^2 \geq 3$, therefore $y_2 = 4 - x^2$ and $y_1 = 3$.

5. Simplify the integrand so that it can be determined.

6. Determine the anti-derivative.

7. Evaluate the integral.

8. State the value of the volume.

b. 1. Identify the region to rotate about the y -axis.

2. The volume created will be completely solid. The region is bounded by $y = 3$ and $y = 4$. The formula for volume of a solid formed by rotation about the y -axis is

$$\pi \int_a^b [f(y)]^2 dy.$$

3. The integrand needs to be expressed in terms of y . Rearrange $y = 4 - x^2$ to make x^2 the subject.

4. Substitute $[f(y)]^2 = 4 - y$ into the volume formula.

5. Determine the anti-derivative.

$$= \pi \int_{-1}^1 \left((4 - x^2)^2 - 3^2 \right) dx$$

$$= \pi \int_{-1}^1 (16 - 8x^2 + x^4 - 9) dx$$

$$= \pi \int_{-1}^1 (7 - 8x^2 + x^4) dx$$

$$= \pi \left[7x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^1$$

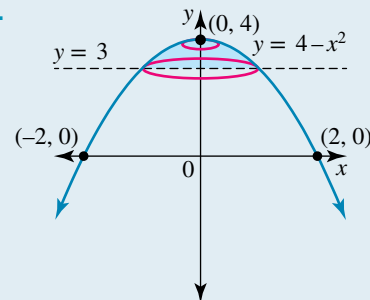
$$= \pi \left[\left(7 - \frac{8}{3} + \frac{1}{5} \right) - \left(-7 + \frac{8}{3} - \frac{1}{5} \right) \right]$$

$$= 2\pi \left(7 - \frac{40}{15} + \frac{3}{15} \right)$$

$$= \frac{136\pi}{15}$$

The volume is $\frac{136\pi}{15}$ cubic units.

b.



$$V = \pi \int_3^4 [f(y)]^2 dy$$

$$y = 4 - x^2$$

$$x^2 = 4 - y$$

$$[f(y)]^2 = 4 - y$$

$$V = \pi \int_3^4 (4 - y) dy$$

$$= \pi \left[4y - \frac{1}{2}y^2 \right]_3^4$$

6. Evaluate the integral.

$$\begin{aligned}
 &= \pi \left[\left(4 \times 4 - \frac{1}{2} \times 4^2 \right) - \left(4 \times 3 - \frac{1}{2} \times 3^2 \right) \right] \\
 &= \pi \left(16 - 8 - 12 + \frac{9}{2} \right) \\
 &= \frac{\pi}{2}
 \end{aligned}$$

7. State the volume.

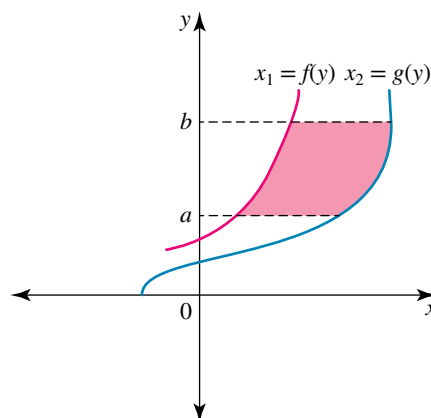
The volume is $\frac{\pi}{2}$ cubic units.

7.5.2 Rotations about the y-axis

It is also possible to rotate two curves about the y-axis. If $x_1 = f(y)$ and $x_2 = g(y)$ and $g(y) \geq f(y)$ over the interval $a \leq y \leq b$.

The solid formed will be hollow in the middle. The volume of the solid formed will be

$$\begin{aligned}
 V &= \int_a^b \pi(x_2^2 - x_1^2) dy \\
 &= \pi \int_a^b ([g(y)]^2 - [f(y)]^2) dy
 \end{aligned}$$



WORKED EXAMPLE 12

Determine the volume formed when the area bounded by the curve $y = x^2 - 4$, the x-axis and the line $x = 3$ is rotated about:

- a. the y-axis b. the x-axis.

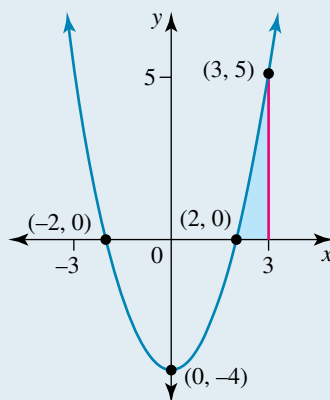
THINK

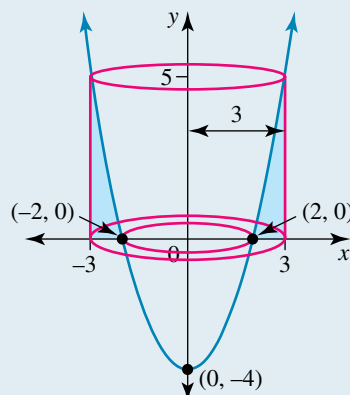
- a. 1. Determine the point of intersection between the curve and the line.

2. Sketch the graph and identify the area to rotate about the y-axis.

WRITE

$$\begin{aligned}
 \text{a. } y &= x^2 - 4 \text{ and } x = 3. \\
 y &= (3)^2 - 4 \\
 &= 9 - 4 \\
 &= 5
 \end{aligned}$$





$$V = \pi \int_0^5 \left([g(y)]^2 - [f(y)]^2 \right) dy$$

3. The volume can be found using

$$V = \pi \int_a^b \left([g(y)]^2 - [f(y)]^2 \right) dy.$$

$a = 0$ and $b = 5$.

4. Rearrange $y = x^2 - 4$ to make x^2 the subject.

$$y = x^2 - 4$$

$$x^2 = y + 4$$

5. The integrand needs to be expressed in terms of y . In the shaded area,

$$3 \geq \sqrt{y + 4}, \text{ therefore } g(y) = 3 \text{ and } f(y) = \sqrt{y + 4}.$$

6. Simplify the integrand so that it can be determined.

$$V = \pi \int_0^5 (9 - (y + 4)) dy$$

$$V = \pi \int_0^5 (9 - (y + 4)) dy$$

$$= \pi \int_0^5 (5 - y) dy$$

7. Determine the anti-derivative.

$$= \pi \left[5y - \frac{1}{2}y^2 \right]_0^5$$

8. Evaluate the integral.

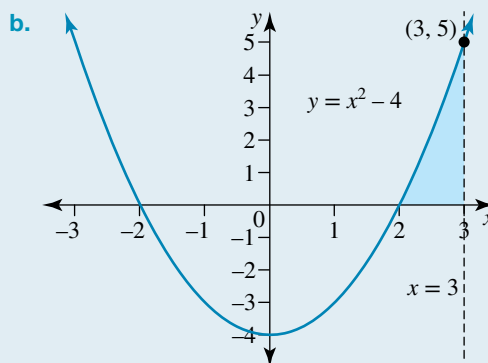
$$= \pi \left[\left(25 - \frac{1}{2} \times 5^2 \right) - 0 \right]$$

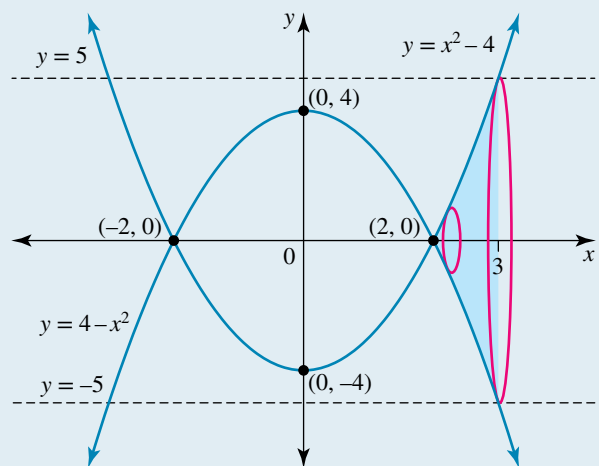
$$= \frac{25\pi}{2}$$

9. State the volume.

The volume is $\frac{25\pi}{2}$ cubic units.

- b. 1. Identify the region to rotate about the x -axis.





2. The volume created will be completely solid. The region is bounded by $x = 2$ and $x = 3$. The formula for volume of a solid formed by rotation about the x -axis is $\int_a^b \pi y^2 dx$.

3. Substitute $f(x) = x^2 - 4$ into the integrand so that it is expressed entirely in terms of x .

4. Simplify the integrand so that it can be determined.

5. Determine the anti-derivative.

6. Evaluate the integral.

7. State the volume.

$$V = \pi \int_2^3 [f(x)]^2 dx$$

$$= \pi \int_2^3 (x^2 - 4)^2 dx$$

$$= \pi \int_2^3 (x^4 - 8x^2 + 16) dx$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x \right]_2^3$$

$$= \pi \left[\left(\frac{1}{5} \times 3^5 - \frac{8}{3} \times 3^3 + 16 \times 3 \right) - \left(\frac{1}{5} \times 2^5 - \frac{8}{3} \times 2^3 + 16 \times 2 \right) \right]$$

$$= \pi \left(\frac{243}{5} - 72 + 48 - \frac{32}{5} + \frac{64}{3} - 32 \right)$$

$$= \frac{113\pi}{15}$$

The volume is $\frac{113\pi}{15}$ cubic units.

7.5.3 Composite figures

Volumes of revolution

If $g(x) \geq f(x)$ over the interval $a \leq x \leq b$, then the volume of the shape formed by

rotating the region about the x -axis is $V = \pi \int_a^b \left([g(x)]^2 - [f(x)]^2 \right) dx$.

If $g(y) \geq f(y)$ over the interval $a \leq y \leq b$, then the volume of the shape formed by

rotating the region about the y -axis is $V = \pi \int_a^b \left([g(y)]^2 - [f(y)]^2 \right) dy$.

It may be necessary to identify a number of smaller regions so that within each region the functions do not intersect.

WORKED EXAMPLE 13

Determine the volume formed when the area between the curves $y = x^2$ and $y = 8 - x^2$ is rotated about:

- a. the x -axis

THINK

- a. 1.** Determine the point of intersection between the two curves.

WRITE

- a.** Let $y_1 = x^2$ and $y_2 = 8 - x^2$.

$$y_1 = y_2$$

$$x^2 = 8 - x^2$$

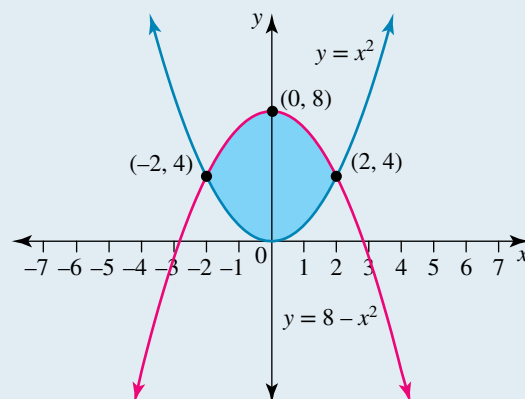
$$2x^2 = 8$$

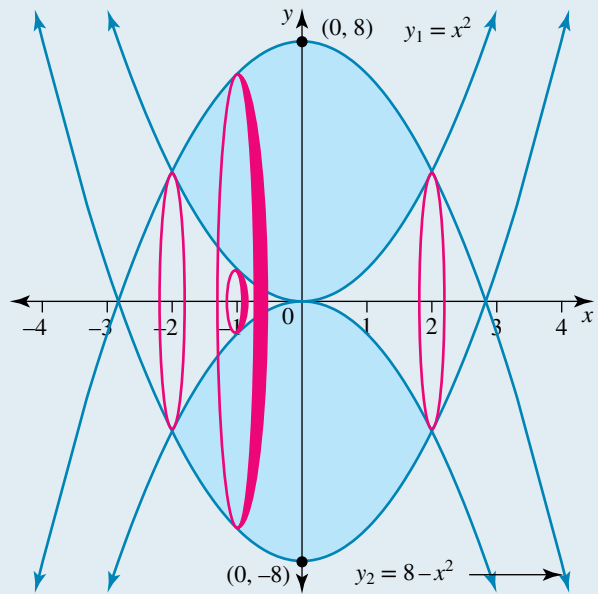
$$x^2 = 4$$

$$x = \pm 2$$

When $x = \pm 2$, $y = 4$.

2. Identify the region to be rotated about the x -axis.





3. Over the region $-2 \leq x \leq 2$, $8 - x^2 \geq x^2$, therefore only one integral is required.
 $a = -2, b = 2$ and

$$V = \pi \int_a^b \left([g(x)]^2 - [f(x)]^2 \right) dx.$$

4. As $8 - x^2 \geq x^2$, $g(x) = 8 - x^2$ and $f(x) = x^2$. Substitute in the integrand.

5. Simplify the integrand so that it can be determined.

6. Determine the anti-derivative.

7. Evaluate the integral.

8. State the volume.

$$V = \pi \int_{-2}^2 \left([g(x)]^2 - [f(x)]^2 \right) dx$$

$$= \pi \int_{-2}^2 \left((8 - x^2)^2 - (x^2)^2 \right) dx$$

$$= \pi \int_{-2}^2 (64 - 16x^2 + x^4 - x^4) dx$$

$$= 16\pi \int_{-2}^2 (4 - x^2) dx$$

$$= 16\pi \left[4x - \frac{1}{3}x^3 \right]_{-2}^2$$

$$= 16\pi \left[\left(4 \times 2 - \frac{1}{3} \times 8 \right) - \left(4 \times -2 - \frac{1}{3} \times -8 \right) \right]$$

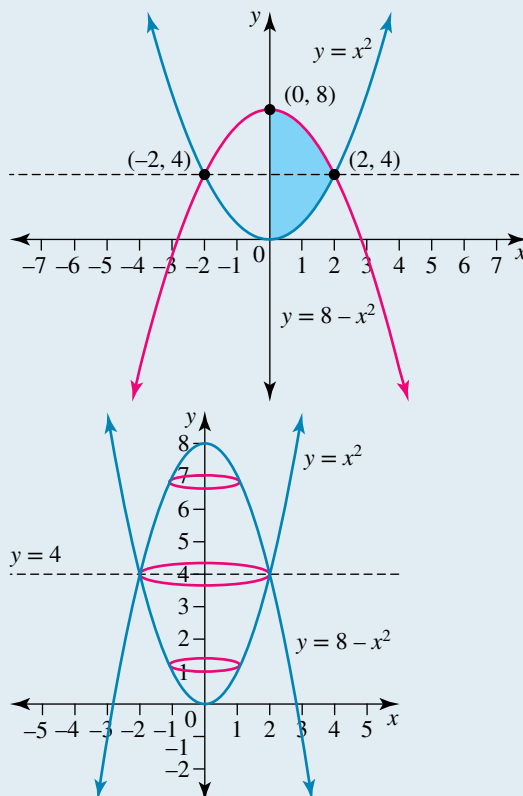
$$= 16\pi \times 2 \left(8 - \frac{8}{3} \right)$$

$$= \frac{512\pi}{3}$$

The volume is $\frac{512\pi}{3}$ cubic units.

- b. 1. Sketch the graph and identify the area to rotate about the y-axis.

b.



2. For $0 \leq y \leq 4$ the region is bounded by the curve $y_1 = x^2$. For $4 \leq y \leq 8$, the region is bounded by the curve $y_2 = 8 - x^2$. In both cases, the volume can be determined using

$$\pi \int_a^b [f(y)]^2 dy.$$

3. The integrand needs to be expressed in terms of y .

Rearrange y_1 and y_2 for x^2 so they can be easily substituted into the formula.

$$y_1 = x^2 \leftrightarrow x^2 = y_1$$

$$y_2 = 8 - x^2 \Rightarrow x^2 = 8 - y_2$$

4. Determine the anti-derivative.

5. Evaluate the integral.

6. State the value of the volume.

$$V = \pi \int_0^4 [f(y)]^2 dy + \pi \int_4^8 [g(y)]^2 dy$$

$$= \pi \int_0^4 y dy + \pi \int_4^8 (8 - y) dy$$

$$\begin{aligned} &= \pi \left[\frac{1}{2} y^2 \right]_0^4 + \pi \left[8y - \frac{1}{2} y^2 \right]_4^8 \\ &= \pi \left(\frac{1}{2} \times 4^2 - 0 \right) + \pi \left[\left(8 \times 8 - \frac{1}{2} \times 8^2 \right) - \left(8 \times 4 - \frac{1}{2} \times 4^2 \right) \right] \\ &= 8\pi + \pi (64 - 32 - 32 + 8) \\ &= 16\pi \end{aligned}$$

The volume is 16π cubic units.

Exercise 7.5 Volumes of revolution

Technology free

1. **WE11** Determine the volume formed when the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$ is rotated about:
 - a. the x -axis
 - b. the y -axis.
2. Determine the volume formed when the area bounded by the curve $y = \sqrt{x}$ and the line $y = 2$ is rotated about:
 - a. the x -axis
 - b. the y -axis.
3. **WE12** Determine the volume formed when the area bounded by the curve $y = x^2 - 9$, the x -axis and the line $x = 4$ is rotated about:
 - a. the y -axis
 - b. the x -axis.
4. Determine the volume formed when the area bounded by the curve $y = \sqrt{x}$, the x -axis and the line $x = 4$ is rotated about:
 - a. the y -axis
 - b. the x -axis.
5. Consider the area bounded by the curve $y = 16 - x^2$ and the line $y = 12$.
 - a. If the area is rotated about the x -axis, determine the volume of the object created.
 - b. If the area is rotated about the y -axis, determine the volume of the object created.
6. Consider the area bounded by the curve $y = 16 - x^2$, the x -axis and the line $x = 5$.
 - a. If the area is rotated about the y -axis, determine the volume of the object created.
 - b. If the area is rotated about the x -axis, determine the volume of the object created.
7. Let R be the region bounded by the curve $y = x^3$, the x -axis and the line $x = 3$.
 - a. If the region R is rotated about the y -axis, determine the volume of the object formed.
 - b. If the region R is rotated about the x -axis, determine the volume of the object formed.
8. The area between the curve $y = \sqrt{2x - 8}$, the x -axis and the line $x = 6$ is rotated about:
 - a. the y -axis
 - b. the x -axis.

Determine the volume of the resulting solid of revolution in each case.
9. a. Determine the volume formed when the area bounded by the curve $y = 4 \cos^{-1} \left(\frac{x}{2} \right)$ and the lines $x = 2$ and $y = 2\pi$ is rotated about the y -axis.
 - b. Determine the volume formed when the area bounded by the curve $y = 2 \sin^{-1} \left(\frac{x}{3} \right)$, the x -axis and the line $x = 3$ is rotated about the y -axis.
10. Determine the volume formed when the area between the curves $y = 2x$ and $y = x^2$ is rotated about:
 - a. the x -axis
 - b. the y -axis.
11. Determine the volume formed when the area between the curves $y = x^2$ and $y = x^3$ is rotated about:
 - a. the x -axis
 - b. the y -axis.
12. a. If the area between two curves $y = 2\sqrt{x}$ and $y = \frac{x^2}{4}$ is rotated about the x -axis, determine the volume formed.
 - b. If the area between two curves $y = \sqrt{ax}$ and $y = \frac{x^2}{a}$ is rotated about the x -axis, determine the volume formed.

13. **WE13** Determine the volume formed when the area between the curves $y = x^2$ and $y = 18 - x^2$ is rotated about:
- the x -axis
 - the y -axis.
14. Determine the volume obtained when the area between the curves $y = x^2 - 4x + 4$ and $y = 4 + 4x - x^2$ is rotated about the x -axis.
15. Determine the volume obtained when the area between the curves $y = 8 \sin\left(\frac{\pi x}{4}\right)$ and $y = 2x^2$ is rotated about the x -axis.
16. **a.** A cylindrical hole of radius 2 cm is cut vertically through the centre of a solid sphere of cheese radius 4 cm. Determine the volume of cheese remaining.
- b.** A cylindrical hole of radius $\frac{a}{2}$ is cut vertically through the centre of a solid sphere of radius a . Determine the volume remaining.
17. **a.** Determine the volume of a torus (a doughnut shaped figure) formed when the area bounded by the circle $x^2 + (y - 8)^2 = 16$ is rotated about the x -axis.
- b.** Determine the volume of a torus (a doughnut shaped figure) formed when the area bounded by the circle $(x - a)^2 + y^2 = r^2$, $a > r > 0$, is rotated about the y -axis.
18. Determine the volume formed when the area bounded by the curve $y = a^2 - x^2$ and the line $y = \frac{a^2}{4}$, where $a > 0$, is rotated about
- the x -axis
 - the y -axis.

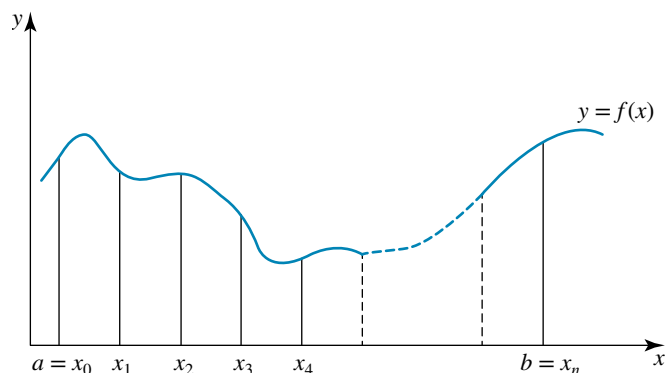


7.6 Approximation using Simpson's rule

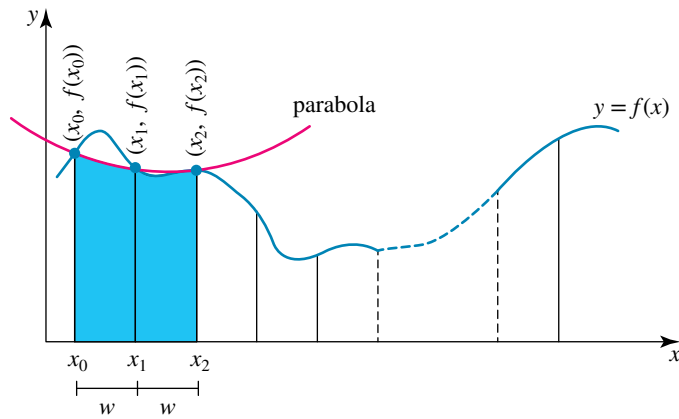
7.6.1 Simpson's rule

In Mathematical Methods, you have used the trapezoidal rule to estimate the area under a curve.

Another way to estimate the area under a curve is to use **Simpson's rule**. In this method, the region between $x_0 = a$ and $x_n = b$ is divided into n intervals, where n is an even number. The width of an interval, w , is found using $w = \frac{b - a}{n}$.

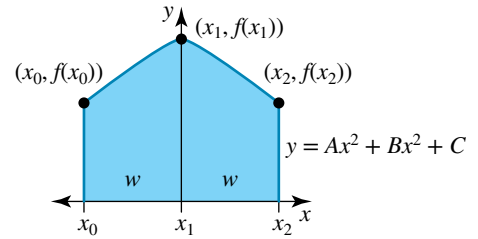


When using the trapezoidal rule, the area is approximated by fitting trapeziums across each interval. With Simpson's rule, the intervals are taken in pairs and a parabola is fitted through the three points.



Simpson's rule is considerably more accurate than the trapezoidal rule.

The derivation of this rule is an interesting exercise in algebra. Assume that the three points lie on the parabola $y = Ax^2 + Bx + C$. We can derive the formula for the area between the parabola and the x -axis over the region $x_0 \leq x \leq x_2$. The area under the parabola will be the same no matter where on the x -axis it is located. Therefore to simplify the algebra, assume that the parabola is centred on the y -axis.



The area under the parabola $y = Ax^2 + Bx^2 + C$ between $x_0 = -w$ and $x_2 = w$ can be found as follows:

$$\begin{aligned}
 A_S &= \int_{x_0}^{x_2} (Ax^2 + Bx + C) dx \\
 &= \left[\frac{1}{3}Ax^3 + \frac{1}{2}Bx^2 + Cx \right]_{-w}^w \\
 &= \left(\frac{1}{3}Aw^3 + \frac{1}{2}Bw^2 + Cw \right) - \left(-\frac{1}{3}Aw^3 + \frac{1}{2}Bw^2 - Cw \right) \\
 &= \frac{2}{3}Aw^3 + 2Cw
 \end{aligned}$$

The point $(x_0, f(x_0))$ lies on the parabola ($x_0 = -w$)

$$f(x_0) = Aw^2 - Bw + C \quad [1]$$

The point $(x_1, f(x_1))$ lies on the parabola ($x_1 = 0$)

$$f(x_1) = C \quad [2]$$

The point $(x_2, f(x_2))$ lies on the parabola ($x_2 = w$)

$$f(x_2) = Aw^2 + Bw + C \quad [3]$$

$$[2] - [1]: f(x_1) - f(x_0) = -Aw^2 + Bw \quad [4]$$

$$[3] - [2]: f(x_2) - f(x_1) = Aw^2 + Bw \quad [5]$$

$$[5] - [4]: f(x_0) - 2f(x_1) + f(x_2) = 2Aw^2 \quad [6]$$

Using [2] and [6], A_s can be simplified as follows:

$$\begin{aligned} A_s &= \frac{w}{3} (2Aw^2) + 2w(C) \\ &= \frac{w}{3} [f(x_0) - 2f(x_1) + f(x_2)] + 2wf(x_0) \\ &= \frac{w}{3} [f(x_0) - 2f(x_1) + f(x_2) + 6f(x_0)] \\ &= \frac{w}{3} [f(x_0) + 4f(x_1) + f(x_2)] \end{aligned}$$

Using a similar method, it can be shown that the area of the next 2 intervals is $\frac{w}{3} [f(x_2) + 4f(x_3) + f(x_4)]$.

The total area can be found by adding the area of each parabola together.

$$\begin{aligned} \int_a^b f(x) &= \frac{w}{3} [f(x_0) + 4f(x_1) + f(x_2)] + \frac{w}{3} [f(x_2) + 4f(x_3) + f(x_4)] + \dots + \frac{w}{3} [f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \\ &= \frac{w}{3} [f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \dots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \\ &= \frac{w}{3} [f(x_0) + 4[f(x_1) + f(x_3) + \dots] + 2[f(x_2) + f(x_4) + \dots] + f(x_n)] \end{aligned}$$

Simpson's rule

Simpson's rule for approximating $\int_a^b f(x)dx$ is

$$\int_a^b f(x) dx \approx \frac{w}{3} [f(x_0) + 4[f(x_1) + f(x_3) + \dots] + 2[f(x_2) + f(x_4) + \dots] + f(x_n)]$$

where $w = \frac{b-a}{n}$.

WORKED EXAMPLE 14

Use Simpson's rule with four strips to determine the approximate value of the definite integral

$$\int_0^{\frac{\pi}{2}} \sin(\sqrt{x}) dx.$$

THINK

- There are 4 strips, so $n = 4$. The integration is over the interval $0 \leq x \leq \frac{\pi}{2}$ so $a = 0$ and $b = \frac{\pi}{2}$. Calculate the width of the interval.

WRITE

$$\begin{aligned} w &= \frac{b-a}{n} \\ &= \frac{\frac{\pi}{2} - 0}{4} \\ &= \frac{\pi}{8} \end{aligned}$$

2. Construct a table of values for x_0 to x_4 and $f(x_0)$ to $f(x_4)$. Technology could be used to generate the $f(x_i)$ values.

n	x_n	$f(x_n) = \sin(\sqrt{x_n})$
0	0	0
1	$\frac{\pi}{8}$	0.586
2	$\frac{2\pi}{8} = \frac{\pi}{4}$	0.775
3	$\frac{3\pi}{8}$	0.884
4	$\frac{4\pi}{8} = \frac{\pi}{2}$	0.950

3. State the formula for Simpson's rule.

$$\text{Area} \approx \frac{w}{3} [f(x_0) + 4[f(x_1) + f(x_3) + \dots] + 2[f(x_2) + f(x_4) + \dots] + f(x_n)]$$

4. Write the formula to approximate the value of the required definite integral.

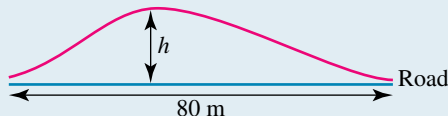
$$\begin{aligned} \text{So } \int_0^{\frac{\pi}{2}} \sin(\sqrt{x}) \, dx \\ &\approx \frac{w}{3} [f(x_0) + 4(f(x_1) + f(x_3)) + 2(f(x_2)) + f(x_4)] \\ &= \frac{1}{3} \times \frac{\pi}{8} [0 + 4(0.586 + 0.884) + 2(0.775) + 0.950] \\ &= 1.097 \end{aligned}$$

5. The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd. Substitute for w and $f(x_i)$ in the formula.

Simpson's rule can also be used to approximate areas and volumes of objects by taking measurements at regular intervals.

WORKED EXAMPLE 15

A road is to be built by cutting through a hillside as shown.



The width of the cutting is 80 m and it has a depth of 50 m. A measurement of the height, h , of the cutting is taken every 10 m. The results are recorded in the table below.



Measurement	Horizontal distance	Height
1	0	0
2	10	2
3	20	4.5
4	30	6
5	40	5
6	50	4
7	60	3
8	70	2.5
9	80	0

Determine the volume, in cubic metres, of rock that is to be removed to build the road.

THINK

1. Consider the shape of the cutting to be a prism and state the appropriate formula for volume.
2. State Simpson's rule to estimate the area of the cross-section.
3. Calculate the width, w , of each strip by identifying the number of strips, n , and the values for a and b .

4. The measurement for h can be thought of as $f(x_i)$ for the formula. The first measurement is $f(x_0)$. The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd.

5. Substitute the estimate for area into the formula for volume.
6. State the solution.

WRITE

Volume = area of cross-section $\times$ width

$$\text{Area} \approx \frac{w}{3} [f(x_0) + 4[f(x_1) + f(x_3) + \dots] + 2[f(x_2) + f(x_4) + \dots] + f(x_n)]$$

Examine the table given in the question. There are nine measurements, however, measurement 1 is the initial measurement and the first strip occurs once measurement 2 is taken.

Therefore $n = 8$, $a = 0$, $b = 80$.

$$\text{So } w = \frac{b - a}{n} = \frac{80 - 0}{8} = 10$$

The width of each strip is 10 m.

$$\begin{aligned} \text{Area} &\approx \frac{10}{3} [f(x_0) + 4(f(x_1) + f(x_3) + f(x_5) + f(x_7)) \\ &\quad + 2(f(x_2) + f(x_4) + f(x_6)) + f(x_8)] \\ &= \frac{10}{3} [0 + 4(2 + 6 + 4 + 2.5) \\ &\quad + 2(4.5 + 5 + 3) + 0] \\ &= 276.\dot{6} \end{aligned}$$

$$\text{Area} \approx 276.67 \text{ m}^2$$

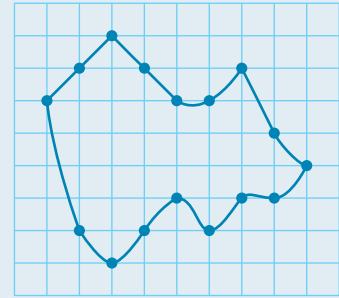
$$\begin{aligned} \text{Volume} &\approx 276.\dot{6} \times 50 \\ &= 13\,833.\dot{3} \end{aligned}$$

Approximately 13 833 cubic metres of rock will need to be removed.

In the examples so far, one of the sides has been a straight line. Simpson's rule can be used to estimate the area of any convex figure.

WORKED EXAMPLE 16

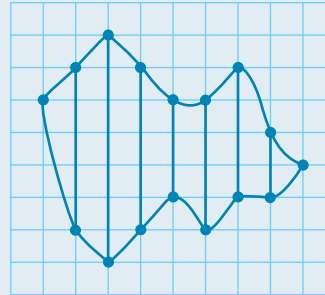
Use Simpson's rule to determine the area of the following if each major square has the dimensions $1\text{ cm} \times 1\text{ cm}$.



THINK

- The shape needs to be divided into an even number of intervals. 8 strips will fit on the grid nicely.
Each interval is 1 unit wide so $w = 1$.

WRITE



- Measure the length of each line and record the results in a table.

n	$f(x_n)$
0	0
1	5
2	7
3	5
4	3
5	4
6	4
7	2
8	0

- State the formula for Simpson's rule to determine the area. Note that the width of each strip is 1 cm.

$$\text{Area} \approx \frac{w}{3} [f(x_0) + 4[f(x_1) + f(x_3) + \dots] + 2[f(x_2) + f(x_4) + \dots] + f(x_n)]$$

- The formula for Simpson's rule groups together $f(x_0)$ and $f(x_n)$, $f(x_j)$ where j is even, and $f(x_k)$ where k is odd.

where $w = 1$.

$$\begin{aligned} &= \frac{1}{3} [0 + 4(5 + 5 + 4 + 2) + 2(7 + 3 + 4) + 0] \\ &= \frac{1}{3} (4 \times 16 + 2 \times 14) \\ &= 30.\dot{6} \end{aligned}$$

- State the area.

The area is approximately 30.67 cm^2 .

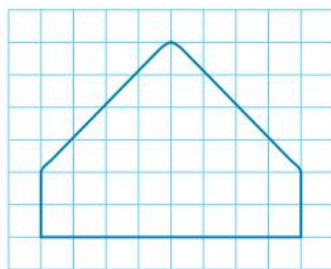
Because Simpson's rule fits a parabola across the top of 2 intervals, there must be an even number of strips, and the area will be exact for determining the area under a parabola. Perhaps unexpectedly, Simpson's rule is also exact for determining the area under a cubic function.

Technology can be used to calculate areas using Simpson's rule. In this spreadsheet, the area between the curve $f(x) = \sqrt{x}$ between $x = 0, x = 16$ and the x -axis with 12 strips is calculated. In the spreadsheet, the values for a, b and n are entered into cells G1, G2 and G3 respectively. The value for w is calculated using $w = \frac{b-a}{n}$. The values for x_i are determined by adding w to the previous value of x_i . For each $x_i, f(x_i)$ is calculated. In the column labelled Simpson, the appropriate multiples of the $f(x_i)$ are applied. The formulas for D3 and D4 can be entered and the pair can be dragged to fill the series. Cell D15 calculates the totals and G15 multiplies the total by $\frac{w}{3}$.

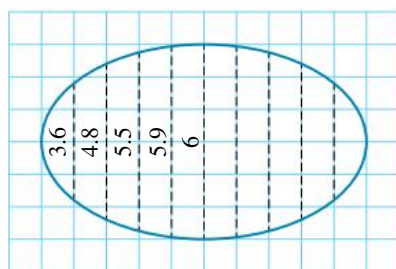
	A	B	C	D	E	F	G
1	i	xi	$f(x_i)$	Simpson		a =	0
2	0	0	0	0		b =	16
3	1	1.333333	1.154701	4.618802		n =	12
4	2	2.666667	1.632993	3.265986		w =	1.333333
5	3	4	2	8			
6	4	5.333333	2.309401	4.618802			
7	5	6.666667	2.581989	10.32796			
8	6	8	2.828427	5.656854			
9	7	9.333333	3.05505	12.2202			
10	8	10.66667	3.265986	6.531973			
11	9	12	3.464102	13.85641			
12	10	13.33333	3.651484	7.302967			
13	11	14.66667	3.829708	15.31883			
14	12	16	4	4			
15				95.71878		Area =	42.54168
16							

	A	B	C	D	E	F	G
1	i	xi	$f(x_i)$	Simpson		a =	0
2	0	=G1	=SQRT(B2)	=C2		b =	16
3	1	=B2+G\$4	=SQRT(B3)	=4*C3		n =	12
4	2	=B3+G\$4	=SQRT(B4)	=2*C4		w =	=(G2-G1)/G3
5	3	=B4+G\$4	=SQRT(B5)	=4*C5			
6	4	=B5+G\$4	=SQRT(B6)	=2*C6			
7	5	=B6+G\$4	=SQRT(B7)	=4*C7			
8	6	=B7+G\$4	=SQRT(B8)	=2*C8			
9	7	=B8+G\$4	=SQRT(B9)	=4*C9			
10	8	=B9+G\$4	=SQRT(B10)	=2*C10			
11	9	=B10+G\$4	=SQRT(B11)	=4*C11			
12	10	=B11+G\$4	=SQRT(B12)	=2*C12			
13	11	=B12+G\$4	=SQRT(B13)	=4*C13			
14	12	=B13+G\$4	=SQRT(B14)	=C14			
15				=SUM(D2:D14)		Area =	=G4*(D15)/3
16							

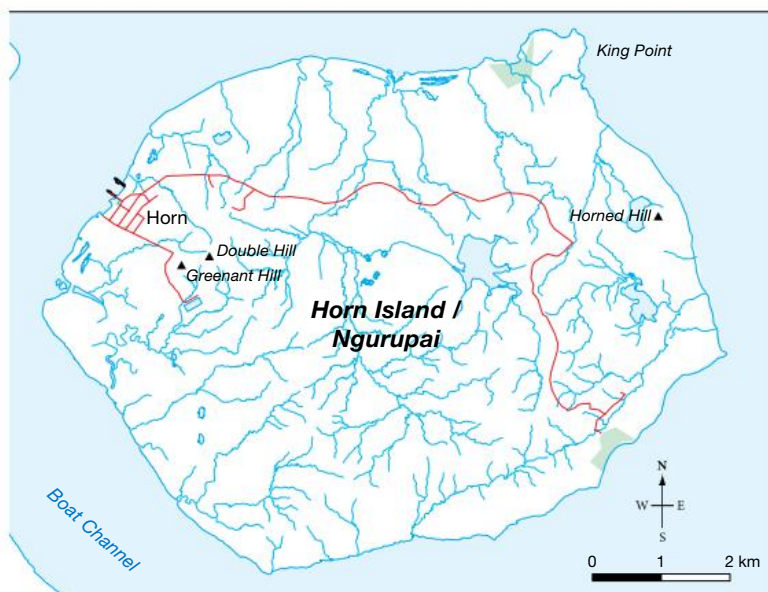
8. **WE16** Use Simpson's rule with four strips to determine the area of the following if each major square has the dimensions $1\text{ cm} \times 1\text{ cm}$.



9. **a.** Use Simpson's rule with 10 strips to determine the approximate area of the following shape. The area of each major square has the dimensions $1\text{ cm} \times 1\text{ cm}$. The numbers shown represent the length (in cm) of the internal vertical distances.



- b.** Compare your answer in part **a** to the result obtained by using the formula for the area of an ellipse $A = \pi ab$, where a is half the length of the ellipse and b is half the width.
10. Use Simpson's rule to estimate the area of Horn Island.



Source: Natural Resources, Mines and Energy, Queensland Government

11. The area bounded by the curve $y = x^2$ between $x = a$ and $x = b$ and the x -axis is given by $\int_a^b x^2 dx$.

Because Simpson's rule uses a parabola to approximate the area under a curve, this method should give an exact value for $\int_a^b x^2 dx$.

Show that if $\int_a^b x^2 dx$ is approximated by Simpson's rule using two strips then it gives the exact value.

7.7 Exponential probability density function

7.7.1 Exponential probability density function

In Mathematical Methods, you explored the graph of the probability density function of the normal distribution. This particular distribution is symmetric, with a mean of μ and a standard deviation of σ . The normal distribution is one example of a probability distribution function, there are many others.

For a real function, f , to be a valid probability distribution, the following must hold:

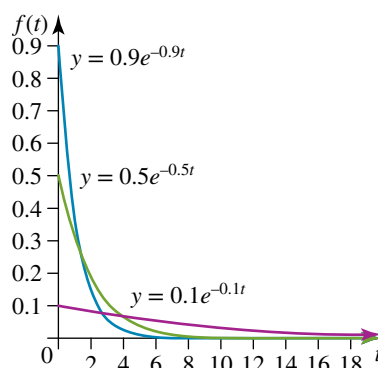
1. f is non-negative, that is $f(t) \geq 0$ for all $t \in \mathbb{R}$
2. the area under the graph of the function is equal to 1, $\int_{-\infty}^{\infty} f(t) dt = 1$

The **exponential probability density function** $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ is an example. The function is

non-negative and

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) dt &= \int_0^{\infty} \lambda e^{-\lambda t} dt \\ &= \left[\frac{\lambda}{-\lambda} e^{-\lambda t} \right]_0^{\infty} \quad \text{as } \lim_{t \rightarrow \infty} e^{-t} = 0 \text{ and } e^0 = 1. \\ &= -e^{-\infty} - (-e^0) \\ &= 1 \end{aligned}$$

In this distribution, t is the wait time for an event to occur and the average number of events in a time period (for example; minute, hour, day) is λ . It can be shown that the average wait time is $\frac{1}{\lambda}$. This has been left for an exercise.



In this diagram, different values for λ are shown. In this instance, $\lambda \in \{0.1, 0.5, 0.9\}$.

WORKED EXAMPLE 17

If the average life of an LED lightbulb is 50 000 hours, determine the probability that a particular lightbulb lasts for the following:

- between 40 000 and 70 000 hours
- more than 60 000 hours.

THINK

- The average life is 50 000. Therefore $\frac{1}{\lambda} = 50\,000$.
- As the distribution is concerned with wait times (waiting for the lightbulb to die), it is appropriate to use $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$.
- $P(40\,000 \leq T \leq 70\,000)$ can be determined by integrating $f(t)$ between $t = 40\,000$ and $t = 70\,000$.

- State the solution.

- An important feature of a continuous distribution is that $P(T > x) \equiv P(T \geq x)$. Thus, $P(T > 60\,000) = P(T \geq 60\,000)$. Remember that $\lim_{t \rightarrow \infty} e^{-t} = 0$.

WRITE

$$\text{a. } \frac{1}{\lambda} = 50\,000$$

$$\lambda = \frac{1}{50\,000}$$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\begin{aligned} P(40\,000 \leq T \leq 70\,000) &= \int_{40\,000}^{70\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} dt \\ &= \left[-\frac{50\,000}{50\,000} e^{\frac{-t}{50\,000}} \right]_{40\,000}^{70\,000} \\ &= \left[-e^{\frac{-t}{50\,000}} \right]_{40\,000}^{70\,000} \\ &= -e^{\frac{-70\,000}{50\,000}} + e^{\frac{-40\,000}{50\,000}} \\ &= -e^{-\frac{7}{5}} + e^{-\frac{4}{5}} \end{aligned}$$

$$P(40\,000 \leq T \leq 70\,000) = e^{-\frac{4}{5}} - e^{-\frac{7}{5}}$$

$$\approx 0.2027$$

$$\begin{aligned} \text{b. } P(T \geq 60\,000) &= \int_{60\,000}^{\infty} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} dt \\ &= \left[-e^{\frac{-t}{50\,000}} \right]_{60\,000}^{\infty} \\ &= -e^{-\infty} + e^{\frac{-60\,000}{50\,000}} \\ &= e^{-\frac{6}{5}} \end{aligned}$$

Alternatively, the problem can be solved using the following process.

$$\begin{aligned} P(T \geq 60\,000) &= 1 - P(T \leq 60\,000) \\ &= 1 - \int_0^{60\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} dt \end{aligned}$$

$$\begin{aligned} P(T \geq 60\,000) &= 1 - P(T \leq 60\,000) \\ &= 1 - \int_0^{60\,000} \frac{1}{50\,000} e^{\frac{-t}{50\,000}} dt \\ &= 1 - \left[-e^{\frac{-t}{50\,000}} \right]_0^{60\,000} \\ &= 1 - \left(-e^{\frac{-60\,000}{50\,000}} + e^0 \right) \\ &= 1 + e^{-\frac{6}{5}} - 1 \\ &= e^{-\frac{6}{5}} \end{aligned}$$

2. State the solution.

$$\begin{aligned} P(T \geq 60\,000) &= e^{-\frac{6}{5}} \\ &\approx 0.3012 \end{aligned}$$

WORKED EXAMPLE 18

The expected wait time for a bus is given by the function $f(t) = \begin{cases} 0.2e^{-0.2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$,

where t is the wait time in minutes.

- Determine the probability that wait time is between 0 and 2 minutes.
- Determine the probability that wait time is between 4 and 6 minutes given that they have already waited 4 minutes.

THINK

- The probability can be calculated by determining the area under the curve between $t = 0$ and $t = 2$.

WRITE

$$\begin{aligned} \text{a. } P(0 \leq T \leq 2) &= \int_0^2 0.2e^{-0.2t} dt \\ &= \left[\frac{0.2}{-0.2} e^{-0.2t} \right]_0^2 \\ &= -e^{-0.4} + e^0 \\ &= 1 - e^{-0.4} \\ &\approx 0.3297 \end{aligned}$$

1. Recall the conditional probability formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$. $P(A \cap B)$ represents the intersection of the two events $P(T \geq 4)$ and $P(T \leq 6)$. Thus $P(A \cap B) = P(4 \leq T \leq 6)$. Determine $P(4 \leq T \leq 6)$.

$$\begin{aligned} \text{b. } P(4 \leq T \leq 6) &= \int_4^6 0.2e^{-0.2t} dt \\ &= \left[\frac{0.2}{-0.2} e^{-0.2t} \right]_4^6 \\ &= -e^{-1.2} + e^{-0.8} \\ &= e^{-0.8} - e^{-1.2} \end{aligned}$$

$P(B)$ is the probability that they have already waited 4 minutes. This means that the wait will be a minimum of 4 minutes. Determine $P(T \geq 4)$. Remember $\lim_{t \rightarrow \infty} e^{-t} = 0$.

2. To determine the probability that the wait time is between 4 and 6 minutes, given that the wait has already been 4 minutes, calculate $\frac{P(4 \leq T \leq 6)}{P(T \geq 4)}$.

$$\begin{aligned} P(T \geq 4) &= \int_4^{\infty} 0.2e^{-0.2t} dt \\ &= \left[\frac{0.2}{-0.2} e^{-0.2t} \right]_4^{\infty} \\ &= -e^{-\infty} + e^{-0.8} \\ &= e^{-0.8} \\ \frac{P(4 \leq T \leq 6)}{P(T \geq 4)} &= \frac{e^{-0.8} - e^{-1.2}}{e^{-0.8}} \\ &= e^{0.8} (e^{-0.8} - e^{-1.2}) \\ &= 1 - e^{-0.4} \\ &\approx 0.3297 \end{aligned}$$

In the example, the probability of a particular wait time is independent of the time that has already been spent waiting. This property of the exponential probability distribution means that the distribution can be referred to as “memoryless”.

For example, severe floods are sometimes referred to as “hundred year floods”, meaning that their likelihood of occurring is approximately once every one hundred years. This expected wait time until the next flood can be modelled using the exponential probability density function. Some people erroneously believe that once the flood has occurred, it should be about one hundred years until the next one, however, the memoryless nature of the distribution means that once a flood has occurred, the likelihood of another flood occurring within the next 10 years, for example, is the same as it was before the flood hit.

studyon

Units 3 & 4 > Area 4 > Sequence 2 > Concept 6

Exponential probability density function Summary screen and practice questions

Exercise 7.7 Exponential probability density function

Technology free

1. Let T be an exponential random variable with a probability density function given by

$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$, $\lambda = 2$. Determine the following:

- a. $P(T \leq 2)$
- b. $P(T \geq 1)$.
2. **WE17** If the average life of an electric light tube is 500 hours, determine the probability that a particular light tube will last for the following:
 - a. between 400 and 600 hours
 - b. more than 700 hours.
3. If a text arrives every 15 seconds on average, (this means that $\lambda = 4$ per minute), calculate the probability of waiting less than or equal to 30 seconds (that is 0.5 minutes) for a text.
4. Accidents occur at an average of 4 per week, (this means that $\lambda = 4$). Calculate the probability that at least 2 weeks will elapse between accidents.

5. **WE18** The expected wait time for a bus is given by the function $f(t) = \begin{cases} 0.2e^{-0.2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$, where t is the wait time in minutes.
- Determine the probability that wait time is between 0 and 5 minutes.
 - Determine probability that a bus will arrive in the next 5 minutes given that you have already waited 10 minutes.
6. The average rate of job requests in a computer support centre is 4 per minute. It can be assumed that the number of requests per minute follows an exponential probability density function.
- Identify the exponential probability density function.
 - Calculate the probability that at least 15 seconds will elapse between requests.
 - Calculate the probability that less than 1 minute will elapse between requests.
 - If no requests have been made in the last 30 seconds, determine the probability that a job will arrive in the next 15 seconds.

Technology active

For questions 7–11 give all answers correct to four decimal places.

- Suppose that the response time T of a help line (the elapse time between the end of a user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected response time equal to 5 seconds.
 - Determine the probability that the response time is at most 10 seconds.
 - Determine the probability that the response time is between 5 and 10 seconds.
- Suppose that calls are received at a 24-hour hotline at a rate of 0.5 calls per day. The number of days, T , between successive calls has an exponential distribution.
Determine the probability that more than two days elapses between calls.
- Calls arrive at an average rate of 12 per hour. Calculate the probability that a call will occur in the next 5 minutes given that you have already waited 10 minutes for a call.
- Suppose that the time between messages from your sister has an exponential distribution with a mean time of 3 days.
 - If you just received a message from her, determine the probability that you will receive the next message within the next 2 days.
 - You realise that you have not received a message in at least 2 days. Determine the probability that she will not message you within the next 2 days.
- The time between emergency patients in a country hospital follows an exponential distribution with an average of 2.1 new patients per day.
 - Identify the exponential probability distribution function for the expected wait time between new patients.
 - The triage nurse Cameron has just started his shift. Determine the probability that a new patient will arrive in the next 15 minutes.
 - Cameron has nearly finished his shift. With 15 minutes remaining, determine the probability that a new patient will arrive before the shift finishes.
 - Teagan works a regular Tuesday to Friday 10 hour shift. Determine the probability that no new patients arrive during one of her shifts.
 - Determine the probability that no new patients arrive during one week of Teagan's shifts.
 - Teagan has observed that in 10% of shifts, a new patient arrives in the first x hours. Determine x correct to 1 decimal place.

7.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

- For each of the following integrals, identify the domain of the integrand and whether the integrand exists.
a. $\int_0^2 \left(x + \frac{1}{x-2}\right) dx$ **b.** $\int_0^3 (e^x + e^{-x})^2 dx$ **c.** $\int_1^5 \frac{x+2}{x^2+3x-10} dx$ **d.** $\int_0^4 \sqrt{32-2x^2} dx$
- Determine the area of the region bounded by the curve $y = x^3 - 8x$ and the x -axis.
- Determine the area of the region bounded by the curve $y = \frac{1}{x^2-1}$, the lines $x = 2$, $x = 3$ and the x -axis.
- Determine the area of the region bounded by the curve $y = x^2 - 4x$ and the line $y = 2x$.
- Determine the area of the region between the curve $y = \frac{4}{9-x^2}$ and the line $y = 2x$ from $x = 1$ to $x = 2$.
- Determine the volume of the solid formed by rotating the line $y = \frac{x}{2}$ about the x -axis from the origin to $x = 4$.
- Determine the volume of the solid formed by rotating the line $y = \frac{x}{4}$ about the y -axis from the origin to $x = 8$.
- Determine the volume of revolution when the region bounded by the curve $y = x^2 - 6x$ and the x -axis is rotated about the x -axis.
- Determine the volume of revolution when the region bounded by the curve $y = -5 \ln(x+1) + 3$ and the coordinate axes is rotated about the y -axis.
- Determine the volume of revolution when the region bounded by the curve $y = \sqrt{2-x}$ and the coordinate axes is rotated about the x -axis.
- Determine the volume of revolution when the region bounded by the curve $y = \sqrt{16-x}$ and the coordinate axes is rotated about the y -axis.
- An exponential probability density function is given by $f(t) = \begin{cases} 2.2e^{-2.2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$.
 - Determine $P(0 \leq T \leq 2)$ correct to four decimal places.
 - Determine $P(T \geq 3)$ correct to four decimal places.

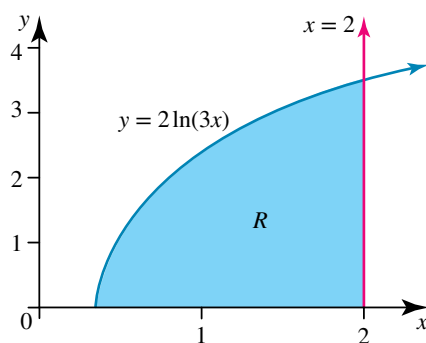
Complex familiar

- Determine the exact volume formed when the area bounded by the curve with the equation $y = x^4$, the x -axis and the line $x = 2$ is rotated about the:
 - x -axis
 - y -axis.
- Consider the functions $f(x) = 5 \sin\left(\frac{3x}{4}\right)$ and $g(x) = \frac{3x}{4}$ where $x \geq 0$. Determine, correct to 3 decimal places, the area between functions f and g .
- The area of a quadrant of a circle is represented by $\int_0^6 \sqrt{36-x^2} dx$. Compare the areas obtained by using an exact calculus method and using Simpson's rule with 6 strips.
- Kayley estimates that the average time between nuisance phone calls is 2 days. If it has been one day without a call, determine the probability that there will be a call in the next 3 hours. Give your answer correct to four decimal places.

Complex unfamiliar

Use the following information to answer questions 17–20.

The region R is bounded by the line $x = 2$, the curve $y = 2\ln(3x)$ and the x -axis.



17. Determine the exact value of the region R .
18. Determine the exact value of the volume formed when the region R is rotated about the y -axis.
19. The line $x = k$ divides the exact value of the region in half. Determine the value of k correct to 3 decimal places.
20. When the region R is rotated about the x -axis it forms a solid of revolution. Determine the volume of the solid.

studyon

Units 3 & 4 Sit exam

Answers

7 Applications of integral calculus

Exercise 7.2 Area between a function and the axes

1. a. $R \setminus \{-3, 3\}$ The integral exists.
 b. $(-2, 2)$ The integral exists.
 c. $(-4, 4)$ The integral does not exist.
 d. R The integral exists.
 e. $R \setminus \{0\}$ The integral exists.
 f. $R \setminus \{-1, 0\}$ The integral exists.
2. a. $\frac{1}{6} \ln\left(\frac{5}{2}\right)$
 b. $-\frac{\pi}{6}$
 c. The integral does not exist.
 d. 1.893
 e. $2 \ln(4)$
 f. $\ln\left(\frac{4}{3}\right)$
3. a. $R \setminus \{-3, -2\}$ The integral exists.
 b. $R \setminus \{1\}$ The integral does not exist.
 c. $R \setminus \{2, 6\}$ The integral does not exist.
 d. R The integral exists.
 e. $\left(-\frac{1}{3}, \frac{1}{3}\right)$ The integral does not exist.
 f. $\left[\frac{1}{2}, \infty\right)$ The integral exists.
4. a. $2 \ln(6)$
 b. The integral does not exist.
 c. The integral does not exist.
 d. $\frac{\pi}{36}$
 e. The integral does not exist.
 f. $\frac{9\sqrt{3}}{5}$
5. a. $\frac{1}{6}$ square units
 b. $\frac{4}{3}$ square units
 c. $\frac{64\sqrt{2}}{3}$ square units
 d. 32 square units
 e. $\frac{256}{3}$ square units
 f. $\frac{1372}{27}$ square units
6. a. 8 square units
 b. $\frac{37}{12}$ square units
 c. $\frac{81}{2}$ square units
 d. $\frac{64}{3}$ square units
7. a. $\frac{1}{2} \ln(2)$ square units
 b. $\ln\left(\frac{3}{2}\right)$ square units
 c. $\frac{11}{2} + \frac{1}{2}e^{-4}$ square units
 d. $3 \ln(3) + \frac{1}{2}(e^2 + e^{-2}) - 3$ square units
8. a. $x > -\frac{1}{2}, \frac{26}{3}$ square units
 b. $R \setminus \left\{-\frac{5}{3}\right\}, \frac{1}{3} \ln\left(\frac{14}{5}\right)$ square units

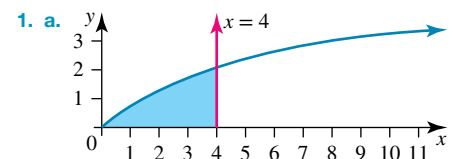
- c. $R \setminus \left\{-\frac{3}{2}\right\}, \frac{1}{35}$ square units
- d. $x < \frac{16}{3}, 6$ square units
9. a. $\frac{1}{2} \ln\left(\frac{21}{5}\right)$ square units
 b. $\frac{\sqrt{2}}{4}$ square units
 c. $\frac{1}{2}(1 - e^{-4})$ square units
 d. 2 square units
10. a. $\ln\left(\frac{9}{4}\right)$ square units
 b. $\ln\left(\frac{3}{2}\right)$ square units
 c. $\ln(8)$ square units
 d. $10 + \frac{3}{2} \ln\left(\frac{7}{3}\right)$ square units
11. $\frac{32}{3}$ square units
12. $\frac{256}{15}$ square units
13. $\frac{4}{3}\sqrt{a^3}$ square units
14. $\frac{8\sqrt{a^5}}{15}$ square units

Exercise 7.3 Area between functions

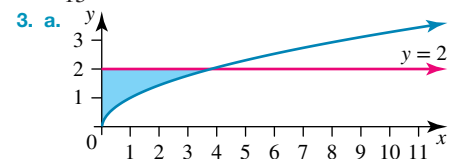
1. a. $\frac{243}{2}$ square units
 b. 36 square units
 c. 288 square units
 d. $\frac{243}{2}$ square units
2. $9 \ln\left(\frac{9}{4}\right) - \frac{1}{2}$ square units
3. a. $\frac{1}{6}$ square units
 b. $\frac{1}{2}$ square units
 c. $\frac{3}{10}$ square units
 d. $\frac{2}{3}$ square units
4. a. $2\sqrt{2}$ square units
 b. 4 square units
 c. $\frac{9}{8}$ square units
 d. $2e - 2e^{-2}$ square units
5. a. $\frac{15}{4} - 2 \ln(4)$ square units
 b. $\frac{40}{27} + \frac{1}{3} \ln\left(\frac{1}{9}\right)$ square units
6. $\frac{16}{3}$ square units
7. $\frac{5}{6}$ square units
8. $\frac{5}{2}$ square units
9. Sample responses can be found in the worked solutions in the online resources.
10. Sample responses can be found in the worked solutions in the online resources.
11. $e - e^{-2} \text{ m}^2$
12. 7.34 square units

13. a. 15.43 square units
b. -0.54
14. a. 4.39 square units
b. 0.49

Exercise 7.4 Volumes of solids of revolution



- b. 8π cubic units
c. $\frac{512\pi}{15}$ cubic units



- b. $\frac{32\pi}{5}$ cubic units
4. 8π cubic units
5. a. 375π cubic units
b. $\frac{125\pi}{27}$ cubic units
6. a. 4π cubic units
b. 6π cubic units
7. $\frac{\pi}{2}$ cubic units
8. $4\pi(e^2 - 1)$ cubic units
9. 10 L
10. 1100 mL
11. a. $\frac{9\pi^2}{4}$ cubic units
b. $\frac{4\pi^2}{3}$ cubic units
12. a. $\frac{768\pi}{5}$ cubic units
b. 6π cubic units
c. $\frac{1296\pi}{5}$ cubic units
13. a. $\frac{81\pi}{2}$ cubic units
b. $\frac{320\pi}{3}$ cubic units
c. $\frac{400\pi}{3}$ cubic units
d. $\frac{4}{3}\pi ab^2$ cubic units
e. $\frac{4}{3}\pi a^2b$ cubic units
- f. If rotating about the x -axis: $\frac{4x^2}{57^2} + \frac{y^2}{22^2} = 1$
If rotating about the y -axis: $\frac{x^2}{22^2} + \frac{4y^2}{57^2} = 1$
- f. 57 780 mm³
15. a. Sample responses can be found in the worked solutions in the online resources.
b. 136 L

16. a. 3099.70 cm³
b. 3183.03 cm³
c. 3141.59 cm³
d. 3223.69 cm³
17. a. $32\pi(3 - 4\ln(2))$ cubic units
b. $\frac{64\pi}{15}$ cubic units
18. $\frac{27776\pi}{3}$ cm³

Exercise 7.5 Volumes of revolution

1. a. $\frac{704\pi}{5}$ cubic units
b. 8π cubic units
2. a. 8π cubic units
b. $\frac{32\pi}{5}$ cubic units
3. a. $\frac{49\pi}{2}$ cubic units
b. $\frac{76\pi}{5}$ cubic units
4. a. $\frac{128\pi}{5}$ cubic units
b. 8π cubic units
c. $\frac{4352\pi}{15}$ cubic units
5. a. $\frac{81\pi}{2}$ cubic units
b. 8π cubic units
6. a. $\frac{383\pi}{15}$ cubic units
b. $\frac{486\pi}{5}$ cubic units
7. a. $\frac{2187\pi}{7}$ cubic units
b. $\frac{416\pi}{15}$ cubic units
8. a. 4π cubic units
b. $4\pi^2$ cubic units
9. a. $\frac{9\pi^2}{2}$ cubic units
b. $\frac{64\pi}{15}$ cubic units
10. a. $\frac{8\pi}{3}$ cubic units
b. $\frac{2\pi}{35}$ cubic units
11. a. $\frac{96\pi}{5}$ cubic units
b. $\frac{3\pi a^3}{10}$ cubic units
12. a. 1296π cubic units
b. 81π cubic units
13. a. $\frac{512\pi}{3}$ cubic units
b. $\frac{192\pi}{5}$ cubic units
14. a. $32\sqrt{3}\pi$ cubic units
b. $\frac{a^3\pi\sqrt{3}}{2}$ cubic units

17. a. 256π cubic units
 b. $2\pi^2 ar^2$ cubic units
18. a. $\frac{11\sqrt{3}\pi a^5}{40}$ cubic units
 b. $\frac{9\pi a^4}{32}$ cubic units

Exercise 7.6 Approximation using Simpson's rule

1. 0.828 square units
2. a. 0.810 square units
 b. 0.810 square units
3. a. 1.105 square units
 b. 1.107 square units
4. a. $\tan^{-1}(2)$ square units
 b. $\tan^{-1}(2) \approx 1.107148 \dots$ This answer is very close to the one obtained in question 3.
5. a. $\frac{9\pi}{2}$ square units
 b. 13.995 square units which is a 1% difference
6. 6300 m^3
7. 793.3 m^2
8. 32 cm^2
9. a. 46.5 cm^2
 b. Simpson's rule is accurate to about 1.3%
10. Note: answers will vary depending on the number of strips used. The area of Horn Island is 53 km^2 .
11. Exact value $= \frac{1}{3}(b^3 - a^3)$. Simpson's rule approximation $= \frac{1}{3}(b^3 - a^3)$.

Exercise 7.7 Exponential probability density function

1. a. $1 - e^{-4}$
 b. e^{-2}
2. a. $-e^{-\frac{6}{5}} + e^{-\frac{4}{5}}$
 b. $e^{-\frac{7}{5}}$
3. $1 - e^{-2}$
4. e^{-8}
5. a. $1 - e^{-1}$
 b. $1 - e^{-1}$
6. a. $f(t) = \begin{cases} 4e^{-4t} & t \geq 0 \\ 0 & t < 0 \end{cases}$
 b. e^{-1}
 c. $1 - e^{-4}$
 d. $1 - e^{-1}$
7. a. 0.8647
 b. 0.2325

8. 0.3679
9. 0.6321
10. a. 0.4866
 b. 0.5134
11. a. $f(t) = \begin{cases} 2.1e^{-2.1t} & t \geq 0 \\ 0 & t < 0 \end{cases}$
 b. 0.0216
 c. 0.0216
 d. 0.4169
 e. 0.0302
 f. 1.2041 hours (≈ 1 hours 12 minutes)

7.8 Review: exam practice

1. a. $R \setminus \{2\}$ The integral does not exist.
 b. R The integral exists.
 c. $R \setminus \{-5, 2\}$ The integral does not exist.
 d. $[-4, 4]$ The integral exists.
2. 32 units^2
3. $\frac{1}{2} \ln\left(\frac{3}{2}\right) \text{ units}^2$
4. 36 units^2
5. $3 + \frac{2}{3} \ln\left(\frac{2}{5}\right) \text{ units}^2$
6. $\frac{16\pi}{3} \text{ units}^3$
7. $\frac{128\pi}{3} \text{ units}^3$
8. $\frac{1296\pi}{5} \text{ units}^3$
9. $\frac{\pi}{2} \left(21 + 5e^{\frac{6}{5}} - 20e^{\frac{3}{5}}\right) \approx 1.82 \text{ units}^3$
10. $2\pi \text{ units}^3$
11. $\frac{8192\pi}{15} \text{ units}^3$
12. a. 0.9877
 b. 0.0014
13. a. $\frac{512\pi}{9} \text{ units}^3$
 b. $\frac{52\pi}{3} \text{ units}^3$
14. Area is 7.873 units^2
15. Exact value: $9\pi \approx 28.2743$. Simpson's rule: 27.99114
16. 0.0606
17. $4 \ln(6) - \frac{10}{3} \text{ units}^2$
18. $\frac{\pi}{9} (72 \ln(6) - 35) \text{ units}^3$
19. 1.411
20. $\frac{8\pi}{3} (3(\ln(6))^2 - 6 \ln(6) + 5) \text{ units}^3$

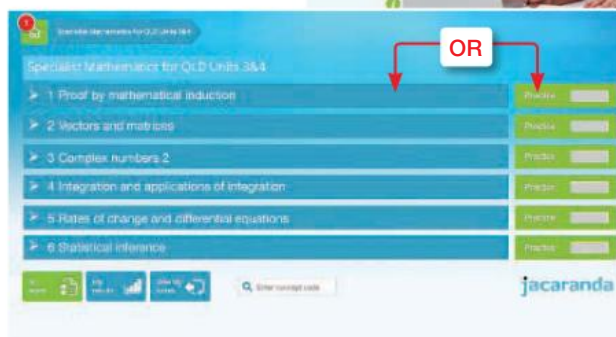
REVISION UNIT 4 Further calculus and statistical inference

TOPIC 1 Integration and applications of integration

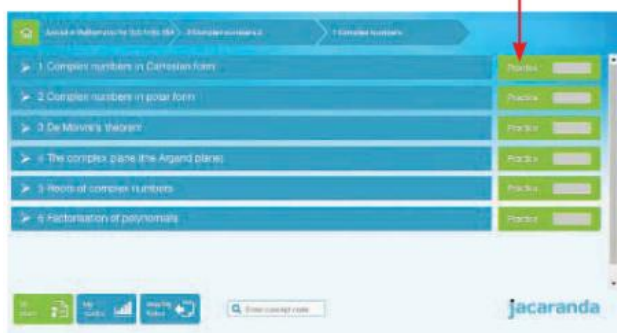
- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



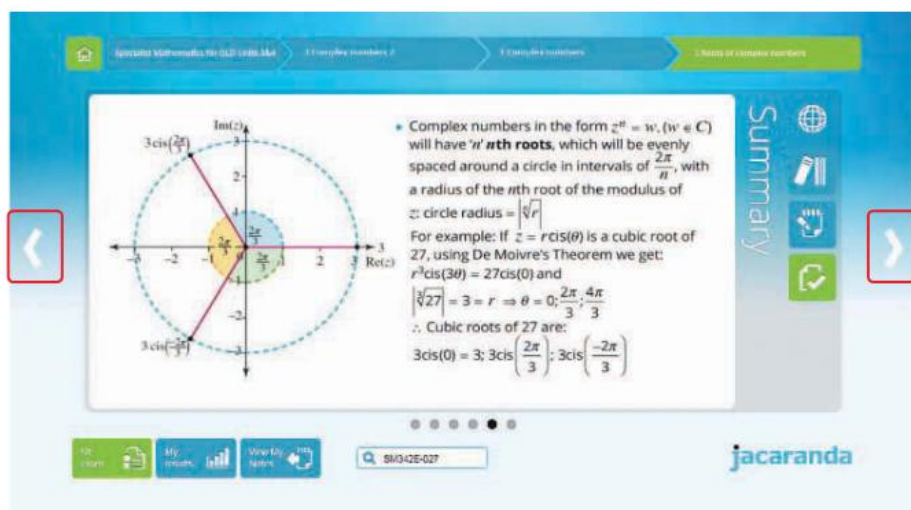
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



8 Rates of change and differential equations

8.1 Overview

The rate of change of one quantity with respect to another quantity is a measure of how the quantities are related or how a change in one quantity will effect a change in the other.

As you have already learned, rates of change form the basis of the study of calculus. Differential equations build on the concepts already covered in calculus; differential and integral.

Modelling the world around us using complex differential equations and sophisticated computers to solve these equations has led to an ever-increasing understanding of our world and the

universe we inhabit. One example of the use of differential equations is in the science of meteorology. Super computers can analyse enormous amounts of weather data, but it is the use of differential equations to model long and short-term patterns that has enabled meteorologists to determine how changes in temperature and moisture in the atmosphere may lead to changes in the weather. This can mean the difference between life and death as communities now have advance warning of extreme weather conditions.



LEARNING SEQUENCE

8.1 Overview

8.2 Implicit differentiation

8.3 Related rates as instances of the chain rule

8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

8.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Implicit differentiation

8.2.1 Introduction to implicit differentiation

Up until now the relationship between x and y has always been explicit: y is the dependent variable and x the independent variable, with y in terms of x written as $y = f(x)$. It is said that y depends upon x and $\frac{dy}{dx}$ can be found directly. For example:

$$\text{If } y = x^2 + 4x + 13, \text{ then } \frac{dy}{dx} = 2x + 4.$$

$$\text{If } y = 3 \sin(2x) + 4e^{-2x}, \text{ then } \frac{dy}{dx} = 6 \cos(2x) - 8e^{-2x}.$$

$$\text{If } y = \sqrt{16 - x^2}, \text{ then } \frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}.$$

$$\text{If } y = \ln(5x + 3), \text{ then } \frac{dy}{dx} = \frac{5}{5x + 3}.$$

There are times, however, when y is not expressed explicitly in terms of x . In these cases there is a functional dependence or a so-called implicit relationship between x and y of the form $f(x, y) = c$, where c is a constant, or $g(x, y) = 0$. For example:

$$x^2 + y^2 = 16$$

$$4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$$

$$e^{-2xy} + 3 \sin(2x - 3y) = c$$

These represent curves as an implicit relation and are not graphs of functions.

In some cases it may be possible to rearrange to make y the subject, but in most cases this is simply not possible. In this implicit form an expression for $\frac{dy}{dx}$ can still be obtained, but it will be in terms of both x and y . This can be obtained by differentiating each term in turn.

For example, $\frac{d}{dx}(x^2) = 2x$, and in general, if n is a constant, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Recall that $\frac{d}{dx}(\text{constant}) = 0$.

Consider $\frac{d}{dx}(y^2)$. To find this, use the chain rule and let $u = y^2$. Then $\frac{d}{dx}(y^2) = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = 2y \times \frac{dy}{dx}$; furthermore, in general $\frac{d}{dx}(y^n) = \frac{d}{dy}(y^n) \frac{dy}{dx} = ny^{n-1} \frac{dy}{dx}$.

This last result is known as **implicit differentiation**; it is just an application of the chain rule. When it is necessary to differentiate a function of y with respect to x , differentiate with respect to y and multiply by $\frac{dy}{dx}$.

WORKED EXAMPLE 1

Given $x^2 + y^2 = 16$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .

THINK

1. Write the equation.

WRITE/DRAW

$$x^2 + y^2 = 16$$

2. Take $\frac{d}{dx}()$ of each term in turn.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

3. Use the results above together with $\frac{d}{dx}(c) = 0$, since the derivative of a constant is zero.

$$2x + 2y \frac{dy}{dx} = 0$$

4. Transpose the equation to make $\frac{dy}{dx}$ the subject.

$$2x = -2y \frac{dy}{dx}$$

5. Cancel the common factor and divide by x . State the final result, giving $\frac{dy}{dx}$ in terms of both x and y .

$$\frac{dy}{dx} = -\frac{x}{y}$$

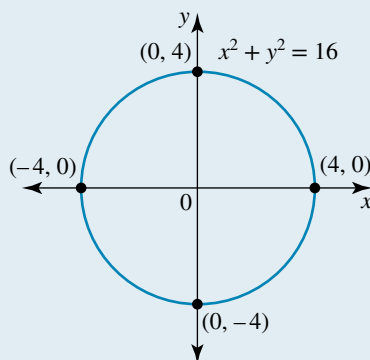
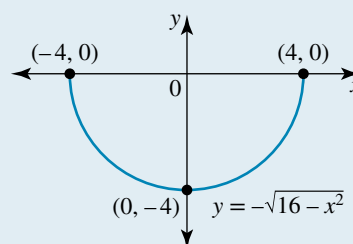
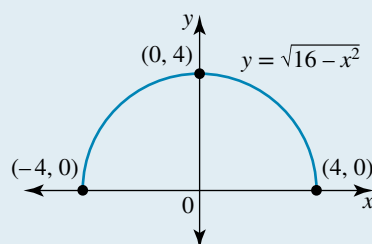
Alternatively, notice that in this particular example it is possible to rearrange the equation to make y the subject

1. Write the equation.
2. Rearrange to make y the subject.
3. There are two branches to the relation, which is a circle with centre at the origin and radius 4. Consider the branch or top half of the circle, which by itself is a function.

$$x^2 + y^2 = 16$$

$$y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$



4. Rearrange to write $\frac{dy}{dx}$ in terms of x , differentiating using the chain rule.

$$\text{Consider } y = \sqrt{16 - x^2}.$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{16 - x^2}}$$

5. However, since $y = \sqrt{16 - x^2}$, express $\frac{dy}{dx}$ in terms of both x and y as before.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Further examples involving implicit differentiation

Sometimes it may be necessary to use the product rule to obtain the required derivatives.

WORKED EXAMPLE 2

a. Determine $\frac{d}{dx}(3xy)$.

b. Given $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$, write down an expression for $\frac{dy}{dx}$ in terms of both x and y .

THINK

a. 1. Write the expression.

2. Use the product rule.

3. Calculate the derivatives.

4. Use the product rule.

5. State the final result.

b. 1. Write the equation.

2. Take $\frac{d}{dx}()$ of each term in turn.

3. Substitute for the result from the second term from part a above and use implicit differentiation on each term.

4. Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.

5. Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.

6. Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y .

WRITE/DRAW

a. $\frac{d}{dx}(3xy)$

$$\frac{d}{dx}(3x \times y) = \frac{d}{dx}(u \times v)$$

Let $u = 3x$ and $v = y$.

$$\frac{du}{dx} = 3 \text{ and } \frac{dv}{dx} = \frac{d}{dy}(y) \frac{dy}{dx} = 1 \times \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{d}{dx}(u \times v) = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx}(3xy) = 3x \frac{dy}{dx} + 3y$$

b. $4x^2 + 3xy - 2y^2 + 5x - 7y + 8 = 0$

$$\frac{d}{dx}(4x^2) + \frac{d}{dx}(3xy) - \frac{d}{dx}(2y^2) + \frac{d}{dx}(5x) - \frac{d}{dx}(7y) + \frac{d}{dx}(8) = 0$$

$$8x + 3x \frac{dy}{dx} + 3y - 4y \frac{dy}{dx} + 5 - 7 \frac{dy}{dx} + 0 = 0$$

$$8x + 3y + 5 = 7 \frac{dy}{dx} + 4y \frac{dy}{dx} - 3x \frac{dy}{dx}$$

$$8x + 3y + 5 = \frac{dy}{dx}(7 + 4y - 3x)$$

$$\frac{dy}{dx} = \frac{8x + 3y + 5}{7 + 4y - 3x}$$

Implicit differentiation with exponential or trigonometric functions

Sometimes it may be necessary to use the derivatives of exponential or trigonometric functions together with implicit differentiation techniques to obtain the required derivative.

WORKED EXAMPLE 3

Given $e^{-2xy} + 3 \sin(2x - 3y) = c$, where c is a constant, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .

THINK

1. Write the equation.
2. Take $\frac{d}{dx}()$ of each term in turn.
3. Consider just the first term and use implicit differentiation.
4. Use the product rule.
5. Substitute back for u .
6. Consider just the second term and use implicit differentiation.
7. Substitute back for v .
8. Substitute for the first and second terms.
9. Expand the brackets.
10. Transpose the equation to get all terms involving $\frac{dy}{dx}$ on one side of the equation.
11. Factor the terms involving $\frac{dy}{dx}$ on the right-hand side of the equation.
12. Divide and state the final result, giving $\frac{dy}{dx}$ in terms of both x and y .

WRITE

$$e^{-2xy} + 3 \sin(2x - 3y) = c$$

$$\frac{d}{dx}(e^{-2xy}) + \frac{d}{dx}(3 \sin(2x - 3y)) = \frac{d}{dx}(c)$$

$$\frac{d}{dx}(e^{-2xy}) = \frac{d}{dx}(e^{-u}) \text{ where } u = 2xy$$

$$\frac{d}{du}(e^{-u}) \frac{du}{dx}$$

$$-e^{-u} \frac{d}{dx}(2xy) = -e^{-u} \left(2y + 2x \frac{dy}{dx} \right)$$

$$\frac{d}{dx}(e^{-2xy}) = - \left(2y + 2x \frac{dy}{dx} \right) e^{-2xy}$$

$$\frac{d}{dx}(3 \sin(2x - 3y)) = \frac{d}{dv}(3 \sin(v)) \frac{dv}{dx}$$

$$\text{where } v = 2x - 3y \text{ so that } \frac{dv}{dx} = 2 - 3 \frac{dy}{dx}.$$

$$\frac{d}{dx}(3 \sin(2x - 3y)) = 3 \cos(v) \left(2 - 3 \frac{dy}{dx} \right)$$

$$\frac{d}{dx}(3 \sin(2x - 3y)) = 3 \cos(2x - 3y) \left(2 - 3 \frac{dy}{dx} \right)$$

$$- \left(2y + 2x \frac{dy}{dx} \right) e^{-2xy} + 3 \cos(2x - 3y) \left(2 - 3 \frac{dy}{dx} \right) = 0$$

$$-2ye^{-2xy} - 2xe^{-2xy} \frac{dy}{dx} + 6 \cos(2x - 3y) - 9 \frac{dy}{dx} \cos(2x - 3y) = 0$$

$$-2ye^{-2xy} + 6 \cos(2x - 3y) = 2xe^{-2xy} \frac{dy}{dx} + 9 \cos(2x - 3y) \frac{dy}{dx}$$

$$-2ye^{-2xy} + 6 \cos(2x - 3y) = (2xe^{-2xy} + 9 \cos(2x - 3y)) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6 \cos(2x - 3y) - 2ye^{-2xy}}{2xe^{-2xy} + 9 \cos(2x - 3y)}$$

8.2.2 Parametric differentiation

In the previous section, we saw that whether the variables x and y are given in the form $y = f(x)$ as an explicit relationship or the form $f(x, y) = 0$ as an implicit relationship, an expression for $\frac{dy}{dx}$ can be found.

In this section, the two variables x and y are connected or related in terms of another variable, called a linking variable or a parameter. Often t or θ are used as parameters.

A parameter is a variable that changes from case to case but in each particular case or instant it remains the same. For example, an expression can be found for $\frac{dy}{dx}$; however, it will be in terms of the parameter by using the chain rule, since $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$, and noting that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$. Alternatively, it may also be possible

to eliminate the parameter from the two parametric equations and obtain an implicit relationship between the two variables x and y . The following examples will illustrate these concepts.

WORKED EXAMPLE 4

- a.** If $x = 3 \cos(t)$ and $y = 4 \sin(t)$, write the gradient $\frac{dy}{dx}$ in terms of t .
- b.** For the parametric equations $x = 3 \cos(t)$ and $y = 4 \sin(t)$, determine an implicit relationship between x and y , and write an expression for the gradient. Hence, verify your answer to part **a**.

THINK

- a. 1.** Differentiate x with respect to t . The dot notation is used for the derivative with respect to t .

- 2.** Differentiate y with respect to t .

- 3.** Use the chain rule to determine $\frac{dy}{dx}$.

- 4.** Substitute for the derivatives.

- 5.** State the gradient in simplest form.

- b. 1.** Write the parametric equations.

- 2.** Express the trigonometric ratios on their own.

- 3.** Eliminate the parameter to find the implicit relationship.

- 4.** Use implicit differentiation on the implicit form.

- 5.** Perform the implicit differentiation.

WRITE

a. $x = 3 \cos(t)$

$$\frac{dx}{dt} = \dot{x} = -3 \sin(t)$$

$$y = 4 \sin(t)$$

$$\frac{dy}{dt} = \dot{y} = 4 \cos(t)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \dot{y} \times \frac{1}{\dot{x}} = \frac{\dot{y}}{\dot{x}}$$

$$\frac{dy}{dx} = \frac{4 \cos(t)}{-3 \sin(t)}$$

$$\frac{dy}{dx} = -\frac{4}{3} \cot(t)$$

b. $x = 3 \cos(t)$ [1]

$$y = 4 \sin(t) \quad [2]$$

$$\cos(t) = \frac{x}{3} \quad [1]$$

$$\sin(t) = \frac{y}{4} \quad [2]$$

$$\text{Since } \cos^2(t) + \sin^2(t) = 1,$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\frac{1}{9} \frac{d}{dx}(x^2) + \frac{1}{16} \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$\frac{2x}{9} + \frac{2y}{16} \frac{dy}{dx} = 0$$

6. Transpose the equation.

$$\frac{2x}{9} = -\frac{y}{8} \frac{dy}{dx}$$

7. Make $\frac{dy}{dx}$ the subject.

$$\frac{dy}{dx} = -\frac{16x}{9y}$$

8. From the implicit differentiation, substitute for the parametric equations.

$$\frac{dy}{dx} = -\frac{16 \times 3 \cos(t)}{9 \times 4 \sin(t)}$$

9. Simplify to verify the given result, as above.

$$\frac{dy}{dx} = -\frac{4}{3} \cot(t)$$

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Units 3 & 4

Area 5

Sequence 1

Concept 1

Implicit differentiation Summary screen and practice questions

Exercise 8.2 Implicit differentiation

Technology free

- WE1** Given $x^3 + y^3 = 27$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .
- Given $\sqrt{x} + \sqrt{y} = 4$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .
- For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y .
 - $y^2 - 2x = 3$
 - $\frac{x^2}{16} - \frac{y^2}{9} = 1$
 - $\frac{x^2}{4} + \frac{y^2}{9} = 1$
 - $4x - 2y - 3x^2 + y^2 = 10$
- WE2**
 - Determine $\frac{d}{dx}(4x^2y^2)$.
 - Given $9x^3 + 4x^2y^2 - 3y^3 + 2x - 5y + 12 = 0$, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .
- Calculate the gradient of the normal to the curve $x^2 - 4xy + 2y^2 - 3x + 5y - 7 = 0$ at the point where $x = 2$ in the first quadrant.
- For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y .
 - $2x^2 + 3xy - 3y^2 + 8 = 0$
 - $x^2 + x^2y^2 - 6x + 5 = 0$
 - $x^3 - 3x^2y + 3xy^2 - y^3 - 27 = 0$
 - $y^3 - y^2 - 3x - x^2 + 9 = 0$
- WE3** Given $2xy + e^{-(x^2+y^2)} = c$, where c is a constant, determine an expression for $\frac{dy}{dx}$ in terms of both x and y .
- If $\sin(3x + 2y) + x^2 + y^2 = c$, where c is a constant, determine $\frac{dy}{dx}$ in terms of both x and y .

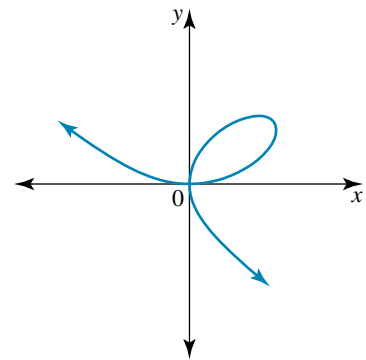
9. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y .
- a. $\frac{y^2 - 2x}{3x^2 + 4y} = 6x$ b. $\frac{x^3 + 8y^3}{x^2 - 2xy + 4y^2} = x^2$
c. $e^{x+y} + \cos(y) - y^2 = 0$ d. $e^{xy} + \cos(xy) + x^2 = 0$
10. For each of the following implicitly defined relations, determine an expression for $\frac{dy}{dx}$ in terms of x and y .
- a. $\ln(2x + 3y) + 4x - 5y = 10$ b. $\ln(3xy) + x^2 + y^2 - 9 = 0$
c. $\frac{1}{x} + 2xy + \frac{1}{y} - 6 = 0$ d. $2y^3 - \frac{\sin(3x)}{\sec(2y)} + x^2 = 0$
11. a. Calculate the gradient of the tangent to the curve $y^2 = x^3(2 - x)$ at the point $(1, 1)$.
b. Calculate the equation of the tangent to the curve $x^3 + 3xy + y^3 + 13 = 0$ at the point $(1, -2)$.
c. For the circle $x^2 + 4x + y^2 - 3y + 6xy - 4 = 0$, calculate the gradient of the tangent to the circle at the point $(2, -1)$.
d. A certain ellipse has the equation $3x^2 + 2xy + 4y^2 + 5x - 10y - 8 = 0$. Calculate the gradient of the normal to the ellipse at the point $(1, 2)$.
12. a. Determine the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point in the fourth quadrant where $x = 3$.
b. Determine the gradient of the tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at the point in the first quadrant where $x = 1$.
c. Determine the gradient of the normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point in the fourth quadrant where $x = 5$.
d. Determine the gradient of the normal to the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$.
13. **WE4** a. If $x = 4 \cos(t)$ and $y = 4 \sin(t)$, determine the gradient $\frac{dy}{dx}$ in terms of t .
b. For the parametric equations $x = 4 \cos(t)$ and $y = 4 \sin(t)$, determine an implicit relationship between x and y , and write an expression for the gradient. Hence, verify your answer to part a.
14. a. Given $x = t^2$ and $y = 2t - t^4$, determine $\frac{dy}{dx}$ in terms of t .
b. For the parametric equations $x = t^2$ and $y = 2t - t^4$, express y in terms of x and find $\frac{dy}{dx}$. Hence, verify your answer to part a.
15. For each of the following, determine $\frac{dy}{dx}$ in terms of the parameter.
- a. The parabola $x = 2t^2$ and $y = 4t$
b. The ellipse $x = 3 \sin(2t)$ and $y = 4 \cos(2t)$
c. The rectangular hyperbola $x = 5t$ and $y = \frac{5}{t}$
d. The hyperbola $x = 3 \sec(t)$ and $y = 4 \tan(t)$
16. In each of the following, a and b are constants. Determine $\frac{dy}{dx}$ in terms of the parameter.
- a. $x = at^2$ and $y = 2at$ b. $x = a \cos(t)$ and $y = b \sin(t)$
c. $x = at$ and $y = \frac{a}{t}$ d. $x = a \sec(t)$ and $y = b \tan(t)$
17. Check your answers to question 16a–d by eliminating the parameter and determining $\frac{dy}{dx}$ by another method.

- 18.** Rene Descartes (1596–1650) was a French mathematician and philosopher who lived in the Dutch republic. He is noted for introducing the Cartesian coordinate system in two dimensions and is credited as the father of analytical geometry, the link between algebra and geometry.

- a.** A curve called the folium of Descartes has the equation

$$x^3 - 3axy + y^3 = 0. \text{ Determine } \frac{dy}{dx} \text{ for this curve, given that } a \text{ is a constant.}$$

- b.** Show that the folium of Descartes can be represented by the parametric equations $x = \frac{3at}{1+t^3}$ and $y = \frac{3at^2}{1+t^3}$, and calculate the gradient of the curve at the point t .

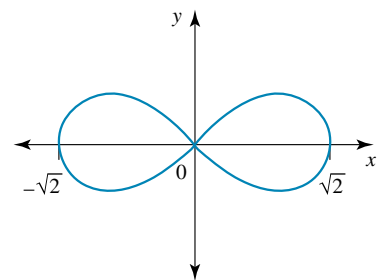


- 19. a.** A curve called a lemniscate has the implicit equation

$$(x^2 + y^2)^2 = 2(x^2 - y^2). \text{ Determine an expression for } \frac{dy}{dx} \text{ in terms of } x \text{ and } y.$$

- b.** Show that the curve can be represented by the parametric equations

$$x = \frac{\sqrt{2} \cos(t)}{\sin^2(t) + 1} \text{ and } y = \frac{\sqrt{2} \sin(t) \cos(t)}{\sin^2(t) + 1}, \text{ and determine } \frac{dy}{dx} \text{ in terms of } t.$$



- 20.** A curve known as the cissoid of Diocles has the equation

$$y^2 = \frac{x^3}{2a - x}.$$

- a.** Determine $\frac{dy}{dx}$ for this curve, given that a is constant.

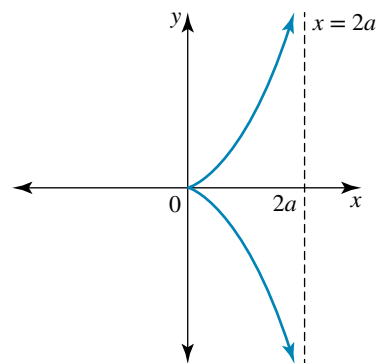
- b.** Show that a parametric representation of the cissoid curve is

$$\text{given by the equations } x = \frac{2at^2}{1+t^2} \text{ and } y = \frac{2at^3}{1+t^2}, \text{ and}$$

$$\text{determine } \frac{dy}{dx} \text{ in terms of } t.$$

- c.** Show that another alternative parametric representation of the cissoid curve is given by the equations $x = 2a \sin^2(t)$ and

$$y = \frac{2a \sin^3(t)}{\cos(t)}, \text{ and determine } \frac{dy}{dx} \text{ in terms of } t.$$



8.3 Related rates as instances of the chain rule

8.3.1 Introduction

When two or more quantities vary with time and are related by some condition, their rates of change are also related. The steps involved in solving these related rate problems are listed below.

1. Define the variables
2. Write down the rate that is provided in the question.
3. Establish the relationship between the variables.
4. Write down the rate that needs to be determined.
5. Use a chain rule or implicit differentiation to relate the variables.

WORKED EXAMPLE 5

A circular metal plate is being heated and its radius is increasing at a rate of 2 mm/s. Calculate the rate at which the area of the plate is increasing when the radius is 30 millimetres.

THINK

1. Define the variables.
2. The radius is increasing at a rate of 2 mm/s. Note that the units also help to determine the given rate.
3. The plate is circular. Write the formula for the area of a circle.
4. Determine the rate at which the area of the plate is increasing.
5. Form a chain rule for the required rate in terms of the given variables.
6. Since one rate, $\frac{dr}{dt}$, is known, we need to determine $\frac{dA}{dr}$.
7. Substitute the given rates into the required equations.
8. Evaluate the required rate when $r = 30$.
9. State the final result with the required units, leaving the answer in terms of π .

WRITE

Let r mm be the radius of the metal at time t seconds.

Let A mm² be the area of the plate at time t seconds.

$$\frac{dr}{dt} = 2 \text{ mm/s.}$$

$$A = \pi r^2.$$

$$\frac{dA}{dt} = ? \text{ and evaluate this rate when } r = 30.$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

$$\begin{aligned} \frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt} \\ &= 2\pi r \times 2 \end{aligned}$$

$$\frac{dA}{dt} = 4\pi r \text{ when } r = 30$$

$$\left. \frac{dA}{dr} \right|_{r=30} = 120\pi \text{ mm}^2/\text{s}$$

The area is increasing at 120π mm²/s.

Decreasing rates

If a quantity has a rate decreasing with respect to time, then the required rate is given as a negative quantity.

WORKED EXAMPLE 6

A spherical balloon has a hole in it, and the balloon's volume is decreasing at a rate of 2 cm³/s. At what rate is the radius changing when the radius of the balloon is 4 cm?

THINK

1. Define the variables.
2. The volume is decreasing at a rate of 2 cm³/s. Notice that the units also help determine the given rate.
3. The balloon is spherical. Write the formula for the volume of a sphere.

WRITE

Let r cm be the radius of the balloon at time t seconds.

Let V cm³ be the volume of the balloon at time t seconds.

$$\frac{dV}{dt} = -2 \text{ cm}^3/\text{s}$$

$$V = \frac{4}{3}\pi r^3$$

4. We need to determine the rate at which the radius is changing.
5. Form a chain rule for the required rate in terms of the given variables.
6. Since one rate, $\frac{dV}{dt}$, is known, we need to determine $\frac{dr}{dV}$.
7. Substitute the given rates into the required equations.
8. Evaluate the required rate when $r = 4$.
9. State the final result in the required units, leaving the answer in terms of π .

$\frac{dr}{dt} = ?$ and evaluate this rate when $r = 4$.

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

Since $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$ and

$$\frac{dr}{dV} = 1 / \frac{dV}{dr} = \frac{1}{4\pi r^2}$$

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{dV} \times \frac{dV}{dt} \\ &= \frac{1}{4\pi r^2} \times -2 \\ &= -\frac{1}{2\pi r^2} \end{aligned}$$

When $r = 4$, $\left. \frac{dr}{dt} \right|_{r=4} = -\frac{1}{32\pi}$.

The radius is decreasing at a rate of $\frac{1}{32\pi}$ cm/s.

Relating the variables

It is often necessary to express a required expression in terms of only one variable instead of two. This can be achieved by finding relationships between the variables.

WORKED EXAMPLE 7

A conical funnel has a height of 25 centimetres and a radius of 20 centimetres. It is positioned so that its axis is vertical and its vertex is downwards. Oil leaks out through an opening in the vertex at a rate of 4 cubic centimetres per second. Calculate the rate at which the oil level is falling when the height of the oil in the funnel is 5 centimetres. (Ignore the cylindrical section of the funnel.)



THINK

1. Define the variables.
2. Write the rate given in the question.
3. Write which rate is required to be determined.

WRITE/DRAW

Let r cm be the radius of the oil in the funnel.
Let h cm the height of the oil in the funnel.

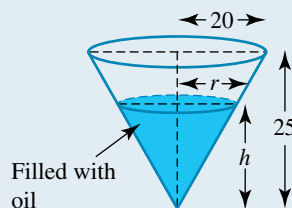
$$\frac{dV}{dt} = -4 \text{ cm}^3/\text{s}$$

$$\frac{dh}{dt} = ? \text{ when } h = 5$$

4. Make up a chain rule for the required rate, in terms of the given variables.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

5. Determine the relationship between the variables. The variables r and h change; however, the height and radius of the funnel are constant, and this can be used to determine a relationship between h and r .



$$V = \frac{1}{3}\pi r^2 h$$

Using similar triangles,

$$\frac{20}{25} = \frac{r}{h}$$

$$r = \frac{4h}{5}$$

6. We need to express the volume in terms of h only.

Substitute into $V = \frac{1}{3}\pi r^2 h$:

$$\begin{aligned} V &= \frac{1}{3}\pi \left(\frac{4h}{5}\right)^2 h \\ &= \frac{16\pi h^3}{75} \end{aligned}$$

7. Determine the rate to substitute into the chain rule.

$$\begin{aligned} \frac{dV}{dh} &= \frac{16\pi h^2}{25} \\ \frac{dh}{dV} &= \frac{25}{16\pi h^2} \end{aligned}$$

8. Substitute for the required rates.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{25}{16\pi h^2} \times -4$$

9. Evaluate the required rate when $h = 5$.

$$\left. \frac{dh}{dt} \right|_{h=5} = \frac{25}{16\pi 5^2} \times -4 = -\frac{1}{4\pi}$$

10. State the final result with the required units, leaving the answer in terms of π .

The height is decreasing at a rate of $\frac{1}{4\pi}$ cm/s.

Determining the required variables

An alternative method to solving related rate problems is to use implicit differentiation.

WORKED EXAMPLE 8

A ladder 3 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The top end of the ladder is slipping down at a constant speed of 0.1 metres per second. Determine the rate at which the lower end is moving away from the wall when the lower end is 1 metre from the wall.

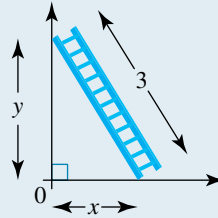


THINK**Method 1: Using the chain rule**

1. Define the variables.
2. The top end of the ladder is slipping down at a constant speed of 0.1 m/s.
3. Apply Pythagoras' theorem.
4. We need to determine the rate at which the lower end is moving away from the wall when the lower end is 1 m from the wall.
5. Construct a chain rule for the required rate in terms of the given variables.
6. Express y in terms of x .
7. Since one rate, $\frac{dy}{dt}$, is known, we need to determine $\frac{dx}{dy}$.
8. Substitute the given rates into the required equation.
9. Evaluate the required rate when $x = 1$.
10. State the final result with the required units, giving an exact answer.

WRITE/DRAW

Let x metres be the distance of the base of the ladder from the wall, and let y metres be the distance of the top of the ladder from the ground.



$$\frac{dy}{dt} = -0.1 \text{ m/s}$$

$$x^2 + y^2 = 3^2 = 9$$

$$\frac{dx}{dt} = ? \text{ when } x = 1.$$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

$$x^2 + y^2 = 9$$

$$y^2 = 9 - x^2$$

$$y = \pm \sqrt{9 - x^2}$$

$$y = \sqrt{9 - x^2} \text{ since } y > 0$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{9 - x^2}}$$

$$\frac{dx}{dy} = -\frac{\sqrt{9 - x^2}}{x}$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{dy} \times \frac{dy}{dt} \\ &= -\frac{\sqrt{9 - x^2}}{x} \times -0.1 \end{aligned}$$

$$\left. \frac{dx}{dt} \right|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$$

$$\left. \frac{dx}{dt} \right|_{x=1} = \frac{\sqrt{2}}{5} \text{ m/s}$$

The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

Method 2: Using implicit differentiation

The first 4 steps are identical to method 1 above.
From this point, we use implicit differentiation to find the required rate.

5. Take $\frac{d}{dt}$ () of each term in turn.
6. Use implicit differentiation.
7. Rearrange to make the required rate the subject.
8. Determine the appropriate values.
9. Substitute in the appropriate values of x and y .
10. State the final result as before.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(9)$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$$

When $x = 1$, $y = \sqrt{8}$ and $\frac{dy}{dt} = -0.1 = -\frac{1}{10}$

$$\left. \frac{dx}{dt} \right|_{x=1} = \frac{\sqrt{8}}{10} = \frac{2\sqrt{2}}{10}$$

$$\left. \frac{dx}{dt} \right|_{x=1} = \frac{\sqrt{2}}{5} \text{ m/s}$$

The lower end is moving away from the wall at a rate of $\frac{\sqrt{2}}{5}$ m/s.

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Units 3 & 4

Area 5

Sequence 1

Concept 2

Related rates as instances of the chain rule Summary screen and practice questions

Exercise 8.3 Related rates as instances of the chain rule

Technology free

1. **WES** A circular oil slick is expanding so that its radius increases at a rate of 0.5 m/s. Determine the rate at which the area of the oil slick is increasing when the radius of the slick is 20 metres.
2. A circular disc is expanding so that its area increases at a rate of $40\pi \text{ cm}^2/\text{s}$. Determine the rate at which the radius of the disc is increasing when the radius is 10 cm.
3. **a.** A square has its sides increasing at a rate of 2 centimetres per second. Calculate the rate at which the area is increasing when the sides are 4 centimetres long.
b. A stone is dropped into a lake, sending out concentric circular ripples. The area of the disturbed water region increases at a rate of $2 \text{ m}^2/\text{s}$. Calculate the rate at which the radius of the outermost ripple is increasing when its radius is 4 metres.



4. A spherical bubble is blown so that its radius is increasing at a constant rate of 2 millimetres per second. When its radius is 10 millimetres, calculate the rate at which:
- its volume is increasing
 - its surface area is increasing.
5. **WE6** A spherical basketball has a hole in it and its volume is decreasing at a rate of $6 \text{ cm}^3/\text{s}$. At what rate is the radius changing when the radius of the basketball is 6 cm?
6. A metal ball is dissolving in an acid bath. Its radius is decreasing at a rate of 3 cm/s. At what rate is the ball's surface area changing when the radius of the ball is 2 cm?
7. A mothball has its radius decreasing at a constant rate of 0.2 millimetres per week. Assume it remains spherical.
- Show that the volume is decreasing at a rate that is proportional to its surface area.
 - If its initial radius is 30 millimetres, calculate how long it takes to disappear.
8. **WE7** A conical vase has a height of 40 cm and a radius of 8 cm. The axis of the vase is vertical and its vertex is downwards. Initially it is filled with water which leaks out through a small hole in the vertex at a rate of $6 \text{ cm}^3/\text{s}$. Calculate the rate at which the water level is falling when the height of the water is 16 cm.
9. A cone is such that its radius is always equal to half its height. If the radius is decreasing at a rate of 2 cm/s, calculate the rate at which the volume of the cone is decreasing when the radius is 4 cm.
10. The sides of an equilateral triangle are increasing at a rate of 2 centimetres per second. When the sides are $2\sqrt{3}$ centimetres, calculate the rate at which:
- its area is increasing
 - its height is increasing.
11. a. Show that the formula for the volume, V , of a right circular cone with height h is given by
- $$V = \frac{1}{3}\pi h^3 \tan^2(\alpha), \text{ where } \alpha \text{ is the semi-vertex angle.}$$
- b. Falling sand forms a heap in the shape of a right circular cone whose semi-vertex angle is 71.57° . If its height is increasing at 2 centimetres per second when the heap is 5 centimetres high, calculate the rate at which its volume is increasing.
12. **WE8** A ladder 5 metres long has its top end resting against a vertical wall and its lower end on horizontal ground. The bottom end of the ladder is pushed closer to the wall at a speed of 0.3 metres per second. Calculate the rate at which the top end of the ladder is moving up the wall when the lower end is 3 metres from the wall.
13. A kite is 30 metres above the ground and is moving horizontally away at a speed of 2 metres per second from the boy who is flying it. When the length of the string is 50 metres, at what rate is the string being released?
14. a. The volume, $V \text{ cm}^3$, of water in a hemispherical bowl of radius r cm when the depth of the water is h cm is given by $V = \frac{1}{3}\pi h^2(3r - h)$.
A hemispherical bowl of radius 10 centimetres is being filled with water at a constant rate of 3 cubic centimetres per second. At what rate is the depth of the water increasing when the depth is 5 cm?
- b. A drinking glass is in the shape of a truncated right circular cone. When the glass is filled to a depth of h cm, the volume of liquid in the glass, $V \text{ cm}^3$, is given by $V = \frac{\pi}{432}(h^3 + 108h^2 + 388h)$. Lemonade is leaking out from the glass at a rate of $7 \text{ cm}^3/\text{s}$. Calculate the rate at which the depth of the lemonade is falling when the depth is 6 cm.



15. a. A rubber flotation device is being pulled into a wharf by a rope at a speed of 26 metres per minute. The rope is attached to a point on the wharf 1 metre vertically above the dingy. At what rate is the rope being drawn in when the dingy is 10 metres from the wharf?



- b. A car approaches the ground level of a 30-metre-tall building at a speed of 54 kilometres per hour. Calculate the rate of change of the distance from the car to the top of the building when it is 40 metres from the foot of the building.



16. The distance, q cm, between the image of an object and a certain lens in terms of p cm, the distance of the object from the lens, is given by $q = \frac{10p}{p - 10}$.

- a. Show that the rate of change of distance that an image is from the lens with respect to the distance of the object from the lens is given by $\frac{dq}{dp} = \frac{-100}{(p - 10)^2}$.

- b. If the object distance is increasing at a rate 0.2 cm/s, how fast is the image distance changing, when the distance from the object is 12 cm?

17. a. When a gas expands without a change of temperature, the pressure P and volume V are given by the relationship $PV^{1.4} = C$, where C is a constant. At a certain instant, the pressure is 1.01×10^5 pascals and the volume is 22.4×10^{-3} cubic metres. The volume is increasing at a rate of 0.005 cubic metres per second. Calculate the rate at which the pressure is changing at this instant.

- b. The pressure P and volume V of a certain fixed mass of gas during an adiabatic expansion are connected by the law $PV^n = C$, where n and C are constants. Show that the time rate of change of volume satisfies $\frac{dV}{dt} = -\frac{V}{nP} \frac{dP}{dt}$.

18. a. A jet aircraft is flying horizontally at a speed of 300 km/h at an altitude of 1 km. It passes directly over a radar tracking station located at ground level. Calculate the rate in degrees per second at which the radar beam to the aircraft is turning when the jet is at a horizontal distance of 30 kilometres from the station.

- b. A helicopter is flying horizontally at a constant height of 300 metres. It passes directly over a light source located at ground level. The light source is always directed at the helicopter. If the helicopter is flying at 108 kilometres per hour, calculate the rate in degrees per second at which the light source to the helicopter is turning when the helicopter is at a horizontal distance of 0.4 kilometres from the light source.



8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

8.4.1 Classification of differential equations

A **differential equation** is an equation involving derivatives. It is of the form $g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0$.

A differential equation contains the function $y = f(x)$ as the dependent variable, x as the independent variable, and various derivatives. In this topic, only differential equations that contain functions of one variable, $y = f(x)$, are considered.

Differential equations can be classified according to their order and degree. The **order** of a differential equation is the order of the highest derivative present. The **degree** of a differential equation is the degree of the highest power of the highest derivative.

A linear differential equation is one in which all variables including the derivatives are raised to the power of 1.

Some examples of differential equations are:

a. $\frac{dy}{dx} = ky$

b. $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

c. $\ddot{x} - t\dot{x} + 2x = t$

d. $\frac{d^2y}{dx^2} + n^2y = 0$

e. $x\left(\frac{dy}{dx}\right)^3 + 3\frac{dy}{dx} + 5y = 0$

f. $D_t^3x = \sqrt{x^2 + 1}$

Note that (a) and (e) are first order; (b), (c) and (d) are second order; and (f) is third order. Equation (e) has a degree of 3, whereas all the others have a degree of 1. Equations (a), (b), (c) and (d) are linear; (e) and (f) are non-linear. Note also that there are many different notations for derivatives; for example, second-order derivatives can be expressed

as $\ddot{x} = \frac{d^2x}{dt^2}$ and $D_t^2x = \frac{d^2x}{dt^2}$.

Differential equations are extremely important in the study of mathematics and appear in almost every branch of science.

Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) are both credited with the discovery of calculus and differential equations in the 1670s and 1680s.

Gottfried Leibniz



8.4.2 Classifying solutions to a differential equation

The solution of a differential equation is usually obtained by the process of integration. Because the integration process produces an arbitrary constant of integration, the solutions of a differential equation are classified as follows.

A **general solution** is one which contains arbitrary constants of integration and satisfies the differential equation.

A **particular solution** is one which satisfies the differential equation and some other initial value condition, also known as a boundary value, that enables the constant(s) of integration to be found.

In general, the number of arbitrary constants of integration to be found is equal to the order of the differential equation. Throughout this course we study and solve special types of first- and second-order differential equations.

8.4.3 Differential equations of the form $\frac{dy}{dx} = f(x)$

Direct integration

In this section we solve first-order differential equations of the form $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$. Differential equations of this form can be solved by direct integration. Hence, it is necessary to be familiar with all the integration techniques studied so far. Anti-differentiating both sides gives $y = \int f(x) dx + c$. This is the general solution, which can be thought of as a family of curves. If we use the given condition $x = x_0$ when $y = y_0$, we can determine the value of the constant of integration c in this particular case, which thus gives us the particular solution.

WORKED EXAMPLE 9

- a.** Determine the general solution to $\frac{dy}{dx} + 12x = 0$.
- b.** Determine the particular solution of $\frac{dy}{dx} + 6x^2 = 0$, $y(1) = 2$.

THINK

- a. 1.** Rewrite the equation to make $\frac{dy}{dx}$ the subject.
- 2.** Anti-differentiate to obtain y .
- 3.** Write the general solution in terms of a constant.
- b. 1.** Rewrite the equation to make $\frac{dy}{dx}$ the subject.
- 2.** Anti-differentiate to obtain y .
- 3.** Express y in terms of x with an arbitrary constant.
- 4.** Substitute and use the given conditions to determine the value of the constant.
- 5.** Substitute back for c and state the particular solution.

WRITE

a. $\frac{dy}{dx} + 12x = 0$

$$\frac{dy}{dx} = -12x$$
$$y = - \int 12x dx$$
$$y = -6x^2 + c$$

b. $\frac{dy}{dx} + 6x^2 = 0$

$$\frac{dy}{dx} = -6x^2$$
$$y = - \int 6x^2 dx$$
$$y = -2x^3 + c$$

$y(1) = 2:$

$$\Rightarrow x = 1 \text{ when } y = 2$$
$$2 = -2(1)^3 + c$$
$$c = 4$$
$$y = 4 - 2x^3$$

Finding particular solutions

In chapter 6, linear substitutions were used to integrate linear expressions. The example presented in Worked example 10 is a review of this process.

WORKED EXAMPLE 10

Solve the differential equation $(4 - 3x)^2 \frac{dy}{dx} + 1 = 0$, $y(1) = 2$.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Anti-differentiate to obtain y .
3. Use index laws to express the integrand as a function to a power.
4. Use a linear substitution. Express dx in terms of du by inverting both sides.

5. Substitute for u and dx .
6. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.
7. Perform the integration process, using $\int u^n du = \frac{1}{n+1} u^{n+1}$.
8. Substitute back for x .
9. Substitute and use the given conditions to determine the value of the constant.

WRITE

$$(4 - 3x)^2 \frac{dy}{dx} + 1 = 0$$

$$(4 - 3x)^2 \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = \frac{-1}{(4 - 3x)^2}$$

$$y = \int \frac{-1}{(4 - 3x)^2} dx$$

$$y = - \int (4 - 3x)^{-2} dx$$

$$\text{Let } u = 4 - 3x.$$

$$\frac{du}{dx} = -3$$

$$\frac{dx}{du} = -\frac{1}{3}$$

$$dx = -\frac{1}{3} du$$

$$y = - \int u^{-2} \frac{-1}{3} du$$

$$y = \frac{1}{3} \int u^{-2} du$$

$$y = -\frac{1}{3} u^{-1} + c$$

$$y = -\frac{1}{3u} + c$$

$$y = -\frac{1}{3(4 - 3x)} + c$$

$$y(1) = 2$$

$$\Rightarrow x = 1 \text{ when } y = 2$$

$$2 = -\frac{1}{3} + c$$

$$c = 2 + \frac{1}{3}$$

$$c = \frac{7}{3}$$

10. Substitute back for c , and state the particular solution. Although this is a possible answer, this result can be simplified.

$$y = \frac{-1}{3(4-3x)} + \frac{7}{3}$$

11. Form the lowest common denominator.

$$y = \frac{-1 + 7(4-3x)}{3(4-3x)}$$

12. Expand the brackets in the numerator; do not expand the brackets in the denominator.

$$y = \frac{-1 + 28 - 21x}{3(4-3x)}$$

13. Simplify and take out common factors which cancel.

$$y = \frac{27 - 21x}{3(4-3x)} \\ = \frac{3(9-7x)}{3(4-3x)}$$

14. State the final answer in simplified form. Note the maximal domain for which the solution is valid.

$$y = \frac{9-7x}{4-3x} \text{ for } x \neq \frac{4}{3}$$

15. Note that as a check, we can use the given condition to check the value of y .

Substitute $x = 1$:

$$y = \frac{9-7}{4-3} = 2$$

Stating the domain for which the solution is valid

As seen in the last example, the **maximal domain** for which the solution is valid is important. When solving differential equations, unless the solution is defined for all values of x , that is for $x \in \mathbb{R}$, we are required to state the largest subset of $\mathbb{R}$ for which the given differential equation and solution are valid.

WORKED EXAMPLE 11

Solve the differential equation $\sqrt{3x-5} \frac{dy}{dx} + 6 = 0$, $y(7) = 2$, stating the largest domain for which the solution is valid.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.

WRITE

$$\sqrt{3x-5} \frac{dy}{dx} + 6 = 0$$

$$\sqrt{3x-5} \frac{dy}{dx} = -6$$

$$\frac{dy}{dx} = \frac{-6}{\sqrt{3x-5}}$$

2. Anti-differentiate to obtain y .

$$y = \int \frac{-6}{\sqrt{3x-5}} dx$$

3. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = -6 \int \frac{1}{\sqrt{3x-5}} dx$$

4. Use index laws to express the integrand as a function to a power.

$$y = -6 \int (3x-5)^{-\frac{1}{2}} dx$$

5. Use a linear substitution. Express dx in terms of du by inverting both sides.

$$\text{Let } u = 3x - 5.$$

$$\frac{du}{dx} = 3$$

$$\frac{dx}{du} = \frac{1}{3}$$

$$dx = \frac{1}{3} du$$

6. Substitute for u and dx .

$$y = -6 \int u^{-\frac{1}{2}} \left(\frac{1}{3} du \right)$$

7. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$y = -2 \int u^{-\frac{1}{2}} du$$

8. Perform the integration process using

$$\int u^n du = \frac{1}{n+1} u^{n+1}.$$

$$y = -4\sqrt{u} + c$$

9. Substitute back for x .

$$y = -4\sqrt{3x-5} + c$$

10. Substitute and use the given conditions to determine the value of the constant.

$$y(7) = 2$$

$$\Rightarrow x = 7 \text{ when } y = 2$$

$$2 = -4\sqrt{16} + c$$

$$c = 18$$

$$y = 18 - 4\sqrt{3x-5}$$

11. Substitute back for c and state the particular solution.

12. Determine the domain for which the solution is valid from the differential equation.

$$\frac{dy}{dx} = \frac{-6}{\sqrt{3x-5}} \text{ for } 3x-5 > 0$$

13. Solve the inequality for x to state the largest domain for which the solution is valid for the given differential equation. State the answer.

$$3x > 5$$

$$x > \frac{5}{3}$$

The solution $y = 18 - 4\sqrt{3x-5}$ is

valid for $x > \frac{5}{3}$.

Solving first-order differential equations involving inverse trigonometric functions

The results $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$ for $x \in (-a, a)$, $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1} \left(\frac{x}{a} \right) + c$ for

$x \in (-a, a)$ and $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$ for $x \in \mathbb{R}$ are used throughout this topic.

WORKED EXAMPLE 12

Solve the differential equation $\sqrt{16-x^2} \frac{dy}{dx} + 2 = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Anti-differentiate to obtain y .
3. Perform the integration process using $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$.
4. Substitute and use the given conditions to determine the value of the constant.
5. Substitute back for c and state the particular solution.
6. Determine the domain for which the solution is valid from the differential equation.
7. Solve the inequality for x to state the largest domain for which the solution and the differential equation is valid. State the answer.

WRITE

$$\sqrt{16-x^2} \frac{dy}{dx} + 2 = 0$$

$$\sqrt{16-x^2} \frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{16-x^2}}$$

$$y = \int \frac{-2}{\sqrt{16-x^2}} dx$$

$$y = 2 \cos^{-1}\left(\frac{x}{4}\right) + c$$

$$y(0) = 0$$

$$\Rightarrow x = 0 \text{ when } y = 0$$

$$0 = 2 \cos^{-1}(0) + c$$

$$c = -2 \cos^{-1}(0)$$

$$c = -2 \times \frac{\pi}{2}$$

$$c = -\pi$$

$$y = 2 \cos^{-1}\left(\frac{x}{4}\right) - \pi$$

$$y = \int \frac{-2}{\sqrt{16-x^2}} dx$$

$$\sqrt{16-x^2} > 0$$

$$x^2 < 16$$

$$|x| < 4$$

The solution $y = 2 \cos^{-1}\left(\frac{x}{4}\right) - \pi$ is valid for $|x| < 4$ or $x \in (-4, 4)$.

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Concept 3

Solving differential equations of the form $\frac{dy}{dx} = f(x)$ Summary screen and practice questions

Exercise 8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

Technology free

1. a. **WE9** Determine the general solution of $\frac{dy}{dx} + 12x^3 = 0$.
b. Determine the particular solution of $\frac{dy}{dx} + 6x = 0$, $y(2) = 1$.
2. a. Determine the general solution of $\frac{dy}{dx} + 12 \cos(2x) = 0$.
b. Solve the differential equation $\frac{dy}{dx} + 6 \sin(3x) = 0$, $y(0) = 0$, and express y in terms of x .
3. Determine the general solution to each of the following.
 - a. $\frac{dy}{dx} - 4x = 3$
 - b. $\frac{dy}{dx} - (3x - 5)(x + 4) = 0$
 - c. $e^{2x} \frac{dy}{dx} + 6 = 2e^{4x}$
 - d. $\sqrt{x^2 + 9} \frac{dy}{dx} - x = 0$
4. **WE10** Solve the differential equation $(5 - 4x)^2 \frac{dy}{dx} + 1 = 0$, $y(1) = 2$.
5. Solve the differential equation $(7 - 4x) \frac{dy}{dx} + 2 = 0$, $y(2) = 3$.
6. **WE11** Solve the differential equation $\sqrt{2x - 5} \frac{dy}{dx} + 1 = 0$, $y(3) = 0$, stating the largest domain for which the solution is valid.
7. Solve the differential equation $\sqrt{x} \frac{dy}{dx} + 2 = 0$, $y(4) = 3$, expressing y in terms of x , and state the largest domain for which the solution is valid.
8. **WE12** Solve the differential equation $\sqrt{64 - x^2} \frac{dy}{dx} - 6 = 0$, $y(4) = 0$, stating the largest domain for which the solution is valid.
9. Solve the differential equation $(16 + x^2) \frac{dy}{dx} + 4 = 0$, $y(4) = \frac{\pi}{4}$, stating the largest domain for which the solution is valid.

For questions 10–18, solve each of the differential equations given and state the maximal domain for which the solution is valid.

10. a. $3x \frac{dy}{dx} - 2x^2 = 5$, $y(1) = 3$
b. $\frac{dy}{dx} = 6(e^{-3x} + e^{3x})$, $y(0) = 0$
11. a. $\frac{dy}{dx} - 4 \sin(2x) = 0$, $y(0) = 2$
b. $\frac{dy}{dx} + 6 \cos(3x) = 0$, $y\left(\frac{\pi}{2}\right) = 5$
12. a. $\frac{dy}{dx} - 8 \sin^2(2x) = 0$, $y(0) = 0$
b. $\frac{dy}{dx} - 12 \cos^2(3x) = 0$, $y(0) = 0$
13. a. $\frac{dy}{dx} = \frac{1}{\sqrt{4x + 9}}$, $y(0) = 0$
b. $\frac{dy}{dx} + \frac{1}{3 - 2x} = 0$, $y(2) = 1$
14. a. $\frac{dy}{dx} = \frac{1}{(3x - 5)^2}$, $y(2) = 3$
b. $\frac{dy}{dx} = \frac{8}{7 - 4x}$, $y(2) = 5$

15. a. $(x^2 + 9)\frac{dy}{dx} - 3x = 0, y(0) = 0$

16. a. $(x^2 + 6x + 13)\frac{dy}{dx} - x = 3, y(0) = 0$

17. a. $\sec(2x)\frac{dy}{dx} + \sin^3(2x) = 0, y(0) = 0$

18. a. $\frac{dy}{dx} + \ln(2x) = 4, y\left(\frac{1}{2}\right) = 1$

b. $\sqrt{x^2 + 4}\frac{dy}{dx} + x = 0, y(0) = 0$

b. $(x^2 - 4x + 9)\frac{dy}{dx} + x = 2, y(0) = 0$

b. $\operatorname{cosec}(3x)\frac{dy}{dx} + 9\cos^2(3x) = 0, y(0) = 0$

b. $e^x \frac{dy}{dx} + x = 5, y(0) = 0$

8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

8.5.1 Invert, integrate and transpose

Solving first-order differential equations of the form $\frac{dy}{dx} = g(y), y(x_0) = y_0$ is studied in this section. In this situation it is not possible to integrate directly. The first step in the solution process is to invert both sides.

From $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, we obtain $\frac{dx}{dy} = \frac{1}{g(y)}$.

Integrate both sides with respect to y to obtain

$$x = \int \frac{1}{g(y)} dy + c.$$

This gives the general solution. The initial condition can be used to find the value of the constant c . The resulting equation must be rearranged to express y in terms of x , which gives the particular solution.

Finding general solutions

Finding a general solution means finding the solution in terms of an arbitrary constant.

WORKED EXAMPLE 13

Determine the general solution to the differential equation $\frac{dy}{dx} - 4\sqrt{y} = 0$.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Invert both sides.
3. Anti-differentiate to obtain x in terms of y .
4. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

WRITE

$$\begin{aligned}\frac{dy}{dx} - 4\sqrt{y} &= 0 \\ \frac{dy}{dx} &= 4\sqrt{y} \\ \frac{dx}{dy} &= \frac{1}{4\sqrt{y}} \\ x &= \int \frac{1}{4\sqrt{y}} dy \\ x &= \frac{1}{4} \int \frac{1}{\sqrt{y}} dy\end{aligned}$$



5. Use index laws to express the integrand as a power.

$$x = \frac{1}{4} \int y^{-\frac{1}{2}} dy$$

6. Perform the integration process using

$$\int u^n du = \frac{1}{n+1} u^{n+1}.$$

$$x = \frac{1}{4} \times \frac{2}{1} y^{\frac{1}{2}} + c$$

7. Simplify.

$$\begin{aligned} x &= \frac{1}{2} y^{\frac{1}{2}} + c \\ &= \frac{1}{2} \sqrt{y} + c \end{aligned}$$

8. Transpose to make y the subject.

$$\begin{aligned} \frac{1}{2} \sqrt{y} &= x - c \\ \sqrt{y} &= 2x - 2c \end{aligned}$$

9. Since c is a constant, $2c$ is also a constant.

$$\text{Let } A = 2c.$$

$$\sqrt{y} = 2x - A$$

10. Square both sides and state the answer in terms of an arbitrary constant A .

$$y = (2x - A)^2$$

Finding particular solutions

Finding particular solutions involves solving the differential equation and expressing y in terms of x , and finding the value of the constant of integration.

WORKED EXAMPLE 14

Solve the differential equation $\frac{dy}{dx} + (4 - 3y)^2 = 0, y(2) = 1$.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.

WRITE

$$\frac{dy}{dx} + (4 - 3y)^2 = 0$$

2. Invert both sides.

$$\frac{dy}{dx} = -(4 - 3y)^2$$

3. Express x as an integral of y .

$$\frac{dx}{dy} = -\frac{1}{(4 - 3y)^2}$$

$$x = \int \frac{-1}{(4 - 3y)^2} dy$$

4. Use index laws to express the integrand as a function to a power.

$$x = - \int (4 - 3y)^{-2} dy$$

5. Use a linear substitution. Express dy in terms of du by inverting both sides.

$$\text{Let } u = 4 - 3y.$$

$$\frac{du}{dy} = -3$$

$$\frac{dy}{du} = -\frac{1}{3}$$

$$dy = -\frac{1}{3} du$$

6. Substitute for u and dy .

$$x = - \int u^{-2} \left(\frac{-1}{3} du \right)$$

7. Use the properties of indefinite integrals to transfer the constant factor outside the front of the integral sign.

$$x = \frac{1}{3} \int u^{-2} du$$

8. Perform the integration process using

$$\int u^n du = \frac{1}{n+1} u^{n+1}.$$

$$x = -\frac{1}{3} u^{-1} + c$$

$$x = -\frac{1}{3u} + c$$

9. Substitute back for y .

$$x = -\frac{1}{3(4-3y)} + c$$

10. Substitute and use the given conditions to determine the value of the constant.

$$y(2) = 1$$

$$\Rightarrow x = 2 \text{ when } y = 1$$

$$2 = -\frac{1}{3} + c$$

$$c = 2 + \frac{1}{3}$$

$$c = \frac{7}{3}$$

11. Substitute back for c .

$$x = -\frac{1}{3(4-3y)} + \frac{7}{3}$$

12. To begin making y the subject, transpose the equation.

$$\frac{1}{3(4-3y)} = \frac{7}{3} - x$$

13. Form a common denominator on the right-hand side.

$$\frac{1}{3(4-3y)} = \frac{7-3x}{3}$$

14. Cancel the common factor and invert both sides.

$$4-3y = \frac{1}{7-3x}$$

15. Rearrange to make y the subject.

$$3y = 4 - \frac{1}{7-3x}$$

16. Express the right-hand side of the equation with a common denominator.

$$3y = \frac{4(7-3x) - 1}{7-3x}$$

17. Expand the brackets in the numerator.

$$3y = \frac{28-12x-1}{7-3x}$$

18. Simplify and take out the common factor.

$$3y = \frac{27-12x}{7-3x}$$

$$3y = \frac{3(9-4x)}{7-3x}$$

19. State the final answer in a simplified form and state the maximal domain.

$$y = \frac{9-4x}{7-3x} \text{ for } x \in R \setminus \left\{ \frac{7}{3} \right\}$$

Determine c or rearrange to make y the subject?

When solving these types of differential equations, it is necessary to determine the constant of integration and also rearrange to make y the subject. Sometimes the order in which we do these operations can make the processes simpler.

WORKED EXAMPLE 15

Solve the differential equation $\frac{dy}{dx} + 4y = 0, y(0) = 3$.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Invert both sides.
3. Integrate both sides.
4. Take the constant factor outside the front of the integral sign.
5. Use $\int \frac{1}{u} du = \ln(|u|) + c$ to express x in terms of y and the constant of integration c .

From this point forward, we have two processes to complete: determine c , and transpose the equation to make y the subject.

Method 1: Determine c first, then transpose to make y the subject.

6. Substitute and use the given conditions to determine the value of the constant.

7. Substitute back for c and take out the common factor.

8. Use the logarithm laws to simplify the expression.

9. Use the definition of the logarithm.

10. Invert both sides again in attempting to make y the subject.

11. Because $e^{-4x} > 0$, the modulus is not needed. State the particular solution to the differential equation.

WRITE

$$\frac{dy}{dx} + 4y = 0$$

$$\frac{dy}{dx} = -4y$$

$$\frac{dx}{dy} = -\frac{1}{4y}$$

$$x = -\int \frac{1}{4y} dy$$

$$x = -\frac{1}{4} \int \frac{1}{y} dy$$

$$x = -\frac{1}{4} \ln(|y|) + c$$

$$y(0) = 3$$

$$\Rightarrow x = 0 \text{ when } y = 3$$

$$2 = -\frac{1}{4} \ln(|3|) + c$$

$$c = \frac{1}{4} \ln(3)$$

$$x = -\frac{1}{4} \ln(|y|) + \frac{1}{4} \ln(3)$$

$$x = \frac{1}{4} [\ln(3) - \ln(|y|)]$$

$$x = \frac{1}{4} \ln\left(\frac{3}{|y|}\right)$$

$$4x = \ln\left(\frac{3}{|y|}\right)$$

$$e^{4x} = \frac{3}{|y|}$$

$$\frac{|y|}{3} = \frac{1}{e^{4x}} \\ = e^{-4x}$$

$$y = 3e^{-4x}$$

Method 2: Make y the subject and then find the constant c .

6. Rearrange to make y the subject.

$$x = -\frac{1}{4} \ln(|y|) + c$$

$$\frac{1}{4} \ln(|y|) = c - x$$

$$\ln(|y|) = 4c - 4x$$

$$\text{Let } B = 4c.$$

$$\ln(|y|) = B - 4x$$

$$|y| = e^{B-4x}$$

$$= e^B e^{-4x}$$

$$\text{Let } A = e^B.$$

$$|y| = A e^{-4x}$$

$$y(0) = 3$$

$$\Rightarrow x = 0 \text{ when } y = 3$$

$$3 = A e^{-0}$$

$$3 = A$$

$$y = 3e^{-4x}$$

7. Since c is a constant, $4c$ is also a constant.

8. Use the definition of the logarithm.

9. Since B is a constant, e^B is also a constant.

10. Substitute and use the given condition to determine the value of the constant.

11. Because $e^{-4x} > 0$, the modulus is not needed.
Substitute for A and state the particular solution to the differential equation.

Stating the domain for which the solution is valid

As discussed in the previous section, the solution to a differential equation should include the largest domain for which the solution is valid.

WORKED EXAMPLE 16

Solve the differential equation $2\frac{dy}{dx} + \sqrt{16 - y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.

2. Invert both sides.

3. Integrate with respect to y .

4. Perform the integration process using $\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$.

WRITE

$$2\frac{dy}{dx} + \sqrt{16 - y^2} = 0$$

$$2\frac{dy}{dx} = -\sqrt{16 - y^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{16 - y^2}}{2}$$

$$\frac{dx}{dy} = \frac{-2}{\sqrt{16 - y^2}}$$

$$x = \int \frac{-2}{\sqrt{16 - y^2}} dy$$

$$x = 2 \cos^{-1}\left(\frac{y}{4}\right) + c$$

5. Substitute and use the given conditions to determine the value of the constant.

$$\begin{aligned}y(0) &= 0 \\ \Rightarrow x &= 0 \text{ when } y = 0 \\ 0 &= 2 \cos^{-1}(0) + c \\ c &= -2 \cos^{-1}(0)\end{aligned}$$

$$c = -2 \times \frac{\pi}{2}$$

$$c = -\pi$$

6. Substitute back for c .

$$x = 2 \cos^{-1} \left(\frac{y}{4} \right) - \pi$$

7. Rewrite the equation.

$$2 \cos^{-1} \left(\frac{y}{4} \right) = x + \pi$$

$$\cos^{-1} \left(\frac{y}{4} \right) = \frac{x + \pi}{2}$$

8. Take the cosine of both sides to make y the subject.

$$\frac{y}{4} = \cos \left(\frac{x}{2} + \frac{\pi}{2} \right)$$

$$y = 4 \cos \left(\frac{x}{2} + \frac{\pi}{2} \right)$$

9. Expand using trigonometric compound-angle formulas.

$$\begin{aligned}y &= 4 \left(\cos \left(\frac{x}{2} \right) \cos \left(\frac{\pi}{2} \right) - \sin \left(\frac{x}{2} \right) \sin \left(\frac{\pi}{2} \right) \right) \\ &= 4 \left(\cos \left(\frac{x}{2} \right) \times 0 - \sin \left(\frac{x}{2} \right) \times 1 \right)\end{aligned}$$

10. State the particular solution.

$$y = -4 \sin \left(\frac{x}{2} \right)$$

11. Determine the domain for which the solution is valid.

$$\cos^{-1} \left(\frac{y}{4} \right) = \frac{x + \pi}{2}$$

The range of $y = \cos^{-1}(x)$ is $[0, \pi]$,

but $|y| < 4$, so

$$0 < \frac{x + \pi}{2} < \pi$$

12. Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.

$$0 < x + \pi < 2\pi$$

$$-\pi < x < \pi$$

The solution $y = -4 \sin \left(\frac{x}{2} \right)$ is valid for $-\pi < x < \pi$.

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Units 3 & 4

Area 5

Sequence 1

Concept 4

Solving differential equations of the form $\frac{dy}{dx} = g(y)$ Summary screen and practice questions

Exercise 8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

Technology free

1. **WE13** Determine the general solution to the differential equation $\sqrt{y}\frac{dy}{dx} + 4 = 0$.
 2. Determine the general solution to the differential equation $\frac{dy}{dx} - \tan(2y) = 0$.
 3. Find the general solution to each of the following.
 - a. $\frac{dy}{dx} = \frac{y^2}{4}$
 - b. $\frac{dy}{dx} = y + 4$
 - c. $\frac{dy}{dx} = \frac{y}{4}$
 - d. $\frac{dy}{dx} = \frac{4}{y^2}$
 4. **WE14** Solve the differential equation $\frac{dy}{dx} + (5 - 4y)^2 = 0$, $y(1) = 2$.
 5. Solve the differential equation $\frac{dy}{dx} + 4y - 7 = 0$, $y(0) = 3$.
 6. **WE15** Solve the differential equation $\frac{dy}{dx} + 3y = 0$, $y(0) = 5$.
 7. Given the differential equation $\frac{dy}{dx} - 5y = 0$, $y(0) = 3$, express y in terms of x .
 8. Solve the following differential equations, expressing y in terms of x .
 - a. $\frac{dy}{dx} + 5y = 0$, $y(0) = 4$
 - b. $\frac{dy}{dx} - 3y = 0$, $y(1) = 2$
 9. Solve the following differential equations, expressing y in terms of x .
 - a. $\frac{dy}{dx} + 2y = 5$, $y(0) = 3$
 - b. $\frac{dy}{dx} - 3y + 4 = 0$, $y(0) = 2$
 10. **WE16** Solve the differential equation $\sqrt{64 - y^2} - 6\frac{dy}{dx} = 0$, $y(0) = 8$, stating the largest domain for which the solution is valid.
 11. Solve the differential equation $16 + y^2 - 4\frac{dy}{dx} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.
- For questions 12–18, solve each of the differential equations given, and where appropriate state the largest domain for which the solution is valid.
12. a. $\frac{dy}{dx} = \sqrt{y}$, $y(1) = 4$
b. $\frac{dy}{dx} = y^2$, $y(1) = 3$
 13. a. $\frac{dy}{dx} = 4e^{2y}$, $y(2) = 0$
b. $\frac{dy}{dx} + 6e^{3y} = 0$, $y(1) = 0$
 14. a. $\frac{dy}{dx} = (5 - 2y)^2$, $y(1) = 3$
b. $\frac{dy}{dx} + (7 - 3y)^2 = 0$, $y(3) = 2$
 15. a. $\frac{dy}{dx} + 6 \operatorname{cosec}\left(\frac{y}{2}\right) = 0$, $y\left(\frac{1}{3}\right) = 0$
b. $\frac{dy}{dx} = 2 \sec(2y)$, $y\left(\frac{1}{8}\right) = \frac{\pi}{12}$
 16. a. $\frac{dy}{dx} - \sqrt{4y + 9} = 0$, $y(0) = 0$
b. $\frac{dy}{dx} - 4y^2 = 9$, $y(0) = 0$
 17. a. $\frac{dy}{dx} + 4y = y^2$, $y(0) = 3$
b. $\frac{dy}{dx} - 3y = y^2$, $y(0) = 6$

18. a. $\frac{dy}{dx} + 7y = y^2 + 12, y(0) = 0$ b. $\frac{dy}{dx} - 6y - y^2 = 8, y(0) = 0$
19. a. If k and y_0 are constants, solve the differential equation $\frac{dy}{dx} + ky = 0, y(0) = y_0$.
 b. Given that a, b and c are non-zero real constants, solve the differential equations:
 i. $\frac{dy}{dx} + ay = b, y(0) = c$ ii. $\frac{dy}{dx} + ay = by^2, y(0) = c$.
20. a. Given that a and b are non-zero real constants, solve the differential equations:
 i. $\frac{dy}{dx} = (ay + b)^2, y(0) = 0$ ii. $\frac{dy}{dx} = b^2y^2 + a^2, y(0) = 0$.
 b. If a and b are constants with $a > b > 0$:
 i. solve the differential equation $\frac{dy}{dx} = (y + a)(y + b), y(0) = 0$ ii. find $\lim_{x \rightarrow \infty} y(x)$.

8.6 Solving differential equations of the form

$$\frac{dy}{dx} = f(x)g(y)$$

8.6.1 Separation of variables

Differential equations of the form $\frac{dy}{dx} = f(x)g(y), y(x_0) = y_0$ are called variables separable equations, as it is possible to separate all the x terms onto one side of the equation and all the y terms onto the other side of the equation.

For $\frac{dy}{dx} = f(x)g(y)$, divide both sides by $g(y)$, since $g(y) \neq 0$. This gives

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

Integrate both sides of the equation with respect to x .

$$\begin{aligned} \int \frac{1}{g(y)} \frac{dy}{dx} dx &= \int f(x) dx \\ \text{Thus, } \int \frac{1}{g(y)} dy + c_1 &= \int f(x) dx + c_2. \\ \int \frac{1}{g(y)} dy &= \int f(x) dx + c, \text{ since } c = c_2 - c_1. \end{aligned}$$

After performing the integration, an implicit relationship between x and y is obtained. However, in specific cases it may be possible to rearrange to make y the subject.

WORKED EXAMPLE 17

Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{x+4}{y^2+4}$.

THINK

1. Write the differential equation.
2. Separate the variables and integrate both sides.
3. Perform the integration and add the constant on one side only.
4. The general solution is given as an implicit equation, as in this case it is impossible to solve this equation explicitly for y .

WRITE

$$\frac{dy}{dx} = \frac{x+4}{y^2+4}$$

$$\int (y^2 + 4) dy = \int (x + 4) dx$$

$$\frac{1}{3}y^3 + 4y = \frac{1}{2}x^2 + 4x + c$$

$$\frac{1}{3}y^3 + 4y - \frac{1}{2}x^2 - 4x = c$$

Finding particular solutions

Finding particular solutions involves solving the differential equation, expressing y in terms of x where possible, and finding the value of the constant of integration.

WORKED EXAMPLE 18

Solve the differential equation $\frac{dy}{dx} + y = 6x^2y$, $y(0) = 1$.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Factor the RHS.
3. Separate the variables and integrate both sides.
4. Perform the integration and add in the constant on one side only.
5. Substitute and use the given conditions to determine the value of the constant.
6. Substitute back for c and use definition of a logarithm to state the solution explicitly as y in terms of x . Note that the modulus is not needed, as $e^{2x^3-x} > 0$.

WRITE

$$\frac{dy}{dx} + y = 6x^2y$$

$$\frac{dy}{dx} = 6x^2y - y$$

$$\frac{dy}{dx} = y(6x^2 - 1)$$

$$\int \frac{1}{y} dy = \int (6x^2 - 1) dx$$

$$\ln(|y|) = 2x^3 - x + c$$

$$y(0) = 1$$

$$\Rightarrow x = 0 \text{ when } y = 1$$

$$\ln(|1|) = 0 + c$$

$$c = 0$$

$$\ln(|y|) = 2x^3 - x$$

$$y = e^{2x^3-x}$$

Stating the domain for which the solution is valid

As previously stated, when solving differential equations it is necessary to state the largest domain for which the solution is valid.

WORKED EXAMPLE 19

Solve the differential equation $\frac{dy}{dx} + 2x\sqrt{16-y^2} = 0$, $y(0) = 4$, stating the largest domain for which the solution is valid.

THINK

1. Rewrite the equation to make $\frac{dy}{dx}$ the subject.
2. Separate the variables and integrate both sides.
3. Perform the integration and add the constant on one side only.
4. Substitute and use the given conditions to determine the value of the constant.
5. Substitute back for c .
6. Take the cosine of both sides to make y the subject.
7. Determine the domain for which the solution is valid.
8. Solve the inequality for x to state the largest domain for which the solution is valid. State the answer.

WRITE

$$\frac{dy}{dx} + 2x\sqrt{16-y^2} = 0, y(0) = 4$$

$$\frac{dy}{dx} = -2x\sqrt{16-y^2}$$

$$\int \frac{-1}{\sqrt{16-y^2}} dy = \int 2x dx$$

$$\cos^{-1}\left(\frac{y}{4}\right) = x^2 + c$$

$$y(0) = 4$$

$$\Rightarrow x = 0 \text{ when } y = 4$$

$$\cos^{-1}(1) = c$$

$$c = 0$$

$$\cos^{-1}\left(\frac{y}{4}\right) = x^2$$

$$\frac{y}{4} = \cos(x^2)$$

$$y = 4\cos(x^2)$$

$$\cos^{-1}\left(\frac{y}{4}\right) = x^2$$

The range of $y = \cos^{-1}(x)$ is $[0, \pi]$,

but $x \neq 0$ and $\frac{1}{\sqrt{16-y^2}}$ is defined for

$$|y| < 4, \text{ so } 0 < x^2 < \pi$$

The solution $y = 4\cos(x^2)$ is valid for $0 < x < \sqrt{\pi}$.

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Units 3 & 4

Area 5

Sequence 1

Concept 5

Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ Summary screen and practice questions

Exercise 8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

Technology free

1. **WE17** Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{x+2}{y^3+8}$.
2. Obtain an implicit relationship of the form $f(x,y) = c$ for $\frac{dy}{dx} = \frac{y^2+4}{x^2y^2}$.
3. Obtain an implicit relationship of the form $f(x,y) = c$ for each of the following differential equations.
 - a. $\frac{dy}{dx} = \frac{x^2+4}{y^2+4}$
 - b. $\frac{dy}{dx} = \frac{xy}{y^2+4}$
 - c. $\frac{dy}{dx} = \frac{x^2y^2}{y^2+4}$
 - d. $\frac{dy}{dx} = \frac{xy^2e^{x^2}}{y^3+8}$
4. **WE18** Solve the differential equation $\frac{dy}{dx} - y = 3x^2y$, $y(0) = 1$.
5. Given the differential equation $\frac{dy}{dx} + y^2 = 2xy^2$, $y(2) = 1$, express y in terms of x .
6. **WE19** Solve the differential equation $\frac{dy}{dx} - 2x\sqrt{64-y^2} = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.
7. Solve the differential equation $2\frac{dy}{dx} - x(16+y^2) = 0$, $y(0) = 0$, stating the largest domain for which the solution is valid.

For questions 8–16, solve each of the given differential equations and express y in terms of x .

8. a. $\frac{dy}{dx} - \frac{y^2}{x} = 0$, $y(1) = 1$
b. $\frac{dy}{dx} + 12y^2 \sin(4x) = 0$, $y(\pi) = 1$
9. a. $\frac{dy}{dx} + \frac{x}{y} = 0$, $y(1) = 2$
b. $\frac{dy}{dx} + 6y^2x^2 = 0$, $y(1) = 3$
10. a. $\frac{dy}{dx} + 18x^3y^2 = 0$, $y(-1) = 2$
b. $\frac{dy}{dx} - \frac{y^2}{x^2} = 0$, $y(1) = 2$
11. a. $\frac{dy}{dx} = y^2e^{2x}$, $y(0) = 1$
b. $\frac{dy}{dx} + 12x^5y^2 = 0$, $y(1) = 2$
12. a. $\frac{dy}{dx} + y = 3x^2y$, $y(0) = 1$
b. $\frac{dy}{dx} + 6x^2y^2 = y^2$, $y(-1) = 2$
13. a. $\frac{dy}{dx} + 2xy^2 = y^2$, $y(1) = 2$
b. $\frac{dy}{dx} + 8x^3y^4 = y^4$, $y(0) = 1$
14. a. $x\frac{dy}{dx} + 2y = y^2$, $y(1) = 1$
b. $x\frac{dy}{dx} - 4y = y^2$, $y(1) = 1$
15. a. $(4+x^2)\frac{dy}{dx} - 2xy = 0$, $y(0) = 1$
b. $\frac{y^2+4}{x^2+9} - \frac{y}{x}\frac{dy}{dx} = 0$, $y(0) = 2$

16. a. $\frac{dy}{dx} - x(25 + y^2) = 0, y(0) = 0$ b. $\frac{dy}{dx} + 4x\sqrt{25 - y^2} = 0, y(0) = 5$
17. For each of the following, use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and hence reduce to a separable differential equation and find the solution.
- a. $x\frac{dy}{dx} + 3y = 4x, y(2) = 1$ b. $x\frac{dy}{dx} - y = 4x, y(1) = 2$

8.7 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

- Show that the gradient of the curve $2x^2 + 3y^2 - 4xy - 9 = 0$ at the point $(-3, -1)$ is equal to $\frac{4}{3}$.
- An ice block has the shape of a cube and has its volume decreasing at a rate of $2 \text{ cm}^3/\text{min}$. Determine the rate at which each of the sides are decreasing in cm/min , when the sides are 2 cm .
- Complete the following.
 - Write an expression for the gradient in terms of both x and y for the curve $2x^3 - 6x^2y^2 + 3y^3 - 2 = 0$.
 - Determine the gradient in terms of x and y if $4x^4 - 7xy - 3y^4 - 8 = 0$.
 - Calculate the coordinates of the points on the curve $2x^2 + 2xy + y^2 = 8$ where the tangent to the curve is horizontal.
- Determine the general solution of the differential equation $\frac{dy}{dx} + 36x = 0$.
- Determine the particular solution of the differential equation $\frac{dy}{dx} = \frac{1}{(7x - 6)^2}, y(0) = 0$.
- Determine the general solution of the differential equation $3\frac{dy}{dx} + y = 0$, (let A be the constant).
- Determine the general solution of the differential equation $\frac{dy}{dx} + 3x^2y^2 = 0$, (let A be the constant).
- Determine the particular solution of the differential equation $\frac{dy}{dx} = y^2 + 4, y(0) = 0$.
- Calculate particular solutions to each of the following differential equations, stating the maximal domain, where appropriate.
 - $\frac{dy}{dx} - x^2 = 4, y(1) = 2$ b. $x^2\frac{dy}{dx} + 4 = 2x, y(1) = 2$
 - $(5x + 3)^2\frac{dy}{dx} + 4 = 0, y(-1) = 2$
- Solve the following:
 - $(x^2 + 9)\frac{dy}{dx} + 3 = 0, y(0) = 2$ b. $\sqrt{9 + x^2}\frac{dy}{dx} + 3x = 0, y(0) = 2$
 - $(9 - x^2)\frac{dy}{dx} + 3 = 0, y(0) = 2$

11. Solve the following:

a. $\frac{dy}{dx} + 9y^2 = 0, y(0) = 1$

b. $\frac{dy}{dx} + \frac{9}{y} = 0, y(0) = 1$

c. $\frac{dy}{dx} + 9y^3 = 0, y(0) = 1$

12. Solve the following:

a. $\frac{dy}{dx} + 4y = 7, y(0) = 2$

b. $\frac{dy}{dx} + (5 - 2y)^2 = 0, y(1) = 2$

c. $\frac{dy}{dx} + \sqrt{9 - y^2} = 0, y(0) = 3$

Complex familiar

13. a. Determine the equation of the normal to the curve $x^3 - 4xy + y^3 - 15 = 0$ at the point $(2, -1)$.

b. For the ellipse $4x^2 + 4xy + 9y^2 - 6y - 2x - 23 = 0$, determine the equation of the tangent at the point where $x = 2$ in the first quadrant.

c. Determine the coordinates of the points on the curve $9x^2 - 54x + 16y^2 + 64y + 1 = 0$ where the tangent to the curve is parallel to the y-axis.

14. While blowing up a balloon a child observes that the radius of the balloon increase at a rate of 2 cm/s. Assuming it remains spherical, when the radius is 6 cm, calculate:

a. the rate at which the volume is increasing

b. that rate at which its surface area is increasing.

15. Solve the following:

a. $\frac{dy}{dx} - y^2 = 4, y(0) = 0$

b. $\frac{dy}{dx} - y^2 = 4y, y(0) = 2$

c. $16\frac{dy}{dx} + e^{4y} = 0, y(0) = 0$

16. Solve the following:

a. $\frac{dy}{dx} - 6x^2y^2 = 0, y(1) = 1$

b. $\frac{dy}{dx} = 4x^3e^{-y}, y(0) = 0$

c. $\frac{dy}{dx} - y^2 = 4xy^2, y(1) = 1$

Complex unfamiliar

17. A spiral curve has the parametric equations $x = t \cos(t)$ and $y = t \sin(t)$.

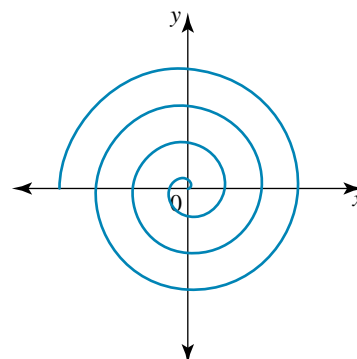
a. Show that $\frac{dy}{dx} = \frac{\sin(t) + t \cos(t)}{\cos(t) - t \sin(t)}$.

b. Show that the curve has the implicit equation

$$x^2 + y^2 = \left(\tan^{-1} \left(\frac{y}{x} \right) \right)^2.$$

c. Using implicit differentiation on $x^2 + y^2 = \left(\tan^{-1} \left(\frac{y}{x} \right) \right)^2$

verify that $\frac{dy}{dx} = \frac{x^3 + xy^2 + y \tan^{-1} \left(\frac{y}{x} \right)}{x \tan^{-1} \left(\frac{y}{x} \right) - yx^2 - y^3}$.



18. A man 2 metres tall is walking at 1.5 m/s. He passes under a light source 6 metres above the ground. Calculate:

- the rate at which his shadow is lengthening
- the speed at which the end point of his shadow is increasing
- the rate at which his head is receding from the light source when he is 8 metres from the light.

19. a. A train leaves a point O and travels south at a constant speed of 30 km/h.
A car that was initially 90 km east of O travels west at a constant speed of 40 km/h.
Let s be the distance in km of the car from the train after a time of t hours.



i. Show that $s = 10\sqrt{25t^2 - 72t + 81}$.

ii. Determine when the train and the car are closest and calculate the closest distance in kilometres between the car and the train.

iii. Calculate the rate at which the distance between the train and the car is changing after 1 hour.

b. Two railway tracks intersect at 60° . One train is 100 km from the junction and moves towards it at 80 km/h, while another train is 120 km from the junction and moves towards the junction at 90 km/h.

i. Show that the trains do not collide.

ii. Calculate the rate at which the trains are approaching each other after 1 hour.

20. Use the substitution $v = \frac{y}{x}$ to show that $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

Hence, reduce the differential equation $x \frac{dy}{dx} + ay = bx$ to a separate differential equation and determine its general solution for the cases when:

a. $a = -1$

b. $a \neq -1$

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Answers

8 Rates of change and differential equations

Exercise 8.2 Implicit differentiation

1. $-\frac{x^2}{y^2}$
2. $-\sqrt{\frac{y}{x}}$
3. a. $\frac{1}{y}$
c. $\frac{9x}{16y}$
4. a. $8x^2y \frac{dy}{dx} + 8xy^2$
5. $-\frac{9}{11}$
6. a. $\frac{4x+3y}{6y-3x}$
c. 1
7. $\frac{y - xe^{-(x^2+y^2)}}{ye^{-(x^2+y^2)} - x}$
8. $-\frac{3 \cos(3x+2y) + 2x}{2(y + \cos(3x+2y))}$
9. a. $\frac{27x^2 + 12y + 1}{y - 12x}$
b. $\frac{4x^3 - 6x^2y - 3x^2 + 8xy^2}{24y^2 + 2x^3 - 8x^2y}$
c. $\frac{2y + \sin(y) - e^{x+y}}{y \sin(xy) - ye^{xy} - 2x}$
d. $\frac{xe^{xy} - x \sin(xy)}{2 + 8x + 12y}$
10. a. $\frac{15y + 10x - 3}{y(2x^2 + 1)}$
b. $-\frac{x(2y^2 + 1)}{y^2(2x^2y - 1)}$
c. $\frac{x^2(1 - 2xy^2)}{3 \cos(3x) \cos(2y) - 2x}$
d. $\frac{6y^2 + 2 \sin(3x) \sin(2y)}{6y^2 + 2 \sin(3x) \sin(2y)}$
11. a. 1
c. $-\frac{2}{7}$
12. a. $3x - 4y - 25 = 0$
c. $\frac{4}{5}$
- b. $-\frac{9x}{4y}$
d. $\frac{3x-2}{y-1}$
 $\frac{27x^2 + 8xy^2 + 2}{5 + 9y^2 - 8x^2y}$
b. $\frac{3-x-xy^2}{x^2y}$
d. $\frac{2x+3}{y(3y-2)}$
13. a. $-\cot(t)$
b. Sample responses can be found in the worked solutions in the online resources.
14. a. $\frac{1-2t^3}{t}$
b. Sample responses can be found in the worked solutions in the online resources.
15. a. $\frac{1}{t}$
c. $-\frac{1}{t^2}$
16. a. $\frac{1}{t}$
c. $-\frac{1}{t^2}$
17. a. $\frac{1}{t}$
c. $-\frac{1}{t^2}$
18. a. $\frac{x^2 - ay}{ax - y^2}$
19. a. $-\frac{x(x^2 + y^2 - 1)}{y(x^2 + y^2 + 1)}$
20. a. $\frac{y^2 + 3x^2}{2y(2a - x)}$
c. $\frac{\sin(t)(2 \cos^2(t) + 1)}{2 \cos^3(t)}$
- b. $-\frac{4}{3} \tan(2t)$
d. $\frac{4}{3} \operatorname{cosec}(t)$
b. $-\frac{b}{a} \cot(t)$
d. $\frac{b}{a} \operatorname{cosec}(t)$
b. $-\frac{b}{a} \cot(t)$
d. $\frac{b}{a} \operatorname{cosec}(t)$
b. $\frac{t(t^3 - 2)}{2t^3 - 1}$
b. $\frac{2 - 3 \cos^2(t)}{\sin(t)(2 + \cos^2(t))}$
b. $\frac{t(t^2 + 3)}{2}$

Exercise 8.3 Related rates as instances of the chain rule

1. 20π m/s
2. 2 cm/s
3. a. $16 \text{ cm}^2/\text{s}$
b. $\frac{1}{4\pi} \text{ m/s}$
4. a. $800\pi \text{ mm}^3/\text{s}$
b. $160\pi \text{ mm}^2/\text{s}$
5. Decreasing at $\frac{1}{24\pi} \text{ cm/s}$
6. Decreasing at $48\pi \text{ cm}^2/\text{s}$
7. a. Sample responses can be found in the worked solutions in the online resources.
b. 150 weeks
8. Decreasing at $\frac{225}{128\pi} \text{ cm/s}$
9. Decreasing at $64\pi \text{ cm}^3/\text{s}$
10. a. $6 \text{ cm}^2/\text{s}$
b. $\sqrt{3} \text{ cm/s}$
11. a. Sample responses can be found in the worked solutions in the online resources.
b. $450\pi \text{ cm}^3/\text{s}$
12. 0.225 m/s
13. $\frac{8}{5} \text{ m/s}$
14. a. $\frac{1}{25\pi} \text{ cm/s}$
b. $\frac{27}{16\pi} \text{ cm/s}$
15. a. 25.87 m/s
b. 12 m/s
16. a. Sample responses can be found in the worked solutions in the online resources.
b. Decreases by 5 cm/s

17. a. Decreases by 31 562.5 Pa/s
b. Sample responses can be found in the worked solutions in the online resources.
18. a. $0.005^\circ/\text{s}$ b. $-2.1^\circ/\text{s}$

Exercise 8.4 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

1. a. $y = c - 3x^4$ b. $y = 13 - 3x^2$
2. a. $y = c - 6 \sin(2x)$ b. $y = 2(\cos(3x) - 1)$
3. a. $y = 2x^2 + 3x + c$
b. $y = x^3 + \frac{7}{2}x^2 - 20x + c$
c. $y = e^{2x} + 3e^{-2x} + c$
d. $y = \sqrt{x^2 + 9} + c$
4. $y = \frac{9x - 11}{4x - 5}, x \in R \setminus \left\{\frac{5}{4}\right\}$
5. $y = 3 + \frac{1}{2} \ln(|7 - 4x|), x \in R \setminus \left\{\frac{7}{4}\right\}$
6. $y = 1 - \sqrt{2x - 5}, x > \frac{5}{2}$
7. $y = 11 - 4\sqrt{x}, x > 0$
8. $y = 6 \sin^{-1}\left(\frac{x}{8}\right) - \pi, x \in (-8, 8)$
9. $y = \frac{\pi}{2} - \tan^{-1}\left(\frac{x}{4}\right), x \in R$
10. a. $y = \frac{1}{3}[5 \ln(|x|) + x^2 + 8], x \in R \setminus \{0\}$
b. $y = 2(e^{3x} - e^{-3x}), x \in R$
11. a. $y = 4 - 2 \cos(2x), x \in R$
b. $y = 3 - 2 \sin(3x), x \in R$
12. a. $y = 4x - \sin(4x), x \in R$
b. $y = 6x + \sin(6x), x \in R$
13. a. $y = \frac{1}{2}(\sqrt{4x + 9} - 3), x > -\frac{9}{4}$
b. $y = 1 + \frac{1}{2} \ln(|2x - 3|), x \in R \setminus \left\{\frac{3}{2}\right\}$
14. a. $y = \frac{10x - 17}{3x - 5}, x \in R \setminus \left\{\frac{5}{3}\right\}$
b. $y = 5 - 2 \ln(|7 - 4x|), x \in R \setminus \left\{\frac{7}{4}\right\}$
15. a. $y = \frac{3}{2} \ln\left(\frac{x^2 + 9}{9}\right), x \in R$
b. $y = 2 - \sqrt{x^2 + 4}, x \in R$
16. a. $y = \frac{1}{2} \ln\left(\frac{x^2 + 6x + 13}{13}\right), x \in R$
b. $y = \ln\left(\frac{3}{\sqrt{x^2 - 4x + 9}}\right), x \in R$
17. a. $y = -\frac{1}{8} \sin^4(2x), x \in R$
b. $y = \cos^3(3x) - 1, x \in R$
18. a. $y = 5x - x \ln(2x) - \frac{3}{2}, x > 0$
b. $y = (x - 4)e^{-x} + 4, x \in R$

Exercise 8.5 Solving differential equations of the form $\frac{dy}{dx} = g(y)$

1. $y = \sqrt[3]{(B - 6x)^2}$
2. $y = \frac{1}{2} \sin^{-1}(Be^{2x})$
3. a. $y = \frac{4}{c - x}$ b. $y = Ae^x - 4$
c. $y = Ae^{\frac{x}{4}}$ d. $y = \sqrt[3]{12x + A}$
4. $y = \frac{15x - 13}{12x - 11}, x \in R \setminus \left\{\frac{11}{12}\right\}$
5. $y = \frac{1}{4}(7 + 5e^{-4x})$
6. $y = 5e^{-3x}$
7. $y = 3e^{5x}$
8. a. $y = 4e^{-5x}$ b. $y = 2e^{3x-3}$
9. a. $y = \frac{1}{2}(5 + e^{-2x})$
b. $y = \frac{2}{3}(2 + e^{3x})$
10. $y = 8 \cos\left(\frac{x}{6}\right), -6\pi < x < 0$
11. $y = 4 \tan(x), -\frac{\pi}{2} < x < \frac{\pi}{2}$
12. a. $y = \frac{1}{4}(x + 3)^2, x \in R$
b. $y = \frac{3}{4 - 3x}, x \in R \setminus \left\{\frac{4}{3}\right\}$
13. a. $y = -\frac{1}{2} \ln(17 - 8x), x < \frac{17}{8}$
b. $y = -\frac{1}{3} \ln(18x - 17), x > \frac{17}{18}$
14. a. $y = \frac{5x - 8}{2x - 3}, x \in R \setminus \left\{\frac{3}{2}\right\}$
b. $y = \frac{7x - 23}{3x - 10}, x \in R \setminus \left\{\frac{10}{3}\right\}$
15. a. $y = 2 \cos^{-1}(3x), x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$
b. $y = \frac{1}{2} \sin^{-1}(4x), x \in \left(-\frac{1}{4}, \frac{1}{4}\right)$
16. a. $y = x^2 + 3x, x \in R$
b. $y = \frac{3}{2} \tan(6x), -\frac{\pi}{12} < x < \frac{\pi}{12}$
17. a. $y = \frac{12}{3 + e^{4x}}, x \in R$
b. $y = \frac{6e^{3x}}{3 - 2e^{3x}}, x \in R \setminus \left\{\frac{1}{3} \ln\left(\frac{3}{2}\right)\right\}$
18. a. $y = \frac{12(e^x + 1)}{4e^x + 3}, x \in R$
b. $y = \frac{4(1 - e^{2x})}{e^{2x} - 2}, x \in R \setminus \left\{\ln(\sqrt{2})\right\}$
19. a. $y = y_0 e^{-kx}$
b. i. $y = \left(c - \frac{b}{a}\right)e^{-ax} + \frac{b}{a}$
ii. $y = \frac{ac}{(a - bc)e^{ax} + bc}$

20. a. i. $y = \frac{b^2x}{1-abx}, x \neq \frac{1}{ab}$
 ii. $y = \frac{a}{b} \tan(abx)$
 b. i. $y = \frac{ab(1 - e^{-(a-b)x})}{ae^{-(a-b)x} - b}$
 ii. $-a$

Exercise 8.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)g(y)$

1. $\frac{y^4}{4} + 8y - \frac{x^2}{2} - 2x = c$
 2. $y - 2 \tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{x} = c$
 3. a. $\frac{1}{3}y^3 + 4y - \frac{1}{3}x^3 - 4x = c$
 b. $\frac{1}{2}y^2 + 4 \ln(|y|) - \frac{1}{2}x^2 = c$
 c. $y - \frac{4}{y} - \frac{1}{3}x^3 = c$
 d. $\frac{1}{2}y^2 - \frac{8}{y} - \frac{1}{2}e^{x^2} = c$
 4. $y = e^{x^3+x}$
 5. $y = \frac{1}{3+x-x^2}, x \in \mathbb{R} \setminus \left\{\frac{1 \pm \sqrt{13}}{2}\right\}$
 6. $y = 8 \sin(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$
 7. $y = 4 \tan(x^2), 0 < x < \frac{\sqrt{2\pi}}{2}$
 8. a. $y = \frac{1}{1 - \ln(|x|)}, x \in \mathbb{R} \setminus \{-e, 0, e\}$
 b. $y = \frac{1}{4 - 3 \cos(4x)}$
 9. a. $y = \sqrt{5 - x^2}, x \in [-5, 5]$
 b. $y = \frac{3}{6x^3 - 5}, x \in \mathbb{R} \setminus \left\{\sqrt[3]{\frac{5}{6}}\right\}$
 10. a. $y = \frac{2}{9x^4 - 8}, x \in \mathbb{R} \setminus \left\{\sqrt[4]{\frac{8}{9}}\right\}$
 b. $y = \frac{2x}{2-x}, x \in \mathbb{R} \setminus \{2\}$
 11. a. $y = \frac{2}{3 - e^{2x}}, x \in \mathbb{R} \setminus \left\{\ln(\sqrt{3})\right\}$
 b. $y = \frac{2}{4x^6 - 3}, x \in \mathbb{R} \setminus \left\{\sqrt[6]{\frac{3}{4}}\right\}$
 12. a. $y = e^{x^3-x}, x \in \mathbb{R}$
 b. $y = \frac{2}{4x^3 - 2x + 3}, 4x^3 - 2x + 3 \neq 0$
 13. a. $y = \frac{2}{2x^2 - 2x + 1}, x \in \mathbb{R}$
 b. $y = \frac{1}{\sqrt[3]{6x^4 - 3x + 1}}, 6x^4 - 3x + 1 \neq 0$

14. a. $y = \frac{2}{1-x^2}, x \in \mathbb{R} \setminus \{\pm 1\}$
 b. $y = \frac{4x^4}{5-x^4}, x \in \mathbb{R} \setminus \{\pm \sqrt[4]{5}\}$
 15. a. $y = \frac{1}{4}(x^2 + 4), x \in \mathbb{R}$
 b. $y = \frac{2}{3}\sqrt{2x^2 + 9}, x \in \mathbb{R}$
 16. a. $y = 5 \tan\left(\frac{5x^2}{2}\right), x \in \left(-\frac{\sqrt{5\pi}}{5}, \frac{\sqrt{5\pi}}{5}\right)$
 b. $y = 5 \cos(2x^2), x \in \left(-\frac{\sqrt{2\pi}}{2}, \frac{\sqrt{2\pi}}{2}\right)$
 17. a. $y = x - \frac{8}{x^3}, x \in \mathbb{R} \setminus \{0\}$
 b. $y = 2x(1 + 2 \ln(|x|)), x \in \mathbb{R} \setminus \{0\}$

8.7 Review: exam practice

- Sample responses can be found in the worked solutions in the online resources.
- Decrease by $\frac{1}{6}$
- a. $\frac{2x(x-2y^2)}{y(4x^2-3y)}$ b. $\frac{16x^3-7y}{7x+12y^3}$
 c. $(2, -4), (-2, 4)$
- $y = c - 18x^2$
- $y = \frac{x}{6(6-7x)}$
- $y = Ae^{-\frac{x}{3}}$
- $y = \frac{1}{c+x^3}$
- $y = 2 \tan(2x)$
- a. $y = 4x + \frac{1}{3}(x^3 - 7), x \in \mathbb{R}$
 b. $y = 2 \ln(|x|) - 2 + \frac{4}{x}, x \in \mathbb{R} \setminus \{0\}$
 c. $y = \frac{12x+8}{5x+3}, x \in \mathbb{R} \setminus \left\{-\frac{3}{5}\right\}$
- a. $y = 2 - \tan^{-1}\left(\frac{x}{3}\right)$
 b. $y = 11 - 3\sqrt{x^2+9}$
 c. $y = 2 + \frac{1}{2} \ln\left(\left|\frac{x-3}{x+3}\right|\right), x \in \mathbb{R} \setminus \{-3, 3\}$
- a. $y = \frac{1}{9x+1}, x \in \mathbb{R} \setminus \left\{-\frac{1}{9}\right\}$
 b. $y = \sqrt{1-18x}, x < \frac{1}{18}$
 c. $y = \frac{1}{\sqrt{18x+1}}, x > -\frac{1}{18}$
- a. $y = \frac{1}{4}(7 + e^{-4x})$
 b. $y = \frac{5x-7}{2x-3}, x \in \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$
 c. $y = 3 \cos(x), 0 \leq x \leq \pi$

13. a. $16y + 5x + 6 = 0$
 b. $10y + 9x - 28 = 0$
 c. $(7, -2), (-1, -2)$
14. a. $288\pi \text{ cm}^3/\text{s}$ b. $96\pi \text{ cm}^2/\text{s}$
15. a. $y = 2 \tan(2x), -\frac{\pi}{4} < x < \frac{\pi}{4}$
 b. $y = \frac{4e^{4x}}{3 - e^{4x}}, x \in \mathbb{R} \setminus \left\{ \frac{1}{4} \ln(3) \right\}$
 c. $y = \frac{1}{4} \ln\left(\frac{4}{x+4}\right), x > -4$
16. a. $y = \frac{1}{3 - 2x^3}$
 b. $y = \ln(x^4 + 1)$
 c. $y = \frac{1}{4 - x - 2x^2}$
17. a. Sample responses can be found in the worked solutions in the online resources.
- b. Sample responses can be found in the worked solutions in the online resources.
- c. Sample responses can be found in the worked solutions in the online resources.
18. a. 0.75 m/s b. 2.25 m/s c. 1.6 m/s
19. a. i. Sample responses can be found in the worked solutions in the online resources.
 ii. $1.44 \text{ h}, 54 \text{ km}$
 iii. Decreasing by 18.86 km/h
- b. i. Sample responses can be found in the worked solutions in the online resources.
 ii. $\frac{220\sqrt{7}}{7} \text{ km/h}$
20. a. $y = x(c + b \ln(|x|)), x \in \mathbb{R} \setminus \{0\}$
 b. $y = \frac{bx}{a+1} + \frac{c}{x^a}$

9 Applications of first-order differential equations

9.1 Overview

Differential equations first came into existence through the study and discovery of the concepts of calculus by Isaac Newton (1643–1727) and Gottfried Leibniz (1646–1716). A differential equation is a mathematical equation that relates a physical quantity with its rate of change. Because such relations are extremely common, differential equations play a major role in engineering, physics, economics and bioess the mass remaining in milligrlogy. In mathematics, differential equations are studied for several purposes, such as determining their solution set and their ability to model many practical applications. In 1822 Jean-Baptiste Fourier (1768–1830) published his work on heat flow in *Théorie analytique de la chaleur* (The Analytic Theory of Heat), in which he based his reasoning on Newton's law of cooling, namely, that the flow of heat between two adjacent molecules is proportional to the small difference of their temperatures. This partial differential equation is now taught to most secondary school and university students of physics. Differential equations play an integral role in modelling virtually every physical, technical or biological process, from celestial motion, to bridge design, to interactions between neurons. Many fundamental laws of physics can be formulated as differential equations. There are many different types of differential equations, however we only consider a limited variety of them in Specialist Mathematics.



LEARNING SEQUENCE

- 9.1 Overview
- 9.2 Growth and decay
- 9.3 Other applications of first-order differential equations
- 9.4 Bounded growth and Newton's law of cooling
- 9.5 Chemical reactions and dilution problems
- 9.6 The logistic equation
- 9.7 Slope fields
- 9.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.2 Growth and decay

9.2.1 Introduction

In Chapter 8, methods of solving first-order differential equations were discussed. In this chapter some applications of first-order differential equations are explored. There are many applications of differential equations in business and science; in fact, they are useful whenever a rate of change needs to be considered.

9.2.2 The law of natural growth

Consider the differential equation $\frac{dy}{dx} = ky$, where k is a constant.

To solve this equation:

Invert both sides of the equation: $\frac{dx}{dy} = \frac{1}{ky}$

Integrate both side with respect y : $\int \frac{dx}{dy} dy = \int \frac{1}{ky} dy$

Separate the variables: $x = \int \frac{1}{ky} dy$

$$x = \frac{1}{k} \int \frac{1}{y} dy$$

Integrate: $x = \frac{1}{k} \ln |y| + c$, where c is a constant of integration and $y \neq 0$.

Rearrange: $\ln |y| = kx + A$, where $A = -kc$ is another constant

Set both sides to base e : $|y| = e^{kx+A}$

$$|y| = e^{kx} \times e^A$$

$$|y| = Be^{kx}, \text{ where } B = e^A$$

$$|y| = Be^{kx}$$

Remove the absolute value sign as e^{kx} is always positive:

Assume that $y(0) = y_0$, then $B = y_0$.

Therefore, the particular solution of the differential equation $\frac{dy}{dx} = ky$, $y(0) = y_0$ is given by $y = y_0 e^{kx}$.

This can easily be verified using differentiation.

If $y = y_0 e^{kx}$, then

$$\begin{aligned}\frac{dy}{dx} &= ky_0 e^{kx} \\ &= k(y_0 e^{kx}) \\ &= ky\end{aligned}$$

Thus, there is a function, $y = y_0 e^{kx}$, whose derivative is proportional to itself. This result gives rise to several applications.

9.2.3 Population growth

The law of natural growth states that the rate of increase of population is proportional to the current population at that time. Let $N = N(t)$ represent the population number of a certain quantity at a time t , where $t \geq 0$. This leads to the equation

$$\begin{aligned}\frac{dN}{dt} &\propto N \\ \Rightarrow \frac{dN}{dt} &= kN, \text{ where } k \text{ is a positive constant.}\end{aligned}$$

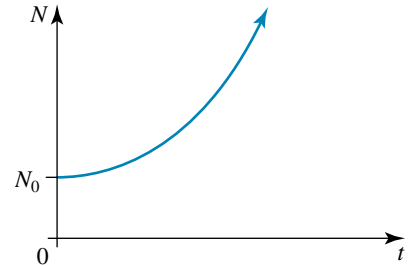
Assuming that the initial population number is $N_0 = N(0)$, the solution of this differential equation is given by $N = N_0 e^{kt}$. This is obtained by simply substituting N for y and t for x .

This equation has been found to model population growth, although it applies over only a limited time frame. Also, it does not include factors for population-changing events such as war or famine.

The equation $N = N_0 e^{kt}$ is called the law of natural growth or exponential growth.

The constant k can be interpreted as the excess birth rate over the death rate, and is called the annual growth rate.

Note that mathematically, N is treated as a continuous variable. However, when these equations are used to model populations, N is really a discrete quantity, the number of individuals in the population. In these situations the final answer may need to be rounded to the nearest whole number.



WORKED EXAMPLE 1

The population of a certain town increases at a rate proportional to the current population. At the start of 2009 the population was 250 000, and at the start of 2017 the population was 750 000.

- Express the population number in millions, N , in terms of t , the time in years after the start of 2009.
- What is the predicted population at the start of 2025?
- In which year does the population reach 5 million?

THINK

- Let $N = N(t)$ represent the population number (in millions) of the town after 2009.
 - In the year 2009, i.e. at $t = 0$, the population was 250 000, or one-quarter of a million. Use this to determine the value of N_0 .
 - In the year 2017 ($t = 8$), the population was 750 000, which is three-quarters of a million. Use this to determine the value of k .
 - Solve the equation for k by taking the natural logarithms of both sides.

WRITE

$$\begin{aligned}\text{a. } \frac{dN}{dt} &\propto N \\ \therefore \frac{dN}{dt} &= kN \\ \Rightarrow N &= N_0 e^{kt} \\ \frac{1}{4} &= N_0 e^0 \\ N_0 &= \frac{1}{4}\end{aligned}$$

$$\text{Substitute } N_0 = \frac{1}{4}:$$

$$N(t) = \frac{1}{4} e^{kt}$$

$$\text{Substitute } t = 8 \text{ and } N = \frac{3}{4}:$$

$$\frac{3}{4} = \frac{1}{4} e^{8k}$$

$$e^{8k} = 3$$

$$8k = \ln(3)$$

$$k = \frac{1}{8} \ln(3)$$

5. Express the population number in millions at a time t years after 2009.

b. 1. Determine the population in 2025, that is, when $t = 16$.

2. State the result.

c. 1. Determine the year by finding the value of t when $N = 5$.

2. Solve the equation for t by taking the natural logarithms of both sides.

3. State the result.

$$N(t) = \frac{1}{4}e^{\frac{t}{8}\ln(3)}$$

b. Substitute $t = 16$:

$$N(16) = \frac{1}{4}e^{\frac{16}{8}\ln(3)}$$

$$N(16) = \frac{1}{4}e^{2\ln(3)}$$

$$= \frac{1}{4}e^{\ln(9)}$$

$$= \frac{1}{4} \times 9$$

$$= 2.25$$

The population in 2025 is predicted to be 2.25 million.

c. $5 = \frac{1}{4}e^{\frac{t}{8}\ln(3)}$

$$20 = e^{\frac{t}{8}\ln(3)}$$

$$\frac{t}{8}\ln(3) = \ln(20)$$

$$t = \frac{8\ln(20)}{\ln(3)}$$

$$\approx 21.815$$

The population reaches 5 million late in the year 2030.

9.2.4 Radioactive decay

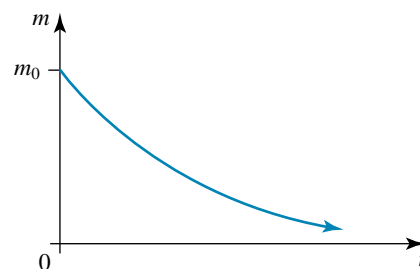
Chemical experiments have shown that when a radioactive substance decays, it changes into another element or an isotope of the same element. Scientists have determined that the rate of decay is proportional to the mass of the radioactive substance present at that time.

Let $m = m(t)$ represent the mass of the radioactive substance at a time, t , where $t \geq 0$.

Then

$$\begin{aligned} \frac{dm}{dt} &\propto -m \\ \Rightarrow \frac{dm}{dt} &= -km, \text{ where } k > 0 \text{ and is called the decay constant.} \end{aligned}$$

The solution of this differential equation is given by $m = m_0e^{-kt}$, where $m_0 = m(0)$. This equation is called the law of exponential decay.



WORKED EXAMPLE 2

The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 30 milligrams of a radioactive substance is present. After 2 hours the experimenter observes that 20% has disintegrated. Determine the amount remaining after a further 3 hours, correct to 3 decimal places.

THINK

1. Let $m = m(t)$ represent the amount of the radioactive substance present in milligrams (mg) after a time t hours.
2. Initially, when $t = 0$, the mass is 30 mg. Use this to determine the value of m_0 .
3. After 2 hours, 20% has disintegrated, so 80% remains of the initial 30 mg. Use this to determine the amount left after 2 hours.
4. Substitute $m_0 = 30$, $t = 2$ and $m = 24$ into the law of exponential decay, $m = m_0 e^{-kt}$, where $m_0 = m(0)$.
5. Solve the equation for k by taking the natural logarithms of both sides.
6. Express the mass remaining in milligrams after t hours.
7. Determine the amount remaining after a further 3 hours, that is, when $t = 5$.
8. State the result.

WRITE

$$\frac{dm}{dt} \propto -m$$

$$\therefore \frac{dm}{dt} = -km$$

$$\Rightarrow m = m_0 e^{-kt} \quad [1]$$

$$m_0 = m(0) = 30$$

$$\text{When } t = 2, m = 0.8 \times m_0:$$

$$m(2) = 0.8 \times 30 \\ = 24$$

$$\text{Substitute } m_0 = 30, t = 2 \text{ and}$$

$$m = 24 \text{ into [1]:}$$

$$24 = 30e^{-2k}$$

$$e^{-2k} = \frac{24}{30}$$

$$-2k = \ln \left(\frac{4}{5} \right)$$

$$k = -\frac{1}{2} \ln \left(\frac{4}{5} \right)$$

$$m(t) = 30e^{\frac{t}{2} \ln \left(\frac{4}{5} \right)}$$

$$\text{Substitute } t = 5:$$

$$m(t) = 30e^{\frac{5}{2} \ln \left(\frac{4}{5} \right)}$$

$$\approx 17.173$$

$$17.173 \text{ mg remains after 5 hours.}$$

9.2.5 Half-lives

For $m = m_0 e^{-kt}$ where $k > 0$, as $t \rightarrow \infty$, $m \rightarrow 0$.

An infinite time is required for all of the radioactive material to disintegrate. For this reason the rate of disintegration is often measured in terms of the **half-life** of the radioactive material. The half-life, T , is the time it takes for half of the original mass to disintegrate.

To determine the half-life, the value of T when $m = \frac{1}{2}m_0$ is required.

Substitute $m = \frac{1}{2}m_0$ into the equation $m = m_0e^{-kt}$.

This gives: $\frac{1}{2}m_0 = m_0e^{-kT}$

$$\frac{1}{2} = e^{-kT}$$

$$e^{kT} = 2$$

Solving for T gives: $T = \frac{1}{k} \ln(2)$

Notice that this half-life formula does not depend upon the initial mass m_0 and is thus independent of the time when observations began. Half-lives for radioactive substances can range from milliseconds to billions of years.

WORKED EXAMPLE 3

It takes 2 years for 30% of a particular radioactive substance to disintegrate. Determine the half-life of the substance, correct to 3 decimal places.

THINK

- Let $m = m(t)$ represent the amount of the radioactive substance present after time t in years.
- Since 30% has disintegrated, 70% remains.
- Substitute $m = 0.7m_0$ and $t = 2$.
- Solve the equation for k by taking the natural logarithms of both sides.
- Determine the half-life.
- State the final result.

WRITE

$$m = m_0e^{-kt}$$

$$\text{When } t = 2, m = 0.7m_0.$$

$$0.7m_0 = m_0e^{-2k}$$

$$0.7 = e^{-2k}$$

$$-2k = \ln(0.7)$$

$$k = -\frac{1}{2} \ln(0.7)$$

$$T = \frac{1}{k} \ln(2)$$

$$T = \frac{1}{-\frac{1}{2} \ln(0.7)} \ln(2)$$

$$T \approx 3.887$$

The half-life of the substance is 3.887 years.

study on

Unit 3 & 4

Area 5

Sequence 2

Concepts 1 & 2

Modelling growth with differential equations Summary screen and practice questions

Modelling decay with differential equations Summary screen and practice questions

Exercise 9.2 Growth and decay

Technology active

1. **WE1** The population of a city increases at a rate proportional to the current population. In 2015 the population was 1.79 million, and in 2019 the population was 2 million.
 - a. Express the population number, N , in terms of t , the time in years after 2015.
 - b. What is the predicted population in 2030?
 - c. When does the population reach 5 million?
2. The number of frogs in a colony increases at a rate proportional to the current number. After 15 months there are 297 frogs present, and this number grows to 523 after 26 months. Determine the initial number of frogs present in the colony.
3.
 - a. The rate of growth of a population of insects is proportional to its present size. A colony of insects initially contains 600 individuals and is found to contain 1300 after 2 weeks. Determine the number of insects after a further 3 weeks.
 - b. Initially there are 10 000 fish in a reservoir. The number of fish in the reservoir grows continuously at a rate of 4% per year. Determine the number of fish in the reservoir after 3 years.
4. The number of possums in a certain area grows at a rate proportional to the current number. In 4 months the number of possums has increased from 521 to 678. Determine the number of possums in the area after a further 5 months.
5. The population of Australia in 2013 was approximately 23 131 000. In 2018 it had grown to 24 992 000.
 - a. What is the predicted population in 2025?
 - b. In what year will the population first exceed 30 million?
6. In 2011 the world population reached 7 billion. In 2018 the world population was approximately 7 632 819 325.
 - a. What is the estimated world population in 2024?
 - b. In what year will the world population reach 10 billion?
7. **WE2** The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. Initially 80 milligrams of a radioactive substance is present, and after 1 hour it is observed that 10% has disintegrated. Determine the amount remaining after a further 2 hours.
8. The rate of decay of a radioactive substance is proportional to the amount of the substance present at that time. After 2 hours, 64 milligrams of a radioactive substance is present, and after 4 hours, 36 milligrams is present. Determine the initial amount of the substance present.
9. **WE3** For a radioactive substance, it takes 3 years for 15% to disintegrate. Determine the half-life of the substance.
10. An isotope of radium has a half-life of 1601 years. What percentage remains after 1000 years?
11.
 - a. The rate of decay of a radioactive substance is proportional to the amount of the substance present at any time. If 1 gram of radioactive material decomposes to half a gram in 22 652 years, how much remains after 40 000 years?
 - b. If 15% of a radioactive element disintegrates in 6 years, determine the half-life of this element.



12. Iodine-131 is present in radioactive waste and decays exponentially to form a substance that is not radioactive. The half-life of iodine-131 is 8 days. If 120 milligrams is considered a safe level, how long will it take 2 grams of iodine-131 to decay to a safe level?
13. Strontium-90 is an unpleasant radioactive isotope that is a byproduct of a nuclear explosion. Strontium-90 has a half-life of 28.9 years. How many years elapse before 60% of the strontium-90 in a sample has decayed?
14. Marie Curie (1867–1934) was a French scientist. She coined the term ‘radioactivity’ and was also the first female scientist to win a Nobel Prize. Unfortunately she died of radiation poisoning as a result of her experiments. In 1944 the element Curium was named after her.
- a. Curium-243 has a half-life of 29.1 years. It decays into plutonium-239 through alpha decay. Determine the percentage of curium-243 that remains in a sample after 10 years.
- b. Plutonium-239 is a silvery metal that is used for the production of nuclear weapons. If 3 micrograms of plutonium-239 decomposes to 1 microgram in 38 213 years, determine the half-life of plutonium-239.
15. Uranium-238 has a half-life of 4.468 billion years. Estimate the percentage of the original amount of uranium-238 that remains in the universe, assuming we began with that original amount 13.8 billion years ago.
16. A biologist is investigating two different types of bacteria. Both types grow at a rate proportional to the number present. Initially 900 type A bacteria are present, and after 10 hours 3000 are present. The number of type B bacteria doubles after 4 hours, and there are 2500 type B bacteria after 8 hours. Determine the time at which equal numbers of type A and type B bacteria are present.
17. The table below shows the comparative population in millions of Brisbane and Melbourne, in certain years.



City	2001	2009
Melbourne	3.339	3.996
Brisbane	1.609	2.004

Assume the growth rates are proportional to the current population and these growth rates continue.

- a. Show that Brisbane’s population growth rate is greater than Melbourne’s population growth rate.
- b. In what year will the population of Melbourne first exceed 8 million?
- c. In what year will the population of Brisbane first exceed 8 million?
- d. In what year will the population of Brisbane exceed the population of Melbourne?
18. In the 1950s, W F Libby and others at the University of Chicago devised a method of estimating the age of organic material based on the decay rate of carbon-14, a radioactive isotope of carbon. Carbon-14 dating can be used on objects ranging from a few hundred years old to 40 000 years old. Carbon-14 obeys the law of radioactive decay. If Q denotes the amount of carbon-14 present at time t years, then

$$\frac{dQ}{dt} = -kQ.$$



- a. If Q_0 represents the amount of carbon-14 present at time t_0 and k is a constant, express Q in terms of Q_0 , t_0 and k .
- b. Given that the half-life of carbon-14 is 5730 years, determine the value of k .

In residual amounts of carbon-14 it is possible to measure $\frac{Q}{Q_0}$ for some wood and plant remains, and

hence determine the elapsed time since the death of these remains, that is, the period during which decay has been taking place. This technique of radiocarbon dating is of great value to archaeologists.

- c. For a particular specimen, $\frac{Q}{Q_0} = \frac{2}{7}$. Determine the age of the specimen.

9.3 Other applications of first-order differential equations

9.3.1 Miscellaneous types

There are many other applications of first-order differential equations that are similar to the growth and decay models.

Investments and money matters

When an amount of money, $\$P$, is invested with continuously compounding interest at $R\%$, the rate at which the accumulated amount grows is proportional to the amount invested. In fact, the proportionality constant is the interest, so $\frac{dP}{dt} = Pr$, where t is the time ($t \geq 0$) and $r = \frac{R}{100}$.

Pressure and height in the atmosphere

The rate of decrease of pressure in the atmosphere, P , with respect to the height above sea level, h , is proportional to the pressure at that height. For a proportionality constant k (such that $k > 0$), $\frac{dP}{dh} = -kP$.

Drug disappearance in the body

For many drugs, the rate of disappearance from the body is proportional to the amount, D , of the drug still present in the body at time t . If k is the proportionality constant ($k > 0$), $\frac{dD}{dt} = -kD$.

Light intensity and depth

When a beam of light passes through a medium, it loses its intensity as it penetrates more deeply. If I is the intensity of light at depth x , the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If k is the proportionality constant ($k > 0$), then $\frac{dI}{dx} = -kI$.

Other applications

There are also applications related to electrical circuits. Applications to kinematics, dynamics and variable forces are considered in Chapter 11.

WORKED EXAMPLE 4

When money is invested in a bank at a constant rate of $R\%$ with continuously compounding interest, the accumulated amount $\$P$ at a time t years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = Pr$.



- a. Assuming an initial investment of $\$P_0$, solve the differential equation to determine the amount $\$P$ after a time t years.
- b. Determine the amount to which $\$10\,000$ will grow in 6 years if invested at 4.5%.

THINK

- a. 1. To solve the differential equation, invert both sides.
The initial investment is $\$P_0$.

2. Because r is a constant, it can be taken outside the integral sign.

3. Integrate.

4. Substitute $t = 0$, $P = P_0$.

5. Rewrite the equation, substituting $c = -\frac{1}{r} \ln(P_0)$, and use logarithm laws to simplify.

6. Rearrange and write as an exponential.

7. Solve for P and state the solution of the differential equation.

- b. 1. Substitute $P_0 = 10\,000$ and $r = 4.5\% = \frac{4.5}{100}$.

2. Determine the amount after 6 years.

3. State the total amount.

WRITE

$$\text{a. } \frac{dP}{dt} = Pr, P(0) = P_0$$

$$\frac{dt}{dP} = \frac{1}{Pr}$$

$$t = \frac{1}{r} \int \frac{1}{P} dP$$

$$t = \frac{1}{r} \ln(|P|) + c$$

$$= \frac{1}{r} \ln(P) + c, \text{ as } P > 0$$

$$0 = \frac{1}{r} \ln(P_0) + c$$

$$c = -\frac{1}{r} \ln(P_0)$$

$$t = \frac{1}{r} \ln(P) - \frac{1}{r} \ln(P_0)$$

$$t = \frac{1}{r} \ln\left(\frac{P}{P_0}\right)$$

$$rt = \ln\left(\frac{P}{P_0}\right)$$

$$\frac{P}{P_0} = e^{rt}$$

$$P = P_0 e^{rt}$$

$$\text{b. } P(t) = 10\,000 e^{\frac{4.5t}{100}}$$

$$P(6) = 10\,000 e^{\frac{4.5 \times 6}{100}}$$

$$= 13099.64$$

The principal has grown to $\$13\,099.64$ after 6 years.

9.3.2 Other population models

In the natural growth model, the growth rate of a population is proportional to the current population. However, population models can have other growth rates. For example, the growth rate may be proportional to a power of the current population other than 1. In these models, the same approaches can be used as for natural growth.

WORKED EXAMPLE 5

The population of a certain town increases at a rate proportional to the square root of the current population. In 2009 the population was 250 000, and in 2017 the population was 640 000.

Express the population number, N , in terms of t , the time in years after 2009. What is the predicted population in 2025?

THINK

1. Let $N = N(t)$ represent the population number of the town after 2009.
2. To solve the differential equation, invert both sides and write in index notation.
3. Integrate both sides with respect to N and multiply through by k .
4. Integrate.
5. Substitute $t = 0$ and $N = 250\,000$ into the equation.
6. Substitute $c = -1000$.
7. Substitute $t = 8$ and $N = 640\,000$ to solve for k .
8. Substitute $k = 75$ into the equation $kt = 2\sqrt{N} - 1000$ and rearrange to make N the subject.
9. State the particular solution of the differential equation.
10. Substitute $t = 16$.
11. State the final result.

WRITE

$$\frac{dN}{dt} = k\sqrt{N}, \quad N(0) = 250\,000$$

$$\frac{dt}{dN} = \frac{1}{k\sqrt{N}}$$

$$\frac{dt}{dN} = \frac{1}{k}N^{-\frac{1}{2}}$$

$$\int 1 dt = \frac{1}{k} \int N^{-\frac{1}{2}} dN$$

$$kt = \int N^{-\frac{1}{2}} dN$$

$$kt = 2N^{\frac{1}{2}} + c$$

$$kt = 2\sqrt{N} + c$$

$$0 = 2\sqrt{250\,000} + c$$

$$c = -1000$$

$$kt = 2\sqrt{N} - 1000$$

$$8k = 2\sqrt{640\,000} - 1000$$

$$8k = 600$$

$$k = 75$$

$$75t = 2\sqrt{N} - 1000$$

$$2\sqrt{N} = (75t + 1000)$$

$$\sqrt{N} = \frac{1}{2}(75t + 1000)$$

$$N(t) = \frac{1}{4}(75t + 1000)^2$$

$$N(16) = \frac{1}{4}(75 \times 16 + 1000)^2$$

$$= 1\,210\,000$$

The predicted population in 2025 is 1 210 000.

9.3.3 Population models with regular removal

Another model that can limit but not bound the population is to remove some of the population at regular intervals. This has applications for situations such as people emigrating from a country or leaving a certain city yearly. The growth rate k still represents the excess of births over deaths and is often given as a percentage.

WORKED EXAMPLE 6

A farm initially has 200 sheep. The number of sheep on the farm grows at a rate of 10% per year. Each year the farmer sells 15 sheep.

- Write the differential equation modelling the number of sheep, N , on the farm after t years.
- Solve the differential equation to express the number of sheep, N , on the farm after t years.
- Hence, determine the number of sheep on the farm after 5 years.



THINK

- Let $N(t)$ be the number of sheep on the farm at time t years. The growth rate is 10%, and 15 are sold each year.
 - Write the differential equation in simplest form, along with the initial condition. The initial number of sheep is 200.
1. To solve the differential equation, invert both sides.
 - Integrate both sides.
 - Perform the integration.
 - Substitute $t = 0$ and $N = 200$.
 - Substitute $c = -10 \ln(50)$ and use log laws to simplify.
 - Rearrange and solve for N .
 - State the solution of the differential equation.

WRITE

$$\begin{aligned}
 \text{a. } k &= 0.1 \\
 \frac{dN}{dt} &= 0.1N - 15 \\
 \frac{dN}{dt} &= \frac{N}{10} - 15 \\
 \frac{dN}{dt} &= \frac{N - 150}{10}, N(0) = 200 \\
 \text{b. } \frac{dt}{dN} &= \frac{10}{N - 150} \\
 t &= \int \frac{10}{N - 150} dN \\
 t &= 10 \ln(|N - 150|) + c \\
 &= 10 \ln(N - 150) + c, \text{ as } N > 150 \\
 0 &= 10 \ln(200 - 150) + c \\
 c &= -10 \ln(50) \\
 t &= 10 \ln(N - 150) - 10 \ln(50) \\
 t &= 10 [\ln(N - 150) - \ln(50)] \\
 \frac{t}{10} &= \ln\left(\frac{N - 150}{50}\right) \\
 \frac{N - 150}{50} &= e^{\frac{t}{10}} \\
 N - 150 &= 50e^{\frac{t}{10}} \\
 N &= 150 + 50e^{\frac{t}{10}} \\
 N &= 50 \left(3 + e^{\frac{t}{10}}\right)
 \end{aligned}$$

c. 1. Substitute $t = 5$.

$$c. N(5) = 50 \left(3 + e^{\frac{1}{2}} \right) \approx 232.44$$

2. State the number of sheep present after 5 years.
Note that we always round down, and express the answer as an integer.

There are 232 sheep present after 5 years.

studyon

Units 3 & 4

Area 5

Sequence 2

Concept 3

Other applications of first-order differential equations Summary screen and practice questions

Exercise 9.3 Other applications of first-order differential equations

Technology active

1. **WE4** When money is invested in a bank at a constant rate of $R\%$ with continuously compounding interest, the accumulated amount $\$P$ at a time t years after the start of the investment satisfies the differential equation $\frac{dP}{dt} = P \times r$. What initial investment is required if $\$10\,000$ is the target in 2 years' time and the interest rate is 5% ?
2. The price of houses in a certain area grows at a rate proportional to their current value. In 2002 a particular house was purchased for $\$315\,000$; in 2018 the house was sold for $\$1\,260\,000$. What was the value of the house in 2010?
3.
 - a. The rate of decrease of air pressure, P , with respect to the height above sea level, h , is proportional to the pressure at that height. The constant of proportionality is k such that $k > 0$. If P_0 is the air pressure at sea level, solve the differential equation to determine the pressure at height h .
 - b. If the air pressure at sea level is 76 cm of mercury, and the pressure at a height of 1 kilometre above sea level is 62.2235 cm of mercury, determine the pressure at a height of 2 kilometres above sea level.
4.
 - a. For many drugs, the rate of disappearance from the body is proportional to the amount of the drug still present in the body. If $D = D(t)$ is the amount of the drug present at time t and k is the constant of proportionality where $k > 0$, solve the differential equation expressing D in terms of t given the initial dosage of D_0 .
 - b. The elimination time for alcohol (the time it takes for a person's body to remove alcohol from their bloodstream) is measured as $T = \frac{1}{k}$. The process varies from person to person, so the value of k varies. For one person the elimination time is 3 hours. For this person, how long will it take the excess level of alcohol in their bloodstream to be reduced from 0.10% to the legal adult driving level of 0.049% ?
5. When light passes through a medium, it loses its intensity as it penetrates deeper. If I is the intensity of light at depth x , the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth, so that $\frac{dI}{dx} = -kI$, where k is a positive constant.
 - a. Solve the differential equation to show that $I = I_0 e^{-kx}$.
 - b. If 5% of light is lost in penetrating 25% of a glass slab, what percentage is lost in penetrating the whole slab?

6. When a rope is wrapped around a pole, the rate of change of tension, T (in newtons), in the rope with respect to the angle, θ (in radians), is proportional to the tension at that instant. The constant of proportionality is the coefficient of friction, μ .
- Given that $\frac{dT}{d\theta} = \mu T$ and the tension when the angle is zero is T_0 , show that $T = T_0 e^{\mu\theta}$.
 - A man holds one end of a rope with a pull of 80 newtons. The rope goes halfway around a tree, and the coefficient of friction in this case is 0.3. What force can the other end sustain?
7. **WES** The population of a country increases at a rate proportional to the square root of the current population. In 2013 the population was 4 million, and in 2018 the population was 9 million. Express the population number, N , in terms of t , the time in years after 2013. What is the predicted population in 2028?
8. The number of insects in a colony increases at a rate inversely proportional to the current number. At first there are 20 insects present, and this number grows to 80 after 5 months. Determine the number of insects present in the colony after a further 11 months.
9. **a.** The variation of resistance, R ohms, of a copper conductor with temperature T degrees Celsius satisfies the differential equation $\frac{dR}{dT} = \alpha R$.
- If the resistance of copper is R_0 at 0°C , show that $R = R_0 e^{\alpha T}$.
 - If $\alpha = 0.004$ per degree Celsius and the resistance at 60°C is 40 ohms, determine the resistance at 30°C .
- b.** The charge, Q units, on a plate of a condenser t seconds after it starts to discharge is proportional to the charge at that instant. If the charge is 500 units after a half-second and falls to 250 units after 1 second, determine:
- the original charge
 - the time needed for the charge to fall to 125 units.
10. **a.** In an electrical circuit consisting of a resistance of R ohms and an inductance of L henries, the current, i amperes, after a time t seconds satisfies the differential equation $L \frac{di}{dt} + Ri = 0$. The initial current is i_0 . Solve the differential equation to determine the current at any time t .
- b.** In an electrical circuit consisting of a resistance of R ohms and a capacitance of C farads, the charge, Q coulombs, at a time t seconds decays according to the differential equation $\frac{dQ}{dt} + \frac{Q}{RC} = 0$. Assuming the initial charge is Q_0 , solve the differential equation to determine the charge at any time t after discharging.
11. **WES** A certain area initially contains 320 rabbits. The number of rabbits in the area grows at a rate of 25% per year. Each year 40 rabbits are culled to try to limit the population.
- Write the differential equation modelling the number of rabbits in the area, N , after t years.
 - Solve the differential equation to express the number of rabbits in the area after t years.
 - Hence, determine the number of rabbits in the area after 8 years.
12. Another area initially contains N_0 rabbits. The number of rabbits in this area grows at a rate of 20% per year. Each year K rabbits are culled to limit the population. After 10 years, there are $6K$ rabbits present.
- Show that $N_0 = K \left(5 + \frac{1}{e^2} \right)$.



13. A farm initially has 200 cows. The number of cows on the farm grows at a rate of 5% per year. Each year the farm sells 5 cows.

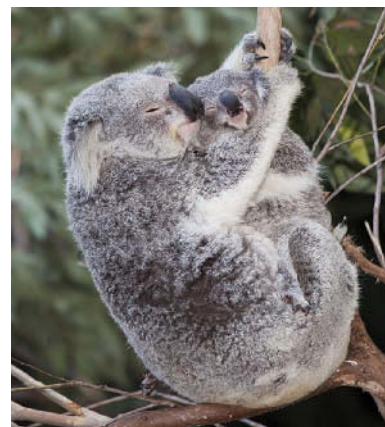
- Write the differential equation modelling the number of cows, N , on the farm after t years.
- Solve the differential equation to express the number of cows on the farm after t years.
- Hence, determine the number of cows on the farm after 3 years.



14. A certain country has a 4% population growth rate, and additionally every year 10 000 immigrate to the country. If in 2019 the population of the country was half a million, estimate the population in 2024.

15. Koalas on a plantation have a population growth rate of 10%. Initially 400 koalas are present.

- Every year 50 koalas are sold to zoos. How many years elapse before the number of koalas is reduced to 200?
- After the population has reached 200, what is the maximum number of koalas the managers of the plantation can sell to zoos each year so that the number of koalas at the plantation will increase above 200?



16. Solve the differential equation $\frac{dN}{dt} = kN^\alpha$, $N(0) = N_0$ for when:

- $\alpha = \frac{1}{2}$
- $\alpha = -1$
- $\alpha = \frac{3}{2}$
- $\alpha = 2$.

17. a. Show that the solution of the differential equation $\frac{dN}{dt} = kN - c$, where k and c are positive constants

and $N(0) = N_0$, is given by $N(t) = \left(N_0 - \frac{c}{k}\right)e^{kt} + \frac{c}{k}$.

- b. Show that:

i. N increases if $N_0 > \frac{c}{k}$

ii. N decreases if $N_0 < \frac{c}{k}$

iii. if $N_0 = \frac{c}{k}$, then N remains stable.

18. The number of bacteria in a culture increases at a rate proportional to the number of bacteria present in the culture at any time. By natural increase, the culture will double in $\ln(8)$ days. If the initial number of bacteria present is N_0 and bacteria are removed from the colony at a constant rate of Q per day, show that the number of bacteria:

- will increase if $N_0 > 3Q$
- will remain stationary if $N_0 = 3Q$
- will decrease if $N_0 < 3Q$.

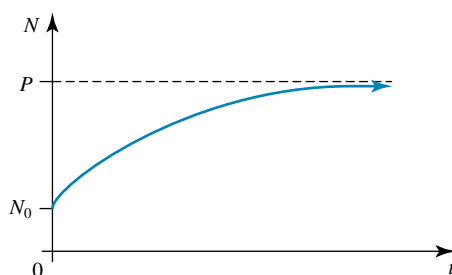
9.4 Bounded growth and Newton's law of cooling

9.4.1 Bounded growth models

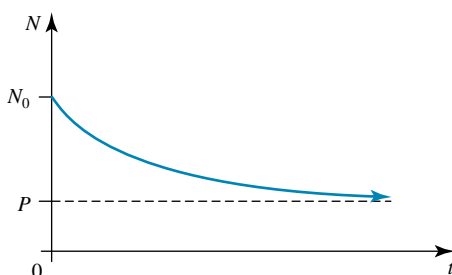
If $N = N(t)$ is the population, then the law of exponential growth states that $\frac{dN}{dt} = kN$, where k is a positive constant. Assuming that the initial population number is $N_0 = N(0)$, the solution of this differential equation is given by $N = N_0 e^{kt}$. This model predicts that, as time increases, the population number increases without bounds; that is, as $t \rightarrow \infty$, $N \rightarrow \infty$. Usually the growth of a quantity is not unlimited but is instead bounded; that is, the quantity approaches some maximum value or equilibrium value, and growth cannot occur above this value. This is called bounded growth. In such cases, the growth rate is proportional to the difference between the population number and the equilibrium value.

The differential equation to model this type of behaviour is $\frac{dN}{dt} = k(P - N)$, where $k > 0$ and $P > 0$. P is called the carrying capacity or the equilibrium value.

Assuming an initial condition $N_0 = N(0)$, if $N_0 < P$, then as $t \rightarrow \infty$, $N \rightarrow P$ from below.



However, if $N_0 > P$, then as $t \rightarrow \infty$, $N \rightarrow P$ from above.



WORKED EXAMPLE 7

The mass of medium-sized dogs are found to follow a bounded growth model. The initial mass of a typical newborn puppy is 0.5 kg, and after 16 weeks its mass has increased to 4.5 kg. It is known that the dog's mass will never exceed 15 kg.

- Determine the mass of the dog after one year.
- Sketch the graph of mass versus time for this model.

THINK

- a. 1. Let $m = m(t)$ be the mass in kg of the dog after t weeks. The equilibrium value is $P = 15$ and the initial mass is 0.5 kg. State the differential equation to model the mass.
2. To solve this differential equation, invert both sides.
3. Separate the variables and multiply by the constant k .
4. Perform the integration.
5. The initial mass of a newborn puppy is 0.5 kg. Use this to determine the value of c .
6. Substitute $c = \ln(14.5)$ and simplify.
7. After 16 weeks the mass is 4.5 kg. Use this to determine the value of k .
8. Leave the equation in terms of k and use the definition of the logarithm to rearrange.
9. Rearrange to make m the subject.
10. State the particular solution of the differential equation.
11. Now determine the mass of the dog after one year, that is $t = 52$ weeks.
12. State the final result.

WRITE/DRAW

a. $\frac{dm}{dt} = k(P - m)$

$$\frac{dm}{dt} = k(15 - m), m(0) = 0.5$$

$$\frac{dt}{dm} = \frac{1}{k(15 - m)}$$

$$t = \frac{1}{k} \int \frac{1}{15 - m} dm$$

$$kt = \int \frac{1}{15 - m} dm$$

$$kt = -\ln(|15 - m|) + c$$

$$kt = -\ln(15 - m) + c \text{ as } 0.5 < m < 15$$

$$m(0) = 0.5:$$

$$0 = -\ln(15 - 0.5) + c$$

$$c = \ln(14.5)$$

$$kt = -\ln(15 - m) + \ln(14.5)$$

$$kt = \ln\left(\frac{14.5}{15 - m}\right)$$

$$m(16) = 4.5:$$

$$16k = \ln\left(\frac{14.5}{15 - 4.5}\right)$$

$$k = \frac{1}{16} \ln\left(\frac{29}{21}\right)$$

$$kt = \ln\left(\frac{14.5}{15 - m}\right)$$

$$e^{kt} = \frac{14.5}{15 - m}$$

$$(15 - m)e^{kt} = 14.5$$

$$15 - m = 14.5e^{-kt}$$

$$m = 15 - 14.5e^{-kt}$$

$$m(t) = 15 - 14.5e^{-\frac{t}{16} \ln\left(\frac{29}{21}\right)}$$

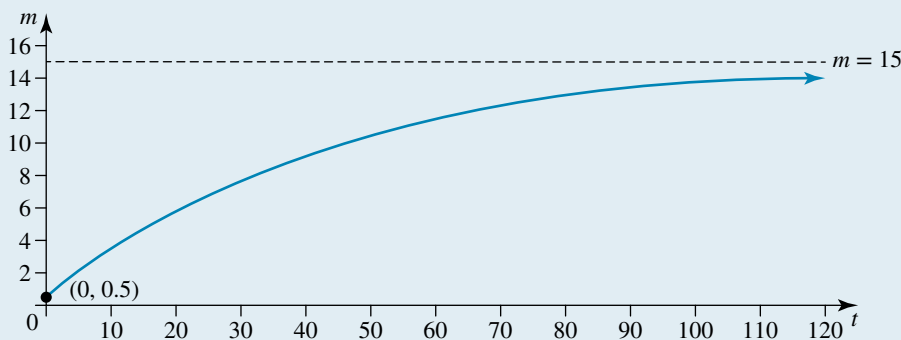
$$m(52) = 15 - 14.5e^{-\frac{52}{16} \ln\left(\frac{29}{21}\right)}$$

$$\approx 9.92$$

The mass of the dog after one year is 9.92 kg.



- b. Sketch the graph of m versus t .



9.4.2 Newton's law of cooling

Newton's law of cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the surrounding medium. Let T denote the temperature of a body after a time t and T_m denote the temperature of the surrounding medium, which is assumed to be constant. Newton's law of cooling can be written as $\frac{dT}{dt} \propto (T - T_m)$ or $\frac{dT}{dt} = k(T - T_m)$, where k is a constant of proportionality. If the body is cooling, then $k < 0$.

To solve this differential equation, let θ be the difference in temperature, so that $\theta = T - T_m$. Because T_m is a constant, $\frac{dT}{dt} = \frac{d\theta}{dt}$; therefore, $\frac{d\theta}{dt} = k\theta$. The particular solution of the differential equation is $\theta = \theta_0 e^{kt}$, where $\theta(0) = \theta_0 = T_0 - T_m$ and T_0 is the initial temperature of the body. Newton's law of cooling is another example of a bounded growth or decay model.

WORKED EXAMPLE 8

A metal ball is heated to a temperature of 200°C and is then placed in a room that is maintained at a constant temperature of 30°C . After 5 minutes the temperature of the ball has dropped to 150°C . Assuming Newton's law of cooling applies, determine:

- the equation for the difference between the temperature of the ball and the temperature of the room
- the temperature of the ball after a further 10 minutes
- how long it will take for the temperature of the ball to reach 40°C .

THINK

- Let T denote the temperature of the ball after a time t minutes. The constant room temperature is $T_m = 30^\circ\text{C}$.
- Let θ be the difference in temperature between the ball and the room.
- Write the solution of the particular differential equation.

WRITE

$$\text{a. } \frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T - 30)$$

$$\theta = T - 30$$

$$\frac{d\theta}{dt} = k\theta$$

$$\theta = \theta_0 e^{kt}$$

4. To determine the value of θ_0 , use the given initial conditions.
5. To determine the value of k , use the other given condition.
6. Substitute in the given values.
7. Rearrange and use the definition of the logarithm.
8. Solve for k .
9. Write the solution of the differential equation.
- b. 1. Substitute $t = 15$ to determine the temperature after a further 10 minutes.
2. State the temperature of the ball at this time.
- c. 1. To determine when the temperature of the ball reaches 40°C , use $T = 40$.
2. Write an equation involving the unknown time.
3. Solve the equation for t , using the definition of the logarithm.
4. State the required result.
- Initially, $t = 0$ and $T_0 = 200^\circ\text{C}$.

$$\begin{aligned}\theta_0 &= T_0 - T_m \\ &= 200 - 30 \\ &= 170\end{aligned}$$
When $t = 5$, $T = 150^\circ\text{C}$, so

$$\begin{aligned}\theta_5 &= 150 - 30 \\ &= 120\end{aligned}$$

$$\theta_0 = 170, \theta = 120 \text{ and } t = 5:$$

$$120 = 170e^{5k}$$

$$\frac{120}{170} = e^{5k}$$

$$5k = \ln\left(\frac{12}{17}\right)$$

$$k = \frac{1}{5} \ln\left(\frac{12}{17}\right)$$

$$\theta(t) = 170e^{\frac{t}{5} \ln\left(\frac{12}{17}\right)}$$
b.
$$\begin{aligned}\theta(15) &= 170e^{\frac{15}{5} \ln\left(\frac{12}{17}\right)} \\ &\approx 59.79\end{aligned}$$
Since $T = \theta + 30$, the temperature of the ball is 89.79°C .
c.
$$\begin{aligned}T &= \theta + 30 \\ 40 &= \theta + 30 \\ \theta &= 40 - 30 \\ \theta &= 10\end{aligned}$$
Substitute $\theta = 10$

$$10 = 170e^{\frac{t}{5} \ln\left(\frac{12}{17}\right)}$$

$$\frac{10}{170} = e^{\frac{t}{5} \ln\left(\frac{12}{17}\right)}$$

$$\frac{t}{5} \ln\left(\frac{12}{17}\right) = \ln\left(\frac{10}{170}\right)$$

$$t = \frac{5 \ln\left(\frac{10}{170}\right)}{\ln\left(\frac{12}{17}\right)}$$

$$\approx 40.67$$
The temperature of the ball reaches 40°C after 40.67 minutes.

study on

Units 3 & 4 > Area 5 > Sequence 2 > Concepts 4 & 5

Bounded growth Summary screen and practice questions

Newton's law of cooling Summary screen and practice questions

Exercise 9.4 Bounded growth and Newton's law of cooling

Technology active

1. **WE7** The mass of a bird is found to follow a bounded growth model. The initial mass of a baby bird is 30 grams, and after 10 weeks its mass is 100 grams. If it is known that the mass of the bird will never exceed 200 grams, determine the mass of the bird after 30 weeks.
2. The number of birds in a flock follows a bounded growth model. Initially there are 200 birds in the flock. After 5 months the number has grown to 800. The number of birds in the flock can never exceed 3000. After how many months are there 1500 birds in the flock?
3.
 - a. The mass of cats are found to follow a bounded growth model. The initial mass of a typical newborn cat is 0.1 kg, and after 30 weeks the cat's mass has increased to 4 kg. If it is known that the cat's mass will never exceed 5 kg, determine the mass of the cat after 40 weeks.
 - b. The mass of toy poodle dogs are found to follow a bounded growth model. The initial mass of a puppy toy poodle is 0.2 kg, and after 14 weeks its mass has increased to 1.3 kg. If it is known that the dog's mass will never exceed 4.5 kg, determine the mass of the toy poodle after one year.
4. The number of fish in a lake follows a bounded growth model. Initially there are 50 fish in the lake, and after 10 months the number of fish has increased to 500. The number of fish in the lake can never exceed 1000.
 - a. Determine the number of fish in the lake after 20 months.
 - b. After how many months are there 900 fish in the lake?
5. A woman is on a diet. Her initial mass is 84 kg, and she knows that her mass will always be above 70 kg. After 10 weeks she has lost 7 kg. Assuming a bounded decay model, what is her total mass loss after 20 weeks?
6. **WE8** A warm can of soft drink at a temperature of 25°C is chilled by placing it in a refrigerator at a constant temperature of 2°C . If the temperature of the drink falls to 22°C in 5 minutes, determine how long it will take for the temperature of the drink to reach 13°C .
7. An iron is preheated to a temperature of 250°C . After 6 minutes the temperature of the iron is 210°C . Brent is ironing in a room that is kept at a constant temperature of 20°C . Brent knows that it is best to iron a cotton T-shirt when the temperature of the iron is between 195°C and 205°C . How long does Brent have to iron the T-shirt?
8.
 - a. A cold can of soft drink is taken from a refrigerator at a temperature of 3°C and placed in a sunroom at a temperature of 30°C . If after 2 minutes the temperature of the can is 4°C , determine its temperature after a further 3 minutes.
 - b. A body at an unknown temperature is placed in a room that has a temperature of 18°C . If after 10 minutes the body has a temperature of 22°C , and after a further 10 minutes its temperature is 20°C , determine the initial temperature of the body.



9. A mother is giving her baby a bath. The bathtub contains hot water that is initially at a temperature of 42°C in a room where the temperature is constant at 20°C . The water cools, and after 2 minutes its temperature is 40°C . The mother knows that babies like to be bathed when the temperature of the water is between 34°C and 38°C , otherwise the bath water is either too hot or too cold. How long should the baby stay in the bath?
10. The temperature of a room is 25°C . A thermometer which was in the room is taken outdoors and in five minutes it reads 15°C . Five minutes later the thermometer reads 10°C . What is the temperature outdoors?
11. A frozen chicken should be thawed before cooking. A freezer is maintained at a constant temperature of -18°C . A chicken is taken out of the freezer at 8:00 am and placed on a kitchen bench. At 11:00 am the temperature of the chicken is 0°C , and at 2:00 pm its temperature is 10°C . At 5:00 pm the chicken has thawed and is placed in an oven which has been preheated to a temperature of 200°C . At 6:00 pm the temperature of the chicken in the oven is 70°C . Determine the temperature of the chicken when it is removed from the oven at 7:00 pm.
12. a. An RL series circuit consists of a resistance, R ohms, and an inductance, L henries, connected to a voltage source, E volts. The rise of current, i amperes, after a time t seconds satisfies the differential equation $L \frac{di}{dt} + Ri = E$.
- Assuming the initial current is zero, determine the current at any time t .
 - Show that the time required for the current to reach half its ultimate value is given by $\frac{L}{R} \ln\left(\frac{1}{2}\right)$.
- b. A capacitor of C farads is charged by applying a steady voltage of E volts through a resistance of R ohms. The potential difference, v volts between the plates satisfies the differential equation $RC \frac{dv}{dt} + v = E$. Assuming the initial voltage is zero, determine the voltage v at any time t .
13. Show that the particular solution of the differential equation $\frac{dN}{dt} = k(P - N)$, $N(0) = N_0 > 0$, where $P > 0$, is given by $N(t) = P + (N_0 - P)e^{-kt}$. Consider both cases when $N_0 > P$ and $N_0 < P$.
14. a. Show that the solution of the differential equation $\frac{dT}{dt} = k(T - T_m)$, where k and T_m are positive constants and $T(0) = T_0$, is given by $T(t) = T_m + (T_0 - T_m)e^{kt}$.
- b. Police discover a dead body in a hotel room at 8:00 am. At that time its temperature is 30.4°C . One hour later the temperature of the body is 29.9°C . The room was kept at a constant temperature of 17°C overnight. If normal body temperature is 37°C , determine the time of death.
15. A refrigerator is kept at a constant temperature of 3°C . A baby's bottle is taken from the fridge, but the bottle is too cold to give to the baby. The mother places the bottle in a saucepan full of boiling water (maintained at 95°C) for 90 seconds. When the bottle is removed from the saucepan, its temperature is 45°C .
- a. Let $H = H(t)$ be the temperature of the bottle at a time t minutes while it is being heated. If $\frac{dH}{dt} = k_1(H - T_h)$, where $k_1 < 0$ for $0 \leq t \leq t_1$, state the values of t_1 and T_h .



- b. The mother knows that the bottle should be at a temperature of 35°C for the baby to drink it. She realises that the bottle is now unfortunately too hot to give to the baby, so she places it back in the fridge to cool. She takes the bottle out of the fridge after 45 seconds. It is now at the correct temperature to give to the baby. Let $C = C(t)$ be the temperature of the bottle at a time t minutes after it is placed back in the fridge. If $\frac{dC}{dt} = k_2(C - T_c)$, where $k_2 < 0$ for $0 \leq t \leq t_2$, state the values of t_2 and T_c .



- c. Determine the ratio of $\frac{k_1}{k_2}$.

16. Let $\$P_0$ be the initial amount of money borrowed from a lender at an annual interest rate of $r\%$. Assume that the interest is compounded continuously and that the borrower makes equal payments of $\$m$ per year to reduce the amount borrowed. If $\$P$ is the amount of money owing at time t , the differential equation modelling the repayments is given by $\frac{dP}{dt} = rP - m$.

- a. If the loan is completely paid off after a time T years, show that $\frac{m}{m - rP_0} = e^{rT}$.
- b. If Jared borrows $\$300\,000$ for a house loan at 6% per annum and pays off the loan after 20 years, determine his monthly repayments and the total amount of interest paid.
- c. Sharon can afford to pay $\$350$ a month over 5 years at 8% per annum compounded continuously. What is the maximum amount she can borrow for a car loan?
- d. Ashley wants to buy a block of land and will have to borrow $\$120\,000$ to do it. He can afford to make monthly repayments of $\$1200$ on the loan, compounded continuously at 6.3% . How long will it take Ashley to pay off the loan?
- e. Ryan pays off a loan of $\$8000$ by paying back $\$150$ per month over 6 years. Assuming the interest is compounded continuously, determine the annual interest rate.

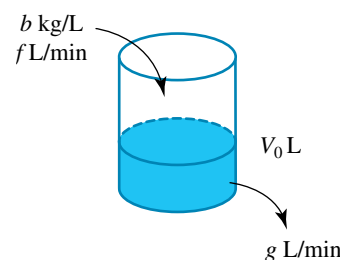


9.5 Chemical reactions and dilution problems

9.5.1 Input–output mixing problems

Consider a tank that initially holds V_0 litres of a solution in which a kilograms of salt have been dissolved. Another solution containing b kilograms of salt per litre is poured into the tank at a rate of f litres per minute. The well-stirred mixture leaves the tank at a rate of g litres per minute. The problem is to determine the amount of salt in the tank at any time t .

Let Q denote the amount of salt in kilograms in the tank at a time t minutes. The time rate of change of Q is equal to the rate at which the salt flows into the tank minus the rate at which the salt flows out of the tank; that is,



Differential equation for mixing problems

$$\frac{dQ}{dt} = \text{inflow rate} - \text{outflow rate}$$

We know b kilograms of salt per litre is being poured into the tank at a rate of f litres per minute; multiplying these together gives the inflow rate of the salt as bf kilograms per minute.

To determine the outflow rate, first we determine the volume $V(t)$ of the tank at time t . The initial volume is V_0 , but f litres per minute flow in, while g litres per minute flow out. Therefore, the volume at time t minutes is given by $V(t) = V_0 + (f - g)t$. To get the required units, we multiply the solution's outflow rate of g litres per minute by the mass and divide by the volume to get the outflow rate of the salt as

$\frac{gQ}{V_0 + (f - g)t}$ kilograms per minute. Therefore, the differential equation is

$$\frac{dQ}{dt} = bf - \frac{Qg}{V_0 + (f - g)t}, \text{ and } Q(0) = a.$$

(Note that concentration, c , is the mass divided by the volume, $c = \frac{\text{mass}}{\text{volume}}$.)

When $g \neq f$, this equation cannot be solved by the techniques of integration studied in this course. However, if we are given a solution, differentiation and substitution can be used to verify the solution.

WORKED EXAMPLE 9

A tank has a capacity of 300 litres and contains 200 litres of water in which 100 kilograms of salt have been dissolved. A salt solution of concentration 2 kilograms is poured into the tank at a rate of 3 litres per minute, and the well-stirred mixture flows out at a rate of 2 litres per minute.

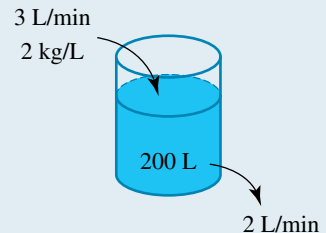
a. State the differential equation for Q , the amount of salt in kilograms in the tank after t minutes.

b. Verify that $Q = 2(200 + t) + \frac{C}{(200 + t)^2}$

is a general solution of the differential equation.

c. Determine the value of C .

d. Determine the concentration of salt in kilograms per litre when the tank overflows.



THINK

- a. 1. The initial volume in the tank is 200 litres; 3 litres per minute flows in and 2 litres per minute flows out. Let $V(t)$ be the volume in litres at time t minutes.
2. The inflow rate is the concentration $\times$ the inflow volume rate.
3. The output flow rate is the outflow concentration $\times$ the outflow volume rate.

WRITE

$$\begin{aligned} \text{a. } V(t) &= 200 + (3 - 2)t \\ &= 200 + t \end{aligned}$$

$$\text{Inflow rate} = 3 \times 2 \text{ kg/min} = 6 \text{ kg/min}$$

$$\begin{aligned} \text{Outflow rate} &= \frac{2Q}{V(t)} \text{ kg/min} \\ &= \frac{2Q}{200 + t} \text{ kg/min} \end{aligned}$$

4. State the required differential equation along with the initial condition.

$$\frac{dQ}{dt} = \text{inflow rate} - \text{output rate}$$

$$= 6 - \frac{2Q}{200 + t} \quad [1]$$

$$Q(0) = 100$$

- b. 1. To verify the given solution, first write the solution in index form.

b. $Q = 2(200 + t) + \frac{C}{(200 + t)^2} \quad [2]$

$$= 2(200 + t) + C(200 + t)^{-2}$$

2. Now differentiate the given solution as the left-hand side.

$$\text{LHS} = \frac{dQ}{dt}$$

$$= 2 - 2C(200 + t)^{-3}$$

$$= 2 - \frac{2C}{(200 + t)^3}$$

3. Substitute for Q into the right-hand side of [1].

$$\text{RHS} = 6 - \frac{2Q}{200 + t}$$

$$= 6 - \frac{2}{200 + t} \left[2(200 + t) + \frac{C}{(200 + t)^2} \right]$$

$$= 6 - \frac{2 \times 2(200 + t)}{200 + t} - \frac{2C}{(200 + t)^3}$$

4. Expand the right-hand side.

$$= 6 - 4 - \frac{2C}{(200 + t)^3}$$

5. Simplify the right-hand side. Because the left-hand side is equal to the right-hand side, we have verified that the given solution does satisfy the differential equation.

$$= 2 - \frac{2C}{(200 + t)^3}$$

- c. 1. To determine the value of C , use the given initial conditions.

- c. Substitute $t = 0$ and $Q = 100$ into [2].

2. Substitute the given values in the given solution.

$$100 = 2 \times 200 + \frac{C}{200^2}$$

3. Solve for C .

$$100 = 400 + \frac{C}{200^2}$$

$$\frac{C}{200^2} = -300$$

$$C = -12\,000\,000$$

- d. 1. Determine the time when the tank overflows. The capacity of the tank is 300 litres.

d. $300 = 200 + t$

The tank overflows when $t = 100$ minutes.

2. Determine the value Q of when the tank overflows.

Substitute for C :

$$Q(t) = 2(200 + t) - \frac{12\,000\,000}{(200 + t)^2}$$

$$Q(100) = 600 - \frac{12\,000\,000}{300^2}$$

$$= 466\frac{2}{3} \text{ kg}$$

3. Determine the concentration when the tank overflows.

$$\text{Concentration} = \frac{\text{mass}}{\text{volume}}$$

$$\frac{Q(100)}{V(100)} = \frac{466\frac{2}{3}}{300}$$

$$\approx 1.556 \text{ kg/litre}$$

4. State the answer.

When the tank overflows, the concentration of salt in the solution is 1.556 kg/litre.

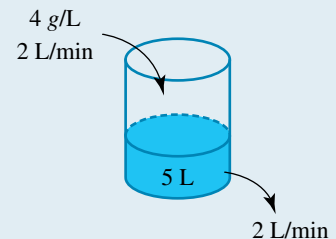
9.5.2 Equal input and output flow rates

In the special case when the inflow rate is equal to the outflow rate, that is, when $f = g$ so the volume remains constant, the differential equation can be solved using the separable integration techniques learnt in a previous chapter. In fact, in this case the differential equation is just another application of a bounded growth or decay model.

WORKED EXAMPLE 10

A tank contains 5 litres of water in which 10 grams of salt have been dissolved. A salt solution containing 4 grams per litre is poured into the tank at a rate of 2 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 2 litres per minute.

- Set up the differential equation for the amount of salt, Q grams, in the tank at time t minutes.
- Solve the differential equation to determine Q at any time t .
- How much salt is in the tank after 5 minutes? What is its concentration then?
- Show that the salt solution can never exceed 20 grams.
- Sketch the graph of Q versus t .



THINK

- The inflow rate is equal to the outflow rate. The volume in the tank remains constant at 5 litres.
 - The inflow rate is inflow concentration $\times$ the inflow volume rate. The output flow rate is the outflow concentration $\times$ the outflow volume rate.
 - State the differential equation.
 - State the differential equation in simplest form, along with the initial condition.
- To solve the differential equation, invert both sides.
 - Use the separation of variables technique, taking the constant factors to one side.
 - Perform the integration.
 - Since $Q(0) = 10$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.

WRITE/DRAW

a. $V(t) = 5$

$$\begin{aligned}\frac{dQ}{dt} &= \text{inflow rate} - \text{output rate} \\ &= 4 \times 2 - \frac{2Q}{V(t)}\end{aligned}$$

$$\frac{dQ}{dt} = 8 - \frac{2Q}{5}$$

$$\frac{dQ}{dt} = \frac{2(20 - Q)}{5}, Q(0) = 10$$

b. $\frac{dt}{dQ} = \frac{5}{2(20 - Q)}$

$$\int \frac{2}{5} dt = \int \frac{1}{20 - Q} dQ$$

$$\frac{2t}{5} = -\ln(|20 - Q|) + c$$

Substitute $t = 0$ when $Q = 10$:

$$0 = -\ln(20 - 10) + c$$

$$c = \ln(10)$$

5. Substitute back for C and simplify using log laws.

6. Use the definition of the logarithm.

Remove the absolute value sign as $e^{\frac{2t}{5}} > 0$

7. Invert both sides.

8. Rearrange and solve for Q .

9. State the solution of the differential equation in simplest form.

- c. Determine the concentration after 5 minutes.

- d. Determine the limit as t approaches infinity.

- e. The graph starts at $Q(0) = 10$ and approaches the horizontal asymptote of $Q = 20$.

$$\frac{2t}{5} = -\ln|20 - Q| + \ln(10)$$

$$\frac{2t}{5} = \ln\left|\frac{10}{20 - Q}\right|$$

$$e^{\frac{2t}{5}} = \frac{10}{20 - Q}$$

$$\frac{20 - Q}{10} = e^{-\frac{2t}{5}}$$

$$20 - Q = 10e^{-\frac{2t}{5}}$$

$$Q = 20 - 10e^{-\frac{2t}{5}}$$

$$Q(t) = 10\left(2 - e^{-\frac{2t}{5}}\right)$$

- c. Substitute $t = 5$:

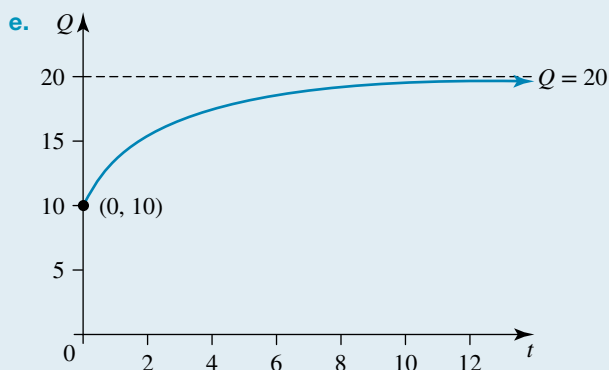
$$\begin{aligned} Q(5) &= 10(2 - e^{-2}) \\ &= 18.646 \text{ grams} \end{aligned}$$

The concentration is

$$\begin{aligned} c &= \frac{Q}{V} \\ &= \frac{18.646}{5} \\ &= 3.73 \text{ grams per litre} \end{aligned}$$

- d. As $t \rightarrow \infty$, $e^{-\frac{2t}{5}} \rightarrow 0$, $\therefore Q \rightarrow 20$.

Therefore, the amount of salt can never exceed 20 grams.



9.5.3 Chemical reaction rates

In a chemical reaction, the rate at which a new substance is formed is proportional to the unused amounts of the reacting substances. Differential equations can be set up and solved to model the amount of the new substance as it forms over time.

WORKED EXAMPLE 11

In a biomolecular chemical reaction between 3 grams of substance A and 6 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B respectively. A and B combine in equal parts to form substance X. Initially no X is present, but after 3 minutes, 3 grams of X have formed.

- Set up the differential equation for the amount of substance X at time t minutes.
- Determine the amount of X present after a further 3 minutes.

THINK

- Let $x = x(t)$ be the amount in grams of substance X formed at time t minutes.
 - The velocity of the reaction is proportional to the product of the unused amounts of A and B, so $\frac{x}{2}$ is used from both A and B.
- To solve the differential equation, invert both sides.
 - Take the constant k to the other side and use separation of variables.
 - To determine the integral on the right-hand side, we need to use partial fractions.
 - Add the partial fractions and simplify.
 - Equate the coefficients and solve for A and B.
 - Substitute back for the partial fraction decomposition.
 - Perform the integration.
 - Take out the common factors and use log laws.
 - Since $x(0) = 0$ and $0 \leq x < 6$, the modulus signs are not needed. Use the initial condition to determine the constant of integration.
 - Substitute back for c and use log laws again.

WRITE

$$\text{a. } [A + B \rightarrow X]$$

$$\frac{dx}{dt} \propto \left(3 - \frac{x}{2}\right) \left(6 - \frac{x}{2}\right)$$

$$\frac{dx}{dt} = k(6 - x)(12 - x), x(0) = 0$$

where k is a constant to be found.

$$\text{b. } \frac{dx}{dt} = \frac{1}{k(6 - x)(12 - x)}$$

$$kt = \int \frac{1}{(6 - x)(12 - x)} dx$$

$$\frac{1}{(6 - x)(12 - x)} = \frac{A}{6 - x} + \frac{B}{12 - x}$$

$$\begin{aligned} \frac{1}{(6 - x)(12 - x)} &= \frac{A(12 - x) + B(6 - x)}{(6 - x)(12 - x)} \\ &= \frac{12A + 6B - x(A + B)}{(6 - x)(12 - x)} \end{aligned}$$

$$12A + 6B = 1 \quad [1]$$

$$A + B = 0 \Rightarrow A = -B \quad [2]$$

$$A = \frac{1}{6}, B = -\frac{1}{6}$$

$$kt = \frac{1}{6} \int \left(\frac{1}{6 - x} - \frac{1}{12 - x} \right) dx$$

$$kt = -\frac{1}{6} \ln(|6 - x|) + \frac{1}{6} \ln(|12 - x|) + c$$

$$kt = \frac{1}{6} \ln\left(\left|\frac{12 - x}{6 - x}\right|\right) + c$$

$$0 = \frac{1}{6} \ln(2) + c$$

$$c = -\frac{1}{6} \ln(2)$$

$$kt = \frac{1}{6} \ln\left(\frac{12 - x}{6 - x}\right) - \frac{1}{6} \ln(2)$$

$$kt = \frac{1}{6} \ln\left(\frac{12 - x}{2(6 - x)}\right) \quad [3]$$

11. We can now determine the value of k by substituting $t = 3$ when $x = 3$.

Substitute $t = 3$ when $x = 3$:

$$3k = \frac{1}{6} \ln\left(\frac{9}{6}\right)$$

$$3k = \frac{1}{6} \ln\left(\frac{3}{2}\right)$$

$$k = \frac{1}{18} \ln\left(\frac{3}{2}\right) \quad [4]$$

12. Attempt to make x the subject, but leave the result in terms of k , since k has a known value.

$$\text{From [3], } 6kt = \ln\left(\frac{12-x}{2(6-x)}\right)$$

13. Use the definition of a logarithm.

$$\frac{12-x}{2(6-x)} = e^{6kt}$$

14. Invert both sides.

$$\frac{12-2x}{12-x} = e^{-6kt}$$

15. Remove the denominator.

$$12-2x = (12-x)e^{-6kt}$$

16. Expand the brackets.

$$12-2x = 12e^{-6kt} - xe^{-6kt}$$

17. Transfer the x to one side and factorise.

$$12-12e^{-6kt} = 2x - xe^{-6kt}$$

$$12(1-e^{-6kt}) = x(2-e^{-6kt})$$

18. State the solution of the differential equation.

$$x = x(t) = \frac{12(1-e^{-6kt})}{2-e^{-6kt}}$$

19. Determine the value of x after a further 3 minutes. When $t = 6$

$$\text{where } k = \frac{1}{18} \ln\left(\frac{3}{2}\right)$$

$$x(6) = \frac{12(1-e^{-36k})}{2-e^{-36k}}$$

20. Use exact values.

$$\text{From [4], } 36k = 2 \ln\left(\frac{3}{2}\right)$$

$$36k = \ln\left(\frac{9}{4}\right)$$

21. Evaluate

$$e^{-36k} = \frac{4}{9}$$

$$x(6) = \frac{12\left(1 - \frac{4}{9}\right)}{2 - \frac{4}{9}}$$

$$= \frac{30}{7}$$

22. State the final result. Note that as $t \rightarrow \infty$, $x \rightarrow 12$.

After 6 minutes, $4\frac{2}{7}$ grams of X have formed.

study on

Units 3 & 4

Area 5

Sequence 2

Concept 6

Chemical reactions and dilution problems Summary screen and practice question

Exercise 9.5 Chemical reactions and dilution problems

Technology active

1. **WE9** A vessel initially contains 200 litres of a salt solution with a concentration of 0.1 kg/litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 1.5 kg/litre is added to the vessel at a rate of 2 litres per minute. The contents of the vessel are kept well stirred.
 - a. Set up the differential equation for Q , the amount of salt in kilograms in the vessel after t minutes.
 - b. Verify that $Q = \frac{3}{2}(200 - t) + C(200 - t)^3$ is a general solution of the differential equation.
 - c. Determine the value of C .
 - d. Determine the concentration of salt after 100 minutes.
2. A vat initially contains 50 litres of a sugar solution. More sugar solution containing b grams per litre is poured into the vat at a rate of 6 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 3 litres per minute.
 - a. Set up the differential equation for Q , the amount of sugar in grams in the vat after t minutes.
 - b. Verify that $Q = 2(50 + 3t) + \frac{C}{50 + 3t}$ is a general solution of the differential equation, and determine the value of b .
3.
 - a. A tank initially contains 50 litres of water. A chemical solution is drawn off at a rate of 5 litres per minute, and at the same time a mixture containing the chemical at a concentration of 3 grams per litre is added to the tank at a rate of 4 litres per minute. The contents of the tank are kept well stirred.
 - i. Set up the differential equation for Q , the amount of the chemical in grams in the tank after t minutes.
 - ii. Verify that $Q(t) = 3(50 - t) + C(50 - t)^5$ is a general solution of the differential equation.
 - b. A container initially contains 30 litres of water. A brine solution of concentration 4 grams per litre is added to the container at a rate of 1 litre per minute. The well-stirred mixture is drawn off at 2 litres per minute.
 - i. Set up the differential equation for Q , the amount of brine in grams in the container after t minutes.
 - ii. Verify that $Q(t) = \frac{2t}{15}(30 - t)$ is the particular solution of the differential equation.
4.
 - a. A vessel initially contains 20 litres of water. A sugar solution with a concentration of 3 grams per litre is added at a rate of 2 litres per minute. The well-stirred mixture is drawn off at 1 litre per minute.
 - i. Set up the differential equation for Q , the amount of sugar in grams in the vessel after t minutes.
 - ii. Verify that $Q(t) = \frac{3t(t + 40)}{20 + t}$ is the particular solution of the differential equation.
 - b. A trough initially contains 20 litres of water. A dye solution with a concentration of 4 grams per litre is added to the trough at a rate of 2 litres per minute. The well-stirred mixture is sent down a drain at a rate of 3 litres per minute.
 - i. Set up the differential equation for Q , the amount of dye in grams in the trough after t minutes.
 - ii. Verify that $Q(t) = \frac{t}{100}(t - 40)(t - 20)$ is a particular solution of the differential equation.
5.
 - a. A vessel initially contains 64 litres of a salt solution with a concentration of 2 grams per litre. The solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt at a concentration of 4 grams per litre is added to the vessel at a rate of 5 litres per minute. The contents of the vessel are kept well stirred.
 - i. Set up the differential equation for Q , the amount of salt in grams in the vessel after t minutes.
 - ii. Verify that $Q = 4(64 + 2t) + C(64 + 2t)^{-\frac{3}{2}}$ is a general solution of the differential equation.
 - iii. Determine the value of C .
 - iv. Determine the concentration when the volume of the vessel is 100 litres.

- b. A vat initially contains 40 litres of water. A brine solution containing b grams per litre is poured into the vat at a rate of 2 litres per minute, and simultaneously the well-stirred mixture leaves the vat at a rate of 5 litres per minute. If Q is the amount of brine in grams in the vessel after a time t minutes and $Q = 3(40 - 3t) + C(40 - 3t)^{\frac{5}{3}}$ is a general solution of the differential equation, determine the value of b .
6. **WE10** A tank initially contains 20 litres of a salt solution that has a concentration of 0.25 grams per litre. The salt solution is drawn off at a rate of 3 litres per minute, and at the same time a mixture containing salt of concentration 4 grams per litre is added to the tank at a rate of 3 litres per minute. The contents of the tank are kept well stirred.
- Set up the differential equation for Q , the amount of salt in grams in the tank after t minutes.
 - Solve the differential equation to determine Q at any time t .
 - Determine the concentration of salt after 1.6 minutes.
 - Show that the amount of salt can never exceed 80 grams.
 - Sketch the graph of Q versus t .
7. A vat contains 15 litres of pure water. A sugar solution containing 5 grams per litre is poured into the vat at a rate of 4 litres per minute, and simultaneously the well-stirred mixture leaves the vat at the same rate.
- Set up the differential equation for the amount of sugar, Q grams, in the vat at time t minutes.
 - Solve the differential equation to determine Q at any time t .
 - After how long is the sugar concentration 4 grams per litre?
8. a. A tank contains 600 litres of water in which 30 kilograms of salt have been dissolved. Water is poured into the tank at a rate of 5 litres per minute, and the mixture is kept uniform by stirring. The mixture leaves the tank at a rate of 5 litres per minute.
- Determine the amount of salt in kilograms in the tank at any time t minutes.
 - Determine the amount of salt in kilograms after 2 hours.
 - How long before there are 15 kilograms of salt in the tank?
- b. A sink contains 50 litres of water for washing dishes, in which 50 grams of detergent have been dissolved. A dishwashing solution containing 3 grams per litre of detergent is poured into the sink at a rate of 4 litres per minute and the mixture is kept uniform. The mixture leaves the sink down the drain at the same rate. Determine the concentration of detergent after 5 minutes.
9. a. A 10 litre urn is full of boiling water. Caterers pour the entire contents of a 300 gram jar of coffee into the urn and mix them thoroughly. While the caterers are making cups of coffee, the coffee is drawn out of the urn at a rate of 0.2 litres per minute, and at the same time boiling water is added at the same rate.
- Lilly likes to drink her coffee when the concentration of the coffee is 25 grams per litre. How long after the process starts should she wait to get her cup of coffee?
 - When the concentration of coffee in the urn falls below 15 grams per litre, more coffee must be added to the urn. How long after the process starts must more coffee be added?
- b. The urn now contains hot water at a temperature of 93°C . Water is poured from the urn to make a cup of coffee. The coffee cup is placed in a room where the temperature is constant at 17°C . After 1 minute the temperature of the coffee is 88°C . Lilly likes to drink her coffee when its temperature is between 50°C and 65°C . How long does Lilly have to drink the coffee?



10. **WE11** In a chemical reaction between 1 gram of substance A and 3 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X, and initially no X is present. After 3 minutes, 1 gram of X has formed.
- Set up the differential equation to determine the amount of substance X at time t minutes.
 - Solve the differential equation to determine the amount of substance X at time t minutes.
 - Determine the amount of X present after a further 3 minutes.
11. In a chemical reaction between 4 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, but after 2 minutes, 3 grams of X has formed.
- Set up the differential equation for the amount of substance X at time t minutes.
 - Solve the differential equation to determine the amount of X in grams present after a time t minutes.
 - How much more time passes before 6 grams of X are present?
 - What is the maximum amount of X that could eventually be formed?
12. In a chemical reaction between 2 grams of substance A and 4 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form substance X. Initially, no X is present, and after 2 minutes, 1 gram of X has formed.
- Set up the differential equation to determine the amount of substance X at time t minutes.
 - Solve the differential equation to determine the amount of substance X at time t minutes.
 - Determine the amount of X present after a further 2 minutes.
 - What is the ultimate amount of substance X that can eventually be formed?
13. In a chemical reaction between 5 grams of substance A and 2 grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. A and B combine in equal parts to form a substance X. Initially, no X is present, and after 3 minutes, 2 grams of X have formed.
- How long before 3 grams of X have formed?
 - After 6 minutes, how much of X has formed?
 - What is the ultimate amount of substance X that can eventually be formed?
14. a. Many chemical reactions follow *Wilhelmy's Law*, which states that the velocity of the reaction is proportional to the concentration of the reacting substance. In such a reaction containing a grams of a reagent, the amount of substance X transformed, x grams, after a time t minutes is given by
- $$\frac{dx}{dt} = k(a - x), \text{ where } a \text{ and } k \text{ are both positive constants.}$$
- If initially there is no X present, show that $x(t) = a(1 - e^{-kt})$.
 - If $a = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes.
- b. In a chemical reaction, equal amounts of A and B combine to form substance X. Initially there are b grams of both A and B present, and no X is present. If x grams of X have formed after t minutes, the velocity of the reaction is proportional to the product of the unused amounts of A and B.
- Given that $\frac{dx}{dt} = k\left(b - \frac{x}{2}\right)^2$ where b and k are both positive constants, show that
- $$x(t) = \frac{2b^2kt}{2 + bkt}.$$
- If $b = 5$ and after 4 minutes 2 grams of X is present, determine the amount of X present after 10 minutes.
 - What is the ultimate amount of substance X that can eventually be formed?
15. a. A vat initially contains V_0 litres of water. A brine solution of concentration b grams per litre is added to the vat at a rate of f liters per minutes. The well-stirred mixture is drawn off at the same rate.
- Set up the differential equation for Q , the amount of brine in grams in the vat after t minutes.
 - Use integration to show that $Q(t) = bV_0\left(1 - e^{-\frac{ft}{V_0}}\right)$ is a particular solution of the differential equation.

- b. A container initially contains V_0 litres of a brine solution in which q_0 grams of brine have been dissolved. A brine solution of concentration b grams per litre is added to the container at a rate of f litres per minute. The well-stirred mixture leaves the container at the same rate.

i. Set up the differential equation for Q , the amount of brine in grams in the container after t minutes.

ii. Use integration to show that $Q(t) = bV_0 + (q_0 - bV_0)e^{-\frac{ft}{V_0}}$ is a particular solution of the differential equation.

16. a. In a trimolecular chemical reaction, three substances, A, B and C, react to form a single substance X. The rate of the reaction is proportional to the product of the unreacted amounts of A, B and C. Substances A, B and C combine in equal parts to form substance X, and initially no X is present.

Given that $\frac{dx}{dt} = k \left(a - \frac{x}{3}\right)^3$ where a is the initial amount of all substances A, B and C present and

$k > 0$, show that $t = \frac{3x(6a - x)}{2a^2k(3a - x)^2}$ for $0 \leq x < 3a$.

- b. In a chemical reaction between a grams of substance A and b grams of substance B, the velocity of the reaction is proportional to the product of the unused amounts of A and B. Substances A and B combine in equal parts to form substance X, and initially no X is present. Given that

$\frac{dx}{dt} = k \left(a - \frac{x}{2}\right) \left(b - \frac{x}{2}\right)$ where $k > 0$ is a constant, x is the amount of substance X formed at a

time t minutes, and $a > b > 0$, show that $x(t) = \frac{2ab \left(1 - e^{-\frac{(a-b)kt}{2}}\right)}{a - be^{-\frac{(a-b)kt}{2}}}$ and deduce that the limiting amount of X present is $2b$.

9.6 The logistic equation

9.6.1 Introduction

In previous sections we considered growth when it is directly proportional to itself; that is, the law of natural growth, $\frac{dN}{dt} = kN$; and also when the growth rate is proportional to the difference between the number and the equilibrium value; that is, bounded growth, $\frac{dN}{dt} = k(P - N)$. In this section we consider growth rates that are proportional to the product of both of these: $\frac{dN}{dt} \propto N(P - N)$.

9.6.2 Logistic growth

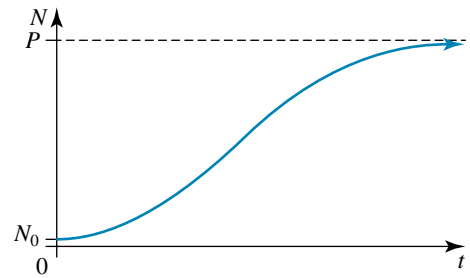
This equation can be written in the form $\frac{dN}{dt} = kN(P - N)$ or $\frac{dN}{dt} = kN \left(1 - \frac{N}{P}\right)$, where k is the growth rate and P is called the carrying capacity, the equilibrium value or the ultimate maximum value. For an initial number of $N(0) = N_0$, assuming $k > 0$ and $P > N_0 > 0$, it can be shown that

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}} = \frac{P}{1 + \left(\frac{P}{N_0} - 1\right)e^{-kt}}. \text{ (Showing this is left as an exercise for you to do$$

yourself.) In this relationship, as $t \rightarrow \infty$, $N \rightarrow P$.

The graph of the logistic equation always has the S shape shown below.

The growth pattern modelled by this type of differential equation is called logistic growth. It was devised by the Belgian mathematician Pierre François Verhulst (1804–1849). His idea was a response to the work of the English scholar Thomas Robert Malthus, who published a paper in 1798, *An Essay on the Principle of Population*, predicting unlimited population growth. Verhulst disagreed with Malthus's model and used his own model to show the characteristics of bounded population growth.



WORKED EXAMPLE 12

A rumour is spreading in a neighbourhood that contains 800 people. The rumour spreads at a rate proportional to the product of the number who have heard the rumour and the number who have not yet heard the rumour. At first only 2 people in the area have heard the rumour, but after 4 days, 15% of the residents of the neighbourhood have heard it.

- Let $N = N(t)$ be the number who have heard the rumour after t days. Write and solve the differential equation.
- Determine the number of residents in the neighbourhood who have heard the rumour after 5 more days.
- After how many days have 50% of the residents in the area heard the rumour?
- Sketch the graph.

THINK

- The growth rate is proportional to the product of N and $(P - N)$.
- Write the differential equation and the initial condition.
- To solve the differential equation, invert both sides.
- Use the separation of variables.
- Use partial fractions on the integrand on the right-hand side.
- Add the partial fractions.
- Write equations that can be solved to determine the values of A and B .
- Solve the equations and state the values of A and B .

WRITE/DRAW

- The maximum number is $P = 800$.

$$\frac{dN}{dt} \propto N(800 - N)$$

$$\frac{dN}{dt} = kN(800 - N), N(0) = 2$$

$$\frac{dt}{dN} = \frac{1}{kN(800 - N)}$$

$$kt = \int \frac{1}{N(800 - N)} dN$$

$$\frac{1}{N(800 - N)} = \frac{A}{N} + \frac{B}{800 - N}$$

$$\begin{aligned} \frac{1}{N(800 - N)} &= \frac{A(800 - N) + BN}{N(800 - N)} \\ &= \frac{N(B - A) + 800A}{N(800 - N)} \end{aligned}$$

From the coefficient of N :

$$B - A = 0 \quad [1]$$

From the term independent of N :

$$800A = 1 \quad [2]$$

From [1], $B = A$.

$$\text{From [2], } A = \frac{1}{800}.$$

$$A = B = \frac{1}{800}$$

9. Substitute for A and B in the integration equation.
 10. Perform the integration.
 11. Because $2 \leq N < 800$, the modulus signs are not needed.
 12. Use the initial condition to determine the constant of integration.
 13. Solve for c and apply log laws.
 14. Substitute back for c and use log laws again.
 15. After 4 days, 15% of the residents have heard the rumour. This is 120 people. Use this to determine k .
 16. Use the definition of the logarithm but leave the result in terms of k , as k has a known value.
 17. Rearrange to make N the subject.
 18. Take out the common factor of N .
 19. State the solution of the differential equation.
- b. 1. Determine the value of N after 5 more days (i.e. when $t = 9$).
2. State the required number, rounding down.
- c. 1. Determine t when 50% of 800 have heard the rumour; that is, when $N = 400$. It is easier to use an earlier equation.

$$kt = \frac{1}{800} \int \left(\frac{1}{N} + \frac{1}{800 - N} \right) dN$$

$$800kt = \ln(|N|) - \ln(|800 - N|) + c$$

$$800kt = \ln(N) - \ln(800 - N) + c$$

Substitute $N = 2$ when $t = 0$:

$$0 = \ln(2) - \ln(800 - 2) + c$$

$$c = \ln(798) - \ln(2)$$

$$= \ln\left(\frac{798}{2}\right)$$

$$= \ln(399)$$

$$800kt = \ln(N) - \ln(800 - N) + \ln(399)$$

$$= \ln\left(\frac{399N}{800 - N}\right)$$

Substitute $N = 120$ when $t = 4$:

$$3200k = \ln\left(\frac{399 \times 120}{800 - 120}\right)$$

$$k = \frac{1}{3200} \ln\left(\frac{1197}{17}\right)$$

$$800k = \frac{1}{4} \ln\left(\frac{1197}{17}\right)$$

$$\frac{399N}{800 - N} = e^{800kt}$$

$$399N = (800 - N)e^{800kt}$$

$$800 - N = 399Ne^{-800kt}$$

$$800 = N + 399Ne^{-800kt}$$

$$800 = N(1 + 399e^{-800kt})$$

$$N(t) = \frac{800}{1 + 399e^{-\frac{t}{4} \ln\left(\frac{1197}{17}\right)}}$$

b. Substitute $t = 9$:

$$N(9) = \frac{800}{1 + 399e^{-\frac{9}{4} \ln\left(\frac{1197}{17}\right)}}$$

$$\approx 778.3767$$

778 people have heard the rumour.

$$c. 800kt = \frac{1}{4} \ln\left(\frac{1197}{17}\right)$$

$$= \ln\left(\frac{399N}{800 - N}\right)$$

Substitute $N = 400$:

$$\frac{t}{4} \ln\left(\frac{1197}{17}\right) = \ln\left(\frac{399 \times 400}{800 - 400}\right)$$

2. Solve for t .

$$t = \frac{4 \ln\left(\frac{399 \times 400}{800 - 400}\right)}{\ln\left(\frac{1197}{17}\right)}$$

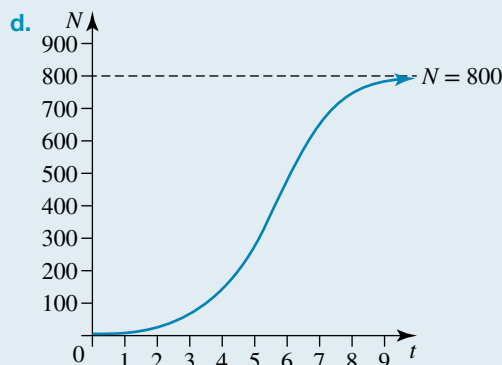
$$= \frac{4 \ln(399)}{\ln\left(\frac{1197}{17}\right)}$$

$$\approx 5.63$$

3. State the answer.

After 5.63 days, 50% of the residents have heard the rumour.

d. Sketch the graph. (Note that as $t \rightarrow \infty, N \rightarrow 800$.)

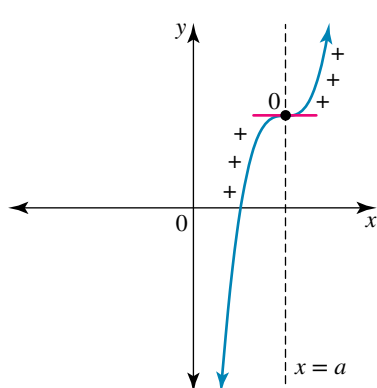


9.6.3 Points of inflection

From your studies in Mathematical Methods, recall that a stationary point is defined as a point on the curve at which the slope of the tangent to the curve is zero; that is, where the gradient $\frac{dy}{dx} = f'(x) = 0$. This concept was used to determine the location of any turning points. To distinguish between a local maximum and a minimum, we use the sign test or the second derivative test.

On a continuous curve, between a local maximum and minimum there is another critical point called an inflection point. At such a point the curve changes from being concave to convex or vice versa. The tangent to the curve at such a point crosses the graph. The rate of change of the gradient is zero at a point of inflection. Mathematically, this can be interpreted as $\frac{d^2y}{dx^2} = f''(x) = 0$. There are two types of inflection points.

Horizontal stationary point of inflection

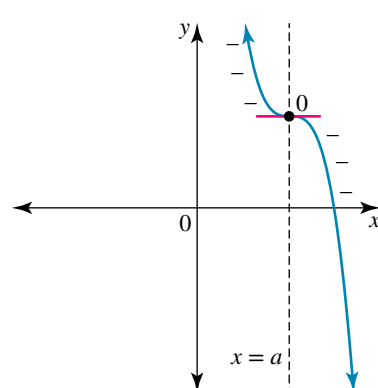


$$x < a, f'(a) < 0$$

$$x = a, f'(a) = 0$$

$$x > a, f'(a) > 0$$

$$f''(a) = 0$$



$$x < a, f'(a) < 0$$

$$x = a, f'(a) = 0$$

$$x > a, f'(a) > 0$$

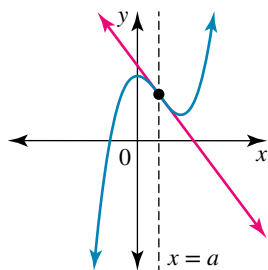
$$f''(a) = 0$$

Inflection point

At $x = a$,

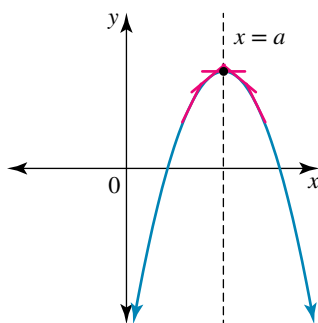
$$f'(a) \neq 0$$

$$f''(a) = 0$$

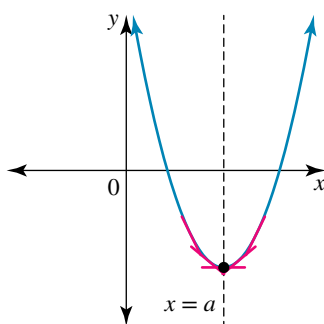


Concavity

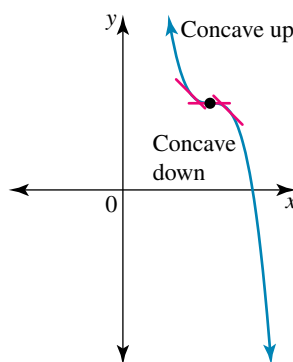
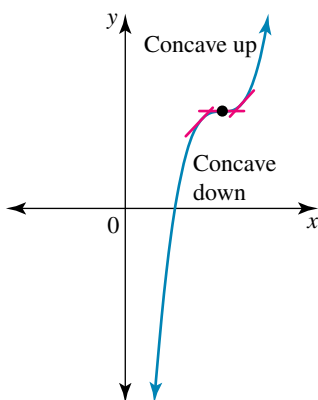
A non-stationary point of inflection is where the tangent to the curve moves from being above the curve to below the curve or vice versa. Curves are also defined as being **concave up** or **concave down**. When a tangent is above the curve at each point, then the derivative function is decreasing and $f''(x) < 0$.



If the tangent to the curve is below the curve at each point, then the derivative function is increasing and $f''(x) > 0$.



At a point of inflection, the tangent will pass through the curve, and the curve will change from concave down to concave up or vice versa on either side of the point of inflection.



An interesting feature of a logistic curve is that it contains a point of inflection corresponding to the point of maximum growth rate. The mathematics and reasoning used to determine the location of this point of inflection will be examined in the following section.

9.6.4 Analysis of the logistic solution

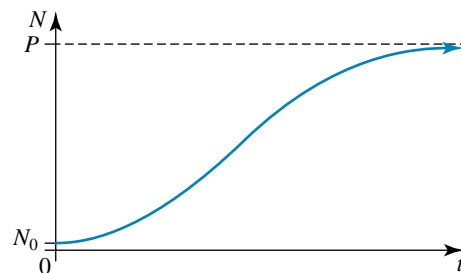
The differential equation for the logistic curve in general is given by $\frac{dN}{dt} = kN \left(1 - \frac{N}{P}\right)$. For an initial number $N(0) = N_0$, assuming $k > 0$ and $P > N_0 > 0$, the solution is given by

$$N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}} = \frac{P}{1 + \left(\frac{P}{N_0} - 1\right)e^{-kt}} \text{ for } t \geq 0.$$

The graph of the logistic curve always has this so-called S-shaped curve. The graph is relatively flat at the ends, so the gradient is close to zero as $N \rightarrow 0$ and $N \rightarrow P$. These lines, $N = 0$ and $N = P$, are in fact horizontal asymptotes.

Because $\frac{dN}{dt} = kN \left(1 - \frac{N}{P}\right)$, we can write $\frac{dN}{dt} = \frac{k}{P}(PN - N^2)$. Taking the derivative again with respect to t , because k and P are constants, and using implicit differentiation or the chain rule, then

$$\begin{aligned} \frac{d^2N}{dt^2} &= \frac{d}{dt} \left(\frac{dN}{dt} \right) \\ &= \frac{k}{P} \frac{d}{dt} (PN - N^2) \\ &= \frac{k}{P} \frac{d}{dt} (PN - N^2) \frac{dN}{dt} \\ &= \frac{k}{P} (P - 2N) \frac{dN}{dt} \end{aligned}$$



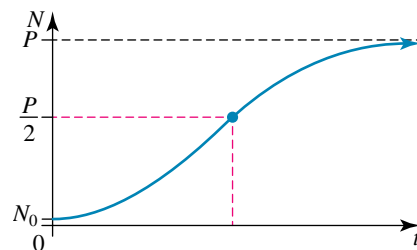
There is a point on the graph at which the gradient or rate of change of growth is a maximum. Assume $k > 0$, $N_0 > 0$ and $\frac{dN}{dt} > 0$. This means that in the equation $\frac{d^2N}{dt^2} = \frac{k}{P} (P - 2N) \frac{dN}{dt}$, the expressions $\frac{k}{P}$ and $\frac{dN}{dt}$ are both positive.

To keep the two sides of the equation equal, if $\frac{d^2N}{dt^2} > 0$, then $P - 2N > 0$. Therefore, $0 < N < \frac{P}{2}$ is where the curve is concave up.

Conversely, if $\frac{d^2N}{dt^2} < 0$, then $P - 2N < 0$.

Therefore, $\frac{P}{2} < N < P$ is where the curve is concave down.

Recall that if the second derivative on a curve is zero, there is a point of inflection. For a logistic curve, the point of inflection occurs when $\frac{d^2N}{dt^2} = 0$; that is, for $N = \frac{P}{2}$, or halfway up the curve.



The point of inflection is the point of maximum growth rate, or where the value is increasing most rapidly. It can be shown in general that when $N = \frac{P}{2}$, the corresponding value for t is given by $\frac{1}{k} \ln \left(\frac{P}{N_0} - 1 \right)$.

WORKED EXAMPLE 13

A logistic equation has the solution $y(x) = \frac{200}{1 + 99e^{-\frac{x}{2}}}$.

- Using differentiation, show that the solution satisfies the differential equation $\frac{dy}{dx} = \frac{y(200 - y)}{400}$.
- Show that $\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80000}$.
- Hence determine the coordinates of the point of inflection.
- Sketch the graph of y versus x .

THINK

- Write the given solution in index form.

- Use the chain rule for differentiation.

- Simplify the expression for $\frac{dy}{dx}$.

- We need to express x in terms of y .

- Express the exponential in terms of y .

- Rewrite the expression for $\frac{dy}{dx}$.

- Replace $99e^{-\frac{x}{2}}$ with $\frac{200 - y}{y}$.

WRITE/DRAW

$$a. y = 200 \left(1 + 99e^{-\frac{x}{2}} \right)^{-1}$$

$$\frac{dy}{dx} = 200 \times -1 \times -\frac{1}{2} \times 99e^{-\frac{x}{2}} \left(1 + 99e^{-\frac{x}{2}} \right)^{-2}$$

$$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}} \right)^2}$$

$$\frac{200}{1 + 99e^{-\frac{x}{2}}} = y \quad [1]$$

$$\frac{1}{1 + 99e^{-\frac{x}{2}}} = \frac{y}{200} \quad [2]$$

$$1 + 99e^{-\frac{x}{2}} = \frac{200}{y}$$

$$99e^{-\frac{x}{2}} = \frac{200}{y} - 1$$

$$99e^{-\frac{x}{2}} = \frac{200 - y}{y} \quad [3]$$

$$\frac{dy}{dx} = \frac{100 \times 99e^{-\frac{x}{2}}}{\left(1 + 99e^{-\frac{x}{2}} \right)^2}$$

Substitute [3] into $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{100 \times \left(\frac{200 - y}{y} \right)}{\left(1 + \frac{200 - y}{y} \right)^2}$$

8. Simplify.

- b. 1. To determine the second derivative, differentiate with respect to x again. However, since $\frac{dy}{dx}$ is in terms of y , use implicit differentiation.

2. Expand the brackets.

3. Take the derivative and leave the constant factor.

4. Take out the common factor of 2 and simplify.

- c. 1. The inflection point occurs when the second derivative is zero.

2. Determine the corresponding x -value.

3. Solve for x .

4. Use the definition of the logarithm.

5. State the coordinates of the inflection point.

- d. Sketch the graph.

$$\frac{dy}{dx} = \frac{y(200 - y)}{400}$$

$$\text{b. } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy} \left(\frac{dy}{dx} \right) \times \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy} \left(\frac{y(200 - y)}{400} \right) \times \left(\frac{y(200 - y)}{400} \right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dy} \left(\frac{1}{400} (200 - y^2) \right) \times \left(\frac{y(200 - y)}{400} \right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{400} (200 - 2y) \times \left(\frac{y(200 - y)}{400} \right)$$

$$\frac{d^2y}{dx^2} = \frac{y(100 - y)(200 - y)}{80\,000}$$

- c. When $\frac{d^2y}{dx^2} = 0$, $y = 0, 100, 200$. However $y \neq 0$, so the graph never crosses the x -axis, and $y \neq 200$. The lines $y = 0$ and $y = 200$ are horizontal asymptotes. Therefore, $y = 100$ is the only possible result.

Substitute $y = 100$:

$$100 = \frac{200}{1 + 99e^{-\frac{x}{2}}}$$

$$1 + 99e^{-\frac{x}{2}} = 2$$

$$99e^{-\frac{x}{2}} = 1$$

$$e^{-\frac{x}{2}} = \frac{1}{99}$$

$$e^{\frac{x}{2}} = 99$$

$$\frac{x}{2} = \ln(99)$$

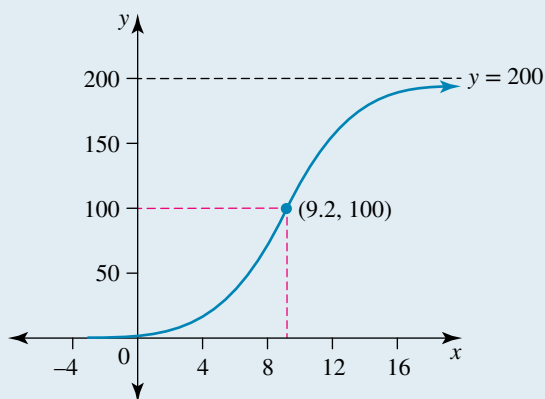
$$x = 2 \ln(99)$$

$$(2 \ln(99), 100) \approx (9.2, 100)$$

- d. For $y = \frac{200}{1 + 99e^{-\frac{x}{2}}}$, $y \rightarrow 200$

as $x \rightarrow \infty$ and the inflection point is at $(2 \ln(99), 100) \approx (9.2, 100)$.





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The logistic equation Summary screen and practice questions

Exercise 9.6 The logistic equation

Technology active

- WE12** A rumour is spreading in an elderly nursing home which contains 100 pensioners. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard the rumour. At first, 4 pensioners in the home have heard the rumour, and after 2 weeks, 20 of the pensioners have heard it.

 - Let $N = N(t)$ be the number who have heard the rumour after t weeks. Write and solve the differential equation.
 - Determine the number of pensioners in the home who have heard the rumour after 3 more weeks.
 - How long is it before 50 pensioners in the home have heard the rumour?
- An infection is spreading in another elderly nursing home which contains 80 pensioners. The infection spreads at a rate proportional to both the number who have the infection and the number who have not yet caught the infection. At first, only 2 pensioners in the home have the infection. After 3 days, 25% of the pensioners have the infection.

 - Let $N = N(t)$ be the number who have the infection after t days. Write and solve the differential equation.
 - Determine the number of pensioners in the home who have the infection after 4 more days.
 - How long is it before 60% of the pensioners in the home have the infection?
- WE13** A logistic equation has the solution $y(x) = \frac{500}{1 + 9e^{-3x}}$.

 - Using differentiation, show that it satisfies the differential equation $\frac{dy}{dx} = \frac{3y(500 - y)}{500}$.
 - Show that $\frac{d^2y}{dx^2} = \frac{9y(250 - y)(500 - y)}{125\,000}$.
 - Hence, determine the coordinates of the point of inflection.
 - Sketch the graph of y versus x .
- A logistic equation has the solution $y(x) = \frac{400}{1 + 199e^{-2x}}$.

 - Using differentiation, show that it satisfies the differential equation $\frac{dy}{dx} = \frac{y(400 - y)}{200}$.

- b. Show that $\frac{d^2y}{dx^2} = \frac{y(200 - y)(400 - y)}{20\,000}$.
- c. Hence, determine the coordinates of the point of inflection.
- d. Sketch the graph of y versus x .
5. A logistic equation has the solution $y(x) = \frac{600}{1 + 99e^{-\frac{x}{3}}}$.
- a. Using differentiation, show that it satisfies the differential equation $\frac{dy}{dx} = \frac{y(600 - y)}{1800}$.
- b. Show that $\frac{d^2y}{dx^2} = \frac{y(300 - y)(600 - y)}{1\,620\,000}$.
- c. Hence, determine the coordinates of the point of inflection.
- d. Sketch the graph of y versus x .
6. An area of the ocean initially contains 500 fish. The area can support a maximum of 4000 fish. The number of fish in the area grows continuously at a rate of 8% per year; that is, the constant of proportionality is 8%.
- a. Write the differential equation for the number of fish, $N(t)$, in the area at a time t years, assuming a logistic growth model.
- b. Solve the differential equation to determine the number of fish in the area after t years.
- c. Hence, determine the number of fish in the area after 5 years.
- d. After how many years will the number of fish reach 3000?
7. In a movie, people in a town are turned into zombies by a touch from an existing zombie. The town contains 360 people, and initially there is only one zombie. After 3 days, there are 60 zombies. The number of zombies in the town grows at a rate proportional to both the number of zombies and the number of people in the town who are not yet zombies.
- a. Write the differential equation for the number of zombies, N , after t days.
- b. After how many days will 75% of the people in the town become zombies?
8. A rumour is spreading in a school with 2000 students. The rumour spreads at a rate proportional to both the number who have heard the rumour and the number who have not yet heard it. Initially, only 2 students have heard the rumour, but after 3 days, 10% of the school has heard it.
- a. Write the differential equation for the number of students, N , in the school after t days who have heard the rumour, assuming a logistic growth model.
- b. After how many days have half the students heard the rumour?
- c. After how many days is the rumour spreading most rapidly?
9. A disease is spreading through an area with a population of 10000. Initially, 4 people in the area have the disease. After 4 days, 50 people have the disease. The disease is spreading at a rate proportional to both the number of people who have the disease and the number of people who do not yet have the disease. After 2 weeks, how many people in the area have the disease?
10. The number of children who contract the common cold from a kindergarten is found to be proportional to those who have the cold and those yet to get the cold. Initially, only 1 child from the kindergarten has the common cold. After 2 days, 5 children have the cold, and after a further, 2 days 15 children have the cold.
- a. How many children from the kindergarten will eventually get the cold?
- b. After how many days is the cold spreading most rapidly?
- c. How many children from the kindergarten have the cold after 6 days?
- d. Sketch the graph of the number of children with the cold against the time in days.



9.7 Slope fields

9.7.1 Introduction

Slope fields, also known as direction fields, are a tool for graphically visualising the solutions to a first-order differential equation. A slope field is simply a graph on the Cartesian coordinate system showing short line segments that represent the slopes of the possible solutions at each point.

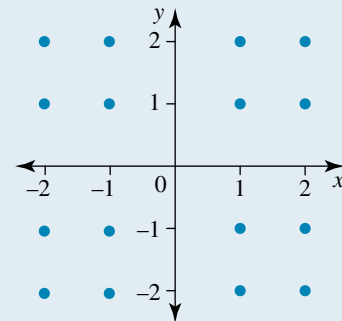
Type 1, $\frac{dy}{dx} = f(x)$

Type 1 first-order differential equations are of the form $\frac{dy}{dx} = f(x)$. In this section you will draw the slope fields for a simple differential equation of this type.

WORKED EXAMPLE 14

Sketch the slope field for the differential equation

$\frac{dy}{dx} = \frac{x}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values
 $x = -2, -1, 0, 1, 2$ on the grid shown.



THINK

1. Make some observations about the slope.
2. Determine the slope at appropriate values of x .
3. Consider another x -value.
4. Consider the negative values of x .
5. Consider the final x -value.

WRITE/DRAW

When $x = 0$ (i.e. along the y -axis), the slope $\frac{dy}{dx}$ is zero. The y -values do not affect the slope.

When $x = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$. A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27^\circ$ with the positive x -axis.

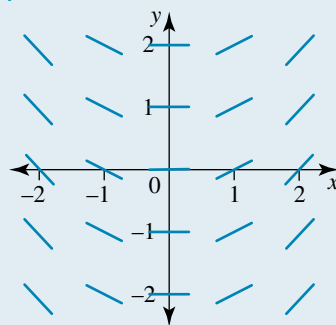
When $x = 2$, the slope is $\frac{dy}{dx} = 1$. A slope of 1 makes an angle of $\tan^{-1}(1) = 45^\circ$ with the positive x -axis.

When $x = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153^\circ$ with the positive x -axis.

When $x = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135^\circ$ with the positive x -axis.

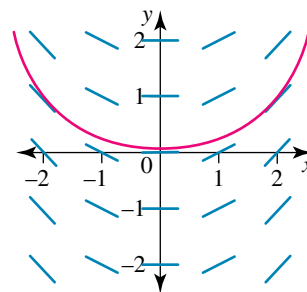
6. Draw short line segments with the slopes found at each of the points.

The slope field in this case is symmetrical about the y -axis.



Note that we obtain these graphical solutions to a general solution of a first-order differential equation; that is, we do not necessarily require a particular condition.

In general, the solutions to the differential equation $\frac{dy}{dx} = \frac{x}{2}$ are a family of curves of the form $y = \frac{x^2}{4} + c$. However, if we joined some of the slopes shown in the answer to Worked example 14 and drew a particular curve passing through the origin, the result would represent the solution curve $y = \frac{x^2}{4}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{x}{2}$, $y(0) = 0$. The slope field simply represents tangents to the solution curve at the points.



Type 2, $\frac{dy}{dx} = f(y)$

Type 2 first-order differential equations are of the form $\frac{dy}{dx} = f(y)$.

WORKED EXAMPLE 15

Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{y}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK

1. Make some observations about the slope.
2. Determine the slope at appropriate values of y .
3. Consider another y -value.

WRITE/DRAW

When $y = 0$ (i.e. along the x -axis), the slope is zero. The x -values do not affect the slope.

When $y = 1$, the slope is $\frac{dy}{dx} = \frac{1}{2}$.

A slope of $\frac{1}{2}$ makes an angle of $\tan^{-1}\left(\frac{1}{2}\right) \approx 27^\circ$ with the positive x -axis.

When $y = 2$, the slope is $\frac{dy}{dx} = 1$.

A slope of 1 makes an angle of $\tan^{-1}(1) = 45^\circ$ with the positive x -axis.

4. Consider the negative values of y .

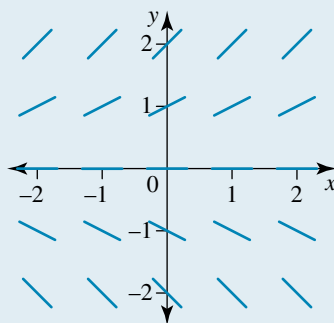
When $y = -1$, the slope is $\frac{dy}{dx} = -\frac{1}{2}$. A slope of $-\frac{1}{2}$ makes an angle of $\tan^{-1}\left(-\frac{1}{2}\right) \approx 153^\circ$ with the positive x -axis.

5. Consider the final y -value.

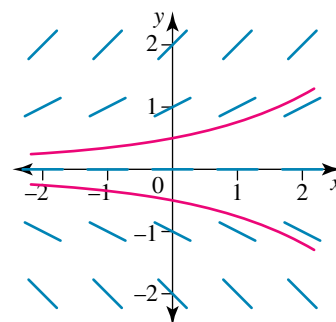
When $y = -2$, the slope is $\frac{dy}{dx} = -1$. A slope of -1 makes an angle of $\tan^{-1}(-1) = 135^\circ$ with the positive x -axis.

6. Draw short line segments with the slopes found at each of the points.

The slope field in this case is symmetrical about the x -axis.



Note that, if we joined the slope field lines in Worked example 15 and drew curves, they would represent the curve $y = y_0 e^{\frac{x}{2}}$, which is the particular solution to the differential equation $\frac{dy}{dx} = \frac{y}{2}$, $y(0) = y_0$. The two curves shown have $y_0 > 0$ and $y_0 < 0$.



Type 3, $\frac{dy}{dx} = f(x, y)$

Type 3 first-order differential equations are of the form $\frac{dy}{dx} = f(x, y)$.

WORKED EXAMPLE 16

Sketch the slope field for the differential equation $y \frac{dy}{dx} + x = 0$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on a grid.

THINK

1. Rearrange the differential equation to make $\frac{dy}{dx}$ the subject.

WRITE/DRAW

$$y \frac{dy}{dx} + x = 0$$

$$y \frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

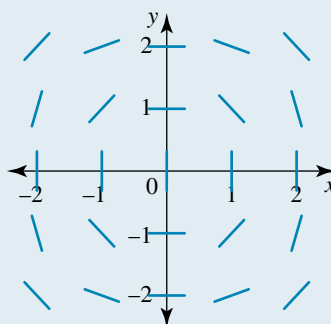
2. We have to evaluate the slope at each of the 25 points on the grid. Rather than substituting point by point, draw up a table.

Substitute $x = 2$ and $y = 2$:
the slope $\frac{dy}{dx} = -\frac{2}{2} = -1$.

3. Complete the values. Note that an undefined slope is one that is parallel to the y-axis.

$y \backslash x$	-2	-1	0	1	2
-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
-1	-2	-1	0	1	2
0	Undef.	Undef.	Undef.	Undef.	Undef.
1	2	1	0	-1	-2
2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

4. Draw the slope field by drawing short line segments with the slope found at each point.



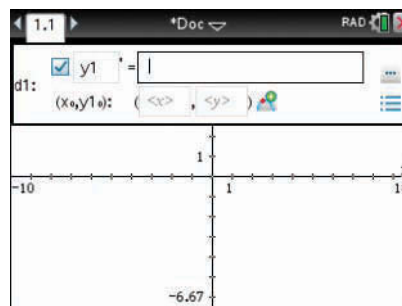
In the answer for Worked example 16, if the line segments were joined, the field would look like a series of circles of varying radii centred at the origin. In fact, if $x^2 + y^2 = r^2$, using implicit differentiation gives $2x + 2y \frac{dy}{dx} = 0$ or $x + y \frac{dy}{dx} = 0$.

9.7.2 Sketching slope fields using technology

Certain software packages such as GeoGebra, Desmos and various CAS capable technologies can produce slope fields. The TI-nspire CX calculator also has the capability to produce a slope field.

Consider Worked example 14 where a slope field was drawn for the differential equation $\frac{dy}{dx} = \frac{x}{2}$. Using the TI-nspire CX calculator we can produce a slope field by completing the steps below.

1. Open a Graphs page
2. Slope fields can be graphed by selecting, MENU
3: Graph Entry/Edit
8: Diff Eq

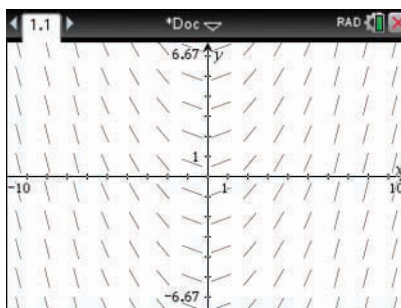


3. Complete the entry line as

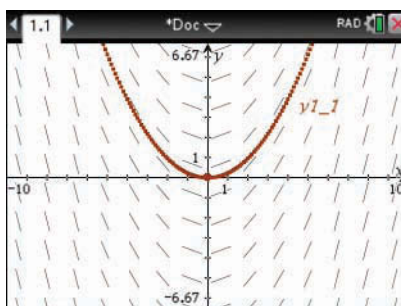
x

2

Followed by ENTER

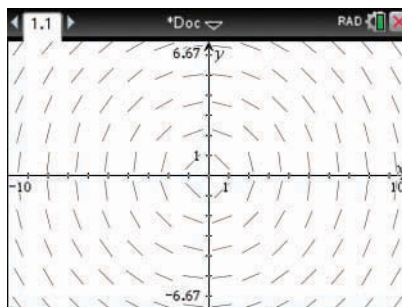
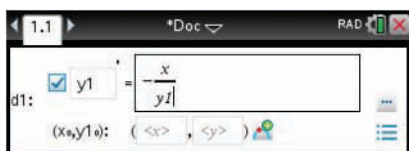


4. A particular solution can be drawn if the initial conditions are known.
5. Assume the initial conditions for the differential equation are (0, 0). Press TAB to view the entry line and insert 0 and 0 into initial conditions coordinate template as shown. Press ENTER to see the particular solution.



Note: if the differential equation is of the form $\frac{dy}{dx} = f(y)$ or $\frac{dy}{dx} = f(x, y)$ the y variable must be entered as y followed by the function number (rather than simply y).

For example, to sketch a slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ you should enter $-\frac{x}{y1}$ into the entry line for $y1' =$ as shown.



9.7.3 Interpreting a slope field

As previously shown technology can be used to draw slope fields and questions involving slope fields are sometimes designed to analyse the fields. There are four common question types:

- Given a differential equation, choose the correct slope field.
- Given a solution of a differential equation, choose the correct slope field.
- Given a slope field, choose the correct differential equation.
- Given a slope field, choose the solution of the differential equation.

When analysing a slope field, the following approaches are useful.

- Determine the values where the slope is zero; that is, the values of x and y for which $\frac{dy}{dx} = 0$.
- Determine values for which the slope is parallel to the y -axis. In this case, the slope is infinite, and if the differential equation is of the form $\frac{dy}{dx} = \frac{g(x, y)}{h(x, y)}$, then the denominator $h(x, y)$ equals 0.
- Determine the values of the slopes along the x - and y -axes.
- Determine if the slopes are independent of x and therefore depend only on the y -value. In this case, the differential equation is of the form $\frac{dy}{dx} = f(y)$.
- Determine if the slopes are independent of y and therefore depend only on the x -value. In this case, the differential equation is of the form $\frac{dy}{dx} = f(x)$. (Differential equations of only one variable of the types

$\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = f(y)$ are called autonomous differential equations.)

- Determine where the slopes are positive and where the slopes are negative.
- The symmetry of the slope field and the slopes in each of the four quadrants can give information on the solution.
- Determine if there is no slope field for certain values. This could indicate a required domain.

Note that we are not solving the differential equation when we analyse a slope field. However, if we are given a solution, we can use implicit differentiation to understand the families of curves that are represented.

WORKED EXAMPLE 17

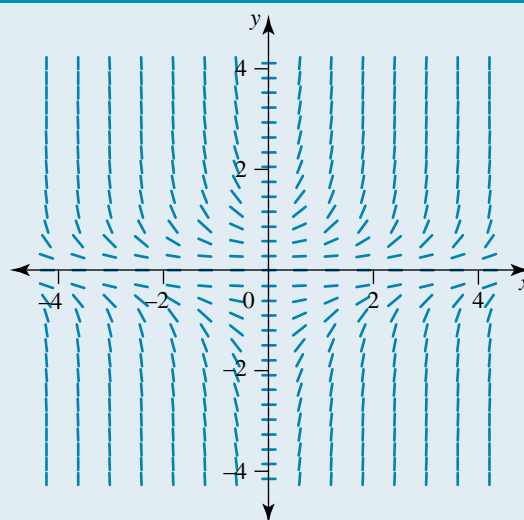
The differential equation that best represents the slope field shown is:

A. $\frac{dy}{dx} = \frac{y^2}{x}$

B. $\frac{dy}{dx} = \frac{x^2}{y}$

C. $\frac{dy}{dx} = yx^2$

D. $\frac{dy}{dx} = y^2x$



THINK

1. Consider the y-axis.
2. Consider the first quadrant.
3. Consider the second quadrant.
4. Consider the third quadrant.
5. Consider the fourth quadrant.
6. State the result.

WRITE

When $x = 0$ (i.e. along the y-axis), the slopes are all zero. Option A is incorrect.

When $x = 1$ and $y = 1$, the slopes are 1. This satisfies B, C and D.

When $x = -1$ and $y = 1$, the slopes are -1 . Option C is incorrect.

When $x = -1$ and $y = -1$, the slopes are 1. Options B and D are still valid.

When $x = 1$ and $y = -1$, the slopes are 1. Option B is incorrect.

Option D, $\frac{dy}{dx} = y^2x$, is the only differential equation that is represented by this slope field.

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Exercise 9.7 Slope fields**Technology free**

For questions 1–6, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

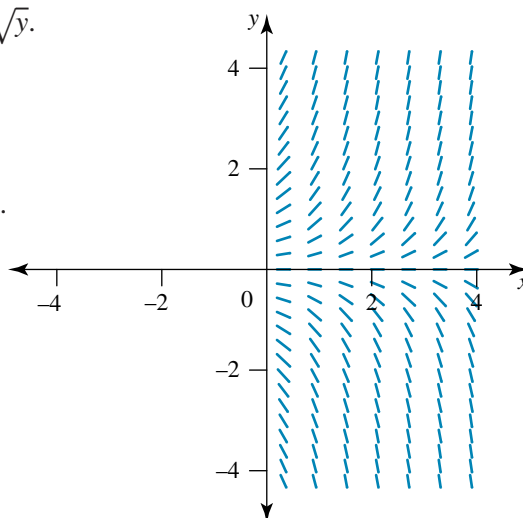
1. **WE14** Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{x}$.
2. Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.
3. **WE15** Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{2}{y}$.
4. Sketch the slope field for the differential equation $\frac{dy}{dx} = 2\sqrt{y}$.
5. **WE16** Sketch the slope field for the differential equation $y\frac{dy}{dx} - x = 0$.
6. Sketch the slope field for the differential equation $\frac{dy}{dx} = xy$.
7. **WE17 MC** The differential equation that best represents the slope field shown is:

A. $\frac{dy}{dx} = \sqrt{xy}$

B. $\frac{dy}{dx} = x\sqrt{y}$

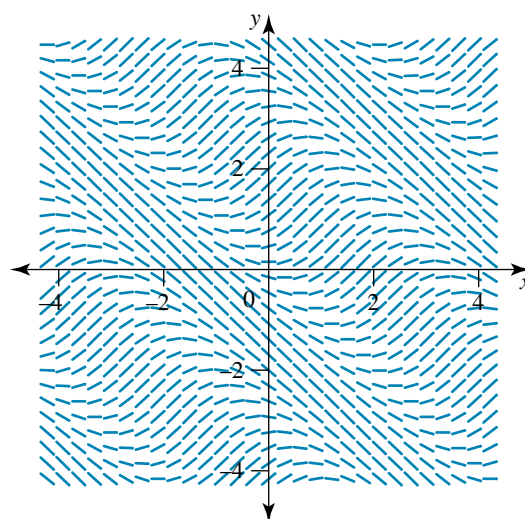
C. $\frac{dy}{dx} = y\sqrt{x}$

D. $\frac{dy}{dx} = \frac{1}{\sqrt{xy}}$



8. **MC** The differential equation that best represents the slope field shown is:

- A. $\frac{dy}{dx} = \cos(x + y)$
- B. $\frac{dy}{dx} = \cos(x - y)$
- C. $\frac{dy}{dx} = \sin(x - y)$
- D. $\frac{dy}{dx} = \sin(x + y)$



For questions 9–13, sketch the fields for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$.

9. a. Sketch the slope field for the differential

equation $\frac{dy}{dx} = \frac{2}{(x-1)^2}$.

- b. i. Sketch the slope field for the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 1}$.

- ii. Solve the differential equation $\frac{dy}{dx} = \frac{1}{x^2 + 1}$, $y(0) = 0$ and sketch the solution curve on the diagram for part i.

10. a. Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2$.

- b. i. Sketch the slope field for the differential equation $\frac{dy}{dx} = y^2 + 1$.

- ii. Solve the differential equation $\frac{dy}{dx} = y^2 + 1$, $y(0) = 0$ and sketch the solution curve on the diagram for part i.

11. Sketch the slope fields for each of the following differential equations.

a. $\frac{dy}{dx} = \frac{-x}{y+1}$

b. $\frac{dy}{dx} = \frac{x-2}{y}$

12. Sketch the slope fields for each of the following differential equations.

a. $\frac{dy}{dx} = (y-1) \cos\left(\frac{\pi x}{2}\right)$

b. $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi x}{2}\right)$

13. a. Sketch the slope field for $\frac{dy}{dx} = x + y$.

- b. Given that $y = Ae^{kx} + Bx + C$ is a solution of the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 0$, determine the values of A, B, C , and k .

- c. Show that the solution curve has a turning point at $(0, 0)$.

- d. On the slope field, sketch the particular solution to the differential equation $\frac{dy}{dx} = x + y$ that passes through the point $(0, 0)$.

14. Match the four differential equations to the four slope fields shown below.

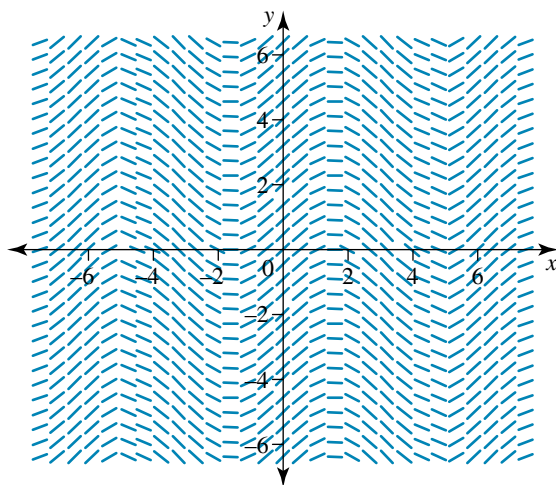
a. $\frac{dy}{dx} + \sin(x) = 0$

b. $\frac{dy}{dx} + \cos(x) = 0$

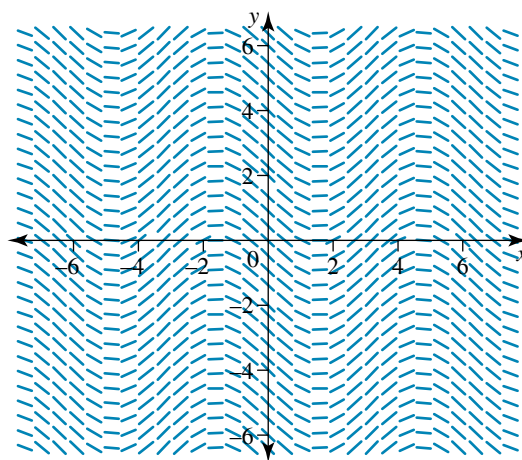
c. $\frac{dy}{dx} - \sin(x) = 0$

d. $\frac{dy}{dx} - \cos(x) = 0$

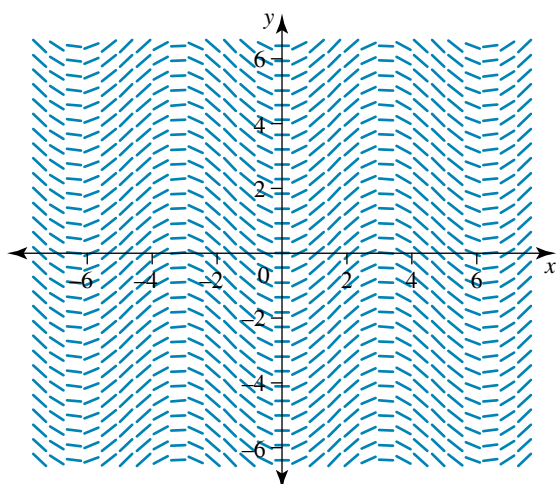
Slope 1



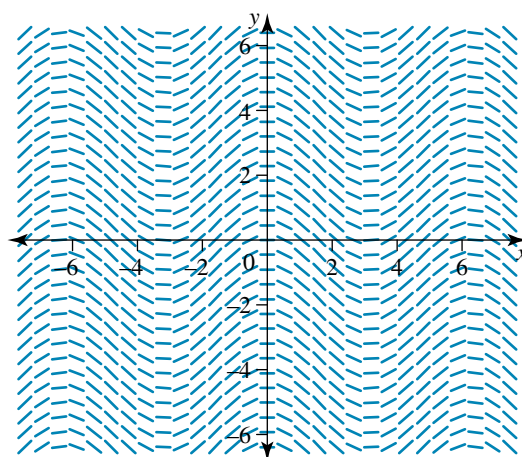
Slope 2



Slope 3



Slope 4



15. Match the four differential equations to the four slope fields shown on the next page.

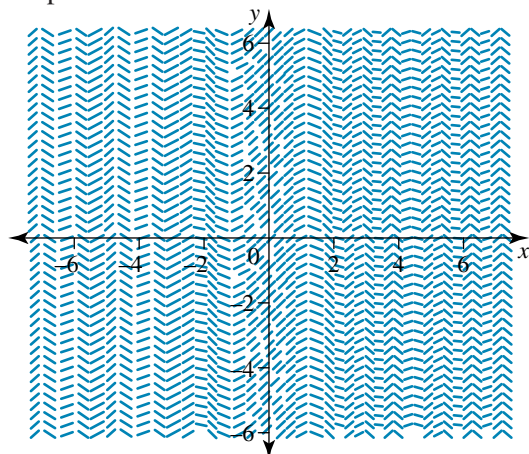
a. $\frac{dy}{dx} = \sin(x^2)$

b. $\frac{dy}{dx} = \cos(x^2)$

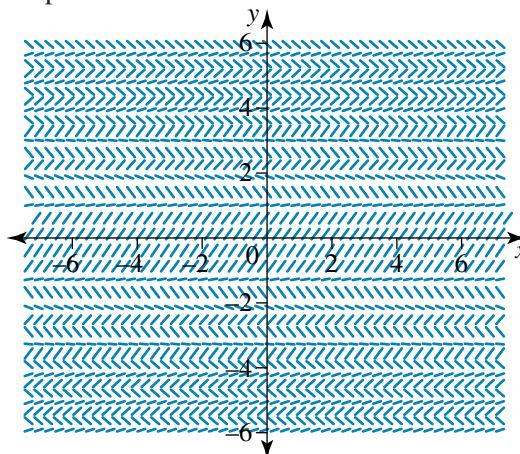
c. $\frac{dy}{dx} = \sin(y^2)$

d. $\frac{dy}{dx} = \cos(y^2)$

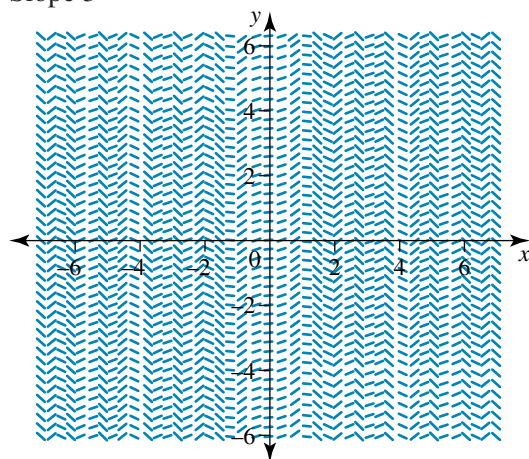
Slope 1



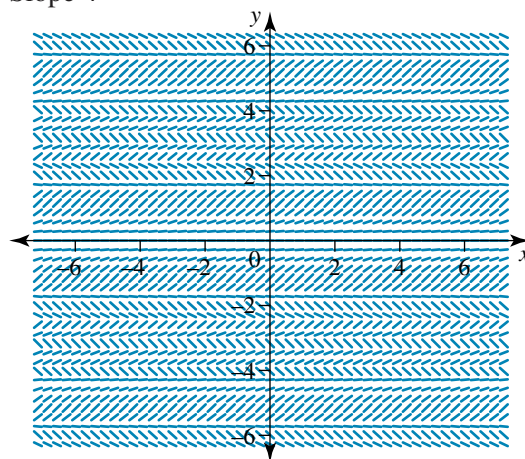
Slope 2



Slope 3



Slope 4



16. Match the four differential equations to the four slope fields shown below.

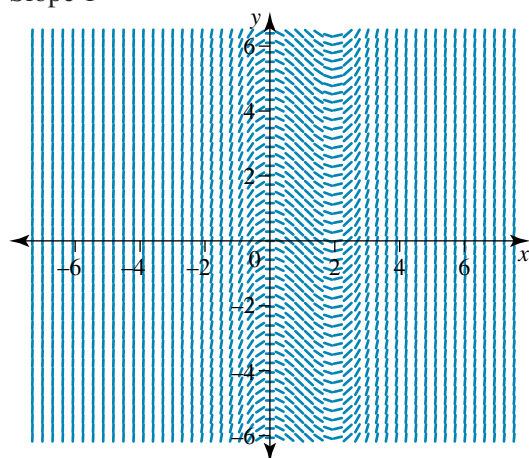
a. $\frac{dy}{dx} = x(x - 2)$

b. $\frac{dy}{dx} = y(y - 2)$

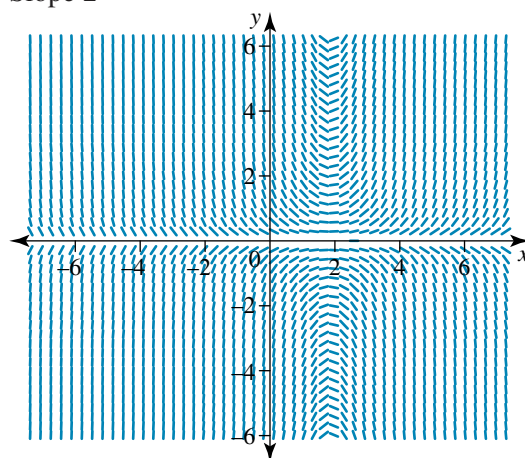
c. $\frac{dy}{dx} = y(x - 2)$

d. $\frac{dy}{dx} = x(y - 2)$

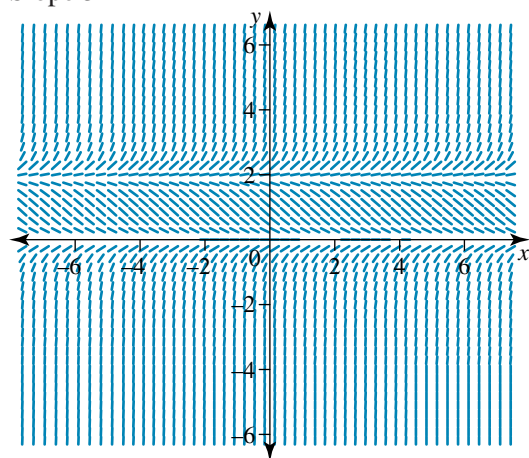
Slope 1



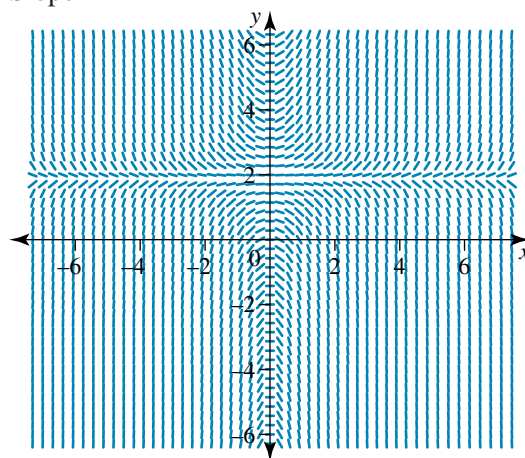
Slope 2



Slope 3



Slope 4



17. Match the four differential equations to the four slope fields shown below.

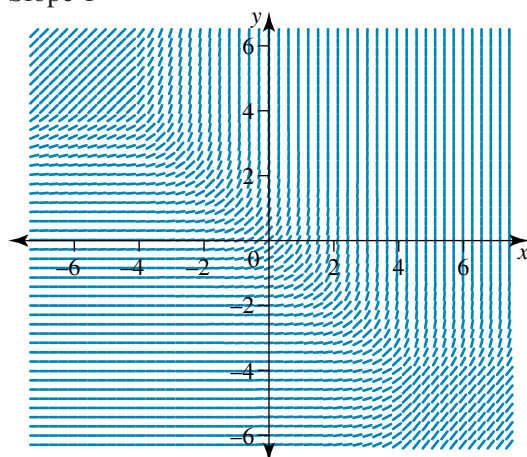
a. $\frac{dy}{dx} = e^{x-y}$

b. $\frac{dy}{dx} = e^{y-x}$

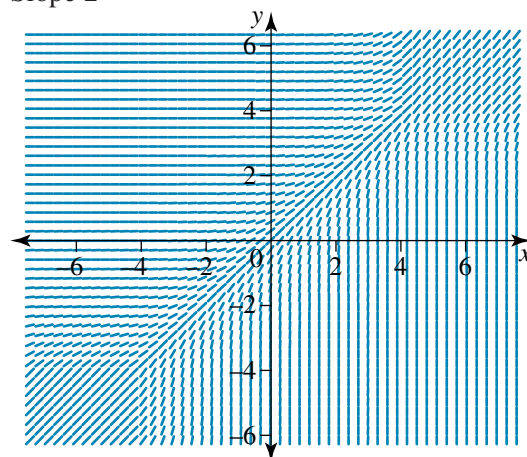
c. $\frac{dy}{dx} = e^{x+y}$

d. $\frac{dy}{dx} = e^{-(x+y)}$

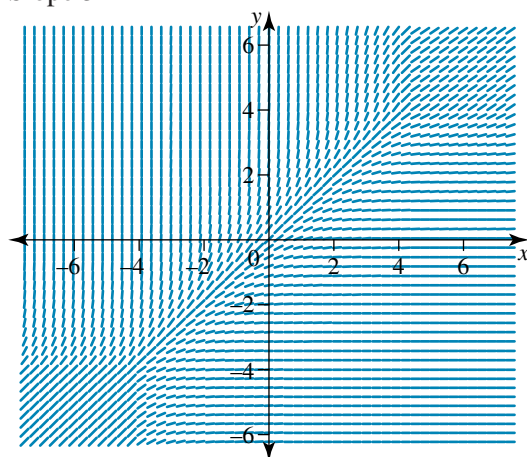
Slope 1



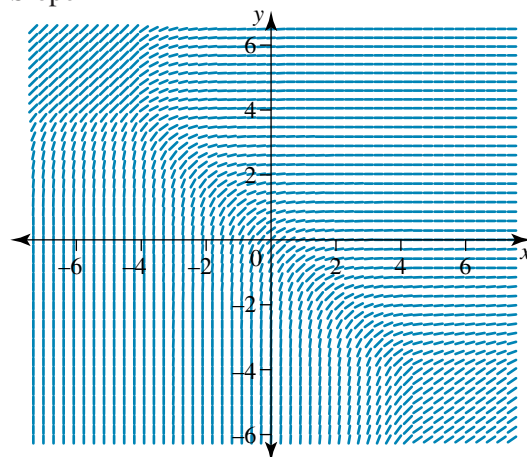
Slope 2



Slope 3



Slope 4



18. Match the four differential equations to the four slope fields shown below.

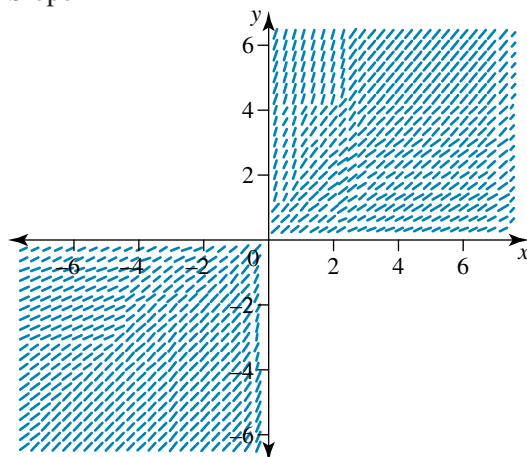
a. $\frac{dy}{dx} = \sqrt{\frac{y}{x}}$

b. $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$

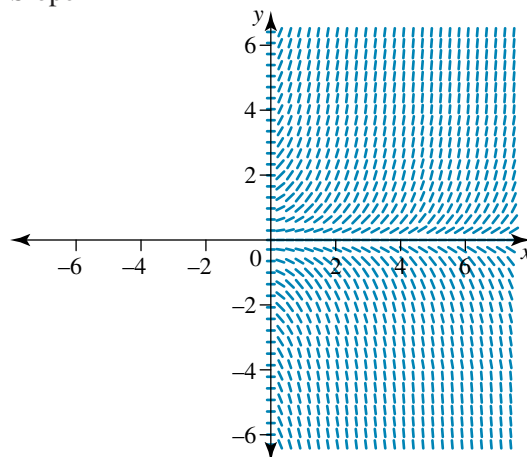
c. $\frac{dy}{dx} = y\sqrt{x}$

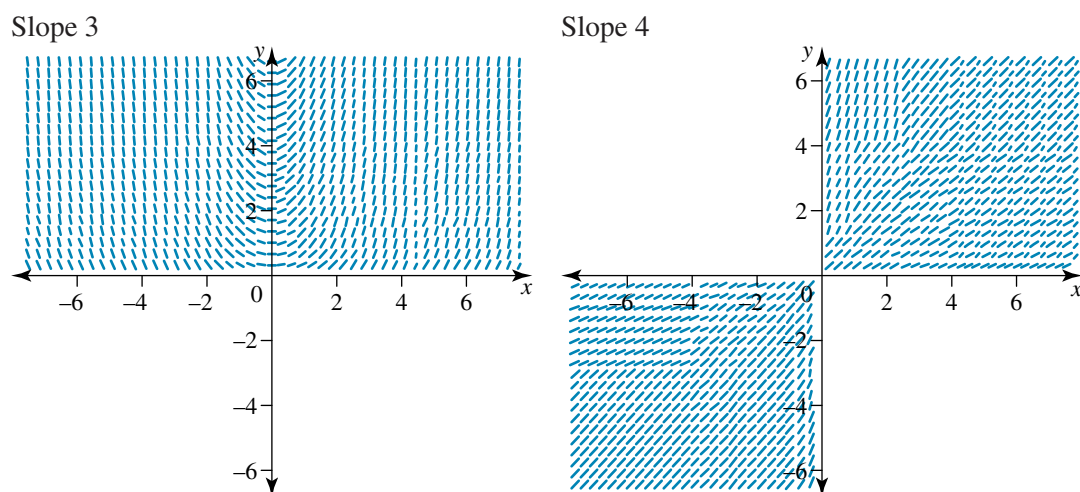
d. $\frac{dy}{dx} = x\sqrt{y}$

Slope 1



Slope 2





9.8 Review: exam practice

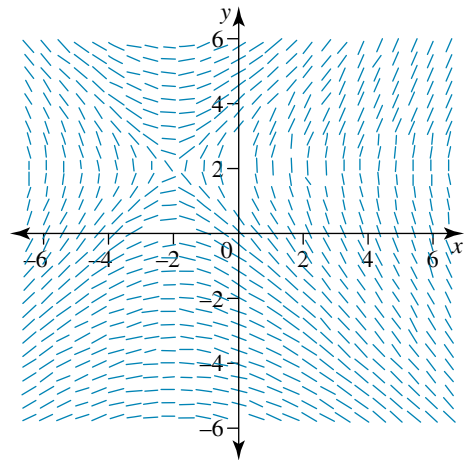
A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. The number of rabbits in a certain area increases at a rate proportional to the number present. If the number of rabbits doubles in four months, how many months elapse before the number of rabbits triples?
2. A radioactive substance has a half-life of 5 years. After 3 years how much of the substance has disintegrated?
3. When light passes through a medium it loses its intensity as it penetrates depths. The rate of loss of intensity with respect to the depth is proportional to the intensity at that depth. If 5% of light is lost in penetrating 40% of a glass slab, calculate the percentage lost penetrating the whole slab (give your answer correct to one decimal place.)
4. A container initially holds 10 litres of water. A salt solution containing 5 kg per litre is poured into the tank at a rate of 6 litres per minute and the solution is kept uniform by stirring. If the mixture flows out at a rate of 4 litres per minute and Q kg is the amount of salt in the tank at time t minutes, write an expression for $\frac{dQ}{dt}$.
5. A cake is taken out of an oven, which has been kept at a constant temperature of 180°C , and placed on a table where the temperature is 18°C . Using Newton's law of cooling, write a differential equation where the temperature $T^\circ\text{C}$, of the cake at a time t minutes which satisfies these conditions.
6. Let $W(t)$ represent the number of wolves in a certain area at time t months. Initially there are 200 wolves present and the number of wolves is increasing at a rate directly proportional to the product of the number of wolves and to $200 - W(t)$. The wolves are also being killed off at 20 per month. If k is a positive constant, write the differential equation for the number of wolves present.
7. If a logistic equation has as its solution $N(t) = \frac{200}{1 + 99e^{-kt}}$ where $k > 0$, write an expression for $\frac{dN}{dt}$.



8. A logistic equation has the solution $N(t) = \frac{500}{1 + 9e^{-kt}}$ where N is the population number of a city in millions at a time t years and k is a positive real constant. For what value of the population is the population increasing most rapidly?
9. **MC** The slope field for a certain differential equation is shown.



The differential equation could be:

- A. $\frac{dy}{dx} = -\left(\frac{x+2}{y-2}\right)$
 B. $\frac{dy}{dx} = \frac{y+2}{x-2}$
 C. $\frac{dy}{dx} = \frac{x-2}{y+2}$
 D. $\frac{dy}{dx} = \frac{x+2}{y-2}$
10. The number of mice in a certain area grows at a rate proportional to the current number.
 Initially there are four mice present and after three months the number of mice has grown to 32.
- Calculate the number of mice in the area after one year.
 - After how many months were there 1000 mice present?
11. The rate of disintegration of a radioactive element is proportional to the amount of the element remaining at any time. If 30% disintegrates in 3 years, find the half-life.
12. When light passes through a medium it loses its intensity as it penetrates depths. If I is the intensity of light at depth x , the rate of loss of intensity with respect to the depth is proportional to the intensity at that depth, so that $\frac{dI}{dx} = -kI$, where k is a positive constant. Solve the differential equation to show that $I = I_0 e^{-kx}$.



Complex familiar

13. A hospital patient is receiving a drug at a constant rate of R milligrams per hour through a drip. At a time t hours the amount of the drug in the patient is x milligrams. The rate of loss of the drug from the patient is proportional to the amount of the drug in the patient. Initially there are no amounts of the drug in the patient. Given that $\frac{dx}{dt} = R - kx$ where k is a given constant, solve the differential equation to express x in terms of t , k and R .
14. A lake has a capacity to hold a maximum of 50 ducks. Initially there are 4 ducks in the lake. The number of ducks in the lake grows at a rate proportional to the difference between the maximum number and the current number of ducks in the lake. After 4 months there are 16 ducks in the lake.
- Write down and solve the differential equation for the number of ducks n in the lake after t months.
 - How many ducks are in the lake after 8 months?



15. A gas BBQ is used for cooking and is outdoors where the surrounding temperature is constant at 15°C . When the gas is ignited the BBQ heats up at a rate 25°C per minute, but also cools according to Newtons law of cooling. When the temperature of the BBQ grill is 75°C , the temperature of the BBQ is increasing at a rate of 45°C per minute.



- Show that differential equation representing the temperature $T^\circ\text{C}$ of the grill on the BBQ, at a time t minutes after the gas has been ignited, satisfies $\frac{dT}{dt} = \frac{60 + T}{3}$.
 - Solve the differential equation.
 - Determine how long it takes the grill of the BBQ to reach a suitable cooking temperature of 185°C .
16. A tank contains 20 litres of water. A salt solution of concentration 2 grams per litre is poured into the tank at a rate of 6 litres per minute, and the well-stirred mixture flows out at a rate of 4 litres per minute.
- Set up the differential equation for Q the amount of salt in grams in the tank after a time t minutes.
 - Verify, by differentiation that $Q = \frac{4t(t^2 + 30t + 300)}{(t + 10)^2}$ is a solution of the differential equation.

Complex unfamiliar

17. A water fountain has a volume of 800 litres. Someone has dissolved 4 kg of detergent into the fountain. To remove the detergent concentration, fresh water is added to the fountain at a rate of 4 litres per minute and the solution is kept uniform by stirring. The mixture leaves the fountain at a rate of 4 litres per minute. Setup and solve the differential equation for Q the amount of the detergent in kg in the fountain at a time t minutes after the fresh water started to be added. How long before there is only 200 grams of detergent in the fountain?
18. Let $N(t)$ be the population number in an environment, which can support a maximum number of P . The population grows at a rate proportional to both the current population and the difference between the maximum number P and the current population. Initially the population number is $\frac{P}{10}$, and initially is growing at a rate $\frac{3P}{100}$.
- Show that the differential equation is given by $\frac{dN}{dt} = \frac{N(P - N)}{3P}$, $N(0) = \frac{P}{10}$.
 - Solve the differential expressing N in terms of t .
 - Show that the population is increasing most rapidly when $t = 3 \ln(9)$.
19. A biologist is studying the growth of two different strands of bacteria. She has two different experimental models, and both start at exactly the same time.

Type 1

The number of type 1 bacteria follows a logistic growth rate. Initially, there are 50 type 1 bacteria present, and it is known that the number of these bacteria cannot rise above 1000. The number of type 1 bacteria is increasing most rapidly before 10 hours and the rate slows down after 10 hours.

- Write and solve the differential equation for the number of type 1 bacteria present at t hours after the experiment starts.

Type 2

The number of type 2 bacteria follows the law of natural growth. Initially, there are 25 type 2 bacteria present, and after 18 hours there are 800 type 2 bacteria present.

- b. Write and solve the differential equation for the number of type 2 bacteria present at t hours after the experiment starts.
- c. After how many hours are there equal numbers of type 1 and type 2 bacteria present? Determine the actual number of each type.
- d. Sketch the graphs for both bacteria on the same set of axes.
20. Given the general logistic equation $\frac{dN}{dt} = kN \left(1 - \frac{N}{P}\right)$, $N(0) = N_0$, where k is a positive constant and $P > N_0 > 0$:
- a. show, using integration, that $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$
- b. show that $\lim_{t \rightarrow \infty} N(t) = P$
- c. show that $\frac{d^2N}{dt^2} = k^2N \left(1 - \frac{2N}{P}\right) \left(1 - \frac{N}{P}\right)$ and hence that there is a point of inflection at $\left(\frac{1}{k} \ln\left(\frac{P}{N_0} - 1\right), \frac{P}{2}\right)$
- d. verify, using differentiation, that $N(t) = \frac{PN_0}{N_0 + (P - N_0)e^{-kt}}$ is a solution of $\frac{dN}{dt} = kN \left(1 - \frac{N}{P}\right)$, $N(0) = N_0$.

study on

Units 3 & 4 Sit exam

Answers

9 Applications of first-order differential equations

Exercise 9.2 Growth and decay

- $N = 1.79 \times 10^6 e^{0.0277t}$
 - 2.71 million
 - 2052
- 137
- 4146
 - 11 275
- 942
- 27 851 607
 - 2029
- 8 220 578 135
 - 2039
- 58.31 mg
- 113.78 mg
- 12.8 years
- 65%
- 0.29 g
 - 25.59 years
- 32.47 days
- 38.2 years
- 78.8%
 - 24 110 years
- 11.76%
- 6.180 hours
- Sample responses can be found in the worked solutions in the online resources.
 - 2039
 - 2059
 - 2147
- $Q(t) = Q_0 e^{-k(t-t_0)}$
 - 1.21×10^{-4}
 - 10 356 years

Exercise 9.3 Other applications of first-order differential equations

- \$9048.37
- \$630 000
- $P(h) = P_0 e^{kh}, k > 0$
 - 50.94 cm Hg
- $D(t) = D_0 e^{-kt}$
 - 2.08 hours
- Sample responses can be found in the worked solutions in the online resources.
 - 18.5%
- Sample responses can be found in the worked solutions in the online resources.
 - 205.31 newtons
- 25 million
- 140
- Sample responses can be found in the worked solutions in the online resources.
 - 35.48 ohms
 - 1000 units
 - 1.5 seconds
- $i(t) = i_0 e^{\frac{Rt}{L}}$
 - $Q(t) = Q_0 e^{-\frac{t}{RC}}$
- $\frac{dN}{dt} = \frac{N-160}{4}, N(0) = 320$
 - $N(t) = 160 \left(1 + e^{\frac{t}{4}}\right)$
 - 1342
- Sample responses can be found in the worked solutions in the online resources.
- $\frac{dN}{dt} = \frac{N-100}{20}, N(0) = 200$

$$b. N(t) = 100 \left(1 + e^{\frac{t}{20}}\right)$$

$$c. 216$$

$$14. 666\,052$$

$$15. a. 11 \text{ years}$$

$$b. 19$$

$$16. a. N(t) = \left(\frac{1}{2}kt + \sqrt{N_0}\right)^2$$

$$b. N(t) = \sqrt{2kt + N_0^2}$$

$$c. N(t) = \frac{4N_0}{(2 - kt\sqrt{N_0})^2}$$

$$d. N(t) = \frac{N_0}{1 - ktN_0}$$

17. Sample responses can be found in the worked solutions in the online resources.

18. Sample responses can be found in the worked solutions in the online resources.

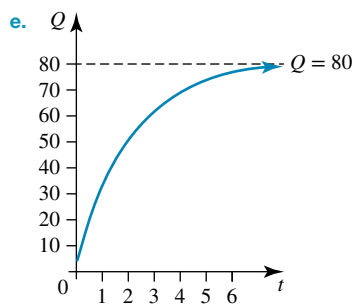
Exercise 9.4 Bounded growth and Newton's law of cooling

- 165.4 g
- 13 months
- 4.4 kg
 - 3.06 kg
- 736
 - 35.1 months
- 10.5 kg
- 26.34 min
- 1.75 min
- 5.43°C
 - 26°C
- 5.27 min
- 5°C
- 108.37°C
- $i(t) = \frac{E}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$
 - Sample responses can be found in the worked solutions in the online resources.
- $v(t) = E \left(1 - e^{-\frac{t}{RC}}\right)$
- Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- 9:28 pm
- $t_1 = \frac{3}{2}, T_h = 95$
 - $t_2 = \frac{3}{4}, T_c = 3$
 - 1.12
- Sample responses can be found in the worked solutions in the online resources.
 - Monthly repayments: \$2146.52, total interest: \$215 164.60
 - \$17 308.20
 - 11.8 years
 - 10.56%

Exercise 9.5 Chemical reactions and dilution problems

- $\frac{dQ}{dt} = 3 - \frac{3Q}{200-t}, Q(0) = 20$
 - Sample responses can be found in the worked solutions in the online resources.
 - $-\frac{7}{200\,000}$
 - 1.15 kg/L
- $\frac{dQ}{dt} = 6b - \frac{3Q}{50+3t}$
 - 2

3. a. i. $\frac{dQ}{dt} = 12 - \frac{5Q}{50 - t}$
 ii. Sample responses can be found in the worked solutions in the online resources.
- b. i. $\frac{dQ}{dt} = 4 - \frac{2Q}{30 - t}, Q(0) = 0$
 ii. Sample responses can be found in the worked solutions in the online resources.
4. a. i. $\frac{dQ}{dt} = 6 - \frac{Q}{20 + t}, Q(0) = 0$
 ii. Sample responses can be found in the worked solutions in the online resources.
- b. i. $\frac{dQ}{dt} = 8 - \frac{3Q}{20 - t}, Q(0) = 0$
 ii. Sample responses can be found in the worked solutions in the online resources.
5. a. i. $\frac{dQ}{dt} = 20 - \frac{3Q}{64 + 2t}, Q(0) = 128$
 ii. Sample responses can be found in the worked solutions in the online resources.
- iii. -2^{16}
 iv. 3.34 g/L
- b. 3
6. a. i. $\frac{dQ}{dt} = \frac{3(80 - Q)}{20}, Q(0) = 5$
 b. $Q(t) = 5 \left(16 - 15e^{-\frac{3t}{20}} \right)$
 c. 1.05 g/L
 d. Sample responses can be found in the worked solutions in the online resources.

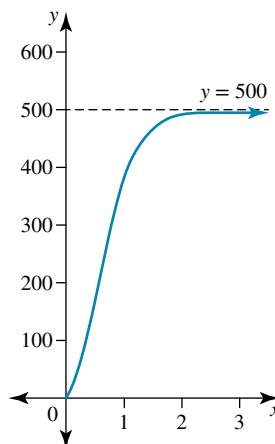


7. a. $\frac{dQ}{dt} = \frac{4(75 - Q)}{15}, Q(0) = 0$
 b. $Q(t) = 75 \left(1 - e^{-\frac{4t}{15}} \right), t \rightarrow \infty, Q \rightarrow 75$
 c. 6.04 min
8. a. i. $30e^{-\frac{t}{120}}$ ii. 11.04 kg iii. 83.18 min
 b. 1.659 g/L
9. a. i. 9.12 min ii. 34.66 min
 b. 5.5 min
10. a. $\frac{dx}{dt} = k(2 - x)(6 - x), x(0) = 0$
 b. $x(t) = \frac{6(1 - e^{-4kt})}{3 - e^{-4kt}}$ where $k = \frac{1}{12} \ln\left(\frac{5}{3}\right)$
 c. $\frac{16}{11}$ g
11. a. $\frac{dx}{dt} = k(8 - x)^2, x(0) = 0$
 b. $x(t) = \frac{24t}{3t + 10}$

- c. 8 minutes (for 10 minutes total after the start of the reaction)
 d. 8 g
12. a. $\frac{dx}{dt} = k(4 - x)(8 - x), x(0) = 0$
 b. $x(t) = \frac{8(1 - e^{-4kt})}{2 - e^{-4kt}}$ where $k = \frac{1}{8} \ln\left(\frac{7}{6}\right)$
 c. $\frac{52}{31}$ g
 d. 8 g
13. a. 6.57 minutes b. $\frac{26}{9}$ g
 c. 4 g
14. a. i. Sample responses can be found in the worked solutions in the online resources.
 ii. 3.61 g
 b. i. Sample responses can be found in the worked solutions in the online resources.
 ii. 3.85 g
 iii. 10 g
15. a. i. $\frac{dQ}{dt} = \frac{f}{V_0}(bV_0 - Q), Q(0) = 0$
 ii. Sample responses can be found in the worked solutions in the online resources.
 b. i. $\frac{dQ}{dt} = \frac{f}{V_0}(bV_0 - Q), Q(0) = q_0$
 ii. Sample responses can be found in the worked solutions in the online resources.
16. Sample responses can be found in the worked solutions in the online resources.

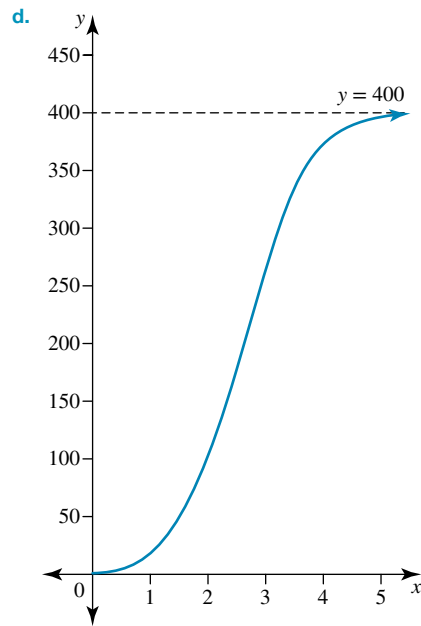
Exercise 9.6 The logistic equation

1. a. $N(t) = \frac{100}{1 + 24e^{-0.8959t}}$
 b. 78
 c. 3.55 weeks
2. a. $N(t) = \frac{80}{1 + 39e^{-0.855t}}$
 b. 72
 c. 4.76 days
3. a, b Sample responses can be found in the worked solutions in the online resources.
 c. $\left(\frac{1}{3} \ln(9), 250 \right)$
 d.



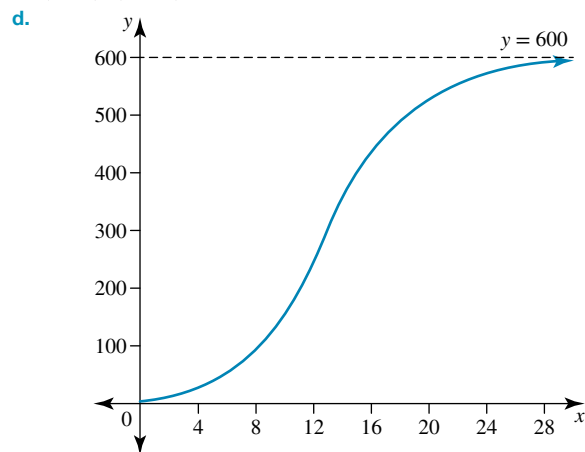
4. **a, b** Sample responses can be found in the worked solutions in the online resources.

c. $\left(\frac{1}{2} \ln(199), 200\right)$



5. **a, b** Sample responses can be found in the worked solutions in the online resources.

c. $(3 \ln(99), 300)$



6. **a.** $\frac{dN}{dt} = \frac{N(4000 - N)}{50\,000}, N(0) = 500$

b. $N(t) = \frac{4000}{1 + 7e^{-\frac{2t}{5}}}$

c. 2054

d. 7.61 years

7. **a.** $\frac{dN}{dt} = kN(360 - N), N(0) = 1$

b. 5 days (4.9) days

8. **a.** $\frac{dN}{dt} = kN(2000 - N), N(0) = 2$

b. 5 days (4.4 days)

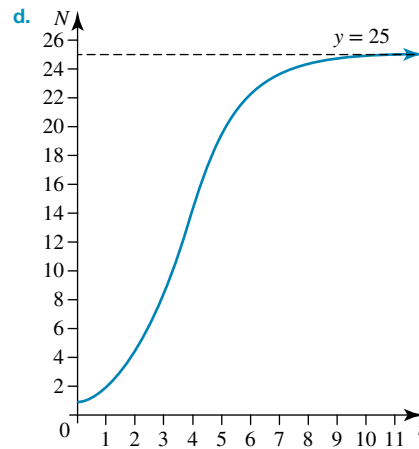
c. 5 days (4.4 days)

9. 7374

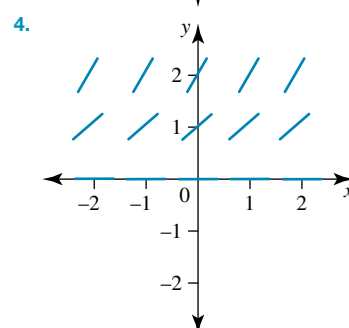
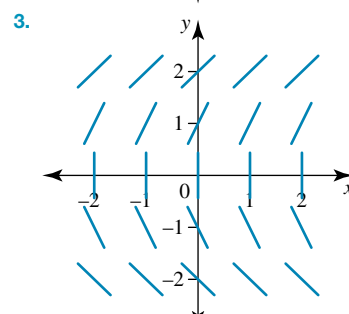
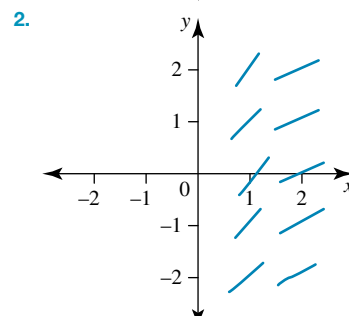
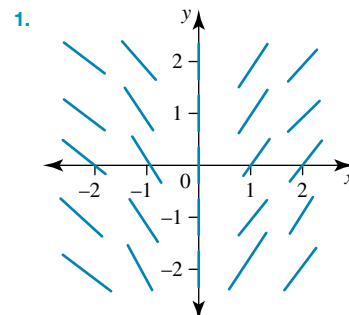
10. **a.** 25

c. 22

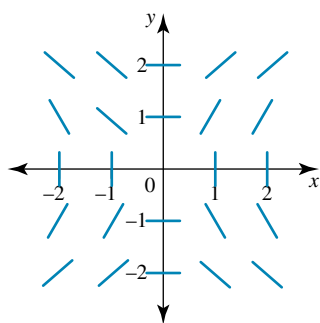
b. 4 days (3.55 days)



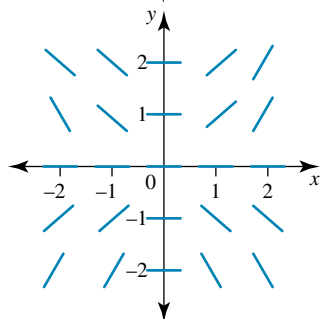
Exercise 9.7 Slope fields



5.



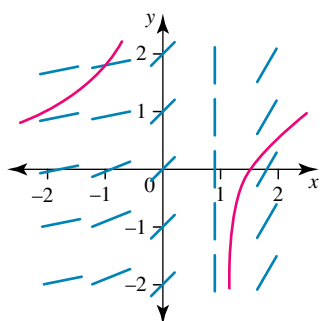
6.



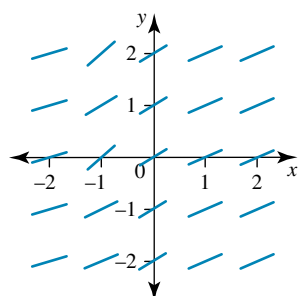
7. C

8. D

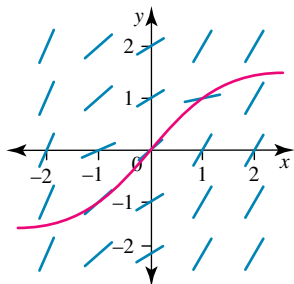
9. a.



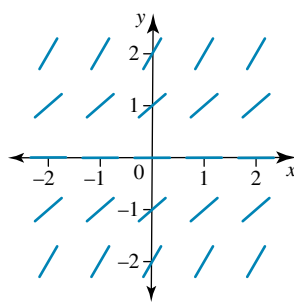
b. i.



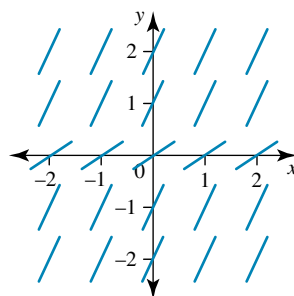
ii. $y = \tan^{-1}(x)$



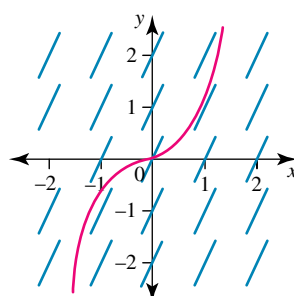
10. a.



b. i.

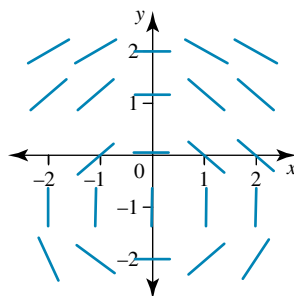


ii.

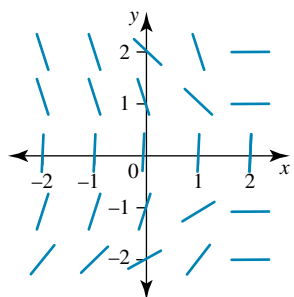


$y = \tan(x)$

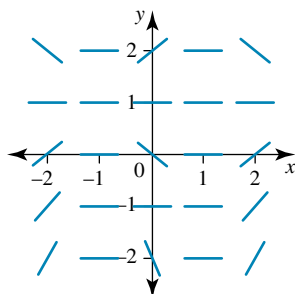
11. a.



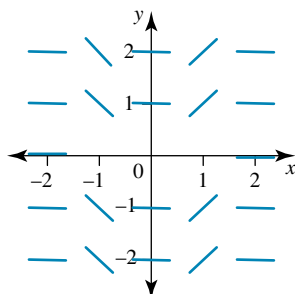
b.



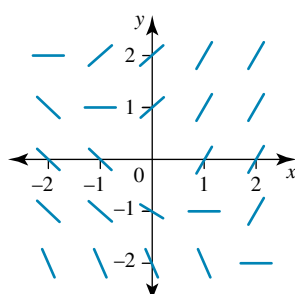
12. a.



b.

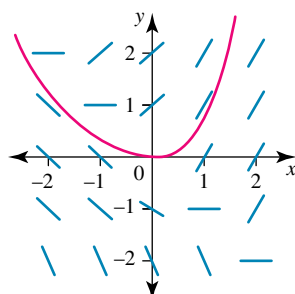


13. a.

b. $A = 1, B = -1, C = -1, k = 1$

c. Sample responses can be found in the worked solutions in the online resources.

d.



14. a. Slope 4	b. Slope 2	c. Slope 3	d. Slope 1
15. a. Slope 3	b. Slope 1	c. Slope 4	d. Slope 2
16. a. Slope 1	b. Slope 3	c. Slope 2	d. Slope 4
17. a. Slope 2	b. Slope 3	c. Slope 1	d. Slope 4
18. a. Slope 1	b. Slope 4	c. Slope 2	d. Slope 3

9.8 Review: exam practice

- $\frac{\ln(81)}{\ln(2)} \approx 6.3$ months
- 34% has disintegrated.
- 12%
- $\frac{dQ}{dt} = 30 - \frac{4Q}{10 + 2t}$
- $\frac{dT}{dt} = k(T - 18), T(0) = 180$ where $k < 0$
- $\frac{dW}{dt} = kW(200 - W) - 20, W(0) = 200$

7. $\frac{dN}{dt} = kN \left(1 - \frac{N}{200}\right), N(0) = 2$

8. 250

9. D

10. a. $4 \times 2^{12} = 16\,384$

b. $t = \log_2(250) \approx 8$ months

11. $T_{\frac{1}{2}} = \frac{-3 \ln(2)}{\ln(0.7)} \approx 5.83$ years

12. Sample responses can be found in the worked solutions in the online resources.

13. $x(t) = \frac{R}{k} (1 - e^{-kt})$

14. a. $n(t) = 2(25 - 23e^{-kt}), k = \frac{1}{4} \ln\left(\frac{23}{17}\right)$

b. $\frac{572}{23} \approx 24$ ducks

15. a. Sample responses can be found in the worked solutions in the online resources.

b. $T(t) = 15 \left(5e^{\frac{t}{3}} - 4\right)$

c. $t = 3 \ln\left(\frac{49}{15}\right) \approx 3.55$ minutes

16. a. $\frac{dQ}{dt} = 12 - \frac{2Q}{10 + t}, Q(0) = 0$

b. Sample responses can be found in the worked solutions in the online resources.

17. $Q(t) = 4e^{-\frac{t}{200}}, 10$ hours

18. a. Sample responses can be found in the worked solutions in the online resources.

b. $N(t) = \frac{P}{1 + 9e^{-\frac{t}{3}}}$

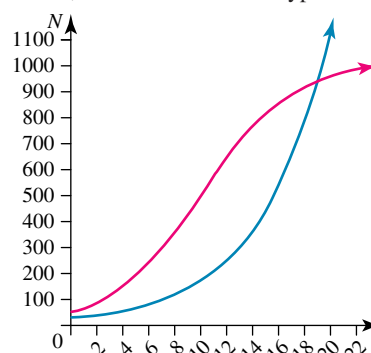
c. Sample responses can be found in the worked solutions in the online resources.

19. a. $n(t) = \frac{1000}{1 + 19e^{-\frac{t}{10} \ln(19)}}$

b. $m(t) = 25e^{\frac{t}{18} \ln(32)}$

c. 18.8 h, 930 bacteria of each type

d.



20. a. Sample responses can be found in the worked solutions in the online resources.

b. Sample responses can be found in the worked solutions in the online resources.

c. Sample responses can be found in the worked solutions in the online resources.

d. Sample responses can be found in the worked solutions in the online resources.

10 Modelling motion 1

10.1 Overview

Galileo Galilei (1564–1642) was a famous Italian scientist and mathematician. His discoveries using a telescope changed the way astronomers looked at the Earth. He advocated for the theory that was revolutionary – that the Earth revolved around the Sun, not the other way around. This theory was considered a heresy by the Roman Catholic Church at that time and Galileo was placed under house arrest for promoting the theory.

Galileo also believed that the natural world could be best understood using mathematics and through his efforts the study of nature moved from a descriptive and qualitative form to one which was more mathematical and experimental.

He also made many contributions to the study of motion. There are many situations of practical significance that involve motion under constant acceleration. The common example of bodies in free fall towards the Earth's surface, regardless of their size, mass or composition, involves motion under constant acceleration — that of acceleration due to gravity.

Galileo illustrated this property by dropping various sized cannon balls from the top of the Leaning Tower of Pisa. Provided the effects of air resistance are minimised (that is, the object should not be too large in area or fall too far) all bodies will fall with constant acceleration.

The acceleration due to gravity depends on the distance from the centre of the Earth, it varies slightly at different places on the Earth's surface — being greater at the poles than at the equator and less at high altitudes. It is usual to state the constant of acceleration due to gravity as 9.8 m/s^2 unless stated otherwise.



LEARNING SEQUENCE

10.1 Overview

10.2 A body in equilibrium under concurrent forces

10.3 Action and reaction forces

10.4 Momentum and resultant force

10.5 Forces on connected particles

10.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 A body in equilibrium under concurrent forces

10.2.1 Statics

Statics is the study of the **equilibrium** of a particle under the action of concurrent coplanar forces.

10.2.2 Some definitions

Particle

A particle is a body or an object with a mass such that all of the mass acts through one point. Large objects such as cars, trains or desks are still regarded as point particles for the purposes of this chapter. It will always be assumed that a particle is uniform or a point particle, so that in problems a square or a rectangle can be drawn to represent the object.

Forces

A force can be defined as a push or a pull. A force is an action which tries to alter the state of the motion of a particle. For example, pushing a car with a certain force may move the car, but if the push is not strong enough, it will not move.

A force has both magnitude and direction and is a vector quantity. The magnitude of the vector $\underline{F}$ is expressed as F (that is, without the vector tilde). The standard unit of force is the newton, N. This is named after Sir Isaac Newton (1642–1727) who postulated several laws of motion.



10.2.3 Equilibrium

A system of concurrent coplanar forces acting on a particle is a set of two-dimensional forces all passing through the same point. The equilibrium of a particle means that the vector sum of all the forces acting on the particle is zero. The forces cancel each other out; the particle is said to be in static equilibrium, and the acceleration of the particle is zero.

When a particle is in equilibrium and is acted upon by only two forces, $\underline{F}_1$ and $\underline{F}_2$, then $\underline{F}_1 + \underline{F}_2 = \underline{0}$ and $\underline{F}_1 = -\underline{F}_2$, so the two forces act in a straight line but in opposite directions. If a particle is in equilibrium and has two forces acting on it that are not in a straight line, then there must be another, or third, force that balances these forces. So when a particle is acted upon by three forces, $\underline{F}_1$, $\underline{F}_2$ and $\underline{F}_3$, and the particle is in equilibrium, the vector sum of these three forces must be zero: $\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0}$. In this case these three concurrent forces can be represented as the sides of a closed triangle, and the vector sum when the forces are placed head to tail is zero.

Trigonometry and Pythagoras' theorem can be used to solve many force problems.

Throughout this chapter answers should be given as exact values where appropriate; otherwise, they should be rounded correct to 2 decimal places. For calculations involving inexact values, retain the decimal places throughout the working, otherwise the final answer will be affected by the rounding process.

WORKED EXAMPLE 1

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. Determine the third force.



THINK

1. Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\vec{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.

2. Because the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.

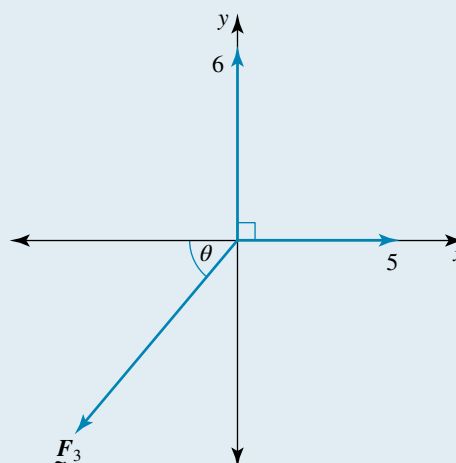
3. Use Pythagoras' theorem to determine the magnitude of the third force.

4. State the magnitude of the third force.

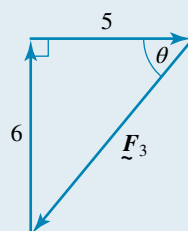
5. Calculate the direction of the third force.

6. Solve for θ .

7. State the final result.

WRITE/DRAW

Redraw the diagram.



$$|\vec{F}_3|^2 = 6^2 + 5^2$$

$$= 61$$

$$|\vec{F}_3| = \sqrt{61}$$

$$\tan(\theta) = \frac{6}{5}$$

$$\theta = \tan^{-1}\left(\frac{6}{5}\right)$$

$$= 50.19^\circ$$

The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of 129.81° clockwise with the force of 5 newtons.

10.2.4 Angles other than right angles

If the forces are not at right angles, then the sine and cosine rules can be used to solve the force problems.

WORKED EXAMPLE 2

A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. Determine the third force by the sum of vectors.

THINK

1. Draw a force diagram with the three forces acting through a point. Because the particle is in equilibrium, the third force, $\underline{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.

2. Because the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.

3. Use the cosine rule to determine the magnitude of the third force.

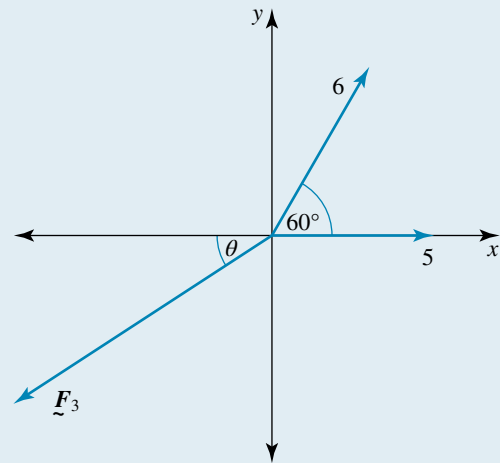
4. State the magnitude of the third force.

5. To calculate the direction of the third force, use the sine rule.

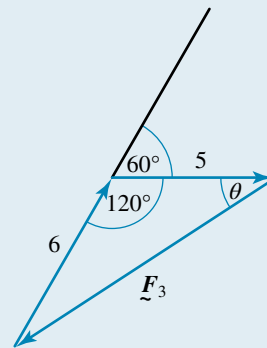
6. Solve for $\sin(\theta)$.

7. Find the angle θ .

8. State the final result.

WRITE/DRAW

Redraw the diagram.



$$|\underline{F}_3|^2 = 6^2 + 5^2 - 2 \times 6 \times 5 \cos(120^\circ) \\ = 91$$

$$|\underline{F}_3| = \sqrt{91}$$

$$\frac{F_3}{\sin(120^\circ)} = \frac{6}{\sin(\theta)}$$

$$\sin(\theta) = \frac{6 \sin(120^\circ)}{F_3}$$

$$= \frac{3\sqrt{3}}{\sqrt{91}}$$

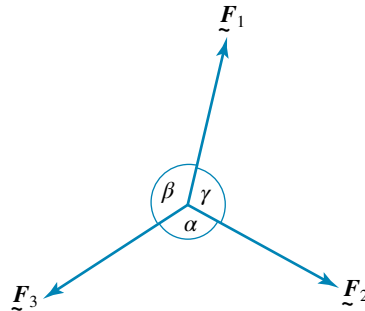
$$\theta = \sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{91}}\right)$$

$$\approx 33^\circ$$

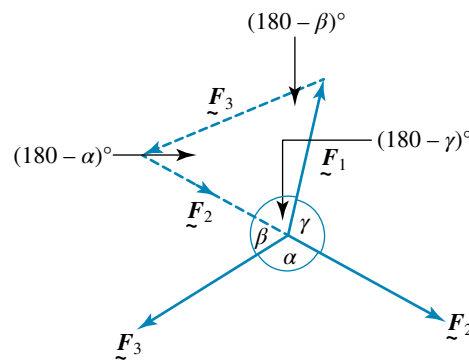
The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° clockwise with the force of 5 newtons.

10.2.5 Lami's theorem

Lami's theorem states that if three concurrent forces act on a body in equilibrium, then the magnitude of each force is proportional to the sine of the angle between the other two forces.



Redraw the diagram with the forces forming a closed triangle. The magnitude of each force is proportional to the sine of the angle between the other two forces.



The sine rule gives $\frac{F_1}{\sin(180^\circ - \alpha)} = \frac{F_2}{\sin(180^\circ - \beta)} = \frac{F_3}{\sin(180^\circ - \gamma)}$, and it is known that $\sin(180^\circ - \alpha) = \sin(\alpha)$ etc. Thus, $\frac{F_1}{\sin(\alpha)} = \frac{F_2}{\sin(\beta)} = \frac{F_3}{\sin(\gamma)}$, which is Lami's theorem.

10.2.6 Other types of forces

Weight force

The mass and weight of a particle are different. The mass of a particle is defined as the amount of substance contained in the body. The mass of a particle is constant and is usually measured in kilograms or grams. Mass is a scalar quantity. Weight is the force with which gravity pulls on a mass. The weight of a particle is a variable quantity and changes with the change in position and location due to the acceleration of the gravity acting on it. Think of a particle on Earth compared to the Moon. The weight (force) of the particle is reduced because gravity is weaker there, but the mass is still the same.

Weight is a vector quantity and is defined to be the downwards force exerted in a gravitational field on a particle. For a particle of mass m kg, this weight force is m kgwt or mg newtons and always acts vertically downwards, where $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity in the Earth's gravitational field.

A weight force acts on all bodies near the Earth's surface. The units for a weight force can be either kilogram-weight or newtons; the International System of Units (SI) for force, is the newton.



The conversion is $1 \text{ kgwt} = g\text{N}$

Tension

A taut string has a tension in it denoted by T .



The tension is the same at all points in the string and this tension is unaltered if the string passes over a smooth hook or pulley.

Note: In this discussion, only light strings are considered, so that the mass of the string can be ignored. Also, strings are assumed to be inextensible, so that when the string is pulled tight it is taut.

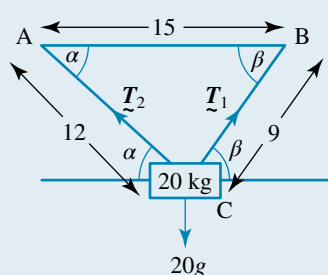
WORKED EXAMPLE 3

A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. Determine the tensions in the strings by the sum of vectors.

THINK

1. Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1 . Include the weight force, $20g$. Note that all the forces in the diagram are in newtons.
2. The triangle is a right-angled triangle.
3. Determine the trigonometric ratios.
4. Since the particle is in equilibrium, the vector sum of three forces is zero and they form a closed triangle.
5. Use Lami's theorem.
6. Since $\sin(90^\circ) = 1$, calculate the tension in the shorter string.

WRITE/DRAW

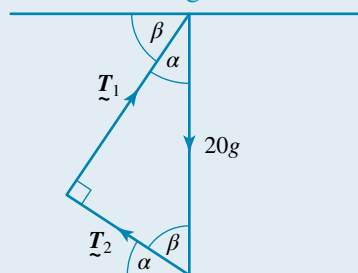


Because $d(AC) = 12$, $d(BC) = 9$ and $d(AB) = 15$, and $12^2 + 9^2 = 15^2$, $\triangle ACB$ is a right-angled triangle. $\angle ACB = 90^\circ$ and $\alpha + \beta = 90^\circ$.

$$\sin(\beta) = \cos(\alpha) = \frac{12}{15} \text{ and}$$

$$\cos(\beta) = \sin(\alpha) = \frac{9}{15}$$

Redraw the diagram.



$$\frac{T_1}{\sin(\beta)} = \frac{T_2}{\sin(\alpha)} = \frac{20g}{\sin(90^\circ)}$$

$$T_1 = 20g \sin(\beta)$$

7. Substitute for the trigonometric ratio and evaluate the tension in the shorter string.

$$T_1 = \frac{20g \times 12}{15}$$

$$= 16g$$

$$= 16 \times 9.8$$

$$= 156.8$$

8. Calculate the tension in the longer string.

$$T_2 = 20g \sin(\alpha)$$

9. Substitute for the trigonometric ratio and evaluate the tension in the longer string.

$$T_2 = \frac{20g \times 9}{15}$$

$$= 12g$$

$$= 12 \times 9.8$$

$$= 117.6$$

10. Note that the lengths of the strings do not represent the magnitudes of the tensions in the strings; however, the tensions act along the line and direction of the strings. State the final result.

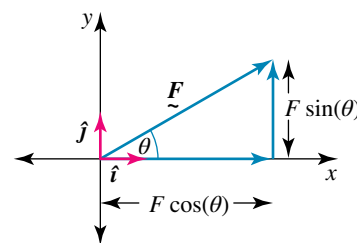
The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

10.2.7 Resolving forces

An alternative method to Lami's theorem and using trigonometry is the method of resolving forces. Consider a force $\vec{F}$ making an angle of θ with the positive x -axis.

Resolving a vector means splitting the vector up into its horizontal and vertical components. Because $\hat{i}$ is a unit vector in the x -direction and $\hat{j}$ is a unit vector in the y -direction, $\vec{F} = F \cos(\theta)\hat{i} + F \sin(\theta)\hat{j}$.

In two dimensions, the equilibrium of a particle means that the vector sum of all the resolved horizontal components of the forces is equal to zero. In the $\hat{i}$ direction, $\sum F_x = 0$. The vector sum of all the resolved vertical components of the forces in a perpendicular direction is equal to zero, from the $\hat{j}$ component: $\sum F_y = 0$. Now we will repeat the first three worked examples, but this time, we will solve them using the method of resolution of vectors.



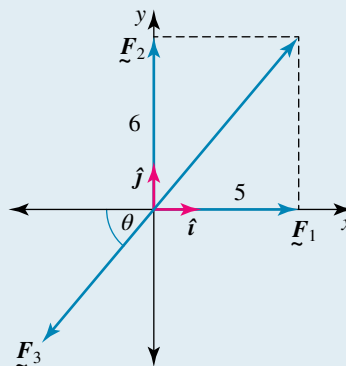
WORKED EXAMPLE 4

A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 5 newtons; another force acts at right angles to the first force and has a magnitude of 6 newtons. By resolving the forces, determine the third force.

THINK

- Draw a force diagram with the three forces acting through a point. Since the particle is in equilibrium, the third force, $\vec{F}_3$, must act in the opposite direction to the vector sum of the two given forces. Let θ be the acute angle the third force makes with the horizontal.

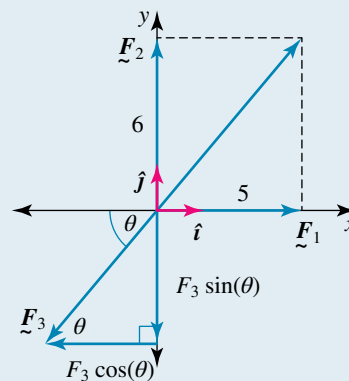
WRITE/DRAW



2. Express the given horizontal force in terms of $\hat{i}$.
3. Express the given vertical force in terms of $\hat{j}$.
4. Resolve the force $\underline{F}_3$ into its horizontal and vertical components.

$$\underline{F}_1 = 5\hat{i}$$

$$\underline{F}_2 = 6\hat{j}$$



5. Because the particle is in equilibrium and by addition of vectors, $\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0}$.
6. By resolving horizontally, the $\hat{i}$ component must be zero.
7. By resolving vertically, the $\hat{j}$ component must be zero.
8. To determine F_3 eliminate θ from [1] and [2].

$$\underline{F}_3 = -F_3 \cos(\theta)\hat{i} - F_3 \sin(\theta)\hat{j}$$

$$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0}$$

$$(5 - F_3 \cos(\theta))\hat{i} + (6 - F_3 \sin(\theta))\hat{j} = \underline{0}$$

$$5 - F_3 \cos(\theta) = 0$$

$$F_3 \cos(\theta) = 5 \quad [1]$$

$$6 - F_3 \sin(\theta) = 0$$

$$F_3 \sin(\theta) = 6 \quad [2]$$

Square both equations.

$$[1]^2: F_3^2 \cos^2(\theta) = 25$$

$$[2]^2: F_3^2 \sin^2(\theta) = 36$$

$$F_3^2 \cos^2(\theta) + F_3^2 \sin^2(\theta) = 25 + 36$$

$$F_3^2 (\cos^2(\theta) + \sin^2(\theta)) = 61$$

$$F_3^2 = 61$$

$$F_3 = \sqrt{61}$$

9. Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.
10. State the magnitude of the third force.
11. To calculate θ , the direction of the third force, divide the equations to eliminate F_3 .
12. Solve for θ .
13. State the final result.

$$\frac{[2]}{[1]} \Rightarrow \tan(\theta) = \frac{6}{5}$$

$$\theta = \tan^{-1}\left(\frac{6}{5}\right)$$

$$= 50.19^\circ$$

The third force has a magnitude of $\sqrt{61}$ newtons and acts at an angle of 129.81° with the force of 5 newtons.

WORKED EXAMPLE 5

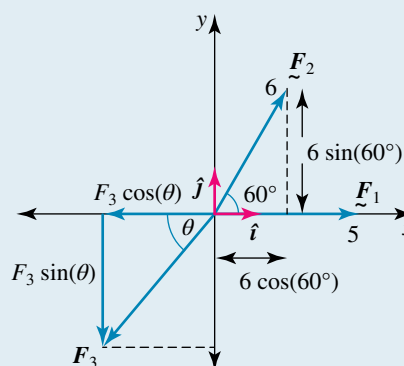
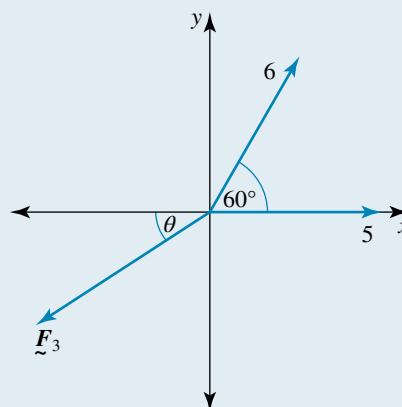
A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 5 newtons; the second force, of magnitude 6 newtons, acts at 60° with the first force. By resolving the forces into components, determine the third force.



THINK

1. Draw a force diagram and resolve the forces into their horizontal and vertical components.

WRITE/DRAW



2. Express the given horizontal force $\underline{F}_1$ in terms of $\hat{i}$.
3. $\underline{F}_2$ has components $F_2 \cos(60^\circ)$ horizontally to the right and $F_2 \sin(60^\circ)$ vertically upwards.
4. Resolve the force $\underline{F}_3$ into its horizontal and vertical components, noting that both are in the opposite directions to $\hat{i}$ and $\hat{j}$.
5. Since the particle is in equilibrium, use addition of vectors.
6. By resolving horizontally, the $\hat{i}$ component must be zero.
7. By resolving vertically, the $\hat{j}$ component must be zero.
8. To determine F_3 eliminate θ from [1] and [2].
9. Now add these equations, since $\cos^2(\theta) + \sin^2(\theta) = 1$.
10. State the magnitude of the force.

$$\underline{F}_1 = 5\hat{i}$$

$$\begin{aligned}\underline{F}_2 &= 6 \cos(60^\circ)\hat{i} + 6 \sin(60^\circ)\hat{j} \\ &= 3\hat{i} + 3\sqrt{3}\hat{j}\end{aligned}$$

$$\underline{F}_3 = -F_3 \cos(\theta)\hat{i} - F_3 \sin(\theta)\hat{j}$$

$$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0}$$

$$(8 - F_3 \cos(\theta))\hat{i} + (3\sqrt{3} - F_3 \sin(\theta))\hat{j} = \underline{0}$$

$$8 - F_3 \cos(\theta) = 0$$

$$F_3 \cos(\theta) = 8 \quad [1]$$

$$3\sqrt{3} - F_3 \sin(\theta) = 0$$

$$F_3 \sin(\theta) = 3\sqrt{3} \quad [2]$$

Square both equations.

$$[1]^2: F_3^2 \cos^2(\theta) = 64$$

$$[2]^2: F_3^2 \sin^2(\theta) = 27$$

$$F_3^2 \cos^2(\theta) + F_3^2 \sin^2(\theta) = 64 + 27$$

$$F_3^2 (\cos^2(\theta) + \sin^2(\theta)) = 91$$

$$F_3^2 = 91$$

$$F_3 = \sqrt{91}$$

11. To calculate θ , the direction of the third force, divide the equations to eliminate F_3 .

$$\frac{[2]}{[1]} \Rightarrow \tan(\theta) = \frac{3\sqrt{3}}{8}$$

12. Solve for θ .

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{3\sqrt{3}}{8}\right) \\ &= 33^\circ\end{aligned}$$

13. State the final result.

The third force has a magnitude of $\sqrt{91}$ newtons and acts at an angle of 147° with the force of 5 newtons.

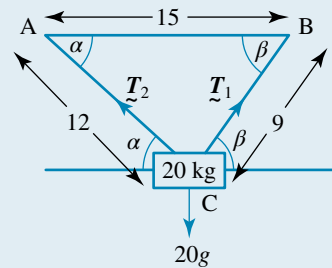
WORKED EXAMPLE 6

A particle of mass 20 kg is supported by two strings of lengths 12 cm and 9 cm. The ends of the string are attached to two fixed points 15 cm apart on the same horizontal level. By resolving the forces into components, determine the tensions in the strings.

THINK

1. Draw the force diagram. Let the two fixed points be A and B and the position of the particle be point C. Let $\angle BAC = \alpha$ and $\angle ABC = \beta$. Let the tension in the 12 cm string be T_2 and let the tension in the 9 cm string be T_1 . Include the weight force $20g$. Note that all the forces in the diagram are in newtons.
2. The triangle is a right-angled triangle.
3. Determine the trigonometric ratios.
4. T_1 has components $T_1 \cos(\beta)$ horizontally to the right and $T_1 \sin(\beta)$ vertically upwards. T_2 has components $T_2 \cos(\alpha)$ horizontally to the left and $T_2 \sin(\alpha)$ vertically upwards.

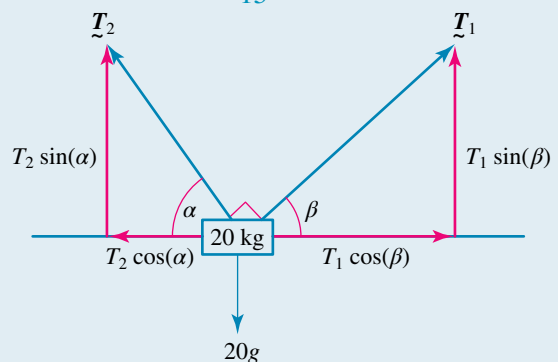
WRITE/DRAW



Because $d(AC) = 12$, $d(BC) = 9$ and $d(AB) = 15$, and $12^2 + 9^2 = 15^2$, $\triangle ACB$ is right angled. $\angle ACB = 90^\circ$ and $\alpha + \beta = 90^\circ$.

$$\sin(\beta) = \cos(\alpha) = \frac{12}{15} \text{ and}$$

$$\cos(\beta) = \sin(\alpha) = \frac{9}{15}$$



5. Resolving horizontally, all the forces horizontally to the right minus all the forces horizontally to the left add to zero.

$$T_1 \cos(\beta) - T_2 \cos(\alpha) = 0 \quad [1]$$

6. Resolving vertically, all the forces vertically upward minus all the forces vertically downwards sum to zero.

$$T_2 \sin(\alpha) + T_1 \sin(\beta) - 20g = 0 \quad [2]$$

7. Substitute for the trigonometric ratios into [1] and [2].

$$\frac{9T_1}{15} - \frac{12T_2}{15} = 0 \quad [1]$$

$$\frac{9T_2}{15} + \frac{12T_1}{15} - 20g = 0 \quad [2]$$

8. From [1], express T_2 in terms of T_1 .

$$T_2 = \frac{3T_1}{4}$$

9. Simplify [2].

$$9T_2 + 12T_1 = 300g$$

10. Substitute for T_2 and solve for T_1 .

$$9\left(\frac{3T_1}{4}\right) + 12T_1 = 300g$$

$$\frac{T_1(27 + 48)}{4} = 300g$$

$$75T_1 = 1200g$$

$$T_1 = \frac{1200g}{75}$$

$$= 16g$$

$$= 16 \times 9.8$$

$$= 156.8$$

12. Substitute for T_1 and evaluate T_2 .

$$T_2 = \frac{3T_1}{4}$$

$$= \frac{3}{4} \times 16g$$

$$= 12g$$

$$= 12 \times 9.8$$

$$= 117.6$$

13. State the final result, noting that the shorter string carries more tension.

The tension in the 9 cm string is 156.8 newtons and the tension in the 12 cm string is 117.6 newtons.

10.2.8 Resolving all the forces

Lami's theorem can only be used when dealing with problems involving three forces. When there are more than three forces acting, resolve the forces.

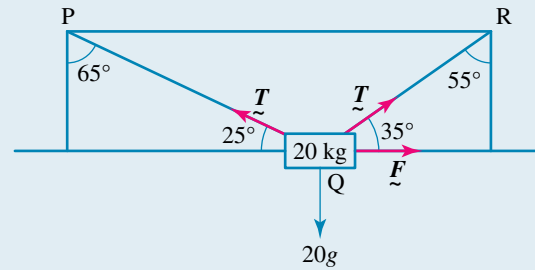
WORKED EXAMPLE 7

A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 20 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of F newtons until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 65° and 55° to the vertical respectively. Determine the value of F and the tension in the strings.

THINK

1. Because the hook is smooth, the tension is unaltered when passing over it, so let the tension in the strings PQ and QR both be T . We need to find the values of T and F . Note that all the forces in the diagram are in newtons.
2. The tension in the string QR has a horizontal component $T \cos(35^\circ)$ and a vertical component $T \sin(35^\circ)$, both in the positive direction.
3. The tension in the string PQ has a horizontal component $T \cos(25^\circ)$ in the negative direction and a vertical component $T \sin(25^\circ)$ in the positive direction.
4. Resolve the forces horizontally.
5. Resolve the forces vertically.
6. Use [2] to factor and solve for T .
7. State the value of the tension in the strings.
8. Use [1] to determine F .
9. Substitute for T .
10. State the value of F .

WRITE/DRAW



QR: horizontal component $T \cos(35^\circ)$,
vertical component $T \sin(35^\circ)$

PQ: horizontal component $T \cos(25^\circ)$ in
the negative direction, vertical component
 $T \sin(25^\circ)$ in the positive direction.

$$F + T \cos(35^\circ) - T \cos(25^\circ) = 0 \quad [1]$$

$$T \sin(35^\circ) + T \sin(25^\circ) - 20g = 0 \quad [2]$$

$$T(\sin(35^\circ) + \sin(25^\circ)) = 20g$$

$$T = \frac{20g}{\sin(35^\circ) + \sin(25^\circ)} \\ \approx 196.75$$

The tension in the strings PQ and QR is
196.75 newtons.

$$F = T \cos(25^\circ) - T \cos(35^\circ) \\ = T(\cos(25^\circ) - \cos(35^\circ))$$

$$F = \frac{20g(\cos(25^\circ) - \cos(35^\circ))}{(\sin(35^\circ) + \sin(25^\circ))}$$

$$F \approx 17.15 \text{ N}$$

studyon

Units 3 & 4 > Area 5 > Sequence 3 > Concept 1

A body in equilibrium under concurrent forces Summary screen and practice questions

Exercise 10.2 A body in equilibrium under concurrent forces

Technology free

1. **WE1** A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. Determine the third force.
2. A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of F newtons; another force acts at right angles to this force and has a magnitude of $2F$ newtons. Determine the third force.

Technology active

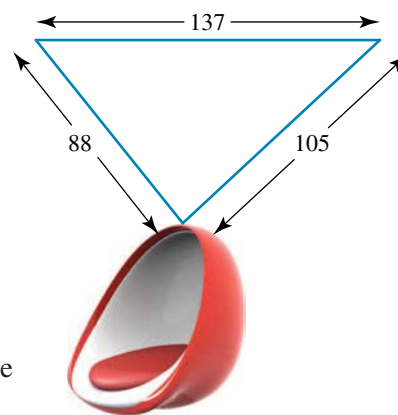
3. **a.** A body is in equilibrium under the action of three forces, $\underline{F}_1 = 3\hat{i} - 4\hat{j}$, $\underline{F}_2 = x\hat{i} + 7\hat{j}$ and $\underline{F}_3 = 6\hat{i} + y\hat{j}$. Determine the values of x and y .
- b.** A body is in equilibrium under the action of three forces, $\underline{F}_1 = -2\hat{i} + 5\hat{j}$, $\underline{F}_2 = 6\hat{i} - 8\hat{j}$ and $\underline{F}_3 = x\hat{i} + y\hat{j}$. Determine the values of x and y and the magnitude of the force $\underline{F}_3$.
4. **a.** A particle is in equilibrium and is acted upon by three forces. One acts horizontally and has a magnitude of 50 newtons; another acts at right angles to this force and has a magnitude of 120 newtons. Determine the third force.
- b.** A particle is in equilibrium and is acted upon by three forces. One acts horizontally and has a magnitude of 30 newtons; another acts at right angles to the 30 newton force and has a magnitude of F_1 newtons. The second force $\underline{F}_2$ acts at an angle of 120° anticlockwise with the 30 newton force. Determine the magnitudes of the forces $\underline{F}_1$ and $\underline{F}_2$.
5. **WE2 & WE5** A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 7 newtons; a second force of magnitude 8 newtons acts at 30° with the first force. Determine the third force
 - a.** by the sum of vectors
 - b.** by resolving the forces into components.
6. A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of F newtons; a second force of magnitude $2F$ newtons acts at 45° with the first force. Determine the third force
 - a.** by the sum of vectors
 - b.** by resolving the forces into components.
7. **WE3 & WE6** A particle of mass 10 kg is supported by two strings of lengths 5 cm and 12 cm. The ends of the string are attached to two fixed points 13 cm apart on the same horizontal level. Determine the tensions in the strings
 - a.** by the sum of vectors
 - b.** by resolving the forces into components.
8. A particle of mass 5 kg is supported by two strings of lengths 7 cm and 24 cm. The ends of the string are attached to two fixed points 25 cm apart on the same horizontal level. Determine the tensions in the strings
 - a.** by the sum of vectors
 - b.** by resolving the forces into components.
9. **WE4** A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of 7 newtons; another force acts at right angles to this force and has a magnitude of 8 newtons. By resolving the forces, determine the third force.
10. A particle is in equilibrium and is acted upon by three forces. One force acts horizontally and has a magnitude of F newtons; another force acts at right angles to this force and has a magnitude of $2F$ newtons. By resolving the forces, determine the third force.
11. **WE7** A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of 5 kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of F newtons, until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 60° and 30° to the vertical respectively. Calculate the value of F and the tension in the strings.
12. A string PQR is tied to two fixed points, P and R, on the same horizontal plane. A mass of m kg is suspended from the string at Q by means of a smooth hook that is pulled aside by a horizontal force of 8 newtons, until the system is in equilibrium. The parts of the string PQ and QR are then inclined at angles of 40° and 20° to the vertical respectively. Determine the value of m and the tension in the strings.
13. **a.** A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 30 newtons; the second force, with magnitude 60 newtons, acts at 40° with the first force. Determine the third force.

- b. A particle is in equilibrium and is acted upon by three forces. The first force acts horizontally and has a magnitude of 120 newtons; the second force, with magnitude 50 newtons, acts at an angle of 140° clockwise with the first force. Determine the third force.
14. a. A particle of mass 4 kg is supported by a string AB attached to a fixed point A and is pulled aside from the vertical by a horizontal force of F newtons acting on the particle at B. If AB makes an angle of 35° with the vertical in the equilibrium position, calculate:
- the value of F
 - the magnitude of the tension in the string.
- b. A child of mass m kg is sitting on a swing that is held in equilibrium by a horizontal force of 200 newtons applied by his father. At this time, the rope attached to the swing makes an angle of 40° with the vertical. Calculate the tension in the rope and the mass of the child.

15. a. A painting of mass 600 g is to be hung on a wall. It is supported by a string whose ends are attached to two fixed points on the wall at the same horizontal level a distance of 80 cm apart. A hook is attached to the painting. When the painting is hanging in equilibrium, the hook is 30 cm directly below the midpoint of the two fixed points. Calculate the tension in the string around the hook.
- b. A string of length $2L$ has its ends fixed to two points at the same horizontal level a distance of L apart. A bead of mass m kg is threaded on the string and hangs at the midpoint of the string. Calculate the tension in the string.



16. a. An egg chair is hanging by two inextensible cables of lengths 88 cm and 105 cm. The other ends of the cable are attached to the ceiling at a distance of 137 cm apart. The shorter cable has a tension of T_1 newtons and the longer cable has a tension of T_2 newtons.



- Show that $T_2 = \frac{88 T_1}{105}$.
 - The longer cable is strong, but the shorter cable will break if the tension in it exceeds 980 newtons. The egg chair has a mass of 10 kg, and a person of mass m kg sits in it. Determine the maximum value of m for which the eggchair remains in equilibrium.
- b. A downlight of mass 250 g is hanging from two cables of lengths 48 cm and 55 cm. The other ends of the cables are attached to the ceiling a distance of 73 cm apart. Show that the tensions in the 48 cm and 55 cm cables have magnitudes of $\frac{55g}{292}$ and $\frac{48g}{292}$ newtons respectively.
17. a. A particle of mass 10 kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $2T$ newtons and the tension in the string is $3T$ newtons. The string makes an angle of θ with the vertical. Determine the angle θ and the value of T .
- b. A particle of mass $2m$ kg is attached to one end of a string, and the other end of the string is attached to a fixed point. When in equilibrium, the mass is pulled aside by a horizontal force of $3mg$ newtons and the tension in the string is T newtons. The string makes an angle of θ with the vertical. Determine the angle θ and express T in terms of m .

18. a. A string ABC is tied to two fixed points, A and C, at the same horizontal level. A mass of 10 kg is suspended at B. When in equilibrium, the strings AB and BC make angles of θ and 70° with AC, and the magnitudes of the tensions in the strings AB and BC are T and $2T$ newtons respectively. Determine the angle θ and the value of T .
- b. A string PQR is tied to two fixed points, P and R, at the same horizontal level. A mass of 4 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of 50° and θ with PR, and the magnitude of the tension in the string PQ is 12 newtons. Determine the angle θ and the tension in the string QR.
19. a. Two points A and B are at the same horizontal level and are a distance of $5L$ apart. A mass of m kg at point C is connected by two strings, such that the lengths AC and BC are $3L$ and $4L$ respectively. Determine the tensions in the two strings AC and BC.
- b. A particle is supported by two strings at right angles to one another and attached to two points at the same horizontal line. Prove that the tensions in the strings are inversely proportional to the lengths of the strings.
20. a. A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of 10 newtons. When the string makes angles of 55° and 35° to the vertical, the system is in equilibrium. Determine the tension in the string and the mass, m .
- b. A string is tied to two points at the same horizontal level. A mass of m kg is suspended from a smooth hook and is pulled across horizontally by a force of F newtons. When in equilibrium, the string makes angles of α and β to the horizontal, where $\beta > \alpha$. Show that

$$\frac{F}{mg} = \frac{\cos(\alpha) - \cos(\beta)}{\sin(\alpha) + \sin(\beta)}.$$

21. a. A string PQR is tied to two fixed points P and R at the same horizontal level. A mass of 5 kg is suspended at Q. When in equilibrium, the strings PQ and QR make angles of α and β with PR respectively. The magnitude of the tensions in the strings PQ and QR are 30 and 35 newtons respectively. Determine the angles α and β .
- b. A string ABC is tied to two fixed points A and C at the same horizontal level. A mass of $7m$ kg is suspended at B. When in equilibrium, the strings AB and BC make angles of α and β with AC respectively. The magnitude of the tensions in the strings AB and BC are $5mg$ and $4mg$ newtons respectively. Determine the angles α and β .

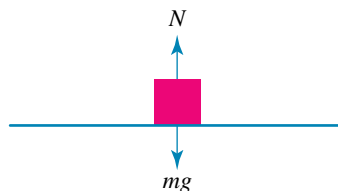
10.3 Action and reaction forces

10.3.1 Other types of forces-Action and reaction

If we think of a computer resting on a desk, the weight force is not the only force acting on the computer, for if the desk were not there, the computer would fall vertically downwards.

Newton's third law of motion states that for every action there is an equal and opposite reaction. That is, if a body exerts a force by resting on a desk, then the desk exerts a force of equal magnitude but in the opposite direction. This type of force is called the normal reaction and will be denoted by N . The upward force is the reaction of the desk on the computer or object.





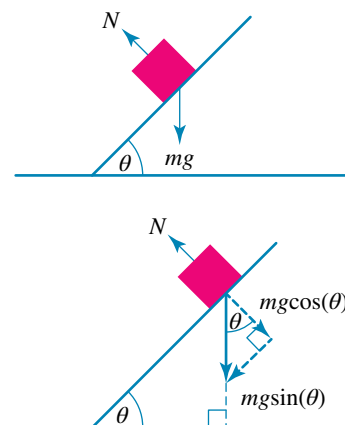
The normal reaction is always perpendicular to the surface and only acts when a body is in contact with another surface.

10.3.2 Inclined planes

Consider a particle of mass m kg on a smooth plane inclined at an angle of θ to the horizontal. A smooth plane means that there are no frictional forces acting. The particle is being acted on by two external forces: the normal reaction N , which is perpendicular to the plane, and the weight force, which acts vertically downwards.

When dealing with problems involving inclined planes, it is usually easier to resolve parallel and perpendicular to the plane rather than horizontally and vertically.

The weight force has components $mg\sin(\theta)$ parallel to and down the plane and $mg\cos(\theta)$ perpendicular to the plane. Note that if these were the only forces acting on the particle, then it could not be in equilibrium. In that situation the particle would slide down the plane.



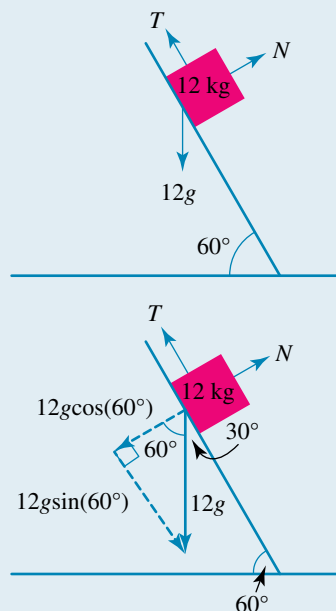
WORKED EXAMPLE 8

A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by a light strap. Calculate the tension in the strap and the reaction of the plane if the strap is parallel to and acting up the line of greatest slope.

THINK

1. Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting up the line of the slope. $12g$ is the weight force and N is the normal reaction. Note that the normal reaction is perpendicular to the plane. All forces in the diagram are in newtons.
2. Resolve the weight force into its components parallel and perpendicular to the plane.

WRITE/DRAW



3. Resolve the forces up the plane. The tension force is up the plane and the component of the weight force acts down the plane.

$$T - 12g\sin(60^\circ) = 0 \quad [1]$$

4. Resolve the forces perpendicular to the plane. The normal reaction and the component of the weight force act perpendicular to the plane.

$$N - 12g\cos(60^\circ) = 0 \quad [2]$$

5. Use [1] to solve for T .

$$\begin{aligned}\text{From [1]:} \\ T &= 12g\sin(60^\circ) \\ &= 12g \times \frac{\sqrt{3}}{2}\end{aligned}$$

6. State the value of T .

The tension in the strap is $6g\sqrt{3}$ newtons.

7. Use [2] to solve for N .

$$\begin{aligned}\text{From [2]:} \\ N &= 12g\cos(60^\circ) \\ &= 12g \times \frac{1}{2} \\ &= 6g\end{aligned}$$

8. State the value of N .

The normal reaction is $6g$ newtons.

Resolve all the forces

When two or more forces have components, it is a common mistake to forget to find the resolved components of *all* the forces both parallel and perpendicular to the plane.

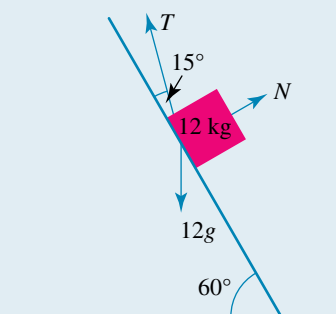
WORKED EXAMPLE 9

A suitcase of mass 12 kg is maintained in equilibrium on a smooth plane inclined at 60° to the horizontal by means of a strap. Calculate the tension in the strap and the reaction of the plane if the strap is inclined at an angle of 15° to the plane.

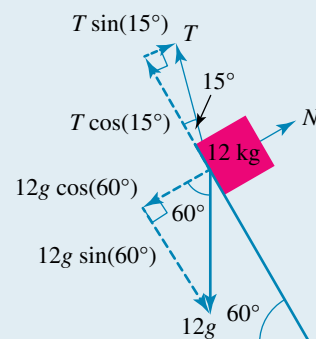
THINK

1. Draw the diagram and mark in the forces on the suitcase. Let T be the tension in the strap, acting at an angle to the plane. $12g$ is the weight force and N is the normal reaction. All forces in the diagram are in newtons.

WRITE/DRAW



- Resolve the weight force and the tension into their components parallel and perpendicular to the plane.



- Resolve the forces up the plane. The component of the tension force is up the plane and the component of the weight force acts down the plane.
- Resolve the forces perpendicular to the plane. The normal reaction and the component of the tension act up and the component of the weight force acts down, perpendicular to the plane.
- Use [1] to solve for T .

$$T \cos(15^\circ) - 12g \sin(60^\circ) = 0 \quad [1]$$

- State the value of T .
- Use [2] to solve for N .

$$N + T \sin(15^\circ) - 12g \cos(60^\circ) = 0 \quad [2]$$

From [1]:

$$T \cos(15^\circ) = 12g \sin(60^\circ)$$

$$T = \frac{12g \sin(60^\circ)}{\cos(15^\circ)}$$

The tension in the strap is 105.44 newtons.

From [2]:

$$N = 12g \cos(60^\circ) - T \sin(15^\circ)$$

$$= 12g \times \frac{1}{2} - 105.44 \times \sin(15^\circ)$$

- State the value of N .

The normal reaction is 31.51 newtons.

Note that, in Worked examples 8 and 9, the normal reaction and tension forces have different values.

10.3.3 Connected particles

When dealing with connected particles or a system involving two or more particles, isolate the particles and resolve around each particle independently.

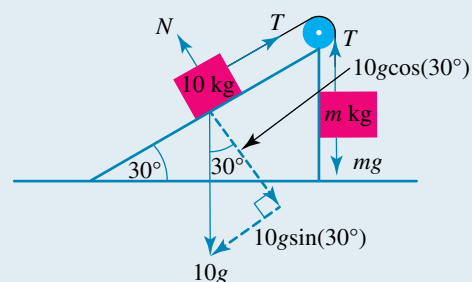
WORKED EXAMPLE 10

A mass of 10 kg lies at rest on a smooth plane inclined at 30° to the horizontal. The mass is held in place by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of m kg that hangs vertically. Determine the value of m .

THINK

- Draw the diagram and mark in all the forces. The pulley is smooth, meaning the tensions on each side are equal. Let the tension in the strings be T . All forces in the diagram are in newtons.

WRITE/DRAW



2. Resolve vertically upwards around the mass m hanging vertically.	$T - mg = 0$
3. Resolve parallel to the plane around the 10 kg mass.	$T - 10g\sin(30^\circ) = 0$
4. Equate the tensions.	$mg = 10g\sin(30^\circ)$
5. Solve for the unknown mass.	$m = 10\sin(30^\circ)$ $= 10 \times \frac{1}{2}$
6. State the value of the mass.	The mass is 5 kg.

If the system has two connected particles in a situation similar to that depicted in Worked example 11, then resolve horizontally and vertically around each particle. This can give four equations, so in these types of problems, there can be up to four unknowns that need to be found.

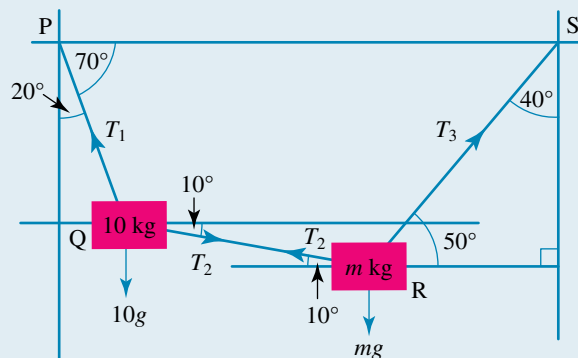
WORKED EXAMPLE 11

PQRS is a rope fastened to a horizontal beam at P and S. A mass of m kg is suspended from R and a mass of 10 kg is suspended from Q. The section of the rope PQ makes an angle of 70° with PS, and section QR is inclined at 10° below the horizontal, while section RS is inclined at an angle of 40° to the vertical. Determine the tensions in all the sections of the rope and the value of m .

THINK

1. Draw the diagram and mark in all appropriate angles. Let the tensions in the strings PQ, QR and RS be T_1 , T_2 and T_3 respectively. We have 4 unknowns: T_1 , T_2 , T_3 and m .

WRITE/DRAW



2. Resolve horizontally around Q using right as the positive direction.
3. Resolve vertically around Q using upwards as positive.
4. Resolve horizontally around R.
5. Resolve vertically around R.
6. Use [1] to express T_2 in terms of T_1 .

$$T_2 \cos(10^\circ) - T_1 \sin(20^\circ) = 0 \quad [1]$$

$$T_1 \cos(20^\circ) - T_2 \sin(10^\circ) - 10g = 0 \quad [2]$$

$$T_3 \cos(50^\circ) - T_2 \cos(10^\circ) = 0 \quad [3]$$

$$T_3 \sin(50^\circ) + T_2 \sin(10^\circ) - mg = 0 \quad [4]$$

$$[1] \Rightarrow T_2 \cos(10^\circ) = T_1 \sin(20^\circ)$$

$$T_2 = \frac{T_1 \sin(20^\circ)}{\cos(10^\circ)}$$

7. Substitute for T_2 into [2].

$$[2] \Rightarrow T_1 \cos(20^\circ) - T_2 \sin(10^\circ) = 10g$$

$$T_1 \cos(20^\circ) - \frac{T_1 \sin(20^\circ) \sin(10^\circ)}{\cos(10^\circ)} = 10g$$

8. Solve for T_1 by taking a common denominator.

$$\frac{T_1 \cos(20^\circ) \cos(10^\circ) - T_1 \sin(20^\circ) \sin(10^\circ)}{\cos(10^\circ)} = 10g$$

$$\frac{T_1 [\cos(20^\circ) \cos(10^\circ) - \sin(20^\circ) \sin(10^\circ)]}{\cos(10^\circ)} = 10g$$

9. Use the trigonometric compound-angle formula:
 $\cos(A) \cos(B) - \sin(A) \sin(B) = \cos(A + B)$

$$T_1 \cos(30^\circ) = 10g \cos(10^\circ)$$

$$T_1 = \frac{10g \cos(10^\circ)}{\cos(30^\circ)}$$

$$T_1 = 111.44$$

10. State the value of T_1 .

The tension in the string PQ is 111.44 newtons.

11. Back substitute and solve for T_2 .

$$T_2 = \frac{10g \cos(10^\circ)}{\cos(30^\circ)} \times \frac{\sin(20^\circ)}{\cos(10^\circ)}$$

$$T_2 = \frac{10g \sin(20^\circ)}{\cos(30^\circ)}$$

$$T_2 = 38.70$$

12. State the value of T_2 .

The tension in the string QR is 38.70 newtons.

13. Use [3] to solve for T_3 .

$$[3] \Rightarrow T_3 \cos(50^\circ) - T_2 \cos(10^\circ) = 0$$

$$T_3 = \frac{T_2 \cos(10^\circ)}{\cos(50^\circ)}$$

14. Substitute for T_2 .

$$T_3 = \frac{10g \sin(20^\circ)}{\cos(30^\circ)} \times \frac{\cos(10^\circ)}{\cos(50^\circ)}$$

$$T_3 = 59.30$$

15. State the value of T_3 .

The tension in the string RS is 59.30 newtons.

16. Use [4] to solve for m .

$$[4] \Rightarrow mg = T_3 \sin(50^\circ) + T_2 \sin(10^\circ)$$

17. Substitute for T_2 and T_3 .

$$mg = 59.30 \sin(50^\circ) + 38.70 \sin(10^\circ)$$

$$mg = 52.147$$

$$m = \frac{52.147}{9.8}$$

$$= 5.32$$

18. State the value of m .

The mass m is 5.32 kg.

study on

Units 3 & 4

Area 5

Sequence 3

Concept 2

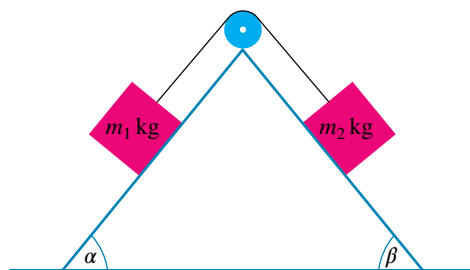
Action and reaction forces Summary screen and practice questions

Exercise 10.3 Action and reaction forces

Technology active

- WE8** A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting up and parallel to the plane. Calculate the magnitude of this force and the reaction of the plane.

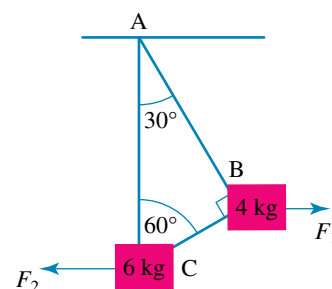
2. A book of mass 200 g is maintained in equilibrium on a smooth inclined plane by a force of 1 N acting up and parallel to the plane. Calculate the inclination of the plane to the horizontal and the reaction of the plane on the book.
3. **WE9** A parcel of mass 5 kg is maintained in equilibrium on a smooth plane inclined at 40° to the horizontal by means of a force acting at an angle of 20° to the plane. Calculate the magnitude of this force and the reaction of the plane on the parcel.
4. A book of mass 200 g is maintained in equilibrium on a smooth inclined plane by a force of 1 N acting at an angle 20° with the plane. Calculate the inclination of the plane to the horizontal and the reaction of the plane on the book.
5. **WE10** A mass of m kg is held at rest on a smooth plane inclined at 60° to the horizontal by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of 4 kg that hangs vertically. Determine the value of m .
6. A mass of M kg is held at rest on a smooth plane inclined at θ to the horizontal by a string that is parallel to the plane and passes over a smooth pulley at the top of the plane. The other end of the string is connected to a mass of m kg that hangs vertically. If $M > m$, show that $\sin(\theta) = \frac{m}{M}$.
7. **WE11** PQRS is a rope fastened to a horizontal beam at P and S. A mass of 5 kg is suspended from R and a mass of m kg is suspended from Q. The section of the rope PQ makes an angle of 60° with PS, and section QR is inclined at 45° below the horizontal, while section RS is inclined at an angle of 30° to the vertical. Determine the tensions in all the sections of the rope and the value of m .
8. PQRS is a rope fastened to a horizontal beam at P and S. A mass of m_1 kg is suspended from Q and a mass of m_2 kg is suspended from R. The section of the rope PQ makes an angle of 55° with PS, and section QR is inclined at 25° below the horizontal. The tensions in the string PQ and RS are 30 and 40 newtons respectively. Determine the tension in QR, the inclination of RS to PS, and the masses m_1 and m_2 .
9. a. Two particles are at rest on two smooth inclined planes which are placed back to back. The planes are inclined at angles of α and β to the horizontal, and the particles are connected by a string which passes over a smooth pulley at the top of the planes. The particle on the plane inclined at an angle of α has a mass of m_1 kg and the particle on the plane inclined at an angle of β has a mass of m_2 kg. Show that



$$\frac{m_1}{m_2} = \frac{\sin(\beta)}{\sin(\alpha)}.$$

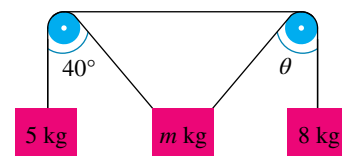
- b. Two bodies are at rest on two smooth inclined planes which are placed back to back. The planes are inclined at angles of 60° and 30° to the horizontal, and the bodies are connected by a string passing over a smooth pulley at the top of the planes. The body on the steeper plane has a mass of 4 kg and the other body has a mass of m kg. Determine the tension in the string and the value of m .
10. Masses of m kg and 5 kg are suspended from two points of a string that has one end attached to a beam. The lower mass of 5 kg is pulled aside by a horizontal force of F newtons until the upper and lower parts of the string are inclined at 20° and 40° respectively to the vertical. Determine the values of:
 - a. F
 - b. the mass m
 - c. the tensions in both sections of the string.
11. Two masses are attached to two different points on a string that hangs at one end from a support. The higher mass is 3 kg; the lower mass is 4 kg and is acted upon by a horizontal force of 20 newtons to the right. Determine the tension in each portion of the string and the angle that each section of the string makes with the vertical.

12. Two masses are attached to two different points of a string that hangs from a fixed point, A. The higher mass of 4 kg at point B is acted upon by a horizontal force of F_1 newtons acting to the right, and the lower mass of 6 kg at point C is acted upon by a horizontal force of magnitude F_2 newtons to the left, such that the lower mass is vertically below A. If AB is inclined at the angle of 30° to the vertical and angle ABC is 90° when the system is in equilibrium, calculate the tension in each portion of the string and the magnitudes of F_1 and F_2 .



13. Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass m_1 kg at point B is acted upon by a horizontal force of 15 newtons acting to the right so that the string AB makes an angle of 40° with the vertical. The lower mass m_2 kg at point C is acted upon by a horizontal force of magnitude 5 newtons to the left such that the lower mass is vertically below A. The angle ABC is 90° when the system is in equilibrium. Calculate the tension in each portion of the string and the values of m_1 and m_2 .

14. Two bodies of mass 5 kg and 8 kg are at the same horizontal level, with the 5 kg mass to the left of the 8 kg mass. Both are hanging vertically and are connected by a light string that passes over two pulleys vertically above both particles and at the same horizontal level. A third body of mass m kg is suspended between the pulleys by the string such that the string on the left makes an angle of 40° to the vertical. If the system is in equilibrium, determine the angle that the string on the right makes with the vertical and the mass of the third body.



15. A string passes over two smooth pegs on the same horizontal level and supports masses of 5 kg at both ends hanging vertically. A mass of 6 kg is then hung from the string midway between the pegs, and this moves down 3 cm.
- If the system is now in equilibrium, calculate the distance between the pegs.
 - If instead the 6 kg mass was replaced with a mass of m kg, write an expression in terms of m for the distance of the mass below the pegs.
 - What would happen if $m = 10$ kg?
16. PQRS is a rope fastened to a horizontal beam at P and S. A mass of 5 kg is suspended from R and a mass of m kg is suspended from Q. If the section of the rope PQ makes an angle of 50° with PS, and QR is 25° below the horizontal while RS is inclined at 40° to the vertical, determine all the tensions in the strings and the value of m .
17. ABCD is a rope fastened to a horizontal beam at A and D. A mass of m kg is suspended from B and a mass of M kg is suspended from C. If the tensions in the strings AB and CD are equal and twice the tension in the string BC, show that the strings AB and CD make equal angles with the vertical. If the string BC is inclined at 60° below the horizontal, calculate the angle that AB makes with the vertical.
18. a. Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass of $2m$ kg at point B is acted upon by a horizontal force of $3F$ newtons acting to the right, and the lower mass of m kg at point C is acted upon by a horizontal force of magnitude F newtons to the left, such that the lower mass is vertically below A. If the angle ABC is 90° when the system is in equilibrium, calculate the angle that AB makes with the vertical.
- b. Two masses are attached to two different points of a string that hangs from a fixed point A. The higher mass m_1 kg at point B is acted upon by a horizontal force of F_1 newtons acting to the right. The lower mass m_2 kg at point C is acted upon by a horizontal force of magnitude F_2 newtons to the left such that the lower mass is vertically below A. If the angle ABC is 90° and AB is inclined at angle of α to the vertical, show that

$$\frac{m_1}{m_2} = \left(\frac{F_1}{F_2} - 1 \right) \cot^2(\alpha) - 1.$$

19. PQR is a rope fastened to a beam at P. A mass of m_1 kg is suspended at Q and a mass of m_2 kg is suspended at R so that m_2 is below and to the right of m_1 . A horizontal force of magnitude F newtons acts to the right at the point R. The section of the rope PQ makes an angle of α with the horizontal, and the section of the rope QR is inclined at an angle of β below the horizontal where $\alpha > \beta$.

Show that $\frac{m_2}{m_1} = \frac{\cos(\alpha) \sin(\beta)}{\sin(\alpha - \beta)}$ and $F = \frac{m_1 g \cos(\alpha) \cos(\beta)}{\sin(\alpha - \beta)}$.

20. PQRS is a rope fastened to a horizontal beam at P and S. A mass of m_1 kg is suspended from Q and a mass of m_2 kg is suspended from R. The sections of the rope PQ and RS make angles of α and γ respectively with PS, and QR makes an angle of β below the horizontal, where $\alpha > \beta > \gamma$.

Show that $\frac{m_2}{m_1} = \frac{\cos(\alpha) \sin(\beta + \gamma)}{\cos(\gamma) \sin(\alpha - \beta)}$.

10.4 Momentum and resultant force

10.4.1 Newton's laws of motion

Dynamics is the study of the causes of the motion of particles.

The study of dynamics is based upon Newton's three laws of motion. Sir Isaac Newton (1642–1727) worked in many areas of mathematics and physics. It is believed that he first started studying the effects of gravity after watching an apple fall from a tree.

He wondered why the apple fell, and he wanted to calculate the speed at which it hit the ground. He was also curious about the stars and the movement of the planets in the skies: he asked, were they falling too? He developed his theory of gravitation in, 1666, and in 1686 he published his three laws of motion in the book *Principia Mathematica Philosophiae Naturalis*. These laws can be stated in modern English as follows.



Newton's first law of motion: Every body continues in a state of rest or uniform motion in a straight line unless acted upon by an external force.

Newton's second law of motion: The resultant force acting on a body is proportional to the rate of change of momentum.

Newton's third law of motion: For every action there is an equal but opposite reaction.

10.4.2 Momentum

The concept of **momentum** is fundamental to Newton's second law of motion. The momentum $\underline{p}$ of a body of mass m moving with velocity $\underline{v}$ is defined by

$$\begin{aligned}\text{momentum} &= \text{mass} \times \text{velocity} \\ \underline{p} &= m\underline{v}\end{aligned}$$

The standard unit for momentum is kilogram metres/second (kg m/s).

Note that momentum is a vector quantity.

WORKED EXAMPLE 12

Determine the magnitude of the momentum of a car of mass 1.4 tonnes moving at 72 km/h.

THINK

1. Determine the mass of the car in SI units; that is, express the mass in kg.
2. Convert the speed into m/s. Multiply by 1000 and divide by 60 and 60 again, since there are 60 minutes in an hour and 60 seconds in a minute.
3. Find the product of the mass and speed.
4. State the magnitude of the momentum.

WRITE

$$m = 1.4 \text{ tonnes}$$

$$m = 1400 \text{ kg}$$

$$v = 72 \text{ km/h}$$

$$v = \frac{72 \times 1000}{60 \times 60}$$

$$= 20 \text{ m/s}$$

$$p = mv$$

$$= 1400 \times 20$$

The magnitude of the momentum is 28 000 kg m/s.

Resultant force, $\underline{R} = m\underline{a}$

Newton's second law also involves the resultant force. Let $\underline{R} = \sum \underline{F}$ be the vector resultant sum of all the forces acting on a body. If the resultant force acting on a body is proportional to the rate of change of momentum, then $\underline{R} \propto \frac{d}{dt}(\underline{p})$. Taking k as the proportionality constant, $\underline{R} = k \frac{d}{dt}(\underline{p})$.

Now substituting for the momentum, $\underline{R} = k \frac{d}{dt}(m\underline{v})$. Differentiating using the product rule gives

$$\underline{R} = k \left[m \frac{d\underline{v}}{dt} + \underline{v} \frac{dm}{dt} \right]. \text{ But since the mass is constant, } \frac{dm}{dt} = 0, \text{ we have } \underline{R} = km \frac{d\underline{v}}{dt} = km\underline{a}, \text{ since the}$$

acceleration $\underline{a} = \frac{d\underline{v}}{dt}$.

We can make $k = 1$ if we use correct and appropriate units. If the mass m is measured in kg, the forces in N and the acceleration in m/s^2 , then $k = 1$. So a force of 1 newton is defined as the force which, when applied to a mass of 1 kilogram, produces an acceleration of 1 m/s^2 .

Newton's laws imply that a force is needed to make an object move. By considering the forces acting on an object we can analyse its motion.

WORKED EXAMPLE 13

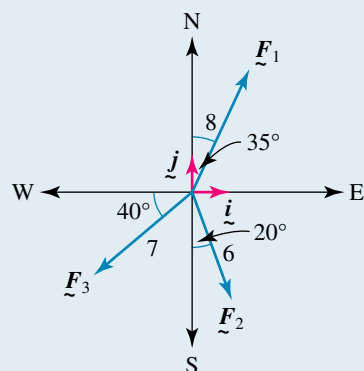
A particle of mass 2 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underline{F}_1$ has a magnitude of 8 newtons and acts in direction N35°E, $\underline{F}_2$ has a magnitude of 6 newtons and acts in direction S20°E, and $\underline{F}_3$ has a magnitude of 7 newtons and acts in direction S50°W. If $\hat{i}$ is a unit vector of magnitude 1 newton acting east and $\hat{j}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.



THINK

1. Draw the force diagram.
2. Resolve the force $\underline{F}_1$ into its components.
3. Resolve the force $\underline{F}_2$ into its components.
4. Resolve the force $\underline{F}_3$ into its components.
5. Determine the resultant force, which is the vector sum of all the forces acting on the body.
6. Use the properties of vector addition.
7. Evaluate the resultant force.
8. Use Newton's second law of motion to find the acceleration, which is a vector.
9. Calculate the magnitude of the acceleration.
10. State the result.

WRITE/DRAW



$$\underline{F}_1 = 8 \sin(35^\circ)\hat{i} + 8 \cos(35^\circ)\hat{j}$$

$$\underline{F}_2 = 6 \sin(20^\circ)\hat{i} - 6 \cos(20^\circ)\hat{j}$$

$$\underline{F}_3 = -7 \cos(40^\circ)\hat{i} - 7 \sin(40^\circ)\hat{j}$$

$$\underline{R} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$$

$$\begin{aligned}\underline{R} &= (8 \sin(35^\circ)\hat{i} + 8 \cos(35^\circ)\hat{j}) \\ &\quad + (6 \sin(20^\circ)\hat{i} - 6 \cos(20^\circ)\hat{j}) \\ &\quad - (7 \cos(40^\circ)\hat{i} + 7 \sin(40^\circ)\hat{j})\end{aligned}$$

$$\underline{R} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$$

$$\begin{aligned}\underline{R} &= (8 \sin(35^\circ) + 6 \sin(20^\circ) - 7 \cos(40^\circ))\hat{i} \\ &\quad + (8 \cos(35^\circ) - 6 \cos(20^\circ) - 7 \sin(40^\circ))\hat{j}\end{aligned}$$

$$\underline{R} = 1.278\hat{i} - 3.584\hat{j}$$

$$\underline{F} = m\underline{a}$$

Since the mass $m = 2$,

$$\underline{R} = 1.278\hat{i} - 3.584\hat{j} = 2\underline{a}$$

$$\underline{a} = 0.639\hat{i} - 1.792\hat{j}$$

$$\begin{aligned}|\underline{a}| &= \sqrt{0.639^2 + 1.792^2} \\ &= 1.90\end{aligned}$$

The magnitude of the acceleration is.
1.90 m/s²

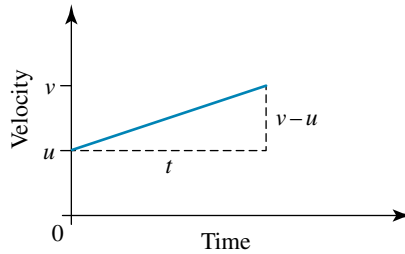
10.4.3 Constant acceleration formulas

If an object is moving due to a constant force, then its acceleration is constant. Rectilinear motion is motion in a straight line, either horizontally or vertically.

When we consider the motion of an object in a straight line with uniform acceleration, a number of rules can be used.

The diagram on the next page, represents the motion of an object with initial velocity u and final velocity v after t seconds.





The gradient of the line is calculated by $\frac{v-u}{t}$. On a velocity–time graph, the gradient is the acceleration of the object. Hence, $a = \frac{v-u}{t}$.

Transposing $a = \frac{v-u}{t}$, we get $v = u + at$ (equation 1).

Alternatively, by antidifferentiating $\frac{dv}{dt} = a$ with respect to t :

$$\int \frac{dv}{dt} dt = \int a dt$$

$$v = at + c$$

When $t = 0$, $c = u$ (initial velocity): $v = at + u$

The area under a velocity–time graph gives the displacement of the object. Therefore, using the rule for the area of a trapezium, we get $s = \frac{1}{2}(u+v)t$ (equation 2), where $\frac{1}{2}(u+v)$ is the average velocity of the object. Displacement is the average velocity of an object multiplied by time.

Substituting equation 1, $v = u + at$, into equation 2, $s = \frac{1}{2}(u+v)t$, gives:

$$s = \frac{1}{2}(u + u + at)t$$

$$s = \frac{1}{2}(2u + at)t$$

$$s = ut + \frac{1}{2}at^2 \quad \text{(equation 3)}$$

Alternatively, by antidifferentiating $\frac{dx}{dt} = v = u + at$ with respect to t ,

$x = ut + \frac{1}{2}at^2 + d$, where d is the initial position.

When $s = x - d$, which is the change in position of the particle from its starting point, $s = ut + \frac{1}{2}at^2$.

Using $v = u + at$ (equation 1) and $s = ut + \frac{1}{2}at^2$ (equation 3):

$$v^2 = (u + at)^2$$

$$v^2 = u^2 + 2uat + a^2t^2$$

$$v^2 = u^2 + 2a(ut + \frac{1}{2}at^2)$$

$$v^2 = u^2 + 2as \quad \text{(equation 4)}$$

If we substitute $u = v - at$ into $s = ut + \frac{1}{2}at^2$, we get

$$s = (v - at)t + \frac{1}{2}at^2,$$

which simplifies to

$$s = vt - \frac{1}{2}at^2 \quad (\text{equation 5})$$

In summary, when considering rectilinear motion of an object in which the acceleration is constant, the following rules can be used.

Constant acceleration formulas

• $v = u + at$	where	$v = \text{velocity (m/s),}$
• $s = ut + \frac{1}{2}at^2$		$u = \text{initial velocity (m/s),}$
• $v^2 = u^2 + 2as$		$a = \text{acceleration (m/s}^2\text{),}$
• $s = \frac{1}{2}(u + v)t$		$t = \text{time (s) and}$
• $s = vt - \frac{1}{2}at^2$		$s = \text{displacement (m)}$

In applying these rules, we must consider the following conditions.

- These quantities only apply when the acceleration is constant.
- Retardation or deceleration implies that acceleration is negative.
- The variable s is the displacement of an object. It is not necessarily the distance travelled by the object.

In solving constant acceleration problems it is important to list the quantities given to determine what is required and which equation is appropriate. It is also important to check units. All quantities must be converted to compatible units such as m and m/s or km and km/h.

Remember that to convert km/h to m/s, we multiply by $\frac{1000}{3600}$ or $\frac{5}{18}$; to convert m/s to km/h, we multiply by $\frac{18}{5}$ or 3.6.

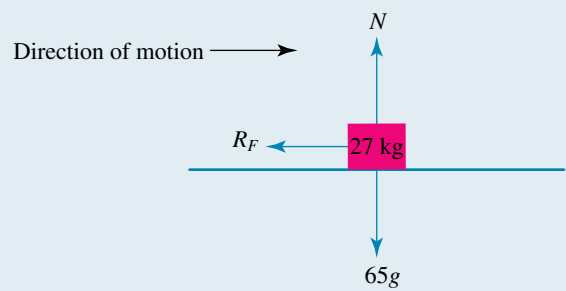
WORKED EXAMPLE 14

A boy of mass 25 kg on a skateboard of mass 2 kg is freewheeling on a level track. His speed decreases from 10 m/s to 5 m/s as he moves a distance of 30 m. Determine the magnitude of the resistant force, which is assumed to be constant.



THINK

1. Draw a force diagram. The only horizontal force acting on the boy and the skateboard is the resistance force R_F , which opposes the direction of motion.
2. Use the constant acceleration formulas.
3. Calculate the acceleration, a , of the boy on the skateboard.
4. Determine the resistance force using Newton's second law.
5. State the magnitude of the resistance force.

WRITE/DRAW

$$u = 10, v = 5, s = 30, a = ?$$

$$\text{Use } v^2 = u^2 + 2as.$$

$$25 = 100 + 2a \times 30$$

$$60a = -75$$

$$a = -1.25 \text{ m/s}^2$$

The acceleration is negative as he is slowing down.

$$R_F = ma$$

$$m = 27 \text{ and } a = -1.25$$

$$\begin{aligned} R_F &= 27 \times -1.25 \\ &= -33.75 \end{aligned}$$

The resistance force is 33.75 newtons.

10.4.4 Newton's third law of motion

Action and reaction pairs are equal and opposite and act on different bodies.

This law has already been considered when including the normal reaction on particles when they are at rest on, or in contact with, a surface. Note that only the external forces acting on a body will be considered in this course, and the effects of compression or the rotational aspects of bodies when forces are applied to them will be ignored.

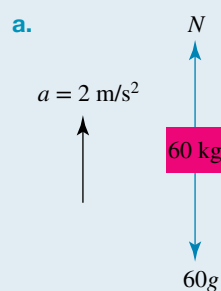
WORKED EXAMPLE 15

A 60 kg student stands in a lift. Calculate the reaction of the lift floor on the student if the lift is:

- a. accelerating upwards at 2 m/s^2
- b. accelerating downwards at 2 m/s^2
- c. moving with constant velocity.

THINK

1. Draw a force diagram, including the direction of motion. Let N newtons be the reaction of the lift floor on the student when the lift travels upwards.

WRITE/DRAW

2. Resolve the forces in the direction of motion using Newton's second law.

3. Calculate the magnitude of the reaction force.

4. State the value of the reaction force.

- b. 1. Draw a force diagram, including the direction of motion. Let N newtons now be the reaction of the lift floor on the student when the lift travels downwards. This value of N is different from the case in part a.

2. Resolve the forces in the direction of motion using Newton's second law.

3. Calculate the magnitude of the reaction force.

4. State the value of the reaction force.

- c. 1. When the lift moves with constant velocity, its acceleration is zero.

2. State the value of the reaction force in this case.

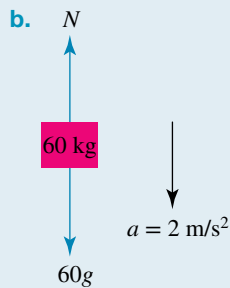
Consider the upwards direction as positive.
The lift is accelerating upwards at 2 m/s^2 and $m = 60$.

$$\underline{F} = m\underline{a}$$

$$N - 60g = 60 \times 2$$

$$\begin{aligned} N &= 60g + 60 \times 2 \\ &= 60(9.8 + 2) \\ &= 708 \end{aligned}$$

The reaction force is 708 N.



Consider the downwards direction as positive.
The lift is accelerating downwards at 2 m/s^2 .

$$\underline{F} = m\underline{a}$$

$$60g - N = 60 \times 2$$

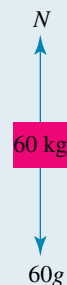
$$\begin{aligned} N &= 60g - 60 \times 2 \\ &= 60(9.8 - 2) \\ &= 468 \end{aligned}$$

The reaction force is 468 N.

- c. The weight force is equal to the reaction force.

$$N = mg$$

$$\begin{aligned} &= 60 \times 9.8 \\ &= 588 \end{aligned}$$



The reaction force is 588 N.

10.4.5 Motion on inclined planes

When a particle is moving along a slope, the forces are resolved parallel and perpendicular to the slope. For inclined plane problems, consider the $\hat{i}$ direction as parallel to the slope. Thus, Newton's second law of

motion states that $\sum F_x = ma$. That is, the vector sum of all the resolved forces in the direction of motion is

equal to ma . The vector sum of all the resolved forces in the direction perpendicular to the motion in the $\hat{j}$ direction, $\sum F_y$, is equal to zero, because the particle is not moving in that direction.

In this section, the term ‘tractive force’ is used. This is a force that moves a body on a surface.

Slopes

When a particle moves up or down a slope, the inclination of the slope is often stated as the vertical distance raised over the distance travelled along the slope. So a slope of 1 in n means that $\sin(\theta) = \frac{1}{n}$, where θ is the inclination of the slope to the horizontal.

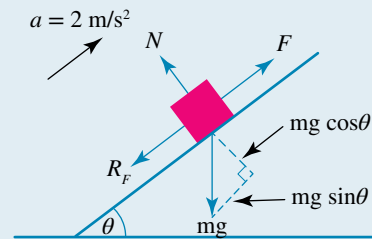
WORKED EXAMPLE 16

A boy of mass 70 kg rides a 10 kg bicycle up a slope of 1 in 5. If the resistance to the motion is one-quarter of the weight force, calculate the tractive force of the tyres when he ascends with an acceleration of 2 m/s^2 .

THINK

1. Draw a force diagram. Let F newtons be the tractive force exerted by the tyres that makes the bike go up the hill. The resistance force R_F is one-quarter of the weight force. Consider the $\hat{i}$ direction as the positive direction going up and parallel to the slope.
2. Resolve the forces upwards in the direction of motion.
3. Solve for the tractive force.
4. Substitute for the given values.
5. Evaluate and determine the value of the required force.
6. State the result.

WRITE/DRAW



The boy and the bike move up the slope with an acceleration of a .

$$F - R_F - mg \sin(\theta) = ma$$

$$F = R_F + mg \sin(\theta) + ma$$

$$R_F = \frac{1}{4} mg \text{ and a slope of 1 in 5 is}$$

$$\text{defined to mean } \sin(\theta) = \frac{1}{5}.$$

$$F = \frac{mg}{4} + \frac{mg}{5} + ma$$

$$F = m \left(\frac{9g}{20} + a \right)$$

The total mass of the boy and the bike is 80 kg and the acceleration is 2 m/s^2 .

$$F = 80 \left(\frac{9 \times 9.8}{20} + 2 \right)$$

$$= 512.80$$

The tractive force is 512.80 N.

10.4.6 Resolution of all the forces

When two or more forces have components, it is important to remember to find the resolved components of *all* the forces parallel and perpendicular to the plane.

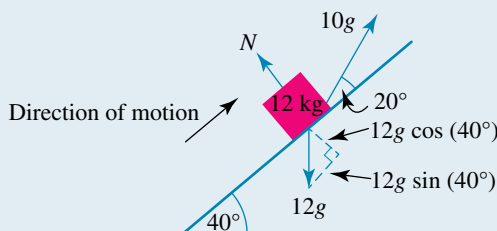
WORKED EXAMPLE 17

A certain strap on a suitcase will break when the tension in it exceeds 10 kgwt. Calculate the greatest acceleration with which a 12 kg suitcase can be dragged up a smooth plane inclined at an angle of 40° to the horizontal when the strap is inclined at 20° to the plane.

THINK

1. Draw the force diagram. Let T newtons be the tension in the strap. Note that $T = 10g$, as all the forces must be in newtons. Recall $1\text{ kgwt} = g\text{ N}$.

WRITE/DRAW



2. Calculate the component of the tension force up and parallel to the plane.
3. Calculate the component of the weight force parallel to the plane.
4. Use Newton's second law to find the acceleration.
5. Resolve all the forces, perpendicular to the plane. However, this equation is not needed in this case.
6. Solve to determine the maximum acceleration.
7. State the acceleration that occurs when the tension is 10g newtons.

The component of the tension in the strap parallel to the plane is $10g \cos(20^\circ)$.

The component of the weight force is $12g \sin(40^\circ)$, which acts down the plane.

The suitcase moves up the plane with an acceleration of a .

$$10g \cos(20^\circ) - 12g \sin(40^\circ) = 12a$$

$$10g \sin(20^\circ) + N - 12g \cos(40^\circ) = 0$$

$$a = \frac{9.8}{12}(10 \cos(20^\circ) - 12 \sin(40^\circ))$$

$$= 1.37$$

The acceleration is 1.37 m/s^2 .

studyon

Units 3 & 4

Area 5

Sequence 3

Concept 3

Momentum and resultant force Summary screen and Practice questions

Exercise 10.4 Momentum and resultant force

Technology active

1. **WE12** Find the magnitude of the momentum of a truck of mass 6 tonnes moving at 96 km/h.
2. Determine the magnitude of the momentum of the Earth as it orbits the Sun. Take the mass of the Earth to be $5.972 \times 10^{24} \text{ kg}$ and the orbital speed to be 30 km/s.
3. **WE13** A particle of mass 3 kg on a smooth horizontal plane is acted upon by three forces in that horizontal plane. $\underline{F}_1$ has a magnitude of 9 newtons and acts in direction N55°E, $\underline{F}_2$ has a magnitude of 7 newtons and acts in direction S40°E, and $\underline{F}_3$ has a magnitude of 6 newtons and acts in direction S55°W. If $\hat{i}$ is a unit vector of magnitude 1 newton acting east and $\hat{j}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.

4. A particle of mass 4 kg on a smooth horizontal plane is acted upon by four forces in that horizontal plane. $\underline{F}_1$ has a magnitude of 12 newtons and acts due north, $\underline{F}_2$ has a magnitude of 8 newtons and acts in direction S60°W, $\underline{F}_3$ has a magnitude of 8 newtons and acts in direction S60°E, and $\underline{F}_4$ has a magnitude of 10 newtons and acts due east. If $\hat{i}$ is a unit vector of magnitude 1 newton acting east and $\hat{j}$ is a unit vector of magnitude 1 newton acting north, determine the magnitude of the acceleration of the particle.
5. **WE14** A car of mass 1200 kg moves on a horizontal road and brakes. Its speed decreases from 16 m/s to 6 m/s while it moves a distance of 25 m. Calculate the magnitude of the resistant breaking force which is assumed to be constant.
6. A woman is riding a bicycle and is moving at a speed of 6 km/h. She freewheels, coming to rest after 5 seconds. If the magnitude of the constant resistance force is 20 newtons, calculate the mass of the woman and the bicycle.
7. **WE15** An 80 kg man stands in a lift. Calculate the reaction of the lift floor on the man if the lift is:
- accelerating upwards at 3 m/s^2
 - accelerating downwards at 3 m/s^2
 - moving with constant velocity.
8. A girl stands in a lift. When the lift accelerates up at 1.5 m/s^2 , the reaction of the lift floor on the girl is 734.5 N. When the lift accelerates down at 1.5 m/s^2 , calculate the reaction of the lift floor on the girl.
9. **WE16** A car of mass 1200 kg ascends a hill of slope 1 in 4. If the resistance to the motion is one-fifth of the weight force, calculate the tractive force of the tyres when the car ascends the hill with an acceleration of 2 m/s^2 .
10. A truck of mass 1.5 tonnes is moving at a speed of 36 km/h down a hill of slope 1 in 6. The driver applies the brakes and the truck comes to rest after 2 seconds. Calculate the braking force, which is assumed to be constant, ignoring any other resistance forces.
11. **WE17** A parcel of mass 5 kg is pushed up a slope inclined at an angle of 40° to the horizontal by a force of $F \text{ N}$ that is inclined at an angle of 15° to the plane. The parcel accelerates at 2 m/s^2 up the slope. Calculate the value of F .
12. A crate of mass 8 kg sits on a smooth plane inclined at an angle of 25° to the horizontal. It is acted upon by a force of $F \text{ N}$ that is inclined at an angle of 20° to the plane.
- If the crate accelerates at 1 m/s^2 up the slope, calculate the value of F .
 - calculate the maximum value of F that can be applied without the crate leaving the plane.
13. **a.** Calculate the resultant force acting on a body of mass 3 kg moving with an acceleration of 2 m/s^2 .
b. Calculate the magnitude of the acceleration when a mass of 2.5 kg is acted upon by a force of $\underline{F} = 3\hat{i} - 4\hat{j} \text{ N}$.
c. Calculate the magnitude of the acceleration when a mass of 2 kg is acted upon by two forces of $\underline{F}_1 = 3\hat{i} - 4\hat{j}$ and $\underline{F}_2 = -7\hat{i} + 7\hat{j} \text{ N}$.
14. **a.** A particle of mass 2 kg is acted upon by two forces. One force has a magnitude of $3\sqrt{2} \text{ N}$ acting in the north-west direction; the other force has magnitude of $4\sqrt{2} \text{ N}$ acting in the south-west direction. Determine the magnitude of the acceleration of the particle.
b. A particle of mass $m \text{ kg}$ is acted upon by two forces. One force has a magnitude of $b \text{ N}$ acting in the north-east direction; the other force has magnitude of $c \text{ N}$ acting in the south-east direction. Determine the magnitude of the acceleration acting on the body.
15. **a.** A particle of mass 5 kg on a smooth horizontal plane is acted upon by three forces. $\underline{F}_1$ has a magnitude of 9 N and acts in direction N55°E, $\underline{F}_2$ has a magnitude of 8 N and acts in direction S40°E, and $\underline{F}_3$ has a magnitude of 7 N and acts in direction S55°W. Determine the magnitude of the acceleration of the particle.



- b. A body on a smooth horizontal plane is acted upon by four forces. $\underline{F}_1$ has a magnitude of 6 N and acts in direction N30°E, $\underline{F}_2$ has a magnitude of $8\sqrt{2}$ N and acts in direction S45°E, $\underline{F}_3$ has a magnitude of 4 N and acts in direction N60°W, and $\underline{F}_4$ has a magnitude of $5\sqrt{2}$ N and acts in direction S45°W. If $\hat{i}$ is a unit vector of magnitude 1 N acting east and $\hat{j}$ is a unit vector of magnitude 1 N acting north, calculate the resultant force acting on the body.
16. a. A constant force of 12 newtons acts on a body of mass 4 kg originally at rest. Calculate:
- the speed of the body after 3 seconds
 - the distance travelled by the body in 3 seconds.
- b. Calculate the average resistance force exerted by the brakes on a car of mass 1.2 tonnes during braking if its speed decreases from 60 km/h to rest:
- in a time of 1.5 seconds
 - in travelling a distance of 15 metres.
17. a. A package of mass 4 kg lies on the floor of a lift that is accelerating upwards at 2 m/s^2 . Calculate the reaction of the lift floor on the package.
- b. A parcel of mass 5 kg rests on the floor of a lift. If the reaction between the box and the floor is 48 N, calculate the downwards acceleration of the lift.
18. A man of mass 75 kg stands in a lift. What is the reaction between his feet and the floor when the lift is:
- ascending with a constant acceleration of 0.5 m/s^2
 - descending with a constant acceleration of 0.5 m/s^2
 - moving with a constant speed of 0.5 m/s ?
19. A boy stands in a lift. When the lift accelerates upwards with a constant acceleration of $a \text{ m/s}^2$, the reaction of the floor on the boy is 502.5 N. When the lift accelerates downwards with a constant acceleration of $a \text{ m/s}^2$, the reaction of the floor on the boy is 477.5 N. Determine the mass of the boy and the value of a .
20. A child of mass 40 kg is on a 2 kg sled on horizontal snow-covered ground. The child and sled are being pulled along the snow by a force of 100 N. Determine the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:
- horizontal
 - at an angle of 35° to the horizontal.
21. a. A child of mass 40 kg is on a 2 kg sled which is being pulled up a snow-covered slope by a force of 300 N. The slope is inclined at an angle of 40° to the horizontal. Determine the acceleration of the child and the magnitude of the normal reaction of the child on the snow if the force is:
- parallel to the slope
 - acting at an angle of 25° to the slope.
- b. A child of mass 40 kg is on a 2 kg sled sliding down a snow-covered slope which is inclined at an angle of 40° to the horizontal. Given that the initial speed is zero and that the friction is negligible, calculate the speed attained when the child and sled have travelled a distance of 20 m down the slope.



22. Calculate the tension in a rope if a 15 kg mass is moved with an acceleration of 2 m/s^2 :
- vertically upwards
 - along a smooth horizontal plane parallel to the rope
 - along a smooth horizontal plane if the rope makes an angle of 15° with the plane
 - up along a smooth plane that is inclined at an angle of 20° to the horizontal, and the rope is parallel to the plane
 - up along a smooth plane that is inclined at an angle of 20° , and the rope makes an angle of 15° with the plane.
23. A parcel of mass $m \text{ kg}$ is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 25 N is applied acting up and parallel to the slope, the parcel accelerates up the slope at 0.8 m/s^2 . When the force is increased to 30 N acting up and parallel to the slope, the parcel accelerates up the slope at 1.8 m/s^2 . Determine the values of m and θ .
24. a. A package of mass $m \text{ kg}$ is on a smooth horizontal surface. When a force of 5 newtons is applied to the parcel acting at an angle of α to the horizontal, the parcel accelerates at 0.5 m/s^2 . When the force is increased to 10 N and is acting at an angle of 2α to the horizontal, the package accelerates at 0.9 m/s^2 . Determine the values of m and α .
- b. A package of mass 5 kg is on a smooth slope inclined at an angle of θ to the horizontal. When a force of 40 N is applied acting at an angle of α to the slope, the package accelerates up the slope at 1.2 m/s^2 . When the force is increased to 50 N, the package accelerates at 3.1 m/s^2 . Determine the values of θ and α .

10.5 Forces on connected particles

10.5.1 Two or more particles

When dealing with connected particles or a system involving two or more particles, isolate the particles and resolve around each independently. In the direction of motion, equate the resultant force to ma where m is the mass of the particle and a the acceleration of the system.

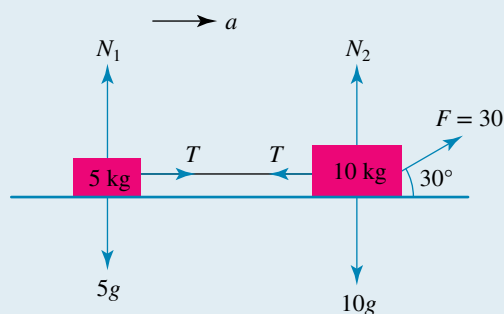
WORKED EXAMPLE 18

Two boxes of masses 5 kg and 10 kg are connected by a light string and lie on a smooth horizontal surface. A force of 30 newtons pulls on the 10 kg box at an angle of 30° to the surface. Determine the acceleration of the system and the tension in the string connecting the boxes.

THINK

- Draw the force diagram. The force pulling the box is 30 N. Let T be the tension in newtons in the connecting string and let $a \text{ m/s}^2$ be the acceleration of the system.

WRITE/DRAW



- Isolate the particles and consider the motion of the first box. It is not necessary to resolve vertically around the boxes in this situation.

Resolving horizontally around the 10 kg box:
 $30 \cos(30^\circ) - T = 10a$ [1]

3. Consider the motion of the second box.
4. To find the acceleration, eliminate the tension in the string.
5. Calculate the common acceleration of the system.
6. Calculate the tension in the string.

Resolving horizontally around the 5 kg box:

$$T = 5a \quad [2]$$

Substitute $T = 5a$ into [1]:

$$30 \cos(30^\circ) - 5a = 10a$$

$$30 \times \frac{\sqrt{3}}{2} = 15a$$

$$a = \sqrt{3}$$

The acceleration of the system is $\sqrt{3} \text{ m/s}^2$.

From [2], $T = 5a$.

The tension is $5\sqrt{3} \text{ N}$.

10.5.2 Particles connected by smooth pulleys

Often particles may be connected by a smooth pulley. In these situations, the tension in the string is unaltered when the string passes over the pulley.

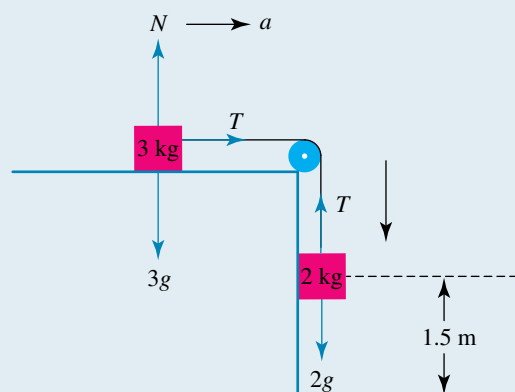
WORKED EXAMPLE 19

A block of mass 3 kg rests on a smooth table, 2 metres from the edge. It is connected by a light inextensible string that passes over a smooth pulley at the edge of the table to a second block of mass 2 kg hanging vertically. Initially the 2 kg block is 1.5 metres above the ground. Calculate the time taken for the 2 kg block to hit the ground after the system is released from rest.

THINK

1. Draw the force diagram and let T newtons be the tension in string.
2. Isolate the particles and consider the motion of the particle hanging vertically.
3. Consider the motion of the particle on the table.
4. To calculate the acceleration, eliminate the tension in the string.

WRITE/DRAW



The 2 kg block moves vertically downwards with an acceleration of $a \text{ m/s}^2$.

$$2g - T = 2a \quad [1]$$

The 3 kg block moves horizontally along the smooth table with an acceleration of $a \text{ m/s}^2$.

$$T = 3a \quad [2]$$

Substitute $T = 3a$ into [1]:

$$2g - 3a = 2a$$

5. Calculate the common acceleration of the system.

$$5a = 2g$$

$$a = \frac{2g}{5}$$

6. Use constant acceleration formulas to find the time for the particle hanging vertically to hit the ground.

$$a = \frac{2g}{5}, u = 0, s = 1.5, t = ?$$

$$\text{Use } s = ut + \frac{1}{2}at^2.$$

7. Calculate the required time.

$$\frac{3}{2} = 0 + \frac{1}{2} \times \frac{2g}{5} t^2$$

$$t^2 = \frac{15}{2g}$$

$$t = \sqrt{\frac{15}{2 \times 9.8}}$$

$$= 0.87$$

8. State the final result.

The time taken is 0.87 seconds.

10.5.3 Atwood's machine

George Atwood (1745–1807) was an English mathematician who invented a machine consisting of masses attached to pulleys to illustrate Newton's laws of motion. He was also a highly skilled chess player.

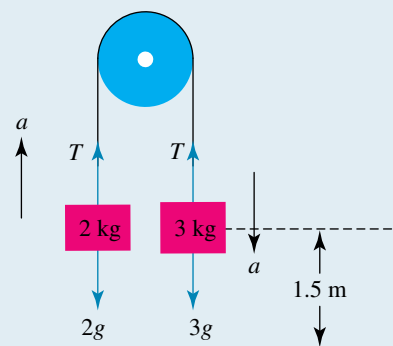
WORKED EXAMPLE 20

Two particles of masses 3 kg and 2 kg are both hanging vertically and are connected by a light inextensible string which passes over a smooth pulley. Initially the 3 kg block is 1.5 metres above the ground. Calculate the time taken for the 3 kg block to hit the ground once the system is released from rest.

THINK

1. Draw the force diagram and let T newtons be the tension in the string. Because the 3 kg mass is heavier, it moves down and the 2 kg mass moves up. Because they are connected, they both move with the same acceleration $a \text{ m/s}^2$.

WRITE



2. Isolate the particles and consider the motion of the 3 kg particle hanging vertically.
3. Consider the motion of the 2 kg particle hanging vertically.

The 3 kg particle moves vertically downwards with an acceleration of $a \text{ m/s}^2$.

$$3g - T = 3a \quad [1]$$

The 2 kg particle moves vertically upwards with an acceleration of $a \text{ m/s}^2$.

$$T - 2g = 2a \quad [2]$$

4. To calculate the acceleration, eliminate the tension in the string.
5. Calculate the common acceleration of the system.
6. Use constant acceleration formulas to calculate the time for the particle hanging vertically to hit the ground.
7. Calculate the required time.

Add [1] and [2]:

$$g = 5a$$

$$a = \frac{g}{5} \text{ m/s}^2$$

$$a = \frac{g}{5}, u = 0, s = 1.5, t = ?$$

$$\text{Use } s = ut + \frac{1}{2}at^2.$$

$$\frac{3}{2} = 0 + \frac{1}{2} \times \frac{g}{5} t^2$$

$$t^2 = \frac{15}{g}$$

$$t = \sqrt{\frac{15}{9.8}}$$

$$= 1.24$$

The time taken is 1.24 seconds.

8. State the final result.

10.5.4 Connected vehicles

When cars, trucks, trains and carriages are connected, they are typically connected by tow bars. It is the tension in the tow bar that enables the secondary object to move with the same acceleration as the primary object, which exerts the force to make the connected objects move.

WORKED EXAMPLE 21

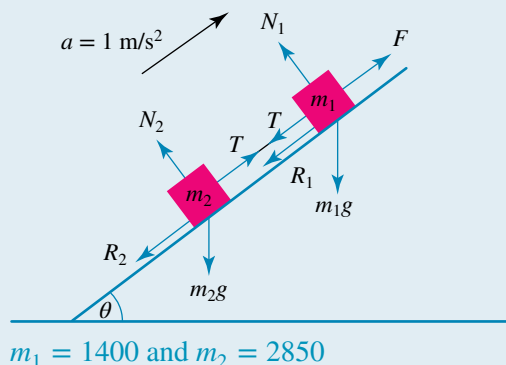
A car of mass 1400 kg is pulling a caravan of mass 2850 kg up an incline of slope 1 in 4. The resistances to the motion of the car and the caravan are both one-quarter of their weight forces. Determine the tension in the tow bar and the tractive force exerted by the engine in the car if the car moves up the incline with an acceleration of 1 m/s^2 .



THINK

1. Draw the force diagram. Let the tension in the tow bar be T newtons and let the tractive force of the car be F newtons. Let the resistances to the car and the caravan be R_1 and R_2 respectively.

WRITE/DRAW



2. Isolate the particles and resolve around each independently.

3. Isolate the particles and resolve around each independently.

4. The values of F and T are to be found.

5. Calculate the tension in the tow bar. Recall that a slope of 1 in 4 means $\sin(\theta) = \frac{1}{4}$. The system accelerates up the slope at 1 m/s^2 , so $a = 1 \text{ m/s}^2$.

6. State the value of the tension in the tow bar. Note that kN is the unit representation of 1000 newtons.

7. Calculate the tractive force.

8. State the value of the tractive force.

9. Alternative methods are possible.

10. Use the system to calculate the value of the tractive force.

Resolving upwards parallel to the slope around the car:

$$F - R_1 - T - m_1 g \sin(\theta) = m_1 a \quad [1]$$

Resolving upwards parallel to the slope for the caravan:

$$T - R_2 - m_2 g \sin(\theta) = m_2 a \quad [2]$$

The resistance forces are one-quarter of the weight forces.

$$R_1 = \frac{m_1 g}{4} \text{ and } R_2 = \frac{m_2 g}{4}$$

Substitute the known values into [2]:

$$T = R_2 + m_2 g \sin(\theta) + m_2 a$$

$$T = \frac{m_2 g}{4} + \frac{m_2 g}{4} + m_2 \times 1$$

$$T = m_2 \left(\frac{g}{2} + 1 \right)$$

$$T = 2850 (4.9 + 1)$$

$$T = 16\,815$$

The tension in the tow bar is 16 815 newtons or 16.815 kN.

Substitute the known values into [1]:

$$F = R_1 + T + m_1 g \sin(\theta) + m_1 a$$

$$F = \frac{m_1 g}{4} + 16\,815 + \frac{m_1 g}{4} + 1 \times m_1$$

$$F = m_1 \left(\frac{g}{2} + 1 \right) + 16\,815$$

$$F = 1400 (4.9 + 1) + 16\,815$$

$$F = 25\,075$$

The tractive force exerted by the car is 25.075 kN.

If we add equations [1] and [2], we can eliminate the tension in the tow bar.

$$F - (R_1 + R_2) - (m_1 + m_2) g \sin(\theta) = (m_1 + m_2) a$$

Doing this is equivalent to resolving around the system as a whole unit rather than considering individual particles.

$$F = (R_1 + R_2) + (m_1 + m_2) g \sin(\theta) + (m_1 + m_2) a$$

$$F = \frac{1}{4} (m_1 + m_2) g + (m_1 + m_2) g \sin(\theta) + (m_1 + m_2) a$$

$$F = (m_1 + m_2) \left[\frac{g}{4} + g \sin(\theta) + a \right]$$

$$F = 4250 \left[\frac{9.8}{4} + \frac{9.8}{4} + 1 \right]$$

$$F = 25\,075 \text{ N as before.}$$

Exercise 10.5 Forces on connected particles

Technology active

- WE18** Two boxes of mass 3 kg and 4 kg are connected by a string and lie on a smooth horizontal surface. A force of F N acts on the 4 kg box at angle of 60° to the surface. The system accelerates at 1.5 m/s^2 . Determine the value of F .
- Two boxes of mass m_1 kg and m_2 kg are connected by a string and lie on a smooth horizontal surface. A force of 18 N acts on the m_1 kg box at angle of 30° to the surface, and the tension in the string joining the two masses is $4\sqrt{3}$ N. The system accelerates at $\sqrt{3} \text{ m/s}^2$. Determine the values of m_1 and m_2 .
- WE19** A block of mass 4 kg rests on a smooth table, 1.5 m from the edge. It is connected by a light inextensible string passing over a smooth pulley at the edge of the table to a second block of mass 3 kg hanging vertically over the edge. Initially the 3 kg block is 1 m above the ground. Calculate the speed with which the 3 kg block hits the ground after the system is released.
- A block of mass m_2 kg rests on a smooth table. It is connected by a light inextensible string that passes over a smooth pulley at the edge of the table to a second block of mass m_1 kg hanging vertically over the edge of the table. When the system is released from rest, the tension in the string is $\frac{24g}{5}$ N and the mass m_1 accelerates down at $\frac{2g}{5} \text{ m/s}^2$. Calculate the values of m_1 and m_2 .
- WE20** Two particles with masses 4 kg and 3 kg are both hanging vertically and are connected by a light inextensible string that passes over a smooth pulley. Initially the 4 kg block is 2 m above the ground. Determine the speed with which the 4 kg block hits the ground when the system is released from rest.
- Two particles with masses m kg and $2m$ kg are both hanging vertically and are connected by a light inextensible string that passes over a smooth pulley. Determine the tension in the string when the system is released from rest.
- WE21** A car of mass 1200 kg is pulling a trailer of mass 950 kg up an incline of slope 1 in 5. The resistances to the motion of the car and the trailer are both one-third of their weight forces. Calculate the tractive force exerted by the engine in the car and the tension in the tow bar if the car moves up the incline with an acceleration of 0.8 m/s^2 .
- A train of mass 10 tonnes is pulling a carriage of mass 5 tonnes up an incline of slope 1 in 3. The resistances to the motion of the train and the carriage are both one-fifth of their weight forces. The tractive force exerted by the engine in the train is 90 kN. Calculate the tension in the tow bar connecting the train and the carriage and the acceleration of the system.
- Two suitcases of mass 12 kg and 15 kg are connected by a string and lie on a smooth horizontal surface. A force of 20 newtons acts on the 12 kg suitcase. Calculate the acceleration of the system and the tension in the string connecting the suitcases if:
 - the force is horizontal
 - the force acts at angle of 40° to the surface.
- Two particles with masses 3 kg and 5 kg both initially at rest are hanging vertically and are connected by a light inextensible string that passes over a smooth fixed pulley. Calculate the acceleration of the system when released and the tension in the string.
 - Two particles with masses m_1 kg and m_2 kg are hanging vertically and are connected by a light inextensible string that passes over a smooth fixed pulley. If $m_2 > m_1$, show that the acceleration of the system is given by $\frac{(m_2 - m_1)g}{m_1 + m_2} \text{ m/s}^2$ and the tension in the string is equal to $\frac{2m_1m_2g}{m_1 + m_2}$ N.

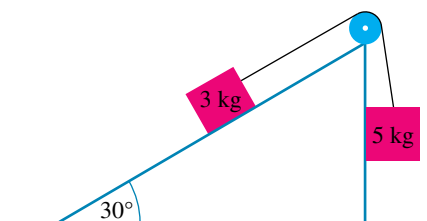
11. a. A mass of 3 kg is lying on a smooth horizontal table and is joined by a light inextensible string to a mass of 5 kg hanging vertically. This string passes over a smooth pulley at the edge of the table. The system is released from rest. Determine the system's acceleration and the tension in the string connecting the masses.
- b. Two particles with masses m_1 kg and m_2 kg are connected by a light inextensible string. Mass m_1 lies on a smooth horizontal table and m_2 hangs vertically. The connecting string passes over a smooth pulley at the table's edge. Show that the tension in the string is $\frac{m_1 m_2 g}{m_1 + m_2}$ N.

12. Two suitcases of mass 12 kg and 15 kg are connected by a string. They both lie on a smooth ramp inclined at an angle of 20° to the horizontal. A force of 120 N acts on the 12 kg suitcase, pulling it up the ramp.



Determine the acceleration of the system and the tension in the string connecting the suitcases if:

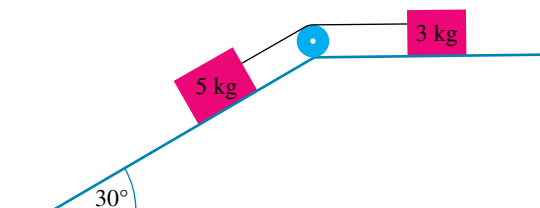
- a. the force is parallel to the ramp
- b. the force acts at an angle of 40° to the ramp.
13. a. A particle of mass 3 kg is placed on the surface of a smooth plane inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass 5 kg hanging freely. Determine the acceleration of the system and the tension force in the string.
- b. A particle of mass m_1 kg is placed on the surface of a smooth plane inclined at an angle of θ to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the plane to a particle of mass m_2 kg hanging vertically. Show that the acceleration of the system is given by



$$\frac{g(m_2 - m_1 \sin(\theta))}{m_1 + m_2} \text{ m/s}^2 \text{ and that the tension force in the string is}$$

$$\frac{m_2 m_1 g (1 + \sin(\theta))}{m_1 + m_2} \text{ N, provided } m_2 > m_1 \sin(\theta).$$

14. a. A particle of mass 5 kg is placed on a slope inclined at an angle of 30° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass 3 kg, which is on a smooth horizontal table. Calculate the tension in the string.

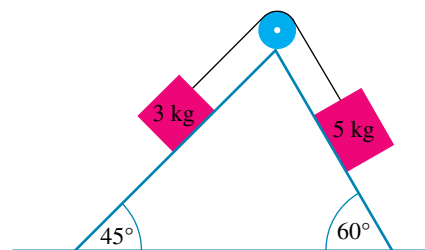


- b. A particle of mass m_2 kg is placed on a slope inclined at an angle of θ to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass m_1 kg, which is on a smooth horizontal table. Show that the tension in the string is given by

$$\frac{m_2 m_1 g \sin(\theta)}{m_1 + m_2} \text{ N.}$$

15. a. Two masses of 3 kg and 5 kg are placed on two smooth inclines of 45° and 60° respectively placed back to back. The masses are connected by a light inextensible string that passes over a smooth pulley at the top of the inclines. Show that the acceleration of the system is given by

$$\frac{g}{16} (5\sqrt{3} - 3\sqrt{2}) \text{ m/s}^2.$$

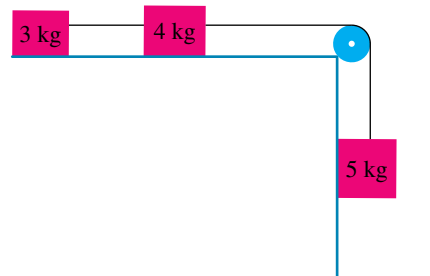


- b. A wedge consists of two smooth inclined planes placed back to back. A particle of mass m_1 rests on the slope inclined at an angle of α to the horizontal, and a particle of mass m_2 rests on the slope inclined at an angle of β to the horizontal. The particles are connected by a light string that passes over a smooth pulley at the common vertex of the two planes. If $\beta > \alpha$ and the mass m_2 moves down the plane, show that the tension in the string is given by

$$\frac{m_1 m_2 g (\sin(\alpha) + \sin(\beta))}{m_1 + m_2} \text{ N.}$$

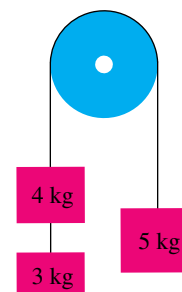
16. a. A truck of mass 8 tonnes pulls a trailer of mass 2 tonnes with an acceleration of magnitude 2 m/s^2 along a level road. The truck exerts a tractive force of magnitude 60 000 newtons, and the total resistance due to wind and other factors of the trailer is 2000 newtons. Determine the resistance to motion of the truck and the tension in the coupling.
- b. A truck of mass 8 tonnes is pulling a trailer of mass 2 tonnes up a hill of slope 1 in 3. The tractive force exerted by the engine is 60 000 newtons, and the resistances to both the truck and the trailer are one-tenth of the respective weight forces. Determine the acceleration of the system and the tension in the coupling between the truck and the trailer.

17. Two masses of 3 kg and 4 kg are connected by a light string and both lie on a smooth horizontal table. The 4 kg mass is connected to another mass of 5 kg hanging vertically by another light inextensible string that passes over a smooth fixed pulley at the edge of the table. The system is released from rest and the 5 kg mass moves downwards.



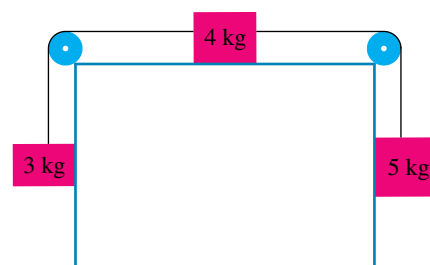
Calculate the tension in the string connecting the:

- a. 3 kg and 4 kg masses
- b. 4 kg and 5 kg masses.
18. Two masses of 3 kg and 4 kg are hanging vertically and are connected by a light string with the 4 kg mass above the 3 kg mass. The 4 kg mass is attached to another string, which passes over a smooth pulley and supports a third mass of 5 kg hanging vertically. The system is released from rest.



Calculate the tension in the string connecting the:

- a. 3 kg and 4 kg masses
- b. 4 kg and 5 kg masses.
19. a. Two masses of 3 kg and 5 kg are hanging vertically over two different edges of a table. They are connected by two light strings that pass over two smooth fixed pulleys at the edges of the table to another 4 kg mass which lies on a smooth horizontal table. The system is released from rest and the 5 kg mass moves downwards.



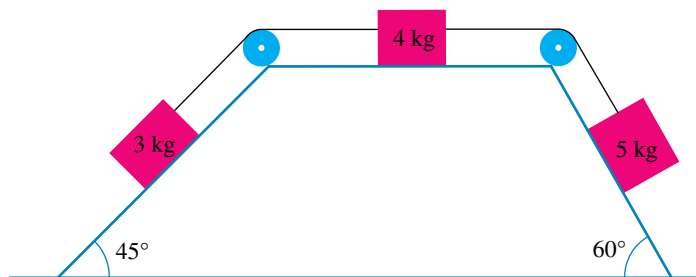
Determine the tension in the string connecting the:

- i. 3 kg and 4 kg masses
- ii. 4 kg and 5 kg masses.

- b. Two masses of m_2 and m_1 kg are hanging vertically over two different edges of a table. They are connected by two light strings that pass over two smooth fixed pulleys at the edges of the table to another mass of M kg which lies on a smooth horizontal table. The system is released from rest. If $m_2 > m_1$, so the mass m_2 moves downwards, show that the acceleration of the system is given by

$$\frac{g(m_2 - m_1)}{m_1 + m_2 + M} \text{ m/s}^2.$$

20. a. A particle of mass 3 kg is placed on a slope inclined at an angle of 45° to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass 4 kg which is on a smooth horizontal table. Another light inextensible string passing over a smooth pulley at a different edge of the table connects the 4 kg mass to 5 kg mass on a slope inclined at an angle of 60° to the horizontal. If the 5 kg mass moves down the slope, show that the tension connecting the 3 kg and 4 kg masses is $\frac{g}{8} (5\sqrt{3} + 9\sqrt{2})$ N and the tension connecting the 4 kg and 5 kg masses is $\frac{g}{24} (35\sqrt{3} + 15\sqrt{2})$ N.



- b. A particle of mass m_1 kg is placed on a slope inclined at an angle of α to the horizontal. It is connected by a light inextensible string passing over a smooth pulley at the top of the slope to a particle of mass M kg which is on a smooth horizontal table. Another light inextensible string passing over a smooth pulley at the edge of the table connects the M kg mass to another mass, m_2 kg, on a slope inclined at an angle of β to the horizontal. Assuming that $m_2 \sin(\beta) > m_1 \sin(\alpha)$ so that the mass m_2 moves down the plane, show that the acceleration of the system is given by

$$\frac{g(m_2 \sin(\beta) - m_1 \sin(\alpha))}{m_1 + m_2 + M} \text{ m/s}^2.$$

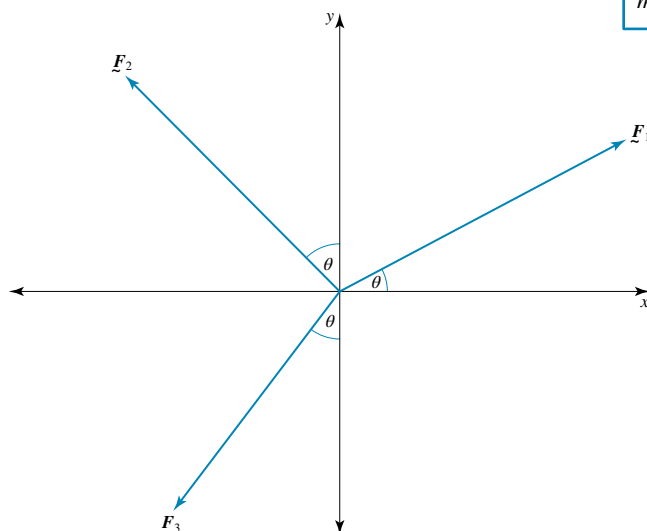
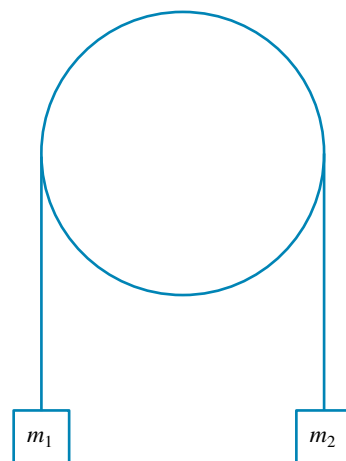
10.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. A particle is acted upon by two forces, one of magnitude 10 newtons acting west, and another of magnitude 8 newtons acting in the direction $N30^\circ E$. Determine the magnitude of the resultant force in newtons, acting on the particle.
2. A box of mass 5 kg is on horizontal ground. A rope making an angle of 30° with the horizontal pulls on the box with a force of 5 kg wt. If the box moves with an acceleration of a m/s² along the ground and the normal reaction of the box on the ground is N newtons, determine the values of a and N .
3. A block of mass m kg, rests on a smooth plane inclined at an angle of θ to the horizontal. It is supported by a force of F kg wt acting at angle of α with the plane. Write an expression for $\frac{F}{mg}$ in terms of the angles θ and α .

4. A constant force of 10 newtons acts on a mass of 5 kg initially moving at 1 m/s. After the mass has moved a distance of 20 metres in a straight line, determine the magnitude of the momentum in kg m/s.
5. A particle is acted upon by three forces. One force has a magnitude of $10\sqrt{2}$ newtons acting in the north-east direction. Another force has magnitude of 10 newtons acts in the west direction and the third force has magnitude of 10 newtons acts south. In what direction does the particle move?
6. A mass of m kg is supported by two light inextensible strings; one string has a length of L and has a tension of T_1 , the other string has a length of $2L$ and has a tension of T_2 . The other ends of the string are attached to a ceiling at a distance of $\sqrt{5}L$ apart. Write an expression for $\frac{T_1}{T_2}$.
7. A man of mass 90 kg is standing in a lift. The lift is moving downwards with an acceleration equal to 0.5 m/s^2 . Calculate the reaction of the lift floor on the man in newtons.
8. Two boxes of masses m_1 and m_2 kg are on a smooth horizontal table and are connected by a taut string that has a tension of T newtons. A force of F newtons acting at an angle of θ to the horizontal is applied to the mass m_1 away from the mass m_2 causing the boxes to move with a constant acceleration. Write an expression for $\frac{T}{F}$.
9. A light inextensible string passes over a smooth pulley. Particles of masses m_1 and m_2 are attached to each end of the string as shown in the diagram. If the mass m_1 accelerates upwards at $\frac{g}{4}$, determine the ratio $\frac{m_2}{m_1}$.
10. **MC** The following diagram shows a particle in equilibrium under the action of three concurrent coplanar forces $\underline{F}_1$, $\underline{F}_2$ and $\underline{F}_3$. The forces $\underline{F}_1$, $\underline{F}_2$ and $\underline{F}_3$ have magnitudes of F_1 , F_2 and F_3 respectively. Which one of the following statements is correct?



- A. $(F_2 - F_3) \cos(\theta) = F_1 \sin(\theta)$
- B. $F_1 \sin(\theta) = (F_2 + F_3) \cos(\theta)$
- C. $F_1 \cos(\theta) = (F_2 - F_3) \sin(\theta)$
- D. $F_1 \cos(\theta) = (F_2 + F_3) \sin(\theta)$

11. A particle is in equilibrium and is acted upon by three forces. One acts horizontally and has a magnitude of 3 newtons, another acts at right angles to the first force and has a magnitude of 4 newtons. Find the magnitude and direction of the third force.
12. A mass of m kg is supported by two light inextensible strings both attached to a ceiling. One string makes an angle of 30° with the ceiling and the other string makes an angle of 45° with the ceiling. The shorter string will break if it is subjected to a tension of more than 12 kgwt. Find the maximum value of the mass if the system is held in equilibrium.

Complex familiar

13. A block of mass 4 kg is at rest on horizontal ground. The block is given an initial push, so that its speed is 3 m/s. The block comes after moving 2 metres along the ground. Find the frictional force between the block and the floor.
14. A block of mass 3 kg is connected by a light inextensible string which passes over a smooth pulley to another mass of M kg, where $M > 3$. Initially the mass M is $\frac{8}{10}$ metres above the ground. When the system is released from rest, the mass M takes one second to hit the ground. Find the value of M .
15. Complete the following.
 - a. A 10 kg parcel is in a lift which accelerates downwards at 2 m/s^2 . Find the reaction of the lift floor on the parcel.
 - b. When the lift accelerates upwards at 2 m/s^2 , the reaction of the lift floor on a different parcel in the lift is 59 newtons. Find the mass of this parcel.
16. A body of mass 6 kg is held on a smooth plane inclined at an angle of 30° to the horizontal by a force of magnitude T newtons.
 - a. Find the value of T if T acts up and parallel to the slope.
 - b. Find the value of T if T acts at an angle of 30° to the slope.

Complex unfamiliar

17. A man pulls a car and a trailer up a hill which is inclined at an angle of 30° to the horizontal. The car has a mass of 1800 kg and the trailer has a mass of 200 kg and the resistance to both the car and trailer is one-tenth of their weight force. The cable is attached to the car which moves up the hill with an acceleration of 0.01 m/s^2 . Find the tension in the tow-bar connecting the car to the trailer and the tension in the cable if the cable makes an angle of 20° with the hill.



18. Complete the following.
 - a. A particle can be supported on a smooth plane whose inclination is θ by either a force P newtons parallel to the plane or a force Q newtons horizontally. Show that $\tan(\theta) = \frac{\sqrt{Q^2 - P^2}}{P}$.
 - b. A particle of mass 2.5 kg moves up a plane inclined at an angle of 30° to the horizontal with an acceleration of $a \text{ m/s}^2$ when a force of F newtons acting parallel to the plane is applied. When the force F newtons acts at an angle of 30° to the plane, the mass moves up the plane with an acceleration of $\frac{a}{5} \text{ m/s}^2$. Find the exact value of F .

19. ABC is a rope fastened to a beam at A. A mass of 2 kg is suspended at B and a mass 3 kg is suspended at C, so that the 3 kg mass is below and to the right of the 2 kg mass. A horizontal force of magnitude 15 newtons acts to the right at the point C. The section of the rope AB makes an angle of α with the horizontal and the section of the rope BC is inclined at an angle of β below the horizontal. Find the tensions in AB and BC and the angles α and β .
20. ABCD is a rope fastened to a horizontal beam at A and D. A mass of 2 kg is suspended from B and a mass of 3 kg is suspended from C. The sections of the rope AB makes an angle of α with AD and BC is inclined at an angle of β below the horizontal. If CD makes an angle of 35° with AD and the tension in the rope CD is 15 newtons, find the tensions in AB and BC and the angles α and β .

study on

Units 3 & 4 Sit exam

Answers

10 Modelling motion 1

Exercise 10.2 A body in equilibrium under concurrent forces

- $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force
- $\sqrt{5}F$ N acting at angle of 116.57° with the F newton force
- a. $x = -9, y = -3$
b. $x = -4, y = 3, |\underline{F}_3| = 5$
- a. 130 N acting at angle of 112.62° with the 50 newton force
b. $|\underline{F}_1| = 30\sqrt{3}, |\underline{F}_2| = 60$
- a. 14.49 N acting at angle of 163.98° with the 7 newton force
b. 14.49 N acting at angle of 163.98° with the 7 newton force
- a. $2.80F$ N acting at angle of 149.64° with the F newton force
b. $2.80F$ N acting at angle of 149.64° with the F newton force
- a. 90.46 N in the 5 cm string and 37.69 N in the 12 cm string
b. 90.46 N in the 5 cm string and 37.69 N in the 12 cm string
- a. 47.04 N in the 7 cm string and 13.72 N in the 24 cm string
b. 47.04 N in the 7 cm string and 13.72 N in the 24 cm string
- $\sqrt{113}$ N acting at angle of 131.19° with the 7 newton force
- $\sqrt{5}F$ N acting at angle of 116.57° with the F newton force
- $49(2 - \sqrt{3})$ N, $49(\sqrt{3} - 1)$ N
- 4.63, 26.60 N
- a. 85.19 N acting at angle of 153.08° with the 30 newton force
b. 87.79 N acting at angle of 158.53° with the 120 newton force
- a. i. $F = 27.45$ N
ii. $T_{AB} = 47.85$ N
b. 311.14 N, 24.32 kg
- a. 4.9 N
b. $\frac{\sqrt{3}mg}{3}$ N
- a. i. Sample responses can be found in the worked solutions in the online resources.
ii. 120.48 kg
b. Sample responses can be found in the worked solutions in the online resources.
- a. $41.81^\circ, 43.83$ N
b. $56.31^\circ, \sqrt{13}mg$ N
- a. $46.84^\circ, 37.56$ N
b. $75.58^\circ, 30.98$ N
- a. $T_{AC} = \frac{4mg}{5}$ N, $T_{BC} = \frac{3mg}{5}$ N
b. Sample responses can be found in the worked solutions in the online resources.
- a. 40.72 N, 5.79 kg
b. Sample responses can be found in the worked solutions in the online resources.
- a. $\alpha = 44.92^\circ, \beta = 52.63^\circ$
b. $\alpha = 55.95^\circ, \beta = 45.58^\circ$

Exercise 10.3 Action and reaction forces

- 31.50 N, 37.54 N
- $30.68^\circ, 1.69$ N
- 33.52 N, 26.07 N
- $28.65^\circ, 1.38$ N
- $\frac{8\sqrt{3}}{3}$
- Sample responses can be found in the worked solutions in the online resources.
- $T_{PQ} = T_{RS} = 35.87$ N, $T_{QR} = 25.36$ N, 1.34 kg
- 18.99 N, $64.52^\circ, m_1 = 1.69$ kg, $m_2 = 4.5$ kg
- a. Sample responses can be found in the worked solutions in the online resources.
b. $2\sqrt{3}g$ N, $m = 4\sqrt{3}$ kg
- a. 41.12 N
b. 6.53 kg
c. $T_{\text{upper}} = 120.21$ N, $T_{\text{lower}} = 63.96$ N
- $T_{\text{upper}} = 71.45$ N at 16.25°
 $T_{\text{lower}} = 44.01$ N at 27.03°
- $T_{AB} = \frac{20\sqrt{3}g}{3}$ N, $T_{BC} = 12g$ N
 $F_1 = \frac{28\sqrt{3}g}{3}$ N, $F_2 = 6\sqrt{3}g$ N
- $T_{AB} = 15.56$ N, $T_{BC} = 6.53$ N
 $m_1 = 0.79$ kg, $m_2 = 0.43$ kg
- $23.69^\circ, 11.16$ kg
- a. 8 cm
b. $\frac{4m}{\sqrt{100 - m^2}}$
c. The system cannot be in equilibrium; therefore, the string breaks.
- $T_{PQ} = 45.98$ N, $T_{QR} = 32.61$ N, $T_{RS} = 45.98$ N, $m = 2.19$ kg
- 14.48°
- a. 39.23°
b. Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.

Exercise 10.4 Momentum and resultant force

- 160 000 kg m/s
- 1.79×10^{29} kg m/s
- 2.62 m/s²
- $\frac{1}{2}\sqrt{29}$ m/s²
- 5280 N
- 60 kg
- a. 1024 N
b. 544 N
c. 784 N
- 539.5 N
- 7692 N
- 9950 N
- 42.96 N
- a. 43.77 N
b. 207.75 N
- a. 6 N
b. 2 m/s²
c. 2.5 m/s²
- a. $\frac{5\sqrt{2}}{2}$ m/s²
b. $\frac{1}{m}\sqrt{b^2 + c^2}$ m/s²

15. a. 1.68 m/s^2
 b. $(6 - 2\sqrt{3})\hat{i} + (3\sqrt{3} - 11)\hat{j} \text{ N}$
16. a. i. 9 m/s
 ii. 13.5 m
 b. i. $13\,333\frac{1}{3} \text{ N}$
 ii. $11111\frac{1}{9} \text{ N}$
17. a. 47.2 N b. 0.2 m/s^2
18. a. 772.5 N b. 697.5 N c. 735 N
19. $m = 50 \text{ kg}$, 0.25 m/s^2
20. a. 2.38 m/s^2 , 411.60 N b. 1.95 m/s^2 , 354.24 N
21. a. i. 0.84 m/s^2 , 315.30 N
 ii. 0.17 m/s^2 , 188.52 N
 b. 15.87 m/s
22. a. 177 N b. 30 N c. 31.06 N
 d. 80.28 N e. 83.11 N
23. $m = 5 \text{ kg}$, $\theta = 25.38^\circ$
24. a. $m = 9.67 \text{ kg}$, $\alpha = 14.75^\circ$
 b. $\theta = 40.77^\circ$, $\alpha = 18.19^\circ$

Exercise 10.5 Forces on connected particles

- 21 N
 - $m_1 = 5 \text{ kg}$, $m_2 = 4 \text{ kg}$
 - 2.90 m/s
 - $m_1 = 8 \text{ kg}$, $m_2 = 12 \text{ kg}$
 - 2.37 m/s
 - $\frac{4mg}{3} \text{ N}$
 - Tractive force $12\,957.33 \text{ N}$, tension 5725.33 N
 - 0.77 m/s^2 , 30 kN
 - a. 0.74 m/s^2 , 11.11 N b. 0.57 m/s^2 , 8.51 N
 - a. 2.45 m/s^2 , 36.75 N
 b. Sample responses can be found in the worked solutions in the online resources.
 - a. 6.13 m/s^2 , 18.38 N
 b. Sample responses can be found in the worked solutions in the online resources.
 - a. 1.09 m/s^2 , 66.67 N b. 0.053 m/s^2 , 51.07 N
 - a. $\frac{7g}{16} \text{ m/s}^2$, $\frac{45g}{16} \text{ N}$
 b. Sample responses can be found in the worked solutions in the online resources.
 - a. $\frac{15g}{16} \text{ N}$
 b. Sample responses can be found in the worked solutions in the online resources.
 - a. Sample responses can be found in the worked solutions in the online resources.
 b. Sample responses can be found in the worked solutions in the online resources.
16. a. $R_{\text{truck}} = 38 \text{ kN}$, $T = 6 \text{ kN}$
 b. 1.75 m/s^2 , $T = 12 \text{ kN}$
17. a. $\frac{5g}{4} \text{ N}$ b. $\frac{35g}{12} \text{ N}$
18. a. $\frac{5g}{2} \text{ N}$ b. $\frac{35g}{6} \text{ N}$
19. a. i. $\frac{7g}{2} \text{ N}$
 ii. $\frac{25g}{6} \text{ N}$
 b. Sample responses can be found in the worked solutions in the online resources.
20. Sample responses can be found in the worked solutions in the online resources.

10.6 Review: exam practice

- $2\sqrt{21}$
- $a = \frac{\sqrt{3}g}{2}$ and $N = \frac{5g}{2}$
- $\frac{F}{mg} = \frac{\sin(\theta)}{\cos(\alpha)}$
- 45 kg m/s
- The particle is in equilibrium
- $\frac{T_1}{T_2} = 2$
- 837 N
- $\frac{T}{F} = \frac{m_2 \cos(\theta)}{m_1 + m_2}$
- $\frac{5}{3}$
- D
- 5 N at an angle of $\frac{\pi}{2} + \tan^{-1}\left(\frac{3}{4}\right)$ with the 3 N force
- $(6\sqrt{2} + 2\sqrt{6}) \text{ kg}$
- 9 N
- 4.5 kg
- a. 960 N b. 5 kg
- a. $3g \text{ N}$ b. $2g\sqrt{3} \text{ N}$
- Tension in tow bar = 1178 newtons . Tension in cable = $12\,536.01 \text{ newtons}$
- a. Sample responses can be found in the worked solutions in the online resources.
 b. $F = \frac{98}{5\sqrt{3} - 2} \text{ N}$
- $T_{AB} = 51.24 \text{ newtons}$, $T_{BC} = 33.01 \text{ newtons}$, $\alpha = 72.98^\circ$, $\beta = 62.97^\circ$
- $T_1 = 42.22 \text{ newtons}$, $T_2 = 24.16 \text{ newtons}$, $\alpha = 73.08^\circ$, $\beta = 59.42^\circ$

11 Modelling motion 2

11.1 Overview

Sir Isaac Newton (1642–1726), was a scientist, mathematician and astronomer. In 1687, he published his work, called *Principia*, which stated the three universal laws of motion and defined the law of universal gravitation. His writings led to many more advancements in the understanding of our universe.

Two of the bodies in our solar system, Neptune and Pluto, were discovered more by mathematicians using pencil and paper than by astronomers using a telescope. Earlier, when Uranus was discovered, mathematicians began to calculate its trajectory and predict its future position. It turned out that the predictions failed to agree with observation. They concluded that another planet was exerting a gravitational pull on Uranus and that this caused the discrepancy between theory and observation. Independently, German and English mathematicians predicted the position and size of the new planet and when astronomers looked at the predicted position, they discovered the new planet, Neptune. Pluto was found in a similar way.

As we continue our study of the area of mathematics that deals with forces and motion, it can be appreciated that the arrangement of planets exerting forces on one another can become very complex. In our course of study, some rules that determine how bodies move will be investigated.



LEARNING SEQUENCE

- 11.1 Overview
- 11.2 Forces that depend on time
- 11.3 Forces that depend on velocity
- 11.4 Forces that depend on displacement
- 11.5 Simple harmonic motion
- 11.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Forces that depend on time

11.2.1 Setting up the equation of motion

In this chapter we consider the motion of an object of constant mass m moving in a straight line and subjected to a system of forces. In previous Chapters, the forces F have been constant, and since $F = ma$, the acceleration a was also constant, so the constant acceleration formulas could be used. However, if the forces are dependent upon time, t , velocity, v , or displacement, x , then the constant acceleration formulas *cannot* be used. In situations where the forces are not constant, a differential equation must be set up and solved.

Note: The notation used for acceleration in this Chapters is two dots above x , that is $\ddot{x}$.

The dot above a variable represents differentiation with respect to time, e.g. $\dot{x} = \frac{dx}{dt}$.

If the resultant force, $F = F(t)$, depends upon time, t , then by Newton's second law of motion, $m\ddot{x} = m \frac{dv}{dt} = F(t)$. Because the mass is constant, the acceleration, $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \ddot{x}$, is a function of time, t , where v is the velocity. Thus,

$$a = \ddot{x} = \frac{dv}{dt} = \frac{F(t)}{m}$$

This equation is called the equation of motion. It can be solved by integrating both sides with respect to t to obtain the velocity as $v = \frac{1}{m} \int F(t) dt$. A constant of integration can be found using an initial condition for v and t . Because $v = \frac{dx}{dt}$, we can integrate again to express the displacement, x , in terms of t . This is another application of solving a second-order differential equation.

WORKED EXAMPLE 1

An object of mass 2 kilograms moves in a straight line and is acted upon by a force of $12t - 24$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 5 metres to the right of the origin, and it comes to rest after 2 seconds. Calculate the distance travelled by the object in the first 2 seconds.

THINK

1. Use Newton's second law of motion.
2. Formulate the differential equation to be solved (the equation of motion).
3. Integrate both sides with respect to t .
4. Perform the integration using basic integration techniques, adding in the first constant of integration.
5. Use the given condition to calculate the first constant of integration.

WRITE/DRAW

$$\begin{aligned} m\ddot{x} &= F(t) \text{ where} \\ m &= 2 \text{ and } F(t) = 12t - 24 \\ 2\ddot{x} &= 12t - 24 \\ \ddot{x} &= \frac{d^2x}{dt^2} = \frac{dv}{dt} = 6t - 12 \\ v &= \int (6t - 12)dt \\ v &= 3t^2 - 12t + c_1 \end{aligned}$$

When $t = 2$, the object is at rest, so $v = 0$.
Substitute to find c_1 :
 $0 = 12 - 24 + c_1$
 $c_1 = 12$

6. Substitute back for the constant of integration.
7. Integrate both sides again with respect to t .
8. Perform the integration, adding in a second constant of integration.
9. Use the given initial condition to calculate the second constant of integration.
10. Substitute back for the constant of integration.
11. Determine the displacement at the required time.
12. State the required result.

$$v = \frac{dx}{dt} = 3t^2 - 12t + 12$$

$$x = \int (3t^2 - 12t + 12) dt$$

$$x = t^3 - 6t^2 + 12t + c_2$$

Initially, when $t = 0$, $x = 5$.

Substitute to find c_2 :

$$5 = 0 + c_2$$

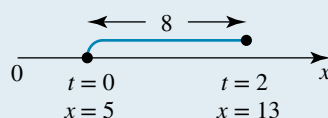
$$c_2 = 5$$

$$x = x(t) = t^3 - 6t^2 + 12t + 5$$

Substitute $t = 2$:

$$\begin{aligned} x(2) &= 8 - 24 + 24 + 5 \\ &= 13 \end{aligned}$$

The object moves from $t = 0$, $x = 5$ to $t = 2$, $x = 13$. The distance travelled is 8 metres.



11.2.2 Integrals involving trigonometric functions

Recall the integrals involving basic trigonometric functions and exponential functions:

$$\int \sin(kx) dx = -\frac{1}{k} \cos(kx)$$

$$\int \cos(kx) dx = \frac{1}{k} \sin(kx)$$

$$\int e^{kx} dx = \frac{1}{k} e^{kx}.$$

WORKED EXAMPLE 2

A particle of mass 5 kilograms moves back and forth along the x -axis and is subjected to a force of $-20 \sin(2t) - 20 \cos(2t)$ newtons at time t seconds. If its initial velocity is 2 metres per second and it starts from a point 3 metres to the left of the origin, express the displacement x metres in terms of t .

THINK

1. Use Newton's second law of motion.
2. Formulate the equation of motion to be solved.

WRITE

$$m\ddot{x} = F(t) \text{ where}$$

$$m = 5 \text{ and}$$

$$F(t) = -20 \sin(2t) - 20 \cos(2t)$$

$$5\ddot{x} = -20 \sin(2t) - 20 \cos(2t)$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4 \sin(2t) - 4 \cos(2t)$$

3. Integrate both sides with respect to t .

$$x = \int (-4 \sin(2t) - 4 \cos(2t)) dt$$

4. Perform the integration using the results

$$v = 2 \cos(2t) - 2 \sin(2t) + c_1$$

$$\int \sin(kx) dx = -\frac{1}{k} \cos(kx) \text{ and}$$

$$\int \cos(kx) dx = \frac{1}{k} \sin(kx) \text{ with } k = 2. \text{ Note that only}$$

one constant of integration is required.

5. Use the given initial condition to calculate the first constant of integration.

Initially, when $t = 0$, $v = 2$:

$$2 = 2 \cos(0) - 2 \sin(0) + c_1$$

$$c_1 = 0$$

6. Substitute back for the constant of integration.

$$v = \frac{dx}{dt} = 2 \cos(2t) - 2 \sin(2t)$$

7. Integrate both sides again with respect to t .

$$x = \int (2 \cos(2t) - 2 \sin(2t)) dt$$

8. Perform the integration, adding in a second constant of integration.

$$x = \sin(2t) + \cos(2t) + c_2$$

9. Use the given initial condition to calculate the second constant of integration.

Initially, when $t = 0$, $x = -3$

(to the left):

$$-3 = \sin(0) + \cos(0) + c_2$$

$$c_2 = -4$$

10. Substitute back for the constant of integration. This is the required result.

$$x = \sin(2t) + \cos(2t) - 4$$

11.2.3 Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that acts opposite to the direction of motion. The force is a function of the time during which the brakes are applied. This can be used to model the equation of motion during this time and determine, for example, the distance travelled while braking.

WORKED EXAMPLE 3

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $32\,000 - 12\,800t$ newtons, where t is the time in seconds after the driver applies the brakes. Calculate:

- the time after which the speed of the car has been reduced to 57.6 km/h
- the distance travelled in this time.

THINK

- Use Newton's second law of motion. The braking force acts opposite to the direction of motion.
- Formulate the equation of motion to be solved.

WRITE/DRAW

a. $m\ddot{x} = -F(t)$ where

$$m = 1600 \text{ and } F(t) = 32\,000 - 12\,800t$$

$$1600\ddot{x} = 12\,800t - 32\,000$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = 8t - 20$$

3. Integrate both sides with respect to t .
4. Perform the integration using the basic integration techniques, adding in the first constant of integration.
5. Use the given initial conditions to calculate the first constant of integration. We need to use correct units.

$$v = \int (8t - 20)dt$$

$$v = 4t^2 - 20t + c_1$$

The initial speed is 90 km/h.

Convert km/h to m/s:

$$\frac{90 \times 1000}{60 \times 60} = 25 \text{ m/s}$$

Initially, when $t = 0$, $v = 25$:

$$25 = 0 + c_1$$

$$c_1 = 25$$

$$v = 4t^2 - 20t + 25$$

The final speed is 57.6 km/h.

Convert km/h to m/s:

$$\frac{57.6 \times 1000}{60 \times 60} = 16 \text{ m/s}$$

Find t when $v = 16$.

$$16 = 4t^2 - 20t + 25$$

$$4t^2 - 20t + 9 = 0$$

$$(2t - 1)(2t - 9) = 0$$

$$t = \frac{1}{2}, \frac{9}{2}$$

$$t = \frac{1}{2}$$

6. Substitute back for the constant of integration.
7. Determine the braking time.
8. The earlier time is the one required as this will be the first instance the car's speed will be reduced from 90 km/h to 57.6 km/h before coming to rest.

9. State the required result.

- b. 1.** Express the velocity in terms of time, t .

2. Integrate both sides again with respect to t .

3. Perform the integration using basic integration techniques, adding in a second constant of integration.

4. Use the given initial condition to calculate the second constant of integration.

5. Express the displacement travelled in terms of time.

6. Determine the distance travelled, D metres, while the car is braking, as in these cases the distance travelled is the displacement. Make the substitution $t = 0.5$ seconds.

b. $v = \frac{dx}{dt} = 4t^2 - 20t + 25$

$$x = \int (4t^2 - 20t + 25)dt$$

$$x = \frac{4t^3}{3} - 10t^2 + 25t + c_2$$

Since we want to find the distance travelled from first braking, assume that when $t = 0$, $x = 0$, so that $c_2 = 0$.

$$x = x(t) = \frac{4t^3}{3} - 10t^2 + 25t$$

$$D = x(t)$$

$$D = \frac{4 \times (0.5)^3}{3} - 10 \times 0.5^2 + 25 \times 0.5$$

$$= 10\frac{1}{6}$$

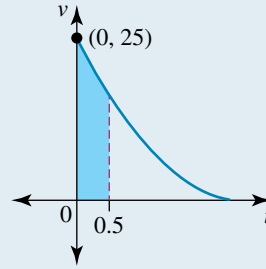


7. State the final result.

8. An alternative method to find the distance travelled can be used.

The distance travelled while the car is braking is $10\frac{1}{6}$ metres.

Since the distance travelled is the area under the velocity–time graph, this distance is given by the definite integral.



$$\begin{aligned} D &= \int_0^{\frac{1}{2}} (4t^2 - 20t + 25) dt \\ &= \int_0^{\frac{1}{2}} (2t - 5)^2 dt \\ &= \left[\frac{1}{6} (2t - 5)^3 \right]_0^{\frac{1}{2}} \\ &= \frac{1}{6} (-4)^3 - \frac{1}{6} (-5)^3 \\ &= 10\frac{1}{6} \end{aligned}$$

studyon

Units 3 & 4 > Area 5 > Sequence 4 > Concept 1

Forces that depend on time Summary screen and practice questions

Exercise 11.2 Forces that depend on time

Technology free

- WE1** An object of mass 3 kilograms moves in a straight line and is acted upon by a force of $6t - 18$ newtons, where t is the time in seconds and $t \geq 0$. Initially the object is 4 metres to the right of the origin, and it comes to rest after 3 seconds. Calculate the distance travelled by the object in the first 3 seconds.
- A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $60t - 60$ newtons, where t is the time in seconds. Initially the particle is at the origin, and after 1 second the particle is 52 metres to the left of the origin. Calculate when the particle first comes to rest.

3. **WE2** A particle of mass 8 kilograms moves back and forth along the x -axis and is subjected to a force of $-16 \sin\left(\frac{t}{3}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts at the origin, express the displacement, x metres, in terms of t .
4. A particle of mass 8 kilograms moves back and forth along the x -axis and is subjected to a force of $-16 \cos\left(\frac{t}{2}\right)$ newtons at time t seconds. If initially the particle is at rest and starts from a point 8 metres from the origin, calculate the displacement of the particle after a time of $\frac{\pi}{2}$ seconds.
5. **WE3** A car of mass 1500 kg is moving along a straight road at a speed of 72 km/h when the driver brakes. The braking force is $180\,000t - 75\,000$ newtons, where t is the time in seconds after the driver applies the brakes. Calculate:
- the time after which the speed of the car has been reduced to 36 km/h
 - the distance travelled in this time.
6. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{320}{(t+2)^3}$ kN, where t is the time in seconds after the driver applies the brakes. Calculate:
- the time taken for the speed of the car to be reduced to 57.6 km/h
 - the distance travelled in this time.
7. **a.** A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $6 - 18t$ newtons, where t is the time in seconds and $t \geq 0$. Initially the body is at rest at the origin. Express x in terms of t .
- b.** A particle of mass 500 grams is moving along the x -axis and at time t seconds is subject to a force of $4e^{2t} - 2$ newtons. Initially the body is at rest at the origin. Express x in terms of t .
8. **a.** A particle of mass 5 kilograms moves in a straight line and is acted upon by a force of $30t - 40$ newtons, where t is the time in seconds. Initially the particle is 3 metres to the right of the origin and moving to the right with a speed of 2 m/s. Determine its displacement after 2 seconds.
- b.** A particle of mass 2 kilograms is moving along the x -axis and at time t seconds is subject to a force of $48t - 12$ newtons. If after 1 second its displacement is 4 metres and its initial velocity is 1 m/s, determine its displacement at any time t seconds.

Technology active

9. A bus of mass 6 tonnes moves in a straight line between two stops. The force acting on the bus as it moves between the two stops is given by $2000 - \frac{200t}{9}$ newtons, where t is the time in seconds after it leaves the first stop. Calculate the distance between the two stops and the time it takes to travel between them.
10. A particle of mass 2 kilograms moves in a straight line so that at time t seconds it is acted upon by a force of $8 - 8e^{-0.1t}$ newtons. If initially it is moving away from the origin with a velocity of 12 metres per second, calculate how far it has travelled in the first 5 seconds.
11. **a.** A particle of mass 4 kilograms moves back and forth along the x -axis and is subjected to a force of $-144 \cos(3t)$ newtons at time t seconds. If initially it is at rest 2 metres from the origin, calculate the furthest distance it reaches from the origin.
- b.** A particle of mass 2 kilograms moves back and forth along the x -axis and is subjected to a force of $-6 \sin\left(\frac{t}{2}\right)$ newtons at time t seconds. If its initial velocity is 6 m/s and the particle starts from a point 4 metres from the origin, determine the greatest distance it reaches from the origin.



12. A particle of mass 800 kg moves in a straight line so that at time t seconds it is retarded by a force of $\frac{640}{(t+5)^3}$ kN. If the initial velocity of the particle is 16 m/s, calculate how far it has travelled in the first 5 seconds.
13. a. A particle is moving in a straight line path and has a constant acceleration of a m/s². If its initial velocity is u m/s and it starts from the origin, show using calculus that its velocity, v m/s, and displacement, s metres, at any time, t seconds, are given by $v = u + at$ and $s = ut + \frac{1}{2}at^2$.
- b. A bullet of mass 15 g is fired vertically upwards with an initial speed of 49 m/s from a height of 2 m above the ground. It is subjected only to the gravitational force. Calculate:
- its height in metres above the ground at any time, t seconds
 - the greatest height above ground level that the bullet reaches.
14. a. A car of mass 1500 kg moves in a straight line. When travelling at 60 km/h the driver applies the brakes. The braking force is $50t$ kN, where t is the time in seconds after the driver applies the brakes. Calculate the distance travelled until the car comes to rest.
- b. A car of mass m kg is travelling at a speed of U m/s along a level road when the driver applies the brakes. If the braking force is kt newtons, where t is the time in seconds after the driver applies the brakes and k is a positive constant, show that the car



comes to rest after a time of $\sqrt{\frac{2mU}{k}}$ seconds and travels a distance of $\frac{2U}{3}\sqrt{\frac{2mU}{k}}$ metres.

15. a. A particle of mass 3 kilograms is acted upon by a horizontal force of $29.4e^{-0.2t}$ newtons at a time t seconds. Its initial velocity is zero and it starts 20 metres to the right of the origin. Determine the displacement at time t .
- b. A body of mass m kg is moving in a straight line path on a horizontal table by a force of be^{-kt} newtons, where b and k are positive constants. If its initial speed is U m/s, show that after a time t seconds its displacement is given by $x = Ut + \frac{b}{mk} \left[t + \frac{1}{k}(e^{-kt} - 1) \right]$ metres.
16. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $\frac{59\,040\sqrt{2}}{\sqrt{(369t+128)^3}}$ kN where t is the time in seconds after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h, and in this time the car has travelled D metres. Determine the values of T and D for this situation.

11.3 Forces that depend on velocity

11.3.1 Setting up the equation of motion

If the force $F = F(v)$ acting on a body of mass m depends upon the velocity, v , then by Newton's second law of motion, $m\ddot{x} = F(v)$. Because $\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v \frac{dv}{dx}$ by the chain rule, there are two possible approaches.

Inverting $\ddot{x} = \frac{dv}{dt} = \frac{F(v)}{m}$ gives $\frac{dt}{dv} = \frac{m}{F(v)}$. Integrating both sides with respect to v gives $t = m \int \frac{1}{F(v)} dv$, since the mass is constant, with initial conditions on t and v . This gives us a relationship between v and t .

Alternatively, inverting $\ddot{x} = v \frac{dv}{dx} = \frac{F(v)}{m}$ gives $\frac{dx}{dv} = \frac{mv}{F(v)}$, and integrating both sides with respect to v gives $x = m \int \frac{v}{F(v)} dv$, with initial conditions on x and v . This gives us a relationship between v and x .

WORKED EXAMPLE 4

A body of mass 5 kg moving in a straight line is opposed by a force of $10(v + 3)$ newtons, where v is the velocity in m/s. Initially the body is moving at 3 m/s and is at the origin. Show that the displacement x at time t is given by $x = 3(1 - t - e^{-2t})$.

THINK

1. Use Newton's second law of motion. The force opposes the direction of motion.
2. Formulate the equation of motion to be solved.
3. Invert both sides or separate the variables.
4. Integrate both sides with respect to v .
5. Perform the integration, placing the first constant of integration on one side of the equation.
6. Use the given initial conditions to calculate the first constant of integration.
7. Substitute back for the first constant of integration.
8. Use the laws of logarithms.
9. Use the definition of logarithms to solve for v .
The modulus signs are not needed since $e^{-2t} > 0$.
10. Use $v = \frac{dx}{dt}$.
11. Integrate both sides with respect to t .
12. Perform the integration, placing the second constant of integration on one side of the equation.

WRITE

$$\begin{aligned}
 m\ddot{x} &= -F(v) \text{ where} \\
 m &= 5 \text{ and } F(v) = 10(v + 3) \\
 5\ddot{x} &= -10(v + 3) \\
 \ddot{x} &= \frac{d^2x}{dt^2} = \frac{dv}{dt} = -2(v + 3) \\
 \frac{dt}{dv} &= -\frac{1}{2(v + 3)} \\
 -2 \frac{dt}{dv} &= \frac{1}{v + 3} \\
 -2t &= \int \frac{1}{v + 3} dv \\
 -2t &= \ln(|v + 3|) + c_1 \\
 \text{Initially, when } t = 0, v = 3: \\
 0 &= \ln(6) + c_1 \\
 c_1 &= -\ln(6) \\
 -2t &= \ln(|v + 3|) - \ln(6) \\
 -2t &= \ln\left(\frac{|v + 3|}{6}\right) \\
 \frac{v + 3}{6} &= e^{-2t} \\
 v + 3 &= 6e^{-2t} \\
 v &= 6e^{-2t} - 3 \\
 v &= \frac{dx}{dt} = 6e^{-2t} - 3 \\
 x &= \int (6e^{-2t} - 3) dt \\
 x &= -3e^{-2t} - 3t + c_2
 \end{aligned}$$

13. Use the given initial conditions to calculate the second constant of integration.

Initially, when $t = 0$, $x = 0$
(it's at the origin):

$$0 = -3 + c_2$$

$$c_2 = 3$$

14. Substitute back for the second constant of integration.

$$x = -3e^{-2t} - 3t + 3$$

15. Factorise and the result is shown.

$$x = 3(1 - t - e^{-2t})$$

11.3.2 Horizontal rectilinear motion

When a driver of a car travelling at some initial speed applies the brakes, the braking force for a short period of time is a resistance force that opposes the direction of motion and is a function of the speed as the brakes are applied. The equation of motion can be modelled during this time, and the distance travelled while braking can be determined.

WORKED EXAMPLE 5

A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The resistance braking force is $6400\sqrt{v}$ newtons where v is the speed in m/s after the driver applies the brakes. Calculate:

- the time taken for the speed of the car to be reduced to 57.6 km/h
- the distance travelled in this time.

THINK

- Use Newton's Second Law of Motion.
The braking force opposes the direction of motion.
- Formulate the equation of motion to be solved.
- Invert both sides or separate the variables.
- Integrate both sides with respect to v .
- Perform the integration, placing the first constant of integration on one side of the equation.
- Use the given initial conditions to calculate the first constant of integration. We need to use correct units.

WRITE

- $m\ddot{x} = -F(v)$ where
 $m = 1600$ and $F(v) = 6400\sqrt{v}$

$$1600\ddot{x} = -6400\sqrt{v}$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -4\sqrt{v}$$

$$\frac{dt}{dv} = -\frac{1}{4\sqrt{v}}$$

$$-4\frac{dt}{dv} = \frac{1}{\sqrt{v}} = v^{-\frac{1}{2}}$$

$$-4t = \int v^{-\frac{1}{2}} dv$$

$$\begin{aligned} -4t + c_1 &= 2v^{\frac{1}{2}} \\ &= 2\sqrt{v} \end{aligned}$$

$$90 \text{ km/h} = 25 \text{ m/s}$$

Initially, when $t = 0$, $v = 25$:

$$\begin{aligned} c_1 &= 2\sqrt{25} \\ &= 10 \end{aligned}$$

7. Substitute back for the constant of integration.
8. Determine the braking time.

$$\begin{aligned}
 10 - 4t &= 2\sqrt{v} \\
 57.6 \text{ km/h} &= 16 \text{ m/s} \\
 \text{Find } t \text{ when } v &= 16: \\
 10 - 4t &= 2\sqrt{16} = 8 \\
 4t &= 2 \\
 t &= \frac{1}{2}
 \end{aligned}$$

9. State the required result.

After 0.5 seconds, the car's speed has been reduced from 90 to 57.6 km/h.

- b. 1. Use the alternative form of the equation for the acceleration.

$$\text{b. } \ddot{x} = v \frac{dv}{dx} = -4\sqrt{v}$$

$$\begin{aligned}
 \frac{dv}{dx} &= -\frac{4\sqrt{v}}{v} \\
 &= -\frac{4}{\sqrt{v}}
 \end{aligned}$$

2. Invert both sides or separate the variables.

$$\begin{aligned}
 -4 \frac{dx}{dv} &= v^{\frac{1}{2}} \\
 -\int 4 dx &= \int v^{\frac{1}{2}} dv
 \end{aligned}$$

3. Perform the integration, placing the second constant of integration on one side of the equation.
4. Use the given initial condition to calculate the second constant of integration.

$$-4x + c_2 = \frac{2}{3}v^{\frac{3}{2}}$$

When $v = 25$, $x = 0$:

$$\begin{aligned}
 c_2 &= \frac{2}{3}\sqrt{25^3} \\
 &= \frac{250}{3}
 \end{aligned}$$

5. Substitute back for the second constant of integration.
6. Determine the distance travelled.

$$\frac{250}{3} - 4x = \frac{2}{3}v^{\frac{3}{2}}$$

Solve for x when $v = 16$:

$$\begin{aligned}
 4x &= \frac{250}{3} - \frac{2}{3}(\sqrt{16})^3 \\
 4x &= \frac{250 - 128}{3} \\
 x &= \frac{61}{6}
 \end{aligned}$$

7. State the distance travelled while braking.

The distance travelled while braking is $10\frac{1}{6}$ metres.

11.3.3 General cases

When a body moves, the drag force is in general proportional to some power of the velocity. That is, the total air resistance and drag forces can be expressed as $F(v) = kv^n$; typical values of n are 1, 2, 3, 4, 5, $\frac{1}{2}$, $\frac{3}{2}$ and so on. Using these and generalising from the last example, some general expressions can be derived.

WORKED EXAMPLE 6

A car is moving along a straight road at a velocity of U m/s when the driver brakes. The braking force in newtons is proportional to the fourth power of the velocity, v , where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2}U$ m/s, and in this time the car has travelled a distance of D metres. Show that

$$\frac{D}{T} = \frac{9U}{14}.$$

THINK

1. Use Newton's second law of motion. The braking force opposes the direction of motion.
2. First obtain a relationship between v and t .
3. Invert both sides or separate the variables.
4. Integrate both sides with respect to v .
5. Perform the integration, placing the first constant of integration on one side of the equation.
6. Use the given initial conditions to calculate the first constant of integration. We are using correct units.
7. Substitute back for the constant of integration.
8. Obtain a relationship between the parameters.

WRITE

Let the mass of the car be m and the proportionality constant be k .

$F \propto v^4$ then $F(v) = kv^4$ and

$$m\ddot{x} = -F(v) = -kv^4$$

$\ddot{x} = -\lambda v^4$ where $\lambda = \frac{k}{m}$ is one single constant.

$$\text{Use } \ddot{x} = \frac{dv}{dt} = -\lambda v^4.$$

$$\frac{dt}{dv} = -\frac{1}{\lambda v^4}$$

$$\int -\lambda dt = \int v^{-4} dv$$

$$-\lambda t + c_1 = -\frac{1}{3}v^{-3}$$

Initially, when $t = 0$, $v = U$:

$$c_1 = -\frac{1}{3}U^{-3}$$

$$c_1 = -\frac{1}{3U^3}$$

$$-\lambda t - \frac{1}{3U^3} = -\frac{1}{3v^3}$$

$$\lambda t + \frac{1}{3U^3} = \frac{1}{3v^3}$$

When $t = T$, $v = \frac{1}{2}U$:

$$\lambda T + \frac{1}{3U^3} = \frac{1}{3\left(\frac{1}{2}U\right)^3}$$

$$= \frac{8}{3U^3}$$

9. Simplify this relationship and express λ in terms of T and U .

$$\lambda T = \frac{8}{3U^3} - \frac{1}{3U^3}$$

$$\lambda T = \frac{7}{3U^3}$$

$$\lambda = \frac{7}{3TU^3}$$

10. Next, obtain a relationship between v and x .

Use $\ddot{x} = v \frac{dv}{dx} = -\lambda v^4$ so that

$$\frac{dv}{dx} = -\lambda v^3$$

11. Invert both sides or separate the variables.

$$\frac{dx}{dv} = -\frac{1}{\lambda v^3}$$

12. Integrate both sides with respect to v .

$$\int -\lambda dx = \int v^{-3} dv$$

13. Perform the integration, placing the second constant of integration on one side of the equation.

$$-\lambda x + c_2 = -\frac{1}{2}v^{-2}$$

14. Use the given initial conditions to calculate the second constant of integration.

Initially (when $t = 0$) $x = 0$ and $v = U$:

$$c_2 = -\frac{1}{2}U^{-2}$$

$$c_2 = -\frac{1}{2U^2}$$

15. Substitute back for the second constant of integration.

$$-\lambda x - \frac{1}{2U^2} = -\frac{1}{2v^2}$$

$$\lambda x + \frac{1}{2U^2} = \frac{1}{2v^2}$$

16. Obtain a relationship between the parameters.

When $x = D$, $v = \frac{1}{2}U$:

$$\lambda D + \frac{1}{2U^2} = \frac{1}{2\left(\frac{1}{2}U\right)^2}$$

$$= \frac{2}{U^2}$$

$$\lambda D = \frac{2}{U^2} - \frac{1}{2U^2}$$

$$\lambda D = \frac{3}{2U^2}$$

$$\lambda = \frac{3}{2DU^2}$$

17. Simplify this relationship and express λ in terms of D and U .

$$\lambda = \frac{7}{3TU^3} = \frac{3}{2DU^2}$$

18. Eliminate λ by equating the two expressions for λ .



19. Simplify the resulting expression.

$$7 \times 2DU^2 = 9TU^3$$

$$14DU^2 = 9TU^3$$

$$\frac{D}{T} = \frac{9U^3}{14U^2}$$

$$\frac{D}{T} = \frac{9U}{14}$$

$$\text{Hence } \frac{D}{T} = \frac{9U}{14} \text{ as required.}$$

11.3.4 Vertical motion

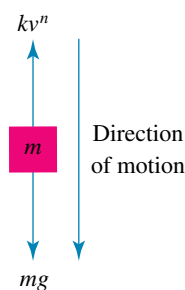
When a body moves vertically, its weight force must be considered as part of its equation of motion.

Downwards motion

Consider a body of mass m moving vertically downwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and acts vertically upwards.

The resistance force is proportional to some power of the body's velocity. Considering downwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = mg - kv^n$. As the body falls, it reaches a so-called **terminal** or **limiting velocity**, v_T . This value can be obtained from $v_T = \lim_{t \rightarrow \infty} v(t)$, or as

it is a constant speed when the acceleration is zero, $\ddot{x} = 0$. Thus, the terminal velocity satisfies $mg - kv_T^n = 0$.

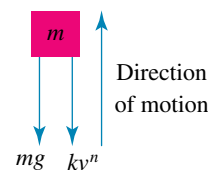


Upwards motion

Consider a body of mass m moving vertically upwards. The forces acting on the body are its weight force, which acts vertically downwards, and the force of air resistance, which opposes the direction of motion and also acts vertically downwards.

The resistance force is proportional to some power of the body's velocity. Considering upwards as the positive direction, the body's equation of motion is given by $m\ddot{x} = -mg - kv^n = -(mg + kv^n)$.

Typical values of n are 1, 2, 3, 4, 5, $\frac{1}{2}$ and $\frac{3}{2}$.



WORKED EXAMPLE 7

A large brick of mass 5 kg is accidentally dropped from a high-rise construction site. As it falls vertically downwards it is retarded by a force of $0.01v^2$ newtons, where v m/s is the speed of the brick at a time t seconds after it was dropped. It has travelled a distance of x metres in this time.

a. Show that while the brick is falling, its

equation of motion is given by $\ddot{x} = \frac{4900 - v^2}{500}$.

b. Show that $x = 250 \ln \left(\frac{4900}{4900 - v^2} \right)$.



- c. Calculate the magnitude of the momentum of the brick after it has fallen a distance of 100 m.
- d. Show that $t = \frac{25}{7} \ln \left(\frac{70 + v}{70 - v} \right)$.
- e. Calculate the terminal velocity of the brick.
- f. Calculate the time taken for the brick to fall a distance of 100 m.

THINK

- a. 1. Use Newton's second law of motion.
2. Formulate the equation of motion to be solved. Simplify and the required result is shown.

- b. 1. To obtain a relationship between v and x , use $\ddot{x} = v \frac{dv}{dx}$ and invert both sides.

2. Integrate both sides with respect to v .

3. Perform the integration, placing the first constant of integration on one side of the equation.
4. Use the given initial conditions to calculate the first constant of integration.
5. Substitute back for the first constant of integration and take out common factors.
6. Use log laws to show the required result.

- c. 1. To calculate the momentum, we first need to determine the speed.
2. Use the definition of the logarithm and transpose to make v the subject.

WRITE

$$\begin{aligned} \text{a. } m &= 5, k = 0.01 \\ m\ddot{x} &= mg - kv^2 \\ 5\ddot{x} &= 5 \times 9.8 - 0.01v^2 \\ 5\ddot{x} &= 49 - \frac{v^2}{100} \\ 5\ddot{x} &= \frac{4900 - v^2}{100} \\ \ddot{x} &= \frac{4900 - v^2}{500} \end{aligned}$$

$$\begin{aligned} \text{b. } \ddot{x} &= v \frac{dv}{dx} = \frac{4900 - v^2}{500} \\ \frac{dx}{dv} &= \frac{500v}{4900 - v^2} \\ x &= 500 \int \frac{v}{4900 - v^2} dv \\ &= -\frac{500}{2} \int \frac{-2v}{4900 - v^2} dv \\ x &= -250 \ln(4900 - v^2) + c_1 \end{aligned}$$

Since the brick was dropped when $t = 0$,
 $x = 0$ and $v = 0$:

$$\begin{aligned} 0 &= -250 \ln(4900) + c_1 \\ c_1 &= 250 \ln(4900) \\ x &= -250 \ln(4900 - v^2) + 250 \ln(4900) \\ &= 250 [\ln(4900) - \ln(4900 - v^2)] \\ x &= 250 \ln \left(\frac{4900}{4900 - v^2} \right) \end{aligned}$$

- c. When $x = 100$, $v = ?$

$$\begin{aligned} 100 &= 250 \ln \left(\frac{4900}{4900 - v^2} \right) \\ e^{0.4} &= \frac{4900}{4900 - v^2} \\ 4900 - v^2 &= 4900e^{-0.4} \\ v^2 &= 4900(1 - e^{-0.4}) \end{aligned}$$

3. Calculate the speed when the brick has fallen this required distance.

In this case choose the positive value of the velocity as the brick is travelling in the same direction as the direction of motion.

4. Determine the magnitude of the momentum.

$$v = 70\sqrt{(1 - e^{-0.4})}$$

$$v = 40.19243 \text{ m/s}$$

$$p = mv$$

$$= 5 \times 40.19243$$

$$= 200.962$$

The momentum of the brick is
200.96 kg m/s.

- d. 1. To obtain a relationship between v and t , use

$$\ddot{x} = \frac{dv}{dt} \text{ and invert both sides.}$$

2. Integrate both sides with respect to v .

3. To determine this integral, use partial fractions. Express the integrand into its partial fractions decomposition.

4. Calculate the values of the constants A and B .

d. $\frac{dv}{dt} = \frac{4900 - v^2}{500}$

$$\frac{dt}{dv} = \frac{500}{4900 - v^2}$$

$$t = \int \frac{500}{4900 - v^2} dv$$

$$\frac{500}{4900 - v^2} = \frac{A}{70 - v} + \frac{B}{70 + v}$$

$$= \frac{A(70 + v) + B(70 - v)}{(70 - v)(70 + v)}$$

$$= \frac{70(A + B) + v(A - B)}{4900 - v^2}$$

Equate the coefficients:

$$v: A - B = 0 \Rightarrow A = B$$

$$\text{constant term: } 70(A + B) = 500$$

$$\text{Since } A = B \Rightarrow 70(A + A) = 500$$

$$140A = 500$$

$$A = \frac{500}{140}$$

$$A = \frac{25}{7}$$

$$\therefore A = B = \frac{25}{7}$$

5. Express the integrand in a form for which we can perform the integration.

6. Perform the integration, placing the second constant of integration on one side of the equation.

7. Use the given initial condition to calculate the second constant of integration.

$$t = \int \frac{500}{4900 - v^2} dv$$

$$= \frac{25}{7} \int \left(\frac{1}{70 + v} + \frac{1}{70 - v} \right) dv$$

$$t = \frac{25}{7} (\ln(|70 + v|) - \ln(|70 - v|) + c_2)$$

When $x = 0$, $t = 0$, $v = 0$:

$$0 = \frac{25}{7} (\ln(70) - \ln(70) + c_2)$$

$$c_2 = 0$$

8. Substitute back for the second constant of integration and use log laws again. The required result is shown.

$$t = \frac{25}{7} (\ln(|70 + v|) - \ln(|70 - v|))$$

$$t = \frac{25}{7} \ln\left(\left|\frac{70 + v}{70 - v}\right|\right)$$

But since $0 \leq v < 70$, the modulus signs are not needed.

$$t = \frac{25}{7} \ln\left(\frac{70 + v}{70 - v}\right)$$

- e. The terminal velocity can be found when the acceleration is zero.

$$\text{e. } \frac{4900 - v^2}{500} = 0$$

$$v^2 = 4900$$

$$v_T = \sqrt{4900}$$

$$v_T = 70$$

The terminal velocity is 70 m/s.

- f. 1. Determine the time taken for the brick to fall the required distance.

$$\text{f. } t = ? \text{ when } v = 40.19243$$

$$t = \frac{25}{7} \ln\left(\frac{70 + 40.19243}{70 - 40.19243}\right)$$

$$= 4.67$$

The time taken to fall is 4.67 s.

2. State the time to fall the required distance.
3. An alternative method to determine the time is to numerically evaluate a definite integral.

$$t = \int_0^{40.19243} \frac{500}{4900 - v^2} dv = 4.67$$

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Forces that depend on velocity Summary screen and practice questions

Exercise 11.3 Forces that depend on velocity

Technology free

- WE4** A body of mass 2 kg moving in a straight line is opposed by a force of $v - 8$ newtons, where v is the velocity in m/s. Initially the body is at rest at the origin. Show that the displacement at time t is given by $x = 8 \left(1 + 2 \left(e^{-\frac{t}{2}} - 1 \right) \right)$.
- A body of mass 6 kg moving in a straight line is acted upon by a resistance force of $2(v - 6)$ newtons, where v is the velocity in m/s. Initially the body is moving at 12 m/s and is at the origin. Show that the displacement at time t is given by $x = 6 \left(t + 3 \left(1 - e^{-\frac{t}{3}} \right) \right)$.
- WE5** A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $320\sqrt{v^3}$ newtons, where v is the speed in m/s after the driver applies the brakes. Calculate:
 - the time taken for the speed of the car to be reduced to 57.6 km/h
 - the distance travelled in this time.

4. A car of mass 1600 kg is moving along a straight road at a speed of 90 km/h when the driver brakes. The braking force is $\frac{369}{100} v^3$ newtons, where v is the speed in m/s after the driver applies the brakes. After a time T seconds, the speed of the car has been reduced to 57.6 km/h and in this time the car has travelled a distance of D metres. Determine the values of T and D for this situation.
5. **WE6** A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the cube of the velocity v , where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2} U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{2U}{3}$.
6. A car moves along a straight road at a speed of U m/s when the driver brakes. The resistance braking force in newtons is proportional to the square root of the velocity v cubed, where v is the velocity in m/s after the driver applies the brakes. After a time T seconds, the velocity of the car has been reduced to $\frac{1}{2} U$ m/s, and in this time the car has travelled a distance of D metres. Show that $\frac{D}{T} = \frac{\sqrt{2}U}{2}$.
7. **WE7** A skydiver of mass 90 kg falls vertically from rest from a plane. While falling vertically downwards he is retarded by a force of $0.1v^2$ newtons, where v m/s is his speed at a time t seconds after falling a distance of x metres.
- Show that while the skydiver is falling, his equation of motion is given by $\ddot{x} = \frac{8820 - v^2}{900}$.
 - Show that $x = 450 \ln\left(\frac{8820}{8820 - v^2}\right)$.
 - Determine the magnitude of the momentum of the skydiver after he has fallen a distance of 150 m.
 - Determine the terminal velocity of the skydiver.
 - Calculate the time taken for the skydiver to fall a distance of 150 m.
8. A body of mass m kg falls vertically. While falling vertically downwards the particle is retarded by a force of kv^2 newtons, where k is a positive constant and v m/s is the speed at a time t seconds. Show that the terminal speed is given by $v_T = \sqrt{\frac{mg}{k}}$ and that $t = \frac{v_T}{2g} \ln\left(\frac{v_T + v}{v_T - v}\right)$.



Technology active

9. **a.** A body of mass 2 kg moving in a straight line is acted upon by a resistance force of $4v$ newtons, where v is the velocity in m/s. Initially the body is moving at 1 m/s and is at the origin. Show that the displacement at time t is given by $x = \frac{1}{2}(1 - e^{-2t})$.
- b.** A body of mass 3 kg moving in a straight line is acted upon by a resistance force of $9(v + 4)$ newtons, where v is the velocity in m/s. Initially the body is moving at 2 m/s and is at the origin. Show that the displacement at time t is given by $x = 2(1 - e^{-3t} - 2t)$.
10. A yacht of mass 500 kg is sailing in a straight line at a speed of 57.6 km/h when the driver disengages the engine. The resistance force is $400\sqrt{v}$ newtons, where v m/s is the speed of the yacht at a time t seconds. Calculate:
- the time taken for the speed of the yacht to be reduced to 14.4 km/h
 - the distance travelled in this time.



- The braking force is $80v^{\frac{3}{2}}$ newtons, where v m/s is the speed of the car at a time t seconds. After a time T s, the speed of the car is 14.4 km/h, and in this time it has travelled a distance of D m. Calculate the values of:

b. D .

- a. its speed, v m/s, at a time t seconds satisfies $v = Ue^{-\frac{kt}{m}}$

b. its displacement, x m, after a time t seconds is given by $x = \frac{mU}{k} \left(1 - e^{-\frac{kt}{m}} \right)$

c. its speed, v m/s, after moving a distance x m is given by $v = U - \frac{kx}{m}$.

- a. its speed, v m/s, after moving a distance of x metres is given by $v = Ue^{-\frac{kx}{m}}$

b. its speed, v m/s, at a time t seconds satisfies $v = \frac{mU}{m + kUt}$.

- $$\text{a. } T = \frac{m}{b} \ln\left(1 + \frac{bU}{a}\right)$$

b. $D = \frac{1}{b} (mU - aT).$

- $$T = \frac{m}{\sqrt{ab}} \tan^{-1} \left(U \sqrt{\frac{b}{a}} \right) \text{ and } D = \frac{m}{2b} \ln \left(1 + \frac{bU^2}{a} \right).$$

- a. Show that the speed at any time, t seconds, is given by $v = \frac{mg}{k} \left(1 - e^{-\frac{kt}{m}} \right)$.

b. Deduce that the limiting (or terminal speed) is given by $\frac{mg}{k}$.

c. Show that when the speed is half the terminal speed, the distance of the body below the point of projection is given by $\frac{0.193m^2g}{k^2}$ metres.

17. During a snow storm, a small block of ice of mass 10 grams falls from the sky.

As it falls vertically downwards it is retarded by a force of $0.002v^2$ newtons, where v m/s is its speed at a time t seconds after falling a distance of x m.

- a. Show that while the ice block is falling, its equation of motion is given by $\ddot{x} = \frac{49 - v^2}{5}$.

- b. Show that $v = 7\sqrt{1 - e^{-0.4x}}$.

- c. Show that $v = \frac{7(1 - e^{-2.8t})}{1 + e^{-2.8t}}$ and hence find the terminal speed of the ice block.

- d. Show that $x = 5 \ln\left(\frac{e^{1.4t} + e^{-1.4t}}{2}\right)$.

18. A ball of mass m kg is projected vertically upwards from ground level with an initial speed of U m/s. While travelling upwards or downwards it is subjected to a force of air resistance equal to kv^2 newtons, where k is a positive constant and v is its velocity in m/s.

- a. Show that the ball reaches a maximum height of $\frac{m}{2k} \ln\left(1 + \frac{kU^2}{mg}\right)$ m.

- b. Show that the time required for the ball to reach its maximum height is given by

$$\sqrt{\frac{m}{kg}} \tan^{-1}\left(U\sqrt{\frac{k}{mg}}\right) \text{ s.}$$

- c. Show that the ball returns to its original point with a speed of $U\sqrt{\frac{mg}{mg + kU^2}}$ m/s.



11.4 Forces that depend on displacement

11.4.1 Setting up the equation of motion

If the force $F = F(x)$ depends upon the displacement x , then the acceleration is effectively a function of the displacement. Because $a = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v \frac{dv}{dx} = \frac{d}{dv} \left(\frac{1}{2}v^2 \right) \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$ by the chain rule, then by Newton's second law of motion, $\ddot{x} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right) = \frac{F(x)}{m}$. Integrating both sides with respect to x gives $\frac{1}{2}v^2 = \frac{1}{m} \int F(x) dx$ with initial conditions on x and v . This gives us a relationship between v and x .

WORKED EXAMPLE 8

A particle of mass 3 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{-(15 - 6x)}{x^3}$ newtons and the particle is at rest at a distance of 5 m to the right of the origin. Determine where else the particle comes to rest.

THINK

1. Use Newton's second law of motion.

WRITE

$$m\ddot{x} = F(x) \text{ where}$$

$$m = 3 \text{ and } F(x) = \frac{-(15 - 6x)}{x^3}$$

2. Formulate the equation of motion to be solved.
Simplify by dividing through by 3.

$$3\ddot{x} = \frac{-(15 - 6x)}{x^3}$$

$$\ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right) = \frac{-(5 - 2x)}{x^3}$$

3. Integrate both sides with respect to x .

$$\frac{1}{2} v^2 = \int \left(\frac{-(5 - 2x)}{x^3} \right) dx$$

4. Express the integrand in index notation.

$$\frac{1}{2} v^2 = \int (-5x^{-3} + 2x^{-2}) dx$$

5. Perform the integration, placing the constant of integration on one side of the equation.

$$\frac{1}{2} v^2 = \frac{5}{2} x^{-2} - 2x^{-1} + c$$

6. Write the expression with positive indices.

$$\frac{1}{2} v^2 = \frac{5}{2x^2} - \frac{2}{x} + c$$

7. Use the given initial conditions to calculate the constant of integration.

When $v = 0$, $x = 5$:

$$0 = \frac{5}{2(5)^2} - \frac{2}{5} + c$$

$$c = \frac{2}{5} - \frac{1}{10} = \frac{3}{10}$$

8. Substitute back for the constant of integration.

$$\frac{1}{2} v^2 = \frac{5}{2x^2} - \frac{2}{x} + \frac{3}{10}$$

9. Form a common denominator.

$$\frac{1}{2} v^2 = \frac{25 - 20x + 3x^2}{10x^2}$$

10. Factorise the quadratic in the numerator.

$$v^2 = \frac{(3x - 5)(x - 5)}{5x^2}$$

11. Express the velocity in terms of x .

$$v = \frac{1}{|x|} \sqrt{\frac{(3x - 5)(x - 5)}{5}}$$

12. Determine the values of x when the particle comes to rest.

If $v = 0$, then $(3x - 5)(x - 5) = 0$,

$$\text{so } x = 5 \text{ or } x = \frac{5}{3}.$$

As we were given $x = 5$ when $v = 0$,
the required solution is $x = \frac{5}{3}$.

13. State the final result.

The particle comes to rest $\frac{5}{3}$ metres
from the origin.

11.4.2 Relationships between time, displacement, velocity and acceleration

So far, we have used integration techniques to obtain relationships between time, t , displacement, x , velocity, v , and acceleration, a . However, we can also use differentiation to obtain relations by using

$$a = \ddot{x} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} = v \frac{dv}{dx} = \frac{d}{dv} \left(\frac{1}{2} v^2 \right) \frac{dv}{dx} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

WORKED EXAMPLE 9

A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{9 - 4x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = -4x$.

THINK

1. Differentiate using the chain rule.

WRITE

$$v = \sqrt{9 - 4x^2} = (9 - 4x^2)^{\frac{1}{2}}$$

$$\begin{aligned}\frac{dv}{dx} &= \frac{1}{2} \times -8x \times (9 - 4x^2)^{-\frac{1}{2}} \\ &= \frac{-4x}{\sqrt{9 - 4x^2}}\end{aligned}$$

2. Use an expression for the acceleration.

$$\begin{aligned}a &= v \frac{dv}{dx} \\ a &= \sqrt{9 - 4x^2} \times \frac{-4x}{\sqrt{9 - 4x^2}}\end{aligned}$$

3. Alternatively, square the velocity, halve it, and differentiate with respect to x .

$$\begin{aligned}a &= -4x \\ v^2 &= 9 - 4x^2 \\ \frac{1}{2}v^2 &= \frac{9}{2} - 2x^2 \\ a &= \frac{d}{dx} \left(\frac{1}{2}v^2 \right) = -4x\end{aligned}$$

11.4.3 Expressing x in terms of t

In some cases it may be possible to rearrange and express the displacement x in terms of t by solving for v and using $v = \frac{dx}{dt}$.

WORKED EXAMPLE 10

A body of mass 5 kg moves in a straight line and is retarded by a force of $20x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 1 m to the right of the origin and the initial velocity of the body is 2 m/s. Express x in terms of t where t is the time in seconds.

THINK

1. Use Newton's second law of motion. The braking force opposes the direction of motion.
2. Formulate the equation of motion to be solved.
3. Integrate both sides with respect to x .

WRITE

$$\begin{aligned}m\ddot{x} &= F(x) \text{ where} \\ m &= 5 \text{ and } F(x) = -20x \\ 5\ddot{x} &= -20x \\ \ddot{x} &= \frac{d}{dx} \left(\frac{1}{2}v^2 \right) = -4x \\ \frac{1}{2}v^2 &= \int -4x dx\end{aligned}$$

4. Perform the integration, placing the constant of integration on one side of the equation.
5. Use the given initial conditions to calculate the first constant of integration.

$$\frac{1}{2}v^2 = -2x^2 + c_1$$

Initially, when $t = 0$, $v = 2$ and $x = 1$:

$$2 = -2 + c_1$$

$$c_1 = 4$$

6. Substitute back for the first constant of integration.
7. Rearrange and solve for v .

$$\frac{1}{2}v^2 = -2x^2 + 4$$

$$v^2 = 8 - 4x^2$$

$$v^2 = 4(2 - x^2)$$

$$v = \pm 2\sqrt{2 - x^2}$$

$$v = \frac{dx}{dt} = 2\sqrt{2 - x^2}, 0 \leq x \leq \sqrt{2}$$

8. Initially, when $t = 0$, $x = 1$ and $v = 2$, $2 > 0$; therefore, we can take the positive root only.
9. Invert both sides.

$$\frac{dt}{dx} = \frac{1}{2\sqrt{2 - x^2}}$$

10. Integrate both sides with respect to x .

$$t = \frac{1}{2} \int \frac{1}{\sqrt{2 - x^2}} dx$$

11. Perform the integration.

$$t = \frac{1}{2} \sin^{-1} \left(\frac{x}{\sqrt{2}} \right) + c_2$$

12. Use the given initial conditions to calculate the second constant of integration.

When $t = 0$, $v = 2$ and $x = 1$:

$$0 = \frac{1}{2} \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) + c_2$$

$$c_2 = -\frac{\pi}{8}$$

13. Substitute back for the second constant of integration.

$$t = \frac{1}{2} \sin^{-1} \left(\frac{x}{\sqrt{2}} \right) - \frac{\pi}{8}$$

14. Rearrange to make x the subject.

$$\sin^{-1} \left(\frac{x}{\sqrt{2}} \right) = 2t + \frac{\pi}{4}$$

$$\frac{x}{\sqrt{2}} = \sin \left(2t + \frac{\pi}{4} \right)$$

15. Expand using compound-angle formula
 $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.

$$\frac{x}{\sqrt{2}} = \sin(2t) \cos \left(\frac{\pi}{4} \right) + \cos(2t) \sin \left(\frac{\pi}{4} \right)$$

$$\frac{x}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sin(2t) + \frac{1}{\sqrt{2}} \cos(2t)$$

16. State the final result.

$$x = \sin(2t) + \cos(2t)$$

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Exercise 11.4 Forces that depend on displacement

Technology free

1. **WE8** A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{3-x}{x^3}$ newtons and the particle is at rest at a distance of 1 m from the origin. Determine where else the particle comes to rest.
2. A particle of mass m kg moves so that its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{1+bx}{x^3}$ newtons and the particle is at rest when $x = 1$ and also when $x = -\frac{1}{5}$. Calculate the value of the constant b .
3. **WE9** A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{4+9x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = 9x$.
4. A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^4$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = 4x^7$.
5. **WE10** A body of mass 8 kilograms moves in a straight line and is retarded by a force of $2x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 8 metres from the origin and at rest. Express x in terms of t where t is the time in seconds.
6. A body of mass 3 kg moves in a straight line and is retarded by a force of $\frac{x}{3}$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin and the initial velocity of the body is 6 m/s. Express x in terms of t where t is the time in seconds.
7.
 - a. A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $12x - 18$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 2 metres per second. Express x in terms of t where t is the time in seconds.
 - b. A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $18x - 30$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is 2 metres from the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.
8.
 - a. A body of mass 2 kilograms moves in a straight line and is acted upon by a force of $2x - 2$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.
 - b. A body of mass 3 kilograms moves in a straight line and is retarded by a force of $12x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 8 metres per second. Express x in terms of t where t is the time in seconds.
9.
 - a. A particle of mass 2 kilograms moves in a straight line and is retarded by a force of $2e^{-2x}$ newtons, where x is its displacement in metres from a fixed origin. Initially the particle is at the origin, and its initial velocity is 1 metre per second. Express x in terms of t where t is the time in seconds.
 - b. A body of mass 3 kilograms moves in a straight line and is acted upon by a force of $6e^{4x}$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at the origin, and the initial velocity of the body is 1 metre per second. Express x in terms of t where t is the time in seconds.

Technology active

10.
 - a. A particle of mass 4 kg is subjected to a force of $\frac{28x-24}{x^3}$ newtons, where x is its displacement in metres from a fixed origin. If the particle is at rest at a distance of 3 m from the origin, determine where else the particle comes to rest.

- b. A particle of mass 2 kg moves so that at a time t seconds, its displacement is x m from a fixed origin. The particle is acted upon by a force of $\frac{20x - 16}{x^3}$ newtons, and the particle is at rest at a distance of 2 m from the origin. Determine where else the particle comes to rest.
11. a. A body of mass 16 kg moves in a straight line and is retarded by a force of $64x$ newtons, where x is its displacement in metres from a fixed origin. If the initial velocity of the body is 16 m/s, determine where it comes to rest.
- b. A car of mass m kg is travelling at a speed of U m/s along a straight level road when the driver applies the brakes. The braking force is kx newtons, where x is the distance travelled in metres after the driver applies the brakes and k is a positive constant. Show that when the car comes to rest it has travelled a distance of $U\sqrt{\frac{m}{k}}$ metres.
12. a. A body is moving in a straight line. Its velocity, v m/s, is given by $v = x^2$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = 2x^3$.
- b. A body is moving in a straight line. Its velocity, v m/s, is given by $v = \sqrt{16 - 25x^2}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = -25x$.
- c. A body is moving in a straight line. Its velocity, v m/s, is given by $v = e^{2x} + e^{-2x}$ when it is x m from the origin at time t seconds. Show that its acceleration, a m/s², is given by $a = 2(e^{4x} - e^{-4x})$.
- d. A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = 3 - 2e^{-2t}$. Show that its acceleration, a m/s², is given by $a = 2(3 - v)$.
13. a. A body of mass 500 grams moves in a straight line and is retarded by a force of $8x$ newtons, where x is its displacement in metres from a fixed origin. Initially the body is at rest 3 metres from the origin. Express x in terms of t where t is the time in seconds.
- b. A block of mass m kilograms moves back and forth along a straight line track. It is subjected to a force that opposes the motion and whose magnitude is proportional to its distance from the origin, O, the constant of proportionality being k . The particle starts from rest when its displacement from O is a metres. Show that if its speed is v metres per second at any time t seconds and displacement x metres, then $v^2 = \frac{k}{m}(a^2 - x^2)$ and $x = a \cos\left(\sqrt{\frac{k}{m}}t\right)$.
14. a. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = x^n$, where n is a constant. Show that its acceleration, a m/s², is given by $a = nx^{2n-1}$.
- b. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = \sqrt{b - n^2x^2}$, where n and b are constants. Show that its acceleration, a m/s², is given by $a = -n^2x$.
- c. A body is moving in a straight line. When it is x m from the origin at time t seconds, its velocity, v m/s, is given by $v = e^{nx} + e^{-nx}$, where n is a constant. Show that its acceleration, a m/s², is given by $a = n(e^{2nx} - e^{-2nx})$.
- d. A body is moving in a straight line. Its velocity, v m/s, at time t seconds is given by $v = b - ne^{-nt}$, where n is a constant. Show that its acceleration, a m/s², is given by $a = -n(v - b)$.
15. a. A block of mass m kg moves along a horizontal table top and is subjected to a resistance force of $\frac{a + bx}{x^3}$ newtons, where a and b are constants. If initially the block is at rest at a displacement of a metres, show that the block next comes to rest again when the displacement is $\frac{-a}{2b + 1}$ metres.

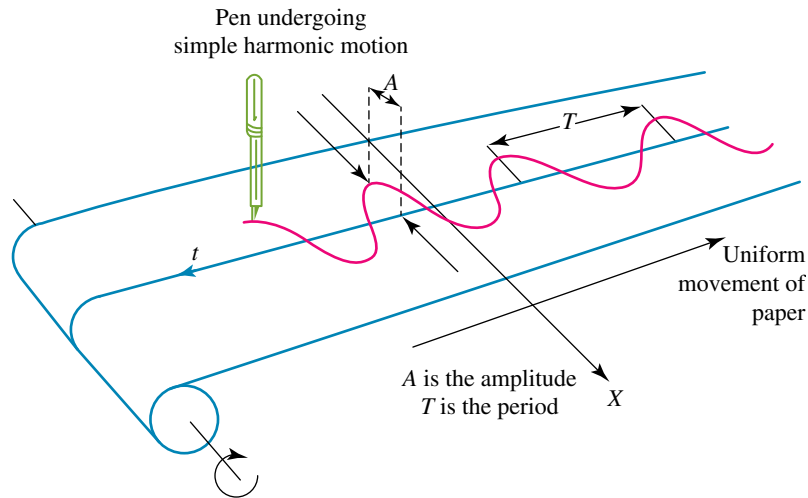
- b. A particle of mass m kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $ma(ax + b)$ newtons, where a and b are non-zero real constants. Initially the particle is $\frac{b}{a}$ m from the origin, moving with a speed of $2b$ m/s. Show that
- $$x = \frac{b}{a}(2e^{at} - 1).$$
16. A particle of mass m kg moves in a straight line and is subjected to a force that opposes its motion. The magnitude of the opposing force is inversely proportional to the cube of the distance from a fixed point, O . The constant of proportionality is k . The particle starts from rest when its distance from O is a m.
- a. Show that its speed, v m/s, is given by $v = \frac{\sqrt{k(a^2 - x^2)}}{\sqrt{m}ax}$, where x m is its distance from O at a time t seconds.
- b. Obtain a relationship between t and x , and hence show that when x has the value of $\frac{\sqrt{7}a}{4}$ m, a time of $\frac{3a^2}{4} \sqrt{\frac{m}{k}}$ s has elapsed.
17. a. A particle of mass 4 kg moves in a straight line and is retarded by a force $\frac{32}{x^2}$ newtons, where x is its distance in metres from a fixed point, O . If the particle starts from rest when its distance from O is 4 m, determine the time taken to travel to the origin. Give your answer correct to 2 decimal places.
- b. A particle of mass m kg moves in a straight line against a central force, that is one whose magnitude is inversely proportional to the square of the distance from a fixed point, O . The constant of proportionality is k . If the particle starts from rest when its distance from O is a metres, show that the speed, v m/s, at a distance x m from O satisfies $v = \sqrt{\frac{2k(a-x)}{m}}$.
18. A particle of mass 4 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $32x(x^2 - 9)$ newtons and initially is 2 m from the origin, moving with a speed of 10 m/s.
- a. Express x in terms of t .
- b. As t approaches infinity, the particle approaches a fixed position. Determine that position.

11.5 Simple harmonic motion

11.5.1 Definition of simple harmonic motion

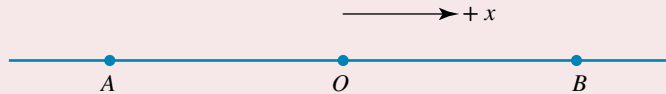
In many situations, objects **oscillate** in a straight line between two fixed points. The motion repeats itself in cycles of equal time intervals or periods. During the motion the particle is subjected to an acceleration which is always of opposite sign to the displacement. This type of motion is known as **simple harmonic motion**. Examples of this type of motion include the swing of a pendulum, the vertical movement of a mass attached to a vertical spring, a bungee jumper (although only for a short time) and a vibrating guitar string. Generally these motions, though periodic, are complex, but in many cases the oscillations are simple and can be mathematically modelled using trigonometric functions.

Consider the diagram where a pen is executing simple harmonic motion in a line perpendicular to the uniform motion of a reel of paper. The trace left by the pen records the displacement in the X-direction about a mean position (the time axis) as a function of time.



Definition of simple harmonic motion

If a particle moves periodically in a straight line between two points A and B as shown below, so that its acceleration is directed towards a fixed point on that line and is proportional to the displacement from the fixed point, then the particle is said to move with simple harmonic motion. Points A and B are the furthest or extreme positions from the centre.



Mathematically this translates to

$$\frac{d^2x}{dt^2} \propto -x$$

$$\frac{d^2x}{dt^2} = -kx, \text{ where } k > 0$$

To make it easier when taking the square root in later calculations, we replace the positive constant k with ω^2 giving,

$$\ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 x.$$

As acceleration is expressed in terms of displacement,

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -\omega^2 x$$

$$\frac{1}{2} v^2 = -\omega^2 \int x dx$$

$$\frac{1}{2} v^2 = -\frac{1}{2} \omega^2 x^2 + c$$

At the point A, let $x = A$, as this is the point at which the particle changes direction, $v = 0$.

$$0 = -\frac{1}{2}\omega^2 A^2 + c$$

$$c = \frac{1}{2}\omega^2 A^2$$

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega^2 A^2$$

$$v^2 = \omega^2 A^2 - \omega^2 x^2$$

$$v^2 = \omega^2 (A^2 - x^2)$$

Therefore $v = -\omega \sqrt{(A^2 - x^2)}$ if the particle is moving in the negative x direction and $v = \omega \sqrt{(A^2 - x^2)}$ if the particle is moving in the positive x direction.

To establish an expression for the displacement of the particle at time t , we can choose either the positive square root above or the negative square root as an expression for v .

Let's take the positive square root.

$$v = \omega \sqrt{(A^2 - x^2)}$$

$$\frac{dx}{dt} = \omega \sqrt{(A^2 - x^2)}$$

$$\frac{dt}{dx} = \frac{1}{\omega \sqrt{(A^2 - x^2)}}, -A < x < A$$

$$t = \frac{1}{\omega} \int \frac{dx}{\sqrt{(A^2 - x^2)}}$$

$$\omega t + \alpha = \sin^{-1} \left(\frac{x}{A} \right)$$

where α is the constant of integration.

Rearranging for x gives

$$x = A \sin(\omega t + \alpha)$$

or $x = A \cos(\omega t + \beta)$ using the complementary trigonometric identities

Note: Selecting $v = -\omega \sqrt{(A^2 - x^2)}$ in the beginning will result in the same equation.

If $x = 0$ when $t = 0$ (The particle is moving through the mean position at $t = 0$), then

$$\omega(0) + \alpha = \sin^{-1} \left(\frac{0}{A} \right) \Rightarrow \alpha = 0. \text{ Hence,}$$

$$\omega t = \sin^{-1} \left(\frac{x}{A} \right)$$

$$\therefore x = A \sin(\omega t)$$

If $x = A$ when $t = 0$ (The particle is at the furthest/extreme point when $t = 0$), then

$$\begin{aligned}
 \omega(0) + \alpha &= \sin^{-1}\left(\frac{A}{A}\right) \\
 \Rightarrow \alpha &= \sin^{-1}(1) = \frac{\pi}{2} \\
 x &= A \sin\left(\omega t + \frac{\pi}{2}\right) \\
 \therefore x &= A \cos(\omega t)
 \end{aligned}$$

Each complete oscillation (period) occurs in T seconds.

$$T = \frac{2\pi}{\omega}$$

The frequency (f) of simple harmonic motion is the number of oscillations or complete cycles made per second. So, $f = \frac{1}{T}$

Simple harmonic motion formulas

If a particle moves with simple harmonic motion, then $\ddot{x} = \frac{d^2x}{dt^2} = -\omega^2x$, where the velocity of the particle is given by the equation $v^2 = \omega^2(A^2 - x^2)$, $-A < x < A$.

If $\frac{d^2x}{dt^2} = -\omega^2x$, then $x = A \sin(\omega t + a)$ or $x = A \cos(\omega t + \beta)$.

If $x(0) = 0$ then $x = A \sin(\omega t)$ and if $x(0) = A$ then $x = A \cos(\omega t)$ where A is the amplitude or furthest distance from the centre.

$T = \frac{2\pi}{\omega}$ gives the period of the motion and $f = \frac{1}{T}$ gives the frequency per second.

WORKED EXAMPLE 11

An object is moving with simple harmonic motion where the period of motion of the object is 4 seconds. If the velocity of the object when passing through $x = 0$ is 10 m/s calculate the amplitude and the acceleration at the furthest position.

THINK

1. Write the rule for the period, T and the value of T .
2. Substitute $T = 4$ into the rule and solve for ω .

WRITE

$$\begin{aligned}
 T &= \frac{2\pi}{\omega} \\
 T &= 4 \\
 4 &= \frac{2\pi}{\omega} \\
 \omega &= \frac{2\pi}{4} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

3. Write the rule for the velocity and the value of v when $x = 0$.

$$v^2 = \omega^2(A^2 - x^2)$$

$$v(0) = 10$$

4. Substitute the values of v and x into the formula and solve for A .

$$10^2 = \left(\frac{\pi}{2}\right)^2 (A^2 - 0^2)$$

$$100 = A^2 \times \frac{\pi^2}{4}$$

$$A^2 = \frac{400}{\pi^2}$$

$$A = \pm \frac{20}{\pi}$$

5. Amplitude is always positive.

$$A = \frac{20}{\pi}, A > 0$$

6. Write the rule for acceleration and substitute into the rule the value of ω .

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$= -\frac{\pi^2}{4} x$$

7. At the extreme position, $x = \pm \frac{20}{\pi}$.

$$\text{When } x = \frac{20}{\pi}$$

$$\frac{d^2x}{dt^2} = -\frac{\pi^2}{4} x$$

$$= -\frac{\pi^2}{4} \times \frac{20}{\pi}$$

$$= -5\pi$$

$$\text{When } x = -\frac{20}{\pi}$$

$$\frac{d^2x}{dt^2} = -\frac{\pi^2}{4} x$$

$$= -\frac{\pi^2}{4} \times -\frac{20}{\pi}$$

$$= 5\pi$$

8. Write the answer.

The amplitude is $\frac{20}{\pi}$ and the

acceleration at the furthest points is

$$\pm 5\pi \text{ m/s}^2$$

WORKED EXAMPLE 12

An object moving with simple harmonic motion has a velocity of 8 m/s and 6 m/s respectively when it is 3 m and 4 m from the centre. Calculate:

- the length of the path
- the period
- the time taken to move between the two points.

THINK

- a. 1. Write the formula for velocity.
 2. When $v = 8, x = 3$.
 When $v = 6, x = 4$.
 Substitute the values for v and x into the rule to obtain two equations.
 3. Divide equation [1] by equation [2].
 4. Cancel down both fractions and cross multiply.
 Rearrange the equation and solve for A .

5. Write the answer.

- b. 1. Using equation (1), substitute $A = 5$ into this equation.

2. Write the formula for T and substitute $\omega = 2$.

- c. 1. Write the formula for x , assuming when $t = 0, x = 0$.

2. Substitute $x = 3$ and solve for t .

Substitute $x = 4$ and solve for t .

3. Calculate the difference to determine the time taken to move between the two points.

WRITE

$$\begin{aligned} \text{a. } v^2 &= \omega^2(A^2 - x^2) \\ 64 &= \omega^2(A^2 - 9) & [1] \\ 36 &= \omega^2(A^2 - 16) & [2] \end{aligned}$$

$$\begin{aligned} \frac{64}{36} &= \frac{\omega^2(A^2 - 9)}{\omega^2(A^2 - 16)} \\ 16(A^2 - 16) &= 9(A^2 - 9) \\ 16A^2 - 256 &= 9A^2 - 81 \\ 7A^2 &= 175 \\ A^2 &= 25 \\ A &= \pm 5, \\ A &= 5, A > 0 \end{aligned}$$

The amplitude is 5 and as the length of the path is $2A$, the length of the path is 10 m.

$$\begin{aligned} \text{b. } 64 &= \omega^2(A^2 - 9) \\ 64 &= \omega^2(25 - 9) \\ 64 &= 16\omega^2 \\ \omega^2 &= 4 \\ \omega &= \pm 2 \\ \therefore \omega &= 2, \omega > 0 \\ T &= \frac{2\pi}{\omega} \\ &= \pi \text{ s} \end{aligned}$$

- c. $x = 5 \sin(2t)$ if $x(0) = 0$

$$\begin{aligned} 3 &= 5 \sin(2t) \\ t &= \frac{1}{2} \sin^{-1} \left(\frac{3}{5} \right) = 0.322 \\ 4 &= 5 \sin(2t) \\ t &= \frac{1}{2} \sin^{-1} \left(\frac{4}{5} \right) = 0.464 \end{aligned}$$

$0.464 - 0.322 = 0.142 \text{ s}$
 The object takes 0.142s to move between the two points.

WORKED EXAMPLE 13

The speed of a particle moving along the x -axis is given by $v^2 = 48 + 32x - 16x^2$ where x is measured in metres.

- Show that the motion is simple harmonic.
- Determine the centre of motion, the amplitude and the period.

THINK

1. To show that the motion is simple harmonic, an expression for the acceleration in terms of x is required. As the speed is in terms of v^2 , use

$$\frac{d^2x}{dt^2} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right).$$

2. A particle exhibiting simple harmonic motion satisfies the equation of the form $\frac{d^2x}{dt^2} = -\omega^2x$, where the centre of motion is $x = 0$.

1. The centre of motion is at the point where the acceleration is zero.
2. To determine the amplitude the extreme points, need to be found. The velocity is zero when the particle has reached one of the extreme points. Solve the equation for $v = 0$.
3. The total distance travelled by the particle in one oscillation is the sum of the magnitudes of the extreme points.
4. The amplitude of the motion is one half of the total distance.
5. Determine the value of ω from the acceleration equation and substitute into the formula for the period.
6. Write the answer.

WRITE

$$\text{a. } \frac{d^2x}{dt^2} = \frac{d}{dx} \left(\frac{1}{2}v^2 \right)$$

$$v^2 = 48 + 32x - 16x^2$$

$$\frac{d}{dx} \left(\frac{1}{2}v^2 \right) = \frac{d}{dx} (24 + 16x - 8x^2)$$

$$\frac{d^2x}{dt^2} = 16 - 16x$$

$$= -16(x - 1)$$

$$\frac{d^2x}{dt^2} = -16(x - 1)$$

$$\frac{d^2x}{dt^2} = -4^2(x - 1)$$

This defines simple harmonic motion about the point $x = 1$.

$$\text{b. } -16(x - 1) = 0$$

$$x = 1$$

The centre of motion is at $x = 1$.

Solve the equation $v^2 = 48 + 32x - 16x^2$, for $v = 0$.

$$48 + 32x - 16x^2 = 0$$

$$3 + 2x - x^2 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \text{ or } x = -1$$

$$3 + |-1| = 4 \text{ m}$$

$$\text{Amplitude} = \frac{4}{2} = 2$$

$\omega^2 = 16$ from the acceleration equation

$$\omega = +4, \omega > 0$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ s}$$

Centre of motion: $x = 1$

Amplitude: $A = 2 \text{ m}$

$$\text{Period: } T = \frac{\pi}{2} \text{ s}$$

Exercise 11.5 Simple harmonic motion

Technology free

- WE11** An object moving with simple harmonic motion has a speed of 5 m/s when passing through its centre of motion. Calculate the speed and acceleration when it is 1.5 m from its centre given that the period is π seconds.
- An object is moving with simple harmonic motion with amplitude 8 m. If the period of motion is 8 m, determine how long it takes to travel 5 m from its centre of motion (mean position).
- The displacement x at time t of a point moving in a straight line is given by $x = A \sin(\omega t + \alpha)$. Write an expression for x if initially $v = 0$ when $x = -5$.
- The acceleration of an object moving with simple harmonic motion and 3 metres from the centre of motion is 6 m/s^2 . If the amplitude of the object is 5 metres, calculate the speed of the particle at the centre and then when it is 2 metres from the centre of motion.
- A particle is moving with simple harmonic motion of period $\frac{2\pi}{3}$ seconds and the maximum velocity is 9 m/s. Calculate the amplitude and the velocity when the particle is 2 m from the centre of motion.
- A particle moving with simple harmonic motion starts from rest at $x = 5$ and after 2 seconds reaches $x = 2.5$. Calculate:
 - an expression for the displacement at time $t \geq 0$
 - the speed at $x = 0$
 - the amplitude, frequency and period
 - the maximum speed
 - the maximum acceleration.

Technology active

- WE12** A particle moving with simple harmonic motion has speeds of $\sqrt{5}$ m/s and 2 m/s respectively at distances of 1 m and 2 m from the centre of motion. Calculate:
 - the length of the path of the object
 - the period
 - the time spent moving between the points.
- An object is moving in simple harmonic motion with a period of $\pi\sqrt{3}$ seconds. If it starts from rest at $x = 12$, calculate:
 - the frequency
 - the amplitude
- A particle moving in simple harmonic motion with period 2π seconds is 2 metres from the centre and moving away with a speed of $4\sqrt{2}$ m/s. Calculate the time taken for the particle to reach the centre.
- WE13** The speed of a particle moving along the x -axis is given by $v^2 = 40 + 30x - 10x^2$.
 - Show that the motion is simple harmonic.
 - Determine the mean position, the amplitude and the period.
- A cork bobs up and down with the waves, rising and falling a total of 2 metres about its mean position. Calculate its greatest velocity and acceleration if the period of the motion is 6 seconds.
- An object moves with simple harmonic motion. If its acceleration at distance d from the mean position is m , prove that the period of the motion is $2\pi\sqrt{\frac{d}{m}}$.
- If the speeds of a point moving with simple harmonic motion at distances 2 m and 3 m from the mean position are 5 m/s and 4 m/s respectively, calculate the period, the amplitude, the maximum speed and the maximum acceleration.

14. If the displacement of a moving point at time t is given by $x = a \cos(nt) + b \sin(nt)$, prove that the motion is simple harmonic. If $a = 3$, $b = 4$, $n = 2$, determine the period and use technology to calculate the amplitude, maximum speed, and maximum acceleration. All distances are in metres and the unit of time is seconds.

11.6 Review: exam practice

A summary of this chapter is available in the Resource section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. A body of mass 2 kg is acted upon by a variable force of $4t$ newtons at a time, t seconds, when its velocity is v m/s. If the body moves in a straight line with an initial velocity of u m/s, then express v in terms of u and t .
2. A particle of mass 2 kg is acted upon by a variable force of v^2 newtons, where v is the velocity of the particle in m/s at time t seconds. The particle moves in a straight line and initially $v = 2$. Write an expression for v in terms of t .
3. A body of mass 3 kg is acted upon by a variable force of $3v$ newtons, and has an acceleration of a m/s², when its velocity is v m/s at a time t seconds. The initial velocity of the body is u m/s. Express v in terms of u and t .
4. A body of mass m kg is moving in a straight line. When it is x metres from the origin it is travelling at v m/s. Given that $v = x^2$, calculate the size of the force acting on the body.
5. An object of mass 4 kg is acted upon by a variable force so that its velocity v m/s is given by $v^2 = 4(4 - x^2)$, where x metres is its displacement from the origin. Initially the object is at rest 2 metres from the origin. Calculate the maximum force in newtons acting on the body.
6. A body of mass m kg moves in a straight line, its velocity is v m/s at a time t seconds. The force acting on the body is $f(v)$ newtons. Given that $v = v_1$ when $t = t_1$ and $v = v_2$ when $t = t_2$, express the difference between t_2 and t_1 as an integral.
7. A particle moves in a straight line such that its velocity v ms⁻¹ is given by $v(x) = \sin\left(\frac{x}{2}\right)$, where x metres is its displacement from the origin. Determine the acceleration of the particle in ms⁻².
8. A particle of mass 2 kg is moving in a straight line. Its acceleration is $-4x$ m/s² where x metres is its displacement from a fixed origin. The particle is at rest where $x = 4$. Calculate the speed of the particle when $x = 2$.
9. An object moving with simple harmonic motion passes through two points P and Q which are 22 cm apart, with the same velocity having taken two seconds to pass between them. After 2 more seconds it returns to Q . Determine the period and the amplitude of the motion.
10. A car of mass m kg is travelling on a level roadway. The engine exerts a constant propulsive force of P newtons and the total resistance to the motion of the car is kv^4 newtons, where k is positive constant and v is its speed in m/s. If the car moves from rest, determine an expression for the distance travelled in metres until it obtains a speed of V m/s.
11. A taxi of mass 2400 kg moves in a straight line stopping at two sets of traffic lights. The force acting on the taxi as it moves between the two traffic lights is given by $1200 - 20t$ newtons where t is the time in seconds after it leaves the first traffic light. Determine the distance and the time it takes to travel between the two traffic lights.



12. A body of mass 10 kilograms moves in a straight line so that at time t seconds it is retarded by a force of $\frac{240}{\sqrt{(3t+1)^3}}$ newtons. If the initial velocity of the body is 16 m/s, calculate how far it has travelled in the first 5 seconds.

Complex familiar

13. A particle of mass 3 kilograms moves back and forth along the x -axis and is subjected to a force of $-48 \cos(2t) - 36 \sin(2t)$ newtons at time t seconds. If its initial velocity is 6 m/s and initially the particle starts from a point 4 metres to the right of the origin, calculate its displacement after $\frac{\pi}{8}$ seconds.
14. A particle of mass 4 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is retarded by a force of $8e^{-x}$ newtons and the particle is at rest at a distance of 2 metres from the origin. Express x in terms of t .
15. A particle of mass 3 kg moves so that at a time t seconds, its displacement is x metres from a fixed origin. The particle is acted upon by a force of $\frac{51x - 36}{x^3}$ newtons and the particle is at rest at a distance of 3 metres from the origin. Determine where else the particle comes to rest.
16. A cyclist of mass 75 kg is riding a 15 kg bike along a level road at a speed of 57.6 km/h when she begins to freewheel. The resistance force is $\frac{672}{\sqrt{v}}$ newtons, where v m/s is her speed after she begins to freewheel. When her speed has decreased to 14.4 km/h she has travelled a distance of D metres in a time of T seconds. Determine the values of D and T .



Complex unfamiliar

17. A high-speed train of mass m kg starts from rest and moves horizontally along a straight-line track. The engine of the train exerts a constant force of F newtons in the direction of motion, but the train is subjected to a resistance force of kv^2 newtons, where v is its speed in m/s. The terminal velocity of the train is 270 km/h.



- a. Show that the equation of motion of the train is given by $m\ddot{x} = F \left(1 - \left(\frac{v}{75} \right)^2 \right)$.
- b. Write in terms of F and m , the distance travelled by the train to reach a speed of 180 km/h.
- c. Also in terms of F and m , calculate the time taken for the train to reach a speed of 180 km/hr.
18. A ball of mass m is projected vertically upwards into the air from ground level, with an initial speed of U . The ball reaches a maximum height of H and falls back to the ground. It hits the ground with a

speed of W . While in the air, going up and down the ball experiences a resistive force of kv^2 where v is the velocity of the ball and k is a constant.

- a. Show that the terminal velocity V_T of the ball as it falls is $\sqrt{\frac{mg}{k}}$.
 - b. Show that as the ball rises its maximum height $H = \frac{V_T^2}{2g} \ln \left(1 + \frac{U^2}{V_T^2} \right)$.
 - c. Show that the time to reach the maximum is $\frac{V_T}{g} \tan^{-1} \left(\frac{U}{V_T} \right)$.
 - d. Show that $\frac{1}{W^2} = \frac{1}{U^2} + \frac{1}{V_T^2}$.
19. A cyclist of mass m kg travelling at a speed U m/s reaches a level section of road and begins to freewheel. He observes that after travelling a distance of D metres in time T seconds along this section of road his speed has fallen to V m/s, where $U > V > 0$. If during the freewheeling his retardation is proportional to the square root of the speed, show that $\frac{D}{T} = \frac{1}{3} (U + \sqrt{UV} + V)$.
 20. A particle is moving in a straight line with simple harmonic motion of period $\frac{2\pi}{\omega}$ and amplitude A .
 - a. Write expressions for its displacement at time t if initially:
 - i. it is at the centre of its motion
 - ii. it is in an extreme position
 - iii. it is at a distance of $\frac{A}{2}$ from the centre and moving (1) away from the centre, (2) towards the centre.
 - iv. In case iii show that it will first reach an extreme position after times (1) $\frac{\pi}{3\omega}$ and (2) $\frac{2\pi}{3\omega}$.
 - b. Express A in terms of V and ω where V is the speed at distance $\frac{A}{3}$ from an extreme position and show that the maximum speed is $\frac{3V}{\sqrt{5}}$.
 - c. If initially the particle is in an extreme position, calculate the next two times it will be distant $\frac{1}{2}A$ from the other extreme position.
 - d. Determine the speed when at the point $\frac{A}{3}$ from one extreme position and determine the time taken to reach this point from the other extreme position.

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Answers

11 Modelling motion 2

Exercise 11.2 Forces that depend on time

- 9 m
- 4 s
- $18 \sin\left(\frac{t}{3}\right)$
- $4\sqrt{2}$ m
- a. 1 s
b. 25 m
- a. $\frac{1}{2}$ s
b. 10 m
- a. $x = t^2 - t^3$
b. $x = 2(e^{2t} - 1) - 2t^2 - 4t$
- a. 1 m to the left of the origin
b. $x = 4t^3 - 3t^2 + t + 2$
- 900 m, 90 s
- 67.39 m
- a. 6 m to the left of the origin
b. 16 m
- 40 m
- a. Sample responses can be found in the worked solutions in the online resources.
b. i. $49t - 4.9t^2 + 2$
ii. 124.5 m
- a. 11.11 m
b. Sample responses can be found in the worked solutions in the online resources.
- a. $x = 49t + 245e^{-0.2t} - 225$
b. Sample responses can be found in the worked solutions in the online resources.
- $T = \frac{1}{2}, D = \frac{400}{41}$

Exercise 11.3 Forces that depend on velocity

- Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- a. $\frac{1}{2}$ s
b. 10 m
- $T = \frac{1}{2}, D = \frac{400}{41}$
- Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- a, b. Sample responses can be found in the worked solutions in the online resources.
c. 4500.17 kg m/s
d. $42\sqrt{5}$ m/s
e. 5.69 s
- Sample responses can be found in the worked solutions in the online resources.
- a. $x = \frac{1}{2}(1 - e^{-2t})$
b. $x = 2(1 - e^{-3t} - 2t)$
- a. 5 s
b. $46\frac{2}{3}$ m
- a. 5 s
b. 40 m
- 12–18. Sample responses can be found in the worked solutions in the online resources.

Exercise 11.4 Forces that depend on displacement

- 3 m to the left of the origin
- 2
- Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- $8 \cos\left(\frac{t}{2}\right)$
- $18 \sin\left(\frac{t}{3}\right)$
- a. $x = \frac{1}{2}(3 + e^{2t})$
b. $x = \frac{1}{3}(5 + e^{3t})$
- a. $x = 1 - e^t$
b. $x = 4 \sin(2t)$
- a. $x = \ln(t + 1)$
b. $x = -\frac{1}{2} \ln(1 - 2t)$
- a. $\frac{1}{2}$ m
b. $\frac{1}{2}$ m
- a. 8 m
b. Sample responses can be found in the worked solutions in the online resources.
- Sample responses can be found in the worked solutions in the online resources.
- a. $x = 3 \cos(4t)$
b. Sample responses can be found in the worked solutions in the online resources.
- 14–16. Sample responses can be found in the worked solutions in the online resources.
- a. 6.28 s
b. Sample responses can be found in the worked solutions in the online resources.
- a. $x = \frac{3(5 - e^{-12t})}{5 + e^{-12t}}$
b. 3 m

Exercise 11.5 Simple harmonic motion

- 4 m/s, -6 m/s^2
- $t = \frac{4}{\pi} \sin^{-1}\left(\frac{5}{8}\right)$
- $x = 5 \cos(\omega t)$
- $5\sqrt{2}$ m/s; $\sqrt{42}$ m/s
- 3 m; $3\sqrt{5}$ m/s
- a. $5 \cos\left(\frac{\pi t}{6}\right)$
b. $\frac{5\pi}{6}$
c. $A = 5, T = 12, f = \frac{1}{12}$
d. $\frac{5\pi}{6}$
e. $\frac{5\pi^2}{36}$
- a. 8 m
b. $2\pi\sqrt{3}$
c. 0.47 s
- a. $\frac{\sqrt{3}}{3\pi}$ vibrations/s
b. 12
- 2.80 s
- a. Sample responses can be found in the worked solutions in the online resources.
b. $x = \frac{3}{2}, A = 2.5, T = \frac{p\sqrt{10}}{5}$
- $v = \frac{\pi}{3}, a = \frac{\pi^2}{9}$

12. Sample responses can be found in the worked solutions in the online resources.

$$13. T = \frac{2\pi\sqrt{5}}{3}, A = \frac{\sqrt{161}}{3}, v = \sqrt{\frac{161}{5}}, a = \frac{3\sqrt{161}}{5}$$

$$14. \pi \text{ s}; 5 \text{ m}; 10 \text{ m/s}; 20 \text{ m/s}^2$$

11.6 Review: exam practice

$$1. v = u + t^2$$

$$2. v = \frac{2}{1-t}$$

$$3. v = ue^t$$

$$4. 2mx^3$$

$$5. 32$$

$$6. t_2 - t_1 = m \int_{v_1}^{v_2} \frac{1}{f(v)} dv$$

$$7. \frac{1}{4} \sin(x)$$

$$8. 4\sqrt{3}$$

$$9. 8 \text{ s}; 11\sqrt{2} \text{ cm}$$

$$10. \int_0^v \frac{mv}{P - kv^4} dv$$

$$11. 1200 \text{ m and } 120 \text{ s}$$

$$12. 32 \text{ m}$$

$$13. \frac{7\sqrt{2}}{2} \text{ m}$$

$$14. x = 2 \ln(t + 1)$$

$$15. 0.4 \text{ m from the origin}$$

$$16. T = 5, D = 53\frac{1}{7}$$

17. a. Sample responses can be found in the worked solutions in the online resources.

$$b. \frac{5625 m}{2F} \ln\left(\frac{9}{5}\right) \text{ m}$$

$$c. \frac{75 m}{2F} \ln(5) \text{ s}$$

18. Sample responses can be found in the worked solutions in the online resources.

19. Sample responses can be found in the worked solutions in the online resources.

$$20. a. i. x = A \sin(\omega t)$$

$$ii. x = A \cos(\omega t)$$

$$iii. (1) x = A \sin\left(\omega t + \frac{\pi}{6}\right), (2) x = A \sin\left(\omega t + \frac{5\pi}{6}\right)$$

- iv. Sample responses can be found in the worked solutions in the online resources.

- b. Sample responses can be found in the worked solutions in the online resources.

$$c. \frac{2\pi}{3\omega}, \frac{4\pi}{3\omega}$$

$$d. t = \frac{1}{\omega} \left(\pi - \sin^{-1}\left(\frac{2}{3}\right) \right)$$

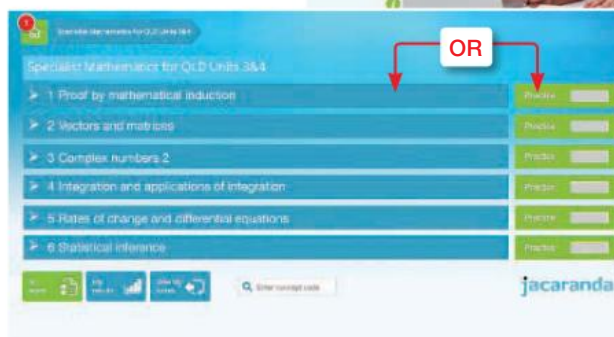
REVISION UNIT 4 Further calculus and statistical inference

TOPIC 2 Rates of change and differential equations

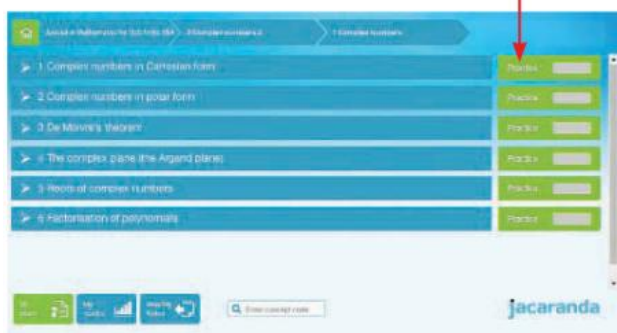
- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.



- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



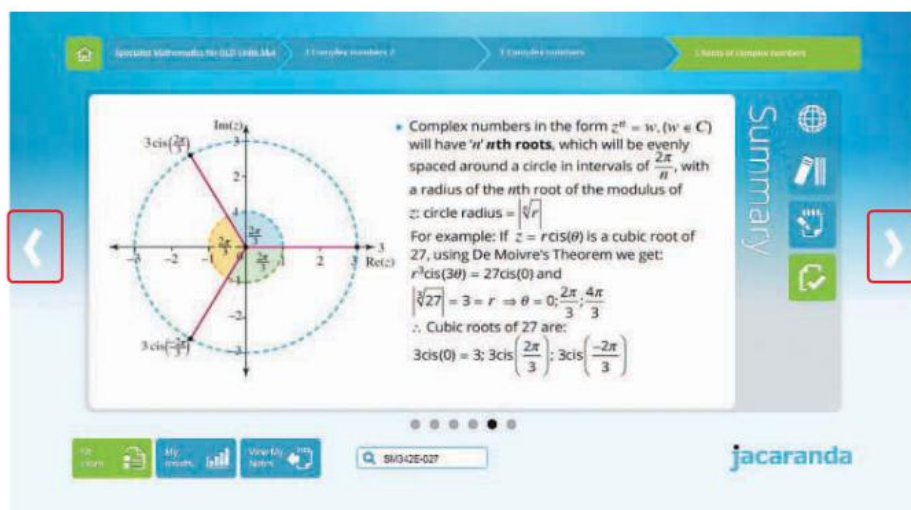
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



12 Statistical inference

12.1 Overview

Confidence intervals, first introduced by the Polish mathematician Jerzy Neyman (1894–1981) in a paper published in 1937, give an indication of the likelihood that a particular sample contains a population parameter. His work on experiments and statistics eventually formed the basis for hypothesis testing and the method adopted by the Federal Drug Authority (FDA) to test new medicines today.



LEARNING SEQUENCE

- 12.1 Overview
- 12.2 Review of continuous random variables and the normal distribution
- 12.3 Sample means and simulations
- 12.4 Confidence intervals
- 12.5 Applications of confidence intervals
- 12.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

12.2 Review of continuous random variables and the normal distribution

12.2.1 Continuous random variables

In Mathematical Methods, you have studied continuous random variables. Remember that a random variable X can be called continuous if its set of possible values is an entire interval of numbers. That is, for some $A < B$, any number x between A and B is possible.

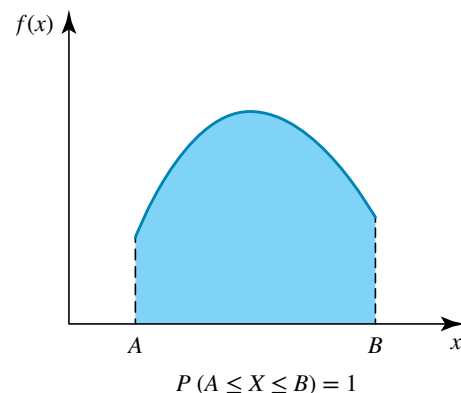
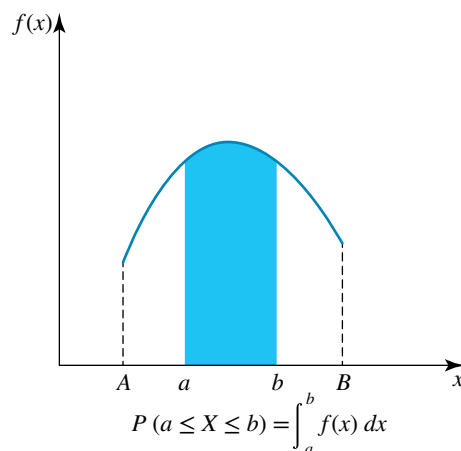
If the measurement scale used for X can be subdivided depending on the accuracy of the equipment available, then X is a continuous variable. For example, length may be measured in kilometres, metres, centimetres, millimeters etc, so the variable length will be a continuous variable. The smallest unit for price of an object is cents, so the variable price will be a discrete variable. When deciding if a variable is discrete or continuous, a useful rule of thumb is: variables that can be counted are discrete and variables that are measured are continuous.

If X is a continuous random variable and X can be any value in the domain $[A, B]$, then $P(A \leq X \leq B) = 1$. It is possible to find a function, called a probability density function $f(x)$ so that

$$P(a \leq X \leq b) = \int_a^b f(x) dx.$$

Note that as $P(A \leq X \leq B) = 1$, then $\int_A^B f(x) dx = 1$.

In Chapter 7, you explored the exponential probability density function. Another probability distribution function is used to represent the normal distribution.



12.2.2 The normal distribution

You have already explored this function in Mathematical Methods, so this is a quick review.

The normal distribution is symmetric with a mean of μ and a standard deviation of σ . The probability density function for the normal distribution is $f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. For a random variable X that is

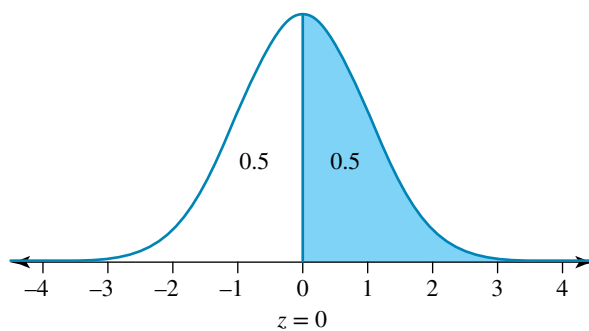
normally distributed with mean μ and variance σ^2 we can write $X \sim N(\mu, \sigma^2)$. If $\mu = 0$ and $\sigma = 1$, then the distribution is called the standard normal distribution. A random variable which has a standard normal distribution is called a standard normal random variable, and is usually denoted by Z . The probability

density function of Z is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ and $P(z_1 < Z < z_2) = \int_{z_1}^{z_2} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \right) dz$. Fortunately, it is

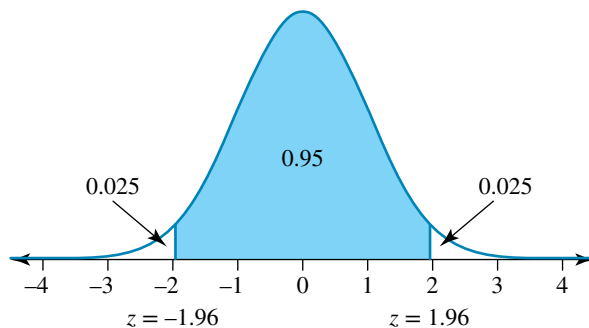
not necessary to determine the value of this integral as the values have been recorded in published tables and are also accessible through a graphics calculator. The values z_1 and z_2 are referred to as the z scores. If the area between the curve and the x -axis is known, then the inverse normal can be used to determine the z score that will correspond with that area.

Because the standard normal distribution is symmetric, $P(z < 0) = P(z > 0) = 0.5$.

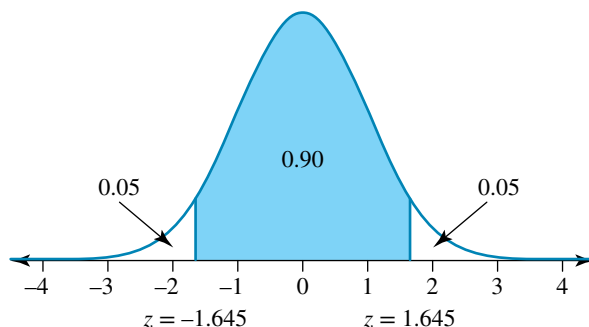
In this chapter, we are going to be particularly interested in the middle 90%, the middle 95% and the middle 99% of the distribution.



Let's begin by considering the middle 95% of the distribution. As the total area under the curve is 1, the area for the middle 95% is 0.95, leaving 0.05 to be shared equally between each end of the distributions (known as the tails). This means that the area of each tail is 0.025. Using the inverse normal for a right tail area of 0.025, results in a z -score of 1.96. As the distribution is symmetric, $P(-1.96 \leq Z \leq 1.96) = 0.95$. This means that for any normal distribution, 95% of the distribution will lie within 1.96 standard deviations of the mean.

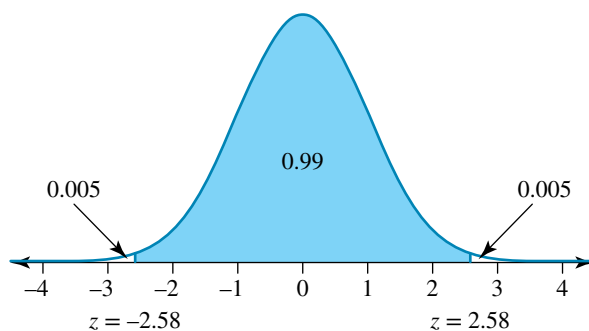


In a similar fashion, the inverse normal for a right tail area of 0.05 can be used to determine that $P(-1.645 \leq Z \leq 1.645) = 0.9$, meaning that 90% of the distribution will lie within 1.645 standard deviations of the mean.



To identify the location of the middle 99% of the distribution, the inverse normal for a right tail area of 0.005 is used to identify that $P(-2.58 \leq Z \leq 2.58) = 0.99$. This means that 99% of the distribution lies within 2.58 standard deviations of the mean.

In a similar fashion, the inverse normal can be used to determine the z scores for any area that we require.



12.2.3 Linear combinations of random variables

Before jumping straight into the field of statistical inference, it is highly important to review some key ideas of probability and statistics. Here we will extend our knowledge of random variables by considering combinations of random variables and the mean and standard deviation of such combinations.

Consider a random variable Y which is a linear function of another random variable X . That is $Y = aX + b$, where a and b are constants. If X is a discrete or continuous random variable, then $Y = aX + b$ would also be a discrete or continuous random variable respectively. Using the definition of the mean,

that is $\mu = E(X) = \sum_{i=1}^n \frac{x_i}{n}$, we can deduce that: $E(Y) = E(aX + b) = aE(X) + b$. The proof of this formula is given below assuming that X is a continuous random variable.

To determine the mean or expected value of a continuous random variable X , we use the formula

$$E(X) = \int_{-\infty}^{\infty} (x \times f(x)) dx.$$

$$\text{Thus } E(Y) = E(aX + b)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} ((ax + b) \times f(x)) dx \\ &= \int_{-\infty}^{\infty} (ax \times f(x)) dx + \int_{-\infty}^{\infty} (b \times f(x)) dx \\ &= a \int_{-\infty}^{\infty} (x \times f(x)) dx + b \int_{-\infty}^{\infty} (f(x)) dx \\ &= aE(X) + b \times 1 \\ &= aE(X) + b \end{aligned}$$

Note that in a continuous distribution $\int_{-\infty}^{\infty} f(x) dx = 1$. That is the area under the curve $y = f(x)$ equals 1.

Similarly, using the formula for the variance of X which states that $\text{Var}(X) = E(X^2) - [E(X)]^2$, we can show that $\text{Var}(Y) = \text{Var}(aX + b) = a^2 \text{Var}(X)$.

There are many other properties of linear combinations of random variables which can be useful when solving problems. Below is a summary of some useful properties for the random variables X and Y .

Expected value and variance formulae

Formulas by definition

$$E(X) = \int_{-\infty}^{\infty} (x \times f(x)) dx \text{ and } \text{Var}(X) = E(X^2) - [E(X)]^2$$

Linear combinations of one random variable

$$E(aX + b) = aE(X) + b$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Linear combinations of two random variables

$$E(aX + bY) = aE(X) + bE(Y)$$

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) \text{ if } X \text{ and } Y \text{ are independent}$$

studyon

Units 3 & 4

Area 6

Sequence 1

Concept 1

Review of continuous random variables Summary screen and practice questions

12.3 Sample means and simulations

12.3.1 Estimating population parameters

In statistics, the entire underlying set of individuals of a group is called the **population**. In mathematics a population refers not just to people, but to any group. The distribution of a population can be summarised by specific values known as **parameters**. If the distribution of data is X , the parameters include:

1. The expected value of the distribution, $E(X)$, which corresponds to the **mean**. That is, $E(X) = \mu$.
2. The **standard deviation**, σ , of the distribution. This parameter gives information about how the data are spread out from the mean. The standard deviation and mean have the same units making them excellent parameters for analysing the distribution.
3. The **variance**, σ^2 , of the distribution can also be used to describe the spread of a distribution. The variance is the square of the standard deviation and therefore has different units compared to the mean and standard deviation. The variance, also denoted by $\text{Var}(X)$, plays a central role in many areas of statistics.

The parameters of mean and standard deviation or variance are most frequently chosen to describe the population.

For very large populations, or where data for the entire population is very difficult and/or expensive to obtain, **samples** can be used to estimate the population parameters. A sample is a set of individuals or events selected from a population for analysis to give estimates of parameters of the whole population. In this section, we are concerned with determining the mean of different samples. As we will see, the size of the sample has an impact on the accuracy of the predictions that can be made. In practice, if the sample size is greater than 30, it is considered a large sample.

WORKED EXAMPLE 1

Dylan is interested in finding the average length of television shows on Netflix. He collects 12 samples and records the length of 10 different shows. All times are recorded in minutes. Use the sample means to estimate mean program length of the population.



Sample	Show 1	Show 2	Show 3	Show 4	Show 5	Show 6	Show 7	Show 8	Show 9	Show 10
1	60	55	60	60	60	5	55	30	30	30
2	60	30	30	60	35	130	35	60	30	35
3	85	30	50	55	60	60	55	25	55	30
4	30	30	30	30	30	30	30	30	30	30
5	30	30	30	30	30	25	25	25	25	25
6	60	60	60	60	60	60	60	30	30	60
7	30	60	60	60	90	60	35	35	35	30

Sample	Show 1	Show 2	Show 3	Show 4	Show 5	Show 6	Show 7	Show 8	Show 9	Show 10
8	55	55	60	35	35	55	55	55	50	55
9	85	60	60	30	30	60	60	60	60	60
10	30	30	30	45	45	45	45	45	45	45
11	70	70	130	35	35	70	70	35	35	70
12	60	105	60	120	60	60	60	60	60	105

THINK

1. Recall the formula for mean.
2. Determine the mean for each sample.

WRITE

$$\bar{x} = \frac{\sum x_i}{n}$$

$$\begin{aligned}\text{Sample 1: } \bar{x} &= \frac{445}{10} \\ &= 44.5\end{aligned}$$

$$\begin{aligned}\text{Sample 2: } \bar{x} &= \frac{505}{10} \\ &= 50.5\end{aligned}$$

$$\begin{aligned}\text{Sample 3: } \bar{x} &= \frac{505}{10} \\ &= 50.5\end{aligned}$$

$$\begin{aligned}\text{Sample 4: } \bar{x} &= \frac{300}{10} \\ &= 30\end{aligned}$$

$$\begin{aligned}\text{Sample 5: } \bar{x} &= \frac{275}{10} \\ &= 27.5\end{aligned}$$

$$\begin{aligned}\text{Sample 6: } \bar{x} &= \frac{540}{10} \\ &= 54\end{aligned}$$

$$\begin{aligned}\text{Sample 7: } \bar{x} &= \frac{495}{10} \\ &= 49.5\end{aligned}$$

$$\begin{aligned}\text{Sample 8: } \bar{x} &= \frac{510}{10} \\ &= 51\end{aligned}$$

$$\begin{aligned}\text{Sample 9: } \bar{x} &= \frac{565}{10} \\ &= 56.5\end{aligned}$$



3. Calculate the average $\bar{x}$.

4. Answer the question.

$$\begin{aligned}\text{Sample 10: } \bar{x} &= \frac{405}{10} \\ &= 40.5\end{aligned}$$

$$\begin{aligned}\text{Sample 11: } \bar{x} &= \frac{620}{10} \\ &= 62\end{aligned}$$

$$\begin{aligned}\text{Sample 12: } \bar{x} &= \frac{750}{10} \\ &= 75\end{aligned}$$

$$\begin{aligned}\text{Average } \bar{x} &= \frac{44.5 + 50.5 + 50.5 + \dots + 75}{12} \\ &= \frac{591.5}{12} \\ &\approx 49.3\end{aligned}$$

An estimate of the mean program length is 49.3 minutes.

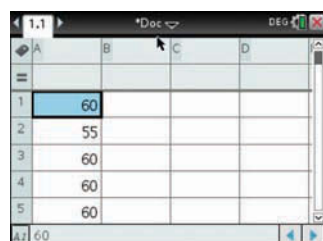
TI | THINK

- On a Home page, select Add Lists & Spreadsheets to: New Document

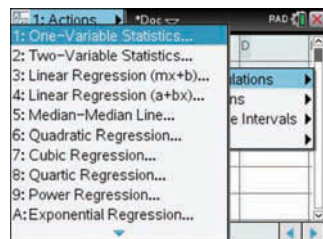
WRITE



- Enter the List 1 data by completing the entry lines
60
55
60
.
.
30
Continue by entering the entire list.



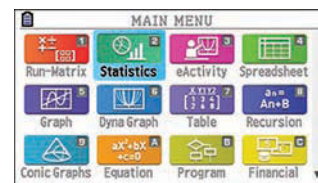
- Select
MENU
4: Statistics
1: Stat Calculations
1: One-Variable
Statistics ...



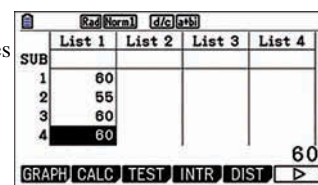
CASIO | THINK

- On a Main Menu screen, select Statistics

WRITE



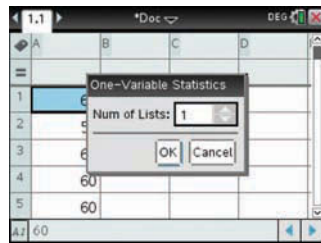
- Enter the List 1 data by completing the entry lines
60
55
60
.
.
30
Continue by entering the entire list.



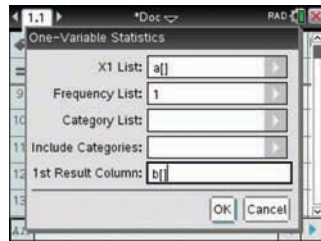
- To ensure the settings are correct, select
SET
In the 1 VAR Freq field, select 1
Press the EXIT button.



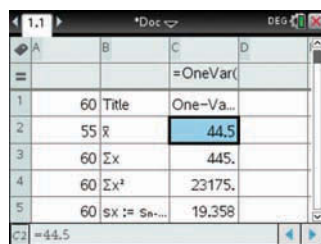
- Set the number of lists as 1 by pressing the OK button.



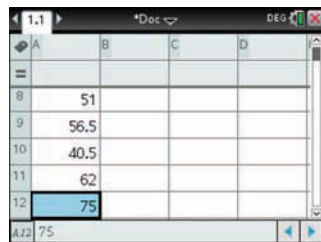
- Complete the entry lines
X1 List: a[]
Frequency List: 1
1st Result Column: b[]



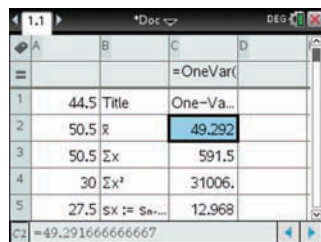
- Press the ENTER button.
The answer appears on the screen.



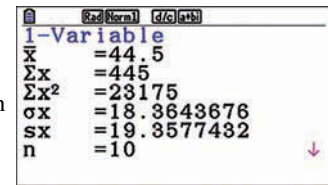
- Repeat this process to determine the mean for each of the 12 samples.
To calculate an estimate of the population mean enter the means in new list as shown.



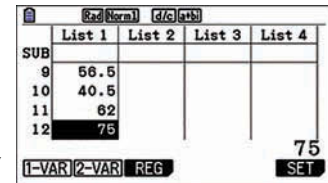
- Press the ENTER button.
The answer appears on the screen.



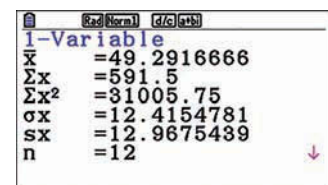
- Press the 1-VAR button.
The answer appears on the screen.
Repeat steps 2-4 for each of the data lists.



- Repeat this process to determine the mean for each of the 12 samples.
To calculate an estimate of the population mean enter the means as a new list as shown.



- Press the 1-VAR button.
The answer appears on the screen.



12.3.2 The distribution of sample means

For purposes of understanding we will examine a very small population and an even smaller sample. Consider the following data points: 23, 42, 12, 21 and 11. The mean of these points is $\mu = 21.8$ and the standard deviation is $\sigma \approx 11.16$.

All the different samples of size 2 from this data set are presented in the table. The mean of a sample is used to give some indication of the likely population mean. The sample mean is given the symbol $\bar{x}$. As each sample provides one possible estimate for the population mean, each estimate is known as a point estimate.

Data points	23, 42	23, 12	23, 21	23, 11	42, 12	42, 21	42, 11	12, 21	12, 11	21, 11
$\bar{x}$	32.5	17.5	22	17	27	31.5	26.5	16.5	11.5	16

As you can see, there is wide variety in the values of $\bar{x}$. This variability will be reduced in larger samples. Examining the average of the sample $\bar{x}$ values in the table, denoted by $\mu_{\bar{x}}$, you will find $\mu_{\bar{x}} = 21.8$. Notice that this is the same as the population mean. While the means of individual samples will vary, the mean of the sample means will be the same as the population mean. That is $\mu_{\bar{x}} = \mu$. When all possible samples are considered, the tendency for the mean of the sample means to be the same as the population mean is known as the **Central Limit Theorem**.

The standard deviation of the sample means can be found using $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{11.16}{\sqrt{2}} \approx 7.89$. As the standard deviation is divided by the square root of the sample size, this measure of variability becomes smaller as the sample size increases.

If the population is normally distributed, then the sample means will also be normally distributed with a mean of μ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$. If the population is not normal, but the samples are sufficiently large (usually $n > 30$), then the sample means will also be normally distributed with a mean of μ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$. This will be explored further in the next exercise.

12.3.3 Verifying the formulae

The original population X has an expected value $E(X) = \mu$ and a variance of $\text{Var}(X) = \sigma^2$. In this instance, we are concerned with the distribution of sample means taken from this population. Each sample has a mean $\bar{x}_i$, and the distribution of these means can be referred to as $\bar{X}$.

We observed that $E(\bar{X}) = \mu$. As each sample comes from the original population, the expected value for the mean of each sample is the same as the population mean, μ . That is $E(X_i) = E(X) = \mu$. (In reality the actual value is $\bar{x}_i$, but we can't expect a random value, we expect it to be μ).

$$\begin{aligned}
 E(\bar{X}) &= E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) \\
 &= \frac{1}{n}E(X_1 + X_2 + \dots + X_n) \\
 &= \frac{1}{n}(E(X_1) + E(X_2) + \dots + E(X_n)) \quad \text{since } E(aX + bY) = aE(X) + bE(Y) \\
 &= \frac{1}{n}(n\mu) \\
 &= \mu
 \end{aligned}$$

In a similar fashion, the variance of the distribution can be proven. It is assumed that the random samples are mutually independent and as each sample comes from the population, it will have the same variance as the population. That is $\text{Var}(X_i) = \text{Var}(X) = \sigma^2$.

$$\begin{aligned}
\text{Var}(\bar{X}) &= \text{Var}\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) \\
&= \frac{1}{n^2} \text{Var}(X_1 + X_2 + \dots + X_n) \quad \text{since } \text{Var}(aX + b) = a^2 \text{Var}(X) \\
&= \frac{1}{n^2} (n\sigma^2) \\
&= \frac{\sigma^2}{n} \\
\text{sd}(\bar{X}) &= \frac{\sigma}{\sqrt{n}}, \text{ recalling that } \text{sd}(X) = \sqrt{\text{Var}(X)}
\end{aligned}$$

WORKED EXAMPLE 2

500 students are studying a Mathematics course at the University of the Sunshine Coast. On a recent exam, the mean score was 67.2 with a standard deviation of 17.

- a. Samples of size 5 are selected and the means determined. Calculate the mean and standard deviation of the distribution of $\bar{X}$.
- b. If the sample size was increased to 30, determine the effect this would have on the mean and standard deviation of the distribution of sample means.

THINK

- a. 1. The mean of the distribution of sample means is the same as the population mean.

2. Write down the formula for standard deviation of the distribution of sample means.

3. Calculate the standard deviation.

- b. 1. The mean is not dependent on sample size.

2. Calculate the standard deviation using $n = 30$.

3. Write your conclusions.

WRITE

$$\begin{aligned}
\text{a. } E(\bar{X}) &= \mu \\
&= 67.2
\end{aligned}$$

$$\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$$

$$\begin{aligned}
\text{sd}(\bar{X}) &= \frac{17}{\sqrt{5}} \\
&\approx 7.6
\end{aligned}$$

$$\begin{aligned}
\text{b. } E(\bar{X}) &= \mu \\
&= 67.2
\end{aligned}$$

$$\begin{aligned}
\text{sd}(\bar{X}) &= \frac{\sigma}{\sqrt{n}} \\
&= \frac{17}{\sqrt{30}} \\
&\approx 3.1
\end{aligned}$$

Increasing the sample size does not change the mean of the distribution, but the standard deviation of the distribution is reduced.

Exercise 12.3 Sample means and simulations

Technology active

1. **WE1** Ronit decides to measure his mean travel time to school. He records the time taken every day for 7 weeks and records the time in minutes, resulting in 7 samples. Use the sample means to estimate mean travel time of the population.

Week	Monday	Tuesday	Wednesday	Thursday	Friday
1	92	43	41	39	35
2	118	81	46	51	38
3	62	48	46	41	49
4	82	48	42	43	41
5	78	51	42	41	38
6	63	62	41	43	44
7	55	41	46	41	32

2. Leesa wants to know the mean movie length for her favourite movie channel. Each day for a week she records the lengths of 8 movies. Her results are recorded in minutes. Use the sample means from each day to estimate mean movie length of the population.

Day	Movie 1	Movie 2	Movie 3	Movie 4	Movie 5	Movie 6	Movie 7	Movie 8
Monday	115	95	105	95	115	100	90	95
Tuesday	95	85	90	90	105	95	75	95
Wednesday	110	95	80	110	95	90	105	80
Thursday	95	100	90	95	105	100	90	85
Friday	105	95	90	100	105	100	90	105
Saturday	90	85	90	110	80	100	90	90
Sunday	105	100	100	95	90	90	90	110

3. **WE2** Every year 500 students apply for a place at Monotreme University. The average enrolment test score is 600 with a standard deviation of $\sqrt{300}$.
 - a. Samples of size 10 are selected and the means determined. Calculate the mean and standard deviation of the distribution of $\bar{X}$.
 - b. If the sample size was increased to 20, determine the effect this would have on the mean and standard deviation of the distribution of sample means.

4. 1000 students are studying a statistics course at Echidna University. On a recent exam, the mean score is 90.5 with a standard deviation of 10.
- Samples of size 15 are selected and the means determined. Calculate the mean and standard deviation of the distribution of $\bar{X}$.
 - If the sample size was increased to 30, determine the effect this would have on the mean and standard deviation of the distribution of sample means.



5. Use the random number generator on your calculator to generate random numbers between 0 and 100. Perform the simulation 40 times and calculate the sample mean.
- Compare your results to those of your classmates. How close were they to the expected value of 50?
 - Calculate the average of $\bar{X}$ for your class. How close is it to 50?

Questions 6–7 refer to the following data set – a simulation of the total obtained when a pair of dice were tossed. There were 8 people who each tossed the dice 10 times.

	Player 1	Player 2	Player 3	Player 4	Player 5	Player 6	Player 7	Player 8
Toss 1	7	6	10	7	11	2	10	8
Toss 2	8	9	8	6	6	11	4	10
Toss 3	2	4	3	10	6	8	8	6
Toss 4	8	5	6	5	11	7	6	2
Toss 5	10	12	6	8	10	8	3	4
Toss 6	4	10	9	5	3	6	5	5
Toss 7	7	6	5	6	9	10	4	2
Toss 8	11	8	9	8	9	9	6	10
Toss 9	4	7	10	10	7	4	12	8
Toss 10	7	8	3	8	10	4	7	11

6. Calculate an estimate of the population mean and compare it to the theoretical mean.
7. Repeat the experiment with your own simulation of dice tosses. Use your results to estimate the population mean.

Questions 8–10 refer to a population with a mean of 73 and a standard deviation of 12.

8. If samples of size 20 are selected, calculate the mean and standard deviation of the distribution of sample means.
9. If the sample size is increased to 30, calculate the mean and standard deviation of the sample means.
10. Determine the sample size that would be needed to reduce the standard deviation of the distribution of sample means to less than 2.

Questions 11–13 refer to a population with a mean of 123 and a standard deviation of 43.

11. If samples of size 25 are selected, calculate the mean and standard deviation of the distribution of sample means.
12. If samples of size 40 are selected, calculate the mean and standard deviation of the distribution of sample means.
13. Determine the sample size that would be needed to reduce the standard deviation of the distribution of sample means to less than 5.
14. We wish to create data points from a skewed population. Use Excel or another similar package to create the data points.
 - a. Use a random number generator to find 70 numbers between 1 and 20. Use the generator to find another 30 numbers between 15 and 20. The distribution should now be skewed to the right. You should now have 100 numbers.
 - b. Use technology to sketch your distribution.

We now want to take samples from the distribution to see what the distribution of sample means might look like when the original population was skewed.

- c. Select random samples of size 10 from your distribution and calculate the sample means.
- d. Use technology to sketch the distribution of sample means. Compare the two sketches and comment on the similarities and difference.

12.4 Confidence intervals

12.4.1 Confidence intervals

We have seen that different samples from one population may have different means. As a sample is used to give an indication of the actual population parameters by making a point estimate, we need to be more specific than '*the population mean is approximately equal to the sample mean*'. Confidence intervals allow us to quantify the interval that the population mean might lie in.

For example, in the data used to explore sample means earlier, the first sample was {23, 42} which had a mean of 32.5. The sample mean provides a **point estimate** for the actual population mean of 21.8. In this section, it will be shown that it is possible to use this sample to calculate a 90% confidence interval of (16.9, 48.1). Notice that the sample mean of 32.5 is in the middle of this interval. The lower limit of 16.9 is 15.6 units below the sample mean and the upper limit of 48.1 is 15.6 units above the sample mean. The 90% confidence interval means that in 90% of the possible samples, the population mean will be in the calculated confidence interval. This will be demonstrated later.

Note: this is not the same as saying that there is a 90% chance that the population mean is in the confidence interval. The population mean is an actual value, in this case 21.8. Therefore, for any given sample, the confidence interval either contains the mean (which it does in this case) or it doesn't contain the mean.

12.4.2 Calculation of confidence intervals

The **confidence interval** for μ can be calculated using the formula $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$. Notice that the interval will be $z \frac{s}{\sqrt{n}}$ on either side of the sample mean $\bar{x}$, where s is the sample standard deviation, n is the sample size and z is the z score for the level of confidence we are interested in.

Earlier in this chapter, we observed that the middle 95% of the standard normal distribution lies within 1.96 standard deviations of the mean. That is $P(-1.96 \leq Z \leq 1.96) = 0.95$. Therefore, if we want to determine a 95% confidence interval, we use $z = 1.96$. (Remember that the distribution of the sample means will have a standard deviation of $\frac{s}{\sqrt{n}}$).

Similarly, z scores of $z = 1.645$ and $z = 2.58$ are used for the 90% and 99% confidence intervals respectively.

Returning to the sample data that we used for investigating sample means. Remember this population has a mean of 21.8. For each sample it is possible to calculate a 90% confidence interval (although not very accurately because of the small sample). As it is a 90% confidence interval, we use $z = 1.645$. For the first sample {23, 42}, $\bar{x} = 32.5$, $s = 13.435$ and $n = 2$. To calculate the lower limit:

$$\begin{aligned}\bar{x} - z \frac{s}{\sqrt{n}} &= 32.5 - 1.645 \times \frac{13.435}{\sqrt{2}} \\ &= 32.5 - 15.6 \\ &= 16.9\end{aligned}$$

To calculate the upper limit:

$$\begin{aligned}\bar{x} + z \frac{s}{\sqrt{n}} &= 32.5 + 1.645 \times \frac{13.435}{\sqrt{2}} \\ &= 32.5 + 15.6 \\ &= 48.1\end{aligned}$$

This sample estimates that the population mean will lie in the interval (16.9, 48.1). Notice that this interval does contain the population mean of 21.8. In a similar fashion, the 90% confidence for each of the 10 possible samples can be calculated. They are recorded in the table shown.

Data points	$\bar{x}$	s	90% confidence interval	Contains Population Mean
23, 42	32.5	13.435	(16.9, 48.1)	Yes
23, 12	17.5	7.778	(8.5, 26.5)	Yes
23, 21	22	1.414	(20.4, 23.6)	Yes
23, 11	17	8.485	(7.1, 26.9)	Yes
42, 12	27	21.213	(2.3, 51.7)	Yes
42, 21	31.5	14.849	(14.2, 48.8)	Yes
42, 11	26.5	21.920	(1, 52)	Yes
12, 21	16.5	6.364	(9.1, 23.9)	Yes
12, 11	11.5	0.707	(10.7, 12.3)	No
21, 11	16	7.071	(7.8, 24.2)	Yes

As you can see, 90% (9 out of 10) of the calculated confidence intervals contain the population mean. This is the meaning of the 90% confidence interval.

Notice that the size of the confidence intervals varies between samples. The smaller the sample standard deviation, the smaller the confidence interval for the sample.

12.4.3 Verifying the formulae

We have learnt that when the sample size is large, that is $n > 30$, the distribution of $\bar{x}$ is normally distributed with a mean of $\mu_{\bar{x}} = \mu$ and a standard deviation of $\frac{\sigma}{\sqrt{n}}$. As a sample is normally selected in order to estimate the population mean and standard deviation, the best estimates for μ and σ are $\bar{x}$ and s respectively. We know that for normal distributions, $z = \frac{x - \mu}{\sigma}$. This means that to find the upper and lower values of z ,

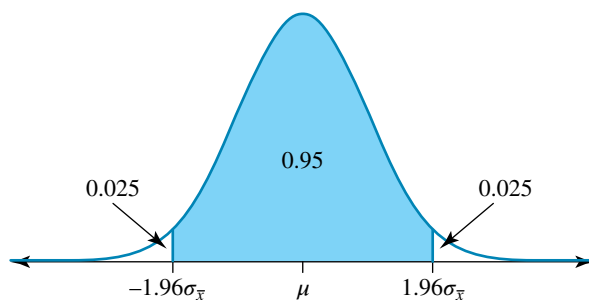
$$z = \frac{x \pm \bar{x}}{\frac{s}{\sqrt{n}}}$$

$$x \pm \bar{x} = \frac{s}{\sqrt{n}} z$$

$$x = \bar{x} \pm \frac{s}{\sqrt{n}} z$$

Rearranging gives $x = \bar{x} \pm z \frac{s}{\sqrt{n}}$.

For a 95% confidence interval, 95% of the distribution is in the middle area of the distribution. This means that the tails combined contain 5% of the distribution (2.5% each). The z score for this distribution is 1.96.



WORKED EXAMPLE 3

After surveying the 20 people in your class, you find that they plan to spend an average of \$4.53 on their lunch today. The standard deviation for your sample was \$0.23. Use these statistics to state a 95% confidence interval for the average amount that will be spent on lunch today.

THINK

- There are 20 people on the class. This is the sample size.
\$4.53 is spent on lunch, this is the sample mean.
The sample standard deviation is \$0.23.
- For a 95% confidence interval, $z = 1.96$.

WRITE

$$n = 20$$

$$\bar{x} = 4.53$$

$$s = 0.23$$

$$z = 1.96$$

3. The confidence interval is $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$.

Use technology to calculate the values of $\bar{x} - z \frac{s}{\sqrt{n}}$

and $\bar{x} + z \frac{s}{\sqrt{n}}$.

4. Identify the 95% confidence interval.

5. State the answer.

$$\bar{x} - z \frac{s}{\sqrt{n}} = 4.53 - 1.96 \times \frac{0.23}{\sqrt{20}} = 4.43$$

$$\bar{x} + z \frac{s}{\sqrt{n}} = 4.53 + 1.96 \times \frac{0.23}{\sqrt{20}} = 4.63$$

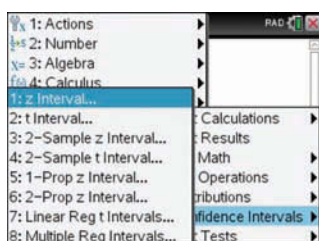
$$\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right) = (4.43, 4.63)$$

The 95% confidence interval for the amount planned to be spent by a student on their lunch is (\$4.43, \$4.63).

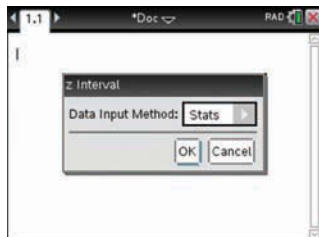
TI | THINK

- On a Calculator page, press MENU then select
6: Statistics
6: Confidence Intervals

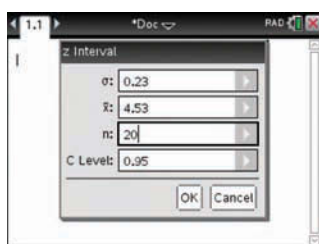
WRITE



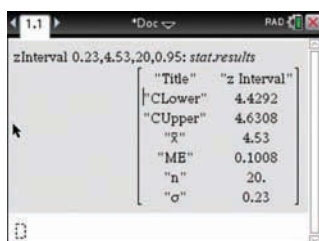
- To set the appropriate data input method, select Stats
Press the OK button.



- Complete the entry lines
0.23
4.53
20
0.95



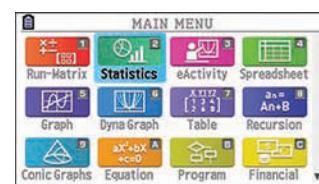
- Press the ENTER or OK button.
The answer appears on the screen.



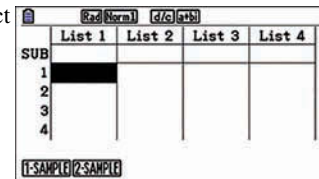
CASIO | THINK

- On a Main Menu screen, select Statistics

WRITE



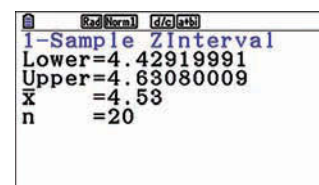
- On a Statistics screen, select
INTR z
1-SAMPLE



- Complete the entry lines
Variable
0.95
0.23
4.53
20



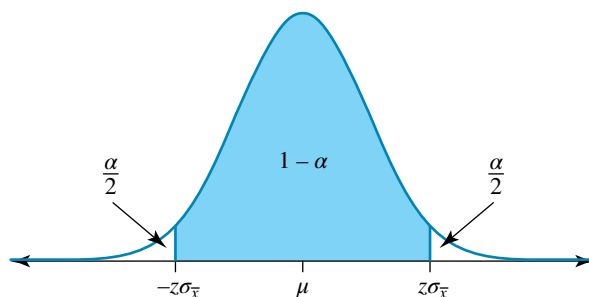
- Press the EXE button.
The answer appears on the screen.



We have previously learnt that for a 90% or 99% confidence interval a z scores of 1.645 or 2.58 are used respectively.

In general, the inverse normal can be used to determine the z score to be used in the confidence interval calculation. Remember that the area under the curve is 1. The confidence interval is located in the middle of the distribution and the area of the right hand tail is used to determine the number of standard deviations from the mean. Because the curve is symmetric, the area of each tail will be equal. If the total tail area is α , then the middle area would be

$1 - \alpha$ and the area of each tail is $\frac{\alpha}{2}$. As the middle area is $1 - \alpha$, we can talk about a $1 - \alpha$ confidence interval. As the area of each tail is $\frac{\alpha}{2}$, the z score that has a tail area of $\frac{\alpha}{2}$ is used.



WORKED EXAMPLE 4

Arabella samples 102 people and finds, with a standard deviation of 0.8, that on average they eat 5.2 cups of vegetables per day. Use these statistics to state an 85% confidence interval for the average daily vegetable consumption.

THINK

- There are 102 people in the sample.
This is the sample size.
The sample mean is 5.2.
The sample standard deviation is 0.8.
- For a 85% confidence interval, determine the area of each tail.
- Use the inverse normal to determine the z score with a tail area of 0.075
- The confidence interval is $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right)$.

Use technology to calculate the values of $\bar{x} - z \frac{s}{\sqrt{n}}$
and $\bar{x} + z \frac{s}{\sqrt{n}}$.
- Identify the 85% confidence interval.
- State the answer.

WRITE

$$\begin{aligned} n &= 102 \\ \bar{x} &= 5.2 \\ s &= 0.8 \end{aligned}$$

85% confidence interval

$$1 - \alpha = 0.85$$

$$\alpha = 0.15$$

$$\frac{\alpha}{2} = 0.075$$

15% will be in the tails

7.5% in each tail

$$z_{0.075} = 1.44$$

$$\begin{aligned} \bar{x} - z \frac{s}{\sqrt{n}} &= 5.2 - 1.44 \times \frac{0.8}{\sqrt{102}} \\ &= 5.09 \end{aligned}$$

$$\begin{aligned} \bar{x} + z \frac{s}{\sqrt{n}} &= 5.2 + 1.44 \times \frac{0.8}{\sqrt{102}} \\ &= 5.31 \end{aligned}$$

$$\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right) = (5.09, 5.31)$$

The 85% confidence interval for the number of cups of vegetables consumed per day is (5.09, 5.31).

When deciding whether to calculate, for example, a 95% or 99% confidence interval, it may seem logical to use the 99% confidence interval as we are more confident that the population mean will lie in the calculated interval. However, this increase in confidence also results in a larger interval. As the confidence interval is $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$, the size of the interval will be $2z \frac{s}{\sqrt{n}}$. In practice, this interval may be quite large and therefore while it is more likely to contain the population mean, the estimate lacks precision. It is possible to reduce the size of the interval without a reduction in confidence by increasing the sample size. A larger value of n will mean that $2z \frac{s}{\sqrt{n}}$ will be smaller and therefore the estimate will be more precise.

The value $\frac{s}{\sqrt{n}}$ is known as the standard error for a confidence interval. Increasing the sample size will decrease the standard error. The value $z \frac{s}{\sqrt{n}}$ is called the **margin of error**. The larger the level of confidence in the interval, the larger the value of z and therefore the margin of error. However, reducing the standard error by increasing n allows for a high level of confidence in the interval with a lower margin of error.

WORKED EXAMPLE 5

In Worked example 3, the average amount spent on lunch was between \$4.43 and \$4.63. Determine the sample size needed to reduce the interval to $\pm \$0.05$ for a 95% confidence interval.

THINK

- The confidence interval formula is $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$, this means that we need $z \frac{s}{\sqrt{n}} = 0.05$.

- Although the sample standard deviation will change with a larger sample, the current value is the best estimate that we have.

- For a 95% confidence interval, $z = 1.96$

- Solve $z \frac{s}{\sqrt{n}} = 0.05$ for n .

WRITE

$$z \frac{s}{\sqrt{n}} = 0.05$$

$$s = 0.23$$

$$z = 1.96$$

$$z \frac{s}{\sqrt{n}} = 0.05$$

$$1.96 \times \frac{0.23}{\sqrt{n}} = 0.05$$

$$1.96 \times \frac{0.23}{0.05} = \sqrt{n}$$

$$\sqrt{n} = 9.016$$

$$n = (9.016)^2$$

$$n = 81.288256$$

At least 82 people would need to be surveyed.

Exercise 12.4 Confidence intervals

Technology active

1. **WE3** Of the 75 Queenslanders you surveyed, the average amount spent on holidays this year was \$2 314 with a standard deviation of \$567. Use these statistics to state a 95% confidence interval, to the nearest dollar, for the average amount spent on holidays by Queenslanders.
2. After surveying 30 swimmers as they entered the local swimming complex, you found that the average distance that they intended to swim was 1.2 km. The sample standard deviation was 0.5 km. Use these statistics to state a 95% confidence interval for the average distance people were intending to swim that day.
3. **WE4** James samples 116 people and finds, with a standard deviation of \$537, that their average car value is \$23 456. Use these statistics to state a 99% confidence interval for the average car value for the population.



4. Charles samples 95 people and finds, with a standard deviation of 5.8 g, that on average, 25.7 g of chocolate are consumed per person per day. Use these statistics to determine a 90% confidence interval for an estimate for the amount of chocolate consumed per person per day.
5. **WE5** For question 1, determine the sample size needed to reduce the interval to $\pm \$100$ at the 95% level of confidence.
6. For question 2, determine the sample size needed to reduce the interval to ± 0.1 km at the 95% level of confidence.
7. You are auditing a bank. From a sample of 50 cash deposits, the mean and standard deviation are calculated to be \$203.45 and \$43.32 respectively.
 - a. Determine a 95% confidence interval for the average cash deposit amount.
 - b. Determine a 90% confidence interval for the average cash deposit amount.
 - c. Calculate the sample size needed to reduce the interval to $\pm \$2$ for a 95% confidence interval.



8. A sample of 40 AA batteries were tested to determine their average lifetime. It was found that they lasted an average of 2314 minutes with a standard deviation of 243 minutes.
 - a. Determine a 95% confidence interval for the average battery life.
 - b. Determine a 99% confidence interval for the average battery life.
 - c. Calculate the sample size needed to reduce the interval to ± 50 minutes for a 95% confidence interval.
9. For her class assignment, Holly records the time to travel to school. She records the times for 30 days and finds the average trip time is 24.6 minutes with a standard deviation of 7.6 minutes.
 - a. Determine a 95% confidence interval for the average time that Holly will take to travel to school.
 - b. Determine a 90% confidence interval for the average time that Holly will take to travel to school.
 - c. Calculate the sample size needed to reduce the interval to ± 2 minutes for a 95% confidence interval.
 - d. Calculate the sample size needed to reduce the interval to ± 2 minutes for a 90% confidence interval.

12.5 Applications of confidence intervals

12.5.1 Using confidence intervals to estimate other confidence intervals

If a sample's confidence interval for a population parameter is known, it is possible to estimate the confidence interval at a different level of significance.

WORKED EXAMPLE 6

If we know that a 99% confidence interval for the average blog length is between 1200 and 2000 words, calculate a 95% confidence interval for the average blog length.



THINK

1. As we are talking about the 99% confidence interval and the 95% confidence interval, 2 different z values will be used. To differentiate between them, call them $z_{0.005}$ and $z_{0.025}$ respectively. The number 0.005 represents the area of one tail for the 99% confidence interval. The number 0.025 represents the area of one tail for the 95% confidence interval.

2. The confidence interval formula is $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$. The mean will be in the centre of the distribution.

3. Use the upper limit of the confidence interval, $\bar{x} + z \frac{s}{\sqrt{n}}$, to calculate $z \frac{s}{\sqrt{n}}$.

4. As this is a 99% confidence interval, $z_{0.005} = 2.58$, solve for $\frac{s}{\sqrt{n}}$.

5. For a 95% confidence interval, $z_{0.025} = 1.96$. Use this value to calculate the confidence interval $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$.

6. State the 95% confidence interval.

WRITE

$$z_{0.005} = 2.58$$

$$z_{0.025} = 1.96$$

$$\begin{aligned}\bar{x} &= \frac{1200 + 2000}{2} \\ &= 1600\end{aligned}$$

$$2000 = \bar{x} + z_{0.005} \frac{s}{\sqrt{n}}$$

$$2000 = 1600 + z_{0.005} \frac{s}{\sqrt{n}}$$

$$z_{0.005} \frac{s}{\sqrt{n}} = 400$$

$$2.58 \times \frac{s}{\sqrt{n}} = 400$$

$$\frac{s}{\sqrt{n}} = \frac{400}{2.58}$$

$$\bar{x} - z \frac{s}{\sqrt{n}} = 1600 - 1.96 \times \frac{400}{2.58}$$

$$= 1600 - 304$$

$$= 1296$$

$$\bar{x} + z \frac{s}{\sqrt{n}} = 1600 + 1.96 \times \frac{400}{2.58}$$

$$= 1600 + 304$$

$$= 1904$$

The 95% confidence interval is between 1296 and 1904 words.

12.5.2 Simulating sampling

When selecting a random sample from a population, every member of the population needs to have the same probability of being selected. This means that sampling options such as surveying a class (sample) to get information about an entire year group (population) does not result in a random sample as every student in the cohort does not necessarily have the same probability of being selected. Options like spinners and random number generators can be used to select members of a sample to ensure that it is random.

Sometimes, to explore how a particular population distribution behaves, it is desirable to model selecting a sample. Software like Excel, Geogebra or even your graphics calculators can be used to model the random sampling and to produce the histograms of the distribution.

Normal distributions

If the population is believed to be normally distributed, then the means of samples of any size n will be normally distributed, with the average value of the sample means being the same as that of the population. Randomly sampling from a normal distribution can be modelled by using the inverse normal option.

WORKED EXAMPLE 7

A population is normally distributed with a mean of 50 and a standard deviation of 5.

- a.** Use technology to model the selection of 10 random samples of size 20.
- b.** For each sample, use technology to calculate the 90% confidence interval for the population mean.
- c.** Use technology to sketch the population distribution and the distribution of sample means.
- d.** Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

THINK

- a. 1.** To model creating a sample of size 20, use a random number generator to generate 20 random numbers between 0 and 1. In Excel, =RAND() will result in a number between 0 and 1.

WRITE

a.	Random #
	0.85583
	0.553912
	0.567864
	0.566725
	0.104497
	0.83723
	0.412178
	0.980235
	0.283082
	0.666841
	0.811105
	0.659588
	0.637259
	0.275291
	0.665903
	0.824447
	0.619637
	0.693842
	0.324964
	0.505637



2. The random numbers generated can be thought of as an area under the normal curve between $-\infty$ and the element chosen to be in the sample. Use the inverse normal option to convert the random number to an element in a normal distribution with a mean of 50 and a standard deviation of 5. In Excel, $\text{NORM.INV}(a, \mu, \sigma) = b$ will return the population value for a population with mean μ and standard deviation σ where $P(Z \leq b) = a$. Generate the sample values.

Random #	InvNormal
0.85583	55.30885
0.553912	50.67776
0.567864	50.85469
0.566725	50.84022
0.104497	43.71832
0.83723	54.91569
0.412178	48.89027
0.980235	60.29314
0.283082	47.13146
0.666841	52.15603
0.811105	54.40988
0.659588	52.05669
0.637259	51.75571
0.275291	47.01555
0.665903	52.14314
0.824447	54.66222
0.619637	51.52264
0.693842	52.53386
0.324964	47.73068
0.505637	50.07065

3. Calculate the sample mean of these new values.

$$\bar{x} = \frac{55.30885 + 50.67776 + \dots + 50.07065}{20}$$

$$\approx 51.43$$

4. For a 90% confidence interval, $z_{0.05} = 1.645$. Calculate the confidence interval $\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}}\right)$. As the population standard deviation is known, use that. Use technology to calculate the values.

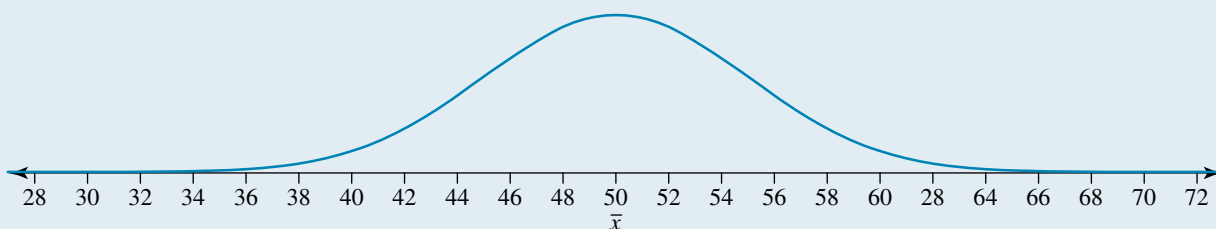
$$\begin{aligned}\bar{x} - z \frac{s}{\sqrt{n}} &= 51.43 - 1.645 \times \frac{5}{\sqrt{20}} \\ &= 51.43 - 1.84 \\ &= 49.59 \\ \bar{x} + z \frac{s}{\sqrt{n}} &= 51.43 + 1.645 \times \frac{5}{\sqrt{20}} \\ &= 51.43 + 1.84 \\ &= 53.27\end{aligned}$$

The confidence interval is (49.59, 53.27).

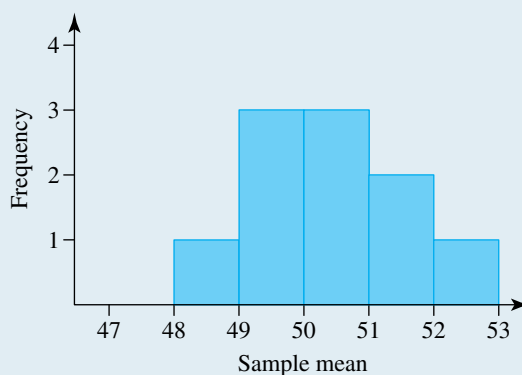
- b. Repeat steps a. 1. through to a. 4. to generate 9 more samples and calculate the confidence intervals.

Sample	Mean	Lower limit	Upper limit
1	51.43437	49.6008	53.26795
2	49.08482	47.25124	50.91839
3	49.30787	47.4743	51.14145
4	51.66994	49.83637	53.50352
5	50.52729	48.69371	52.36087
6	48.0063	46.17272	49.83987
7	52.59666	50.76308	54.43023
8	50.53343	48.69985	52.367
9	50.84502	49.01145	52.6786
10	49.86264	48.02907	51.69622

- c. 1. As the normal distribution is a known distribution, software like the probability option of Geogebra can be used to create a graph of the normal distribution with a mean of 50 and a standard deviation of 5.



2. Use technology to sketch the distribution of the sample means. For example, this can be done using the statistics option in Geogebra.



- d. Identify the samples where the 90% confidence intervals contain the population mean of 50. Comment on the distributions using your understanding of confidence intervals.
- d. Samples 1, 2, 3, 4, 5, 8, 9 and 10 contain the population mean. As this is a 90% confidence interval, 90% of all possible samples should contain the population mean. From the 10 samples taken, 8 of them contain the population mean, a result close to the expected one. The average of all of the sample means is 50.39 which is very close to the population mean of 50. While the mean of the distribution of sample means is very close to the mean of the population, the spread of the distribution is smaller than that of the original population.

Other distributions

If the population distribution is not normal, it is still possible to simulate the selection of random samples. If the sample size is large enough (usually at least 30), then the distribution of the sample means will still be normal, so the formula for confidence intervals can still be used.

WORKED EXAMPLE 8

A population is distributed so that the $P(X \leq b) = \int_0^b 0.5e^{-0.5x} dx$.

- a. The population mean can be found by calculating $\mu = \int_0^{\infty} 0.5xe^{-0.5x} dx$. Determine the population mean.

- b. The population variance can be found by calculating $\sigma^2 = \int_0^{\infty} 0.5(x - \mu)^2 e^{-0.5x} dx$.

Determine the population standard deviation.

- c. Use technology to model the selection of 10 random samples of size 40.
- d. For each sample, use technology to calculate the 90% confidence interval for the population mean.
- e. Use technology to sketch the population distribution and the distribution of sample means.
- f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

THINK

- a. 1. Use integration by parts to determine the integral. Let $u = x$ and $\frac{dv}{dx} = 0.5e^{-0.5x}$.

Determine $\frac{du}{dx}$ and v .

2. Determine the integral using the formula

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

3. State the population mean.

- b. 1. Using the given formula determine the variance. Substitute $\mu = 2$ into the integrand.

2. Use integration by parts to determine the integral. Let $u = (x - 2)^2$ and $\frac{dv}{dx} = 0.5e^{-0.5x}$

3. Determine the integral.

WRITE

a. $u = x$ $\frac{dv}{dx} = 0.5e^{-0.5x}$
 $\frac{du}{dx} = 1$ $v = -e^{-0.5x}$

$$\begin{aligned}\mu &= \int_0^{\infty} 0.5xe^{-0.5x} dx \\ &= \int u \frac{dv}{dx} dx \\ &= uv - \int v \frac{du}{dx} dx \\ &= [-xe^{-0.5x}]_0^{\infty} - \int_0^{\infty} (-e^{-0.5x}) dx \\ &= [-xe^{-0.5x}]_0^{\infty} + \int_0^{\infty} e^{-0.5x} dx \\ &= 0 + [-2e^{-0.5x}]_0^{\infty} \\ &= 0 + 2 \\ &= 2\end{aligned}$$

$$\mu = 2$$

b. $\sigma^2 = \int_0^{\infty} 0.5(x - \mu)^2 e^{-0.5x} dx$
 $= \int_0^{\infty} 0.5(x - 2)^2 e^{-0.5x} dx$
 $u = (x - 2)^2$ $\frac{dv}{dx} = 0.5e^{-0.5x}$
 $\frac{du}{dx} = 2(x - 2)$ $v = -e^{-0.5x}$

$$\begin{aligned}\sigma^2 &= \int_0^{\infty} 0.5(x - 2)^2 e^{-0.5x} dx \\ &= \int u \frac{dv}{dx} dx \\ &= uv - \int v \frac{du}{dx} dx \\ &= [-(x - 2)^2 e^{-0.5x}]_0^{\infty} - \int_0^{\infty} -2(x - 2) e^{-0.5x} dx \\ &= 0 + 2^2 e^0 + 2 \int_0^{\infty} (x - 2) e^{-0.5x} dx \\ &= 4 + 2 \int_0^{\infty} (x - 2) e^{-0.5x} dx\end{aligned}$$



4. Use integration by parts to complete the integral.

$$\begin{aligned} u &= x - 2 & \frac{dv}{dx} &= e^{-0.5x} \\ \frac{du}{dx} &= 1 & v &= -2e^{-0.5x} \end{aligned}$$

$$\begin{aligned} \sigma^2 &= 4 + 2 \left(\left[-2(x-2)e^{-0.5x} \right]_0^\infty - \int_0^\infty -2e^{-0.5x} dx \right) \\ &= 4 + 2 \left(0 + 2(-2)(1) + \left[-4e^{-0.5x} \right]_0^\infty \right) \\ &= 4 + 2(-4 + 0 + 4) \\ &= 4 \end{aligned}$$

5. The standard deviation is denoted by σ . Determine the standard deviation by taking the square root of the variance.

The standard deviation $\sigma = \sqrt{4} = 2$.

- c. 1. Use a random number generator to generate 40 random numbers to represent 40 possible random probabilities. In Excel the command =RAND() will achieve this. *Note:* not all 40 values have been listed.

- c. Sample Numbers generated (a total of 40 numbers are generated for each sample):

Random #
0.038742
0.42023
0.283778
0.430065
0.654831
0.141399
0.923711
0.990582
0.240471
0.231727
⋮

2. Let a particular random number generated be a . The random number generated could be thought of $P(x \leq b) = a$. This means that the area under the curve $0.5e^{-0.5x}$ between $x = 0$ and $x = b$ is a .
Solve $\int_0^b 0.5e^{-0.5x} dx = a$ for b .

$$\begin{aligned} a &= \int_0^b 0.5e^{-0.5x} dx \\ &= \left[-e^{-0.5x} \right]_0^b \\ &= -e^{-0.5b} + e^0 \\ &= 1 - e^{-0.5b} \\ 1 - a &= e^{-0.5b} \\ -0.5b &= \ln(1 - a) \\ b &= -2 \ln(1 - a) \end{aligned}$$

3. Use technology to calculate $-2 \ln(1 - a)$ for each random number a that is generated. These values represent the value chosen to form the sample.

Sample data (a total of 40 numbers are generated for each sample).

Random #	Sample
0.038742	0.079024
0.42023	1.090246
0.283778	0.66753
0.430065	1.124464
0.654831	2.127445
0.141399	0.304903
0.923711	5.146463
0.990582	9.330223
0.240471	0.550112
0.231727	0.527219
$\vdots$	$\vdots$

- d. 1. In order to determine the confidence interval, the mean must be determined. Calculate the sample mean of these new values.

$$\begin{aligned} \text{d. } \bar{x} &= \frac{0.079024 + 1.090246 + 0.66753 + \dots}{40} \\ &\approx 2.27 \end{aligned}$$

2. For a 90% confidence interval, $z_{0.05} = 1.64$. Calculate the confidence interval

$$\left(\bar{x} - z \frac{s}{\sqrt{n}}, \bar{x} + z \frac{s}{\sqrt{n}} \right).$$

As the population standard deviation of 2 is known, use that. Use technology to calculate the values.

$$\begin{aligned} \bar{x} - z \frac{s}{\sqrt{n}} &= 2.27 - 1.64 \times \frac{2}{\sqrt{40}} \\ &= 2.27 - 0.52 \\ &= 1.75 \end{aligned}$$

$$\begin{aligned} \bar{x} + z \frac{s}{\sqrt{n}} &= 2.27 + 1.64 \times \frac{2}{\sqrt{40}} \\ &= 2.27 + 0.52 \\ &= 2.79 \end{aligned}$$

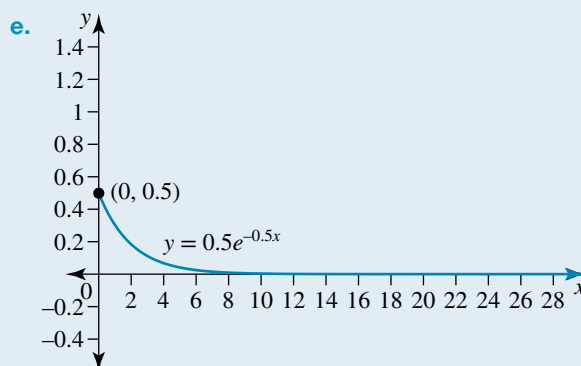
The confidence interval is (1.75, 2.79).



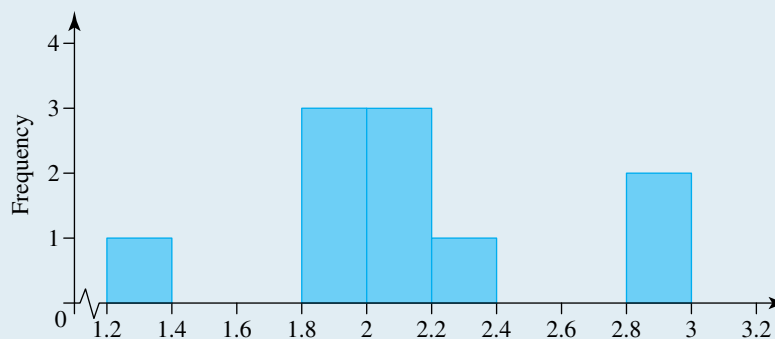
3. Repeat steps c. 1. through to d. 2. to generate 9 more samples and calculate the confidence intervals.

Sample	Mean	Lower limit	Upper limit
1	2.271128	1.752515	2.789742
2	2.100404	1.581791	2.833835
3	2.18984	1.671226	2.92327
4	1.821862	1.303248	2.555292
5	1.366244	0.84763	2.099674
6	2.998408	2.479794	3.731838
7	2.156755	1.638142	2.890186
8	2.911023	2.392409	3.644453
9	1.82039	1.301776	2.55382
10	1.889885	1.371272	2.623316

- e. 1. Use technology to sketch the probability density function $0.5e^{-0.5x}$.



2. Use technology to sketch the distribution of the sample means. For example, this can be done using the statistics option in Geogebra.



- f. Identify the samples where the 90% confidence intervals contain the population mean of 2. Comment on the distributions using your understanding of confidence intervals.
- f. Confidence intervals from samples 1, 2, 3, 4, 5, 7, 9 and 10 contain the population mean. As this is a 90% confidence interval, 90% of all possible samples should contain the population mean. From the 10 samples taken, 8 of them contain the population mean, a result close to the expected one. The average of all of the sample means is 2.15 which is very close to the population mean of 2. While the population is negatively skewed, the distribution of the sample means is beginning to look like the normal distribution with a mean of 2.

12.5.3 Confirming normality

In Worked examples 7 and 8, sketches of the distributions of sample means were used to demonstrate that the distributions are approximately normal. It is possible to use a method called normal probability plots to obtain a scatter plot that should be roughly linear if the data is normal.

To determine if a set of data is roughly normal, rank order the data. The cumulative probability, p_i associated with each rank ordered point of data, x_i , can be calculated using $p_i = \frac{i - 0.5}{n}$, where i is the rank order and n is the number of data points. The associated z value, z_i , is then calculated for each p_i . If the data is approximately normal, graphing z_i against x_i will be roughly linear if the x_i data is approximately normal.

WORKED EXAMPLE 9

Use a normal probability plot to confirm that the sample means generated in Worked example 7 are approximately normally distributed.

THINK

1. The sample means need to be rank ordered. Sort the means from smallest to largest and allocate a rank order value to each.

WRITE

Means (rank ordered)	Rank
48.0063	1
49.08482	2
49.30787	3
49.86264	4
50.52729	5
50.53343	6
50.84502	7
51.4343	8
51.66994	9
52.59666	10

2. Calculate the cumulative probability for each point of data using $p_i = \frac{i - 0.5}{n}$.

Means (x_i)	Rank (i)	p_i
48.0063	1	0.05
49.08482	2	0.15
49.30787	3	0.25
49.86264	4	0.35
50.52729	5	0.45

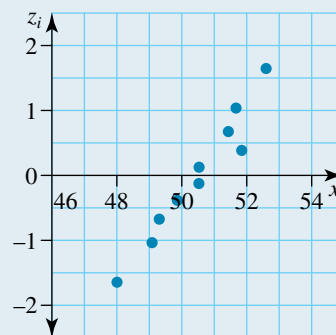
Means (x_i)	Rank (i)	p_i
50.53343	6	0.55
50.84502	7	0.65
51.4343	8	0.75
51.66994	9	0.85
52.59666	10	0.95

3. Calculate the standard inverse normal for each p_i value. This can be completed in Excel using $\text{=NORM.INV}(p_i, \mu, \sigma)$.

$\text{=NORM.INV}(0.05, 0, 1) = -1.64485$
 $\text{=NORM.INV}(0.15, 0, 1) = -1.03643$
 $\vdots$

Means (x_i)	Rank (i)	p_i	z_i
48.0063	1	0.05	-1.64485
49.08482	2	0.15	-1.03643
49.30787	3	0.25	-0.67449
49.86264	4	0.35	-0.38532
50.52729	5	0.45	-0.12566
50.53343	6	0.55	0.125661
50.84502	7	0.65	0.38532
51.4343	8	0.75	0.67449
51.66994	9	0.85	1.036433
52.59666	10	0.95	1.644854

4. Sketch z_i against x_i . If the x_i values are approximately normal, then the scatterplot should be approximately linear.



As the scatterplot is approximately linear, the distribution of sample means is approximately normal.

Exercise 12.5 Applications of confidence intervals

Technology active

- WE6** If we know that a 99% confidence interval for the average television commercial break length is between 2 and 3 minutes, calculate a 95% confidence interval for the average television commercial break length.
- A sample of feature movies has a 90% confidence level for the average running time of (85, 90) minutes. Calculate a 99% confidence interval for the average running time length.
- If the average song length of the top 100 songs in an online music store has a 95% confidence interval of (215.171, 238.695) minutes, calculate the 99% confidence interval.



- On a particular day, the average wait time for a ride has a 95% confidence interval of (9.11, 10.44) minutes. Calculate the 90% confidence interval for the sample.
- WE7 & WE9** A population is normally distributed with a mean of 16 and a standard deviation of 2.
 - Use technology to model the selection of 10 random samples of size 20.
 - For each sample, use technology to calculate the 90% confidence interval for the population mean.
 - Use technology to sketch the population distribution and the distribution of sample means.
 - Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
 - Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.
- A population is normally distributed with a mean of 72 and a standard deviation of 7.
 - Use technology to model the selection of 10 random samples of size 20.
 - For each sample, use technology to calculate the 90% confidence interval for the population mean.
 - Use technology to sketch the population distribution and the distribution of sample means.
 - Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
 - Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.
- X is the height of Emperor penguins and $X \sim N(122, 53.29)$.
 - Use technology to model the selection of 10 random samples of size 15.
 - For each sample, use technology to calculate the 95% confidence interval for the population mean.

- c. Use technology to sketch the population distribution and the distribution of sample means.
 - d. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
 - e. Use a normal probability plot to confirm that the sample means generated are approximately normally distributed.
8. **WE8** A population is distributed so that the $P(X \leq b) = \int_0^b 4e^{-4x} dx$.
- a. The population mean can be found by calculating $\mu = \int_0^\infty 4xe^{-4x} dx$. Determine the population mean.
 - b. The population variance can be found by calculating $\sigma^2 = \int_0^\infty 4(x - \mu)^2 e^{-4x} dx$. Determine the population standard deviation.
 - c. Use technology to model the selection of 10 random samples of size 40.
 - d. For each sample, use technology to calculate the 90% confidence interval for the population mean.
 - e. Use technology to sketch the population distribution and the distribution of sample means.
 - f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
9. A population is distributed so that the $P(X \leq b) = \int_0^b 0.25e^{-0.25x} dx$.
- a. The population mean can be found by calculating $\mu = \int_0^\infty 0.25xe^{-0.25x} dx$. Determine the population mean.
 - b. The population variance can be found by calculating $\sigma^2 = \int_0^\infty 0.25(x - \mu)^2 e^{-0.25x} dx$. Determine the population standard deviation.
 - c. Use technology to model the selection of 10 random samples of size 40.
 - d. For each sample, use technology to calculate the 90% confidence interval for the population mean.
 - e. Use technology to sketch the population distribution and the distribution of sample means.
 - f. Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.

12.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au

Simple familiar

1. 1500 students are enrolled in an online University course. In the first quiz, the average score (out of a possible 20) was 15.75 with a standard deviation of 3.1.
 - a. If samples of size 20 are selected, calculate the mean and standard deviation for the distribution of $\bar{X}$.
 - b. If the sample size was increased to 30, calculate the mean and standard deviation for the distribution of $\bar{X}$.
2. A population has a mean of 58.9 and a standard deviation of 5.2.
 - a. If samples of size 20 are selected, calculate the mean and standard deviation for the distribution of $\bar{X}$.
 - b. Calculate the sample size needed to reduce the standard deviation to less than 1.
3. In a population of size 20 000, the mean score is 72.3 with a variance of 33. Samples of size 40 are selected from this population.
 - a. Calculate the mean value of the distribution of sample means.
 - b. Calculate the standard deviation of the distribution of sample means.
4. After sampling his local farm's tomatoes, Giovanni, finds that a 90% confidence interval for the mass of one tomato is (113 g, 117 g). What does this mean?

5. A population has a mean of 19 and a standard deviation of 4. Determine the size of the sample needed if the standard deviation of the sample means is to be less than 0.5.
6. A population has a mean of 89 and a variance of 20. Determine the size of the sample needed if the variance of the sample means is to be less than 2.
7. A population is believed to be normally distributed with a mean of 45 and a standard deviation of 4.3. A sample of 15 has a mean of 39.5. Determine a 99% confidence interval for the population mean.
8. A sample of 38 students has a mean of 45.14 and a standard deviation of 12. Determine a 95% confidence interval for the population mean.
9. Breanna is trying to decide on a mobile phone plan. She needs to estimate the number of texts that she makes per month to assist with this decision. Over the year, she has made an average of 150 texts per month with a standard deviation of 15.1. Determine a 95% confidence interval for the number of texts that she can expect to make each month.
10. If a 99% confidence interval for the average cost of a tray of mangoes is (\$17.75, \$19.85) determine a 95% confidence interval for the cost of a tray of mangoes.
11. The average time to run 3 km has a 95% level of confidence of (10, 13) minutes. Determine the 90% confidence interval.
12. If a 95% confidence interval for the temperature of the water at Caloundra in December is (24.1°C, 27.4°C) determine a 99% confidence interval for the average temperature.

Complex familiar

13. In 2018, 52704 runners completed the New York City Marathon with an average time of 4h:38m:24s and standard deviation of 0h:58m:03s. Determine a 95% confidence interval for the average completion time of the New York City Marathon.
14. Michael is recording data about the amount of time it takes his employees to solve a help desk ticket. After analysing 100 tickets, he finds that a confidence interval for the average time is (195.3, 272.6) minutes. If the standard deviation of his sample is 150 minutes, determine the type of confidence interval Michael used in his investigation.
15. Marina records the masses of newborn babies born during her shifts in January. She uses the sample standard deviation of 1.23 kg to calculate a 95% confidence interval of (3.20 kg, 3.62 kg). Determine the number of babies in her sample.
16. From a sample of 30 students, a 95% confidence interval is calculated to be (18.5, 52.7). When an additional 20 students are included in the sample, the new sample mean is increased to 37.3. If the population standard deviation was used in the calculations, determine a 99% confidence interval for the population mean using the entire sample.

Complex unfamiliar

17. A continuous random variable is said to have a uniform distribution on the interval $[A, B]$ if the

$$\text{probability density function is } f(x; A, B) = \begin{cases} \frac{1}{B - A}, & A \leq X \leq B \\ 0, & \text{otherwise} \end{cases}.$$

- a. Sketch the distribution.
- b. Use $\mu = \int_A^B x f(x) dx$ to determine the expected value of the distribution.
- c. Compare the sketch and your calculated expected value and comment on the reasonableness of your solution for μ .
- d. Use $\sigma^2 = \int_A^B (x - \mu)^2 f(x) dx$ to determine the variance of the distribution.

18. Consider the uniform distribution described in question 17 where $A = 3$ and $B = 7$.
- Calculate the mean and standard deviation of the distribution.
 - Use technology to model the selection of 10 random samples of size 20 and determine the 90% confidence interval for each sample.
 - Use technology to sketch the population distribution and the distribution of sample means.
 - Comment on the number of distributions of the sample means and the number of confidence intervals that contain the population mean.
19. In analysing traffic flow, headway is the time between one car passing a fixed point and the next car beginning to pass the point. If X is the headway time between two consecutive variables in minutes, the probability density function of X can be obtained using $f(x) = \begin{cases} 0.25e^{-0.25(x-0.5)}, & x \geq 0.5 \\ 0, & x < 0.5 \end{cases}$.
- Use $\mu = \int_{0.5}^{\infty} x f(x) dx$ to determine the expected population mean.
 - Use $\sigma^2 = \int_{0.5}^{\infty} (x - \mu)^2 f(x) dx$ to determine the population standard deviation.
 - Use technology to model the selection of 10 random samples of size 40 and determine the 90% confidence interval for each sample.
 - Use technology to sketch the normal probability plot for the distribution of sample means. Comment on the normality of the distribution.
20. Gamma functions form interesting skewed distributions. For $\alpha > 0$, the gamma function $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$. A continuous random variable X is said to have a gamma distribution if the probability density function of X is $f(x; \alpha, \beta) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha)}, & x \geq 0 \\ 0, & x < 0 \end{cases}$. Different values of α and β produce different density functions. For this question, assume $\alpha = 2$ and $\beta = 1$.
- Determine $\Gamma(\alpha)$.
 - Use $\mu = \int_0^{\infty} x f(x; \alpha, \beta) dx$ to determine the expected population mean.
 - Use $\sigma^2 = \int_0^{\infty} (x - \mu)^2 f(x; \alpha, \beta) dx$ to determine the expected population standard deviation.
 - Use technology to model the selection of 10 random samples of size 40 and determine the 90% confidence interval for each sample. Note, the Gamma probability density function is an inbuilt function in Excel, and the formula =GAMMA.INV(probability, alpha, beta) can be used.
 - Use technology to sketch the population distribution and the distribution of sample means.
 - Use technology to sketch the normal probability plot for the distribution of sample means. Comment on the normality of the distribution.

study on

Units 3 & 4 Sit exam

Answers

12 Statistical inference

Exercise 12.3 Sample means and simulations

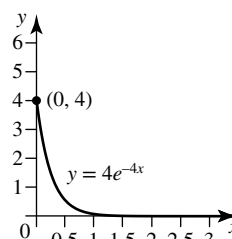
- 51.5 minutes
- 95.9 minutes
- a. $\mu_{\bar{x}} = 600$, $\sigma_{\bar{x}} = 5.48$
b. $\mu_{\bar{x}} = 600$, $\sigma_{\bar{x}} = 3.87$
- a. $\mu_{\bar{x}} = 90.5$, $\sigma_{\bar{x}} = 2.58$
b. $\mu_{\bar{x}} = 90.5$, $\sigma_{\bar{x}} = 1.83$
- Answers will vary. The expected average is 50 but it is likely that results will be different to this.
- $E(\bar{X}) = 7.09$. Theoretical mean is 7
- Answers will vary.
- $E(\bar{X}) = 73$, $sd(\bar{X}) = 2.68$
- $E(\bar{X}) = 73$, $sd(\bar{X}) = 2.19$
- 36
- $E(\bar{X}) = 123$, $sd(\bar{X}) = 8.6$
- $E(\bar{X}) = 123$, $sd(\bar{X}) = 6.8$
- 74
- Answers will vary. The initial distribution should be skewed to the right and the distribution of sample means should be symmetrical.

Exercise 12.4 Confidence intervals

- (\$2186, \$2442)
- (1.02 km, 1.38 km)
- (\$23 327, \$23 585)
- (24.7 g, 26.7 g)
- 124
- 97
- a. (\$191.44, \$215.46)
b. (\$193.37, \$213.53)
c. 1803
- a. (2239, 2389) minutes
b. (2215, 2413) minutes
c. 91
- a. (21.9, 27.3) minutes b. (22.3, 26.9) minutes
c. 56 d. 40

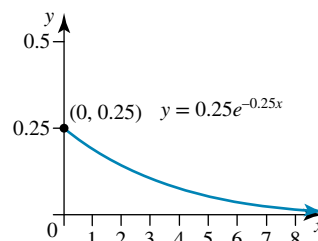
Exercise 12.5 Applications of confidence intervals

- (2.12, 2.88) minutes
- (83.6, 91.4) minutes
- (211.450, 242.416) minutes
- (9.22, 10.33) minutes
- Answers will vary. Distribution of sample means should have a mean of 16. Normal probability plot should be approximately linear.
- Answers will vary. Distribution of sample means should have a mean of 72. Normal probability plot should be approximately linear.
- Answers will vary. Distribution of sample means should have a mean of 122. Normal probability plot should be approximately linear.
- a. 0.25
b. 0.25
c, d. Answers will vary.
e. Population



Distribution of sample means will vary.

- The mean of the sample means should be approximately equal to 0.25 and the distribution of sample means should be approximately normal.
- 4
- 4
- Answers will vary.
- Population

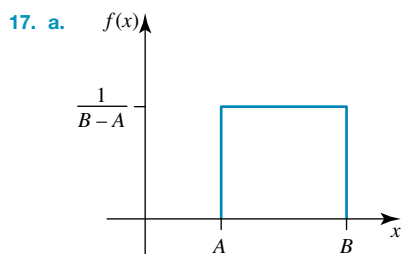


Distribution of sample means will vary.

- The mean of the sample means should be approximately equal to 4 and the distribution of sample means should be approximately normal.

12.6 Review: exam practice

- a. $E(\bar{X}) = 15.75$
 $sd(\bar{X}) = 0.69$
b. $E(\bar{X}) = 15.75$
 $sd(\bar{X}) = 0.57$
- a. $E(\bar{X}) = 58.9$
 $sd(\bar{X}) = 1.16$
b. The sample size needs to be at least 28.
- a. $E(\bar{X}) = 72.3$
b. $Var(\bar{X}) = 0.908$
- A 90% confidence interval means that 90% of the confidence intervals calculated from samples of the same size as Giovanni's will contain the population mean weight.
- 65
- 11
- (36.64, 42.36)
- (41.32, 48.96)
- (141.46, 158.54)
- (\$18.00, \$19.60)
- (10.24, 12.76) minutes
- (23.58°C, 27.92°C)
- (4h:37m:54s, 4h:38m:54s)
- Michael used a 99% confidence interval for his investigation.
- 132
- (19.86, 54.74)



b. $\mu = \frac{B + A}{2}$

- c. As every member of the population has the same chance of being selected, the expected value should be in the middle of the distribution. Therefore the expected value is $\frac{B + A}{2}$.

d. $\sigma^2 = \frac{(B - A)^2}{12}$

18. a. $\mu = 5, \sigma = \frac{2\sqrt{3}}{3}$

- b. Sample responses can be found in the Worked solutions in the online resources.
 c. Answers will vary. The mean of the sample means should be approximately 5.
 d. Answers will vary. Approximately 90% of the sample confidence intervals should contain the population mean of 5. Even though the initial population is

uniform, the distribution of the sample means should look like a normal distribution if enough samples are chosen.

19. a. 4.5 minutes

- b. 4 minutes

- c. Sample responses can be found in the Worked solutions in the online resources. To generate the sample elements. If a is a random number generated.

$b = 0.5 - 4 \ln(1 - a)$ where b is an element in the sample.

- d. Answers will vary. The mean of the sample means should be approximately 4.5.

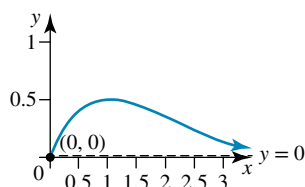
20. a. 1

- b. 2

- c. $\sqrt{2}$

- d. Sample responses can be found in the Worked solutions in the online resources.

- e. Population looks like



Distribution of sample means will vary.

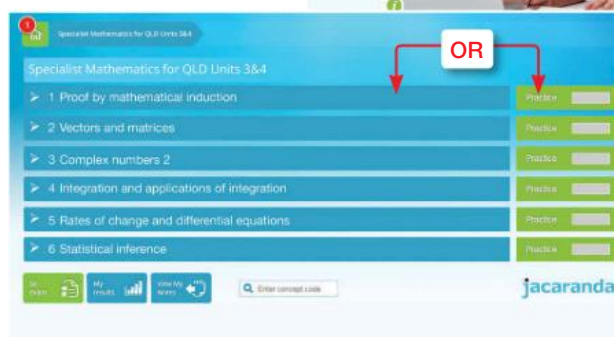
- f. Results will vary. The distribution should be normal.

REVISION UNIT 4 Further calculus and statistical inference

TOPIC 3 Statistical inference

- For revision of this entire topic, go to your **studyON** title in your bookshelf at www.jacplus.com.au.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Specialist Mathematics for Queensland Units 3 & 4* to see the entire course divided into syllabus topics.
- Select the **area** you are studying to navigate into the sequence level **OR** select **Practice** to answer all practice questions available for each area.



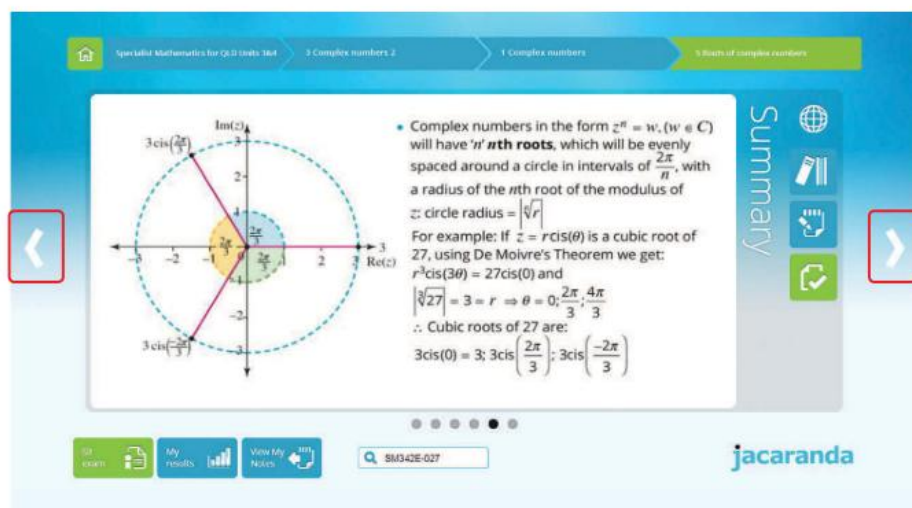
- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.



- Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practice questions at the concept level for a more granular set of questions.



PRACTICE ASSESSMENT 3

Specialist Mathematics: Unit 4 examination

Unit 4: Further calculus and statistical inference

Topic 1: Integration and applications of integration

Topic 2: Rates of change and differential equations

Topic 3: Statistical inference

Conditions

Technique	Response Type	Duration	Reading
Technology active	Short response	120 minutes	5 minutes
Resources		Instructions	
• QCAA formula sheet • Notes not permitted • Non-CAS graphics calculator		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to two decimal places.	

Criterion	Marks allocated	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	60	

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Question 1 (8 marks)

a. $\int \frac{1}{\sqrt{5-2x}} dx$

b. $\int \frac{x}{(5x - 6)^2} dx$

The velocity of a particle moving in a straight line is given by $v(x) = 1 + 2 \ln(x)$, $x > 0$, where x is the position at any time t .

Question 3 (3 marks)

Determine $P(T \geq 3)$ if the probability density function is $f(t) = 0.2e^{-0.2t}$

Question 4 (6 marks)

- a. Use Simpson's rule with 6 strips to estimate $\int_0^{12} \sin\left(\left(\frac{x}{2\pi}\right)^{1.5}\right) dx$
- b. Evaluate the reasonableness of your estimate by comparing it to the value obtained by calculating the area with technology.

Determine the following integrals

a. $\int x \sin\left(\frac{x}{2}\right) dx$

b. $\int e^x \cos(2x) dx$

c. $\int \frac{4x + 3}{(x + 2)(x - 3)} dx$

d. $\int \frac{1}{\sqrt{9-x^2}} dx$

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins or other markings on the paper.

If a sample of size 40 with a mean of 34.6 and a standard deviation of 5.3 is drawn from a population, determine the 95% confidence interval for the population mean.

Question 7 (7.5 marks)

- The time period between stops.
- The distance between the stops.
- The maximum velocity of the bus and when it occurs.
- Explain what is occurring when the velocity is negative.
- Justify the reasonableness of your results by considering the validity of the given model.

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Question 8 (4.5 marks)

The region bounded by the curve $y = \ln(x - 1)$, the line $x = 4$ and the x -axis is rotated about the y -axis. Determine the volume of the solid formed.

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Question 9 (12 marks)

Assuming that the growth rate is doubled while in Holly's solution, determine when should she return the bacteria if Matthew is to find the correct number when he returns.

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

PRACTICE ASSESSMENT 4

Specialist Mathematics: Units 3 & 4 examination

Unit

Unit 3: Mathematical induction, and further vectors, matrices and complex numbers

Unit 4: Further calculus and statistical inference

Topic

Unit 3: Topic 1: Proof by mathematical induction

Topic 2: Vectors and matrices

Topic 3: Complex numbers 2

Unit 4: Topic 1: Integration and applications of integration

Topic 2: Rates of change and differential equations

Topic 3: Statistical inference

Conditions

Technique	Response Type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response Short response	Paper 1: 90 minutes Paper 2: 90 minutes	5 minutes 5 minutes
Resources		Instructions	
• QCAA formula sheet • Notes not permitted • Non-CAS graphics calculator permitted for Paper 2 ONLY		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to two decimal places.	

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	34	36	

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Paper 1 – Technology free

Part A: Simple familiar — total marks: 21

Question 1 (8 marks)

The acceleration of a particle travelling in a straight line is given by $\frac{dv}{dt} = v + 2$, where v is the velocity at any time t .

When $t = 0$, the particle is at rest at the origin. Determine, as a function of time:

- a. the velocity
- b. the acceleration
- c. the position.

This image shows a blank sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Question 2 (6 marks)

A point is moving according to the parametric equations $\begin{cases} x = 6t + 1 \\ y = 3t^2 - 3 \end{cases}, t \geq 0.$

- Determine the Cartesian equation that describes the motion.
- Calculate $\frac{dy}{dx}$ when $t = 1$.

[illegible]

[illegible]

Question 3 (7 marks)

If $z_1 = 1 + i$ is a root of $P(z) = z^4 + 2z^3 + 7z^2 - 18z + 26$

- Use the conjugate root theorem to identify a root, z_2 of the polynomial.
- If $P(z) = (z - z_1)(z - z_2)Q(z)$, determine $Q(z)$.
- Determine the factors of $Q(z)$.

[illegible]

Part B: Complex familiar — total marks: 6

Question 4 (6 marks)

Prove by induction that $9^n - 3$ is divisible by 6 for $n \geq 1$.

[illegible]

Question 5 (7 marks)

$\frac{dx}{dt} = kx(a - x)$ where x is the amount of product present, k is a positive constant and a is the initial amount of the catalyst.

If the initial amount of product is x_0 , solve the differential equation to express x as a function of t .

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Paper 2 – Technology active

Part A: Simple familiar – total marks: 21

Question 1 (4 marks)

If a sample of 30 students at a university has an average age of 29.7 years with a standard deviation of 4.2 years, determine the 99% confidence for the average age of a student at the university.

Question 2 (5 marks)

Use matrix algebra to solve the system of equations.

$$w + x - y + 2z = 13$$

$$2w + 3x + y - 2z = -12$$

$$-w + 2x + y - z = -5$$

$$-2w - x + 3y - 2z = -18$$

Question 3 (4 marks)

The wait time at a bus stop is modelled by the probability density function $f(t) = 0.1e^{-0.1t}$ minutes.

- a. Determine the expected wait time.
- b. Determine the probability that the wait is less than 5 minutes.

Question 4 (8 marks)

A plane passes through the points $A(1, 1, 0)$, $B(2, -1, -4)$ and $C(-1, 3, 2)$.

- a. Determine vector $\overrightarrow{AB}$ on the plane.
- b. Determine vector $\overrightarrow{AC}$ on the plane.
- c. Determine a vector perpendicular to the plane.
- d. Determine the equation of the plane and evaluate the reasonableness of your solution.

Question 5 (8 marks)

$$T = \begin{array}{ccc|c} & \begin{array}{c} \text{From} \\ A \quad B \quad C \end{array} & & \begin{array}{c} \text{To} \\ A \\ B \\ C \end{array} \\ \begin{array}{c} 0.6 \\ 0.1 \\ 0.3 \end{array} & \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.5 \end{bmatrix} & & \end{array}$$

- Initially, 15%, 45% and 40% of shoppers shop at A , B and C respectively. Calculate the proportion in each shop 1 week later.
- If a shopper shops in store A in the first week, determine the probability that they shop in store C two weeks later.
- It is found that long term the proportion of people in each shop is constant. Determine the long-term distribution of shoppers.

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

Question 6 (7 marks)

Determine how long it will take for the radii of the circles to touch.

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

GLOSSARY

| notation a symbol used to represent ‘is exactly divisible by’.

altitude angle the angle φ between the line segment $\overline{OP}$ and the xy -plane, where $-90^\circ \leq \varphi \leq 90^\circ$.

angular velocity the rate at which the position vector of a rotating particle changes with respect to time

Argand plane a plane used to represent complex numbers using a horizontal real axis and a vertical imaginary axis.

argument of z the angle that a line segment of a complex number on an Argand plane makes with the positive real axis.

azimuth angle the anti-clockwise angle θ formed from the positive x -axis to the projection of $\underline{u}$ onto the xy -plane, where $0^\circ \leq \theta \leq 360^\circ$.

Cartesian equation of the path the relationship between x and y in a parametric equation.

Cartesian form from the origin to point (x, y) , the Cartesian form of vector $\underline{u}$ is $u = x\hat{i} + y\hat{j}$.

Central Limit Theorem the theory that as all possible samples of a population are considered, the sample mean tends to be the same as that of the population.

circular motion the motion of a particle along a circular path.

complex conjugate the conjugate, $\bar{z}$, of a complex number z has the opposite sign of the imaginary component.

Multiplication or addition of a complex number and its conjugate results in a real number.

complex number system an extension of the real number system that includes numbers that involve i , the imaginary unit.

component form expressing a vector in terms of the standard unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

concave down describes a parabola that opens downwards.

concave up describes a parabola that opens upwards.

confidence interval an interval of values within which the true value of the population mean is expected to lie between.

consistent in linear equations, a system that has at least one solution.

cross product the multiplication of two vectors which results in a vector.

De Moivre's theorem states that if $z = r\text{cis}(\theta)$ then $z^n = r^n\text{cis}(n\theta)$ for $n \in \mathbb{Z}$.

Deciphering the decoding of a message into a form of a plain text message.

degree the highest power of the highest derivative present in the equation.

diagonal matrix a square matrix where both the elements above and the elements below the leading diagonal are all zero.

differential equation an equation involving derivatives in the form $g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) = 0$

digraph a network diagram that has directed edges connecting a set of vertices.

direction the course along which an object moves.

dominance vector the result of the rows of a dominance matrix being added together.

dot (scalar) product the multiplication of two vectors which results in a scalar.

eigenvalue a solution, λ , of the equation $AX = \lambda X$.

eigenvector a non-zero vector (matrix) X , such that $AX = \lambda X$, so that λ is an eigenvalue of A .

Enciphering the coding of a message into a form that is unreadable to a third party.

equilibrium for a particle, where the vector sum of all the forces acting on a particle is equal to zero.

explicit relationship the elimination of a parameter from the two parametric equations to find an equation in the form $y = f(x)$.

Exponential probability density function An exponential random variable X with parameter λ , where $\lambda > 0$, has a

$$\text{probability density function given by } f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The mean of X is given by $E(X) = \frac{1}{\lambda}$.

Gaussian elimination a method used to solve systems of linear equations by using simple row operation procedures.

general solution contains arbitrary constants of integration and satisfies the differential equation.

half-life the time it takes for half the original mass of a radioactive material to disintegrate.

implicit differentiation the computation of the derivative of an implicit function without explicitly determining the function.

implicit relationship a relationship obtained between the two parametric equations in the form $f(x, y) = 0$.

inconsistent in linear equations, a system that has no solutions.

inductive step assuming that since a formula works once for $n = 1$, it will also work for $n = k$. The next step is proving that the formula is also true $n = k + 1$.

initial statement the first step in a proof by induction, where it is proved that the formula works for $n = 1$.

inner product the multiplication of two vectors which results in a scalar, also known as the dot or scalar product.

invertible matrix a matrix that has an inverse, also referred to as a non-singular matrix.

Kinematics the study of motion.

Lami's theorem states that if three concurrent forces act on a body in equilibrium, then the magnitude of each force is proportional to the sine of the angle between the other two forces.

Leslie matrix a matrix used to determine the growth of populations from one year to the next.

limiting velocity the maximum speed that a body can reach in freefall when the resistance force acting on the body is proportional to the gravitational force, resulting in acceleration equalling zero.

linear equations an equation in which the highest power of any variable is 1.

lower triangular matrix a square matrix where all the elements above the leading diagonal are zero.

magnitude the size of an object.

margin of error the distance between the sample estimate and the endpoints of the confidence interval.

Markov chain the sequence of month to month variations in the state of a system.

Matrix algebra the process of finding solutions to matrix equations.

maximal or **implied domain** the domain over which a relation or function is defined.

mean the average value of a set of numbers. Also corresponds to the expected value, $E(X)$, of the distribution of a population.

modulus-argument the polar form of a complex number.

momentum the momentum $\underline{p}$ of a body of mass m moving with a velocity $\underline{v}$ is defined by $\underline{p} = m\underline{v}$.

Newton's law of cooling the law states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the temperature of the medium surrounding the body. It can be written as, $\frac{dT}{dt} \propto (T - T_m)$ or $\frac{dT}{dt} = k(T - T_m)$.

normal the perpendicular vector defined by vectors $\underline{a}$ and $\underline{b}$.

order the highest derivative present in the differential equation

orthogonal at right angles

oscillate the repeated motion of an object moving in a straight line between two fixed points.

parameter in algebra, a varying constant in a common equation used to describe a family of polynomials. In statistics, a characteristic of a population.

parametric equation an equation written to model the path of a particle, where time is introduced as a third variable.

particular solution satisfies the differential equation and some initial value condition, known as the boundary value, that enables the constant(s) of integration to be found.

plane a flat surface.

plane curve the collection of x and y points obtained as t varies.

point estimate a specific value that is an estimate of a parameter of a population based on sampling. For example, if the value of the sample mean is used as an estimate of the population mean, then it is called a point estimate.

polar form a way to represent complex numbers using a modulus and an argument.

population the group that you want information about.

position vector the directed line segment joining a point P , with the origin.

principal value the angle $\text{Arg}(z)$ where $\text{Arg}(z) \in (-\pi, \pi]$.

projection the image of one figure or object upon another.

proof by mathematical induction the process of proving that a certain property or formula is true for every value of N .

rectilinear motion motion in a straight line.

reduced row-echelon form a matrix that has ones in the leading diagonal and all other elements in the matrix are zeros.

resolution of vectors the process of expressing a vector in component form

right-handed coordinate system the orientation of an axis in a diagram, where the fingers on your right-hand curl from the x -axis to the y -axis, and your thumb is pointed in the direction of the positive z -axis.

roots the solutions to an equation.

row-echelon form a matrix that has ones in the leading diagonal and zeros under the leading diagonal.

samples the group within the population that the information is being collected from.

scalar a number that has no direction, and when multiplied with a vector, it produces another vector.

scalar resolute the scalar resolute of $\underline{b}$ on $\underline{a}$, indicates how much of $\underline{b}$ is in the direction of $\underline{a}$.

scalar triple product the product of vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, defined as $\underline{a} \cdot (\underline{b} \times \underline{c})$.

simple harmonic motion the oscillation of a particle subjected to acceleration that is always of opposite sign to the displacement.

Simpson's rule A method for approximating $\int_a^b f(x) dx$. The interval $[a, b]$ on the x -axis is divided into an even number of equal subintervals, n , each of width w where $w = \frac{b-a}{n}$.

singular matrix a matrix that has no inverse and has a zero determinant.

slope fields a graphical view of the solutions of a first-order differential equation, where the derivative at a certain point is represented by the midpoint of a line segment of the corresponding slope.

solid of revolution a solid figure created by the 360° rotation of a function with defined boundaries around an axis.

sphere a round solid figure with every point on its surface equidistant from its centre.

standard deviation a measure of spread. $SD(X) = \sigma = \sqrt{\text{Var}(X)}$. The larger the standard deviation, the more spread out the data. If the standard deviation is small, the data is clumped about the mean.

standard unit vectors the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ that are used to express a point's position relative to the origin.

state the proportion of any of the possible outcomes of a Markov chain.

steady state the long-term behaviour of a system.

summation sign the Greek letter sigma, $\sum$, used to represent the addition of multiple terms.

symmetric equation of a line the Cartesian equation of a line three-dimensional space.

system of linear equations a finite list of linear equations.

terminal velocity the maximum speed that a body can reach in freefall when the resistance force acting on the body is proportional to the gravitational force, resulting in acceleration equalling zero.

torque the turning force placed on a object defined by $\text{Torque} = |\underline{r} \times \underline{F}|$, where $\underline{F}$ is the force and $\underline{r}$ is the vector from the point of where the force is applied to the point about which the turning is considered.

transition matrix a matrix that helps in the calculations of the transitions between states from one time period to the next.

triangular matrix a square matrix where either all the elements above or all the elements below the leading diagonal are zero.

triple product the product of vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, defined as $\underline{a} \cdot (\underline{b} \times \underline{c})$, often referred to as the scalar triple product.

uniform circular motion the motion of a particle along a circular path that is travelling at a constant speed.

unique solution where there is only one possible solution to an equation.

unit vector a vector with a magnitude of 1, given by $\hat{u} = \frac{\underline{u}}{|\underline{u}|}$.

upper triangular matrix a square matrix where all the elements below the leading diagonal are zero.

variance a measure of spread. $\text{Var}(X) = \sigma^2 = E(X^2) - E[(X)]^2$

vector equation of a plane the vector equation of a plane is given by the equation $(\underline{r} - \underline{a}) \cdot \underline{n} = 0$

vector function equations written in the form $\underline{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, where the vector location of a particle is a function of t .

vector product the multiplication of two vectors which results in a vector. Also known as the cross product.

vector used to describe a quantity that has both a magnitude and a direction.

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