NEW SENIOR

Mathematics

ADVANCED + EXTENSION 1 YEAR 11 | 4TH EDITION

NSW STAGE 6

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NEW SENIOR MATHEMATICS
ADVANCED + EXTENSION 1
YEAR 11, FOURTH EDITION

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NSW STAGE 6

NEW SENIOR Mathematics

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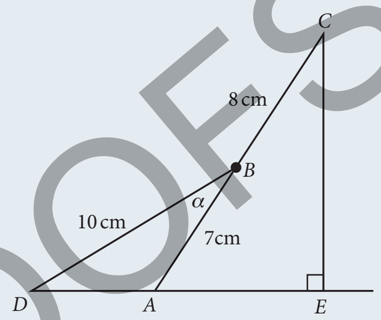
CHAPTER 9
Further trigonometry

9.1 TRIGONOMETRY IN THREE DIMENSIONS

With three dimensional problems, once you have created a 3D diagram, it is important to break it down into its 2D parts and do your calculations based on those simpler diagrams.

Example 1

In this diagram, $AC = 15\,\text{cm}$, $AB = 7\,\text{cm}$, $DB = 10\,\text{cm}$ and $\angle DBA = \alpha$.
Find the perpendicular distance CE of C from DA in terms of α and evaluate it for $\alpha = 25°$.



Solution

Use the cosine rule in ΔDBA: $DA^2 = 10^2 + 7^2 - 2 \times 10 \times 7\cos\alpha$

$DA^2 = 149 - 140\cos\alpha$

$\therefore\quad DA = \sqrt{149 - 140\cos\alpha}$ [1]

Use the sine rule in ΔDBA: $\dfrac{10}{\sin(\angle DAB)} = \dfrac{DA}{\sin\alpha}$

$\therefore\quad \sin(\angle DAB) = \dfrac{10\sin\alpha}{DA}$ [2]

In ΔCAE: $CE = 15\sin(\angle CAE)$ [3]

Now: $\angle DAB = 180° - \angle CAE$

Hence: $\sin(\angle DAB) = \sin(\angle CAE)$

Using [1], [2] and [3]: $CE = \dfrac{150\sin\alpha}{\sqrt{149 - 140\cos\alpha}}$

For $\alpha = 25°$: $CE = \dfrac{150\sin 25°}{\sqrt{149 - 140\cos 25°}}$

$= 13.48$

Perpendicular distance = 13.48 cm

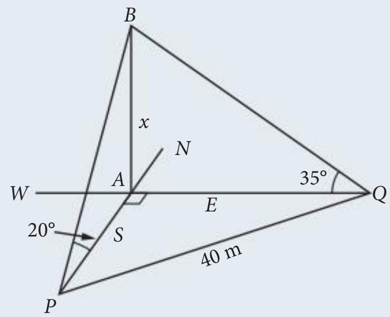
Example 2

From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20°; from a point Q due east of the tower, the angle of elevation is 35°. The distance from P to Q is 40 metres. Find the height of the tower.

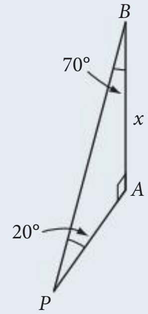
Solution

(Trigonometric values are rounded to 3 decimal places.)

Draw a 3D diagram to represent this information. Let the height of the tower be x m.



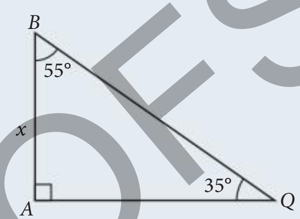
There are three right-angled triangles: ΔPAB and ΔQAB in the vertical plane, and ΔPAQ in the horizontal plane. Draw each right-angled triangle separately as you use it.



In ΔPAB: $\tan 20° = \dfrac{x}{PA}$ **OR** $\tan 70° = \dfrac{PA}{x}$

$\therefore \quad PA = \dfrac{x}{\tan 20°} \qquad PA = x\tan 70°$

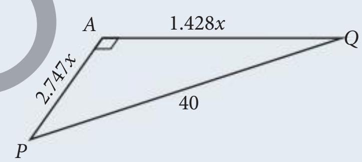
$\therefore \quad PA = 2.747x \qquad PA = 2.747x$



In ΔQAB: $\tan 35° = \dfrac{x}{QA}$ **OR** $\tan 55° = \dfrac{QA}{x}$

$\therefore \quad QA = \dfrac{x}{\tan 35°} \qquad QA = x\tan 55°$

$\therefore \quad QA = 1.428x \qquad QA = 1.428x$



In ΔPAQ, use Pythagoras' theorem:

$$(2.747x)^2 + (1.428x)^2 = 40^2$$
$$7.546x^2 + 2.039x^2 = 1600$$
$$9.585x^2 = 1600$$
$$x = \sqrt{\frac{1600}{9.585}}$$
$$x = 12.920$$

The height of the tower is 12.92 m.

To avoid errors due to rounding, it is better if you can avoid using your calculator until the last step. Without rounding, the calcuation is:

$$(x\tan 70°)^2 + (x\tan 55°)^2 = 1600$$
$$x^2(\tan^2 70° + \tan^2 55°) = 1600$$
$$x^2 = \frac{1600}{\tan^2 70° + \tan^2 55°}$$
$$x = \frac{40}{\sqrt{\tan^2 70° + \tan^2 55°}} \qquad x = 12.918 \approx 12.92\text{ m}$$

EXERCISE 9.1 TRIGONOMETRY IN THREE DIMENSIONS

1 A and B are the bases of two vertical towers each 30 m high. B is due north of A. From a point P that is due east of A and in the same horizontal plane, the angles of elevation of the tops of the towers A and B are 45° and 36° 52′ respectively. Calculate the distance:

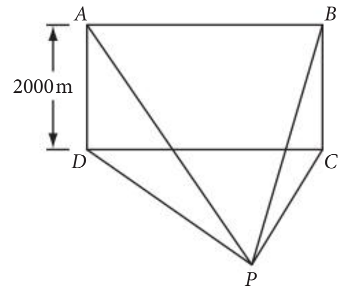
(a) from P to the bases of the two towers **(b)** between the two towers.

2 A, B, C, D are four points equally spaced on the circumference of a horizontal circle of radius 3 cm. P is a point above the circle such that $PA = PD = 4$ cm and $PB = PC = 6$ cm. Calculate the size of $\angle APC$.

3 The angle of elevation of the top of a building from a point P due east of it is 40°, and from a point Q due south of P is 30°. If the distance from P to Q is 20 metres, find the height of the building.

4 From a point A, the angle of elevation of the top of a tower due north of it is 20°. From point B, due east of the tower, the angle of elevation is 18°. A and B are 100 m apart. Show that the height h of the tower is given by $h = \dfrac{100}{\sqrt{\tan^2 72° + \tan^2 70°}}$.

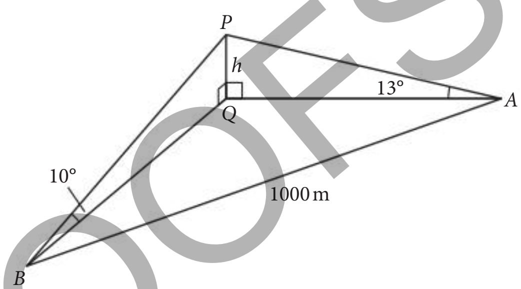
5 A and B represent successive positions of an aircraft flying horizontally at an altitude of 2000 metres. From a point P the angles of elevation of A and B are 15° and 25° respectively. If $\angle DPC = 60°$, calculate the distance AB.



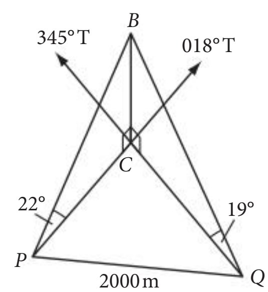
6 The elevation of a hill at a point P due east of it is 42° and at a point Q due south of it is 22°. If the distance from P to Q is 400 metres, find the height of the hill.

7 A tower PQ of height h metres has an angle of elevation of 13° at a point A due east of it. From another point B, the bearing of the tower is 050° T and the angle of elevation is 10°. The points A and B are 1000 metres apart on the same level as the tower's base Q.

(a) Show that $\angle AQB = 140°$.
(b) Consider the triangle APQ and show that $AQ = h \tan 77°$.
(c) Find a similar expression for BQ.
(d) Use the cosine rule in the triangle AQB to calculate h to the nearest metre.

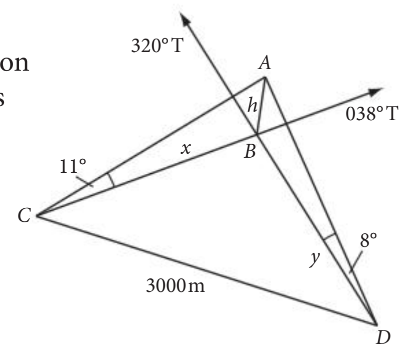


8 Two observers P and Q, who are 2000 metres apart, take the bearing and elevation of a balloon B at the same instant. The observer P finds the bearing to be 018° T and the angle of elevation to be 22°, while the observer Q finds the bearing to be 345° T and the angle of elevation to be 19°. Find the height BC of the balloon to the nearest metre.

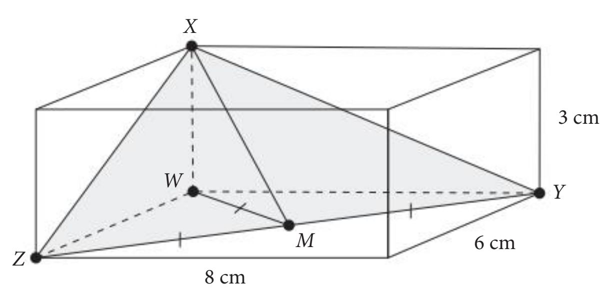


9 AB is a tower. CD is a road. A person walking along the road observes the tower AB from C at a bearing of 038° T and with an angle of elevation of 11°. After travelling 3000 m along the road to D, the person observes the tower from D at a bearing of 320° T and an angle of elevation of 8°. By letting $CB = x$, $BD = y$ and $AB = h$:

(a) Show that $\angle CBD = 78°$.
(b) Express x in terms of h.
(c) Express y in terms of h.
(d) Using the cosine rule with ΔCBD, find the value of h correct to one decimal place.



10 The diagram below shows plane XYZ inside a rectangular prism. M is the midpoint of the base diagonal YZ. Find, correct to the nearest degree, the size of angle WMX.

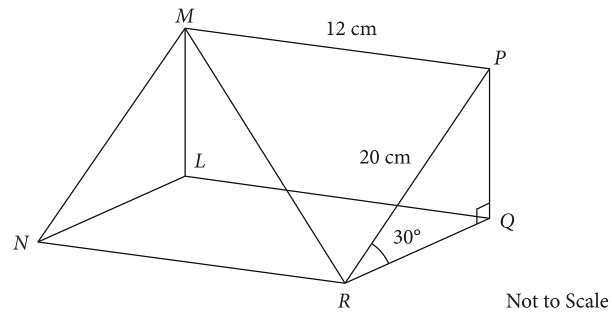


11 The diagram shows a triangular prism.

Triangle PQR is a cross-section of the prism.

$PR = 20$ cm, $MP = 12$ cm, $\angle PRQ = 30°$, $\angle PQR = 90°$

Calculate the size of the angle that the line MR makes with the plane $RQLN$. Give your answer correct to 1 decimal place.

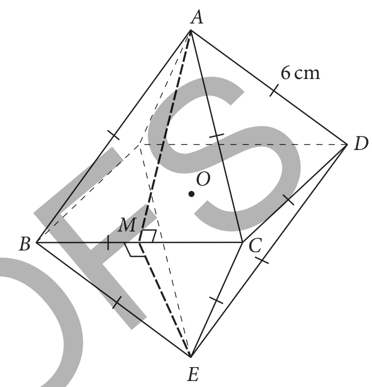


12 Two identical right pyramids with square bases and equilateral triangular faces are joined to form a regular octahedron.

The following diagram shows a regular octahedron with edge length 6 cm and centre at O.

AM and ME are perpendicular heights of ΔABC and ΔBCE respectively and point M is the midpoint of BC.

Find the size of $\angle AME$, to the nearest degree.



13 Douglas and Sandra visit Sydney Harbour on New Year's Eve to watch the fireworks.

Douglas is standing outside the Sydney Opera House and Sandra is on a yacht in the harbour.

They can both see the fireworks at the top of the Sydney Harbour Bridge.

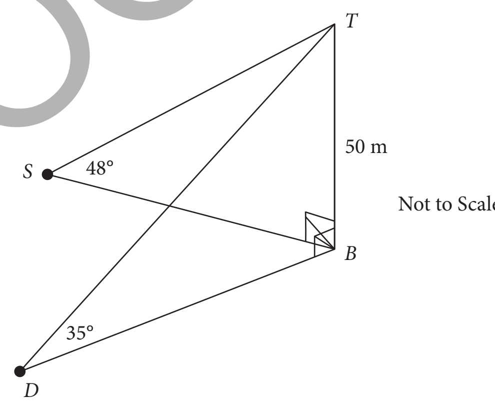
From Douglas's position (D), the Sydney Harbour Bridge (B) is at a bearing of 78°, and the angle of elevation of the top of the bridge (T) is 35°.

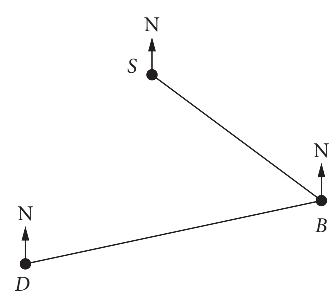
From Sandra's position (S), the Sydney Harbour Bridge is at a bearing of 140°, and the angle of elevation of the top of the bridge is 48°.

The top of the bridge is 50 m above the base of the bridge.

(a) Using appropriate diagrams and reasoning, show that $\angle SBD = 62°$.

(b) Find the distance between Sandra and Douglas, correct to the nearest metre.



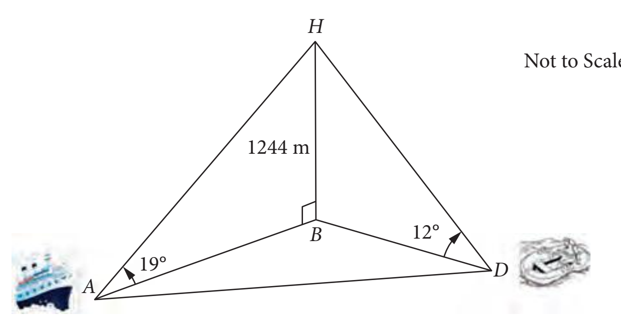


14 Daniel is in a life raft, D, and Amy is in a cabin cruiser, A, searching for him.

They are in contact by mobile telephone. Daniel tells Amy that he can see the top of the mountain, Mt Harris, H.

From Daniel's position, the mountain has a bearing of 340° and the angle of elevation to the top of the mountain is 12°.

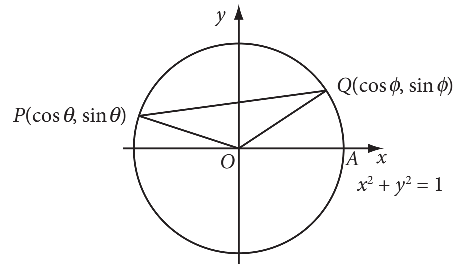
Amy can also see Mt Harris. From her position, the mountain has a bearing of 70° and the top of the mountain has an angle of elevation of 19°. The top of Mt Harris is 1244 m above sea level.



Find the distance of the life raft from Amy's position. Give your answer correct to the nearest metre.

9.2 SUM AND DIFFERENCE OF TWO ANGLES

You can derive formulae for the trigonometric functions of sums and differences of angles, i.e. formulae for $\cos(\theta \pm \phi)$, $\sin(\theta \pm \phi)$ and $\tan(\theta \pm \phi)$, in terms of trigonometric functions of the angles θ and ϕ.



First, obtain a formula for $\cos(\theta - \phi)$, then deduce formulae for the others from it.

Starting from A, mark a unit circle to show $\angle AOP = \theta$ and $\angle AOQ = \phi$ (where $\theta > \phi$ for convenience), as shown. The coordinates of P and Q are thus $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ respectively, and $\angle POQ = \theta - \phi$.

Using the distance formula: $PQ^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2$

$= 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$ [a]

Using the cosine rule in ΔPOQ: $PQ^2 = 1 + 1 - 2\cos(\angle POQ)$

$= 2 - 2\cos(\theta - \phi)$ [b]

Equating [a] and [b]: $2 - 2\cos(\theta - \phi) = 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$

$\therefore$ $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ [1]

This formula [1] is true for all values of θ and ϕ.

By writing $(-\phi)$ in place of ϕ, remembering that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, you also have:

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ [2]

Similarly, with $(90° - \theta)$ in place of θ in [1]:

$$\cos[90° - (\theta + \phi)] = \cos(90° - \theta)\cos\phi + \sin(90° - \theta)\sin\phi$$

And using the complementary angles formulae, i.e. $\cos(90° - \alpha) = \sin\alpha$ and $\sin(90° - \alpha) = \cos\alpha$:

$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ [3]

Writing $(-\phi)$ in place of ϕ in [3]: $\sin(\theta - \phi) = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$

$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$ [4]

Combining these for the tangent functions: $\tan(\theta + \phi) = \dfrac{\sin(\theta + \phi)}{\cos(\theta + \phi)}$

$= \dfrac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$ [5]

And for $(\theta - \phi)$: $\tan(\theta - \phi) = \dfrac{\sin(\theta - \phi)}{\cos(\theta - \phi)}$

$= \dfrac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$ [6]

Example 3

Find the expansion for each expression, simplifying where possible.

(a) $\sin(3x+2y)$ (b) $\cos(2\alpha+\beta)$ (c) $\tan(A+45°)$

Solution

(a) $\sin(3x+2y) = \sin 3x \cos 2y + \cos 3x \sin 2y$

(b) $\cos(2\alpha+\beta) = \cos 2\alpha \cos\beta - \sin 2\alpha \sin\beta$

(c) $\tan(A+45°) = \dfrac{\tan A + \tan 45°}{1 - \tan A \tan 45°} = \dfrac{1+\tan A}{1-\tan A}$

Example 4

Simplify each expression.

(a) $\sin(2\alpha+\beta)\cos\beta - \cos(2\alpha+\beta)\sin\beta$ (b) $\cos(2\theta-3\alpha)\cos 2\theta + \sin(2\theta-3\alpha)\sin 2\theta$

Solution

By recognising the form of the equation, the two-angle expansion can be used in reverse:

(a) $\sin(2\alpha+\beta)\cos\beta - \cos(2\alpha+\beta)\sin\beta = \sin[(2\alpha+\beta)-\beta] = \sin 2\alpha$

(b) $\cos(2\theta-3\alpha)\cos 2\theta + \sin(2\theta-3\alpha)\sin 2\theta = \cos[(2\theta-3\alpha)-2\theta]$
$= \cos(-3\alpha) = \cos 3\alpha$

Alternatively, the expressions on the LHS can be expanded and like terms collected. For example, the solution to **(a)** becomes:

$$\begin{aligned}\sin(2\alpha+\beta)\cos\beta - \cos(2\alpha+\beta)\sin\beta &= [\sin 2\alpha\cos\beta + \cos 2\alpha \sin\beta]\cos\beta - [\cos 2\alpha\cos\beta - \sin 2\alpha \sin\beta]\sin\beta \\ &= \sin 2\alpha\cos^2\beta + \cos 2\alpha\sin\beta\cos\beta - \cos 2\alpha\cos\beta\sin\beta + \sin 2\alpha\sin^2\beta \\ &= \sin 2\alpha[\cos^2\beta + \sin^2\beta] = \sin 2\alpha\end{aligned}$$

Example 5

If θ and ϕ are acute angles and $\sin\theta = \dfrac{3}{5}$ and $\tan\phi = \dfrac{24}{7}$, find, without using a calculator, the exact value of the following expressions:

(a) $\sin(\theta+\phi)$ (b) $\cos(\theta-\phi)$ (c) $\tan(\theta-\phi)$

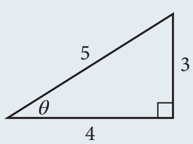
Solution

Draw right-angled triangles for each ratio and use Pythagoras' theorem to find the third side.

$\sin\theta = \dfrac{3}{5}$

$\cos\theta = \dfrac{4}{5}$

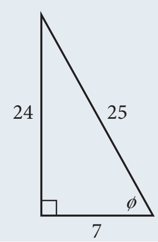
$\tan\theta = \dfrac{3}{4}$



$\tan\phi = \dfrac{24}{7}$

$\sin\phi = \dfrac{24}{25}$

$\cos\phi = \dfrac{7}{25}$



(a) $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$

$$=\frac{3}{5}\times\frac{7}{25}+\frac{4}{5}\times\frac{24}{25}$$

$$=\frac{117}{125}$$

(b) $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$

$$=\frac{4}{5}\times\frac{7}{25}+\frac{3}{5}\times\frac{24}{25}$$

$$=\frac{100}{125}$$

$$=\frac{4}{5}$$

(c) $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$

$$=\frac{\frac{3}{4}-\frac{24}{7}}{1+\frac{3}{4}\times\frac{24}{7}}$$

$$=\frac{21-96}{28+72}$$

$$=-\frac{75}{100}$$

$$=-\frac{3}{4}$$

EXERCISE 9.2 SUM AND DIFFERENCE OF TWO ANGLES

1 Expand:

(a) $\sin(A+2B)$ (b) $\sin(2x-y)$ (c) $\cos(2x-3y)$

(d) $\cos(2\theta+60°)$ (e) $\tan(\theta+\alpha)$ (f) $\tan(A-135°)$

2 Simplify:

(a) $\sin A\cos(A-B)+\cos A\sin(A-B)$ (b) $\cos(\theta+\alpha)\cos(\theta-\alpha)+\sin(\theta+\alpha)\sin(\theta-\alpha)$

(c) $\sin 2A\cos A-\cos 2A\sin A$ (d) $\cos 60°\cos 30°-\sin 60°\sin 30°$

(e) $\dfrac{\tan\theta-\tan 20^\circ}{1+\tan 20^\circ\tan\theta}$ (f) $\dfrac{\tan 2\alpha+\tan\alpha}{1-\tan 2\alpha\tan\alpha}$ (g) $\sin(2A+B)\cos(A+B)-\cos(2A+B)\sin(A+B)$

(h) $\cos(3\theta+\alpha)\cos(2\theta+\alpha)+\sin(3\theta+\alpha)\sin(\theta+\alpha)$ (i) $\dfrac{\tan 3x-\tan x}{1+\tan 3x\tan x}$

3 The expression $\dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$ simplifies to:

A $\tan A+\tan B+\tan C$ B $\tan(A-B+C)$ C $\tan(A+B-C)$ D $\tan(A+B+C)$

4 (a) Find the exact value of $\sin 38°\cos 22°+\cos 38°\sin 22°$.

(b) Find the exact value of $\dfrac{\tan 119^\circ+\tan 16^\circ}{1-\tan 119^\circ\tan 16^\circ}$.

(c) Find the exact value of $\cos 165°$. (d) Expand and simplify $\sin(x+40°)+\sin(x-40°)$.

5 Write the expansion of $\cos(A+B)$. From this, deduce the expansion of $\cos(A-B)$.

6 Write the expansion of $\cos(\theta-\phi)$. Write $(90°-\theta)$ in place of θ to deduce the expansion of $\sin(\theta+\phi)$.

7 If θ and ϕ are angles between 0° and 90°, $\sin\theta=\frac{3}{5}$, $\tan\phi=\frac{7}{24}$, find the following without using a calculator.

(a) $\sin(\theta-\phi)$ (b) $\cos(\theta+\phi)$ (c) $\tan(\theta-\phi)$

8 If $\tan A=4$, $\tan B=\frac{3}{5}$, and A and B are acute angles, then $A-B=\ldots$

A 45° B 30° C 60° D 135°

9 If $\tan\alpha=\frac{4}{3}$ and $\cos\beta=\frac{12}{13}$, where $0<\beta<\alpha<90°$, evaluate the following without using a calculator.

(a) $\sin 2\alpha$ (b) $\tan 2\alpha$ (c) $\cos(\alpha-\beta)$

10 (a) Using the expansion of $\sin(A+B)$, prove that $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) Using the expansion of $\tan(A+B)$, prove that $\tan 75° = 2+\sqrt{3}$.

11 Find the value (in simplest surd form) of:

(a) $\cos 75°$ (b) $\tan 15°$ (c) $\cos 15°$

12 Use the expansion of $\sin(A+B)$ to evaluate $\sin 195°$.

13 Given that $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, find the value of $\sin(A+B)$ if A and B are both acute angles.

A $-\frac{107}{140}$ B $-\frac{21}{221}$ C $\frac{107}{140}$ D $\frac{171}{221}$

14 Given that $\tan\alpha = \frac{1}{2}$, which of the following is the exact value of $\tan\left(\alpha + \frac{\pi}{3}\right)$?

A $\frac{\sqrt{3}-2}{2\sqrt{3}+1}$ B $\frac{\sqrt{3}+2}{2\sqrt{3}-1}$ C $\frac{1-2\sqrt{3}}{2+\sqrt{3}}$ D $\frac{1+2\sqrt{3}}{2-\sqrt{3}}$

15 If $\sin A = t$ and $\cos B = t$, where $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$, then $\cos(A+B)$ is equal to:

A 0 B $\sqrt{1-t^2}$ C $1-2t^2$ D $-2t\sqrt{1-t^2}$

9.3 DOUBLE ANGLE FORMULAE

Using the sum-and-difference-of-two-angles formula [2] (see previous section), writing θ in place of ϕ:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \quad [2]$$

$$\therefore \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

From $\sin^2\theta + \cos^2\theta = 1$:

$$\cos 2\theta = 2\cos^2\theta - 1 \quad [7]$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

Similarly, using formula [3] and writing θ in place of ϕ:

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \quad [3]$$

$$\therefore \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad [8]$$

And using formula [5], writing θ in place of ϕ:

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} \quad [5]$$

$$\therefore \tan(\theta+\theta) = \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},\ \tan\theta \neq \pm 1 \quad [9]$$

The double angle formulae can be used in many different ways.

For example, as $4\theta = 2\times 2\theta$, therefore $\sin 4\theta = 2\sin 2\theta\cos 2\theta$.

This result can be further simplified by again using the double angle formulae to obtain an expression in terms of θ:

$$\sin 4\theta = 2\times 2\sin\theta\cos\theta\times(\cos^2\theta - \sin^2\theta) = 4\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta).$$

Also, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, so it is possible to express a function of an angle in terms of half the angle.

Example 6

If $\tan A = -\frac{3}{4}$, $90° < A < 180°$, and $\cos B = \frac{5}{13}$, $0° < B < 90°$, write the exact value of the following.

(a) $\sin(A - B)$ **(b)** $\cos 2A$ **(c)** $\tan(A + B)$ **(d)** $\sin 2B$

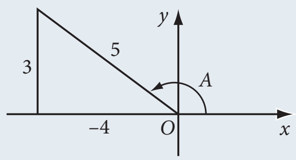
Solution

Draw diagrams to show the given ratio for each angle, then use the diagrams to find the other ratios for the angle.

$\sin A = \frac{3}{5}$

$\cos A = -\frac{4}{5}$

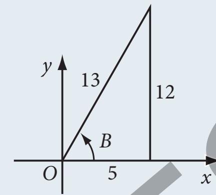
$\tan A = -\frac{3}{4}$



$\sin B = \frac{12}{13}$

$\cos B = \frac{5}{13}$

$\tan B = \frac{12}{5}$



(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{63}{65}$$

(b) $\cos 2A = \cos^2 A - \sin^2 A$

$$= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

(c) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{-\frac{3}{4} + \frac{12}{5}}{1 + \frac{3}{4} \times \frac{12}{5}} = \frac{33}{56}$$

(d) $\sin 2B = \sin B \cos B$

$$= 2 \times \frac{12}{13} \times \frac{5}{13}$$

$$= \frac{120}{169}$$

Example 7

(a) Prove that $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$.

(b) Prove that $\dfrac{\sin 2\alpha + \sin\alpha}{1 + \cos 2\alpha + \cos\alpha} = \tan\alpha$.

Solution

(a) $\text{LHS} = \dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta}$

$$= \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos(3\theta - \theta)}{\frac{1}{2}\sin 2\theta}$$

$$= \frac{2\cos 2\theta}{\sin 2\theta}$$

$$= 2\cot 2\theta = \text{RHS}$$

(b) **Method 1**

$$\text{LHS} = \frac{\sin 2\alpha + \sin\alpha}{1 + \cos 2\alpha + \cos\alpha}$$

$$= \frac{2\sin\alpha\cos\alpha + \sin\alpha}{1 + 2\cos^2\alpha - 1 + \cos\alpha}$$

$$= \frac{\sin\alpha\left(2\cos\alpha + 1\right)}{\cos\alpha\left(2\cos\alpha + 1\right)}$$

$$= \frac{\sin\alpha}{\cos\alpha} \quad \text{if } 2\cos\alpha + 1 \neq 0$$

$$= \tan\alpha = \text{RHS}$$

Method 2

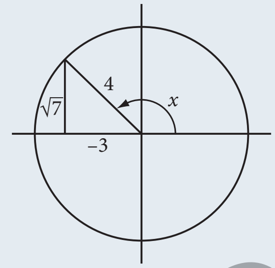
In the denominator, use formula [7] to directly replace $1 + \cos 2\alpha$ with $2\cos^2\alpha$.

Example 8

If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} \le x \le \pi$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$

Solution

From the diagram (drawn to show the given ratio for the angle):



(a) $\sin x = \frac{\sqrt{7}}{4}$

(b) $\sin 2x = 2\sin x \cos x$

$$= 2 \times \frac{\sqrt{7}}{4} \times \left(-\frac{3}{4}\right)$$

$$= -\frac{3\sqrt{7}}{8}$$

Example 9

Simplify:

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$ **(b)** $\sin\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}-\phi\right)$ **(c)** $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

Solution

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$ (using formula [2] from page 75)

$$= \cos\frac{\pi}{2} = 0$$

(b) $\sin\left(\frac{\pi}{2}-\theta\right)+\sin\left(\frac{\pi}{2}-\phi\right) = \cos\theta + \cos\phi$

(c) $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right) = \frac{1}{2}\times 2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

$= \frac{1}{2}\sin 2\left(\frac{\pi}{4}-x\right)$ (using double-angle formula [8] from page 75)

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}-2x\right) = \frac{1}{2}\cos 2x$$

EXERCISE 9.3 DOUBLE ANGLE FORMULAE

1 **(a)** By writing $\sin 3\theta$ as $\sin(2\theta + \theta)$, write $\sin 3\theta$ in terms of $\sin\theta$.
(b) Hence write $\cos 3\theta$ in terms of $\cos\theta$. **(c)** Hence write $\tan 3\theta$ in terms of θ.

2 If $\sin\theta = \frac{3}{4}$, $90° < \theta < 180°$, evaluate (in surd form):
(a) $\sin 2\theta$ **(b)** $\cos 2\theta$ **(c)** $\tan 2\theta$. **(d)** In which quadrant is 2θ?

3 Simplify:
(a) $\frac{\sin 2A}{1+\cos 2A}$ **(b)** $\frac{1}{2}\sin 2\theta \tan\theta$ **(c)** $\cos^2 2\theta - \sin^2 2\theta$ **(d)** $\cos^2 30° - \sin^2 30°$
(e) $\sin 4x \cos 4x$ **(f)** $1 + \cos(180° + 2\theta)$ **(g)** $\sin x \cos x \cos 2x$ **(h)** $2\sin 2x \cos 2x$
(i) $(\sin\theta + \cos\theta)^2$ **(j)** $(\sin A - \cos A)^2$ **(k)** $\frac{2\tan\theta}{1-\tan^2\theta}$ for $\theta = 22.5°$ **(l)** $\sin^2 50° + \sin^2 40°$
(m) $\sin(45° - x)\cos(45° - x)$ **(n)** $\frac{1-\cos 2\theta}{1+\cos 2\theta}$ **(o)** $2\cos^2 3x - 1$

4 If $\sin\theta = \frac{3}{5}, \frac{\pi}{2} \le \theta \le \pi$ and $\tan\phi = \frac{7}{24}, 0 \le \phi \le \frac{\pi}{2}$, find the value of:

(a) $\sin(\theta - \phi)$ (b) $\cos(\theta - \phi)$ (c) $\tan(\theta - \phi)$

5 Simplify:

(a) $1 + \tan^2\left(\frac{\pi}{2} - \alpha\right)$ (b) $1 - \cos^2(\pi + \theta)$ (c) $\sin\theta\cos\left(\frac{\pi}{2} - \theta\right) + \cos\theta\sin\left(\frac{\pi}{2} - \theta\right)$

(d) $2\cos^2\frac{\pi}{6} - 1$ (e) $1 - \sin\theta\cos\left(\frac{\pi}{2} - \theta\right)$ (f) $\sin(\pi - \theta)\cos\phi - \cos(\pi - \theta)\sin\phi$

6 Given that $\sin\theta = \frac{9}{41}$ and $\frac{\pi}{2} < \theta < \pi$, which of the following fractions is equivalent to $\sin 2\theta$?

A $-\frac{720}{1681}$ B $-\frac{81}{1681}$ C $\frac{81}{1681}$ D $\frac{720}{1681}$

7 What is the value of $\sin 2A$ given $\tan A = -\frac{5}{12}$ and $-\frac{\pi}{2} < A < 0$?

A $\frac{60}{169}$ B $-\frac{60}{169}$ C $\frac{120}{169}$ D $-\frac{120}{169}$

8 If $\sin x = \frac{7}{25}$ and x is an obtuse angle, find the value of $\tan 2x$.

A $\frac{527}{336}$ B $-\frac{527}{336}$ C $\frac{336}{527}$ D $-\frac{336}{527}$

9 Find a simplified exact value of $\sin\left(\frac{\pi}{8}\right)$.

9.4 USING IDENTITIES TO SIMPLIFY EXPRESSIONS AND PROVE RESULTS

When working with some trigonometric equations, you will also need to use the Pythagorean identities learnt in the Mathematics Advanced course:

$\sin^2\theta + \cos^2\theta = 1$	$\sin^2\theta = 1 - \cos^2\theta$	$\cos^2\theta = 1 - \sin^2\theta$
$\sec^2\theta - \tan^2\theta = 1$	$\sec^2\theta = 1 + \tan^2\theta$	$\tan^2\theta = \sec^2\theta - 1$
$\mathrm{cosec}^2\theta - \cot^2\theta = 1$	$\mathrm{cosec}^2\theta = 1 + \cot^2\theta$	$\cot^2\theta = \mathrm{cosec}^2\theta - 1$

Example 10

Prove that $\frac{\cos 2\theta + \sin 2\theta - 1}{\cos 2\theta - \sin 2\theta + 1} = \tan\theta$.

Solution

$$\begin{aligned}
\text{LHS} &= \frac{\cos 2\theta + \sin 2\theta - 1}{\cos 2\theta - \sin 2\theta + 1} \\
&= \frac{1 - 2\sin^2\theta + 2\sin\theta\cos\theta - 1}{2\cos^2\theta - 1 - 2\sin\theta\cos\theta + 1} \\
&= \frac{2\sin\theta(\cos\theta - \sin\theta)}{2\cos\theta(\cos\theta - \sin\theta)} \\
&= \frac{\sin\theta}{\cos\theta} \\
&= \tan\theta = \text{RHS}
\end{aligned}$$

Notice how two different expansions for $\cos 2\theta$ are used in Example **10** above. To decide which expansion is the best to use in each part you must consider the −1 in the numerator and the +1 in the denominator. The aim is to remove these constants by using the appropriate form.

Using $\cos 2\theta = \cos^2\theta - \sin^2\theta$ would have made the question more complicated. Try this to see for yourself.

EXERCISE 9.4 USING IDENTITIES TO SIMPLIFY EXPRESSIONS AND PROVE RESULTS

Prove the following identities (questions **1** to **21**):

1 $\dfrac{\sin A + \cos A \tan B}{\cos A - \sin A \tan B} = \tan(A + B)$

2 $\dfrac{\sin 2\theta \cos\theta - \cos 2\theta \sin\theta}{\cos 2\theta \cos\theta + \sin 2\theta \sin\theta} = \tan\theta$

3 $\dfrac{\tan A - \tan B}{\tan A + \tan B} = \dfrac{\sin(A - B)}{\sin(A + B)}$

4 $\sin(\theta + \alpha)\sin(\theta - \alpha) = \sin^2\theta - \sin^2\alpha$

5 $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2$

6 $\tan(45° + A) + \tan(45° - A) = \dfrac{2}{\cos 2A}$

7 $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

8 $\dfrac{\sin 2\theta + 1}{\cos 2\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

9 $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} + \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\sec 2\theta$

10 $\dfrac{1 - \cos x}{\sin x} = \tan\dfrac{x}{2}$

11 $\dfrac{\sin A + \sin(90° - A) + 1}{\sin A - \sin(90° - A) + 1} = \cot\dfrac{A}{2}$

12 $\dfrac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} = \dfrac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}$

13 $\sin(A + B + C) = \sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B - \sin A\sin B\sin C$

What is the resulting identity if C is replaced by $(90° - B)$?

14 $\cos(A + B + C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \cos B\sin C\sin A - \cos C\sin A\sin B$

What is the resulting identity if B is replaced by $(90° - C)$?

15 $\tan(\theta + \alpha)\tan(\theta - \alpha) = \dfrac{\tan^2\theta - \tan^2\alpha}{1 - \tan^2\theta\tan^2\alpha}$

16 $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = 1 - \dfrac{1}{2}\sin 2\theta$

17 $\dfrac{2\cos\frac{\theta}{2} - 1 - \cos\theta}{2\cos\frac{\theta}{2} + 1 + \cos\theta} = \dfrac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}}$

18 $\cot(x + y) = \dfrac{\cot x\cot y - 1}{\cot x + \cot y}$

19 $\dfrac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta$

20 $\tan(\theta + 45°)\tan(\theta - 45°) = -1$

21 $\dfrac{1 - \tan\theta\tan 2\theta}{1 + \tan\theta\tan 2\theta} = 4\cos^2\theta - 3$

22 If $\tan A = \dfrac{p}{q}$, express the following in terms of p and q.

(a) $q\sin A\cos A + p\sin^2 A$ **(b)** $p\sin 2A + q\cos 2A$

23 If A, B and C are the angles of a triangle, prove that $\cos A\cos B - \sin A\sin B + \cos C = 0$.

24 Given that $\sin 18° = \dfrac{1}{4}\left(\sqrt{5} - 1\right)$, find $\cos 36°$ in surd form.

25 Find $\tan x$ in terms of $\tan\theta$ if $\tan\theta = \dfrac{\cos(\theta + x)}{\cos(\theta - x)}$.

26 Three points P, Q, R are in a horizontal plane. Angles RPQ and RQP are α and β respectively. If PQ is x units in length, show that the perpendicular distance y from R to PQ is given by $y = \dfrac{x\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$.

27 Using the double-angle formula for $\tan\theta$, find $\tan 22.5°$ in simplest surd form.

28 $\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)$ simplifies to:

A $\cos 2x$ **B** $\sin 2x$ **C** $\cos x$ **D** $\sin x$

29 If $\tan\theta=\frac{3}{5}$ and $\pi<\theta<\frac{3\pi}{2}$, find the value of: **(a)** $\sin\theta$ **(b)** $\cos\theta$ **(c)** $\cos 2\theta$

30 If $\frac{\pi}{2}\le x\le\pi$ and $\cos x=-\frac{5}{6}$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$ **(c)** $\tan 2x$

31 If $\operatorname{cosec}\alpha=-\frac{17}{8}$ and $\pi<\alpha<\frac{3\pi}{2}$, find the value of: **(a)** $\cot\alpha$ **(b)** $\tan 2\alpha$

32 Prove the following.

(a) $2\cos\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}-x\right)=\cos 2x$ **(b)** $\tan\left(\theta+\frac{\pi}{4}\right)\tan\left(\theta-\frac{\pi}{4}\right)=-1$

(c) $\left(\sec^2\theta-1\right)\tan\left(\frac{\pi}{2}-\theta\right)=\tan\theta$ **(d)** $\tan\left(x+\frac{3\pi}{4}\right)=\frac{\tan x-1}{\tan x+1}$

(e) $\dfrac{1-\sin\left(\frac{\pi}{2}-2x\right)}{\sin 2x}=\tan x$ **(f)** $\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)=2\sec 2A$

33 If $\tan x=\frac{5}{4}$, $\tan y=\frac{1}{9}$ and $0<y<x<\frac{\pi}{2}$, prove that $x-y=\frac{\pi}{4}$.

34 Simplify: **(a)** $\dfrac{2\tan\theta}{1-\tan^2\theta}$ where $\theta=\frac{7\pi}{8}$ **(b)** $2\cos^2 3x-1$ where $x=\frac{2\pi}{9}$

35 By expanding each term on the left-hand side, prove that $\sin\left(\theta+\frac{\pi}{6}\right)\sin\left(\theta-\frac{\pi}{6}\right)=\sin^2\theta-\frac{1}{4}$.

36 If $0\le\theta\le\frac{\pi}{2}$, prove that $\tan\theta=\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$. Hence show that the exact value of $\tan\frac{\pi}{8}$ is $\sqrt{2}-1$.

37 **(a)** By writing expansions for $\sin(A+B)$ and $\sin(A-B)$, find a simplified expression for $\sin(A+B)+\sin(A-B)$.

(b) By writing $\theta=A+B$ and $\phi=A-B$, find an expression for $\sin\theta+\sin\phi$ as the product of two trigonometric functions.

38 If $\sec\theta-\tan\theta=\frac{3}{5}$, show that $\sin\theta=\frac{8}{17}$. (Hint: Use t formulae.)

39 If $4\tan(\alpha-\beta)=3\tan\alpha$, prove that $\tan\beta=\dfrac{\sin 2\alpha}{7+\cos 2\alpha}$.

40 Use the factors of x^3-y^3 to show that $\cos^6\theta-\sin^6\theta=\left(1-\frac{1}{4}\sin^2 2\theta\right)\cos 2\theta$.

41 If $\tan\theta=t$, express $\sin 2\theta$ and $\cos 2\theta$ in terms of t. Find the values of t for which $(k+1)\sin 2\theta+(k-1)\cos 2\theta=k+1$.

42 If A, B and C are successive terms of an arithmetic series, prove that $\sin A+\sin C=2\sin B\cos(B-A)$.

43 If $\cos\theta=\dfrac{l^2-m^2}{l^2+m^2}$ and $0<\theta<\frac{\pi}{2}$, express $\tan\theta$ and $\sin 2\theta$ in terms of l and m.

44 If $\tan\alpha=k\tan\beta$, show that $(k-1)\sin(\alpha+\beta)=(k+1)\sin(\alpha-\beta)$.

45 Show that $4\sin\theta\sin\left(\theta-\frac{\pi}{3}\right)\sin\left(\theta-\frac{2\pi}{3}\right)=\sin 3\theta$.

9.5 OVERVIEW OF TRIGONOMETRIC EQUATIONS

Equations of the form $\sin\theta = \sin\alpha$

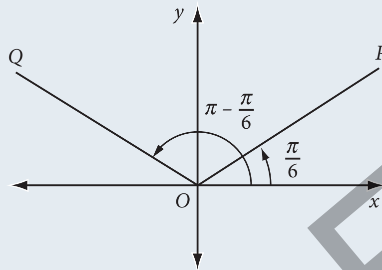
Example 11

Find all values of θ for which $\sin\theta = \frac{1}{2}$.

Solution

$$\sin\theta = \frac{1}{2}$$

$$\therefore \ \sin\theta = \sin\frac{\pi}{6}, \sin\left(\pi - \frac{\pi}{6}\right), \sin\left(2\pi + \frac{\pi}{6}\right), \ldots$$



Consider a coordinate diagram.

$$\angle XOP = \frac{\pi}{6}$$

$$\angle XOQ = \pi - \frac{\pi}{6}$$

The ray OP defines an infinite number of angles in the first quadrant. If you rotate OP about the origin (either clockwise or anticlockwise), then during each revolution it is along the original ray OP once.
Each full rotation increases the angle by 2π, so you find that OP is the terminal ray defining the angles:

- $\frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, \ldots$ for anticlockwise rotation
- $-2\pi + \frac{\pi}{6}, -4\pi + \frac{\pi}{6}, -6\pi + \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi + \frac{\pi}{6}$ where $n = 0, \pm 2, \pm 4, \ldots$ [1]

or: $n \times 180° + 30°$ (in degrees)

Similarly, the terminal ray OQ defines an infinite number of angles:

$\pi - \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, 5\pi - \frac{\pi}{6}, \ldots$ for anticlockwise rotation

$-\pi - \frac{\pi}{6}, -3\pi - \frac{\pi}{6}, -5\pi - \frac{\pi}{6}, \ldots$ for clockwise rotation.

This result can be summarised as: $n\pi - \frac{\pi}{6}$ where $n = \pm 1, \pm 3, \pm 5, \ldots$ [2]

or: $n \times 180° - 30°$ (in degrees)

Statements [1] and [2] can be written together as:

$$\theta = n\pi + (-1)^n \frac{\pi}{6}$$

or: $\theta = n \times 180° + (-1)^n \times 30°$ (in degrees)

Note: $(-1)^n$ is 1 when n is zero or even, and is -1 when n is odd.

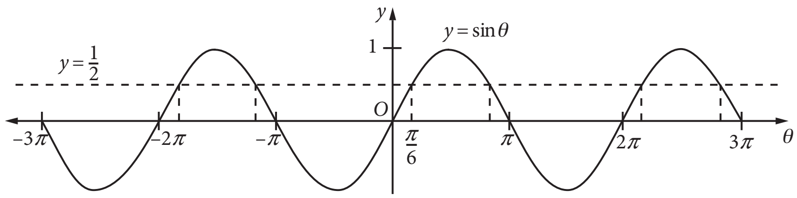
Following Example **11** above, the general solution to the equation $\sin\theta = \sin\alpha$ can be written as:

$\theta = n\pi + (-1)^n\alpha$ (in radians)

$\theta = n \times 180° + (-1)^n\alpha°$ (in degrees) for any integer n.

The general solution of trigonometric equations has not been included in this course. This material is included to show why you have to be very careful to consider all possible results in the given domain when solving trigonometric equations.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \sin\theta$ and $y = \frac{1}{2}$ (from Example **11**), as shown in the following diagram.



From symmetry, you can observe that the line $y = \frac{1}{2}$ intersects the sine curve at values of θ that are $\frac{\pi}{6}$ units to the right of $n\pi$ where $n = 0, \pm2, \pm4, \ldots$ and that are $\frac{\pi}{6}$ units to the left of $n\pi$ when $n = \pm1, \pm3, \ldots$

Both of these solutions are contained in the statement: $\theta = n\pi + (-1)^n \frac{\pi}{6}$

$$\text{or: } \theta = \begin{cases} n\pi + \frac{\pi}{6}, & n \text{ even or zero} \\ n\pi - \frac{\pi}{6}, & n \text{ odd} \end{cases}$$

Example 12

Solve $\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ for $0 \le \theta \le 2\pi$.

Solution

$\sin\left(\theta + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$\sin\theta < 0$ in third and fourth quadrants: $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{5\pi}{4}, \sin\frac{7\pi}{4}$

$$\theta + \frac{\pi}{4} = \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\theta = \pi, \frac{3\pi}{2}$$

Equations of the form $\cos\theta = \cos\alpha$

Example 13

Find all angles θ for which $\cos\theta = \frac{1}{2}$.

Solution

$\cos\theta = \frac{1}{2}$

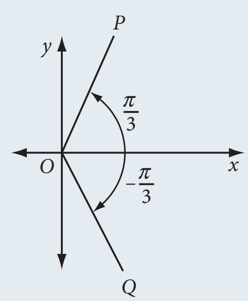
i.e. $\cos\theta = \cos\frac{\pi}{3}, \cos\left(2\pi - \frac{\pi}{3}\right), \cos\left(2\pi + \frac{\pi}{3}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 4\pi + \frac{\pi}{3}, \ldots$

and OQ defines the angles: $-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 4\pi - \frac{\pi}{3}, \ldots$

These results can be summarised as: $\theta = 2n\pi \pm \frac{\pi}{3}$ where n is any integer

or: $\theta = n \times 360° \pm 60°$ (in degrees)

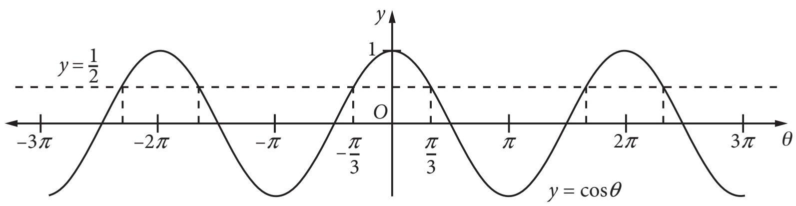


Following Example **13**, above, the general solution to the equation $\cos\theta = \cos\alpha$ can be written as:

$\theta = 2n\pi \pm \alpha$ (in radians)

$\theta = n \times 360° \pm \alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \cos\theta$ and $y = \frac{1}{2}$ (from Example **13**), as shown in the following diagram.



From symmetry, you can observe that the line $y = \frac{1}{2}$ intersects the cosine curve at values of θ that are $\frac{\pi}{3}$ units to the left and right of $2n\pi$ where $n = 0, \pm 2, \pm 4, \ldots$ i.e. $\theta = 2n\pi \pm \frac{\pi}{3}$.

Example 14

Solve $2\cos(3x + 30°) + \sqrt{3} = 0$ for $0° \le x \le 360°$.

Solution

$2\cos(3x + 30°) = -\sqrt{3}$

$\cos(3x + 30°) = -\frac{\sqrt{3}}{2}$

$\cos\theta < 0$ in second and third quadrants: $\cos(3x + 30°) = \cos 150°, \cos 210°, \ldots$

As $0° \le x \le 360°$, thus $0° \le 3x \le 3 \times 360°$, so two more revolutions are needed.

$3x + 30° = 150°, 210°, 510°, 570°, 870°, 930°$

$3x = 120°, 180°, 480°, 540°, 840°, 900°$

$x = 40°, 60°, 160°, 180°, 280°, 300°$

Equations of the form $\tan\theta = \tan\alpha$

Example 15

Find all the angles for which $\tan\theta = 1$, where θ is in radians.

Solution

$\tan\theta = 1$

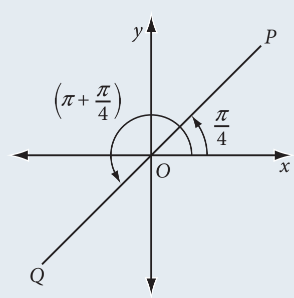
i.e. $\tan\theta = \tan\frac{\pi}{4}, \tan\left(\pi + \frac{\pi}{4}\right), \tan\left(2\pi + \frac{\pi}{4}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, \ldots$

and OQ defines the angles: $\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}, \ldots$

These results can be summarised as: $\theta = n\pi + \frac{\pi}{4}$ where n is any integer

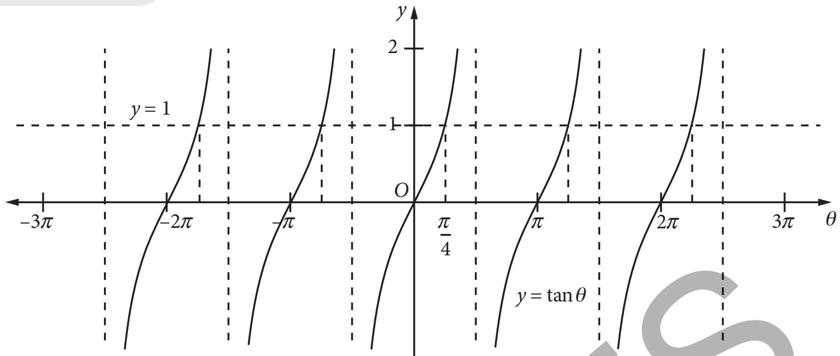
or: $\theta = n \times 180° + 45°$ (in degrees)



Following Example **15**, the general solution to the equation $\tan\theta = \tan\alpha$ can be written as:

$\theta = n\pi + \alpha$ (in radians)
$\theta = n \times 180° + \alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \tan\theta$ and $y = 1$ (from Example **14**), as shown in the diagram at right.



From symmetry, you can observe that the line $y = 1$ intersects the tangent curve at values of θ, which are $\frac{\pi}{4}$ units to the right of $n\pi$ where $n = 0, \pm1, \pm2, \ldots$

Example 16

Solve $\tan x = 3\cot x$ for $-\pi \le x \le \pi$.

Solution

$\tan x = 3\cot x$

$\tan x = \dfrac{3}{\tan x}$

$\tan^2 x = 3$

$\tan x = \pm\sqrt{3}$

Solution is in all four quadrants: $x = -\pi + \dfrac{\pi}{3}, -\dfrac{\pi}{3}, \dfrac{\pi}{3}, \pi - \dfrac{\pi}{3}$

$x = -\dfrac{2\pi}{3}, -\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

EXERCISE 9.5 OVERVIEW OF TRIGONOMETRIC EQUATIONS

1 Solve for $0 \le x \le 2\pi$.

(a) $\sin x = 1$ (b) $\cos x = 0$ (c) $\tan x = -1$ (d) $\sqrt{3}\operatorname{cosec} x = 2$ (e) $\sec x = -2$

(f) $\cot x = \sqrt{3}$ (g) $2\sin\left(x - \dfrac{\pi}{6}\right) + 1 = 0$ (h) $\cos\dfrac{x}{2} = 1$ (i) $2\sin^2 x = 1$ (j) $\sin x = 0.3894$

2 The solution to $4\cos^2 x - 1 = 0$ for $0 \le x \le 2\pi$ is:

A $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ B $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ C $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$ D $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

3 Solve for $-\pi \le x \le \pi$.

(a) $\cos^2 x - 2\cos x + 1 = 0$ (b) $\sin^2 x = \sin x$ (c) $\cos 2x = \sin x$
(d) $\sin^2 x = 1 - \cos x$ (e) $\cos 2x = 2 + \cos x$ (f) $\tan 2x = \cot x$
(g) $\cos 2x = \cos x$ (h) $2\sin x = \sec x$ (i) $\tan^2 x = \tan x$

4 The solution to $2\cos(2x - 60°) = \sqrt{3}$ for $0° \le x \le 180°$ is:

A $x = 45°$ B $x = 145°$ C $x = 30°, 330°$ D $x = 15°, 145°$

5 Solve for $0 \le \theta \le 2\pi$.

(a) $\sqrt{2}\sin 2\theta + 1 = 0$ (b) $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ (c) $\cos 2\theta \cos\frac{\pi}{6} - \sin 2\theta \sin\frac{\pi}{6} = 0.5$

(d) $\tan\theta = \sin 2\theta$ (e) $\tan\theta = \cot\theta$ (f) $\sin 3\theta + \sin\theta = 0$ (g) $\sin 4\theta - \sin 2\theta = 0$

6 Solve for $-\pi \le \theta \le \pi$.

(a) $\cos 3\theta = \cos\theta$ (b) $2\cos 2\theta = 4\cos\theta - 3$ (c) $3\tan 2\theta = 2\tan\theta$

(d) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ (e) $2\cos\left(2\theta - \frac{\pi}{3}\right) = \sqrt{3}$ (f) $2\sin^2\theta + \cos\theta = 1$

7 Solve for $0° \le x \le 360°$.

(a) $4 + \sin x = 6\cos^2 x$ (b) $\sin x = \cos x$ (c) $1 + 2\cos^2 x = 5\sin x$

8 Solve for $0 \le \theta \le 2\pi$.

(a) $\tan^3\theta - \tan\theta = 0$ (b) $\tan\theta = \sin\theta$ (c) $\sec 2\theta = \operatorname{cosec} 2\theta$

(d) $\sin 2\theta = \tan\theta$ (e) $\sin 3\theta = \sin 2\theta$

9.6 SIMPLE TRIGONOMETRIC EQUATIONS

There is no general method for solving trigonometric equations and inequalities, but there are certain standard procedures and types, as illustrated in the examples below.

Example 17

Solve the equation $3\sin 2\theta = 1.5$, $0 \le \theta \le \pi$.

Solution

$3\sin 2\theta = 1.5$

$\therefore \quad \sin 2\theta = 0.5$

$0 \le \theta \le \pi$ means that $0 \le 2\theta \le 2\pi$:

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence: $\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

This solution can be checked graphically by using graphing software to find the intersection of $y = 3\sin 2\theta$ and $y = 1.5$. Over the domain $0 \le \theta \le \pi$ these functions intersect only twice. If the domain is increased, there will be two more intersections for each domain increase of π units.

Example 18

Solve the equation $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

Solution

Let $\theta = 2x - \frac{\pi}{6}$: $\cos\theta = \frac{\sqrt{3}}{2}$

Hence: $\theta = \ldots -\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{11\pi}{6}, \ldots$

From the limits: $0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$

But $2x = \theta + \frac{\pi}{6}$: $0 \le \theta + \frac{\pi}{6} \le 4\pi$

$-\frac{\pi}{6} \le \theta \le 4\pi - \frac{\pi}{6}$

$\therefore \quad \theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi + \frac{\pi}{6}, 2\pi + \frac{11\pi}{6}$

$$\therefore\ 2x-\frac{\pi}{6}=-\frac{\pi}{6},\frac{\pi}{6},\frac{11\pi}{6},2\pi+\frac{\pi}{6},2\pi+\frac{11\pi}{6}$$
$$2x=0,\frac{\pi}{3},2\pi,\frac{7\pi}{3},4\pi$$
$$x=0,\frac{\pi}{6},\pi,\frac{7\pi}{6},2\pi$$

Example 19

Find the values of x for which $\cos 2x\le\frac{1}{\sqrt{2}}$, $0\le x\le 2\pi$.

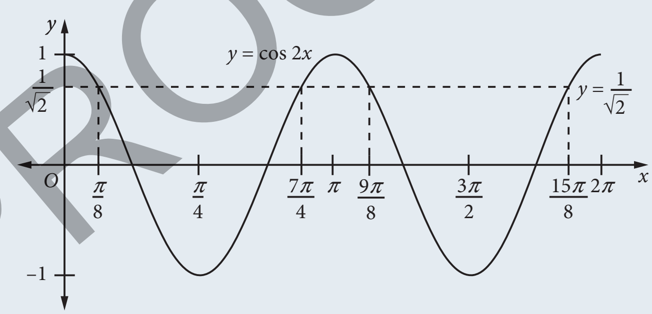
Solution

First solve the equation, then solve the inequality graphically.

Solve the equation: $\cos 2x=\frac{1}{\sqrt{2}}$ $\quad(0\le x\le 2\pi)$

$$2x=\frac{\pi}{4},\frac{7\pi}{4},2\pi+\frac{\pi}{4},2\pi+\frac{7\pi}{4}\quad\text{(as } 0\le 2x\le 4\pi\text{, around the circle twice)}$$
$$x=\frac{\pi}{8},\frac{7\pi}{8},\frac{9\pi}{8},\frac{15\pi}{8}$$

Now sketch the graph of $y=\cos 2x$ for $0\le x\le 2\pi$, showing the line $y=\frac{1}{\sqrt{2}}$ also:



Use the diagram to find where the graph of $y=\cos 2x$ is on or below the line $y=\frac{1}{\sqrt{2}}$.

Hence $\cos 2x\le\frac{1}{\sqrt{2}}$ for $\frac{\pi}{8}\le x\le\frac{7\pi}{8}$ and for $\frac{9\pi}{8}\le x\le\frac{15\pi}{8}$.

EXERCISE 9.6 SIMPLE TRIGONOMETRIC EQUATIONS

1 Solve for values of θ and x between 0 and 2π inclusive:

(a) $\sin\theta=\frac{\sqrt{3}}{2}$ (b) $\tan x=-1$ (c) $\cos x=-0.5$ (d) $\sqrt{3}\tan\theta=1$

(e) $\sin 2\theta=-\frac{1}{2}$ (f) $\operatorname{cosec}\theta=-2$ (g) $\cot 2x=\sqrt{3}$ (h) $\sec 2\theta=\sqrt{2}$

2 Solve between 0° and 360°: (a) $\cos x=0.4$ (b) $4\tan 2\theta+3=0$

3 The solution to $\sqrt{2}\sin 2\theta+1=0$ for $0\le\theta\le 2\pi$ is:

A $\frac{5\pi}{4},\frac{7\pi}{4}$ B $\frac{5\pi}{8},\frac{7\pi}{8}$ C $\frac{5\pi}{8},\frac{7\pi}{8},\frac{13\pi}{8},\frac{15\pi}{8}$ D $\frac{5\pi}{4},\frac{7\pi}{4},\frac{13\pi}{4},\frac{15\pi}{4}$

4 Solve for $-\pi\le x\le\pi$: (a) $2\cos 2x+1=0$ (b) $\sqrt{2}\sin 2x-1=0$

5 Solve between 0 and 2π inclusive:

(a) $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ (b) $\tan\left(\theta-\frac{\pi}{3}\right)=-\sqrt{3}$ (c) $\cos\left(2x+\frac{\pi}{3}\right)=\frac{1}{2}$ (d) $2\sin\left(2x-\frac{\pi}{6}\right)=1$

(e) $\tan\left(2\theta-\frac{\pi}{4}\right)+1=0$ (f) $2\cos\left(2x-\frac{\pi}{3}\right)=\sqrt{3}$ (g) $\sin 2\theta=-\cos\frac{7\pi}{4}$ (h) $\sin x=\cos x$

6 Solve for $0°<x<360°$: (a) $\cos(2x+60°)=0.7242$ (b) $5\sin(2x-70°)+4=0$

7 If $0 \le x \le 2\pi$, the solution to $\sin x \le \frac{\sqrt{3}}{2}$ is:

A $x \le \frac{\pi}{3}$ **B** $x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **C** $0 \le x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **D** $0 \le x \le \frac{\pi}{3}$ or $\frac{2\pi}{3} \le x \le 2\pi$

8 If $0 \le x \le 2\pi$, solve: **(a)** $\sin x \ge \frac{1}{2}$ **(b)** $\cos x < \frac{1}{2}$ **(c)** $\sin x > 0$ **(d)** $\cos x > \frac{\sqrt{3}}{2}$
(e) $\sin x < 1$ **(f)** $\cos x > 0$ **(g)** $\tan x > 1$ **(h)** $\sqrt{2}\cos x > -1$

9 Solve for $0 < x < \pi$: **(a)** $\sin 2x \ge \frac{1}{2}$ **(b)** $\cos 2x \le 0$

10 Solve for $-\pi < x < \pi$: **(a)** $2\sin 2x \le \sqrt{3}$ **(b)** $2\cos 2x > -1$

9.7 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

You have already solved simple trigonometric equations, you can now use the angle formulae found in this chapter to solve harder equations.

Example 20

Solve the equation $\sin 2x = 3\cos x$, $0 \le x \le 2\pi$.

Solution

As $\sin 2x = 2\sin x \cos x$: $2\sin x \cos x = 3\cos x$

$$\cos x(2\sin x - 3) = 0$$

$$\therefore \cos x = 0 \text{ or } \sin x = 1.5$$

Because $|\sin x| \le 1$, the only solution is $\cos x = 0$.

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Example 21

Solve the equation $\cos 2x \cos\alpha - \sin 2x \sin\alpha = -0.5$, $0 \le x \le 2\pi$, where $\alpha = \frac{\pi}{6}$.

Solution

Use the expansion of $\cos(A + B)$ to simplify the LHS:

$\cos 2x \cos\alpha - \sin 2x \sin\alpha = \cos(2x + \alpha)$

Hence, as $\alpha = \frac{\pi}{6}$: $\cos\left(2x + \frac{\pi}{6}\right) = -0.5$

$$2x + \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}$$

$$2x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{5\pi}{2}, \frac{19\pi}{6}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$$

EXERCISE 3.7 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

1 Solve: (a) $\cos 2\theta = \cos\theta$, $0 \le \theta \le 2\pi$ (b) $2\cos 2\theta = 4\cos\theta - 3$, $0 \le \theta \le 2\pi$
(c) $3\tan 2\theta = 2\tan\theta$, $0 \le \theta \le 2\pi$ (d) $\tan\theta + 2\cot\theta = 3$, $0° \le \theta \le 360°$

2 The solution to $5\sin x = 2\sec x$ for $0° \le x \le 180°$ is:

A $x = 11°47'$ or $78°13'$
B $x = 23°34'$ or $156°26'$
C $x = 26°34'$ or $63°26'$
D $x = 53°8'$ or $126°52'$

3 Solve: (a) $\cos 2x \cos\frac{\pi}{6} - \sin 2x \sin\frac{\pi}{6} = \frac{1}{2}$, $0 \le x \le 2\pi$ (b) $\sin 2x \cos\frac{\pi}{3} + \cos 2x \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $-\pi \le x \le \pi$

4 The solution to $\sin\theta = \cos 2\theta$ for $0 \le \theta \le 2\pi$ is:

A $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$
B $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$
C $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
D $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

5 Solve $\tan\theta = \sin 2\theta$, $0 \le \theta \le 2\pi$.

6 Solve $\cos^2\theta = 2\cos^2\frac{\theta}{2}$, $0° \le \theta \le 360°$.

7 Solve $\sin 3x \cos x - \cos 3x \sin x = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

8 Solve $\tan 2\theta = 2\tan\theta$, $0 \le \theta \le 2\pi$.

9 (a) Prove that $\tan 4\theta - \tan\theta = \dfrac{\sin 3\theta}{\cos 4\theta \cos\theta}$.
(b) Hence, solve $\tan 4\theta = \tan\theta$ for $0 \le \theta \le \pi$.

10 (a) By considering $\tan 3A$ as $\tan(2A + A)$, show that $\tan A \tan 2A \tan 3A = \tan 3A - \tan 2A - \tan A$.
(b) Hence, solve $\tan 3A - \tan 2A - \tan A = 0$, where $0 < A \le \pi$.

11 (a) Show that $\dfrac{\sin 2\theta \cos\theta - \sin\theta \cos 2\theta}{\sin 6\theta \cos 2\theta - \sin 2\theta \cos 6\theta} = \frac{1}{4}\sec\theta \sec 2\theta$.
(b) If $\frac{1}{4}\sec\theta \sec 2\theta = -\frac{1}{4}$, $\theta, 0 \le \theta \le 2\pi$

12 (a) Show that the solution of $\cos 3x - \sin 2x = 0$, for $0 < x < \frac{\pi}{2}$ is given by $\sin x = \dfrac{\sqrt{5}-1}{4}$.
You may use the identity: $\cos 3x = 4\cos^3 x - 3\cos x$.
(b) Without using a calculator, verify that $x = \frac{\pi}{10}$ is a solution to $\cos 3x = \sin 2x$.
(c) Using the results obtained in parts (a) and (b), prove that $\sin\frac{\pi}{5}\cos\frac{\pi}{10} = \frac{\sqrt{5}}{4}$.

9.8 SOLVING TRIGONOMETRIC EQUATIONS USING THE AUXILIARY ANGLE METHOD

The **auxiliary angle** method of solving trigonometric equations involves changing an equation of the form $a\sin x \pm b\cos x = c$ into the form $r\sin(x \pm \alpha) = c$, which is then easier to solve. In this form, α is called the auxiliary angle.

This method can also be used to change $a\cos x \pm b\sin x = c$ into the form $r\cos(x \mp \alpha) = c$. In both cases, the constants a, b, r and α are positive real numbers.

For example, to express $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$:

$$\begin{aligned} \text{Let } a\sin x + b\cos x &= r\sin(x+\alpha) \\ &= r(\sin x\cos\alpha + \cos x\sin\alpha) \\ &= r\sin x\cos\alpha + r\cos x\sin\alpha \end{aligned}$$

This is an identity, so the coefficients of $\sin x$ and $\cos x$ on each side must be the same.

i.e. $a = r\cos\alpha$

$b = r\sin\alpha$

$\therefore\ a^2 + b^2 = r^2(\cos^2\alpha + \sin^2\alpha)$

Hence: $r^2 = a^2 + b^2$

$r = \sqrt{a^2 + b^2}$ because r is a positive real number.

From the coefficients, there is also $\cos\alpha = \frac{a}{r}$ and $\sin\alpha = \frac{b}{r}$.

As a and b are positive constants, so $\cos\alpha$ and $\sin\alpha$ are also positive. This also means that α is in the first quadrant (i.e. it is an acute angle), such that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{b}{a}$.

Hence, the auxiliary angle method gives:	$a\sin x + b\cos x = r\sin(x + \alpha)$
which then allows you to obtain:	$a\sin x - b\cos x = r\sin(x - \alpha)$
	$a\cos x + b\sin x = r\cos(x - \alpha)$
	$a\cos x - b\sin x = r\cos(x + \alpha)$

In each case, $r = \sqrt{a^2 + b^2}$ and α is an angle in the first quadrant such that $\tan\alpha = \frac{b}{a}$.

Example 22

Express: **(a)** $\sqrt{3}\sin x - \cos x$ in the form $r\sin(x - \alpha)$ **(b)** $3\cos x - 4\sin x$ in the form $r\cos(x + \alpha)$.

Solution

(a) $\sqrt{3}\sin x - \cos x = r\sin(x - \alpha)$

$= r(\sin x\cos\alpha - \cos x\sin\alpha)$

$= r\sin x\cos\alpha - r\cos x\sin\alpha$

Equate coefficients of $\sin x$ and $\cos x$: $r\cos\alpha = \sqrt{3}$ [1]

$r\sin\alpha = 1$ [2]

$[1]^2 + [2]^2$: $r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r^2 = 4$

$r = 2$ (as $r > 0$)

Hence from [1] and [2]: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

As $\cos\alpha$ and $\sin\alpha$ are both positive, α is in the first quadrant, such that $\tan\alpha = \frac{1}{\sqrt{3}}$, i.e. $\alpha = \frac{\pi}{6}$.

From the first equation: $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

(b) $3\cos x - 4\sin x = r\cos(x + \alpha)$

$a = 3, b = 4$: $r = \sqrt{3^2 + 4^2} = 5$

$\tan\alpha = \frac{4}{3}$: $\alpha = 53^\circ 8'$

$\therefore\ 3\cos x - 4\sin x = 5\cos(x + 53^\circ 8')$

Example **1** illustrates two different auxiliary angle methods that may be used. You should practise both.

Important uses of the auxiliary angle method

- Writing $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y = a\sin x + b\cos x$ much easier.
- Writing $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$ allows you to solve equations of the type $a\sin x \pm b\cos x = c$.

Example 23

Sketch the graph of $y = \sqrt{3}\sin x - \cos x$, $0 \le x \le 2\pi$.

Solution

Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$. Hence: $y = 2\sin\left(x - \frac{\pi}{6}\right)$

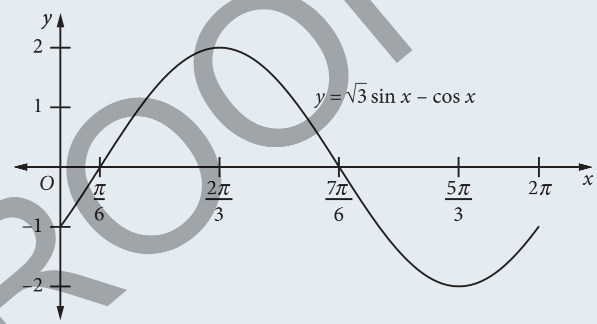
At the endpoints of the domain, $x = 0$ and $x = 2\pi$: $y = 2\sin\left(-\frac{\pi}{6}\right) = 2\sin\left(2\pi - \frac{\pi}{6}\right) = -1$

The greatest value of y is 2, where:

$$1 = \sin\left(x - \frac{\pi}{6}\right)$$
$$x - \frac{\pi}{6} = \frac{\pi}{2}$$
$$x = \frac{2\pi}{3}$$

The least value of y is -2, where:

$$-1 = \sin\left(x - \frac{\pi}{6}\right)$$
$$x - \frac{\pi}{6} = \frac{3\pi}{2}$$
$$x = \frac{5\pi}{3}$$



The graph crosses the x-axis where: $\sin\left(x - \frac{\pi}{6}\right) = 0$

$$x - \frac{\pi}{6} = 0, \pi$$
$$x = \frac{\pi}{6}, \frac{7\pi}{6}$$

Example 24

Solve the following equations.

(a) $\sqrt{3}\sin x - \cos x = 1$, $0 \le x \le 2\pi$

(b) $8\cos x + 6\sin x = -3$, $0° \le x \le 360°$

Solution

(a) Example 1 **(a)** has already shown that $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$.

$$\therefore \quad 2\sin\left(x - \frac{\pi}{6}\right) = 1$$
$$\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$
$$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$x = \frac{\pi}{3}, \pi$$

(b) Use $a\cos x + b\sin x = r\cos(x - a)$.

$$8\cos x + 6\sin x = -3$$
$$a = 8,\ b = 6: \quad r = \sqrt{8^2 + 6^2} = 10$$
$$\tan\alpha = \frac{6}{8} = 0.75 \quad \text{so} \quad \alpha = 36°\,52'$$
$$\therefore \quad 10\cos(x - 36°\,52') = -3$$
$$\cos(x - 36°\,52') = -0.3$$
$$x - 36°\,52' = 107°\,27',\ 252°\,33'$$
$$x = 144°\,19',\ 289°\,25'$$

EXERCISE 9.8 SOLVING TRIGONOMETRIC EQUATIONS USING THE AUXILIARY ANGLE METHOD

1 Express each of the following in the form $r\sin(x+\alpha)$.

(a) $\sin x+\cos x$ (b) $3\sin x+\sqrt{3}\cos x$
(c) $5\sin x+12\cos x$, $0°<\alpha<90°$ (d) $2\sin x+\cos x$, $0°<\alpha<90°$

2 Express each of the following in the form $r\sin(x-\alpha)$.

(a) $\sin x-\sqrt{3}\cos x$ (b) $2\sin x-3\cos x$, $0°<\alpha<90°$
(c) $2\sin x-\cos x$, $0°<\alpha<90°$ (d) $3\sin x-3\cos x$

3 Express each of the following in the form $r\cos(x-\alpha)$.

(a) $\cos x+\sin x$ (b) $24\cos x+7\sin x$, $0°<\alpha<90°$
(c) $2\cos x+2\sqrt{3}\sin x$ (d) $3\cos x+2\sin x$, $0°<\alpha<90°$

4 Express each of the following in the form $r\cos(x+\alpha)$.

(a) $\cos x-\sin x$ (b) $\sqrt{3}\cos x-\sin x$
(c) $8\cos x-\sin x$, $0°<\alpha<90°$ (d) $5\cos x-3\sin x$, $0°<\alpha<90°$

5 Which expression is equivalent to $8\sin x-15\cos x$?

A $17\cos(x-61°56')$ B $17\sin(x-61°56')$
C $17\cos(x+61°56')$ D $17\sin(x+61°56')$

6 Find **(i)** the maximum and **(ii)** the minimum value of the following expressions. Also find the smallest positive values of x for which the maximum and minimum occur.

(a) $\sin x-\sqrt{3}\cos x$ (b) $\cos x+\sin x$ (c) $2\sqrt{3}\cos x-2\sin x$
(d) $5\sin x+12\cos x$ (answer in degrees)

7 Solve:

(a) $\cos x+\sin x=1$, $0\le x\le 2\pi$ (b) $\cos x+\sqrt{3}\sin x=2$, $0\le x\le 2\pi$
(c) $3\cos x+2\sin x=\sqrt{13}$, $0°\le x\le 360°$ (d) $3\sin x-\sqrt{3}\cos x=\sqrt{3}$, $0\le x\le 2\pi$
(e) $6\sin x+8\cos x=-5$, $0°\le x\le 360°$ (f) $4\cos x+3\sin x=-1$, $0°\le x\le 360°$
(g) $\cos x-\sqrt{3}\sin x=2$, $0\le x\le 2\pi$ (h) $\cos x-\sin x=-1$, $-\frac{\pi}{2}\le x\le\frac{3\pi}{2}$
(i) $3\sin x+4\cos x=-2$, $-180°\le x\le 180°$ (j) $\sqrt{2}\sin x-\cos x=1.5$, $0°\le x\le 360°$

8 Sketch the graph of $f(x)=\sqrt{3}\cos x-\sin x$, $0\le x\le 2\pi$. Use your sketch to find the values of x for which:

(a) $f(x)=1$ (b) $f(x)>1$

9 Sketch the graph of $f(x)=\cos x+\sin x$, $-\pi\le x\le\pi$. Use your sketch to find the values of x for which:

(a) $f(x)=-1$ (b) $f(x)\ge -1$

10 Solve $6\sin 2x+4\sin x\cos x-3\sin x=0$ for $0\le x\le\pi$.

11 Which of the following is equivalent to $\sin x-\sqrt{3}\cos x$?

A $2\sin\left(x-\frac{\pi}{3}\right)$ B $2\sin\left(x-\frac{\pi}{6}\right)$ C $2\sin\left(x+\frac{\pi}{3}\right)$ D $2\sin\left(x+\frac{\pi}{6}\right)$

9.9 SOLVING QUADRATIC TRIGONOMETRIC EQUATIONS

Example 25

(a) Solve the equation $\tan^2\theta + \tan\theta - 2 = 0,\ 0° < \theta < 360°$.

(b) Solve the equation $\cos^2 x = 2\cos x,\ -\pi \le x \le \pi$.

Solution

(a) Factorise: $(\tan\theta - 1)(\tan\theta + 2) = 0$

$\tan\theta = 1$ **or** -2

$\theta = 45°, 225°$ **or** $116°34', 296°34'$

Solution is $\theta = 45°, 116°34', 225°, 296°34'$.

(b) Rearrange: $\cos^2 x - 2\cos x = 0$

Factorise: $\cos x(\cos x - 2) = 0$

$\therefore\ \cos x = 0$ **or** 2

Because $|\cos x| \le 1$, the only solution is $\cos x = 0$.

$\therefore\ x = -\frac{\pi}{2}, \frac{\pi}{2}$

Example 26

Solve the equation $\sec^2 x - 2\tan x = 4$ for $0 \le x \le 2\pi$.

(Trigonometric values rounded to 3 d.p. where necessary.)

Solution

The trigonometric functions are different, but they can be linked by the identity $\sec^2 x = 1 + \tan^2 x$:

$$\sec^2 x - 2\tan x = 4$$

$$1 + \tan^2 x - 2\tan x = 4$$

$$\tan^2 x - 2\tan x - 3 = 0$$

$$(\tan x - 3)(\tan x + 1) = 0$$

$\tan x = -1$ **or** 3

$$x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 1.249, \pi + 1.249$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}, 1.249, 4.391$$

EXERCISE 9.9 SOLVING QUADRATIC TRIGONOMETRIC EQUATIONS

1 Solve for values between 0 and 2π inclusive:

(a) $\tan^2 x - 1 = 0$ **(b)** $\sin^2 x - \sin x = 0$ **(c)** $\cos^2\theta - 2\cos\theta + 1 = 0$

(d) $\sqrt{3}\tan^2 x + \tan x = 0$ **(e)** $4\sin^2\theta = 1$ **(f)** $\sin^2 x - \sin x\cos x = 0$

2 If $0 \le \theta \le 2\pi$, the solution to $2\cos^2\theta - 1 = 0$ is:

A $\frac{\pi}{4}, \frac{7\pi}{4}$ **B** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **C** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **D** $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

3 Solve for values between 0 and 2π inclusive:

(a) $2\cos^2\theta - 3\cos\theta - 2 = 0$ (b) $2\cos^2\theta + \sin\theta = 1$

(c) $2\sin^2\theta - 3\cos\theta = 2$ (d) $(2\cos x + 1)(\sin x - 1) = 0$

4 Solve for $0 < x < 360°$: (a) $2\tan^2 x + \tan x = 15$ (b) $5\cos^2 x + 2\sin x = 2$

5 If $0 \le \theta \le 2\pi$, then the solution to $3\sin^2\theta - 4\cos\theta + 1 = 0$ is (approximately or exactly):

A $\theta = 0.841, 5.442$ B $\theta = 1.969, 4.315$ C $\theta = 2.301, 3.983$ D $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

6 Solve for $0 \le \theta \le 2\pi$: (a) $3\tan^3\theta - 3\tan^2\theta - \tan\theta + 1 = 0$ (b) $\cos^3\theta - 2\cos^2\theta + \cos\theta = 0$

9.10 SOLVING EQUATIONS USING ANGLE FORMULAE

More complex trigonometric equations will be considered in this section.

Example 27

Solve the equation $2\sin\left(x + \frac{5\pi}{6}\right) = \sin x$, for $0 \le x \le 2\pi$.

Solution

$$2\left(\sin x\cos\frac{5\pi}{6} + \cos x\sin\frac{5\pi}{6}\right) = \sin x$$

$$2\sin x \times \left(-\frac{\sqrt{3}}{2}\right) + 2\cos x \times \frac{1}{2} = \sin x$$

$$-\sqrt{3}\sin x + \cos x = \sin x$$

$$(1 + \sqrt{3})\sin x = \cos x$$

$$\tan x = \frac{1}{\sqrt{3}+1}$$

$$x = 0.3509, \pi + 0.3509$$

$$x = 0.351, 3.493$$

Example 28

Solve for $0 \le x \le 2\pi$: (a) $4\cos x = \operatorname{cosec} x$ (b) $\cos 4x - \cos 2x = 0$

Solution

(a)

$$4\cos x = \operatorname{cosec} x$$

$$4\cos x = \frac{1}{\sin x}$$

$$4\sin x\cos x = 1$$

$$2\sin 2x = 1$$

$$\sin 2x = 0.5$$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

(b)

$$\cos 4x - \cos 2x = 0$$

$$2\cos^2 2x - 1 - \cos 2x = 0$$

$$2\cos^2 2x - \cos 2x - 1 = 0$$

$$(\cos 2x - 1)(2\cos 2x + 1) = 0$$

$$\cos 2x = 1, \cos 2x = -0.5$$

$$2x = 0, 2\pi, 4\pi \quad \textbf{or} \quad 2x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$$

$$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$$

This equation could also have been solved by writing $\cos 4x = \cos 2x$.

Example 29

(a) Use the expansion of $\sin(2\theta + \theta)$ to obtain an expression for $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence find the roots of $4x^3 - 3x + 0.5 = 0$.

Solution

(a) $\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(b) Let $x = \sin\theta$: $4\sin^3\theta - 3\sin\theta + 0.5 = 0$

$3\sin\theta - 4\sin^3\theta = 0.5$

$\sin 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

Hence the roots are $x = \sin\frac{\pi}{18}, \sin\frac{5\pi}{18}, \sin\frac{25\pi}{18}$.

Example 30

(a) Show that $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$.

(b) By using suitable substitutions for A and B, show that $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.

(c) Hence solve $\sin 2x + \sin 4x = \sin 6x$ for $0 \le x \le \pi$.

Solution

(a) LHS $= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B$

(b) Let $x = A + B$ and $y = A - B$.

Adding these equations: $A = \frac{x+y}{2}$ Subtracting the equations: $B = \frac{x-y}{2}$

Substitute these results in **(a)**: $\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

(c) Use the result in **(b)** on the LHS: $\sin 2x + \sin 4x = 2\sin 3x\cos(-x) = 2\sin 3x\cos x$

Use the double angle formula $\sin 2x = 2\sin x\cos x$ on the RHS: $\sin 6x = 2\sin 3x\cos 3x$

$2\sin 3x\cos(-x) = 2\sin 3x\cos 3x$

$2\sin 3x\cos x = 2\sin 3x\cos 3x$ (cos x is an even function, so $\cos(-x) = \cos x$)

$2\sin 3x(\cos x - \cos 3x) = 0$

$\sin 3x = 0$ **or** $\cos x - \cos 3x = 0$

$3x = 0, \pi, 2\pi, 3\pi$ **or** $\cos 3x = \cos x$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$ **or** $3x = x, 2\pi - x, 2\pi + x, 4\pi - x$

$2x = 0, 4x = 2\pi, 2x = 2\pi, 4x = 4\pi$

$x = 0, \frac{\pi}{2}, \pi$

Solution is $x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$.

Example 31

Solve the equation $\cos 4x + \sin 3x = 0$ for $0 \le x \le \pi$.

Solution

Rewrite equation: $\cos 4x = -\sin 3x$

Sine is an odd function, so: $-\sin 3x = \sin(-3x)$: $\cos 4x = \sin(-3x)$

Use $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to rewrite equation: $\cos 4x = \cos\left(\frac{\pi}{2} + 3x\right)$

$$4x = \frac{\pi}{2} + 3x,\ 2\pi - \left(\frac{\pi}{2} + 3x\right),\ 2\pi + \left(\frac{\pi}{2} + 3x\right),\ 4\pi - \left(\frac{\pi}{2} + 3x\right),\ 6\pi - \left(\frac{\pi}{2} + 3x\right),\ 8\pi - \left(\frac{\pi}{2} + 3x\right)$$

$$x = \frac{\pi}{2},\ 7x = \frac{3\pi}{2},\ x = \frac{5\pi}{2},\ 7x = \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}.$$

$$x = \frac{\pi}{2}, \frac{3\pi}{14}, \frac{\pi}{2}, \frac{11\pi}{14}, \frac{15\pi}{14}.$$

As $0 \le x \le \pi$, the solution is $x = \frac{3\pi}{14}, \frac{\pi}{2}$ **or** $\frac{11\pi}{14}$.

EXERCISE 9.10 SOLVING EQUATIONS USING ANGLE

1 Solve for $0 \le x \le 2\pi$:

(a) $\sin\left(x + \frac{\pi}{3}\right) = \cos x$ (b) $\sin\left(x + \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} - x\right)$ (c) $2\sin\left(x + \frac{\pi}{6}\right) = \sin x$

(d) $\sin\left(x + \frac{\pi}{4}\right) = \cos\left(x - \frac{\pi}{3}\right)$ (e) $4\tan\left(x - \frac{\pi}{6}\right) = \tan\left(x - \frac{\pi}{3}\right)$ (f) $\cot x + \cot\left(x + \frac{\pi}{4}\right) = 3$

2 Solve for $0 \le x \le 2\pi$:

(a) $2\cos x = \operatorname{cosec} x$ (b) $4\sin x = \sec x$ (c) $4\cos x = \sqrt{3}\operatorname{cosec} x$

(d) $3\sin x = \sec x$ (e) $\tan x = \operatorname{cosec} 2x$ (f) $\sin 2x + \cos 2x = 1$

3 Solve for $0 \le x \le 2\pi$, using the double angle formulae:

(a) $\cos 2x - \cos x = 0$ (b) $\cos 2x - \sin x = 0$ (c) $\sin 4x + \sin 2x = 0$ (d) $\tan 4x + \tan 2x = 0$

4 Solve $0 \le x \le 2\pi$:

(a) $3\sin 2x + 4\sin^2 x - 2 = 0$ (b) $4(\sin^2 2x - \sin^2 x) = 1$ (c) $\tan 2x = \tan x$ (d) $\tan 2x + 3\tan x = 0$

5 Solve $0 \le x \le \pi$:

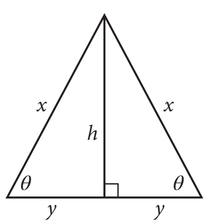
(a) $\cos^2 x - \sin^2 x = 1$ (b) $\tan x + \cot x = 2$ (c) $\tan x + 2\cot x + \sec x = 0$

6 Solve $0 \le x \le 2\pi$:

(a) $3\sin x + 4\cos x = 5$ (b) $\cos x + 3\sin x = 2$ (c) $10\tan x - 2\sec x = 5$ (d) $\sin 2x = 1 + \cos 2x$

7 Show that if $a^2 + b^2 < c^2$, the equation $a\cos\theta + b\sin\theta = c$ has no real roots.

8 The equal sides of an isosceles triangle are x cm and the third side is $2y$ cm. The equal angles are each θ and the height of the triangle is h cm, as shown.
If the perimeter of the triangle is four times the height, find the size of the angles of the triangle to the nearest minute.



9 (a) Use the expansion of $\cos(2\theta + \theta)$ to obtain an expression for $\cos 3\theta$ in terms of $\cos\theta$.

(b) Hence find the roots of $8x^3 - 6x - \sqrt{3} = 0$.

10 **(a)** Use the expansion of $\tan(2\theta + \theta)$ to obtain an expression for $\tan 3\theta$ in terms of $\tan\theta$.

(b) Use this result to show that $\tan 15° = 2 - \sqrt{3}$. Justify your answer.

(c) Use the expansion of $\tan(45° + 30°)$ to find the exact value of $\tan 75°$. Compare this answer to the other result obtained in **(b)**.

11 Solve for $0 \le x \le \pi$.

(a) $\cos 4x = \cos 2x$ **(b)** $\tan 2x = \cot x$ **(c)** $\sin 3x = \sin x$ **(d)** $\cos 3x = \sin 2x$

12 Solve for $0 \le \theta \le \pi$.

(a) $\cos 3\theta = \sin\left(\frac{\pi}{4} - \theta\right)$ **(b)** $\sin 2\theta = \cos\left(\theta - \frac{\pi}{4}\right)$ **(c)** $\cos 2\theta = \sin\left(\theta + \frac{\pi}{4}\right)$ **(d)** $\sin\left(\theta - \frac{\pi}{3}\right) = \cos 2\theta$

13 Solve for $0 \le x \le \pi$.

(a) $\sin 3x + \sin x = 0$ **(b)** $\sin 2x + \cos 3x = 0$ **(c)** $\tan 2x + \cot 3x = 0$

14 **(a)** Given that $2\sin\theta\cos\phi = \sin(\theta + \phi) + \sin(\theta - \phi)$, show that $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

(b) Hence solve for $0 \le \theta \le \pi$:

(i) $\sin 3\theta = \sin 5\theta + \sin\theta$ **(ii)** $\sin 2\theta + \sin 3\theta + \sin 4\theta = 0$

(iii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$ **(iv)** $\sin 5\theta + \sin 3\theta = \sin 4\theta + \sin 2\theta$

15 **(a)** Given that $2\cos\theta\cos\phi = \cos(\theta + \phi) + \cos(\theta - \phi)$, show that $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$.

(b) Hence solve for $0 \le x \le 2\pi$:

(i) $\cos 5x + \cos x = \cos 3x$ **(ii)** $\cos x + \cos 3x = \cos 5x + \cos 7x$ **(iii)** $\cos 5x + \cos x + \cos 7x + \cos 3x = 0$

CHAPTER REVIEW 9

1 Simplify: **(a)** $\dfrac{\tan\theta + \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{6}\tan\theta}$ **(b)** $\dfrac{\sin 2\theta - \sin\theta}{\cos 2\theta - \cos\theta + 1}$

2 Solve $2\tan 2x - 1 = 0$ for $0° < x < 360°$.

3 Simplify:

(a) $\sin(\theta + \phi)\cos\phi - \cos(\theta + \phi)\sin\phi$ **(b)** $\dfrac{2\tan\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}$ **(c)** $\sin x\cos x\cos 2x\cos 4x$

4 **(a)** Show that $\cos(A + B) = \cos A\cos B(1 - \tan A\tan B)$.

(b) Suppose that $0 < A < \frac{\pi}{2}$ and $0 < B < \frac{\pi}{2}$. Show by deduction that if $\tan A\tan B = 1$ then $A + B = \frac{\pi}{2}$.

5 Show that: **(a)** $\dfrac{\cos\theta}{1 + \sin\theta} = \sec\theta - \tan\theta$ **(b)** $\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$, given that $\cos 2\theta \ne -1$.

6 Use the expansion of $\tan 2A$ to show that the exact value of $\tan 22.5° = \sqrt{2} - 1$. Hence find the exact value of $\tan 11.25°$.

7 Solve the following equations for $0 \le x \le \pi$.

(a) $\cos 3x = \cos 2x\cos x$ **(b)** $\cos 3x + \cos 5x + \cos 7x = 0$

8 Solve for $-\pi \le x \le \pi$.

(a) $\cos x - \sin x = 1$ **(b)** $\sin 4x - \sin 2x = 0$ **(c)** $\cos x - \sqrt{3}\sin x = 1$

9 **(a)** Express $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) - 2\cos\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(b) Hence, or otherwise, solve $2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) - 2\cos\theta = 1$ for $0 < \theta < 2\pi$.

10 (a) Express $3\sin x + 4\cos x$ in the form $r\sin(x+\alpha)$ where $0 \le \alpha \le \frac{\pi}{2}$.

(b) Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \le x \le 2\pi$. Give answer(s) to two decimal places.

(c) Write the general solution for $3\sin x + 4\cos x = 5$.

11 Use $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$ to find the exact value of $\tan\frac{\pi}{8}$.

12 Show that the cubic equation $8x^3 - 6x + 1 = 0$ can be reduced to the form $\cos 3\theta = -\frac{1}{2}$ by substituting $x = \cos\theta$. From this, deduce the following:

(a) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

(b) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} = 6 + \sec\frac{\pi}{9}$

(c) $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

(d) $\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} = 33$

13 It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Use this result to solve $\cos 3\theta + \cos 2\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$.

14 (a) Expand $\cos(2A+B)$ and hence prove that $\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$.

(b) By writing $x = k\cos\theta$ and giving k a suitable value, use the formula proved in part **(a)** to find the three roots of the equation $27x^3 - 9x = 1$. Hence write the value of the product $\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$.

15 If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation $x^3 - (a+1)x^2 + (c-a)x - c = 0$, show that $\alpha + \beta + \gamma = n\pi + \frac{\pi}{4}$.

16 Solve $\sin x = \cos 5x$ for $0 < x < \pi$.

17 (a) Find A and B in terms of x and y such that $\sin x + \sin y = 2\sin A\cos B$.

(b) Find the solution of $\sin\theta + \sin 2\theta + \sin 3\theta = 0$ for $0 \le \theta \le \pi$.