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(a division of Pearson Australia Group Pty Ltd)
20 Thackray Road, Port Melbourne, Victoria 3207
PO Box 460, Port Melbourne, Victoria 3207

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The *Pearson Mathematics 7–10* series is an adaptation of the *Heinemann Mathematics 7–10 series*.
First published 2011 by Pearson Australia
2014 2013 2012 2011
10 9 8 7 6 5 4 3 2 1

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Publisher: Dirk Strasser
Project Editors: Helen Withycombe and Andrea Davison
Editor: Liz Waud
Designer: Glen McClay
Typeset by: Nikki M Group
Copyright & Pictures Editors: Jacqui Liggett, Marg Barber and Julia Weaver
Desktop Operator: Helene Loos
Series Cover Design: Glen McClay, Miranda Costa, Kim Ferguson, Jo Groud
Illustrators: Wendy Gorton, Marc Lynch, Marty Schneider, Sunset Digital
Printed in China

National Library of Australia Cataloguing-in-Publication entry

Title:	Pearson mathematics 10–10A / David Coffey … [et. al].
Edition:	1st ed.
ISBN:	9781442529656 (student book)
Target Audience:	For secondary school age.
Subjects:	Mathematics—Textbooks.
	Mathematics—Problems, exercises, etc.
Other Authors:	Coffey, David, 1956–
Dewey Number:	510

Pearson Australia Group Pty Ltd ABN 40 004 245 943

Acknowledgements

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PEARSON mathematics

Student Book

The Year 10 student book provides maximum flexibility for teachers by including both the 10 and 10A material in the same book. All the 10A material is clearly labelled, enabling teachers to follow the 10 course or the 10–10A course or to customise what their Year 10 students cover. The student book has been structured according to the latest research on how students learn mathematics and on how to avoid common misunderstandings, making it easy for teachers to cater for mixed abilities and to ask good questions.

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The teacher companion makes lesson preparation easy by combining full-colour textbook pages with teaching strategies, ideas for class activities and fully worked solutions in a series of chapter booklets.

AUSTRALIAN CURRICULUM

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The Australian Curriculum: Mathematics aims to ensure that students:

- *are confident, creative users and communicators of mathematics, able to investigate, represent and interpret situations in their personal and work lives and as active citizens*
- *develop an increasingly sophisticated understanding of mathematical concepts and fluency with processes, and are able to pose and solve problems and reason in* Number and Algebra, Measurement and Geometry, *and* Statistics and Probability
- *recognise connections between the areas of mathematics and other disciplines, and appreciate mathematics as an accessible and enjoyable discipline to study.*

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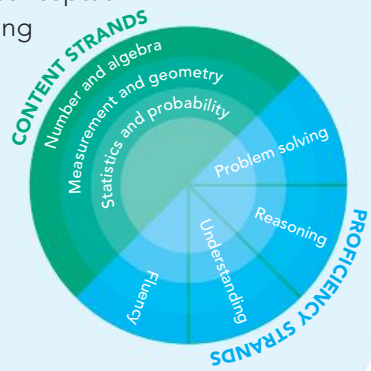
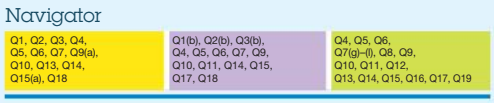
Pearson Mathematics shares these aims.

Pearson Mathematics is a compelling research-based series, written by experienced and practising Australian teachers with the support of Australia's leading mathematics education experts. It has been specifically designed to scaffold students' fluency development, conceptual understanding, reasoning and problem-solving skills, in addition to addressing the Australian Curriculum's General capabilities and Cross-curriculum priorities.



Differentiation

Pearson Mathematics has been written and designed with the needs of the full-ability spectrum of students in Australian classrooms in mind.



Navigator—Three optional graded pathways through *every* Exercise.

Recall—Each chapter begins with a review of assumed and necessary knowledge, with Recall Worksheets available for each skill for those students needing extra revision.

Challenge—Every Mixed Review has a full page challenge section for early finishers and those students who need to be extended.

Scaffolded Exercises—All Exercises are split into the Australian Curriculum Proficiency Strands: Fluency, Understanding and Reasoning. They have been carefully paced to help students build skills, develop deep conceptual understanding and apply learning. Every Exercise also has Open-ended questions to encourage students' creative thinking and ability to communicate mathematics effectively.

Cross-curriculum priorities

Aboriginal cultu

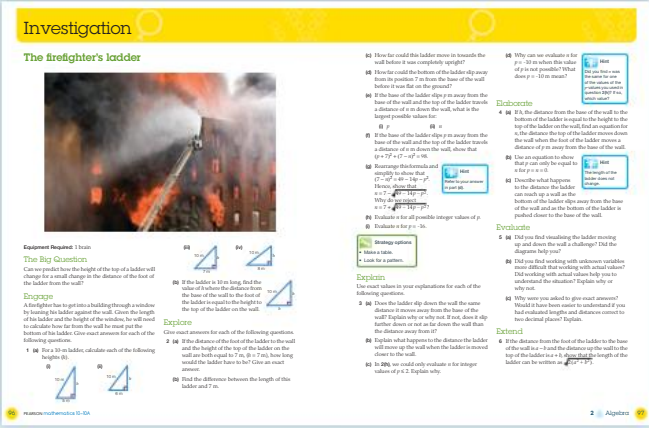
Intercultural understa

Sustainability

Asia and Australia's en

Encouraging inquiry and problem solving

Problem solving


Investigation

The firefighter's ladder

The Big Question

Engage

Explore

Explain

Elaborate

Evaluate

Extend

The Australian Mathematics Curriculum focuses on students developing problem-solving skills, in particular the ability to employ mathematical strategies to solve problems efficiently and to be able to respond to unfamiliar situations. **Pearson Mathematics** is designed to develop these skills.

Problem solving toolbox

Twelve key problem-solving strategies are included inside the front cover of the Student Book and these strategy options are referenced throughout investigations and problem solving.

Outside the Square: Problem solving

These are located throughout every chapter and give students ample opportunity to develop their problem-solving skills and logical reasoning.

Outside the Square

Problem solving

Investigations

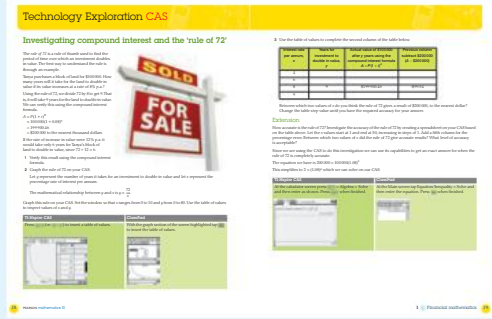
Investigation

These are scaffolded to be accessible to all learners and structured following a 5e+ format.

Technology Explorations

Technology Exploration

Pearson Mathematics understands the importance the Australian Curriculum places on students developing ICT literacy. Technology Explorations help develop students' deeper understanding of the mathematical concepts covered.


Technology Exploration CAS

Investigating compound interest and the 'rule of 72'

Making teachers' lives easier

Regular revision and reinforcement

Pearson Mathematics has a broad range of cumulative and chapter-based revision.

Half-time—A mid-chapter review

Chapter review—An end-of-chapter review

Mixed review—Cumulative revision for every third chapter

Engaging students

Pearson Mathematics has been designed to capture students' interest, with the incorporation of Maths 4 Reals and Home Pages.

Maths 4 Reals

Scenarios that help students make connections with mathematics in the real world.

Home Pages

Anticipate the 'Why learn this?' question and provide a motivating entry into a new topic, featuring an interview about mathematics in the workplace.

Make a table
Act it out
Look for a pattern
Solve a simpler probl
Draw a picture
Test all possible combinations
Work backwards

lem

1 Financial mathematics

Stuck for a gift idea? Talk to Naomi

Name: Naomi Simson
Job: CEO of online gifting website RedBalloon
Qualifications: Bachelor of Commerce
Awards: Winner of 2008 Telstra Business Woman of the Year for Innovation and finalist in the 2009 Veuve Clicquot Business Woman Award

What do learning to fly a helicopter, didgeridoo lessons or dinner in the Daintree Rainforest have in common? They are all experiences that can be purchased as gifts on the website RedBalloon.

Naomi Simson, the founder and CEO of RedBalloon has a vision to change gifting in Australia forever. RedBalloon allows people to give the gift of an experience that will create vivid memories and stories to tell long after it is given. The business began as a side hobby for a full-time mother in 2001 and has grown to employ 50 people. With 225 000 gifts given last year, Naomi has needed skills in administration, marketing, sales, customer service and strong money awareness.

"You can't run a business without financial skills because these skills enable you to understand how money moves around a business. To run a successful business you must be able to read a balance sheet and a cash flow forecast. You have to understand profit and loss and how money works."

Why learn this?

Financial formulas are used by employers, banks, accounting firms, the tax office and other institutions. Understanding how these formulas are used empowers the individual. A person can apply financial formulas to check entitlements or to make the best choices between loans or investments offered by different institutions.

Financial formulas also have applications in seemingly unrelated areas like population growth in geography, bacterial growth in medicine or radioactive decay in chemistry.

After completing this chapter you will be able to:

- use and extend the simple interest formula to develop the compound interest formula
- use the compound interest formula to solve problems involving different rates and different compounding periods
- use recursive techniques
- compare effective and nominal interest rates and solve related problems
- adapt the interest formulas to solve problems of appreciation, depreciation, and growth and decay.

1 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 Convert the following percentages to decimals.

Worksheet R1.1

(a) 17% **(b)** 6% **(c)** 19.2%

(d) 4.8% **(e)** 2.01% **(f)** $\frac{1}{4}$ %

2 Calculate:

Worksheet R1.2

(a) 15% of \$20 000 **(b)** 6% of \$10 000

(c) 9.5% of \$350 000 **(d)** $7\frac{1}{2}$ % of \$45 000

3 **(a)** Increase 40 by:

Worksheet R1.3

(i) 100% **(ii)** 50% **(iii)** 25% **(iv)** 200%

(b) Decrease 60 by:

(i) 50% **(ii)** 30% **(iii)** 10% **(iv)** 75%

4 Evaluate:

Worksheet R1.4

(a) 7^2 **(b)** 2^3 **(c)** 10^6

(d) $(0.6)^2$ **(e)** $\left(\frac{1}{2}\right)^5$ **(f)** $(1.1)^4$

5 Convert the following to a decimal number of years. Assume a year has 52 weeks or 365 days.

Worksheet R1.5

(a) 6 months **(b)** 3 months **(c)** 8 years and 9 months

(d) 26 weeks **(e)** 13 weeks **(f)** 146 days

Key Words

adjusted value	exponential growth	residual value
appreciation	flat rate	salvage value
compound interest	inflation rate	scrap value
constants	initial cost	simple interest
cumulative depreciation	interest	straight-line depreciation
depreciation	nominal interest rate	total possible depreciation
depreciation value	per annum	unit cost depreciation
effective interest rate	prime cost	variables
expected life	principal	written-down value
exponential decay	reducing balance depreciation	

1.1 Interest

Borrowing and lending

There are many situations in which individuals, organisations or communities borrow money from, or lend money to, others. Here are a few examples:

- An individual may borrow money from a bank to buy a house, a car or to start a business.
- An investor who puts money in a bank account is lending money to the bank. The bank in turn uses the investor's money to lend to others.
- A government can borrow money from individuals by selling government bonds. This money can then be used to pay for some of the services they provide.

The original amount of money that is borrowed or loaned is called the **principal**.

The lender has provided a service for which they can expect to be paid. The borrower eventually repays the lender the principal and an extra agreed amount called the **interest**. The interest is usually expressed as a percentage **per annum** (p.a.), which is 'for each year' in Latin.

Calculating simple interest

Simple interest is calculated on the original principal. The simple interest (I) depends on three things: the principal (P), the interest rate per annum (R), and the time in years (T). When dealing with financial calculations we often want to know the total amount that needs to be repaid by us, or paid to us, at the end of the contract. In the case of simple interest, the total amount is the sum of the principal and the interest.

The simple interest formula:

$$I = PRT$$

where I = amount of simple interest
P = principal
R = rate of interest (p.a.) as a decimal
T = time in years

Also:

$$A = P + I$$

where A = total amount repaid on the loan or collected from the investment
P = principal
I = simple interest

Simple interest is sometimes referred to as flat rate interest.

Worked Example 1 WE1

$20 000 is borrowed for 5 years at an interest rate of 9.5% p.a. If the interest is calculated as simple interest, find:

(a) the amount of interest charged over the period of the loan

(b) the total amount to be repaid at the end of the loan.

Thinking		Working	
(a) 1	Identify what needs to be found.	(a)	$I = ?$
2	Write down the simple interest formula.		$I = PRT$
3	Identify the values of P, R and T.		$P = 20\,000$, $R = 9.5\% = 0.095$, $T = 5$
4	Substitute these values into the formula and calculate I.		$I = 20\,000 \times 0.095 \times 5$ $= \$9500$
(b) 1	Identify what needs to be found.	(b)	$A = ?$
2	Write down the formula.		$A = P + I$
3	Identify the values of P and I.		$P = 20\,000$, $I = 9500$
4	Substitute these values into the formula and calculate the value of A.		$A = 20\,000 + 9500$ $= \$29\,500$

Given values for any three of these variables we can work out the value of the fourth variable.

Worked Example 2

WE2

Find the information that is missing in each of the following simple interest situations.

(a) \$200 interest is paid on an investment of \$5000 after 6 months.

(b) \$350 interest is paid on an investment at 4% p.a. after 9 months.

(c) \$1250 interest is paid on an investment of \$3000 invested at 5% p.a.

Thinking		Working	
(a) 1	Write out values for the known variables to identify the unknown variable. (Remember the rate is expressed as a decimal and the time is expressed in years.)	(a)	$I = \$200$ $P = \$5000$ $R = ?$ $T = \frac{6}{12} = 0.5$
2	Substitute the values, rearrange the formula and solve for the unknown.		$I = PRT$ $200 = 5000 \times R \times 0.5$ $R = \frac{200}{5000 \times 0.5}$ $= 0.08$
3	Write the answer.		The rate is 8% p.a.
(b) 1	Write out values for the known variables to identify the unknown variable. (Remember the rate is expressed as a decimal and the time is expressed in years.)	(b)	$I = \$350$ $P = ?$ $R = 0.04$ $T = \frac{9}{12} = 0.75$

2	Substitute the values, rearrange the formula and solve for the unknown.	$I = PRT$ $350 = P \times 0.04 \times 0.75$ $P = \dfrac{350}{0.04 \times 0.75}$ $= 11\,666.67$
3	Write the answer.	The principal is \$11 666.67
(c) 1	Write out values for the known variables to identify the unknown variable. (Remember the rate is expressed as a decimal and the time is expressed in years.)	(c) $I = \$1250$ $P = \$3000$ $R = 0.05$ $T = ?$
2	Substitute the values, rearrange the formula and solve for the unknown.	$I = PRT$ $1250 = 3000 \times 0.05 \times T$ $T = \dfrac{1250}{3000 \times 0.05}$ $= 8.333$
3	Write the answer.	The time is $8\frac{1}{3}$ years, which is 8 years and 4 months.

Compound interest

You may be familiar with the expression 'compounding the problem'. This means an original problem is made worse and worse again. **Compound interest** is based on the idea of simple interest but is designed to make the amount of interest grow bigger and bigger by increasing the principal.

Compound interest achieves this growth with two simple steps:

- Calculating, at the end of the time period, the interest on the principal for that time period.
- Adding this interest to the principal to make the new principal for the next time period.

Consider the situation of \$1000 invested for 4 years at 6% p.a. compounded annually:

Year	Principal	Interest calculation at the end of the year $I = P \times R \times T$	Amount of investment at the end of the year $A = P + I$
1	\$1000	$\$1000 \times 0.06 \times 1 = \$60$	$\$1000 + \$60 = \$1060$
2	\$1060	$\$1060 \times 0.06 \times 1 = \$63.60$	$\$1060 + \$63.60 = \$1123.60$
3	\$1123.60	$\$1123.60 \times 0.06 \times 1 = \$67.416$	$\$1123.60 + \$67.416 = \$1191.016$
4	\$1191.016	$\$1191.016 \times 0.06 \times 1 = \$71.46096$	$\$1191.016 + \$71.46096 = \$1262.47696$

For practical reasons the answer is finally rounded to \$1262.48. It is important to avoid rounding any numbers until the end of the calculation because rounding errors can occur if we round too early.

Also, the value of this investment after 4 years of simple interest would be only \$1240. In this case, compounding the investment earned an extra \$22.48.

Now look at the table carefully and note the following:

1 The total value of the investment at the end of the first year is 106% of \$1000. This is the same as multiplying the initial principal by 1.06.

2 Each year, the total value is 1.06 times the total value at the end of the previous year (or the beginning of the current year).

Using these observations we can rewrite the last column of the table in a way that shows the structure more clearly.

Year	Amount owed at the end of each year (in dollars)
1	$1000 \times 1.06 = 1000 \times (1.06)^1$
2	$1000 \times (1.06)^1 \times 1.06 = 1000 \times (1.06)^2$
3	$1000 \times (1.06)^2 \times 1.06 = 1000 \times (1.06)^3$
4	$1000 \times (1.06)^3 \times 1.06 = 1000 \times (1.06)^4$

Note that the index number is the same as the year number.

If A_n = the total amount owed after n years, then for the above problem we can write:

$$A_n = 1000(1 + 0.06)^n$$

We can write the following formulas for loans or investments with an annual compounding period.

When the interest is compounded annually:

$$A = P(1 + r)^n$$

where A = amount of the loan or investment after n years

P = principal

r = rate of interest (p.a.) as a decimal

n = time in years

Also:

$$I = A - P$$

where I = compound interest owed or earned after a certain number of years

A = amount of the loan or investment after the same number of years

P = principal

One of the important differences to note between the simple interest formula and the compound interest formula is that the simple interest formula gives us the amount of interest, whereas the compound interest formula gives us the total amount at the end of the compounding period.

Worked Example 3 WE3

$12 500 is borrowed for 7 years at an interest rate of 15% p.a. If the interest is compounded annually, find:

(a) the total value of the loan after 7 years

(b) the amount of interest charged over the period of the loan.

Thinking		Working	
(a)	1	Identify what needs to be found.	(a) $A = ?$
	2	Write down the annual compound interest formula.	$A = P(1 + r)^n$
	3	Identify the values of P, r and n.	$P = 12\,500$, $r = 15\% = 0.15$, $n = 7$
	4	Substitute these values into the formula and calculate the value of A.	$A = 12\,500(1 + 0.15)^7$ $= 12\,500 \times 1.15^7$ $= \$33\,250.25$ (rounded to the nearest cent)
(b)	1	Identify what needs to be found.	(b) $I = ?$
	2	Write down the formula.	$I = A - P$
	3	Identify the values of A and P.	$A = 33\,250.25$, $P = 12\,500$
	4	Substitute these values into the formula and solve for the value of I.	$I = 33\,250.25 - 12\,500$ $= \$20\,750.25$

Note: To evaluate numbers with powers like 1.15^7 we could enter $1.15 \times 1.15 \times 1.15 \times 1.15 \times 1.15 \times 1.15 \times 1.15$ but it is more efficient to use the 'index key' which may look like x^y, a^x or y^x.

Appreciation

Increase in cost or value is called **appreciation**. Loans and investments are not the only things that appreciate. Land, houses, collectibles (antiques/memorabilia), salaries and cost of food are also examples of things that usually appreciate over time. The rate of increase in prices within the economy is called the **inflation rate**. The simple interest formula can be used to calculate a **flat rate** of appreciation. The compound interest formula can be used to predict a future cost or value if the amount of appreciation increases with each year (compounds annually).

1.1 Interest

Navigator

Answers page 716

Q1 (a)–(e), Q2 (a) (c) (e), Q3 (a)–(e), Q4, Q5, Q7, Q8, Q9, Q10, Q14	Q1 (a)–(j), Q2 (a) (d) (f), Q3 (a)–(j), Q4, Q6, Q8, Q9, Q10, Q11, Q12, Q14	Q1 (h)–(n), Q2 (b) (d) (f), Q3 (h)–(n), Q4, Q6, Q9, Q10, Q11, Q12, Q13, Q14, Q15

If necessary, round answers to the nearest cent.

Fluency

WE1

1 For each of the following loans the interest is calculated as simple interest. Find:

(i) the amount of interest charged over the period of the loan

(ii) the total amount to be repaid at the end of the loan.

(a) \$40 000 borrowed for 1 year at 8% p.a.

(b) \$30 000 borrowed for 1 year at 9% p.a.

(c) \$30 000 borrowed for 5 years at 7% p.a.

(d) \$40 000 borrowed for 12 years at 5% p.a.

(e) \$150 000 borrowed for 25 years at 7.5% p.a.

(f) \$240 000 borrowed for 30 years at 6.5% p.a.

(g) \$300 000 borrowed for 1.1 years at 6.75% p.a.

(h) \$350 000 borrowed for 1.1 years at 7.25% p.a.

(i) \$13 000 borrowed for 5 years at $9\frac{1}{2}$ % p.a.

(j) \$15 800 borrowed for 4 years at $10\frac{3}{4}$ % p.a.

(k) \$45 000 borrowed for $12\frac{1}{2}$ years at 5% p.a.

(l) \$36 000 borrowed for $5\frac{1}{2}$ years at 7% p.a.

(m) \$10 000 borrowed for 2 years and 3 months at 15% p.a.

(n) \$16 000 borrowed for 3 years and 9 months at 17% p.a.

2 Find the information that is missing in each of the following simple interest situations. WE2

(a) \$2400 interest is paid on an investment at 20% p.a. after 3 years.

(b) \$7200 interest is paid on an investment at 15% p.a. after 6 years.

(c) \$25 200 interest is paid on an investment of \$20 000 after 9 years.

(d) \$40 788 interest is paid on an investment of \$30 900 after 12 years.

(e) \$65 000 interest is paid on an investment of \$100 000 invested at 6.5% p.a.

(f) \$135 000 interest is paid on an investment of \$150 000 invested at 7.5% p.a.

3 For each of the loans in Question **1** the interest is now compounded annually. Find: WE3

(i) the total amount to be repaid at the end of the loan

(ii) the amount of interest charged over the period of the loan.

4 Lily invested \$20 000 for 5 years at an interest rate of 5.25% p.a. Jarrah invested \$45 000 for 7 years at an interest rate of 9.5% p.a.

(a) Calculate the interest earned on each investment assuming:

(i) annual compound interest (ii) simple interest.

(b) Compare the compounded interest with the simple interest.

5 Calculate the total amount owing on a loan of \$7000 after 2 years, if the 16% interest p.a. is:

(a) compounded annually (b) calculated as simple interest.

6 \$4245 at 6.5% p.a. compound interest, compounded annually over 8 years, will amount to:

A $\$4245(1 + 0.065)$

B $\$4245(1 + 0.065) \times 8$

C $\$4245(1 + 0.065)^8$

D $\$4245(1 + 0.65)^8$

Understanding

7 At what simple percentage rate will an initial investment of \$7500 grow to a total investment of \$8400 in 10 years?

8 If the simple interest on an investment over 18 years at 4% is \$93, what was the amount of the original investment?

Hint

Use $A = P + I$ to find the simple interest first. Then use $I = PRT$ to find R.

Hint
In some cases you may need to use 'guess and check' to solve the equation.

9 Complete the following table assuming an annual compounding period.

Amount (*A*)	Principal (*P*)	Interest rate p.a. (*R*)	Time in years (*T*)
\$1359.81		8%	13
	\$24 600	12.2%	5
\$1 048 209.60	\$1 000 200	4.8%	
\$151.71		3%	20
\$60.50	\$50		2

10 For each of the following appreciation questions:

(i) state whether the problem should be modelled by the simple interest formula or by the compound interest formula

(ii) solve the problem, rounding answers to the nearest whole number or dollar.

(a) Martina currently gets \$10 pocket money a week. She has been promised a \$4 increase each year. What is the promised rate of increase? How much more pocket money will she get in 3 years than she is getting now?

(b) An antique clock appreciates in value by 11% p.a. If it was valued 10 years ago at \$350 how much is it worth now?

(c) Yang paid \$20 000 for a new motorbike. If the inflation rate is 6.5% p.a., what is the estimated cost of a similar new motorbike in 4 years' time?

(d) Rhea has been awarded an annual salary increase rate of 7% p.a. Her current salary is \$42 000 p.a. Calculate how much she can expect to earn in 3 years' time if the same rate continues.

(e) Ethel was given a \$500 ring 30 years ago. Each year it appreciated at a flat rate of 5%. What is it worth now?

(f) In 1993, a house was purchased for \$214 500. It was sold in 2010 for \$652 000. What was the yearly average rate of appreciation?

11 Costs in a business are growing at an annually compounding rate of 8% p.a. Currently they run at \$780 per week but 7 years ago they were:

A $\dfrac{\$780}{1.08^7}$ **B** $\$780 \times 1.08^7$ **C** $\$780 \times 1.08^7 - 780$ **D** $\dfrac{\$780}{1.8^7}$

Hint
Either choose a number for the amount Ahmed has to invest or let *P* be this amount.

Reasoning

12 Ahmed has some money to invest. He wants to at least quadruple his money.

(a) How long will it take if he invests at a simple interest rate of 3% p.a.?

(b) How long will it take if he invests at an annually compounded interest rate of 3%?

Give your answers in whole years.

Hint
Write two equations with *P* and *R* in them. Then use these two equations to write $\dfrac{51\,398.55}{48\,951}$ in terms of *P* and *R*.

13 A certain sum of money \$*P* is deposited into a bank account at an annually compounding interest rate, *R*. After 2 years it has grown to \$48 951 and after 3 years it has grown to \$51 398.55.

What is the interest rate?

Open-ended

14 Find values for the amount of an investment and the annual interest rate, given that the investment compounds annually for 2 years, to give a final value within \$100 of \$15 000.

15 Calum's car was written off as a result of hail damage in a sudden and severe local storm. His interstate cousin Jessica also had to replace her car after flood damage. Both needed to borrow the same amount of money. Calum was able to get a flat rate loan at 10% p.a. for 8 years. Jessica will end up paying the same amount of interest as Calum but her loan has a simple interest rate between 10% and 20% p.a. and will need to be repaid over a shorter period of time. Find possible values for the interest rate and period of Jessica's loan.

Outside the Square — Problem solving

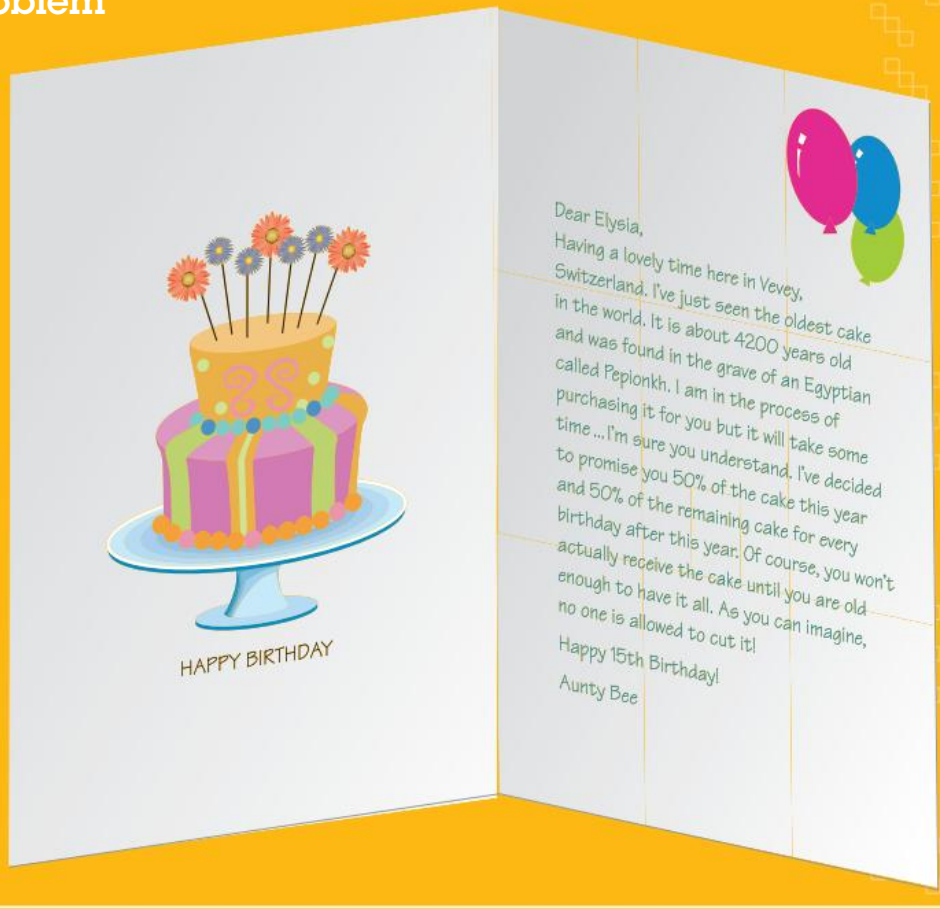
The oldest cake problem

1 What percentage of the cake has been promised to Elysia by her 21st birthday?

2 How old will Elysia be when she finally gets Aunty Bee's present?

Strategy options

- Draw a diagram.
- Solve a simpler problem.



Technology Exploration Excel

Equipment required: 1 brain, 1 computer with Excel

Versions of this Exploration for other technologies are available in Pearson Reader.

Comparing simple and compound interest

Calculating simple interest

1 Open a new spreadsheet. We will be using a principal of $1000 invested at a rate of 10% p.a. However, on our spreadsheet we will use the decimal equivalent of 10%. The interest will be calculated at the end of each year, so our time period will be 1. The screenshot below shows you where to put these values as well as some headings to make the information more relevant. It is important that you put the values in the cells indicated, as these are the cells referenced for calculations later on.

2 Enter the year numbers under the heading of 'Year' in Column A. This can be done by entering 0 in cell A6, then the formula '=A6+1' in cell A7. Copy this formula down the column by clicking on the lower right-hand corner of the cell and dragging down. Type the headings 'Simple Interest' and 'Total Amount' in Columns B and C and the values of 0 and 1000 next to 'Year 0'. In the screenshot shown below both columns B and C have been widened to make the headings fit. To do this put the cursor between the column headings B and C, the cursor will change to a cross with horizontal arrows, and drag it to the size required. Repeat this process for the other column.

B7 | fx =$C$1*$C$2*$C$3

	A	B	C	D
1	Rate		0.1	
2	Principal		1000	
3	Period		1	
4				
5	Year	Simple Interest	Total Amount	
6	0	0	1000	
7	1	100		
8	2			
9	3			
10	4			
11	5			

3 To calculate the interest earned after one year, enter the formula '=$C$1*$C$2*$C$3' into cell B7. We use a '$' symbol to 'fix' a particular cell, meaning that these cell addresses will still be used when the formula is copied down the column. Copy the formula down the column as described in Step **2**.

4 To add the interest to the total amount, enter the formula '=B7+C6' into cell C7. Copy this formula down the column. Now put the heading 'Amount of Interest' in cell D5 and 0 in D6. To calculate the total interest put the formula '=D6+B7' into cell D7 and then copy it down the column as before. Again, this column needs to be widened.

(a) What is the total amount of the investment after 8 years?

(b) How much of this total amount is interest?

Compounding interest

5 Extend the table with the headings 'Year' in E5, 'Compound Interest' in F5, and 'Total Amount' in G5. Manually enter or use a formula to enter the number of the year from 0 to 8, and an initial value of 0 for interest and 1000 for the total amount.

6 This time, we will add the interest earned after each year to the principal, and then use this amount as the new principal to calculate the interest for the following year. To do this, enter a new formula in the interest column, as shown in the screenshot below.

F7 | fx =G6*$C$3*$C$1

	A	B	C	D	E	F	G
1	Rate		0.1				
2	Principal		1000				
3	Period		1				
4							
5	Year	Simple Interest	Total Amount	Amount of Interest	Year	Compound Interest	Total Amount
6	0	0	1000	0	0	0	1000
7	1	100	1100	100	1	100	
8	2	100	1200	200	2		
9	3	100	1300	300	3		
10	4	100	1400	400	4		
11	5	100	1500	500	5		
12	6	100	1600	600	6		
13	7	100	1700	700	7		
14	8	100	1800	800	8		
15							

7 To add the interest to the total amount, enter the formula '=F7+G6' into cell G7. Highlight both cells F7 and G7, put the cursor in the lower right-hand corner of G7 and drag down to row 14.

(a) What is the total amount of the investment after 8 years?

(b) How much of this total amount is interest?

(c) How much extra interest is calculated when it is calculated using a new amount each time, instead of the original principal?

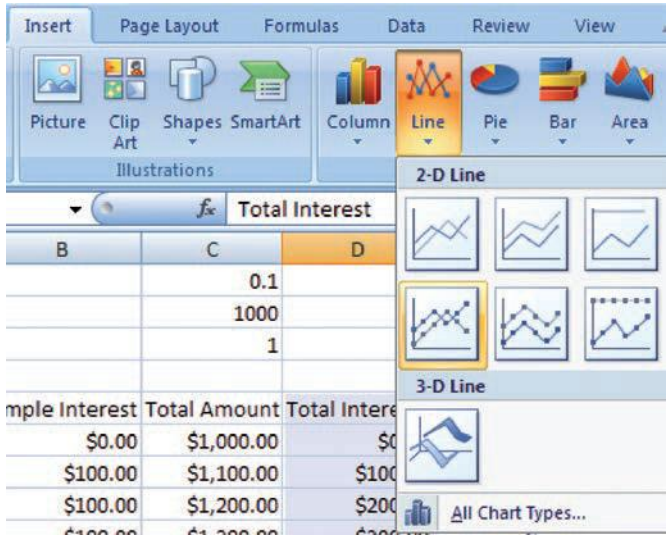
(d) (i) Explain the difference in the interest formula used in Step **3**, and the one used in Step **6**.

(ii) Why was it necessary to modify the formula?

8 Now put the heading 'Amount of Interest' in cell H5 and use a suitable formula to fill the values into this column.

9 As we are working with money, format the entries in your table to show two decimal places by highlighting relevant cells and selecting 'Currency' from the dropdown menu above the 'Number' option under the 'Home' tab.

10 The difference between the two types of interest can be clearly observed on a line graph. Highlight the 'Amount of Interest' columns (D and H). (Hold down the 'Ctrl' key to highlight more than one column.) You only need to highlight rows 5 to 14. Go to the 'Insert' Tab and select the line graph shown in the following screenshot.



11 Add axis labels by clicking on the 'Layout' tab and using the Axis Titles dropdown menu. Add a title by using the Chart Title dropdown box.

(a) Describe and then explain the shape of the simple interest graph.

(b) Describe and then explain the shape of the compound interest graph.

Taking it further

12 To calculate compound interest, we calculated 10% of the principal and added it on. We could also have calculated 110%, by multiplying by 1.1. Continuing to multiply each new amount obtained by 1.1 would give the total amount after each year. Create a third table in your spreadsheet with the headings of 'Year' and 'Total Amount'. Enter Year numbers of 0 to 8 in the first column, and 1000 as the first value of the 'Total Amount' column. Under this, enter a formula that multiplies this initial principal of $1000 by 1.1. Copy this formula down the column. What do you notice about the values generated by the formula?

1.2 Compound interest—the general formula

In the last section, we always used an annual compounding period when dealing with compound interest. This is not always the case in practical situations. For instance, the interest on a loan may compound daily, monthly, quarterly or every half year. A simple modification to our compound interest rule will allow us to deal with these different compounding periods. The important thing is to ensure that our compounding period and our interest rate are expressed using the same time period. So, if we have a monthly compounding period we will need a monthly interest rate.

Consider a loan at 12% p.a. If the compounding period is a:

- quarter (three months), the interest rate for each period is $\frac{12}{4}\% = 3\%$ (or 0.03) per period and there will be 4 periods per year
- month, the interest rate for each period is $\frac{12}{12}\% = 1\%$ (or 0.01) per period and there will be 12 periods per year
- day, the interest rate for each period is $\frac{12}{365}\% \approx 0.0329\%$ (or 0.000 329) per period and there will be 365 periods per year.

We would say the quarterly rate was 3%, the monthly rate 1% and the daily rate $\frac{12}{365}\%$. We keep the daily rate in fraction form so as not to introduce an error. We can round off to an appropriate number of decimal places at the end of our calculations.

Compound interest formula (general case):

$$A = P(1 + r)^n$$

where A = amount owing or earned at the end of the complete process
P = principal (original amount borrowed or loaned)
r = interest rate for the period, expressed as a decimal
n = number of compounding periods

Using the compound interest formula

This more general form of the compound interest formula can now be used to solve problems in a variety of situations.

Worked Example 4 WE4

Clive borrowed $12 000 to buy a lathe for his factory. He agreed to pay interest at a rate of 15% p.a. and to repay the loan in full in 3 years. Calculate the amount to be repaid if the interest is compounded:

(a) annually **(b)** half-yearly **(c)** quarterly.

Thinking		Working
(a) **1**	Identify what needs to be found.	(a) $A = ?$
2	Write the compound interest formula.	$A = P(1 + r)^n$
3	Identify the values of the other variables. Compounding every year means the number of time periods in a year is 1.	$P = 12\,000$ $r = 0.15$ $n = 3 \times 1 = 3$
4	Substitute these values into the formula and calculate the value of A.	$A = 12\,000 \times (1.15)^3$ $= \$18\,250.50$
(b) **1**	Identify what needs to be found.	(b) $A = ?$
2	Write the compound interest formula.	$A = P(1 + r)^n$
3	Identify the values of the other variables. Compounding half-yearly means the number of time periods in a year is 2.	$P = 12\,000$ $r = \frac{0.15}{2} = 0.075$ $n = 3 \times 2 = 6$
4	Substitute these values into the formula and calculate the value of A.	$A = 12\,000 \times (1.075)^6$ $= \$18\,519.62$
(c) **1**	Identify what needs to be found.	(c) $A = ?$
2	Write the compound interest formula.	$A = P(1 + r)^n$
3	Identify the values of the other variables. Compounding quarterly means the number of time periods in a year is 4.	$P = 12\,000$ $r = \frac{0.15}{4} = 0.0375$ $n = 3 \times 4 = 12$
4	Substitute these values into the formula and calculate the value of A.	$A = 12\,000 \times (1.0375)^{12}$ $= \$18\,665.45$

The more compounding periods there are, the greater the total amount to be repaid (the greater the value of the investment).

1.2 Compound interest —the general formula

Navigator

Answers page 716

Q1, Q2, Q3, Q4, Q5, Q9 (a) (b), Q10	Q1, Q3, Q4, Q5, Q6, Q8, Q9 (a) (b), Q10	Q1, Q4, Q5, Q6, Q7, Q8, Q9, Q11

If necessary, round answers to the nearest cent.

Fluency

WE4

1 Calculate the amount to be repaid after 2 years on a loan of \$16 000 if the 14% interest p.a. is compounded:

(a) annually **(b)** half-yearly **(c)** quarterly.

2 Calculate how much an investor would repay on a debt of \$8000 after $1\frac{1}{2}$ years if the 12% interest p.a. is compounded:

(a) half-yearly **(b)** quarterly, i.e. every 3 months.

3 If a loan of \$5800 is made at 16% p.a. compounded every half year, over 6 years the debt would grow to:

A $\$5800(1 + 0.16)^6$ **B** $\$5800(1 + 0.08)^{12}$

C $\$5800(1 + 0.16)^{12}$ **D** $\$5800(1 + 0.8)^{12}$

4 If the loan in Question **3** were compounded quarterly, the interest accrued would be:

A $\$5800(1 + 0.04)^6$ **B** $\$5800(1 + 0.16)^6 - \$5800$

C $\$5800(1 + 0.04)^{24} - \$5800$ **D** $\$5800(1 + 0.4)^{24} - \$5800$

Understanding

5 Calculate how much interest (to the nearest cent) is added over 10 years to an account paying 9% p.a. interest on an initial sum of \$45 800 if the interest is compounded:

(a) half-yearly **(b)** quarterly **(c)** monthly.

6 How much more will an investor get on an investment of \$32 000 over 4 years in an account offering 8.8% p.a. if the interest is compounded weekly rather than annually? (Assume 52 weeks in a year.)

7 Calculate the interest in Question **6** if it were to be compounded daily.

8 Martina won $5000 in the lottery and decided to invest it with a view to taking a trip to Paris sometime in the future. Martina was given two options when she invested the money. She could have an interest rate of 4.5% p.a. compounded quarterly or she could have 4.4% p.a. compounded daily. Which is the better deal, and by how much, if Martina invests the money for 2 years?

Reasoning

9 **(a)** $20 000 is to be invested for 3 years. Calculate the interest earned on each of the following to find which would give the best return.

(i) 5% p.a. compounded quarterly

(ii) 4.5% p.a. compounded every 2 months

(iii) 4% p.a. compounded monthly

(b) Do you think the size of the investment would alter which option in part **(a)** would give the best return?

(c) Prove your answer to part **(b)** algebraically by redoing part **(a)** with $\$P$ in place of $20 000.

Open-ended

10 'The more frequent the compounding, the greater the interest.' Explain what this statement means. Use a principal of $10 000 invested at 8% p.a. for 5 years and three different compounding periods (such as weekly, quarterly or monthly) to help you demonstrate your answer. Express the percentage increase correct to one decimal place.

11 Tyson is having trouble understanding the difference between simple and compound interest. He can't see why compound interest is higher than simple interest after the first compounding period if the principal, interest rate and overall time period are the same. How would you explain the difference to Tyson? Use an example of $4000 invested for 3 years at 5% p.a. to help you explain your answer.

Technology Exploration CAS

Equipment required: 1 brain, 1 TI-Nspire CAS or ClassPad CAS

Investigating compound interest and the 'rule of 72'

The *rule of 72* is a rule of thumb used to find the period of time over which an investment doubles in value. The best way to understand the rule is through an example.

Tanya purchases a block of land for \$100 000. How many years will it take for the land to double in value if its value increases at a rate of 8% p.a.?

Using the rule of 72, we divide 72 by 8 to get 9. That is, it will take 9 years for the land to double in value. We can verify this using the compound interest formula.

$$\begin{aligned} A &= P(1+r)^n \\ &= 100\,000(1+0.08)^9 \\ &\approx 199\,900.46 \\ &\approx \$200\,000 \text{ to the nearest thousand dollars} \end{aligned}$$

If the rate of increase in value were 12% p.a. it would take only 6 years for Tanya's block of land to double in value, as $72 \div 12 = 6$.

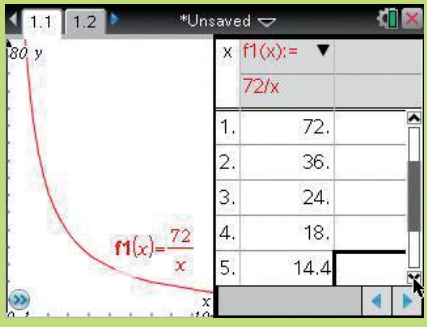
1. Verify this result using the compound interest formula.
2. Graph the rule of 72 on your CAS.

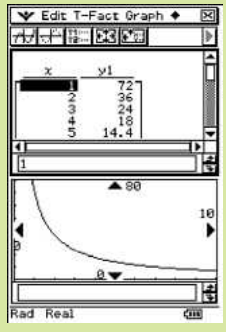
   Let y represent the number of years it takes for an investment to double in value and let x represent the percentage rate of interest per annum.

   The mathematical relationship between y and x is $y = \dfrac{72}{x}$.

Set the window so that x ranges from 0 to 10 and y from 0 to 80. Use the table of values to inspect values of x and y.

Using the TI-Nspire CAS	Using the ClassPad
Press ctrl T (or ctrl T) to insert a table of values.	With the graph section of the screen highlighted, tap to insert the table of values.





3 Use the table of values to complete the second column of the table below.

Interest rate per annum, x	Years for investment to double in value, y	Actual value of \$100 000 after y years using the compound interest formula $A = P(1 + r)^n$	Previous column subtract \$200 000 (A – \$200 000)
4			
6			
8	9	\$199 900.46	-\$99.54
9			

Between which two values of x do you think the rule of 72 gives a result of \$200 000, to the nearest dollar? Change the table step value until you have the required accuracy for your answer.

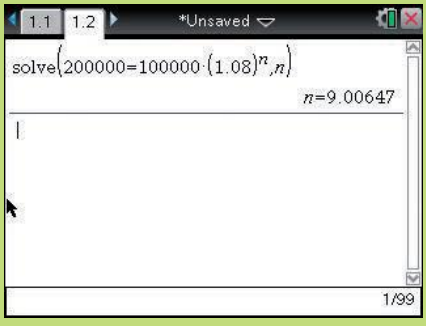
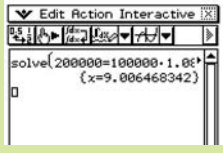
Taking it further

How accurate is the rule of 72? Investigate the accuracy of the rule of 72 by creating a spreadsheet on your CAS, based on the table above. Let the x-values start at 1 and end at 50, increasing in steps of 1. Add a fifth column for the percentage error. Between which two values of x did the rule of 72 give accurate results? What level of accuracy is acceptable?

As we are using the CAS to do this investigation, we can use its capabilities to get an accurate answer.

The equation we have is $200\,000 = 100\,000(1.08)^n$.

This simplifies to $2 = (1.08)^n$, which we can solve on our CAS.

Using the TI-Nspire CAS	Using the ClassPad
At the calculator screen press (menu) > Algebra > Solve and then enter as shown. Press (enter) when finished.	At the Main screen tap Action Equation/Inequality > solve and then enter the equation. Press (EXE) when finished.
	

Investigation

Using recursion

Equipment required: 1 brain

The Big Question

How do you program a robotic vacuum cleaner?

Engage

A recursive process requires two instructions:

1 Where to start.

2 A repeated rule for getting from one step to the next.

A repeated rule is also called a recursive rule.

Here are some examples of recursion:

- The set of Natural numbers starts with the number 1. The recursive rule to get the next number is to add 1 to the previous number.
- The sequence: 2, 3, 5, 9, 17, 33, … starts with the number 2. The recursive rule to get the next number is to double the previous number and subtract 1.
- The sequence: 10, 14, 12, 13, 12.5, 12.75, … starts with the numbers 10 and 14. The recursive rule to get the next number is to average the last two numbers.

1 For each of the following show that the number sequence or technique is recursive.

(a) 42, 21, 10.5, 5.25, 2.625, …

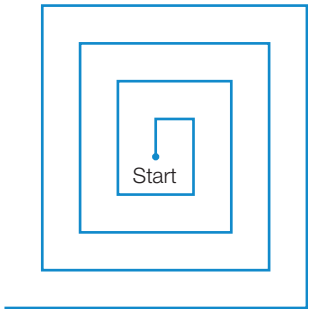
(b) 1, 2, 5, 26, 677, 458 330, …

(c) Calculating the total of an investment of $5000 at 10% p.a. over several years where the interest compounds annually.

Explore

A robotic vacuum cleaner is a flying saucer shape with a centred circular suction disc underneath.

The diagram below shows the spiral cleaning path of the robot. It can be programmed to follow this path using recursion.



2 If you walked this path what would you do at the end of each section?

Strategy options

- Break problem into manageable parts.
- Act it out.
- Look for a pattern.

Explain

3 Explain how a robotic vacuum cleaner can be programmed to follow the spiral cleaning path explored above (remember to specify a distance *and* direction). Organise your explanation under the headings:

- Instructions for getting started
- Instructions for the recursive step

You may like to use diagrams to make your explanation as clear as possible and to double check that your instructions work.

Elaborate

4 **(a)** Consider the distance travelled by the robot (at each step) when following your instructions above. Then complete this table:

Number of times the recursive step is repeated (n)	Distance travelled in this step (d_n)
0	
1	
2	
3	
4	
5	

(b) Find a formula relating n and d_n.

Evaluate

5 Is it possible to achieve a thorough cleaning of the floor with the recursion program you've developed? Explain your thinking.

Extend

The robot has completed the recursive step 16 times. It needs to cover the same area again and return to the start. Write instructions for a program that will achieve this.

Compound interest—further applications

1.3

The compound interest formula $A = P(1 + r)^n$ uses four variables. If we are given values for any three of these variables we can find the value of the fourth variable.

Finding the principal

Worked Example 5

WE5

Aunt Eva leaves Heidi a legacy of some deposit stock that was invested for 10 years at 5.11% p.a. compounded daily. The value of the cheque received was $21 669.68. Calculate the initial deposit. Give your answer rounded to the nearest dollar. (Ignore the occurrence of leap years in the 10-year period.)

Thinking	Working
1 Identify what needs to be found.	$P = ?$
2 Write the compound interest formula.	$A = P(1 + r)^n$
3 Identify the values of the other variables. Compounding daily means the number of time periods in a year is 365.	$A = 21\,669.68$ $r = \frac{0.0511}{365}$ $n = 10 \times 365 = 3650$
4 Substitute these values into the formula and simplify.	$21\,669.68 = P\left(1 + \frac{0.0511}{365}\right)^{3650}$
5 Transpose the equation to find P.	$P = \frac{21\,669.68}{\left(1 + \frac{0.0511}{365}\right)^{3650}}$
6 Evaluate P.	$P = \$13\,000.00595$ So, the principal was $13 000.

Note: The exact value of r is used in this example. This is to reinforce the idea, mentioned previously, that we should only round an answer at the very end of our calculations, otherwise we run the risk of introducing errors into our answers. Make sure you know how to use your calculator efficiently so you can complete calculations like those shown.

Finding the rate of interest

Worked Example 6 WE6

In the first year of his business Jack had takings of \$228 000. They grew to \$520 000 after 12 years, increasing at the same rate every year. At what annual percentage rate has his business been growing? Round your answer to two decimal places.

Thinking	Working
1 Identify what needs to be found.	$r = ?$
2 Write the compound interest formula.	$A = P(1 + r)^n$
3 Identify the values of A, P and n. (Compounding every year means the number of time periods in a year is 1.)	$A = 520\,000$, $P = 228\,000$, $n = 12 \times 1 = 12$
4 Substitute these values into the formula and simplify.	$520\,000 = 228\,000(1 + r)^{12}$
5 Transpose the equation to find r. *Remember*: Don't round until the end. (The inverse operation to finding a power of 12 is the 12th root. See calculator note below.)	$\frac{520\,000}{228\,000} = (1 + r)^{12}$ $\sqrt[12]{\frac{520\,000}{228\,000}} = (1 + r)$ $r = \sqrt[12]{\frac{520\,000}{228\,000}} - 1$ $r = 0.071\,122\,251\,37$
6 Multiply by 100 to convert to a percentage and round the answer to the appropriate number of decimal places.	$r = 7.11\%$ p.a. The business has been growing at a rate of 7.11% p.a.

Note: The key for calculating roots on a scientific calculator may look like $\sqrt[y]{x}$ and involve these steps:

1. Enter a value for x (the number under the root sign). In the above example, enter the expression 520 000 ÷ 228 000.
2. Press the root key $\sqrt[y]{x}$.
3. Enter a value for y. In the above example enter the number 12.
4. Press =.

You need to know how to perform this operation on your calculator.

Finding the time (or the number of compounding periods)

Worked Example 7 WE7

How many years will it take a \$452 stamp to be worth \$1000 if it increases in value by 7.5% p.a.? Assume that interest is compounding annually. Give your answer to the nearest whole number of years.

Thinking	Working
1 Identify what needs to be found.	$n = ?$
2 Write down the formula.	$A = P(1 + r)^n$
3 Identify the values of A, P and r.	$A = 1\,000$, $P = 452$, $r = 0.075$
4 Substitute these values into the formula and simplify.	$1000 = 452(1 + 0.075)^n$ $1000 = 452(1.075)^n$
5 Transpose the equation, as far as you can, to find n.	$\frac{1000}{452} = (1.075)^n$
6 Use 'guess and check' to try to find the solution.	$2.212\,389\,381 = (1.075)^n$ If $n = 4$, $(1.075)^4 = 1.335\,469$ If $n = 10$, $(1.075)^{10} = 2.061\,031$ If $n = 12$, $(1.075)^{12} = 2.381\,779$ If $n = 11$, $(1.075)^{11} = 2.215\,608$
7 Answer the question.	It will take 11 years for the stamp to increase its worth to \$1000.

In this example, the compounding was done annually. However, a similar process is used if the compounding period is not a year. The only difference is that after the value of n has been found it needs to be divided by the number of compounding periods in a year to find the number of years. So, if we had $n = 28$ and we had quarterly compounding then this would represent $\frac{28}{4} = 7$ years.

Also, we used a guess and check process to find the value of n. If you have a CAS you could use the Solve function to get an exact answer which you would then need to interpret.

1.3 Compound interest—further applications

Navigator

Q1, Q2, Q3, Q5, Q6, Q8	Q1 (b)–(d), Q2 (b)–(d), Q3 (a)–(b), Q4, Q5, Q6, Q8	Q1 (c)–(d), Q2 (c)–(d), Q3 (c), Q4, Q5, Q6, Q7, Q9

Answers page 717

Fluency

1 In each of the following, use the compound interest formula to find the principal.

WE 5

(a) Calculate how much would need to be invested at 8% p.a. compounded each half year to accumulate to \$9600 in 6 years. Round your answer to the nearest dollar.

(b) How much would Li have to deposit in order to receive \$10 000 in 7 years if she places her money in an account that pays 8.8% interest, compounded quarterly? Round your answer to the nearest dollar.

(c) How much would Sylvester have to deposit to receive \$8000 in 5 years' time if he places money in an account that pays 7.8% interest, compounded half yearly? Round your answer to the nearest cent.

(d) A deposit accumulates to \$4500 in 9 months at 12% p.a. compounded quarterly. The initial deposit, in dollars, was:

A $\frac{4500}{1.3^3}$ **B** $\frac{4500}{1.03^3}$ **C** $\frac{4500}{1.12^{3/4}}$ **D** 4500×1.04^3

WE6

2 In each of the following, use the compound interest formula to find the annual interest rate.

(a) Sales of \$26 900 grow to \$78 000 in 11 years. Calculate the percentage growth p.a. rounded to one decimal place. Assume the growth compounds annually.

(b) Costs of \$6200 increase to \$8000 in 8 years. Calculate the percentage increase p.a. rounded to two decimal places. Assume annual compounding.

(c) Calculate the rate of interest p.a. which would allow \$7900 to accumulate to \$9000 in 5 years if interest is compounded each half year. Round your answer to one decimal place.

(d) \$2800 accumulates to \$4500 in 3 years. The percentage rate of interest if it is compounded annually is:

A $\sqrt[3]{\frac{4500}{2800}} - 1$ **B** $\left(\sqrt[3]{\frac{4500}{2800}} - 1\right) \times 100$

C $\sqrt[3]{\frac{4500}{2800}} \times 100$ **D** $\frac{4500}{2800} \times 100$

WE7

3 In each of the following, use the compound interest formula and the method of guess and check (or a CAS) to find the number of years. In each case, assume that interest is compounding annually.

(a) How many years will it take for a \$2400 sapphire ring to be worth \$8000 if it increases in value by 10.6% p.a.? Round your answer to the nearest whole number of years.

(b) How many years will it take for a \$40 000 block of land to be worth \$90 000 if it increases in value by 8.5% p.a.? Round your answer to the nearest whole number.

(c) How many years will it take for a \$3400 porcelain dinner set to be worth \$5000 if it increases in value by 9.7% p.a.? Round your answer to the nearest tenth of a year.

Understanding

4 \$5000 is invested for 4 years with the interest compounded quarterly. For the first 2 years the interest rate is 4% p.a. and for the other 2 years it is 4.8% p.a.

(a) How much will the investment be worth at the end of the 4 years?

(b) How much interest will be earned?

(c) Would the investor be better off to invest for the whole 4-year period at 4.4% p.a.? How much interest would this investment earn?

(d) How much interest would be earned if the first 2 years were at 4.8% p.a. and the rest at 4% p.a.?

(e) The conditions of investment change and interest is now compounded annually. What would the interest rate need to be to earn the same amount of interest as found in part **(b)**?

(f) The interest rate is now 5% p.a. compounded annually. After how many years will the investment be at least as great as the result shown in part **(a)**?

5 Pene's savings are in a bank account that attracts 3.5% p.a. If her savings are to grow from \$10 000 to over \$20 000 in 20 years, what is the smallest number of compounding periods per year required?

Reasoning

6 A building society offered an interest rate of 7.25% p.a. on any investment over \$3000. Mary deposited \$3600 and received \$5944 after 7 years.

(a) How often was the interest compounded?

(b) What simple interest rate would have given the same return in the same time? Round your answer to one decimal place.

(c) Mary was a little disappointed because she'd calculated a return of \$5971. Her calculations were based on an incorrect assumption. What was that assumption?

7 If savings in an account are to be *at least* tripled over 20 years with an interest rate of 5.55% p.a., how often should the interest be compounded? Give your answer in months.

Open-ended

8 Jacob and Hafida are planning an overseas trip that will cost \$20 000. They have decided to pool their money and invest it at 6.5% p.a. compounding quarterly for 1 year. They did the following calculations:

How much do we need to invest?

$P = ?,\ A = 20\,000,\ n = 4,\ r = 0.065$

$$20\,000 = P\left(1 + \frac{0.065}{4}\right)^4$$

$$20\,000 = P(1.02)^4$$

$$20\,000 = P \times 1.08$$

$$P = \frac{20\,000}{1.08}$$

$$P = 18\,518.52$$

Checking:

$A = ?,\ P = 18\,518.52,\ n = 4,\ r = 0.065$

$$A = 18518.52\left(1 + \frac{0.065}{4}\right)^4$$

$$A = 18518.52(1 + 0.02)^4$$

$$A = 20\,045.04$$

When they collect their money a year later, they are almost \$250 below their target. Where did they go wrong?

9 Give different rates and compounding periods that will result in savings being increased by at least 50% over 4 years.

Half-time 1

Round answers to the nearest cent, unless told otherwise.

Ex.1.2

1 If a loan of \$7200 is made at 17% p.a. compounded quarterly over 5 years, the debt would grow to:

A $7200(1+0.17)^5$ **B** $7200\left(1+\frac{0.17}{4}\right)^{20}$ **C** $7200\left(1+\frac{0.17}{4}\right)^{5}$ **D** $7200(1+0.17)^{20}$

Ex.1.3

2 Calculate how much would need to be invested at 6% p.a. compounded monthly to accumulate to \$14 000 in 8 years.

Ex.1.1

3 Calculate the total amount owing on a loan of \$6000 after 3 years if the 14% interest p.a. is compounded annually.

Ex.1.3

4 \$100 000 grows to \$400 000 in 1 year. The rate of growth is compounding every 6 months. Find the value of the annual interest rate.

Ex.1.2

5 How much interest is added over 8 years to an account paying 8.25% p.a. interest on the initial sum of \$55 700 if the interest is compounded:

(a) half-yearly **(b)** monthly **(c)** daily.

Ex.1.3

6 An item originally valued at \$100 000 increased in value by a total of 55.133% over a number of years. Each year it appreciated at a rate of 5% of its value the previous year. Find the number of years, rounded to the nearest whole number.

Ex.1.1

7 Catherine and Ken invested \$20 000 for 6 years at an interest rate of 7.5% p.a.

(a) Calculate the value of the interest earned on the investment if it was compounded annually.

(b) Calculate the value of the interest earned on the investment if simple interest was paid instead.

(c) Compare the compounded interest with the simple interest.

Round your answers to the nearest dollar.

Ex.1.1

8 A painting was purchased 11 years ago for \$26 900. It has just been sold for \$78 000. Calculate the flat rate of appreciation p.a. Round your answer to one decimal place.

Comparing interest rates

1.4

In the last section we transposed the compound interest formula to find the principal, the annual interest rate and the number of years for an investment. We will use these transposition skills to compare the amount of interest earned in the first year of an investment under various compounding periods.

We will make these comparisons using a principal of \$10 000 and an annual interest rate of 6%.

If the compounding period is 1 year, then $A = 10\,000(1 + 0.06)^1 = 10\,600$. However, we are interested only in the amount of interest earned, which is calculated using $I = A - P$. In this case this gives \$600 interest.

If the compounding period is 6 months, then $A = 10\,000(1 + 0.03)^2 = 10\,609$, which represents \$609 interest. So, the percentage return is:

$$\frac{609}{10\,000} \times 100 = 6.09\%$$

The following table summarises this information for a number of compounding periods, and includes the simple interest calculation as well. This serves to remind us that simple interest over 1 year is the same as compound interest, compounded annually, for 1 year.

Case	Number of compounding periods	Interest in the first year	Percentage return
A	0 (simple interest)	\$600	$\frac{600}{10\,000} \times 100 = 6\%$ p.a.
B	1 (annual)	\$600	$\frac{600}{10\,000} \times 100 = 6\%$ p.a.
C	2 (6 monthly)	\$609	$\frac{609}{10\,000} \times 100 = 6.09\%$ p.a.
D	4 (quarterly)	\$613.64	$\frac{613.64}{10\,000} \times 100 \approx 6.14\%$ p.a.
E	12 (monthly)	\$616.78	$\frac{616.78}{10\,000} \times 100 \approx 6.17\%$ p.a.

In each of these cases the **nominal interest rate** is 6% p.a. but we can see the amount of interest earned is not a constant. To help make comparisons easier we like to calculate the **effective interest rate**. In case C above this is 6.09% p.a., in Case D it is 6.14% p.a. and for case E it is 6.17% p.a. This is the rate that would give the amount if simple interest was applied for 1 year.

$$R_{ef} = \left(\frac{I_1}{P}\right) \times 100$$

where R_{ef} = effective interest rate or real interest rate (p.a.) as a percentage
I_1 = amount of interest after the first year
P = principal

We can see that the more frequently we compound the interest, the higher the effective interest rate.

Worked Example 8 WE8

Find the effective rate of interest (R_{ef}) if \$25 000 is invested for 1 year at the following rates. Where necessary round your answers to three decimal places.

(a) a simple rate of 9.5% p.a.

(b) an annually compounding rate of 9.5% p.a.

(c) a monthly compounding rate of 9.5% p.a.

(d) a weekly compounding rate of 9.5% p.a.

Thinking	Working
(a) For simple interest, the effective rate is the same as the nominal rate.	(a) $R_{ef} = 9.5\%$ p.a.
(b) For annual compounding, the effective rate is the same as the nominal rate.	(b) $R_{ef} = 9.5\%$ p.a.
(c) 1 Calculate the total amount after the first year. Record your answer to four decimal places.	(c) $A = P(1+r)^n$ $= 25\,000\left(1+\frac{0.095}{12}\right)^{12}$ $= 27\,481.1896$
2 Calculate the amount of interest earned in the first year.	$I = A - P$ $= 27\,481.1896 - 25\,000$ $= 2481.1896$
3 Calculate the effective interest rate.	$R_{ef} = \frac{I}{P} \times 100$ $= \frac{2481.1896}{25\,000} \times 100$ $= 9.9247$
4 Round your answer to three decimal places.	The effective interest rate is 9.925% p.a.
(d) 1 Calculate the total amount after the first year. Record your answer to four decimal places.	(d) $A = P(1+r)^n$ $= 25\,000\left(1+\frac{0.095}{52}\right)^{52}$ $= 27\,489.0887$
2 Calculate the amount of interest earned in the first year.	$I = A - P$ $= 27\,489.0887 - 25\,000$ $= 2489.0887$
3 Calculate the effective interest rate.	$R_{ef} = \frac{I}{P} \times 100$ $= \frac{2489.0887}{25\,000} \times 100$ $= 9.9563$

4	Round your answer to three decimal places.	The effective interest rate is 9.956% p.a.

The effective interest rate is useful for comparing situations with different nominal rates and different compounding periods.

A formula for converting nominal rates to effective rates

A simple formula for converting nominal interest rates to effective interest rates is shown below. We multiply by 100 at the end to convert the rate to a percentage.

$$R_{ef} = \left[\left(1 + \frac{R}{n}\right)^n - 1\right] \times 100$$

where R_{ef} = effective interest rate (p.a.) as a percentage
R = nominal interest rate as a decimal
n = number of compounding periods in 1 year

This is useful because it requires only the nominal interest rate and the number of compounding periods in a year. We do not need to know the principal, nor do we need to calculate the amount of interest earned in the first year.

Worked Example 9

WE 9

A loan attracts an interest rate of 23% compounded daily. Find:

(a) the nominal rate, R, and the number of compounding periods, n

(b) the effective rate, R_{ef}, using the values found in **(a)**.

Thinking		Working
(a) 1	The nominal rate is the stated rate.	(a) $R = 23\%$
2	The number of compounding periods will be the number of days in a year.	$n = 365$
(b) 1	Convert the nominal rate to a decimal.	(b) $R = 23\% = 0.23$
2	Write down the formula.	$R_{ef} = \left[\left(1 + \frac{R}{n}\right)^n - 1\right] \times 100$
3	Substitute the decimal value of R and the value of n into the formula.	$R_{ef} = \left[\left(1 + \frac{0.23}{365}\right)^{365} - 1\right] \times 100$
4	Calculate R_{ef} as a percentage.	$= 0.2585 \times 100$ $= 25.85\%$ p.a.

1.4 Comparing interest rates

Navigator

Answers page 717

Q1 (a)–(d), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q11	Q1 (a)–(c) (e), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11	Q1 (a) (b) (e) (f), Q2 (d)–(f) ,Q6, Q7, Q9, Q10, Q11, Q12

Fluency

WE8

1 For each of the following find the effective rate of interest (R_{ef}).

(a) \$40 000 invested at a simple interest rate of 7% p.a.

(b) \$10 000 invested at an annual compounding rate of 9.4% p.a.

(c) \$100 000 invested at a monthly compounding rate of 12.75% p.a.

(d) \$35 000 borrowed at a weekly compounding rate of 19.7% p.a.

(e) \$20 000 borrowed at a fortnightly compounding rate of 16.5% p.a.

(f) \$200 000 invested at a rate of $6\frac{1}{4}$ % p.a. compounded every 5 days.

WE9

2 For each of the following find:

(i) the nominal rate, R, and the number of compounding periods, n

(ii) the effective rate, R_{ef}, using the values found in **(i)**.

(a) 7.3% p.a. compounded 5 times a year

(b) 15.5% p.a. compounded monthly

(c) 21% p.a. compounded quarterly

(d) $16\frac{1}{4}$ % p.a. compounded every 4 months

(e) daily compounding rate of $20\frac{1}{3}$ % p.a.

(f) half-yearly compounded rate of 5.25% p.a.

3 The effective interest rate that corresponds to 5% p.a. compounded every 2 months is:

A $\left[\left(1+\frac{0.05}{2}\right)^2-1\right]\times 100$

B $\left[\left(1+\frac{0.05}{6}\right)^6-1\right]\times 100$

C $\left[\left(1+\frac{0.5}{6}\right)^6-1\right]\times 100$

D $\left[\left(1+\frac{5}{2}\right)^2-1\right]\times 100$

Understanding

4 Two banks are currently offering the following rates on their investment savings accounts:

Q Bank: 15% p.a. compounded quarterly
D Bank: 14.9% p.a. compounded daily

(a) Find the effective rates being offered by the two banks.

(b) Which bank is offering the better deal?

5 Ned needs to buy a new laptop. He is planning to take out a personal loan with an interest rate of 20.5% p.a. compounded quarterly. The alternative is to use his credit card that has an interest rate of 20.1% compounded daily.

(a) What is the effective rate of this personal loan?

(b) What is the credit card's effective rate?

(c) Which of the two options should Ned choose and why?

6 A bed can be purchased on a 'get it now, pay in 12 months' plan.

The interest payment options are different depending on whether the purchase is made online or instore. The online purchase attracts a flat rate interest of 25% p.a. The interest on the instore purchase is 23% p.a. compounded daily.

(a) What is the real or effective interest rate charged on an instore purchase?

(b) Which is the cheaper option?

Reasoning

7 The following represent investment options. In each case, choose the better option *without calculating the effective rate* and justify your choice.

(a) 12% p.a. compounded monthly
15% p.a. compounded daily

(b) 10.5% p.a. compounded weekly
10.5% p.a. compounded monthly

(c) 11% p.a. compounded daily
12% p.a. compounded daily

(d) 15% p.a. flat rate for 2 years
15% p.a. compounded annually for 2 years

8 The options in Question **7** now represent loans. Will your choice of option change? Justify your answer.

9 Inez and Sam need to choose between the following lucrative investments:

Option A yields 35.2% p.a. compounded daily.
Option B yields 42% p.a. compounded annually.
Option C yields 40% p.a. compounded quarterly.

Which option should they choose and why?

10 Simon has saved the 10% deposit for a new car. The car company can offer him an interest-free loan on the remaining balance for 6 months and then 15% p.a. compounded daily. Alternatively, he could borrow the money elsewhere for a flat rate of 9% p.a.

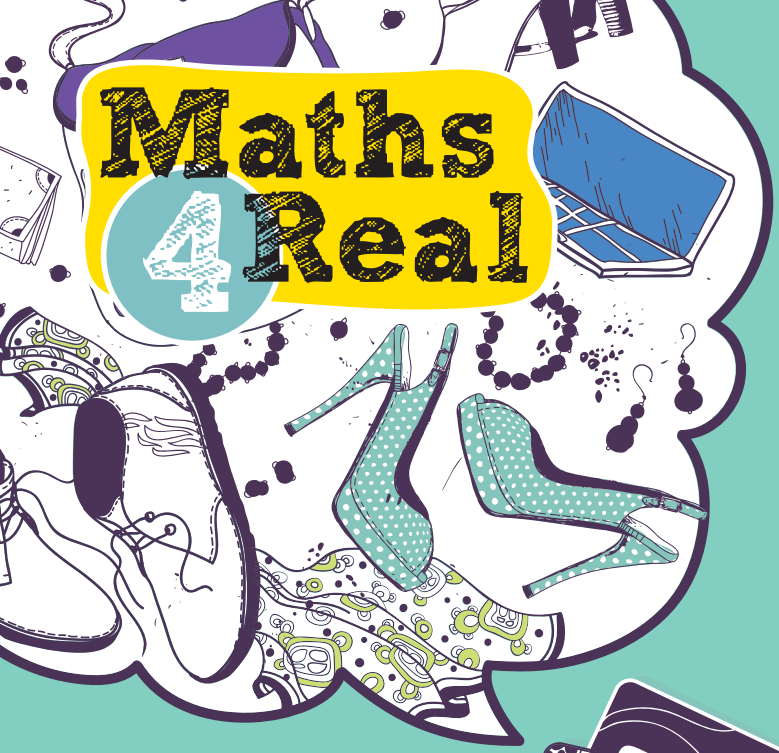
(a) If he pays out the loan plus interest in a year, which option will be cheaper?

(b) If he pays out the loan plus interest in two years, which option will be cheaper?

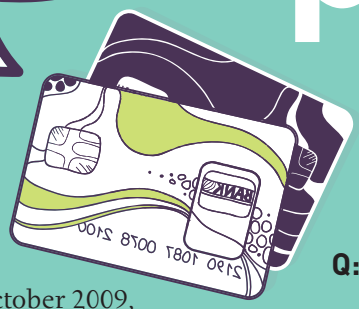
Open-ended

11 'Nominal interest rate' and 'simple interest rate' are the same thing. Explain with an example why this statement is untrue.

12 Two investments have different nominal interest rates and different compounding periods. However, the effective interest rate for both is exactly 46.41% p.a. What might be the nominal rates and respective compounding periods for these two investments?



Fantastic plastic



The convenience of credit cards, coupled with the interest charged on them, has caused many people to accumulate enormous debts that they are unable to repay. In October 2009, the total amount owed by Australians on credit cards was a staggering $45 billion. There were approximately 16 million credit cards in circulation, and the average amount owing on a card was $3141.

Q: What is credit?

A: 'Credit' is a way of buying goods or services without actually using cash. When you use a credit card to make a purchase, the bank pays the cost, and then sends you a bill for that amount (plus any other credit purchases) at the end of the month.

Q: What happens if you don't pay your credit card bill by the due date?

A: If you do not pay that bill in full by the due date, the bank will charge you interest—a certain percentage of the cost of your purchases is added onto the bill. Even if you pay the minimum amount on the bill, interest is still charged on the remaining amount.

Q: What is a debit card?

A: A debit card can be used in the same way as a credit card, except that it requires you to have the money already in a bank account, which is transferred to the store's account at the time of the purchase. You are using money you already have, instead of borrowing it from the bank.

1 List some of the advantages and disadvantages of using a debit card instead of a credit card.

Q: How is credit card interest calculated?

A: All credit cards have an Annual Percentage Rate of interest (APR). The interest rate varies from card to card, but is usually somewhere between 15% and 20%. Periodically (usually once a month), the bank will issue the credit card holder with a statement. The statement lists all of the purchases that have been made with the card during the statement period, and shows the total amount owing. Interest is calculated at the end of the period, and added to the account.

To calculate credit card interest banks use the following process:

- The balance (amount owing) at the end of each day of the statement period is determined. These daily balances are then added up and divided by the number of days in the statement period to get an 'Average Daily Balance' or ADB.
- A daily interest rate is calculated by dividing the annual rate by 365.
- The ADB is multiplied by the daily interest rate, then by the number of days in the statement period to obtain the interest for that period.

2 Jamie is 18 years old and has just successfully applied for a credit card with an APR of 18%. His statement after his first month is shown below. Use the process outlined to calculate the interest on Jamie's account that would be charged if he does not pay the balance in full by the due date. (See if you can find a short cut to your calculations.)

Eastpac Bank

Account Number: 3478 4421 0097 1749

Statement Period: 14/5/11 to 13/6/11

Date	Purchase	Amount	Balance
	Opening balance		$0.00
14 May	2 movie tickets	$34.00	$34.00
19 May	1 pair jeans and 2 T-shirts	$92.90	$126.90
23 May	Camera	$181.00	$307.90
30 May	Laptop	$799.00	$1106.90
2 June	Dinner with friends	$33.00	$1139.90
7 June	Phone bill	$54.96	$1194.86
11 June	Petrol	$68.00	$1262.86
	Closing balance		$1262.86

Interest-free periods

Many credit cards offer an 'interest free' period of up to 55 days. This is usually the length of time between the first day of the statement period and the due date for repayment. If the total balance of the account is paid off before the due date, interest will not be charged provided that the previous month's statement was also paid in full before the due date, so no interest is carried over.

Minimum monthly repayments

The bank does not require you to pay the full balance shown on your statement. Instead, it calculates a 'minimum monthly repayment' amount, which you must pay by the due date (or risk a 'late payment fee'). This amount is about 5% of the balance. However, not all of it goes towards paying off the principal of the debt (the actual amount owing). Some of the minimum repayment amount is interest. To calculate how much is interest:

- find the monthly interest percentage rate by dividing the APR by 12
- calculate this monthly interest percentage amount of the minimum repayment. This represents the interest in the repayment. The rest is the amount that will actually be deducted from your credit card balance (the principal).

3 (a) Calculate Jamie's minimum monthly repayment, if it is 5% of the closing balance.

(b) Calculate how much of the minimum repayment amount is interest.

(c) If Jamie only pays the mimimum repayment amount, by how much will his credit card balance be reduced?

(d) Discuss how only paying the minimum amount each month can create a problem with credit card debt.

Research

Gather some information on three or four of the many different credit cards available. Find out the minimum requirements needed to apply for the card (such as age and income), the interest rate, any annual or monthly fees, whether there is an interest-free period, and any other special rewards or deals offered with the card.

Present your findings as a visual display. Which card would you recommend to someone who would like a credit card, but does not want a growing debt? What additional advice would you give them in this area?

1.5 Depreciation

Depreciation is the process by which items decrease in value. Most of the objects around us are disposable items that we wear out or consume and so they depreciate in value. This includes clothes, most furniture, cars, computers, televisions, buildings, mobile phones, hardware and machines of all sorts. This section will focus on developing models for calculating depreciation. The formulas involved are closely related to those used for appreciation introduced in Section 1.1, which is the opposite process to depreciation.

Straight-line depreciation

Straight-line depreciation or flat-rate depreciation applies when items lose a constant amount of value each year. This amount of loss is measured in dollars and is known as the **depreciation value**. The rate of this depreciation can be written as a percentage or decimal. Depreciation over a number of years is known as **cumulative depreciation**. Another important term is **initial cost** or **prime cost** which refers to the original purchase price of the item. Finally, the value of the item after depreciation is known as the **written-down value** or **adjusted value** (previously called the 'book value'). With straight-line depreciation, the written-down value will eventually be zero.

Consider this situation:

A company buys a truck for \$75 000. Each year it depreciates at 20% of its prime cost.

The depreciation each year is 20% of \$75 000
$= 0.2 \times 75\,000$
$= \$15\,000$

The cumulative depreciation after 3 years will be 20% of $\$75\,000 \times 3$
$= 0.2 \times 75\,000 \times 3$
$= \$45\,000$

The written-down value after 3 years will be $\$75\,000 - \$45\,000$
$= \$30\,000$

After another 2 years the written-down value will reach zero.

When dealing with straight-line depreciation:

- cumulative depreciation = prime cost × rate of depreciation × time in years
- written-down value = prime cost − cumulative depreciation

These relationships resemble the formulas for simple interest. We can make this connection more formal by writing the formula in a particular way.

The straight-line depreciation formula is:

$$D_T = PRT$$

where D_T = cumulative depreciation after T years, in dollars
P = prime cost, in dollars
R = rate of depreciation (p.a.) as a decimal and T = time in years

Also: $$V_T = P - D_T$$
where V_T = written-down value after T years, in dollars

Worked Example 10 WE10

The initial cost of a washing machine 2 years ago was $1950. It depreciates at 18.5% p.a. Assuming straight-line depreciation what is the:

(a) cumulative depreciation since the time of purchase

(b) current written-down value of the machine?

Thinking		Working
(a) **1**	Identify what needs to be found.	(a) $D_2 = ?$
2	Write down the formula for straight line depreciation.	$D_T = PRT$
3	Identify the values of P, R and T.	$P = 1950$, $R = 18.5\% = 0.185$, $T = 2$
4	Substitute these values into the formula, and calculate D_2.	$D_2 = 1950 \times 0.185 \times 2$ $= \$721.50$
(b) **1**	Identify what needs to be found.	(b) $V_2 = ?$
2	Write down the formula.	$V_T = P - D_T$
3	Identify the values of P and D_2.	$P = 1950$, $D_2 = 721.50$
4	Substitute these values into the formula, and calculate V_2.	$V_2 = 1950 - 721.50$ $= \$1228.50$

Reducing balance depreciation

Reducing balance depreciation applies when items depreciate by a smaller amount each year. Each year, the amount of depreciation is calculated as a percentage of the previous year's value. As the value of the item decreases so the corresponding depreciation also decreases. With reducing balance depreciation the written-down value will never reach zero.

Consider again the situation of the company that purchased a truck for $75 000 with a depreciation of 20% p.a. but this time the depreciation is calculated on the written-down value each year and not on the prime cost.

Initial value = $75 000

Depreciation year 1 = 20% of $75 000
= 0.20 × $75 000
= $15 000

Written-down value year 1 = $75 000 – $15 000
= $60 000

Depreciation year 2 = 0.20 × $60 000
= $12 000

Written-down value year 2 = $60 000 – $12 000
= $48 000

Depreciation year 3 = 0.20 × $48 000
= $9600

Written-down value year 3 = $48 000 – $9600
= $38 400

This can be set out neatly in a spreadsheet for the first 5 years as follows:

	A	B	C	D
1	Year no.	Depreciation	Cumulative depreciation	Written-down value
2	0			75000
3	1	15000	15000	60000
4	2	12000	27000	48000
5	3	9600	36600	38400
6	4	7680	44280	30720
7	5	6144	50424	24576
8				

If you are experienced in the use of spreadsheets you might like to try to work out the formulas used to create this spreadsheet.

The spreadsheet view makes it easier for us to verify that the written-down value is 80% of the previous written-down value. So, instead of calculating $0.20 \times \$75\,000$ and subtracting that from \$75 000 to get \$60 000 we could just calculate $0.80 \times \$75\,000$ to get \$60 000.

The last column in the spreadsheet can be rewritten in the form shown in the following table that will make it easier for us to see the connection to compound interest.

Year	Written-down value ($)
0	75 000
1	$0.8 \times 75\,000$
2	$0.8 \times (0.8 \times 75\,000) = (0.8)^2 \times 75\,000$
3	$0.8 \times (0.8 \times (0.8 \times 75\,000)) = (0.8)^3 \times 75\,000$
4	$0.8 \times (0.8 \times (0.8 \times (0.8 \times 75\,000))) = (0.8)^4 \times 75\,000$
5	$0.8 \times (0.8 \times (0.8 \times (0.8 \times (0.8 \times 75\,000)))) = (0.8)^5 \times 75\,000$

So, after T years the written-down value is $75\,000(0.8)^T$, which could, of course, be written as $75\,000(1 - 0.2)^T$. This is similar to the compound interest formula, except there is a subtraction sign instead of an addition sign in the brackets.

The reducing balance depreciation formula is:

$$V_T = P(1 - R)^T$$

where V_T = written-down value after T years, in dollars
P = prime cost, in dollars
R = rate of depreciation (p.a.) as a decimal
T = time in years

The above formula is based on the assumption that the depreciation is an annual event, as is usually the case. If it was not an annual event we could alter the formula in the same way we altered the compound interest formula to take account of this.

Worked Example 11

WE11

The prime cost of a lawn mower 7 years ago was \$5600. If it depreciates at 9% p.a. assuming reducing balance depreciation, calculate the:

(a) current written-down value of the mower

(b) cumulative depreciation since the time of purchase.

Thinking		Working	
(a) 1	Identify what needs to be found.	(a)	$V_7 = ?$
2	Write down the formula for reducing balance depreciation.		$V_T = P(1 - R)^T$
3	Identify the values of P, R and T.		$P = 5600,\ R = 9\% = 0.09,\ T = 7$
4	Substitute these values into the formula and calculate V_7.		$V_7 = 5600(1 - 0.09)^7$ $= \$2893.86$
(b) 1	Identify what needs to be found.	(b)	$D_7 = ?$
2	Write down the formula.		$D_T = P - V_T$
3	Identify the values of P and V_7.		$P = 5600,\ V_7 = 2893.86$
4	Substitute these values into the formula and calculate D_7.		$D_7 = 5600 - 2893.86$ $= \$2706.14$

'Straight-line depreciation' can also be expressed as:

'depreciation at a percentage of its prime cost or initial value or purchase price every year'.

'Reducing balance depreciation' can also be expressed as:

'depreciation at a percentage of its reduced value every year'.

Unit cost depreciation

The **total possible depreciation** of an item is its purchase cost less any resale value or **scrap value** that it may still have at the end of its useful life. Scrap value is also known as **salvage value** or **residual value**.

The **expected life** of an item can be expressed as a unit of time (years/months/weeks/hours) or a unit of work like 'pages printed' for a printer or 'photos taken' for a camera.

The **unit cost depreciation** is a measure of how much the item depreciates in dollars per unit of its expected life.

Total possible depreciation = prime cost – scrap value

$$\text{Unit cost depreciation} = \frac{\text{total possible depreciation}}{\text{expected life}}$$

Depreciation (after a certain amount of use or work)
= Unit cost depreciation × number of units of work done

Worked Example 12

WE 12

A laptop is expected to work for 6000 hours before it has a resale or scrap value of \$100. The purchase price is \$1300. Find:

(a) the total possible depreciation

(b) the unit cost depreciation

(c) the depreciation after 1000 hours of use

(d) the written-down value after 1000 hours of use.

Thinking		Working	
(a)	1	Write down the formula.	(a) Total possible depreciation = prime cost − scrap value
	2	Identify the prime cost and scrap value and substitute into the formula.	Prime cost = \$1300 Scrap value = \$100 Total possible depreciation = \$1300 − \$100
	3	Calculate the total depreciation.	= \$1200
(b)	1	Write down the formula.	(b) Unit cost depreciation $= \dfrac{\text{total possible depreciation}}{\text{expected life}}$
	2	Identify the total depreciation and expected life and substitute into the formula.	Total depreciation = \$1200 Expected life = 6000 hours Unit cost depreciation $= \dfrac{1200}{6000}$
	3	Calculate the unit cost depreciation.	$= 0.2$ The laptop decreases in value by 20 cents for every hour of work.
(c)	1	Write down the formula.	(c) Depreciation after use = unit cost depreciation × number of units of work done
	2	Identify the unit cost depreciation and work done, and substitute into the formula.	Unit cost depreciation = \$0.20 per hour, number of units of work done = 1000
	3	Calculate the depreciation after the given use.	Depreciation after use $= 0.2 \times 1000$ = \$200
(d)	1	Write down the formula.	(d) $V = P - D$
	2	Identify the values of P (original value) and D (accumulated depreciation) and substitute into the formula.	P = \$1300, D = \$200 V = \$1300 − \$200
	3	Calculate V, the written-down value after the given use.	V = \$1100

1.5 Depreciation

Navigator

Q1 (a)–(f), Q2 (a)–(f), Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q19	Q1 (b)–(e) (g), Q2 (b)–(e) (g), Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q14, Q15, Q19	Q1 (b) (c) (f) (h), Q2 (b) (c) (f) (h), Q3, Q6, Q7, Q10, Q11, Q12, Q15, Q16, Q17, Q18, Q19, Q20

Answers page 717

Fluency

1 For each of the following, assume straight-line depreciation. Find: WE10

(i) the cumulative depreciation since the time of purchase

(ii) the current written-down value.

Round all answers to the nearest cent.

(a) The initial price 5 years ago was \$30 000. $R = 7\%$ p.a.

(b) The prime cost 12 years ago was \$47 500. $R = 5\%$ p.a.

(c) The purchase price 20 years ago was \$100 000. $R = 5\%$ p.a.

(d) The initial cost 23 years ago was \$790 000. $R = 4.25\%$ p.a.

(e) The prime cost 4 years ago was \$145 000. $R = 10.5\%$ p.a.

(f) The initial price $5\frac{1}{2}$ years ago was \$300 000. $R = 6.5\%$ p.a.

(g) The purchase price 6 months ago was \$47 500. $R = 20\frac{3}{4}\%$ p.a.

(h) The prime cost 18 months ago was \$152 700. $R = 33\frac{1}{4}\%$ p.a.

2 For each of the above, assume reducing balance depreciation. Find: WE11

(i) the current written-down value

(ii) the cumulative depreciation since the item was purchased.

Round all answers to the nearest cent.

3 (a) A machine is expected to work for 3000 hours before it has a resale or scrap value of \$150. The purchase price is \$6150. Find: WE12

(i) the total possible depreciation

(ii) the unit cost depreciation

(iii) the depreciation after 10 hours of use

(iv) the written-down value after 10 hours of use.

(b) A machine is expected to work for 2000 hours before it has no scrap value. The purchase price is \$1000. Find:

(i) the total possible depreciation

(ii) the unit cost depreciation

(iii) the depreciation after 1000 hours of use

(iv) the written-down value after 1000 hours of use.

Understanding

4 What would be the total depreciation after 6 years on a \$780 workstation if it is depreciated at 11.25% p.a. of its prime cost?

5 A car costs \$24 000 new. The value of the car in 3 years, if it depreciates at 25% p.a. of its prime cost each year, will be:

A $\$\dfrac{24\,000 - 24\,000 \times 0.75}{3}$

B $\$24\,000 - 24\,000 \times 0.25 \times 3$

C $\$24\,000 \times 0.75^3$

D $\$24\,000 \times 0.25 \times 3$

6 Calculate the value of a computer (initial cost \$5600) after 4 years if it is depreciated at 15% p.a of its prime cost each year.

7 Calculate the depreciation in each year, cumulative depreciation and written-down or adjusted value for each of the first 3 years if a \$45 000 forklift is depreciated at 20% of its prime cost each year.

8 Which of the answers in Question **5** would be correct if the depreciation was calculated on the car's written-down value at the start of each year?

9 Calculate the value of a computer (cost \$5600) after 4 years if it is depreciated at 15% p.a. of its written-down value at the start of each year. Compare this answer with that obtained for Question **6**.

10 How much would the total depreciation be after 6 years on a \$780 workstation if it is depreciated at 15% p.a. of its written-down value at the start of each year?

11 Calculate the depreciation, cumulative depreciation and written-down value for each of the first 3 years if a \$45 000 forklift is depreciated at 20% p.a. of its written-down value at the start of each year. Compare this answer with that obtained in Question **7**.

12 A \$7500 machine has a scrap value of \$300, and is guaranteed for 11 000 cycles. In a year when it operates for 2630 cycles the amount of depreciation, in dollars, is:

A $\dfrac{7500 - 2630}{11\,000} \times 300$

B $\dfrac{11\,000 - 7500}{2630} \times 300$

C $\dfrac{7500 - 300}{11\,000} \times 2630$

D $\dfrac{7500}{11\,000} \times 2630$

13 **(a)** The amount of depreciation (in dollars) for a \$5000 machine with no scrap value, guaranteed for 8000 hours, in its first year when it works for 1350 hours, is:

A $\dfrac{8000}{1350} \times 5000$

B $\dfrac{8000}{5000} \times 1350$

C $\dfrac{5000}{6650} \times 1350$

D $\dfrac{5000}{8000} \times 1350$

(b) At the end of the first year the written-down value (in dollars) of the machine described in part **(a)** is:

A $5000 - \dfrac{5000}{8000} \times 1350$

B $6350 - \dfrac{8000}{5000} \times 1350$

C $8000 - \dfrac{5000}{8000} \times 1350$

D $8000 - \dfrac{1350}{8000} \times 5000$

14 **(a)** A \$10 000 soft-drink bottler with no scrap value is guaranteed for 120 000 bottles. Calculate the depreciation in its first year when it fills 47 000 bottles.

(b) Calculate the value of the soft-drink bottler after the first year.

Reasoning

15 Calculate the value of an $11 500 screen printer with a scrap value of $200 guaranteed for 90 000 prints when it is used 12 400 times.

16 Calculate the value of a $3500 photocopier with a scrap value of $400 guaranteed for 600 000 copies after it produces 210 000 copies.

17 A camera costs $3500, has a scrap value of $900 and is guaranteed for 8000 photos. Calculate the value after the first and second year of its life if in the 2 years it takes 450 and 1800 photos respectively.

18 A $30 000 car depreciates at 20% p.a. of its written-down value at the start of each year every year for the first 3 years. An $80 000 car depreciates at 25% p.a. of its written-down value at the start of each year every year for the first 3 years. After that time, both cars depreciate at 10% p.a. of their written-down value at the start of each year for every subsequent year.

(a) What is the difference in their values:

(i) at purchase **(ii)** after 1 year **(iii)** after 2 years **(iv)** after 3 years?

(b) When will the difference in their values be under $3000?

Open-ended

19 An item was initially purchased for $10 000. It is now worth only $3000. Assuming that it depreciated at the same rate of its prime cost every year, what was the possible rate of decrease and corresponding elapsed time?

20 Repeat Question **19** assuming the depreciation followed a reducing-balance model.

Outside the Square — Problem solving

Weird and wonderful numbers

When you've been up and down the stairs several times, take time out and try to answer these questions:

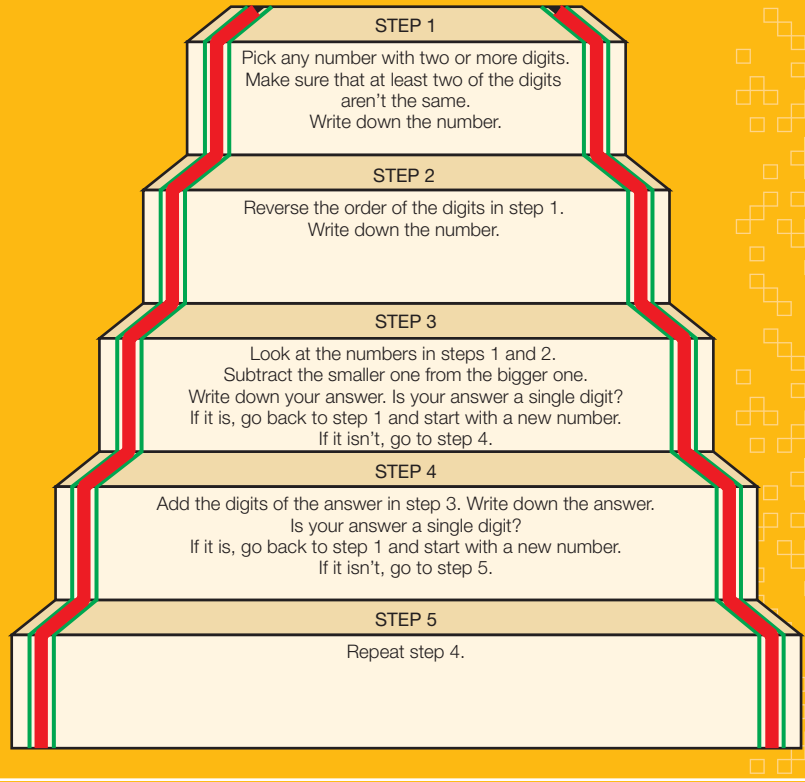
1 What did you find at the bottom of the stairs? What number(s) sent you back to the top?

2 Why is there a special condition attached to step one?

Strategy options
- Act it out.

3 Will you ever find something else at the bottom of the stairs?

Strategy options
- Seek an exception.
- Solve a simpler problem.



1.6 Growth and decay

Growth

The formula $A = P(1 + r)^n$ has been applied to compound interest and appreciation questions. This formula can also be used as a model for calculations in areas of human activity such as population growth. Other examples include the spread of bacteria and viruses. These examples have the same pattern of growth as compound interest. It is known as **exponential growth** and is characterised by an increase in growth with each successive time period.

Exponential growth is modelled by the formula $A = P(1 + r)^n$.

A is an accumulated total after a certain number of time periods, n.

P is an initial number. Sometimes, it is useful to think of P as the value of A when $n = 0$.

r is the growth rate. r is applied to the accumulated total from one time period to the next.

Always round down when whole number answers are required for discrete quantities (quantities that are counted) for exponential growth.

Worked Example 13 WE13

The population of a district starts at 32 000. What will the population be in 7 years:

(a) if it increases by 4% each year **(b)** if it has a growth rate of 1.2% per month?

Thinking	Working
(a) 1 Identify what needs to be found.	(a) $A = ?$
2 Identify the values of P, r and n.	$P = 32\,000$ $r = 4\%$ per year $= 0.04$ $n = 7$ years
3 Write down the growth formula.	$A = P(1 + r)^n$
4 Substitute the values into the formula and calculate the answer.	$A = 32\,000(1 + 0.04)^7$ $= 42\,110$
(b) 1 Identify what needs to be found.	(b) $A = ?$
2 Identify the values of P, r and n and convert the unit for time so it is consistent with the unit for rate.	$P = 32\,000$ $r = 1.2\% = 0.012$ per month $n = 7$ years $= (7 \times 12) = 84$ months
3 Write down the growth formula.	$A = P(1 + r)^n$
4 Substitute the values into the formula and calculate the answer.	$A = 32\,000(1 + 0.012)^{84}$ $= 87\,159.158\,51$
5 Round down to the nearest whole number. (Here, population is a discrete quantity.)	The population will be 87 159.

Decay

The formula $V_T = P(1 - r)^n$ has been applied to depreciation questions. A formula of the same structure can also be used as a model for a variety of problems. The decay of a radioactive material follows this model. Sporting tournaments based on elimination by knockout also follow this model. What these examples have in common is the same pattern of decline that was observed with depreciation. It is known as **exponential decay** and is characterised by something decreasing by a smaller amount with each successive time period.

Exponential decay is modelled by the formula $A = P(1 - r)^n$.

A is a decreased total after a certain number of time periods, n.

P is an initial number. Sometimes it is useful to think of P as the value of A when $n = 0$.

r is the decay rate. r is applied to the decreased total from one time period to the next.

Always round up when whole number answers are required for discrete quantities for exponential decay.

Worked Example 14 WE14

The initial number of subscribers to a magazine was 12 469. How many subscribers will there be after 9 months if the number of subscribers decreases by:

(a) 1% each month **(b)** 0.25% per day?

Assume 30 days in a month.

Thinking		Working
(a) **1**	Identify what needs to be found.	(a) $A = ?$
2	Identify the values of P, r and n.	$P = 12\,469$ $r = 1\%$ per month $= 0.01$ $n = 9$ months
3	Write down the decay formula.	$A = P(1 - r)^n$
4	Substitute the values into the formula and calculate the answer.	$A = 12\,469(1 - 0.01)^9$ $= 11\,391$
(b) **1**	Identify what needs to be found.	(b) $A = ?$
2	Identify the values of P, r and n and change the unit for time so it is consistent with the unit for rate.	$P = 12\,469$ $r = 0.25\%$ per day $= 0.0025$ $n = 9$ months $= (9 \times 30) = 270$ days
3	Write down the decay formula.	$A = P(1 - r)^n$
4	Substitute the values into the formula and calculate the answer.	$A = 12\,469(1 - 0.0025)^{270}$ $= 6343.308\,039$
5	Round up to the nearest whole number. (Here, the number of subscribers is a discrete quantity.)	The number of subscribers will be 6344.

Combining the growth and decay formulas

The exponential growth formula has this basic structure:

Total	=	Initial number	×	Number greater than 1	Time
A	=	P	×	$(1 + r)^n$	

The exponential decay formula has this basic structure:

Total	=	Initial number	×	Number between 0 and 1	Time
A	=	P	×	$(1 - r)^n$	

The above structures are very similar and can be combined into one general formula:

The formula for exponential growth or decay can be written as:

$$Y = ka^X$$

where Y = the total after X time periods
k = the initial number (when $X = 0$)
a is the growth factor. It is the number you multiply by to get from one value of Y to the next
$a > 1$ for growth or $0 < a < 1$ for decay

Note that for a particular problem:

- X and Y are **variables**. Variables take on different values and as one changes, so does the other.
- k and a are **constants** (fixed numbers).

This form of the growth/decay formula is particularly useful in questions such as the example below, where the rate, r, is not given directly.

Worked Example 15 WE15

The number of bacterial cells (N) in a kitchen sink is increasing with time (T) hours. The bacteria count for the first 3 hours is shown below.

***T* (hours)**	0	1	2	3	4	5	6
N	6	12	24	48			

(a) Copy and complete the table.

(b) Find the formula that models this growth.

(c) Use the formula to predict the number of bacteria cells in the sink after 3 hours and 20 minutes.

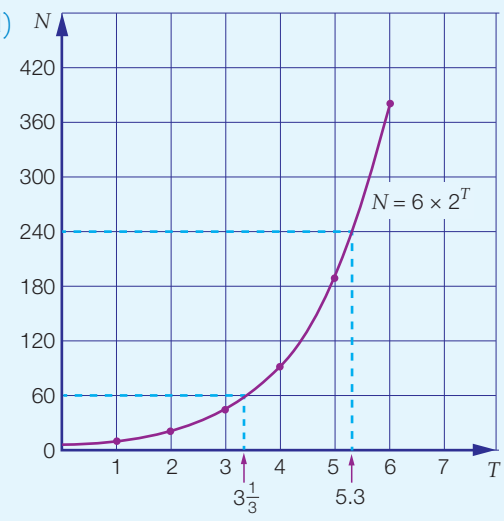
(d) Use the table of values to plot a graph of N against T. Join the points with a smooth curve.

(e) Use the graph to predict the approximate time when there will be 240 bacteria cells in the sink.

Thinking	Working
(a) See the doubling pattern in the values of N and complete the table.	(a)

4	5	6
96	192	384

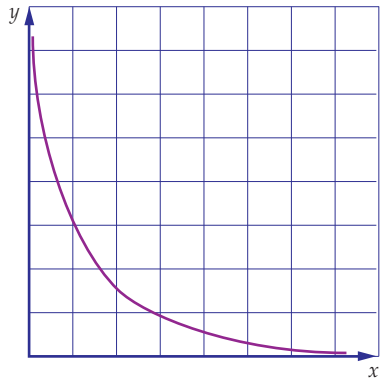
Thinking	Working
(b) 1 Identify the variables.	(b) X is T = time in hours Y is N = total number of bacteria cells
2 Identify the constants. (k is the initial value of N.) (a is the number that you multiply by to get from one value of N to the next.)	$T = 0$, $N = 6$ so $k = 6$ $a = \frac{12}{6} = 2$
3 Write down the general form of the growth equation.	$Y = ka^X$
4 Substitute the variables and constants for this question into the general form.	$N = 6 \times 2^T$
(c) 1 Convert hours and minutes into hours.	(c) $T = 3$ hours 20 minutes $= 3\frac{1}{3}$ hours
2 Substitute into the formula and find N.	$N = 6 \times 2^{3.3333...}$ $= 60.4762$ ≈ 60 bacteria cells
(d) Plot points, using T for the horizontal axis and N for the vertical axis.	(d)



Thinking	Working
(e) Using the graph, draw a horizontal line from the given values on the vertical axis to the curve. Then draw a vertical line from this point on the curve to the horizontal axis to give the corresponding value.	(e) Approximate time = 5.3 hours

Note: The formula $N = 6 \times 2^T$ could be written in the other form of the growth formula as $N = 6(1 + 1)^T$. From this form it can be seen that $R = 1 = 100\%$. Doubling the value of N in the table, as was done in part **(a)**, is the same as increasing the value of N by 100%. Another way of saying this is that the bacterial cells are increasing at a rate of 100% per hour.

The graph in part **(d)** is called an exponential growth curve. It increases slowly at first and then very rapidly or exponentially. An exponential decay curve has a similar shape, but curves downwards as its values decrease rapidly, then more slowly as shown.



1.6 Growth and decay

Navigator

Answers page 718

Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9	Q1 (a)–(c), Q2 (a)–(c), Q3, Q5, Q6, Q7, Q9	Q1 (d)–(f), Q2 (d)–(f), Q3, Q6, Q7, Q9, Q10

Fluency

WE13

1 **(a)** If the population in a certain town is 10 200 and has a growth rate of 6% each year, what is the estimated population after 12 years?

(b) A colony of sea birds increases at a rate of 90% a year. If there are 8000 birds in the colony to start with, how many will be after 1 year?

(c) The population of birds in an aviary has a growth rate of 150% p.a. If the aviary originally has 30 birds, how many birds will there be after 4 years?

(d) A house was purchased $7\frac{1}{2}$ years ago for \$190 000. Houses in this area have grown in value by 22.6% p.a. What is the estimated value of the house now, to the nearest dollar?

(e) A population increases at a rate of 1.2% per month. If the initial population is 200 000, what will be the population after 16 months?

(f) Joseph currently has 976 friends on Facebook and the number of friends is growing at a rate of 5% each year. How many friends can he expect to have after 21 months?

WE14

2 **(a)** The population of an endangered bird is declining at a rate of 25% a year. If there are 176 birds left now, how many will there be after 5 years?

(b) The value of a dishwasher is decreasing at a rate of 20% p.a. If it's worth \$1400 now, how much will it be worth after 2 years?

(c) At the start of a plague the population in a medieval town is 10 200 and has a death rate of 6% each day. Estimate the population after 8 days.

(d) A population decreases at a rate of 1.2% per month. If the initial population is 200 000, what will be the population after 16 months?

(e) An eBay seller noticed that the number of internet sales has been declining at a rate of 7% a month. She sold 2370 items this January. If the trend continues, how many can she expect to sell in the month of July (in 6 months' time)?

(f) Cigarette sales in this country have shown a 2.98% p.a. decay rate in recent decades. If a new company sold 5 million packets of cigarettes in their first year, how many packets can they expect to sell in their fourth year? (*Note*: The fourth year is in 3 years' time.)

3 The following table shows the number of staff, S, employed by a company after T years.

WE15

T	0	1	2	3	4	5	6
S	5	15	45	135			

(a) Complete the table.

(b) Find the formula that models this growth.

(c) Use the formula to predict the number of staff employed by the company after 10 years and 3 months.

(d) Use the table of values to plot a graph of S against T. Join the points with a smooth curve.

(e) Use the graph to predict approximately when the company will have 78 employees.

Understanding

4 (a) If the population of Growthland starts at 15 000 and increases by 5% each year, the population in 6 years' time will be:

A 4500 B 19 500 C 20 101 D 90 000

(b) A population decreases at the rate of 2% per month. If the intial population was 130 000, the population after 1 year is closest to:

A 98 800 B 102 013 C 127 400 D 164 871

5 Kate invests \$5000 in a bank that offers 10% interest per year.

(a) What will her investment be worth after 1 year?

(b) What percentage of her original investment will her bank balance be at the end of the first year?

(c) Express this percentage as a decimal.

(d) What will Kate's investment be worth after 2 years?

(e) What is the first year's balance multiplied by to obtain the second year's?

(f) Complete a table of values showing Kate's bank balance at the end of each year (up to 10 years).

(g) Write an equation relating the amount Kate has in the bank to the number of years of investment.

(h) Plot a graph of this exponential relationship.

(i) Use the graph to predict what Kate's investment will be worth after $3\frac{1}{2}$ years. Check this result with a calculator.

(j) Use the graph to predict when Kate's investment will exceed \$8000.

(k) How long would Kate need to have her money in the bank to double her investment?

6 The value of a car, currently estimated at \$45 000, depreciates at 5% each year.

(a) What percentage of the current value will the car be worth after 1 year?

(b) Express this percentage as a decimal.

(c) Find the equation that describes the relationship between the value of the car and the number of years of depreciation.

(d) Complete a table of values for this exponential relationship. (Consider up to 5 years.)

(e) Plot the graph of this exponential relationship on graph paper.

(f) If the car is to be sold after $2\frac{1}{2}$ years, use the graph to predict what price the owner should expect to sell it for. Use a calculator to check your result.

(g) When should the car be sold to expect a price of \$36 000?

7 Sibel was born in 1995 with an inheritance worth 1 million dollars. The money was unfortunately invested in a fund that has been decreasing by half each year.

(a) Find an equation that models the value of the inheritance (I) after T years.

(b) How much will Sibel's inheritance be worth when she is 20 years old?

(c) When will the inheritance be worth \$7812.50?

Reasoning

8 When Angie and Rob married, some of their wedding cake was kept to celebrate the anniversary of their wedding each year for the next 6 years. On their first anniversary the family decided to eat a third of the cake. At each subsequent anniversary, it became a tradition to eat a third of the cake remaining from the previous year. There was 5 kg of cake left at the end of the wedding.

(a) Find the equation for the amount of cake (A) left after the nth anniversary.

(b) On the sixth and final anniversary, the remaining cake will be divided equally between 62 people. Find the mass of each slice. Round your answer to the nearest gram.

Open-ended

9 The population of a small town started with 10 people. After some time it reached 7290. Give some possibilities for the number of years elapsed and the % p.a. rate of increase.

10 Ana, Ernie and Janet are looking through the newspaper for something they can use in their Year 10 Science presentation. They come across this report:

> Scientists have become aware of a radioactive element known as Xradium in the soil of Nuketown that decays at a rate of 15% every 60 years. It is believed that Nuketown will be unfit for human habitation for about the next 850 years, by which time Xradium will be down to about 12% of its current level.

Janet is convinced the reporter made a mistake. She does the following calculation:

$6\frac{2}{3} \times 15\% = 100\%$ so it should only take about $6\frac{2}{3} \times 60 = 400$ years.

Ernie says: "You can't work it out like that, Janet! You've got to use the decay formula."

Ana says: "How are you supposed to use the decay formula without knowing how much of the stuff is in the soil right now?"

Is there an error in the newspaper report? Help the three students to resolve their problem.

Chapter review 1

D.I.Y. Summary

Key Words

adjusted value	exponential growth	residual value
appreciation	flat rate	salvage value
compound interest	inflation rate	scrap value
constants	initial cost	simple interest
cumulative depreciation	interest	straight-line depreciation
depreciation	nominal interest rate	total possible depreciation
depreciation value	per annum	unit cost depreciation
effective interest rate	prime cost	variables
expected life	principal	written-down value
exponential decay	reducing balance depreciation	

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 The original amount of money that is borrowed or loaned is called the _____.

2 Interest rates are normally expressed with p.a. after them. This stands for _____ and means for every year.

3 _____ is always higher than _____ after the first compounding period.

4 Increase in cost or value is called _____ and decrease in cost or value is called _____.

5 The stated interest rate is known as the _____. The adjusted rate that takes into account the effect of any compounding in the first year is called the _____ .

6 The amount of depreciation on a certain item is \$10 each year for 7 years. This is an example of _____.

7 Depreciation that has a decreasing value from one year to the next is called _____.

8 If the depreciation is \$5 in the first year and \$4 in the second year, the _____ over the first two years is \$9.

9 The value of an item after depreciation is known as its _____ or _____.

10 The formula $V_T = P - D_T$ has three _____, V_T, P and D_T. In this formula, P stands for the _____or _____.

11 Any value that an item still has at the end of its useful life is known as _____ , _____ or _____.

12 The initial purchase price less any scrap value is the _____ .

13 This value divided by the expected life gives the _____.

14 The growth of bacteria is an example of _____. The opposite process is called _____.

Fluency

1 (a) The amount an investment of \$8200 amounts to after 2 years, if 9.5% p.a. interest is compounded annually, is closest to: Ex. 1.1

A \$1558 B 1632 C \$7790 D \$9832

(b) The amount an investment of \$4000 amounts to after 2 years, if 3.5% p.a. interest is compounded annually, is closest to:

A \$285 B \$4192 C \$4280 D \$4285

2 Michael borrowed \$14 000 to buy a new piano. He agreed to pay interest at a rate of 16% p.a. and to repay the loan in full after 4 years. Calculate the amount to be repaid if the interest is compounded: Ex. 1.2

(a) half-yearly (b) quarterly (c) monthly.

3 Helga deposited some money into a savings account. The account guaranteed a return of 9.25% p.a. compounded daily. Calculate the initial deposit if the savings had grown to \$47 638.38 after 5 years. Give your answer rounded to the nearest dollar. Ex. 1.3

4 Find the effective interest rate for each of the following: Ex. 1.4

(a) \$38 000 invested at a simple rate of 8.7% p.a.

(b) \$50 000 invested at an annually compounding rate of 9.5% p.a.

(c) the interest owed on a loan of \$700 after 1 year was \$28

(d) 17.4% p.a. compounded every 2 months

5 The initial price 5 years ago was \$60 000. The rate of depreciation is 7% p.a. Ex. 1.5

(a) Assuming straight-line depreciation, find the current written-down value.

(b) Assuming reducing balance depreciation, find the current written-down value.

Round all answers to the nearest cent.

6 The population of a small rural community has been declining at a rate of 1% a year. If there were 461 residents in 2006, estimate the population in 2016. Ex. 1.6

Understanding

7 Simple interest is paid on an investment of \$12 500 and yields \$19 687.50 in 5 years. What is the rate of interest being paid?

8 Nina's salary has increased by 6% p.a. each year over the past 3 years to \$70 500 p.a. Calculate her salary 3 years ago, to the nearest dollar.

9 Assume an inflation rate of 5.6% p.a. In 4 years, an ice-cream, currently \$1.20, will cost:

A $\$1.20(1.014)^4$ B $\$1.20(1.056)^4$ C $\$1.20(1.014)^{16}$ D $\$1.20(1.56)^4$

10 Calculate the rate of interest p.a. (rounded to one decimal place) that would allow \$6700 to accumulate to \$10 000 in 5 years if interest is compounded each quarter.

11 A dining suite was purchased $3\frac{1}{2}$ years ago for \$4600. It depreciated at a fixed percentage of its reduced value each year and has just sold for \$2800. The rate of depreciation is closest to:

A 15.25% p.a. B 13.22% p.a. C 13% p.a. D 11.18% p.a.

12 Copy and complete the following.

(a) If the prime cost of a photocopier is $10 800 and it has a scrap value of $500, then the total possible depreciation is $_____. If it has an expected life of 206 000 pages, then the unit cost depreciation is $_____per page. After it copies 125 000, the depreciation is $_____ and the written-down value is $_____.

(b) If the prime cost of an item is $52 500 and it has a scrap value of $750 then the total possible depreciation is $_____. If it has an expected life of 5000 days, then the unit cost depreciation is $_____per day. After it does a year's work, the depreciation is $_____ and the written-down value is $_____.

13 The population of Metown increases by 1.1% each year. What was the population 6 years ago if its current population is 68 000?

14 The temperature inside a room decreases by 3% every 15 minutes. Find the temperature at 7 p.m. if it is 26°C at 5 p.m. Round your answer to one decimal place.

Reasoning

15 How many years will it take for an investment of $10 000 to increase by at least 50% if the interest rate of 10% p.a. is compounded half-yearly? Give your answer to the nearest tenth of a year.

16 The following represent loan options. In each case, choose the better option *without calculating the effective rate* and justify your choice.

(a)	11% p.a. compounded quarterly	14% p.a. compounded monthly
(b)	10% p.a. compounded daily	10% p.a. compounded monthly
(c)	17% p.a. compounded daily	17.5% p.a. compounded daily
(d)	12% p.a. flat rate for 2 years	12% p.a. compounded annually for 2 years

17 One pair of breeding rats can become a colony of 2000 in one year. Estimate the population rate of growth per month. Give your answer rounded to one decimal place.

18 A new city restaurant had only 12 customers for lunch on its opening day. Business improved and the number of lunchtime customers followed an exponential growth curve. After 2 years the restaurant had reached its capacity of 120. What was the growth rate per month over these first 2 years? Round your answer to the nearest whole number.

2 Algebra

What would you do if you had control of the traffic lights?

Name: Miles Brown
Job: Communication Technical Services Officer
Qualifications: Advanced Diploma of Electronic Engineering

When 000 FIRE is called, there is an extensive network of equipment that needs to be activated and personnel that need to be alerted. Who makes sure that all the technology is in place so that an emergency can be handled as smoothly as possible?

Miles Brown works as a Communication and Technical Services Officer at the CFA (Country Fire Authority) and does exactly that.

"We're in charge of all the maintenance and installation of the communication and signalling systems for the CFA, from repairing radios and telephone equipment, to when there's a fire call; all the things that make the lights go on, sirens go on, doors open, and traffic lights change.

There's a lot of rural travel, working outdoors and physical tasks. Sometimes you can be running cabling, spending a day up in a roof or crawling around under a building. It's very hands on. That's one of the reasons that I chose it."

Miles uses algebra to calculate the resistance of electronic circuits and measure the signal strength of radios.

Why learn this?

Algebra gives us a shorthand way to describe and generalise patterns that occur in the real world. Algebraic techniques allow us to manipulate algebraic expressions into different forms to compare these patterns. In this chapter, you will acquire skills so that you can solve more complex equations and use variables to model a greater variety of situations.

After completing this chapter you will be able to:

- expand binomial products, expressing answers in simplest form
- factorise quadratic trinomials
- recognise and use special products to factorise expressions
- rearrange formulas to make a different variable the subject
- simplify algebraic expressions involving fractions.

2 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 How many terms are there in each expression?

(a) $x^2 + 6x + 9$ **(b)** $16x^2 - y^2$ **(c)** $xy + 2y + 4x + 8$

Worksheet R2.1

2 What is the coefficient of each of these terms?

(a) $4x$ **(b)** $-6ab$ **(c)** y **(d)** $-12x^2y$

Worksheet R2.2

3 Simplify:

(a) $3 \times a + 5 \times b$ **(b)** $8 \times 3x - 8 \times y$

Worksheet R2.3

4 Expand and simplify if possible:

(a) $3(x + 2)$ **(b)** $5(x - 3)$ **(c)** $3x + 2(x - 4)$

Worksheet R2.4

5 Expand and simplify if possible:

(a) $-2(x + 3)$ **(b)** $-8(x - 5)$ **(c)** $-(x + 2)$

Worksheet R2.5

6 Simplify:

(a) $(3x)^2$ **(b)** $(2y)^2$ **(c)** $(7k)^2$

Worksheet R2.6

7 Simplify:

(a) $(4x)^2 - 2x^2$ **(b)** $p^2 - (3p)^2$ **(c)** $(-y)^2 + y^2$

Worksheet R2.7

8 Rearrange to make x the subject.

(a) $y = 2x + 3$ **(b)** $2a = \dfrac{x - 4b}{c}$ **(c)** $s = \dfrac{5x - at^2}{3}$

Worksheet R2.8

9 Find two numbers that:

(a) multiply to give 21 and add to give 10 **(b)** multiply to give 24 and add to give 14

(c) multiply to give 20 and add to give -9 **(d)** multiply to give -18 and add to give 3

(e) multiply to give -20 and add to give -8 **(f)** multiply to give -40 and add to give -3.

Worksheet R2.9

10 Simplify the following.

(a) $\dfrac{2}{3} + \dfrac{1}{4}$ **(b)** $\dfrac{5}{6} - \dfrac{3}{8}$ **(c)** $\dfrac{a}{3} + \dfrac{4a}{3}$

Worksheet R2.10

11 Simplify the following.

(a) $\dfrac{3}{8} \times \dfrac{4}{9}$ **(b)** $\dfrac{x}{10} \times \dfrac{5}{x}$ **(c)** $\dfrac{5}{6} \div \dfrac{10}{9}$

Worksheet R2.11

Key Words

algebraic fractions	factorise by grouping in pairs	Perfect Square
binomial product	factorising	quadratic expression
completing the square	highest common factor (HCF)	quadratic trinomial
Difference of Two Squares	monic quadratic	

2.1 Expanding brackets

Expanding brackets

The Distributive Law is used to expand brackets.

$a(b + c) = ab + ac$
$a(b - c) = ab - ac$

Worked Example 1

Expand and simplify each of the following.

(a) $3(4x - 2)$ **(b)** $x(x + 4) - 2(x - 1)$

Thinking	Working
(a) Multiply all terms inside the brackets by the term in front of the brackets.	(a) $3(4x - 2)$ $= 12x - 6$
(b) **1** Expand both sets of brackets by multiplying every term inside the brackets by the term in front of the brackets. Change all signs when multiplying by a negative number.	(b) $x(x + 4) - 2(x - 1)$ $= x^2 + 4x - 2x + 2$
2 Simplify by collecting like terms.	$= x^2 + 2x + 2$

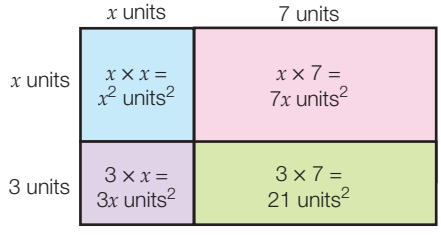
Expanding binomial products

A **binomial product** is a product that has two terms in each expression. $(x + 3)(x + 7)$ is an example of a binomial product. Remember that a multiplication sign is not required between the factors to indicate a product. We can understand how to find the expansion of binomial products by considering the rectangle below.

If the length of a rectangle is $(x + 7)$ units and the width is $(x + 3)$ units then an expression for the area is:

$A = lw$
$A = (x + 7)(x + 3)$ units^2

We can separate the $x + 7$ into a length of x units and a length of 7 units. We separate $(x + 3)$ in a similar way.



We can add the areas of the four smaller rectangles and simplify like terms.

$A = x^2 + 7x + 3x + 21$
$A = x^2 + 10x + 21$ units^2

We can equate the two expressions that represent the area of the rectangle.

$(x + 7)(x + 3) = x^2 + 10x + 21$

This is called the extended distributive law.

To help us expand, we can use FOIL. This tells us how to multiply the terms:

First terms in each bracket
Outside terms in each bracket
Inside terms in each bracket
Last terms in each bracket

and then add them all together.

To expand binomial products, multiply each term in the second bracket by each term in the first bracket and add them together.

$$(a + b)(c + d) = \overset{F}{ac} + \overset{O}{ad} + \overset{I}{bc} + \overset{L}{bd}$$

Worked Example 2

WE2

Expand and simplify $(x + 5)(2x - 3)$

Thinking	Working
1 Multiply each term in the second bracket by each term in the first bracket. $(x + 5)(2x - 3)$	$(x + 5)(2x - 3)$ $= 2x^2 - 3x + 10x - 15$
2 Simplify by collecting like terms.	$= 2x^2 + 7x - 15$

Expanding special binomial products

There are some expansions that are helpful to expand by rule:

The **Difference of Two Squares** rule:

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2\\ &= a^2 - b^2\end{aligned}$$

The **Perfect Square** rules:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

$$\begin{aligned}(a - b)^2 &= (a - b)(a - b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

These rules are used to expand special binomial products:

$(a - b)(a + b) = a^2 - b^2$ (Difference of Two Squares)
$(a + b)^2 = a^2 + 2ab + b^2$ (Perfect Square)
$(a - b)^2 = a^2 - 2ab + b^2$ (Perfect Square)

Worked Example 3 WE3

Expand and simplify:

(a) $(x-3)(x+3)$ **(b)** $(3x+1)^2$

Thinking	Working
(a) Identify that it has the form of a special product and apply the Difference of Two Squares rule.	(a) $(x-3)(x+3) = (x)^2 - (3)^2$ $= x^2 - 9$
(b) Identify that it has the form of a special product and apply the Perfect Square rule.	(b) $(3x+1)^2 = (3x)^2 + 2 \times 3x \times 1 + (1)^2$ $= 9x^2 + 6x + 1$

Expanding three factors

If we have three or more factors, we first multiply any two factors together and simplify the result. We then repeat this process until the expression is fully expanded. Look for special product expansions to expand more easily.

Worked Example 4 WE4

Expand and simplify.

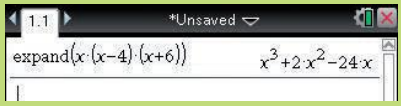
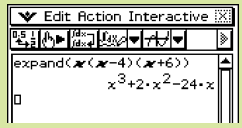
(a) $x(x-2)(x+3)$ **(b)** $(x+3)(x+1)(x-4)$ **(c)** $(x-1)(x-5)(x+1)$

Thinking	Working
(a) 1 Expand and simplify the binomial product.	(a) $x(x-2)(x+3)$ $= x(x^2 + 3x - 2x - 6)$ $= x(x^2 + x - 6)$
2 Multiply by the single factor.	$= x^3 + x^2 - 6x$
(b) 1 Look for special product expansions. As there are none, choose any two brackets to expand. (Here, we will expand the first two brackets first and simplify their product.)	(b) $(x+3)(x+1)(x-4)$ $= (x^2 + x + 3x + 3)(x-4)$ $= (x^2 + 4x + 3)(x-4)$
2 Expand the two remaining brackets by multiplying each term in the second bracket by each term in the first.	$= x^3 - 4x^2 + 4x^2 - 16x + 3x - 12$
3 Simplify the expansion.	$= x^3 - 13x - 12$
(c) 1 Identify that two factors have the form of a special product and use this to expand these factors. Rearrange the expansion if necessary. (Here, two factors multiply to give the difference of two squares $(x-1)(x+1) = x^2 - 1$).)	(c) $(x-1)(x-5)(x+1)$ $= (x-1)(x+1)(x-5)$ $= (x^2 - 1)(x-5)$
2 Expand the two remaining brackets.	$= x^3 - 5x^2 - x + 5$

Using your CAS to check expansions

The algebra component of your CAS makes checking algebraic expansions very easy.

You should already be able to work out by hand that the expansion of $x(x-4)(x+6)$ is x^3+2x^2-24x. On your CAS you use the expand function to find the expansion.

Using the TI-Nspire CAS	Using the ClassPad
From the home screen, open a new document. You don't need to save this so select 'No' when asked to save and press enter twice. Press (menu) > Algebra > Expand and then enter the expression, being careful to include the correct number of brackets and a multiplication sign (×) between each set of brackets in the expression. 	From the menu, select Main. Tap Interactive > Transformation > Expand. A dialogue box will appear. Enter the expression then tap OK. 

Expand the following and check the expansions using your CAS.

1 $3x(x-2)(x+7)$ **2** $(x-2)(x+8)(x+2)$ **3** $(x+1)(x-4)(x+11)$

If you have expanded the expressions correctly you should see these answers on your screen.

1 $3x^3+15x^2-42x$ **2** $x^3+8x^2-4x-32$ **3** $x^3+8x^2-37x-44$

2.1 Expanding brackets

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5, Q6 Column 1, Q7, Q9, Q11, Q13, Q14, Q17	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5, Q6 Column 2, Q7, Q8, Q10, Q12, Q13, Q14, Q15, Q18	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5, Q6 Column 3, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q15, Q16, Q18, Q19

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Fluency

WE1

1 Expand and simplify each of the following.

(a) $4(x+3)$ **(b)** $8(x+2)$ **(c)** $12(x+11)$

(d) $2(6x-1)$ **(e)** $7(2x-5)$ **(f)** $x(5x-4)$

(g) $-2(x+1)$ **(h)** $-x(x-7)$ **(i)** $-x(3x-6)$

(j) $-3x(x+6)$ **(k)** $-2x(x-9)$ **(l)** $-4x(x+2)$

(m) $x(x+2)+3(x+2)$ **(n)** $x(2x+1)+2(2x+1)$ **(o)** $x(5x+2)+3(5x+2)$

(p) $2x(x-3)-5(x+6)$ **(q)** $5x(2x+5)-4(3x-8)$ **(r)** $4x(3x-1)-6(2x+1)$

WE2

2 Expand and simplify each of the following.

(a) $(x+1)(x+3)$ (b) $(x+4)(x+2)$ (c) $(x+5)(x+4)$

(d) $(x+8)(x-3)$ (e) $(x+7)(x-9)$ (f) $(x-3)(x+7)$

(g) $(x-1)(x-7)$ (h) $(x-4)(x-5)$ (i) $(x-9)(x-3)$

(j) $(2x+3)(x-2)$ (k) $(4x-5)(x+3)$ (l) $(7x+1)(x-4)$

(m) $(3x-1)(2x-5)$ (n) $(5x-3)(7-x)$ (o) $(6x-5)(2-3x)$

WE3

3 Expand and simplify each of the following, using the rules for special products.

(a) $(2-x)(2+x)$ (b) $(7-x)(7+x)$ (c) $(6+x)(6-x)$

(d) $(2x+1)(2x-1)$ (e) $(3x+4)(3x-4)$ (f) $(5-3x)(5+3x)$

(g) $(x+6)^2$ (h) $(x+11)^2$ (i) $(8+x)^2$

(j) $(x-3)^2$ (k) $(x-7)^2$ (l) $(x-12)^2$

(m) $(2x+5)^2$ (n) $(3x+2)^2$ (o) $(7x-2)^2$

WE4

4 Expand and simplify each of the following.

(a) $x(x+6)(x+4)$ (b) $x(x-1)(x+5)$ (c) $x(x+1)(x-4)$

(d) $-x(x-3)(x+2)$ (e) $-3x(x-2)(x-1)$ (f) $-4x(x-8)(x+3)$

(g) $(x+4)(x+2)(x+5)$ (h) $(x+2)(x-5)(x-1)$ (i) $(x-3)(x-6)(x-2)$

(j) $(x-3)(x+3)(x+5)$ (k) $(x+7)(x-2)(x+2)$ (l) $(2x+1)(x+4)(x-4)$

(m) $(x-3)(x+4)(x+4)$ (n) $(x+2)(x+2)(x-1)$ (o) $(x-6)(x+5)(x-6)$

Understanding

5 The expression $(x+1)(x+3)(x-2)$ is *not* the same as:

A $(x+3)(x+1)(x-2)$ B $(x-3)(x+1)(x+2)$

C $(x+1)(x-2)(x+3)$ D $(x-2)(x+1)(x+3)$

6 Expand and simplify the following.

(a) $(4x-3y)(2x+5y)$ (b) $(5x+2y)(7x+6y)$ (c) $(3x-4y)(5x-8y)$

(d) $(2x+7y)(2x+7y)$ (e) $(7x+6y)(7x+6y)$ (f) $(5x+8y)(5x+8y)$

(g) $(4x-y)(4x-y)$ (h) $(6x-5y)(6x-5y)$ (i) $(9x-7y)(9x-7y)$

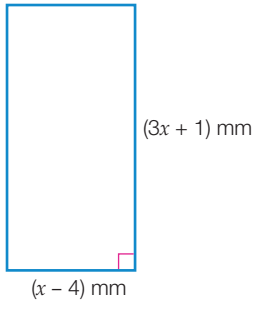
(j) $(5y-x)(5y+x)$ (k) $(3x-y)(3x+y)$ (l) $(2x+3y)(2x-3y)$

(m) $x(2x-1)(3x+5)$ (n) $x(3x+2)(5x-6)$ (o) $3x(4x+7)(2x+3)$

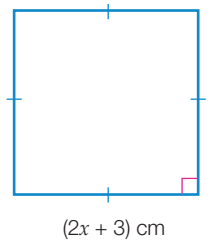
(p) $-x(3x-4)(5x+4)$ (q) $-2x(5x+3)(3x+4)$ (r) $-4x(2x-3)(3x-5)$

7 Find the area, in terms of x, of each of these figures.

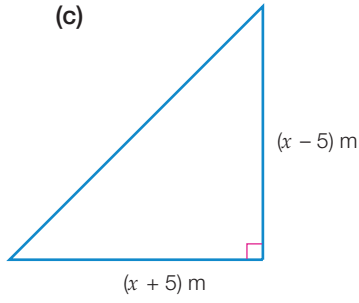
(a)



(b)



(c)



8 **(a)** Expand $x(x + a)(x + b)$.

(b) Use your answer to **(a)** to:

(i) write down the coefficient of x^2

(ii) write down the coefficient of x

(iii) show that $x(x + a)(x + b) = x^3 - 4x^2 - 12x$, when $a = -6$ and $b = 2$.

(c) For the expression $x(x + 25)(x - 4)$:

(i) write down the values of a and b

(ii) use the expansion found in **(a)** to expand $x(x + 25)(x - 4)$.

9 **(a)** If x is the second of three consecutive even numbers, write an expression for the product of the three numbers.

(b) Expand the expression in **(a)**.

(c) Why is it simpler to make x the second and not the first number?

10 **(a)** Draw a rectangle of length $(x + a)$ units and width $(x + b)$ units.

(b) Write an expression for the area of the rectangle in **(a)** in factorised form.

(c) Expand and simplify your expression in **(b)**.

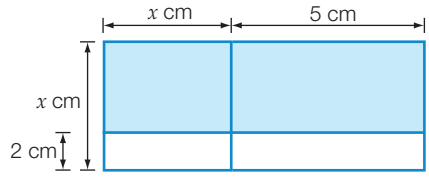
(d) Draw a square of side length x units. Extend its length by a units and the width by b units to form four small rectangles.

(e) Add the areas of the four smaller rectangles and compare this answer to part **(c)**.

11 **(a)** Write down the dimensions of the shaded rectangle.

(b) Write the binomial product used to find the area of the shaded region, then expand and simplify.

(c) Find the area of the square of side x cm and the rectangle of length 5 cm and width x cm.



(d) Find the unshaded area, writing it as the sum of the areas of the two rectangles.

(e) Use your answers in **(c)** and **(d)** to find the area of the shaded region in simplest form.

(f) Compare your answers in **(b)** and **(e)**.

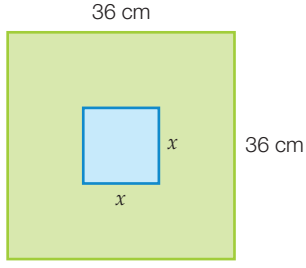
Reasoning

12 A square green tile of side length 36 cm has a small blue square of side length x cm painted on it.

(a) Write the expression (in cm^2) for the area of the tile that is not blue.

(b) Write this expression as a product of factors.

(c) Use this product to find the area that remains green if the blue square has a side of 6 cm.



13 **(a)** Expand and simplify $(x - a)^2 - a^2$.

(b) Write down the expansion for $x(x - 2a)$.

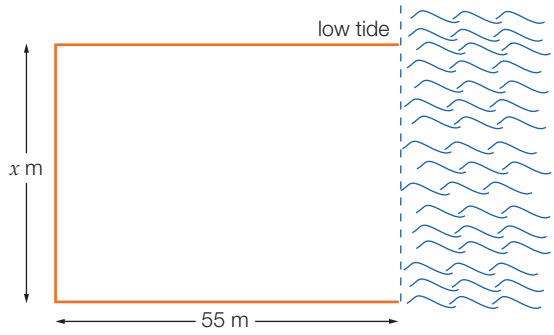
(c) What can you conclude about $(x - a)^2 - a^2$ and $x(x - 2a)$ from your answers in **(a)** and **(b)**?

(d) A rectangular lawn has a length of x m and a width of $(x - 2a)$ m. A square lawn of length $(x - a)$ m has a square-based fountain of side length a m in the centre. If the price of turfing the lawn is based only on its area, is one lawn more expensive to lay with turf than the other? Justify your answer.

(e) If the square lawn has side length 40 m, the fountain has a side length of 3 m and the price of turf is \$12/m^2, what is the total cost of laying the lawn?

14 The Holmes family own a property at the beach. Their land is x m wide and extends down to the water's edge. At low tide, the length of the land is 55 m, as shown in the diagram at right.

As the tide comes in, the length is reduced by y m.

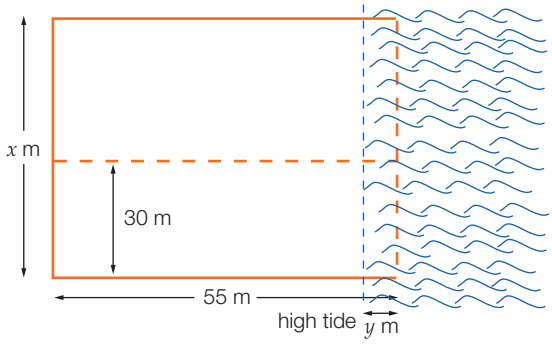


(a) Write an expression for the length of the land at any time of day.

(b) Write an expression for the area of the land at any time of day.

(c) What is the area of the land at low tide?

(d) The family have decided to subdivide the land by reducing their land width by 30 m. Write an expression for the area of the remaining land at any time of the day.

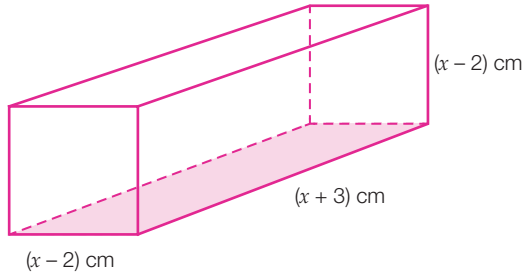


(e) Expand the expression.

(f) If the tide comes in 10 m at high tide, what is the area of land at high tide?

(g) What is the difference between the area of land at low tide before the subdivision and at high tide after the subdivision?

15 Consider this box.



(a) Write an expression (in cm^3) for the volume of the box in factorised form.

(b) Find the volume when x is 4 cm.

(c) Show that the volume of the box is 72 cm^3, when $x = 5$.

(d) Write the expression (in cm^3) for the volume in expanded form. Use this to calculate the volume when $x = 5$.

(e) State the dimensions (length, width and height) of the box when the volume is 72 cm^3.

(f) What is the smallest possible integer value of x so that the box exists?

(g) Find the volume of the box when x is the value found in part **(f)**.

16 Decide whether $(x^2 + a^2)^3$ can equal zero for some value(s) of x and a other than $x = 0$, $a = 0$.

Open-ended

Hint
A binomial expression contains two terms.

17 Find two different sets of values for a, b, c and d so that the expanded, simplified form of $(ax + b)(cx + d)$ is a binomial expression.

18 **(a)** Expand and simplify $\frac{1}{2}(x + y)^2 + \frac{1}{2}(x - y)^2$.

(b) Find two different sets of values for x and y so that the expression obtained in part **(a)** is the square of a whole number.

19 Find two different whole numbers representing x so that the value of the expression $(4 - 2x)(x + 3)(5x - 10)$ is:

(a) positive **(b)** negative **(c)** zero.

Factorising using common factors

2.2

Factorising is the opposite or reverse process to expanding. In the previous section we converted expressions from factorised form (with brackets) to expanded form (without brackets).

For example,

$$5(x+3) = 5 \times x + 5 \times 3$$
$$= 5x + 15$$

factorised form → expanded form

In other words, $5(x+3)$ can be expanded to $5x + 15$. In reverse, $5x + 15$ can be factorised to $5(x+3)$. This involves finding the **highest common factor (HCF)** of the terms and writing it outside a set of brackets that contains the remaining terms.

A common factor of -1 may also be taken out.

Worked Example 5

WE5

Factorise:

(a) $15xy - 9y$ **(b)** $8x^2 + 12x$ **(c)** $-12a^2b - 9ab^2$

Thinking	Working
(a) Find the HCF of the terms ($3y$). Write it in front of the brackets. Divide each term by the HCF and write the quotient inside the brackets.	(a) $15xy - 9y$ $= 3y(5x - 3)$
(b) Find the HCF of the terms ($4x$). Write it in front of the brackets. Divide each term by the HCF and write the quotient inside the brackets.	(b) $8x^2 + 12x$ $= 4x(2x + 3)$
(c) Find the HCF of the terms ($3ab$). Write it in front of the brackets. Take out the factor of -1 also. Divide each term by the negative of the HCF and write the quotient inside the brackets. Remember to change signs inside the brackets.	(c) $-12a^2b - 9ab^2$ $= -3ab(4a + 3b)$

You can check your answer by expanding the brackets in your answer to get back to the original expression.

Sometimes the HCF is an expression of two or more terms.

Worked Example 6

WE 6

Factorise $k(m+3)+5(m+3)$.

Thinking	Working
Find the HCF of the terms. (Here, it is $(m+3)$.) Write it in front of the brackets. Divide each term by the HCF and write this quotient inside the brackets.	$k(m+3)+5(m+3)$ $=(m+3)(k+5)$

Factorising four terms by grouping in pairs

Expressions with four terms may **factorise by grouping in pairs**. Take out a common factor from each pair to give two terms. Then take out a common factor from each term.

We sometimes refer to this as a 'two plus two' factorisation.

Worked Example 7

WE 7

Factorise each of the following expressions by grouping in pairs.

(a) $2ax-8ay-cx+4cy$ **(b)** $uw+vx+xu+vw$

Thinking	Working
(a) 1 Factorise the first two terms by finding the HCF. (Here, it is $2a$.) Factorise the next two terms by finding the HCF. (Here, it is $-c$.) Don't forget to change signs when required.	(a) $2ax-8ay-cx+4cy$ $=2a(x-4y)-c(x-4y)$
2 Find the new HCF and write it outside the brackets. (Here, it is $(x-4y)$.) Divide each term by the HCF and write the quotient inside the brackets.	$=(x-4y)(2a-c)$
(b) 1 Rearrange the pairs so there is a common factor in each pair.	(b) $uw+vx+xu+vw$ $=uw+xu+vw+vx$
2 Find and take out the HCF from the first two terms (here, it is u) and the HCF from the next two terms (here, it is v).	$=u(w+x)+v(w+x)$
3 Find the new HCF and write it outside the brackets. (Here, it is $(w+x)$.) Divide each term by the HCF and write the quotient inside the brackets.	$=(w+x)(u+v)$

2.2 Factorising using common factors

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q6, Q7, Q8, Q9, Q11, Q13	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13	Q1 Column 3, Q2 Column 2, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13

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Fluency

1 Factorise each expression. WE5

(a) $4x + 8$ **(b)** $18x - 15$ **(c)** $24x - 9$

(d) $xy + 7x$ **(e)** $12kx + 21k$ **(f)** $32ab + 20b$

(g) $7x^2y + 7y^2$ **(h)** $9x^2 + 3xy$ **(i)** $48m^2 + 8m$

(j) $-6x - x^2$ **(k)** $-4x + x^2$ **(l)** $-24a^2 + 10a$

(m) $6ab - 18a^2b$ **(n)** $40x^2y + 16xy^2$ **(o)** $10cd^2 - 2cd$

2 Factorise each expression. WE6

(a) $x(k + 1) + 4(k + 1)$ **(b)** $a(x - 4) + 6(x - 4)$

(c) $m(n - 7) + 2(n - 7)$ **(d)** $x(p + 3) - 9(p + 3)$

(e) $n(q - 2) - 7(q - 2)$ **(f)** $x(u - 4) - 5(u - 4)$

(g) $d(c + 5) + (c + 5)$ **(h)** $a(d + 2) - (d + 2)$

(i) $k(m + 7) - (m + 7)$ **(j)** $x(x - 3) + 5(x - 3)$

3 Factorise each of the following expressions by grouping in pairs. WE7

(a) $xy + 3y + 5x + 15$ **(b)** $ab + 7b + 2a + 14$ **(c)** $mn + kn + 4m + 4k$

(d) $xy - y + 3x - 3$ **(e)** $3a - 3b + ac - bc$ **(f)** $km - 4m + 2k - 8$

(g) $15ab - 10b - 12a + 8$ **(h)** $6ab - 4b - 15a + 10$ **(i)** $8mn - 6n - 12m + 9$

(j) $vx - vy - ax + ay$ **(k)** $pq - rq - np + nr$ **(l)** $gh - fh - dg + df$

(m) $m^2n + np + m^2t + pt$ **(n)** $a^2b + ab^2 - ac - bc$ **(o)** $d^2k - dk - dr + r$

(p) $27xy - 10 + 18y - 15x$ **(q)** $36a + 5c - 4ac - 45$ **(r)** $25vw + 5v - 15w - 3$

(s) $32a^3c + 15b - 12abc - 40a^2$ **(t)** $21d^2 + 20fh - 12dh - 35df$ **(u)** $m^2nr - 20p + 4mpr - 5mn$

4 Answer TRUE or FALSE to each of these statements.

(a) $8a + 4ab = 4a(2 + b)$ **(b)** $2x - 6xy = 2x(1 - y)$

(c) $mn + 3n - 2m - 6 = (m - 3)(n + 2)$ **(d)** $x^2 - 4x + x - 4 = (x - 4)(x + 1)$

(e) $3x^2 + 15x - 2x - 10 = (x - 2)(3x + 5)$ **(f)** $4x^2 + 20x - 3x - 15 = (x - 5)(4x + 3)$

5 When factorised, $16f^2g - 12gf$ becomes:

A $2fg(8f - 6)$ **B** $4f^2g(4g - 3f)$ **C** $16fg(f - 12)$ **D** $4fg(4f - 3)$

6 The total area of the two shapes shown is:

A $(a + 4)(b - c)$ **B** $(a + c)(b - 4)$ **C** $(a + b)(c - 4)$ **D** $(a - c)(b + 4)$

7 $10dk + 25d - 2k - 5$ factorises to:

A $(2k + 5)(5d - 1)$ **B** $5d(2k + 5)$ **C** $(2k - 5)(5d + 1)$ **D** $(10k + 1)(d - 5)$

Understanding

8 Mandy has $24y$ cents. She buys x kg of potatoes that cost 36 cents per kilogram. Express (in cents) in factorised form how much money Mandy has now.

9 Factorise the expression for the square of a given number subtracted from eight times that number.

10 Five small bulldog clips are added to a box containing p small bulldog clips. Five large bulldog clips are added to another box that contains p large bulldog clips. Each small clip costs a cents and each large clip costs b cents.

(a) Give the expression (in cents) for the combined cost of the clips.

(b) Factorise the expression in part **(a)**.

(c) If the cost of one large clip is twice the cost of one small clip write a factorised expression for the total cost of the clips in terms of a and p.

11 **(a)** Draw and label a rectangle that has a length of p m and a width of r m.

(b) The length is reduced by a m. Show this on your diagram and find the new area as the difference of two areas.

(c) The width of this shortened rectangle is now increased by b m. Show this on your diagram. Label the dimensions and find an expression for the added area.

(d) Show that the area of the newly formed rectangle is $(pr - ar + pb - ab)$ m^2.

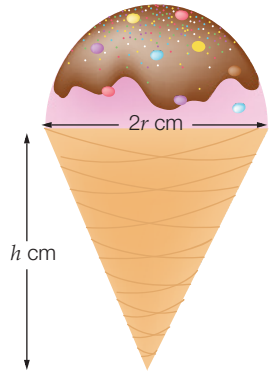
(e) Find the area as a product of the new length and width.

(f) Comment on your answers to parts **(d)** and **(e)**.

Reasoning

12 The volume of a sphere of radius r is $\frac{4\pi r^3}{3}$ and the volume of a cone of base radius r and height h is $\frac{\pi r^2 h}{3}$.

Renata is holding an ice-cream cone, with a scoop of ice-cream on top. The ice-cream can be considered to be hemispherical (half a sphere).



(a) Give an expression for the total volume of ice-cream (m^3), including the ice-cream in the cone and the scoop on top, if the ice-cream filled the cone.

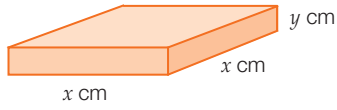
(b) Factorise the expression in part **(a)**.

(c) If $r = 3$ cm, $h = 10$ cm and $\pi = 3.142$, what is the total volume? (Give your answer correct to two decimal places.)

(d) Assume that there is no ice-cream inside the cone, just the scoop on top, and Renata lets it melt inside the cone. If the height of the cone is $2r$, use your expression in part **(b)** to show that the total volume is twice the volume of the cone and therefore all the melted ice-cream will fill the cone exactly.

Open-ended

13 A paver has the shape and dimensions shown.



(a) Write an expression (in cm^2) for the total surface area.

(b) Factorise the expression in part **(a)**.

(c) Find two sets of values for x and y so that the volume (in cm^3) has numerically the same value as the surface area. The values do not have to be integers.

Factorising monic quadratic expressions

2.3

A **quadratic expression** has the form of $ax^2 + bx + c$.

In a quadratic expression the highest power of the variable is 2. In a **monic quadratic** expression, the coefficient of the squared variable is 1. For example, $x^2 + 5x - 4$, $6 - 3a^2$, h^2 and $4n^2 + 7n$ are all quadratic expressions. Of these, only $x^2 + 5x - 4$ and h^2 are monic quadratic expressions.

Factorising quadratic expressions with four terms

In the previous section we factorised four terms by grouping in pairs into binomial products. If the first term is a squared variable and the two middle terms contain the same variable and the last term is a constant we may be able to use grouping in pairs to factorise these expressions into binomial products.

Worked Example 8

WE 8

Factorise $x^2 - 4x + 5x - 20$.

Thinking	Working
1 Write down the expression. Check if there is a factor common to all terms. (None.)	$x^2 - 4x + 5x - 20$
2 Group 'in pairs' by looking for a common factor in the first and the second pair of terms ($x^2 - 4x$ has a common factor of x) ($5x - 20$ has a common factor of 5). Take out the factor from each pair.	$= x(x - 4) + 5(x - 4)$
3 Take out the factor common to both groups ($x - 4$).	$= (x - 4)(x + 5)$

Factorising quadratic trinomials

A **quadratic trinomial** consists of three terms. For example, $x^2 + 5x - 4$ is a quadratic trinomial. The monic quadratic trinomials that we will factorise in this section will all be in the form of $ax^2 + bx + c$ where $a = 1$. Not all quadratic expressions in this form will factorise into binomial products.

Consider $x^2 + 3x - 5x + 5$:

We can take out a common factor from each pair to give $x(x + 3) - 5(x - 1)$ but it will not factorise any further. Why do some expressions with four terms factorise fully into binomial products whereas others do not?

Consider these expressions:

$$x^2 - 5x + 2x - 14 = x(x-5) + 2(x-7)$$
$$x^2 - 4x - 3x - 11 = x(x-4) - (3x+11)$$
$$x^2 + 3x - 7x - 20 = x(x+3) - (7x+20)$$

None of these expressions factorise into binomial products.

Consider these expressions:

$$x^2 - 7x + 2x - 14 = x(x-7) + 2(x-7) = (x-7)(x+2)$$
$$x^2 + x + 6x + 6 = x(x+1) + 6(x+1) = (x+1)(x+6)$$
$$x^2 - 3x - 2x + 6 = x(x-3) - 2(x-3) = (x-3)(x-2)$$

All of these expressions factorise into binomial products.

In these expressions, the coefficients of the two middle terms multiply to give the constant term:

$-7 \times 2 = -14$ $\quad$ $1 \times 6 = 6$ $\quad$ $-3 \times -2 = 6$

This means that we can use the method of grouping in pairs by splitting the middle term of a monic trinomial into two parts as long as the factors of the constant term are the coefficients of the two middle terms.

Consider:

$$x^2 - 5x - 14 = x^2 - 7x + 2x - 14 = x(x-7) + 2(x-7) = (x-7)(x+2)$$
$$x^2 + 7x + 6 = x^2 + x + 6x + 6 = x(x+1) + 6(x+1) = (x+1)(x+6)$$
$$x^2 - 5x + 6 = x^2 - 3x - 2x + 6 = x(x-3) - 2(x-3) = (x-3)(x-2)$$

Sometimes a quadratic trinomial has a common factor that needs to be taken out before the monic trinomial can be factorised.

Worked Example 9 WE9

Factorise these quadratic trinomials by splitting the middle term.

(a) $p^2 - 11p + 30$ $\quad$ **(b)** $3q^2 + 6q - 24$

Thinking	Working
(a) 1 Find the factors of the constant term (30) that add to give the coefficient of the middle term (-11). (Here, the numbers are -5 and -6.)	(a) $p^2 - 11p + 30$ $\quad$ $\left.\begin{array}{l} \times \rightarrow +30 \\ + \rightarrow -11 \end{array}\right\}$ -5, -6
2 Rewrite the middle term as two terms using the two numbers found as the coefficients of the variable.	$= p^2 - 5p - 6p + 30$
3 Factorise by removing a common factor from each pair using 'two and two' grouping.	$= p(p-5) - 6(p-5)$
4 Take out the common factor.	$= (p-5)(p-6)$

(b) 1	Factorise by taking out the HCF first. (HCF = 3)	(b) $3q^2 + 6q - 24$ $= 3(q^2 + 2q - 8)$
2	Find the factors of the constant term inside the brackets (-8) that add to give the middle term inside the brackets (+2). (Here, the numbers are +4 and -2.) Rewrite the middle term as two terms using the number found as the coefficients of the variable.	$= 3(q^2 + 4q - 2q - 8)$ $\left.\begin{array}{l} \times \rightarrow -8 \\ + \rightarrow +2 \end{array}\right\}$ -2, +4
3	Factorise by removing a comon factor from each pair using 'two and two' grouping inside the brackets. Add another set of brackets.	$= 3[q(q + 4) - 2(q + 4)]$
4	Factorise further by taking out the common factor	$= 3(q + 4)(q - 2)$

Using the cross method to factorise monic trinomials

This method is a quick way to factorise monic trinomials once you understand the factorisation process.

- Draw a pair of crossed arrows and write the variable at the beginning of each arrow.

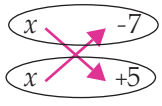
- Identify a factor pair for the constant term.

  For the expression $x^2 - 2x - 35$, the constant term -35 has factor pairs of -35, 1; -7, 5; -5, 7 and -1, 35.

- Trial a factor pair by writing them at the end of each arrow.

  For $x^2 - 2x - 35$, select a pair of factors -7 and 5. Follow the arrows to write the two terms as a sum or difference. If this does not give the middle term, try a new pair of factors.

- Circle the top and the bottom of the arrows as shown. Each circle contains a binomial factor.



  $-7x + 5x = -2x$

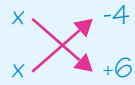
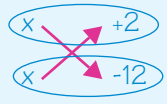
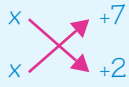
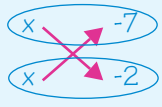
- Write the factorisation.

  $x^2 - 2x - 35 = (x - 7)(x + 5)$

Worked Example 10 WE10

Factorise the following quadratic expressions using the cross method.

(a) $x^2 - 10x - 24$ **(b)** $x^2 - 9x + 14$

Thinking	Working
(a) 1 Choose a pair of factors that multiply to give the constant term. If the constant term is negative, choose factors opposite in sign (try $-4 \times 6 = -24$). Use the cross method to see if they add to give the correct middle term ($6x - 4x = 2x$). This pair gives an incorrect middle term, so try another pair.	(a) $x^2 - 10x - 24$  $6x - 4x = 2x$ (incorrect in magnitude and sign)  $-12x + 2x = -10x$ (correct in magnitude and sign)
2 Write down the factorisation.	$= (x + 2)(x - 12)$
(b) 1 Choose a pair of factors that multiply to give the constant term. If the constant term is positive, choose factors that have the same sign (try $7 \times 2 = 14$). Use the cross method to see if they add to give the correct middle term ($7x + 2x = 9x$). If this pair gives an incorrect middle term, try another pair of factors.	(b) $x^2 - 9x + 14$  $2x + 7x = 9x$ (incorrect in sign)  $-2x - 7x = -9x$ (correct in magnitude and sign)
2 Write down the factorisation.	$= (x - 2)(x - 7)$

2.3 Factorising monic quadratic expressions

Navigator

Answers page 721

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q9, Q10	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q9, Q10	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency

WE8

1 Factorise each expression by grouping in pairs.

(a) $x^2 + 2x + 4x + 8$ **(b)** $x^2 - 3x + 6x - 18$ **(c)** $x^2 - x + 7x - 7$

(d) $x^2 + 5x - 2x - 10$ **(e)** $x^2 - 4x - 3x + 12$ **(f)** $x^2 + 3x - 4x - 12$

(g) $x^2 - 2x + 4x - 8$ **(h)** $x^2 + x - 3x - 3$ **(i)** $x^2 - 2x - 8x + 16$

2 Factorise these quadratic trinomials by splitting the middle term. WE9

(a) $x^2 + 3x + 2$ (b) $y^2 + 9y + 14$ (c) $a^2 - 10a + 24$

(d) $k^2 + 11k + 24$ (e) $m^2 - 2m - 15$ (f) $x^2 - x - 30$

(g) $p^2 - 10p + 16$ (h) $d^2 - 5d + 4$ (i) $y^2 - 5y + 6$

(j) $m^2 - 4m - 12$ (k) $c^2 - 10c + 21$ (l) $g^2 + 2g - 35$

(m) $2y^2 + 14y + 24$ (n) $3m^2 + 18m + 24$ (o) $5x^2 - 5x - 100$

(p) $-4a^2 - 8a + 12$ (q) $-3k^2 + 12k + 63$ (r) $-5x^2 - 20x + 105$

3 Factorise these quadratic trinomials using the cross method. WE10

(a) $z^2 - 8z + 15$ (b) $k^2 + 5k - 6$ (c) $z^2 + 10z + 21$

(d) $t^2 - t - 20$ (e) $r^2 + r - 30$ (f) $t^2 - 13t + 22$

(g) $z^2 - z - 42$ (h) $m^2 - 9m + 20$ (i) $y^2 + 7y - 60$

(j) $n^2 - 3n - 28$ (k) $p^2 + 9p - 36$ (l) $a^2 - 14a + 48$

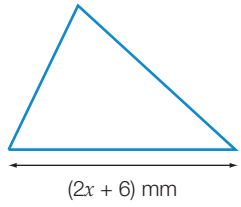
4 If $A = 4x^2 + 6x + 29$ and $B = 1 + 9x + 5x^2$, then $A - B$ when factorised becomes:

A $(x - 4)(x + 7)$ B $(x + 4)(7 - x)$ C $(x + 7)(4 - x)$ D $(x - 7)(4 - x)$

Understanding

5 A rectangle has a width of $(x + 3)$ cm. Find the length of the rectangle if its area is $(2x^2 + 10x + 12)$ cm^2.

6 A triangle has a base length of $(2x + 6)$ mm. Find the height of the triangle if its area is $(3x^2 - 3x - 36)$ mm^2.



Reasoning

7 A rectangular table has an area of $(x^2 + 30x - 1800)$ cm^2.

(a) If the length of the table (cm) is one of the factors of the expression, find an expression for the width.

(b) If the length of the table is 210 cm, find the width.

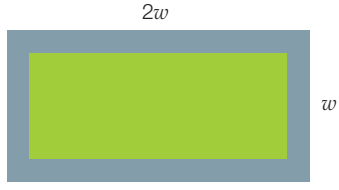
(c) If the width is 1 m, what is the length?

8 A rectangular block of land is twice as long as its width w m. A concrete path of uniform width is to be constructed around the inside border of the block and will occupy an area of $(6w - 4)$ m^2.

(a) What is the area left unconcreted?

(b) Factorise this area.

(c) If the factors represent the length and width of the unconcreted area, find the width of the path.



Hint

Rewrite your answer in **(b)** as the product of two algebraic factors by subtracting the same amount from the original length and width.

Open-ended

9 Write two monic quadratic trinomials that will factorise, and factorise them.

10 If $x^2 + 5x - 24$ represents a rectangular area:

(a) write down two possible sets of dimensions for the rectangle

(b) evaluate the area for each set of dimensions.

2.4 Factorising using special products

Some quadratic expressions can be factorised easily if you can recognise a special product. The special products we have dealt with are the Difference of Two Squares and the Perfect Squares.

The Difference of Two Squares rule

The Difference of Two Squares rule can be used to factorise quadratic expressions.

$(a + b)(a - b) = a^2 - b^2$

By reversing this expansion:

$a^2 - b^2 = (a + b)(a - b)$

Worked Example 11 WE11

Factorise these expressions using the Difference of Two Squares rule.

(a) $d^2 - 81$ **(b)** $180 - 5y^2$ **(c)** $4k^2 - 9m^2$

Thinking	Working
(a) 1 Look for common factors. (There are none.)	(a) $d^2 - 81$
2 Use the Difference of Two Squares rule to factorise the expression. (Here, $a = d$, $b = 9$.)	$= (d + 9)(d - 9)$
(b) 1 Look for a common factor and take it out. (Here, 5 is a common factor.)	(b) $180 - 5y^2$ $= 5(36 - y^2)$
2 Use the Difference of Two Squares rule to factorise the expression. (Here, $a = 6$ and $b = y$.)	$= 5(6 + y)(6 - y)$
(c) 1 Look for common factors. (There are none.)	(c) $4k^2 - 9m^2$
2 Express both terms as squares.	$= (2k)^2 - (3m)^2$
3 Use the Difference of Two Squares rule to factorise the expression. (Here, $a = 2k$ and $b = 3m$.)	$= (2k - 3m)(2k + 3m)$

Worked Example 12

WE 12

Factorise $(x+3)^2 - 9$.

Thinking	Working
1 Look for common factors. (There are none.)	$(x+3)^2 - 9$
2 Write the expression as the difference of two squares.	$= (x+3)^2 - 3^2$
3 Use the Difference of Two Squares rule to factorise the expression. ($a = x+3$, $b = 3$)	$= [(x+3) - 3][(x+3) + 3]$
4 Simplify by adding or subtracting like terms.	$= x(x+6)$

Surds can be used to factorise quadratics using the Difference of Two Squares rule.

A surd is an irrational number such as $\sqrt{2}$ or $\sqrt[3]{4}$. It does not have an exact decimal equivalent. Here we can use surds to write an integer such as 7 as a square number $(\sqrt{7})^2 = 7$.

Worked Example 13

WE 13

Factorise these expressions. (Your answers will include surds.)

(a) $x^2 - 3$ **(b)** $10 - 5x^2$

Thinking	Working
(a) 1 Look for common factors. (There are none.)	(a) $x^2 - 3$
2 Express both terms as squares.	$= x^2 - (\sqrt{3})^2$
3 Use the Difference of Two Squares rule. ($a = x$, $b = \sqrt{3}$)	$= (x - \sqrt{3})(x + \sqrt{3})$
(b) 1 Look for common factors and factorise.	(b) $10 - 5x^2$ $= 5(2 - x^2)$
2 Express both terms as squares.	$= 5[(\sqrt{2})^2 - x^2]$
3 Use the Difference of Two Squares rule. ($a = \sqrt{2}$, $b = x$)	$= 5(\sqrt{2} - x)(\sqrt{2} + x)$

Worked Example 14

WE 14

Factorise these expressions.

(a) $(x+7)^2 - (x+2)^2$ **(b)** $(x-5)^2 - 7$

Thinking		Working	
(a) 1	Look for common factors. (There are none.)	(a)	$(x+7)^2-(x+2)^2$
2	Use the Difference of Two Squares rule. ($a = x + 7$, $b = x + 2$)		$=[(x+7)-(x+2)][(x+7)+(x+2)]$
3	Simplify. Take care with negatives.		$=(x+7-x-2)(x+7+x+2)$
4	Add or subtract like terms.		$=5(2x+9)$
(b) 1	Look for common factors. (There are none.)	(b)	$(x-5)^2-7$
2	Express both terms as squares.		$=(x-5)^2-(\sqrt{7})^2$
3	Use the Difference of Two Squares rule to factorise the expression.		$=(x-5+\sqrt{7})(x-5-\sqrt{7})$

Worked Example 15

WE15

Factorise these expressions.

(a) $x^2 - y^2 + 5x - 5y$

(b) $25m^2 - n^2 - 5m - n$

Thinking		Working	
(a) 1	Look for common factors. (There are none.)	(a)	$x^2-y^2+5x-5y$
2	Use the Difference of Two Squares rule to factorise the first two terms ($x^2 - y^2$) and take out a common factor of the last two terms ($5x - 5y$).		$=(x^2-y^2)+(5x-5y)$ $=(x-y)(x+y)+5(x-y)$
3	Take out the common factor ($x - y$).		$=(x-y)(x+y+5)$
(b) 1	Look for common factors. (There are none.)	(b)	$25m^2-n^2-5m-n$
2	Use the Difference of Two Squares rule to factorise the first two terms ($25m - n^2$) and take out a common factor of the last two terms ($-5m - n$) remembering to change the sign inside the brackets.		$=(5m+n)(5m-n)-1(5m+n)$ $=(5m+n)(5m-n-1)$

Note that in these examples there are three terms in one of the factors, so these factorisations are not binomial products.

The Perfect Square rules

$(a + b)^2 = a^2 + 2ab + b^2$ and
$(a - b)^2 = a^2 - 2ab + b^2$

By reversing these expansions we get:
$a^2 + 2ab + b^2 = (a + b)^2$ and
$a^2 - 2ab + b^2 = (a - b)^2$

We can use these rules to factorise many quadratic trinomials.

Worked Example 16 WE16

Factorise these expressions using a Perfect Square rule.

(a) $x^2 + 10x + 25$ **(b)** $40x^2 - 120x + 90$

Thinking	Working
(a) 1 Look for common factors. (There are none.)	(a) $x^2 + 10x + 25$
2 Factorise using a Perfect Square rule.	$= x^2 + 2(x)(5) + 5^2$ $= (x + 5)^2$
(b) 1 Look for common factors and factorise. (HCF = 10)	(b) $40x^2 - 120x + 90$ $= 10(4x^2 - 12x + 9)$
2 Factorise further using a Perfect Square rule.	$= 10[(2x)^2 - 2(2x)(3) + 3^2]$ $= 10(2x - 3)^2$

Special product rules:

$a^2 - b^2 = (a + b)(a - b)$ (Difference of Two Squares)

$a^2 + 2ab + b^2 = (a + b)^2$ (Perfect Square)

$a^2 - 2ab + b^2 = (a - b)^2$ (Perfect Square)

2.4 Factorising using special products

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5 Column 1, Q6 Column 1, Q7, Q8, Q10 Column 1, Q11, Q13, Q15, Q18	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5 Column 2, Q6 Column 2, Q7, Q8, Q9, Q10 Column 2, Q11, Q12, Q13, Q15, Q18	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5 Column 2, Q6 Column 3, Q7, Q8, Q9, Q10 Column 3, Q11, Q12, Q14, Q15, Q16, Q17, Q19

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Fluency

1 Factorise these expressions using the Difference of Two Squares rule. WE11

(a) $p^2 - 4$ **(b)** $d^2 - 100$ **(c)** $9 - k^2$

(d) $3p^2 - 75$ **(e)** $80 - 5m^2$ **(f)** $8k^2 - 32$

(g) $(2y)^2 - 49$ **(h)** $25 - (4x)^2$ **(i)** $(6m)^2 - (5n)^2$

(j) $9x^2 - 16$ **(k)** $4 - 25a^2$ **(l)** $81d^2 - 64$

2 Factorise these expressions. WE12

(a) $(x + 2)^2 - 9$ **(b)** $(a - 1)^2 - 25$ **(c)** $16 - (k + 5)^2$

(d) $36 - (p + 1)^2$ **(e)** $1 - (2y - 3)^2$ **(f)** $1 - (3x - 2)^2$

(g) $5(x + 3)^2 - 5y^2$ **(h)** $7(a + 5)^2 - 7b^2$ **(i)** $3(x - 4)^2 - 12$

WE13

3 Factorise these expressions. (Your answers will include surds.)

(a) $x^2 - 2$ (b) $7 - p^2$ (c) $4k^2 - 5$

(d) $9p^2 - 2$ (e) $3a^2 - 33$ (f) $2b^2 - 6$

WE14

4 Factorise these expressions.

(a) $(x + 2)^2 - (y + 1)^2$ (b) $(a + 2)^2 - (a + 3)^2$ (c) $(2k + 3)^2 - (m - 3)^2$

(d) $(x + 4)^2 - (x - 4)^2$ (e) $(2x + 1)^2 - (x + 3)^2$ (f) $(5x - 4)^2 - (x - 7)^2$

(g) $(x + 2)^2 - 3$ (h) $(y - 3)^2 - 6$ (i) $(x + 8)^2 - 5$

(j) $(x - 5)^2 - 13$ (k) $(x - 1)^2 - 19$ (l) $(2x + 1)^2 - 2$

WE15

5 Factorise these expressions.

(a) $x^2 - 9 + 4x + 12$ (b) $a^2 - 16 + 5a - 20$

(c) $x^2 - y^2 + 7x - 7y$ (d) $9a^2 - b^2 + 6a + 2b$

(e) $x^2 - 16y^2 + 2x - 8y$ (f) $4p^2 - q^2 - 6p - 3q$

WE16

6 Factorise each expression using a Perfect Square rule.

(a) $x^2 + 6x + 9$ (b) $x^2 + 4x + 4$ (c) $x^2 - 14x + 49$

(d) $x^2 - 8x + 16$ (e) $x^2 + 2x + 1$ (f) $x^2 + 12x + 36$

(g) $4x^2 + 20x + 25$ (h) $9x^2 + 24x + 16$ (i) $25x^2 - 10x + 1$

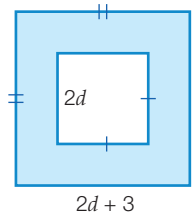
Understanding

7 When factorised, $(2a - 1)^2 - (a + 1)^2$ becomes:

A $3(a - 1)(a + 1)$ B $3a(a - 2)$

C $(a - 1)(a + 1)$ D $(2a - 1)(a + 1)$

8 When factorised, the expression for the blue area shown becomes:



A $4d(d + 3)$ B $(4d + 3)(d + 1)$

C $(2d + 3)(d + 1)$ D $3(4d + 3)$

9 When factorised, $\left(\frac{a}{k}\right)^2 - \left(\frac{b}{2k}\right)^2$ is:

A $\frac{1}{4k^2}(2a - b)(2a + b)$ B $\frac{1}{2k^2}(a - b)(a + 2b)$

C $\frac{1}{4k^2}(a - 3b)(a + 3b)$ D $\frac{1}{k^2}(a - b)(a + b)$

10 Factorise these expressions.

(a) $8m^2 - 200n^2$ (b) $6x^2 - 24r^2$ (c) $5a^2 - 45c^2$

(d) $27a^2 - 48b^2$ (e) $72x^2 - 98y^2$ (f) $175t^2 - 28p^2$

(g) $54k^2 - 24m^2$ (h) $2x^2y^2 - 32z^2$ (i) $3p^2q^2 - 12n^2$

11 Factorise these expressions.

(a) $16c^2 - 25d^2 + 10d + 8c$ (b) $k^2 - 64 + km - 8m$

(c) $81 - 49a^2 - 27d + 21ad$ (d) $16 - 9y^2 - 16w + 12wy$

(e) $25m^2 - 9n^2 - 20m - 12n$ (f) $144p^2 - 49r^2 - 12pq + 7rq$

Reasoning

12 The difference between the squares of two consecutive odd numbers is 56.

(a) If x and y are the two numbers and x is greater than y, write the equation describing the information.

(b) Factorise the left side of the equation in part **(a)**.

(c) Explain why $x = y + 2$.

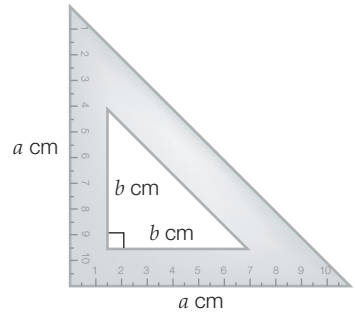
(d) Substitute $y + 2$ for x in the equation in part **(b)** and simplify.

(e) Solve the equation to obtain the value for y. Hence, state the value of x.

13 **(a)** Write the expression (in cm^2) for the shaded area of the set square shown in factorised form.

(b) What is the shaded area if $a = 3b$?

(c) What are the dimensions of the instrument if the shaded area is 100 cm^2 given $a = 3b$?



14 **(a)** Factorise $a^2 - b^2$.

(b) How many factors do you have?

(c) Find $\dfrac{a^2 - b^2}{a - b}$, $a \neq b$.

(d) Write $a^4 - b^4$ as the difference of two squares.

(e) Factorise $a^4 - b^4$ fully.

(f) How many factors did you have?

(g) Find $\dfrac{a^4 - b^4}{a - b}$, $a \neq b$.

(h) Write $a^8 - b^8$ as the difference of two squares.

(i) Factorise $a^8 - b^8$ fully using your answer from **(e)**.

(j) Find $\dfrac{a^8 - b^8}{a - b}$, $a \neq b$.

(k) Use the pattern to write an expression for $\dfrac{a^{16} - b^{16}}{a - b}$, $a \neq b$.

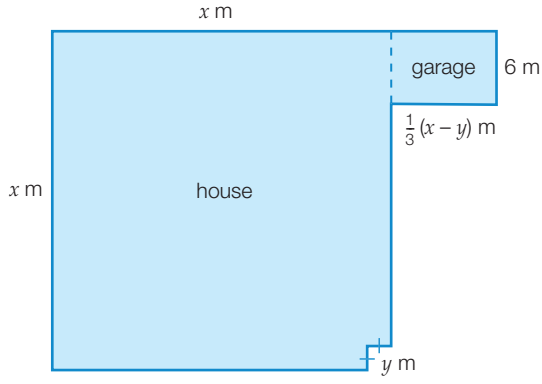
(l) Use the expression in **(k)** to show that $2^{16} - 1 = 3 \times 5 \times 17 \times 257$.

15 The shape of a house with its garage is shown.

(a) Write the expression (in m^2) for the area of the house.

(b) Write the expression (in m^2) for the area of the garage.

(c) Write the expression (in m^2) for the total area in factorised form.



Hint
Can $k = 0$?
Can k be a fraction?

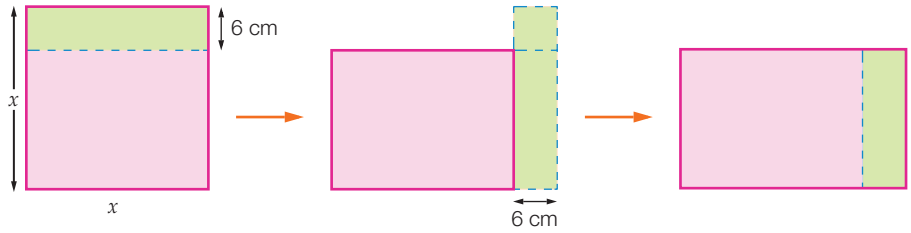
16 **(a)** If $a = m - n$, $b = 2\sqrt{mn}$ and $c = m + n$, show that $c^2 = a^2 + b^2$.

(b) Show that if $p = kr$ then the expression $t^2 = p^2 - r^2$ becomes $t^2 = r^2(k - 1)(k + 1)$.

(c) What values of k would make $t = 0$?

(d) What values of k are not possible? Justify your answer.

17 A square sheet of plywood is made into a rectangle by cutting 6 cm from one end and joining it to the other. The overhanging edge is then trimmed.



(a) If the length of the sides of the original sheet of plywood were x cm, find the area of the square sheet of plywood.

(b) What area has been trimmed off the rectangle?

(c) Use your answers in **(a)** and **(b)** to find the area of the rectangular sheet of plywood.

(d) Factorise your answer.

(e) Use the diagrams above to find the length and width of the rectangle obtained in terms of x.

(f) What is the relationship of the answer in **(d)** to the length and width of the rectangle?

Open-ended

18 Write two trinomial expressions that have a difference that is a perfect square.

19 Write an expression with four terms so that two terms are the difference of the squares of two variables and the other two terms are a multiple of the sum of these variables. Factorise your expression.

Outside the Square — Problem solving

Playground swings

The length of the chains of a playground swing that takes t seconds for one swing is given by $l = \frac{5t^2}{M}$.

1 What is the difference (in m) in the chain lengths of two swings that take t_1 seconds and t_2 seconds, respectively, to make one swing? (Assume t_1 is greater than t_2.) Express your answer in factorised form.

2 If $t_1 - t_2 = \frac{3M}{10}$ and $t_1 + t_2 = \frac{M}{5}$ then find the difference in the chain lengths.

Strategy options
- Look for a pattern.
- Have I seen a similar problem?

Half-time 2

1 Expand and simplify. **Ex. 2.1**

(a) $(x + 2)(x + 6)$ **(b)** $(x - 5)(x + 7)$

(c) $(3x - 2)(x + 5)$ **(d)** $(2x - 1)(x + 3)$

2 Factorise each of the following. **Ex. 2.3**

(a) $x^2 + 2x - 15$ **(b)** $x^2 - 14x + 24$ **(c)** $x^2 - 3x - 40$

3 Factorise each of the following. **Ex. 2.3**

(a) $x^2 + 5x + 7x + 35$ **(b)** $x^2 - 3x - 4x + 12$ **(c)** $x^2 + 7x - 3x - 21$

4 Factorise each of the following. **Ex. 2.4**

(a) $x^2 + 14x + 49$ **(b)** $x^2 - 18x + 81$ **(c)** $9x^2 - 30x + 25$

5 Factorise each of the following by taking out a common factor. **Ex. 2.2**

(a) $7x + 14y$ **(b)** $18x^2 + 24xy$

(c) $3(x - 2) + y(x - 2)$ **(d)** $3(x + 5) + 2y(x + 5)$

6 Expand and simplify. **Ex. 2.1**

(a) $(x - 3)(x + 3)$ **(b)** $(x + 8)(x - 8)$

(c) $(3x + y)(3x - y)$ **(d)** $(6x - 5y)(6x + 5y)$

7 Expand and simplify. **Ex. 2.1**

(a) $x(x + 2)(x + 3)$ **(b)** $3(x - 2)(x + 2)$

(c) $(3x - 4)(7x + 19)(3x + 3)$ **(d)** $6x(2x - 6)(4x - 1)$

8 Factorise each of the following. **Ex. 2.4**

(a) $x^2 - 9$ **(b)** $9x^2 - 36y^2$

(c) $8x^2 - 50y^2$ **(d)** $27 - 48x^2$

9 The expression $acx^2 + adx + bcx + bd$ can be factorised to obtain: **Ex. 2.3**

A $(bx + a)(cx + d)$ **B** $(ax + c)(bx + d)$

C $(dx + b)(ax + c)$ **D** $(ax + b)(cx + d)$

10 The expression $a^2 - b^2 + 3a + 3b$ can be factorised to obtain: **Ex. 2.3**

A $(a - b)(a - b - 3)$ **B** $(a - b)(a + b - 3)$

C $(a + b)(a - b + 3)$ **D** $(a + b)(a + b - 3)$

11 The expression $a^2b^2 - c^2 + abd - cd$ can be factorised to obtain: **Ex. 2.3**

A $(ad + b)(cd - b)$ **B** $(ab - c)(ab + c + d)$

C $(ac - d)(a + b + c + d)$ **D** $(cd - b)(ab - b)$

2.5 Factorising by completing the square

Factorising four terms by grouping 'three and one'

Not all expressions with four terms are factorised by grouping in pairs. Often a group of three terms and one term are used if the three terms form a perfect square, and the fourth term is subtracted. We can factorise the first three terms using a Perfect Square rule, then use the Difference of Two Squares rule to complete the factorisation.

Worked Example 17 WE17

Factorise $x^2 + 4x + 4 - y^2$ by grouping 'three and one'.

Thinking	Working
1 Are there any common factors? (No) Check if it can be factorised by grouping in pairs. (No)	$x^2 + 4x + 4 - y^2$
2 Identify a perfect square and group the three terms.	$= (x^2 + 4x + 4) - y^2$
3 Factorise using the Perfect Square rule.	$= (x + 2)^2 - y^2$
4 Use the Difference of Two Squares rule to complete the factorisation.	$= (x + 2 - y)(x + 2 + y)$

Completing the square

In Section 2.3 we saw that not all trinomials can be factorised using binomial products.

If we try to factorise the quadratic trinomial $x^2 + 6x - 2$, we cannot find two numbers that multiply to give -2 and add to 6. This method of factorising cannot be used here, so an alternative needs to be tried. The **completing the square** method of factorising combines the use of both the Perfect Square and the Difference of Two Squares rules as shown below.

Consider $x^2 + 6x - 2$ again.

- Add a third term to the first two terms to make a perfect square. For $x^2 + 6x - 2$, we add 9 to obtain $(x^2 + 6x + 9) - 2$.
- Subtract the amount added to keep the expression equivalent. We then subtract 9 to obtain $(x^2 + 6x + 9) - 2 - 9$.

A perfect square has been obtained by adding 9.

Subtract 9 to keep the whole expression equivalent to $x^2 + 6x - 2$.

$x^2 + 6x - 2$ can be rewritten as $(x^2 + 6x + 9) - 9 - 2$

$$x^2 + 6x - 2 = (x + 3)^2 - 11$$

This is now a 'three and one' expression and we can use a Perfect Square rule and the Difference of Two Squares rule to factorise. Remember to use surds.

$$\begin{aligned} x^2 + 6x - 2 &= (x^2 + 6x + 9) - 11 \\ &= (x + 3)^2 - (\sqrt{11})^2 \\ &= (x + 3 - \sqrt{11})(x + 3 + \sqrt{11}) \end{aligned}$$

To complete the square of $x^2 + bx$, divide b, the coefficient of x by 2 to obtain $\frac{b}{2}$ then add $\left(\frac{b}{2}\right)^2$ to make a perfect square. Subtract $\left(\frac{b}{2}\right)^2$ to keep the expression equivalent.

Worked Example 18

WE18

Factorise $x^2 - 8x + 9$ by completing the square.

Thinking	Working
1 Are there any common factors? (No)	$x^2 - 8x + 9$
2 By comparing the first two terms to $x^2 + bx$ identify the value of b. ($b = 8$) Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the square and keep the expression equivalent. (In this case $\left(\frac{b}{2}\right)^2 = 16$, so add and subtract 16.)	$= (x^2 - 8x + 16) - 16 + 9$
3 Factorise the perfect square and simplify.	$= (x - 4)^2 - 7$
4 Use the Difference of Two Squares rule to complete the factorisation.	$= (x - 4)^2 - (\sqrt{7})^2$ $= (x - 4 - \sqrt{7})(x - 4 + \sqrt{7})$

Always look for common factors that can be taken out of all terms. Here, we will only complete the square of a monic quadratic trinomial.

Worked Example 19

WE19

Factorise $3x^2 + 12x - 30$ by completing the square.

Thinking	Working
1 Take out a common factor. (In this case 3)	$3x^2 + 12x - 30$ $= 3(x^2 + 4x - 10)$
2 By comparing the first two terms to $x^2 + bx$ identify the value of b. ($b = 4$) Add and subtract $\left(\frac{b}{2}\right)^2$. Use additional brackets to separate the common factor. (In this case $\left(\frac{b}{2}\right)^2 = 4$ so add and subtract 4.)	$= 3[(x^2 + 4x + 4) - 4 - 10]$
3 Factorise the perfect square and simplify keeping the additional brackets.	$= 3[(x + 2)^2 - 14]$

4 Write as the Difference of Two Squares using surds.	$= 3[(x+2)^2 - (\sqrt{14})^2]$
5 Factorise using the Difference of Two Squares rule inside the additional brackets.	$= 3[(x+2-\sqrt{14})(x+2+\sqrt{14})]$
6 Remove the additional brackets as they are no longer necessary.	$= 3(x+2-\sqrt{14})(x+2+\sqrt{14})$

2.5 Factorising by completing the square

Navigator

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Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q9, Q10	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q9, Q10	Q1 Column 2, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency

WE17

1 Factorise these expressions by grouping 'three and one'.

(a) $x^2 + 8x + 16 - y^2$
(b) $y^2 - 4y + 4 - k^2$
(c) $y^2 + 8y + 16 - m^2$
(d) $m^2 - 10m + 25 - 4n^2$
(e) $x^2 + 2x + 1 - 9y^2$
(f) $q^2 + 6q + 9 - 16p^2$
(g) $a^2 - 6ab + 9b^2 - 16$
(h) $x^2 - 10xy + 25y^2 - 36$
(i) $4a^2 - 12a + 9 - 16b^2$
(j) $9c^2 + 30c + 25 - 4d^2$

WE18

2 Factorise these expressions by completing the square.

(a) $x^2 + 6x + 2$
(b) $x^2 - 10x + 4$
(c) $x^2 - 8x - 3$
(d) $x^2 + 20x + 77$
(e) $x^2 - 12x + 21$
(f) $x^2 + 2x - 8$
(g) $x^2 - 14x - 2$
(h) $x^2 - 4x - 2$
(i) $x^2 + 16x + 47$
(j) $x^2 + 3x + 1$
(k) $x^2 + 7x + 9$
(l) $x^2 - x - 4$
(m) $x^2 - 5x + 5$
(n) $x^2 + 5x - 2$
(o) $x^2 + x - 5$
(p) $x^2 - 9x + 13$
(q) $x^2 - 3x - 2$
(r) $x^2 - 7x - 1$

WE19

3 Factorise each expression by completing the square.

(a) $2x^2 + 4x - 10$
(b) $4x^2 - 16x - 4$
(c) $10x^2 - 80x - 30$
(d) $3x^2 - 6x - 12$
(e) $5x^2 + 20x - 5$
(f) $7x^2 + 42x + 14$

4 The expression $2m^2 - 12m + 4$ when factorised is:

A $2(m - 6 - \sqrt{7})(m - 6 + \sqrt{7})$
B $2(m + 7 - \sqrt{3})(m + 7 + \sqrt{3})$
C $2(m - 6 - \sqrt{2})(m - 6 + \sqrt{2})$
D $2(m - 3 - \sqrt{7})(m - 3 + \sqrt{7})$

Understanding

5 Factorising $x^2 + x - \frac{5}{16}$ gives:

A $(x - \frac{1}{4})(x + \frac{5}{4})$
B $(x - \frac{5}{16})(x + 1)$
C $(x - \frac{5}{4})(x + \frac{1}{16})$
D $(x + \frac{1}{4})(x - \frac{5}{4})$

6 A seagull is diving for a fish. Its height above sea level , in metres, at time t seconds is given by the expression $t^2 - 8t + 14$.

(a) How high is the seagull at $t = 0$?

(b) Factorise the expression by completing the square.

(c) How long is the time interval between $t = 4 - \sqrt{2}$ and $t = 4 + \sqrt{2}$, rounded to two decimal places.

(d) Explain what happens when $t = 4 \pm \sqrt{2}$.

(e) Explain what the seagull is doing between $t = 4 - \sqrt{2}$ and $t = 4 + \sqrt{2}$.

7 The profit made by a newspaper publisher in the first 8 months is given by $P = -5x^2 + 40x - 5$ where x is the time in months.

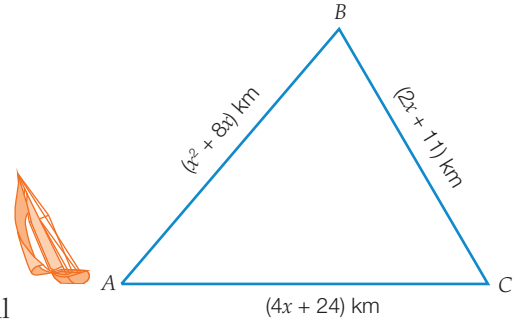
(a) Factorise the expression for P.

(b) Show that $P = 0$ when $x = 4 \pm \sqrt{15}$.

(c) Show what happened to the company in the first 8 months by substituting different values for x into the expression.

Reasoning

8 A yacht sails a leg of a race by going in a straight course from point A to point B for $(x^2 + 8x)$ km. From point B it travels to point C for $(2x + 11)$ km, then back to point A for $(4x + 24)$ km, as shown in the diagram.



(a) Write the simplified expression (in km) for the total distance the yacht sailed.

(b) Factorise the expression in **(a)**.

(c) If the factorised expression also represents the product of the length and width of a rectangle and if the length is 15 km, find the value of x. Give your answer as an exact distance.

(d) Find the length of the race, rounded to two decimal places, by substituting the exact value for x into the expression for the total distance found in **(a)**.

9 The number of earthworms in a compost heap after d days is given by $100 - 4d - d^2$.

(a) How many earthworms were there originally?

(b) Factorise the expression for the number of worms.

(c) Use your factorisation to show that when $d = 2(\sqrt{26} - 1)$, no worms remain.

(d) Evaluate $2(\sqrt{26} - 1)$ and round to two decimal places to find on which day the number of earthworms is reduced to zero.

(e) A second colony of earthworms is now in another compost heap. Their population is given by $64 - 2n - n^2$.

(i) Factorise the expression.

(ii) Use this factorisation to show that no earthworms remain when $n = \sqrt{65} - 1$.

(f) Use your answers to **(c)** and **(e) (ii)** to decide which colony stayed longer in their respective compost heaps.

Open-ended

10 **(a)** Write the expression $x^2 - 2ax + 2b$ in the form $(x - a)^2 + A$.

(b) Write an expression for A in terms of a and b.

(c) **(i)** What is the relationship between a and b if the expression is to be a perfect square?

(ii) Find a value for a and b that produces a perfect square.

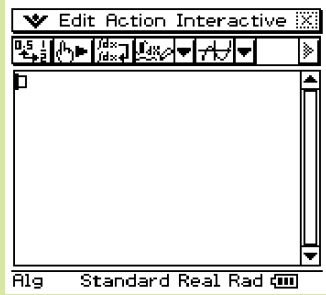
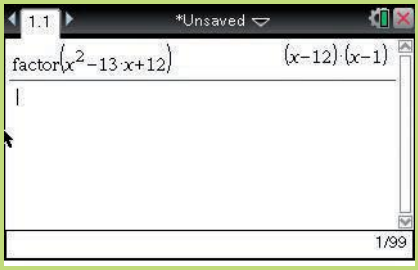
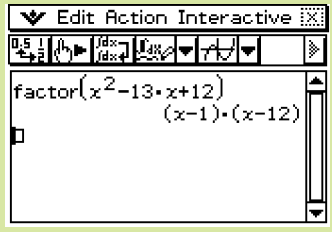
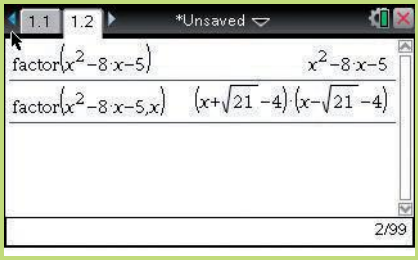
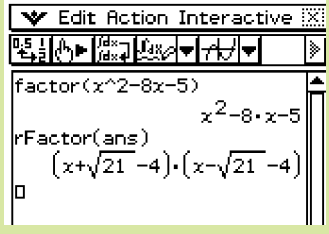
Technology Exploration CAS

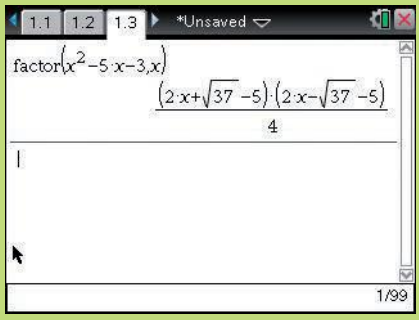
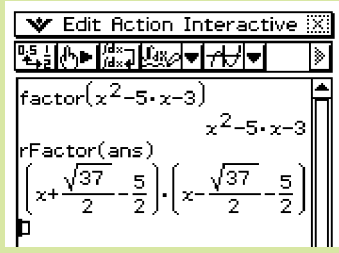
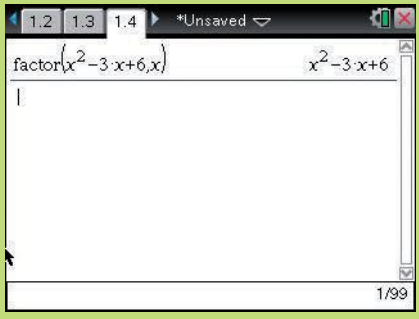
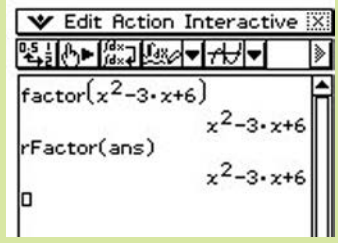
Equipment required: 1 brain, a TI-Nspire CAS or ClassPad CAS

Factorisation involving surds

Factorisation by hand is often time consuming especially when the factorisation involves surds. We will use a CAS calculator to factorise the following expressions (if possible) quickly and accurately.

(a) $x^2 - 13x + 12$ **(b)** $x^2 - 8x - 5$ **(c)** $x^2 - 5x - 3$ **(d)** $x^2 - 3x + 6$

Using the TI-Nspire CAS	Using the ClassPad
Open a new calculator page. Press (on) and (tab) until the first icon on the bottom row is highlighted as shown below. Then press (enter).	Tap on Main and make sure Standard is displayed at the bottom of the screen. If it isn't, just tap decimal and it will change. 
Press (menu) > Algebra > Factor and press (enter). Enter expression **(a)** to be factorised inside the brackets. Press (enter). 	Tap Interactive > Transformation > factor. In the dialogue box that appears, enter expression **(a)**. Tap OK.  The expression will now be factorised as shown.
Repeat the instructions above for expression **(b)**. It does not factorise with rational factors. If the variable is included though, the expression does now factorise. Note that the factorisation now contains surds. 	Repeat the instructions above for expression **(b)**. It does not factorise with rational factors. However, if rFactor is used instead of factor, the expression does now factorise. Note that the factorisation now contains surds. 

Using the TI-Nspire CAS	Using the ClassPad
Repeat the instructions for expression **(c)**, remembering to include the variable. Note that the factorisation contains fractions and surds.  	Repeat the instructions for expression **(c)**. It does not factorise using factor, so try using rFactor again. Note that the factorisation contains fractions and surds. 
Repeat the instructions for expression **(d)**, remembering to include the variable.  Note that the expression does not factorise.	Repeat the instructions for expression **(d)**. It does not factorise using factor, so try using rFactor again.  Note that the expression does not factorise.

Use your CAS to factorise the following expressions, if possible.

1. $x^2 - 6x + 2$
2. $x^2 - 7x - 13$
3. $x^2 - 17x - 84$
4. $x^2 - 5x + 12$

Taking it further

5. Use your CAS to find three monic quadratic expressions that:
   - **(a)** have rational factors
   - **(b)** contain surds but have no fractions when factorised
   - **(c)** contain surds and fractions
   - **(d)** do not factorise.

2.6 Algebraic fractions

Multiplication and division

To simplify a fraction, we need to look for common factors in the numerator (top) and the denominator (bottom) of the fraction. We can simplify by dividing the numerator and denominator by these common factors. These rules also apply to **algebraic fractions** (expressions involving variables in fraction form).

Worked Example 20 WE20

Simplify $\frac{4a}{6}$.

Thinking	Working
1 Find the factor common to the numerator and denominator (2) and simplify the fraction by dividing both by this factor.	$\frac{4a}{6}$ $= \frac{\cancel{4}^{2}a}{\cancel{6}_{3}}$
2 Write the fraction in its simplified form.	$= \frac{2a}{3}$

A common factor may be an algebraic term or expression.

Worked Example 21 WE21

Simplify $\frac{(x+2)(x+4)}{(x-1)(x+2)}$.

Thinking	Working
1 Write down the expression and look for common factors. (Here, $(x+2)$ is common to the numerator and denominator.)	$\frac{(x+2)(x+4)}{(x-1)(x+2)}$
2 Simplify the fraction by dividing the common factors into both the numerator and the denominator.	$= \frac{\cancel{(x+2)}^{1}(x+4)}{(x-1)\cancel{(x+2)}_{1}}$ $= \frac{x+4}{x-1}$

The above worked example cannot be simplified further as there are no other common factors. We cannot cancel the x in both the numerator and denominator because x is *not* a factor of $x+4$ or $x-1$.

We can also simplify when multiplying or dividing two or more fractions. We may need to factorise them first to identify any common factors. As well as cancelling common factors within fractions, we can also cancel a factor that is common to the numerator of one fraction and the denominator of the other.

Worked Example 22

WE 22

Simplify:

(a) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)} \times \dfrac{(x-2)(x+1)}{(x+5)(x+3)}$

(b) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)} \div \dfrac{(x+5)(x+3)}{(x-2)(x+1)}$

Thinking	Working
(a) 1 Write down the expression and identify common factors of the numerators and denominators. (Here, $(x+5)$ and $(x+1)$.)	(a) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)} \times \dfrac{(x-2)(x+1)}{(x+5)(x+3)}$
2 Simplify by dividing the common factors into both the numerators and the denominators. A factor can be divided into the denominator of one fraction and the numerator of another.	$= \dfrac{(x+2)\overset{1}{\cancel{(x+5)}}}{(x-1)\underset{1}{\cancel{(x+1)}}} \times \dfrac{(x-2)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+5)}}(x+3)}$
3 Multiply the remaining numerators and denominators. Leave the fraction in factorised form.	$= \dfrac{(x+2)(x-2)}{(x-1)(x+3)}$
(b) 1 Write down the expression.	(b) $\dfrac{(x+2)(x+5)}{(x-1)(x+1)} \div \dfrac{(x+5)(x+3)}{(x-2)(x+1)}$
2 Convert the division into a multiplication by inverting the second fraction (turning it upside-down) and simplify by dividing the common factors into both the numerators and denominators.	$= \dfrac{(x+2)(x+5)}{(x-1)(x+1)} \times \dfrac{(x-2)(x+1)}{(x+5)(x+3)}$ $= \dfrac{(x+2)\overset{1}{\cancel{(x+5)}}}{(x-1)\underset{1}{\cancel{(x+1)}}} \times \dfrac{(x-2)\overset{1}{\cancel{(x+1)}}}{\underset{1}{\cancel{(x+5)}}(x+3)}$
3 Multiply the remaining numerators and denominators. Leave the fraction in factorised form.	$= \dfrac{(x+2)(x-2)}{(x-1)(x+3)}$

Worked Example 23

WE 23

Simplify $\dfrac{4x^2-12x}{x^2-25} \times \dfrac{x^2+7x+10}{2x^2+4x}$.

Thinking	Working
1 Factorise expressions where possible. ($4x^2-12x$: take out a common factor of $4x$.) (x^2-25: apply the Difference of Two Squares rule.) ($x^2+7x+10$: factorise the quadratic trinomial.) ($2x^2+4x$: take out a common factor of $2x$.)	$\dfrac{4x^2-12x}{x^2-25} \times \dfrac{x^2+7x+10}{2x^2+4x}$ $= \dfrac{4x(x-3)}{(x-5)(x+5)} \times \dfrac{(x+2)(x+5)}{2x(x+2)}$

2 Identify common factors in the numerators and the denominators. (Here, they are x, 2, $(x+2)$ and $(x+5)$.) Divide the common factors into the numerators and the denominators.	$= \dfrac{\overset{2\ 1}{\cancel{4x}}(x-3)}{(x-5)\underset{1}{\cancel{(x+5)}}} \times \dfrac{\overset{1}{\cancel{(x+2)}}\overset{1}{\cancel{(x+5)}}}{\underset{1\ 1}{\cancel{2x}}\underset{1}{\cancel{(x+2)}}}$
3 Multiply the remaining numerators and denominators.	$= \dfrac{2(x-3)}{x-5}$

Worked Example 24

WE24

Simplify $\dfrac{x^2+x-12}{3x^2+12x} \div \dfrac{x^2-9}{12x}$.

Thinking	Working
1 Change the division into a multiplication by inverting the second fraction.	$\dfrac{x^2+x-12}{3x^2+12x} \div \dfrac{x^2-9}{12x}$ $= \dfrac{x^2+x-12}{3x^2+12x} \times \dfrac{12x}{x^2-9}$
2 Factorise where possible. (x^2+x-12: factorise the quadratic trinomial.) ($3x^2+12x$: take out a common factor of $3x$.) ($4x^2-9$: apply the Difference of Two Squares rule.)	$= \dfrac{(x-3)(x+4)}{3x(x+4)} \times \dfrac{12x}{(x-3)(x+3)}$
3 Identify common factors in the numerators and denominators. (Here, they are $(x-3)$, $(x+4)$, 3 and x.) Simplify by dividing these factors into the numerators and the denominators.	$= \dfrac{\overset{1}{\cancel{(x-3)}}\overset{1}{\cancel{(x+4)}}}{\underset{1\ 1}{\cancel{3x}}\underset{1}{\cancel{(x+4)}}} \times \dfrac{\overset{4\ 1}{\cancel{12x}}}{\underset{1}{\cancel{(x-3)}}(x+3)}$
4 Multiply the remaining numerators. Multiply the remaining denominators. Note that it is not possible to cancel any further.	$= \dfrac{4}{x+3}$

When multiplying (or dividing) algebraic fractions, remember to:

1. Factorise if possible.
2. Change a division into a multiplication by inverting the second fraction.
3. Identify common factors in the numerators and denominators of the fractions and divide the numerators and denominators by these common factors.
4. Simplify by multiplying the remaining numerators and the remaining denominators.

Addition and subtraction

We can use the rules for addition and subtraction of fractions to deal with algebraic fractions.

Worked Example 25 WE25

Simplify:

(a) $\frac{x}{4}+\frac{x}{5}$ **(b)** $\frac{x+7}{4}-\frac{x-3}{6}$

Thinking	Working
(a) 1 Find the lowest common denominator. (Here, the LCD of 4 and 5 is 20.)	(a) $\frac{x}{4}+\frac{x}{5}$
2 Find equivalent fractions using the lowest common denominator.	$=\frac{5x}{20}+\frac{4x}{20}$
3 Rewrite the expression as a single fraction.	$=\frac{5x+4x}{20}$
4 Simplify the numerator by adding like terms.	$=\frac{9x}{20}$
(b) 1 Find the lowest common denominator (here, the LCD of 4 and 6 is 12) and write the equivalent fractions with this (multiply the first fraction by $\frac{3}{3}$ and the second by $\frac{2}{2}$).	(b) $\frac{x+7}{4}-\frac{x-3}{6}$ $=\frac{3(x+7)}{12}-\frac{2(x-3)}{12}$
2 Rewrite the expression as a single fraction.	$=\frac{3(x+7)-2(x-3)}{12}$
3 Expand brackets in the numerator. (Notice that $-2\times-3=+6$.)	$=\frac{3x+21-2x+6}{12}$
4 Simplify the numerator by adding or subtracting like terms.	$=\frac{x+27}{12}$

When adding (or subtracting) algebraic fractions:

1. Find the lowest common denominator (LCD). If there is a factor that is common to both denominators, only use it *once* in the common denominator.
2. Rewrite each fraction as an equivalent fraction with this common denominator.
3. Add (or subtract) the new numerators.

2.6 Algebraic fractions

Navigator

Answers page 723

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Fluency

WE20

1 Simplify the following.

(a) $\frac{8x}{10}$ (b) $\frac{6k}{42}$ (c) $\frac{25p}{45}$

(d) $\frac{4(a-b)}{2}$ (e) $\frac{9(x+1)}{6}$ (f) $\frac{14(y-2)}{8}$

(g) $\frac{5y(a-4)}{10}$ (h) $\frac{8x(x+5)}{36}$ (i) $\frac{18x(x-4)}{24}$

WE21

2 Simplify the following.

(a) $\frac{(x+1)(x+6)}{(x-5)(x+1)}$ (b) $\frac{(x-3)(x+2)}{(x-3)(x-2)}$ (c) $\frac{(2x-3)(x+4)}{(x+4)(2x+1)}$

(d) $\frac{(3x+1)(x-2)}{(x-7)(3x+1)}$ (e) $\frac{(a+2)(b+5)}{(b+3)(a+2)}$ (f) $\frac{(x+7)(y-2)}{(y-2)(x-1)}$

WE22

3 Simplify the following.

(a) $\frac{3}{x-2} \times \frac{x-2}{9}$ (b) $\frac{15}{5-2x} \times \frac{5+2x}{5}$

(c) $\frac{x-1}{x-4} \div \frac{x-1}{x+4}$ (d) $\frac{x+4}{x+3} \div \frac{x+4}{x-3}$

(e) $\frac{x(x+7)}{2(x-3)} \div \frac{4x}{x-3}$ (f) $\frac{7x(x-4)}{10x} \div \frac{2(x-4)}{5}$

(g) $\frac{(x+1)(x-3)}{(x-4)(x+7)} \times \frac{(x+7)(x+2)}{(x-1)(x+1)}$ (h) $\frac{(x-4)(x+4)}{(x+3)(x-4)} \times \frac{(x-2)(x+3)}{(x+1)(x-2)}$

(i) $\frac{(3x-2)(x+5)}{(2x+3)(x-6)} \div \frac{(x+2)(x+5)}{(2x+3)(x-3)}$ (j) $\frac{(5x-3)(2x+3)}{(x-4)(x+7)} \div \frac{(2x+3)(5x-4)}{(x-3)(x+7)}$

WE23

4 Simplify the following. Remember to factorise first.

(a) $\frac{x^2+11x+24}{x+3}$ (b) $\frac{x^2-12x+27}{x-3}$

(c) $\frac{x+2}{x^2-3x-10}$ (d) $\frac{x+5}{x^2-2x-35}$

(e) $\frac{8x^2-2}{8x+4}$ (f) $\frac{3x^2-12}{x+2}$

(g) $\frac{4x^2+12x+9}{4x+6}$ (h) $\frac{9x^2+30x+25}{6x+10}$

(i) $\frac{x^2+2x-3}{x^2-5x-24}$ (j) $\frac{x^2+3x-4}{x^2+4x-5}$

(k) $\dfrac{x^2-7x-8}{x^2-10x+16}$

(l) $\dfrac{x^2-7x+6}{x^2-2x-24}$

(m) $\dfrac{3x+6}{4x-24}\times\dfrac{8x-48}{6x+12}$

(n) $\dfrac{5x+10}{2x-8}\times\dfrac{3x-12}{9x+18}$

(o) $\dfrac{x^2-5x}{3x+3}\times\dfrac{x+1}{2x^2+6x}$

(p) $\dfrac{4x^2+8x}{x+5}\times\dfrac{2x-14}{x^2+2x}$

(q) $\dfrac{x^2-x-12}{x^2+9x+18}\times\dfrac{x^2+4x-12}{x^2-2x-8}$

(r) $\dfrac{x^2+7x+10}{x^2+x-20}\times\dfrac{x^2-7x+12}{x^2-5x-14}$

5 Simplify the following. WE24

(a) $\dfrac{x^2-4}{x^2-9}\div\dfrac{4x+8}{12x-36}$

(b) $\dfrac{x^2-25}{x^2-49}\div\dfrac{3x-15}{5x+35}$

(c) $\dfrac{x^2-36}{x^2+6x-7}\div\dfrac{x^2-5x-6}{x^2-1}$

(d) $\dfrac{x^2-6x-16}{x^2-x-30}\div\dfrac{x^2-64}{x^2-5x-6}$

(e) $\dfrac{x^2+6x+9-y^2}{4x+6}\div\dfrac{2x^2+6x-2xy}{10x^2+15x}$

(f) $\dfrac{x^2-2x+1-4y^2}{12x^2-5x-2}\div\dfrac{3x-3+6y}{28x+7}$

6 Simplify the following. WE25

(a) $\dfrac{x}{2}+\dfrac{x}{3}$

(b) $\dfrac{x}{5}+\dfrac{x}{2}$

(c) $\dfrac{x}{6}+\dfrac{x}{4}$

(d) $\dfrac{2x}{3}-\dfrac{x}{6}$

(e) $\dfrac{4x}{5}-\dfrac{x}{15}$

(f) $\dfrac{2x}{3}-\dfrac{5x}{4}$

(g) $\dfrac{4x}{3}-\dfrac{2y}{4}$

(h) $\dfrac{5x}{7}-\dfrac{3y}{4}$

(i) $\dfrac{11x}{8}-\dfrac{8y}{9}$

(j) $\dfrac{x+6}{2}+\dfrac{x+2}{3}$

(k) $\dfrac{x-3}{4}+\dfrac{x+4}{8}$

(l) $\dfrac{x-7}{3}+\dfrac{x+5}{9}$

(m) $\dfrac{3x+2}{6}-\dfrac{x+4}{4}$

(n) $\dfrac{3x-8}{4}-\dfrac{x-3}{10}$

(o) $\dfrac{4x-1}{9}-\dfrac{x-6}{12}$

(p) $\dfrac{3x+2y}{4}+\dfrac{5x-y}{3}$

(q) $\dfrac{7x-2y}{3}+\dfrac{x-3y}{5}$

(r) $\dfrac{8x-3y}{4}+\dfrac{5x+7y}{6}$

(s) $\dfrac{4x-3y}{5}-\dfrac{3x-2y}{6}$

(t) $\dfrac{5x-2y}{4}-\dfrac{x+y}{2}$

(u) $\dfrac{7x+3y}{2}-\dfrac{2x-y}{3}$

7 If $4x+3y=5$, then $\dfrac{x+1}{3}+\dfrac{y+1}{4}$ simplifies to:

A $\dfrac{5}{12}$ B $\dfrac{5}{7}$ C 1 D $\dfrac{7}{12}$

Understanding

8 **(a)** Factorise and simplify as much as possible the expression

$$\frac{9b^2-9a^2}{6b^2+3bc+2ab+ac}\div\frac{3a+3b}{2b+c}.$$

(b) If $a=5$ and $b=7$, show that the expression in part **(a)** takes the value $\dfrac{3}{13}$.

9 (a) Show that $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

(b) Write $a^6 - b^6$ as the difference of two cubes.

(c) Use the result from (b) to show that $\frac{a^6 - b^6}{a^2 - b^2} = a^4 + a^2b^2 + b^4$.

10 A circular disc of diameter $2w$ m is cut from a square piece of timber. The side lengths of the timber are the same as the diameter of the disc.

(a) Write an expression for the area of timber wasted.

(b) What fraction of the area of the uncut timber is the area of the timber wasted? Give your answer in simplified form.

Reasoning

11 The density of an object of mass m kg and volume v m^3 is given by $d = \frac{m}{v}$ kg/m^3

(a) What is the density of a sphere of mass m kg and radius r m if its volume is $\frac{4\pi r^3}{3}$?

(b) A golf ball has density d_1, radius r_1 and volume v_1. A basketball has density d_2, radius r_2 and volume v_2. Show that $\frac{d_2}{d_1} = \left(\frac{m_2}{m_1}\right)\left(\frac{r_1}{r_2}\right)^3$.

(c) If the basketball's mass is 20 times greater than the mass of the golf ball and its radius is 5 times greater than the radius of the golf ball, what is the value of $\frac{d_2}{d_1}$?

Open-ended

12 Write the sum of two terms with different denominators that can be added to give $\frac{x+2}{3}$.

Outside the Square — Problem solving

How to square two-digit numbers quickly—without a calculator

We start with the Difference of Two Squares rule:

$$a^2 - b^2 = (a - b)(a + b)$$

Adding b^2 to both sides we get:

$$a^2 - b^2 + b^2 = b^2 + (a - b)(a + b)$$

So $$a^2 = b^2 + (a - b)(a + b)$$

This expression can be used to find the square of a two-digit number quickly, without a calculator! In the expression:

a is the number to be squared

b can be any number (we choose a convenient value).

For example, to find 19^2, we have $a = 19$ and choose $b = 1$ (you'll see why in a moment).

Start with $a^2 = b^2 + (a - b)(a + b)$

Substitute $a = 19$ and $b = 1$.

$$\begin{aligned} 19^2 &= 1^2 + (19 - 1)(19 + 1) \\ &= 1^2 + (18)(20) \\ &= 1^2 + 360 \\ &= 361 \end{aligned}$$

Notice how the choice of 1 for b results in one of the brackets containing a multiple of 10, thus making multiplication by the contents of the other pair of brackets easier.

Use the method described above to calculate the following. Show the steps involved.

(a) 29^2 (b) 59^2

(c) 79^2 (d) 99^2

Strategy options

- Look for a pattern.
- Break problem into manageable parts.

Rearranging formulas

2.7

A formula is an equation showing the relationship between two or more variables or unknowns. The equation $T = 2\pi\sqrt{\frac{l}{g}}$ is an example of a formula used in physics. We can rearrange this formula to make l or g the subject ($l =$ or $g =$). To rearrange formulas like this we must apply the same inverse operations to both sides, using the same methods as we do for solving equations.

Worked Example 26

WE 26

(a) Rearrange $V = \frac{1}{3}\pi r^2 h$ to make r the subject of the formula.

(b) Find r when $V = 27\pi$ and $h = 4$ are subtituted into the rearranged formula.

Thinking	Working
(a) 1 Perform any inverse operations needed to make the required variable the subject of the formula. (Here, we multiply by 3 and divide by πh.)	(a) $V = \frac{1}{3}\pi r^2 h$ $3V = \pi r^2 h$ $\frac{3V}{\pi} = r^2 h$ $r^2 = \frac{3V}{\pi h}$
2 Take the square root of both sides.	$r = \pm\sqrt{\frac{3V}{\pi h}}$
(b) Write the formula, substitute the values for V and h and simplify.	(b) $r = \pm\sqrt{\frac{3V}{\pi h}}$ $= \pm\sqrt{\frac{3 \times 27\cancel{\pi}}{\cancel{\pi} \times 4}}$ $= \pm\sqrt{\frac{81}{4}}$ $= \pm\frac{9}{2}$ $= \pm 4\frac{1}{2}$

Equations involving roots

If equations involve roots, we square both sides.

Worked Example 27 WE27

(a) Rearrange $T = 2\pi\sqrt{\frac{l}{g}}$ to make **(i)** l and **(ii)** g the subject of the formula.

(b) **(i)** Find l when $T = 0.6\pi$ and $g = 10$ are substituted into the rearranged formula.

(ii) Find g when $T = \frac{4\pi}{7}$ and $l = 0.8$ are substituted into the rearranged formula.

Thinking	Working
(a) **1** Perform any inverse operations needed to isolate the square root on one side of the equation. (Here, we divide by 2π.)	(a) $T = 2\pi\sqrt{\frac{l}{g}}$ $\frac{T}{2\pi} = \sqrt{\frac{l}{g}}$
2 Square both sides of the equation.	$\left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$
3 Perform any inverse operations needed to make the required variable the subject of the formula.	$l = g \times \left(\frac{T}{2\pi}\right)^2$ (i) $l = \frac{gT^2}{4\pi^2}$ (ii) $g = \frac{4\pi^2 l}{T^2}$
(b) **(i)** Substitute values into the formula for l and simplify.	(b) (i) $l = \frac{gT^2}{4\pi^2}$ $= \frac{10 \times (0.6\pi)^2}{4\pi^2}$ $= \frac{10 \times \overset{0.09}{\cancel{0.36}}\cancel{\pi^2}}{\cancel{4}\cancel{\pi^2}}$ $= 0.9$
(ii) Substitute values into the formula for g and simplify.	(ii) $g = \frac{4\pi^2 l}{T^2}$ $= \frac{4 \times \pi^2 \times 0.8}{\left(\frac{4\pi}{7}\right)^2}$ $= \frac{\cancel{4}\cancel{\pi^2} \times 49 \times \overset{0.2}{\cancel{0.8}}}{\cancel{16}\cancel{\pi^2}}$ $= 9.8$

Equations involving powers

The inverse operation of squaring is taking the square root. The solution to $x^2 = 9$ is $x = 3$ or $x = -3$, because $(3)^2 = (-3)^2 = 9$. Therefore when taking the square root, we must include ±

In many cases, only the positive square root is an acceptable answer. For example, time, area, volume and length can only have positive values.

2.7 Rearranging formulas

Navigator

Q1 (a)–(c), Q2 (a)–(c), Q3, Q4, Q5, Q6, Q9	Q1 (d)–(f), Q2 (d)–(f), Q3, Q4, Q5, Q7, Q8, Q9	Q1 (g)–(i), Q2 (g)–(i), Q3, Q4, Q6, Q7, Q8, Q9, Q10

Answers page 724

Fluency

1 **(i)** Rearrange the following, to make the variable in each bracket the subject of the formula. WE26

(ii) Find the value of this variable when the given values are substituted into the rearranged formula.

(a)	$A = s^2$	$[s]$	$A = 27$
(b)	$K = \frac{1}{2}mv^2$	$[v]$	$K = 20$ and $m = 8$
(c)	$F = \frac{mv^2}{r}$	$[v]$	$F = 9$, $r = 7$ and $m = 6$
(d)	$F = \frac{Gm_1m_2}{r^2}$	$[r]$	$F = 16$, $G = 18$, $m_1 = 6$ and $m_2 = 5$
(e)	$g_1 = \frac{r_2^2}{r_1^2}g_2$	$[r_2]$	$g_1 = 9.8$, $g_2 = 10$ and $r_1 = 20$
(f)	$P = \frac{V^2}{R}$	$[V]$	$P = 9$ and $R = 7$
(g)	$P = I^2R$	$[I]$	$P = 10$ and $R = 4.5$
(h)	$E = \frac{1}{2}cv^2$	$[v]$	$E = 96$ and $c = 6$
(i)	$A = \frac{\pi D^2}{4}$	$[D]$	$A = 9\pi$

2 **(i)** Rearrange the following, to make the variable in each bracket the subject of the formula. WE27

(ii) Find the value of this variable when the given values are substituted into the rearranged formula.

(a)	$A = \sqrt{A_x + A_y}$	$[A_y]$	$A = 9$ and $A_x = 43$
(b)	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$[c]$	$b = 9$, $a = 4$ and $x = -1$
(c)	$l = \frac{3r}{\sqrt{T}}$	$[T]$	$r = 8$ and $l = 3$
(d)	$x = \sqrt{y - z}$	$[z]$	$y = 4$ and $x = 10$
(e)	$x = \sqrt{\frac{y}{z}}$	$[z]$	$y = 7$ and $x = 5$

(f) $x = \sqrt{\dfrac{a+b}{c}}$ [b] $c = 3$, $x = 2$ and $a = -4$

(g) $x = \sqrt{y - z}$ [y] $x = 4$ and $z = 6$

(h) $x = \sqrt{\dfrac{a+b}{c}}$ [c] $a = 4$, $b = -1$ and $x = -3$

(i) $x = \sqrt{\dfrac{y}{z}}$ [y] $x = 4$ and $z = 6$

3 Which of the following expressions is equivalent to $S = \dfrac{4r^2}{\sqrt{t}}$?

A $t = \dfrac{16r^2}{S^2}$ B $r = \pm\sqrt{\dfrac{St}{4}}$ C $t = \dfrac{16r^4}{S^2}$ D $r = \dfrac{S\sqrt{t}}{4}$

Understanding

4 The formula for the area of a circle is $A = \pi r^2$.

(a) Rearrange this equation to make r the subject.

(b) Explain why there is no ± sign required.

(c) Find the radius, r, when the area, A, is 40 cm^2.

5 The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(a) Rearrange this equation to make r the subject.

(b) Explain why there is no ± sign required.

(c) Find the radius, r, when the volume, V, is 125 cm^3.

6 The formula for the surface area of a cone is $V = \frac{1}{3}\pi r^2 h$. By first rearranging this equation to change the subject, find:

(a) the height of a cone with volume 312 m^3 and radius 14 m

(b) the radius of a cone with volume 160 cm^3 and height 8 cm.

Reasoning

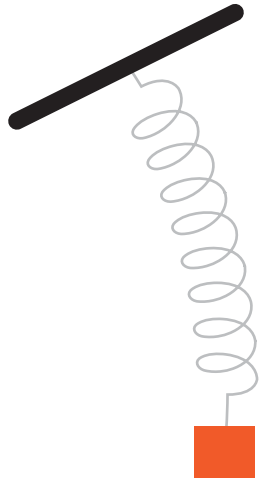
7 The formula $v^2 = u^2 + 2as$ is regularly used in physics to find the final velocity, v, of an object that has initial velocity, u, acceleration, a, and displacement, s. (Displacement is the distance of the object from its starting point.)

(a) Calculate the initial velocity of a car that stops in 42 m, with acceleration -20 m/s^2.

(b) What does a negative value for the acceleration mean in this situation?

8 The formula used to determine the frequency, f, of a spring, given the spring constant, k, and mass of the object, m, is $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$.

Frequency is measured in Hertz and is the number of cycles (up and down) per unit of time (oscillations per second).
A spring with an object of mass 3.5 kg has a frequency of 3 Hertz.



(a) Calculate the spring constant for this spring.

(b) An object of mass 7.2 kg is attached to the same spring. Calculate the new frequency.

Open-ended

9 Elizabeth was changing the subject of the formula $g = 4h^2 - 9k$ to h and has completed the following working.

$$g = 4h^2 - 9k$$

$$g + 9k = 4h^2 \qquad \text{Line 1}$$

$$h^2 = \frac{g + 9k}{4} \qquad \text{Line 2}$$

$$h = \sqrt{\frac{g + 9k}{4}} \qquad \text{Line 3}$$

$$h = \frac{\sqrt{g + 9k}}{2} \qquad \text{Line 4}$$

(a) Identify the line with the mistake.

(b) Explain why this is mathematically inaccurate.

10 **(a)** Make up a formula of your own with at least three variables. One of the variables, that is not the subject of this formula, must be squared. Rearrange the formula to make this variable the subject.

(b) Make up a formula of your own with at least three variables. Take the square root of one of the variables that is not the subject of this formula. Rearrange the formula to make this variable the subject.

Outside the Square — Problem solving

Gravity

The formula $v = \sqrt{2gx}$ is used to calculate the velocity, v, of an object that has already fallen x metres, where g is acceleration due to gravity. Gravity is different depending on the planet that you are on, as shown in the table.

Planet	Acceleration due to gravity (m/s^2)
Mercury	3.59
Venus	8.87
Earth	9.81
Mars	3.77
Jupiter	25.95
Saturn	11.08

1 On which planet would it take the longest amount of time for an object to fall 100 m?

Hint
The slowest velocity gives the longest time.

2 According to this formula, does it take longer for a feather or a heavy boulder to fall?

3 How much higher do I have to drop a ball when on Mars, so that the velocity at which it hits the ground is the same as if it was dropped on Earth?

Strategy options
- Make a table.
- Work backwards.

Investigation

The firefighter's ladder

Equipment required: 1 brain, calculator

The Big Question

Can we predict how the height of the top of a ladder will change for a small change in the distance of the foot of the ladder from the wall?

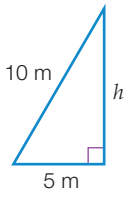
Engage

A firefighter has to get into a building through a window by leaning his ladder against the wall. Given the length of his ladder and the height of the window, he will need to calculate how far from the wall he must put the bottom of his ladder.

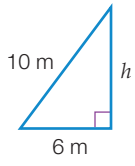
Give exact answers for each of the following questions.

1 **(a)** For a 10 m ladder, calculate each of the following heights (h).

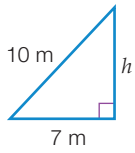
(i)



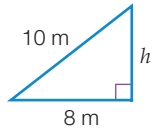
(ii)



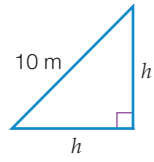
(iii)



(iv)



(b) If the ladder is 10 m long, find the value of h, when the distance from the base of the wall to the foot of the ladder is equal to the height to the top of the ladder on the wall.



Explore

Give exact answers for each of the following questions.

2 **(a)** If the distance of the foot of the ladder to the wall and the height of the top of the ladder on the wall are both equal to 7 m, ($h = 7$ m), how long would the ladder have to be?

(b) Find the difference between the length of this ladder and 7 m.

(c) How far could this ladder move in towards the wall before it was completely upright?

(d) How far could the bottom of the ladder slip away from its position 7 m from the base of the wall before it was flat on the ground?

(e) If the base of the ladder slips p m away from the base of the wall and the top of the ladder travels a distance of n m down the wall, what is the largest possible values for:

(i) p (ii) n

(f) If the base of the ladder slips p m away from the base of the wall and the top of the ladder travels a distance of n m down the wall, show that $(p + 7)^2 + (7 - n)^2 = 98$.

(g) (i) Rearrange this formula and simplify to show that $(7 - n)^2 = 49 - 14p - p^2$.

(ii) Hence, show that $n = 7 - \sqrt{49 - 14p - p^2}$.

(iii) Why do we reject $n = 7 + \sqrt{49 - 14p - p^2}$?

Hint
Refer to your answer in part **(e)(ii)**.

(h) Evaluate n for all possible integer values of p.

(i) Evaluate n for $p = -16$.

Strategy options
- Make a table.
- Look for a pattern.

Explain

Use exact values in your explanations for each of the following questions.

3 (a) Does the ladder slip down the wall the same distance it moves away from the base of the wall? Explain why or why not. If not, does it slip further down or not as far down the wall than the distance away from it?

(b) Explain what happens to the distance the ladder will move up the wall when the ladder is moved closer to the wall.

(c) In **2(h)**, we could only evaluate n for integer values of $p \leq 2$. Explain why.

(d) Why can we evaluate n for $p = -16$ m when this value of p is not possible? What does $p = -16$ m mean?

Hint
Did you find n was the same for one of the values of the p-values you used in Question **2(h)**? If so, which value?

Elaborate

4 (a) If h, the distance from the base of the wall to the bottom of the ladder, is equal to the height to the top of the ladder on the wall, find an equation for n, the distance the top of the ladder moves down the wall when the foot of the ladder moves a distance of p away from the base of the wall.

(b) Use an equation to show that p can only be equal to n for $p = n = 0$.

Hint
The length of the ladder does not change.

(c) Describe what happens to the distance the ladder can reach up a wall as the bottom of the ladder slips away from the base of the wall and as the bottom of the ladder is pushed closer to the base of the wall.

Evaluate

5 (a) Did you find visualising the ladder moving up and down the wall a challenge? Did the diagrams help you?

(b) Did you find working with unknown variables more difficult than working with actual values? Did working with actual values help you to understand the situation? Explain why or why not.

(c) Why were you asked to give exact answers? Would it have been easier to understand if you had evaluated lengths and distances correct to two decimal places? Explain.

Extend

6 If the distance from the foot of the ladder to the base of the wall is $a - b$ and the distance up the wall to the top of the ladder is $a + b$, show that the length of the ladder can be written as $\sqrt{2(a^2 + b^2)}$.

Maths 4 Real

Staying alive

Even at the speed limit of 60 km/h, it takes longer to bring a car to a stop than you might think. The stopping distance of a car is a combination of the reaction distance and the braking distance.

Reaction distance

When driving a motor vehicle, the time taken to react to a perceived hazard (such as a pedestrian, animal, another vehicle stopping suddenly) is called the *reaction time*. This time depends on the driver's age, experience, physical fitness and whether the driver is distracted, or under the influence of alcohol or drugs. Reaction time is usually between 1 and 2 seconds.

The reaction distance d_r is the distance travelled by the vehicle during the reaction time. It depends on both the length of the reaction time and the velocity (speed) v of the vehicle.

$d_r = v \times t_r$

1 **(a)** Calculate the reaction distance (in metres) for the following speeds and reaction times.

Convert the speeds from km/h to m/s.
1 km/h = 3.6 m/s

(i) $v = 60$ km/h, $t_r = 0.8$ s **(ii)** $v = 60$ km/h, $t_r = 1.6$ s

(iii) $v = 65$ km/h, $t_r = 0.8$ s **(iv)** $v = 65$ km/h, $t_r = 1.6$ s

(v) $v = 100$ km/h, $t_r = 0.8$ s **(vi)** $v = 100$ km/h, $t_r = 1.6$ s

(b) Use your answers to part **(a)** to answer the following.

(i) For a reaction time of 1.6 seconds, what is the difference in the reaction distance at a speed of 65 km/h, compared to 60 km/h?

(ii) What effect does doubling the reaction time have on the reaction distance at 100 km/h?

(c) If we were to graph d_r for a certain reaction time over a number of different speeds, would we produce a linear graph or a curve? Explain why. (Use some of the values calculated in **(a)** to draw a graph if you aren't sure).

Braking distance

The *braking distance* travelled by the vehicle is measured from when the brakes are applied to when the vehicle comes to a stop. The vehicle can be considered to be undergoing a constant rate of deceleration while braking. (Deceleration is the opposite of acceleration, i.e. the speed is decreasing instead of increasing.) The braking distance also depends on the road surface, weather conditions, and the condition of the tyres and brakes.

The formula $V_f^2 = V_0^2 - 2ad$ is used to calculate the final velocity of a vehicle (V_f) after it has undergone deceleration. V_0 is the initial velocity, a is the rate of deceleration and d is the braking distance travelled during deceleration. V_f will be zero when the car has stopped.

2 **(a)** Substitute $V_f = 0$ into the above equation, then rearrange it to make d the subject.

In this form, you can see that braking distance is proportional to the square of the velocity. This means a small increase in speed can have a big effect on braking distance.

(b) What would be the shape of the graph produced if braking distance (d) were graphed against velocity, (V_0)? Explain why.

3 Calculate the braking distance given a maximum deceleration of 10 m/s^2 and velocity of:

(a) 25 km/h **(b)** 50 km/h **(c)** 60 km/h

(d) 65 km/h **(e)** 80 km/h **(f)** 120 km/h

Total stopping distance

Total stopping distance = reaction time + braking distance

4 **(a)** Combine your formulas for reaction distance (Question **1**) and braking distance (Question **2**) to create a formula for total stopping distance.

(b) Using a driver reaction time of 1.6 seconds and a deceleration rate of 10 m/s^2, calculate the total stopping distance for a vehicle travelling at:

(i) 50 km/h **(ii)** 60 km/h

(iii) 65 km/h **(iv)** 100 km/h.

(c) Describe the effect of doubling the speed (from 50 km/h to 100 km/h) on:

(i) reaction distance **(ii)** braking distance.

(d) Victoria's Transport Accident Commission (TAC) ran a safety campaign with the slogan 'Wipe off 5'. Describe the effect of lowering speed from 65 km/h to 60 km/h on:

(i) reaction distance **(ii)** braking distance.

5 Speed limits in residential areas have been reduced from 60 km/h to 50 km/h and to 40 km/h around schools. What effect does this have on the total stopping distance when both ends of the reaction time are considered? (Assume 10 m/s^2 maximum deceleration.)

Research

6 Research the effects of alcohol or drugs on driver reaction time.

7 Find some car crash statistics for your state or local area. How many of them involved a fatality (death)? In how many crashes was speed a factor? In how many crashes was alcohol involved?

8 Many state road authorities advise drivers to leave a 'safe braking distance' between their car and the car in front. Find the recommended distance. Given the braking distances you have calculated in this task, do you think the recommended distance is adequate?

Chapter review 2

D.I.Y. Summary

Key Words

algebraic fractions	factorise by grouping in pairs	Perfect Square
binomial product	factorising	quadratic expression
completing the square	highest common factor (HCF)	quadratic trinomial
Difference of Two Squares	monic quadratic	

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 $ax^2 + bx + c$ is called a _____. When the coefficient of x^2 is 1 the expression is a _____.

2 The first thing to do to factorise $50x^2 - 2y^2$ is to take out the _____. We can then use the _____ method to factorise the expression.

3 To factorise $x^2 - 4x - 6$ as $(x - 2 - \sqrt{10})(x - 2 + \sqrt{10})$ we use a method called _____.

4 We _____ to write the expression $2ax + 2ay + 3x + 3y$ as a product of factors $(2a + 3)(x + y)$.

5 $(3t - 2)^2$ is an example of a _____.

6 $(3a + b)(4 - 2a)$ is called a _____.

7 $x^2 - 9$ is an example of the _____.

Fluency

1 Expand and simplify the following. Ex. 2.1

(a) $5(2x + 7)$ (b) $x(x - 3) + 2(x - 3)$

(c) $(x + 6)(x + 4)$ (d) $(4x - 1)(2x + 3)$

2 Expand and simplify the following. Ex. 2.1

(a) $(x + 6)^2$ (b) $(2x - 3)^2$

(c) $(5 - x)(5 + x)$ (d) $(3x + 7)(3x - 7)$

3 Expand and simplify the following. Ex. 2.1

(a) $4(x - 7)(x + 2)$ (b) $-3x(x + 8)(x - 3)$

(c) $(x - 1)(x + 5)(x - 6)$ (d) $(2x + 3)(2 - x)(x + 4)$

4 Factorise these expressions. Ex. 2.2

(a) $7x + 21$ (b) $12ab - 8a$ (c) $18x^2 + 24x$

5 Factorise these expressions. Ex. 2.2

(a) $d(c - 4) + 6(c - 4)$ (b) $x(x + 2) - 3(x + 2)$

(c) $5a(a + 1) - 2(a + 1)$ (d) $2y(3y - 5) - 7(3y - 5)$

6 Factorise these expressions. Ex. 2.3

(a) $x^2 + 11x + 30$ (b) $x^2 - 10x - 24$

7 Factorise each of these quadratic trinomials. Ex. 2.4

(a) $x^2 + 10x + 25$ **(b)** $x^2 - 18x + 81$

(c) $9x^2 - 24x + 16$ **(d)** $4x^2 + 4x + 1$

8 Factorise these expressions. Ex. 2.4

(a) $x^2 - 9$ **(b)** $25 - y^2$ **(c)** $4x^2 - y^2$

(d) $49x^2 - 16$ **(e)** $36x^2 - 81y^2$ **(f)** $64 - 9x^2$

(g) $(2x - 5)^2 - 4$ **(h)** $(x - 5)^2 - (x + 8)^2$ **(i)** $(x + 2)^2 - (x - 3)^2$

(j) $x^2 - 3$ **(k)** $(x + 7)^2 - 5$ **(l)** $(x - 4)^2 - 7$

9 Factorise these expressions. Ex. 2.5

(a) $x^2 + 6x - 2$ **(b)** $x^2 - 10x + 23$

(c) $x^2 - 4x - 7$ **(d)** $x^2 + 8x + 13$

10 Simplify the following. Ex. 2.6

(a) $\frac{18x}{45}$ **(b)** $\frac{2(a-3)}{6}$ **(c)** $\frac{10x(x-1)}{8x}$ **(d)** $\frac{12ab}{4b}$

11 Simplify the following. Ex. 2.6

(a) $\frac{(x+4)(x-3)}{(x-3)(x+2)}$ **(b)** $\frac{(2x-1)(x+1)}{(x+7)(x-7)} \times \frac{(3x+7)(x+7)}{(x-1)(2x-1)}$

(c) $\frac{4x^2 - 4x + 1}{2x - 1}$ **(d)** $\frac{2x^2 - 8x}{x^2 + 3x} \div \frac{x^2 - 16}{x^2 + 8x + 15}$

12 Simplify the following. Ex. 2.6

(a) $\frac{2x}{3} + \frac{x}{2}$ **(b)** $\frac{x+1}{5} + \frac{x-2}{3}$ **(c)** $\frac{3x-2}{10} - \frac{x+5}{6}$

(d) $\frac{6x}{7} + \frac{3y}{2}$ **(e)** $\frac{x+5}{4} + \frac{y+2}{3}$ **(f)** $\frac{x-6}{5} - \frac{y-3}{10}$

13 Rearrange each of the following to make the pronumeral in brackets the subject. Ex. 2.7

(a) $A = lb$ $[b]$ **(b)** $A = 2\pi\sqrt{\frac{l}{g}}$ $[l]$

(c) $SA = 4\pi r^2$ $[r]$ **(d)** $a = x^2y - z$ $[x]$

Understanding

14 If $9x^2 - 12x + 4$ is written as $(ax + b)^2$, state the values of a and b.

15 In January, John's weight was x kg. In April his weight was $(x + a)$ kg and in September it was $(x + b)$ kg.

(a) If the product of John's weight in April and September is $x^2 + x - 30$, determine the values of a and b.

(b) By how many kilograms had John's weight increased by April?

(c) What was John's overall weight gain/loss by the end of September?

(d) What happened to his weight from April to September?

16 The heights in centimetres of four people are ax, $2x$, ay and $2y$.

(a) Write down the expression for their total height.

(b) Factorise the expression in part **(a)**.

(c) If $x = 65$, $y = 75$ and $a = 3$, use your factorised expression to find the average height of the four people.

17 **(a)** Which of the following methods of factorising is the most appropriate for each of the expressions **(i)** to **(vi)** below?

A factorising using common factors

B factorising using the Difference of Two Squares rule

C factorising by grouping in pairs

D factorise by grouping (3 and 1)

E factorising quadratic trinomials (multiply to give … , add to give …)

F factorising by completing the square

(i) $x^2 + 5x - 24$

(ii) $25a^2 - b^2$

(iii) $x^2 + 6x + 9 - y^2$

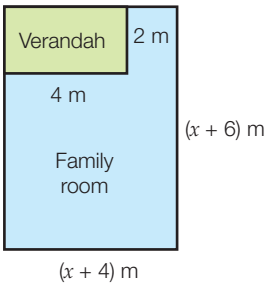
(iv) $x^2 + 12x + 34$

(v) $16x^2 - 25x$

(vi) $4x - 4y + x^2 - y^2$

(b) Factorise each expression in part **(a)**.

18 A house has had a rectangular extension added. The extension has a length of $(x + 6)$ m and a width of $(x + 4)$ m. An area 2 m wide and 4 m long is to be an untiled verandah. The remainder is to be a tiled family room. The cost of tiling a family room is \$18 per square metre.



(a) Write an expression for the area of the family room in factorised form.

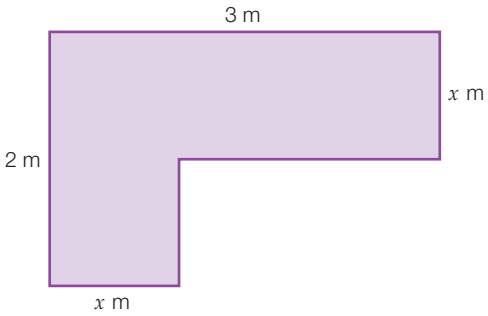
(b) Write an expression for the cost of tiling the family room.

(c) Find the cost of tiling the family room if x is 2 m.

19 Find, in expanded form, the square of the sum of $4x$ and $3y$.

Reasoning

20 **(a)** Write an expression (in m^2) for the area of the kitchen bench surface shown at right. Simplify the expression as far as possible.



(b) Factorise the expression for the area.

(c) Find the area of the kitchen bench surface when the width of the bench, x, is:

(i) 1 m

(ii) 0.5 m

(iii) 0.75 m

21 A biscuit box has a length 14 cm greater than its height and a width 9 cm greater than its height.

(a) Draw a diagram showing this information. Use h to represent height.

(b) Write an expression (in factorised form) to represent the volume of the biscuit box.

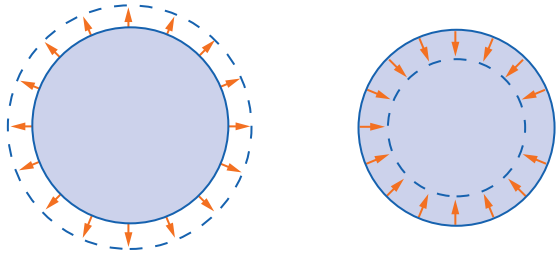
(c) Expand and simplify the expression for volume.

(d) Use your answer for part **(b)** to find the volume of the biscuit box if its height is 6 cm.

(e) Use your answer for part **(c)** to find the volume of the biscuit box if its height is 6 cm.

(f) Which form of the volume expression (factorised or expanded) did you find easier to work with? Explain your answer.

22 A circular disc of metal expands when it is heated. When it is heated to a particular temperature, its radius increases by 2 mm. When cooled to a particular temperature below room temperature, the disc contracts and its radius decreases by 2 mm. (Do not substitute for π in any parts of this question.)



Disc expands when heated Disc contracts when cooled

(a) If the disc has a radius of x mm at room temperature (20°C), write down its area at room temperature.

(b) In terms of x, what is the radius of the heated disc?

(c) Hence, write an expression for the area of this larger disc.

(d) In terms of x, what is the radius of the cooled disc?

(e) Hence, write an expression for the area of this smaller disc.

(f) Write an expression for the difference between the area of the larger disc and that of the smaller disc.

(g) Is there a common factor that can be taken out of this expression for the difference in area? What is it?

(h) Factorise this expression for the difference in area.

(i) If the original disc (at 20°C) had a radius of 10 mm, find the difference in area between the larger and smaller discs.

(j) If the difference in area between the larger and smaller discs is 136π mm^2, find the radius of the disc at 20°C.

23 An exercise book is $(8x + 3)$ cm long and $(2y + 7)$ cm wide.

(a) Write an expression (in cm^2) for the area of the book.

(b) What is the expression for the area if $y = 4x - 5$?

(c) Expand and simplify the expression in part **(b)**.

(d) **(i)** In terms of x, what value of y will result in the expression for the area being a perfect square?

(ii) Expand and simplify the perfect square in **(i)**.

24 The number of private cars observed during a period of x days is modelled by the expression $x^2 + p^2x$ where p is the number of passengers. The number of taxis observed with p passengers during y days is given by $p^2y - y^2$.

(a) Write the expression for the total number of cars and taxis.

(b) Factorise the expression in part **(a)**.

(c) Explain why x must be greater than y if the number of passengers is zero.

(d) **(i)** How many vehicles were there in total if $x = y$?

(ii) It is found that $2px$ of these vehicles were white. Write, in factorised form, the number of non-white vehicles.

3 Measurement

Maths on the big screen

Name: Stuart Williams
Job: Audio Visual Systems Designer
Qualifications: Bachelor of Electronic Engineering

If you designed your own home theatre system, could you connect it all together to make sure everything works? Stuart Williams can. Stuart works for Insight Systems, a company that designs and installs audio visual systems into homes, schools, universities, offices and even courtrooms. Stuart works with projectors, speakers, screens, interactive whiteboards and videoconferencing systems, which are usually linked to control panels by a complex system of cables.

"It's like a big jigsaw puzzle. I enjoy the challenge of putting it all together. I need to think logically and include every little detail, so that I can produce a useful schematic for the technicians who are installing the system." (A schematic is an accurate scale diagram that shows how everything is connected together.)

Stuart uses a variety of mathematical skills in his daily work, including using formulas to calculate the current flowing through an amplifier, reading scale drawings, applying ratios and using Pythagoras' theorem to calculate the height and width of a TV or projection screen.

Why learn this?

Not all shapes are rectangles, and not all solids are rectangular boxes. We need to be able to calculate the surface area and volume of pyramids, cones, spheres, and complex solids made up of combinations of these. These shapes are commonly used in the design and manufacture of everyday objects, such as car components, toys, furniture and even buildings. Volume and capacity are also used to describe properties of solids and liquids such as density and concentration.

After completing this chapter you will be able to:

- use area formulas of basic shapes to find the area of composite shapes
- calculate the surface area, volume and capacity of prisms and cylinders
- calculate the surface area, volume and capacity of pyramids, cones and spheres
- calculate the density of solids and liquids, and find the concentration of dissolved substances. 10A

3 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 Use your calculator to evaluate the following.

(a) 25^2 **(b)** 3.1^3 **(c)** $\sqrt{11.9025}$

Worksheet R3.1

2 Given the formula $A = \frac{1}{2}(a + b)h$:

Worksheet R3.2

(a) find A if $h = 4$, $a = 3$ and $b = 5$

(b) find h if $A = 30$, $a = 7$ and $b = 5$

(c) find b if $A = 48$, $h = 6$ and $a = 9$.

3 Complete the following unit conversions.

Worksheet R3.3

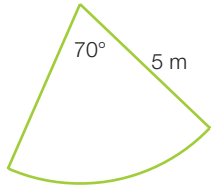
(a) 4.1 m = _____ cm **(b)** 4.25 km = _____ cm **(c)** 40 m^2 = _____ cm^2

(d) 578 cm^2 = _____ m^2 **(e)** 8.25 km^2 = _____ m^2 **(f)** 600 mL = _____ L

(g) 20 cm^3 = _____ mL **(h)** 30.8 m^3 = _____ L **(i)** 7250 cm^3 = _____ m^3

4 Find each of the following values in the sector shown, rounded to two decimal places.

Worksheet R3.4



(a) the arc length

(b) the perimeter

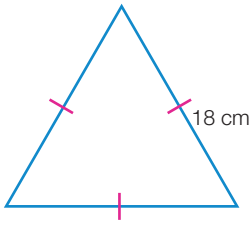
(c) the area

5 **(i)** Use Pythagoras' theorem to find the perpendicular height of each of the following shapes, rounded to two decimal places.

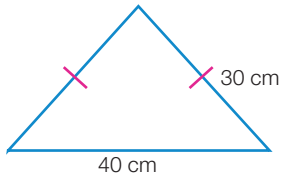
Worksheet R3.5

(ii) Use the height to calculate the area of each shape.

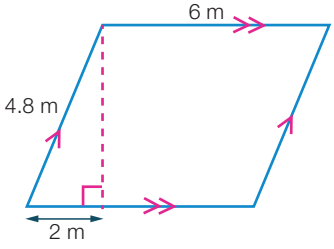
(a)



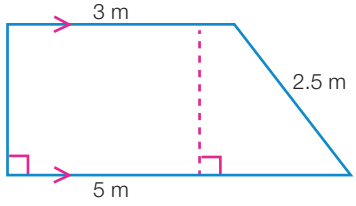
(b)



(c)



(d)



Key Words

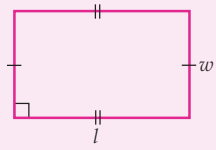
capacity	cylinder	pyramid	tapered solid
composite shape	net	sphere	uniform cross-section
cone	prism	surface area	
10A			
concentration	density		

3.1 Area

The area of circles, sectors, triangles and many different quadrilaterals can be found by using the formulas shown below.

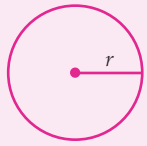
Area formulas for common shapes

Rectangle



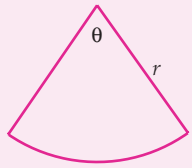
$A = lw$

Circle



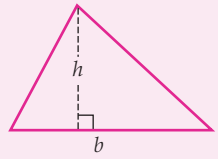
$A = \pi r^2$

Sector



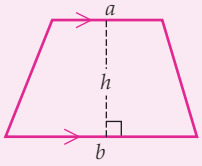
$A = \frac{\theta}{360} \times \pi r^2$

Triangle



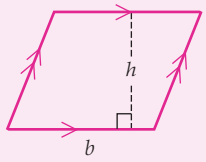
$A = \frac{1}{2}bh$

Trapezium



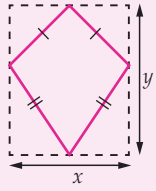
$A = \frac{1}{2}(a + b)h$

Parallelogram



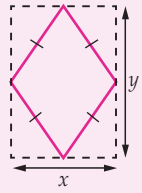
$A = bh$

Kite



$A = \frac{1}{2}xy$

Rhombus



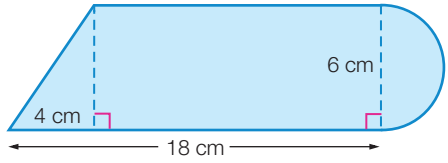
$A = \frac{1}{2}xy$

Finding the area of a **composite shape** involves adding or subtracting areas of simpler shapes.

Worked Example 1

WE1

Find the area of this shape rounded to two decimal places.



Thinking	Working
1 Divide the composite shape into smaller shapes with areas that can be calculated.	
2 Write a sum with the appropriate formula for the area of each part.	$A = \frac{1}{2}bh + lw + \frac{\pi r^2}{2}$
3 Use the information given to determine any missing measurements.	$l = 18 - 4 = 14$ cm $r = 6 \div 2 = 3$ cm
4 Substitute the measurements into each formula. Simplify the calculation where possible before using a calculator.	$A = \frac{1}{\cancel{2}_1} \times \cancel{4}^2 \times 6 + 14 \times 6 + \frac{\pi \times 3^2}{2}$ $= 2 \times 6 + 84 + \frac{9\pi}{2}$ $= 96 + \frac{9\pi}{2}$

5 Evaluate, rounding your answer to the specified number of decimal places and including the correct unit.	$= 110.14\ cm^2$ (2 d.p.)

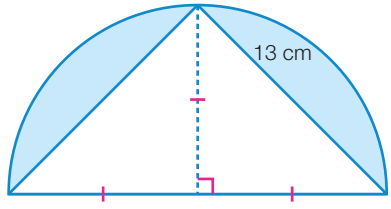
Rounding and accuracy

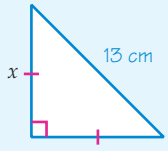
You should note in the above Worked Example that the calculation was simplified before a calculator was used. (Simplifying involves cancelling common factors and/or performing simple whole number multiplications.) The final answer that results is only rounded once, at the very end. This ensures that the most accurate final answer is obtained. Using rounded answers as you work through a problem means you accumulate, or build up, a small amount of error each time, which results in an inaccurate final answer. In order to use your calculator efficiently for these calculations, you will need to apply the correct order of operations, making use of the bracket keys (and) and the answer key ANS, which recalls the previous answer.

Worked Example 2

WE2

Find the shaded area in the diagram, rounding your answer to two decimal places. (You may need to use Pythagoras' theorem to calculate the lengths of some unknown sides.)



Thinking	Working
1 Redraw the triangle that includes the unknown side.	
2 Label the unknown side x, and use Pythagoras' theorem to find its exact value.	$x^2 + x^2 = 13^2$ $2x^2 = 13^2$ $x^2 = \frac{13^2}{2}$ $x = \sqrt{\frac{13^2}{2}}$ $x = \sqrt{84.5}\ cm$
3 Find the shaded area by subtracting the area of the unshaded part from the overall shape.	
4 Write the required area as the difference of two areas, using appropriate formulas for each part.	$A = \frac{1}{2}\pi r^2 - \frac{1}{2}bh$ $b = 2h$ $A = \frac{1}{2}\pi r^2 - \frac{1}{\cancel{2}_1} \times {}^1\cancel{2}h \times h$ $= \frac{1}{2}\pi r^2 - h^2$

5 Substitute the values into the formulas. (Here, r and h are equal to x. The value of x was determined in Step **2**.)

$$= \frac{1}{2} \times \pi \times (\sqrt{84.5})^2 - (\sqrt{84.5})^2$$

$$= \frac{84.5\pi}{2} - 84.5$$

$$= 84.5\left(\frac{\pi}{2} - 1\right)$$

6 Evaluate, rounding your answer to the specified number of decimal places and include the correct unit of area.

$$= 48.23 \text{ cm}^2 \text{ (2 d.p.)}$$

Area of annulus

An annulus is a ring shape created by two concentric circles (circles that have the same centre). To find the area, we subtract the area of the smaller circle from the area of the larger one.

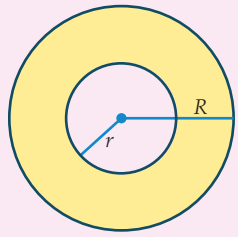
Area of an annulus

$A = \pi R^2 - \pi r^2$

$A = \pi(R^2 - r^2)$ (factorised form)

$= \pi(R + r)(R - r)$ (factorising using the Difference of Two Squares rule—a useful form to work with)

Use capital and lower case versions of a letter to indicate the same dimension in two figures that have the same shape. Here, R is used to represent the radius of the larger circle and r for the radius of the smaller circle.



3.1 Area

Navigator

Answers page 725

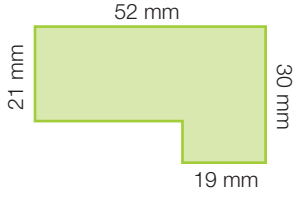
Q1 (a)–(d), Q2 Columns 1 & 2, Q3, Q5 (a) (b) (g), Q8, Q9, Q11, Q14	Q1, Q2 Columns 2 & 3, Q3, Q4, Q5 Column 1, Q6, Q7, Q8, Q9, Q11, Q13, Q14	Q1 (c)–(f), Q2 Columns 2 & 3, Q4, Q5 Column 2, Q6, Q7, Q9, Q10, Q12, Q13, Q14, Q15

Fluency

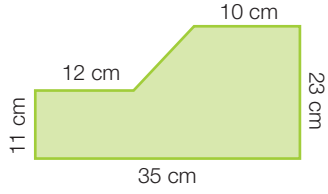
WE1

1 Find the area of each of the following shapes. Round your answer to two decimal places where necessary.

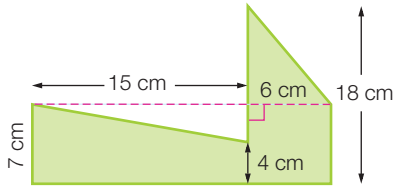
(a)



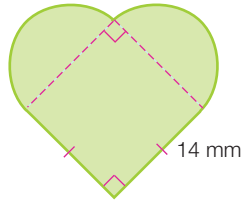
(b)



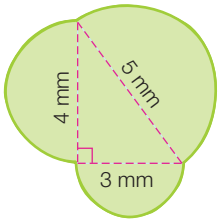
(c)



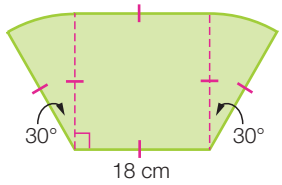
(d)



(e)

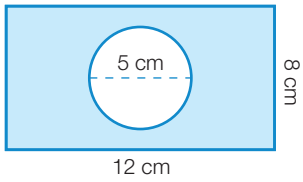


(f)

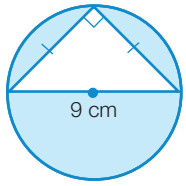


2 Find the shaded area in each of the following, rounding your answer to two decimal places. (You may need to use Pythagoras' theorem to calculate the lengths of some unknown sides.) WE2

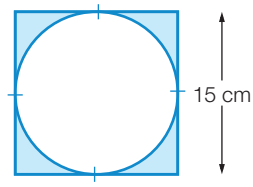
(a)



(b)



(c)



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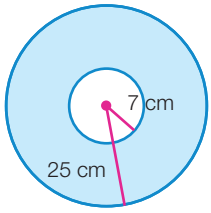
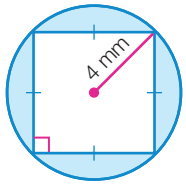
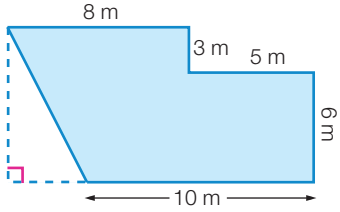


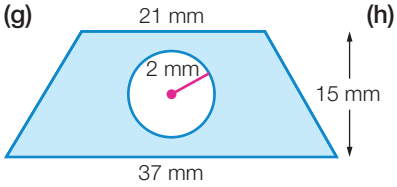
(e)



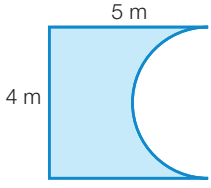
(f)



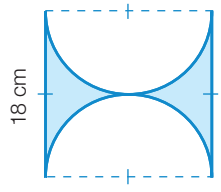
(g)



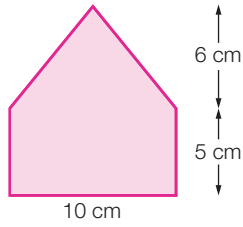
(h)



(i)



3 The area of this composite shape in cm^2 can be found by:



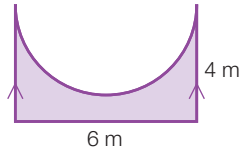
A $(10 \times 5) + (10 \times 6)$

B $(10 \times 11) + (10 \times 6)$

C $(10 \times 5) + (\frac{1}{2} \times 10 \times 6)$

D $(\frac{1}{2} \times 10 \times 5) + (\frac{1}{2} \times 10 \times 6)$

4 The area of this composite shape in m^2 can be found by:



A $(6 \times 4) + (\frac{1}{2} \times \pi \times 6^2)$

B $(6 \times 4) - (\frac{1}{2} \times \pi \times 6^2)$

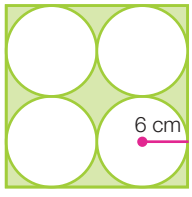
C $(6 \times 4) - (\frac{1}{2} \times \pi \times 3^2)$

D $(6 \times 4) - (\frac{1}{2} \times \pi \times 32)$

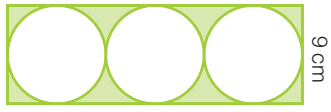
Understanding

5 Find the shaded area of each of the following, rounded to two decimal places.

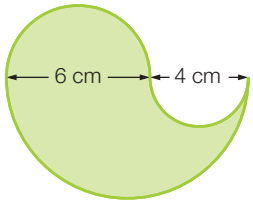
(a)



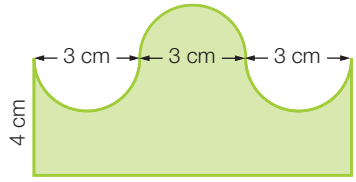
(b)



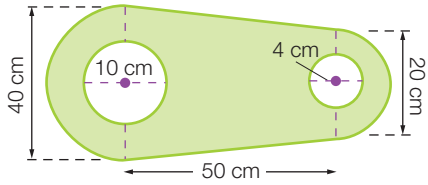
(c)



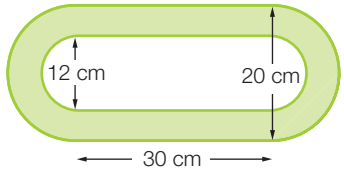
(d)



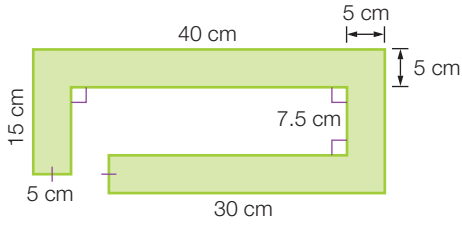
(e)



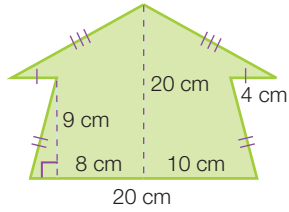
(f)



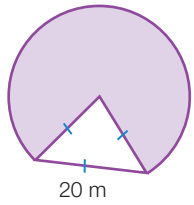
(g)



(h)



6 Find the area of both the sector and the triangle. (You will need to use Pythagoras' theorem to calculate the height of the triangle.) Write both answers correct to the nearest m^2.



Hint

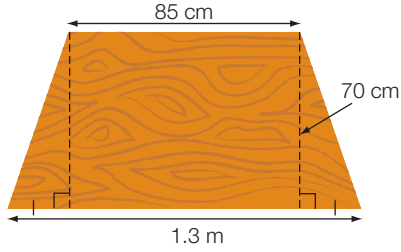
Drawing a diagram can help you visualise the problem.

7 The Department of Roads has a machine that lays bitumen road at a rate of 20 metres in length every 15 minutes. The bitumen is 6 m wide and the machine works 7 hours per day. The department is building a road 12.5 km long.

(a) How long will it take to complete the road? State your answer in both hours and days, correct to two decimal places.

(b) What will be the total area of bitumen laid by the machine?

8 The tables in Joshua's classroom are trapezoidal in shape, as shown at right. The surface of each table needs to be laminated. If there are 13 tables in the classroom, what area (m^2) of laminate is required? Round your answer to two decimal places.

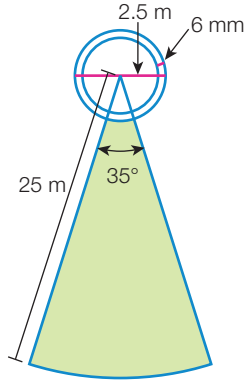


9 Elspeth has a circular swimming pool that is 7 m in diameter. She is installing a path 1 m wide all around the pool. What will be the area of this path, rounded to two decimal places?

10 The throwing area for the Olympic shot put event is shown in the diagram. The shot put is thrown from a circle that is 2.5 m in diameter to the outside of the rim. The rim is 6 mm wide (athletes cannot touch the rim when throwing). To be 'legal' a throw must land in the landing zone shown shaded in green. Calculate, correct to two decimal places:

(a) the area of the rim of the circle

(b) the area of the landing zone.



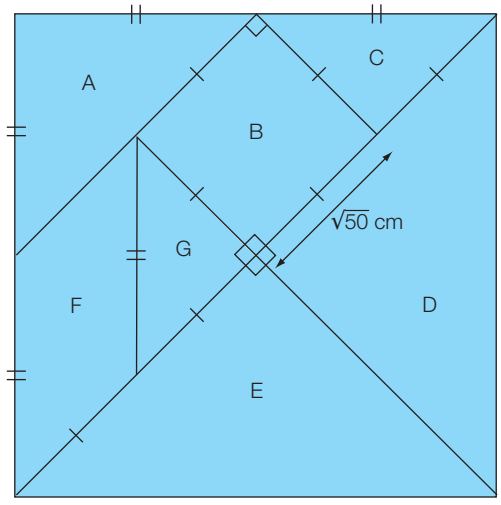
Reasoning

11 Only two rectangles with integer dimensions (side lengths are whole numbers) have the same numerical value for their area and perimeter. Find both rectangles.

12 What is the radius of a circle with an area equal to that of a square 8 m wide? Round your answer to two decimal places.

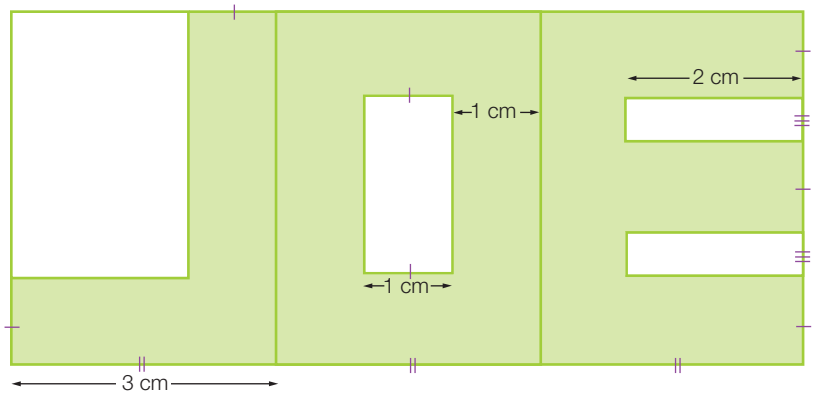
13 Find the area of each piece of the tangram puzzle shown at right.

(Remember $\sqrt{50} \times \sqrt{50} = 50$.)



Open-ended

14 **(a)** To label his school folder, Joe cuts out the letters of his name from a 9 cm by 4 cm piece of coloured paper as shown below. What area of coloured paper do the letters cover?

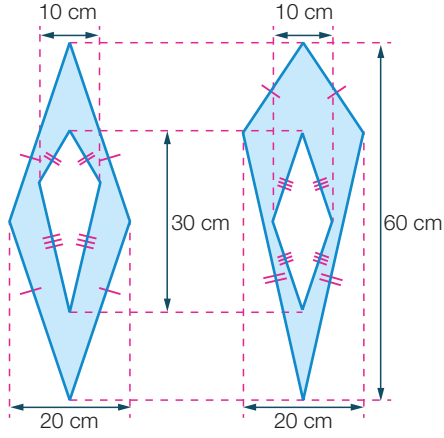


(b) Use the 9 cm by 4 cm rectangle in part **(a)** for another set of three letters. Calculate the area of paper covered by these letters. Is the area covered greater than, less than, or about the same as that covered in part **(a)**?

15 **(a)** Without doing any calculations explain why the shaded area in both shapes is equal.

(b) Draw another pair of shapes with a similar relationship to the pair shown.

(c) Draw as many such pairs of shapes as you can.



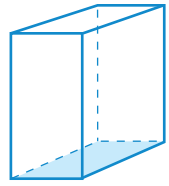
3.2 Surface area of prisms and cylinders

Prisms

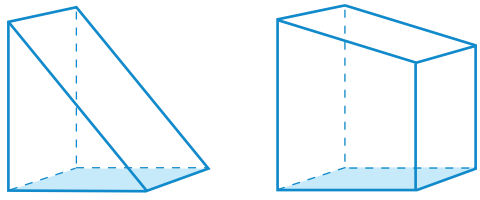
A **prism** is a solid with straight edges and a **uniform cross-section**. This means that the shape on one end (called the base) continues all the way through to the other end.

The **surface area** (SA) of a prism is the total area of all its external surfaces.

Some prisms have rectangular surfaces, such as the box shown.



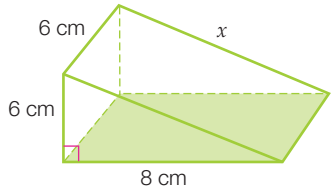
Other prisms have surfaces that are combinations of rectangles, triangles and other shapes, such as those shown here.

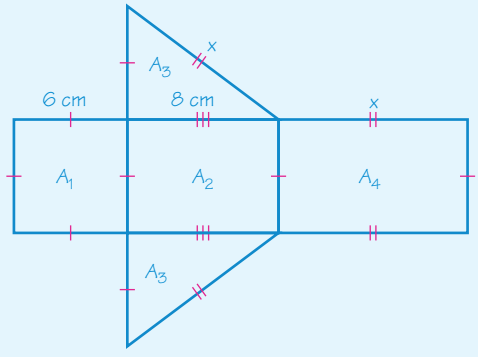


To calculate the surface area of a prism, it is often useful to draw a **net**—a 2D (flat) diagram that clearly shows each of the faces of the prism. If we imagine a prism as a box, we could create the net by cutting along the edges and laying it out flat. A net is shown in the first step of the Worked Example below.

Worked Example 3 WE3

Find the surface area of the triangular prism.



Thinking	Working
1 Draw a net of the solid that shows each of the individual faces that make up the surface. Draw each face as a flat shape. Label each of the different areas A_1, A_2, A_3 … etc. Label identical faces in the same way.	

2 Use Pythagoras' theorem to find the required length.	$x^2 = 6^2 + 8^2$ $= 100$ $x = 10$ cm
3 Write the surface area as the sum of the areas of each face. Replace the labels (A_1, A_2 etc.) with the area formulas for each shape.	$SA = A_1 + A_2 + 2 \times A_3 + A_4$ $= l^2 + lw + \cancel{2}^{1} \times \frac{bh}{\cancel{2}_{1}} + LW$ $= l^2 + lw + bh + LW$
4 Substitute values and evaluate individual areas. Check that values are reasonable.	$= 6^2 + 8 \times 6 + 8 \times 6 + 10 \times 6$ $= 36 + 48 + 48 + 60$
5 Write your answer with the correct units of area.	$= 192$ cm^2

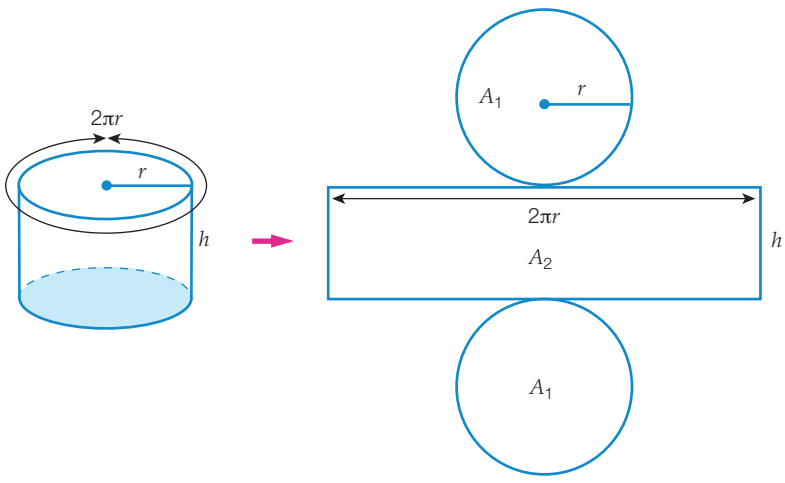
When we have two of the same type of shape, we can use upper and lower case letters to distinguish between their dimensions. In the above Worked Example, the area of the smaller rectangle was written as *lw*, and the area of the larger rectangle as *LW*.

Cylinders

A **cylinder** also has a uniform cross-section, but it is not strictly a prism. The surface area of cylinders can also be found by considering their nets.

The surface area of a cylinder is the sum of the areas of the two circular faces at each end and the area of the curved surface. The area of each circular face is $A = \pi r^2$.

When flattened, the curved surface is a rectangle with its length equal to the circumference of the circles on the ends: $C = 2\pi r$. Its width is equal to the height of the cylinder, h. The area of the curved surface is $A = 2\pi r \times h$.

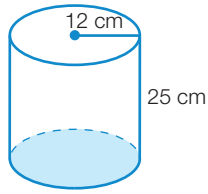


$$\begin{aligned} SA &= 2 \times A_1 + A_2 \\ &= 2 \times \pi r^2 + 2\pi r \times h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r(r + h) \qquad \text{(factorised form)} \end{aligned}$$

Worked Example 4

WE4

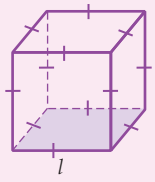
Find the surface area of this cylinder, rounded to two decimal places.



Thinking	Working
1 The surface area of a cylinder is the combined area of the two circular ends and the curved middle section. Write down the formula that shows this.	$SA = 2\pi r^2 + 2\pi rh$
2 State values for r and h.	$r = 12$ cm $\quad h = 25$ cm
3 Substitute r and h into the formula and simplify the calculation.	$SA = 2 \times \pi \times 12^2 + 2 \times \pi \times 12 \times 25$ $= 288\pi + 600\pi$ $= 888\pi$
4 Evaluate, rounding the answer to two decimal places.	$= 2789.73$ cm^2 (2 d.p.)

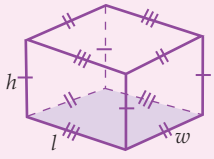
Formulas for the surface area of prisms and cylinders

Cube



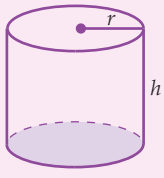
SA = area of 6 squares
$= 6l^2$

Rectangular prism



SA = area of 6 rectangles
$= 2lw + 2wh + 2lh$
$= 2(lw + wh + lh)$ (factorised form)

Cylinder



SA = area of 2 circles and area of curved surface
$= 2\pi r^2 + 2\pi rh$
$= 2\pi r(r + h)$ (factorised form)

3.2 Surface area of prisms and cylinders

Navigator

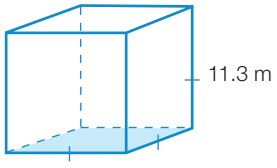
Q1, Q2, Q4, Q5, Q7	Q1 Column 2, Q2, Q3, Q4, Q5, Q6, Q7	Q1 Column 2, Q2, Q3, Q5, Q6, Q7, Q8

Answers page 726

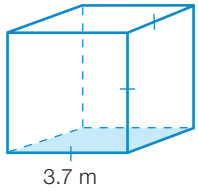
Fluency

1 Find the surface area of each solid, rounded to two decimal places where necessary. WE3

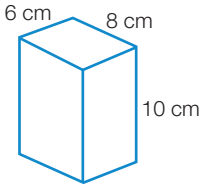
(a)



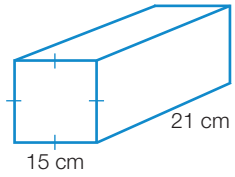
(b)



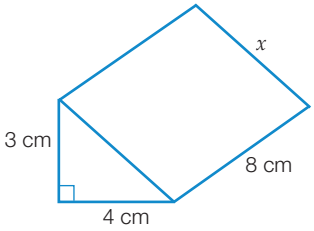
(c)



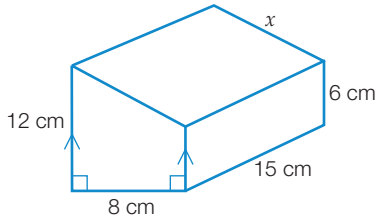
(d)



(e)

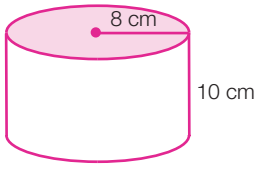


(f)

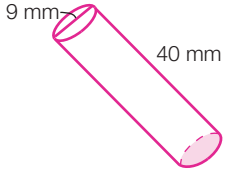


2 Find the surface area of each cylinder, rounded to two decimal places. WE4

(a)

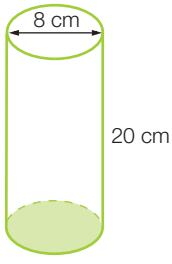


(b)



3 The calculation that gives the surface area of each of the following solids is:

(a)



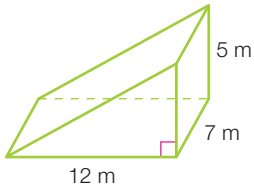
A $2 \times \pi \times 8(8 + 20)$

B $2 \times \pi \times 4(8 + 20)$

C $2 \times \pi \times 8(4 + 20)$

D $2 \times \pi \times 4(4 + 20)$

(b)



A $5 \times 12 + 7(5 + 12 + 17)$

B $\frac{5 \times 12}{2} + 7(5 + 13)$

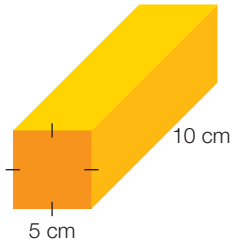
C $\frac{5 \times 12}{2} + 7(5 + 12 + 13)$

D $5 \times 12 + 7(5 + 12 + 13)$

Understanding

4 Amanda has made a set of wooden building blocks as a present for her little brother. She built ten of each type of block and now she has to paint them. What is the total area that she has to paint?

Block A



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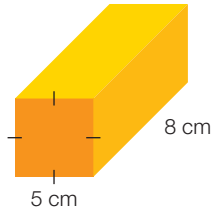

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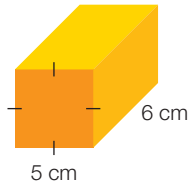
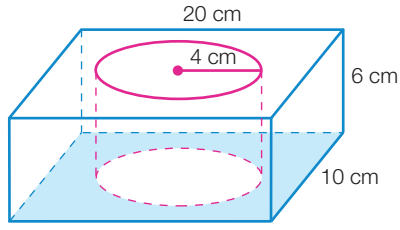
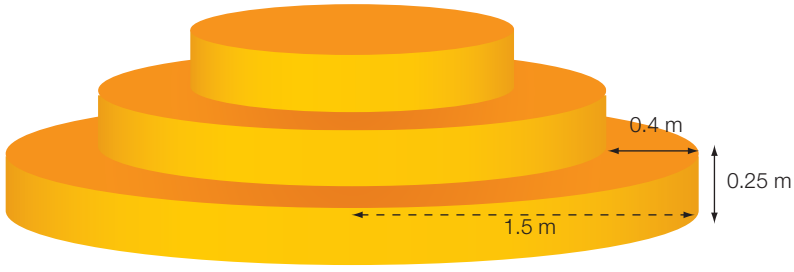


5 The diagram shows a plastic casing that has been specially made to surround a water pipe. Find the surface area of the casing, rounded to two decimal places.



Reasoning

6 A circular podium has three steps as shown. The base of the podium has a radius of 1.5 m and each step has a width of 0.4 m. Each step is 0.25 m higher than the previous one. Calculate, correct to two decimal places, the amount of carpet needed to cover the visible surfaces of the podium.



Open-ended

7 **(a)** Write the formula for the surface area of a rectangular prism.

(b) By how many times does the surface area of the rectangular prism increase if all three of its dimensions are doubled? Demonstrate your answer with two suitable prisms, one of which has its length, width and height double that of the other one.

8 **(a)** Choose two variables to represent the radius and height of a cylinder. Find the surface area of this cylinder. Leave your answer in terms of π.

(b) Double the height of the cylinder and find the new surface area, leaving your answer in terms of π.

(c) Double the radius of the original cylinder and find the new surface area, leaving your answer in terms of π.

(d) Explain why your answers to **(b)** and **(c)** are not the same.

Investigation

Prisms unwrapped

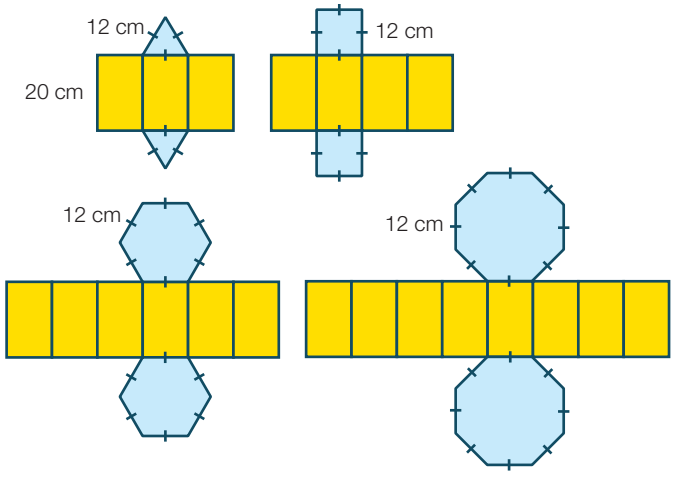
Equipment required: 1 brain, a ruler, maths shape template such as a Mathomat® (optional)

The Big Question

Can we develop a general formula for the surface area of any prism?

Engage

The diagrams below show the nets of some well-known prisms. The base of each prism is a regular polygon (recall that 'regular' means all sides and angles are equal).



1 Sketch the shape of each prism. One way to do this is to draw two of the base shapes, one behind and slightly above the other. Make sure each base is oriented the same way. Then, join the matching vertices with straight lines. Name each prism (e.g. the first is an equilateral triangle-based prism).

2 For each net diagram, what is the shape of the middle section (shown in yellow)?

Explore

3 If each of the prisms above has a length of 20 cm and the side length of each base is 12 cm:

(a) find the area of the middle section of each net

(b) how can you find this area by doing only one simple calculation?

4 (a) To help answer the Big Question, draw nets of prisms of height 20 cm and bases as below:

(i) A right-angled triangle with side lengths of 6, 8 and 10 cm.

(ii) A rectangle with dimensions of 11 cm × 15 cm.

(b) (i) What is the same about the dimensions of the nets you have drawn and the nets shown on this page?

(ii) What is different about the dimensions of the nets you have drawn and the nets on this page?

Strategy options

- Draw a diagram.
- Look for a pattern.

Explain

5 Find the lengths of the 'middle section' rectangle for each net drawn. How is the length of this rectangle found from the dimensions of the base?

Elaborate

6 Copy and complete:

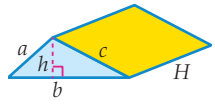
(a) The surface area of any prism can be found as two times the area of _____ + the area of a rectangle whose length is the _____ of the base and whose width is the _____ of the prism.

(b) Write the above statement as a formula. Make sure you define each variable.

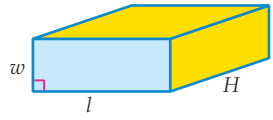
7 State your answer to the Big Question. Describe any special cases (such as when the base of the prism is a regular polygon).

8 Write a formula for the perimeter and area of the base for each of the following prisms. Hence, write a formula for the surface areas.

(a)



(b)



Evaluate

9 Do you think you will use what you have found in this investigation as a shortcut for surface area calculations? Why or why not?

Extend

10 A polyhedron is a solid with faces that are all polygons. Five special polyhedra, know as the 'Platonic solids' have faces that consist only of one type of regular polygon. They are the tetrahedron, cube, octahedron, dodecahedron and icosahedron. Find out what these solids look like and write a formula for their surface area.

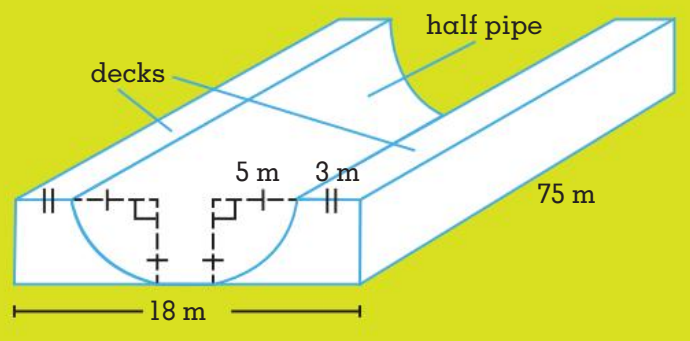
Maths 4 Real

SNOWBOA

Equipment required: calculator

Snowboarding terrain parks are similar to skateboard parks except that the surfaces are covered in hard-packed snow instead of concrete.

A group of professional snowboarders are designing their ideal terrain park. Here is their design for a 'half pipe' (not drawn to scale).

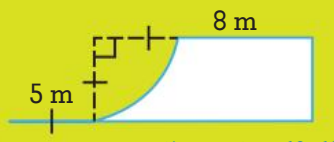


1 Find the area of the surface of the halfpipe that needs to be covered by snow (i.e. the total area of the two decks and the surface in between). Write your answer to the nearest m^2.

Each person in the group is given a 75 m by 18 m rectangle of horizontal space on which to build a feature of their own design. The cross-section of each feature is shown. Each feature is 75 m long.

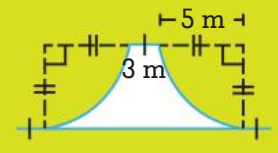
Quarter pipe

Riders 'air' from it and perform tricks while airborne



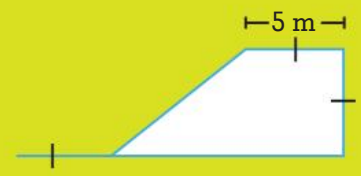
Spines

Two quarter pipes back-to-back



Flat bank

An angled wall to traverse



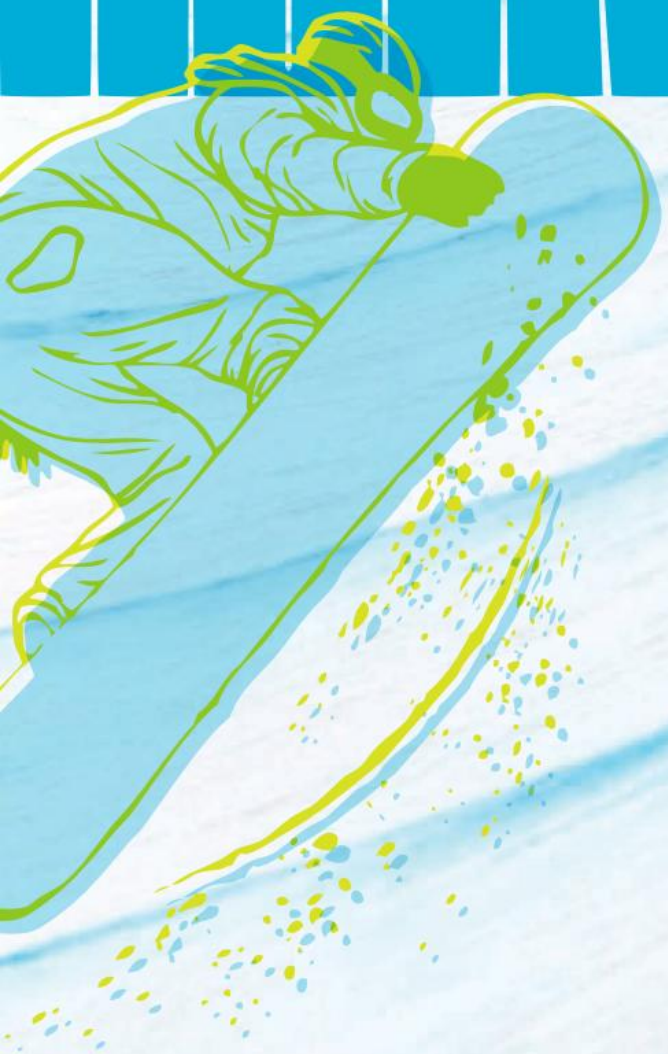
2 Calculate the area of the surface that needs to be covered in snow for each of the three features. (You will need to calculate dimensions that are not shown, which may require techniques such as Pythagoras' theorem.) Write each answer to the nearest m^2.

Research

3 Snowboarders that are keen to practise can purchase terrain kits to build a feature in their own backyards. Some snowboarders design and build their own features from recycled materials. Design a feature that would fit in either your backyard or a space in your local park or school oval. Find the area of surface that would be covered in snow for your feature.

Torah Jane Bright is a champion Australian snowboarder. Born in New South Wales, she turned pro at age 14 and in 2010 won a gold medal at the Winter Olympics in Vancouver, Canada. Torah is a master of the 'half pipe' —a half cylinder with two horizontal platforms (or 'decks').

RIDING TERRAIN PARK



3.3 Volume of prisms and cylinders

Prisms

Prisms and cylinders have a uniform cross-section. Because of this, the volume of these solids is found by multiplying the area of the cross-sectional shape on the end (the base) by the height of the solid, where the height is perpendicular to the base.

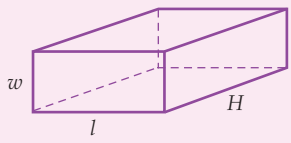
Note that the base is not necessarily the face that the prism is sitting on. The base is whichever face has the uniform cross-section.

The volume of solids with constant cross-section is:

$$V = AH$$

where A is the area of the cross-section
H is the perpendicular height of the solid.

Rectangular prism

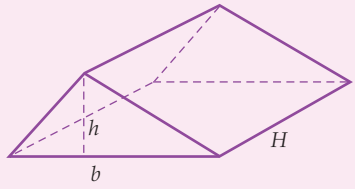


$V = AH$

$A = lw$

$V = lwH$

Triangular prism

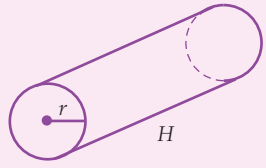


$V = AH$

$A = \frac{1}{2}bh$

$V = \frac{1}{2}bhH$

Cylinder



$V = AH$

$A = \pi r^2$

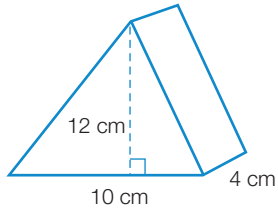
$V = \pi r^2 H$

In the formula for the triangular prism above we used the lower case h for the height of the triangular-shaped base, and the capital H for the height of the prism.

Worked Example 5

WE 5

Find the volume of the triangular prism.



Thinking	Working
1 State the general formula.	$V = AH$
2 Substitute the formula for the area of the base.	$= \frac{1}{2}bhH$
3 Substitute the values.	$= \frac{1}{2} \times 10 \times 12 \times 4$
4 Write the answer with the correct unit of volume.	$= 240\ cm^3$

Capacity

The **capacity** of a container refers to the volume of a substance (usually liquid or gas) that the container can hold.

The unit used to measure capacity is the litre (L).

1000 millilitres (mL) = 1 litre (L)

1000 litres = 1 kilolitre (kL)

$1\ cm^3 = 1\ mL$

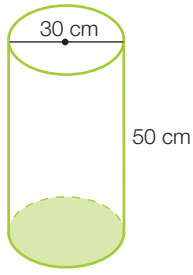
$1000\ cm^3 = 1\ L$

$1\ m^3 = 1000\ L = 1\ kL$

Worked Example 6

WE 6

What is the capacity of this cylindrical tank? Answer in litres, rounded to two decimal places.



Thinking	Working
1 Calculate the volume of the cylinder.	$V = A \times H$ $= \pi r^2 H$ $r = \frac{30}{2}$ $= 15\ cm$ $V = \pi \times 15^2 \times 50$ $= 11\,250\pi$ $= 35\,342.92\ cm^3$ (2 d.p.)
2 Convert the volume to a capacity ($1000\ cm^3 = 1\ L$) and round your answer to two decimal places.	Capacity = 35.34 L (2 d.p.)

3.3 Volume of prisms and cylinders

Navigator

Answers page 726

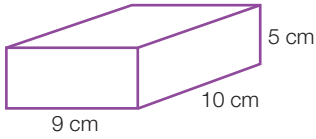
Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Fluency

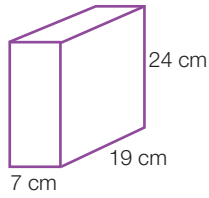
WE5

1 Find the volume of each solid with constant cross-section. Round to two decimal places, if necessary.

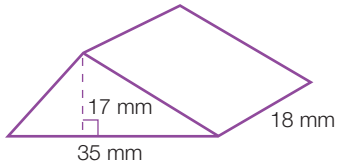
(a)



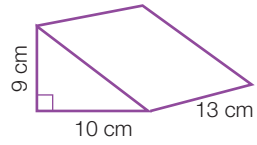
(b)



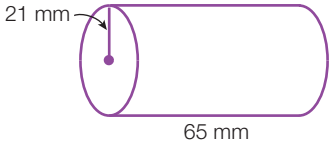
(c)



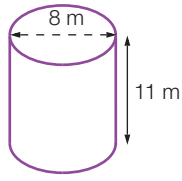
(d)



(e)

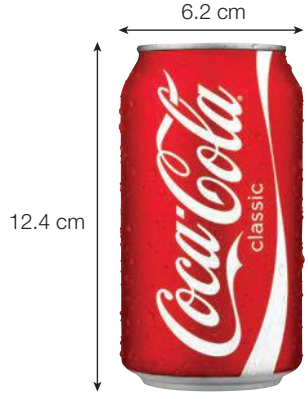


(f)



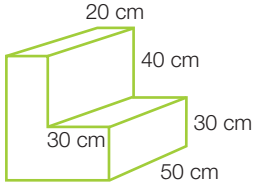
WE6

2 Find the capacity of the soft drink can in mL, rounded to two decimal places. Assume that the can is a regular cylinder.

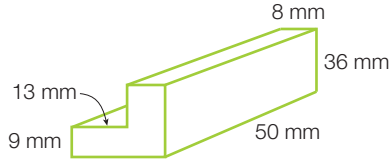


3 Find the volume of each solid with constant cross-section. Round to two decimal places, if necessary.

(a)



(b)



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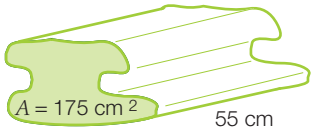
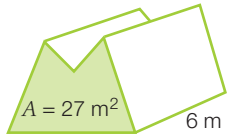
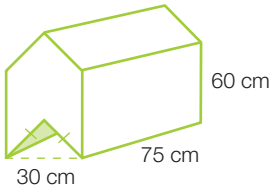


(d)



(e)



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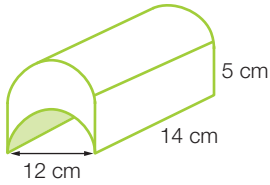
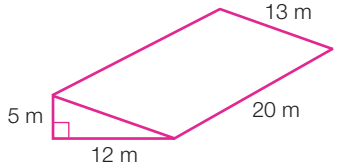


Understanding

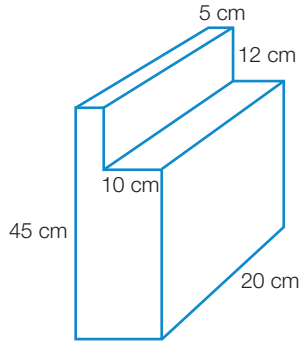
4 **(a)** The capacity of this prism in kL can be found by:



A $12 \times 5 \times 20$ **B** $\frac{1}{2} \times 5 \times 12 \times 20$

C $(\frac{1}{2} \times 5 \times 12) + (20 \times 12)$ **D** $\frac{1}{2} \times 12 \times 20 \times 5 \times 13$

(b) The capacity of this prism in mL can be found by:



A $5 \times 12 \times 20 + 10 \times (45 - 12) \times 20$

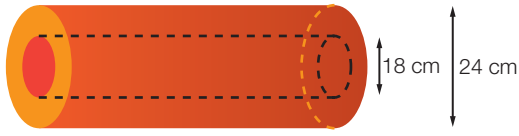
B $5 \times 12 \times 20 + 10 \times 45 \times 20$

C $5 \times 45 \times 20 + 10 \times 12 \times 20$

D $(5 + 10) \times 45 \times 20 - 10 \times 12 \times 20$

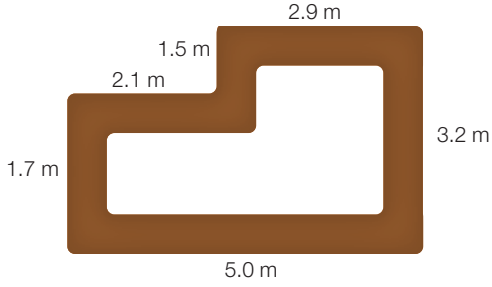
5 Ingrid needs 25 concrete columns for her new verandah. Each column will be cylindrical in shape with a base diameter of 28 cm and a height of 85 cm. What is the total volume of concrete required? Give your answer in cubic metres, correct to two decimal places. ($1\ m^3 = 1\,000\,000\ cm^3$)

6 A 3.5 m long plastic pipe is to be used for drainage. If it has an outer diameter of 24 cm and an inner diameter of 18 cm, find the volume of plastic needed to construct it. Express your answer in m^3 correct to two decimal places.

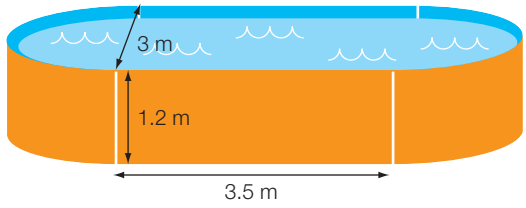


Reasoning

7 Gael is building an extension onto her house. She has just finished digging a series of trenches. All the trenches are 85 cm deep and 55 cm wide. The plan for the trenches is shown in the diagram. What was the volume of dirt removed? Round your answer to two decimal places.



8 Terry's swimming pool is rectangular with two semicircular ends. How many litres of water does it hold when full? State your answer to the nearest whole number.



Open-ended

9 A rectangular box has a volume of 200 cm^3.

(a) Find three possible whole number sets of dimensions that give the correct volume.

(b) Find three sets of dimensions with at least one non-whole number that give the correct volume.

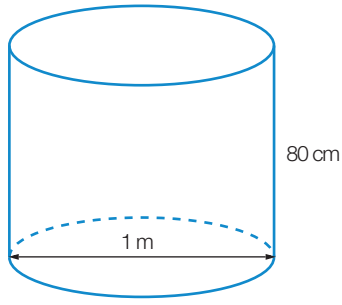
(c) Calculate the surface area of the box for each of the three sets of dimensions found in **(a)**.

10 Anita is attempting to find the volume of the cylinder in cm^3 correct to two decimal places.

(a) Identify the two mistakes in the working shown.

(b) Find the correct answer.

(c) Give Anita two points to write in her notes so she can avoid similar mistakes in the future.



$$\begin{aligned} V &= AH \\ &= \pi r^2 H \\ &= \pi \times 1^2 \times 80 \\ &\approx 251.33 \text{ cm}^3 \end{aligned}$$

Outside the Square — Problem solving

Cola cost crisis

Fizzy Cola comes in cylindrical cans of diameter 5 cm and height 15 cm. The cost of one can is \$2.35. The manufacturers have created a promotional, 'super-sized' can, which is the same height but twice as wide. The sales manager of Fizzy Cola is wondering at what price she should sell the promotional can. Her assistant thinks that the new can is 'twice as big' as the original can, so it must hold 'twice as much' cola and should cost 'twice the price'.

Strategy options

- Break problem into manageable parts.
- Draw a diagram.
- Have I seen a similar problem?

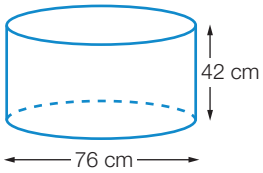
1 Suggest a price for the 'super-sized' can based on a comparision of its volume with the original can.

2 Is the assistant's reasoning correct? Why or why not?

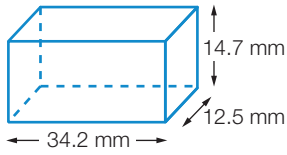
Half-time 3

1 Find the total surface area of the following solids. Round your answers to the nearest whole number. **Ex.3.2**

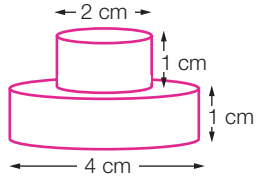
(a)



(b)

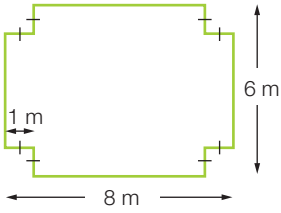


2 Find the exact surface area and exact volume of this composite shape. ('Exact' means writing your answer in terms of π.) **Ex.3.2, 3.3**

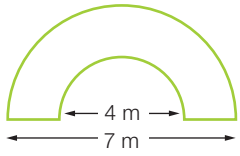


3 Find the area of the following shapes, rounded to two decimal places, if necessary. **Ex.3.1**

(a)

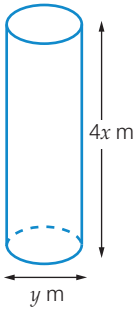


(b)

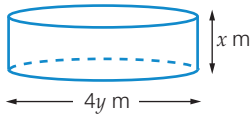


4 Which of the following solids, **A**, **B** or **C**, has the greatest volume? Justify your answer by evaluating the exact volume of each one. **Ex.3.3**

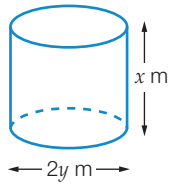
A



B

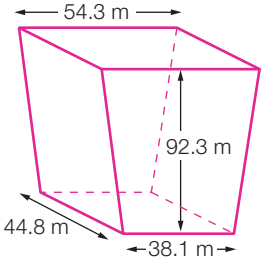


C

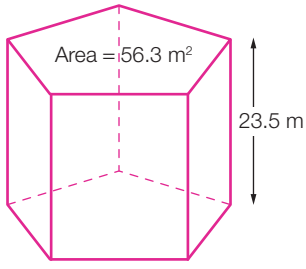


5 Find the volume of the following solids, rounded to two decimal places. **Ex.3.3**

(a)



(b)



6 A cylindrical tank at a milk processing plant has a diameter of 2.6 m and a height of 3.4 m. How many 250 mL cartons could be completely filled from one tank full of milk? **Ex.3.3**

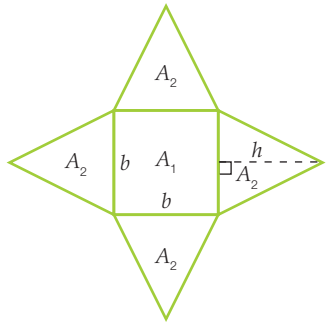
3.4 Surface area of tapered solids and spheres

10A

Tapered solids do not have a uniform cross-section. The cross-section gets smaller as you move from the base of the solid to the top. The shape of the cross-section is always a similar shape to the base—the shape is the same but the size decreases. The tapered solids that we will consider here are **pyramids** and **cones**.

Pyramids

A pyramid is made up of a base and triangular sides that meet at a point, called the vertex. The number of triangles depends on the number of sides of the base shape. A pyramid with a square base has four triangles, whereas a pyramid with a pentagonal base has five triangles, and so on. The net of a square-based pyramid is shown here. The surface area is found by adding the individual areas of the base and sides. For a square based pyramid:

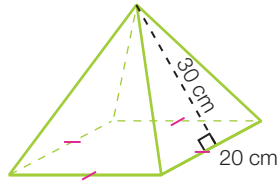


$$SA = A_1 + 4 \times A_2$$
$$= b^2 + \cancel{4}^{2} \times \frac{bh}{\cancel{2}_{1}}$$
$$= b^2 + 2bh$$

Worked Example 7

WE7

Find the surface area of this square-based pyramid.



Thinking	Working
1 The surface area of a square-based pyramid is the combined area of the base and the four triangular faces.	$SA = b^2 + 2bh$
2 State the values of b and h.	$b = 20$ cm $h = 30$ cm
3 Substitute b and h into the formula and evaluate.	$SA = 20^2 + 2 \times 20 \times 30$ $= 400 + 1200$
4 Write the answer.	$= 1600$ cm^2

Cones

The surface area of a cone is the sum of the area of the circular base of radius, r, and the area of the sloping curved surface. The distance along this sloping surface from the base to the vertex is called the slant height, s.

When flattened, the curved surface is a sector of a circle that wraps around the base. This sector has a radius of s and an arc length of $2\pi r$ (i.e. the circumference of the circular base).

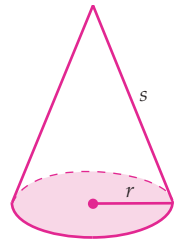
To find the area of the curved surface, we first write the arc length in terms of r and s.

$$\text{Arc length} = \frac{\theta}{360} \times 2\pi \times s$$

$$\cancel{2\pi} r = \frac{\theta}{360} \times \cancel{2\pi} \times s \qquad \text{(Arc length is equal to circumference)}$$

$$r = \frac{\theta}{360} \times s$$

$$\frac{r}{s} = \frac{\theta}{360}$$

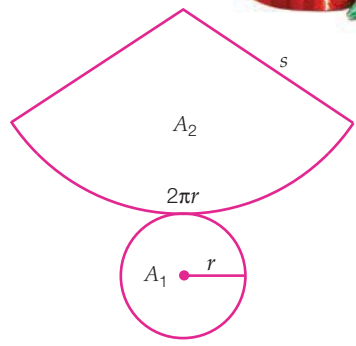


We use this expression in the formula for the area of the sector:

$$\text{Area} = \frac{\theta}{360} \times \pi s^2 \qquad (s = \text{radius of sector})$$

$$= \frac{r}{\cancel{s}} \times \pi \times s^{\cancel{2}}$$

$$A = \pi r s$$



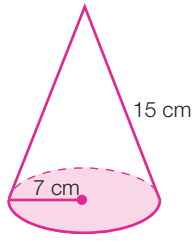
The area of the curved surface is $A = \pi r s$.

So
$$SA = A_1 + A_2$$
$$= \pi r^2 + \pi r s$$
$$= \pi r(r + s) \qquad \text{(factorised form)}$$

Worked Example 8

WE8

Find the surface area of this cone, rounded to two decimal places.

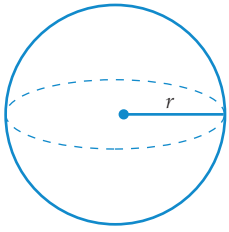


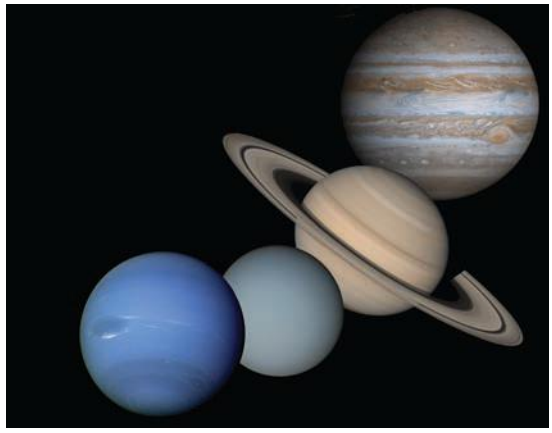
Thinking	Working
1 Write the formula for surface area.	$SA = \pi r^2 + \pi r s$
2 State the values of r and s.	$r = 7$ cm $s = 15$ cm
3 Substitute for r and s into the formula and simplify.	$SA = \pi \times 7^2 + \pi \times 7 \times 15$ $= 49\pi + 105\pi$ $= 154\pi$
4 Evaluate, rounding the answer to two decimal places.	$= 483.81$ cm^2 (2 d.p.)

Sphere

A **sphere** has only one single curved surface. The surface area of a sphere is equivalent to four times the area of a circle of the same radius.

$SA = 4\pi r^2$

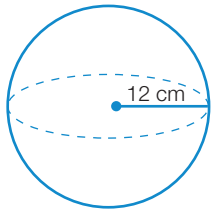




Worked Example 9

WE9

Find the surface area of this sphere, rounded to two decimal places.



Thinking	Working
1 Write the formula for surface area.	$SA = 4\pi r^2$
2 Substitute for r and simplify.	$= 4 \times \pi \times 12^2$ $= 576\pi$
3 Evaluate, rounding the answer to two decimal places.	$= 1809.56\ cm^2$ (2 d.p.)

Composite solids

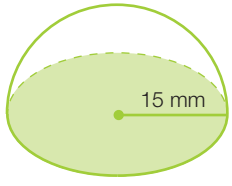
The surface area of any solid is the total area of the individual surfaces—curved or otherwise.

Worked Example 10

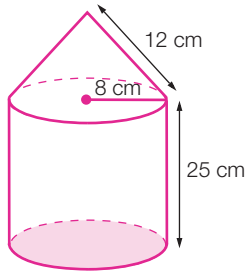
WE10

Find the surface area of each of the solids, correct to two decimal places.

(a)



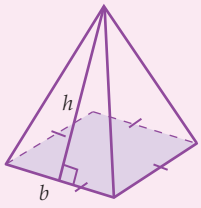
(b)



Thinking		Working
(a) 1	Identify the individual areas. (A hemisphere is half a sphere.)	(a) $SA = A_{\text{hemisphere}} + A_{\text{circle}}$
2	Write a formula for the sum of the areas. Simplify if possible.	$= \frac{1}{\cancel{2}_1} \times {}^{2}\cancel{4}\pi r^2 + \pi r^2$ $= 2\pi r^2 + \pi r^2$ $= 3\pi r^2$
3	Substitute for r.	$= 3 \times \pi \times 15^2$ $= 675\pi$
4	Evaluate, rounding the answer to two decimal places.	$= 2120.58 \text{ mm}^2$ (2 d.p.)
(b) 1	Identify the individual areas.	(b) $SA = A_{\text{cone (curved)}} + A_{\text{cylinder (curved)}} + A_{\text{circle}}$
2	Write a formula for the sum of the areas.	$= \pi rs + 2\pi rh + \pi r^2$
3	Substitute for r, s and h.	$= \pi \times 8 \times 12 + 2 \times \pi \times 8 \times 25 + \pi \times 8^2$
4	Simplify by finding each area in terms of π.	$= 96\pi + 400\pi + 64\pi$ $= 560\pi$
5	Evaluate, rounding the answer to two decimal places.	$= 1759.29 \text{ cm}^2$ (2 d.p.)

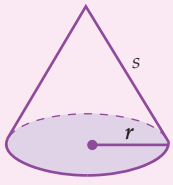
Formulas for the surface area of tapered solids and spheres

Square-based pyramid



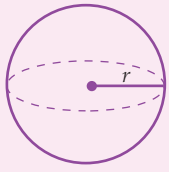
SA = area of square base and area of 4 triangles
$= b^2 + 4 \times \frac{1}{2}bh$
$= b^2 + 2bh$

Cone



SA = area of circle and area of curved surface
$= \pi r^2 + \pi rs$
$= \pi r(r + s)$ (factorised form)

Sphere



SA = area of curved surface
$= 4\pi r^2$

3.4 Surface area of tapered solids and spheres

10A

Navigator

Answers page 726

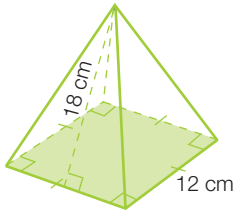
Q1, Q2, Q3, Q4 Column 1, Q6, Q7, Q9, Q11, Q12	Q1, Q2, Q3, Q4 Column 2, Q5, Q6, Q7, Q8, Q10, Q11, Q12	Q1 (b), Q2 (b), Q3 (b), Q4, Q5, Q7, Q8, Q9, Q10, Q12, Q13, Q14

Fluency

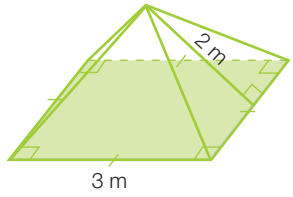
WE7

1 Find the surface area of each pyramid, rounded to the nearest whole number.

(a)



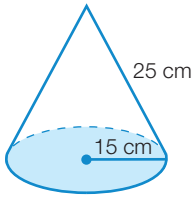
(b)



WE8

2 Find the surface area of each cone, rounded to two decimal places.

(a)



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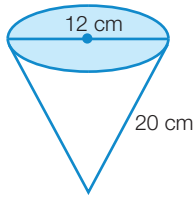
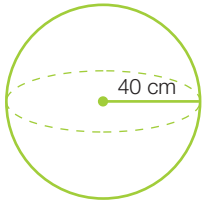


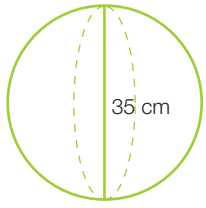
WE9

3 Find the surface area of each sphere, rounded to two decimal places.

(a)



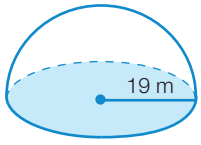
(b)



WE10

4 Find the surface area of each solid, rounded to two decimal places.

(a)



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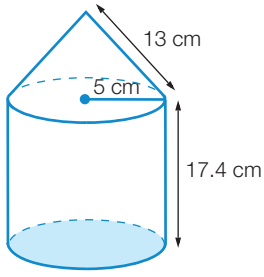
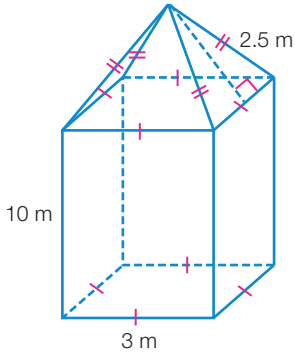
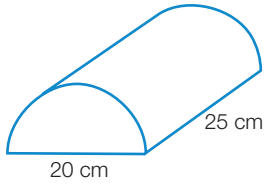


(c)



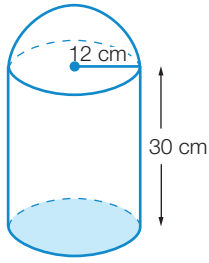
(d)



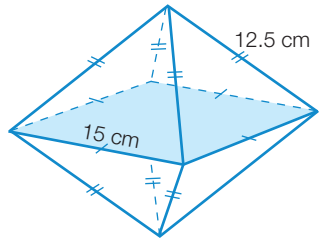
Hint

Use Pythagoras' theorem to calculate the heights of the triangles.

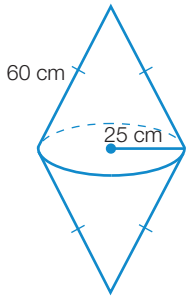
(e)



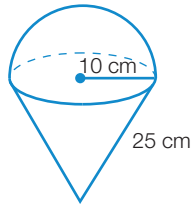
(f)



(g)

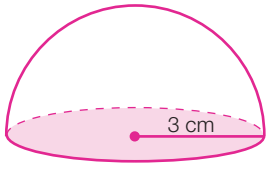


(h)



5 The calculation that gives the surface area of each of the following solids is:

(a)



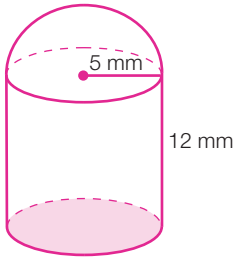
A $4 \times \pi \times 3^2$

B $\frac{1}{2} \times 4 \times \pi \times 3^2$

C $\frac{1}{2} \times 4 \times \pi \times 3^2 + \pi \times 3^2$

D $4 \times \pi \times 3^2 + \pi \times 3^2$

(b)



A $4 \times \pi \times 12^2 + 2 \times \pi \times 5 \times 12 + \pi \times 12^2$

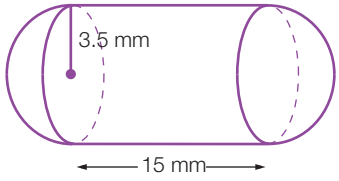
B $2 \times \pi \times 5^2 + \pi \times 5 \times 12 + \pi \times 5^2$

C $4 \times \pi \times 5^2 + 2 \times \pi \times 5 \times 12 + 2 \times \pi \times 5^2$

D $\frac{1}{2} \times 4 \times \pi \times 5^2 + 2 \times \pi \times 5 \times 12 + \pi \times 5^2$

Understanding

6 Find, correct to two decimal places, the surface area of a capsule that has the dimensions shown below.



7 A conical party hat has a vertical height of 18 cm. The circumference of the circular base is 43 cm. Calculate the area of cardboard required to make the hat (assume no overlap). Round your answer to the nearest whole number.

Hint

Determine the radius, then the slant height.

Hint
Draw a labelled diagram.

8 A pyramid in Egypt has a square base that covers an area of 44 100 m^2 and a height perpendicular to the base of 130 m. Calculate the total surface area of the four triangular faces of the pyramid, to the nearest square metre.

Reasoning

9 Marcus has a new tenpin bowling ball and he wants to sew together a protective bag for it. He has a rectangular piece of material 80 cm long and 40 cm wide. The bowling ball has a diameter of 30 cm.

(a) Show that the area of the material is sufficient to cover the surface area of the bowling ball.

(b) Even though Marcus has a sufficient area of material, explain why he should consider using a longer piece of material.

10 (a) Calculate the slant height of a cone of radius 25 cm and surface area 4398.23 cm^2, to the nearest centimetre.

(b) Calculate the radius of a cone with a surface area of 1809.56 cm^2 if the slant height is three times the radius.

Open-ended

11 Find two possible sets of dimensions (radius and slant height) for a cone with a surface area between 200 cm^2 and 300 cm^2.

12 Explain why a cone cannot have a radius and a slant height with the same length.

13 A cone has its height doubled, but its radius stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its total surface area.

(b) Evaluate its total surface area after its height is doubled.

(c) Choose another two values to represent another cone and evaluate its total surface area.

(d) Double the height of this cone and evaluate the new total surface area.

(e) Does doubling the height double the surface area?

14 A cone has its radius doubled, but its height stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its total surface area.

(b) Evaluate its total surface area after its radius is doubled.

(c) Choose another two values to represent another cone and evaluate its total surface area.

(d) Double the radius of this cone and evaluate the new total surface area.

(e) Does doubling the radius double the surface area?

Outside the Square — Problem solving

Egyptian engineering

Egypt 2020 BC. Pharaoh Mentuhotep is constructing two massive stone obelisks on either side of the entrance to a temple. The obelisks are two tall columns that have four trapezoidal faces—they are narrower at the top than at the bottom. Each of the these faces will be covered in hieroglyphics.

At the top of the obelisk sits a small pyramid-shaped cap, called a 'pyramidion'. The four exposed faces of the pyramidion will be covered in gold, to reflect the Sun.

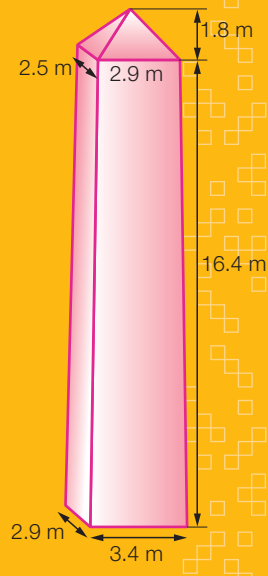
Mentuhotep has given you, his chief engineer, a labelled drawing of what the obelisk should look like. Your job is to determine the area to be covered in:

(a) gold

(b) hieroglyphics.

Strategy options

- Break problem into manageable parts.
- Draw a diagram.
- Have I seen a similar problem?



3.5 Volume of tapered solids and spheres

10A

Tapered solids

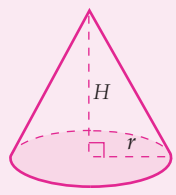
Cones and pyramids take up exactly one-third of the space of a cylinder or prism with corresponding base and height.

The volume of a tapered solid is:

$$V = \frac{1}{3}AH$$

where A is the area of the cross-section at the base
H is the perpendicular height of the solid.

Cone

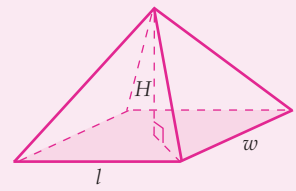


$$V = \frac{1}{3}AH$$

$$A = \pi r^2$$

$$V = \frac{1}{3}\pi r^2 H$$

Rectangular-based pyramid



$$V = \frac{1}{3}AH$$

$$A = lw$$

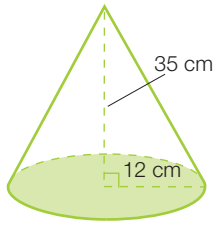
$$V = \frac{1}{3}lwH$$

Worked Example 11

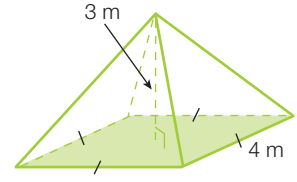
WE11

Find the volume of each of the tapered solids, rounded to two decimal places if necessary.

(a)



(b)

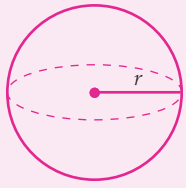


Thinking	Working
(a) 1 Write the general formula.	(a) $V = \frac{1}{3}AH$
2 Write the specific formula for a cone.	$V = \frac{1}{3}\pi r^2 H$
3 Substitute the values and simplify.	$= \frac{1}{3} \times \pi \times 12^2 \times 35$ $= 1680\pi$
4 Evaluate the answer, rounding to two decimal places, and write with the correct unit of volume.	$= 5277.88\ cm^3$ (2 d.p.)
(b) 1 State the general formula.	(b) $V = \frac{1}{3}AH$
2 Substitute the formula for the area of the base.	$= \frac{1}{3}l^2 H$
3 Substitute the values.	$= \frac{1}{\cancel{3}} \times 4^2 \times \cancel{3}$
4 Write the answer with the correct unit of volume.	$= 16\ m^3$

Spheres

Like all circular shapes, the formula for the volume of a sphere involves the value of pi (π).

Volume of a sphere

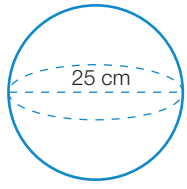


$$V = \frac{4}{3}\pi r^3$$

Worked Example 12

WE 12

Find the volume of the sphere, rounded to two decimal places.

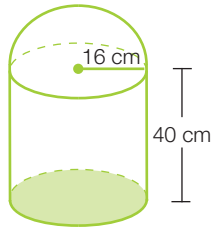


Thinking	Working
1 Write the formula.	$V = \frac{4}{3}\pi r^3$
2 Find the value of the radius.	$r = \frac{25}{2} = 12.5\ cm$
3 Substitute for r and evaluate.	$V = \frac{4}{3} \times \pi \times 12.5^3$
4 Evaluate the answer, rounding to two decimal places and write with the correct unit of volume.	$= 8181.23\ cm^3$ (2 d.p.)

Composite solids are made up of combinations of simpler solids. We can find their volume by adding or subtracting the volumes of the simpler solids.

Worked Example 13 WE13

Find the volume of this solid, rounded to two decimal places.



Thinking	Working
1 Identify the simple solids involved.	$V = V_{cylinder} + V_{hemisphere}$
2 Write the specific volume formula for each solid.	$= \pi r^2 H + \frac{1}{2} \times \frac{4}{3}\pi r^3$
3 Substitute the values of r and H.	$= \pi \times 16^2 \times 40 + \frac{1}{\cancel{2}_1} \times \frac{\cancel{4}^2}{3} \times \pi \times 16^3$
4 Simplify before evaluating.	$= 10\,240\pi + \frac{8192}{3}\pi$
5 Evaluate, checking that values are reasonable. Round the answer to the required number of decimal places and write with the correct unit of volume.	$= 40\,748.55 \text{ cm}^3$ (2 d.p.)

3.5 Volume of tapered solids and spheres 10A

Navigator

Answers page 727

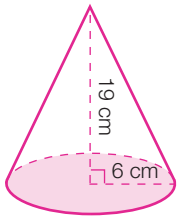
Q1, Q2, Q3, Q4, Q7, Q8, Q9, Q10, Q13	Q1, Q2, Q3, Q5, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14	Q1, Q2, Q3, Q5, Q6, Q8, Q10, Q11, Q12, Q14, Q15

Fluency

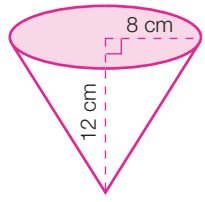
WE11

1 Find the volume of each of these tapered solids, rounded to two decimal places.

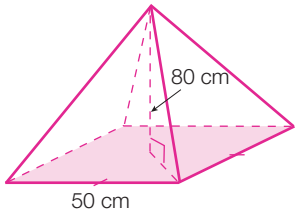
(a)



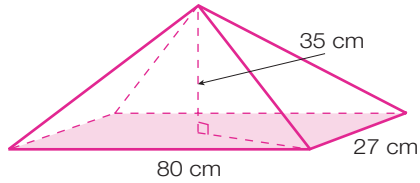
(b)



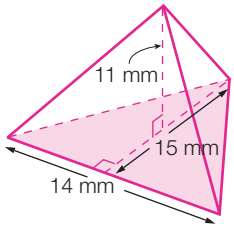
(c)



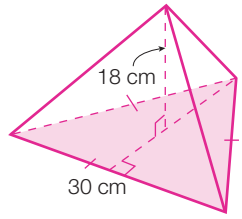
(d)



(e)

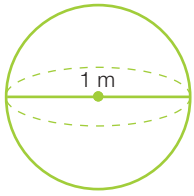


(f)

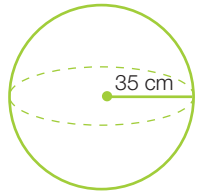


2 Find the volume of each sphere, rounded to two decimal places. WE12

(a)

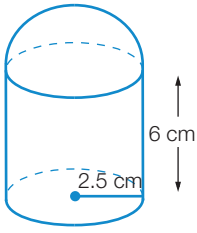


(b)

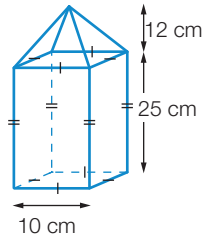


3 Find the volume of each of these solids, rounded to two decimal places. WE13

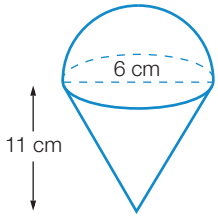
(a)



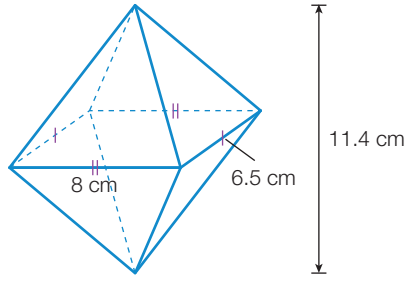
(b)



(c)



(d)



Understanding

4 Earth has an approximate diameter of 12 800 km. If Earth is assumed to be spherical, what is its volume to the nearest 1000 km^3?

5 When the radius of a sphere is doubled:

A the volume is doubled

B the volume is 4 times larger

C the volume is 6 times larger

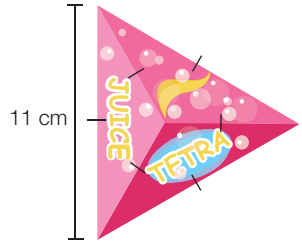
D the volume is 8 times larger

6 The radius of a cone 25 cm high with a volume of 37 969.9 cm^3 is closest to:

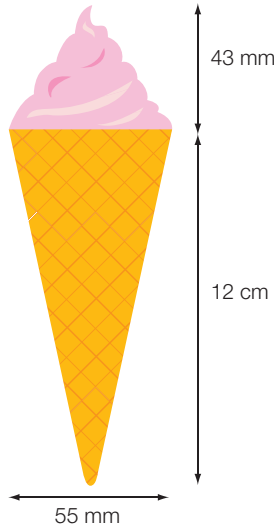
A 38 cm **B** 121 cm **C** 363 cm **D** 1450 cm

7 What is the difference between the volume of a cone of height 20 cm with a base diameter of 30 cm, and a pyramid of height 20 cm with a square base 30 cm wide? State your answer correct to two decimal places.

8 This 'tetra-pak' of frozen juice is a triangular-based pyramid. Each face is an equilateral triangle of side length 11 cm. If the height of the tetra-pak is 9 cm, find its capacity. State your answer correct to the nearest mL.



9 Anica buys an ice-cream cone at her local shop. The soft-serve ice-cream fills the cone and forms a conical (cone-shaped) peak as shown in the diagram. Calculate the amount of ice-cream Anica receives to the nearest mL.



Reasoning

10 Ball bearings are tiny metal balls used to reduce the friction between the moving parts of machines. How many ball bearings with a radius of 3 mm can be made from a solid cube of metal with a side length of 2 cm?

11 The table at right gives the diameters of the eight planets.

(a) How many times is Jupiter's diameter bigger than Mercury's (to the nearest whole number)?

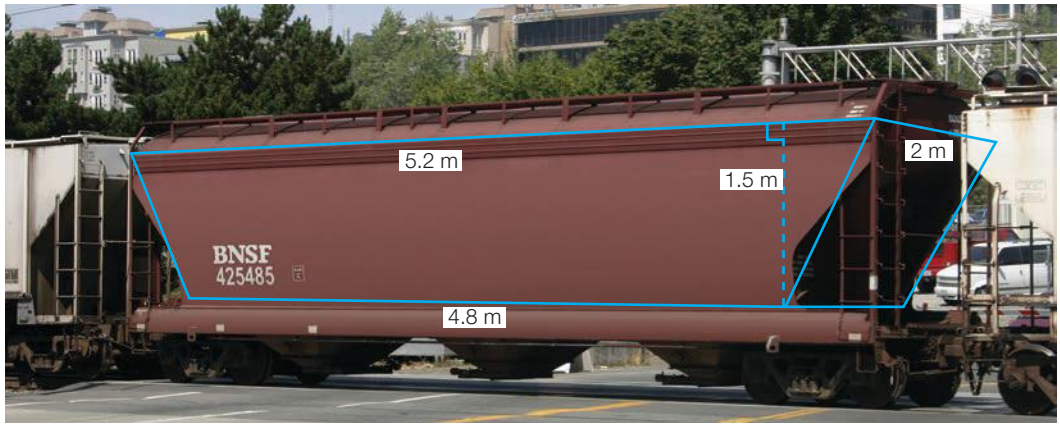
(b) How many of Mercury would fit into Jupiter?

(c) Explain why your answers to parts **(a)** and **(b)** are so different.

(d) Calculate how many Earths would fit into each of the larger planets.

Planet	Diameter (km)
Mercury	4 900
Venus	12 200
Earth	12 800
Mars	6 800
Jupiter	138 000
Saturn	113 000
Uranus	47 000
Neptune	50 000

12 Six wheat silos are situated at a country railway siding, as shown on the following page. The silos are all full and the grain will be moved to the city in railway trucks, as shown in the second diagram. How many trucks will be needed to move all the grain to town? The silos are large cylinders with a cone on top. The trucks can be assumed to be trapezoidal prisms.



Open-ended

13 **(a)** Choose two values to represent the base length and the height of a square-based pyramid. Calculate the pyramid's volume.

(b) Double the length of the base and recalculate the volume.

(c) How many times greater is the volume of the second pyramid than the first?

(d) Repeat **(a)**–**(c)** with a new base and height. What can you conclude?

14 A cone has its height doubled, but its radius stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its volume.

(b) Evaluate its volume after its height is doubled.

(c) Choose another two values to represent another cone and evaluate its volume.

(d) Double the height of this cone and evaluate the new volume.

(e) Can you find a relationship between the volume of a cone after the height is doubled and its original volume? Show the relationship, if there is one, or explain why a relationship does not exist.

15 A cone has its radius doubled, but its height stays the same.

(a) Choose two values to represent the height and radius of the original cone and evaluate its volume.

(b) Evaluate its volume after its radius is doubled.

(c) Choose another two values to represent another cone and evaluate its volume.

(d) Double the radius of this cone and evaluate the new volume.

(e) Can you find a relationship between the volume of a cone after the radius is doubled and its original volume? Show the relationship, if there is one, or explain why a relationship does not exist.

Investigation

The great golf ball box dilemma

Equipment required: 1 brain, thin cardboard, scissors and glue (optional)

You are the packaging designer for 'Supreme Sports', a company that manufactures sports equipment. Your manager has asked you to design the packaging for a set of six golf balls. The packaging must be as space-efficient (minimal wasted space) and cost-effective as possible (as little material used as possible). It should also look appealing and be easy to handle, pack and stack.

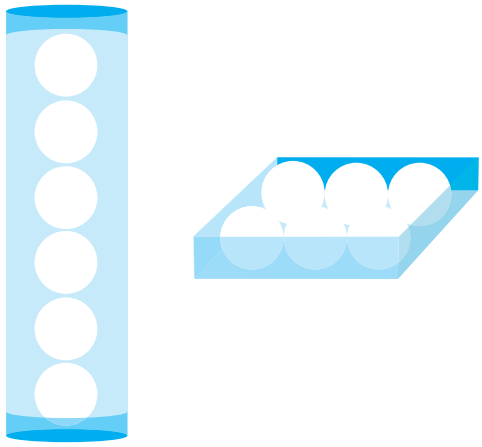
The Big Question

What is the best way to package six golf balls?

Engage

1 The diameter of a standard size golf ball is 4.27 cm. Calculate the volume of six golf balls.

2 You have come up with two basic designs: a long cylindrical box, packing the balls end to end, and a rectangular box holding two rows of three golf balls.



(a) Assuming that the balls are touching each other, determine the minimum length and diameter of the cylinder. Allow 1 cm at the top and 0.5 cm around the outside of the balls for clearance space.

(b) Assuming that the balls are touching each other, determine the minimum length and width of the rectangular prism. Allow 1 cm extra length, 1 cm extra width and an extra 1 cm in height, for clearance space.

Explore

3 (a) For the cylindrical and the rectangular boxes, calculate the volume of:

(i) the box

(ii) the empty space when the balls are inside.

(b) Find the volume of empty space as a percentage of the total volume.

4 (a) Draw a net for both the cylinder and the rectangular prism using the dimensions from Question **2**. In order for the nets to be constructed, you will need to include a minimal number of tabs (flaps) so that the sides can be stuck down. Make these tabs 1 cm wide.

(b) Determine the surface area of each net.

(c) Calculate the cost of constructing each design if the cardboard used to make them costs \$0.005/cm^2.

Strategy options

- Break problem into manageable parts.
- Make a table.
- Have I seen a similar problem?

Explain

5 Which design is more space-efficient? Use your results to justify your choice.

6 Which design is more cost-efficient? Use your results to justify your choice.

Elaborate

7 Considering the criteria of space and cost efficiency, which design would you recommend that the company use?

8 The packaging should be easy to handle, pack and stack. How well does your chosen design fit these criteria?

Evaluate

9 **(a)** Comment on the accuracy of your calculations and identify any potential sources of error.

(b) Do you think the extra room allowed for clearance space at the ends and sides was reasonable and realistic? If not, suggest some more reasonable measurements.

Extend

10 Use the nets you drew in Question **4** to construct a prototype of one or both of the designs. Design a company logo and use it to decorate the packaging.

11 Your manager likes your cylindrical design but thinks it would be even more efficient if the ends were rounded into the shape of a hemisphere, so that half a golf ball could fit inside.

(a) Modify your design as your manager suggested. Recalculate the required length of the packaging, allowing the same clearance space as before.

(b) Recalculate the volume and surface area of this new packaging. Is it more efficient than the other two designs?

12 Come up with a new packaging design that is an entirely different shape to those already considered. Is this design more space- or cost-efficient than the others?

3.6 Applications of volume

10A

We can use volume to calculate other important properties of solids, liquids or gases.

Density

If two substances have the same volume, the heavier of the two has the greater density. For example, a litre of water has a mass of 1 kg. A litre of cooking oil has a mass of less than 1 kg, so cooking oil floats on water.

The **density** of a substance is the amount of mass within a given unit of volume.

A hollow object, such as a ping-pong ball, is made up almost entirely of air so its total mass is a lot less than the same volume of liquid. A ping-pong ball will float on any liquid.

This density-reducing 'hollowness' explains why large steel ships float in water.

$$\text{Density} = \frac{\text{mass}}{\text{volume}} \text{ or } d = \frac{m}{V}$$

where d = density in g/cm^3 (g/mL) or kg/m^3 (kg/kL)
m = mass in grams (g) or kilograms (kg)
V = volume in centimetres cubed (cm^3, or mL) or metres cubed (m^3 or kL)

Worked Example 14 WE14

A rock has a mass of 80 g and volume of 25 cm^3. Find its density.

Thinking	Working
1 Write the formula.	$d = \frac{m}{V}$
2 Substitute the values with their units.	$= \frac{80\text{ g}}{25\text{ cm}^3}$
3 Evaluate the answer and write it with the correct units (g/cm^3).	$= 3.2\text{ g/cm}^3$

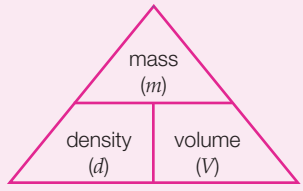
Density is an example of a rate. It measures the change in one quantity (mass) compared to a fixed amount of another quantity (volume). A triangle diagram is helpful in rearranging the density formula to find mass and volume.

Cover the quantity you wish to find. If the other two quantities are side by side, find their product. If the two quantities are placed one over the other, find the quotient.

$d = \frac{m}{V}$

$m = d \times V$

$V = \frac{m}{d}$



Worked Example 15

WE15

Find the mass of 400 cm^3 of a block of metal with a density of 1.6 g/cm^3.

Thinking	Working
1 Write the formula.	$d = \frac{m}{V}$
2 Rearrange the formula to make m the subject.	$m = d \times v$
3 Substitute the values and evaluate.	$m = 1.6 \times 400$
4 Write the answer giving the correct units (grams, as density was given in g/cm^3).	$= 640\ g$
5 Check by recalculating the density.	Check: $\frac{640\ g}{400\ cm^3} = 1.6\ g/cm^3$

Worked Example 16

WE16

Find the volume of a 120 g piece of gold with a density of 19.3 g/cm^3 and round your answer to two decimal places.

Thinking	Working
1 Write the formula.	$d = \frac{m}{V}$
2 Rearrange the formula to make V the subject.	$V = \frac{m}{d}$
3 Substitute the values and evaluate.	$V = \frac{120}{19.3}$
4 Write the answer giving the correct units (cm^3, as density was given in g/cm^3).	$= 6.22\ cm^3$ (2 d.p.)
5 Check by recalculating the density.	Check: $\frac{120\ g}{6.22\ cm^3} = 19.3\ g/cm^3$ (1 d.p.)

Concentration

The amount of one substance that is dissolved in a unit amount of solution is the **concentration**. The dissolved substance is called the solute and the liquid it is dissolved in is called the solvent. For example, the concentration of sugar dissolved in water is the number of grams of sugar in a litre of solution (g/L). This is another example of a rate.

The amount of solute and solution may each be a volume or a mass, hence mL/L or mg/kg are possible units of concentration. In medical preparations, g/mL is a common unit.

$$\text{Concentration} = \frac{\text{amount of solute (mass or volume)}}{\text{amount of solution (mass or volume)}}$$

Note that the concentration is the amount of solute compared to the final solution, not the amount of solvent added.

Worked Example 17 WE17

A 3 L jug of cordial solution is made with 150 mL of cordial concentrate. Find the concentration of the cordial in mL/L.

Thinking	Working
1 Write the formula.	$\text{Concentration} = \frac{\text{amount of solute}}{\text{amount of solution}}$
2 Substitute the amounts with their units.	$= \frac{150\text{ mL}}{3\text{ L}}$
3 Simplify the fraction so that the denominator is 1 L. (Here we divide numerator and denominator by 3.)	$= \frac{50\text{ mL}}{1\text{ L}}$
4 Write the concentration as a rate.	The concentration of the cordial is 50 mL/L

Worked Example 18 WE18

6 g of salt is dissolved in water to make 120 g of saline solution. Find the concentration of the solution in g/100g.

Thinking	Working
1 Write the formula.	$\text{Concentration} = \frac{\text{amount of solute}}{\text{amount of solution}}$
2 Substitute the amounts with their units.	$= \frac{6\text{ g}}{120\text{ g}}$
3 Perform the division to convert to an amount per gram and then multiply by 100 to find the amount per 100 g.	$= \frac{6}{120} \times 100$
4 Write the concentration as a rate.	Concentration is 5 g/100 g

3.6 Applications of volume

10A

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 (a), Q7, Q8, Q11, Q13	Q1, Q2, Q3, Q5, Q6, Q7, Q9, Q12, Q13	Q1, Q2, Q3, Q6, Q7, Q9, Q10, Q11, Q12, Q13

Answers page 727

Fluency

1 For each of the following, find the density. WE14

(a) a metal sphere of volume 9 cm^3, and mass 63.9 g

(b) 20 cm^3 of liquid that has a mass of 128 g

(c) a solid of volume 24 cm^3 and mass 19.2 g

2 For each of the following, calculate the mass and state answers correct to two decimal places. WE15

(a) A lump of metal with a volume of 800 cm^3 and a density of 7.8 g/cm^3.

(b) 450 mL of a liquid that has a density of 1.08 g/mL.

(c) 2 m^3 of a gas that has a density of 1.98 kg/m^3.

3 Calculate the volume of each of the following, rounded to two decimal places. WE16

(a) A 150 g piece of lead with a density of 11.3 g/cm^3.

(b) 486 g of a liquid with a density of 0.7 g/mL.

(c) 13 kg of wood that has a density of 0.85 kg/m^3.

4 2.5 L of sugar solution contains 50 g of sugar. What is its concentration in g/L? WE17

5 500 g of weedkiller solution contains 12 g of active ingredient. Find its concentration in g/100 g. WE18

6 (a) A saline solution is made from 2.5 L of water and 28 g of salt. The concentration of salt in g/L is closest to:

A 28 B 14 C 12 D 11

(b) A solution of copper sulfate is made from 1350 L of water and 1.6 kg of copper sulfate. The concentration of copper sulfate in g/L is closest to:

A 0.8 B 1 C 800 D 1000

Understanding

7 The Goulburn River carries sediment from erosion in its catchment. Barry takes a 5 L sample of water from the river. From this he extracts 38 g of sediment (1000 mg = 1 g). What is the concentration of sediment in the water, in mg/mL?

8 Marlon is making up 5 L of cordial for a party. The instructions on the cordial (concentrate) bottle say to add 1 cup of cordial to 4 cups of water.

(a) If Marlon uses a standard 250 mL cup, how many cups of each of the cordial concentrate and water are needed?

(b) What is the concentration of cordial concentrate in the final solution? Write your answer in mL/L.

Hint

How many litres in 1 m^3?

9 **(a)** Crude oil is measured in units called 'barrels'. 1 barrel is equivalent to 159 litres. Calculate the mass of 1 barrel of crude oil if the density of the crude oil is 825 kg/m^3.

(b) Petrol has a density of 737 kg/m^3. A tanker truck used for transporting petrol has a capacity of 30 kL. Calculate the mass of petrol in a full tanker truck. Round your answer to the nearest tonne.

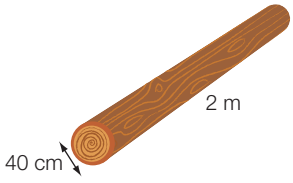
10 Jay's circular swimming pool has a diameter of 7 m and a depth of 1.6 m, but Jay only fills it to 20 cm below the rim. Calculate, correct to two decimal places, the chlorine concentration in g/L if he adds 2.1 kg of chlorine.

Reasoning

11 The density of water is 1 g/cm^3. An object will *float* in water if its density is *less* than 1 g/cm^3.

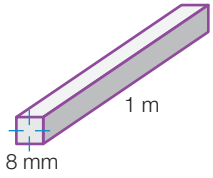
(a) By calculating the densities of the six objects below, determine which will float in water.

A wooden log



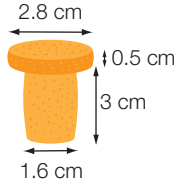
Mass: 100 kg

A steel rod



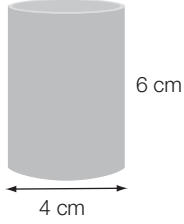
Mass: 0.5 kg

A cork



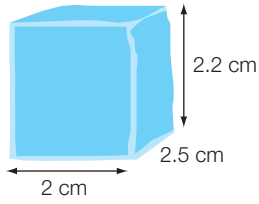
Mass: 2 g

An aluminium cylinder



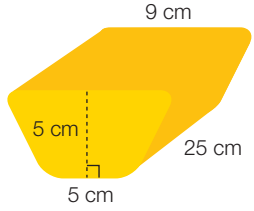
Mass: 200 g

An ice block



Mass: 10 g

A gold bar



Mass: 16 kg

(b) Find the mass of each object if it was made of lead, which has a density of 11.35 g/cm^3. Write your answers correct to the nearest gram.

12 High phosphorus levels in water can lead to blue-green algal blooms. The town of Evanshill takes its water from Lake Farmer, which has phosphorus levels contributing to blue-green algal blooms. This means the water supply to the town must be shut down. Two main sources of phosphorus are:

- Johnstone's abattoir, which discharges 5 ML (1 ML = 1 000 000 L) per day with a phosphorus concentration of 10 mg/L
- the Arvon River, which has an average discharge of 200 ML per day at a phosphorus concentration of 0.2 mg/L.

(a) Which source is contributing more phosphorus per day?

(b) What is the actual concentration of phosphorus contributed to Lake Farmer each day in mg/L? Round your answer correct to two decimal places.

Open-ended

13 The Environment Protection Authority monitors the level of the common bacteria *E. coli* at swimming beaches. If the level on any day is above 1000 organisms/100 mL, the beach is closed. Tess collects a 200 mL water sample from a beach. An examination of the sample shows that in 80 mL there are 700 *E. coli* organisms.

(a) What is the concentration of *E. coli* in the sample in organisms/100 mL?

(b) Is the concentration in the sample high enough to close the beach?

(c) List three sample sizes and the number of *E. coli* that would be high enough to close the beach.

Outside the Square — Problem solving

A pinch of salt

The salt concentration in the ocean is about 0.035 g/cm^3. How much salt could be extracted from a cubic kilometre of sea water?

Strategy options

- Break problem into manageable parts.
- Have I seen a similar problem?

Chapter review 3

D.I.Y. Summary

Key Words

capacity	cylinder	pyramid	tapered solid
composite shape	net	sphere	uniform cross-section
cone	prism	surface area	
10A			
concentration	density		

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 To find the area of a _____, divide the shape into simple shapes, calculate the individual areas and then add them together.

2 Adding together the individual areas of each face of a solid will give the _____.

3 The volume of a _____ or _____ is found by multiplying the area of its uniform cross-section by its height.

4 The volume and surface area of a _____, cylinder and _____ each have unique formulas involving π.

5 Write the formula for area of a rectangle, parallelogram, triangle, kite, trapezium, circle and sector.

6 Volume is measured in m^3 but _____ is measured in L.

7 Write each of the formulas for the surface area and volume of a square-based pyramid, cone and sphere.

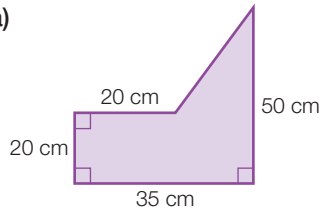
8 The formula for _____ is $\frac{\text{mass}}{\text{volume}}$. 10A

9 The _____ of a dissolved substance in a solution can be expressed in many different units. These include mL/L, g/100 g, mg/L or g/L. 10A

Fluency

1 Find the shaded area of each of these shapes, rounded to two decimal places. Ex.3.1

(a)



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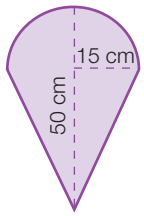

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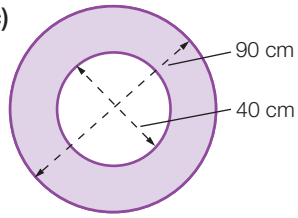
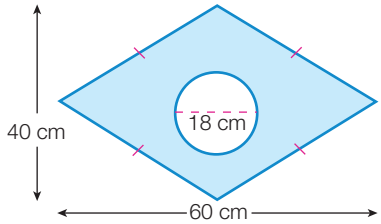


2 The area of this shape correct to two decimal places is: Ex.3.1

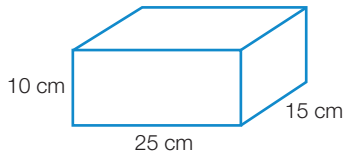
A 182.12 cm^2 B 945.53 cm^2

C 1143.45 cm^2 D 2145.53 cm^2



3 Find the surface area of each of these solids, rounded to two decimal places. Ex 3.2

(a)



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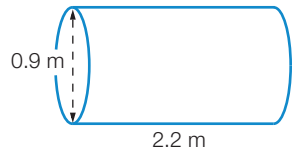
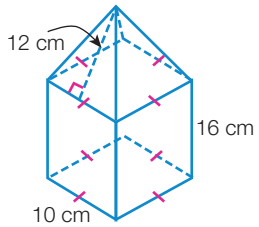


4 The surface area of this solid is equal to: Ex 3.4 10A

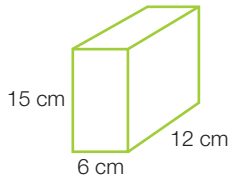
A 340 cm^2 **B** 840 cm^2

C 980 cm^2 **D** 1180 cm^2



5 Find the volume of each of these solids, rounded to two decimal places. Ex 3.3

(a)



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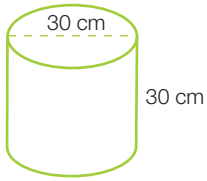

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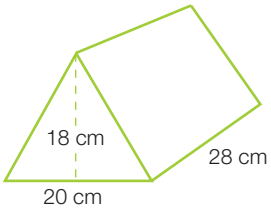
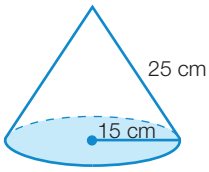


6 Find the surface area of each of these solids, rounded to two decimal places. Ex 3.4 10A

(a)



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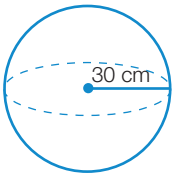
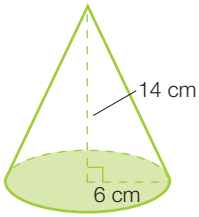
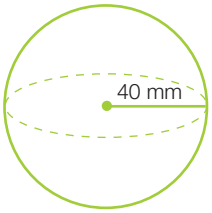


7 Find the volume of each of these solids, rounded to two decimal places. Ex 3.5 10A

(a)



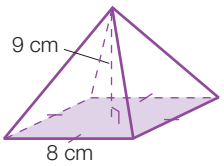
(b)



8 The volume of this solid is: Ex 3.5 10A

A 192 cm^3 **B** 208 cm^3

C 288 cm^3 **D** 576 cm^3

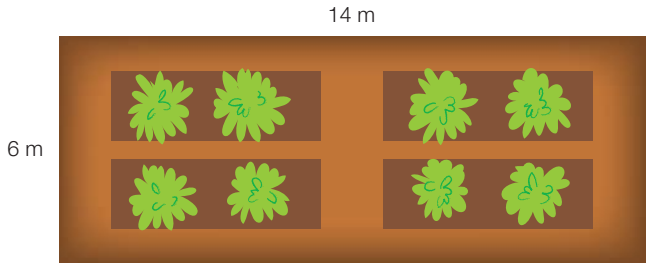


9 A rectangular block of wood with the dimensions of 1 m × 2 m × 2 m has a mass of 120 kg. What is its density? Ex 3.6 10A

10 A certain antiseptic should be mixed with warm water for best effects. If 30 mL should be mixed with 120 mL of water, express this as a concentration in mL/L. Ex 3.6 10A

Understanding

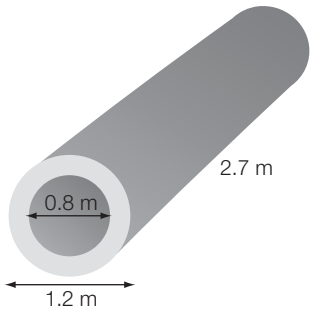
11 The diagram shows four garden beds, each 5.4 m long and 1.8 m wide, surrounded by paving bricks. Find the number of bricks needed to pave the area (to the nearest 100) if a brick measures 225 mm by 112 mm.



12 **(a)** Find the capacity of a cylindrical petrol tank of height 3.2 m and diameter 2.5 m. Express your answer to the nearest litre.

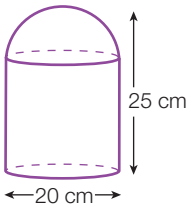
(b) The dimensions of a cylindrical rainwater tank are diameter 1760 mm and height 2100 mm. Calculate the capacity of the tank to the nearest litre.

13 Find the volume of material used to make this pipe in m^3, correct to two decimal places. 10A

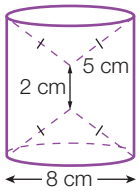


14 Find the surface area and volume of the following shapes rounded to the nearest whole number. Use Pythagoras' theorem where necessary.

(a)



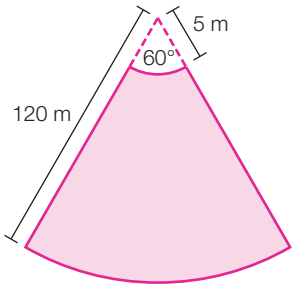
(b)



15 A snowball-throwing court has the measurements shown.

(a) Calculate the area in which the thrower has to land the snowball, to the nearest square metre.

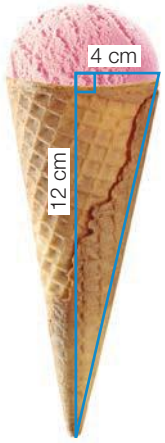
(b) Find the perimeter of the shaded area, to the nearest metre.



16 **(a)** Find the volume of a 200 g block of Styrofoam if its density is 0.1 g/cm^3. 10A

(b) What mass of salt should be dissolved in 300 mL of pure water to give a concentration of 50 g/L?

17 If the inside of this ice-cream cone is completely full of ice-cream, find the total volume of the ice-cream sitting in and on top of the cone, to the nearest cubic centimetre. (Assume the shape of the ice-cream on top is a hemisphere.) 10A

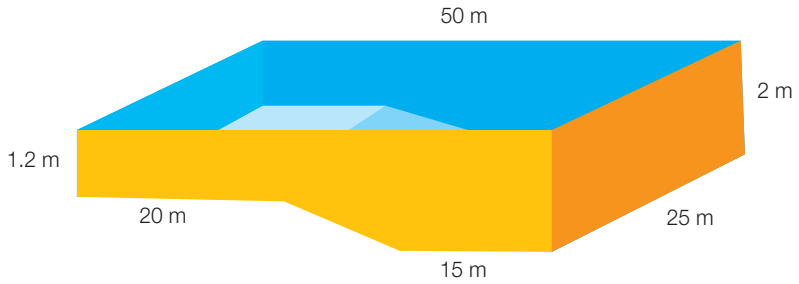


18 Ayse is inflating balloons with helium gas for a birthday party. Each balloon is roughly spherical in shape and about 40 cm in diameter. If she has to inflate two dozen balloons, find the total capacity (in litres) of helium required, rounded to the nearest whole number. 10A

19 The density of the material used to construct the pipe in Question **13** is 5.8 g/cm^3. Find the mass of the pipe. 10A

Reasoning

20 The local council swimming pool is shaped as shown.



(a) Find the capacity of the pool in both litres and kilolitres.

(b) It is proposed to build a pool that is 2 m deep for its entire length. If its other dimensions remain the same, how much extra water will be required for the new pool?

21 A cylindrical container has radius 7 cm and height 10 cm. What must be the height of a rectangular container 7 cm long and 10 cm wide if the two containers are to have the same volume? Write your answer correct to two decimal places.

22 Engineers wish to investigate building a tunnel through a mountain rather than having to build a road over the top. The road will cost $45 000 for every 10 m length that is laid, whether it is over the top or through the tunnel.

(a) The tunnel would be a semicircle of radius 10 m and would be 400 m in length. Calculate the volume of earth to be removed, to the nearest cubic metre.

(b) Find the cost (in whole dollars) of tunnelling and removal of dirt if this cost is estimated to be $150/m^3.

(c) Calculate the surface area of the curved ceiling/wall, to the nearest square metre. (Assume that the entire floor will become road.)

(d) Find the cost (in whole dollars) of fortifying the ceiling wall if the cost is $100/m^2.

(e) Find the cost (in whole dollars) of laying the road through the tunnel.

(f) What is the total cost (in whole dollars) of construction of the road through the tunnel?

The road over the top of the mountain would be 1.4 km long.

(g) Find the cost of laying the road over the top of the mountain.

(h) Which is the cheaper option and by how much?

23 Mert owns a large pool initially containing 1000 L of fresh water. He Mert adds 100 L of salt water, with a concentration of 52 g/L, to the pool, causing the water to become salty. Assuming the salt spreads evenly throughout the pool, what is the concentration of salt in the pool? Write your answer correct to two decimal places. 10A

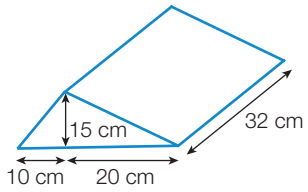
Mixed review A

Fluency

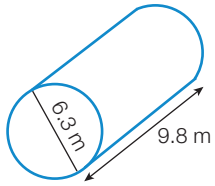
1 Find the interest earned, to the nearest cent, on an investment of \$2500 at 6.5% p.a. compounded monthly for 5 years. Ex. 1.1

2 Find the surface area of each of the following solids. Where necessary, give your answer correct to two decimal places. Ex. 3.2

(a)



(b)



3 Expand and simplify the following. Ex. 2.1

(a) $(x - y)(3x - 2y)$ (b) $(3a - 6b)(2a - 5b)$ (c) $(5t + 4h)(3t - 9h)$

4 Find the effective rate of interest if \$10 000 is invested for 1 year at the following rates. Ex. 1.4

(a) 7% simple interest (b) 7% compounded annually

(c) 7% compounded quarterly (d) 7% compounded monthly

5 Fully factorise the following expressions. Ex. 2.4

(a) $x^2 - 4x + 4$ (b) $x^2 - 9$ (c) $x^2 + 14x + 49$

(d) $100x^2 - 25$ (e) $4xy^2 + 2xy - 8x$ (f) $4x^2 + 32x + 64$

6 Simplify the following. Ex. 2.6

(a) $\frac{x}{5} - \frac{3x}{7}$ (b) $\frac{x+3}{5} + \frac{x+5}{6}$

7 Expand and simplify. Ex. 2.1

(a) $6(x + 11) - x(x + 11)$ (b) $4(x + 6) - x(x + 6)$

(c) $2x(x + 4) - 3(x + 4)$ (d) $7y(y - 1) + 2(y - 1)$

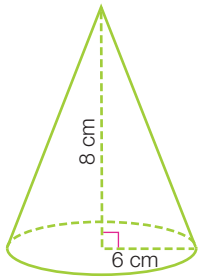
8 Find, to the nearest cent, the total amount owing on a loan of \$4600 for 5 years if the 3.7% p.a. interest is compounded: Ex. 1.2

(a) annually (b) quarterly (c) monthly.

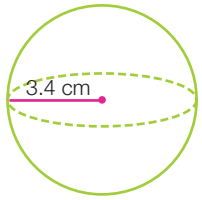
Assume no repayments are made in the time period.

9 Find the volume of each of the following solids. Round your answers to two decimal places. Ex. 3.5 10A

(a)



(b)



10 Factorise each expression. Ex 2.3

(a) $x^2 - 6x - 27$ **(b)** $x^2 - 7x + 10$ **(c)** $6x^2 - x - 2$

11 If the total mass of dissolved solids in a 30 mL sample of lake water is found to be 5 g, what is the concentration in g/L? State your answer correct to one decimal place. Ex 3.6 10A

12 Expand and simplify: Ex 2.2

(a) $(5x - 1)^2$ **(b)** $(4x - 3)^2$ **(c)** $(8 - 3x)^2$ **(d)** $(6 - 5x)^2$

Understanding

13 Alana has \$5000 to invest for 3 years. Which is a better value arrangement?

Option A: 7.5% compounded quarterly Option B: 6.8% compounded monthly

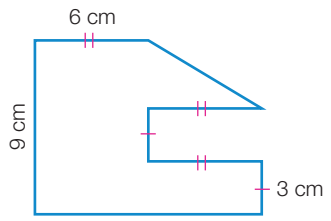
14 Calculate, to the nearest cent, the value after 4 years of a \$2300 laptop if it is depreciated in the following ways.

(a) 15% straight-line depreciation **(b)** 15% reducing balance depreciation

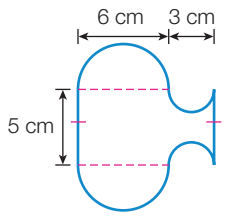
15 A cube has a volume of 512 cm^3. Calculate its surface area.

16 Find the area of each of the following shapes. Where necessary, give your answer correct to two decimal places.

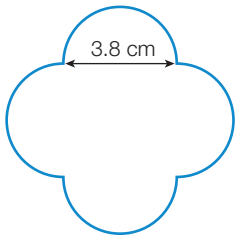
(a)



(b)



(c)

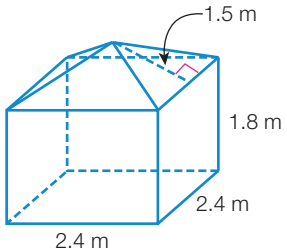


Reasoning

17 Scott wishes to erect a greenhouse. The support structure shown is made of metal. Using the plan shown, calculate correct to one decimal place: 10A

(a) the amount of glass needed for the walls and roof

(b) the length of metal needed for the support structure.



18 A square piece of carpet of side length x is extended by 5 m in length and by 3 m in width.

(a) Write an algebraic expression for the area of the extended carpet, in factorised form.

(b) Expand your answer to **(a)** and simplify by collecting like terms.

(c) If $x = 4$, find the area of:

(i) the original **(ii)** the extended piece of carpet.

(d) If the original piece of carpet cost \$128, suggest a comparable price for the extended piece of carpet.

19 How many mL of tea tree oil should be added to 180 mL of water to make a 10% solution? 10A

Challenge A

1 Calculate 1000 – 1 + 3 – 5 + 7 … to 102 terms (numbers in the sequence).

2 Damo has \$733 to spend in the airport shop before flying home. If he spent all his money on four different kinds of gifts for his 20 friends and relatives, priced at \$18, \$23, \$52 and \$69, how many of each gift did he buy?

3 A music store had a sale on CDs. For every two CDs that you bought at the normal price, you could purchase a third CD for \$1. If 12 CDs were bought for \$120, then the regular price for a CD was:

A \$15.00 **B** \$14.50 **C** \$14.00 **D** \$10.00

Hint
Remember that integers are positive or negative whole numbers or zero.

4 How many integer values of x satisfy $\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$?

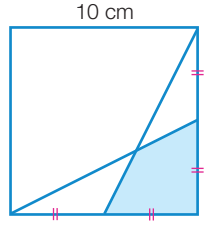
A 1 **B** 2 **C** 3 **D** 4

5 p, q and r are three different positive integers, and $(pq)^2 = pqr$.

Which of the following is a possible value for r?

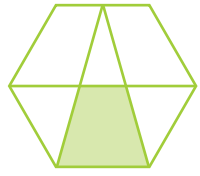
A 1 **B** 3 **C** 4 **D** 6

6 Find the area of the shaded region in the diagram.



7 A square fits tightly in an equilateral triangle with its base on the triangle's base. What percentage of the area of the triangle does it occupy?

8 Find the area of the shaded trapezium shown if the regular hexagon has a side length of 40 cm.

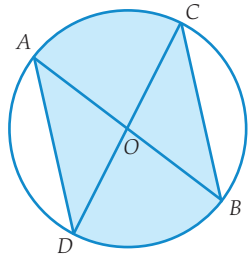


9 In the diagram, AB and CD are diameters of a circle of radius 4 cm.

If AB is perpendicular to CD, the area of the shaded region is:

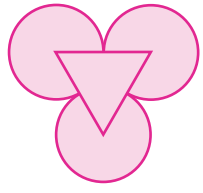
A $8\pi + 4$ **B** $8\pi + 8$

C $16 + 4\pi$ **D** $16 + 8\pi$



10 What is the side length of a square that just fits symmetrically within an isosceles triangle of sides 5 cm, 10 cm and 10 cm?

11 Find the volume of stone in a pillar, 6 m high, the cross-section of which is based on a equilateral triangle of 30 cm side, having three circular sectors drawn with their centres as the vertices of the equilateral triangle. Give your answer in m^2 correct to three decimal places.



Linear relationships 4

Maths saves the planet!

Name: Zoe Ryan
Job: Forest Carbon Specialist
Qualifications: Bachelor of Forest Science

Developing countries with tropical rainforests have had a high rate of deforestation in the past. However, a change in attitude over the past few years and a growing interest in the environment is halting the devastation. Zoe Ryan is a Forest Carbon Specialist. Her job is to measure the amount of carbon stored in a forest to calculate the amount of greenhouse gases that would be emitted if they were cut down.

About a third of Zoe's time is spent abroad, travelling to tropical regions in countries such as Borneo, Papua, Peru, Ecuador and Mozambique. She has to travel on canoes through croc-infested waters to get to some of the forests. As Zoe explains: "It's really wild; you wouldn't find these places in a Lonely Planet travel guide … there's always something totally unexpected, you just have to let go and go with it".

In the forest, Zoe takes measurements of the trees and uses sampling in order to measure the carbon stock in the forest. After the quantity of avoided emissions is calculated, it is then audited and verified by an independent third party. Carbon credits are then sold in the carbon market to companies who want, or need, to reduce their carbon emissions debit. Deforestation contributes about 15% of the world's greenhouse gas emissions. Getting companies to buy carbon credits means that the forest stays protected.

Knowing how important it is to sustainably manage the forest, it is crucial for Zoe to be accurate with her maths: "I can see that the more samples we put in, the lower our error, so the more carbon we're able to sell because our precision is greater. I feel like I'm genuinely doing something that's good for society, and it's also an adventure".

Why learn this?

A variety of professions use linear relationships and systems of linear equations to represent, analyse and solve a variety of problems, and manipulate numbers to find quantities. Linear equations are used extensively in all branches of science and technology: computer science, biology, chemistry, physics, anthropology and all the engineering professions. Other professionals such as economists, bankers, architects, stockbrokers and photographers also use linear equations to interpret and explain information, and to predict future results.

After completing this chapter you will be able to:

- solve linear equations
- find the gradient of the line joining two points or from an equation
- sketch linear graphs
- find parallel and perpendicular equations to given lines
- solve simultaneous equations
- find the solution sets for linear equations.

Recall 4

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

Worksheet R4.1

1 If $P = 2t - 7$, find P when:

(a) $t = 0$ **(b)** $t = 5$

(c) $t = -4$ **(d)** $t = 2$

Worksheet R4.2

2 Solve the following linear equations.

(a) $2a - 3 = 11$ **(b)** $\frac{b}{4} + 7 = 13$

(c) $3(2c - 5) = 45$ **(d)** $\frac{2d + 6}{5} = 4$

Worksheet R4.3

3 Plot these coordinate points on a Cartesian plane.

(a) $A(4, 3)$ **(b)** $B(0, 2)$

(c) $C(-1, 2)$ **(d)** $D(-2, -3)$

Worksheet R4.4

4 Copy and complete the table of values below for each of the following equations, then plot each graph for $-2 \le x \le 2$ on a Cartesian plane.

x	-2	-1	0	1	2
y					

(a) $y = x + 3$ **(b)** $y = x - 2$

(c) $y = 2 - x$ **(d)** $y = 3x$

Worksheet R4.5

5 Sketch the graphs of:

(a) $y = 3$ **(b)** $x = 2$

Worksheet R4.6

6 Write down **(i)** the gradient and **(ii)** the y-intercept of the graphs for each of the following equations.

(a) $y = 3x + 5$ **(b)** $y = 3 - 2x$ **(c)** $2x + 3y = 4$

Worksheet R4.7

7 Find the gradient of the line joining the points with coordinates:

(a) (1, 3) and (4, -2) **(b)** (-3, -6) and (2, -1)

Worksheet R4.8

8 **(a)** Sketch the graphs of $x = 1$ and $y = 3$ on the same set of axes.

(b) Find the point of intersection of the two graphs.

Key Words

coincident lines	independent variable	linear inequality	sketch
dependent variable	linear equation	linear relationship	unique equations
gradient	linear graph	simultaneous equations	

Linear equations

4.1

A **linear relationship** exists between two variables if the graph of the relationship is a **linear graph** (a straight line). In a linear relationship, a uniform change in one variable, the **independent variable**, produces a uniform change in the other, the **dependent variable**. We commonly use x and y as the variables because these represent the axes of the Cartesian plane on which linear relationships are graphed. However, any variables can be used. A linear relationship is described by a **linear equation**. If we know the value of one of the variables we can solve the equation to find the corresponding value of the other variable. By performing inverse operations in the opposite direction, we form equivalent equations, which lead to a solution. Linear equations have only one solution.

Worked Example 1

WE1

Solve each of the following equations to find the value of the variable.

(a) $5x - 6 = 34$

(b) $\frac{3a}{2} + 4 = -1$

Thinking	Working
(a) 1 Use inverse operations in the opposite order to find the value of the variable. (Here, add 6 to both sides and then divide both sides by 5.)	(a) $5x - 6 = 34$ $5x = 40$ $x = 8$
2 Check your answer by substituting the value of the variable into the equation.	$\text{LHS} = 5x - 6$ $= 5 \times 8 - 6$ $= 34$ $= \text{RHS}$
(b) 1 Use inverse operations in the opposite order to find the value of the variable. (Here, subtract 4 from both sides, multiply both sides by 2 and then divide both sides by 3.)	(b) $\frac{3a}{2} + 4 = -1$ $\frac{3a}{2} = -5$ $3a = -10$ $a = \frac{-10}{3}$ $= -3\frac{1}{3}$
2 Check your answer by substituting the value of the variable into the equation.	$\text{LHS} = \frac{3a}{2} + 4$ $= \frac{3 \times -3\frac{1}{3}}{2} + 4$ $= \frac{3 \times -\frac{10}{3}}{2} + 4$ $= -\frac{10}{2} + 4$ $= -5 + 4$ $= -1$ $= \text{RHS}$

Equations where the unknown appears more than once

To solve equations of the type $5a - 3 = 2a + 6$ where the unknown appears on both sides of the equation, we must first add or subtract a variable term on both sides of the equation so that the variable appears on one side of the equation only (usually the LHS). If the equations involve brackets, we can expand the brackets using the Distributive Law $a(b + c) = ab + ac$.

Worked Example 2 WE2

Solve each of the following equations to find the value of the variable.

(a) $5a - 3 = 2a + 6$ **(b)** $2a - 5 = 6a + 1$ **(c)** $4(k + 1) = 3(6 - k)$

Thinking	Working
(a) 1 Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only. (Here, subtract $2a$ from both sides.)	(a) $5a - 3 = 2a + 6$ $3a - 3 = 6$
2 Use inverse operations in the opposite order to find the value of the variable. (Here, add 3 to both sides and then divide both sides by 3.)	$3a = 9$ $a = 3$
3 Check your answer by substituting the value of the variable into each side of the equation.	LHS $= 5a - 3$ $= 5 \times 3 - 3$ $= 15 - 3$ $= 12$ RHS $= 2a + 6$ $= 2 \times 3 + 6$ $= 6 + 6$ $= 12$ LHS = RHS
(b) 1 If the coefficient of the variable term on the RHS of the equation is greater than the coefficient of the variable on the LHS, it is advisable to rewrite the equation interchanging sides. (Here, as $6 > 2$, interchange sides.)	(b) $2a - 5 = 6a + 1$ $6a + 1 = 2a - 5$
2 Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only.	$4a + 1 = -5$
3 Use inverse operations in the opposite order to find the value of the variable.	$4a = -6$ $a = -\frac{6}{4}$
4 Simplify your solution if necessary.	$= -\frac{3}{2}$
5 Check your answer by substituting the value of the variable into each side of the equation.	LHS $= 2a - 5$ $= 2 \times -\frac{3}{2} - 5$ $= -3 - 5$ $= -8$ RHS $= 6a + 1$ $= 6 \times -\frac{3}{2} + 1$ $= -9 + 1$ $= -8$ LHS = RHS

(c)	1	Expand brackets.	(c) $4(k+1) = 3(6-k)$ $4k + 4 = 18 - 3k$
	2	Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only. (Here, add $3k$ to both sides.)	$7k + 4 = 18$
	3	Use inverse operations in the opposite order to find the value of the variable. (Here, subtract 4 from both sides and then divide both sides by 7.)	$7k = 14$ $k = 2$
	4	Check your answer by substituting the value of the variable into both sides of the equation.	LHS $= 4(k+1)$ RHS $= 3(6-k)$ $= 4(2+1)$ $= 3(6-2)$ $= 4 \times 3$ $= 3 \times 4$ $= 12$ $= 12$ LHS = RHS

To solve an equation containing fractions, we find the lowest common multiple (LCM) of the denominators, called the lowest common denominator (LCD). We then multiply both sides of the equation by the LCD. This simplifies the equation by eliminating the fractions. The equation can now be solved as shown previously.

Worked Example 3

WE3

Solve each of the following equations to find the value of the variable.

(a) $\frac{2x+1}{3} = \frac{x-3}{5}$

(b) $\frac{x}{3} - 2 = \frac{x-1}{4}$

Method 1: Multiplying by the LCD

		Thinking	Working
(a)	1	Find the LCD of all fractions in the equation.	(a) LCD = 15
	2	Multiply every term in the equation by the LCD. Insert brackets if required.	$\frac{2x+1}{3} = \frac{x-3}{5}$ $\frac{15(2x+1)}{3} = \frac{15(x-3)}{5}$
	3	Divide the denominator of each term into the LCD.	$5(2x+1) = 3(x-3)$
	4	Expand any brackets.	$10x + 5 = 3x - 9$
	5	Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only. (Here, subtract $3x$ from both sides.)	$7x + 5 = -9$

6	Use inverse operations in the opposite order to find the value of the variable.	$7x = -14$ $x = -2$
7	Check the value of your solution by substituting it into both sides of the original equation.	$\text{LHS} = \frac{2 \times -2 + 1}{3} = \frac{-3}{3} = -1$ $\text{RHS} = \frac{-2 - 3}{5} = \frac{-5}{5} = -1$ LHS = RHS
(b) 1	Find the LCD of all fractions in the equation.	(b) LCD = 12 $\frac{x}{3} - 2 = \frac{x-1}{4}$
2	Multiply every term in the equation by the LCD. Insert brackets if required.	$\frac{12x}{3} - 12 \times 2 = \frac{12(x-1)}{4}$
3	Divide the denominator of each term into the LCD.	$4x - 24 = 3(x - 1)$
4	Expand any brackets.	$4x - 24 = 3x - 3$
5	Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only. (Here, subtract $3x$ from both sides.)	$x - 24 = -3$
6	Use inverse operations in the opposite order to find the value of the variable.	$x = 21$
7	Check the value of your solution by substituting it into both sides of the original equation.	$\text{LHS} = \frac{21}{3} - 2 = 7 - 2 = 5$ $\text{RHS} = \frac{21 - 1}{4} = \frac{20}{4} = 5$ LHS = RHS

Alternatively, we can express all fractions with a common denominator (LCD) and then multiply both sides of the equation by the LCD.

Method 2: Writing all terms as fractions using the LCD

Thinking	Working
(a) 1 Find the LCD of all fractions in the equation.	(a) LCD = 15
2 Write all terms in the equation as fractions using the LCD as the denominator. Use brackets if required. (Here, $\frac{1}{3} = \frac{5}{15}$ and $\frac{1}{5} = \frac{3}{15}$.)	$\frac{2x+1}{3} = \frac{x-3}{5}$ $\frac{5(2x+1)}{15} = \frac{3(x-3)}{15}$
3 Multiply every term on both sides of the equation by the denominator (LCD) to eliminate the fraction.	$5(2x+1) = 3(x-3)$
4 Expand any brackets.	$10x + 5 = 3x - 9$
5 Add or subtract a term containing the variable on both sides of the equation so that the variable appears on one side of the equation only. (Here, subtract $3x$ from both sides.)	$7x + 5 = -9$
6 Use inverse operations in the opposite order to find the value of the variable.	$7x = -14$ $x = -2$
7 Check the value of your solution by substituting it into both sides of the original equation.	LHS $= \frac{2 \times -2 + 1}{3} = \frac{-3}{3} = -1$ RHS $= \frac{-2-3}{5} = \frac{-5}{5} = -1$ LHS = RHS
(b) 1 Find the LCD of all fractions in the equation.	(b) LCD = 12
2 Write all terms in the equation as fractions using the LCD as the denominator. Use brackets if required. (Here, $\frac{1}{3} = \frac{4}{12}$, $2 = \frac{24}{12}$ and $\frac{1}{4} = \frac{3}{12}$.)	$\frac{x}{3} - 2 = \frac{x-1}{4}$ $\frac{4x}{12} - \frac{24}{12} = \frac{3(x-1)}{12}$
3 Multiply every term on both sides of the equation by the denominator (LCD) to eliminate the fraction.	$4x - 24 = 3(x-1)$

4	Expand any brackets.	$4x - 24 = 3x - 3$
5	Use inverse operations in the opposite order to find the value of the variable.	$x = 21$
6	Check the value of your solution by substituting it into both sides of the original equation.	$\text{LHS} = \frac{21}{3} - 2 = 7 - 2 = 5$ $\text{RHS} = \frac{21-1}{4} = \frac{20}{4} = 5$ $\text{LHS} = \text{RHS}$

Writing worded information as equations

When mathematical information is given in words we need to write it in the form of an equation before we can solve it.

Worked Example 4

WE 4

Write an equation for the following and solve it to find the unknown numbers.

The sum of three consecutive numbers is 21.

Thinking	Working
1 Use a variable to define one of the unknowns.	Let the first number be x.
2 'Consecutive' means 'following' so add 1 to x to get the second number and add 2 to x for the third number.	Second number would be $x + 1$. Third number would be $x + 2$.
3 'Sum' tells us to add the three unknown numbers to give the total. (Here, it is 21.)	$x + (x + 1) + (x + 2) = 21$
4 Add like terms to simplify.	$3x + 3 = 21$
5 Use inverse operations in the opposite order to find the value of the variable.	$3x = 18$ $x = 6$
6 Check your solution by substituting it back into the equation.	$\text{LHS} = 6 + (6 + 1) + (6 + 2)$ $= 6 + 7 + 8$ $= 21$ $= \text{RHS}$
7 State the answer in words.	The three consecutive numbers are 6, 7 and 8.

In this case, it would have been simpler to let the middle number equal x. The three numbers would then have been $x - 1$, x and $x + 1$. Try this for yourself.

4.1 Linear equations

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q15, Q18	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 (d)–(f), Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q15, Q16, Q18, Q19	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 (d)–(f), Q6, Q8, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

Answers page 730

Fluency

1 Solve the following equations to find the value of the given variable. WE1

(a) $4x + 3 = 15$ (b) $4x + 1 = 33$ (c) $3c + 7 = -2$

(d) $5a - 3 = 17$ (e) $7x - 2 = -44$ (f) $7d - 5 = -26$

(g) $6 - 5m = -4$ (h) $9 - 4n = 5$ (i) $-2 - 3x = 1$

(j) $19 = 6f + 7$ (k) $-9 = 4e - 1$ (l) $-3 = 5p - 4$

(m) $\frac{x}{8} - 5 = -5$ (n) $\frac{g}{7} + 2 = 5$ (o) $\frac{h}{4} - 7 = -3$

(p) $\frac{2k}{3} + 1 = 3$ (q) $\frac{5x}{3} - 2 = 8$ (r) $\frac{7y}{2} + 3 = 3$

(s) $3 - \frac{4x}{5} = 2$ (t) $6 - \frac{2x}{3} = 4$ (u) $4 - \frac{5x}{7} = -1$

2 Solve the following equations to find the value of the given variable. WE2

(a) $3a + 7 = a + 9$ (b) $9d + 4 = 2d - 3$ (c) $5e + 2 = 7 - e$

(d) $7c - 1 = 4c + 5$ (e) $4b - 3 = 2b + 5$ (f) $4f - 5 = 8 - 2f$

(g) $g - 11 = 5 - g$ (h) $2h + 3 = 7 - 3h$ (i) $4k + 13 = 7 - 2k$

(j) $2 - 3x = 2x + 7$ (k) $3 - 4x = 2x - 3$ (l) $-6 - x = 2x + 1$

(m) $2(x + 3) = 6x$ (n) $3(2x + 1) = 18x$ (o) $5(7x + 2) = 65x$

(p) $3(m + 2) = m - 5$ (q) $2(3d - 5) = 2d + 7$ (r) $4(2d - 3) = 6d + 2$

(s) $5(a - 2) = 3(a + 2)$ (t) $7(a + 2) = 2(2a - 1)$ (u) $4(2g - 3) = 5(3g - 4)$

(v) $7(2t + 4) = 4(7 - 2t)$ (w) $5(2h + 3) = 4(3h + 1)$ (x) $2(4j + 3) = 3(1 - j)$

3 Solve the following equations to find the value of the given variable. WE3

(a) $\frac{a+7}{3} = \frac{a-3}{5}$ (b) $\frac{b-2}{2} = \frac{b+1}{3}$ (c) $\frac{2c-6}{3} = \frac{c+7}{4}$

(d) $\frac{3d+1}{2} = \frac{d-3}{4}$ (e) $\frac{2x+1}{3} = \frac{x-3}{5}$ (f) $\frac{5y-2}{3} = \frac{y+3}{4}$

(g) $\frac{3-2m}{3} = \frac{5-4m}{4}$ (h) $\frac{1-4k}{4} = \frac{3k-7}{2}$ (i) $\frac{2-3t}{4} = \frac{t-5}{3}$

(j) $\frac{a+7}{3} + 2 = -4$ (k) $\frac{p-3}{4} - 1 = 2$ (l) $\frac{m-8}{5} - 2 = -9$

(m) $\frac{3p+1}{2} - \frac{2p}{5} = 6$ (n) $\frac{3n-1}{5} + \frac{n}{2} = 2$ (o) $\frac{3t+2}{4} - \frac{2t}{3} = -1$

(p) $\frac{4m-3}{7} + \frac{2m+5}{2} = 3$ (q) $\frac{2k+3}{2} + \frac{5-k}{3} = 2$ (r) $\frac{2x+3}{4} - \frac{3x-5}{3} = 2$

WE4

4 Write an equation for each of the following and solve it to find the unknown numbers.

(a) 6 is added to x to give a result of 13.

(b) 7 is subtracted from y to give a result of -2.

(c) Twice a certain number is 17.

(d) One-third of a certain number is 12.

(e) Two more than three times a number is 23.

(f) Five less than four times a number is -3.

5 (a) The equation $3a + 5 = -1$ has the solution:

A $a = \frac{4}{3}$ B $a = -\frac{1}{2}$ C $a = -2$ D $a = -18$

(b) The equation $\frac{d}{7} + 3 = -2$ has the solution:

A $d = 7$ B $d = -\frac{5}{7}$ C $d = -7$ D $d = -35$

(c) The equation $\frac{2x}{3} - 4 = 4$ has the solution:

A $x = -48$ B $x = 0$ C $x = 3$ D $x = 12$

6 (a) $3x + 8 = 2 - x$ has the solution:

A $x = -3$ B $x = -\frac{3}{2}$ C $x = \frac{2}{3}$ D $x = 3$

(b) $7(y - 3) = 5(15 - y)$ has the solution:

A $y = 9$ B $y = 8$ C $y = 7$ D $y = 0$

(c) $\frac{3m - 2}{2} = \frac{m + 11}{3}$ has the solution:

A $m = 0$ B $m = 2$ C $m = 4$ D $m = 7$

Understanding

7 The sum of four consecutive numbers is 38. What numbers are they?

8 Find three consecutive even numbers whose sum is 126.

Hint
Add 2 to get the next even number.

9 Sam divided his bag of jubes so that he and his friends Nikki and Minh could share them. He gave himself twice as many as he gave Nikki, and Minh received three times as many as Nikki. If the bag contained 42 jubes, how many did each person receive?

10 If the perimeter of the rectangular block of land shown in the photograph below is 300 m, find the length and width of the block.

11 Suppose that p represents Penny's age.

(a) How old will she be in 7 years' time?

(b) Write an expression to represent twice her age in 7 years' time.

(c) What is Penny's age now if twice her age in 7 years' time is 48?

12 The width of a rectangular block of land is $\frac{2}{7}$ of its length. If its perimeter is 252 m, find its length.

13 To travel to a concert, Olivia and her friends decided to walk to the train station. The trip on the train covered 6 times the distance they had walked. They then rode in a taxi for the final 2 km to the concert venue. Altogether they travelled 12.5 km. How far did Olivia and her friends walk, and how far did they travel by train?

14 Ross is twenty years older than his niece Krystal. Krystal has worked out that in five years' time her uncle will be twice as old as she will be. Find Krystal's age.

15 If $\frac{3}{5}$ of a number is subtracted from $\frac{2}{3}$ of the same number, the result is 3. Find the number.

Reasoning

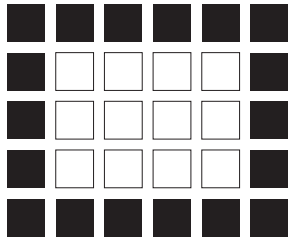
16 Amy is twice as old as Lucy. Three years from now the sum of their ages will be 42. How old is Amy?

17 Two cars start at the same time from the same point and travel in opposite directions. One car is travelling 10 km/h faster than the other. In 3 hours they are 288 km apart. Find the speed of each car.

Open-ended

18 Write three different equations, each having a solution of -2, where at least three mathematical operations are used.

19 **(a)** Mrs Grey wants Tim the tiler to lay square tiles to make a bordered rectangle with white tiles on the interior and black tiles on the border. For his first attempt Tim comes up with the following design, which contains 18 black and 12 white tiles. If the rectangle remains 3 white tiles wide, but the number of white tiles in the length increases to l, how many black tiles will be required for the border?



(b) **(i)** Draw a different rectangular design that uses 12 white tiles.

(ii) How many black tiles would now be needed?

(iii) If the number of white tiles in the width remains constant, but the number of white tiles in the length increases to t, how many black tiles will be required for the border?

(c) Choose a different number of white tiles to make a rectangular pattern and find the number of black tiles (n) needed to make a border.

Technology Exploration CAS

Equipment required: 1 brain, a TI-Nspire CAS or ClassPad CAS

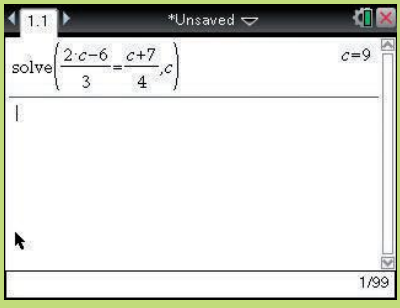
Solving equations

Equations such as $\frac{2c-6}{3} = \frac{c+7}{4}$ can be quite a challenge for our algebra skills. However, we can use our CAS to solve these quite easily.

Using the TI-Nspire CAS

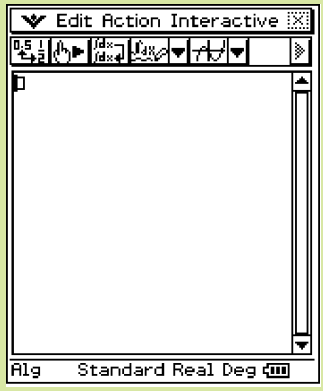
1 Open a new Calculator page. Press (on) and (tab) until the first icon on the bottom row is highlighted, as shown below. Then press (enter).

2 Press (menu) > Algebra > Solve > (enter). To get a fraction template press (ctrl) > (÷). Enter the equation in the normal way. Press (tab) to move around in the fraction. After you have entered the right-hand side of the equation press (,) > [C] > ()) > (enter).

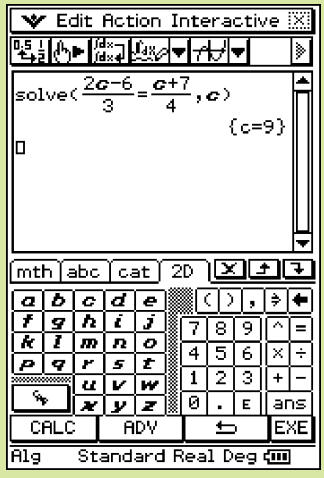


Using the ClassPad

1 Tap on Main and make sure Standard is displayed at the bottom of the screen. If it isn't, just tap Decimal and it will change.

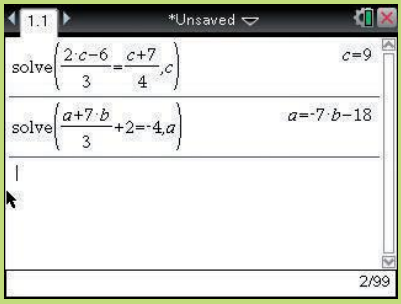
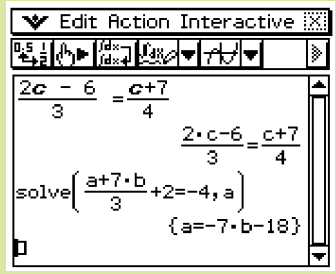
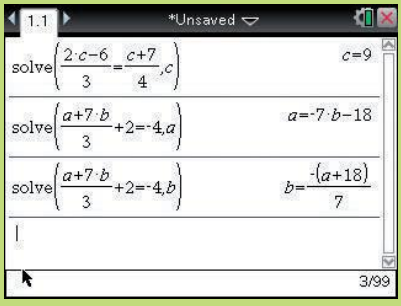
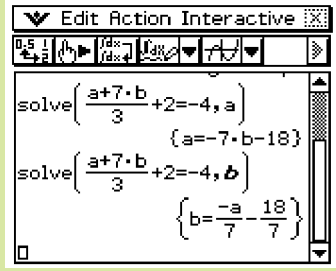


2 Tap Action > Equation/Inequality > solve and then enter the equation. Tap (Keyboard) > [2D] > [▤] to get a fraction template. Tap [VAR] for the variables, but don't use the [abc] letters. When you have finished the equation tap [,] then c from the [VAR] list and finally (]) before you tap (EXE).



For both calculators it is vital that you indicate the variable for which you want to solve. This seems unnecessary in the previous example as there is only one variable, but in cases like $\frac{a+7b}{3} + 2 = -4$ we can solve for either a or b.

Equations where we solve to find one variable in terms of the other are called literal equations.

Using the TI-Nspire CAS	Using the ClassPad
Repeat the previous instructions, using a as the variable for which we wish to solve.  	Repeat the previous instructions, using a as the variable. 
Repeat the instructions again, using b as the variable for which we wish to solve.  	Repeat the instructions again, using b as the variable for which we wish to solve.  For all of these equations you can use Interactive > Equation/Inequality > solve and enter the equation in the dialog box using brackets and the division sign instead of the fraction template.

1 Solve the following equations for the given variable using your CAS.

(a) $\frac{3t+5}{4} = \frac{4t-5}{3}$ **(b)** $\frac{8t-7}{3} = 4t+7$ **(c)** $3a - \frac{4a}{5} = 9$

(d) $\frac{6p}{5} - 4 = 2p - 8$ **(e)** $\frac{2k-3}{4} = \frac{3k-2}{2}$ **(f)** $\frac{p}{2} + \frac{p}{10} = \frac{2p}{5} + 4$

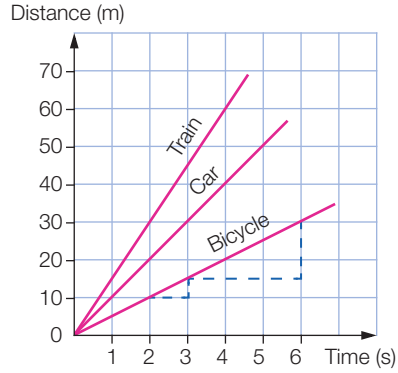
2 Solve the following equations for both variables shown.

(a) $\frac{a}{x-3} = \frac{2a-1}{2x+1}$ **(b)** $\frac{m+6n}{3} = \frac{2m+5n}{4}$ **(c)** $\frac{t+3}{2v} + \frac{t+5}{5v} = 8$

(d) $\frac{a+1}{3b} + \frac{a+1}{4b} = 9$ **(e)** $f - g = \frac{6f+g}{10}$ **(f)** $\frac{1}{p}(q-2) - \frac{1}{p}(2q-1) = 5$

4.2 Gradient

The **gradient** of a line connecting two points is the rate at which the lines rises or falls. The gradient of a line is sometimes referred to as the slope of the line. Consider the distance–time graph of three different modes of transport to the right.



From 0 to 2 seconds, the distance travelled by the bicycle is 10 m. Another way of expressing this is that the bicycle travels 10 m in 2 seconds or is travelling at an average speed of 5 m/s. What distance is travelled by the bicycle from 2 to 3 seconds? From 3 to 6 seconds? Does the average speed change?

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

From 2 to 3 seconds:

$$\text{average speed} = \frac{15 - 10 \text{ m}}{3 - 2 \text{ s}} = \frac{5 \text{ m}}{1 \text{ s}} = 5 \text{ m/s}$$

From 3 to 6 seconds:

$$\text{average speed} = \frac{30 - 15 \text{ m}}{6 - 3 \text{ s}} = \frac{15 \text{ m}}{3 \text{ s}} = 5 \text{ m/s}$$

Similarly, in the first 2 seconds the car travels 20 m; that is, the average speed of the car is 10 m/s. The train travels 30 m in 2 seconds, so its average speed is 15 m/s.

A constant speed produces a straight-line graph. The steeper the line, the greater the speed.

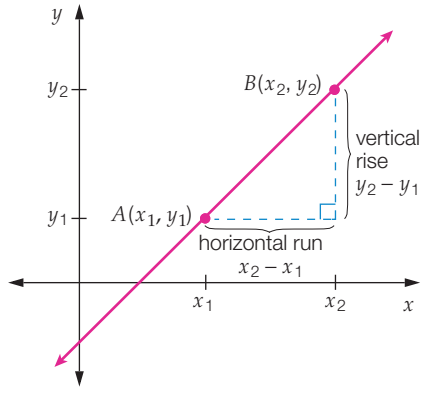
The measure of a line's steepness is called the gradient or slope and is often represented by m. The gradient of a distance–time graph gives the speed.

> The gradient is the same at any point along a straight line.
>
> The steeper the line, the greater the magnitude of the gradient.

The gradient of a line can be found by considering any two points (x_1, y_1) and (x_2, y_2) on the line.

$$\text{Gradient of line } AB = \frac{\text{vertical rise}}{\text{horizontal run}} = \frac{\text{difference in } y\text{-coordinates}}{\text{difference in } x\text{-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) is one point and (x_2, y_2) is another point.



Usually (x_1, y_1) is the point to the left of (x_2, y_2) so that $x_2 - x_1$ is positive. If (x_2, y_2) is lower than (x_1, y_1), then $y_2 - y_1$ will be negative and the gradient will therefore be negative.

However, as $\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$ it makes no difference which point we label (x_1, y_1) or (x_2, y_2).

The gradient of a line is the gradient of any interval within that line. The gradient is constant at any point on the line. The gradient of a line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

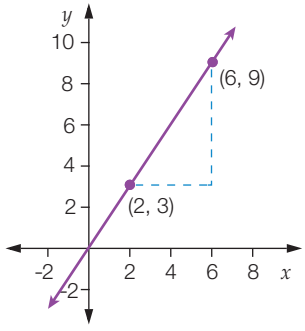
- The gradient is positive if the y-values increase as the x-values increase.
- The gradient is negative if the y-values decrease as the x-values increase.
- A horizontal line has zero gradient.
- A vertical line has an undefined gradient.
- Lines with the same gradient are parallel, $m_A = m_B$.
- Two lines are perpendicular if the product of their gradients is -1. One gradient is the negative reciprocal of the other. $m_A = -\frac{1}{m_B}$. Horizontal and vertical lines are perpendicular.

Worked Example 5

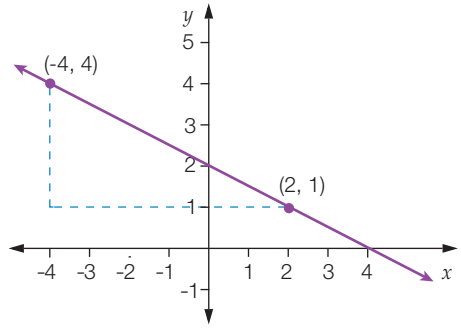
WE5

Find the gradient of each of the following lines through the given points:

(a) (2, 3) and (6, 9)



(b) (-4, 4) and (2, 1)



Thinking	Working
(a) **1** Identify (x_1, y_1) and (x_2, y_2).	(a) Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (6, 9)$.
2 Find the gradient of the line using: $m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{9 - 3}{6 - 2}$ $= \frac{6}{4}$ $= \frac{3}{2}$

(b) **1**	Identify (x_1, y_2) and (x_2, y_2).	(b) Let $(x_1, y_1) = (-4, 4)$ and $(x_2, y_2) = (2, 1)$.
2	$m = \frac{\text{rise}}{\text{run}}$ (As there is a decrease of 3 units in y for an increase of 6 units in x, the vertical 'rise' is -3.)	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{1-4}{2-(-4)}$ $= \frac{-3}{6}$ $= -\frac{1}{2}$

Finding the equation of a line

If we know the gradient and a point on the line, we can find the equation of the line by equating gradients.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If (x, y) is any point on the line, then the gradient can be written as $m = \frac{y - y_1}{x - x_1}$. By substituting given values for m and (x, y), we can rearrange and simplify to obtain a linear equation through the point (x, y) with the given gradient.

Worked Example 6

WE 6

Find the equation of a line with:

(a) a gradient of 2 that passes through the point (3, -5)

(b) a gradient of -4 that passes through the point (-4, 2).

Thinking		Working	
(a) **1**	Write the formula for equating gradients.	(a)	$m = \frac{y - y_1}{x - x_1}$
2	Substitute values for the gradient and the point into the formula.		$2 = \frac{y - -5}{x - 3}$
3	Rearrange the equation and simplify.		$2(x - 3) = y + 5$ $y = 2x - 6 - 5$
4	Write an equation in $y = mx + b$ form.		$y = 2x - 11$
(b) **1**	Write the formula for equating gradients.	(b)	$m = \frac{y - y_1}{x - x_1}$
2	Substitute values for the gradient and the point into the formula.		$-4 = \frac{y - 2}{x - -4}$
3	Rearrange the equation and simplify.		$-4(x + 4) = y - 2$ $y - 2 = -4x - 16$
4	Write an equation in $y = mx + b$ form.		$y = -4x - 14$

4.2 Gradient

Navigator

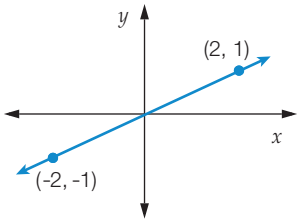
Q1, Q2, Q3, Q4 Column 1, Q5, Q6, Q7, Q8, Q9, Q13	Q1, Q2, Q3, Q4 Column 2, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13	Q1, Q2, Q3, Q4 Column 3, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Answers page 731

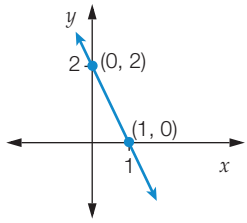
Fluency

1 Find the gradient of each of the following lines through the given points. WE5

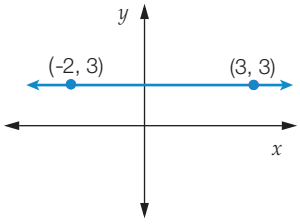
(a) (-2, -1) and (2, 1)



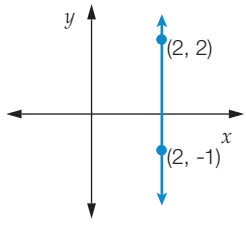
(b) (0, 2) and (1, 0)



(c) (-2, 3) and (3, 3)



(d) (2, -1) and (2, 2)



2 Find the equation of a line with: WE6

(a) a gradient of 6 that passes through the point (1, 3)

(b) a gradient of -2 that passes through the point (4, -1)

(c) a gradient of 3 that passes through the point (-4, -3)

(d) a gradient of -5 that passes through the point (3, -7).

3 Use the results from Questions **1** to answer the following questions.

(a) If a line is sloping up to the right, does it have a positive or a negative gradient?

(b) If a line is sloping down to the right, does it have a positive or a negative gradient?

(c) What is the gradient of a horizontal line?

(d) What is the gradient of a vertical line?

4 Find the gradient of the line joining each of the following pairs of points.

(a) (2, 4) and (5, 5) **(b)** (0, 3) and (2, 7) **(c)** (4, 10) and (5, 6)

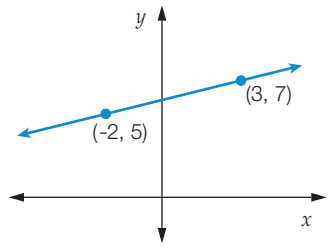
(d) (7, 5) and (9, 2) **(e)** (-2, 1) and (2, -4) **(f)** (-1, -3) and (0, -5)

(g) (0, 0) and (2, 6) **(h)** (2, 1) and (-2, -7) **(i)** (-2, 5) and (5, -2)

(j) (-4, 4) and (2, 0) **(k)** (-3, 2) and (2, 2) **(l)** (5, -4) and (-7, -4)

(m) (-8, 5) and (2, 1) **(n)** (-1, -1) and (6, 0) **(o)** (2, 5) and (2, 7)

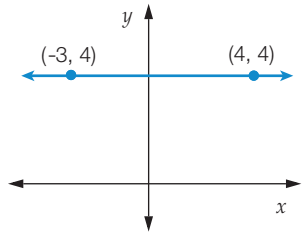
5 **(a)** The gradient of the line shown in the graph is:



A $\frac{5}{2}$ **B** 2

C $\frac{2}{5}$ **D** $-\frac{2}{5}$

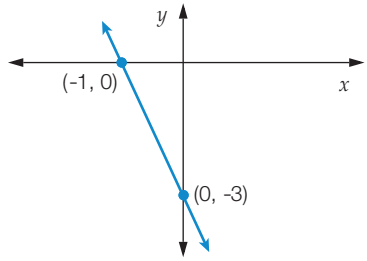
(b) The gradient of the line shown in the graph is:



A 8 **B** 0

C -7 **D** undefined

(c) The gradient of the line shown in the graph is:



A $\frac{1}{3}$ **B** $-\frac{1}{3}$

C 3 **D** -3

Understanding

6 **(a)** Use the results from Question **4** to find a pair of points that lie on:

(i) a horizontal line

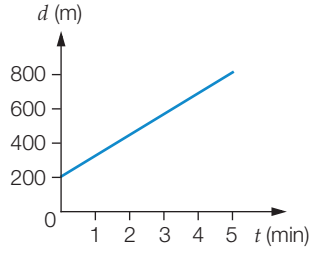
(ii) a vertical line

(iii) the line with the steepest negative gradient

(iv) the line with the smallest positive gradient.

(b) Use the results from Question **4** to find two pairs of points that would give two parallel non-horizontal lines.

7 Alex was 200 m from home and decided to walk to the local shop. This graph shows Alex's distance from home at (d) any particular time (t) on his 5-minute walk to the shop.



(a) How far was the shop from Alex's home?

(b) How far did Alex walk in 5 minutes?

(c) Find the gradient of this graph by first identifying two points that the line passes through.

(d) Find Alex's speed on the walk.

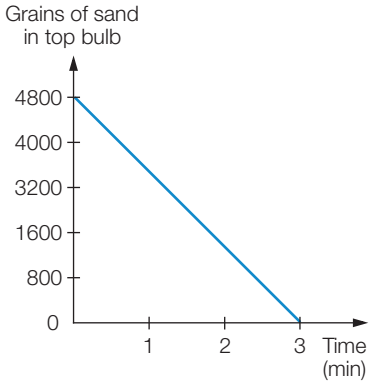
(e) How far was Alex from home after 2 minutes of walking?

(f) How far was Alex from home after 3 minutes of walking?

(g) Write down the relationship or rule to find the distance from home (d) after t minutes.

(h) The graph of Alex's trip is a straight line. What does this tell you about Alex's walking speed?

8 An egg-timer contains grains of sand which pour through the neck of the timer from the top bulb to the bottom in 3 minutes.



(a) How many grains of sand are in the egg-timer?

(b) How many grains of sand are left in the top bulb after 2 minutes?

(c) Find the gradient of this graph by identifying two points that the line passes through.

(d) Find the rate of change of sand grains with respect to time. In other words, what is the change in the number of sand grains left in the top bulb each minute?

(e) What does the negative represent in your answer to part **(c)**?

(f) Write down the relationship or rule to find the number of sand grains left in the top bulb (S) after t minutes.

(g) Use your rule to confirm your answer to **(b)**.

(h) What are the restrictions on the values that t can take?

Reasoning

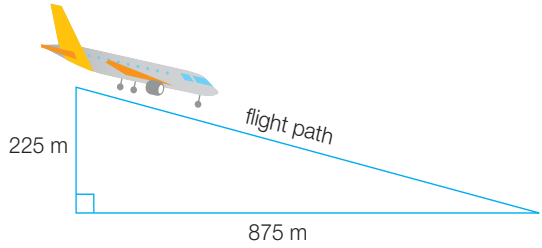
9 **(a)** Find the equation of the line that passes through (2, 4) and (4, 9):

(i) using the point (2, 4) as (x_1, y_1) **(ii)** using the point (4, 9) as (x_1, y_1)

(b) Did you get the same equation for parts **(i)** and **(ii)**? What conclusion can you make from your answers to parts **(i)** and **(ii)**?

10 The gradient of the line joining the points (2, 1) and (6, a) is 4. Find the value of a.

11 A passenger jet is landing on the flight path shown below. What is the gradient of the flight path?



Hint

Use the vertex of the right angle of the triangle as (0, 0).

12 Show that the points (-1, -5), (2, 4) and (5, 13) lie on the same straight line. This means that the points are collinear.

Open-ended

13 Identify three different sets of coordinates that when connected by a straight line produce each of the following gradients:

(a) -3 **(b)** -1 **(c)** $-\frac{1}{2}$ **(d)** 2 **(e)** 5

14 Write down the coordinates of two points such that a line joining them has the same gradient as the line joining (3, 5) and (7, -7).

4.3 Sketching linear graphs

The graph of a linear relationship is a linear graph. We need to locate only two coordinate points in order to **sketch** a straight-line graph.

The equation of a linear graph can be written in the form:

$$y = mx + b$$

where m is the gradient and b is the y-coordinate of the y-intercept $(0, b)$.

Sketching linear graphs using the y-intercept and gradient

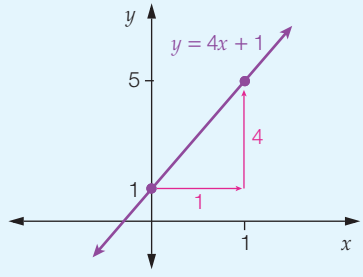
This method involves locating the point at which the graph crosses the vertical y-axis and then using the gradient to locate a second point. Once two points are known, the straight line can be drawn through them.

The gradient $= \frac{\text{rise}}{\text{run}}$ can be used to identify the rise—the number of steps up or down, and the run—the number of steps left or right, to reach the next point.

Sometimes, the equation needs to be reorganised to turn it into the form $y = mx + b$.

Worked Example 7 WE7

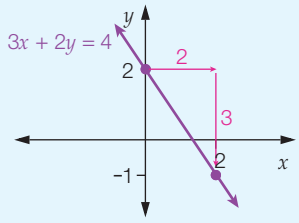
Find the y-intercept and the gradient and then sketch the graph of the line with equation $y = 4x + 1$.

Thinking	Working
1 Use the equation to identify the gradient (m) and the y-intercept $(0, b)$. Use gradient $= \frac{\text{rise}}{\text{run}}$ to identify the rise and the run. (Here, the rise is 4 and the run is 1.)	$y = 4x + 1$ has a y-intercept at $(0, 1)$. Gradient is 4, so the rise = 4, and the run = 1.
2 Rule axes for the Cartesian plane, label the x-axis and y-axis, then mark the y-intercept.	
3 Starting at the y-intercept, move to the right the distance of the run and move up the distance of the rise. (Here, we move 1 unit to the right and 4 units up to find the second point (1, 5).)	
4 Rule a line through the two points.	

Worked Example 8

WE 8

Find the y-intercept and gradient, and then sketch the graph of the line with equation $3x + 2y = 4$.

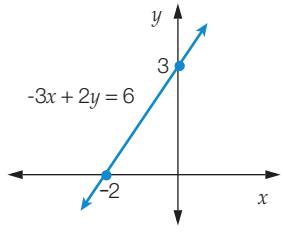
Thinking	Working
1 Rearrange into the form $y = mx + b$.	$3x + 2y = 4$ $2y = -3x + 4$ $y = \frac{-3x}{2} + 2$
2 Use the equation to identify the y-intercept $(0, b)$ and gradient (m).	y-intercept is $(0, 2)$. Gradient is $-\frac{3}{2}$, so the rise is -3 and the run is 2.
3 Draw and label a set of axes. Starting at the y-intercept, move to the right the distance of the run and move down the distance of the rise. A negative rise means a move down. (Here, we move 2 units to the right and 3 units down to find the second point (2, -1).)	
4 Rule a line through the two points.	

Sketching linear graphs using x- and y-intercepts

If a linear equation is not in the form of $y = mx + b$, it is often useful to find the x- and y-intercepts and use them as the major features of a linear graph. For example, $-3x + 2y = 6$.

For this equation, the x-intercept is (-2, 0) and the y-intercept is (0, 3).

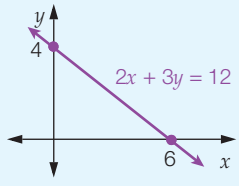
The y-value is zero at the x-intercept and the x-value is zero at the y-intercept. Knowing this helps us to calculate the intercept values.



Worked Example 9

WE 9

Find the x- and y-intercepts for $2x + 3y = 12$ and sketch the graph.

Thinking	Working
1 Find the x-intercept by substituting $y = 0$ into the equation.	When $y = 0$, $2x + 3 \times 0 = 12$ $2x = 12$, so $x = 6$ x-intercept is $(6, 0)$.
2 Find the y-intercept by substituting $x = 0$ into the equation.	When $x = 0$, $2 \times 0 + 3y = 12$ $3y = 12$, so $y = 4$ y-intercept is $(0, 4)$.
3 Mark in the x-intercept and y-intercept and rule a straight line through them.	

Note that for a sketch graph, only the x- and y-intercepts need to be shown on the axes; no other values for x and y need to be shown.

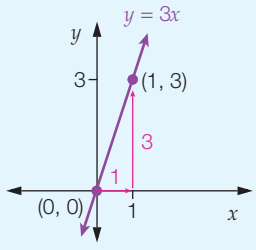
- To find the x-intercept, substitute $y = 0$ in the equation.
- To find the y-intercept, substitute $x = 0$ in the equation.

Sketching linear graphs that pass through the origin

If the straight line passes through the origin, then the method of finding the x- and y-intercepts does not give sufficient information to sketch the graph. The origin gives only one coordinate point (0, 0), so a second point is needed to be able to sketch the graph. We can find the second point by using the gradient. Taking the origin as the y-intercept we use the gradient to find a second point as we did in the gradient and y-intercept method.

Worked Example 10 WE10

Sketch the graph of $y = 3x$, showing the y-intercept and one other point.

Thinking	Working
1 Use the gradient to identify the rise and run.	Gradient = 3, so rise = 3 and run = 1.
2 Because the value of b in the equation is zero, plot the first point at the origin (0, 0).	
3 Find a second point by using the gradient. (Here, the rise is 3 and the run is 1.)	
4 Sketch the graph.	

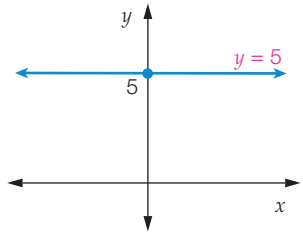
Lines parallel to the axes

Other linear relationships between x and y can be of the form:

- $y = a$, where a is a constant, is the equation of a horizontal line parallel to the x-axis. The gradient of these lines is zero.
- $x = b$, where b is a constant, is the equation of a vertical line parallel to the y-axis. The gradient of these lines is undefined.

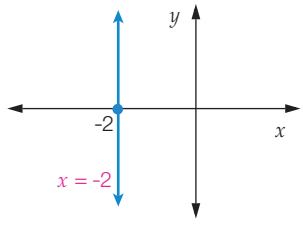
For example, $y = 5$ means that y always has the value 5, no matter what value x is given. Some coordinate points that lie on the line are (2, 5), (0, 5) and (-3, 5).

The graph of $y = 5$ is a straight line parallel to the x-axis.



Similarly, for $x = -2$, x will always have the value -2, no matter what value of y is chosen. Some coordinate points that lie on this line are (-2, 4), (-2, -1) and (-2, 0).

The graph of $x = -2$ is a straight line parallel to the y-axis.



4.3 Sketching linear graphs

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7 (a)–(h), Q8, Q9, Q10, Q16, Q17, Q18	Q1, Q2, Q3, Q4, Q5, Q6, Q7 (c)–(j), Q8, Q9, Q10, Q12, Q13, Q15, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7 (e)–(l), Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Answers page 731

Fluency

1 Find the y-intercept and the gradient and then sketch a graph of each line. WE7

(a) $y = 3x + 2$ (b) $y = 4x - 3$ (c) $y = -2x + 1$ (d) $y = -3x + 2$

2 Find the y-intercept and the gradient and then sketch a graph of each line. WE8

(a) $x + 2y = -4$ (b) $x + 3y = 3$ (c) $3x + 4y = 8$

(d) $2x + 5y = -5$ (e) $2x - 3y = -1$ (f) $4x - 2y = 3$

3 Find the x- and y-intercepts and then sketch a graph of each line. WE9

(a) $2x + 3y = 6$ (b) $4x + y = 4$ (c) $x - 2y = 4$

(d) $x - 3y = 9$ (e) $4x + y = 2$ (f) $3x + y = 1$

4 Sketch graphs of the following, showing the intercept on the axes and one other point for each. WE10

(a) $y = 4x$ (b) $y = x$ (c) $y = -2x$ (d) $y = -3x$

5 (a) The equation of a linear graph with a y-intercept of 2 and a gradient of -3 is:

A $y = 3x + 2$ B $x + 3y = 2$ C $x - 3y = 2$ D $3x + y = 2$

(b) The equation of a linear graph with a gradient of $\frac{1}{2}$ and a y-intercept of -3 is:

A $2x - y = 6$ B $x - 2y = 6$ C $x - 2y = -6$ D $2x - y = -6$

(c) The equation of a linear graph with a y-intercept of $\frac{2}{5}$ and a gradient of $-\frac{1}{4}$ is:

A $5x + 20y = 8$ B $20x + 5y = 8$ C $y = \frac{2x}{5} - \frac{1}{4}$ D $\frac{2x}{5} - y + \frac{1}{4} = 0$

6 When using the y-intercept and gradient to sketch a graph, first mark in the y-intercept and then use the gradient to locate a second coordinate point.

(a) A line has a gradient of $\frac{2}{3}$. From the y-intercept, we move:

A 2 units to the right and 3 units up B 2 units to the right and 3 units down

C 3 units to the right and 2 units up D 3 units to the right and 2 units down

(b) A line has a gradient of $-\frac{3}{4}$. From the y-intercept, we move:

A 4 units right and 3 units up B 3 units right and 4 units up

C 4 units right and 3 units down D 3 units left and 4 units down

(c) A line has a gradient of 4. From the y-intercept, we move:

A 4 units right and 4 units up B 1 unit right and 4 units up

C 4 units right and 4 units down D 1 unit right and 4 units down

7 Sketch graphs of the following, showing the intercepts on the axes.

(a) $y = 1$ (b) $y = 4$ (c) $y = -5$ (d) $y = -2$

(e) $y = 0$ (f) $y = \frac{1}{2}$ (g) $x = 2$ (h) $x = 5$

(i) $x = -3$ (j) $x = -4$ (k) $x = \frac{5}{2}$ (l) $x = 0$

8 (a) Two coordinate points that lie on the line $2x + y = 4$ are:

A (4, 0) and (0, 3) B (-4, 0) and (0, -2)

C (-2, 0) and (0, -4) D (2, 0) and (0, 4)

(b) Two coordinate points that lie on the line $y = 6x$ are:

A (0, 6) and (6, 0) B (0, 0) and (6, 6)

C (2, 12) and (0, 0) D (2, 0) and (0, 3)

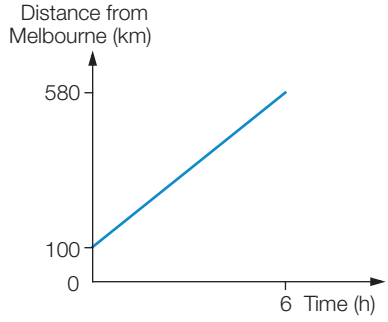
(c) Two coordinate points that lie on the line $x = -3$ are:

A (0, 0) and (1, 3) B (-3, 1) and (1, -3)

C (-3, 0) and (0, 0) D (-3, -5) and (-3, 0)

Understanding

9 (a) Consider the graph shown at right. Write down the intercept with the vertical axis.



(b) Find the gradient of this graph.

(c) Write an equation linking the distance from Melbourne in kilometres (D) and the time in hours (T) in the form $y = mx + b$.

(d) What constant speed was maintained throughout the journey?

(e) If the journey started 250 km from Melbourne and a constant speed of 100 km/h was maintained, what equation could be written to represent the relationship between D and T in the form $y = mx + b$.

(f) A particular trip from Melbourne is represented by the equation $4D - 372T = 30$. What constant speed was maintained throughout the journey?

10 Darren has borrowed \$200 from his parents and has agreed to repay \$40 each week.

(a) After 3 weeks, how much does Darren still owe his parents?

(b) Write an equation linking the amount Darren owes his parents (A) with the number of weeks (n).

(c) Find the amount owed when $n = 0$.

(d) Find the number of weeks when $A = 0$.

(e) Use your answers from **(c)** and **(d)** to sketch the graph of this relationship. (Put A on the vertical axis, and n on the horizontal axis.)

Reasoning

11 The annual yachting regatta of the Mathematical Club is to take place on Lake Tetrahedra. Starting and finishing at the marina, which can be considered to be coordinate point (0, 0), the yachts travel around six buoys in the order from 1–6, located at various positions on the lake. The first buoy is located at the point (2, 4). The second buoy is located at the point (-8, 8). The third buoy is located 10 units directly below the second buoy. The fourth buoy is located at the point (-3, 11). The fifth buoy is located 5 units to the right of the fourth buoy. The sixth buoy is located 4 units to the right of the first buoy.

(a) Plot each point on a Cartesian plane.

(b) Sketch the paths the yachts will take travelling to each buoy and back to the marina.

(c) Determine the equation of each path taken by the yachts between the:

(i) start of the race at the marina and the first bouy

(ii) first and second bouys

(iii) second and third bouys

(iv) third and fourth bouys

(v) fourth and fifth bouys

(vi) fifth and sixth bouys

(vii) sixth bouy and the finish of the race at the marina.

(d) The race referee's motor launch breaks down at the point (4, 7.5). Will this interfere with the paths taken by the yachts?

12 After her bath, Anne removes the plug to empty it. The water flows out of the bathtub at a rate of 0.75 L/s. There were originally 225 L of water in the bath.

(a) Write an equation relating the volume of water left in the bath (V) with the time in seconds (T).

(b) How long does it take to empty the bath?

13 Since the beginning of August, a local reservoir has been losing water at a constant rate. The Water Resources Department estimates that on 12 August the reservoir held 200 million kL of water, and on 21 August it held 164 million kL.

(a) Develop a linear equation relating V, the volume of water remaining (millions of kL), and n, the number of days since the beginning of August (when the reservoir was full).

(b) Draw a graph corresponding to the linear equation developed above to find the maximum capacity of the reservoir.

(c) Why should the graph only be drawn in the first quadrant?

14 A cubic block of ice has an edge of 10 cm. It melts so that its volume decreases at a constant rate (k) and the block remains a cube.

(a) What is the volume of the cube before it starts to melt?

(b) Write an equation relating (V) the volume of the cube after it has been melting for (t) minutes at a constant rate (k).

(c) What is the volume of the cube if it has an edge of 5 cm after 50 minutes?

(d) Find the value of k.

(e) What does the sign of k tell you about what is happening to the ice cube?

(f) Substitute the value of k found in part **(d)** to rewrite the equation found in part **(b)**.

(g) Draw a graph corresponding to the linear equation developed above to find how long it would take for the block of ice to completely melt.

Hint

The change in volume = kt.

15 At the start of the year, Ringo had $600 in the bank. He has been saving nothing and spending $20 a week. In the same time, Sophia, his sister, had $100 in the bank. She has been saving $30 a week and spending nothing.

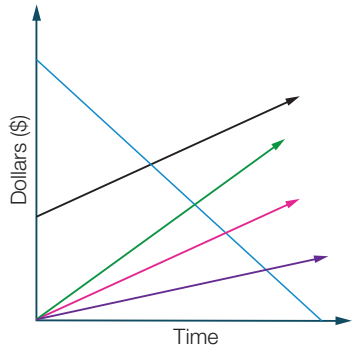
(a) Develop two linear equations relating the amounts of money R and S that Ringo and Sophia each have in their bank accounts, to t, the number of weeks since the start of the year.

(b) Draw a graph corresponding to the linear equations to identify how many weeks it will take for Sophia to have twice the amount of money in the bank as her brother.

Open-ended

16 Write three different linear equations that each have the same gradient, but with a different y-intercept. What property of the three lines indicates that they have the same gradient? Explain.

17 Linear equations are used every day to model real-life situations. The graph shows five lines that model the costs of five different mobile phone plans. Match each graph with a description below and provide an explanation behind each choice.



Note:

- The specific values of gradients and y-intercepts are not needed for this activity.
- The connection fee (also referred to as flagfall) is the fixed cost when a call is made.
- The call rate is the number of cents per minute that is charged based on the duration of the call.

(1) This graph is for the mobile phone plan that has no connection costs and the cost of the call is dependent on the length of the call.

(2) This graph models the cost of the phone plan with no connection fee and the rate at which the call is charged is half the rate of phone plan graph (1).

(3) This graph models the amount of credit remaining on a pre-paid monthly mobile phone plan.

(4) This graph models the cost of a phone plan with a connection fee for every call made and the call rate is equivalent to (1).

(5) This graph models the cost of the phone plan with no connection fee as in (1) but the rate at which the call is charged is one and a half times greater.

18 Sketch three different graphs that have gradients between -1 and -2, and y-intercepts between 1 and 4.

Outside the Square — Problem solving

Odd one out

Four of these points lie on a single straight-line. Which one is the odd one out?

(2, 5) (-5, -7) (2, 4) (5, 13) (9, 21)

Strategy options

- Draw a diagram.
- Look for a pattern.
- Test all possible combinations.

Half-time 4

1 Sketch the graph of the following, showing the intercepts on the axes. Ex 4.3

(a) $y = 3x + 2$ (b) $y = 5 - x$

(c) $y = 2x - 3$ (d) $y = 6 - 3x$

2 Calculate the gradient of the line passing through each pair of points. Ex 4.2

(a) (1, 3) and (5, 5) (b) (2, 4) and (6, -4)

(c) (-5, -3) and (5, 2) (d) (-4, 8) and (-2, -8)

3 Sketch the graph of the following, showing the intercepts on the axes. Ex 4.3

(a) $2x - 3y = 6$ (b) $5x + 2y = 10$

(c) $3x - y = 6$ (d) $3x + 4y = 12$

4 The Celsius temperature scale is used in Australia, whereas the Fahrenheit temperature scale is used in the USA. The freezing point of water is 0° and the boiling point is 100° in the Celsius scale. The freezing point of water is 32° and the boiling point is 212° in the Fahrenheit scale. Show that the formula to convert a Fahrenheit temperature to a Celsius temperature can be expressed as $C = \frac{5}{9}(F - 32)$. Ex 4.1

5 Sketch the graph of the following, showing the intercept with the axes and one other point. Ex 4.3

(a) $y = -3x$ (b) $y = 4x$

(c) $x + 2y = 0$ (d) $3y = 5x$

6 Solve each of the following equations. Ex 4.1

(a) $6x + 2 = 14$ (b) $3(x + 5) = 21$

(c) $\frac{2x - 3}{7} = 5$ (d) $3x + 4 = 16 - x$

(e) $\frac{2x + 4}{5} = \frac{3x - 3}{3}$ (f) $\frac{3x + 2}{2} - \frac{x + 10}{7} = 5$

7 Sketch the graph of each of the following. Ex 4.3

(a) $y = 4$ (b) $x = -2$

(c) $y = -3$ (d) $x = 8$

8 Find the equation of the line with a: Ex 4.2

(a) gradient of 4 passing through the point (-1, 5)

(b) gradient of -1 passing through the point (3, -7).

Maths 4 Real

Mobile phone plans

There are just as many mobile phones as people in Australia and there are many different telecommunication companies that are after your dollar. There are two main types of mobile phone plans.

Each type of mobile plan has its pros and cons. A wide range of services are available within these two types of plans such as 'off-peak' and 'peak' call rates, which depend on the time of day. Generally, the peak rate applies from 8 a.m. to 8 p.m. and the off-peak rate applies from 8 p.m. to 8 a.m. There is also a 'flagfall', or connection fee for most calls. To make it more complex, some plans have 'free' calls and/or text messages to other phones on their network. Because of the huge number of products available, it can be difficult to choose the best plan.

1. **Pre-paid credit**—You don't have a regular monthly plan, but purchase credit in advance of your use. The credit is generally for a defined period of time, after which it expires. The credit is used for particular services such as calls and text messages. When you have spent your credit, you are unable to make additional calls or text messages.
2. **Post-paid plan or cap**—You enter a formal contract for a defined period (usually 24 or 36 months) in which you pay the same amount of money each month for a defined quota of specific services such as calls, texts and internet data. If you don't use your defined quota, you still pay the monthly contract amount. However, if you exceed this quota you are charged on top of your monthly contract amount.

Company	Access fee per month	Call rate (per 30 seconds)	Off-peak calls (per 30 seconds)	Text message	Flagfall	Bonus
Tesla	$0	45c	30c	30c	0c	Nil
Big Phone	$10	30c	20c	20c	70c	$5 of free calls per month
4 Mobile	$20	20c	10c	10c	73c	$10 of free calls per month
Crazy Joe's	$45	10c	5c	5c	75c	$20 of free calls per month

Which is the best deal?

Your parents have agreed to buy you a mobile phone on the condition that you select a plan that costs no more than $50 per month. You do some research on the internet and obtain the above post-paid mobile phone plans from the major telecommunication companies in your area.

1. You estimate that you would make 180 minutes of calls during peak time, 30 minutes of calls during the off-peak time and 100 text messages during a month. Use this estimate to calculate the cost of each plan.
2. Do any of the plans cost $50 or less? Which is the best plan for your estimated usage? Explain your answer.
3. Upon further consideration you decide that you will make all calls during peak time. Create a linear equation that models the cost for each plan if all the calls are made during peak time, taking into account the access fee.
4. Use each of the equations to determine the cost of each plan if 4 hours of peak-time calls were made in the month.
5. Graph the four linear equations on the same set of axes (with time t along the horizontal axis and cost c along the vertical axis).
6. Compare and contrast each of the graphs to identify which plan would be good for a low user (e.g. 60 minutes of peak time calls per month), a moderate user (e.g. 180 minutes of peak time calls per month) and a heavy user (e.g. more than 240 minutes of peak-time calls per month).

Investigation

Breaking even

Equipment required: 1 brain, graph paper

The Big Question

Angelo's Ice-Cream Parlour sells ice-cream in a plain cone for \$3.50 and in a waffle cone for \$4. Angelo orders 200 plain cones and 100 waffle cones each day. He usually sells 220 ice-cream cones per day. Some days he earns a lot more money than on other days and he wonders if there is a certain number of each type of cone that will maximise the amount of money he earns. (The amount he earns is called revenue.) What advice should Angelo be given to solve his problem?

Engage

1 **(a)** On a set of axes, sketch the graph of $y = 10 - x$.

(b) Plot the following points and decide which lie:

(i) on the line

(ii) below the line

(iii) above the line.

(1, 9) (1, 6) (3, 5) (3, 7) (6, 4)
(6, 8) (4, 7) (4, 2) (2, 8) (2, 3)

(c) Shade the region above the line. Check by substitution that all points given in **(b)** that lie in this shaded region satisfy the inequality $y > 10 - x$.

(d) Shade differently the region below the line. Check by substitution that all points given in **(b)** that lie in this shaded region satisfy the inequality $y < 10 - x$.

2 On a new set of axes, sketch the graphs of $x + 2y = 6$ and $4x + 3y = 12$.

(a) Shade $x + 2y < 6$ and shade differently $4x + 3y > 12$.

(b) Show by substitution into the inequalities that (4, 0) lies in the region where the shadings overlap.

Explore

3 To understand Angelo's problem and find a solution, complete the following.

(a) On the same set of axes, sketch the graphs of:

(i) $x + y = 220$ **(ii)** $x = 200$ **(iii)** $y = 120$

(b) Shade the region that is common to the graphs of:

(i) $x + y \leq 220$ **(ii)** $x \leq 200$ **(iii)** $y \leq 120$

(c) This shaded region is called the feasible region. Find the coordinates of all the vertices of this region and label these points on your graph.

(d) Substitute the coordinates of each vertex into the expression $3.5x + 4y$ and determine which vertex gives the maximum value of this expression.

Strategy options

Look for a pattern.

Draw a diagram.

Test all possible combinations.

Explain

4 (a) Explain the meaning of $x + y \leq 220$ in relation to Angelo's ice-creams.

(b) Why did Angelo graph $x \leq 200$ and $y \leq 120$?

(c) Explain why the shaded region is called the feasible region.

(d) Explain what was calculated when the vertices of the feasible region were substituted into the expression $3.5x + 4y$.

Elaborate

5 Use the graphs you sketched and your calculations to advise Angelo about the numbers of each type of cone he should sell to maximise his revenue.

Evaluate

6 (a) How confident are you that you have given Angelo the correct advice? Justify your answer.

(b) What did you learn by completing Questions **1**, **2** and **3**? How did this help you to solve the problem?

(c) How did drawing graphs help you to advise Angelo?

(d) Do you think you could have solved this problem using another method? If so, explain how.

Extend

7 Angelo finds that the days he earns the most revenue are not the days he makes the most profit, as the waffle cones are more expensive for him to buy than the plain cones. If a plain cone with ice-cream costs him \$0.80, a waffle cone with ice-cream costs him \$1.50 and he has overheads (other costs such as wages, electricity and rent) of \$580 per day:

(a) write an equation to show the total cost (C) of producing x plain cones and y waffle cones

(b) write an equation to show the profit, P, of producing x plain cones and y waffle cones.

Hint
Profit = Revenue – Cost

8 (a) If $P = 0$, sketch the graph of the equation in **7 (b)**.

(b) Sketch the graph of $x + y = 220$ on the same axis.

(c) Write down the coordinates of the point of intersection of these two graphs.

(d) What information does this point give Angelo?

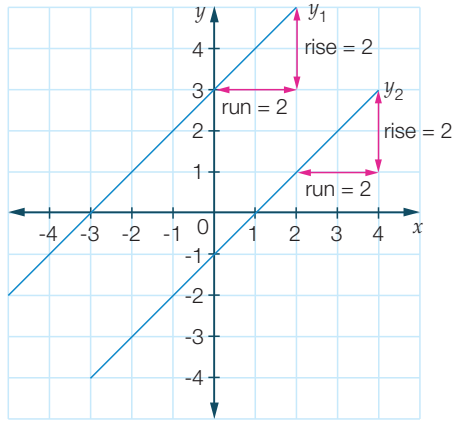
(e) Advise Angelo about the number of plain cones and waffle cones he should sell to make the largest profit.

4.4 Parallel and perpendicular lines

Parallel and perpendicular lines have specific properties associated with their gradients.

Parallel lines

Lines that are parallel to each other will never intersect because they have the same slope or gradient. In the graph shown, we can see that the rise and run of both lines is identical. Hence, they both have the same gradient, which in this case is 1.



Gradient of $y_1 = \frac{\text{rise}}{\text{run}} = \frac{2}{2} = 1$ and y-intercept = 3

Therefore, the equation is:

$y_1 = x + 3$

Gradient of $y_2 = \frac{\text{rise}}{\text{run}} = \frac{2}{2} = 1$ and y-intercept = -1

Therefore, the equation is:

$y_2 = x - 1$

Two lines that are parallel have the same gradient: $m_1 = m_2$.

Worked Example 11 WE11

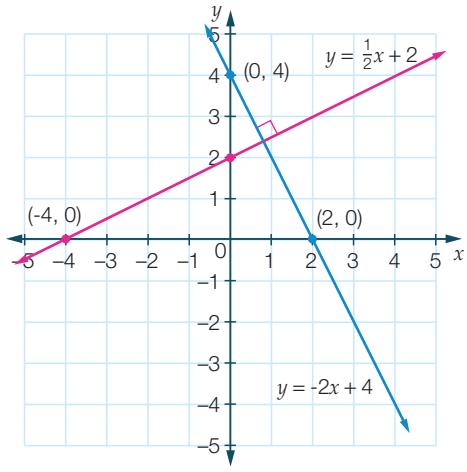
Show that the line with equation $2y = 4x - 9$ is parallel to the line with equation $3y - 6x + 5 = 0$.

Thinking	Working
1 Rearrange both equations using inverse operations to get the equations in the form $y = mx + b$.	$2y = 4x - 9$ $y = 2x - \frac{9}{2}$ $m_1 = 2$ $3y - 6x + 5 = 0$ $3y = 6x - 5$ $y = 2x - \frac{5}{3}$ $m_2 = 2$
2 Compare the gradient values to show that $m_1 = m_2$.	$m_1 = m_2$ Therefore, the lines are parallel.

Perpendicular lines

Lines that are perpendicular intersect at right angles.

The line $y = -2x + 4$ has an x-intercept of (2, 0) and a y-intercept of (0, 4). If these points are rotated 90° anticlockwise about the origin they will be positioned at (0, 2) and (-4, 0). Joining these points gives us a line perpendicular to $y = -2x + 4$.



The gradient of this line is $\frac{1}{2}$ because it has a rise of 2 and a run of 4.

The y-intercept is (2, 0) so the equation of this line is $y = \frac{1}{2}x + 2$.

In the equation $y = -2x + 4$, the gradient $m_1 = -2$.
In the equation $y = \frac{1}{2}x + 2$, the gradient $m_2 = \frac{1}{2}$.

When we multiply m_1 by m_2 we obtain -1.

$$m_1m_2 = -2 \times \frac{1}{2}$$
$$= -1$$

When lines are perpendicular, the product of their gradients is -1.

$$m_1m_2 = -1$$

Therefore, the gradients m_1 and m_2 of perpendicular lines are the negative reciprocal of each other.

$$m_1 = -\frac{1}{m_2} \text{ and } m_2 = -\frac{1}{m_1}$$

Worked Example 12

WE 12

Show that the line with equation $y = 3x - 10$ is perpendicular to the line with equation $6y + 2x + 12 = 0$.

Thinking	Working
1 Rearrange the equations using inverse operations so that both are in the form $y = mx + b$.	$y = 3x - 10$ $6y + 2x + 12 = 0$ $6y = -2x - 12$ $y = -\frac{1}{3}x - 2$
2 Identify the gradients m_1 and m_2, and multiply to determine whether $m_1 \times m_2 = -1$.	$m_1 = 3,\ m_2 = -\frac{1}{3}$ $m_1m_2 = 3 \times -\frac{1}{3}$ $= -1$ Therefore, the lines are perpendicular.

4.4 Parallel and perpendicular lines

Navigator

Answers page 734

Q1, Q2, Q3 Column 1, Q4 Column 1, Q5, Q6, Q7, Q8, Q9, Q10, Q17	Q1, Q2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q15, Q17	Q1, Q2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q8, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Fluency

WE11

1 (a) Show that the line with equation $5x - 2y = 7$ is parallel to the line with equation $2y = 5x + 11$.

(b) Show that the line with equation $10y = 5x + 3$ is parallel to the line with equation $3x - 6y = 2$.

WE12

2 (a) Show that the line with equation $4x + 3y = 7$ is perpendicular to the line with equation $8y = 6x - 2$.

(b) Show that the line with equation $2x - 5y - 1 = 0$ is perpendicular to the line with equation $4y + 10x = 3$.

3 Find the gradient of each of the following pairs of lines and determine if they are parallel.

(a) $y = 3x - 10$ and $y = 3x + 10$

(b) $y = 2x + 5$ and $y = -2x + 6$

(c) $y = \frac{1}{2}x - 4$ and $y = \frac{1}{2}x + 5$

(d) $y + 4x = 4$ and $y = 4x - 8$

(e) $y - 5x = 9$ and $y = 9x + 5$

(f) $y - 2x = 9$ and $y - 2x - 4 = 0$

(g) $2y + 6x - 4 = 0$ and $y + 3x - 8 = 0$

(h) $4y + 12x - 4 = 0$ and $3y + 12x - 9 = 0$

(i) $-2y + 8x - 4 = 0$ and $4y - 16x - 8 = 0$

(j) $5x + 3y = 15$ and $3x - 15y = 10$

4 Find the gradient of each of the following pairs of lines and determine if they are perpendicular.

(a) $y = 2x - 10$ and $2y + x + 12 = 0$

(b) $y - 4x = 10$ and $4y + 2x + 12 = 0$

(c) $y = \frac{1}{3}x$ and $6y + 2x + 12 = 0$

(d) $6y - 3x = 24$ and $2y + 4x + 12 = 0$

(e) $-4y + 3x = 16$ and $8y + 12x + 12 = 0$

(f) $5y - 4x = 25$ and $8y + 5x + 12 = 0$

5 Which of the following lines is parallel to $y = 4x + 12$?

A $y = -\frac{1}{4}x + 12$ B $4y = 4x + 12$ C $2y - 8x + 6 = 0$ D $2y + 8x + 12 = 0$

6 Which of the following lines is perpendicular to $y = 3x - 9$?

A $y = 3x + 9$ B $6y = 9x + 12$ C $y = -3x - 9$ D $9y + 3x = 27$

Understanding

7 Show that the line joining (-2, 2) and (4, 8) is parallel to $y = x + 10$.

8 Show that the line joining (-2, 5) and (4, -7) is parallel to $2x + y - 10 = 0$.

9 Show that the line joining (-4, 1) to (0, 4) is perpendicular to $y = -\frac{4}{3}x - 2$.

10 Show that the line joining (4, 5) to (9, 8) is perpendicular to $y = -\frac{5}{3}x + 10$.

Reasoning

11 **(a)** Which two of the following lines are parallel?

(b) Which two of the following lines are perpendicular?

(i) $y = 3x + 3$

(ii) $3x + y = 3$

(iii) $3x - y = 3$

(iv) $3x + 3y = 2$

(v) $3y = x + 3$

12 Find the equation of the line through (2, -3) that is perpendicular to $y = -\frac{1}{3}x + 11$.

13 Find the equations of the lines through (0, 5) that are parallel and perpendicular to $2x + y - 10 = 0$.

14 Show that the points $A(-1, -1)$, $B(1, -4)$ and $C(2, 1)$ form a right-angled triangle ABC.

15 Show that the points $A(-3, 1)$, $B(-1, 3)$, $C(3, 1)$ and $D(1, -1)$ form the vertices of a parallelogram.

16 Show that the points $A(-4, 3)$, $B(-2, -2)$, $C(3, 0)$ and $D(1, 5)$ form the vertices of a square.

Open-ended

17 Identify three different straight-line equations that are:

(a) parallel to $3x - 6y + 9 = 0$

(b) perpendicular to $4x - 12y - 18 = 0$.

Outside the Square — Problem solving

Area of a triangle

Find the area (in square units) of the triangle formed by the three lines whose equations are:

1 $y - x - 6 = 0$

2 $x - 2y = 3$

3 $x = -y + 6$

Strategy options

- Draw a diagram.
- Break problem into manageable parts.

Technology Exploration CAS

Equipment required: 1 brain, a TI-Nspire CAS or ClassPad CAS

Versions of this Exploration for other technologies are available in Pearson Reader.

The art of linear relationships

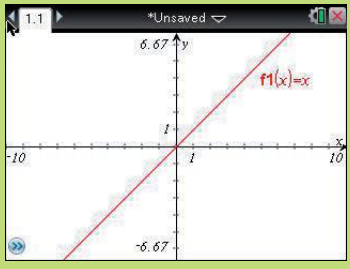
The following designs were all created by using linear relationships. Your task is to work out the equations in order to recreate the designs. You may work alone or in small groups. You can attempt the designs in any order. Once you have worked out a design, write down the equations used. After you have finished these designs, make up your own. Challenge a friend to reproduce your designs. Instructions for drawing a graph are shown below.

Using the TI-Nspire CAS

Press the home page and, under Scratchpad, select B: Graph. Press (enter), Press (menu), select Graph type > 1:Function

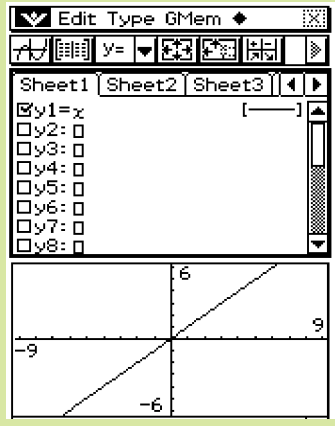
Adjust your view window settings using (menu) > 4 Window /Zoom and press 1 Window Settings.

At the bottom of your screen, enter your equation. Press (menu).



Using the ClassPad

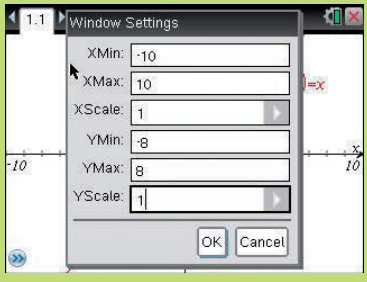
From the Menu tap Graph&Tab... . Input your equations and check the box on the LHS of each one. Adjust your View window using the icon. Tap the graph icon.



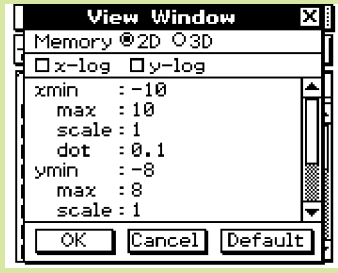
The window settings for each group of designs are shown.

Group A Designs

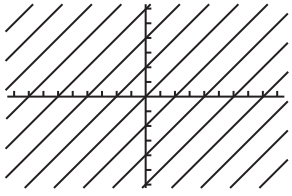
Using the TI-Nspire CAS



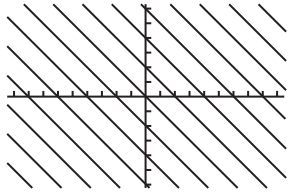
Using the ClassPad



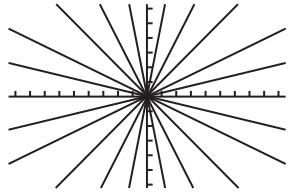
(a)



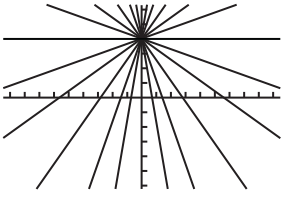
(b)



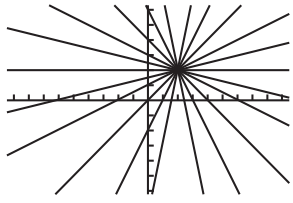
(c)



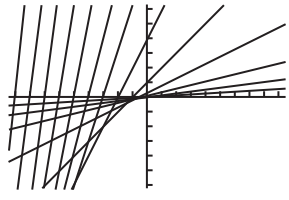
(d)



(e)

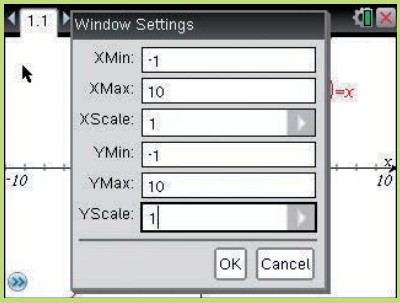


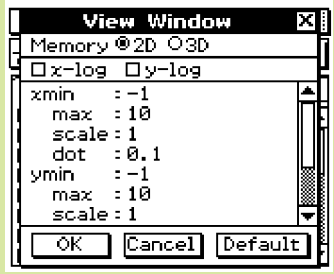
(f)



Group B Designs

Using the TI-Nspire CAS	Using the ClassPad





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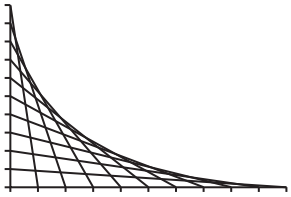

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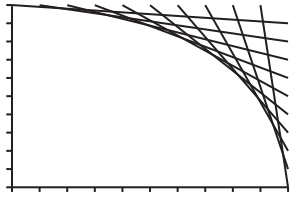

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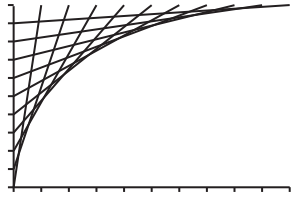

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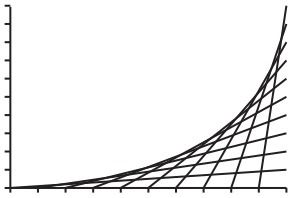

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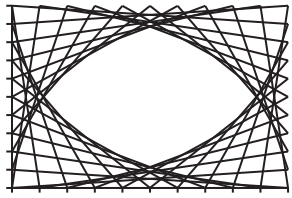

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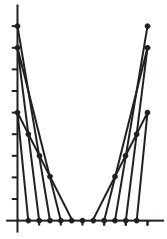
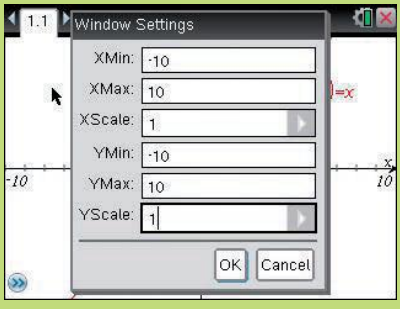
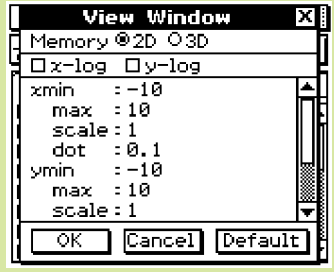


Group C Designs

Using the TI-Nspire CAS	Using the ClassPad





(a)

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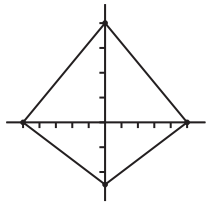

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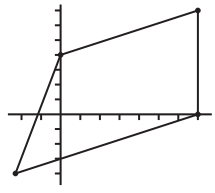

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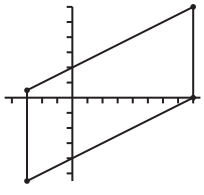

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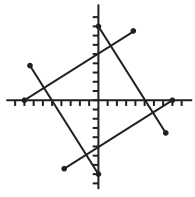

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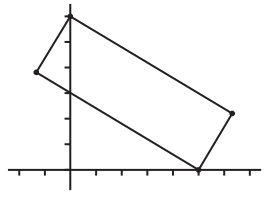


4.5 Linear inequalities

Inequality symbols

$<$ means less than

$>$ means greater than

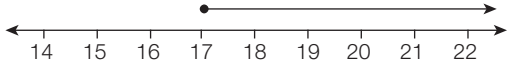
$\leq$ means less than or equal to

$\geq$ means greater than or equal to

What is an inequality?

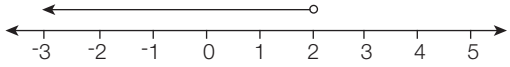
An equation is a statement of equality, such as $x = 17$. A **linear inequality** is a statement such as $x \geq 17$, which means that x can be any value greater than or equal to 17.

On a number line, $x \geq 17$ can be represented as:



An arrow is used to indicate that the inequality represents all the numbers beyond this point. A closed circle is drawn at 17 to show that 17 is included.

Similarly, on a number line $x < 2$ can be represented as:



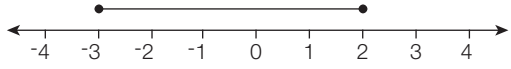
An open circle has been drawn at 2 to indicate that 2 is not included.

If the inequality includes the end point, a closed circle is used; these are inequalities that contain $\geq$ or $\leq$.

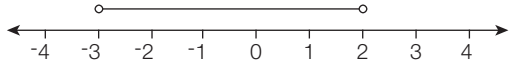
If the inequality does not include the end point, an open circle is used; these are inequalities that contain $>$ or $<$.

Sometimes, we need to represent a group of numbers between two values.

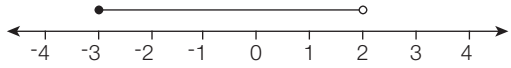
$-3 \leq x \leq 2$ represents all the values of x between, and including, -3 and 2 and is represented on the number line on the right. The closed circles at the end points indicate that -3 and 2 are included.



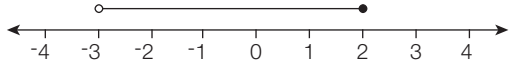
If the end values are not included, we would write $-3 < x < 2$ and this is represented on the number line on the right. The open circles at the end points indicate that -3 and 2 are not included.



If only the lower end of the interval is included we could write $-3 \leq x < 2$ and this is represented on the number line on the right. The closed circle indicates that -3 is included. The open circle indicates that 2 is not.



If only the upper end of the interval is included we could write $-3 < x \leq 2$ and this is represented on the number line on the right. The closed circle indicates that 2 is included. The open circle indicates that -3 is not.



Worked Example 13

WE13

Represent the values of x on a number line for each of the following.

(a) $x \leq 5$ **(b)** $x > -2$ **(c)** $3 < x \leq 7$

Thinking	Working
(a) Because x also equals 5, draw a closed circle at 5 and draw the arrow to the left.	(a) $x \leq 5$
(b) Because -2 is not included in the group of values required, draw an open circle at -2 and draw the arrow to the right.	(b) $x > -2$
(c) The values between 3 and 7 are required as well as 7, so draw an open circle at 3 but a closed circle at 7. Join these with a line.	(c) $3 < x \leq 7$

Just as we can have linear equations, such as $2x - 5 = 8$, we can also have linear inequalities such as $7x - 5 \leq 57$.

To solve an inequality

We solve linear inequalities using the same inverse operations in the opposite order as we use to solve linear equations. Unlike linear equations, inequalities have a range of solutions and we need to be careful when multiplying or dividing by a negative number.

Consider $4 < 8$.

If we multiply both sides by -1, what happens to the inequality sign?

To keep the inequality correct, the sign needs to be reversed.

$-4 > -8$

When we multiply or divide both sides of the inequality by a negative number, the inequality sign is reversed.

$-2x < 8$

$x > \frac{8}{-2}$ (inequality is reversed)

$x > -4$

Worked Example 14

WE14

Solve for x in each of the following inequations.

(a) $x + 3 \geq 12$ **(b)** $4x - 3 < -7$

Thinking	Working
(a) 1 Deconstruct the inequality with an inverse operation (-3 from both sides).	(a) $x + 3 \geq 12$
2 State the solution.	$x \geq 9$

(b) 1	Deconstruct the inequality with inverse operations (+ 3, ÷ 4).	(b) $4x - 3 < -7$ $4x < -4$
2	State the solution.	$x < -1$

Worked Example 15

WE15

Solve for x in each of the following.

(a) $-4x \leq 20$ (b) $9 - 2x \geq 5$

Thinking		Working
(a) 1	Deconstruct the inequality with inverse operations (÷ -4). When dividing by a negative number, reverse the inequality sign.	(a) $-4x \leq 20$
2	State the solution.	$x \geq -5$
(b) 1	Deconstruct the inequalities with inverse operations (– 9, ÷ -2). When dividing by a negative number, reverse the inequality sign.	(b) $9 - 2x \geq 5$ $-2x \geq -4$
2	State the solution.	$x \leq 2$

4.5 Linear inequalities

Navigator

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Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q5, Q6, Q7, Q8, Q9, Q10 Column 1, Q11 Column 1, Q13, Q15	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7, Q8, Q9, Q10 Column 2, Q11 Column 2, Q13, Q14, Q16	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q8, Q9, Q10 Column 3, Q11 Column 3, Q12, Q13, Q14, Q15, Q16

Fluency

WE13

1 Represent the values of x on a number line for each of the following.

(a) $x \geq 5$ (b) $x \leq 8$ (c) $x \geq -2$

(d) $x < -4$ (e) $x > -5$ (f) $x > 7$

(g) $x \leq 0$ (h) $x \geq 2$ (i) $x \leq -5$

(j) $x > 1$ (k) $x > 0$ (l) $x < -2$

(m) $-2 \leq x \leq 4$ (n) $1 \leq x \leq 6$ (o) $0 < x < 5$

(p) $-3 < x < -1$ (q) $-4 \leq x < -2$ (r) $-4 < x \leq 0$

2 Solve for x in each of the following. WE14

(a) $x + 6 > -4$ **(b)** $x - 3 < -8$ **(c)** $x + 12 \leq 2$

(d) $\frac{x}{5} \leq 7$ **(e)** $\frac{x}{9} \geq -1$ **(f)** $\frac{x}{7} > -4$

(g) $3x < -16$ **(h)** $5x > -27$ **(i)** $9x \leq -81$

(j) $7x - 3 \geq 53$ **(k)** $4x + 11 \leq 3$ **(l)** $3x + 24 > 0$

(m) $\frac{4x - 1}{5} > 3$ **(n)** $\frac{2x + 5}{3} < 9$ **(o)** $\frac{3x + 2}{7} \leq 2$

(p) $\frac{3x}{4} + 1 \leq -5$ **(q)** $\frac{7x}{11} - 2 > 5$ **(r)** $\frac{4x}{5} - 10 > -2$

3 Solve for y in each of the following. WE15

(a) $-3y \geq 15$ **(b)** $-8y \leq 48$ **(c)** $-5y \leq 25$

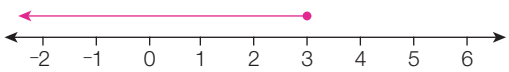
(d) $-y < -12$ **(e)** $-2y > -42$ **(f)** $-3y > -36$

(g) $1 - 2y \leq 17$ **(h)** $8 - 5y \geq -2$ **(i)** $5 - 3y \leq 17$

(j) $5(2 - 3y) \geq 11$ **(k)** $4(3 - 7y) \leq -2$ **(l)** $3(5 - 2y) \geq -4$

(m) $\frac{8 - 3y}{5} > -1$ **(n)** $\frac{14 - 6y}{3} < -3$ **(o)** $\frac{12 - 5y}{4} > -2$

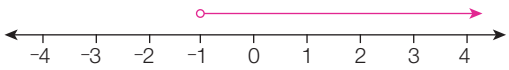
4 **(a)**



This number line represents:

A $x < 3$ **B** $x \leq 3$ **C** $x \geq 3$ **D** $x > 3$

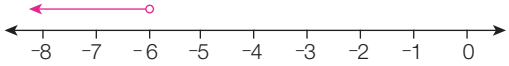
(b)



This number line represents:

A $x < -1$ **B** $x \leq -1$ **C** $x \geq -1$ **D** $x > -1$

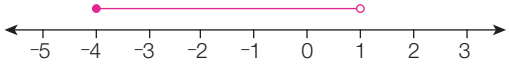
(c)



This number line represents:

A $x < -6$ **B** $x \leq -6$ **C** $x \geq -6$ **D** $x > -6$

(d)



This number line represents:

A $-4 < x \leq 1$ **B** $-4 \geq x > 1$ **C** $-4 \leq x \leq 1$ **D** $-4 \leq x < 1$

5 Which alternative is *not* a possible solution to the inequality given?

(a) $2x + 3 \geq 7$

A $x = 1$ **B** $x = 2.1$ **C** $x = 3$ **D** $x = 5.2$

(b) $5x - 2 < -9$

A $x = -5$ **B** $x = -3$ **C** $x = -2$ **D** $x = 0$

(c) $11 - 4x < -1$

A $x = 10$ **B** $x = 7$ **C** $x = 4$ **D** $x = 2$

(d) $\frac{6x + 1}{2} > 5x - 2$

A $x = -2$ **B** $x = -1$ **C** $x = 1$ **D** $x = 2$

Understanding

6 The reserve price at an auction of a house is \$270 000. The house is sold at auction.

(a) Represent the selling price (P) on a number line.

(b) Write down the linear inequality for the selling price (P).

(c) Write down the inequality for the selling price (P) given that the house sold for less than \$320 000.

Hint
The reserve price is the smallest amount that the house could be sold for.

7 The speed limit on a particular freeway is 100 km/h. Write down an inequality that represents all possible legal speeds on that freeway.

8 A sales representative needs to sell at least five vacuum cleaners a week to receive a bonus commission. Write down an inequality for the number of vacuum cleaners (n) the sales representative needs to sell if he wants to earn a bonus.

9 For the wool and other materials to make seven hand-knitted jumpers, Lucy and Lindsay spent \$385. Write down an inequality to show the selling price (j) for each jumper if they wish to make a profit.

10 Find the solutions for each of the following.

(a) $5x - 3 \geq 3 - 4x$ (b) $2x + 7 \leq 4 - 9x$ (c) $6x + 2 < 3x + 2$

(d) $2(x + 10) > 10 - 3x$ (e) $15x - 4 \leq 4(3x + 2)$ (f) $5(2x - 3) \geq 8x + 9$

(g) $\frac{2x+1}{7} > \frac{x-3}{4}$ (h) $\frac{3x-4}{2} < \frac{4x+7}{5}$ (i) $\frac{2-3x}{2} \leq \frac{5x-16}{3}$

11 Find the solutions for each of the following.

(a) $y + 6 \leq 2y + 7$ (b) $20 - 3y > 5 + 2y$ (c) $7y - 4 \geq 3 + 10y$

(d) $13 - y < 5y + 11$ (e) $4 - \frac{y}{3} \leq -6$ (f) $7 - \frac{y}{6} \geq 1$

(g) $-3 - \frac{5y}{12} < 2$ (h) $\frac{-7y}{3} + 2 > 0$ (i) $\frac{3y+2}{11} > \frac{2y-5}{3}$

Hint
When multiplying or dividing by a negative, the inequality sign reverses.

Reasoning

12 A shopkeeper can fit a total of only 120 2-litre cartons of milk (both full cream and low fat) into the refrigerator. He knows that he needs at least 50 2-litre cartons of full cream milk and that no more than 40 2-litre cartons of low fat milk will be needed.

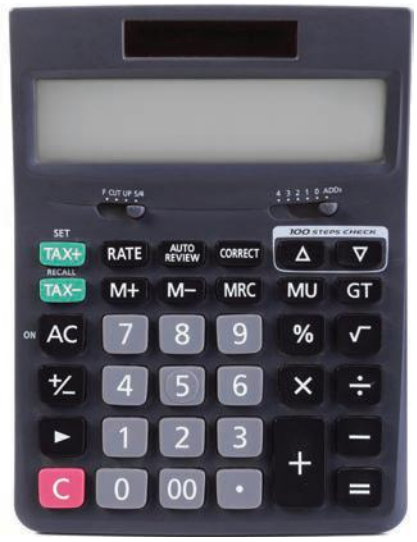
(a) Let f represent the number of cartons of full cream milk and l represent the number of cartons of low fat milk. Write down the two inequalities to show the number of cartons of each type of milk required by the shopkeeper and show them on a number line.

(b) Write down an inequality to show the number of cartons of milk in the refrigerator.

13 Tony has spent \$428 to buy 20 calculators wholesale.

(a) Write an inequality to represent the price (p) he could sell each calculator for and make a profit.

(b) Solve the inequality to find the minimum price that he could sell the calculators for and still make a profit.



14 At the local milk bar, Sue spends \$3.45 on milk and bread. With the change from her \$5 note, she buys some liquorice straps at 40c each.

(a) Write an inequality to represent the number of liquorice straps (n) she could buy.

(b) Solve the inequality to find the maximum number of liquorice straps she could buy.

Open-ended

15 Draw a number line showing numbers that could be greater than or equal to a negative number, but less than a positive number.

16 **(a)** Make up an inequality, with the variable on both sides of the inequality, that will produce solutions that are all negative.

(b) Make up an inequality, with the variable on both sides of the inequality, that will produce solutions that are all positive.

Outside the Square — Problem solving

Win ratio

A tennis player computes her win ratio by dividing the number of matches she has won by the total number of matches she has played. At the start of a weekend, her win ratio is exactly 0.5 or $\frac{1}{2}$. Over the course of the weekend, she plays four matches, winning three and losing one. At the end of the weekend, her overall win ratio is greater than 0.503. What is the greatest number of matches she could have won before the weekend began?

Hint

Let the number of wins at the start of the weekend be w. Given the win ratio was 0.5 or $\frac{1}{2}$, what is the relationship between the number of wins, w, and the total number of games, n?

Strategy options

- Break problem into manageable parts.
- Test all possible combinations.

4.6 Simultaneous equations

You have learnt how to solve an equation to find an unknown quantity. For example, you are able to find a in the equation $3a + 2 = 17$. But many problems involve more than one unknown quantity.

For example, suppose the McKillop and Morris families go to the cinema. The McKillop family pays \$48 for their tickets (two adults and three children) whereas the Morris family pays \$44 for theirs (one adult and four children). The eldest son in the Morris family wants to pay for his own ticket, but no one can remember the individual ticket prices for adults and children. Can they be worked out?

Let a = cost of an adult's ticket and c = cost of a child's ticket.

For the McKillop family: 2 adult's tickets + 3 child's tickets = \$48
so $2a + 3c = 48$ (1)

For the Morris family: 1 adult's ticket + 4 child's tickets = \$44
so $a + 4c = 44$ (2)

We now have two equations involving the unknowns a and c. These are called **simultaneous equations**. To solve this problem, we need to find the values of a and c that make *both* equations true.

Solving simultaneous equations graphically

If we sketch the graphs of two linear equations, the point of intersection will lie on both graphs and will satisfy both equations. There will only be one point of intersection and therefore the equations will have only one solution. Therefore, graphs of linear equations can be used to solve simultaneous equations.

However, there are two important exceptions to this: parallel and coincident lines.

Parallel lines

Parallel lines do not intersect. As there is no point of intersection, the equations of lines that are parallel have no solution. Any equations that have the same gradient, but are not multiples of each other, will be parallel and will therefore have no solution. For example, $y = 2x + 4$ and $y = 2x + 6$ are parallel. No point lies on both lines and therefore no point satisfies both equations. Thus, the simultaneous equations have no solution.

Coincident lines

If one equation is a multiple of the other, the gradients will be the same, but the equations do not represent parallel lines. The graph of one line will be identical to the graph of the other, because the equations represent the same line. These linear graphs are called **coincident lines**. Therefore, there are an infinite number of solutions to these type of equations because every point on either line satisfies both equations.

For example, $2x - y = 5$ and $4x - 2y = 10$ are coincident lines and have an infinite number of solutions.

Solving unique simultaneous equations

Equations that give non-parallel and non-coincident graphs are called **unique equations** and have only one point of intersection and therefore only one solution.

Because the point of intersection satisfies both equations, graphs can be used to find the solution to simultaneous equations.

Method 1: Solving simultaneous equations graphically

Such equations are called simultaneous equations because we wish to solve both of them at the same time. Simultaneous equations can be solved graphically or algebraically. The solution of simultaneous equations gives values for each variable that satisfy all equations. We can solve for any number of variables as long as we have the same number of unique equations. In this section, we will deal with two variables only.

Consider the cinema problem: $2a + 3c = 48$ (1)
$a + 4c = 44$ (2)

Both these equations can be graphed on the same set of axes. For accuracy, graph paper should be used.

For $2a + 3c = 48$:

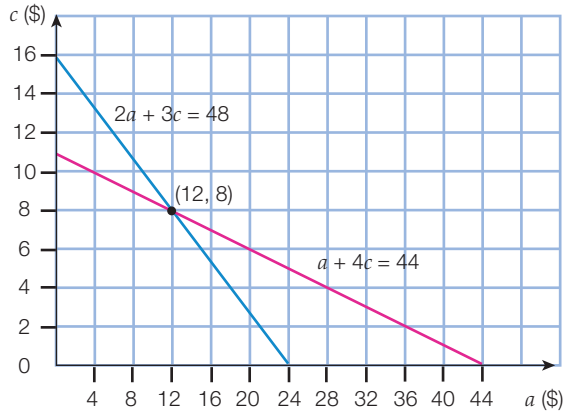
When $a = 0$, $3c = 48$
$c = 16$
y-intercept (0, 16)

When $c = 0$, $2a = 48$
$a = 24$
x-intercept (24, 0)

For $a + 4c = 44$:

When $a = 0$, $4c = 44$
$c = 11$
y-intercept (0, 11)

When $c = 0$, $a = 44$
x-intercept (44, 0)



From the graph it can be seen that the two lines cross at the point (12, 8). This means that $a = 12$, $c = 8$ satisfies both equations. We can check this by substituting these values for a and c back into our original equations.

For $2a + 3c = 48$:

$$\begin{aligned} \text{LHS} &= 2 \times 12 + 3 \times 8 \\ &= 24 + 24 \\ &= 48 \\ &= \text{RHS} \end{aligned}$$

For $a + 4c = 44$:

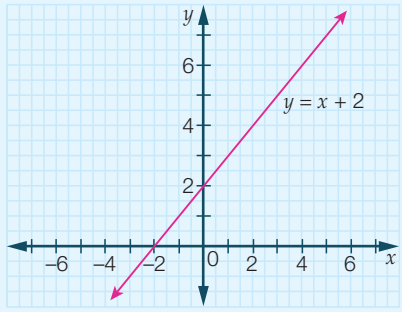
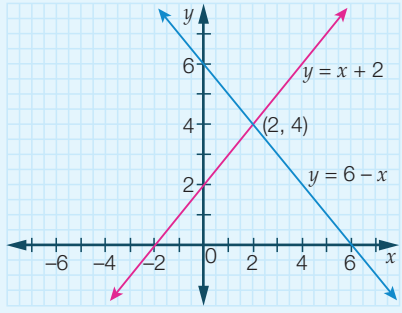
$$\begin{aligned} \text{LHS} &= 12 + 4 \times 8 \\ &= 12 + 32 \\ &= 44 \\ &= \text{RHS} \end{aligned}$$

So, an adult's ticket to the cinema costs \$12 and a child's ticket costs \$8.

Worked Example 16 WE16

(a) Sketch the pair of linear equations $y = x + 2$ and $y = 6 - x$ and find the point of intersection.

(b) Hence, find the solution to $y = x + 2$ and $y = 6 - x$.

Thinking	Working
(a) **1** Sketch the graph of one of the equations ($y = x + 2$).	(a) 
2 Sketch the graph of the other equation ($y = 6 - x$) ensuring that the point of intersection can be seen clearly. You may have to extend your graphs to see the point where the lines cross.	
3 Find the coordinates of the point of intersection on your graph.	
4 Write down the coordinates of the point of intersection.	Point of intersection = (2, 4)
(b) The point of intersection gives the solution to the equations	(b) $x = 2$, $y = 4$ is the solution to the equations $y = x + 2$ and $y = 6 - x$

Notice also that the point of intersection (2, 4) satisfies both equations.

$y = x + 2$
$= 2 + 2$ when $x = 2$
$= 4$

$y = 6 - x$
$= 6 - 2$ when $x = 2$
$= 4$

Method 2: Solving simultaneous equations algebraically

Using algebra can be a more accurate and efficient way of solving simultaneous equations than the graphical method. The graphical method does not work for more complex simultaneous equations with more than three unknowns and it is difficult to get exact solutions if one or both of the values at the point of intersection is not a whole number, unless technology is used.

Simultaneous equations can be solved algebraically using either elimination or substitution.

Solving simultaneous equations by elimination

We usually use elimination when both the variables are on the same side of the equation.

We eliminate one of the variables by adding or subtracting the two equations or multiples of the two equations. When one of the variables has a coefficient of the same magnitude we can add or subtract the equations.

Worked Example 17

WE17

Solve the following pairs of simultaneous equations algebraically using the elimination method.

(a) $3x + 5y = -3$
$x - 5y = 19$

(b) $4x + 5y = -6$
$4x - 3y = 10$

Thinking		Working	
(a) 1	Write the equations and label them.	(a) $3x + 5y = -3$ $x - 5y = 19$	(1) (2)
2	Eliminate the variable with the same coefficient by either adding or subtracting. (Here, we use equations (1) and (2) to eliminate y.) $\begin{array}{rl} & 3x + 5y = -3 \\ + & x - 5y = 19 \\ \hline & 4x + 0 = 16 \end{array}$	$4x = 16$	(1) + (2)
3	Solve the equation obtained for the remaining variable.	$x = 4$	
4	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $x = 4$ into equation (1).)	$3x + 5y = -3$ $3 \times 4 + 5y = -3$ $12 + 5y = -3$ $5y = -15$ $y = -3$	(1)
5	Check the solution by substituting the values obtained into the other original equation.	$x - 5y = 19$ LHS $= x - 5y$ $= 4 - 5 \times -3$ $= 4 + 15$ $= 19$ $=$ RHS	(2)
6	Write the solution for both variables.	$x = 4$, $y = -3$ or $(x, y) = (4, -3)$	
(b) 1	Write the equations and label them.	(b) $4x + 5y = -6$ $4x - 3y = 10$	(1) (2)
2	Eliminate the variable with the same coefficient by either adding or subtracting. (Here, we eliminate x by subtracting equation (2) from (1).)	$8y = -16$	(1) – (2)
3	Solve the equation obtained for the remaining variable.	$y = -2$	
4	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $y = -2$ into equation (1).)	$4x + 5y = -6$ $4x + 5 \times -2 = -6$ $4x - 10 = -6$ $4x = 4$ $x = 1$	(1)

5	Check the solution by substituting the values obtained into the LHS of the other original equation. (Here, we substitute $x = 1$ and $y = -2$ into $4x - 3y$.) The value obtained should equal the value of the RHS.	$\text{LHS} = 4x - 3y$ $= 4 \times 1 - 3 \times -2$ $= 4 + 6$ $= 10$ $= \text{RHS}$
6	Write the solution for both variables.	$x = 1, y = -2$ or $(x, y) = (1, -2)$

However, not all pairs of simultaneous equations can be solved by simply adding or subtracting the two equations to eliminate one of the variables. If the coefficients of neither variable are equal in magnitude, we need to make the coefficients of one of the variables equal by multiplying by the LCM.

Worked Example 18

WE18

Solve the following simultaneous equations algebraically using the elimination method.

(a) $3x + 2y = 5$
$4x - 3y = -16$

(b) $2x + 5y = 9$
$x + 3y - 5 = 0$

Thinking		Working		
(a) 1	Write the equations and label them.	(a) $3x + 2y = 5$ $4x - 3y = -16$	(1) (2)	
2	Choose the variable you wish to eliminate and multiply one or both equations by different numbers so the coefficient of that variable in both equations is the lowest common multiple (LCM). Label your new equations. (Here, we multiply (1) by 3 and (2) by 2 so that the coefficient of y is 6 in both equations.)	$9x + 6y = 15$ $8x - 6y = -32$	(3) (4)	(1) × 3 (2) × 2
3	Eliminate the variable with the same coefficient by either adding or subtracting. (Here, we add (3) and (4) to eliminate y.)	$17x = -17$	(3) + (4)	
4	Solve the equation obtained for the remaining variable.	$x = -1$		
5	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $x = -1$ into (1).)	$3x + 2y = 5$ $3 \times -1 + 2y = 5$ $-3 + 2y = 5$ $2y = 8$ $y = 4$	(1)	
6	Check the solution by substituting the values obtained into the LHS of the other original equation. (Here, we substitute $x = -1$ and $y = 4$ into $4x - 3y$.) The value obtained should equal the value of the RHS.	$\text{LHS} = 4x - 3y$ $= 4 \times -1 - 3 \times 4$ $= -4 - 12$ $= -16$ $= \text{RHS}$		
7	Write the solution for both variables.	$x = -1, y = 4$ or $(x, y) = (-1, 4)$		

(b) 1	Write the equations and label them.	(b) $2x + 5y = 9$ $x + 3y - 5 = 0$	(1) (2)	
2	Rearrange one or both equations so that the variables are on the left-hand side of the equation and the numbers are on the right.	$2x + 5y = 9$ $x + 3y = 5$	(1) (3)	
3	Choose the variable you wish to eliminate and multiply one or both equations by different numbers so that the coefficient of that variable in both equations is the LCM. Label your new equations. (Here, we multiply (3) by 2 to obtain $2x$ in both equations.)	$2x + 6y = 10$ $2x + 5y = 9$	(4) (1)	(3) × 2
4	Eliminate the variable with the same coefficient by either adding or subtracting. (Here, we subtract (1) from (4).)	$y = 1$		(4) − (1)
5	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $y = 1$ into (2).)	$x + 3y - 5 = 0$ $x + 3 \times 1 - 5 = 0$ $x - 2 = 0$ $x = 2$	(2)	
6	Check the solution by substituting the values obtained into the LHS of the other original equation. (Here, we substitute $x = 2$ and $y = 1$ into $2x + 5y$.) The value obtained should equal the value of the RHS.	LHS $= 2x + 5y$ $= 2 \times 2 + 5 \times 1$ $= 4 + 5$ $= 9$ $=$ RHS		
7	Write the solution for both variables.	$x = 2, y = 1$ or $(x, y) = (2, 1)$		

Useful strategies

It doesn't matter which variable you choose to eliminate when solving simultaneous equations. You will get the same solution for both variables.

- If you only need to change one equation to get the same coefficient of one of the variables, eliminate that variable.
- If the signs in front of one of the variables are opposite, eliminate that variable.
- If the signs in front of each of the variables are the same, choose the variable to eliminate where the coefficients are smallest in magnitude.
- If the signs in front of each of the variables are the same, multiplying one equation by a negative number and then adding the equations is often easier than multiplying both by a positive number and then subtracting the equations.

Solving simultaneous equations by substitution

This method is commonly used when one or both equations are written with one of the variables as the subject of the equation (e.g. $y = 3x + 5$ as in gradient intercept form or $x = 4 - 3y$).

Worked Example 19

WE 19

Solve the following pairs of simultaneous equations algebraically using the substitution method.

(a) $y = 2x$
$x + y = 18$

(b) $2x + 3y = -4$
$x = y + 3$

Thinking		Working	
(a) 1	Write the equations and label them.	(a) $y = 2x$ $x + y = 18$	(1) (2)
2	Find the equation that has one variable as the subject of the equation and substitute for this variable into the other equation. (Here, we substitute $2x$ for y in (2).)	Substituting $2x$ for y in (2): $x + (2x) = 18$	(3)
3	Simplify the equation obtained and solve the equation for the remaining variable.	$3x = 18$ $x = 6$	
4	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $x = 6$ into (1).)	$y = 2x$ $= 2 \times 6$ $= 12$	(1)
5	Check the solution by substituting the values obtained into the LHS of the other original equation. (Here, we substitute $x = 6$ and $y = 12$ into $x + y$.) The value obtained should equal the value of the RHS.	LHS $= x + y$ $= 6 + 12$ $= 18$ $=$ RHS	
6	Write the solution for both variables.	$x = 6$, $y = 12$ or $(x, y) = (6, 12)$	
(b) 1	Label the equations	(b) $2x + 3y = -4$ $x = y + 3$	(1) (2)
2	Find the equation that has one variable as the subject of the equation and substitute for this variable into the other equation.	Substituting $y + 3$ for x in (1): $2(y + 3) + 3y = -4$	(3)
3	Simplify the equation obtained. (Here, we substitute $(y + 3)$ for x in (1).)	$2y + 6 + 3y = -4$ $5y + 6 = -4$ $5y = -10$	
4	Solve the equation for the remaining variable.	$y = -2$	
5	Find the value of the eliminated variable by substituting the value of the variable obtained into one of the original equations. (Here, we substitute $y = -2$ into (2).)	$x = y + 3$ $= -2 + 3$ $= 1$	(2)

6	Check the solution by substituting the values obtained into the LHS of the other original equation. (Here, we substitute $x = 1$ and $y = -2$ into $2x + 3y$.) The value obtained should equal the value of the RHS.	$\text{LHS} = 2x + 3y$ $= 2 \times 1 + 3 \times -2$ $= 2 - 6$ $= -4$ $= \text{RHS}$
7	Write the solution for both variables.	$x = 1, y = -2$ *or* $(x, y) = (1, -2)$

Using simultaneous equations

Simultaneous equations often involve worded problems. We need to identify the unknowns, define them with a variable, write the information as equations and then then solve them simultaneously. Consider this problem:

A furniture store is having a sale on its tables and chairs. If a table costs \$60 and a chair costs \$25, and a total of 15 tables and chairs were sold for \$480, how many tables and chairs were sold?

The number of tables and the number of chairs sold are the two unknowns. We define this by letting T = number of tables sold and C = number of chairs sold.

We can then write the equations:

$T + C = 15$ (1)

$60T + 25C = 480$ (2)

We can then solve these equations to find that 3 tables and 12 chairs are sold.

$60T + 25C = 480$ (2)

$25T + 25C = 15 \times 25$ (3) (1) × 25
$= 375$

$35T = 480 - 375$ (4) (2) – (3)
$= 105$

$T = 3$

$3 + C = 15$
$C = 12$

Check that you have solved the problem, by substituting into equation (2).

$\text{LHS} = 60T + 25C$
$= 60 \times 3 + 25 \times 12$
$= 180 + 300$
$= 480$
$= \text{RHS}$

Three tables and twelve chairs were sold.

In general, when solving problems using simultaneous equations, these steps need to be followed:

1. Identify the unknowns. Define each with a variable.
2. Express the information given as equations involving the unknowns. (If there are two unknowns, i.e. two pieces of information required, there must be two equations.)
3. Solve the equations simultaneously either using elimination, substitution or graphically.
4. Check your solution by substitution.
5. Express the final answer to the problem in words.

4.6 Simultaneous equations

Navigator

Answers page 736

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q19	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q19	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5, Q6, Q8, Q9, Q10, Q13, Q14, Q15, Q16, Q17, Q18, Q20, Q21

Equipment required: graph paper for Question **1**

Fluency

WE16

1 **(i)** Sketch each pair of linear equations on graph paper and find the point of intersection.

(ii) Hence, find the solution for each pair of simultaneous equations.

(a) $y = x + 3$, $y = 5 - x$

(b) $y = x - 7$, $y = 3 - x$

(c) $x - y = 6$, $2x - y = 8$

(d) $3x - y = 6$, $x - 2y = 2$

(e) $y = 2x$, $y = 3x - 4$

(f) $y = 6 - 2x$, $y = 4x$

(g) $x = -2$, $y = 3x$

(h) $y = -x$, $y = 3$

(i) $y = x + 2$, $y = x + 4$

WE17

2 Solve the following pairs of simultaneous equations using the elimination method.

(a) $x + y = 5$, $x - y = 3$

(b) $x + y = 1$, $x - y = 9$

(c) $2x + 3y = 9$, $5x - 3y = 12$

(d) $3x + 2y = 13$, $9x - 2y = 47$

(e) $3x + 2y = 9$, $-3x + y = 0$

(f) $7x + y = 8$, $-x + y = -8$

(g) $-2x + y = 7$, $2x + 3y = 5$

(h) $2x - 3y = -2$, $5x - 3y = 4$

(i) $-3x + y = -3$, $-3x - 2y = -12$

WE18

3 Solve the following pairs of simultaneous equations using the elimination method.

(a) $3x + 2y = 1$, $-2x + y = -3$

(b) $5x + 2y = -7$, $x + 7y = 25$

(c) $2x + 6y = 4$, $4x + 3y = -1$

(d) $2x + 5y = -1$, $4x + 3y = 5$

(e) $3x + 2y - 1 = 0$, $-2x + y + 3 = 0$

(f) $4x + 3y + 9 = 0$, $2x - 5y + 11 = 0$

(g) $3x - 2y = 7$, $2x - 3y = 3$

(h) $5x - 2y = -23$, $2x + 7y = 22$

(i) $4x + 2y - 6 = 0$, $-3x - 3y + 6 = 0$

(j) $\frac{x}{2} + y = 0$, $x - 3y = 10$

(k) $x + \frac{y}{4} = 8$, $2x - y = -2$

(l) $5x - 6y = 21$, $\frac{x}{3} + \frac{y}{2} = 5$

(m) $7x - y = -25$, $\frac{2x}{5} - \frac{y}{2} = 3$

(n) $\frac{2x}{3} + y + 10 = 0$, $4x - \frac{y}{4} + 10 = 0$

(o) $\frac{5x}{6} + \frac{3y}{5} + 2 = 0$, $7x + 3y + 27 = 0$

WE19

4 Solve the following pairs of simultaneous equations using the substitution method.

(a) $y = 2x$, $x + 5y = 22$

(b) $x + 3y = -30$, $y = 3x$

(c) $x = -2y$, $y = 3x - 14$

(d) $2x + y = 7$, $x = 6 - y$

(e) $x = y + 1$, $3x - y = 7$

(f) $y = 7 - 3x$, $4x - 3y = 18$

(g) $y = 1 - 2x$, $5x + 2y = 0$

(h) $x = 2 - 7y$, $2y = x - 11$

(i) $y = 6 - 3x$, $x = 2y - 19$

5 **(a)** For simultaneous equations $y = 2x - 5$ and $y = 3 - 2x$, the solution is:

A (0, -5) **B** (0, 3) **C** (2, -1) **D** (2, 1)

(b) For simultaneous equations $x + 2y = 8$ and $3x - y = 3$, the solution is:

A (6, 1) **B** (3, 6) **C** (4, 2) **D** (2, 3)

6 **(a)** Adding which pair of equations below will eliminate y?

A $x + y = 4$, $2x + y = 3$ **B** $3x + 2y = 6$, $-3x + y = 0$ **C** $x - y = 7$, $3x - y = 7$ **D** $2x - 3y = 4$, $x + 3y = 2$

(b) Adding which pair of equations in part **(a)** will eliminate x?

7 **(a)** $5x - 6y = 4$ (1)
$-3x + 8y = 2$ (2)

To eliminate x in the pair of equations above, we need to:

A multiply (1) by 3 and add to 5 times (2)

B multiply (2) by 3 and add to (1)

C multiply (1) by -3 and add to 5 times (2)

D multiply (1) by -3 and add to 5 times (2)

(b) $9x + 7y = 3$ (1)
$5x + 3y = -1$ (2)

To eliminate y in the pair of equations above, we need to obtain a new equation formed by:

A $3 \times (1) + 7 \times (2)$ **B** $5 \times (1) + 9 \times (2)$

C $3 \times (1) - 7 \times (2)$ **D** $5 \times (1) - 9 \times (2)$

8 **(a)** Simultaneous equations $y = x + 3$ and $2x - y = -11$ have the solution:

A $x = 3, y = 6$ **B** $x = -5, y = 1$ **C** $x = -5, y = -8$ **D** $x = -8, y = -5$

(b) Simultaneous equations $x = 2y$ and $4x - 7y - 3 = 0$ have the solution:

A (-3, -6) **B** (2, 4) **C** (3, 6) **D** (6, 3)

Understanding

9 The fruit below cost \$14.05.

The fruit below cost \$8.50.

How much did each pineapple and each kiwifruit cost? Assume that all pineapples are the same price and all kiwi fruit are the same price.

10 Three salad rolls and four pies cost \$20.56 altogether. A salad roll costs \$1.16 more than a pie. How much does each cost?

11 The sum of x and y is 12. When y is subtracted from $3x$ the result is 16. Find x and y.

12 The sum of two numbers is 38 and their difference is 24. What are the numbers?

Reasoning

13 Two books have a total of 500 pages. If the number of pages in one book is 12 pages less than 3 times the number of pages in the other, how many pages are there in each book?

14 A farmer uses 140 m of fencing wire to enclose a rectangular paddock. If the length of the paddock is 10 m longer than the width, find the dimensions of the paddock.

15 After buying a book, Libby received 18 coins totalling $1.65 in change. If she was given only 5- and 10-cent coins, how many of each did she receive?

16 Five less than twice Michelle's present age is equal to Yesmin's present age. Two years ago, twice Yesmin's age was the same as 10 years more than triple Michelle's age. Find the present ages of the two girls.

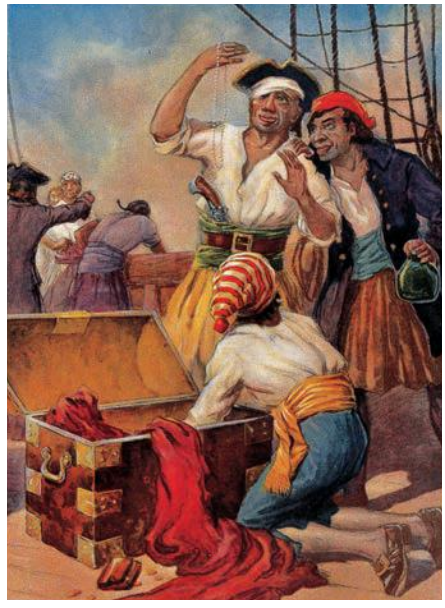
17 Wes and Tony live in two cities 360 km apart. They leave their houses at the same time by car and travel towards each other at 80 km/h and 100 km/h, respectively.

(a) Write an equation for each journey representing the distance from Wes's house (D) at any time (t).

(b) How much time will pass before they meet on the highway?

18 It takes a barge 3 hours to travel 27 km upstream against the current. On its return journey, with the current, the barge can travel the same distance downstream and an additional 3 km in only 2 hours. Find the speed of the boat and the current.

19 The pirate Captain Greybeard left careful instructions for his crew to locate his buried treasure. Using the old oak tree as the origin, north–south as the y-axis, east–west as the x-axis and each unit on the number plane as 20 paces, the solution of four different pairs of simultaneous equations would lead to the burial site.



The first point, A, is at the intersection of the two lines $x = -3$ and $y = -7$. The second point, B, is at the intersection of $2x - y = 7$ and $2x - 3y = -3$. The third point, C, is at the intersection of $2y + x = 0$ and $2y = x - 8$. The final point, D, is at the intersection of $y = 7x$ and $x + y = 8$.

(a) Find the coordinates of each of the points A, B, C and D.

(b) Plot the points A, B, C and D on the same number plane and join AB and CD.

(c) Find the coordinates of the intersection of the two lines, which will give the location of the buried treasure.

(d) Determine the equations of the lines AB and CD and solve algebraically to show that the coordinates found in **(c)** are the solution of these equations.

Open-ended

20 Make up three different pairs of simultaneous equations, each having the same solution, where $x = 3$ and $y = 4$.

21 Emmanuel has three numbers. When he adds them together in pairs he gets three different totals: 11, 17 and 22. What are the three numbers Emmanuel had to start with? Can you describe a method that would enable you to work out the three numbers given any three totals?

Chapter review 4

D.I.Y. Summary

Key Words

coincident lines	independent variable	linear inequality	sketch
dependent variable	linear equation	linear relationships	unique equations
gradient	linear graph	simultaneous equations	

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 A _____ indicates how two quantities that change at a constant rate are related to one another.

2 A _____ can be solved by performing inverse operations on both sides of the equation.

3 The measure of a line's steepness is called the _____.

4 _____ show that the relationship between two variables is a straight line.

5 For a _____, only two points need to be shown.

6 Two linear equations that contain the same variables and are not parallel or coincident can be solved using _____.

7 $3x + 4y = 7$ and $8y = 14 - 6x$ are examples of _____.

Fluency

1 Solve the following equations. Ex. 4.1

(a) $2c - 7 = 9$

(b) $\frac{2x-3}{4} + 6 = -1$

(c) $4v - 5 = v + 4$

(d) $\frac{3a+1}{2} - \frac{2a+3}{5} = 1$

2 The gradient of the line joining the points (2, 9) and (5, -6) is: Ex. 4.2

A 1 B 5 C $-\frac{1}{5}$ D -5

3 Find the gradient and y-intercept for each of the following. Ex. 4.2

(a) $y = 5x + 2$ (b) $y = -2x$ (c) $2x - 5y = 10$ (d) $4x + y = 8$

4 Graph the following lines by finding the y-intercept and gradient. Ex. 4.3

(a) $y = 2x - 5$ (b) $y = -2x$ (c) $x + 5y = 10$ (d) $y = 4x$

5 Sketch graphs of the following, showing the intercepts on the axes. Ex. 4.3

(a) $4x - 5y = 20$ (b) $3x + 7y = 42$ (c) $y = 3 - 5x$ (d) $y = -2x + 6$

6 Show whether or not each pair of lines is parallel, using their gradients. Ex. 4.4

(a) $x + 2y = 7$ and $x + 2y = 3$

(b) $x - y + 6 = 0$ and $3x - 2y + 4 = 0$

(c) $4x + y = 6$ and $4x + y + 6 = 0$

(d) $y = 4x - 3$ and $8x - 2y = 0$

7 Show whether or not each pair of lines is perpendicular, using their gradients.

Ex.4.4

(a) $2x + 2y = 4$ and $2x + y = 8$

(b) $x - y + 9 = 0$ and $2x + 2y - 3 = 0$

(c) $2x - 4y = 9$ and $4x + 2y + 6 = 7$

(d) $3x + 2y = 8$ and $3y = 2x + 4$

8 Find the solution set for the following linear inequalities.

Ex.4.5

(a) $9x \geq 21$

(b) $\frac{3x}{5} + 2 > -5$

(c) $7 - 6x < 2$

9 Solve the following simultaneous equations using the elimination method.

Ex.4.6

$5x - 2y = 8$

$3x - 4y = -12$

10 Solve the following simultaneous equations using the substitution method.

Ex.4.6

$6x - 5y = -21$

$y = 2x + 5$

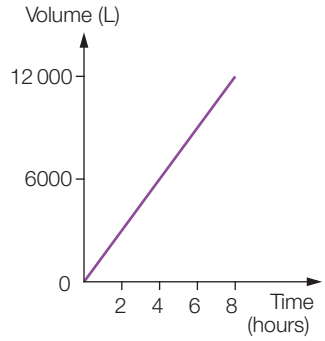
Understanding

11 In an enclosure at the zoo, there are twice as many wombats as emus. Altogether, they have a total of 90 legs. How many wombats are there?

12 To earn extra money, Martin works part-time in a butcher's shop. In a particular week during the school holidays he worked twice as many hours on Tuesday as he did on Monday but only $\frac{3}{4}$ the hours on Wednesday compared to Monday. For these 3 days he received \$127.50. If he is paid \$8.50 per hour, how many hours did Martin work on Monday?

13 A swimming pool has been filled with water.

The graph shows the volume of water (in litres) in the pool after a number of hours as it filled.



(a) How long did it take for the pool to contain 12 000 L of water?

(b) How many litres of water flowed into the pool each hour?

(c) What is the gradient of the graph?

(d) Write down the relationship for the volume (V) in terms of the time (T).

14 Manny, a motorcycle courier, notes his distance from the centre of Melbourne on a delivery trip:

Time (min)	0	10	20	30	40	50	60
Distance from Melbourne (km)	20	26	32	38	44	50	56

(a) Find an equation describing Manny's distance from Melbourne (d km) in terms of the time spent travelling (t minutes).

(b) Draw a graph of Manny's distance from Melbourne as a function of the time spent travelling.

(c) What does the gradient of the graph represent?

(d) What does the intercept with the vertical d-axis represent?

(e) Find Manny's distance from Melbourne after he has been travelling for 14 minutes.

(f) How long does it take for Manny to be 48 km from Melbourne?

15 What is the equation of the line that passes through (6, 3) and is perpendicular to $y - x - 1 = 0$?

16 What is the equation of the line that passes through (8, -2) and is parallel to $y - 5x = -1$?

17 Find the solution set for the following linear inequalities.

(a) $\frac{3x-4}{2} < \frac{2x+1}{3}$

(b) $3(2x - 4) \geq 4(3x + 1)$

18 Twice one number added to three times another number equals -3. If the first number is tripled then subtracted from 8 and the result is divided by 2, the second number is obtained. Find the two numbers.

19 Find the values of a and b in the following equations given that the lines are perpendicular and intersect on the y-axis.

$ax + 3y = 1$

$by = -x - 1$

Reasoning

20 Houses for sale at auction often have an EPR (Expected Price Range) in the advertisement. This means that it is expected that the house will be sold for a price somewhere between the values given.

(a) If the EPR for a house is advertised as \$570 000–\$620 000, write this as an inequality for the selling price of the house (p).

(b) If the deposit must be 10% of the price paid, write an inequality to show how much deposit is needed if the price falls within the EPR.

(c) If the house sold for more than \$50 000 above the highest value in the EPR, write an inequality to show how much the house sold for.

(d) If the house sold for less than \$50 000 above the highest value in the EPR, write an inequality to show how much the house sold for.

21 Three times the distance (d) between two towns subtracted from 823 km would take 4 hours travelling at an average speed (s). However, five times the same distance subtracted from 1071 km would only take 3 hours travelling at the same average speed. How far is the distance (d) between the two towns and what is the average speed (s)?

5 Trigonometry

Hands-on maths!

Name: Gavin Noble
Job: Carpenter
Qualifications: Certificate III Carpentry

Carpenters construct and maintain buildings and furniture made of wood and other products. They need good mathematical skills to ensure that they have the right amount of supplies, to build structures that are safe and to avoid delaying other tradesmen who rely on them. As a carpenter, Gavin erects frames, hangs doors and installs kitchens. He also does home renovations, which includes pouring concrete, laying bricks and plastering walls. "I like the variety of it and have always enjoyed working with timber and with my hands."

Carpenters need to use mathematical skills when measuring, estimating, reading plans and scale diagrams, and producing quotes. It is crucial that carpenters are accurate with their measurements. "I use volume to work out the amount of concrete. For plaster you can either use area to find the number of square metres required or to work out the number of sheets needed."

Building roofs can be very complex. Carpenters can either use trigonometry or a roofing reference book containing the relevant measurements to help them with their work.

Of maths at school, Gavin says: "I found maths hard. I find maths easier when I know why I am doing it. I would enjoy doing a maths course now, because I understand where I would use it in my job".

Why learn this?

Trigonometry has many practical applications. Pilots and ship navigators use sophisticated equipment that uses trigonometry to plot their course and determine their position. They also need to be able to complete these calculations by hand. Surveyors use trigonometry to map an area of land and to measure air space and areas of water. Medical imaging such as CAT scans are three-dimensional images of the body that are created using complex mathematics that includes trigonometry. Engineers routinely use trigonometry to determine the magnitude of quantities such as the force on the supports in a building or a bridge. This enables them to decide the strength of the steel beams required.

After completing this chapter you will be able to:

- use trigonometry to find unknown side lengths and angles in right-angled triangles
- use angles of elevation and depression to solve problems using trigonometry
- use conventional and true bearings to describe location
- use bearings to solve trigonometric problems
- select and use appropriate measurements and trigonometric ratios to solve problems in two dimensions.

5 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 Solve these equations for x. Express your answer as a decimal, correct to the number of decimal places (d.p.) given in brackets next to the equation.

Worksheet R5.1

(a) $3x = 24.5$ (2 d.p.)

(b) $\frac{x}{20} = 0.7825$ (1 d.p.)

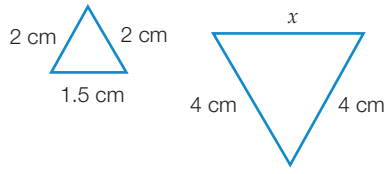
(c) $\frac{12}{x} = 2.307\,02$ (2 d.p.)

(d) $0.345 = \frac{16}{x}$ (3 d.p.)

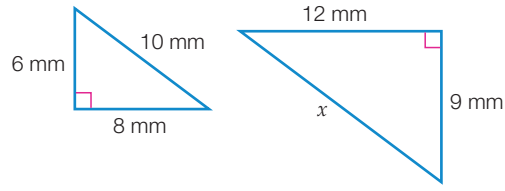
2 Determine the length of the missing side x in each pair of similar triangles.

Worksheet R5.2

(a)



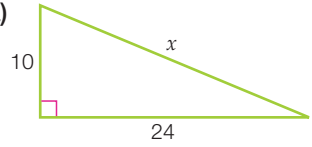
(b)



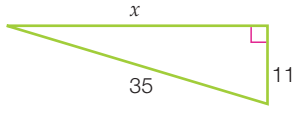
3 Use Pythagoras' theorem to calculate the value of x in each triangle. Round your answer to two decimal places where necessary.

Worksheet R5.3

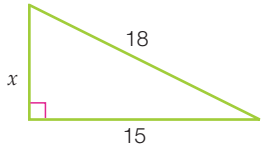
(a)



(b)



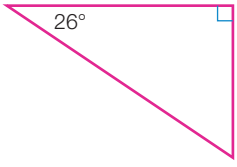
(c)



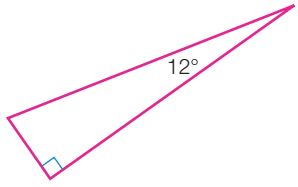
4 Copy each right-angled triangle shown below and label the hypotenuse, the opposite side and the adjacent side for the angle given.

Worksheet R5.4

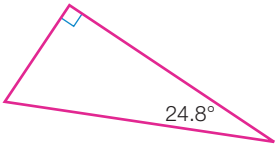
(a)



(b)



(c)



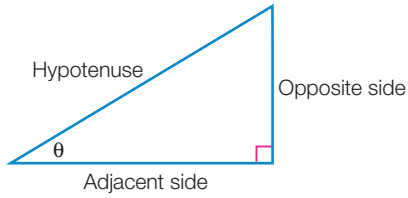
(d)

Key Words

adjacent side	hypotenuse	tangent
angle of depression	minute	trigonometric ratio
angle of elevation	opposite side	true bearing
compass bearing	second	
cosine	sine	

5.1 The trigonometric ratios

The sides of a right-angled triangle can be named using the position of an angle as a reference. This angle is often referred to as θ (the Greek letter theta) but any symbol can be used. The longest side of a right-angled triangle is the **hypotenuse**. The hypotenuse is always opposite the right angle. The side directly opposite the angle θ is the **opposite side** and the side next to θ is the **adjacent side**.



We can use pairs of side lengths to calculate three **trigonometric ratios** (we write the ratios in their fraction forms).

The ratio $\frac{\text{Opposite}}{\text{Hypotenuse}}$ is referred to as the **sine** of θ and abbreviated to sin θ.

The ratio $\frac{\text{Adjacent}}{\text{Hypotenuse}}$ is referred to as the **cosine** of θ and abbreviated to cos θ.

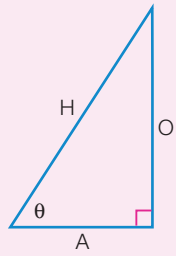
The ratio $\frac{\text{Opposite}}{\text{Adjacent}}$ is referred to as the **tangent** of θ and abbreviated to tan θ.

In a right-angled triangle:

$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$ or $\frac{O}{H}$

$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$ or $\frac{A}{H}$

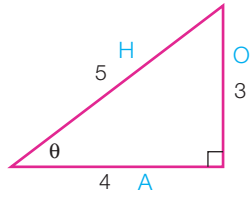
$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$ or $\frac{O}{A}$



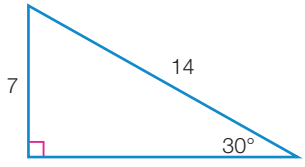
The phrase 'SOH-CAH-TOA' can help you remember which two sides belong to each ratio.

In the triangle shown at right, the sine of the given angle, θ, is $\frac{3}{5}$ or 0.6. We write this as $\sin\theta = 0.6$.

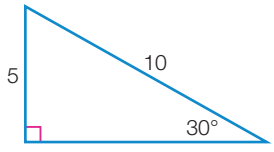
Also in this triangle, $\cos\theta = \frac{4}{5} = 0.8$ and $\tan\theta = \frac{3}{4} = 0.75$.



For a given angle θ, the values of each trigonometric ratio are the same, even if the side lengths of the triangle change. This is because two right-angled triangles with a common angle (θ) are similar triangles.



$\sin 30^\circ = \frac{7}{14} = \frac{1}{2}$



$\sin 30^\circ = \frac{5}{10} = \frac{1}{2}$

Using the ratios

Because the value of each of the trigonometric ratios is unique for a particular angle θ, we can use a ratio to find the length of an unknown side, if we know the value of θ and the length of one other side. Alternatively, if we can calculate the ratio of two known sides, we can use it to determine an angle.

To identify which trigonometric ratio to use:

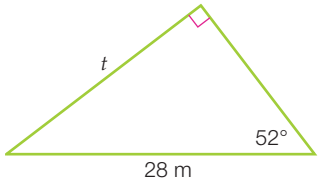
1. Label each side of the triangle with O, A and H, relative to the angle considered.
2. Identify which two sides are involved.
3. Decide which ratio links the two sides (SOH-CAH-TOA is helpful here).

Worked Example 1

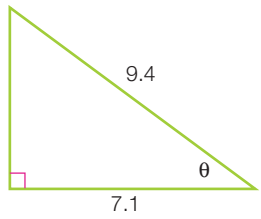
WE1

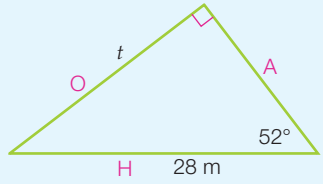
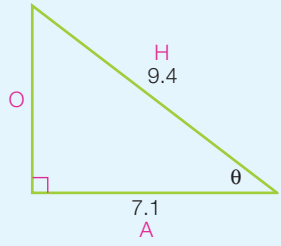
Label the Hypotenuse, Opposite and Adjacent sides on these right-angled triangles and hence decide which trigonometric ratio we could use to find the unknown.

(a)



(b)



Thinking	Working
(a) 1 Label the sides of the triangle O, A and H.	(a) 
2 Choose the ratio that connects the known and unknown values. (Here, we are given the values of H and θ and want to find O, so we will use sin θ to find *t*.)	$\sin\theta = \frac{O}{H}$ $\sin 52° = \frac{t}{28}$
(b) 1 Label the sides of the triangle O, A and H.	(b) 
2 Choose the ratio that connects the known and unknown values. (Here, we are given the values of A and H and θ is unknown, so we will use cos θ to find θ.)	$\cos\theta = \frac{A}{H}$ $\cos\theta = \frac{7.1}{9.4}$

Using a scientific calculator or a CAS

A scientific calculator or a CAS can also be used to find sine, cosine and tangent for a given angle.

A scientific calculator has sin, cos and tan keys.

The TI-Nspire CAS has a 'trig' (trig) key. This brings up all of the ratios. Select the one you want by using the cursor, then press 'enter'.

On the Casio ClassPad, bring up the keyboard with the Keyboard key. The TRIG keys are found under the mth tab.

Make sure that your calculator is in 'degree' mode for the calculations in this chapter.

Worked Example 2

WE2

Use a calculator to find each of the following values, rounded to four decimal places.

(a) sin 29° **(b)** cos 58° **(c)** tan 60.5°

Thinking	Working
(a) **1** Enter the information into the calculator. Press the sin key, followed by the value of the angle, then the '=' sign.	(a)
2 Round the answer to four decimal places.	sin 29° = 0.4848 (4 d.p.)
(b) **1** Enter the information into the calculator. Press the cos key, followed by the value of the angle, then the '=' sign.	(b)
2 Round the answer to four decimal places.	cos 58° = 0.5299 (4 d.p.)
(c) **1** Enter the information into the calculator. Press the tan key, followed by the value of the angle, then the '=' sign.	(c)
2 Round the answer to four decimal places.	tan 60.5° = 1.7675 (4 d.p.)

Angles in degrees, minutes and seconds

The units we use to measure angles are degrees (°). One degree can be broken up into smaller units called **minutes** (′) and **seconds** (″). There are 60 seconds in a minute and 60 minutes in a degree.

That is, $60' = 1°$ and $60'' = 1'$

So $3600'' = 1°$

An angle of 24°30′ is equal to 24.5°, because $\frac{30'}{60'} = 0.5$.

An angle of 57°45′18″ is equal to 57.755°, because $\frac{45}{60} = 0.75$ and $\frac{18}{3600} = 0.005$.

Most scientific calculators have a key marked [°, ″] or [D°M′S], which converts angles in degrees, minutes and seconds to decimal degrees (a degree measurement with a decimal part).

To change 24°30′ into decimal degrees using a scientific calculator, press:

[2] [4] [°, ″] [3] [0] [°, ″] [=]

The screen will show 24.5. Pressing [°, ″] switches between the two forms.

If your calculator works differently, consult the instruction booklet.

On the Casio ClassPad CAS, tap the 'Interactive' menu and select the 'Transformation' option. To enter an angle in degrees, minutes and seconds, choose 'dms' and enter the values. Tap 'OK' and the decimal values of the angle will appear on screen. To convert an angle in decimal degrees to degrees, minutes and seconds, choose the 'toDMS' option under 'Transformations'. Enter the decimal degrees, tap 'OK' and the value in degrees, minutes and seconds will appear on screen.

To enter an angle in degrees, minutes and seconds on the TI-Nspire CAS, enter the number of degrees, then press [ctrl] [⌨] to access a symbol menu. Use the cursor to select the degrees symbol, which is found on the right-hand side of the top row. The minutes symbol is the third from the left in the second row—enter this symbol twice for seconds. To convert an angle in degrees, minutes and seconds to decimal degrees, press [⌨] to bring up a list of actions, scroll down to 'DD' ('Decimal Degrees') and press enter twice (pressing the [D] key takes you straight to the D section of the list). To convert an angle in decimal degrees to degrees, minutes and seconds, select ▸DMS from the same list and press enter twice.

The following Worked Examples use the keys from a scientific calculator.

Worked Example 3 WE3

Convert the following angles to decimal degrees, rounded to two decimal places.

(a) 56°24′ **(b)** 23°16′45″

Thinking	Working
(a) 1 Enter the information into the calculator. Press the following keys: [5] [6] [°, ″] [2] [4] [°, ″] [=] [°, ″]	*(a)*
2 State the answer correct to two decimal places.	*56°24′ = 56.40°*
(b) 1 Enter the information into the calculator. Press the following keys: [2] [3] [°, ″] [1] [6] [°, ″] [4] [5] [°, ″] [=] [°, ″]	*(b)*
2 Round the answer to two decimal places.	*23°16′45″ = 23.28° (2 d.p.)*

Worked Example 4 WE4

Use a calculator to find each of the following values, rounded to four decimal places.

(a) sin 46°36′ **(b)** cos 6°12′34″

Thinking	Working
(a) 1 Enter the information into the calculator. Press the following keys: sin 4 6 °′″ 3 6 °′″ =	(a)
2 Round the answer to four decimal places.	sin 46°36′ = 0.7266 (4 d.p.)
(b) 1 Enter the information into the calculator. Press the following keys: cos 6 °′″ 1 2 °′″ 3 4 °′″ =	(b)
2 Round the answer to four decimal places.	cos 6°12′34″ = 0.9941 (4 d.p.)

5.1 The trigonometric ratios

Navigator

Answers page 739

Q1, Q2 Columns 1 & 2, Q3, Q4, Q5, Q6, Q10, Q11, Q14, Q15, Q16	Q1, Q2 Column 2, Q3, Q4, Q6, Q7, Q8, Q10, Q11, Q13, Q14, Q15, Q16, Q17	Q1, Q2 Column 3, Q3 (d)–(f), Q4 (d)–(f), Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17

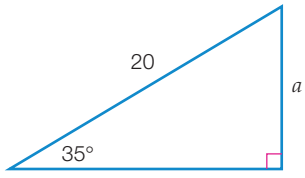
Equipment required: ruler and protractor for Questions **6**, **14–16**

Fluency

WE1

1 Label these right-angled triangles and hence decide which trigonometric ratio should be used to find the unknown in each case.

(a)



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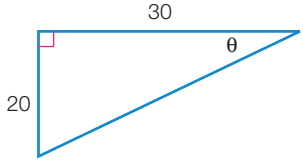
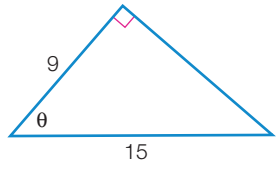


(c)



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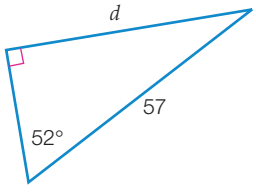

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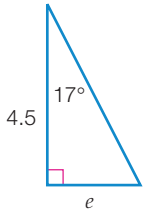

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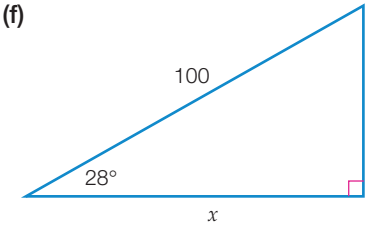


2 Use your calculator to find the following values, rounded to four decimal places. WE2

(a) sin 23° (b) cos 17° (c) tan 39°

(d) cos 5° (e) sin 41° (f) cos 49°

(g) tan 19.7° (h) cos 23.5° (i) tan 43.65°

3 Convert these angles to decimal degrees. Round your answers to two decimal places if necessary. WE3

(a) 15°30′ (b) 35°16′ (c) 23°10′

(d) 76°47′50″ (e) 44°59′20″ (f) 8°45′9″

4 Calculate the following, rounding values to four decimal places. WE4

(a) sin 25°47′ (b) tan 88°30′ (c) cos 64°17′

(d) tan 36°30′46″ (e) sin 20°25′39″ (f) cos 81°44′16″

Understanding

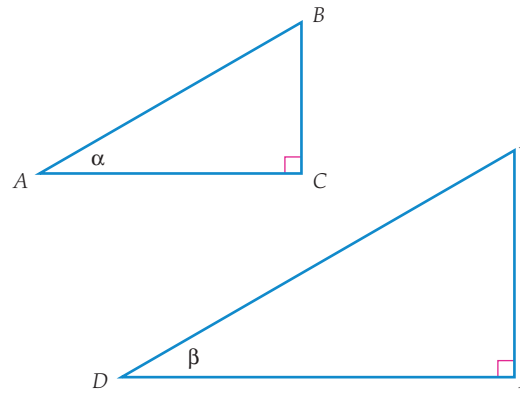
5 (a) Using a calculator, and evaluating the result to four decimal places, the value of sin 37°34′ is:

A 0.6064 B 0.6065 C 0.6097 D 0.7951

(b) Using a calculator, and evaluating the result to four decimal places, the value of cos 17.25° is:

A 0.2965 B 0.3105 C 0.9542 D 0.9550

6 Measure the side lengths of triangles *ABC* and *DEF* to the nearest millimetre.



(a) Calculate the ratios of the following pairs of sides. Round your answers to one decimal place.

(i) $\frac{DF}{AC}$ (ii) $\frac{EF}{BC}$ (iii) $\frac{DE}{AB}$

(b) Calculate the value for sin α.

(c) Calculate the value for sin β.

(d) Are these sine ratios about the same? Why?

(e) Measure the sizes of angles α and β with a protractor.

(f) Would the cosine ratios be the same for each of the triangles?

(g) Would the tangent ratios be the same for each of the triangles?

(h) Give a reason for your answers to **(f)** and **(g)**.

7 (a) Calculate the value of:

(i) sin 35° (ii) cos 35° (iii) tan 35° (iv) $\frac{\sin 35°}{\cos 35°}$

(b) Comment on your results.

8 If sin θ = 0.755 (3 d.p.) and cos θ = 0.656 (3 d.p.), use your result from Question **7** to calculate the value of tan θ, to three decimal places.

9 The cosine of an angle is always less than 1 because:

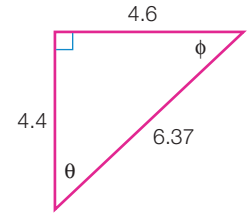
A the Adjacent side is always shorter than the Opposite side

B the Adjacent side is always shorter than the Hypotenuse

C the Adjacent side is always longer than the Hypotenuse

D the Adjacent side is always longer than the Opposite side.

Reasoning



10 **(a)** Calculate the following values using the triangle shown, rounded to three decimal places:

(i) $\sin\theta$ **(ii)** $\cos\phi$

(b) Explain your results.

11 A right-angled triangle has an angle of 54° as one of its acute angles.

(a) Using three different pronumerals to represent the side lengths, draw a diagram of the triangle, including the measurement of the other angle.

(b) Write down values of sin 54° and cos 54° using the pronumerals that represent the side lengths.

(c) Find the values of sin 54° and cos 54° using a calculator, and round to four decimal places.

(d) Use your answers to **(c)** to show that the value of $(\sin 54°)^2 + (\cos 54°)^2$ equals 1. (*Note*: This is normally written as $\sin^2 54° + \cos^2 54°$.)

(e) Use your answers to **(b)** and Pythagoras' theorem to show that $(\sin 54°)^2 + (\cos 54°)^2 = 1$.

12 The value of the tangent ratio for each of the two base angles in a right-angled isosceles triangle is 1. Explain why.

13 Only one of the trigonometric ratios can have a value greater than 1. Name it, and explain why this is not possible for the other two ratios.

Open-ended

14 **(a)** Draw a right-angled triangle with sides of any length.

(i) Carefully measure all the angles of the triangle.

(ii) Carefully measure all the sides of the triangle.

(iii) Find the sine, cosine and tangent ratios for the two acute angles, rounded to two decimal places.

(iv) Write down as many relationships between the angles and the ratios as you can find.

(b) Draw a second triangle with the same angles, but with different side lengths (i.e. a similar triangle). Repeat steps **(ii)** and **(iii)** from part **(a)** for this triangle.

(c) Compare the values of sin, cos and tan that you calculated for the two triangles. How similar are they? Did you expect them to be the same?

(d) Compare the values of the three ratios you calculated for each angle to the values produced by your calculator. Are they the same? Suggest a reason for your answer.

Hint

Which values of O and A give a ratio of $\frac{7}{10}$?

15 A right-angled triangle has a tangent ratio of $\frac{7}{10}$ for one angle.

(a) Use the tangent ratio to draw the triangle. Measure the length of the hypotenuse.

(b) Calculate the other ratios, rounded to four decimal places.

16 Angle θ in a right-angled triangle has a sine ratio of 0.4. Draw three different triangles that could contain angle θ. Use Pythagoras' theorem to help you find any unknown side lengths.

17 **(a)** Find two angles θ and ϕ, so that $\sin\theta = \cos\phi$.

(b) What is the relationship between θ and ϕ if $\sin\theta = \cos\phi$ for all values of θ and ϕ?

Finding lengths

5.2

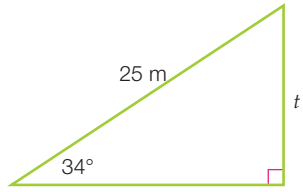
For any right-angled triangle, we can find an unknown side length if we know the size of one of the acute angles and the length of one side. We use the fact that the trigonometric ratios of the side lengths (sin, cos and tan) are constant and unique for a particular angle, θ.

For example, if we know the angle θ and the hypotenuse H, we can find the length O of the opposite side using $\sin\theta = \frac{O}{H}$. We substitute values for $\sin\theta$ and H to create an equation that we can solve to find the length of the unknown side.

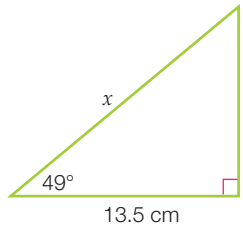
Worked Example 5

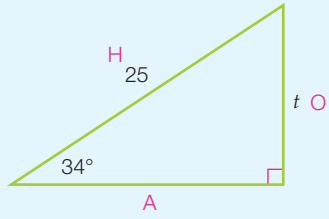
WE 5

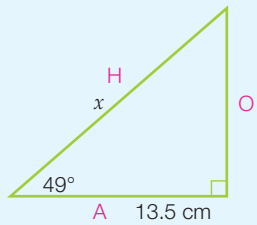
(a) Find the value of t, rounded to two decimal places.



(b) Find the value of x, rounded to two decimal places.



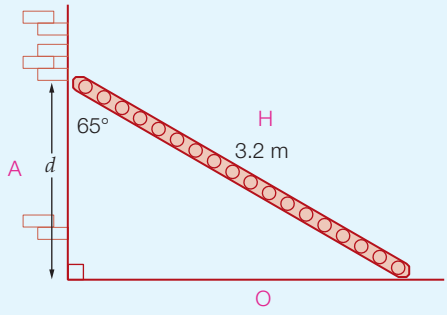
Thinking	Working
(a) **1** Copy the triangle and label the sides O, A and H.	(a) 
2 Write down the known information.	$\theta = 34°$ $O = t$ $H = 25$
3 Identify the trigonometric ratio that connects the known information.	$\sin\theta = \frac{O}{H}$
4 Substitute the known values.	$\sin 34° = \frac{t}{25}$
5 Make the unknown the subject of the equation. (Here, we multiply both sides by 25.)	$t = 25 \times \sin 34°$
6 Use your calculator to evaluate the answer and round it to two decimal places.	$t = 13.98$ m (2 d.p.)

(b) 1	Copy the triangle and label the sides O, A and H.	(b) 
2	Write down the known information.	$\theta = 49°$ $A = 13.5$ $H = x$
3	Identify the trigonometric ratio that connects the known information.	$\cos\theta = \frac{A}{H}$
4	Substitute the known values.	$\cos 49° = \frac{13.5}{x}$
5	Make the unknown the subject of the equation. (Here, we multiply both sides by x, then divide both sides by cos 49°.)	$x \cos 49° = 13.5$ $x = \frac{13.5}{\cos 49°}$
6	Use your calculator to evaluate the answer and round it to two decimal places.	$x = 20.58$ cm (2 d.p.)

Worked Example 6

WE 6

A 3.2 m ladder rests against a wall so that its angle with the wall is 65°. How high up the wall does the ladder reach? State your answer correct to the nearest centimetre.

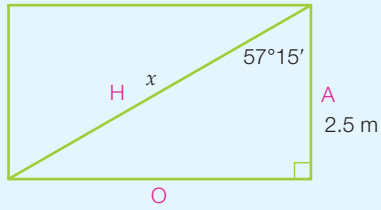
Thinking	Working
1 Draw a diagram that includes a right-angled triangle. (The diagram does not need to be to scale.) Identify the position of the given angle, then label O, A and H. Identify and label the unknown side. (Here, we label the unknown vertical distance d.)	
2 Write down the known information.	$\theta = 65°$ $H = 3.2$ m $A = d$
3 Identify the trigonometric ratio that connects the known information.	$\cos\theta = \frac{A}{H}$
4 Substitute the known values.	$\cos 65° = \frac{d}{3.2}$

5 Make the unknown the subject of the equation. (Here, we multiply both sides by 3.2.)	$d = 3.2 \times \cos 65°$
6 Use your calculator to evaluate the answer and round to two decimal places.	$d = 1.35$ (2 d.p.)
7 State the answer in a sentence.	The ladder rests 1.35 m up the wall.

Worked Example 7

WE7

A rectangular wooden frame has a diagonal piece of wood joining the opposite corners for support. The vertical height of the frame is 2.5 m. The angle between the diagonal and vertical part of the frame is 57°15′. How long is the diagonal piece of the frame?

Thinking	Working
1 Draw a diagram that includes a right-angled triangle. Identify the position of the angle and label it. Label the known sides with their values and the unknown side with a pronumeral. Label the sides O, A and H.	
2 Write down the known information.	$\theta = 57°15'$ $A = 2.5$ m $H = x$
3 Identify the trigonometric ratio that connects the known information.	$\cos\theta = \frac{A}{H}$
4 Create an equation by substituting the known values.	$\cos 57°15' = \frac{2.5}{x}$
5 Make x the subject of the equation. (Here, we multiply both sides by x, then divide by cos 57°15′.)	$x \cos 57°15' = 2.5$ $x = \frac{2.5}{\cos 57°15'}$
6 Use your calculator to evaluate the answer, rounding it to two decimal places.	$x = 4.62$ (2 d.p.)
7 State the answer in a sentence.	The diagonal of the frame is 4.62 m long.

To find an unknown value in a right-angled triangle:

1. Draw the right-angled triangle, showing all known information and the unknown value.
2. Label the sides O, A and H, and write down the known information.
3. Identify which trigonometric ratio to use. (SOH-CAH-TOA is helpful here.)
4. Substitute the known values, form an equation and solve for the unknown.

5.2 Finding lengths

Navigator

Answers page 740

Q1, Q2, Q3, Q5, Q7, Q9, Q11, Q12, Q13, Q14	Q1 Columns 2 & 3, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q11, Q12, Q14	Q1 Column 3, Q2, Q3, Q5, Q6, Q8, Q9, Q10, Q12, Q13, Q14

Fluency

WE5

1 Find the value of the unknown length in each of the following. Round your answers to two decimal places.

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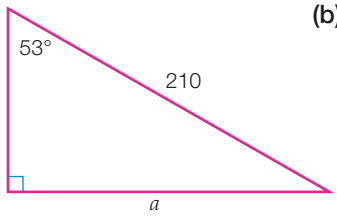

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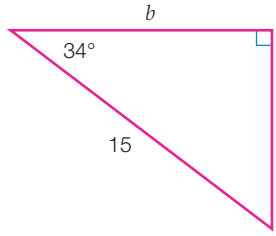

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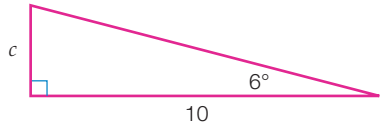

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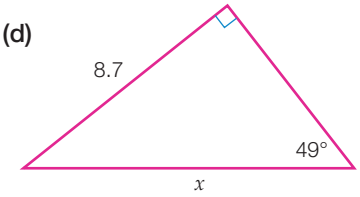

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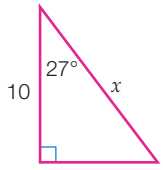

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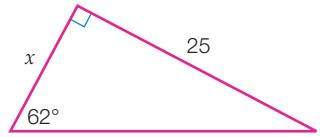

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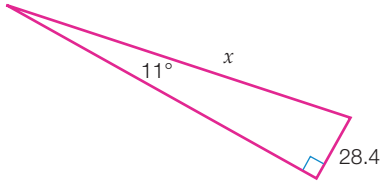

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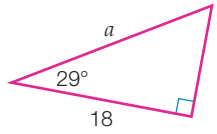

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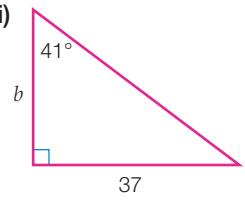

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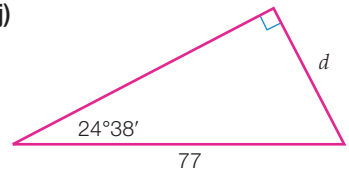

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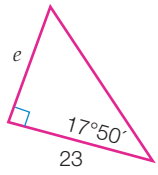

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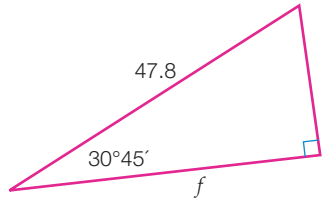

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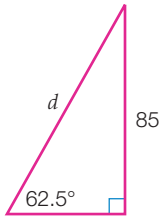

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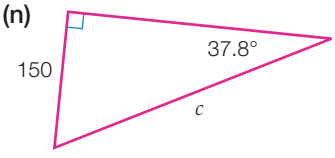

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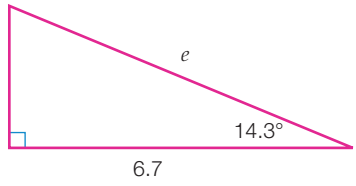


WE6

2 A 3 m rope is stretched from the top of a vertical tent pole and pegged into the ground so that it makes an angle of 50° with the ground. How far away from the pole is the rope pegged? Give your answer to the nearest centimetre.

WE7

3 A rectangular gate has diagonal metal supports joining the opposite corners. The supports make an angle of 25°18′ with the horizontal sections of the frame. If the gate is 1.6 m high, how long is one of the diagonal supports?

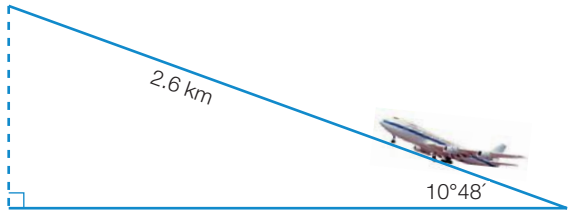
4 (a) A plane takes off and rises steadily at an angle of 10°48′ to the horizontal. The altitude of the plane after it has travelled along a flight path of 2.6 km (to the nearest metre) is:

A 472 m

B 487 m

C 2554 m

D 13 875 m



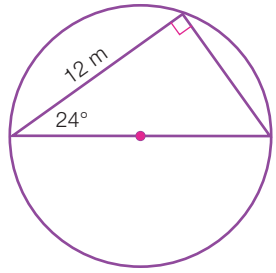
(b) The radius of the circle at right is closest to:

A 6.57 m

B 13.14 m

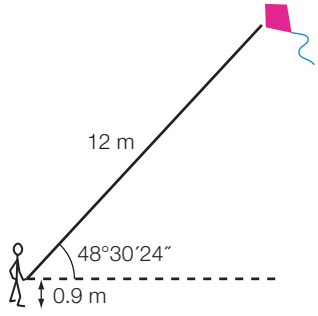
C 14.75 m

D 29.50 m



Understanding

5 Jessica is flying a kite attached to a string 12 m long. She is holding the string at a height of 0.9 m from the ground. How high is the kite if it makes an angle of 48°30′24″ with the horizontal? Round your answer to two decimal places.

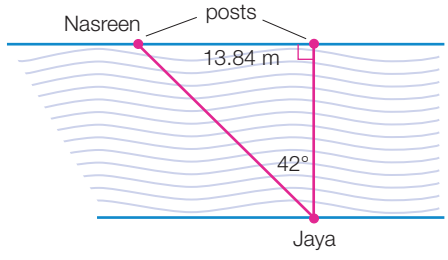


6 To measure a tall building, Charlie stands 150 m from the base of the building and holds an inclinometer (a device for measuring angles) to his eye, as shown in the diagram. He looks along the top of the inclinometer and tilts it up until he can see the top of the building. The inclinometer measures an angle of 58°. This is the angle between Charlie's line of sight to the top of the building and the horizontal. Charlie's eye-level is 1.63 m from the ground.

(a) Draw a diagram that shows all the known information.

(b) Calculate the height of the building, correct to one decimal place.

7 Jaya wants to calculate the width of a river. She finds that lines joining her point of observation to the two posts on the opposite bank are 42° apart. Her sister Nasreen, who is on the opposite bank, measures the distance between the two posts to be 13.84 m. How wide is the river?



8 A mine shaft 350 m long is to be drilled at an angle of 21°24′ to the vertical. How far directly below the ground will the end of the shaft be, rounded to two decimal places?

9 A road sign is in the shape of an equilateral triangle. The line from one vertex to the midpoint of the opposite side is 34.6 cm long. How long is each side of the road sign?

Reasoning

10 Members of a stunt team wish to make a daring climb along a rope, one end of which has been attached to the top of a 23 m building and the other end joined to the top of a nearby building 38 m high. The rope is inclined at an angle of 49°30′ to the horizontal. Answer the following correct to two decimal places.

(a) How far along the rope will each member of the stunt team need to climb to reach the other building?

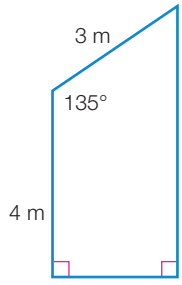
(b) One of the stunt team wants to drop a ball so that a man standing midway between the two buildings can catch it. From how far along the rope does she need to drop it, and how far will it fall if the man catches it 1.8 m above the ground?

Hint

Formula for the area of a trapezium:

$A = \frac{1}{2}(a + b)h$

11 James is trying to decide which room to use for a party. He has two rooms in mind but needs to use the larger one. One room is a rectangular shape with length 4 m and width 3 m. The other room is in the shape of a trapezium as shown in the diagram.

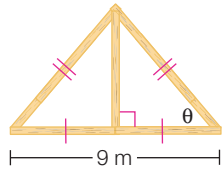


(a) Calculate the area of each of the rooms.

(b) Which room is larger?

Open-ended

12 The 'pitch angle' of the roof of a house is the angle that the roof makes with the horizontal. It is usually between 10° and 30°. If the length of the base of this roof frame is 9 m, find two possible values for the:

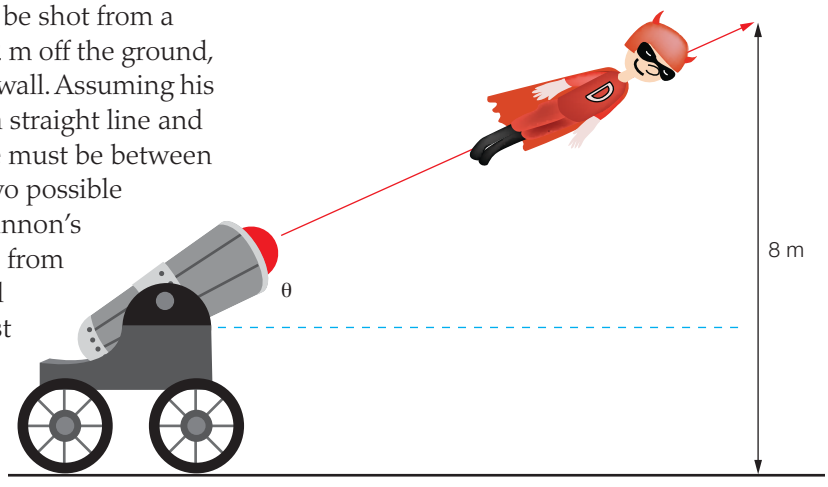


(a) height of the roof at its tallest point

(b) total length of wood needed to build the frame shown.

13 A right-angled triangle ABC has the right angle at B. If the side opposite to angle A has a length of 10 cm, write down at least three possible lengths of the hypotenuse if angle A is between 35° and 40°.

14 Daredevil Dan will be shot from a cannon that sits 1.2 m off the ground, to just clear an 8 m wall. Assuming his flight path will be a straight line and the trajectory angle must be between 30° and 50°, find two possible distances for the cannon's horizontal distance from the base of the wall if this distance must be between 8 m and 10 m.



Finding angles

5.3

So far we have used sine, cosine or tangent ratios to find an unknown side. Because the values of the trigonometric ratios are unique for each angle we can find a particular angle if we know one of its trigonometric ratios. For example, we may want to find the angle θ whose sine is 0.72.

That is, if $\sin\theta = 0.72$, $\theta = ?$

To find θ we need to 'undo' the sine ratio to find the angle that corresponds to the sine ratio with that particular value. Calculators have an INV, 2nd F or SHIFT key. This key, used before the sin key, applies the *inverse* sine ratio, which we write as $\sin^{-1}$.

That is, $\theta = \sin^{-1}(0.72)$.

To find θ on a calculator, press SHIFT sin . 7 2 =.

The screen will show 46.054 480 44. So θ, to the nearest degree, is 46°. Make sure your calculator is in degree mode. (A 'D' or 'DEG' will be showing on the screen.)

On the Casio ClassPad CAS, these inverse ratios are found by pressing the Keyboard key, then selecting the TRIG keys under the mth tab.

On the TI-Nspire CAS, press the trig key, then use the cursor to select the desired option.

Worked Example 8 WE8

If $\cos a = 0.6434$, find:

(a) a in decimal degrees rounded to two decimal places

(b) a in degrees, minutes and seconds.

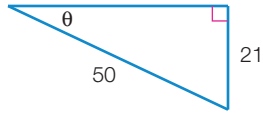
Thinking	Working
(a) 1 Make sure that your calculator is in degree mode. Enter the information into the calculator. Press the following keys: SHIFT cos . 6 4 3 4 =	(a) Screen shows 49.954 181 3
2 Round the answer to two decimal places.	$a = 49.95°$ (2 d.p.)
(b) 1 Convert the answer from part **(a)** to degrees, minutes and seconds. Press °' ''.	(b) Screen shows 49°57′15.05
2 Round the answer to the nearest second.	$a = 49°57'15''$ (nearest second)

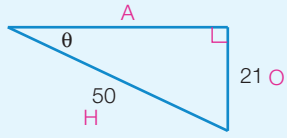
If we know the lengths of any two sides in a right-angled triangle, we can calculate the size of any angle in it by identifying and calculating the trigonometric ratio of the two sides.

Worked Example 9

WE 9

Find the value of θ in degrees, minutes and seconds.

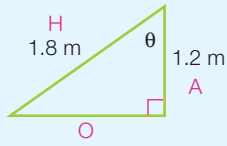


Thinking	Working
1 Label the sides of the triangle.	
2 Decide which trigonometric ratio connects the given values.	$\sin\theta = \frac{O}{H}$
3 Substitute the given values.	$\sin\theta = \frac{21}{50}$
4 Use the inverse of the ratio to find θ.	$\theta = \sin^{-1}\left(\frac{21}{50}\right)$
5 On your calculator enter SHIFT sin (2 1 ÷ 5 0) =	
6 Convert the answer to degrees, minutes and seconds. Press °′″.	$\theta = 24°50'4.51''$
7 Round the answer to the nearest second.	$\theta = 24°50'5''$ (nearest second)

Worked Example 10

WE 10

The top of a 1.8 m ladder that is leaning against a wall, reaches a height of 1.2 m.
Draw a diagram that shows this situation, then use it to calculate the angle that the top of the ladder makes with the wall, correct to the nearest degree.

Thinking	Working
1 Draw a diagram to represent the situation. Identify and label the given side lengths and the unknown angle.	 $\theta = ?$ $A = 1.2$ m $H = 1.8$ m
2 Decide which trigonometric ratio connects the given values.	$\cos\theta = \frac{A}{H}$
3 Substitute the given values.	$\cos\theta = \frac{1.2}{1.8}$
4 Use the inverse of the ratio to find θ.	$\theta = \cos^{-1}\left(\frac{1.2}{1.8}\right)$
5 On your calculator enter SHIFT cos (1 . 2 ÷ 1 . 8) =	$\theta = 48.189\,685\,11$
6 Round the answer to the nearest degree.	$\theta = 48°$ (nearest degree)
7 State the answer.	The ladder makes an angle of approximately 48° with the wall.

5.3 Finding angles

Navigator

Q1, Q2, Q3, Q4, Q5, Q7, Q10, Q11, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q10, Q11, Q13	Q2 (a) (c) (e), Q3, Q5, Q6, Q8, Q9, Q11, Q12, Q13

Answers page 741

Fluency

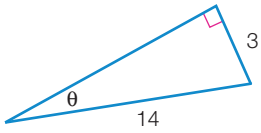
1 Find the value of each pronumeral **(i)** in decimal degrees rounded to two decimal places, **(ii)** in degrees, minutes and seconds. WE8

(a) $\cos a = 0.1538$ **(b)** $\sin b = 0.6719$ **(c)** $\tan c = 2.3085$

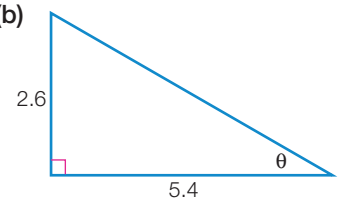
(d) $\cos k = 0.556$ **(e)** $\tan m = 4.687$ **(f)** $\sin p = 0.9715$

2 Find the value of θ in each triangle in degrees, minutes and seconds. WE9

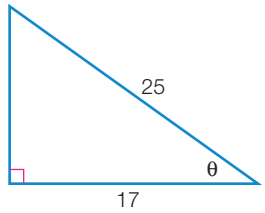
(a)



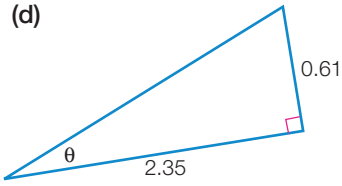
(b)



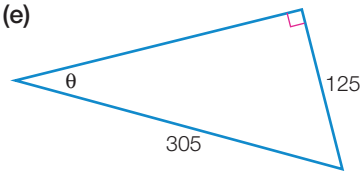
(c)



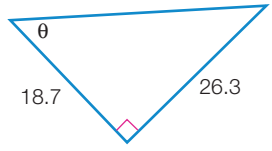
(d)



(e)



(f)



3 A tree that is 8.4 m tall makes a shadow of 12 m. WE10

(a) Draw a diagram to show this situation.

(b) To the nearest degree, calculate the angle that the Sun makes with the ground to form the shadow.

Understanding

4 **(a)** The value of the pronumeral a in the equation $\tan a = 0.6395$, to the nearest degree is:

A 32° **B** 33° **C** 40° **D** 50°

(b) The value of θ in degrees, minutes and seconds in the equation $\sin \theta = 0.8938$ is:

A 26°38′43″ **B** 41°47′25″ **C** 63°21′17″ **D** 63°35′47″

5 On a slope, a railway rises 35 cm for every 50 m of track. Find the angle at which the track is inclined to the horizontal, to two decimal places.

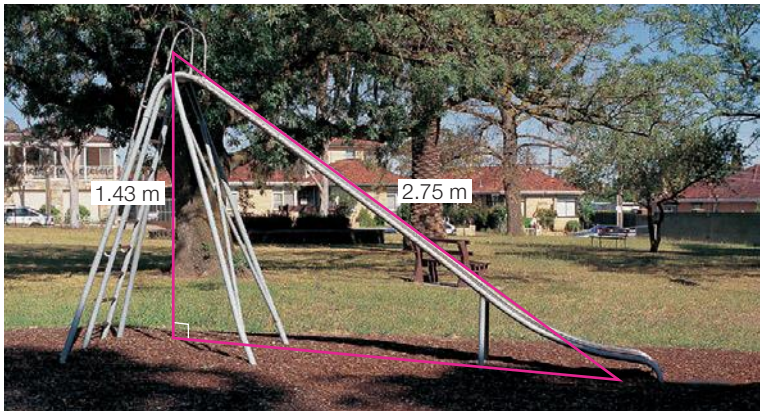
6 Scott is dragging a heavy box along the floor holding the end of a 1.5 m piece of rope. The rope is attached to the top of the box and makes an angle θ with the horizontal. Scott is exactly 1.0 m in front of the box. To the nearest degree, what angle does the rope make with the horizontal?

7 Find the angle of inclination, θ, of the Sun's rays when a 9.5 m high telephone pole casts a shadow 7 m long, as shown in the photograph at right. Write your answer in degrees, minutes and seconds.

8 A 2.75 m long children's slide has a ladder that reaches 1.43 m vertically above the ground. To the nearest degree, the angle the slide makes with the horizontal is:

A 27° **B** 31°

C 42° **D** 58°



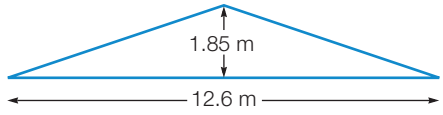
Reasoning

9 A furniture removal company uses a ramp to wheel furniture into its vans. The back of a van is 82 cm above the ground and the ramp is 1.5 m long.

(a) To the nearest degree, what is the angle of the ramp with the ground?

(b) How much longer does the ramp need to be so that the angle is 30°?

10 A symmetrical roof frame of height 1.85 m has a span of 12.6 m as shown in the diagram at right.



(a) Find the angle at the apex of the roof, to the nearest degree. (The apex is the highest point on the roof.)

(b) Find the total length of material needed to make the frame.

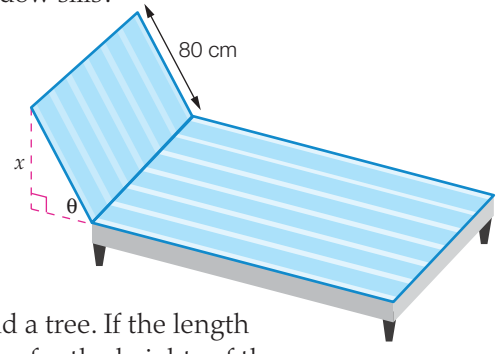
11 A 10 m ladder is to be used to reach different window sills in a building.

(a) If the heights of the window sills range from 2 m to 6 m above the ground, calculate the range of angles that the ladder makes with the ground.

(b) Is it practical to use the ladder to reach all the window sills?

Open-ended

12 A sun lounge can be adjusted from a flat, lying position through to an upright, sitting position. The adjustable back is 80 cm long. Choose five different vertical heights that the back could be raised to. For each height (x) calculate the angle the back makes with the horizontal (θ).



13 Bilal is 1.85 m tall and is standing near a lamp post and a tree. If the length of his shadow is 94 cm, use your own reasonable values for the heights of the lamp post and the tree to determine the length of their shadows at the same time.

Investigation

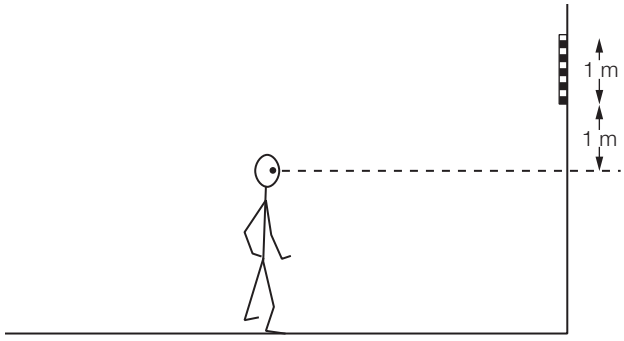
Viewing artwork—where should you stand?

Equipment required: 1 brain, scientific calculator or a CAS

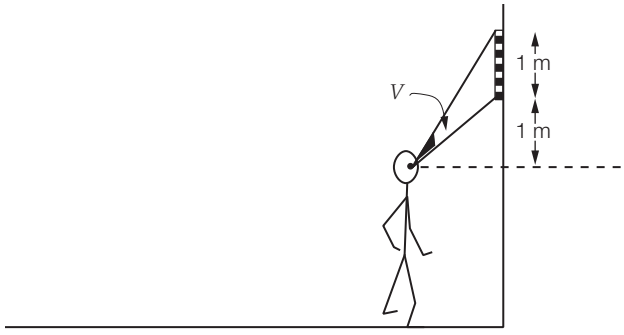
The Big Question

How far should you stand from artwork to have the maximum viewing angle?

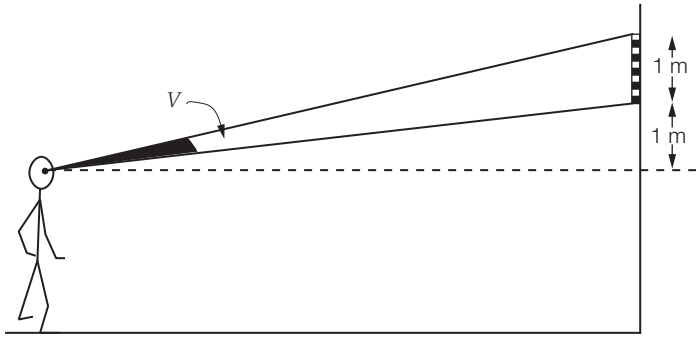
Imagine you are at an art gallery to look at a famous work of art. The painting is 1 m tall, and is mounted 1 m above eye-level on a wall as shown.



If you stand too close to the painting, the viewing angle V is too small to permit a good view (and your neck quickly gets sore).

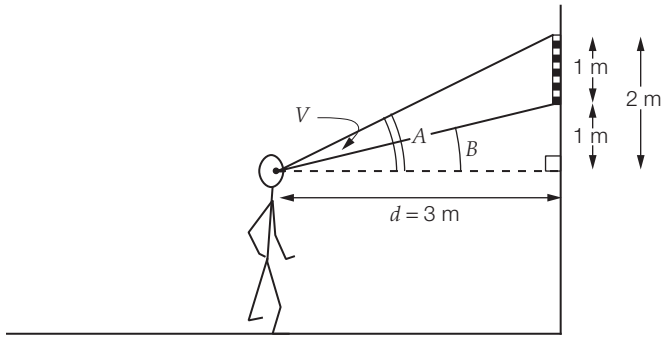


If you stand too far away the viewing angle is also too small, and you don't see any detail in the painting.



Engage

The viewing angle V is the difference between angles A and B on the diagram below.



$V = A - B$.

1 Use the two triangles and the measurements in the diagram to calculate A, B and then V if the person viewing the artwork is standing a distance d of 3 m from the wall.

Explore

2 By changing the value of d, find the largest possible value for V. Try at least five values for d.

Strategy options

- Draw a table.
- Look for a pattern.
- Test all possible combinations.

Explain

3 Describe any pattern or trend you can see in the values of angles A, B, and V as you move closer to or further away from the wall.

Elaborate

4 Use different values of d to find the best viewing position for:

(a) a 0.75 m tall painting mounted 1.2 m above eye-level

(b) a 1.5 m tall painting mounted 0.8 m above eye-level

(c) a 63 cm tall painting mounted 50 cm above eye-level.

5 Summarise your findings and answer the Big Question.

Evaluate

6 (a) How confident are you that you have found the largest value for V?

(b) What could you do to improve your answer?

7 As you calculated different values of V, did you find any rules or shortcuts to use with your calculator that made the task more efficient?

Extend

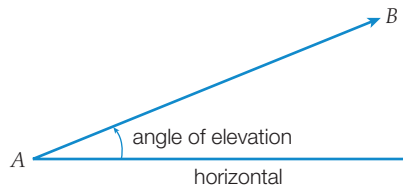
8 Investigate how a person's height affects where they should stand to gain the maximum viewing angle (i.e. where a short person should stand compared to a tall person).

Angles of elevation and depression

5.4

The **angle of elevation** from A to B is the angle that the line AB makes with the horizontal.

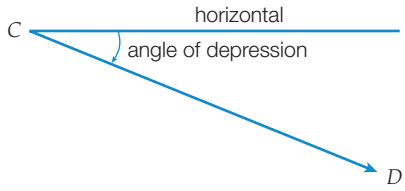
If you were positioned at A, you would need to *raise* your eyes from a horizontal line of sight to view an object at B.



The **angle of depression** from C to D is the angle the line CD makes with the horizontal.

If you were positioned at C, you would need to *lower* your eyes from a horizontal line of sight to view an object at D.

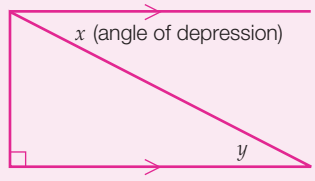
Remember: Elevation—look up, Depression—look down.



The angle of depression may not always be inside the triangle that we draw. However, we can use alternate angles to find an angle within the triangle.

x and y are alternate angles on parallel lines

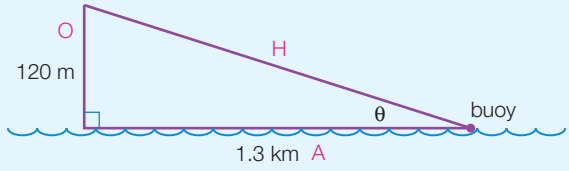
$\therefore x = y$



Worked Example 11

WE11

A marker buoy used to guide ships is 1.3 km away from the base of a 120 m cliff. To the nearest degree, what is the angle of elevation from the buoy to the top of the cliff? Answer correct to the nearest degree.

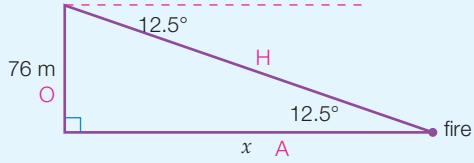
Thinking	Working
1 Draw a diagram that includes a right-angled triangle. Label the sides O, A and H.	
2 Decide which trigonometric ratio connects the given values.	$\theta = ?$ $O = 120$ m $A = 1.3$ km $\tan\theta = \frac{O}{A}$

3 Substitute the given values and form an equation. Remember to have distances in the same units.	$\tan\theta = \frac{120}{1300}$
4 Use the inverse of the ratio to find θ.	$\theta = \tan^{-1}\left(\frac{120}{1300}\right)$
5 Use the calculator to evaluate.	$\theta = 5.273\,89\ldots$
6 Round the answer to the nearest degree.	$\theta = 5°$ (nearest degree)
7 State the answer.	The angle of elevation from the buoy to the top of the cliff is approximately 5°.

Worked Example 12

WE12

A ranger at the top of a lookout tower spots a small fire at an angle of depression of 12.5°. If the tower is 76 m high, calculate the distance from the base of the tower to the fire. Round your answer to the nearest metre.

Thinking	Working
1 Draw a diagram that includes a right-angled triangle. Label the sides O, A and H.	
2 Use alternate angles to find and label the angles of depression and elevation.	
3 Decide which trigonometric ratio connects the given values.	$\theta = 12.5°$ $O = 76$ m $A = x$ $\tan\theta = \frac{O}{A}$
4 Substitute the given values and form an equation.	$\tan 12.5° = \frac{76}{x}$
5 Make *x* the subject.	$x\tan 12.5° = 76$ $x = \frac{76}{\tan 12.5°}$
6 Use a calculator to evaluate and round the answer to the nearest metre.	$x = 343$ m (nearest metre)
7 State the answer in a sentence.	The fire is approximately 343 m from the base of the tower.

5.4 Angles of elevation and depression

Navigator

Q1, Q2, Q3, Q4, Q6, Q8, Q12, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14

Answers page 741

Fluency

1 Hayden is standing 1.2 km away from the base of a 305 m tall building. To the nearest degree, what is the angle of elevation from the point where Hayden is standing to the top of the building? WE11

2 A person in a hot-air balloon 150 m above the ground spots a parked car at an angle of depression of 16.5°. Calculate the distance of the car to the point that is directly underneath the balloon to the nearest metre. WE12

3 During the day, Kristy notices that a post that is 3 m tall casts a shadow of 2.3 m. What is the angle of elevation of the Sun at that time? Answer to the nearest degree.

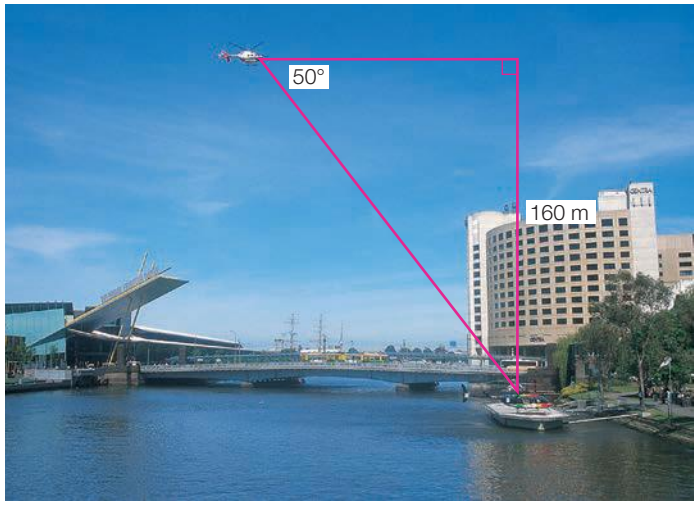
4 The angle of depression from the top of a pine tree to a point on the ground 50 m away is 38°. Find the height of the tree to the nearest metre.

5 A traffic signal 3.95 m high casts a shadow across the road. If the angle of elevation of the Sun is 24.8°, find the length of the shadow, correct to two decimal places.

6 A helicopter is flying at a height of 160 m above a river. The pilot can see another helicopter on a pontoon along the river located at an angle of depression of 50°. Find, to the nearest metre:

(a) the distance from the first helicopter to the second along the line of sight

(b) how far the first helicopter must fly at the same altitude to be directly over the second.



7 (a) A railway rises 400 cm for every 2 km of track. From the lower end of the track, the angle of elevation (to two decimal places) is:

A 0.11° B 11.54° C 25° D 78.69°

(b) From a hot-air balloon, the angle of depression to a marked landing site is 6°50′30″. If the observation deck in the balloon is 190 m above the ground, the horizontal distance the balloon needs to travel to be above the landing site is closest to:

A 191.23 m B 1.58 km C 1.67 km D 1.68 km

Understanding

8 Rachel uses an inclinometer (a device for measuring angles) to measure the angle of elevation from her eye level to the top of a billboard 25 m away. She obtains a value of 42°. She also measures the distance from the ground to her eye level and finds it to be 153 cm. Find the height of the billboard to the nearest metre.

9 A television antenna is mounted on the top of a nearby hill. From a position 4.5 km away, the angle of elevation to the bottom of the antenna is 17°30′, whereas the angle of elevation to the top of the antenna is 17°50′. Find the height of the antenna in metres, correct to two decimal places.

10 A yacht is 1.8 km away from the base of a 120 m cliff. The deck of the yacht is 3 m above sea level. To the nearest degree, what is the angle of elevation from the deck of the yacht to the top of a 4.2 m flagpole on the edge of the cliff?

Reasoning

11 Peter views a stationary beetle on the ground, 2.6 m away, at an angle of depression of 32°. Peter's eyes are 15 cm below the top of his head. He is standing upright.

(a) Find Peter's height to the nearest centimetre.

(b) The beetle moves directly ahead to a new position. If the distance from Peter to the beetle along his line of sight is 4.7 m, find the new angle of depression, correct to one decimal place.

12 Sonya is standing 20 m from a hot-air balloon that is rising vertically at a speed of 1.5 m/s. When the balloon's position is at eye-level, she follows its path for 7 seconds.

(a) Calculate the angle of elevation of the balloon from Sonya's line of sight after 7 seconds.

(b) If Sonya is 175 cm tall, what is the height of the balloon above the ground after 7 seconds?

Open-ended

13 A flagpole 24 m high is to be supported by a cable that is 18 m long when pulled taut. It must be fastened at least halfway up the pole at an angle of elevation between 45° and 60°.

(a) Find three possible angles of elevation for the cable.

(b) For each angle of elevation, find the distance from the flagpole that the cable must be fastened.

14 From the top of a 20 m lookout, Ranger Sam spots a hiker 200 m due east of his position, heading due east away from him at a constant speed of 50 m/min. Ranger Sam notes the angle of depression of the hiker from his position every minute for a period of one hour. Find two values for the angles of depression that Ranger Sam recorded.

Outside the Square — Problem solving

Outback mail run

In outback Australia, remote houses and communities receive their mail by plane.

The pilot flying the mail plane is cruising at an altitude of 4500 m when she looks out of the left window and sees one house at an angle of depression of 17°. Out of the right window, she spots another house at an angle of depression of 13°.

How far apart are the two houses, to the nearest kilometre?

Strategy options

- Draw a diagram.
- Have I seen a similar problem?
- Break problem into manageable parts.

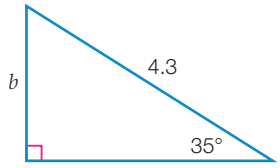
Half-time 5

1 A ladder of length 3 m reaches 1.85 m up a vertical wall. The ladder is moved closer to the wall and now reaches 2.10 m up the wall. What is the change in the angle the ladder makes with the ground? Ex.5.3

2 An ant travels 76 cm in a straight line, turns 90°, follows a straight path for 80 cm and then travels directly to its starting point. The smallest interior angle of the triangle formed is: Ex.5.3

A 40.2° **B** 43.5° **C** 44.6° **D** 45.5°

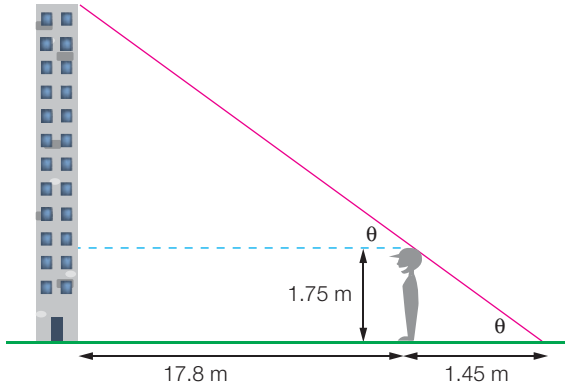
3 Which calculation can be used to find the length of side b? Ex.5.1



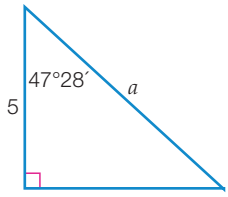
A $b = \dfrac{4.3}{\cos 35°}$ **B** $b = 4.3 \times \sin 35°$

C $b = \dfrac{\tan 35°}{4.3}$ **D** $b = 4.3 \times \cos 35°$

4 Find the angle θ in the small triangle and use the result to calculate the height of the building to the nearest centimetre. Ex.5.3



5 The value of a in the triangle on the right, rounded to two decimal places, is: Ex.5.2



A 2.41 **B** 3.39

C 3.68 **D** 7.40

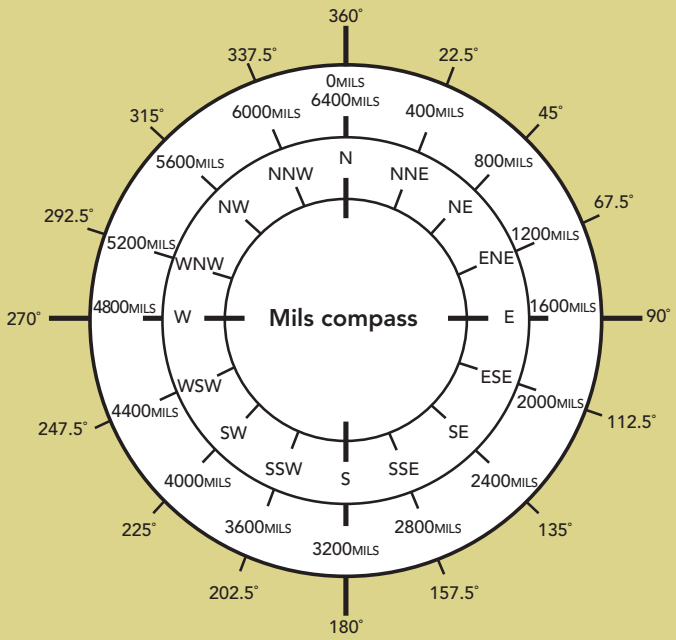
6 From the top of a lookout, a ranger spots a small scrub fire at an angle of depression of 6°14′. How far is the fire from the base of the lookout if the eye-level of the ranger is 35 m above the ground? Give your answer correct to the nearest metre. Ex.5.4

Maths 4 Real

NAVIGATION

THE DEGREE SYSTEM VS THE MILS SYSTEM

The compass that most of us are familiar with has the cardinal directions of north, south, east, west and angles given in degrees. However, this is not the only type of compass. Military forces throughout the world commonly use a compass that measures in 'mils' rather than degrees. The *mils system* gives greater accuracy in directions and makes it easier to estimate distances.



There are 6400 mils in a full 360° degree rotation.

To convert from mils to degrees: multiply by $\frac{360}{6400}$

To convert from degrees to mils: multiply by $\frac{6400}{360}$

1 Convert the following degrees to mils (round to one decimal place where necessary).

(a) 180° **(b)** 225° **(c)** 165°

2 Convert the following mils to degrees (round to one decimal place where necessary).

(a) 5400 mils **(b)** 900 mils **(c)** 40 mils

3 **(a)** How many mils are there in an anti-clockwise turn between south-east and north-east?

(b) How many mils are there in a clockwise turn between south and north-west?

(c) What is the direction in mils of the north arrow?

GRID BEARING VS MAGNETIC BEARING

The position of the magnetic north pole is not fixed but moves slowly, due to changes in Earth's molten iron core. It is important that navigators take this into account when using a compass and a map, as the compass may point in a different direction (**magnetic north**) than the map (**grid north**).

There are three different 'norths':

Magnetic north (MN)—the direction the compass points; towards the Earth's magnetic North Pole

Grid north (GN)—the direction indicated by the vertical grid lines on a map

True north (TN)—the direction to the Earth's geographic North Pole from the location

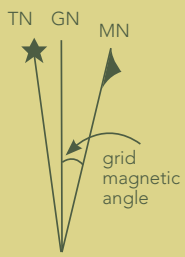
Other important words when describing direction are:

Grid magnetic angle—the angle between grid north and magnetic north

Grid bearing (GB)—a bearing measured on a map

Magnetic bearing (MB)—a bearing measured by a compass

Magnetic north can have an easterly or westerly variation from true north depending on where you are located. This variation is called the grid magnetic angle. It ranges from about 5° west of true north in parts of Western Australia to about 15° east of true north in eastern Australia. The line through the map below indicates where the magnetic variation changes from a westerly variation to an easterly variation.



4 Use the following sample map information to answer the questions below.

Sample topographical map information for a location with easterly variation

Grid magnetic angle = 129 mils (8.4°) in 2010.

Magnetic north is correct for 2010 and moves easterly by 2 mils every 6 years.

(a) By how many mils will the grid magnetic angle change each year?

(b) What is the grid magnetic angle for 2013?

(c) What will be the grid magnetic angle for 2023?

It is important that navigators are able to convert from magnetic bearings to grid bearings in order to synchronise their use of maps and compasses.

For a location with an *easterly* magnetic variation:

To convert from *grid bearing to magnetic bearing* subtract the grid magnetic angle

GB → MB = GM – grid magnetic angle

To convert from *magnetic bearing* to *grid bearing* add the grid magnetic angle

MB → GB = MB + grid magnetic angle

For Questions **5** and **6** use the map information in Question **4** and assume the year is 2013.

5 Convert the following grid bearings to magnetic bearings.

(a) 5460 mils **(b)** 4360 mils
(c) 0200 mils **(d)** 0020 mils

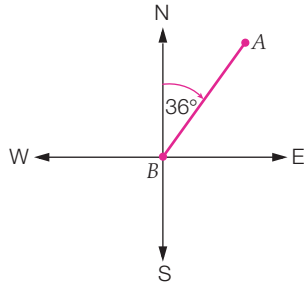
6 Convert the following magnetic bearings to grid bearings.

(a) 0300 mils **(b)** 3245 mils
(c) 1834 mils **(d)** 6300 mils

5.5 Bearings

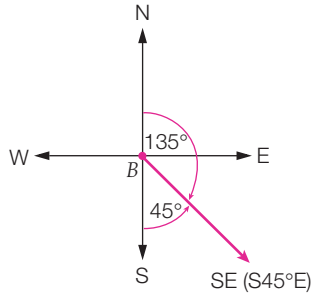
The direction of point A from point B can be expressed in two ways:

True bearings give the angle in degrees in a clockwise direction from north. The true bearing of A from B is 036°T. We use the letter T to denote a true bearing. True bearings are always written with three digits, so the bearing of 36° is written as 036°T.



Compass bearings are directions given in terms of turning a number of degrees east or west of north or south (whichever direction is closer to the position). The compass bearing of A from B is N36°E.

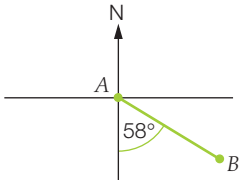
The direction of southeast (SE) is shown here. As a compass bearing, it is written as S45°E. As a true bearing, it is written as 135°T. Compass bearings are sometimes called conventional bearings.



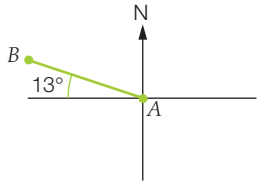
Worked Example 13 WE13

Find **(i)** the compass bearing and **(ii)** the true bearing of point B from point A, for each of the following.

(a)



(b)



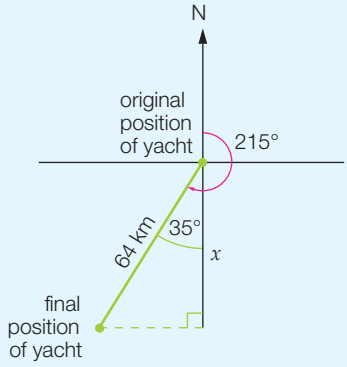
Thinking	Working
(a) (i) 1 Decide whether the point is closer to north or to south. Write either 'N' or 'S'. (Here, it is closer to south so we write 'S'.)	(a) (i) S58°E
2 Decide whether the point is closer to east or to west. Leave a space, then write either 'E' or 'W'. (Here, it is closer to east so we write 'E'.)	
3 From the direction you determined in Step **1**, how many degrees to the east or west do you need to turn? Write this number between the two letters. (Here, we turn 58° east of south.)	

	(ii) Find the angle of turn from north in a clockwise direction to the line joining the point to the centre of the compass. (Here, we find the difference between the compass bearing and 180°.)	(ii) $180 - 58 = 122°T$
(b) **(i)**	**1** Decide whether the point is closer to north or to south. Write either 'N' or 'S'. (Here, it is closer to north so we write 'N'.) **2** Decide whether the point is closer to east or to west. Leave a space, then write either 'E' or 'W'. (Here, it is closer to west so we write 'W'.) **3** From the direction you determined in Step **1** how many degrees to the east or west do you need to turn? Write this number between the two letters. (Here, we turn 77° west of north.)	(b) (i) N77°W
	(ii) Find the angle of turn from north in a clockwise direction to the line joining the point to the centre of the compass. (Here, we add 13° to 270°.)	(ii) $270 + 13 = 283°T$

Worked Example 14

WE 14

A yacht sails 64 km on a true bearing of 215°T. How far south is the yacht from its original position? Round the answer to two decimal places.

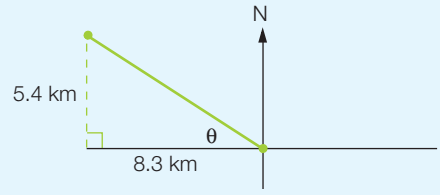
Thinking	Working
1 Draw a diagram showing the known information. Let x be the distance required. Use the given bearing to show a triangle containing the angle required. (215° – 180° = 35°)	
2 Write down the known and unknown values and identify the trigonometric ratio that connects them.	$\theta = 35°$ $H = 64$ km $A = x$ $\cos\theta = \frac{A}{H}$

3 Form an equation by substituting the known and unknown values into the ratio.	$\cos 35° = \frac{x}{64}$
4 Make x the subject.	$x = 64 \times \cos 35°$
5 Evaluate using a calculator and round the answer to two decimal places.	$x = 52.43$ km (2 d.p.)
6 State the answer.	The yacht is 52.43 km south from its original position.

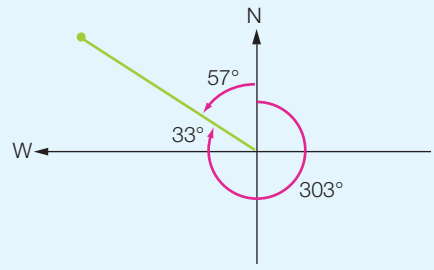
Worked Example 15

WE15

On a hiking trip, Sarah walks 8.3 km west and then 5.4 km north. Calculate her true bearing and compass bearing from her original position, to the nearest degree.

Thinking	Working
1 Draw a diagram showing the known information. Let θ be the unknown angle.	
2 Write down the known and unknown values and identify the trigonometric ratio that connects them.	$O = 5.4$ km $A = 8.3$ km $\theta = ?$ $\tan\theta = \frac{O}{A}$
3 Form an equation by substituting the known values into the ratio.	$\tan\theta = \frac{5.4}{8.3}$
4 Find θ using the inverse of the ratio and round the answer to the nearest degree.	$\theta = \tan^{-1}\left(\frac{5.4}{8.3}\right)$ $\theta = 33°$ (nearest degree)
5 Express the value as a true bearing by adding all the angle values clockwise from north. To find the compass bearing, start at north or south (whichever is closer) and use complementary angles to find the angle required to turn east or west. (Here, we start at north and turn (90 – 33)° west.) Use your diagram to help.	True bearing $= 270° + 33°$ $= 303°$T Compass bearing $= 90° - 33°$ $= 57°$ $=$ N57°W

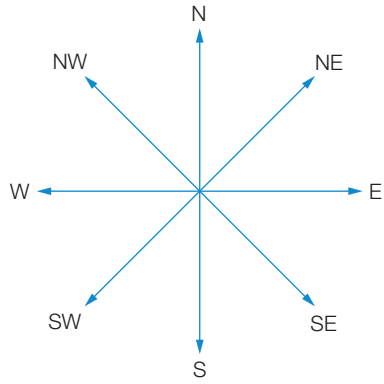
6 State the answer.



True bearing = 303°T
Compass bearing = N 57°W

The directions exactly in between the four compass points of north, east, south and west have the following bearings.

northeast (NE) is 45°T or N45°E
southeast (SE) is 135°T or S45°E
southwest (SW) is 225°T or S45°W
northwest (NW) is 315°T or N45°W



5.5 Bearings

Navigator

Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q11, Q14	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9, Q11, Q13, Q14, Q15	Q1, Q2, Q3, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14, Q15

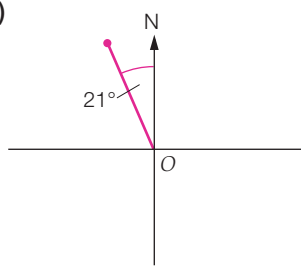
Answers page 741

Fluency

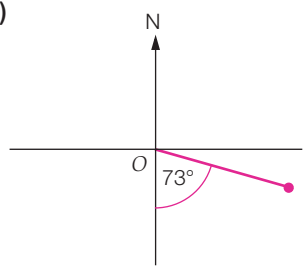
1 Find **(i)** the compass bearing and **(ii)** the true bearing for each of the following points from *O*.

WE13

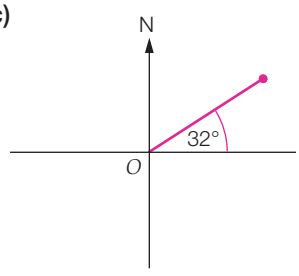
(a)



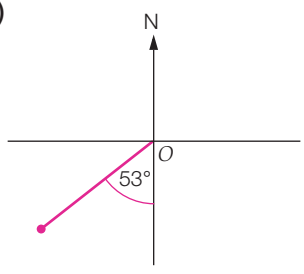
(b)



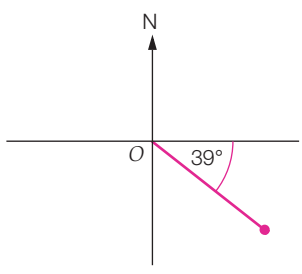
(c)



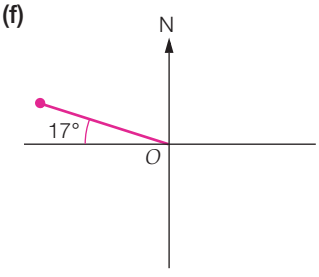
(d)



(e)



(f)



WE14

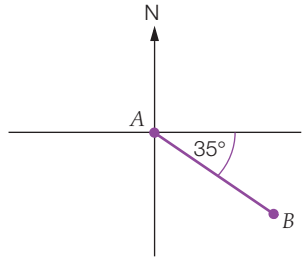
2 A ship sails for 150 km on a bearing of N40°E. How far north of its starting point is it, to the nearest kilometre?

WE15

3 (a) Pedro sails his yacht 2.5 km east and then 4.7 km south. To the nearest degree, state Pedro's true bearing and compass bearing from his original position.

(b) A taxi driver drives 630 m south and then 410 m east. To the nearest degree, what is the bearing of the taxi from its starting position? Give your answer as a compass bearing and a true bearing.

4 (a) The compass bearing of point *B* from point *A*, as shown in the diagram, is:



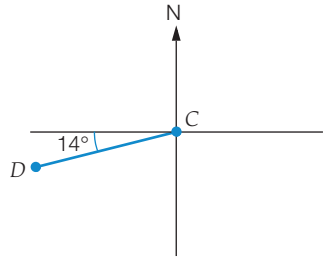
A N35°E

B S55°E

C N35°W

D S35°E

(b) The true bearing of point *D* from point *C*, as shown in the diagram, is:



A 076°T

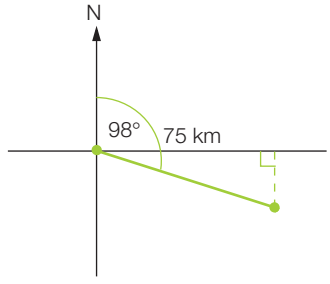
B 194°T

C 256°T

D 284°T

Understanding

5 A helicopter travelling on a true bearing of 098°T is now 75 km east of its original position. How far has the helicopter travelled, to two decimal places?



6 A plane flies on a true bearing of 280°T at 600 km/h after taking off at 1300 hours. To the nearest kilometre, how far north of its starting position is it at 1530 hours?

7 Jack drives north at 80 km/h for 2 hours. He then turns and drives east at the same speed for 3 hours.

(a) To the nearest degree, what is Jack's true bearing from his original position?

(b) If Jack could drive from his current position straight back to his starting position:

(i) how far would he travel, to the nearest kilometre

(ii) what true bearing should he follow?

8 A hiker walks 5 km on a true bearing of 025°T from camp *A* to camp *B*.

Round your answers to two decimal places.

(a) How far east has she walked from *A* to *B*?

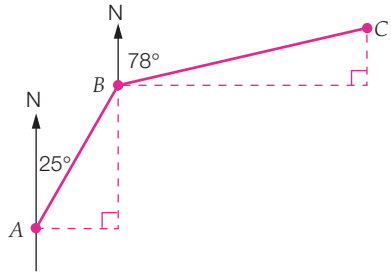
She then continues for another 6 km on a true bearing of 078°T from camp *B* to camp *C*.

(b) **(i)** How far east of camp *B* is she now?

(ii) How far east of camp *A* is she now?

(c) **(i)** How far north of camp B is she now?

(ii) How far north of camp A is she now?



Warning

Round your final answers, but do not use rounded answers in further calculations. Use the original unrounded answers in further calculations.

9 Sasha is running an orienteering course. He runs 750 m to the first checkpoint on a bearing of S53°W. At the checkpoint he is instructed to run another 600 m on a bearing of N68°W to the next checkpoint.

(a) To the nearest metre, how far west is Sasha from his starting position?

(b) Is Sasha's current position north or south of his starting position? How far north or south?

10 A toy robot walks 16 m on a bearing of S62°W, and then turns and walks another 23 m NW. How far west is the robot from its starting point? Round your answer to two decimal places.

Reasoning

11 A crew row their boat from point *A* to point *B*, a distance of 200 m on a bearing of S55°E. They then row due north a further 440 m to point *C*, as shown in the diagram.

(a) Find, to the nearest metre:

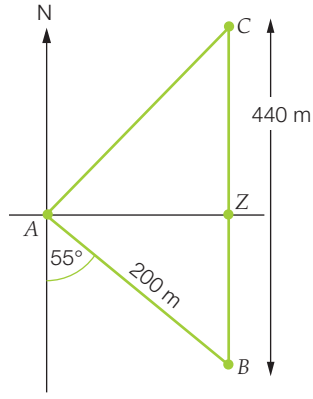
(i) the distance from *B* to *Z*

(ii) the distance from *A* to *Z*.

(b) Find the true bearing of *C* from *A*, to the nearest degree.

Finally, the crew row back from *C* to *A*.

(c) What is the total distance rowed by the crew, to the nearest metre?



Hint

Pythagoras' theorem is helpful for finding missing sides in a right-angled triangle.

12 **(a)** From his home, Minh jogs 2.7 km NE and then 7.4 km SE. To the nearest degree, find his true bearing from his starting position.

(b) If Minh then jogs 6.9 km west, how far south is he from his home? Round your answer to two decimal places.

13 An air force jet takes off from an airfield and flys on a bearing of N37°E. The pilot is asked by ground staff to alter her course, 600 km into the trip, by turning 120° in a clockwise direction.

(a) State the new compass bearing of the jet after it has travelled 600 km.

(b) The jet travels for a further 800 km on its new bearing. How far east has it travelled from its starting position, to the nearest kilometre?

(c) How far north or south of its position at take-off, is the current position of the jet, to the nearest kilometre?

(d) What compass bearing should the jet now follow to return to the airstrip?

(e) How far will it travel on its return, if it follows the bearing you determined in **(d)**? Answer to the nearest kilometre.

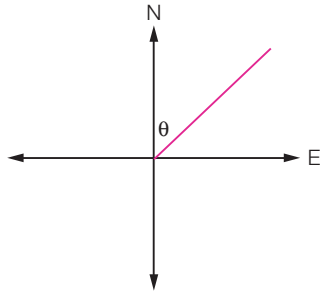
Open-ended

14 Choose a true bearing θ less than 90° and state its true bearing and compass bearing after it is:

(a) reflected about the east-west line (the x-axis of the Cartesian plane)

(b) reflected about the north-south line (the y-axis of the Cartesian plane)

(c) rotated 180°.



15 Design a sailing course around five buoys labelled A, B, C, D and E. On each leg of the course, write down the true bearing to be taken by the sailors competing in the race. Remember that the course finishes at the starting point.

Outside the Square — Problem solving

Seeing the light

1 A ship leaves its dock, which is 3 km due west of a lighthouse, and travels NE at a constant speed of 15 km/h.

(a) To the nearest minute, when is the ship due north of the lighthouse?

(b) At this time, how far is the ship from the lighthouse?

(c) How far has the ship travelled?

2 Rework these answers if the ship's bearing is N20°E.

Strategy options

- Draw a diagram.
- Break problem into manageable parts.

Mixed two-dimensional problems

5.6

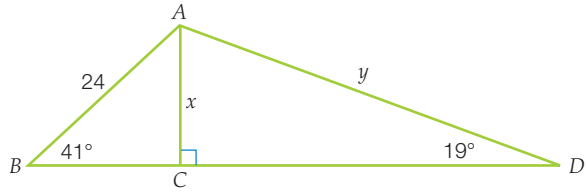
Situations arise when more than one trigonometric ratio is required to find an unknown side or angle. In such cases, it is important to determine which ratio is used first.

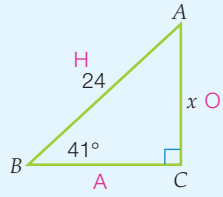
It is important that we use unrounded answers when using these answers in further calculations. This avoids the accumulation of errors due to rounding. Always use the unrounded answer if it is required in further calculations. Use the calculator's memory key to store values, if necessary, and then round off once only for your answer.

Worked Example 16

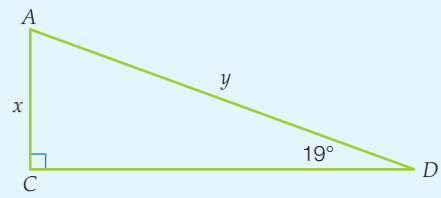
WE16

Find the values of x and y, rounded to two decimal places.



Thinking	Working
1 Identify the triangle with two known values. Use this one first to find the value of an unknown. (Here, we will use ΔABC to find x.) Draw the triangle and label it with known and unknown values. Label the sides O, A and H.	
2 Write down the known and unknown values, then identify the trigonometric ratio that connects them.	$\theta = 41°$ $H = 24$ $O = x$ $\sin\theta = \frac{O}{H}$
3 Form an equation by substituting values into the ratio.	$\sin 41° = \frac{x}{24}$
4 Make x the subject.	$x = 24 \times \sin 41°$
5 Perform the calculations. Do not round your answer.	$x = 15.745\,416\,7$

6 Draw the second triangle and show all the known and unknown values.



7 Write down the known and unknown values, then identify the trigonometric ratio that connects them.

$\theta = 19°$
$O = 15.745\,416\,7$
$H = y$

$$\sin\theta = \frac{O}{H}$$

8 Form an equation by substituting values into the ratio.

$$\sin 19° = \frac{x}{y}$$

9 Make y the subject of the equation. Solve using the previously found, unrounded value of x.

$y \times \sin 19° = x$

$$y = \frac{x}{\sin 19°}$$

$$y = \frac{15.745\,416\,7}{\sin 19°}$$

10 Solve for y using a calculator.

$y = 48.362\,889\,55$

Round answers to two decimal places.

$x = 15.75$ units $\quad y = 48.36$ units (2 d.p.)

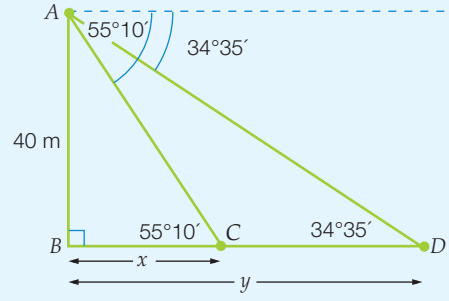
Worked Example 17

WE17

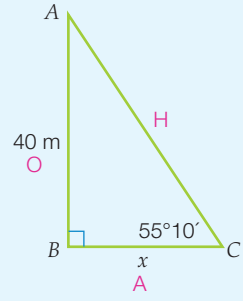
Thea sights two swimmers (both on the same bearing from her) at angles of depression of 34°35′ and 55°10′ from her vantage point on the cliff. If her eye-level is 40 m above the water level, find the distance between the two swimmers, rounded to two decimal places.

Thinking	Working

1 Draw and label a diagram showing all the known and unknown values. (Here, Thea is at point A, the base of the cliff is point B, and the two swimmers are at points C and D.) Label the required unknown values with pronumerals. Use alternate angles to find the angles within the triangles.



2 Draw one of the two triangles out separately. Identify the sides O, A and H.



3 Write down the known and unknown values, then identify the trigonometric ratio that connects them.

$\theta = 55°10'$
$O = 40$ m
$A = x$

$$\tan\theta = \frac{O}{A}$$

4 Form an equation by substituting values into the ratio.

$$\tan 55°10' = \frac{40}{x}$$

5 Make x the subject of the equation. (Here, we multiply both sides by x, then divide by tan 55°10′.)

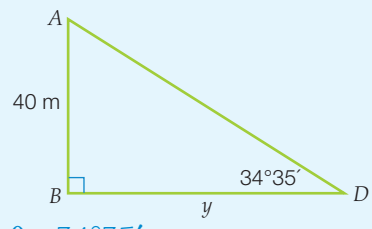
$$x = \frac{40}{\tan 55°10'}$$

6 Evaluate using a calculator. Do not round the answer.

$$x = 27.835\,250\,33$$

7 Draw the second triangle separately. Identify the sides O, A and H.

8 Write down the known and unknown values, then identify the trigonometric ratio that connects them.



$\theta = 34°35'$
$O = 40$ m
$A = y$

$$\tan\theta = \frac{O}{A}$$

9 Form an equation by substituting values into the ratio.

$$\tan 34°35' = \frac{40}{y}$$

10 Make y the subject of the equation. (Here, we multiply both sides by y, then divide by tan 34°35′.)

$$y = \frac{40}{\tan 34°35'}$$

11 Evaluate using a calculator. Leave this value on the screen.

$$y = 58.019\,401\,28$$

12 Find the difference between the two calculated side lengths. (Here, it is $y - x$.)

$$y - x = 30.184\,150\,95$$

13 Round the answer to two decimal places.

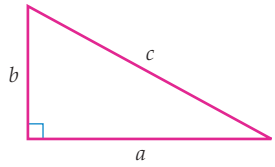
$CD = 30.18$ m (2 d.p.)

14 State the answer.

The distance between the swimmers is approximately 30.18 m.

Using Pythagoras' theorem

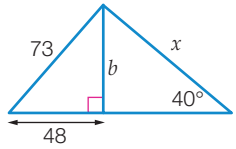
In some cases it is easier to use Pythagoras' theorem as part of the calculation. The theorem states that for a right-angled triangle, $c^2 = a^2 + b^2$, where c is the length of the hypotenuse and a and b are the lengths of the two shorter sides.

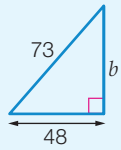
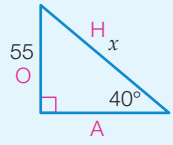


Worked Example 18

WE18

Calculate the value of x, rounded to two decimal places.



Thinking	Working
1 Identify the triangle with two known side lengths. Draw it separately, and then use Pythagoras' theorem to calculate the unknown side length.	 $c^2 = a^2 + b^2$, where $c = 73$, $a = 48$ $73^2 = 48^2 + b^2$ $b^2 = 73^2 - 48^2$ $b^2 = 3025$ $b = 55$
2 Draw out the second triangle. Identify the sides O, A and H. Write down the known and unknown values, including those you just calculated. Use the appropriate trigonometric ratio to solve for the unknown length. Round the answer to two decimal places.	$\theta = 40°$ $H = x$ $O = 55$  $\sin\theta = \frac{O}{H}$ $\sin 40° = \frac{55}{x}$ $x = \frac{55}{\sin 40°}$ $x = 85.56$ units

Note that in the previous Worked Example we could have used trigonometry to calculate the value of b in the first triangle, by using the two given sides to calculate the angle θ, then using the angle and a side to calculate b. However, this involves two steps, which is a less efficient method than using Pythagoras' theorem.

5.6 Mixed two-dimensional problems

Navigator

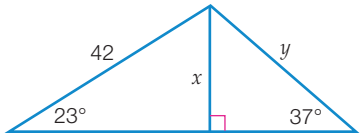
Q1, Q2, Q3, Q4 (a) (b), Q5, Q6, Q7, Q8, Q10, Q12, Q15, Q19	Q1, Q2, Q3, Q4 (c) (d), Q6, Q7, Q9, Q10, Q12, Q13, Q14, Q15, Q16, Q19	Q1 (b) (d) (f), Q2, Q3, Q4, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q15, Q16, Q17, Q18, Q19

Answers page 741

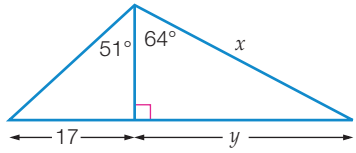
Fluency

1 Find the values of x and y, rounded to two decimal places. WE16

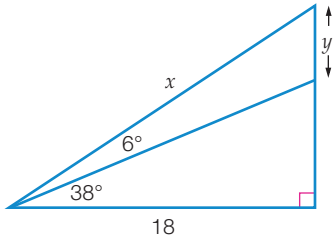
(a)



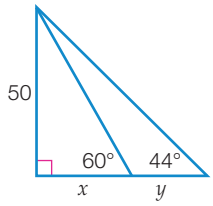
(b)



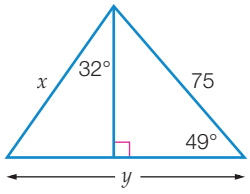
(c)



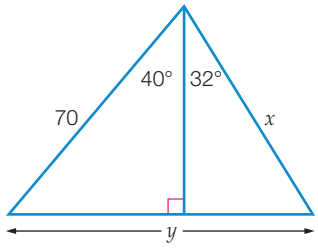
(d)



(e)



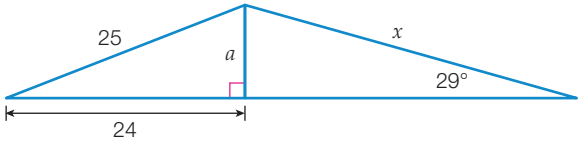
(f)



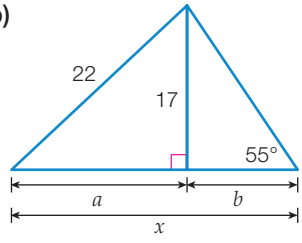
2 From the top of a lighthouse 75 m above sea level, two whales on the same bearing can be seen in the ocean at angles of depression of 28°40′ and 46°25′. Find the distance between the two whales, rounded to two decimal places. WE17

3 Calculate the value of the x in the following, rounded to two decimal places. WE18

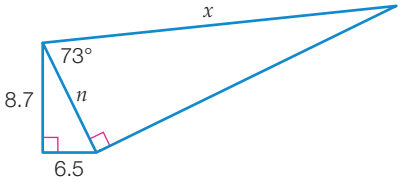
(a)



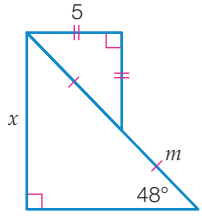
(b)



(c)

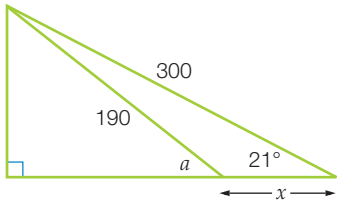


(d)

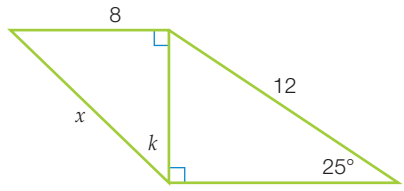


4 Find the values of the pronumerals. Round your answers to one decimal place.

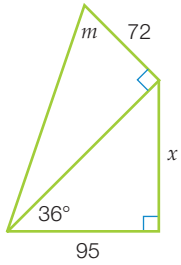
(a)



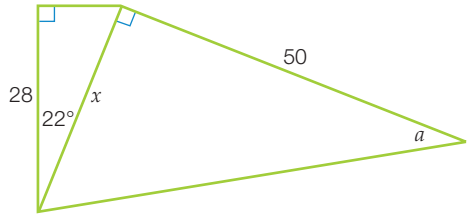
(b)



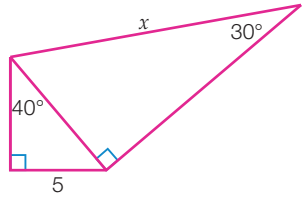
(c)



(d)



5 The length of x in the diagram is closest to:



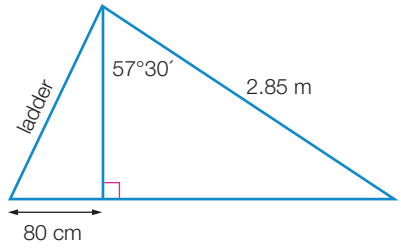
A 15.56

B 17.96

C 31.11

D 42.04

6 A slide in a playground has a vertical support and a ladder as shown in the diagram. The slide is 2.85 m long and makes an angle of 57°30′ with the vertical support. If the base of the ladder is 80 cm away from the support, the length of the ladder, to two decimal places, is:

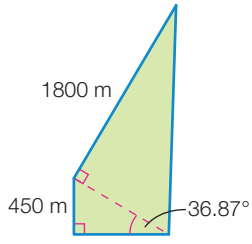


A 0.37 m **B** 0.42 m

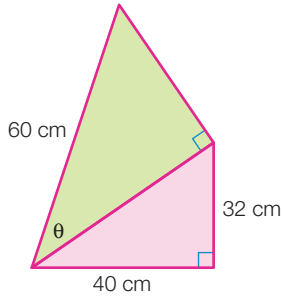
C 1.53 m **D** 1.73 m

Understanding

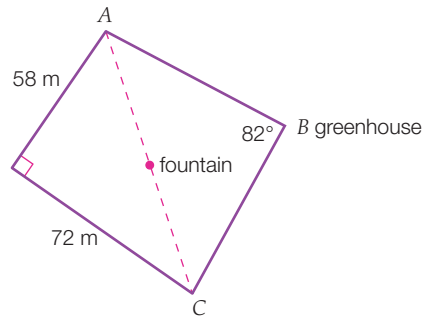
7 A farm whose perimeter shape is shown requires wire fencing costing \$5 per metre. Calculate the total cost of fencing, to the nearest dollar.



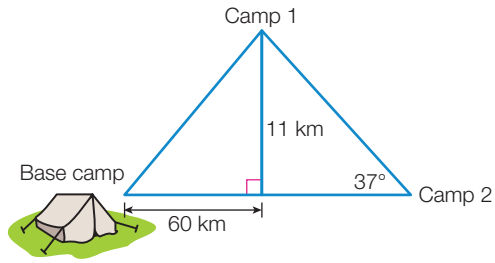
8 A red tile and a green tile are to be joined as part of a mosaic design. What angle θ is required? Express your answer in degrees and minutes.



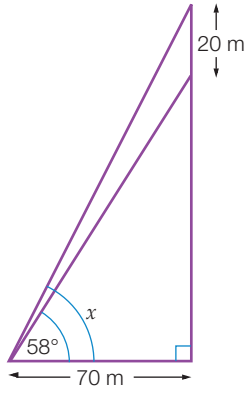
9 A surveyor is hired to peg out the garden shown. Shape *ABC* will be an isosceles triangle and the fountain will be positioned midway between *A* and *C*. To the nearest metre, what will be the length of the direct path from the fountain to the greenhouse?



10 Miguel left Base camp and trekked to Camp 1, then to Camp 2 and finally back to Base camp. To the nearest kilometre, how far did he walk in total?



11 When Liz is 70 m away from a building she notices a stuntman at an angle of elevation of 58° from her horizontal line of sight. He is climbing the side of the building. What will be the angle of elevation after the stuntman has climbed a further 20 m? Round your answer to the nearest degree.



12 A 4.5 m ladder leaning against a wall makes an angle of 64°5′ with the ground. If the ladder is moved 83 cm down the wall, what angle does it now make with the ground, in degrees, minutes and seconds?

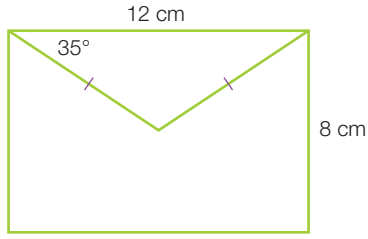
13 A pilot is landing a plane. He views two lights positioned in a line on the runway at angles of depression of 18° and 22°. If the plane is 300 m above the runway, find the distance between the two lights. State your answer correct to two decimal places.

Reasoning

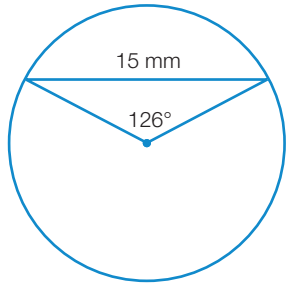
Hint

Draw a vertical line from the bottom of the flap to the top of the envelope.

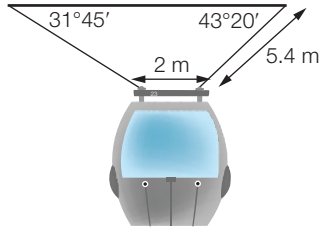
14 An envelope has a length of 12 cm and a width of 8 cm. The back flap is an isosceles triangle with an edge that makes an angle of 35° with the top of the envelope. Find the distance between the bottom of the flap and the bottom of the envelope. Round your answer to two decimal places.



15 A chord 15 mm long forms an angle of 126° at the centre of a circle. Find the length of the diameter of the circle, correct to two decimal places.



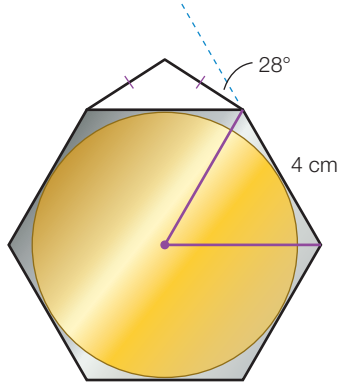
16 A cable car is supported by two wires attached to a horizontal cable. One of the wires is 5.4 m long and makes an angle of 43°20′ with the horizontal cable. The other wire makes an angle of 31°45′ with the cable. If the cable car is 2 m long, how far apart along the cable are the wires, correct to two decimal places?



Hint

The sum of the interior angles of an n-sided polygon is $(n - 2) \times 180°$.

17 A jeweller has designed a medal in the shape of a circle mounted on a regular hexagon. The circular inset is made from gold. The medallion will hang by a chain passing through the triangular shape shown at the top of the medal.



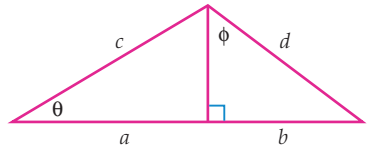
(a) Use the triangle marked on the medal to find the radius of the circular inset.

(b) Use your answer to **(a)** to find the area of gold required to make the circular inset. Round your answer to the nearest square centimetre.

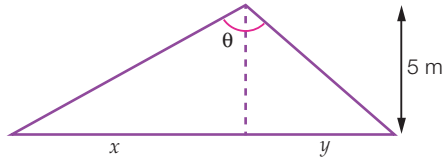
(c) What length of metal is required for the two sides of the triangular hanging top? Round your answer to two decimal places.

Open-ended

18 Determine values for a, b, c, d, θ and ϕ so that $a > b$ and $c > d$.



19 The cross-section of a roof (not necessarily symmetrical in shape) of height 5 m is shown. Determine values for x, y and θ so that the base has a length between 20 m and 25 m, and θ is greater than 100°.



Chapter review 5

D.I.Y. Summary

Key Words

adjacent side	compass bearing	minute	sine	true bearing
angle of depression	cosine	opposite side	tangent	
angle of elevation	hypotenuse	second	trigonometric ratio	

Copy and complete the following using words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 The _____ ratio is obtained by dividing the adjacent side by the hypotenuse.

2 The _____ ratio is obtained by dividing the opposite side by the hypotenuse.

3 The _____ ratio is obtained by dividing the opposite side by the adjacent side.

4 When a direction is given with a measurement east or west from north or south, this is a _____.

5 A _____ always has three digits and shows the number of degrees from north in a clockwise direction.

6 One degree is divided into sixty equal parts. Each part is called a _____.

7 There are 360 _____s in one degree.

8 The angle from the horizontal as you look up is called an _____. The angle from the horizontal as you look down is called an _____.

Fluency

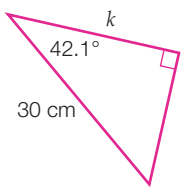
1 Use a calculator to find the following values, rounded to four decimal places. Ex. 5.1

(a) sin 35° (b) tan 51.3° (c) cos 6.045°

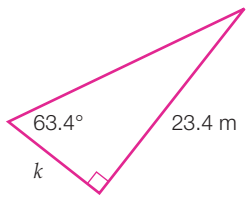
(d) tan 29°45′ (e) sin 16°24′ (f) cos 73°6′40″

2 Find the value of k in each of these triangles, rounded to two decimal places. Ex. 5.2

(a)



(b)



3 For each of the following trigonometric ratios, find the value of θ in: Ex. 5.3

(i) decimal degrees, rounded to two decimal places

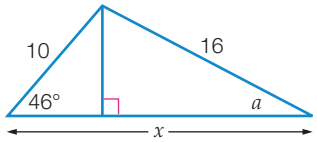
(ii) degrees, minutes and seconds.

(a) $\cos\theta = 0.74$ (b) $\tan\theta = 2.645$ (c) $\sin\theta = 0.4609$

4 A mast of height 5 m is on top of a building whose base is 100 m away from a drain on the road. The angle of elevation of the top of the mast from the drain is 34°. Calculate the height of the building, rounded to two decimal places. Ex. 5.4

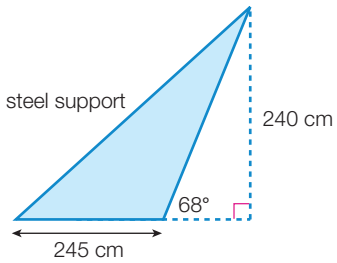
5 A plane flies due north for 70 km, then due west for a further 145 km. What is the compass bearing of the plane from its starting point, to the nearest degree? Ex. 5.5

6 Find the values of the pronumerals in this diagram. Round your answers correct to one decimal place.



Ex. 5.6

7 The sculpture, 'A study in triangles', is supported by a rigid steel rod. What is the length of the steel support, rounded to the nearest centimetre?



Ex. 5.6

8 A conveyor belt in an underground mine is 650 m long and moves to the surface at an angle of 14°48′ from the horizontal. The depth of the mine is:

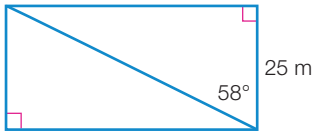
A 2544.57 m **B** 672.30 m **C** 628.44 m **D** 166.04 m

Ex. 5.6

9 The length of the diagonal in this rectangle is:

A 13.25 m **B** 29.48 m

C 40.01 m **D** 47.18 m



Ex. 5.6

Understanding

10 A balloon on the end of a string attached to the ground is being blown by the wind. If the balloon is 12.75 m above the ground and the string makes an angle of 48° to the horizontal, find the length of the string, rounded to two decimal places.

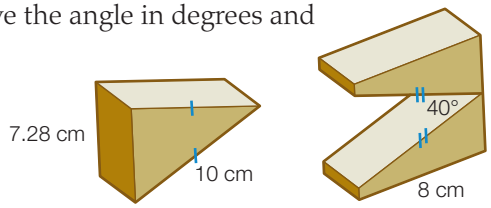
11 A ramp is 9.4 m long and joins two levels, one 3.1 m higher than the other. What angle does the ramp make with the lower level, rounded to one decimal place?

12 A lifeguard on watch at a beach spots a shark at an angle of depression of 9°. If the lifeguard views the shark from a height (to his eye-level) of 16 m above sea-level, find the horizontal distance between the lifeguard and the shark to the nearest metre.

13 A speedboat travels 26 km on a true bearing of 065°T. Find how far the boat is to the east of its starting point. Round your answer to two decimal places.

14 An aeroplane leaves the airport and flies 35 km in the direction N70°E. It then flies due east for a further 38 km. What is the aeroplane's distance and compass bearing from the airport? Round the distance to the nearest kilometre and give the angle in degrees and minutes.

15 Decide whether the wedge shown will fit exactly into the cavity in the block of wood. Give reasons for your answer.

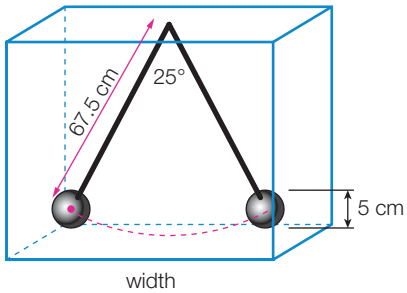


Reasoning

16 Amos leaves home and cycles for a distance of 35 km on a bearing of N30°E. He changes direction and cycles in a direction of S60°E before heading home on a bearing of S49°W. To the nearest kilometre, find the total distance Amos travelled.

17 For an equilateral triangle, explain two different ways the height can be found using trigonometric ratios.

18 A case is to be constructed to hold a pendulum. The shaft of the pendulum is connected to the centre of the bob and is 67.5 cm in length. The diameter of the bob is 5 cm. If the pendulum swings through an angle of 25°, what must the minimum width of the case be? Round your answer to two decimal places.



Advanced trigonometry

10A

6

In the real world not everything is parallel

Name: Michele Russo
Job: Surveyor
Qualifications: Bachelor of Geometric Engineering (Hons)

Surveyors are the 'police' of the geospatial world. Armed with technology such as a GPS and a theodolite, and skills in bearings, trigonometry and triangulation, they segregate land into 'parcels'. Taking precise measurements within certain set error ranges is an essential skill. People like Michele Russo, a graduate surveyor with the Department of Environment and Sustainability, help to resolve disputes between neighbours about the ownership of land.

"As a surveyor, my job is to re-establish title boundaries on the ground. Whenever a development goes through, such as a big subdivision, I go out and locate existing boundaries using 3D measurements. In the real world not everything is parallel. We need to check how well a parcel of land fits in with the surrounding land and the accuracy of that."

Why learn this?

The world is not flat. We live in a three-dimensional world and need to be able to calculate measurements that cannot be physically measured. Trigonometry gives us the skills to determine angles, sides and areas of all types of triangles, and is used extensively by engineers, surveyors and scientists. Tides and daily temperatures follow a cyclic or repeating pattern that can be modelled with trigonometric graphs. The solutions to trigonometric equations can give us vital information about situations in which a cyclic pattern can be detected.

After completing this chapter you will be able to:

- use trigonometry to solve three-dimensional problems
- use the unit circle to define trigonometric functions
- understand the periodic nature of the trigonometric functions
- graph trigonometric functions with and without technology
- use the sine and cosine rules to solve for unknowns in triangles
- apply trigonometry to finding the area of a triangle.

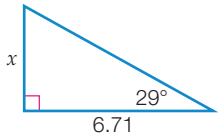
Recall 6

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

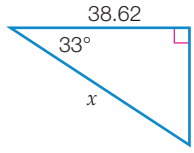
Worksheet R6.1

1 Calculate the value of x and round to two decimal places.

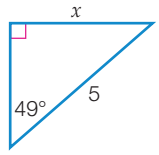
(a)



(b)



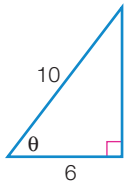
(c)



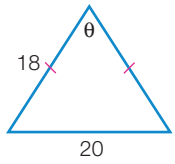
Worksheet R6.2

2 Determine the size of the angle θ, in degrees and minutes.

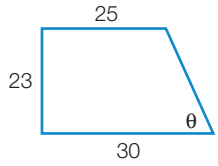
(a)



(b)



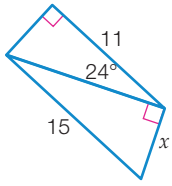
(c)



Worksheet R6.3

3 Find the length of x. Round your answer to two decimal places.

(a)



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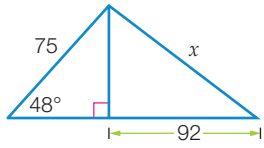


Worksheet R6.4

4 Jake is flying a kite on the end of a 20 m rope. If the rope has an angle of elevation from Jake's line of sight of 69°, how high is the kite above the ground, given the distance from the ground to Jake's eye-level is 1.65 m? Round your answer to two decimal places.

Worksheet R6.5

5 A hiker leaves her campsite and walks 10 km on a bearing of 143°T, then walks 6 km on a bearing of 039°T. Calculate the following and round your answers to two decimal places.

(a) How far is she from her campsite?

(b) What bearing must she take to walk directly back to the camp?

Key Words

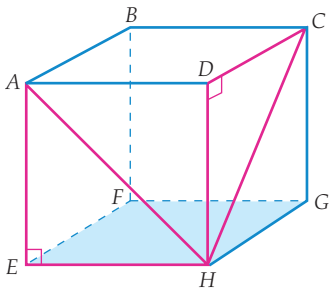
amplitude	coterminal angles	quadrant
asymptote	equivalent angle	sine rule
cosine rule	period	unit circle

Solving three-dimensional problems

6.1

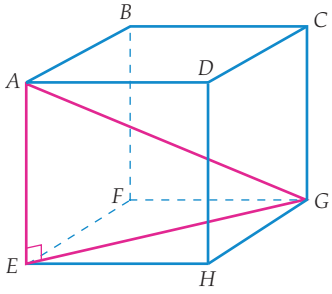
A three-dimensional problem involving right-angled triangles may require the use of any combination of the sine, cosine and tangent ratios, and Pythagoras' theorem. A diagram drawn to appear three-dimensional (3D) has right-angled triangles drawn as two-dimensional (2D) figures. Note that right angles may not appear as exactly 90° on your page in a 3D representation.

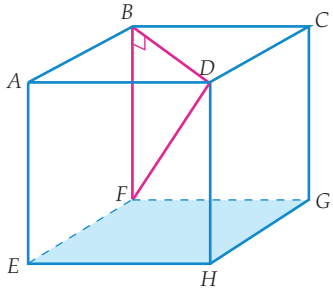
Consider this cube:



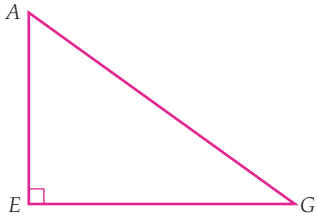
We can find right-angled triangles such as ΔAEH and ΔCDH quite easily.

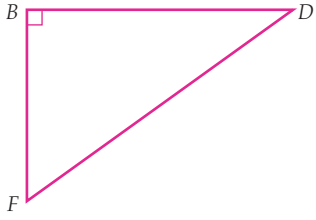
Within the cube we can locate right-angled triangles such as ΔAEG and ΔDBF where the hypotenuse of the triangle is the diagonal connecting opposite vertices of the cube. (Can you find others?)





Always redraw the right-angled triangle you wish to solve as a separate two-dimensional diagram.





You may need to solve more than one right-angled triangle in the three-dimensional diagram to obtain the unknown you require.

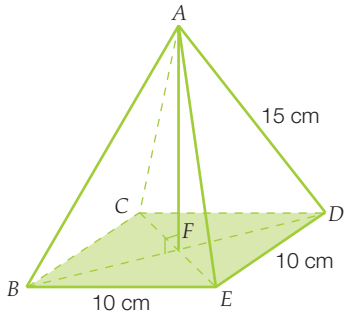
Worked Example 1

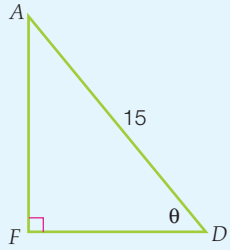
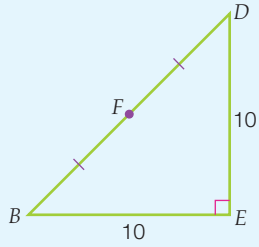
WE1

A right pyramid *ABCDE* stands on a square base of edge length 10 cm. The sloping edges are each of length 15 cm. *AF* is the perpendicular height.

(a) Find the angle that a sloping edge makes with the base of the pyramid, rounded to the nearest degree.

(b) Hence, find the height of the pyramid, rounded to two decimal places.



Thinking	Working
(a) **1** Let θ be the angle required. Choose a triangle to work with. We could use any of the four possible triangles. (Here, we will use Δ*ADF* to find θ.) Redraw this triangle, labelling known and unknown values.	(a) 
2 Identify which values need to be found first. (Here, we need to find *DF* before we can find θ.) Redraw another triangle that includes the unknown value that we need to find. (Here, we will use Δ*BDE* to find *BD* as *DF* is half of *BD*.)	
3 Use Pythagoras' theorem to find the unknown side.	$(BD)^2 = 10^2 + 10^2$ $(BD)^2 = 200$ $BD = \sqrt{200}$
4 Use the side length found in the second triangle to find the unknown side length in the first. (Here, *DF* is half of *BD*.) Do not evaluate your answer. Store the unrounded answer in your calculator.	$DF = \frac{\sqrt{200}}{2}$
5 Using the first triangle (Δ*ADF*), write down the known and unknown values then identify which trigonometric ratio connects them. (Here, we use the two side lengths of Δ*ADF* to find θ.)	In Δ*ADF*: $H = 15$ $A = \frac{\sqrt{200}}{2}$ $\theta = ?$ $\cos\theta = \frac{A}{H}$

6	Form an equation by substituting values into the ratio.	$\cos\theta = \frac{DF}{15}$
7	Use the inverse of the ratio to find θ and substitute the stored unrounded value of the unknown length into the formula. (Here, we substitute for *DF*.)	$\theta = \cos^{-1}\left(\frac{DF}{15}\right)$ $\theta = \cos^{-1}\left(\frac{\sqrt{200}}{30}\right)$
8	Perform the calculation and store the value for θ in your calculator's memory.	$\theta \approx 61.874\,494\,3$
9	Round the answer to the nearest degree.	$\theta = 62°$
10	State the answer.	A sloping edge makes an angle of about 62° with the base of the pyramid.

(b)	1	Using your first triangle, form an equation for the height. (Here, we need *AF*.)	(b) $\sin\theta = \frac{O}{H}$ $\sin\theta = \frac{AF}{15}$
	2	Make the height the subject of the equation and substitute the stored value for θ.	$AF = 15\sin\theta$
	3	Perform the calculation.	$AF = 15\sin 61.874\,494\,3°$ $AF = 13.228\,756\,56$
	4	Round the answer to two decimal places.	$AF = 13.23$ (2 d.p.)
	5	State the answer.	The height of the pyramid is about 13.23 cm.

To solve a three-dimensional problem involving right-angled triangles:

1. Draw a diagram that represents the problem.
2. Identify right-angled triangles that contain the unknown you want or that will help you find it.
3. Redraw these right-angled triangles.
4. Solve these triangles using trigonometric ratios and/or Pythagoras' theorem.

6.1 Solving three-dimensional problems

Navigator

Answers page 742

Q1, Q2, Q3, Q5, Q9, Q12, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q9, Q10, Q11, Q12, Q15, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

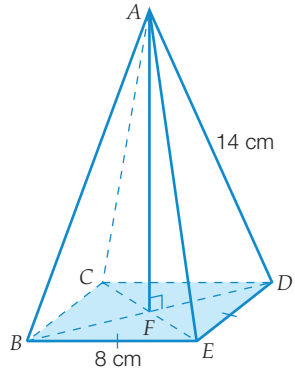
Fluency

WE1

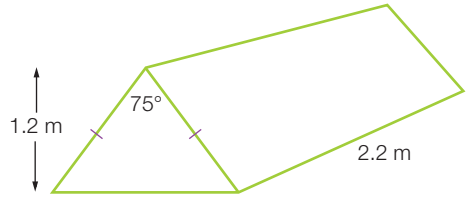
1 A right pyramid *ABCDE* stands on a square base of edge length 8 cm. The sloping edges are each 14 cm long. *AF* is the perpendicular height.

(a) Find the angle that a sloping edge makes with the base of the pyramid, rounded to the nearest degree.

(b) Hence, find the height of the pyramid to two decimal places.



2 A tent has the dimensions shown in the diagram. Find the area of the groundsheet needed to fit inside the tent. Round your answer to two decimal places.

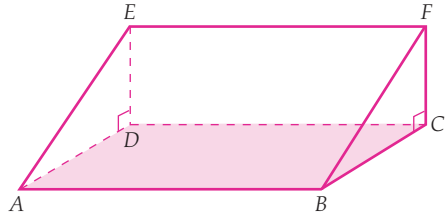


Understanding

3 **(a)** To find the angle that plane *ABFE* makes with plane *ABCD*, we could calculate:

A $\angle FCB$ **B** $\angle BFC$

C $\angle DAB$ **D** $\angle DAE$



(b) To find the length *AF*, we could use the triangle:

A *ABC* **B** *AEF* **C** *BCF* **D** *ABD*

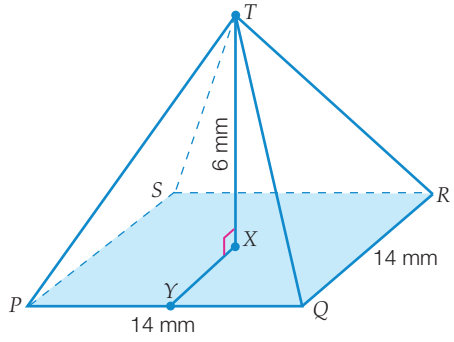
For Questions **4** to **11**, unless stated otherwise, round angles to one decimal place and round other answers to two decimal places.

4 *PQRST* is a square-based pyramid with a base of side length 14 mm and a height of 6 mm. The point *X* on the base is directly below *T* at the apex, and point *Y* is midway along *PQ*.

(a) Find the angle *TXY*.

(b) What angle does the sloping face *PQT* make with the base?

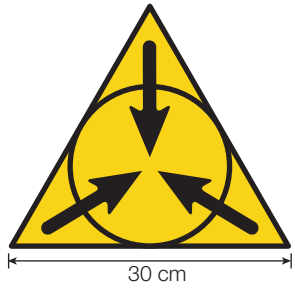
(c) Find the length of *TY*.



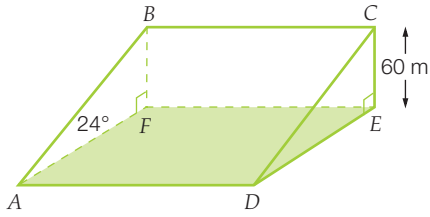
(d) Find the length of a sloping edge of the pyramid.

(e) Find the angle that a sloping edge makes with the base of the pyramid.

5 A new road sign warning of merging traffic consists of a circle inside an equilateral triangle. What is the radius of the circle?



6 A hill *ABCD* rises at an angle of 24° to the horizontal. The hill is 60 m high.



(a) If you walked straight up the hill along *AB*, what distance would you travel?

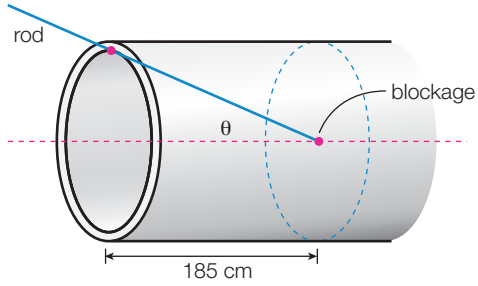
(b) What horizontal distance would you cover walking along *AB*?

(c) Your friend walks along a track from *A* to *C*, taking 30 minutes to reach the top. Horizontally, the distance she covers is 510 m. How fast does she walk in m/min?

(d) Find the angle of the track *AC* to the horizontal.

(e) Gradient represents the steepness of a slope and can be calculated by using the formula $\frac{\text{rise}}{\text{run}}$ or $\frac{\text{vertical distance}}{\text{horizontal distance}}$. Calculate the gradients of *AB* and *AC*, and hence determine which track is steeper.

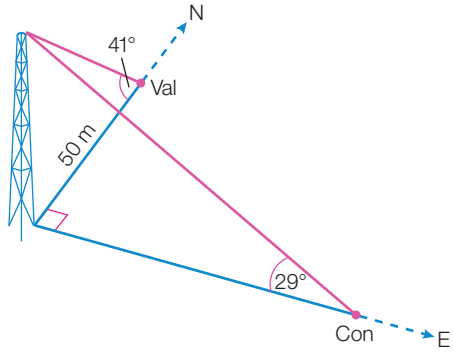
7 A plumber inserts a metal rod into a cylindrical pipe whose inner radius is 9 cm and discovers that there is a blockage in the centre of the inner circle of the pipe at the point shown.



(a) What is the maximum length of the rod that can be inserted inside the pipe to reach the blockage?

(b) What is the maximum angle θ possible? Give your answer in degrees, minutes and seconds.

8 Val is 50 m due north of a communications tower. The top of the tower is at an angle of elevation of 41° from the ground at Val's observation point. From Con's observation point, due east of the tower, the angle of elevation of the top of the tower is 29°.



Find:

(a) the height of the tower

(b) the distance from Con to the foot of the tower

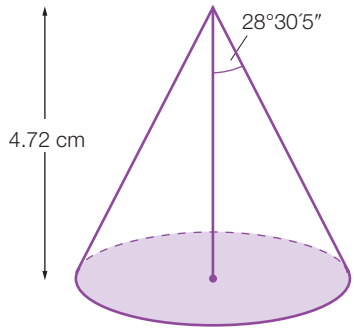
(c) the distance between Val and Con

(d) Con's true bearing from Val's observation point.

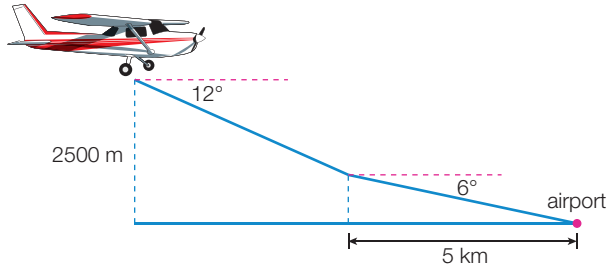
9 The height of a vertical cone is 4.72 cm, and its semi-vertical angle is 28°30′5″. Find:

(a) the slant height of the cone

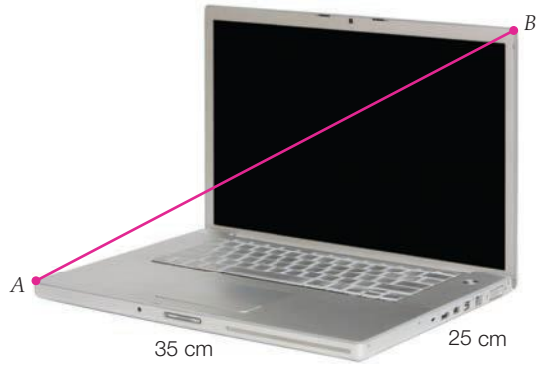
(b) the diameter of the cone.



10 A light plane at a height of 2500 m begins to descend at an angle of 12° to the horizontal. When the plane is 5 km from the airport, the descent angle is reduced to 6°. What horizontal distance from the airport is the plane when it is 2500 m above the ground? Round your answer to the nearest metre.



11 A laptop sends information via infrared from a transmitter at point *A* to a receiver at point *B*. When the screen is at right angles to the keyboard, the distance from the receiver to the transmitter must be no more than 50 cm. Is this true for the laptop shown?



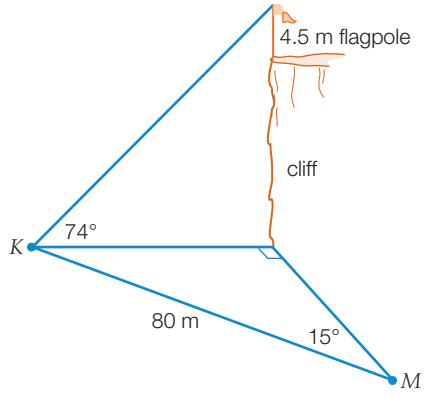
Reasoning

12 A passenger on a cruise ship is positioned 42 m above the water level and is admiring a tropical island in the distance. One end of the island is at a true bearing of 028° and has an angle of depression of 11°, while the other end is at a true bearing of 118° and has an angle of depression of 19°. How long is the island to the nearest metre?

13 A ship's navigator needs to find the height of a cliff. She takes measurements from two different positions in the water, as shown in the diagram. There is a flagpole on top of the cliff.

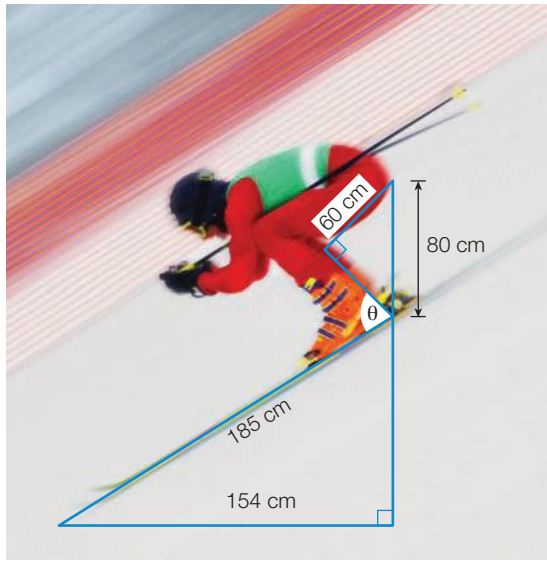
(a) If the flagpole is known to be 4.5 m tall, find the height of the cliff.

(b) What is the distance from the top of the flagpole to the navigator at position *K*?

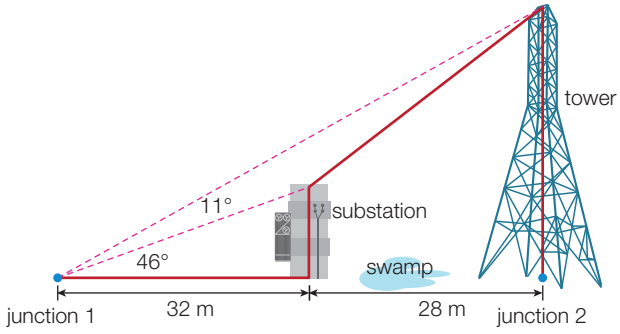


14 Liam wants to construct a rectangular desktop with a diagonal length of 70 cm. When the desktop is tilted at 15° to the horizontal, it is raised 8 cm. Find the area of the desktop, to the nearest square centimetre.

15 To minimise a skier's air resistance, the angle θ shown must be less than 77°. Decide if this is true for the skier shown.



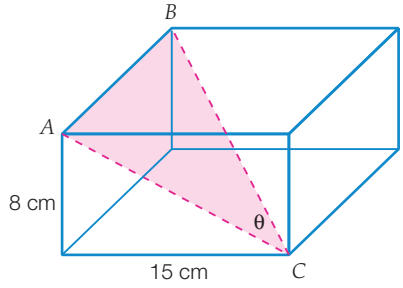
16 Electric cable (shown in red) will be laid underground from junction 1 to the base of the substation. It will continue vertically up the wall of the substation. From the top of the substation it will continue over the swamp to the top of the tower and down to the ground to junction 2. What is the total length of cable required? Round your answer to the nearest metre.



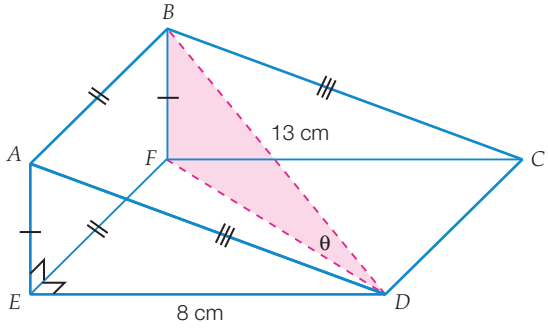
Open-ended

17 A jewellery box in the shape of a rectangular prism has the dimensions shown. Using three different values for θ obtain the area of triangle *ABC* and calculate the ratio $\frac{\text{area of triangle } ABC}{\text{volume of box}}$.

What can you conclude about your results?



18 A wedge of cheese in the shape of a triangular prism has the dimensions shown. By choosing two values for the angle θ, calculate the following, correct to two decimal places.



(a) Find the height, *BF*, of the wedge.

(b) Find the lengths *DF*, *EF* and *BC*.

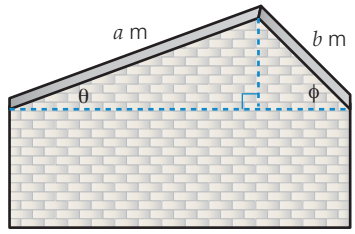
(c) Calculate the area of rectangle *ABCD*.

(d) What effect did changing the value of θ have on the area of *ABCD*?

(e) Calculate the total area of plastic needed to wrap the cheese (no overlap).

(f) For which value of θ was the least plastic used? Compare this to your answer in **(d)**.

19 The cross-section of a roof is shown. a and b are the lengths of the roofing, and θ and ϕ are the angles each roofline makes with the horizontal.



(a) Show that $\frac{a}{b} = \frac{\sin\phi}{\sin\theta}$.

(b) If $a = 2b$ and angles θ and ϕ must be greater than 20° and less than 60°, state one pair of possible values for the angles θ and ϕ. State your answer to the nearest degree.

Outside the Square — Problem solving

The maths challenge

You are in one of two 'tribes' that are competing on a desert island. Both tribes are led to the edge of a deep, fast flowing river, where the weekly challenge is issued. Each tribe is given a large protractor, a 10 m long measuring tape, and some instructions, written on a piece of bark. You crowd around the bark with your fellow tribe members. It reads:

Calculate the height of the tree directly across the river from where you are standing. Use only the equipment you have been given, plus any stick, stone, leaf or vine from the jungle around you. The first team to accurately determine the height of the tree wins double food rations for the week.

P.S. Two triangles could be useful.

'That's impossible!' moan half the tribe members. 'We don't even know the width of the river!'

Describe the method and calculations your tribe could use to calculate the height of the tree and win the challenge.

Strategy options

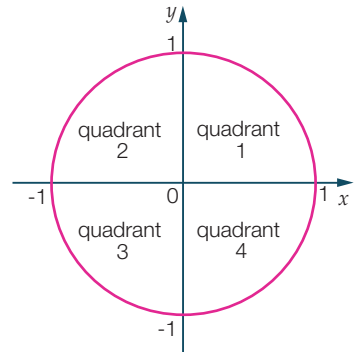
- Make a model
- Act it out.
- Break problem into manageable parts.

The unit circle

6.2

A **unit circle** has a radius of 1 and a centre at the origin (0, 0) of the Cartesian plane. We divide the unit circle into four **quadrants** as shown on the diagram.

The quadrants are numbered anticlockwise starting from quadrant 1, where both x- and y-coordinates are positive.



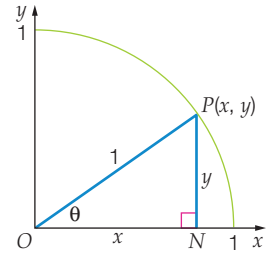
So far, we have used trigonometric ratios for quadrant 1 angles to find angles or sides of right-angled triangles. We will use our knowledge of trigonometric ratios and the unit circle to find and use trigonometric ratios for angles of any size.

In the diagram below, the right-angled triangle *NOP* has been drawn inside the unit circle with its radius as the hypotenuse. From triangle *NOP*, we can define two trigonometric ratios:

- sine θ, abbreviated to sin θ
- cosine θ, abbreviated to cos θ.

$$\sin\theta = \frac{O}{H} = \frac{NP}{OP} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{ON}{OP} = \frac{x}{1} = x$$



In the first quadrant, the angle θ varies from 0° to 90°.

As the angle changes so do the values of sin θ and cos θ.

When θ is 0°, the point *P* will be on the horizontal axis, so the length of *PN* will be 0 and the length of *ON* will be 1. So, sin 0° = 0 and cos 0° = 1.

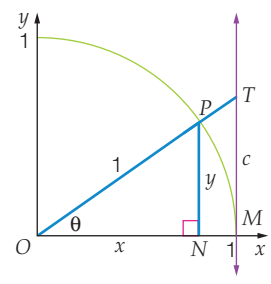
When θ is 90°, the point *P* will be on the vertical axis, so the length of *PN* will be 1 and the length of *ON* will be 0. So, sin 90° = 1 and cos 90° = 0.

We need to look at a slightly different quadrant 1 triangle for our definition of tangent θ, which is abbreviated to tan θ. In the diagram shown here, we have extended *OP* until it touches the line that is a tangent to the circle at *M* and parallel to the y-axis at the point *T*. The length of *TM* is marked here as *c*.

From triangle *MOT*, we can define the trigonometric ratio:

- tangent θ, abbreviated to tan θ.

$$\tan\theta = \frac{O}{A} = \frac{TM}{OM} = \frac{c}{1} = c = \text{length of } TM$$



Unlike sine and cosine ratios, tangent ratios do not have an upper limit.

When $\theta = 0°$, $\tan\theta = 0$, but we can see that as θ increases to 90°, the length of TM increases also. When $\theta = 90°$, TM will be parallel to the vertical axis and be of an unmeasurable, infinite length.

Therefore, tan 90° is undefined.

Alternatively,

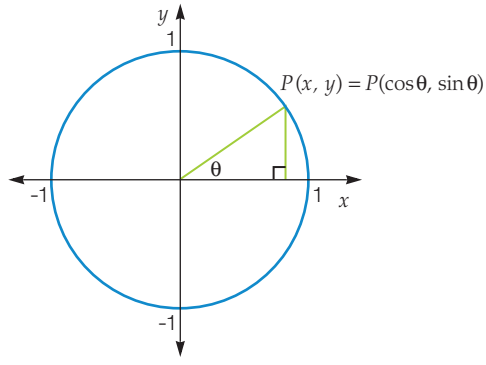
$$\text{From triangle } NOT\text{: } \tan\theta = \frac{O}{A}$$
$$= \frac{NP}{ON}$$
$$= \frac{y}{x}$$
$$\tan 90° = \frac{1}{0}$$

Therefore, tan 90° is undefined.

In the first quadrant, all trigonometric ratios are positive.

The angle θ determines the position of a point P on the circumference of the unit circle. $P(x, y)$ can now be written as $P(\cos\theta, \sin\theta)$, where:

- $\cos\theta$ is the x-coordinate of P
- $\sin\theta$ is the y-coordinate of P
- $\tan\theta = \frac{y}{x}$
  $= \frac{\sin\theta}{\cos\theta}$



In quadrant 1, as θ increases from 0° to 90°:

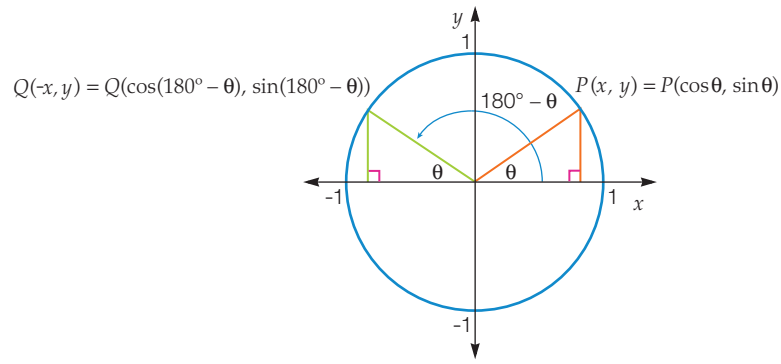
- $\sin\theta$ increases from 0 to 1
- $\cos\theta$ decreases from 1 to 0
- $\tan\theta$ increases from 0 to an undefined positive ratio.

Equivalent angles

The unit circle can also be used to find trigonometric ratios for angles in the other three quadrants using **equivalent angles**. By drawing a congruent right-angled triangle in each quadrant we can determine trigonometric ratios for angles that are not acute.

Angles in the second quadrant

In the second quadrant, a right-angled triangle with an angle of θ measured clockwise from the negative direction of the x-axis will produce an obtuse angle of $(180 - \theta)$ measured anticlockwise from the positive direction of the x-axis. This triangle is a reflection in the y-axis of the original triangle in the first quadrant.



We can use this triangle to determine the trigonometric ratios in quadrant 2.

$$\sin(180° - \theta) = \frac{O}{H} = \frac{y}{1} = y = \sin\theta$$

$$\cos(180° - \theta) = \frac{A}{H} = \frac{-x}{1} = -x = -\cos\theta$$

$$\tan(180° - \theta) = \frac{O}{A} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta$$

In the second quadrant, only the sin ratio is positive.

$\sin(180° - \theta) = \sin\theta$ $\quad$ $\cos(180° - \theta) = -\cos\theta$ $\quad$ $\tan(180° - \theta) = -\tan\theta$

In the second quadrant, the angle θ increases from 90° to 180°. As the angle changes, so do the values of sin θ, cos θ and tan θ.

We have already seen that sin 90° = 1, cos 90° = 0 and tan 90° is undefined.

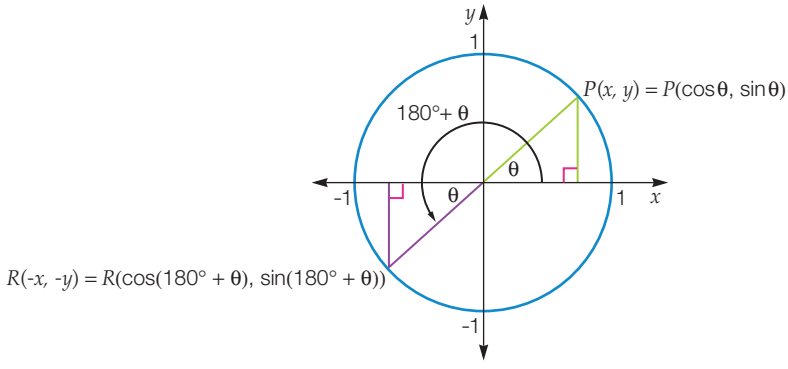
When θ is 180°, the point Q will be on the horizontal axis, so the length of QN will be 0 and the length of ON will be -1. So, sin 180° = 0, cos 180° = -1 and tan 180° = 0

In quadrant 2, as θ increases from 90° to 180°:

- sin θ decreases from 1 to 0
- cos θ decreases from 0 to -1
- tan θ decreases from an undefined positive ratio to 0.

Angles in the third quadrant

In the third quadrant, a right-angled triangle with an angle of θ measured anticlockwise from the negative direction of the x-axis will produce a reflex angle of (180° + θ) measured anticlockwise from the positive direction of the x-axis. This triangle is a 180° rotation of the original triangle in the first quadrant.



We can use this triangle to determine the trigonometric ratios in quadrant 3.

$$\sin(180° + \theta) = \frac{O}{H} = \frac{-y}{1} = -y = -\sin\theta$$

$$\cos(180° + \theta) = \frac{A}{H} = \frac{-x}{1} = -x = -\cos\theta$$

$$\tan(180° + \theta) = \frac{O}{A} = \frac{-y}{-x} = \frac{y}{x} = \tan\theta$$

In the third quadrant, only the tan ratio is positive.

$\tan(180° + \theta) = \tan\theta$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$

In the third quadrant the angle θ varies from 180° to 270°.

As the angle changes so do the values of sin θ, cos θ and tan θ.

We have already seen that sin 180° = 0, cos 180° = -1 and tan 180° is 0.

When θ is 270°, the point *R* will be on the vertical axis, so the length of *RN* will be -1 and the length of *ON* will be 0.

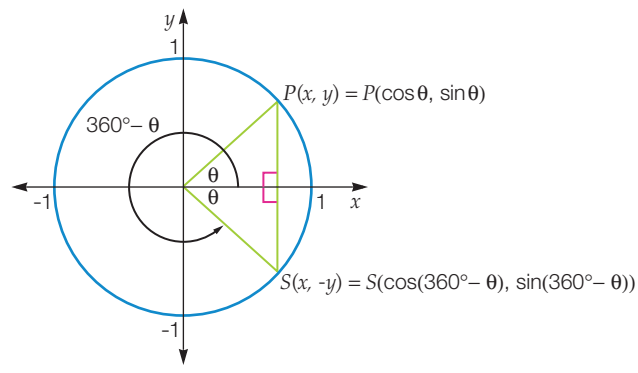
∴ sin 270° = -1, cos 270° = 0 and tan 270° is undefined.

In quadrant 3, as θ increases from 180° to 270°:

- sin θ decreases from 0 to -1
- cos θ increases from -1 to 0
- tan θ decreases from 0 to an undefined negative ratio.

Angles in the fourth quadrant

In the fourth quadrant, a right-angled triangle with an angle of θ measured clockwise from the positive direction of the *x*-axis will produce a reflex angle of (360° – θ) measured anticlockwise from the positive direction of the *x*-axis. This triangle is a reflection in the *x*-axis of the original triangle in the first quadrant.



We can use this triangle to determine the trigonometric ratios in quadrant 4.

$$\sin(360° - \theta) = \frac{O}{H} = \frac{-y}{1} = -y = -\sin\theta$$

$$\cos(360° - \theta) = \frac{A}{H} = \frac{x}{1} = x = \cos\theta$$

$$\tan(360° - \theta) = \frac{O}{A} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta$$

In the fourth quadrant, only the cos ratio is positive.

$\cos(360° - \theta) = \cos\theta$ $\sin(360° - \theta) = -\sin\theta$ $\tan(360° - \theta) = -\tan\theta$

In the fourth quadrant the angle θ varies from 270° to 360°.

As the angle changes so do the values of sin θ, cos θ and tan θ.

We have already seen that sin 270° = -1, cos 270° = 0 and tan 270° is undefined.

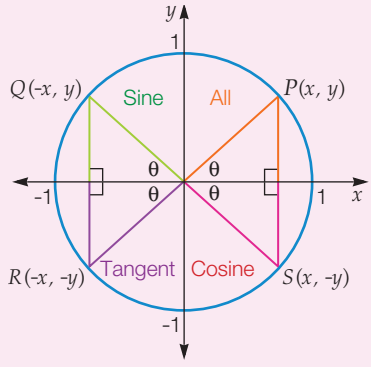
When θ is 360°, the point *P* will be back to its original position so sin 360° = 0, cos 360° = 1 and tan 360° = 0

In quadrant 4, as θ increases from 270° to 360°:

- sin θ increases from -1 to 0
- cos θ increases from 0 to 1
- tan θ increases from an undefined negative ratio to 0.

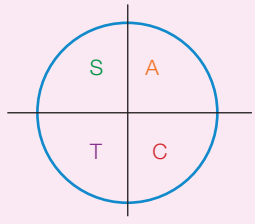
Summary for trigonometrical ratios for angles in other quadrants

	Angle			
	0°	90°	180°	270°
sin θ	0	1	0	-1
cos θ	1	0	-1	0
tan θ	0	undefined	0	undefined



- In quadrant 1, all the ratios are positive as both *x* and *y* are positive. (ALL, A)
- In quadrant 2, only the sine ratio is positive. (SINE, S)
- In quadrant 3, only the tangent ratio is positive as both *x* and *y* are negative. (TANGENT, T)
- In quadrant 4, only the cosine ratio is positive. (COSINE, C)

This can be remembered by using the diagram shown often called a CAST diagram. You can make up your own mnemonic (a word or phrase that you can use to help you remember).



Worked Example 2

WE2

State the quadrant in which each of the following angles is found.

(a) 35° **(b)** 265° **(c)** 135° **(d)** 342°

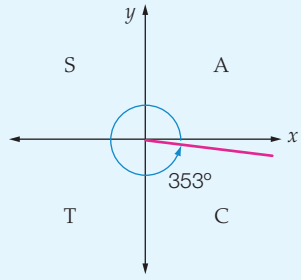
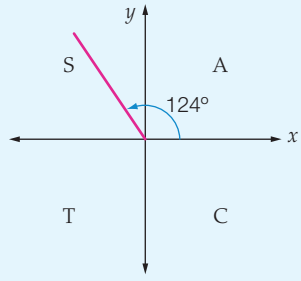
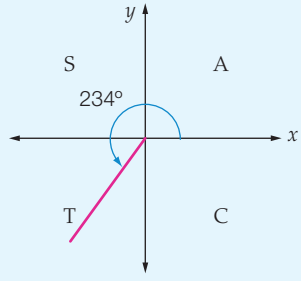
Thinking	Working
(a) 35° is between 0° and 90°.	(a) quadrant 1
(b) 265° is between 180° and 270°.	(b) quadrant 3
(c) 135° is between 90° and 180°.	(c) quadrant 2
(d) 342° is between 270° and 360°.	(d) quadrant 4

Worked Example 3

WE3

For each of the following relationships, find the equivalent trigonometric ratio in quadrant 1, taking care to include the appropriate sign.

(a) tan 353° **(b)** sin 124° **(c)** cos 234°

Thinking	Working
(a) 1 Draw a CAST diagram and locate the given angle. (Here, the angle is in quadrant 4.)	(a) 
2 Find the equivalent angle in quadrant 1.	$(360 - 353)° = 7°$
3 Write the answer, taking care to use the appropriate sign for the quadrant. (tan is negative in quadrant 4)	$\tan 353° = -\tan 7°$
(b) 1 Draw a CAST diagram and locate the given angle. (Here, the angle is in quadrant 2.)	(b) 
2 Find the equivalent angle in quadrant 1.	$(180 - 124)° = 56°$
3 Write the answer, taking care to use the appropriate sign for the quadrant. (sin is positive in quadrant 2)	$\sin 124° = \sin 56°$
(c) 1 Draw a CAST diagram and locate the given angle. (Here, the angle is in quadrant 3.)	(c) 
2 Find the equivalent angle in quadrant 1.	$(234 - 180)° = 54°$
3 Write the answer, taking care to use the appropriate sign for the quadrant. (cos is negative in quadrant 3)	$\cos 234° = -\cos 54°$

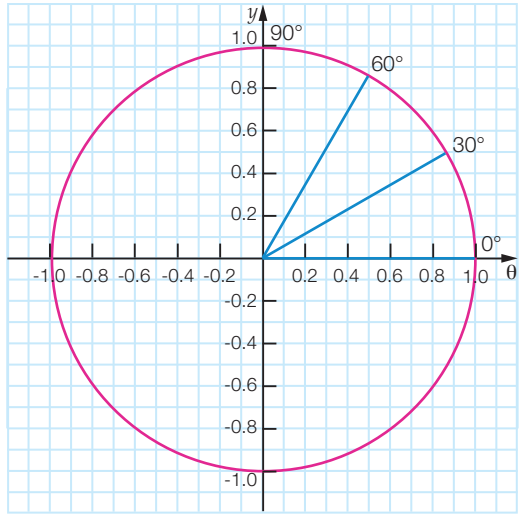
Trigonometric graphs

We can use the unit circle to sketch trigonometric graphs. This involves using symmetry properties and using coordinate values of the unit circle.

In a unit circle, the radius is 1 unit in length and equal to the hypotenuse of a right-angled triangle of horizontal length x and vertical height y. Therefore, $\sin\theta$ is the vertical height of a point on the circumference of the circle given by the y-coordinate, and $\cos\theta$ is the horizontal length of the point given by the x-coordinate.

Worked Example 4 WE4

Use the unit circle to sketch the graph of the equation $y = \cos\theta$ for values of θ ranging from 0° to 360°.



Thinking | **Working**

1 As $\cos\theta = x$, find the x-coordinates of the points on the unit circle for the angles 0°, 30°, 60° and 90° in the unit circle and use these values to complete a table of values for $y = \cos\theta$ for $0° \le \theta \le 90°$.

θ	0°	30°	60°	90°
cos θ	1	0.87	0.5	0

2 Reflect the angles 0°, 30°, 60° and 90° in the y-axis of the unit circle to find angles in the second quadrant. Find the x-coordinates of the points on the unit circle for these angles and use them to complete a table of values for $y = \cos\theta$ for $90° \le x \le 180°$.

θ	120°	150°	180°
cos θ	-0.5	-0.87	-1

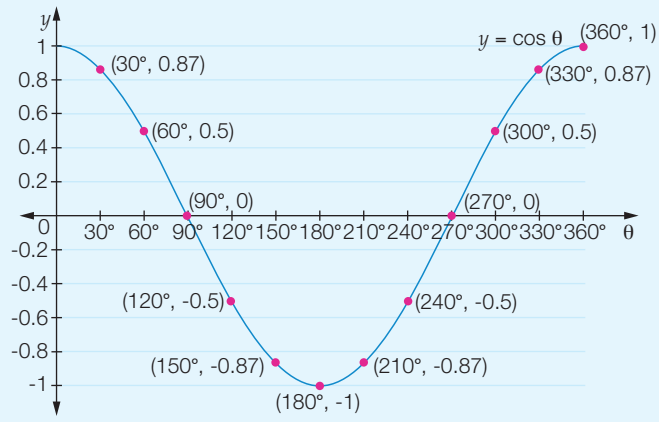
3 Reflect the angles found in step **2** in the x-axis of the unit circle to find angles in the third quadrant. Find the x-coordinates of the points on the unit circle for these angles and use them to complete a table of values for $y = \cos\theta$ for $180° \le x \le 270°$.

θ	210°	240°	270°
cos θ	-0.87	-0.5	0

4 Reflect the angles found in step **3** in the y-axis of the unit circle to find angles in the fourth quadrant. Find the x-coordinates of the points on the unit circle for these angles and use them to complete a table of values for $y = \cos\theta$ for $270° < \theta \le 360°$.

θ	300°	330°	360°
cos θ	0.5	0.87	1

5 Plot the points from the table of values found in Steps **1**, **2**, **3** and **4**, and join the points with a smooth curve.



Angles of any magnitude

Up to now, only angles from 0° to 360° have been used. Notice that $y = 1$ when $\theta = 0°$ and $\theta = 360°$ for $y = \cos\theta$. This is because both angles represent the same position on the unit circle. In fact, if θ is any given angle, every multiple of 360° added to θ or subtracted from θ represents the same point on the unit circle.

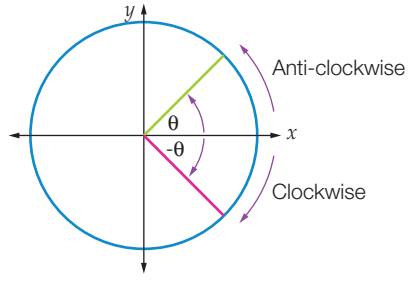
For example, θ values of 25°, 25° + 360°, 25° + 720°, 25° – 360° and 25° – 720°, all occupy the same position on the unit circle and have the same trigonometric ratio values.

$\sin 385° = \sin 25°$

$\cos 745° = \cos 25°$

$\tan(-335°) = \tan 25°$

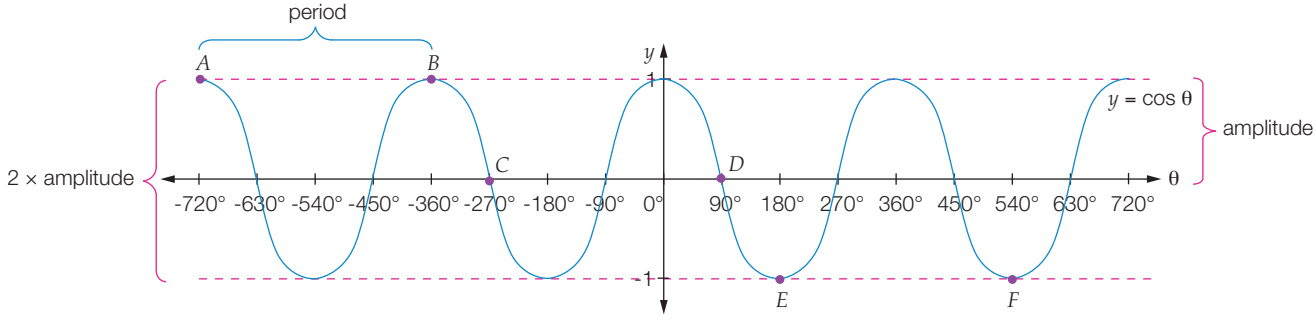
Note that, by convention, positive angles are measured anti-clockwise from the positive direction of the x-axis, whereas negative angles are measured clockwise from the positive direction of the x-axis. (-70° is 70° measured clockwise from the positive direction of the x-axis.)



The shape of trigonometric graphs keeps repeating in cycles infinitely in both directions, so they are called cyclic graphs.

Period

The **period** of a trigonometric graph is the horizontal distance it takes to complete one full cycle of the graph. For example, when the graph of $y = \cos\theta$ from Worked Example 4 is extended by using values of θ from -720° to 720°, its graph looks like the one shown below. The period of the graph is the horizontal distance, between point A and point B or between point C and point D or between point E and point F. For $y = \cos\theta$, the period is 360°.



Amplitude

The **amplitude** of a trigonometric graph is half the distance between the minimum y-value and the maximum y-value. From the graph of $y = \cos\theta$ it can be seen that the minimum value is -1 and the maximum value is 1. The distance between them is 2 units, so the amplitude is half of this distance.

Therefore, the amplitude $= \frac{1}{2} \times 2$

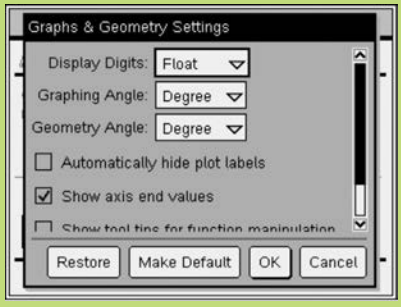
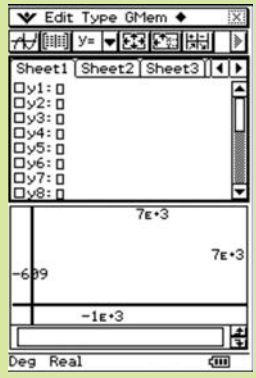
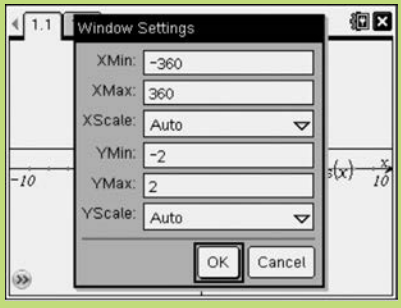
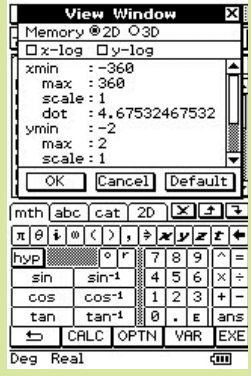
$= 1$

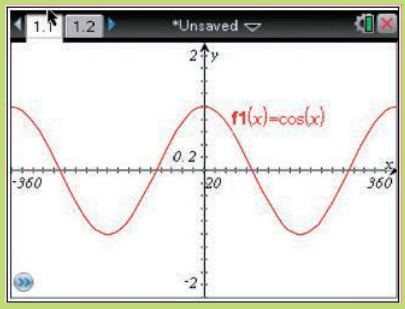
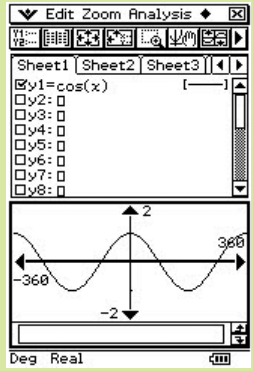
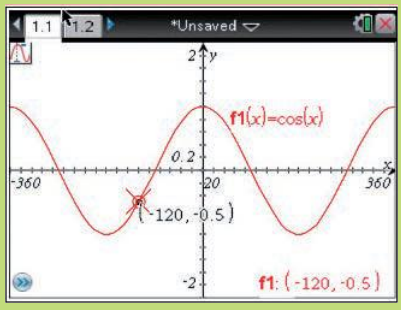
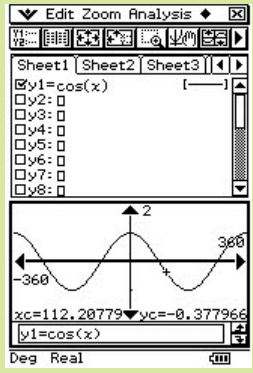
The graphs of $\sin\theta$ and $\cos\theta$ always cycle between a minimum value of -1 and a maximum value of 1.

You cannot find the amplitude of the tangent graph because it has no bounds on its minimum or maximum value.

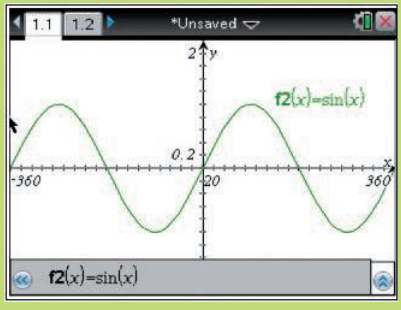
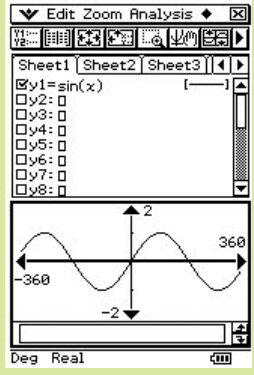
Using technology to sketch trigonometric graphs

Drawing trigonometric graphs by hand and reading values off a graph is inaccurate and time consuming. Here we will use a CAS to draw the trigonometric graph $y = \cos\theta$ and find values from the graph. A CAS will use x to represent θ.

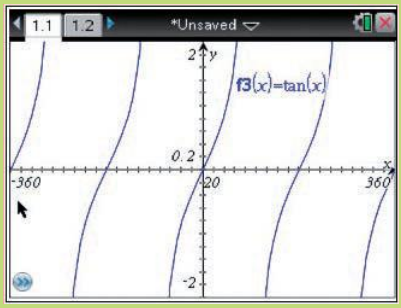
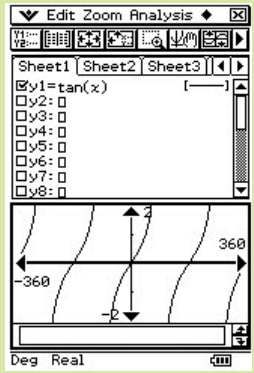
Using the TI-Nspire CAS	Using the ClassPad
Press (on) > (or (home)) > Settings (or Settings and Status) > Settings > Graphs & Geometry > (enter) (or (enter)). Use the central control panel to select the settings shown below.  	Tap (Graph&Tab...) from the Menu and make sure you are in degree mode. If you can see Rad or Gra at the bottom left of the screen tap it to change it to Deg. 
Press (tab) down to OK and press (enter) (or (enter)). At a new Graphs screen press (trig) and then scroll across and press cos (or (cos⁻¹/cos)). You should notice that, at the bottom of the screen, you now have $f1(x)$=cos(. Complete the expression by pressing [X] (or (x)) > (()) (or (()). Change the window settings to get a proper view of the graph. Press (menu) > Window/ Zoom > Window Settings > (enter) (or (enter)) and change the values to those shown below.  	Enter cos(x)as y1. Press (Keyboard) > [TRIG] > cos. This will give you y1= cos(. Complete the expression by pressing (x) > (()) on the keypad. Press (Keyboard) again to turn it off. Check the y1 box. Change the window settings to get a proper view of this graph. Tap [ ] and change the values to those shown below. (The dot value does not have to match.) Now tap OK. 

Using the TI-Nspire CAS	Using the ClassPad
Press (tab) to OK. Press (enter) (or (≈enter)) and the graph will appear on the screen.  	Tap [Analysis] and the graph will appear on the screen and the graph will appear on the screen. 
Press (menu) > Trace > Graph Trace > (enter) (or (≈enter)) and a cursor will appear on the graph that you can move, using ◀ and ▶ on the central control panel, to read values off the graph.  	Tap Analysis > Trace and a cursor will appear on the graph that you can move, using the ◀ and ▶ on the central donut, to read values off the graph. 

Now that we have drawn $y = \cos\theta$ as $f1(x) = \cos(x)$, we can draw $y = \sin\theta$ as $f2(x) = \sin(x)$ in the same way. You should notice that $y = \sin\theta$ is exactly the same shape as $y = \cos\theta$ but translated 90° to the right.

Using the TI-Nspire CAS	Using the ClassPad
 	

Now draw $y = \tan\theta$ as $f3(x) = \tan(x)$.

Using the TI-Nspire CAS	Using the ClassPad
	

The graph of tan θ is quite different to those of sin θ and cos θ. This is because it is not bounded by a maximum value of 1 and a minimum value of -1 and it has points where it is undefined. This means that the graph is not continuous at these points (there are breaks in the graph). A tangent graph approaches **asymptotes**, where the graph is undefined. Asymptotes are lines that the graph approaches but never touches or crosses. The equations of the asymptotes are $\theta = -270°$, $\theta = -90°$, $\theta = 90°$ and $\theta = 270°$. Asymptotes occur every 180° and the graph repeats between these asymptotes, so the period of $y = \tan\theta$ is 180°.

	$y = \cos\theta$	$y = \sin\theta$	$y = \tan\theta$
period	360°	360°	180°
amplitude	1	1	undefined
asymptotes	none	none	$(90 + 180n)°$, where $n = 0, 1, 2, \ldots$

6.2 The unit circle

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q11	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12

Answers page 742

Equipment required: a CAS would be useful for Questions **8** and **9**

Fluency

1 State the quadrant in which each of the following angles is found. WE2

(a) 355° **(b)** 222° **(c)** 111° **(d)** 102°

(e) 191° **(f)** 23° **(g)** 263° **(h)** 97°

(i) 286° **(j)** 342° **(k)** 156° **(l)** 260°

2 For each of the following, find the equivalent trigonometric ratio in quadrant 1, taking care to include the appropriate sign. WE3

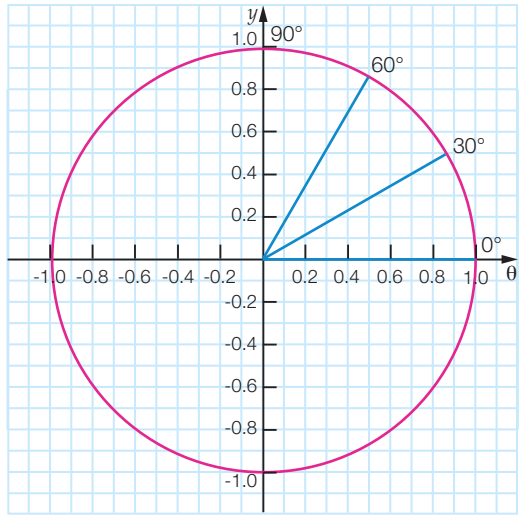
(a) sin 135° **(b)** cos 224° **(c)** tan 111° **(d)** sin 264°

(e) cos 321° **(f)** tan 123° **(g)** sin 263° **(h)** cos 97°

(i) tan 186° **(j)** sin 342° **(k)** cos 166° **(l)** tan 207°

WE4

3 Use the unit circle shown to sketch the graph of the equation $y = \sin\theta$ for values of θ ranging from 0° to 360°



4 (a) Which of the following groups of angles are all found in quadrant 3?

A 213°, 182° and 156°

B 187°, 245° and 194°

C 98°, 124° and 156°

D 342°, 276° and 199°

(b) Find the equivalent trigonometric ratio to sin 25°.

A sin 205°

B sin 345°

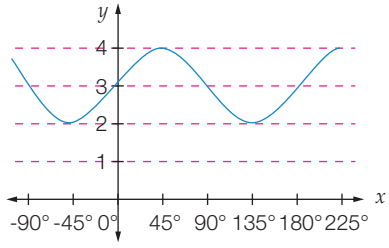
C -sin 155°

D sin 155°

5 The period and the amplitude of the trigonometric relationship shown is:

A 90°, 1 B 90°, 2

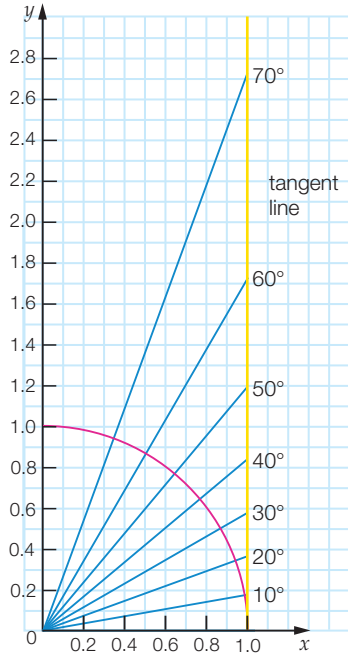
C 180°, 1 D 180°, 2



Understanding

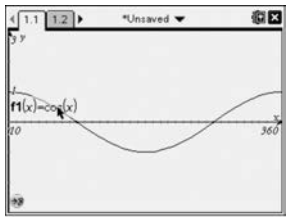
6 (a) Use the unit circle with the tangent line shown in the diagram on the right to sketch the graph of the equation $y = \tan x$ for values of x ranging from 0° to 70° in intervals of 10°.

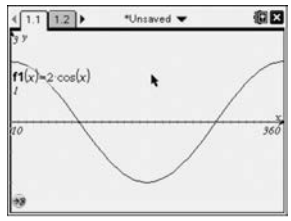
(b) What do you think is happening to the value of $\tan x$ as the value of x gets closer to 90°?

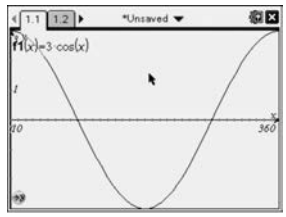


7 Jay knows the values of sine, cosine and tangent when the angles are 30°, 45° and 60°. For which other angles less than 360° can he find trigonometric ratios using what he already knows?

8 Consider the three cosine graphs shown below.

(i) 

(ii) 

(iii) 

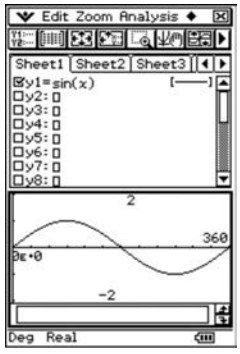
(a) State the period for each graph.

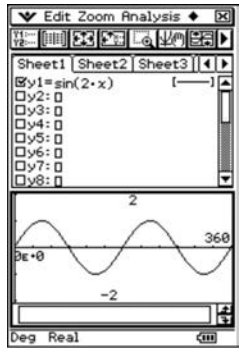
(b) State the amplitude for each graph.

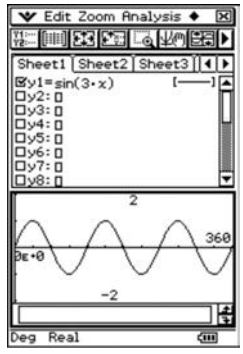
(c) What conclusion can you draw about the effect of the a in $y = a \times \cos x$?

(d) Sketch what you think $y = 4 \cos x$ would look like. Check your answer using a CAS if you can.

9 Consider the three sine graphs shown below.

(i) 

(ii) 

(iii) 

(a) State the period for each graph.

(b) State the amplitude for each graph.

(c) What conclusion can you draw about the effect of the n in $y = \sin(nx)$?

(d) Sketch what you think $y = \sin(6x)$ would look like. Check your answer using a CAS if you can.

Reasoning

10 At a harbour, a siren sounds every time the tide is at 5 m. Since tides are based on sine curves, and the siren first sounds when the angle for the sine ratio is 24°, calculate the angle when the siren will sound next.

Open-ended

11 Write down two angles that are 180° apart. For each of your chosen angles, write down the sign for the ratios: sin, cos and tan. Use a calculator to check that you are correct.

12 **(a)** Explain why $\cos(-\theta) = \cos\theta$ for any angle θ.

(b) Explain why $\sin(-\theta) = -\sin\theta$ for any angle θ.

(c) Using your answers to **(a)**, **(b)** and the relationship $\tan(\theta) = \dfrac{\sin\theta}{\cos\theta}$, explain why $\tan(-\theta) = -\tan\theta$.

6.3 Solving trigonometric equations

Just as we can solve equations such as $3x - 5 = 2$ to get $x = \frac{7}{3}$, so can we solve equations that involve trigonometric ratios.

Using inverse trigonometric ratios to solve equations

We can find a solution to the trigonometric equation $\sin x = 0.5$ by using the inverse sin function on a calculator. This process is the same whether we use a scientific or CAS calculator. By finding $\sin^{-1}(0.5)$, we will get the solution $x = 30°$, which is an angle in quadrant 1. However, we know that $\sin(180° - \theta) = \sin\theta$ so $x = 150°$, a quadrant 2 angle, is also a solution.

In fact, there is an infinite number of solutions, since adding multiples of 360° to each solution produces more solutions. So $x = 30° + 360° = 390°$ and $x = 150° + 360° = 510°$ are another two solutions.

Worked Example 5 WE5

Find all solutions for x that satisfy each of the following equations:

(a) $2\cos x - 1 = 0$ for $0° \le x \le 360°$

(b) $3 - \tan x = 2$ for $0° \le x \le 180°$

Thinking	Working
(a) 1 Rewrite the equation to make the trigonometric ratio the subject of the equation. (Here, $\cos x$ is the subject.)	(a) $2\cos x - 1 = 0$ $2\cos x = 1$ $\cos x = 0.5$
2 Use the appropriate inverse trigonometric function on your calculator to find the solution in quadrant 1. (Here, we use $\cos^{-1}$.)	$x = \cos^{-1}(0.5)$ $x = 60°$
3 Use a CAST diagram to identify other quadrants where the ratio is positive and find an equivalent angle. (Here $\cos x$ is positive in quadrant 4. Subtract the first quadrant angle from 360° to find the equivalent angle.)	$\cos x$ is positive in quadrants 1 and 4. The equivalent angle of 60° in quadrant 4 is $360° - 60° = 300°$.
4 State the solutions.	$2\cos x - 1 = 0$ has solutions $x = 60°, 300°$ for $0° \le x \le 360°$

(b) 1	Rewrite the equation to make the trigonometric ratio the subject of the equation. (Here, tan x is the subject.)	(b) $3 - \tan x = 2$ $-\tan x = -1$ $\tan x = 1$
2	Use the appropriate inverse trigonometric function on your calculator to find the solution in quadrant 1. (Here, we use $\tan^{-1}$.)	$x = \tan^{-1}(1)$ $x = 45°$
3	Use a CAST diagram to identify other quadrants where the ratio is positive and find an equivalent angle. (Here tan x is positive in quadrant 3. Add the first quadrant angle to 180° to find the equivalent angle.)	Tangent is positive in quadrant 1 and 3. The equivalent angle of 45° in quadrant 3 is $180° + 45° = 225°$.
4	Check that the solutions are within the given values for x. Reject any solutions outside these values.	225° is not in $0° \le x \le 180°$ so it is not a solution.
5	State the solutions.	$3 - \tan x = 2$ has the solution $x = 45°$ for $0° \le x \le 180°$.

Writing coterminal angles

From Worked Example 5, the trigonometric equation $\cos x = 0.5$ has solution $x = 60°$ in quadrant 1 and $x = 300°$ in quadrant 4. However, as we have observed, there can be an infinite number of solutions, as adding or subtracting multiples of 360° to an angle produces **coterminal angles** that return us to the same position on the unit circle.

By moving in a anticlockwise direction to add multiples of 360° to 60° in quadrant 1, we have an infinite number of solutions:

$x = 60°, x = 60° + 360°, x = 60° + 2 \times 360°, x = 60° + 3 \times 360°, \ldots$

By moving in a clockwise direction, to subtract multiples of 360° from 60° in quadrant 1, we obtain even more solutions.

$x = 60° - 360°, x = 60° - 2 \times 360°, x = 60° - 3 \times 360°, \ldots$

All the solutions, using angles coterminal with 60°, can be written in a concise way as $x = (60 \pm 360n)°, n = 0, 1, 2, 3, \ldots$

In the same way, by using angles coterminal with $x = 300°$ in quadrant 4, we set another lot of solutions that can be written concisely as $x = (300 \pm 360n)°, n = 0, 1, 2, 3, \ldots$

Worked Example 6 WE6

Determine all solutions to the equation $\sin x = 0.5$.

Thinking	Working
1 Draw a CAST diagram and identify the quadrants where the trigonometric ratio has the correct sign. (Here, it is positive.)	sin x is positive in quadrants 1 and 2.

2 Calculate the solutions for x-values in the range 0° to 360°.	In quadrant 1, sin θ is positive $\sin x = 0.5$ $x = \sin^{-1} 0.5$ $x = 30°$ In quadrant 2, the equivalent angle to 30° is 180° − 30° = 150°. The solutions in the range 0° to 360° are $x = 30°$, $x = 150°$.
3 Find all solutions by adding and subtracting multiples of 360°.	$x = (30 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$ and $x = (150 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$ or $x = ((2n + 1)180 - 30)°$ $n = 0, 1, 2, 3, \ldots$

In the next example, a calculator is used to find an approximate solution.

Worked Example 7

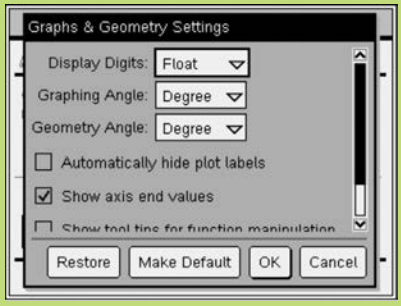
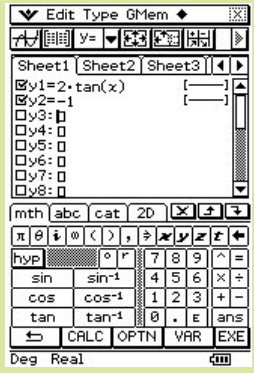
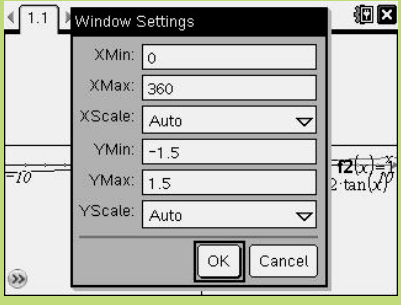
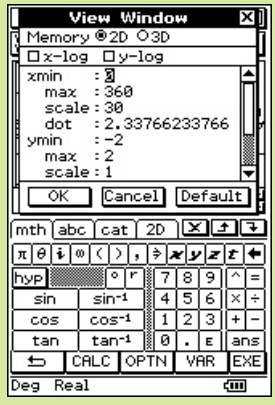
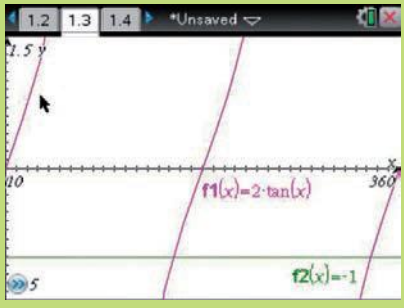
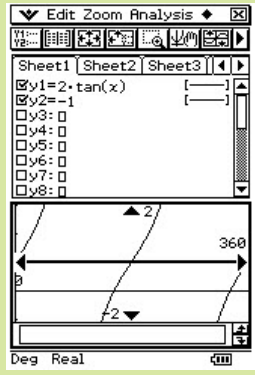
WE7

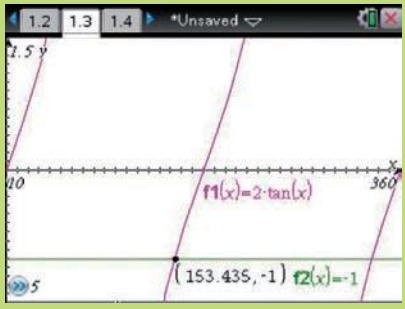
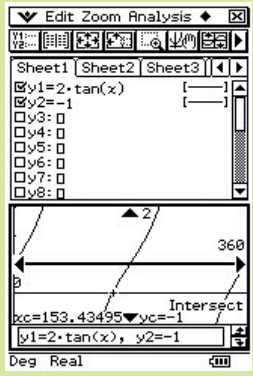
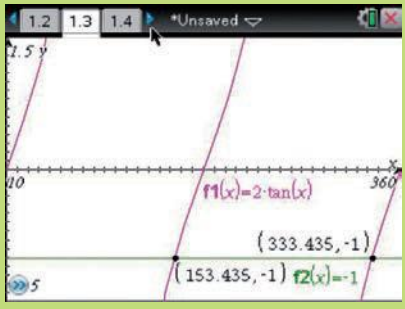
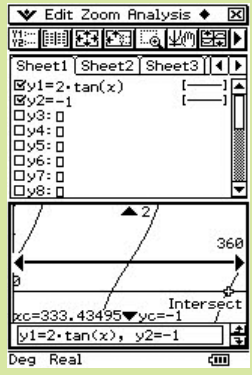
Determine all solutions to the equation $2 \tan x + 1 = 0$. Write your answers correct to two decimal places.

Thinking	Working
1 Rewrite the equation to make the trigonometric ratio the subject of the equation.	$2 \tan x + 1 = 0$ $2 \tan x = -1$ $\tan x = -\frac{1}{2}$
2 Draw a CAST diagram and identify the quadrants where the trigonometric ratio has the correct sign. (Here, it is negative.) Write down the quadrants in which the solutions will be found.	tan x is negative in quadrants 2 and 4.
3 Find the solution for the x-value for the positive ratio in quadrant 1.	In quadrant 1, tan θ is positive $x = \tan^{-1}\left(\frac{1}{2}\right)$ $x = 26.565°$
4 Calculate the solutions for x-values in the range 0° to 360° taking into account the correct quadrants for the sign of the given ratio.	In quadrant 2, the equivalent angle is $(180 - 26.565)° = 153.435°$ $= 153.44°$ (2 d.p.) In quadrant 4, the equivalent angle is $(360 - 26.565)° = 333.435°$ $= 333.44°$ (2 d.p.) The solutions for $0° \le x \le 360°$ are $x = 153.44°$ and $x = 333.44°$
5 Find all solutions by adding and subtracting multiples of 360°.	$x = (153.44 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$ and $x = (333.44 \pm 360n)°$, $n = 0, 1, 2, 3, \ldots$
6 As $y = \tan \theta$ has a period of 180°, write the solution more simply.	$x = (153.44 + 180n)°$, where $n = 0, 1, 2, 3, \ldots$

Using technology to solve trigonometric equations

We can get solutions to these trigonometric equations using our CAS. There are several methods that can be used but we will look at just one of them—a graphical solution. We will use the equation from Worked Example 7 to demonstrate that the process works. The equation is $2\tan(x) + 1 = 0$. We will transpose this to become $2\tan(x) = -1$ and graph the two sides separately.

Using the TI-Nspire CAS	Using the ClassPad
Make sure your CAS is set to degree mode. Press (on) (or (home)) > Settings (or Settings and Status) > Settings > Graphs & Geometry > (enter) (or (enter)). Use the central contol panel to make the following settings.  Press (tab) down to OK and press (enter) (or (enter)). Now select a graph page . from the main screen. At the bottom of the graph screen, enter $2\tan(x)$ as $f1(x)$ and -1 as $f2(x)$.	Tap Graph&Tab from the Menu to get a new graph screen. Make sure your CAS is set to degree mode. If you can see Rad or Gra at the bottom of the screen, tap it to change it to Deg. If there are equations already present, tap Edit > Clear All and tap OK to confirm your choice. Then enter $2\tan(x)$ as y1 and -1 as y2. Check the boxes next to both equations. 
We need to change the window settings to see the graph properly. Press (menu) > Window/Zoom > Window Settings > (enter) (or (enter)) and make sure you have the following values entered.  	Change the window settings to get a proper view of this graph by tapping . Change to the values shown below. 
Press (tab) down to OK and press (enter) (or (enter)). The graph will be displayed on the screen, showing some points of intersection.  	Now tap and the graph will be displayed on the screen, showing some points of intersection. 

Using the TI-Nspire CAS	Using the ClassPad
To find a point of intersection, press (menu) > Analyze Graph > Intersection > (enter) (or (enter)) and move the cursor over to the left of the first point of intersection and press (enter) (or (enter)). Then move to the right of the point of intersection and press (enter) (or (enter)) again. The point of intersection will appear. You may need to press (enter) (or (enter)) again to make it stay on the screen.  You can confirm that this is the same as the second quadrant value we obtained earlier.	To find the points of intersection, tap Analysis > G-Solve > Intersect and the first point of intersection appears on the screen.  You can confirm this as the same as the second quadrant value we obtained earlier.
We then repeat the process to find the fourth quadrant solution.  	To obtain the next solution, tap the ▶ on the central donut. 

6.3 Solving trigonometric equations

Navigator

Answers page 743

Q1, Q2, Q3, Q4, Q7, Q8, Q11	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q12	Q1, Q2, Q3, Q5, Q6, Q9, Q10, Q12

Fluency

WE5

1 Find all solutions for x, that satisfy each of the equations:

(a) $2\cos x + 1 = 0$ for x for $0° \le x \le 360°$

(b) $4 - 2\sin x = 3$ for x for $0° \le x \le 180°$

2 Determine all solutions to the equation $2\cos x + 1 = 0$.

WE6

3 Determine all solutions to the equation $4\sin x - 1 = 0$. Write all answers correct to two decimal places.

WE7

4 The solutions to $4 - 5\sin x = 0$ for x for $0° \le x \le 180°$ are closest to:

A 127°, 127° **B** 53°, 127° **C** 37°, 143° **D** 53°, 307°

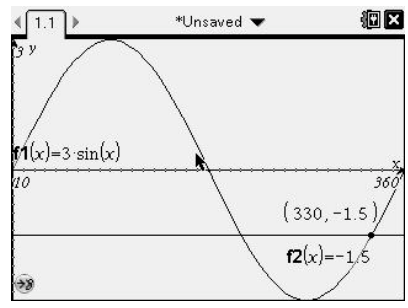
5 Use a calculator to find the solutions for $0° \le x \le 360°$ for each of the following equations. Where necessary, round your answers to two decimal places.

(a) $4\sin x - 3 = 0$ **(b)** $5\tan x + 2 = 0$ **(c)** $3 - 5\cos x = 0$ **(d)** $\frac{3\sin x}{2} - 1 = 0$

Understanding

6 The depth D metres below sea level of a submarine after T minutes is given by the trigonometric equation $D = 4\cos T - 4$ where $T \ge 0$. When will be the next time the submarine surfaces ($D = 0$) if it was at the surface at 3.10 p.m. initially?

7 The graph shown at right could be used to solve which of the following trigonometric equations?



A $3\sin(x) - 1.5 = 0$

B $1.5 - 3\sin(x) = 0$

C $3\sin(x) + 1.5 = 0$

D $3\sin(x) = 1.5$

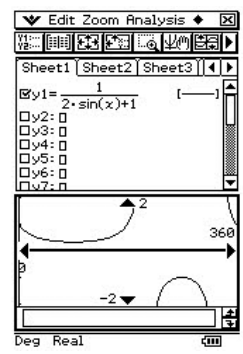
8 Considering the graph shown in Question **7** the solutions shown are in:

A quadrants 1 and 2 **B** quadrants 2 and 3

C quadrants 3 and 4 **D** quadrants 1 and 4.

Reasoning

9 A function is defined as $y = \frac{1}{2\sin(x) + 1}$. Its graph is shown at right.



The graph appears to be undefined for two values of x in the domain [0°, 360°].

(a) What would cause this graph to be undefined?

(b) Find the values of x for which the graph is undefined in the domain [0°, 360°].

10 Find all solutions to the equation $\tan(x) = \sin(x)$ for $0° \le x \le 360°$.

Hint

Simplify the equation using $\tan(x) = \frac{\sin(x)}{\cos(x)}$ to find an equivalent equation and then factorise it using $\sin(x)$ as a common factor.

Open-ended

11 For each of the sine, cosine and tangent ratios, write at least one equation similar to those in Question **5** and solve it for $0° \le x \le 360°$ using a calculator.

12 **(a)** Write any trigonometric equation that has solutions in:

(i) quadrants 1 and 3

(ii) quadrants 3 and 4.

(b) How do you know which trigonometric ratio to use in each of parts **(i)** and **(ii)**?

Half-time 6

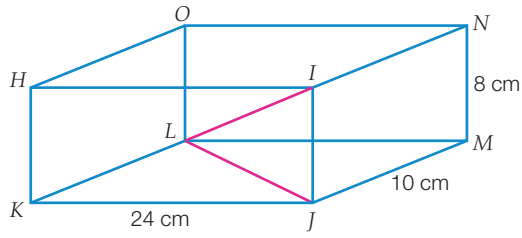
Ex. 6.1

1 A rectangular prism is shown.

(a) Calculate $\angle ILJ$, by first finding the length of LJ.

(b) Calculate the length of IL.

Round both answers to two decimal places.



Ex. 6.3

2 Find all values for x, correct to one decimal place, that satisfy the following equations and conditions:

(a) $3\sin x - 2 = 0,\ 0° \le x \le 180°$

(b) $5\tan x + 1 = 3,\ 0° \le x \le 360°$

Ex. 6.2

3 Find the equivalent first quadrant ratio for each of the following, taking care to include the appropriate sign.

(a) $\sin 135°$ **(b)** $\cos 98°$ **(c)** $\tan 200°$

(d) $\sin 247°$ **(e)** $\cos 300°$ **(f)** $\tan 339°$

Ex. 6.1

4 A right pyramid $ABCDE$ has a square base of side length 9 cm. The sloping edges are 15 cm in length.

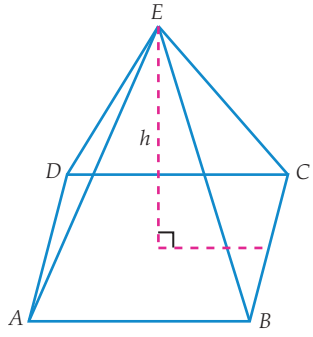
(a) Find the angle that the sloping edge makes with the base of the pyramid, to the nearest degree.

(b) Using your answer to **(a)**, or any other method, find the height of the pyramid, h. State your answer correct to one decimal place.

(c) The side length of the square base of the pyramid is increased by 3 cm, but the length of the sloping edges remains the same.

(i) Will the angle you found in **(a)** increase or decrease?

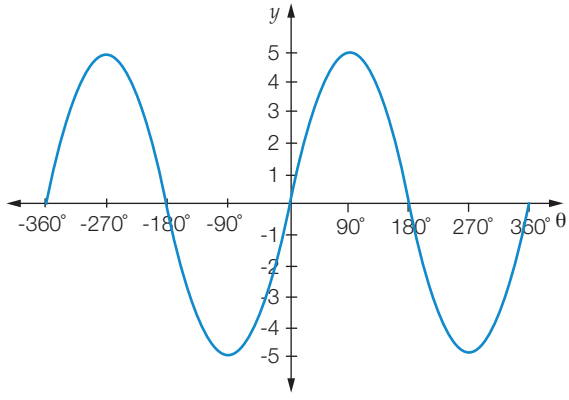
(ii) By how many degrees will it increase or decrease?



Ex. 6.2

5 The period and amplitude of the trigonometric graph shown is:

A 360°, 10 **B** 180°, 5 **C** 360°, 5 **D** 180°, 10



Ex. 6.3

6 Determine all solutions to the following equations. State your solution correct to two decimal places, where necessary.

(a) $\cos x = 0.5$ **(b)** $3\sin x - 1 = 0$

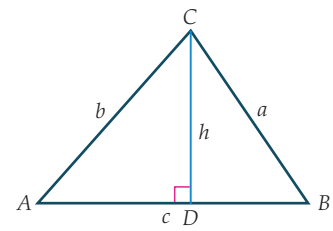
The sine and cosine rules

6.4

The trigonometric ratios $\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ and $\tan\theta = \frac{O}{A}$ apply to right-angled triangles. For situations that involve a triangle that is not right angled, the **sine rule** or **cosine rule** may be used.

Sine rule

Consider a triangle with vertices at A, B and C and side lengths a, b, c, where a is opposite angle A, b is opposite angle B and c is opposite angle C. It consists of triangle CAD and triangle DBC, each of height h.



$\sin A = \frac{h}{b}$ (from ΔCAD)

$\therefore h = b\sin A$ (1)

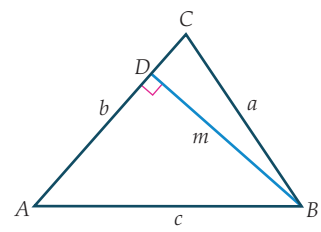
$\sin B = \frac{h}{a}$ (from ΔDBC)

$\therefore h = a\sin B$ (2)

$b\sin A = a\sin B$ equating (1) and (2)

$\frac{\sin A}{a} = \frac{\sin B}{b}$ (3) rearranging this expression

Triangle ABC can also be divided as shown.



$\sin A = \frac{m}{c}$ (from ΔBAD)

$\therefore m = c\sin A$ (4)

$\sin C = \frac{m}{a}$ (from ΔDCB)

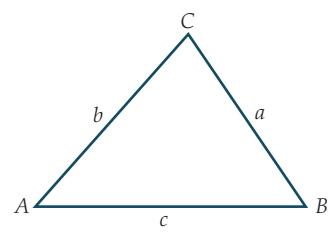
$\therefore m = a\sin C$ (5)

$c\sin A = a\sin C$ equating (4) and (5)

$\frac{\sin A}{a} = \frac{\sin C}{c}$ (6) rearranging this expression

From (3) and (6) we have the sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



The sine rule (also known as the law of sines or the sine formula) is sometimes given in reciprocal form: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

The sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Use to find an unknown angle when two sides and one angle are known.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

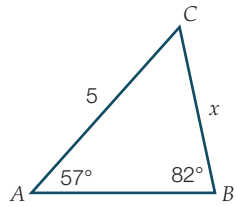
Use to find an unknown side when two angles and one side are known.

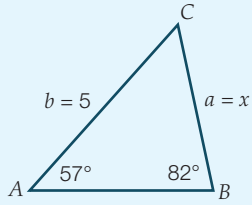
Only two equated expressions can be used at any one time.

Worked Example 8

WE 8

Calculate the value of x, correct to two decimal places.

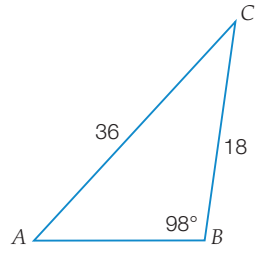


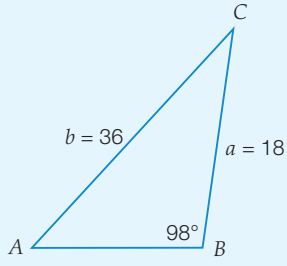
Thinking	Working
1 Draw and label the diagram with the information provided.	
2 Determine the rule to be used. (Here, we will use the sine rule.)	$\frac{a}{\sin A} = \frac{b}{\sin B}$
3 Substitute the information given.	$\frac{x}{\sin 57°} = \frac{5}{\sin 82°}$
4 Rearrange to make the unknown value the subject.	$x = \frac{5 \times \sin 57°}{\sin 82°}$
5 Use a calculator to evaluate the unknown to the required number of decimal places.	$x = 4.23$ units (2 d.p.)

Worked Example 9

WE 9

Calculate the value of $\angle CAB$ in degrees, rounded to two decimal places.

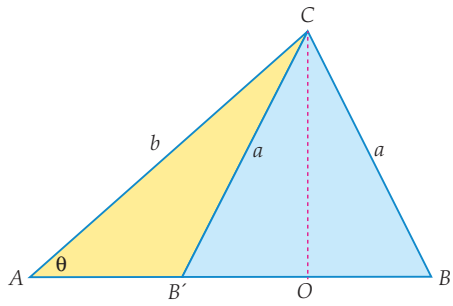


Thinking	Working
1 Draw and label the diagram with the information provided.	
2 Determine the rule to be used. (Here, we will use the sine rule.)	$\frac{\sin A}{a} = \frac{\sin B}{b}$

3 Substitute the information given.	$\frac{\sin A}{18} = \frac{\sin 98°}{36}$
4 Make the unknown value the subject.	$\sin A = \frac{18 \times \sin 98°}{36}$ $A = \sin^{-1}\left(\frac{18 \times \sin 98°}{36}\right)$
5 Use a calculator to evaluate the unknown to the required number of decimal places.	$\angle CAB = 29.68°$ (2 d.p.)

The ambiguous case of the sine rule

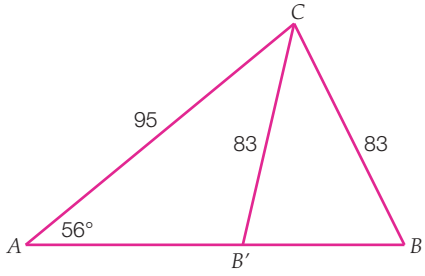
Consider ΔCAB, in which we know two sides a, b and a non-included angle θ. By reflecting side BC about the vertical line OC, we have triangle CAB' that also satisfies the information and $\angle AB'C = 180° - \angle ABC$.

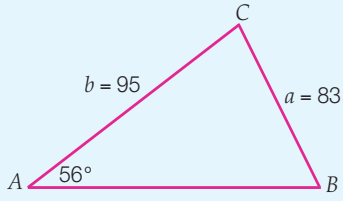


Worked Example 10

WE10

Calculate the values of $\angle ABC$ and $\angle AB'C$, correct to two decimal places.

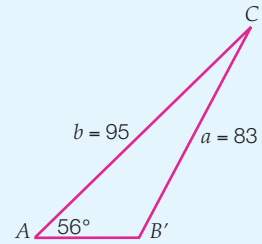


Thinking	Working
1 Draw a labelled diagram for the acute-angled triangle and choose the rule to be used.	 $\frac{\sin A}{a} = \frac{\sin B}{b}$
2 Substitute the information given.	$\frac{\sin 56°}{83} = \frac{\sin B}{95}$
3 Rearrange the equation to make the unknown value the subject.	$B = \sin^{-1}\left(\frac{95 \times \sin 56°}{83}\right)$

4 Use a calculator to evaluate the value of $\angle ABC$ to the required number of decimal places.

$\angle ABC = 71.60°$

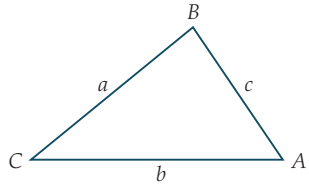
5 Draw a separate labelled diagram for the obtuse-angled triangle.



6 Find $\angle AB'C$ by subtracting the acute angle $\angle ABC$ found above from 180°.

$\angle AB'C = 180° - 71.60°$
$= 108.40°$

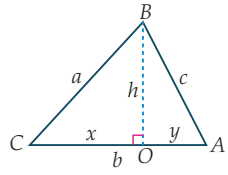
Cosine rule



To find the length c in a triangle using the sine rule you need to know $\angle C$ and one of the other angles $\angle A$ or $\angle B$ and its corresponding side length a or b.

If only $\angle C$ is known and side lengths a and b, the cosine rule can be used.

To derive the formula we obtain expressions for c from the triangle below. Draw the perpendicular height, h, from B to O on side b.



$b = x + y$

$y = b - x$

Squaring both sides,

$y^2 = b^2 - 2bx + x^2$ (1)

Using Pythagoras' theorem in ΔAOB,

$c^2 = h^2 + y^2$ (2)

In ΔCOB, $h^2 = a^2 - x^2$ (3)

Substituting for h^2 using equation (3) in equation (2)

$c^2 = a^2 - x^2 + y^2$ (4)

Substituting for y^2 using equation (1) in equation (4)

$c^2 = a^2 - x^2 + b^2 - 2bx + x^2$

$c^2 = a^2 + b^2 - 2bx$ (5)

In ΔCOB, $x = a \cos C$ (6)

Substituting for x using equation (6) in equation (5)

$c^2 = a^2 + b^2 - 2ab \cos C$

This simplifies to Pythagoras' theorem when $C = 90°$

Similarly, it can be shown that:

$b^2 = a^2 + c^2 - 2ac \cos B$

and $a^2 = b^2 + c^2 - 2bc \cos A$

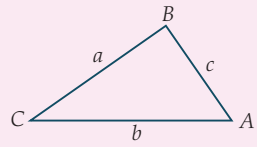
The cosine rule

Depending on the angle to be used, the cosine rule can be written in three ways.

$c^2 = a^2 + b^2 - 2ab\cos C$

$b^2 = a^2 + c^2 - 2ac\cos B$

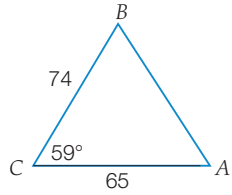
$a^2 = b^2 + c^2 - 2bc\cos A$

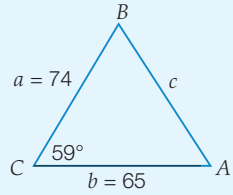


Worked Example 11

WE11

Calculate the length of AB correct to two decimal places.

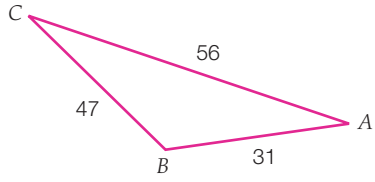


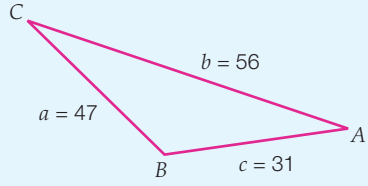
Thinking	Working
1 Draw and label the triangle with the information provided.	
2 Determine the rule to be used. (Here, we use the cosine rule.)	$c^2 = a^2 + b^2 - 2ab\cos C$
3 Substitute the information given into the rule. Do not evaluate e^2.	$c^2 = 74^2 + 65^2 - 2 \times 74 \times 65 \times \cos 59°$ $c^2 = 9701 - 9620\cos 59°$
4 Calculate the value to the required number of decimal places.	$c = 68.89$ units (2 d.p.)

Worked Example 12

WE12

Calculate the value of $\angle ACB$ in degrees correct to two decimal places.



Thinking	Working
1 Draw and label the triangle with the information provided.	

2 Determine the rule to be used. (Here, we use the cosine rule.)

$$c^2 = a^2 + b^2 - 2ab\cos C$$

3 Substitute the information given into the rule.

$$31^2 = 47^2 + 56^2 - 2 \times 47 \times 56 \times \cos C$$

4 Simplify and make cos θ the subject. Do not calculate the cos θ value.

$$961 = 5345 - 5264 \times \cos C$$

$$-4384 = -5264 \times \cos C$$

$$\cos C = \frac{-4384}{-5264}$$

5 Solve to find the angle in the required format using the inverse of cos θ.

$$\angle ACB = \cos^{-1}\left(\frac{-4384}{-5264}\right)$$

$$\angle ACB = 33.61°$$

6.4 The sine and cosine rules

Navigator

Answers page 743

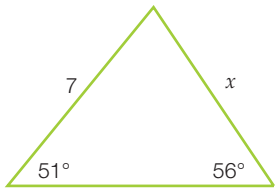
Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q12, Q13, Q18	Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9, Q11, Q12, Q13, Q15, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q8, Q10, Q12, Q14, Q15, Q16, Q17, Q18

Fluency

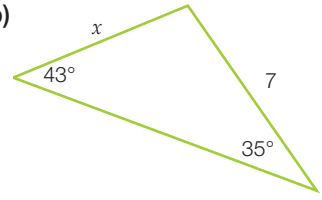
WE8

1 Calculate the value of x correct to two decimal places.

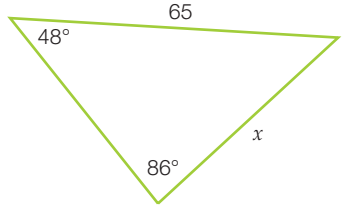
(a)



(b)



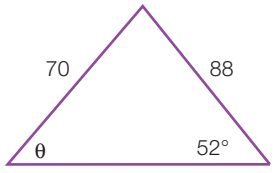
(c)



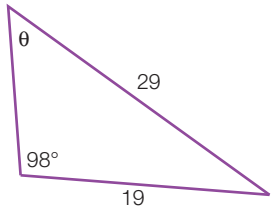
WE9

2 Calculate the value of θ in degrees correct to two decimal places.

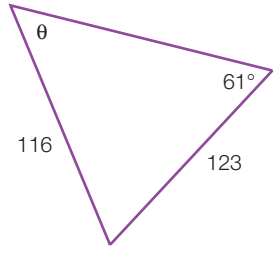
(a)



(b)



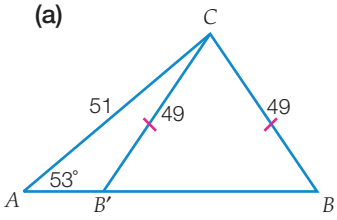
(c)



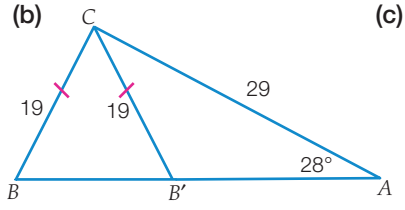
WE10

3 Calculate the values of ∠ABC and ∠$AB'C$ correct to two decimal places.

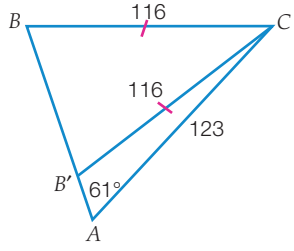
(a)



(b)



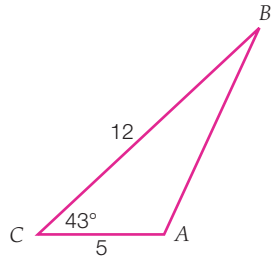
(c)



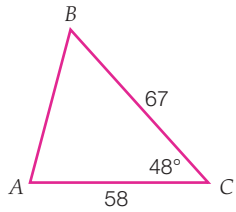
4 Calculate the length AB correct to two decimal places.

WE11

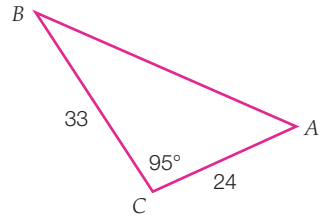
(a)



(b)



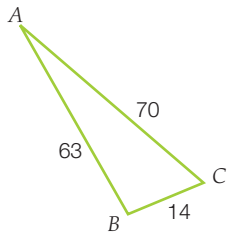
(c)



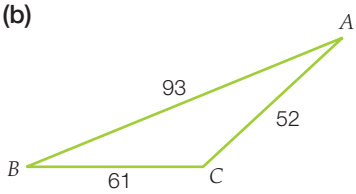
5 Calculate the value of angle $\angle ACB$ correct to two decimal places.

WE12

(a)



(b)



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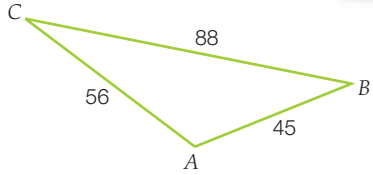
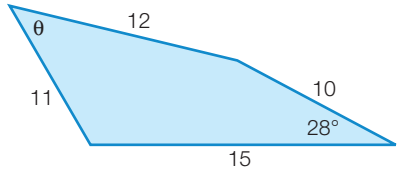
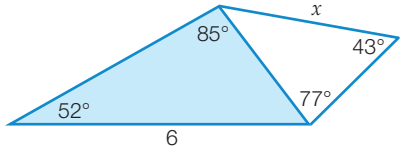


Understanding

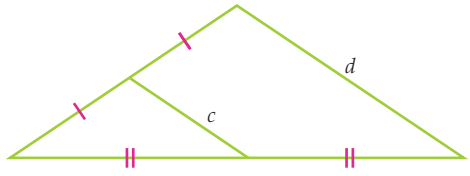
6 Calculate the size of angle θ to the nearest degree.



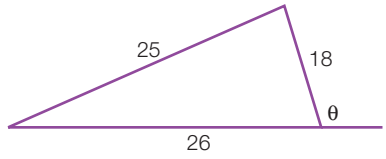
7 Calculate the value of x and round to two decimal places.



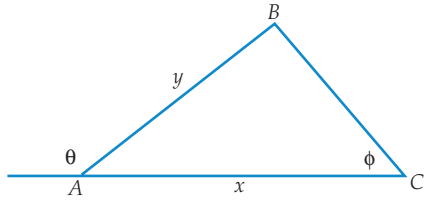
8 Use the cosine rule to show that $d = 2c$.



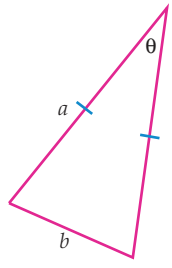
9 Calculate the value of θ, giving your answer as a decimal correct to two decimal places.



10 Show that $x = \dfrac{y\sin(\theta - \phi)}{\sin\phi}$.



11 For the triangle shown, which is the correct statement?



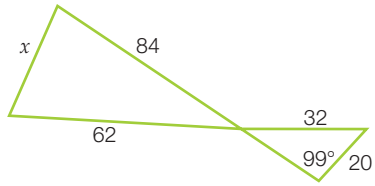
A $a^2 = b^2(2 - \cos\theta)$

B $b^2 = 2a^2(1 + \cos\theta)$

C $a^2 = b^2(2 + \cos\theta)$

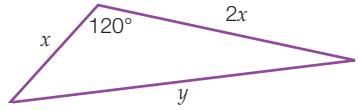
D $b^2 = 2a^2(1 - \cos\theta)$

12 Calculate the value of x rounded to two decimal places.

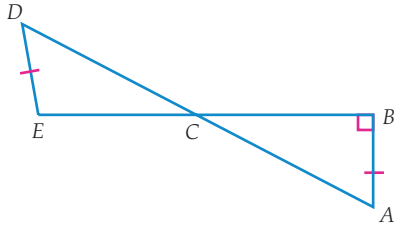


13 The value of y is:

A $7x^2$ **B** $\sqrt{7x}$ **C** $\sqrt{7x^2}$ **D** $49x$



14 Show that $\sin D = \dfrac{EC}{AC}$.



Reasoning

15 Show that the sine rule reduces to $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$ when applied to a right-angled triangle.

16 Apply the cosine rule to an equilateral triangle to show that each interior angle is 60°.

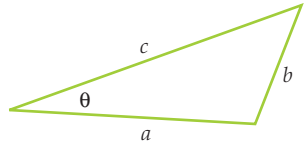
Open-ended

17 **(a)** Using the cosine formula $c^2 = a^2 + b^2 - 2ab\cos C$, show that if $a = b$ and $c = ka$, then $\cos C = \dfrac{2 - k^2}{2}$.

(b) State two values for k and use them to evaluate $\cos\theta$ rounded to two decimal places.

(c) What are the minimum and maximum possible values for k?

18 Determine the angle θ in the following triangle if sides b and c are positive integer multiples of a.



Outside the Square — Problem solving

Trig chimps

Equipment needed: 1 brain, calculator

For all calculations round your answers to two decimal places.

A certain group of monkeys are renowned for their longevity.

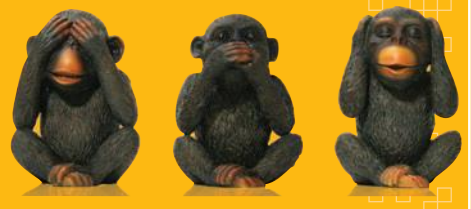
The ages of three chimps, Syn, Coss and Ten satisfy the following conditions.

- The sum of their ages is 158.
- Coss's age is a multiple of 13 and is an odd number.
- Expressed in degrees, Ten's age is in quadrant 2.
- Expressed in degrees, Syn's age and Coss's age are in quadrant 1.
- The tangent of Ten's age is 15.08 less than the cosine of Coss's age.
- The two digits after the decimal point in the cosine of Coss's age are consecutive digits.

What are the ages of the three chimps?

Strategy options

- Guess and check.
- Test all possible combinations.

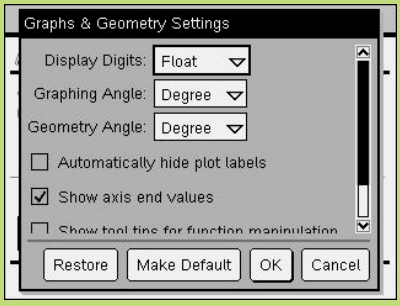
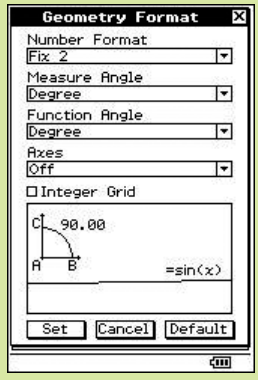
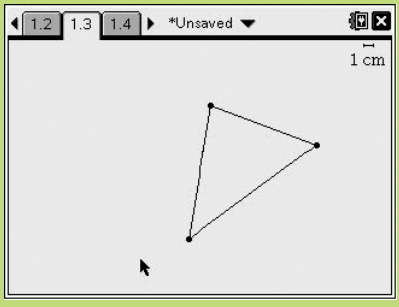
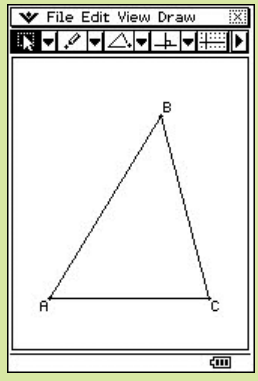


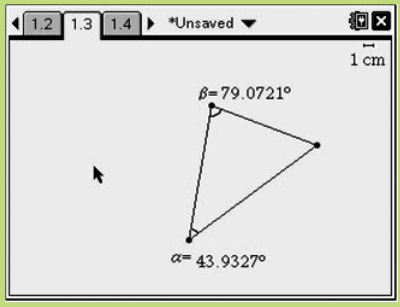
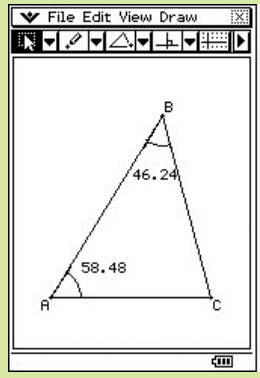
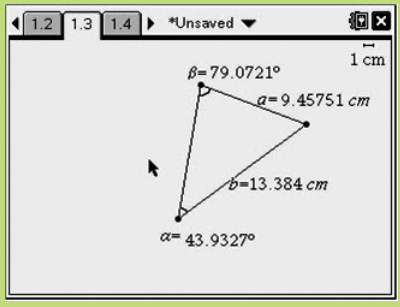
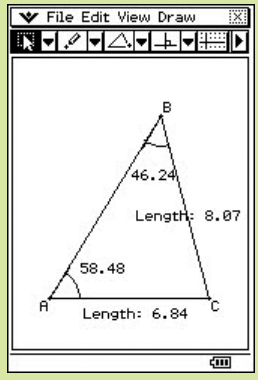
Technology Exploration CAS

Equipment required: 1 brain, 1 TI-Inspire or ClassPad CAS

Demonstrating the sine and cosine rules

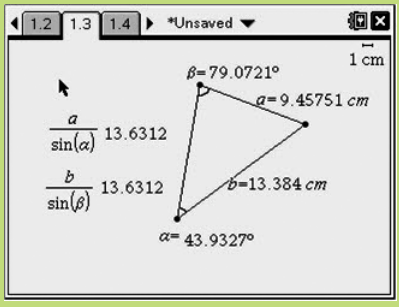
The sine rule states $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ for any triangle. In this notation, the angle A is opposite the side labelled a, and so on. With the help of our CAS we are going to demonstrate that this rule is true.

Using the TI-Nspire CAS	Using the ClassPad
Press (on) (or (⌂)) > Settings (or Settings & Status) > Settings > Graphs & Geometry > (enter) (or (enter)) and make sure you have the following settings in place. 	From the menu, tap Geometry to open a new Geometry page. Tap > Geometry Format and make sure the following settings are in place. 
Press (tab) down to OK and press (enter) (or (enter)) and then select Geometry. This will give you a blank screen. Press (menu) > Shapes > Triangle > (enter) (or (enter)) and then press (enter) (or (enter)) where you want each vertex of the triangle to appear. Don't make your triangle too small, as we want to put a number of measurements on it. Then press (esc) to stop drawing. 	Tap SET to confirm and then tap to choose a triangle to draw. Tap the screen and draw out a rectangle where you want the shape to appear. Make it reasonably big, as you need to show some measurements on it. 

Using the TI-Nspire CAS	Using the ClassPad
We want to measure any two of the angles. To measure an angle press menu > Measurement > Angle > enter (or enter) and then press enter (or enter) at the three vertices, in order, that form one of the angles. Repeat this process to measure the second angle. Press esc to stop measuring angles. You can grab and move these measurements if you wish. We also want to label them. To do this press menu > Actions > Text > enter and move the cursor over the first angle. Press enter (or enter). The text box should now show the angle measurement. Press ctrl > (or ctrl >) and label this angle α. Press = then press enter (or enter). Repeat this process to label the other angle β. 	We want to measure two angles on this; the one at *A* and the one at *B*. Tap on *AC* followed by a tap on *AB* and then tap Draw > Attached Angle and the angle should appear. Then tap somewhere on the screen and repeat the procedure to get the angle at *B*. (You will have different angles to those shown here.)  Tap somewhere on the screen (this needs to be done after every operation).
We now need to find the lengths of the sides opposite the angles we have marked. It is important to measure the correct sides. Press menu > Measurement > Length > enter (or enter) and press enter (or enter) at each vertex that forms a side opposite one of the angles you measured. You may need to press tab to indicate it is a side you want to measure. Measure the side opposite the other angle you measured. Label these sides *a* and *b*, making sure *a* is opposite α. 	We now need to get the length of *BC*. Tap somewhere on *BC* to select it and then tap Draw > Measurement > Length and the length will appear near the top of the screen. Drag it down near *BC* so you know which length it is later. Then repeat for *AC*. 

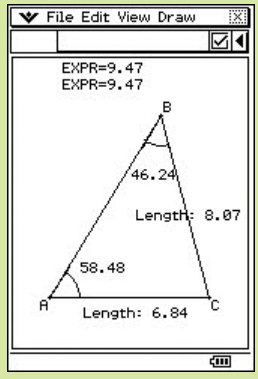
Using the TI-Nspire CAS

To do our calculations, select (menu) >Actions > Text and (enter) (or (≈enter)). In the new text box, enter the expressions $\frac{a}{\sin(\alpha)}$ and then press (menu) > Actions > Calculate and move the pointer over $\frac{a}{\sin(\alpha)}$ and press (enter) (or (≈enter)). A message comes up to select the first value. Move the pointer over to the value for a and press (enter) (or (≈enter)) and then to the value for α and press (enter) (or (≈enter)). Move the pointer beside the expression and press (enter) (or (≈enter)). the value should appear. Repeat this process to find $\frac{b}{\sin(\beta)}$.

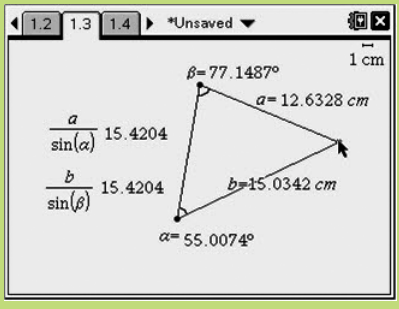


Using the ClassPad

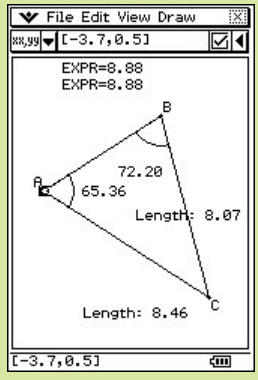
Now we need to do our calculations. Tap the ▶ on the right-hand side at the top to open a box where you can write the expressions. Then tap Draw > Expression and you will notice each of your measurements has a number attached to it. Tap on the number for a side, press (÷), press (Keyboard) > (TRIG) > sin then tap on the number on the angle opposite the side you selected. Close the brackets. Close the (Keyboard). (You will have entered something that looks like @3/sin(@1) Now press (EXE). (You must make sure the numbers match your screen.) Tap somewhere on the screen and then repeat the procedure to get the other value you need. At the top of the screen EXPR = will show the result of each calculation. If you have done this properly the two values should be the same.



Having pressed (esc) to stop calculating now move the pointer to one of the vertices of the triangle and press (☝) to choose the point and drag it to another position. The values for the two expressions should remain the same, demonstrating the sine rule is correct.



Tap somewhere on the screen and then tap on A and drag it to a new position on the screen. The values for the two expressions should remain the same, demonstrating the sine rule is correct.



Now that you have demonstrated the sine rule, use your CAS to demonstrate the cosine rule.

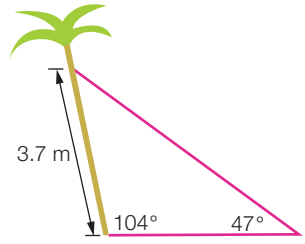
$c^2 = a^2 + b^2 - 2ab\cos C$

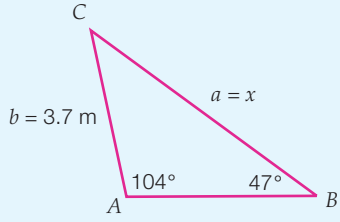
6.5 Applications of the sine and the cosine rules

The formulas from the previous section can be applied to a range of practical situations. In surveying, for example, a total station is an electronic instrument used to measure angles of rotation and distances. This sophisticated equipment has limitations. It needs a clear line of sight to the object, which must have reflective properties for the reflected signal to be recorded.

Worked Example 13 WE13

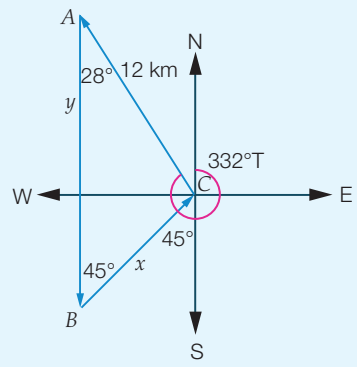
A tree growing at an angle of 104° to the horizontal is in danger of falling. It has been supported by a rope tied to the tree 3.7 m along the tree trunk and pegged to the ground at an angle of 47° as shown in the diagram. Find the length of rope used correct to one decimal place.



Thinking	Working
1 Draw a triangle and show the information on the diagram.	
2 Choose the expression to be used and the values to be used.	$\frac{a}{\sin A} = \frac{b}{\sin B}$ $a = x$, $\angle A = 104$, $b = 3.7$, $\angle B = 47$
3 Substitute the information given.	$\frac{x}{\sin 104°} = \frac{3.7}{\sin 47°}$
4 Rearrange to make the unknown value the subject.	$x = \frac{3.7 \times \sin 104°}{\sin 47°}$
5 Calculate the value to the required number of decimal places.	$x = 4.9$
6 State the answer.	The length of rope is 4.9 m.

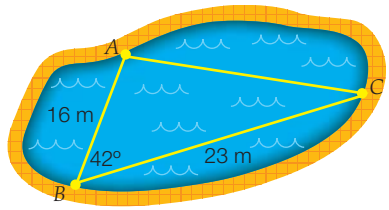
Worked Example 14 WE14

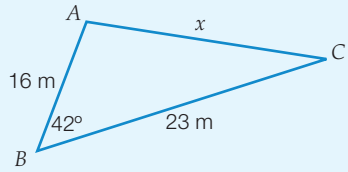
A helicopter leaves base and flies for 12 km on a bearing of 332° T. It then heads due south for a certain distance before flying back to base on a bearing of 45° T. What is the total flight distance? Round your answer to two decimal places.

Thinking	Working
1 Draw a diagram showing the information and identify the unknown.	
2 Use the true bearing and geometry to find triangle angles. (Here, $\angle A$ is obtained using alternate angles and $\angle C$ by $180° - (\angle A + \angle B)$.)	$360 - 332 = 28°$ $\angle A = 28°$ $\angle B = 45°$ $\angle C = 180 - (28 + 45) = 107°$
3 Identify given information and any unknowns.	$a = x$ $b = 12$ $c = y$
4 Decide which formula to use. (Here, we use the sine rule.)	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
5 Substitute the information given to find the first unknown.	$\frac{x}{\sin 28°} = \frac{12}{\sin 45°}$
6 Rearrange to make the unknown value the subject.	$x = \frac{12 \times \sin 28°}{\sin 45°}$
7 Use a calculator to evaluate the unknown value. (Do not round.)	$x = 7.967\,196\,615$
8 Substitute the information given to find the second unknown.	$\frac{y}{\sin 107°} = \frac{12}{\sin 45°}$
9 Rearrange to make the unknown the subject.	$y = \frac{12 \times \sin 107°}{\sin 45°}$
10 Use a calculator to evaluate the unknown value. (Do not round.)	$y = 16.229\,029\,87$
11 Calculate the perimeter using the unrounded values.	Perimeter $= 12 + 16.229\,029\,87 + 7.967\,196\,615$ $= 36.196\,226\,49$
12 State the answer to the required number of decimal places.	The total flight is 36.20 km.

Worked Example 15 WE15

A walkway is to be constructed across a pond from A to C. There is an existing walkway of length 16 m from A to B. A rope measuring 23 m is attached to point B and point C. The angle between the rope and the existing walkway is 42°. Calculate the length of the new walkway correct to two decimal places.



Thinking	Working
1 Draw a diagram showing the information and identify the unknown.	
2 Identify the given information and any unknowns.	$b = x$, $a = 23$, $c = 16$, $B = 42°$
3 Decide which formula to use. (Here, we use the cosine rule.)	$b^2 = a^2 + c^2 - 2ac\cos B$
4 Substitute the information given.	$x^2 = 23^2 + 16^2 - 2 \times 23 \times 16 \times \cos 42°$
5 Use a calculator to evaluate the unknown value. (Do not round.)	$x^2 = 238.045\,408\,4$ $x = 15.428\,720\,25$
6 State the answer to the required number of decimal places.	The required length for the walkway is 15.43 m.

6.5 Applications of the sine and the cosine rules

Navigator

Answers page 744

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q17	Q1, Q2, Q3, Q6, Q7, Q11, Q12, Q14, Q15, Q16, Q17, Q18	Q1, Q2, Q3, Q8, Q9, Q12, Q13, Q15, Q16, Q17, Q18

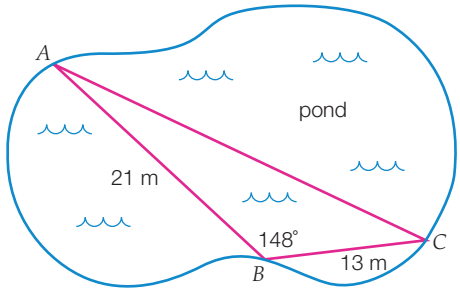
Fluency

WE13

1 Part of the ship's rigging shown forms a triangle with base angles of 80° and 66° and rigging of length 5.3 m. Calculate the length of the section of the mast in the triangle shown correct to two decimal places.

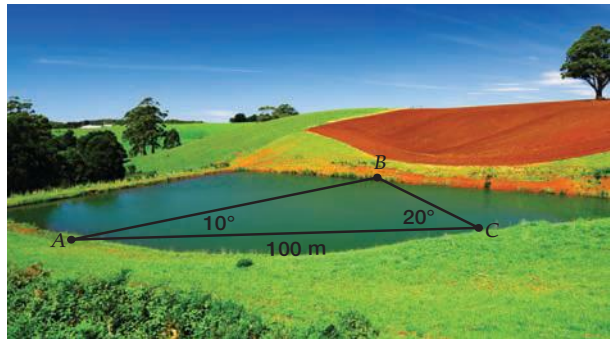
2 Postman Pete leaves the post office and makes deliveries for 2.5 km on Aiden Rd, which is on a bearing of 072°T from his starting point. Pete then heads due north on Sari Boulevard for a certain distance before returning to the post office on a bearing of 210°T. Find the total distance of Postman Pete's route rounded to two decimal places. WE14

3 A walkway is to be constructed across a pond from A to C. There is an existing walkway of length 21 m from A to B. A rope measuring 13 m is attached to point B and point C. The angle between the rope and the existing walkway is 148°. Calculate the length of the new walkway correct to two decimal places. WE15



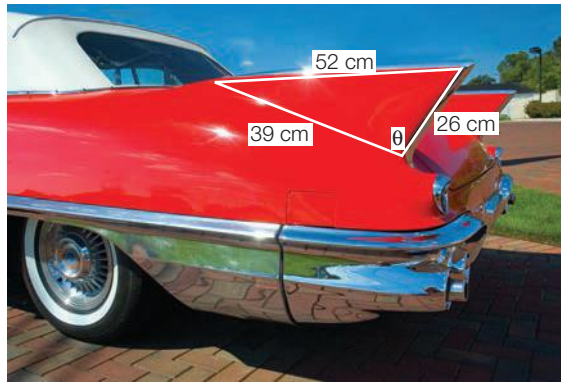
4 A boomerang is mounted on a triangular frame as shown. If the two shorter sides of the frame are 15 cm and 17 cm respectively and the included angle is 134°, calculate the perimeter of the frame, rounded to two decimal places.

5 A pipe is laid across a section of a water dam from A to B. A 100 m rope is attached from point A to point C and the angles between the rope and AB, and the rope and CB are measured. If $\angle A$ is 10° and $\angle C$ is 20°, what length of pipe is required? State the answer to the nearest metre.



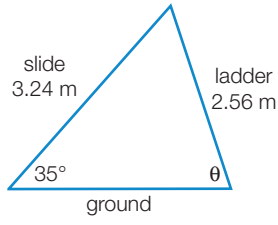
Understanding

6 The fins of this classic car are triangular with the dimensions of 26 cm, 39 cm and 52 cm as shown. For aerodynamic stability, it is recommended that the angle θ be less than 105°. Is this true for this vehicle?

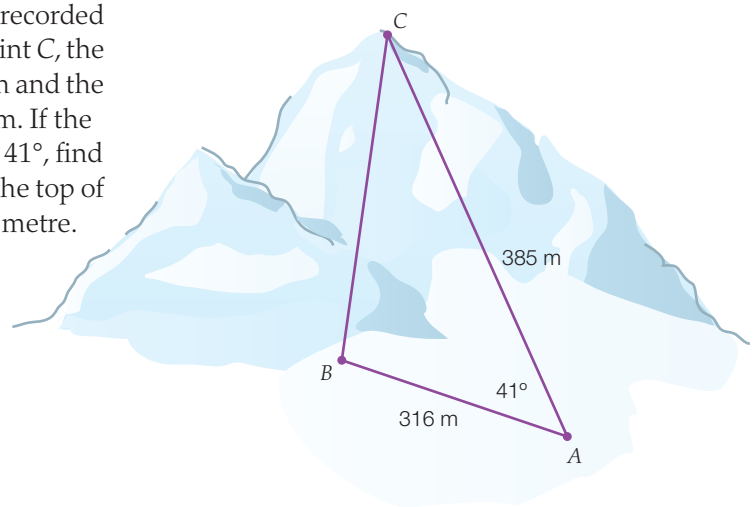


7 Rangers Tim and Zoe are 10 km apart, with Tim due west of Zoe. They notice a fire that is N50°E of Tim and N35°W of Zoe. Decide which ranger is closer to the fire by finding the distance each ranger is from the fire.

8 A children's slide is of length 3.24 m and makes an angle of 35° with the ground. If the ladder is 2.56 m, what angle θ does it make with the ground? Round your answer to two decimal places.



9 A ski lift is to be constructed at point B on a mountain. A surveyor recorded the distance from A to the point C, the top of the mountain, as 385 m and the distance from A to B as 316 m. If the angle between AB and AC is 41°, find the distance between B and the top of the mountain to the nearest metre.



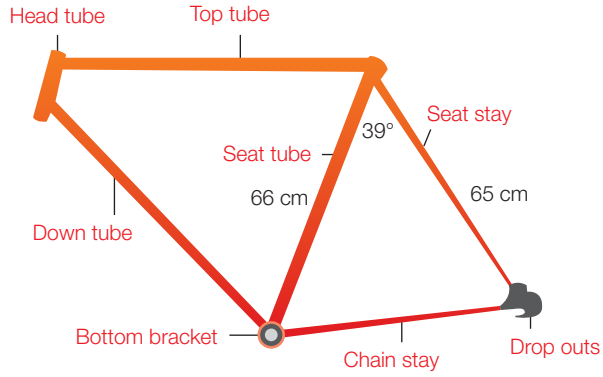
10 To the nearest cm, the length of the chain stay in the bicycle frame is:

A 44 cm

B 47 cm

C 51 cm

D 62 cm



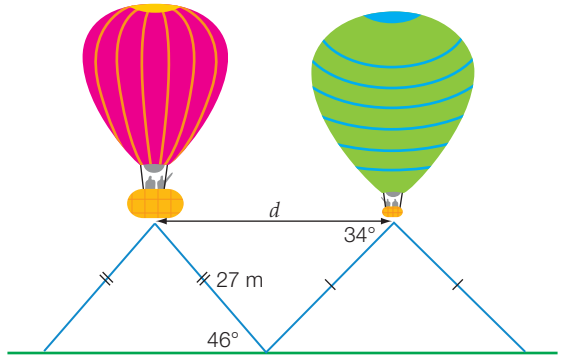
Reasoning

11 Ken and Sofia set out in different directions from their campsite. After Ken walks 2150 m on a bearing of 316° T and Sofia walks for 2000 m, the distance between them is 2696 m. What are the two possible bearings that Sofia could have used? Give your answer correct to the nearest degree.

12 Two hot air balloons are stationary at the same height above the ground. They are secured by ropes as shown.

(a) Calculate the horizontal distance between the base of each balloon, correct to two decimal places.

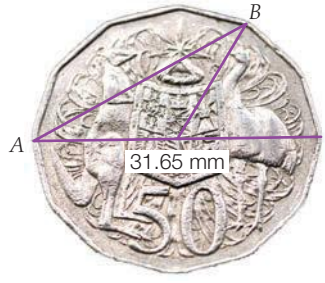
(b) What is the total length of rope required to secure the balloons?



13 The longest diagonal of a 50 cent coin is 31.65 mm. Calculate the length AB correct to two decimal places.

Hint

A 50c coin has the shape of a regular dodecagon (a 12-sided polygon).



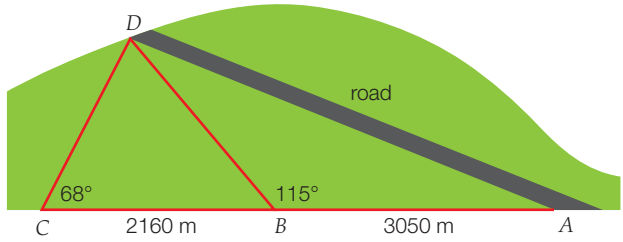
14 The dial on a sundial casts a shadow as shown. The dial is 24 cm long, the angle between the dial and the base is 38° and there is 15 cm between the top of the dial and the end of the shadow. Find the length of the shadow correct to two decimal places.

Hint
Be sure to find all possible solutions.

15 A pendant is in the shape of a scalene triangle and its two smaller interior angles are 25° and 28°. If the longest side is of length 4 cm, the lengths of the two other sides (in cm) correct to two decimal places are:

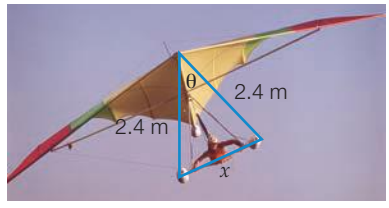
A 2.26, 2.35 **B** 2.12, 2.35 **C** 2.12, 2.18 **D** 2.11, 2.35

16 A new highway is being built across a paddock, from point A to point D as shown on the diagram. This highway currently runs in a straight line from A to C through B, 3050 m from A. The angle between AB and BD is 115°. The distance from B to C is a further 2160 m and the angle between BC and CD is 68°. Find the length of the new highway AD to the nearest metre.

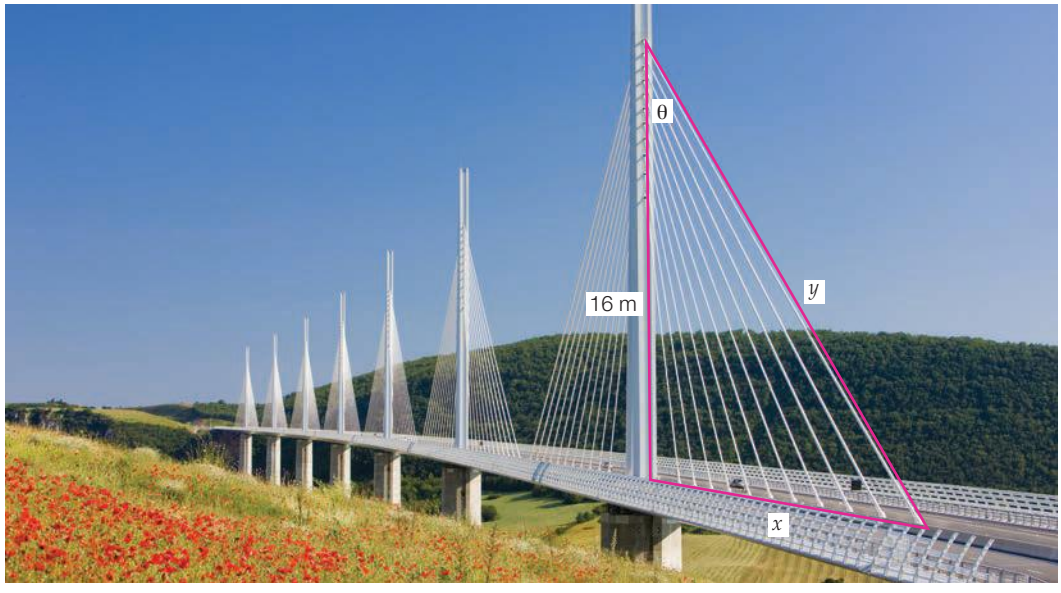


Open-ended

17 For maximum strength and stability, the angle θ between the support wires in the hang glider must be no more than 40° and the distance x between the wheels must be between 80 cm and 1.2 m. Determine two sets of allowable values for θ and x.



18 The length y of one of the bridge's support cables is less than 40 m and the angle θ is in the range $15° < \theta < 75°$. By choosing reasonable values for θ, find two possible values for the length x and the value of y for each x-value.



6.6 Area of a triangle using trigonometry

The area of any triangle can be found using the formula $A = \frac{1}{2}bh$ where b is the base and h is the perpendicular height.

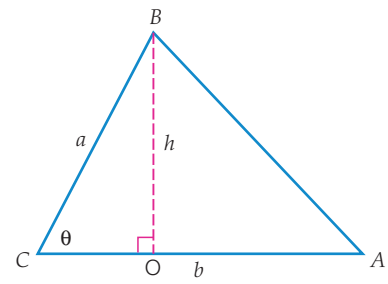
If h is not known but two sides and the angle included between them are known, a new formula can be used.

In ΔBOC, $h = a\sin\theta$ so we can substitute $a\sin\theta$ for h in the area formula.

$$A = \frac{1}{2} \times b \times h$$

$$= \frac{1}{2} \times b \times a\sin\theta$$

$$= \frac{1}{2}ab\sin\theta$$

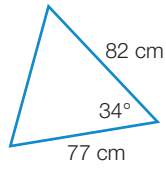


Area of a triangle

$A = \frac{1}{2}ab\sin\theta$, where θ is the angle included between sides a and b.

Worked Example 16 WE16

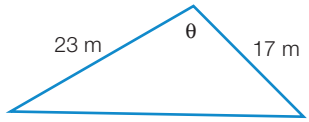
Calculate the area of the triangle correct to two decimal places.



Thinking	Working
1 Identify the given information. (Here, we have a, b and θ.)	$a = 77$ cm, $b = 82$ cm, $\theta = 34$
2 Write the formula for the area of the triangle.	Area $= \frac{1}{2}ab\sin\theta$
3 Substitute the given values into the formula.	$= \frac{1}{2} \times 77 \times 82 \times \sin 34°$
4 Use a calculator to evaluate the answer. (Do not round.)	$= 1765.371\,996$
5 State the answer to the required number of decimal places.	The area is 1765.37 cm^2

Worked Example 17 WE17

Find the size of the angle required if the area of the triangle shown is 195 m^2. Round the answer to the nearest degree.



Thinking	Working
1 Identify the given information. (Here, we have a, b and θ.)	area = 195 m^2, $a = 23$ m, $b = 17$ m
2 Write the formula for the area of the triangle.	$\text{Area} = \frac{1}{2}ab\sin\theta$
3 Substitute the given values into the formula.	$195 = \frac{1}{2} \times 23 \times 17 \times \sin\theta$
4 Use a calculator to evaluate the answer. (Do not round.)	$\sin\theta = \frac{195 \times 2}{23 \times 17}$ $= 0.9974$ $\theta = \sin^{-1}(0.9974)$ $= 85.901\,340\,36$
5 State the answer to the nearest degree.	The angle is 86°

6.6 Area of a triangle using trigonometry

Navigator

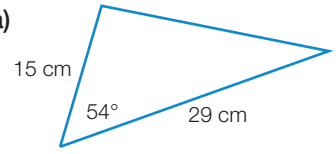
Q1, Q2, Q3, Q4, Q5, Q6, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q12, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13

Answers page 744

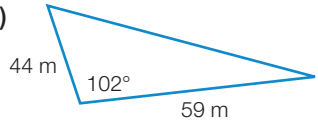
Fluency

1 Calculate the area of each of the following triangles correct to two decimal places. WE16

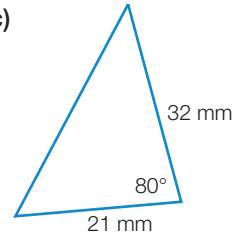
(a)



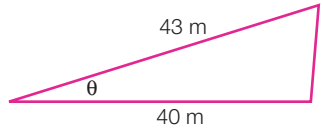
(b)



(c)

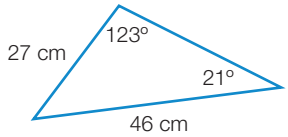


2 Find the size of the angle required if the area of the triangle shown is 529 m^2. Write the answer correct to the nearest degree. Assume θ is an acute angle. WE17

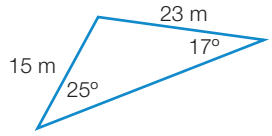


3 Calculate the area of the following triangles correct to two decimal places.

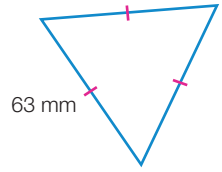
(a)



(b)

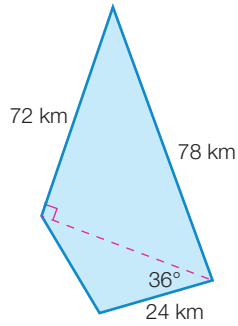


(c)



Understanding

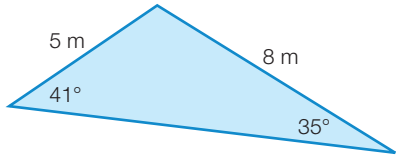
4 Calculate the area of the shape correct to the nearest square kilometre.



Hint
How many equilateral trinagles fit together to make a hexagon?

5 The ends of the cells of a bee's honeycomb are regular hexagons. If each edge length is 1.5 mm, what is the area of the end of one cell? Round your answer to two decimal places.

6 The area of the triangle in square metres is closest to:



A 11.47 **B** 13.12

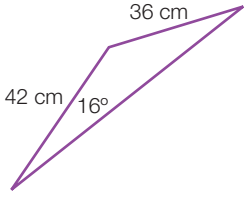
C 19.41 **D** 20.00

Reasoning

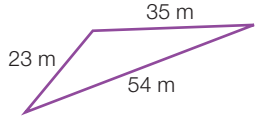
Hint
Are there any ambiguous cases?

7 Calculate the area of the following triangles correct to two decimal places.

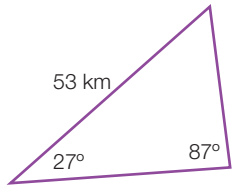
(a)



(b)



(c)

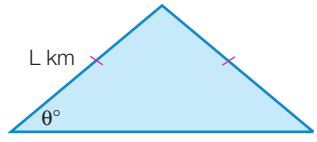


Hint
Divide this triangle into half and use trigonometry to find the base and the height of the small triangles.

8 A plot of land is in the shape of an isosceles triangle.

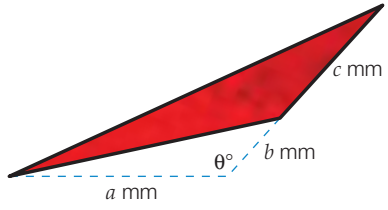
(a) Show that the area of land is $L^2 \sin\theta \cos\theta$ km^2.

(b) What is the area when $L = 5.6$ km and $\theta = 45°$.



9 (a) Show that the area of the section of red stained glass, shown in the diagram on the right, is $\frac{1}{2}ac\sin\theta$ mm^2.

(b) For what value of θ will the area be a maximum? What is this area?



10 The sail shown, consisting of two layers of cotton fabric, is to be replaced at a cost of \$135 per square metre. If the length of the sail along the mast is 3.8 m and the two angles at the base of the sail are 85° and 57° as shown in the diagram, calculate the total replacement cost of the sail. Answer to the nearest dollar.

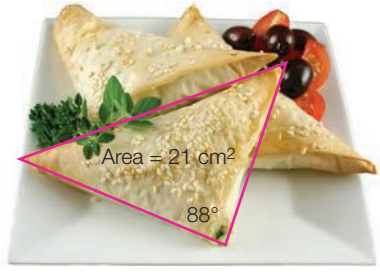
Hint

Use the sine or cosine rule to find the unknown sides or angles.

11 **(a)** In the formula for the area of a triangle, $\frac{1}{2}ab\sin\theta$, what is the range of values possible for the angle θ?

(b) Explain why $\sin\theta$ cannot take on values less than zero.

12 The shape of the cheese savoury can be approximated by an isosceles triangle. The angle between the two equal sides is 88° and the area covered by the savoury is 21 cm^2. If the perimeter is to be covered with a strip of cheese flavouring, what length will be used? Round your answer to the nearest centimetre.



Open-ended

13 Find values for the lengths of two equal sides of an isosceles triangle and the size of the angle between them, so that the area is a square number. The triangle must not be right angled.

Outside the Square — Problem solving

Triangles to the max

Servane has a square piece of cardboard, which is 30 cm × 30 cm. She wants to cut out as many small triangles as possible that have one side of 12 cm, another of 15 cm and the angle between them is 30°. Using area formulas she calculates that she will get 20 triangles. Her friend, Sophie, disagrees and says that 16 triangles are the most she can cut. Is either of them correct? Explain your answer. Draw a diagram to show the maximum number of triangles you could cut out from a piece of cardboard with the same dimensions.

Strategy options

- Make a model.
- Act it out.

Chapter review 6

D.I.Y. Summary

Key Words

amplitude	coterminal angles	quadrant
asymptote	equivalent angle	sine rule
cosine rule	period	unit circle

Copy and complete the following using words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 One half the difference between the minimum value and the maximum value of the graph of a trigonometric ratio is the _____.

2 The _____ relates the angle between two sides of a triangle and the side length opposite the angle.

3 Angles in quadrant 1 that have the same values for trigonometric ratios in the other quadrants are _____.

4 The trigonometric ratios are derived from the _____.

5 The smallest interval after which the graph of a trigonometric relationship repeats is the _____.

6 The _____ can be used when two angles in a triangle and a side length opposite one of the angles is known.

7 Each of the four sections of the unit circle is a _____.

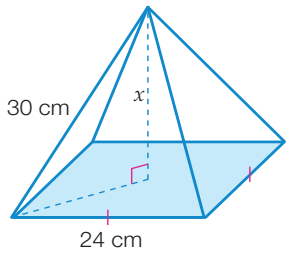
8 Each _____ on the graph $y = \tan\theta$ is a vertical line for which $\tan\theta$ is not defined.

9 _____ are in the same position on the unit circle so their trigonometric ratios are exactly the same.

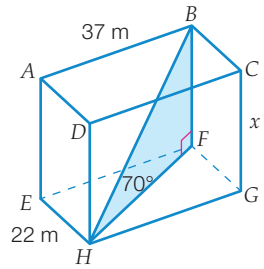
Fluency

1 Calculate the length of x in each shape. Round your answers to two decimal places. Ex. 6.1

(a)



(b)



2 To the nearest centimetre, what is the length of the longest possible timber pole that can fit in a shed in the shape of a rectangular-based prism of length 24 m, width 10 m and height 4 m? Ex. 6.1

3 In which quadrant are each the following angles. Ex. 6.2

(a) 37° (b) 274° (c) 93° (d) 182°

4 Find the equivalent ratio to each of the following in quadrant 1 using the correct sign. Ex. 6.2

(a) cos 173° (b) sin 254° (c) tan 312°

5 The equivalent trigonometric ratio in quadrant 1 of cos 253° is:

A cos 17° **B** -cos 73° **C** -cos 107° **D** cos 37°

Ex. 6.2

6 Find all solutions to $\tan x + 1 = 0$ for $0° \leq x \leq 360°$.

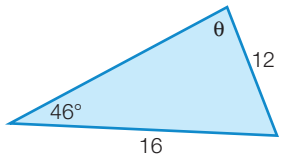
Ex. 6.3

7 Determine all solutions to the equation $4 \cos x - 1 = 0$. Round your answers to two decimal places.

Ex. 6.3

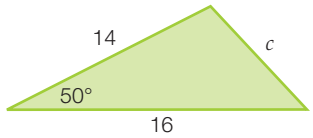
8 In degrees, the angle θ shown is closest to:

A 70.42 **B** 72.10 **C** 73.56 **D** 75.78



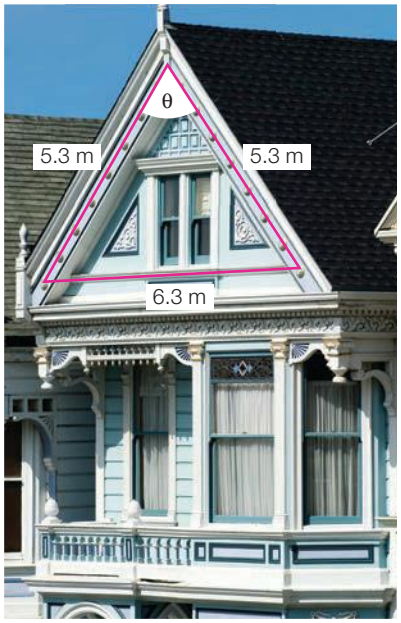
Ex. 6.4

9 Calculate the value of c rounded to two decimal places.



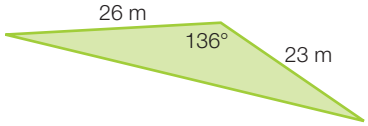
Ex. 6.4

10 A roof consists of rooflines of 5.3 m as shown in the diagram. The rooflines determine the pitch, which is the angle formed where the rooflines meet. Calculate the pitch θ of the roof correct to the nearest degree.



Ex. 6.5

11 Calculate the area of the shape rounded to two decimal places.



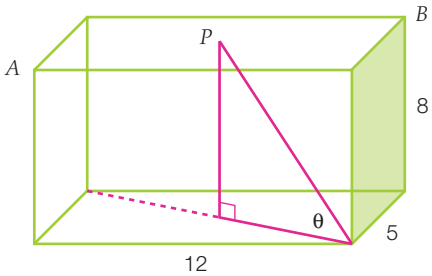
Ex. 6.6

Understanding

12 P is the midpoint of AB. The angle θ is closest to:

A 22.6° **B** 33.7°

C 42.2° **D** 50.9°



13 State the period and the amplitude, where appropriate, of each relationship.

(a) $y = 7 \sin (2x)$

(b) $y = -6 \cos (5x)$

(c) $y = 3 \tan (4x)$

14 The period of $\tan nx$ is 15°. The value of n must be:

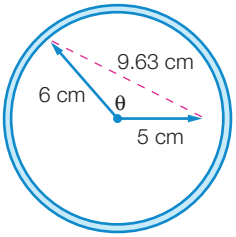
A 10 **B** 12 **C** 15 **D** 24

15 Sketch the graph of $y = 2 \cos (x)$ for $-180° \leq x \leq 180°$ using the graph of the basic function $y = \cos (x)$ as a guide.

16 **(a)** In which quadrant is 400°?

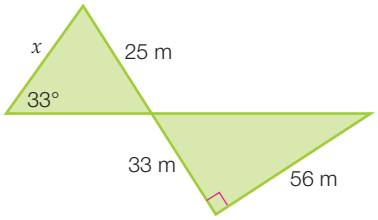
(b) What is the first angle measured clockwise from the positive x-axis that will be coterminal with 400°?

17 The lengths of the hour and minute hand of a clock face are 5 cm and 6 cm respectively. To the nearest degree, what is the size of the obtuse angle formed between the hands when the tips of the hands are 9.63 cm apart.



18 Calculate the area of an isosceles triangle where the angle between the two equal sides each of length 10 cm is 67°. State your answer correct to two decimal places.

19 Calculate the value of x rounded to two decimal places.



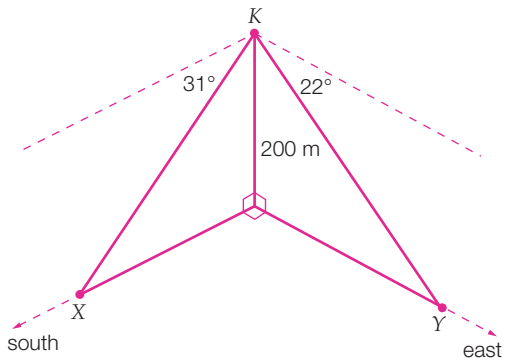
Reasoning

20 The longest drinking straw that can fit into a cylindrical soft-drink can is 15.7 cm. If the can has a radius of 36 mm, find:

(a) the angle, correct to one decimal place, that the straw makes with the base of the can

(b) the height of the can rounded to two decimal places.

21 From a point K on top of a 200 m high building, the angle of depression to a point X on the ground due south of K is 31° and the angle of depression to a point Y on the ground due east of K is 22°.



(a) Find, correct to two decimal places, the distance between:

(i) the building and point X

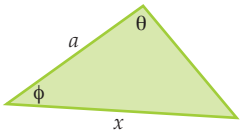
(ii) the building and point Y.

(b) Hence, find the distance between X and Y rounded to two decimal places. (Use Pythagoras' theorem.)

(c) What is the true bearing of Y from X, to the nearest degree?

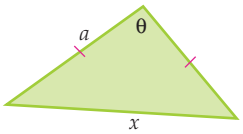
22 Which angles in the unit circle do not have equivalent angles? Use angles from 0° to 360°.

23 **(a)** **(i)** For given values for a, θ and ϕ explain how the sine rule can be used to find the value of x in the triangle at right.



(ii) Find the value of x when $\phi = 41°$, $\theta = 93°$ and $a = 10$. Write your answer correct to two decimal places.

(b) **(i)** For given values for a and θ, explain how the sine rule can be used to find the value of x in the triangle at right.



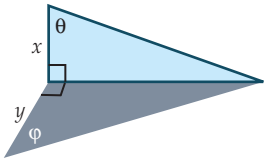
(ii) Find the value of x when $\theta = 93°$ and $a = 10$. Write your answer correct to two decimal places.

24 A fishing boat leaves port on a bearing of 136°T. Following this bearing, it drops x m of netting. The vessel then travels north-east, dropping y m of netting along the way. Finally, it heads directly back to its starting point, dropping netting as it moves. The expression that gives the area in square metres enclosed by the netting is:

A $\frac{1}{2}xy\sin 44°$ **B** $\frac{1}{2}xy\sin 45°$

C $\frac{1}{2}xy\sin 46°$ **D** $\frac{1}{2}xy\sin 89°$

25 The shadow of a triangular sculpture is shown. If $y = 1.5x$, show that $2\tan\theta = 3\tan\varphi$.



Mixed review B

Fluency

1 Find the interest earned on an investment of \$6000 at 4.2% p.a. for 3 years, where the interest is compounded: Ex.1.2

(a) annually (b) half-yearly (c) quarterly.

2 A painting has increased in value, solely due to the effects of inflation, from \$2500 to \$3100 over 5 years. Find the average annual inflation rate. State your answer correct to one decimal place. Ex.1.3

3 Expand and simplify each of the following. Ex.2.1

(a) $(3x-4)(x+2)$ (b) $(2x+5)(3x-1)$ (c) $(4x-3)(2x-5)$

4 Factorise the following expression fully. Ex.2.2

(a) $(3x-7)(2x+6)-5(3x-7)$ (b) $2x(3x-2)(4x+5)+2x(3x-2)$

5 Factorise the following quadratic trinomials. Ex.2.3

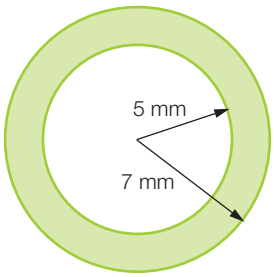
(a) x^2-3x-4 (b) $x^2-14x+45$ (c) $x^2-7x+12$

6 Simplify the following. Ex.2.6

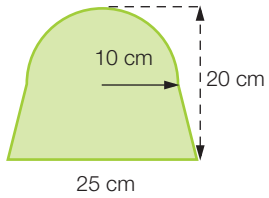
(a) $\frac{1}{3x}-\frac{2}{x}$ (b) $\frac{3x}{x+1}-\frac{2}{x-1}$ (c) $\frac{2x+1}{x}+\frac{3x-3}{3x}$

7 Find the shaded area of the following shapes correct to two decimal places. Ex.3.1

(a)



(b)



8 Find the total surface area, correct to two decimal places, of a shape consisting of a cube of side length 7 cm with a pyramid of perpendicular height 4 cm sitting on top of it. Ex.3.4

9 Find the volume of a square pyramid with base side length 12 cm and perpendicular height 15 cm. Ex.3.5

10 A metal sphere has a volume of 12 cm^3. If its mass is 76.8 g, what is its density? Ex.3.6 10A

11 Solve each of the following equations. Ex.4.1

(a) $\frac{2x+3}{7}=3$ (b) $\frac{4(3x-2)}{5}=4$ (c) $2(4x+5)=26$

(d) $7(2x-1)=21$ (e) $6x+2=7x$ (f) $8x-9=7x-4$

(g) $4(x+5)=2x-4$ (h) $\frac{2(2x-4)}{3}=4$

12 Calculate the gradient of the line that passes through each pair of coordinates. Ex.4.2

(a) (-3, 0) and (3, 9) (b) (-4, -1) and (-2, 5)

(c) (0, 6) and (4, -2) (d) (-3, 2) and (-1, 6)

13 Identify which of the following points lie on each given line? Ex. 4.3

(a) (-2, -1) or (2, 8) $y = 3x + 5$

(b) (1, -1), (-2, -3) or (-3, 5) $y = -2x + 1$

(c) (0, -2), (4, 1) or (-8, 0) $y = \frac{1}{4}x + 2$

14 Graph each of the following lines using either the gradient and y-intercept method or the x- and y-intercepts method. Ex. 4.3

(a) $y = -2x + 1$ **(b)** $y = 3x + 3$ **(c)** $y = \frac{1}{4}x - 4$ **(d)** $y + 6x = 9$

15 Determine whether each pair of lines is perpendicular. Write 'Yes' or 'No' and justify your answer with appropriate calculations. Ex. 4.4

(a) $2x + 4y = 8$ and $6x = 3y - 3$ **(b)** $2x - y = 3$ and $x + 2y = 8$

(c) $5x + 2y = 8$ and $5y = 2x + 10$ **(d)** $3y = 4x - 2$ and $4y = 3x - 4$

16 Determine whether each pair of lines is parallel. Write 'Yes' or 'No' and justify your answer with appropriate calculations. Ex. 4.4

(a) $6x - 3y = 9$ and $3x = 3y - 3$ **(b)** $x - y = 8$ and $x + y - 2 = 0$

(c) $2x + 5y = 7$ and $5y = 2x - 8$ **(d)** $10y = 3x + 2$ and $6x - 20y + 4 = 0$

17 Solve the following inequalities. Ex. 4.5

(a) $2x - 1 \leq 4x + 2$ **(b)** $\frac{4x - 1}{3} > \frac{3x + 5}{2}$

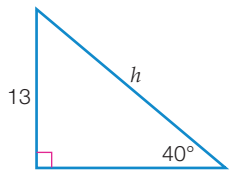
18 Solve the following simultaneous equations. Ex. 4.6

(a) $3x - 5y = 7$
$x + 5y = -11$

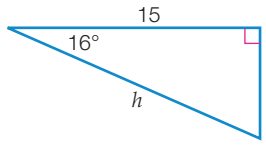
(b) $2x - 3y = 17$
$y = 3x - 22$

19 Find the values of the pronumerals, correct to two decimal places. Ex. 5.2

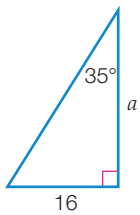
(a)



(b)



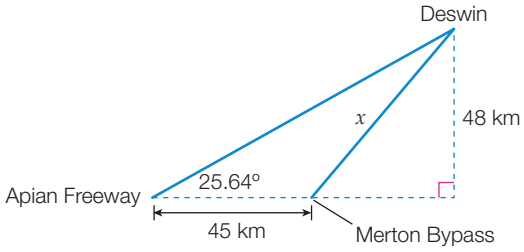
(c)



20 A man standing 8 m from a flagpole looks up to the top of it. Through what angle of elevation does he look (to two decimal places) if the top is 16 m higher than his eye-level? Ex. 5.4

21 A scout walks 3 km on a true bearing of 050°T and then 5 km on a true bearing of 077°T. What is the scout's distance from the starting position (to two decimal places)? Ex. 5.5

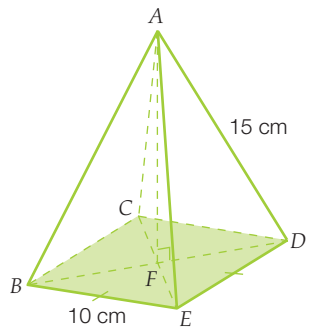
22 A proposed section of highway that was to connect the Apian Freeway to the town of Deswin will now connect the Merton Bypass to Deswin as shown. What will be the length, x km, of the new section of highway? Write your answer to the nearest kilometre. Ex. 5.6



23 A right pyramid *ABCDE* stands on a square base of edge length 10 cm. The sloping edges are each 15 cm long. *AF* is the perpendicular height.

(a) Find, correct to one decimal place, the angle that a sloping edge makes with the base of the pyramid.

(b) Hence, find the height of the pyramid, correct to one decimal place.

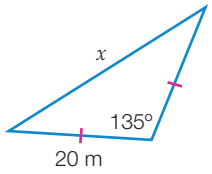


Ex 6.1 10A

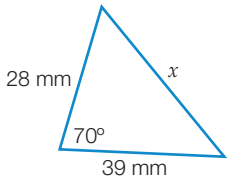
24 Find the value of x in the following triangles correct to two decimal places.

Ex 6.5 10A

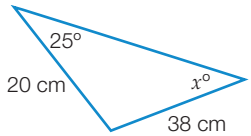
(a)



(b)



(c)



25 Find the area of each of the triangles in Question **24** correct to two decimal places.

Ex 6.6 10A

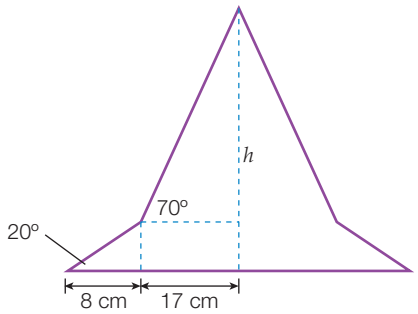
Understanding

26 The height of a particular young pine tree is found to increase in a linear manner each month in the first few years after planting. What was the initial height of the tree, if after 2 months the height is 34 cm, and after 5 months the height is 52 cm?

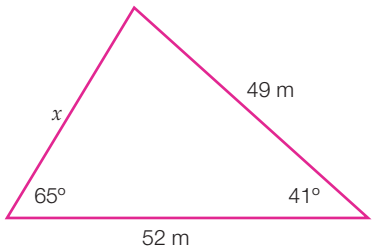
27 The sides of a rectangle differ by 4 m. If all sides are increased by 2 m, the area increases by 52 m^2. What are the dimensions of the original rectangle?

28 Determine whether the following points are collinear: $A(1, -1)$, $B(3, 3)$, $C(6, 9)$.

29 A witch's hat, used on roads to divert traffic, rises from the outside of the brim at an angle of 20° for a horizontal distance of 8 cm then rises to an apex at an angle of 70° for a further horizontal run of 17 cm. Find h, the height of the witch's hat, correct to two decimal places.



30 In the diagram given, the value of x cannot be found using which of the following equations. 10A



A $\dfrac{x}{\sin 41°} = \dfrac{49}{\sin 65°}$

B $\dfrac{x}{\sin 41°} = \dfrac{52}{\sin 74°}$

C $x^2 = 52^2 + 49^2 - 2 \times 52 \times 49 \cos 41°$

D $\dfrac{x}{\sin 41°} = \dfrac{52}{\sin 65°}$

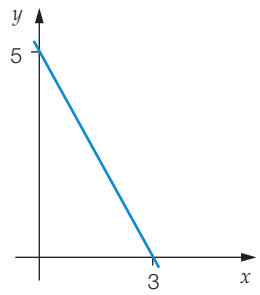
Reasoning

31 Prove that a right-angled isosceles triangle is produced by connecting the points $A(-1, 3)$, $B(5, -1)$ and $C(0, -2)$ and then find its area.

32 Prove that a rectangle is produced by connecting the points $A(-6, 0)$, $B(3, -6)$, $C(5, -3)$ and $D(-4, 3)$ and then find its area?

33 A quadrilateral has the angles $x°$, $(2x + 3)°$, $(2x - 3)°$ and $(x + 6)°$. Find the size of each angle.

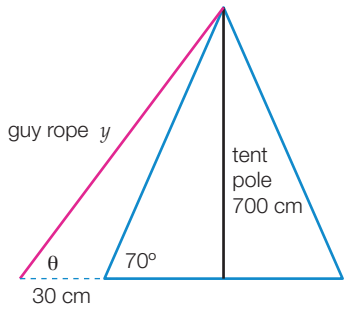
34 Find the equation of the line that is perpendicular to the line shown on the diagram, such that the perpendicular line passes through the point (-2, 3).



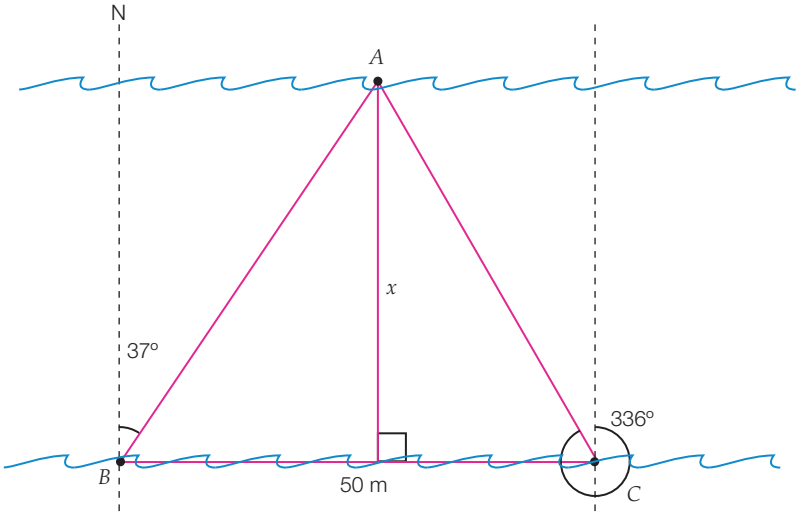
35 The sides of a tent of height 700 cm make an angle of 70° with the ground. A guy rope used to hold the tent pole upright is secured 30 cm from the base of the tent.

(a) Find the angle it makes with the ground correct to two decimal places.

(b) Find the length of the guy rope to the nearest centimetre.

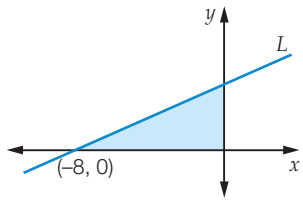


36 A point A on one side of a river is at a bearing of 037°T from point B on the other side of the river. From point C, 50 m due east of point B, the bearing of point A is 336°T. How wide is the river, in metres, correct to two decimal places?



Challenge B

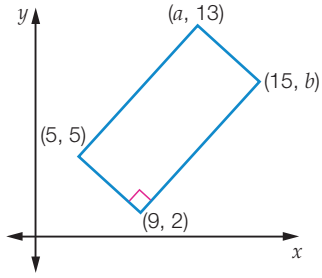
1 The line L crosses the x-axis at (-8, 0). The area of the shaded region is 16. What is the slope of the line L?



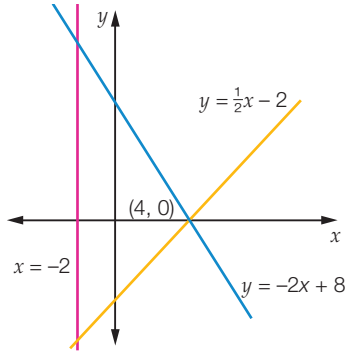
A $-\frac{1}{2}$ **B** $\frac{1}{2}$

C 2 **D** 4

2 In the rectangle shown, what is the value of $a - b$?



3 The lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ meet at (4, 0), as shown. The area of the triangle formed by these two lines and the line $x = -2$ is:



A 15 **B** 27

C 36 **D** 45

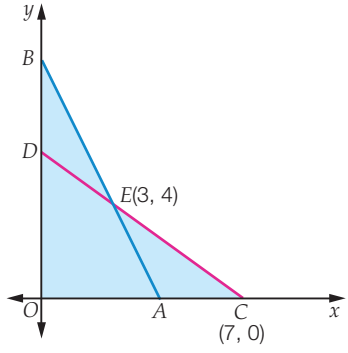
4 ΔABC has vertices $A(0, 0)$, $B(9, 0)$ and $C(0, 6)$. The points P and Q lie on the side AB such that $AP = PQ = QB$. Similarly, the points R and S lie on the side AC so that $AR = RS = SC$. The vertex C is joined to each of the points P and Q. The vertex B is joined to R and S.

(a) Find the equation of the line through the points R and B.

(b) Find the equation of PC.

(c) The line segments PC and RB intersect at X, and the line segments QC and SB intersect at Y. Show, by calculation, that the points A, X and Y lie on the same straight line.

5 A line with slope -3 intersects the positive x-axis at A and the positive y-axis at B. A second line intersects the x-axis at $C(7, 0)$ and the y-axis at D. The lines intersect at $E(3, 4)$.



(a) Find the slope of the line through C and E.

(b) Find the equation of the line through C and E, and the coordinates of the point D.

(c) Find the equation of the line through A and B, and the coordinates of the point B.

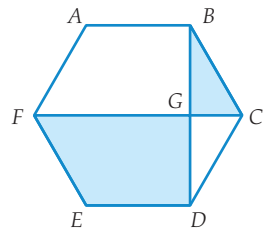
(d) Determine the area of the shaded region.

6 The point Y lies on the line $y = 5x + 3$. The coordinates of the point Z are (3, -2). If X is the midpoint of YZ, then X must lie on the line:

A $y = \frac{5x}{2} - \frac{7}{2}$ **B** $y = 5x + 1$ **C** $y = \frac{5x}{2} + \frac{1}{2}$ **D** $y = 5x - 7$

Challenge B

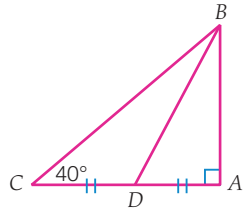
7 In the regular hexagon *ABCDEF*, the lines *BD* and *CF* intersect at *G*. The ratio of the area of quadrilateral *DEFG* to ΔBCG is:



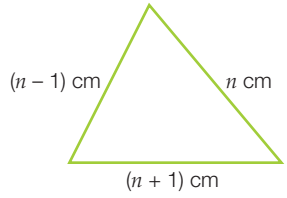
A $3\sqrt{3}:1$ **B** $5:1$

C $4:1$ **D** $2\sqrt{3}:1$

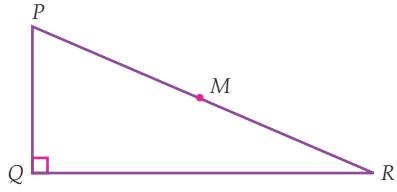
8 A flagpole *AB* is 20 metres tall. From a point *C* on flat ground, the angle of elevation to the top of the flagpole is 40°. If $CD = DA$, calculate the size of $\angle ADB$ rounded to the nearest degree.



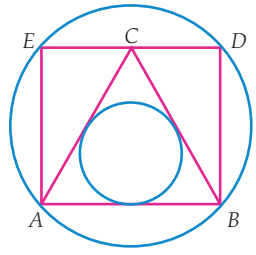
9 Find the area of this triangle if *n* is a whole number and the area of the triangle is a whole number of square centimetres. All the sides are between 10 cm and 20 cm. You may wish to use Heron's formula for the area of a triangle $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ and *a*, *b* and *c* are the sides of the triangle.



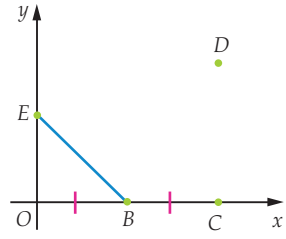
10 Triangle *PQR* is right-angled at *Q* and has side lengths $PQ = 14$ and $QR = 48$. *M* is the midpoint of *PR*. Draw the interval *MS*, where *S* is on *PQ*, such that $MS \parallel RQ$. Find the value of the cosine of $\angle MQP$ correct to two decimal places.



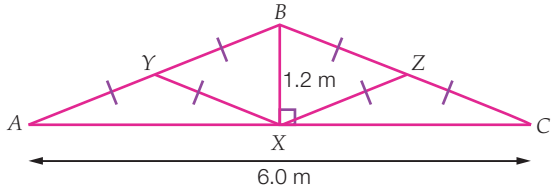
11 ΔABC is an equilateral triangle. The radius of the circle just touching the sides of *ABC* is 1 unit. A larger circle is drawn through the vertices of the rectangle *ABDE*. Calculate the diameter of the larger circle.



12 In the diagram, *E* has coordinates (0, 2), and *B* lies on the positive *x*-axis so that $BE = 7$. The point *C* lies on the positive *x*-axis so that $BC = OB$. If *D* lies in the first quadrant such that $\angle CBD = 30°$ and $\angle BCD = 90°$, what is the length of *ED* correct to two decimal places?



13 The diagram shows a roof truss. Triangle *ABC* is isosceles with $AB = BC$. The midpoints of *AB* and *BC* are *Y* and *Z* respectively. $AC = 6.0$ m and $BX = 1.2$ m. Calculate the length of *AB* correct to one decimal place. Find the total length of timber required to make the truss.



Calculate the size of $\angle BAX$ and hence, find the size of $\angle BYX$. Round both angles to the nearest degree.

Statistics

7

Maths boosts sales

Name: Adam Driussi
Job: Director of The Quantium Group, an actuarial consulting firm
Qualifications: Bachelor of Economics (Actuarial Studies), Fellow of the Institute of Actuaries of Australia and Graduate Diploma in Applied Finance and Investment

What do you do with a razor sharp analytical mind and great communication skills? You become an actuary!

Actuaries analyse large quantities of data and provide strategic advice to help businesses solve a wide range of practical problems. Adam Driussi is director of an actuarial consulting firm. He says, "I love coming to work every day and having to solve a different problem to the day before. The fact that you tend to get paid very well to do it is a very nice bonus."

As an actuary, Adam and his team might analyse a client's customer database. Based on what they discover they will make a suggestion as to how a business can market to their customers more effectively. Adam explains: "For example, eBay communicates with most of its customers via email. So we analysed the data to help eBay develop email campaigns that promoted new items that are relevant to each of eBay's millions of customers—as opposed to promoting the same items to every customer. The result was more than double the number of people responding to the campaigns and buying new items from eBay."

Adam loved maths at school. "My maths skills form the foundation for the problem solving and logic that I need to do my job every day. Without that foundation you could never solve the problems you get posed as an actuary."

Why learn this?

We live in the 'Information Age', but we need to organise and analyse the data to determine if we can draw conclusions or use it to make decisions. Statistics are used by businesses and the media to give credibility to ideas they would like you to believe are true. A knowledge of the processes used to gather and present data will help you distinguish credible fact from unlikely fiction.

After completing this chapter you will be able to:

- find five-figure summaries and use them to draw box plots
- use parallel box plots as one way to compare data sets
- construct and comment on data displays involving bivariate data where the independent variable is time
- interpret and analyse data found in the media
- calculate the mean and standard deviation as a pair of values used to compare data sets 10A
- draw and interpret lines of best fit. 10A

Recall 7

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

Worksheet R7.1

1 Find the mean, median (middle number), mode (most common number) and range (difference between largest and smallest numbers) for each of these sets of data.

(a) The number of siblings for your five best friends:
2 4 5 1 4

(b) The shoe size of your nine young cousins:
3.5 4.5 5.0 5.5 5.0 4.5 5.5 5.0 3.5

Worksheet R7.2

2 Classify each of the following as being nominal, discrete or continuous data.

(a) Number of goals scored by a hockey team each week.

(b) Height of the members of your maths class.

(c) Size of blouses on a rack.

Worksheet R7.3

3 The heights (in centimetres) of 20 students are as follows:

179.7 167.3 168.6 164.7 182.8 160.5 179.2 163.0 177.7 171.3
169.8 173.2 164.7 172.1 165.6 171.9 164.8 166.7 169.1 166.7

Round off the heights to the nearest cm and:

(a) using intervals of 5 cm, draw a stem plot

(b) find the median.

Worksheet R7.4

4 The distance jumped, in cm, by each competitor in the long jump was recorded and is shown in the frequency table.

(a) Find an estimate for the value of the mean.

(b) In which class interval does the median fall?

Class interval (kg)	Frequency
5.75–<6.00	4
6.00–<6.25	6
6.25–<6.50	9
6.50–<6.75	13
6.75–<7.00	5
7.00–<7.25	3

Key Words

bivariate data	interquartile range	percentile
box plot	ogive	primary data
cumulative frequency curve	outlier	quantile
dependent variable	parallel box plot	range
five-number summary	parallel dot plot	scatter plot
independent variable	percentage cumulative frequency curve	secondary data
10A		
extrapolation	line of best fit	standard deviation
interpolation	regression line	variance

Cumulative frequency curves

7.1

In previous years you have learnt how to find measures of centre, mean and median for data that is grouped or ungrouped and discrete or continuous. Here we will find the median of large sets of data that are discrete or continuous. The method involves drawing a **cumulative frequency curve**, which is sometimes also called an **ogive**. As the name suggests this curve is based on the frequency of data. It can be used to find the number of results that are less than a given value and so enables us to find the value below which a given proportion of results lie.

The median of n results is the $\left(\frac{n+1}{2}\right)$th result.

We need to learn these methods so we can deal with data that is presented to us in grouped data form. Wherever practical we would prefer to use the original data set rather than group the data, as the grouped data values will not be as accurate as those obtained from the original data.

Worked Example 1

WE1

The following frequency table lists the weights of 50 packing crates, measured in kilograms. Find the median by constructing a cumulative frequency curve.

Class interval (kg)	Frequency
45–<50	3
50–<55	13
55–<60	8
60–<65	10
65–<70	4
70–<75	12

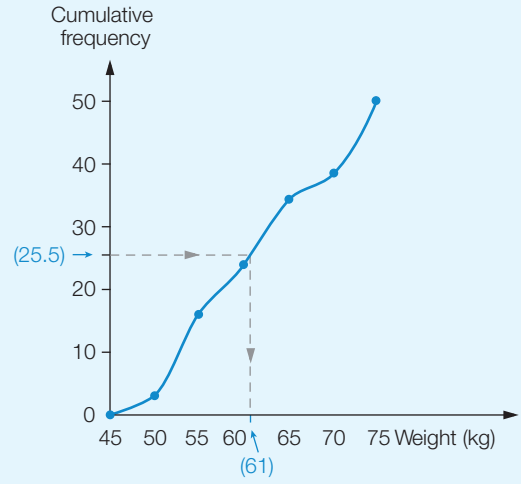
Thinking

Working

1 Use the data grouped in class intervals to construct a cumulative frequency table. In the cumulative frequency column, the first interval always begins at zero and the values for each interval are found by adding together all of the values in the previous intervals. Note that the class interval is now less than a given value—this has changed from a frequency table.

Class interval (kg)	Cumulative frequency
<45	0
<50	3
<55	16
<60	24
<65	34
<70	38
<75	50

2 Plot the cumulative frequency curve. Choose an appropriate scale for both axes.



3 The median of n results is the $\left(\frac{n+1}{2}\right)$th result. There are 50 results, so $n = 50$.

The median $= \left(\frac{50+1}{2}\right)$th

$= 25.5$th result

4 From 25.5 on the cumulative frequency scale, draw a horizontal line to the curve, then a vertical line from this point of contact down to the weight axis. The value obtained from the weight axis is the median.

The median is about 61 kg.

The method used in the example above is not as accurate as obtaining the median directly from the raw, ungrouped data, but for high frequencies this approach is convenient and the error is relatively insignificant.

Discrete data can also be grouped and a cumulative frequency graph can be drawn.

Worked Example 2

WE2

The following table shows the data on the number of components produced during a shift. Find the median by constructing a cumulative frequency curve.

Class interval	Frequency
220–259	4
260–299	9
300–339	10
340–379	7
	$\Sigma f = 30$

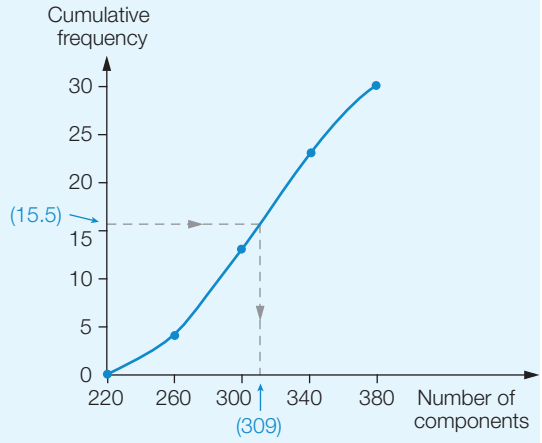
Thinking

Working

1 Draw up the cumulative frequency table. Find the number of results <220, <260, <300 etc.

Number of components	Cumulative frequency
<220	0
<260	4
<300	13
<340	23
<380	30

2 Draw a cumulative frequency curve.



3 Calculate the term required to find the median, i.e. the $\left(\frac{n+1}{2}\right)$th result.

Median $= \left(\frac{30+1}{2}\right)$th result

$= 15.5$th result

4 Read off the result from the graph.

The median is about 309 components per shift.

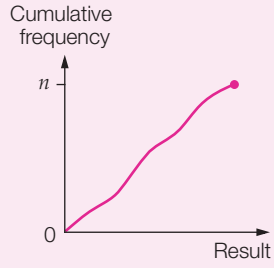
Check that the last value in the cumulative frequency column is the same as the sum of the frequencies (Σf) for both continuous and discrete data.

To find the median of data grouped in class intervals, follow these steps:

1 Construct a cumulative frequency table.

2 Plot the cumulative frequency curve.

3 Find the median by reading the $\left(\frac{n+1}{2}\right)$th result from the curve.



7.1 Cumulative frequency curves

Navigator

Q1, Q2, Q3, Q4, Q5, Q7	Q1, Q2, Q4, Q5, Q6, Q7	Q1 Column 2, Q2, Q4, Q5, Q6, Q7

Answers page 746

Fluency

1 Find the median for each of the following grouped data sets by constructing a cumulative frequency curve.

WE1

(a)

Class interval	Frequency
0–<10	4
10–<20	5
20–<30	8
30–<40	11
40–<50	7

(b)

Class interval	Frequency
8–<10	1
10–<12	4
12–<14	6
14–<16	9
16–<18	7

(c)

Class interval	Frequency
60–<70	9
70–<80	12
80–<90	19
90–<100	17
100–<110	13

(d)

Class interval	Frequency
250–<300	37
300–<350	58
350–<400	26
400–<450	16
450–<500	9
500–<550	4

WE2

2 **(a)** The frequency table on the right shows the number of students present each day at a particular school during first term. Find the median number of students present by constructing a cumulative frequency curve.

Class interval	Frequency
450–459	2
460–469	7
470–479	8
480–489	15
490–499	10
500–509	8

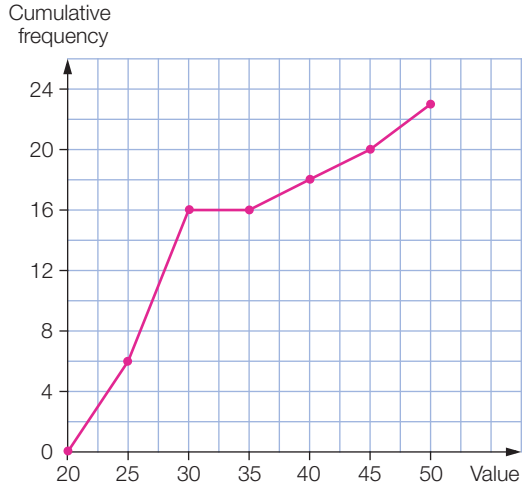
(b) The frequency table on the right shows the number of points scored by each team in one round of a rugby competition. Find the median score by constructing a cumulative frequency curve.

Class interval	Frequency
0–<10	2
10–<20	3
20–<30	3
30–<40	4
40–<50	1
50–<60	1

3 The frequency table on the right lists the lengths of fish caught, measured in centimetres. Find the median length by constructing a cumulative frequency curve.

Class interval (cm)	Frequency
30–<35	7
35–<40	10
40–<45	5
45–<50	12
50–<55	13
55–<60	4

4 The following questions relate to the cumulative frequency graph shown on the right.



(a) The number of values in the data set is:

A 23 **B** 24

C 26 **D** 50

(b) The median is closest to:

A 12 **B** 27

C 28 **D** 29

(c) The class interval that has a frequency of zero is:

A 20–<25 **B** 30–<35

C 30–<36 **D** 31–<36

Understanding

5 The number of people boarding each of 25 successive trains at a station was as follows:

6	16	21	29	39	46	42	29	21	29	35	21	17
10	8	18	17	26	7	9	12	19	26	20	21	

(a) Find the median of this raw data.

(b) Group the data into the intervals 1–10, 11–20 etc. to draw a cumulative frequency curve and find the median.

(c) On how many trains was the number of people boarding less than 18? Compare the answer obtained from the raw data with the one obtained using a cumulative frequency curve.

Reasoning

6 There is another way to estimate the median for grouped data. It uses the likely location of the median within the class interval in which the median lies. If we know that the median is the third of eight results in the class interval 20–<25, we say that the median is approximated by the value $20 + \frac{3}{8} \times 5 \approx 21.9$. We would be unlikely to go beyond even one decimal place in a calculation such as this.

(a) Use this method to find an approximation for the median for Question **1(b)**.

(b) Compare this with the value obtained using the cumulative frequency curve.

(c) Use this method to find an approximation for the median for Question **2(a)**.

(d) Compare this with the value obtained using the cumulative frequency curve.

(e) What do you think about this alternative method? List some good features and some bad features of the method.

Open-ended

7 Use the following information to construct a discrete data set with at least 19 results and then calculate the mean.

(a) median = 39, odd number of results

(b) median = 72.5, even number of results

Outside the Square — Problem solving

Product codes

Most products that you buy have a barcode attached to them. This barcode represents the European Article Number (EAN) and is a 13-digit unique code for that product. As part of the checking system to ensure the correct numbers are scanned by a barcode reader, the 13th digit is actually a check digit.

The following are all legal product codes:

9316090060604	9337090060609
9326090060603	9338090060606
9336090060602	9339090060603

1 In the first column of barcodes, apart from the check digit, which is the only digit that changes? What do you notice about the sum of the two digits that are changing?

2 Predict the final digit in the code 934609006060*.

3 In the second column of barcodes, apart from the check digit, which is the only digit that is changing? Do these digits follow the same pattern as those in the first column of barcodes?

Not every column acts in the same way. In fact the even columns all act in the same way and the odd columns all act in the same way. We will place the very first EAN-13 into a table to show what happens.

× 1	× 3	× 1	× 3	× 1	× 3	× 1	× 3	× 1	× 3	× 1	× 3	× 1
9	3	1	6	0	9	0	0	6	0	6	0	4

4 Calculate the sum of these multiplied values.

5 Do the same for each of the EAN-13 codes given above. What conclusion can you reach?

6 Now calculate the final digit in each of the following EAN-13 codes:

(a) 933919006060*

(b) 933929006060*

(c) 933928006060*

(d) 933927006060*

Strategy options

- Work backwards.
- Test all possible combinations.

7.2 Box plots

Percentiles and quantiles

Finding the median is not the only thing we can find from a cumulative frequency curve. By converting grouped data to a proportion (percentage) and drawing a **percentage cumulative frequency curve** we can find the value below which a percentage of the results occur. For example, we can find the value below which 70% of the results lie. We call this value the 70th **percentile** or, written as a decimal, the 0.7 **quantile**.

Worked Example 3 WE3

The following table shows the data in class intervals from a survey of the weights of packing crates. Use the data to:

(a) draw a cumulative frequency table and then a percentage cumulative frequency table

(b) draw a percentage cumulative frequency curve

(c) evaluate the 80th percentile and the 0.3 quantile.

Weight (kg)	Frequency
45–<50	2
50–<55	3
55–<60	3
60–<65	6
65–<70	4
70–<75	2

Thinking | **Working**

(a) 1 Complete a cumulative frequency table.

(a)

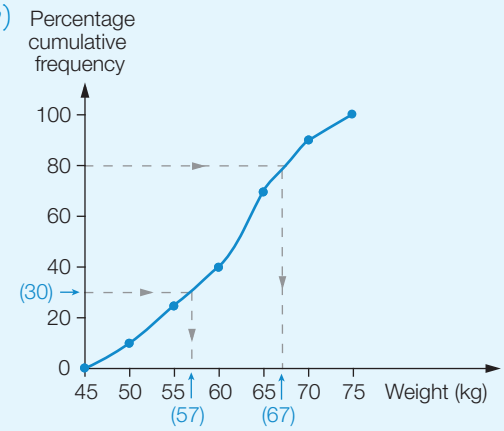
Weight (kg)	Cumulative frequency
<45	0
<50	2
<55	5
<60	8
<65	14
<70	18
<75	20

2 Draw up a table of percentage cumulative frequencies. Each percentage cumulative frequency is obtained by dividing the cumulative frequency by the total number of results (20) and multiplying by 100.

Weight (kg)	Cumulative frequency	Percentage cumulative frequency
<45	0	$\frac{0}{20} \times 100 = 0$
<50	2	$\frac{2}{20} \times 100 = 10$
<55	5	$\frac{5}{20} \times 100 = 25$
<60	8	$\frac{8}{20} \times 100 = 40$
<65	14	$\frac{14}{20} \times 100 = 70$
<70	18	$\frac{18}{20} \times 100 = 90$
<75	20	$\frac{20}{20} \times 100 = 100$

(b) Plot the percentage cumulative frequency against weight. The percentage cumulative frequency curve has a similar shape to that of the cumulative frequency curve, but it has a different vertical scale.

(b)



(c) In the graph of the percentage cumulative frequency curve, find the 80% point on the vertical axis. Draw a line horizontally from this point until it meets the graph. Then draw a line vertically down and read off the result. (See the dashed lines drawn on the graph.) Similarly, convert the 0.3 quantile to a percentage ($0.3 \times 100 = 30\%$) and then find the 30% point on the vertical axis.

(c) The 80th percentile is 67 kg.

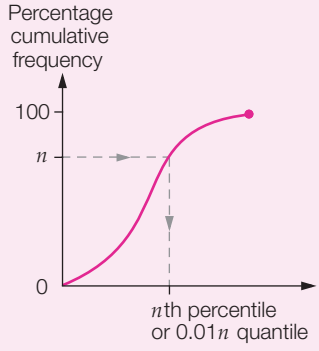
The 0.3 quantile is the 30% mark and is 57 kg.

A cumulative frequency is the number of results below a given value.

$$\text{Percentage cumulative frequency} = \frac{\text{cumulative frequency}}{\text{total number of results}} \times 100\%$$

For any percentage cumulative frequency graph, n% of results lie below the nth percentile.

Percentile and quantile values can be read from a percentage cumulative frequency graph. The nth percentile (or $0.01n$ quantile) is the value on the horizontal axis corresponding to n% on the vertical axis, where $0 \le n \le 100$.



Range and interquartile range

You have seen previously that mean and median are measures of the location (or central tendency) of a set of data. How accurately these statistics describe location depends on the variation in the results, or in other words, on how spread out they are.

There are three commonly used measures of spread (or variation): **range**, **interquartile range** and standard deviation. Standard deviation is related to the mean and is covered in Section 7.7.

Measures of spread

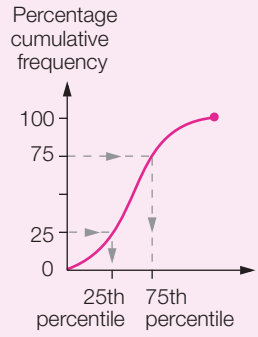
Range is the difference between the largest and smallest results in a set of data.

Range = maximum – minimum

Interquartile range is the difference between the upper quartile (Q_U) and the lower quartile (Q_L).

Interquartile range can be found by reading values from a percentage cumulative frequency curve.

$\text{IQR} = Q_U - Q_L$
$= \text{75th percentile} - \text{25th percentile}$



If you have the raw data set then the interquartile range can be found by breaking the data into two equal subsets. The lower quartile is the median of the lower half of the data and the upper quartile is the median of the upper half of the data.

The quartiles can be written as Q_1 and Q_3.

In summary:

lower quartile = $Q_L = Q_1$ = 25th percentile = 0.25 quantile

median = Q_2 = 50th percentile = 0.50 quantile

upper quartile = $Q_U = Q_3$ = 75th percentile = 0.75 quantile

You have previously learnt how to find the interquartile range of a set of discrete data. Now we will extend our understanding of interquartile range and learn to apply it to grouped data. We then apply our knowledge of calculating median, range and interquartile range to draw box plots.

Calculating interquartile range and drawing box plots

The following is a list of the heights (cm) of the players on the 2011 playing list for Collingwood in the AFL.

177	193	189	184	193	193	176	188	191	174	183	185	181	194	195	186
200	204	195	185	180	204	176	191	183	190	181	204	187	193	196	181
189	181	185	172	184	189	189	187	192	194	185	193	192	183	191	204

The first step in finding the median, quartiles, range and interquartile range is to put the data in order from smallest to largest. When this is done we have the following data list.

172	174	176	176	177	180	181	181	181	181	183	183	183	184	184	185
185	185	185	186	187	187	188	189	189	189	189	190	191	191	191	192
192	193	193	193	193	193	194	194	195	195	196	200	204	204	204	204

From this list we can immediately read off:

$$\text{range} = 204 - 172 = 32$$

To find the median we need to identify the middle value, or values. There are 48 values so the median is between the 24th and 25th values. These have been circled in the ordered data list.

$$\text{median} = \frac{189 + 189}{2} = 189$$

To find the lower quartile we look at the lower half of the ordered data set, or the first 24 values.

172	174	176	176	177	180	181	181	181	181	183	183	183
184	184	185	185	185	185	186	187	187	188	189		

Here there are 24 values so the lower quartile, the median of this lower half, is between the 12th and 13th values. Again, these have been circled.

$$\text{lower quartile} = \frac{183 + 183}{2} = 183$$

To find the upper quartile we look at the upper half of the ordered data set, or the last 24 values.

189	189	189	190	191	191	191	192	192	193	193	193	193
193	194	194	195	195	196	200	204	204	204	204		

Here there are 24 values so the upper quartile, the median of this upper half, is between the 12th and 13th values shown, which is the 36th and 37th values of the full ordered list. Again, these have been circled.

$$\text{upper quartile} = \frac{193 + 193}{2} = 193$$

From this we can calculate:

$$\begin{aligned}\text{interquartile range} &= \text{upper quartile} - \text{lower quartile}\\ &= 193 - 183\\ &= 10\end{aligned}$$

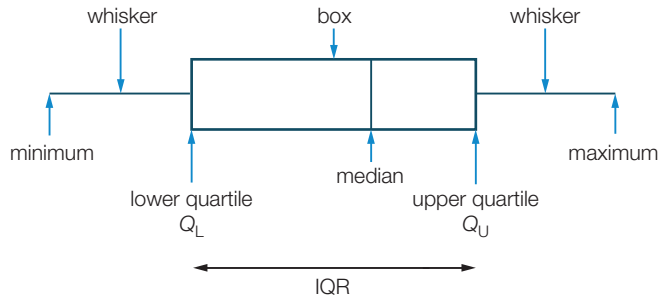
Calculating interquartile range and drawing box plots

When frequencies are large, or we are using grouped secondary data, the interquartile range can be obtained using the percentage cumulative frequency curve to find Q_U and Q_L. Q_U is the 75th percentile (*upper quartile*) and Q_L is the 25th percentile (*lower quartile*). The upper quarter of results occur above the upper quartile, whereas the lower quarter of results occur below the lower quartile. Note that the median may be found on the percentage cumulative frequency curve by evaluating the 50th percentile.

A **box plot**, also known as a box-and-whisker plot, is a graphical summary of five key pieces of data known as the **five-number summary**:

- minimum value
- lower quartile Q_L (25th percentile)
- median (50th percentile)
- upper quartile Q_U (75th percentile)
- maximum value

The central part of the diagram, between the two quartiles, is the box, and the single lines extending to the minimum and maximum values are the whiskers.



Worked Example 4

WE4

The weights of 20 packing crates (kg) are shown in the frequency table.

(a) Calculate the range and interquartile range.

(b) Draw a box plot to represent the data.

Weight (kg)	Frequency
45–<50	2
50–<55	3
55–<60	3
60–<65	6
65–<70	4
70–<75	2

Thinking

(a) 1 Range can be calculated by taking the smallest result from the largest. (We need to estimate these values by using the end points.)

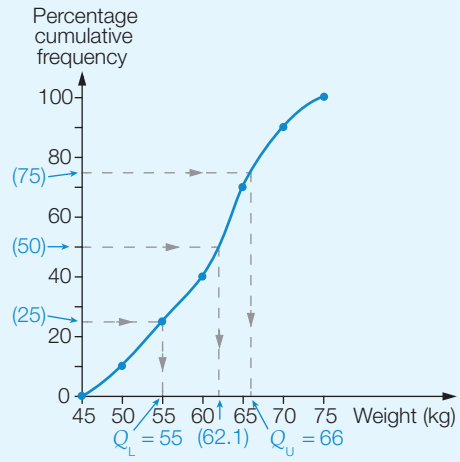
2 Construct a percentage cumulative frequency table.

Working

(a) Range = largest result − smallest result
= 75 − 45
= 30

Weight (kg)	Cumulative frequency	Percentage cumulative frequency
<45	0	$\frac{0}{20} \times 100 = 0$
<50	2	$\frac{2}{20} \times 100 = 10$
<55	5	$\frac{5}{20} \times 100 = 25$
<60	8	$\frac{8}{20} \times 100 = 40$
<65	14	$\frac{14}{20} \times 100 = 70$
<70	18	$\frac{18}{20} \times 100 = 90$
<75	20	$\frac{20}{20} \times 100 = 100$

3 Draw a percentage cumulative frequency curve.



4 Read off Q_U (75th percentile) and Q_L (25th percentile).

$Q_U = 66$
$Q_L = 55$

5 Calculate the interquartile range (IQR).

$IQR = 66 - 55$
$= 11$

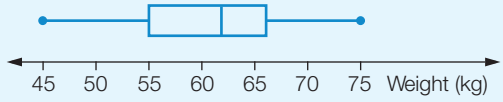
6 State the answers.

The range of weights is 30 kg and the interquartile range is 11 kg.

(b) 1 Read the 50th percentile for the median.

(b) median = 62.1

2 Draw a box plot using the median and the information found in part **(a)**.



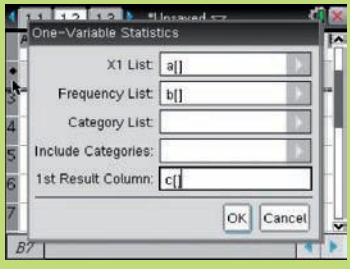
Using technology to find the five-number summary

We can use technology, such as a scientific calculator, a CAS or a spreadsheet, to find the five-number summary whether we have the raw data list or a grouped data table. We will look in some detail at how a CAS can be used to find these values. The grouped data from Worked Example 4 will be used for this demonstration.

Using the TI-Nspire CAS

Go to a new Lists & Spreadsheet screen and enter the class midpoints in column A. (The midpoint of 45–<50 is $\frac{45+50}{2} = 47.5$) Then enter the frequencies in column B.

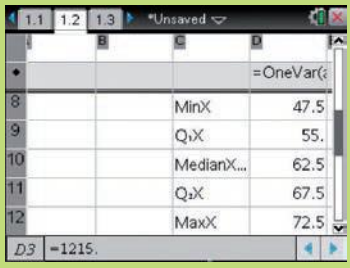
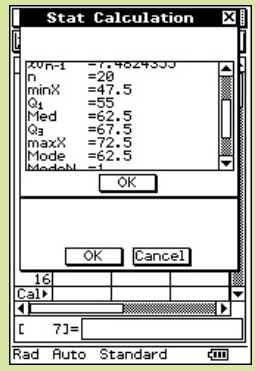
Press (menu) > Statistics > Stat Calculations > One-Variable Statistics > (enter) and press (enter) to confirm there is only one list of numbers. Make the dialog screen look like the following.



Using the ClassPad

From Main choose Statistics and enter the class midpoints in list1. (The midpoint of 45–<50 is $\frac{45+50}{2} = 47.5$) Then enter the frequencies in list2.

Then tap Calc > One-Variable and fill in the dialog box as shown. Use the dropdown menus to get the lists required.

Using the TI-Nspire CAS	Using the ClassPad
(tab) down to highlight OK and press (enter). A whole list of values has now appeared on the screen. If you scroll down you can see the following values. 	Tap OK and you can then see a list of values on the screen. If you scroll down you can see the following values. 
We can see that the values obtained are close to, but not exactly the same as, the values obtained from the graph.	We can see that the values obtained are close to, but not exactly the same as, the values obtained from the graph.

If we had the raw data we could just enter each value individually in the first column.

Outliers

When one or more of the surveyed results lie outside a certain range they may be considered as **outliers**. In any particular data set an outlier may be smaller than most of the other data values, or it might be bigger than most of the other data values, or there might be outliers at both ends of the data range. The identification of an outlier is a subjective thing. Unless you have good reason to believe that the potential outlier is the result of error in the collection process or perhaps the result of a poor definition of what should be included in the data set, then the value should be included in your calculations. In general, the median is not affected by extreme values, whereas the mean can suffer slightly depending on how far away the outlier falls from the rest of the data set. It is important to acknowledge that a data set may contain outliers and to comment on their possible origin.

Outliers may arise from a number of circumstances. For example, equipment failure on an assembly line in a factory may affect the number of components produced per shift, or an injured football player on a given day may obtain fewer kicks than his season's average.

If you draw a graph, such as a dot plot, you may be in a better position to decide whether or not a value is an outlier.

When you use a CAS to draw a box plot you are given the option of having outliers marked in a special way. To enable this to occur a mathematical formula is used to determine which values are outliers. Using this formula, a value is an outlier if it is more than $1.5 \times \text{IQR}$ above Q_U or more than $1.5 \times \text{IQR}$ below Q_L. If you wanted to check your own calculations in this way then you would need to calculate the interquartile range.

Worked Example 5 WE5

Consider the following results from a survey of the number of cans of baked beans produced on an assembly line per hour:

244 246 247 248 242 24 248 248 247 247 246 244 242 242 246

Identify the outlier and suggest a possible reason for it.

Thinking	Working
1 Scan the data list. Does any value stick out as far removed from the other values?	The result 24 is more than 200 less than the next smallest result; it is clearly an outlier.
2 Give possible reasons for the outlier.	Machine breakdown, operator error, operator injury or result incorrectly recorded could all be reasons for this outlier.

Worked Example 6

WE6

Determine if there are any outliers in this list of the weights of 20 packing crates (kg):

60.4	53.9	61.1	62.3	50.4	57.3	45.7	59.4	68.6	65.5
60.8	67.9	57.6	62.9	65.8	70.0	73.7	49.6	54.8	62.7

Thinking	Working
1 The data list needs to be ordered.	Ordered data list: 45.7 49.6 50.4 53.9 54.8 57.3 57.6 59.4 60.4 60.8 61.1 62.3 62.7 62.9 65.5 65.8 67.9 68.6 70.0 73.7
2 Scan the ordered data list to see if any values stick out as potential outliers.	The data list seems relatively evenly spread out; no outlier is obvious.
3 Calculate the IQR by finding the upper and lower quartiles.	$Q_L = \frac{54.8 + 57.3}{2} = 56.05$ $Q_U = \frac{65.5 + 65.8}{2} = 65.65$ $IQR = Q_U - Q_L$ $= 65.65 - 56.05$ $= 9.6$
4 Calculate the cut-off values for the outliers (more than $1.5 \times IQR$ above Q_U or more than $1.5 \times IQR$ below Q_L).	$1.5 \times IQR = 1.5 \times 9.6$ $= 14.4$ $Q_L - 1.5 \times IQR = 56.05 - 14.4$ $= 41.65$ $Q_U + 1.5 \times IQR = 65.65 + 14.4$ $= 80.05$
5 Using this measure, are there any outliers?	None of the values lie outside the designated range, confirming the original view that there are no outliers.

If we were presented with secondary data in grouped format we would need to use the percentage cumulative frequency curve to estimate the values for the quartiles, and hence the interquartile range.

As more tools are added to your mathematical tool chest you will need to decide which tool to use. Technology will not always be the best choice; you need to think about the effort required to get the correct answers.

Showing outliers

When drawing a box plot that includes an outlier, or some outliers, we highlight these values by drawing them as dots instead of stretching the whisker all the way out to the outlier value. As an example, consider the following data set:

3, 5, 5, 6, 6, 7, 7, 12, 20

For this data set:

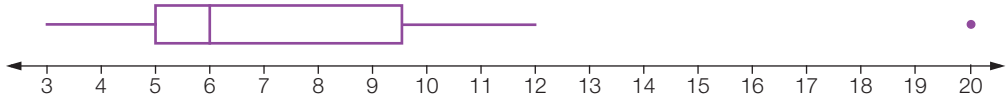
Min = 3, $Q_L = 5$, Median = 6, $Q_U = 9.5$, Max = 20, IQR = 4.5

$1.5 \times \text{IQR} = 6.75$

$Q_L - 1.5 \times \text{IQR} = 5 - 6.75 = -1.75$

$Q_U + 1.5 \times \text{IQR} = 9.5 + 6.75 = 16.25$

The outliers are less than -1.75 or greater than 16.25.



The right-hand whisker is extended to 12, which is the highest value that is not an outlier. The outlier, 20, is shown on the box plot as a dot.

7.2 Box plots

Navigator

Answers page 748

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q12, Q13, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q11, Q12, Q14

Fluency

WE3

1 Draw cumulative frequency and percentage cumulative frequency curves to represent the following data, and find the 60th percentile for each data set.

(a)

Class interval	Frequency
15–<16	4
16–<17	5
17–<18	7
18–<19	3
19–<20	4

(b)

Class interval	Frequency
150–<155	1
155–<160	2
160–<165	3
165–<170	4
170–<175	3

WE4

2 For each given data set, calculate **(i)** the range and **(ii)** the interquartile range, then **(iii)** draw a box plot.

(a)

Class interval	Frequency
0–<10	4
10–<20	5
20–<30	8
30–<40	11
40–<50	7

(b)

Class interval	Frequency
60–69	9
70–79	12
80–89	19
90–99	17
100–109	13

3 The number of points scored by a basketballer in a season of games is:

32	30	16	18	24	16
28	26	19	18	8	26
34	36	41	18	16	14
17	21	26	36	54	8

Without grouping the data, identify any outliers and suggest possible reasons for them.

WE5

4 The following is a list of the heights, measured to the nearest centimetre, of the players at the local football club.

WE6

155	167	189	156	178	179	190	200	206	196	187	156
166	178	168	178	168	170	167	159	167	160	163	181
191	157	156	187	157	159	156	161	153	173	174	168
177	164	175	185	186	164	171	170	180	141	167	159

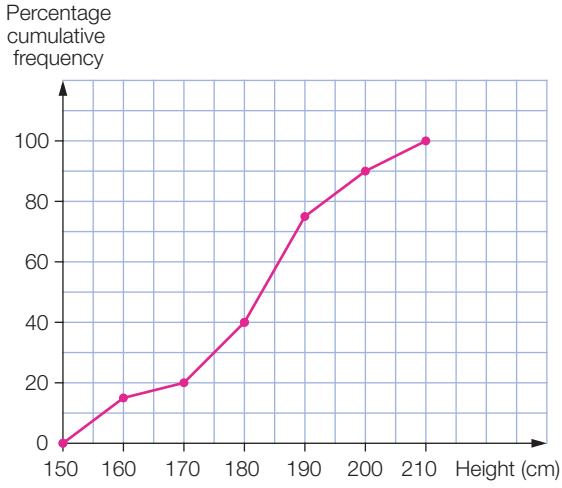
(a) Find the range of the heights.

(b) Calculate the interquartile range by finding the upper and lower quartiles.

(c) Calculate the values that are the cut-off values at either end of the data set after which the results would be considered outliers.

(d) Are there any outliers in the data set?

5 Each question below refers to the percentage cumulative frequency graph shown at right, which represents the heights of players in a football club.



(a) The 30th percentile is closest to:

A 170 **B** 172.5

C 175 **D** 184

(b) The 0.8 quantile is closest to:

A 170 **B** 188

C 191 **D** 194

(c) The interquartile range is closest to:

A 16 **B** 18

C 22 **D** 50

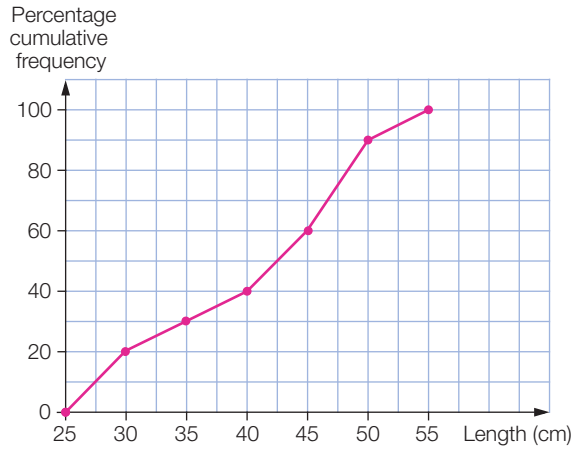
6 For the percentage cumulative frequency graph shown at right, outliers would be:

A results less than 32 cm

B results greater than 63 cm

C results less than 16.5 cm and greater than 63 cm

D none



Understanding

7 The distances run (km) by athletes training for a marathon are summarised in this frequency table.

Class interval	Frequency
10–<15	4
15–<20	6
20–<25	13
25–<30	16
30–<35	11
35–<40	9

(a) Draw a cumulative frequency curve for this data.

(b) How many of the distances run were less than 17 km?

(c) Draw a percentage cumulative frequency curve.

(d) Find the 20th percentile and the 0.8 quantile.

8 The data below shows the distances travelled (km) by car in a week for 30 employees.

895	245	697	1098	573	1569	1296	1880	972	843
1108	1037	986	1265	1398	1107	1275	1117	1023	1062
1197	1047	1284	1402	438	569	974	1328	1194	1073

(a) Calculate the range of the data.

(b) Calculate the interquartile range by finding the upper and lower quartiles.

(c) Draw a box plot and identify any outliers.

9 A survey of the number of components produced on an assembly line per shift gave the following results:

223	246	288	295	374	278
286	251	299	370	345	313
368	312	309	333	357	328
347	281	299	304	305	346
321	287	315	305	274	228
963					

(a) Is it possible to have an outlier that is very high as well as one that is very low?

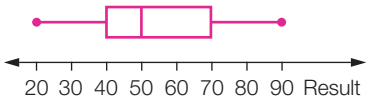
(b) Suggest possible explanations for the 963 components produced in the last shift.

10 For each of the following box plots:

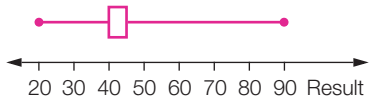
(i) indicate the values below which, and above which, outliers would lie

(ii) decide whether there are any outliers in the data set displayed.

(a)



(b)



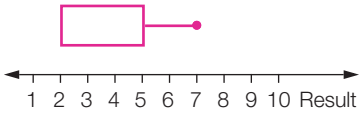
Reasoning

11 For each of the following box plots:

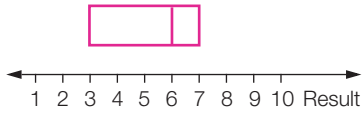
(i) construct a data set of at least 20 values that matches it

(ii) explain the reasoning you used to do this, including any assumptions made and alternative actions that might have been taken.

(a)



(b)



Open-ended

12 Consider this table, which shows the annual earnings for all the employees of a company.

Annual income	Frequency
\$35 000–<\$40 000	17
\$40 000–<\$45 000	24
\$45 000–<\$50 000	18
\$50 000–<\$55 000	16
\$55 000–<\$60 000	1
\$60 000–<\$65 000	1
\$65 000–<\$70 000	2
\$70 000–<\$75 000	1
\$75 000–<\$80 000	1
\$80 000–<\$85 000	1

(a) Calculate estimates for the mean and median annual incomes.

(b) Calculate estimates for the range and interquartile range.

(c) Are there any outliers in the data set? If so, list them.

(d) Remove any outlier(s) from the results and recalculate the mean, median, range and interquartile range.

(e) Comment on your results.

13 **(a)** Draw a cumulative frequency graph with an interquartile range of 35.

(b) Draw a percentage cumulative frequency graph with an interquartile range of 16.

14 **(a)** Construct a data set with at least 20 values that has an outlier at the upper end of the data set. The outlier is to be less than $Q_U + 2 \times \text{IQR}$.

(b) Construct a data set with at least 20 values that has an outlier at the lower end of the data set. The outlier is to be greater than $Q_L - 2 \times \text{IQR}$.

Outside the Square — Problem solving

Draw that plot

A data set has the following characteristics:

- 50th percentile is 56
- 75th percentile is 80
- IQR is 50
- Range is 80

Draw a box plot to represent this information. Which of the five key values are you certain about? Give the maximum and minimum values possible for the values about which you are unsure.

Strategy options

- Work backwards.
- Draw a diagram.

Technology Exploration CAS

Equipment required: 1 brain, 1 TI-Nspire CAS or Casio ClassPad CAS, 1 large sheet of cardboard and thick marker pens for the whole class

Versions of this Exploration for other technologies are available in Pearson Reader.

Investigating data with box plots

1 Copy the table, and in the column headed 'Estimate of age' write down your best guess for the age in years of each person.

Name	Estimate of age	Actual age	Error
Kylie Minogue			
Denzel Washington			
Queen Elizabeth II			
Russell Crowe			
Nicole Kidman			
Arnold Schwarzenegger			
Miley Cyrus			
Stephanie Rice			
Shane Warne			
Scarlet Johansson			
Zac Efron			

2 In the next column, write down the actual ages of these people. Your teacher will give you their actual ages. Subtract the actual ages in column 3 from your estimated ages in column 2 and write down the answer in column 4 (Error). Below is an example.

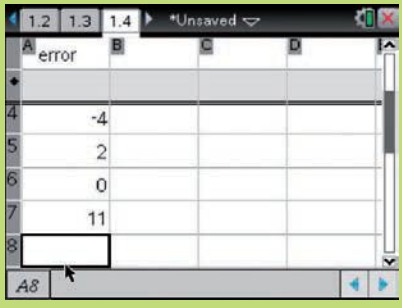
Name	Estimate of age	Actual age	Error
Joe Bloggs	38	35	3
Susie Lee	58	60	-2

3 We will use our CAS to draw a box plot that identifies any outliers in the error data.

Using the TI-Nspire CAS

Go to the Lists & Spreadsheet screen and scroll up to the very top of column A (right next to the A) and name the column. Then scroll back down and enter your error data one row at a time.

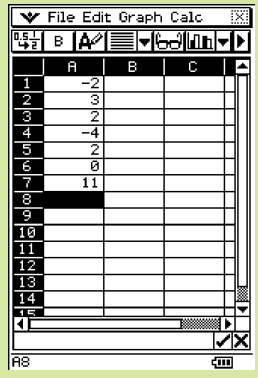
(The work that follows uses the sample data -2, 3, 2, -4, 2, 0, 11.)

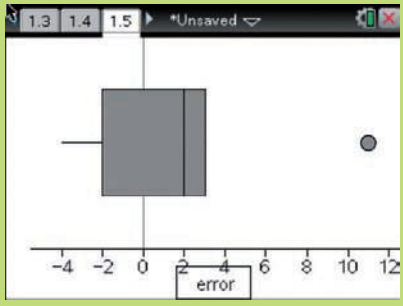
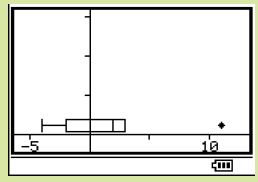
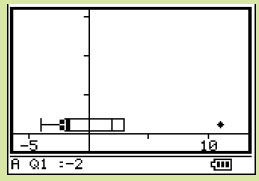


Using the ClassPad

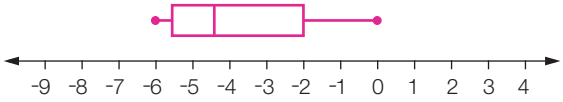
From the menu, choose the Spreadsheet screen and enter your error data one row at a time.

(The work that follows uses the sample data -2, 3, 2, -4, 2, 0, 11.)



Using the TI-Nspire CAS	Using the ClassPad
Now insert a new screen on which to draw the box plot, press ctrl I (or ctrl I) and choose Add Data & Statistics. Press tab to get to the bottom of the screen and choose 'error' as the variable. Then press menu > Plot Type > Box Plot > enter (or enter) and the box plot is displayed with the outlier shown as a dot.  You can move your finger over the touchpad (or use the ◀ and ▶ on the central donut) to move the pointer to reveal the key values for the box plot.	Now tap near the A heading to shade all of column A, then tap Graph > Box Whisker and the graph is drawn.  You can see that the outlier is clearly marked as an individual point. If the outlier is not shown as a dot then tap Series > Show Outliers. If you put the cursor over the key points and tap EXE you will see the actual value. 

4 Make a neat sketch of your box plot in your exercise book together with values for Xmin, Q_1, Median, Q_3 and Xmax. Use a ruler to create an appropriate scale marked in one-unit increments. Your plot should look something like the diagram shown below.



Interpreting box plots

1 Use your box plot to find the range and the IQR of your data.

2 Does your box plot indicate that your guesses were, on the whole, underestimated or overestimated? Why?

3 Your teacher will coordinate the following activity. On a single large sheet of cardboard (i.e. one sheet for the whole class), draw a scale marked in units of 1 with 0 at the centre. The scale should be placed at the top of the sheet of cardboard. Using a thick marker pen, neatly draw your box plot on the cardboard. Make sure that all of the five key values of the box plot are aligned properly with the scale. Print your name next to your box plot.

After all of the box plots have been drawn, answer these questions.

(a) Who was the best estimator of the ages of the famous people? How do you decide?

(b) Who was the worst?

(c) Which plots had median values closest to zero?

(d) Who was the most consistent estimator? How can you tell? Is a consistent estimator always a good estimator?

(e) Who were inconsistent estimators? Why?

(f) Were there any box plots with extreme results? Discuss.

Technology Exploration GeoGebra

Equipment required: 1 brain, 1 computer with GeoGebra

Exploring grouped data

Open the GeoGebra program. You will see seven menu options (File, Edit etc.) at the top of the screen. Below these are eleven icons called tools. Tools will be referred to by the number from the left-hand side. By clicking on the small arrow in the bottom right-hand corner of the tool icon, a dropdown list of more tools appears. The arrow turns red when you hover the cursor over it. If you hover over a tool, the tool's name and how to use it will appear in the top right-hand corner of the screen. Note that some versions of GeoGebra differ in the name that they associate with a tool, however, the icon remains the same. The names used in this activity are taken from GeoGebra Webstart.

- Click on the View menu. Select 'Spreadsheet View'. Resize the 'Spreadsheet View' by placing the cursor over the join. When it becomes a double arrow, click and drag to the right. Leave five columns visible.
- Click on the Options menu and select 'Labelling', then 'No New Objects'.
- If a larger font is required, click on the Options menu and select 'Font Size'. Choose an appropriate size from the list provided.

Using GeoGebra to display and analyse randomly generated grouped data

A data set is said to be discrete if its values are distinct and separate, i.e. they can be counted (1, 2, 3, …). We will generate discrete data in the range from 20 to 110, using class intervals of 10.

A data set is said to be continuous if its values can take on any value within a given interval, i.e. they can be measured, such as for heights, weights and time. We will create a set of continuous data as follows:

- Using icon 10, select the 'Slider' tool . Click at the top of the 'Graphics View' where you want the slider to go. In the dialog box, name the slider "lower". Set the minimum as 20 and the maximum as 110 and the increment at 10. Click 'Apply'.
- In the 'Input' bar, type "upper=lower+10". Press Enter.
- In the 'Input' bar, type "(lower, upper)". Press Enter. This has created interval information. A point *A* will appear in the 'Algebra View'. Right click on it and select 'Trace to Spreadsheet'. Right click on it again and deselect 'Show Object'.
- Type "(lower, randomBetween[2,12])" in the 'Input' bar and press Enter. This will generate a random number between 2 and 12 for each class interval, which we will use as the frequency. A point *B* will appear in the 'Algebra View'. Right click on it and select 'Trace to Spreadsheet'. Right click on it again and deselect 'Show Object'.
- Click on the 'Select' tool . Move the point on the slider *slowly* across to the right. Notice that the Spreadsheet will fill with data. Do not move the slider back. When done, right click on it and deselect 'Show object'.
- If extra data appears in columns C and D, select it as shown and press 'Delete'. Note that the data in column D will be different to what you have.

	A	B	C	D
1	20	30	20	7
2	30	40	30	4
3	40	50	40	12
4	50	60	50	3
5	60	70	60	6
6	70	80	70	2
7	80	90	80	6
8	90	100	90	2
9	100	110	100	9
10	110	120	110	12
11			110	10
12			110	4
13			110	6

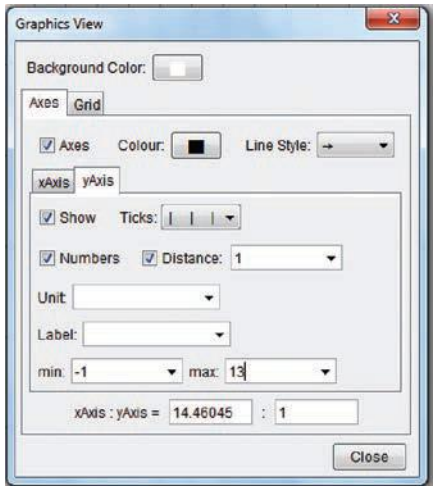
- The information in column C is a repeat of the lower boundary information for the class interval as in column A. Right click on the grey box at the top of column C and select 'Clear Column'.

- Column C will now become the Class Centre. In cell C1 type "=(A1+B1)/2" and then press Enter.
- Click inside cell C1 and then click and hold on the blue square on the bottom of the cell.

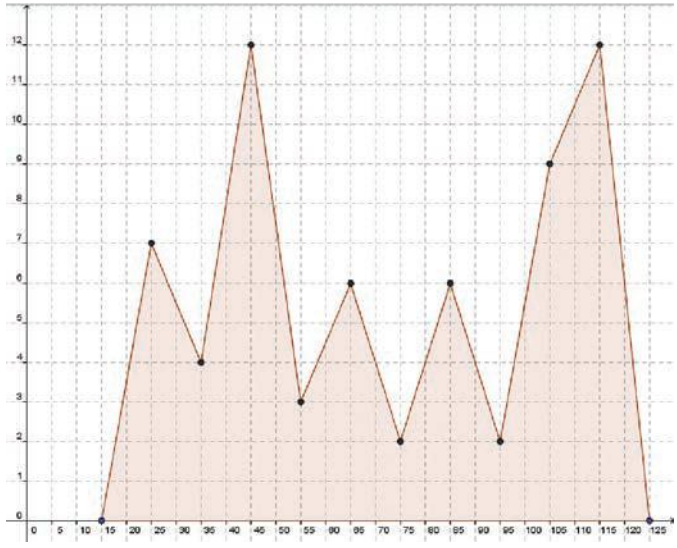
	A	B	C
1	20	30	25

Drag it down to row 10.

- Column D represents the frequencies. What is your highest frequency?
- Right click on the 'Graphics View' and select 'Graphics View' at the bottom of the popup box. In the dialog box, on the xAxis tab, tick 'Distance' and set it at 5, set 'min:' to -12 and 'max:' to 130. Select the yAxis tab, tick 'Distance' and set it to 1, set the minimum to -1 and the maximum to 13. Press Close.



- Click on column C (class centre) and while holding down the 'Ctrl' key, click on column D (frequency). Right click in the blue area and select 'Create List of Points'. Points will appear on the 'Graphics View' and these will form the basis of the frequency polygon.
- Create a start and end point for your polygon. In the 'Input' bar, type "(15,0)" and press Enter. Then type "(125,0)" and press Enter.
- From icon 5, select the 'Polygon' tool . Starting from point (15,0), click on each point from left to right until you reach point (125,0). Finish by clicking on the start point (15,0). Note that all points should be used and the outer edges of the polygon should never cross.
- You can add labels and titles to your graph using the 'Text' tool, from icon 10. Click where you want the text to appear and type the text inside the dialog box. To edit text, right click on it and select 'Object Properties'. Use the text and colour tabs to edit your font.



1 From your graph:

(a) Is it possible to give an estimate for the median? What is your estimate?

(b) What do you think is the range of the data?

We will now construct a frequency histogram for your data.

- Right click on the polygon and deselect 'Show Object'.
- In the 'Input' bar, type "BarChart[20,120,{<list of frequencies in column D separated by commas>}]" and press Enter. For example, for the above polygon the input looks like "BarChart[20,120,{7,4,12,3,6,2,6,2,9,12}]"

2 From your graph:

(a) Is it possible to give an estimate for the median? What is your estimate?

(b) What do you think is the range of the data?

(c) Did any of your answers change? If so, what might be a reason for any differences?

Calculating the mean

We can use the spreadsheet to calculate the mean of the data.

- In cell D11, type "=sum[D1:D10]" and press Enter. This will sum the frequencies or the number of scores.
- In cell E1, type "= C1*D1" and press Enter. This will sum all the scores that have a class centre of 25. Click inside cell E1, and then click and hold on the blue square on the bottom of the cell. Drag it down to row 10. This will sum the scores for each class centre.
- In cell E11, type "=sum[E1:E10]" and press Enter. This will be the sum of all the scores.
- In cell E12, type "=E11/D11" and press Enter. This will divide the sum of the scores by the number of scores and give the mean.

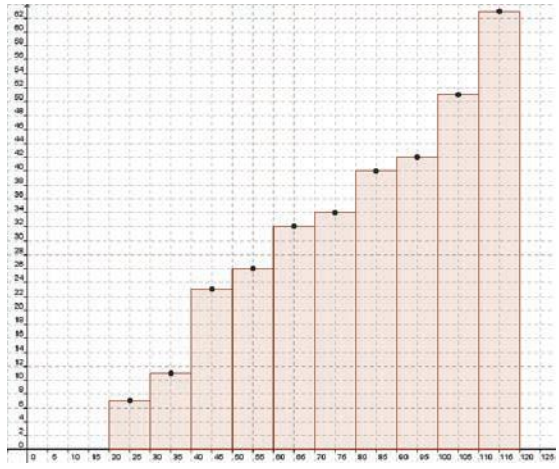
Taking it further: Finding the median and quartiles

Recall that column D represents the frequencies of each class. We will now create a cumulative frequency column that successively sums the frequencies as you progress down the column. Before we start we need to clear the 'Graphics View':

- In the 'Algebra View' clear the small circle next to the letter *a*. a = 630 (Note: The number will differ from what you have.) This should clear the histogram from the page.
- Use the 'Select' tool . Use it to draw a box that covers all the points on the screen. Press 'Delete'.
- In cell F1, type "=D1" and press Enter. This will reproduce the first frequency.
- In cell F2, type "=F1+D2" and press Enter. This will add the first two frequencies.
- Click inside cell F2 and then click and hold on the blue square on the bottom of the cell. Drag it down to row 10.

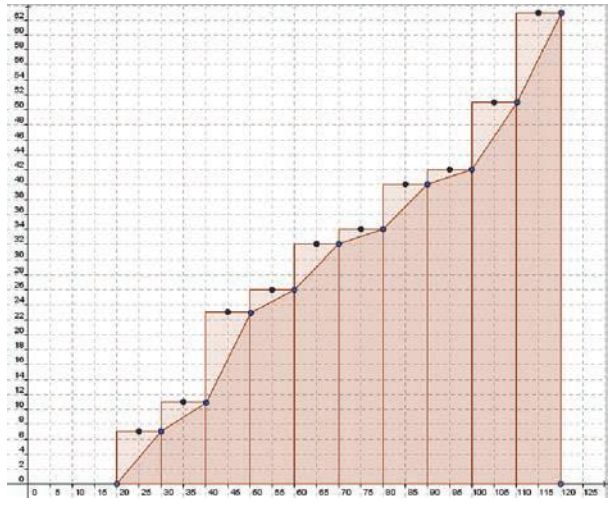
3 What does the value in cell F10 represent?

- Right click on the 'Graphics View' and select 'Graphics View' at the bottom of the popup box. On the yAxis tab, change the 'Distance' to 2 and 'max' value to one more than the value in F10 and tap Close.
- Create a cumulative frequency histogram. In the 'Input' bar, type "barChart[20,120,{<values from column F separated by commas>}]", e.g. for the chart below the input was "barChart[20,120,{7,11,23,26,32,34,40,42,51,63}]". Press Enter.



4 How does the height of the column relate to the frequency of the class?

- From icon 5 select the 'Polygon' tool . Starting on the horizontal axis at point (0, 20), click in the top right-hand corner of each column. When at the top, return to the horizontal axis, click on (0, 120) then on the start point (20, 0).



5 **(a)** How many scores do *you* have? (*Hint:* It is the sum of the frequencies.)

(b) How does it relate to your cumulative frequency graph?

The median is the middle score when the scores are listed in order.

- In the 'Input' bar, type "y=<value from question 5>/2" and press Enter. This will place a horizontal line halfway up the vertical axis.
- From icon 2, select the 'Intersect Two Objects' tool, and click on the point where the line intersects with the polygon. If you find it hard to select the correct point, use 'Ctrl'+'z' to undo, use the scroll wheel on the mouse to zoom in (or from the last icon, select the 'Zoom In' tool) and try again.

6 (a) What are the coordinates of this point? (*Hint:* By hovering over the point it will tell you what letter it is. Look for it in the 'Algebra View'.)

(b) At what horizontal position is this point?

(c) What does the horizontal axis represent?

(d) What do you think the median score is?

- In the 'Input' bar, type"y=<value from question 5>/4" and press Enter. This will place a horizontal line a quarter of the way up the vertical axis.
- Find the point of intersection.

7 (a) What are the coordinates of this point? (*Hint:* By hovering over the point it will tell you what letter it is. Look for it in the 'Algebra View'.)

(b) At what horizontal position is this point?

(c) What do you think this value represents?

- In the 'Input' bar, type "y=(3/4)*<value from question 5>" and press Enter. This will place a horizontal line three-quarters of the way up the vertical axis.
- Find the point of intersection.

8 (a) What are the coordinates of this point? (*Hint:* By hovering over the point it will tell you what letter it is. Look for it in the 'Algebra View'.)

(b) At what horizontal position is this point?

(c) What do you think this value represents?

Box plot

9 (a) What is the value of the lowest score?

(b) What is the value of the score one-quarter of the way through the data (i.e. the value of the lower quartile)?

(c) What is the value of the score that is halfway through the data (i.e. the value of the median)?

(d) What is the score that is three-quarters of the way through the data (i.e. the value of the upper quartile)?

(e) What is the highest score?

- In the 'Input' bar, type "boxPlot[-4,2,<lowest score>,<lower quartile>,<median>,<upper quartile>,<highest score>]". This will place a box plot underneath the horizontal axis.

10 Discuss any differences and similarities between the two graphs.

7.3 Comparing data sets

When faced with a mass of data we need some skills to interpret the data correctly and so enable us to make reasonable conclusions based on that data. We have made use of two measures of centre or location: mean and median, and two measures of spread: range and interquartile range.

Statisticians tend to match some of these measures in particular ways. The mean is often, linked with the range, whereas the median is often linked with the interquartile range.

The linking makes sense as both the mean and range are affected by any data change, whereas the median and interquartile range are not affected if the change is not in the middle portion of the data set.

Worked Example 7 WE7

For each of the following sets of data:

SET A:	1	1	1	2	3	3	3	4	4	4	5	5	5	5	6
	6	6	6	7	7	7	8	8	8	8	8	9	9	9	
SET B:	1	1	1	2	3	3	3	4	4	4	5	5	5	5	6
	6	6	6	7	7	7	8	8	8	8	8	9	9	19	

(a) find the mean

(b) find the median

(c) find the range

(d) calculate the interquartile range

(e) use your findings to compare the two sets of data.

Thinking	Working
(a) Add all the data values in each set and divide by the total number of data values to find the mean.	(a) Set A mean $= \frac{158}{29}$ $= 5.48$ Set B mean $= \frac{168}{29}$ $= 5.79$
(b) Find the middle value of each set.	(b) Set A median = 6 Set B median = 6
(c) Find the difference between the maximum and minimum values.	(c) Set A range $= 9 - 1$ $= 8$ Set B range $= 19 - 1$ $= 18$

(d) Find Q_L and Q_U for each set and find the difference to find the IQR.	(d) Set A $Q_L = 3.5$ Set A $Q_U = 8$ Set A IQR $= 8 - 3.5$ $= 4.5$ Set B $Q_L = 3.5$ Set B $Q_U = 8$ Set B IQR $= 8 - 3.5$ $= 4.5$
(e) Compare the calculations for each statistic.	(e) The two sets have the same median, but the mean of Set B is slightly higher than Set A. The two sets have the same interquartile range, but the range of Set B is greater than Set A. This is because Set B has the score of 19, which affects the mean and the range, but not the median (middle score) nor the interquartile range (spread of the middle half of the data).

Box plots are useful when comparing data sets. In particular, we make use of what we call a **parallel box plot**, which is two, or more, box plots drawn on the same set of axes. From the parallel box plot we can easily compare the median, range and IQR of the data sets as well as decide which data set is more consistent. It is important that the scale used covers all values in both sets.

In the same way, a **parallel dot plot** can also be useful. When drawing a parallel dot plot it is common to put the scale in twice; it effectively serves as the dividing line between the two dot plots.

Worked Example 8

WE 8

The heights (cm) of the students in two Year 8 classes were recorded.

8A:	148	149	150	150	153	154	154	154	155	155	156	156	156
	157	157	158	158	158	158	158	160	161	163	165	168	
8B:	150	151	152	152	152	153	153	154	155	155	155	156	156
	156	156	157	157	158	160	160	160	161	161	162	163	

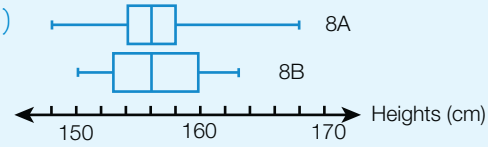
(a) Find the median height for each class.

(b) Find the range for each class.

(c) Find the interquartile range for each class.

(d) Draw a parallel box plot for both classes.

(e) Compare the statistics for the two classes.

Thinking	Working
(a) Find the middle value of each class.	(a) 8A median = 156 8B median = 156
(b) Find the difference between the maximum and minimum values.	(b) 8A range = 168 – 148 = 20 8B range = 163 – 150 = 13
(c) Find Q_L and Q_U for each set and find the difference to find the IQR.	(c) 8A Q_L = 154 8A Q_U = 158 8A IQR = 158 – 154 = 4 8B Q_L = 153 8B Q_U = 160 8B IQR = 160 – 153 = 7
(d) Use the statistics to draw a parallel box plot.	(d) 
(e) Compare the calculations for each statistic.	(e) The medians for the two classes are the same but the range for 8A is much larger than for 8B. However, the IQR for 8A is much smaller than the IQR for 8B.

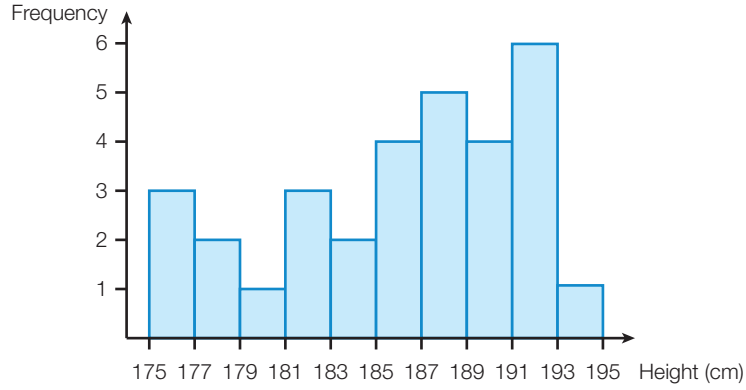
Comparing different data representations

When we are investigating a data set, presenting the data in different forms can give us a deeper understanding of the data. We will examine the following data set that gives the height (cm) of each of the players on the 2011 playing list of the Melbourne Storm in the NRL.

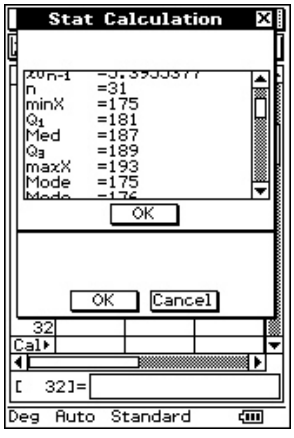
188 178 181 180 189 188 188 185 192 191 188 181 189 193 176 183 192 189
183 186 182 192 192 178 192 176 189 187 186 185 175

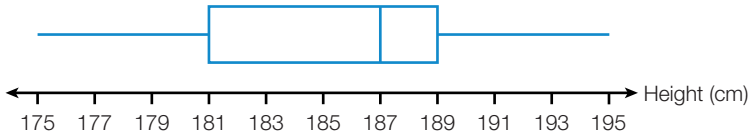
We will produce a histogram, a box plot and a dot plot to represent the data. To construct a histogram we need to know the range so we can decide on the bin size for the grouping. Here we have: range = 193 – 175 = 18. On the basis of this we would make the bin size 2, with a first class interval of 175 –<177. We need to put the grouping in this format as the data is continuous.

Height (cm)	Frequency
175–<177	3
177–<179	2
179–<181	1
181–<183	3
183–<185	2
185–<187	4
187–<189	5
189–<191	4
191–<193	6
193–<195	1

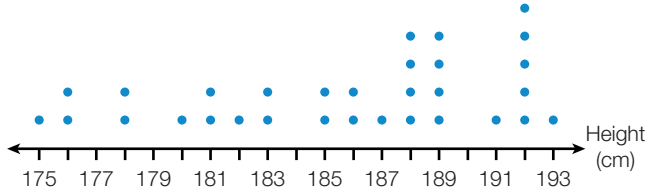


For the box plot we need the five-number summary. This has been calculated using a CAS, as shown, but you could do it by hand if you wished.





The dot plot does not need to group the data as the range is not too big.



Both the histogram and the dot plot show there are more values at the right-hand end of the range as opposed to the left-hand end. This is shown in the box plot by the median being pushed towards the right-hand side. The dot plot provides more information than the histogram as we have retained every data element. In some cases, where the range is quite big, the dot plot would also need to show grouped values. However, if at all possible, it is better to show every data value on the dot plot.

7.3 Comparing data sets

Navigator

Answers page 750

Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q12	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12	Q1, Q2, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12

Fluency

WE7

1 Calculate the following values for each pair of data sets below. Where necessary round your answers to two decimal places.

(i) mean **(ii)** median **(iii)** range **(iv)** interquartile range

(a) Set A: 22 22 24 25 26 27 28 28 29 30
31 32 35 35 36 37 38 39 40

Set B: 12 22 24 25 26 27 28 28 29 30
31 32 35 35 36 37 38 39 40

(b) Set A: 44 44 46 47 47 48 48 48 51
52 52 52 54 55 58 58 58 60

Set B: 24 44 46 47 47 48 48 48 51
52 52 52 54 55 58 58 58 80

(c) Set A: 11 12 12 12 13 13 13 13 14 14 15
16 17 18 18 18 21 22 24 24 24 26

Set B: 11 12 12 12 13 13 13 13 13 13 13
16 17 18 18 18 21 22 24 24 24 26

(d) Make a few comments about what you have found by completing these calculations.

WE8

2 The heights (cm) of the students in two Year 8 classes were recorded.

8C: 145 146 147 147 149 150 150 151 152 153 154 155 155
156 157 158 158 159 160 162 163 163 164 166 169

8D: 147 148 149 150 150 150 152 153 154 154 155 155 155
156 156 156 157 157 158 160 160 160 161 161 162

(a) Find the median height for each class.

(b) Find the range for each class.

(c) Find the interquartile range for each class.

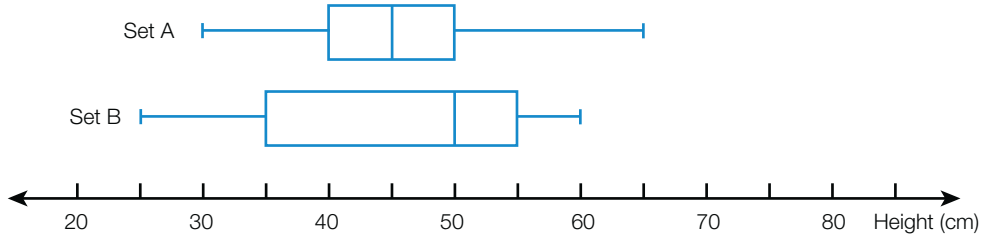
(d) Draw a parallel box plot for both classes.

(e) Compare the statistics for each class.

3 **(a)** For each data set in the following parallel box plot, find the:

(i) median **(ii)** range **(iii)** IQR.

(b) Compare the statistics for the two data sets.



4 The following data list shows the weight (kg) of each player on the 2011 playing list for the Brisbane Broncos in the NRL.

104	106	83	95	110	105	94	110	101	93	104	100
109	108	99	93	90	86	95	106	106	85	105	91
105	89	94	100	98	106	110	111	88	100	96	

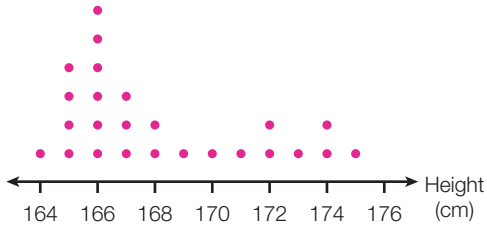
(a) Draw a histogram of the heights. Use a class interval of 5, with the first interval being 80–<85.

(b) Calculate the five-number summary for the data and use it to construct a box plot.

(c) Draw a dot plot of the data.

(d) Make some comments about what the three displays tell you about the data set.

5 **(a)** The students in a class had their heights (cm) recorded. The following dot plot shows the results.

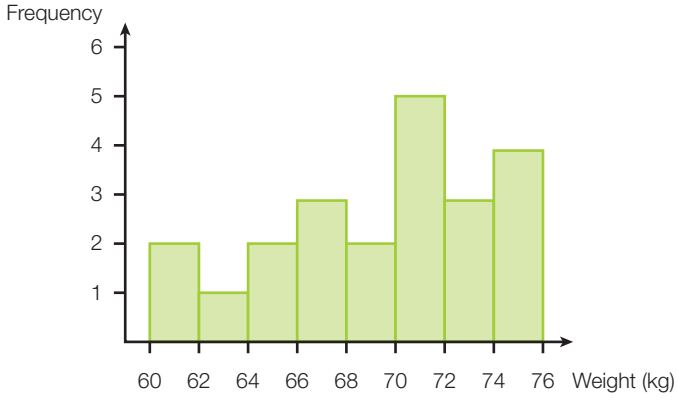


The interquartile range for this data set is closest to:

A 5 **B** 5.5

C 7.5 **D** 11

(b) The histogram below shows the weights, in kg, of the students in another class.



The median of this data set is likely to be closest to:

A 66 **B** 70 **C** 74 **D** 75

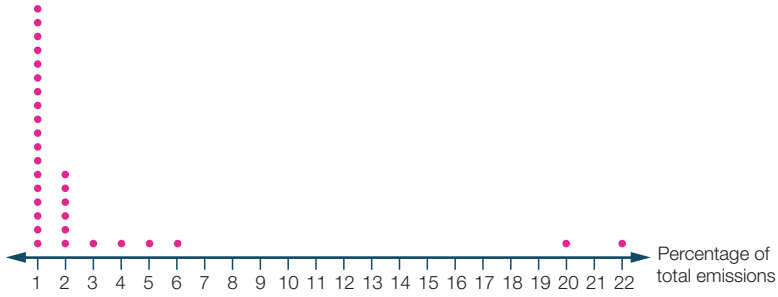
Understanding

6 Three classes sat the same mathematics test, marked out of 100. The results for the three classes are summarised in the following table.

Class	Mean	Median	Range	IQR
A	64.32	68	72	27.5
B	64.12	63	31	19
C	60.92	63	55	38.5

Which class do you think performed best on the test?

7 The following dot plot shows the percentage of the world's total emissions of carbon dioxide for 30 countries, which contribute at least 1% of the total emissions when rounded to the nearest whole number.

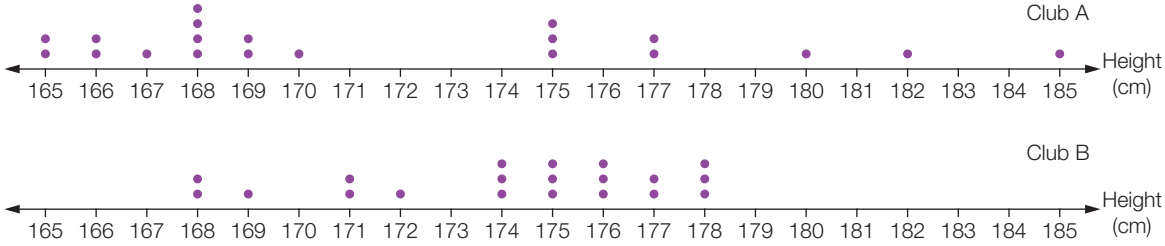


(a) Calculate the **(i)** mean and **(ii)** median for the data.

(b) Calculate the **(i)** range and **(ii)** interquartile range for the data.

(c) Which pair of values better describe the data set: mean and range; or median and interquartile range? Give a reason to support your answer.

8 The following parallel dot plot shows the heights of players at two netball clubs who play in the same competition. The heights have been rounded to the nearest centimetre.



(a) Which club appears to have the taller players?

(b) Calculate the mean and median of the given data for the two clubs.

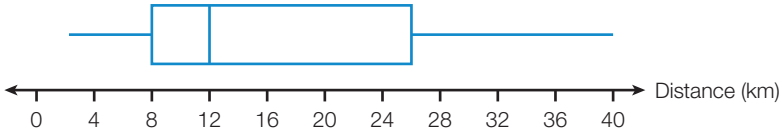
(c) Calculate the range and IQR for each of the two clubs.

(d) Draw a parallel box plot for the data.

(e) Do you still agree with your answer to part **(a)**? Use statistics to justify your answer.

(f) Compare the shape of the parallel dot plot with that of the parallel box plot.

9 A school collected information about the distance (km) that students travel to school. The results were published only in box plot form, as shown below.



(a) Write a sentence or two describing what information the box plot gives you about the travel distances for the students at this school.

(b) What would you estimate to be the mean distance travelled by the students? Describe how you made your estimation.

(c) **(i)** Draw a histogram that might represent the data collected. Use a bin size of 4 with a first class interval of 0–<4, and so on. Information was collected from 160 students.

(ii) Show that your histogram supports the values shown in the box plot.

(iii) Use your histogram to estimate the mean distance. How does this value compare to your estimate in **(b)**?

Reasoning

10 The following table gives the height (cm) and weight (kg) of each of the players on the 2010 playing lists of the AFL teams Adelaide Crows and Essendon.

Adelaide

Height:	200	184	201	185	201	185	184	178	179	178
	194	192	182	187	195	188	180	198	185	192
	181	185	191	181	195	191	196	195	182	188
	176	191	196	184	193	183	192	178	180	194
	193	191	182	190	189	179	181			
Weight:	99	85	100	85	99	84	82	81	80	80
	93	97	75	82	90	86	84	102	84	87
	82	83	99	76	87	74	98	92	79	89
	79	82	98	86	89	75	88	82	77	92
	86	87	80	87	84	81	80			

Essendon

Height:	191	182	197	190	182	184	178	188	186	184
	182	188	187	182	186	191	185	193	201	185
	197	190	194	196	201	191	173	197	198	173
	180	184	189	182	192	187	190	180	201	187
	178	180	185	186	205	182				
Weight:	100	83	94	92	81	83	77	81	83	81
	77	88	74	80	86	94	74	89	108	85
	89	90	89	100	105	91	72	95	93	68
	86	86	87	85	81	90	86	84	105	84
	80	76	78	95	89	87				

(a) Find the mean heights and weights for each club.

(b) Find the median heights and weights for each club.

(c) Find the range and IQR for the heights and weights for each club.

(d) Draw a parallel box plot for the heights of players in each club.

(e) Draw a parallel dot plot for the weights of players in each club.

(f) Which team is taller overall? Use your statistics to justify your decision.

(g) Which team is heavier overall? Use your statistics to justify your decision.

11 The following table shows the percentage of the Australian population, and the percentage of the Aboriginal and Torres Strait Islander (ATSI) population, in particular age ranges.

	0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44
Australia	6.7	6.3	6.5	6.9	7.6	7.6	6.9	7.2	6.9
ATSI	12.3	11.2	11.3	11.2	9.5	7.7	6.3	6.2	5.9

	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–85	85+
Australia	7.0	6.5	5.9	5.4	4.0	3.1	2.3	1.7	1.2
ATSI	5.1	4.2	3.3	2.3	1.5	1.0	0.6	0.3	0.2

(a) Draw cumulative frequency graphs for the Australian population and for the ATSI population.

(b) Use the cumulative frequency curves to estimate the values for the five-number summary for each population. For the purposes of this work you should use 90 as the maximum value.

(c) Draw a parallel box plot showing the two populations.

(d) Write a couple of sentences comparing the two populations.

Open-ended

12 Collect two sets of data from students in your class or year level. Use the data to calculate some statistics. Compare the data sets using the statistics you have found.

Outside the Square — Problem solving

What do you mean?

Task 1

It was reported that most Australians have more than the average number of legs.

How can this be true?

Task 2

Each week, for 4 weeks in a row, a market research company interviewed people asking questions about the performance of the Prime Minister. The results are as follows:

Week 1:	47 approved	42 disapproved	11 were unsure
Week 2:	68 approved	57 disapproved	7 were unsure
Week 3:	112 approved	106 disapproved	8 were unsure
Week 4:	77 approved	68 disapproved	7 were unsure

What percentage approval would you say the Prime Minister enjoys? Give some explanation for why you think your answer is justified.

Strategy options

- Act it out.
- Make a table.

Scatter plots and data investigations

7.4

A large-scale data investigation can be quite a challenge. You first need to decide if you want to use **primary data** (data you collect yourself) or **secondary data** (data that has already been collected by somebody else). This decision will largely be made by the type of investigation you wish to undertake. If you want to research community interest in some topic of current interest such as 'Should a new train line be built that links all the capital cities in Australia?', then you will need to collect your own data. If you want to look at weather patterns over the last 10 years then you will need to use secondary data, presumably from the Bureau of Meteorology.

If you are collecting your own data then you need to decide if a census or a sample is better. A census collects data from the whole population, whereas a sample collects data from a small proportion of the population. You must make sure that you choose a random sample to avoid a biased result. Using random numbers generated by a calculator or computer or picking numbers out of a hat are a couple of ways to pick a random sample.

If you use secondary data you need to establish if you can rely on only one source. Is there any likelihood of bias in the data you use? Using data published by a small business association on the need for tax cuts for small business may not be reliable. Other sources of data would be needed to ensure the data was unbiased.

Worked Example 9

WE9

Choose whether the following examples represent primary data or secondary data.

(a) You are doing a project on water storages, so you collect data from the newspaper on the levels of the dams in your state over the past 12 months.

(b) You survey youth in your area to find out what facilities they would like the council to provide.

Thinking	Working
(a) Has this material been collected by someone else? If yes, it is secondary data.	(a) This is secondary data.
(b) Has this material been collected by someone else? If no, it is primary data.	(b) This is primary data.

Identifying relationships

We now want to see if there is a relationship existing between the variables for which we have been collecting data. In this chapter we deal with **bivariate data**, which means we have two variables. However, we do need to be careful about jumping to unfounded conclusions. Just because it looks as if there is a strong connection between the variables does not mean that one was caused by the other. It may very well be that there is a strong relationship between the variables *head circumference* and *ability to recall times table*s when we are dealing with primary school students of varying age. It would be dangerous to conclude that a student's ability to recall times tables is caused by their head circumference. It is more likely to be related to age—as a student gets older their head gets bigger and they get better at their times tables.

Scatter plots

When we are dealing with bivariate data we often draw a scatter plot to help us look for any connections that might exist between the variables. A **scatter plot** is a series of points drawn on a pair of axes. For instance, we might know the height and weight of a group of people. Let's assume we have decided to plot height on the horizontal axis and weight on the vertical axis.

Worked Example 10 WE10

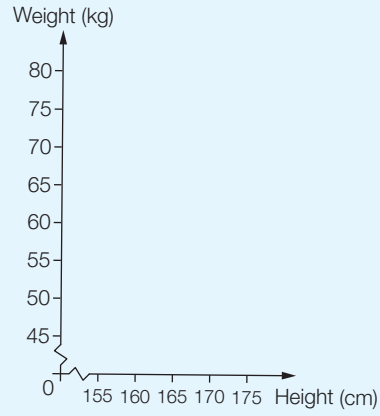
The following table represents the heights (cm) and weights (kg) of 10 students.

Height (cm)	156	167	157	155	161	173	163	162	161	172
Weight (kg)	45	55	61	49	67	75	71	75	72	77

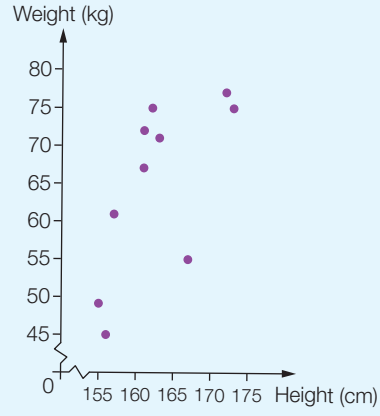
Draw a scatter plot of the data with height on the horizontal axis.

Thinking | **Working**

1 Draw a set of axes, labelled appropriately, with scales stretching far enough to fit all the data. (If you leave out some of the axis values, indicate this on the axis.)



2 Mark each of the points on the graph.

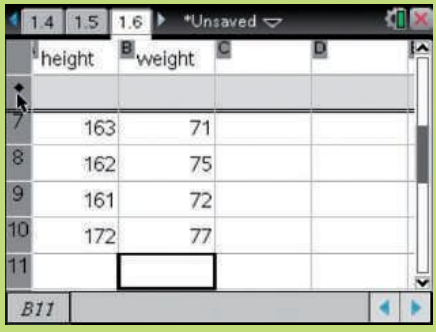
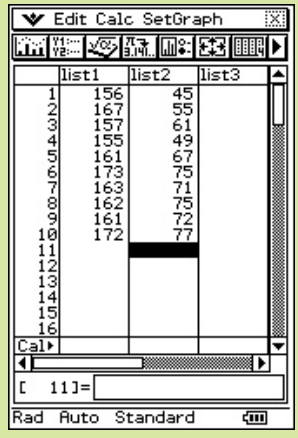
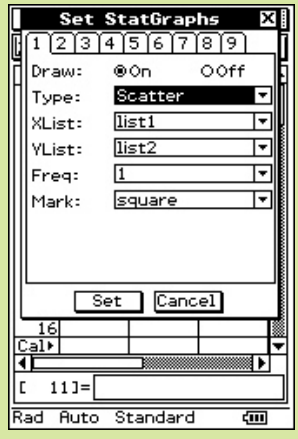
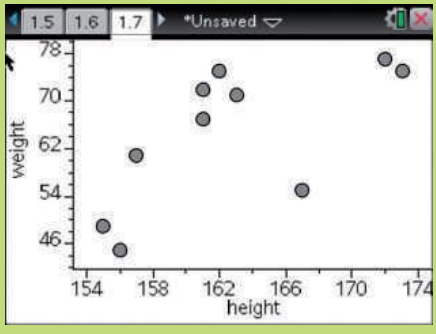
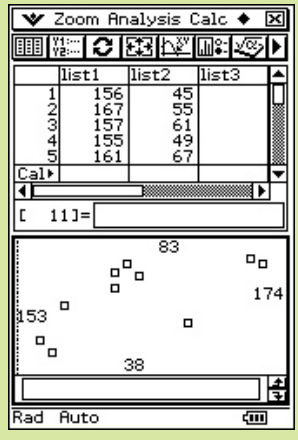


When we look at the scatter plot we can see there is a trend in the placement of the dots. In general, taller students weigh more. So, we might say there is a relationship established between the two variables *height* and *weight* for the students who have been measured. In this case, we might reach the tentative conclusion that as height increases so does weight. However, we would need to examine a lot more data before we made a stronger statement than this.

It is important to note that with bivariate data it is possible for two coordinate pairs to have the same value for either axis. In graphical terms this means we can plot points that would make a vertical line or a horizontal line.

You probably can tell that it can be quite tedious drawing a scatter plot. Technology makes it much easier for us to see the scatter plot and any trends that might be apparent from it.

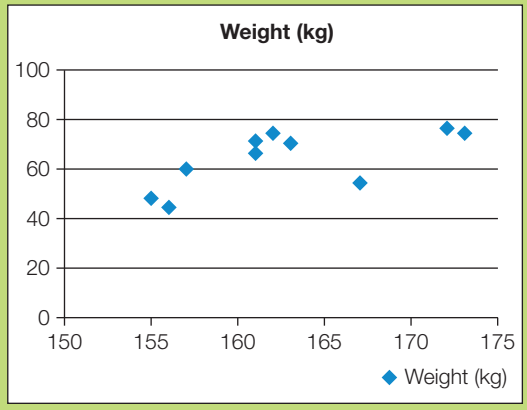
Using the same data as above, we will look at using a CAS to produce the scatter plot.

Using the TI-Nspire CAS	Using the ClassPad
On a Lists & Spreadsheet screen, enter the data in the normal way, putting height in column A and weight in column B. However, this time you need to name the columns so scroll right to the top of column A and type 'height' beside the A. In a similar way, name column B 'weight'. 	At the Statistics screen, enter the data in the normal way, putting height in list1 and weight in list2. 
Now press ctrl > I to insert a new page, choosing Add Data & Statistics. Press tab and select height as the variable for the horizontal axis. Press tab again and choose weight as the variable for the vertical axis. The scatter plot will then appear.	Now tap SetGraph > Setting and fill in the dialog box as shown. 
	Tap SET and then tap and the graph will appear. 

Using Excel

You can also draw a scatter plot using a spreadsheet program such as Excel.

Once the data has been entered press Insert > Scatter and accept the first option. This produces the graph shown on the right. While it has many failings as a mathematical graph, it is good enough for the purpose of checking for any trends in the data. You could make the graph look a lot better from a mathematical point of view, but that is not our priority at this stage.



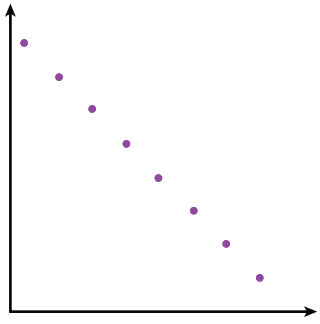
Describing relationships

The scatter plot makes it easier for us to see any relationships that might exist between the variables we are examining. In some cases, the points on the scatter plot look like they lie close to a straight line. If one variable increases as the other decreases we might talk about a negative linear relationship being displayed, whereas if both increase or decrease then we might talk about a positive linear relationship. The following diagrams give some examples of possible linear relationships and the strength of those relationships.

Perfect negative linear relationship

Description: As one variable increases so the other decreases. There is a high degree of order in the placement of the points.

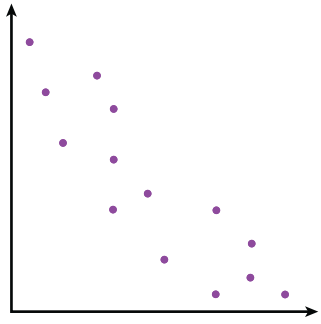
Example: Length and width of a rectangle for a fixed perimeter



Strong negative linear relationship

Description: As one variable increases so the other generally decreases. There may be a few points that don't fit this pattern exactly.

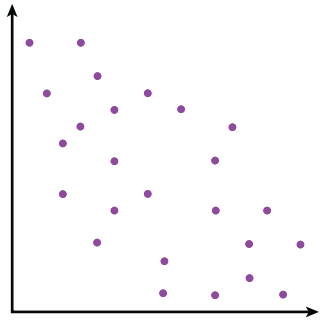
Example: Age of a second-hand car and its value (excluding vintage cars)



Weak negative linear relationship

Description: There is some trend evident of one variable increasing as the other decreases, although there would certainly be a number of points that don't fit this pattern.

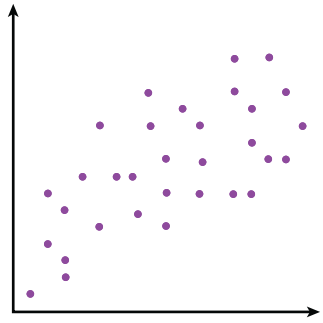
Example: Average number of hours spent keeping fit/week and the number of annual visits to a doctor



Weak positive linear relationship

Description: There is some trend evident of either both variables increasing or both variables decreasing, although there would certainly be a number of points that don't fit this pattern.

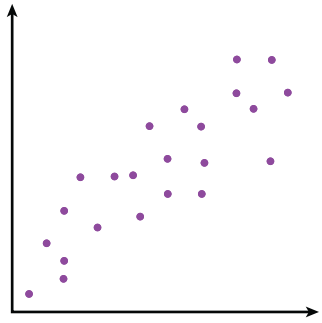
Example: Money spent on a holiday and money spent buying a car



Strong positive linear relationship

Description: As one variable increases so the other generally increases. There may be a few points that don't fit this pattern exactly.

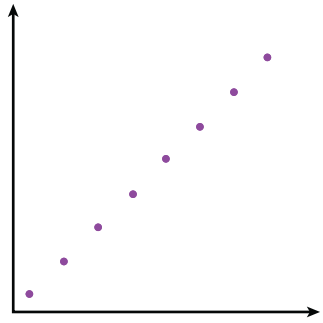
Example: Smoking and lung cancer



Perfect positive linear relationship

Description: As one variable increases so does the other. There is a high degree of order in the placement of the points.

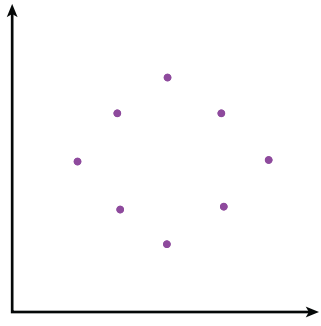
Example: Volume and height of water in a tank



No linear relationship

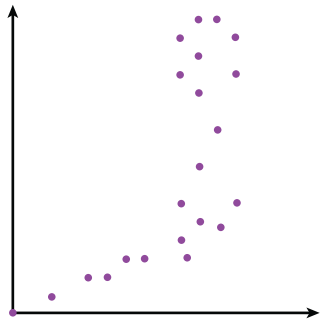
Description: There is no linear pattern evident in the placement of the points.

Example: People who like cycling and people who like playing a musical instrument



Non-linear trends

An apparent trend will not always be linear. In this scatter plot, a curve would be a more appropriate choice. At this level you are simply expected to be able to decide whether the trend is linear or non-linear and, if it is linear, whether the strength of the relationship between the variables is strong, moderate or weak.



7.4 Scatter plots and data investigations

Navigator

Answers page 752

Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q10, Q11, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11, Q13	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13

Fluency

WE9

1 Choose whether the following examples represent primary data or secondary data.

(a) You sit on a street corner and record the colour of the cars passing by.

(b) You send an SMS to 20 friends asking them whether or not they are watching a particular show on television.

(c) You write down the scores for all the weekend's football matches from Monday's newspaper.

(d) Your teacher asks the class how many brothers and sisters they have and records the information on the board.

(e) You go to the website of your favourite football team and write down the height of each player.

WE10

2 The following table represents the heights (cm) and weights (kg) of 10 football players. Draw a scatter plot of the data. Put height on the horizontal axis. Make a comment about the type of relationship, if any, that is displayed by the data.

Height (cm)	203	186	184	189	186	197	188	181	182	185
Weight (kg)	103	84	84	86	89	97	87	82	80	86

3 Are the following examples of a census or a sample?

(a) You ask every student in Year 10 their favourite movie.

(b) You ask the first thirty people who leave the canteen what they purchased.

(c) You send a questionnaire to every family in your school, but only 35% of the forms are returned.

(d) You ask every junior member of your cricket club how long they have been playing cricket.

4 You are conducting a survey on the topic 'The best team sport played in Australia' and are considering using the following questions. Decide whether each question is appropriate or inappropriate, and give a reason.

(a) How often do you participate in team sport?

(b) How often do you watch team sport on television?

(c) Which of the following team sports do you think is the best? Australian Rules Football, Rugby League, Rugby Union, Cricket, Basketball

(d) Do you think too much Government money is spent supporting Olympic athletes?

5 Decide whether or not the following sampling methods would be likely to return biased results. Give a reason for each answer.

(a) You stand in a shopping centre and ask people who smile at you.

(b) You go to the local petrol station and ask people who are filling their car with diesel their opinions about environmental matters.

(c) You put a survey form in the letterbox of one hundred houses in your neighbourhood asking about support for the renovation of the local hall. You include a freepost envelope for the return of the survey.

(d) You find numbers in a telephone directory and call two hundred numbers to find out voting intentions at the next State election.

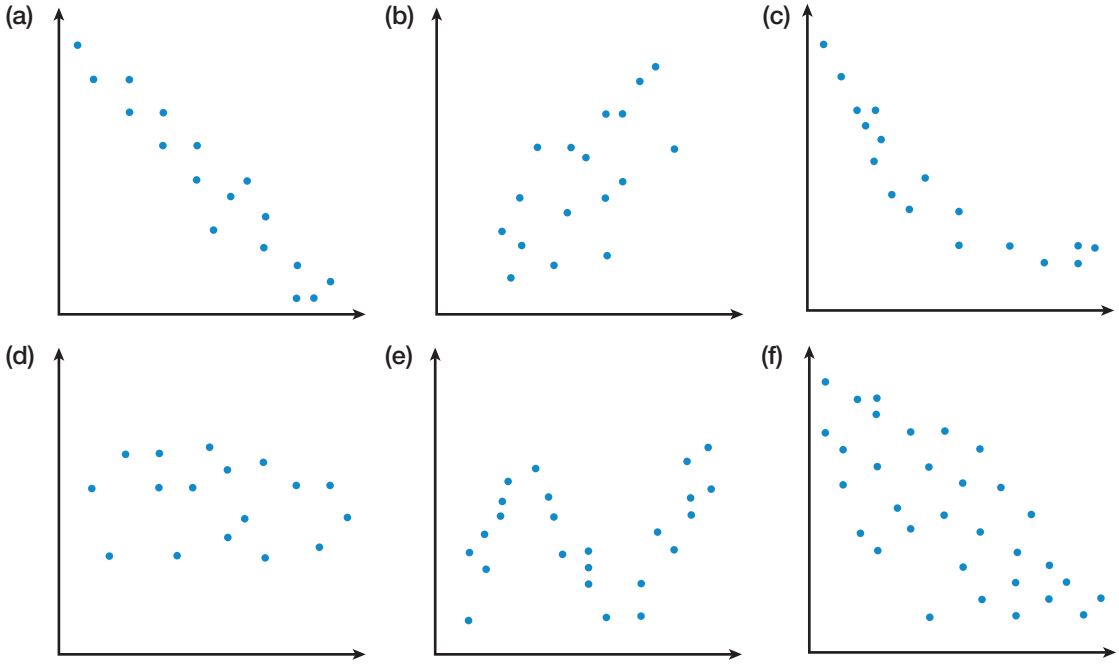
6 The data below show the marks obtained by 10 pupils taking physics and mathematics tests.

Pupil	A	B	C	D	E	F	G	H	I	J
Physics mark (out of 50)	42	38	44	41	46	31	25	30	33	36
Mathematics mark (out of 70)	66	60	65	59	65	50	38	58	60	60

(a) Draw a scatter plot of the data with the physics mark on the horizontal axis.

(b) Comment on any relationship evident between the marks.

7 For each of the following scatterplots state whether the trend is linear or non-linear. If linear, also state whether the trend is positive or negative and give an indication of strength.



Understanding

8 You are a member of the Student Representative Council (SRC) at your school which has 1230 students. A suggestion has been made that a good quality coffee machine should be installed in the school canteen. You have been given the task of conducting an investigation into this matter and need to report back to the SRC within 4 weeks.

(a) You decide that students and teachers are the stakeholders and parents need not be involved in the decision. How will this decision make your job easier?

(b) Why should you conduct a sample of the students?

(c) Why should you conduct a sample of the teachers even though there are only 110 of them?

(d) Your friend, who is very good at mathematics, suggests that you should conduct a stratified random sample. She says that this means that the number of students sampled would be allocated in proportion to some criteria such as year level or gender or both and the proportion of teachers to students sampled would be in proportion to the number of teachers and students in the school.

Explain why this is a good suggestion.

(e) Make up at least three of the questions that would form part of your survey instrument. Test them on a few of your friends to see if there are any flaws in them.

9 The following table shows the width and length, both measured in centimetres, of 88 bivalves of the same species collected on a beach.

W **(cm)**	1.3	1.4	1.4	1.5	1.9	2.1	2.1	2.1	2.2	2.2	2.3	2.4	2.4	2.5
L **(cm)**	1.7	1.9	1.8	1.9	2.3	3.5	2.8	2.7	3.0	2.7	3.0	3.2	2.9	3.5

W **(cm)**	2.5	2.6	2.6	2.6	2.7	2.8	2.8	2.9	2.9	2.9	2.9	3.0	3.0	3.0
L **(cm)**	3.3	3.5	3.4	3.4	3.2	3.8	3.5	3.8	3.9	3.8	3.7	3.8	4.0	3.7

W **(cm)**	3.1	3.4	3.4	3.5	3.6	3.6	3.6	3.6	3.7	3.9	3.9	4.1	4.1	4.1
L **(cm)**	4.1	4.4	4.3	4.5	4.6	4.8	4.9	4.5	4.9	5.4	5.2	5.3	5.4	5.2

W **(cm)**	4.2	4.2	4.2	4.2	4.3	4.7	4.8	4.8	4.8	4.9	5.0	5.0	5.1	5.2
L **(cm)**	5.4	5.5	5.6	5.2	5.6	6.1	6.6	6.1	6.0	6.1	6.6	6.2	6.6	6.4

W **(cm)**	5.2	5.4	5.6	5.6	5.8	5.8	5.8	5.9	6.0	6.0	6.0	6.2	6.2	6.3
L **(cm)**	6.8	6.3	7.0	7.9	7.4	7.4	6.9	7.9	7.7	7.8	7.3	8.5	7.5	7.6

W **(cm)**	6.5	6.6	6.6	6.7	6.7	6.7	6.7	6.9	6.9	7.0	7.1	7.1	7.1	7.2
L **(cm)**	9.3	8.1	8.0	8.2	9.2	8.0	7.9	8.6	8.1	9.5	9.1	9.1	9.4	8.4

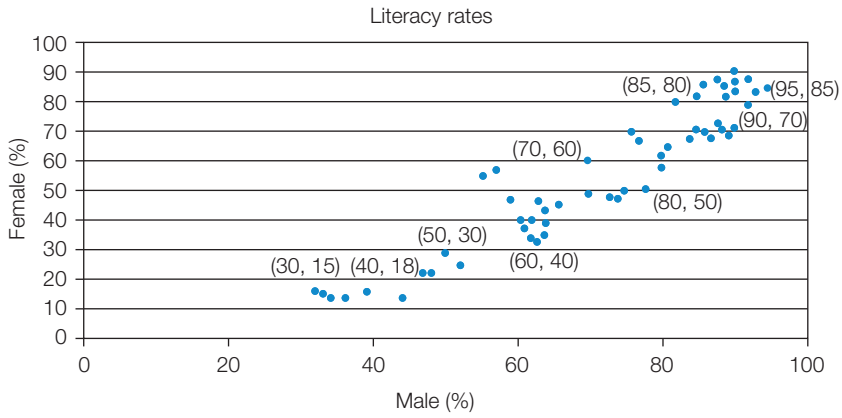
W **(cm)**	7.3	7.3	7.4	7.5
L **(cm)**	9.3	8.9	8.9	8.6

(a) Use a CAS, spreadsheet or any other form of technology to draw a scatter plot.

(b) Comment on any relationship evident between the measurements.

Reasoning

10 A search of the internet was conducted on literacy rates for countries with literacy rates regarded as low or medium. The following graph was produced as part of the report.

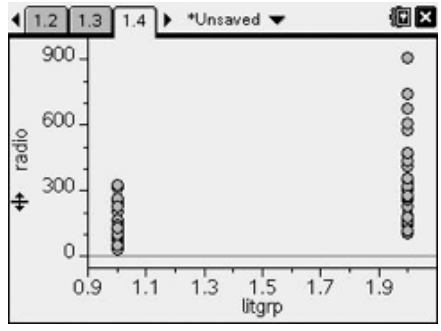


(a) Is the data primary or secondary?

(b) Describe any relationship you can see between male and female literacy rates. As part of this answer state whether you think, in general, male or female literacy rates are higher.

(c) Could your conclusions in **(b)** be applied to countries regarded as having high literacy rates?

11 The internet search of literacy rates also produced data regarding the number of radios per 1000 people in the population. From countries with a literacy rate listed as low (litgrp of 1) or medium (litgrp of 2) the following graph was produced.



(a) What information can you gather from this graph?

(b) The student doing the investigation had meant to construct a box plot. Given the following statistics, construct a parallel box plot for the data.

	Min	Q_L	Median	Q_U	Maximum
Low literacy	34	79	130	226	329
Medium literacy	107	178	306	434	907

(c) Use your parallel box plot to help you make a few statements about the two data sets.

(d) There is one data pair that looks like an outlier. Which pair is this?

Open-ended

12 Suggest a topic for which a census would be appropriate and which you could realistically undertake. Construct the survey instrument that you would use.

13 Suggest a topic for which a sample would be appropriate and which you could realistically undertake. Construct the survey instrument that you would use.

Half-time 7

Ex. 7.2

1 Complete the following tasks for the data shown in the table.

(a) Find the range.

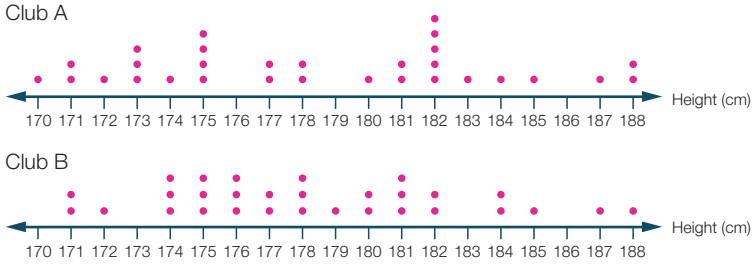
(b) Find the interquartile range.

(c) Draw a box plot.

Class interval	Frequency
10–<15	9
15–<20	15
20–<25	12
25–<30	17
30–<35	13
35–<40	11

Ex. 7.3

2 The following parallel dot plot shows the heights of players at two hockey clubs who play in the same competition. The heights have been rounded to the nearest centimetre.



(a) Which club appears to have the taller players?

(b) Calculate the mean, median, range and IQR for each club.

(c) Draw a parallel box plot for the data.

(d) Reassess your answer for part **(a)**. Justify your assessment using the statistics you have calculated.

(e) Which statistic is the most helpful in making your assessment?

Ex. 7.1

3 Construct a cumulative frequency curve and use it to find an estimate for the median of the following data set.

Class interval	Frequency
0–<20	12
20–<40	19
40–<60	21
60–<80	32
80–<100	19

Ex. 7.4

4 The following table gives the head circumferences, in cm, and the lengths, in cm, of 12 babies at birth. Draw a scatter plot of the data. Put head circumference on the horizontal axis. Make a comment about the type of relationship, if any, that is displayed by the data.

Head circumference (cm)	38	35	39	37	34	32
Length (cm)	51	50	54	53.5	50.5	47.5

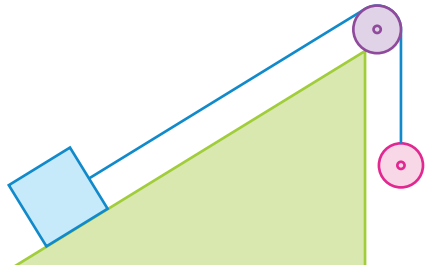
Head circumference (cm)	33	35	35	37	39.5	36
Length (cm)	50	53	49	51	51	50.5

Time-related data

7.5

When we are dealing with two variables in statistics we refer to the data as bivariate data. Often one of the variables causes the other variable to change. The variable that causes the other to change is called the **independent variable**, and the one that changes is called the **dependent variable**.

As an example, if we were conducting an experiment to see the effect that different weights had on the distance a block moved up an inclined plane, the mass of the weight would be the independent variable and the distance moved up the plane would be the dependent variable. It is the weight that causes the block to move up the plane, it is not the distance moved up the plane that causes the mass of the weight to change.



In some cases, it is difficult to choose between the two variables being studied. In cases like these we generally take the first variable given to be the variable for the horizontal axis, which is where the independent variable is always located. In statistics, if time is one of the variables, it is almost always the independent variable.

We will use scatter plots as a way of examining time-related data. When we have large data sets it may be appropriate to use technology such as CAS or a spreadsheet program such as Excel to draw the scatter plot. However, for smaller data sets you should draw the scatter plot by hand.

Worked Example 11

WE11

The following table shows the temperatures, in °C, for Jerusalem in Israel over a 24-hour period from 8 a.m. one morning to 7 a.m. the next morning.

Time	8 a.m.	9 a.m.	10 a.m.	11 a.m.	noon	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.
Temperature (°C)	18.4	21	21.5	22.3	23.3	24.7	24.9	25.3	24.2	23.2	22	20.4

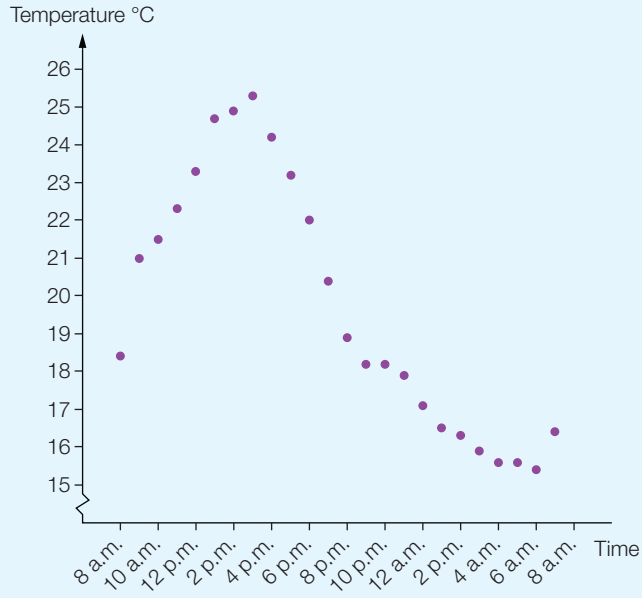
Time	8 p.m.	9 p.m.	10 p.m.	11 p.m.	mid-night	1 a.m.	2 a.m.	3 a.m.	4 a.m.	5 a.m.	6 a.m.	7 a.m.
Temperature (°C)	18.9	18.2	18.2	17.9	17.1	16.5	16.36	15.9	15.6	15.6	15.4	16.5

(a) Draw a scatter plot to show these temperatures.

(b) What is the maximum temperature shown on the graph?

(c) Is this necessarily the maximum daily temperature? If not, what might that maximum temperature have been?

(d) Do the temperatures follow a similar pattern to what you would expect for where you live?

Thinking	Working
(a) Draw the scatter plot in the usual way. To be able to observe the changes it will be necessary to leave out some of the vertical axis.	(a) 
(b) Read the maximum value from the table.	(b) The maximum temperature recorded is 25.3 °C
(c) Where else might the maximum have occurred?	(c) If this is not the absolute maximum it is likely to have been in the preceding hour, as the temperature falls away more quickly after 3 p.m. than it was rising before.
(d) Interpret the graph in relation to your own place of residence.	(d) It doesn't really matter where you live, it is hotter during the day than at night.

7.5 Time-related data

Navigator

Answers page 754

Q1, Q2, Q3, Q5, Q9, Q10	Q1, Q2, Q3, Q4, Q6, Q7, Q9	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9

Equipment required: a CAS or Excel would be useful for Questions **3–5**

Fluency

WE11

1 The table below gives the wind speeds, in km/h, for Quebec in Canada over a 24-hour period.

Time	mid-night	1 a.m.	2 a.m.	3 a.m.	4 a.m.	5 a.m.	6 a.m.	7 a.m.	8 a.m.	9 a.m.	10 a.m.	11 a.m.
Wind speed (km/h)	9	9	13	7	6	6	4	4	6	4	7	6

Time	noon	1 p.m.	2 p.m.	3 p.m.	4 p.m.	5 p.m.	6 p.m.	7 p.m.	8 p.m.	9 p.m.	10 p.m.	11 p.m.
Wind speed (km/h)	6	7	9	7	11	13	11	13	11	9	9	11

(a) Draw a scatter plot to show the wind speed.

(b) What seems unusual about the 2 a.m. recording?

(c) Describe what is happening to the wind speed between 4 p.m. and 8 p.m.

2 For the following pairs of variables, state which one you think is the independent variable. Give a brief reason for your answer.

(a) Test score and number of hours spent studying.

(b) Height, measured in cm, and age, measured in months, for children less than 3 years of age.

(c) Age at death and average number of cigarettes smoked per day.

(d) Amount of rubbish collected, measured in kg, and the number of people in a household.

3 The following data set indicates the number of bicycles that crossed northbound on the Sydney Harbour bridge for 29 consecutive Fridays.

332 372 367 370 361 398 387 363 447 331 530 454 460 428 283
515 444 498 481 500 396 440 402 383 339 325 402 517 416

Draw a scatter plot to represent the data. Use your CAS, if you have one, or draw the graph by hand. Comment on the graph produced.

Understanding

4 The following table gives the number of 10-year-old Australians in June of each year from 1971 to 2009.

1971	260 439	1981	282 850	1991	252 907	2001	275 465
1972	261 046	1982	280 163	1992	254 099	2002	275 707
1973	258 136	1983	266 496	1993	258 857	2003	275 806
1974	256 359	1984	257 190	1994	258 625	2004	276 100
1975	244 424	1985	248 865	1995	259 391	2005	278 212
1976	242 216	1986	242 895	1996	261 345	2006	273 908
1977	244 927	1987	239 390	1997	259 506	2007	274 506
1978	248 919	1988	241 462	1998	261 417	2008	274 009
1979	261 286	1989	242 343	1999	264 986	2009	276 519
1980	262 714	1990	243 448	2000	271 193		

(a) Use Excel, or some other suitable form of technology, to draw a scatter plot displaying the data.

(b) Comment on the trend seen in the number of 10-year-old Australians over this time period.

5 The following table gives the number of road deaths per 100 000 population for South Australia in the period 1975–2008.

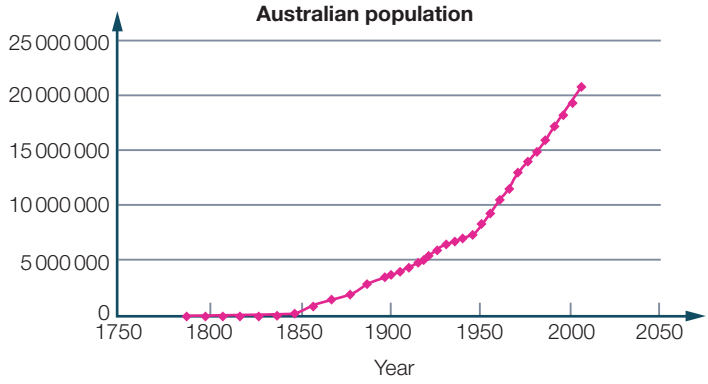
Year	1975	1980	1985	1990	1992	1993	1994	1995	1996	1997	1998	1999
Rate	26.79	20.56	19.54	15.78	11.33	14.92	10.84	12.32	12.28	9.99	11.28	10.08
Year	2000	2001	2002	2003	2004	2005	2006	2007	2008			
Rate	11.03	10.12	10.12	10.25	9.02	9.53	7.46	7.83	6.18			

(a) Use some form of technology to assist you in drawing a scatter plot to represent the data.

(b) Describe any trends you see in the data.

Reasoning

6 The graph on the right shows the estimated Australian population from 1788 until 2006.



(a) Describe the trend you see in the graph.

The figures prior to 1961 do not include the Aboriginal population. Estimates of the size of the Aboriginal population prior to European settlement range from 300 000 to 1 000 000.

(b) How do you think the inclusion of the Aboriginal population prior to 1961 would affect the graph?

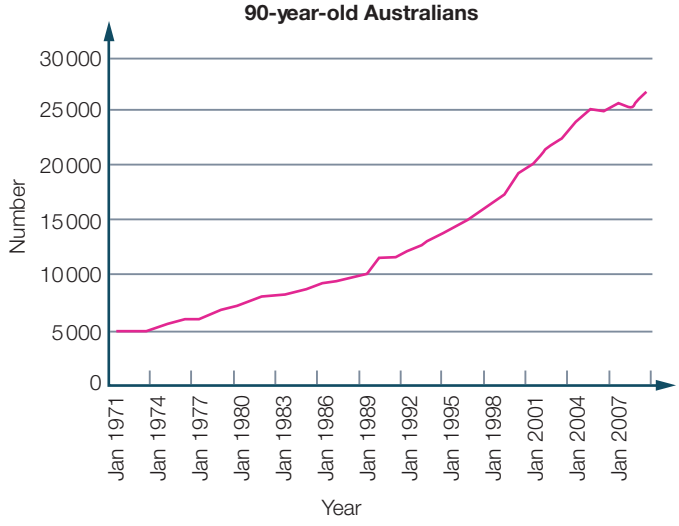
(c) Why does the graph seem to go along the horizontal axis for the first 50 years or so?

(d) How do you think the early figures were collected?

(e) Which figures do you think are more reliable, the early ones or the later ones?

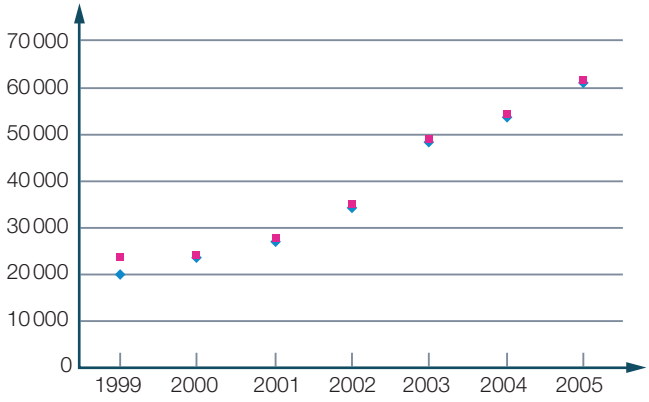
(f) How do we now estimate the Australian population?

7 The graph below shows the number of 90-year-old Australians from 1971 through to 2009. Comment on the trend you see in the graph and make some attempt to explain why this is the case. For what other ages would you expect to see this type of trend?



Open-ended

8 Sonika produced the graph below for the information given in the table. Suggest how she could improve her graph.



Mercedes-Benz E320 used car prices (August 2009)

Year	Elegance	Avantgarde
1999	19 700	23 600
2000	23 400	24 100
2001	26 700	27 600
2002	34 000	35 000
2003	48 200	49 000
2004	53 500	54 400
2005	60 800	61 800

9 The table below gives some information about extinct and endangered species in Australia. These figures relate to the period after European settlement of Australia.

	Extinct	Critically endangered	Endangered	Vulnerable	Total
Total plant species and subspecies	49	81	523	665	1318
Mammals	27	4	33	55	119
Birds	23	6	41	61	131
Reptiles	0	2	14	37	53
Frogs	4	2	14	12	32
Fish	1	3	16	25	45
Invertebrates	0	16	12	6	34
Total animal species and subspecies	55	33	130	196	414

Source: www.climatechange.gov.au/publications/biodiversity/~/media/publications/biodiversity/biodiversity-vulnerability-assessment-lowres.ashx

Of the 27 mammal species that are extinct, 10 of these have occurred since 1900. Some other information about Australia's biodiversity is as follows:

- 82% of our mammals, 45% of our land birds, 89% of our reptiles and 93% of our frogs occur nowhere else
- estimates of probable global losses of species are as high as 25% over the next 30 years
- 90% of native vegetation in the eastern temperate zone has been removed or replaced by introduced crops
- 50% of rainforests have been cleared
- the proportion of Australia covered by woodland or forest has been reduced by more than one-third.

Your task is to produce some graphs to indicate how you think Australia's biodiversity has changed since European settlement. You may wish to research some more information.

10 Write a story about Scott's car journey in his 4WD that lasted at least 8 hours and during which time he travelled at least 550 km. Draw a line graph to represent the story you write.

7.6 Statistics in the media

We frequently read statistics in newspapers and magazines and hear them discussed and reported on the television in news bulletins. However, we need to think carefully about what we hear and read because things are not always as they seem. Some of the issues that may arise include:

- insufficient evidence being provided to support the conclusions made
- percentages used without an indication of the actual size of the survey
- raw figures used that are based on very different populations
- oversimplification of the results.

This is not to say everything we read or hear is incorrect, but it is a good idea to think critically about the key statistical issues involved. As an example, consider the two tables that follow, which accompanied a newspaper article.

Dressing with danger			
% of clothes that cause injuries		% of injuries caused by clothes treated at Victorian hospitals	
Shoes	48.1	Fracture, excluding tooth	18.8
Jumper, jersey, cardigan, shirt	13.0	Sprain/strain	17.2
Buttons	9.2	Open wounds	13.3
Jeans/pants	8.3	Foreign body	9.6
Socks	7.4	Burn	9.0
Clothes, not specified	5.8	Superficial	8.4
Zippers	4.7	Muscle/tendon injury	5.2
Nightwear	3.5	Dislocation	4.9
		Eye	2.9
		Bites	1.2
		Other	9.5

Source: Victorian Injury Surveillance and Applied Research System

At first glance, it appears that clothing injuries could be quite a serious problem. However, when we consider how many injuries in total get treated at Victorian hospitals, very few of them would be caused by clothing.

7.6 Statistics in the media

Navigator

Q1, Q2, Q3, Q4, Q9, Q10	Q1, Q3, Q4, Q5, Q6, Q8, Q9, Q10	Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10

Answers page 755

Fluency

This article appeared in the *Age* on 21 January 2010.

NBA hotshot Bogut closes in on Shark

Having made his presence felt in the National Basketball Association, Andrew Bogut is starting to cast his considerable shadow over the rest of Australia's sporting stars in money matters.

The Milwaukee Bucks centre has shot up from fifth to second in BRW's latest list of the nation's top 50 sports earners, behind occasional golfer Greg Norman.

The 25-year-old's five-year, $US60 million ($A65 million) contract extension has kicked in for this NBA season and Bogut has responded with his most productive season, averaging 15.1 points and 10.4 rebounds a game.

BRW listed Bogut as earning $14 million a year, hot on Norman's ($15 million) tail, even if the two-time British Open champion's wealth has long been coming from off the golf course. Harry Kewell remained our best-paid football player despite his earnings dipping from $10.5 million to $9.5 million at Turkish club Galatasaray.

Formula One ace Mark Webber ($9 million) was next best, one of four involved in motor sports that made the top 10, ahead of motocross star Chad Reed ($8.8 million), NASCAR driver Marcos Ambrose ($5.44 million) and 2007 MotoGP world champion Casey Stoner ($5 million).

Geoff Ogilvy ($6.34 million) is the highest full-time golfer on this list, while Lleyton Hewitt has dropped out of the top 10, his earning power shrinking from $6.6 million to $2.88 million.

Australia's cricketers remain the best-paid locally based sportsmen. Skipper Ricky Ponting ($4.26 million) is the sport's top earner, despite never having really cashed in on the lucrative Indian Premier League.

Fast bowler Brett Lee ($3.52 million) has shown that playing power does not always equate to earning power, having been rated as the nation's second-best-paid cricketer despite missing the Ashes and home summer. Vice-captain Michael Clarke was ranked 21st with $2.54 million and Michael Hussey ($1.30 million) was at 41. Wild child Andrew Symonds made the list at 50 with $1.22 million.

TOP 5 MONEY-EARNERS IN AUSTRALIAN SPORT	
1 Greg Norman (golf)	$15m
2 Andrew Bogut (basketball)	$14m
3 Harry Kewell (soccer)	$9.5m
4 Mark Webber (formula one)	$9m
5 Chad Reed (motocross)	$8.8m

Source: *The Age*, 21 Jan 2010, Sport 9

1 (a) Draw up a table and include all of the named Australian sporting stars and estimate their positions in the Top 50 Earners list. Why is it difficult to be precise about some of the positions?

(b) What percentage drop in earnings did Harry Kewell suffer when moving clubs?

(c) What percentage slide did Lleyton Hewitt suffer in his earnings?

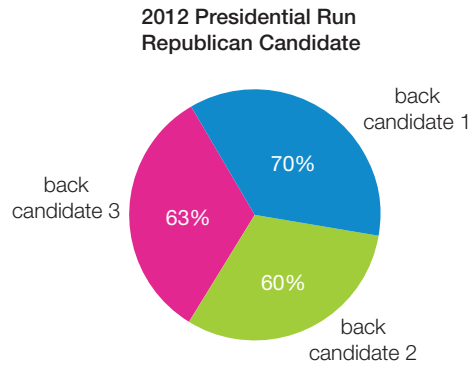
(d) How do you think the figure of $14m for Andrew Bogut was arrived at?

The *Herald Sun*, reporting on the same issue, noted that "Samantha Stosur was the second highest earning tennis player, with $1.53 million".

(e) Where would this place Samantha Stosur in your Top 50?

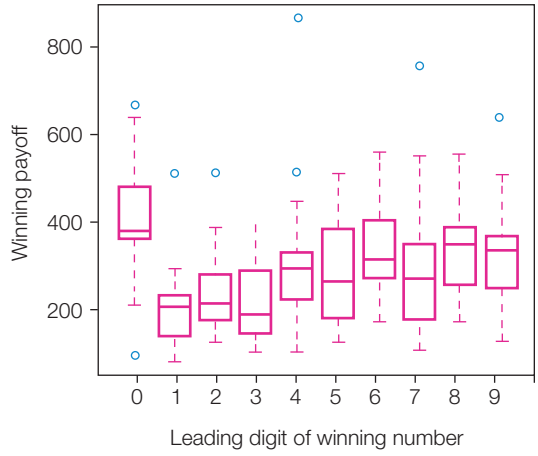
2 (a) In the early stages of the nomination process for the Republican Party leading up to the 2012 election for the US President, an American news network produced a graph similar to the one shown here.

What problems can you see with this graph? Why do you think the graph has been drawn this way? Could you draw an accurate graph from the information provided?



(b) Some graphs provide a simple and accurate visual representation of information. This box plot shows the winnings ($) in the New Jersey Pick-It Lottery. In this lottery you pick any three-digit number you like and the total prize pool is divided by the number of winning entries.

What do you think this graph is telling you and why do you think that is the case?



Understanding

3 The following article by John Rolfe appeared in the *Daily Telegraph* on 19 January 2010.

Why car insurers are ripping us off?

INSURERS have slugged Sydney drivers an extra $100 to make up for poor returns on investments—even though crash and theft rates have fallen dramatically.

Premiums have increased at up to 10 times the rate of inflation, figures obtained by the *Daily Telegraph* show. The price of the average greenslip rose 14% last year and comprehensive cover increased 5%. Yet Sydney's overall price inflation was just 1.3% during that time.

When premium increases are compared to crash and theft rates the differences are even more stark. The most recent accident statistics show a 6% annual fall, while car robberies declined 9%.

The *Daily Telegraph's* investigation exposes the cost of cover is no longer driven by risk, but by financial markets. Insurers make money by investing customers' premiums before that money is paid out in claims.

KPMG insurance group head Brian Greig said: "The returns they were able to generate pre-global financial crisis were substantially higher than they are now. When that (investment) lever is a little flat they look at other levers to generate acceptable returns, because sharemarket analysts are always on to them to achieve that."

The main "lever" is premium increases.

"Insurers are using premiums to compensate for lower investment returns," said Damian Smith, CEO of insurance comparison service RateCity.

Until the GFC, "premium growth wasn't that important", Mr Smith said. "But now insurers are using premiums to try to make up for the investments."

Data from the Motor Accidents Authority of NSW shows the average greenslip purchased during the three months to September 2009 cost $390, up $49 on a year earlier. Before the GFC took effect in 2007, compulsory third party insurance had fallen to $319, lower than 2004 levels.

"The increase seen over the last two reporting periods is directly related to the impact of the GFC," MAA spokesman Christian Fanker said.

Figures from Insurance Statistics Australia—the source the insurance industry uses—reveal comprehensive cover increased 4.9% last financial year. For a Sydney household driving a late-model family car, this would add more than $50 to premiums.

This means the total increase imposed on most drivers exceeds $100.

The number of crashes on NSW roads fell to 9692 per million cars in 2008—down 9.2% from 10,531 in 2007, RTA statistics show. There were 42 833 crashes in 2008. Official crash data for 2009 has not been made public.

(a) According to the article, how much did the average 'greenslip' cost in the 3-month period to September 2009?

(b) How much did the same cover cost a year earlier?

(c) What percentage change is this? Does this agree with what the article claims is the increase?

(d) If the cost had increased at the stated inflation rate, what should the premium have been?

(e) The article claims comprehensive premiums have increased by more than $50. Assuming it is exactly $50, what does this make the old and new costs for comprehensive cover?

(f) What is the average annual increase in greenslip cost from 2007 to 2009?

(g) Why does the article mention the number of crashes?

4 The following article by Beth Rivers appeared in *PerthNow* on 10 January 2010.

Perth's most prized suburbs

CANNING Vale was in demand in 2009, with more houses sold in that suburb than any other, but Guildford delivered the biggest median house price rise.

Landgate Valuation Services' figures show Canning Vale was way out in front with 506 houses sold, ahead of Thornlie with 340 and Ellenbrook with 330.

Real Estate Institute of WA president Alan Bourke said the result was to be expected because of the first homebuyer's boost stimulating outer-metropolitan suburbs. The best performing suburbs in terms of median price rises last year were Guildford, Mt Richon and Victoria Park.

From 2008 to 2009, Guildford saw a rise in its median house price of 17.26 per cent, from $565 000 to $662 500. The median in Mt Richon increased 16.86 per cent from $422 500 to $493 750, and Victoria Park's median increased 15.65 per cent, from $575 000 to $665 000.

The suburbs whose median prices were hit the hardest were North Beach, with its median price falling 35.38 per cent ($1.22 million to $790 000), Ascot, which fell by 32.14 per cent ($766 250 to $520 000), and Mt Lawley's median price fell by 28.28 per cent ($1.22 million to $875 000). Mr Bourke said North Beach was just one example of the many top-end suburbs "hurt" during 2009.

"There was more activity in the bottom end of the market so the median moved down (in the top end)", he said. "There was also a lot of land released across the road from the Ascot Racecourse, which would have pushed the median price down there."

Peppermint Grove retained its crown as being Perth's most expensive suburb, with a median house price of $4.2 million, while Medina, near Kwinana, was the least expensive suburb with a median of $230 000. Most recent REIWA data showed more than 31,500 properties were sold last year.

About 80% were houses and 20 per cent multi-residential, including apartments and units.

Best performing suburbs			
Median sale price		2008	2009
1	Guildford	$565 000	$662 500
2	Mount Richon	$422 500	$493 750
3	Victoria Park	$575 000	$665 000
4	Darlington	$515 000	$590 000
5	Orelia	$297 500	$339 000
6	Maylands	$670 000	$754 000
7	South Fremantle	$825 000	$925 000
8	Stoneville	$457 500	$511 250
9	Fremantle	$795 000	$880 000
10	Joondanna	$633 000	$697 500

Worst performing suburbs			
Median sale price		2008	2009
1	North Beach	$1 222 500	$790 000
2	Ascot	$766 250	$520 000
3	Mount Lawley	$1 220 000	$875 000
4	Mosman Park	$1 522 500	$1 150 000
5	Forrestdale	$520 000	$395 000
6	Applecross	$2 075 000	$1 602 500
7	Perth	$875 000	$690 000
8	Marmion	$885 000	$698 750
9	Kallaroo	$660 000	$545 000
10	Swanbourne	$1 600 000	$1 322 500

Suburbs by volume sold		2008
1	Canning Vale	506
2	Thornlie	340
3	Ellenbrook	330
4	Clarkson	277
5	Gosnells	275
6	Ballajura	256
7	Baldivis	238
8	Port Kennedy	235
9	Butler	224
10	Tapping	218

(a) Calculate the percentage increase for each of the 10 best performing suburbs from 2008 to 2009.

(b) Calculate the percentage decrease for each of the 10 worst performing suburbs from 2008 to 2009.

(c) Make some comment about the Perth housing market, based on these figures.

(d) What further information would make the Suburbs by volume sold table more useful to us?

(e) According to Mr Bourke "There was more activity in the bottom end of the market so the median moved down (in the top end)". What do you think this means? Why does one affect the other?

(f) Why do these figures refer to the median house price? What would be the problems with using either the mean or the mode to represent house prices?

5 The following article appeared in *The Age* in May 2011.

Peer guide for parents

Michelle Griffin
17 May 2011

WHEN Australian teenagers tell their parents that everyone else is allowed to go to the shopping centre alone or stay up late, their parents will finally have a guide to what other families are really saying.

The average age at which teenagers are allowed to make their own decisions is tabled in a survey released today by the federally funded parenting advice website Raising Children. Most children can choose their own clothes by the time they are $12\frac{1}{2}$, but the average age for lifting a teenager's curfew is $17\frac{1}{2}$—six months off legal adulthood.

While it's not possible to tell parents exactly what age to allow teenagers to make decisions, the website's content director, parenting expert Warren Cann, hopes the survey gives parents a point of comparison. "If you find yourself well below or above the norms, you might want to think why," he says.

The survey asked parents what they believed was the right age for children to be allowed to make decisions. Mr Cann believes parents may yield to child pressure earlier. "Kids tend to think they're ready two years before their parents want them to be," he says. The survey suggests the average age for getting a mobile phone is 14, but telecommunications research says teenagers typically get their first mobile at 13. The survey of 360 parents found a strong resistance to allowing younger teenagers open access to websites and social media. The average age of allowing them open access is $15\frac{1}{2}$. On average, parents don't want their children on Facebook or other social media before $14\frac{1}{2}$, even though Facebook allows them to register from 13, and many children as young as eight have active pages.

Primary school teacher and mother of three Simone Traynor allowed her oldest son, Jacob, now 17, to use Facebook from 14. He is allowed to use it even when doing his homework, as long as he maintains good marks in VCE. Ms Traynor's 15-year-old son is now allowed to log on but she will not allow her 12-year-old daughter Sonja to register for an account. Ms Traynor said: "Brandon is not that interested. Sonja's got plenty of real-life social activity, and it seems more vulnerable [for her]: girls can be nasty with these things."

Mother of two Kristin has allowed her daughter, now 16, on Facebook since she was 12, but polices her children's internet time through a net nanny, by reading their internet histories and by setting time limits. "She has to spend an hour off all screens before bed—no phone, iPod, iPad, laptop, nothing. She can read a book or come and have a conversation."

The survey was commissioned to mark the launch of the Raising Children's teenage section, which was built to respond to a growing demand for advice on coping with adolescents.

(a) How many people took part in this survey?

(b) Do you think this is enough people to justify the media interest in the results?

(c) To participate in this survey you needed to visit the website and access the survey online. What type of survey is this?

(d) One of the questions in the survey was: "At what age do you think adolescent boys/girls should be allowed to make their own decisions about the following issues?" A list of issues then followed. Do you think the results for the survey might have been different if the question was: "At what age did you allow your adolescent boys/girls to make their own decisions about the following issues?"

(e) One of the findings of the survey was reported as follows.

"The average age at which teenagers were permitted to wield a mobile phone was 14. This is 12 months older than 13, at which the telcos say is the average age children start using a mobile phone."

Why do you think there is such a difference in these reported ages?

(f) How much confidence would you have, in general, with the findings and conclusions of this survey?

Reasoning

6 The following article by Anna Caldwell appeared in the *Courier Mail* on 2 January 2010.

Coast residents betting on Mermaid strip for lotto win

HOPEFUL Queenslanders hankering for a new-year windfall—or perhaps an upgrade to the homes and lifestyles of the rich and famous—have swarmed newsagencies ahead of tonight's $30 million Gold Lotto megadraw.

And at no place in the state is there a firmer belief that the next big win is just a ticket away than at Golden Sands News at Mermaid Beach on the Gold Coast.

The multi-purpose shop is wedged between two very lucky newsagents. About 1 km in one direction is Nobby Beach News, where a punter took out the $20 million *Oz Lotto* in August and about 1 km in the other direction is Lucky Mermaid News where a couple won $53 million on *Oz Lotto* in June.

Figures released by Golden Casket have crowned the glitter strip Queensland's number one lotto hot spot for first division prize wins for the second year running.

Sixteen lucky first division winners on the Gold Coast shared in a mammoth $106.37 million in 2009.

Darrell Hiscock, who owns Golden Sands News, said that everyone had been tipping his shop to offer up the next golden ticket.

Mathew Graham, from Nobby News, said people stopped by the suburb just because they believed that visiting the Mermaid strip enhanced their chances of winning millions.

(a) Why are people tipping Golden Sands News as the seller of the winning *Gold Lotto* ticket?

(b) In your opinion, is there a statistical basis for their confidence?

(c) What other information would you like to have access to before you made a statement like this about where a winning ticket is likely to be sold?

Other people look at lists of previously drawn numbers and use these as the basis for the numbers they select. Some people look at the numbers most frequently drawn, whereas others look at those that have not been drawn for the longest time.

(d) Try to explain the reasoning both of these groups use to justify their thinking. Do you support either argument yourself?

7 The following fictional article is of a type often seen around election time. Read it and answer the questions that follow.

Poll – voters back Smith

Mr John Smith's return to the Federal Liberal leadership appears to have buoyed the Coalition's [Liberal + National parties] electoral chances, with one in four voters saying they are at least 'a bit more likely' to vote Liberal at the next federal election. Mr Smith was clearly preferred by female voters (47% support) over the leader of the Australian Labor Party, Mr Roger Casey (41%).

Men preferred Mr Casey (54%) to Mr Smith (45%). Support for the ALP among men was 47%, compared to the Coalition's (45%), while from women the Coalition attracted 47% support compared to the ALP's 44%. The survey also asked voters to identify those characteristics they believed Mr Casey or Mr Smith possessed. The results of the survey are summarised below.

SUNDAY BUGLE POLL

Voting intentions	%	Preferred PM	%
ALP	46	Casey	48
Coalition	46	Smith	46
Other/Don't know	8	Neither/Don't know	6

SUNDAY BUGLE POLL: RATING THE LEADERS

	Casey (%)	Smith (%)		Casey (%)	Smith (%)
Decisive	68	53	Sincere	22	55
Forward thinking	68	48	Too arrogant	75	15
Good for Australia	43	55	Honest	20	62
Weak	10	30	Capable	72	72
A good leader	66	43	None of the above	1	2

Source: Corio Market Research 1030 voters from Vic., NSW. Poll done by telephone.

(a) Do the survey results agree with the heading 'Voters back Smith'? Discuss.

(b) Comment on the sample size—is it appropriate for a conclusion?

(c) The survey was done by telephone—is this the best way to do such a survey? Discuss.

(d) Why have the results for choice of preferred PM been separated into male and female opinions?

(e) Why don't the totals of male and female voters' opinions add up to 100%?

(f) Discuss the categories that describe Roger Casey and John Smith, such as decisive and forward thinking. Comment on the appropriateness of these categories. Can you add some of your own?

(g) Is this survey just about the preferred Prime Minister or is it about something else?

8 The Australian Bureau of Statistics releases more than just statistics. They also conduct some analysis of the data and produce reports such as the one below.

DEMOGRAPHIC, SOCIAL AND ECONOMIC CHARACTERISTICS: LIFE EXPECTANCY

This article is part of a comprehensive series released as The Health and Welfare of Australia's Aboriginal and Torres Strait Islander Peoples.

KEY MESSAGES

At the national level for 2005-2007, the gap between Aboriginal and Torres Strait Islander and non-Indigenous life expectancy was 11.5 years for males and 9.7 years for females.

Life expectancy at birth for Aboriginal and Torres Strait Islander males is estimated to be 67.2 years, and 72.9 years for females.

This topic provides an overview of estimates of Aboriginal and Torres Strait Islander life expectancy. Data presented are from ABS experimental life tables for Aboriginal and Torres Strait Islander Australians and provide context for the detailed health and welfare information presented in other topics in this release.

LIFE EXPECTANCY

Life expectancy is a summary measure of mortality. It represents the estimated number of years a newborn baby would live if current mortality patterns remained constant throughout the baby's lifetime (see Box 1.7).

At the national level for 2005-2007:

- Life expectancy at birth for Aboriginal and Torres Strait Islander males is estimated to be 67.2 years, 11.5 years less than life expectancy at birth for non-Indigenous males (78.7 years).
- Life expectancy at birth for Aboriginal and Torres Strait Islander females is estimated to be 72.9 years, 9.7 years less than life expectancy at birth for non-Indigenous females (82.6 years).
- Life expectancy at birth differs across the states and territories, with the highest life expectancy estimates in New South Wales (69.9 years for males and 75.0 years for females) and lowest in the Northern Territory (61.5 years for males and 69.2 years for females). As shown in table 1.6, differences in life expectancy at birth estimates between non-Indigenous and Aboriginal and Torres Strait Islander Australians are greatest in the Northern Territory (14.2 years for males and 11.9 years for females) and Western Australia (14.0 years for males and 12.5 years for females), and lowest in New South Wales (8.8 years for males and 7.5 years for females).

1.6 LIFE EXPECTANCY AT BIRTH (a), by Indigenous status 2005–2007

	Life Expectancy at Birth			
	Indigenous years	Non-Indigenous years	Total (b) years	Difference Between Non-Indigenous and Indigenous Life Expectancy at Birth (c) years
			Males	
NSW	69.9	78.7	78.5	8.8
Qld	68.3	78.6	78.4	10.4
WA	65.0	79.0	78.7	14.0
NT	61.5	75.7	72.0	14.2
Aust (d)	**67.2**	**78.7**	**78.5**	**11.5**
			Females	
NSW	75	82.5	82.4	7.5
Qld	73.6	82.5	82.3	8.9
WA	70.4	82.9	82.5	12.5
NT	69.2	81.2	77.6	11.9
Aust (d)	**72.9**	**82.6**	**82.4**	**9.7**

(a) Due to significant changes in methodology, estimates of life expectancy at birth for 2005-2007 are not comparable to previously published estimates.

(b) Estimates of life expectancy at birth for the total population presented in this publication differ from estimates in ABS 2007, Deaths, Australia, 2006, cat. no. 3302.0.

(c) Differences are based on unrounded estimates.

(d) Includes all states and territories.

Source: ABS 2009, Experimental Life Tables for Aboriginal and Torres Strait Islander Australians, 2005-2007, cat. no. 3302.0.55.003, ABS, Canberra

(a) Why do you think only three States and the Northern Territory are recorded separately in the article above?

(b) According to the information in the article, what was the life expectancy for non-Indigenous males and females at this time?

(c) Why do you think there are such differences in life expectancy for Aboriginal and Torres Strait Islanders compared to non-Indigenous Australians?

Another Government body, the Australian Institute of Health and Welfare, contributes the following additional information.

Australia appears to compare poorly with other countries for Indigenous life expectancy at birth. For example, Registered Indians (Aboriginal people) in Canada have an estimated life expectancy of about 6-8 years below that of the non-Aboriginal population (Tjemka, M. 2002. Statistics Canada, cat. no. 82-003). In New Zealand, Maori life expectancy is 8-9 years below that of the non-Maori population (www.stats.govt.nz). Although comparing these very different populations is perhaps over-simplifying the situation, it does illustrate that the Australian Indigenous life expectancy gap between non-Indigenous persons is approximately twice as large as in comparable developed nations.

Unfortunately, it is difficult to accurately assess the life expectancy gap between Indigenous and non-Indigenous persons. This is due to the fact that the population counts of Indigenous persons in Australia both over time and between jurisdictions is inaccurate, and that Indigenous persons are often undercounted when recording deaths.

(d) This extract mentions a couple of problems that arise when dealing with the issue of the life expectancy of the Australian Indigenous population. What are these problems?

(e) Why would you want to attempt to make comparisons with Canada, New Zealand and the United States?

The Australian Bureau of Statistics can provide further data to assist in our analysis.

In 2009, the median age at death of Aboriginal and Torres Strait Islander Australian males varied across the states and territories, from 48 years in South Australia to 57 years in New South Wales, compared with 67 years in the Northern Territory to 79 years in South Australia for non-Indigenous males. Similarly, the median age at death of Aboriginal and Torres Strait Islander Australian females was lower than for non-Indigenous females, 53 years in South Australia to 66 years in New South Wales compared with 72 years in Northern Territory to 85 years in South Australia.

3.6 MEDIAN AGE AT DEATH, Indigenous status (a) 2004–2009

	NSW	Qld	SA	WA	NT	Total (b)
Males						
Indigenous						
2004	55.8	53.7	49.5	50.0	43.8	51.2
2005	54.3	51.1	42.4	52.8	45.8	50.4
2006	59.3	55.6	50.4	47.9	45.4	52.4
2007	58.1	54.7	50.5	53.3	45.9	53.1
2008	59.9	53.2	49.0	51.2	52.1	53.8
2009	57.2	53.2	48.0	50.6	48.3	52.5
Non-Indigenous						
2004	77.0	76.2	77.6	76.3	63.0	76.8
2005	77.2	76.4	77.9	76.6	63.7	76.9
2006	77.8	76.7	78.3	76.9	64.7	77.4
2007	78.1	77.1	78.7	76.9	64.6	77.7
2008	78.5	77.3	79.2	77.1	66.3	78.0
2009	78.4	77.2	79.3	77.3	66.6	78.0

	NSW	Qld	SA	WA	NT	Total (b)
Females						
Indigenous						
2004	62.7	57.9	53.5	63.6	54.0	60.1
2005	65.8	59.5	47.5	57.8	50.4	57.9
2006	64.8	57.0	59.3	57.0	55.3	59.0
2007	63.0	59.5	58.3	59.3	55.7	59.5
2008	63.8	62.3	53.5	64.0	56.0	60.5
2009	65.9	62.6	53.0	58.7	55.4	61.3
Non-Indigenous						
2004	82.8	82.5	83.3	82.3	71.3	82.7
2005	83.1	82.6	83.7	83.2	70.5	83.1
2006	83.5	83.1	84.1	83.1	75.0	83.4
2007	83.7	83.3	84.3	83.4	69.3	83.6
2008	84.2	83.7	84.6	84.1	75.7	84.1
2009	84.1	83.4	84.6	83.6	71.8	83.9

(a) Care should be exercised when comparing median age at death of Indigenous and non-Indigenous Australians. See Commentary.

(b) Data are for NSW, Qld, WA, SA and NT combined, based on state or territory of usual residence. Victoria, Tasmania and the Australian Capital Territory are excluded due to small numbers of registered Indigenous deaths.

(f) Write a couple of sentences describing the conclusion you would draw from this additional information.

Open-ended

9 Write your own media release that includes statistics in the body of the article as well as in a table.

10 The following advertisement appeared in the Victorian press in 2010.

The Victorian Government developed this advertisement in 2010 with funds from the Community Support Fund.

Write a paragraph or two about this advertisement. You might like to include comment on the following questions or take a different angle altogether.

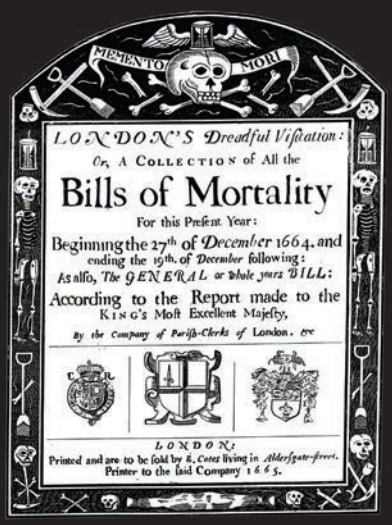
What do you think about this advertisement? What do you think about the statistics that are quoted in the advertisement? How would they have been calculated? Do you think this would be effective in reducing problem gambling?

Maths 4 Real

John Graunt
–father of statistics

John Graunt is regarded as one of the first people to make use of statistical information and to draw conclusions from his analysis of the data collected. He collected information about a range of situations from the weekly Bills of Mortality produced in the city of London.

During the time that John Graunt collected and analysed his statistics, London was suffering an outbreak of the bubonic plague every few years. It is thought that this disease was carried to Europe by rats on merchants' ships as trade with Asia expanded towards the end of the Middle Ages.


MEMENTO MORI

LONDON'S Dreadful Visitation:
Or, A COLLECTION of All the
Bills of Mortality
For this Present Year:
Beginning the 27th of December 1664. and ending the 19th. of December following:
As also, The GENERAL or Whole years BILL:
According to the Report made to the KING'S Most Excellent Majesty,
By the Company of Parish-Clerks of London. &c

LONDON:
Printed and are to be sold by E. Cotes living in Aldersgate-street. Printer to the said Company 1665.

The bubonic plague was sometimes referred to as the Black Death, a reference to the way the skin of a sufferer turned black. People were employed to go through the streets collecting dead bodies on carts and taking them away for disposal. It was soon observed that contact with the sufferer greatly increased the chances of an individual falling victim to the disease themself.

The table that follows is a reproduction of part of Graunt's data regarding the plague. As can be seen, there were a great number of deaths associated with the plague. The table is expressed in the form in which Graunt produced it. As the standards of written English have changed over time, a little explanation is necessary.

'Anno 1592' means in the year 1592.

'Whereof of the Plague' means died as a result of the plague.

'Christned in the said year' gives the number of babies christened in the particular year.

For that time this gives a relatively accurate estimate of the number of births, as most babies were christened.

There have been in London, within this Age, four Times of great Mortality, that is to say, the years 1592, and 1593, 1603, 1625, and 1636.

There died *Anno* 1592 from *March* to *December*	25 886	Anno 1603 within the same space of time were Buried	37 294
Whereof of the *Plague*	11 503	Whereof of the *Plague*	30 561
Anno 1593	17 844	*Anno* 1625 within the same space	51 758
Whereof of the *Plague*	10 662	Whereof of the *Plague*	35 417
Christned in the said year	4021	Anno 1636 from *April* to *December*	23 359
		Whereof of the Plague	10 400

1 Draw a graph to represent the total deaths and deaths from plague for each year given. Make sure that the graph type you choose enables you to compare the years.

2 Which year was the worst for deaths from the plague as compared with deaths from all other causes? (This is not necessarily the year with the greatest number of plague deaths.)

The following extract shows some further data collected by Graunt. This time we are told about the number of burials not associated with the plague ('Buried in all'), the number of deaths associated with the plague ('Besides of the *Plague*') and the approximate number of children born (Christned) for each of the years 1604–1661.

Anno Dom.	Buried in all	Besides of the Plague	Christned	Anno Dom.	Buried in all	Besides of the Plague	Christned
1604	4323	896	5458	1628	7740	3	8564
1605	5948	444	6504	1629	8771	0	9901
1606	5796	2124	6614	1630	9237	1317	9315
1607	5670	2352	6582	1631	8288	274	8524
1608	6748	2262	6845	1632	9527	8	9584
1609	7545	4240	6388	1633	8392	0	9997
1610	7486	1803	6785	1634	10 899	1	9855
1611	6716	627	7014	1635	10 651	0	1034
1612	7778	64	6986	1636	12 959	10 400	9522
1613	7503	16	6846	1637	8681	3082	9160
1614	7367	22	7208	1638	13 261	363	10 311
1615	7850	37	7682	1639	9548	314	10 150
1616	8063	9	7985	1640	11 321	1450	10 850
1617	8280	6	7747	1641	11 767	1375	10 670
1618	9596	18	7735	1642	11 999	1274	10 370
1619	7999	9	8127	1643	12 216	996	9410
1620	9691	21	7845	1644	9441	1492	8104
1621	8112	11	8039	1645	9608	1871	7966
1622	8943	16	7894	1646	10 415	2365	7163
1623	11 095	17	7945	1647	10 462	3597	7332
1624	12 199	11	8299	1648	9283	611	6544
1625	18 848	35 417	6983	1649	10 499	67	5825
1626	7401	134	6701	1650	8749	15	5612
1627	7711	4	8408	1651	10 804	23	6071

3 Draw a graph that shows three pieces of information for each year: the number of deaths in total, the number of deaths associated with the plague, and the number of deaths not associated with the plague.

4 Calculate the difference between the number of births, as approximated by the Christned figures, and the total number of deaths for each of the years. Display this in a table.

5 Draw a graph to represent these figures; call it *Change in population*.

6 The number of births per year, as approximated by the figures in the Christned column, seems to fluctuate greatly. Can you offer some reasons for this? You may need to do some further research to answer this question.

Taking it further

Complete some research on the life and work of John Graunt. What conclusions did he draw from his data? How was his data used? Use your research to make a poster, PowerPoint slideshow or movie.

Investigation

Newspaper polls

Equipment required: 1 brain, newspapers, access to the internet

Some newspapers conduct a daily poll on an issue of current interest. A couple of examples are shown below, taken from the *Herald Sun*.

Should politicians be banned from shopping centres?		Should Australia abandon the Union Jack and create a new flag?	
Total: 1695 votes		Total: 16 404 votes	
YES	NO	YES	NO
1266 votes	429 votes	5561 votes	10 843 votes
74.6%	25.4%	33.9%	66.1%

The Big Question

Are these polls reliable? To answer this you will have to consider what reliable means. This is not always a simple matter and you need to address the following issues:

- Is the question asked a fair question?
- Is the polling method reasonable?
- If the question was asked again the next day, would a similar result be expected?

Sometimes questions are phrased in such a way as to lead to a particular answer. In the polling industry this is known as a push poll.

Engage

To participate in these polls you can either call a telephone vote line or vote online.

1. What type of poll, of those you have studied, are these closest to in their design?
2. Why do you think there is such a difference in the total number of votes for the two examples shown? Is this likely to influence you in how reliable you think the results might be?

Explore

See if you can find some similar examples online or in another newspaper.

Explain

3. Is there a problem with these polls from the point of view of how the sample is obtained?
4. Do a statistical analysis of the results obtained in these two polls, or on a couple of other similar polls that you have found for yourself. Do the results appear to be reasonable?

Evaluate

5. Do you think the wording of the questions should be changed? If so, write the question the way you would like it to be asked.
6. Why do you think newspapers conduct these polls on a daily basis?

Extend

7. Find out more about push polling. You might like to Google the term 'push polling'.
8. Make up some examples of these one-question polls for yourself, based on some issues of local or current interest. Decide how you would go about conducting your poll so that the results obtained are reliable.

Strategy options

- Break the problem into manageable parts.
- Have I seen a similar problem?

Standard deviation

10A

7.7

Traditionally, the most commonly used measure of spread was the **standard deviation**. The reason we have not looked at it previously is that it can be time consuming to calculate without technology.

One of the criticisms of the IQR is that it does not take into account all of the data. When there are outliers this is possibly a good thing, but when the whole data set is relatively uniform it would seem sensible to use them all in the calculation of a value for the measure of spread.

Consider the following data set:

5, 6, 7, 10, 10, 12, 14, 16

The mean of the data is 10. We can now calculate how far each of these values is away from the mean. This is referred to as the score's deviation from the mean. The deviation values are:

-5, -4, -3, 0, 0, 2, 4, 6

If you add these together you get 0, which is not very helpful. In fact, you will *always* get 0. (You might like to think about why this is true.) To avoid this situation we square the deviations before adding them.

The squared deviation values are:

25, 16, 9, 0, 0, 4, 16, 36

These add to 106. We then divide by the number of scores, in this case 8, to find the average deviation from the mean, which is called the **variance**. In this case the variance is 13.25. We then take the square root of the variance to find the standard deviation. We take the square root to reverse the effect of the squaring we did earlier. The standard deviation is approximately 3.64. We use the symbol σ to represent the standard deviation.

$$\text{Variance} = \frac{(\text{sum of deviations from mean})^2}{\text{number of values}}$$

$$\text{Standard deviation} = \sqrt{\text{variance}}$$

Worked Example 12

WE 12

Find the standard deviation, rounded to two decimal places, for the following data:

2, 3, 3, 3, 5, 7, 8, 11, 12

Thinking	Working
1 Find the mean of the data set.	Mean: $\frac{2+3+3+3+5+7+8+11+12}{9} = 6$
2 Find the deviations from the mean and then square them.	Deviations: -4, -3, -3, -3, -1, 1, 2, 5, 6 Square of deviations: 16, 9, 9, 9, 1, 1, 4, 25, 36

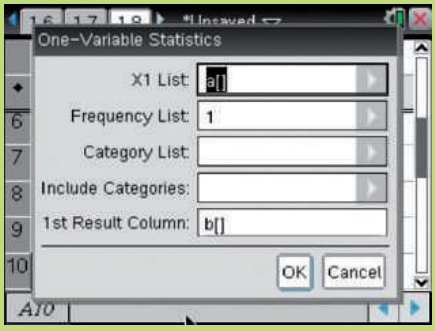
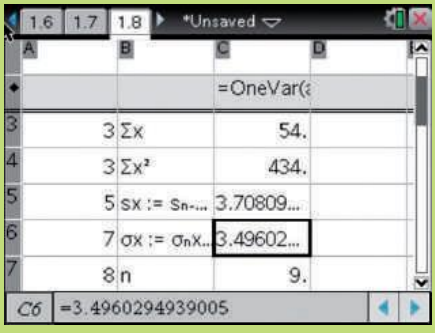
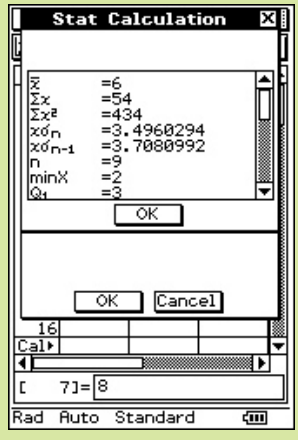
3 Add the squares of the deviations and divide by the number of values to write an expression for the variance.

$$\text{Variance} = \frac{110}{9}$$

4 Take the square root of the variance to give the standard deviation.

$$\text{Standard deviation} = \sqrt{\frac{110}{9}} = 3.50 \text{ (to 2 d.p.)}$$

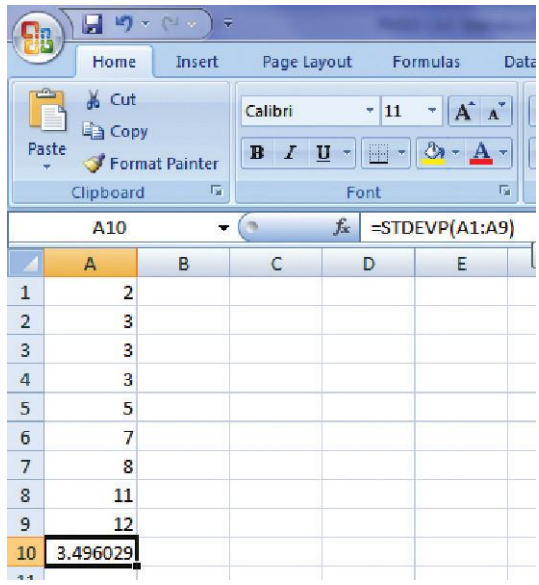
This is a rather involved calculation and it would not be practical to do this if we had hundreds of data values. This is where technology comes into play. Many scientific calculators are able to calculate the standard deviation and a spreadsheet, such as Excel, has a built-in standard deviation function. A CAS can be used as shown below. We will use the data from Worked Example 12 to demonstrate the process.

Using the TI-Nspire CAS	Using the ClassPad
Go to a Lists & Spreadsheet screen and enter the data in column A. Then press (menu) > Statistics > Stat Calculations > One-Variable Statistics > (enter) and then press (enter) to confirm there is only one list. Make sure the next dialog screen is filled in as shown below. 	From Menu select Statistics and enter the data in list1. Now tap Calc > One-Variable and make sure the dialog box looks like the one below.
Then (tab) down to OK and press (enter). You may then need to scroll through the list to find the value we require. It is σx. 	Then tap OK and a screen of statistics will appear, as shown below. 
This is 3.50, rounded to two decimal places, just as we found when we did the calculation by hand.	From this screen, the value we are interested in is $x\sigma_n$, which is 3.50 rounded to two decimal places, just as we found when we did the calculation by hand.

When dealing with the standard deviation we are often also interested in the value of the mean. If we were doing the calculation by hand we need the mean before we can proceed. In the list of statistics calculated by your CAS the mean is represented by $\bar{x}$.

If you do not have access to a CAS but do have access to a spreadsheet such as Excel, you can also find the standard deviation. In Excel you need to use the function *stdevp* which tells Excel you are dealing with a population.

You can calculate the mean in Excel using the function *average*.

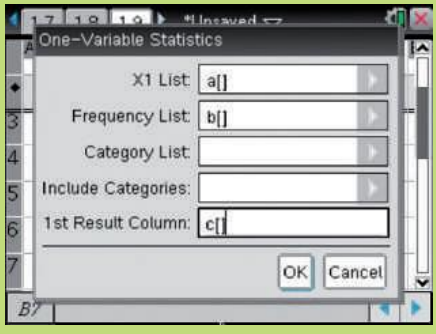


If you have a large data set that is set out in a frequency table you can still use your CAS to simplify the calculations. If the data is grouped you just use the class centre as the value in the first column. As an example, we will find the standard deviation for the following list of times taken by students to complete the 100 m sprint at the School Sports Carnival.

Time (s)	Frequency
10.50–<10.75	2
10.75–<11.00	4
11.00–<11.25	7
11.25–<11.50	3
11.50–<11.75	5
11.75–<12.00	10

Using the TI-Nspire CAS

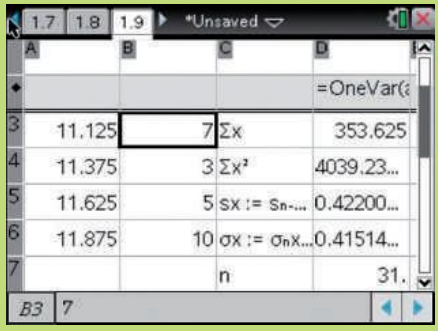
Enter the class centre values in column A and the frequencies in column B. Then press (menu) > Statistics > Stat Calculations > One-Variable Statistics > (enter) and then press (enter) to confirm there is only one list. Make sure the next dialog screen is filled in as shown below.

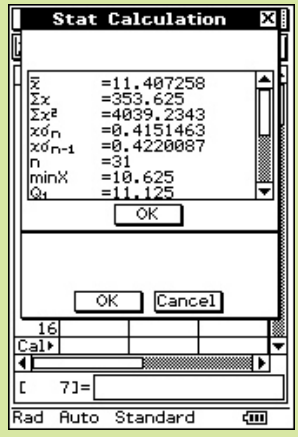


Using the ClassPad

Enter the class centre values in list1 and the frequencies in list2. Now tap Calc > One-Variable and make sure the dialog box looks like the one below.

Using the TI-Nspire CAS	Using the ClassPad
Then (tab) down to OK and press (enter). You may then need to scroll through the list to find the value we require. Once again it is σx. Along the way you might like to confirm that n = 31, the sum of the frequencies.	Then tap OK and a screen of statistics will appear, as shown below. You may then need to scroll through the list to find the value we require. Once again it is $x\sigma_n$. Along the way you might like to confirm that n = 31, the sum of the frequencies.





Interpreting the standard deviation

For two data sets that have the same mean, the smaller the value of the standard deviation the more closely packed is the data set. The data set with the smaller standard deviation is not necessarily bigger, stronger, longer (or whatever the variable represents) but just more consistently located with respect to the mean value.

As the mean also uses all of the data values it makes sense to link it with the standard deviation. In the same way, the median is not affected by extreme values, so effectively does not use them, which is why it is often linked with the IQR.

Worked Example 13 WE13

The table below shows the test results for 15 students in English and mathematics. Both tests were scored out of 50. Find the mean and standard deviation for each data set and use these values to compare them. Round the values to two decimal places, where necessary.

English	42	44	31	20	25	34	45	50	44	12	13	23	44	47	42
Maths	38	34	42	39	40	44	43	27	34	27	40	32	31	22	23

Thinking	Working
1 Calculate the mean and standard deviation.	English: mean = 34.4 std dev = 12.39 Maths: mean = 34.4 std dev = 6.98
2 Comment on the statistical values calculated.	The two tests had the same mean score but the standard deviation was much less for the mathematics test. This means the scores for the mathematics test were more consistent. However, we can see that in the English test both the best and worst results, overall, were recorded.

Now that you have some more statistical analysis tools at your disposal you will have to make more decisions about which values to use. If you have access to a CAS then the real difference between using the mean and standard deviation compared to the median and IQR is that the mean and standard deviation use all the data values whereas the median and IQR effectively use only the middle 50% of the data values. If there are outliers it may be better to use the median and IQR. If the data set seems relatively uniformly distributed then it may be better to use the mean and standard deviation.

7.7 Standard deviation

10A

Navigator

Q1, Q2, Q3, Q6, Q8, Q9	Q1, Q2, Q3, Q4, Q5, Q8	Q1, Q2, Q3, Q4, Q5, Q7, Q8

Answers page 757

Fluency

1 Find the standard deviation, rounded to two decimal places, for the following data: WE12

11 12 15 15 17 19 21 24 26 29

2 The table below shows the test results for 15 students in English and mathematics. Both tests were scored out of 50. Find the mean and standard deviation for each data set and use these values to compare them. Express the values rounded to two decimal places, where necessary. WE13

English	40	40	34	23	24	39	46	47	41	19	23	23	34	37	45
Maths	33	44	32	49	36	38	41	29	37	29	41	35	30	22	24

3 Find the mean and standard deviation, rounded to two decimal places where required, for the grouped data shown below. The data represents the distances, in km, travelled to school by a group of students.

Distance (km)	Frequency
0–<2	22
2–<4	28
4–<6	15
6–<8	7
8–<10	4
10–<12	2

4 The following table shows the heights, in m, of the ten highest mountains in each of Australia, New Zealand, Brazil and Sri Lanka.

Australia	2220	2185	2178	2166	2132	2048	2047	2045	2050	1979
New Zealand	3754	3500	3440	3279	3201	3183	3176	3163	3160	3157
Brazil	2973	2890	2882	2861	2849	2798	2791	2770	2734	2033
Sri Lanka	2524	2243	2240	2100	2100	2076	2036	2016	2010	1896

(a) Calculate the range for each country.

(b) Calculate the standard deviation, rounded to two decimal places, for each country.

(c) Write a sentence or two about what you have found.

Understanding

5 (a) Find the standard deviation, rounded to two decimal places, for each of the following data sets:

(i) 13 14 14 15 15 15 16 16 19 20 22 25 26 28

(ii) 13 14 14 15 15 15 16 16 19 20 22 25 26 48

(iii) 3 14 14 15 15 15 16 16 19 20 22 25 26 28

(iv) 3 14 14 15 15 15 16 16 19 20 22 25 26 48

(b) Write a sentence or two about what this tells you.

6 You have recorded a data set as: 23 45 46 47 55 57 58 59 62 65 and calculated the mean and standard deviation. When checking your work you discover that the '23' should have been '43'. Changing this will:

A increase both the mean and standard deviation

B increase the mean and decrease the standard deviation

C decrease the mean and increase the standard deviation

D decrease both the mean and the standard deviation.

Reasoning

7 The screenshot shows some information taken from an MP3 playlist. You will need to be careful how you deal with this data. Give your answers correct to the nearest second.

Genre	Time	Genre	Time
Rock	2:07	Pop	2:25
Blues	3:25	Soundtrack	2:12
RnB	2:26	Rock	4:15
Blues	5:27	R&B	3:59
Rock	3:35	Blues	2:33
Country & F...	2:16	Blues	15:30
Rock	5:28	Rock	2:47
Rock	4:02	Rock	5:30
Blues	3:31	Rock	3:32
Blues	2:51	Rock	12:28
Rock	3:26	R&B	4:36
Rock	2:34	Rock	3:30
Pop	3:46	Pop	2:15
Classic Rock	4:01	Blues	3:17
Grunge	3:50	Country & F...	2:48
Pop	3:13	RockNRoll	3:05
Pop	2:25	Pop	3:00
Pop	2:30	Rock	3:51
Blues	2:09	Rock	3:43
Rock	3:31	R&B	3:56
Rock	5:51	Soundtrack	5:56
RockNRoll	1:59	Rock	2:52
Rock	3:32	R&B	4:10
Pop	3:49	Pop	3:52
Blues	4:20	Pop	2:40
Blues	2:46	Rock	3:34
Rock	3:20	Blues	6:00
Rock	3:32	R&B	4:47
Rock	3:32	Rock	5:55
Blues	4:07	Classical	3:57
Rock	8:33	Rock	3:42
Rock	8:53	Pop	4:14

(a) Find the mean and standard deviation for songs listed as *Rock* (*Rock*, *Classic Rock*, *RockNRoll*).

(b) Find the mean and standard deviation for songs listed as *Pop*.

(c) Find the mean and standard deviation for songs listed as neither *Rock* nor *Pop*.

(d) Draw a parallel box plot showing the three categories *Rock*, *Pop* and *Other*. Use decimal time values here.

(e) Write a couple of sentences comparing the three categories. Is there any evidence to support the view that *Rock* songs are of a different length to *Pop* songs, for instance?

Hint

Change the time in seconds to decimal parts of a minute.

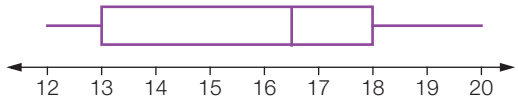
Open-ended

8 (a) Construct a data set with at least 20 values that has a mean of between 20 and 22, a range of at least 15 and a standard deviation of between 2 and 4.

(b) Write a couple of sentences about how you went about this task.

(c) What effect did the condition about the range have on this task?

9 Construct a data set, of at least 20 values, that would fit the following box plot and then find the mean and standard deviation of your data set.



Outside the Square Problem solving

Keeping score

From her attempts to shoot 40 points in the season's first basketball game, Angela scored only 10 points. This gave her a shooting average of 25%. In the second game she had the chance to score 30 points but only scored enough to raise her shooting average over the two games to 50%.

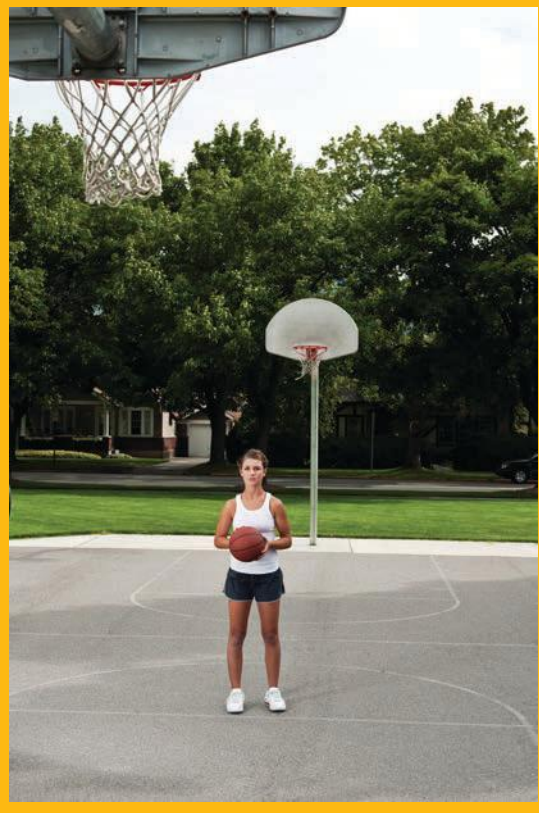
(a) How many points did she score in the second game?

(b) What was her shooting average for the second game?

(c) What is the greatest shooting average Angela could have had after the second game, assuming her maximum in the second game was still 30 points?

Strategy options

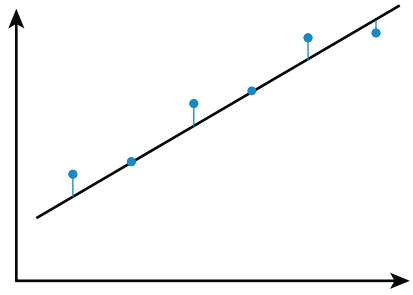
- Guess and check.
- Work backwards.



7.8 Lines of best fit 10A

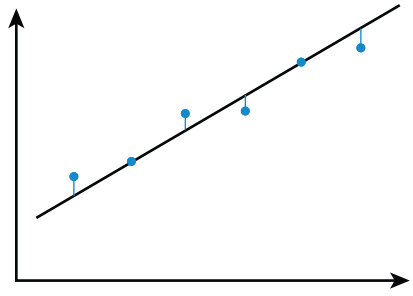
Now that we have drawn some scatter plots of bivariate data it is time to do something a bit more mathematical with them. We would like to be able to use the data we have collected and graphed to make predictions about other data pairs. To do this we need to draw what we call a **line of best fit** and find its equation. There are a number of ways we can do this.

We can draw the line of best fit approximately by eye.



The line should be positioned so that the square of the lengths of the blue lines is a minimum. The line shown in the first diagram is almost certainly not the line of best fit for the data shown!

The line drawn through the same data points in the second diagram is a much better. Can you see why? One way to achieve this is to try to have the same number of points above and below the line.



Excel or a CAS can be used to draw an exact line of best fit. This line of best fit is called a least squares line of best fit as it minimises the vertical distance between each point and the line of best fit. These values are squared so that distances from points above the line do not cancel out the distances from the points below the line. A least squares line of best fit is also known as a **regression line**.

Excel or a CAS can also be used to find the equation of the regression line.

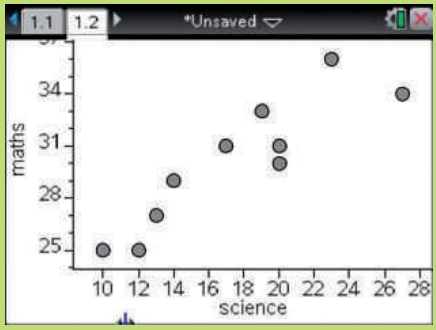
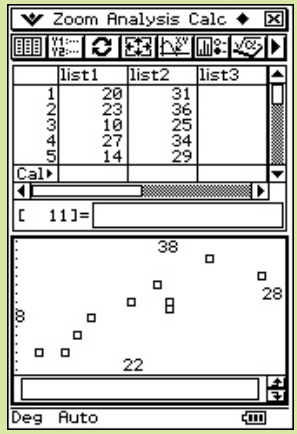
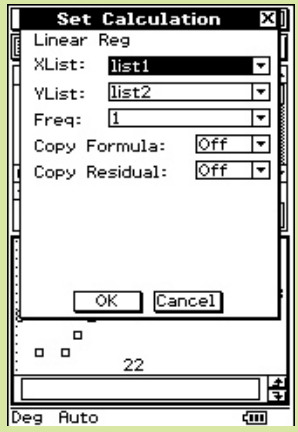
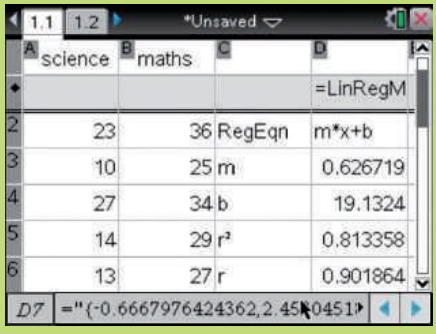
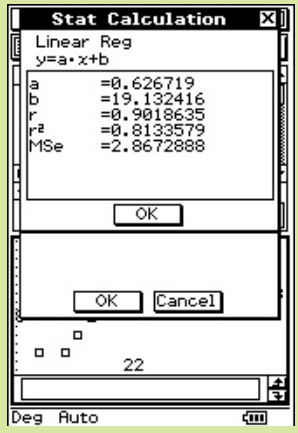
Let's imagine we have the following table of data that shows the marks obtained by 10 pupils taking science and mathematics tests.

Pupil	A	B	C	D	E	F	G	H	I	J
Science mark (out of 30)	20	23	10	27	14	13	12	20	17	19
Mathematics mark (out of 40)	31	36	25	34	29	27	25	30	31	33

We can use CAS to draw the scatter plot and get the equation of the least squares line of best fit.

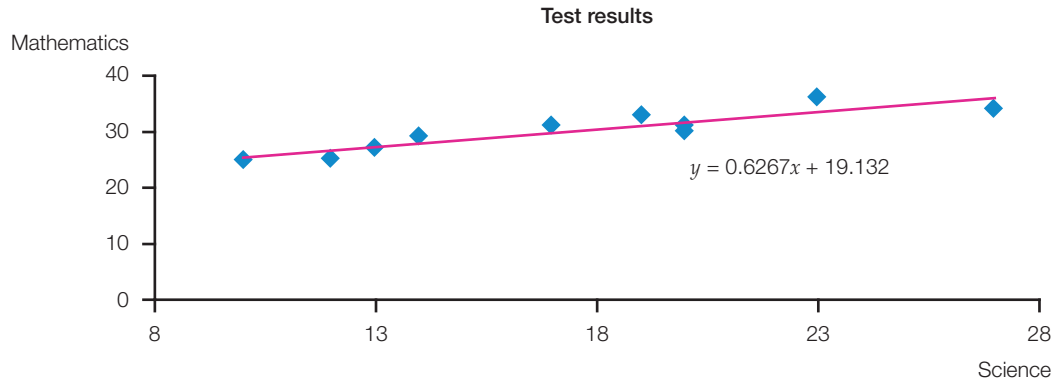
Before we draw our graph we need to decide which variable to place on which axis. At this stage, we will put the variable from the top row of the table of values on the horizontal axis. In this case, that will be the science mark.

A word of warning is appropriate at this stage. Once we have drawn the scatter plot, either by hand or using technology, we need to inspect it and decide whether it would be appropriate to draw a straight line over the points. If it doesn't look like the dots get close to a straight line then we shouldn't draw one.

Using the TI-Nspire CAS	Using the ClassPad
On a Statistics screen, enter the data in the normal way. Remember to name the columns as you want to draw a graph. Press ctrl I (or ctrl I) to insert a new page and choose Add Data & Statistics. Press tab and choose Science. Then press tab again and choose Maths. The scatter plot is then displayed. 	On the Statistics screen, enter the values in the normal way and draw a scatter plot.  Then tap Calc > Linear Reg and fill in the dialog box as shown. 
Now return to the data screen by pressing ctrl ◀ (or ctrl ◀). Then press menu > Statistics > Stat Calculations > Linear Regression ($mx+b$) and make X List Science and Y List Maths. Then tab down and press OK. You get a screen of results, as shown below.  From this screen we see that the equation is $y = 0.627x + 19.132$, so the mathematics mark = 0.627 × science mark + 19.132.	When you click OK you get a series of values that are important.  From this screen we see that the equation is $y = 0.627x + 19.132$, so the mathematics mark = 0.627 × science mark + 19.132.

If you don't have a CAS but have access to a spreadsheet, you can still get the equation of the line of best fit.

Enter the values and draw a scatter plot in the normal way. The sheet shown has been tidied up a bit with the addition of axes labels and a new title but these features do not affect your ability to get the equation of the line of best fit. To do this, right-click one of the points of your scatter plot and click Add Trendline. Click beside Linear, as well as Display Equation on Chart and then Close. You should get a screen like the one shown below.



This gives us the same result as we obtained using CAS:

mathematics mark = 0.627 × science mark + 19.132

Making predictions

Now that we have the equation of the line of best fit we can use it to make predictions. For instance, if another student scored 22 for science we could predict their mathematics mark by substituting the known value into the equation.

$$\begin{aligned}\text{mathematics mark} &= 0.627 \times \text{science mark} + 19.132\\ &= 0.627 \times 22 + 19.132\\ &= 32.926\end{aligned}$$

For a student who scored 22 for science, the predicted mathematics mark is 33.

If we had the mathematics mark for another student we cannot just substitute this into the reorganised equation to predict their science mark. We must put the mathematics results on the horizontal axis and find a new equation to use. Your technology makes this switch of variables quite simple.

science mark = 1.298 × mathematics mark − 21.564

If another student scored 29 for mathematics we could predict their science mark by substituting the known value into the equation.

$$\begin{aligned}\text{science mark} &= 1.298 \times \text{mathematics mark} - 21.564\\ &= 1.298 \times 29 - 21.564\\ &= 16.078\end{aligned}$$

For a student who scored 29 for mathematics, the predicted science mark is 16.

In both of the previous examples, the value has been within the range of known data values for that variable. For science scores the range was 10–27 and for mathematics scores it was 25–36. When making predictions if we use a value from within the known range we are using a technique called **interpolation**. If we use a value from outside this range, say the science score was 6, we call this **extrapolation**. In general, we have more confidence in values obtained using interpolation than in those obtained using extrapolation. This is because we cannot be certain that the same trend will be evident outside the known range.

7.8 Lines of best fit

10A

Navigator

Q1, Q2, Q3, Q4, Q8, Q9 | Q1, Q2, Q3, Q5, Q7, Q8 | Q1, Q2, Q3, Q5, Q6, Q7, Q8

Answers page 757

Equipment required: a CAS or access to a spreadsheet

Fluency

1 Find the equation of the line of best fit, using Excel or some other form of technology, for each of the following sets of data. Round your answers to three decimal places, where necessary. In each case, the required rule is of the form $y =$.

(a)

x	0	1	2	3	4	5	6
y	4	16	29	42	54	65	76

(b)

x	0	2	5	7	10	12	14
y	-7	8	60	92	150	170	200

(c)

x	1	3	7	11	13	16	20
y	45	28	18	0	-5	-15	-45

(d)

x	3	6	8	10	13	15	18
y	26	12	1	-12	-24	-30	-48

2 It is believed that the height of a male at 30 months is about half his full adult height. (This is called a 'rule of thumb'.) The table below gives the height (cm) for 15 boys measured at 30 months of age and again at 19 years of age (taken to be their full adult height).

30-month height (cm)	89.0	89.9	90.3	90.8	90.9	91.0	91.1	91.2
19-year height (cm)	178.0	177.1	179.6	181.8	184.0	180.5	182.0	183.1

30-month height (cm)	91.2	91.9	92.9	93.3	94.7	95.4	96.1
19-year height (cm)	180.1	185.1	182.0	186.3	187.4	187.9	189.4

(a) Draw a scatter plot of the data, putting the 30-month height on the horizontal axis.

(b) Draw, by eye, a line of best fit over your scatter plot.

(c) Do you think that the 30-month height is a good predictor of full adult height?

(d) The rule of thumb says that a 94 cm 30-month-old boy will grow to a full height of 188 cm. What height does your line of best fit predict? Give your answer correct to the nearest cm.

3 **(a)** Using the line of best fit equation $d = 23.2t - 4.6$, the interpolated value for d when $t = 40$ is closest to:

A 59 **B** 88 **C** 923 **D** 928

(b) **(i)** For the following table of values, use technology to find the equation of the line of best fit. Give your answer in the form $q =$.

p	0	3	5	12	15
q	2	7	9	15	21

(ii) Assuming the relationship continues to be linear, the extrapolated value for q when $p = 25$ is closest to:

A 32 **B** 30 **C** 28 **D** 25

Understanding

4 The following table gives the national running records for various distances for men for 10 different countries.

100 m (s)	10.39	10.31	10.17	12.18	10.35	10.11	10.98	10.60	10.61	10.46
200 m (s)	20.81	20.06	20.22	23.20	20.65	20.38	21.82	21.42	20.96	20.66
5000 m (min)	14.04	13.28	13.55	16.70	13.42	13.34	14.16	13.77	13.32	13.10

(a) Draw a scatter plot with the 100 m on the horizontal axis and the 200 m on the vertical axis.

(b) Does there appear to be a connection between the times for the 100 m and the 200 m?

(c) Draw a scatter plot with the 100 m on the horizontal axis and the 5000 m on the vertical axis.

(d) Does there appear to be a connection between the times for the 100 m and the 5000 m?

(e) Can you explain what you have found?

5 An engineer conducted an experiment modelling the force needed to set an object in motion. Her experiment consisted of different-sized blocks of wood of mass (m kg) being set in motion on a horizontal surface by exerting a force (F N). The size of the force needed to move each block of wood was recorded and the following results were obtained.

Mass m (kg)	1	2	3	5	7.5	10	12.5	15	20
Force F (N)	5	9	15	24	38	50	60	70	100

(a) Use technology to find the equation of the linear line of best fit, if this is appropriate for this data. Round your answer to two decimal places.

(b) The ratio $\frac{F}{m}$ is proportional to the *coefficient of friction* between the blocks of wood and the horizontal surface. What is the value of this ratio?

(c) Find the force, rounded to two decimal places, needed to move a block of wood of mass:

(i) 4 kg **(ii)** 11 kg **(iii)** 18.5 kg.

(d) Assuming the relationship between the force and the mass continues to hold for heavier blocks of wood, find the force needed to move a block of wood of mass:

(i) 50 kg **(ii)** 200 kg **(iii)** 1 tonne.

6 A chemist conducted an experiment in which the solubility (S units) of sodium chloride in water was tested for different temperatures (T°C). The results of the experiment are recorded in the following table.

Temperature T (°C)	0	10	20	30	40	50	60	70	80	90	100
Solubility S (units)	35.0	35.4	36.1	36.5	37.1	37.7	38.2	38.6	38.9	39.4	40.0

(a) Use technology to find the equation of the line of best fit when temperature is on the horizontal axis. Round your answer to two decimal places.

(b) Find the solubility of sodium chloride in water at:

(i) 25°C **(ii)** 78°C **(iii)** 120°C

(c) Use technology to find the equation of the line of best fit when solubility is on the horizontal axis.

(d) Find the temperature at which sodium chloride has the following solubilities in water:

(i) 36.0 units **(ii)** 38.5 units **(iii)** 45.0 units

(e) What is the potential problem with the answers to **(b) (iii)** and **(d) (iii)**?

Reasoning

7 A complex spring system is tested under compression and the results obtained are shown in the table below.

Compression Δx (mm)	0	20	40	60	80	100	120	140
Applied force F (N)	0	1.8	3.9	6.1	8.2	9.9	11.9	14.1

Compression Δx (mm)	160	180	200	220	240	260	280	300
Applied force F (N)	16.1	17.9	20.0	26.7	32.9	38.4	44.6	50.2

(a) Use technology to draw a scatter plot of the data, with compression on the horizontal axis. You should be able to see that the data splits into two groups. At which compression value does this occur?

(b) Find an equation for each of the two groups of data. Round your answers to three decimal places.

(c) Find the applied force required, rounded to one decimal place, to produce a compression of:

(i) 75 mm **(ii)** 150 mm **(iii)** 225 mm

(d) Breaking the data into the same two groups, find the equations of the lines of best fit if applied force was on the horizontal axis.

(e) Find the compression, to the nearest whole number, if the spring system has an applied force of:

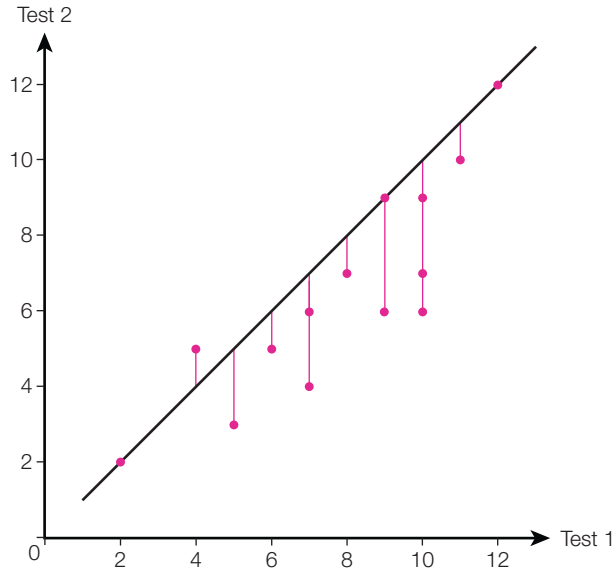
(i) 5 N **(ii)** 30 N **(iii)** 45 N

(f) The spring system has a maximum compression of 500 mm with an applied force of 90.0 N. Have we found an appropriate equation for this part of the graph? Explain why, listing any assumptions you have made. (Determine a new equation if necessary.)

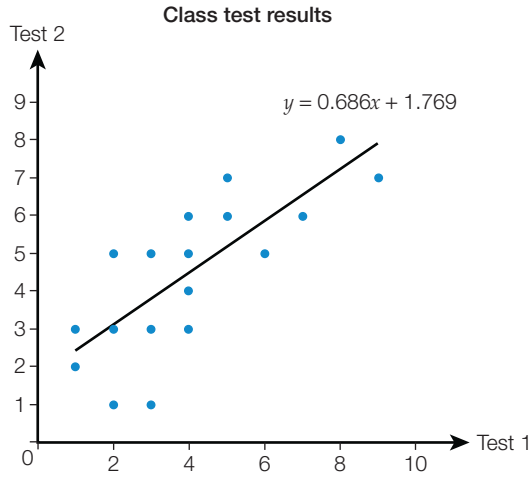
Open-ended

8 (a) What mistake has Emily made when drawing her line of best fit?

(b) Redraw the graph in your exercise book (the coordinate pairs are all whole number pairs) and draw the line of best fit correctly.



9 A scatter plot has been drawn with its line of best fit drawn over it. The equation of the line of best fit has been given. Make up three questions you could ask using this information. The data refers to results on two mathematics tests.



Outside the Square — Problem solving

Making the data fit

Equipment required: a spreadsheet such as Excel or a CAS would be useful

You have drawn a scatter plot and found that the equation of the line of best fit is $y = 3x - 4$, with the values rounded off to the nearest whole number.

You are to find a data set of at least 10 number pairs that would produce such a line of best fit.

Resist the temptation to make them all fit the line perfectly!

Strategy options

- Look for a pattern
- Work backwards.

Chapter review 7

D.I.Y. Summary

Key Words

bivariate data	interquartile range	percentile
box plot	ogive	primary data
cumulative frequency curve	outlier	quantile
dependent variable	parallel box plot	range
five-number summary	parallel dot plot	scatter plot
independent variable	percentage cumulative frequency curve	secondary data
10A		
extrapolation	line of best fit	standard deviation
interpolation	regression line	variance

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 Data used from the internet is _____.

2 The difference between the upper quartile and the lower quartile is called the _____.

3 Another name for a cumulative frequency curve is _____.

4 Data collected from your own observations is _____.

5 A _____ is a graph that shows the minimum, lower quartile, median, upper quartile and maximum values for a data set.

6 The value below which 80% of values lie on a cumulative frequency curve can also be referred to as the 0.8 _____.

7 The difference between the highest observed value and the lowest is called the _____.

8 If a value is more than $1.5 \times \text{IQR}$ above Q_U or more than $1.5 \times \text{IQR}$ below Q_L then it is an _____.

9 When we use a regression line to predict a value within the known range we are using _____. If the value is outside the known range we are using _____. 10A

10 The _____ is the square root of the _____. 10A

Fluency

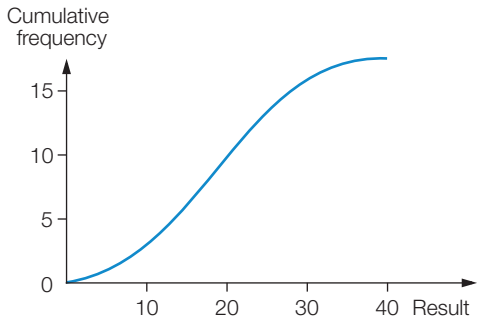
1 For each of the following cumulative frequency curves: Ex.7.1, 7.2

(i) read off the median

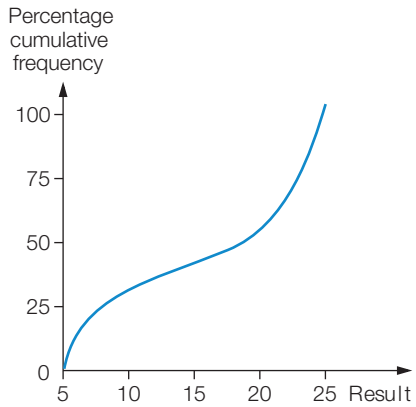
(ii) read off the interquartile range

(iii) draw a box plot.

(a)



(b)



2 For the set of data shown at right: Ex.7.2

(a) find the median by drawing a percentage cumulative frequency curve

(b) read off the interquartile range

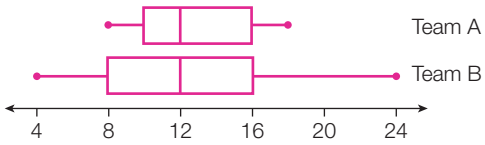
(c) draw a box plot.

Class interval	Frequency
120–<125	5
125–<130	17
130–<135	32
135–<140	54
140–<145	24
145–<150	13

3 A frog in training for the big jump-off achieves the following jump lengths (in centimetres). On the basis of these jumps, how long would a jump need to be to be considered an outlier? Ex.7.2

19 25 17 22 20 19 16 15 17 21

4 You have been working in a group on an analysis task. One member of the group has produced the following parallel box plot for the data that has been collected. Your task is to write the first draft of the analysis. This should include a comment about which team you think has been more successful. Ex.7.3



5 For each of the following occupations, state whether you think they deal with primary data or secondary data. Ex.7.4

(a) Meteorologist

(b) Television weather forecaster

(c) Insurance broker

(d) Cancer research scientist

6 Find, rounded to two decimal places, the standard deviation for the following data sets. Ex.7.7 10A

(a) 3 4 4 5 6 7 7 7 8 9 10 11 11 12 12 12 14 15 16

(b)

Number of flaws	Frequency
0	2
1	19
2	16
3	15
4	7

(c)

Weight (m)	Frequency
0–<5	6
5–<10	11
10–<15	15
15–<20	8
20–<25	2

7 Find the equation of the line of best fit, using a spreadsheet or some other form of technology, for each of the following sets of data. Round your answers to three decimal places, where necessary. In each case, the required rule is of the form $y =$.

Ex 7.8 10A

(a)

x	0	1	2	3	4	5	6
y	3	7	13	20	26	30	35

(b)

x	0	2	5	7	10	12	14
y	18	15	11	5	0	-6	-11

Understanding

8 Consider the following data:

72.8 64.8 55.4 44.3 61.8 52.9 62.0 73.6 34.5 72.3
65.9 57.7 64.4 55.3 45.2 71.2 69.6 54.2 59.9 51.1

(a) Group the data into intervals of 10 (i.e. 30–<40, 40–<50 etc.) and draw a cumulative frequency curve.

(b) Calculate the five-number summary for the data.

(c) Find the median from the cumulative frequency curve and compare it with the median calculated from the raw data.

9 The rainfall (in millimetres) recorded on each day of a month in a particular location was:

12 15 17 16 25 2 44 3 2 0
4 6 8 10 0 11 3 6 3 8
6 15 15 17 0 22 17 21 30 33

(a) Calculate the range for the rainfall figures.

(b) Calculate the five-number summary for the data.

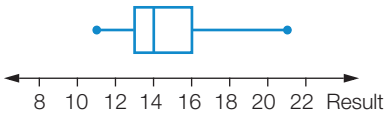
(c) Draw a box plot to represent the data.

10 For each of the following box plots:

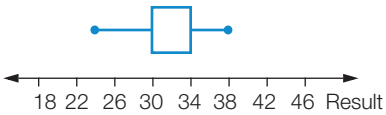
(i) indicate what values would cause results to be considered as outliers

(ii) decide whether there are any outliers in the data set displayed.

(a)



(b)



Reasoning

11 Some of the data available from the Bureau of Meteorology website for Perth (Metro) is shown below.

Temperature (°C)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Annual
Mean maximum	30.9	31.3	29.5	25.6	22.4	19.3	18.3	18.8	20.0	22.9	26.2	28.8	24
Mean minimum	17.9	18.0	16.5	13.5	10.5	8.4	7.8	8.1	9.5	11.2	14.1	16.2	12

(a) Find the mean difference in temperature (°C) for each month.

(b) Draw a scatter plot showing mean maximum temperature, mean minimum temperature and mean difference in temperature.

(c) Describe the temperature pattern for Perth (Metro) using the graph as an aid.

(d) Could you use this graph to describe the temperature pattern for Kalgoorlie, in inland West Australia? Give a reason for your answer.

(e) The daily maximum and minimum temperatures in Perth for June 2010 are given.

Minimum:	10.9	6.8	5.6	4.8	9.9	5.7	6.5	6.6	8.7	7.3	9.6	6.9
	4.3	2.2	2.3	6.1	12.0	9.7	10.8	13.1	6.4	9.4	8.2	10.8
	11.1	6.4	3.1	6.3	8.4	5.2						
Maximum:	16.2	17.9	17.6	17.0	16.9	14.3	15.5	15.9	15.1	16.3	16.1	16.0
	13.4	13.8	14.0	16.9	16.3	16.5	17.0	17.3	16.4	17.3	18.2	18.2
	15.2	14.2	14.5	14.6	11.9	14.0						

How did June 2010 compare to the overall data?

12 The number of colony forming units per 100 mL of water of two different bacteria taken from a particular lake at weekly periods is recorded in the following table. The table also shows the maximum temperature (°C) for the day of recording. 10A

Fecal coliform (CFU/100 mL)	350	73	180	150	200	130	2200	6300
***E. coli* (CFU/100 mL)**	350	90	120	330	200	120	1600	5800
Temp. (°C)	15.3	14.6	16.8	16.8	18	24	19.6	21.7

Fecal coliform (CFU/100 mL)	360	1200	560	1700	19	430	320	87
***E. coli* (CFU/100 mL)**	300	980	630	1800	16	270	270	160
Temp. (°C)	20.1	19.4	22.1	24.1	22	23	19	18

(a) In how many different ways can the data be grouped?

Use your CAS to:

(b) draw a scatter plot with fecal coliform on the x-axis and *E. coli* bacteria on the y-axis

(c) find the equation of the line of best fit, assuming the data is linear.

13 The following fictional article contains information about the opinions of voters on the introduction of tolls on certain roads around Melbourne.

Toll roads – poll reaction

The State Government is facing a big electoral backlash over its City Link toll roads according to a *Sunday Bugle* poll conducted by Corio Market Research Centre in marginal seats along the route.

The poll found that at least half the voters surveyed avoid paying the tolls on the Tullamarine Freeway and Monash Freeway.

Residents of the affected areas have increased worries about traffic on their local suburban roads growing massively. The poll shows that the Government could lose three of the four seats in which the survey was conducted.

SUNDAY BUGLE POLL: TOLL ROADS

How marginal electorates would vote . . .

	Government %	**Opposition %**
Tullamarine	33	47
Essendon	43	43
Oakleigh	38	38
Knox	46	32

. . . and who avoid tolls

Tullamarine	59%
Essendon	50%
Oakleigh	52%
Knox	49%

Source: Telephone poll of 1200 voters conducted by Corio Market Research Centre

(a) Explain what is meant by the heading. Do the survey results support this 'big electoral backlash'?

(b) Add up the percentages for Government and Opposition voters in the Tullamarine electorate. Why don't they add up to 100%? Why is this important to the judgment of the poll?

(c) Comment on the sample size. Is it appropriate for a conclusion?

(d) The poll was done by telephone. Which sampling methods would you use to get the best indication of how voters in each electorate are reacting to City Link toll roads?

(e) What is the significance of the four different locations?

14 At his head office, the owner of a company holds a record of the salaries paid to his employees. Annual salaries 3 years ago and today are given in the following table.

Salary range (× $10 000)	Number of employees (3 years ago)	Number of employees (currently)
1–<3	12	9
3–<5	15	10
5–<7	9	9
7–<9	5	8
9–<11	3	7
11–<13	0	5

(a) Represent the above data by drawing a graph that emphasises the difference between salaries paid 3 years ago and those paid today.

(b) Contrast the means, medians and interquartile ranges for the two sets of data.

(c) Draw a box plot for each set of data.

(d) Describe, in your own words, the change in salaries over the past 3 years.

(e) How much more is the company paying in salaries today than it was 3 years ago?

(f) By what percentage has the average salary risen?

15 The grouped frequency table below shows the age distribution of the employees of a local company. The age given is the nearest whole number of years on 1 January of the current year. So, if a worker was 22 years and 5 months old they would be recorded as 22, whereas a worker 22 years and 7 months old would be recorded as 23. 10A

Age at 1 Jan (years)	16–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55
Frequency	25	30	32	28	33	42	55	10

(a) Estimate the mean age, to one decimal place, giving a reason for your estimate.

(b) Calculate the mean and standard deviation for the ages of the employees, giving your answers rounded to two decimal places, where appropriate.

(c) Draw a cumulative frequency curve and use it to find the five-number summary.

(d) Draw a box plot of the data.

(e) Use your cumulative frequency graph to estimate the number of employees who are:

(i) under 32 **(ii)** over 48.

8 Geometry

Want to play and create computer games for a living?

Name: Paul Seedy
Job: Lead Computer Games Programmer
Qualifications: Bachelor of Electronic Engineering

As a Lead Games Programmer at Tantalus, Paul Seedy gets to create video games for Microsoft Xbox, Sony PlayStation and Nintendo DS.

"I love programming, and to be able to apply that to the field of video games, it's the perfect combination. It's software for entertaining people."

Building video games involves a lot of problem solving and mathematics. Programmers use mathematics to simulate a virtual world with 3D animation, physics simulators and collision detection. Creating life-like movements requires skills in geometry and trigonometry to generate points, vectors, planes, curved surfaces, curved lines and splines. If programmers get something wrong the player's character may 'fall through the world into the abyss'.

When Paul thinks back to learning mathematics at school, he remembers asking himself, "Am I going to be using this in the real world? The answer for me was 'Yes', I can't believe how much of it I am using. Particularly in video games development you'll use everything you've learnt: algebra, calculus, matrices and probability. Programming video games is mathematics."

Why learn this?

Geometry is used daily by almost everyone. In geometry, we explore spatial sense and geometric reasoning. We encounter geometry everywhere: in nature, space, architecture, engineering, machines, robotics, vehicles, astronomy, art, sculpture, sports, cars and much more.

After completing this chapter you will be able to:

- determine what information is needed to prove triangles are congruent or similar
- find unknown sides in pairs of congruent or similar triangles
- apply geometric theorems, techniques and constructions
- apply congruent triangle results to establish properties of special triangles and quadrilaterals
- construct arguments to prove geometric results
- determine properties of triangles and quadrilaterals using deductive reasoning
- prove and apply angle and chord properties of circles. 10A

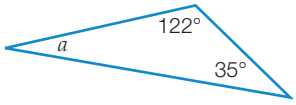
8 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

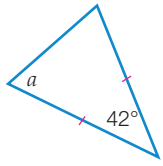
1 Find the value of the variable in each of the following.

Worksheet R8.1

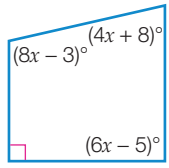
(a)



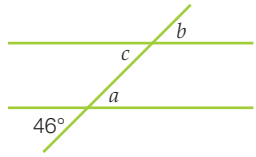
(b)



(c)



2 Find the value of the variables in this diagram.

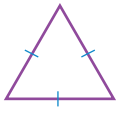


Worksheet R8.2

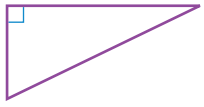
3 State whether the following triangles are equilateral, scalene, isosceles or right-angled.

Worksheet R8.3

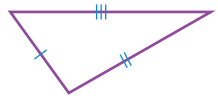
(a)



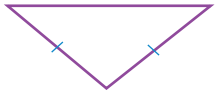
(b)



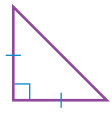
(c)



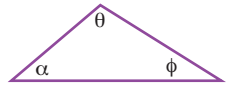
(d)



(e)



(f)

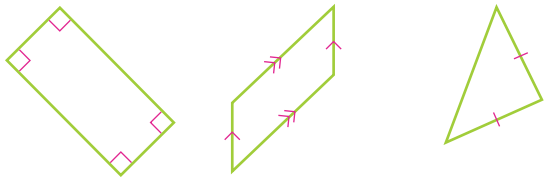


4 Match the name to its shape.

Worksheet R8.4

A rectangle **B** scalene triangle **C** isosceles triangle

D parallelogram **E** trapezium



Key Words

bisect	included angle	property	similar
congruent	perpendicular	prove	theorem
definition	perpendicular bisector		

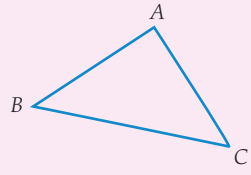
10A

angle at the centre	arc	major segment	segment
angle at the circumference	chord	minor arc	subtend
angle in a semicircle	equidistant	minor segment	subtended
angles in the same segment	major arc	sector of a circle	

8.1 Congruent and similar triangles

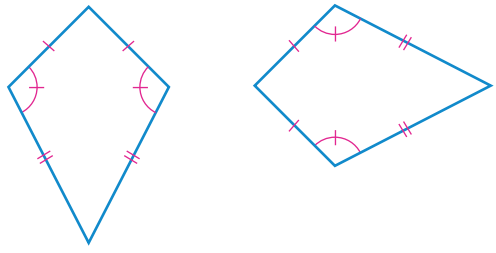
Important notation in geometric diagrams:

- Sides with the same markings are equal in length.
- Sides with the same number of arrows are parallel.
- The symbol $\parallel$ means 'is parallel to'.
- The symbol $\perp$ means 'is perpendicular to'.
- The symbol $\equiv$ means is 'congruent to'.
- The symbol $\therefore$ means 'therefore'.
- The symbol $|||$ means 'is similar to'.
- The triangle with vertices A, B and C is written as ΔABC.
- Angles are denoted using the symbol $\angle$; the angle at vertex A is written as $\angle BAC$. They can also be given letter names such as α, β, ϕ.
- Intervals are named according to their end points; the interval between A and B is written as AB.



Congruency

Congruent figures are identical in size and shape, but their orientation may be different.

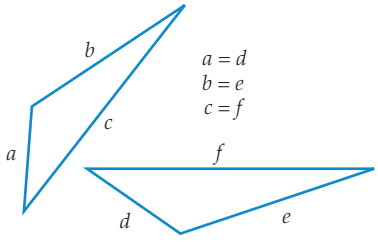


Tests for congruent triangles

Triangles are congruent if they are the same shape and size. Triangles are congruent if any one of these four sets of conditions is met:

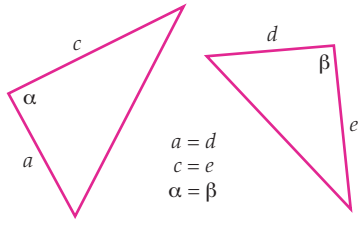
1 Side Side Side (SSS)

The three pairs of sides have the same length.



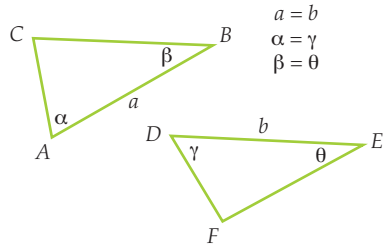
2 Side Angle Side (SAS)

Two pairs of sides have the same length and the **included angle** is the same.



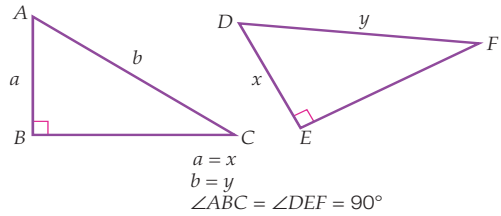
3 Angle Angle Side (AAS)

Two pairs of angles are the same and one pair of matching sides is the same length.



4 Right angle Hypotenuse Side (RHS)

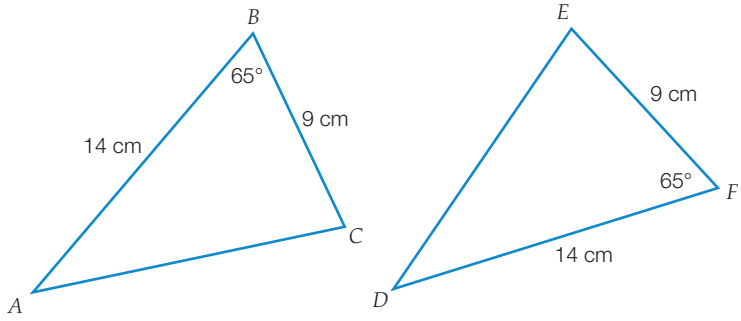
If both triangles contain a right angle, the hypotenuse in both is the same and another pair of sides is the same length.



Worked Example 1

WE1

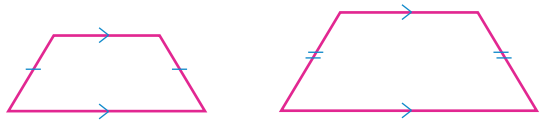
Show that ΔABC and ΔDFE are congruent, stating the congruency test used.



Thinking	Working	
1 List the sides and angles.	$AB = DF$	(both 14 cm)
	$\angle ABC = \angle DFE$	(both 65°)
	$BC = EF$	(both 9 cm)
2 Justify your answer.	The triangles are congruent due to SAS.	

Similarity

Similar shapes are those where all side lengths have undergone the same dilation. Similar figures have the same shape but are different sizes. All matching sides of one figure have been enlarged or reduced by the same factor to form the second figure. This factor is called the 'dilation factor'.



When a figure is dilated:

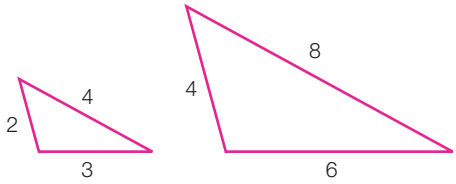
- matching angles are the same
- matching side lengths are in the same ratio.

Tests for similar triangles

Triangles are similar if one is a proportional dilation of the other. They have the same shape but are different sizes. Two triangles are similar if they satisfy any of the following conditions:

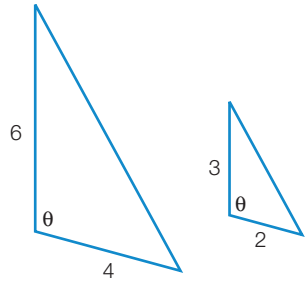
1 Side Side Side (SSS)

If the three sides of one triangle are proportional to the three sides of another triangle.



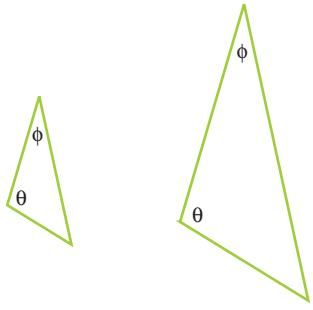
2 Side Angle Side (SAS)

If two sides of one triangle are proportional to two sides of another triangle, and the included angles are equal.



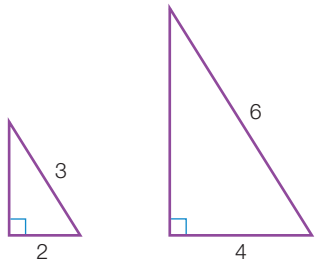
3 Angle Angle Angle (AAA)

If two angles of one triangle are equal to two angles in another triangle. All three angles are therefore equal.



4 Right Angle Hypotenuse Side (RHS)

If the hypotenuse and one side of a right-angled triangle are proportional to the hypotenuse and one side of another right-angled triangle.



Note: These tests are similar to the four tests for congruence, the difference being that where congruence requires that matching sides be the same length, similarity requires that matching sides be proportional.

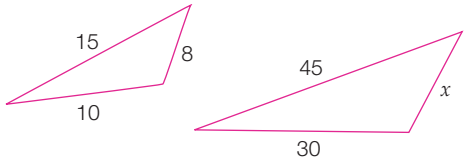
Finding lengths using similarity

If two triangles are similar we can form an equation to solve for the unknown length using the ratio of matching sides. We need to identify the matching sides to equate the ratios.

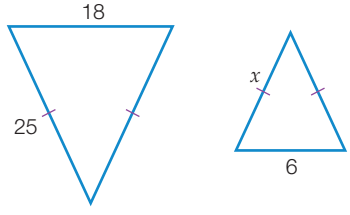
Worked Example 2 WE2

Given that these pairs of triangles are similar, find x.

(a)



(b)



Thinking	Working
(a) **1** Identify matching sides and equate the two ratios.	(a) $x:8 = 45:15$
2 Write the ratios as an equation with x as a numerator and solve for x.	$\frac{x}{8} = \frac{45}{15}$ $x = \frac{45}{15} \times 8$ $x = 24$ units
(b) **1** Identify matching sides and equate the two ratios.	(b) $x:25 = 6:18$
2 Write the ratios as an equation with x as a numerator and solve for x.	$\frac{x}{25} = \frac{6}{18}$ $x = \frac{6}{18} \times 25$
3 Write the length as an exact measurement. Do not approximate with decimals unless instructed to do so.	$x = 8\frac{1}{3}$ units

To find the length of an unknown side in a similar triangle:

1. Find two pairs of matching sides that include the unknown side.
2. Equate the two ratios.
3. Solve the equation to find the exact value of the unknown, unless told otherwise.

8.1 Congruent and similar triangles

Navigator

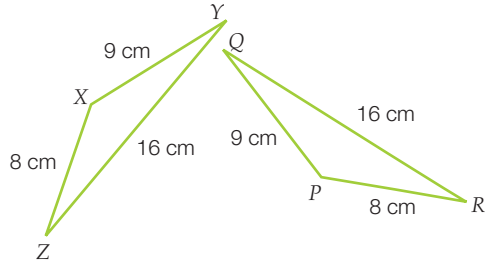
Q1 Column 1, Q2 Column 1, Q3, Q4, Q5, Q6, Q7, Q8 Column 1, Q10, Q12, Q14, Q15	Q1 Column 2, Q2 Column 2, Q3, Q4, Q5, Q6, Q7, Q8 Column 2, Q9, Q10, Q12, Q14, Q15	Q1 Column 2, Q2 Column 2, Q3, Q4, Q5, Q6, Q7, Q8 Column 2, Q9, Q10, Q11, Q12, Q13, Q15

Answers page 761

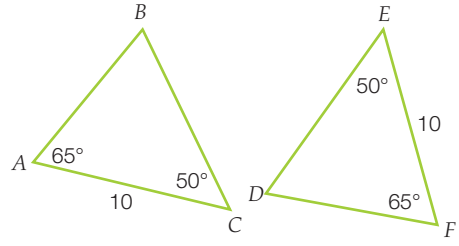
Fluency

1 Show that the following pairs of triangles are congruent, stating the congruency test used. WE1

(a)



(b)



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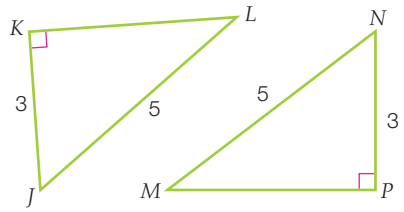
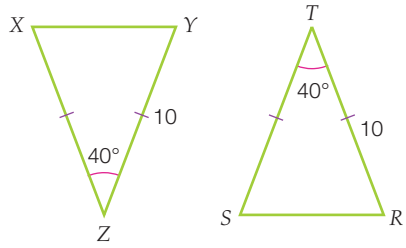
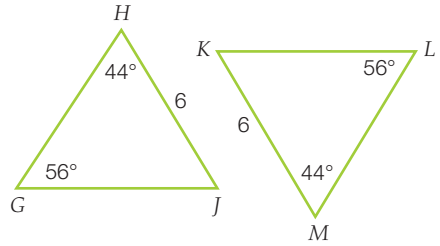


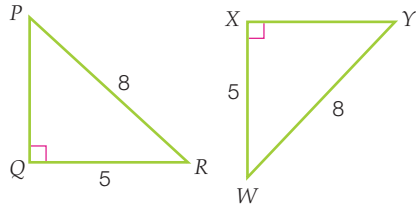
(d)



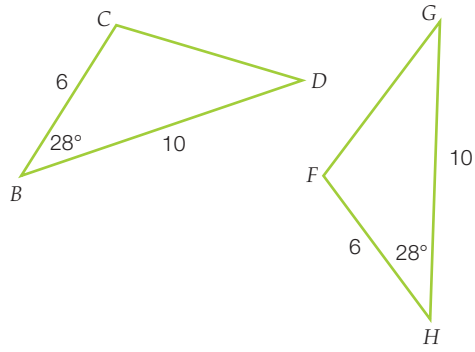
(e)



(f)



(g)



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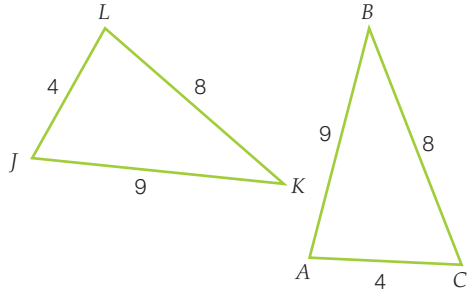
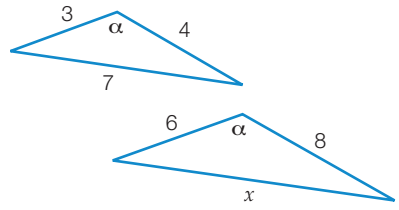


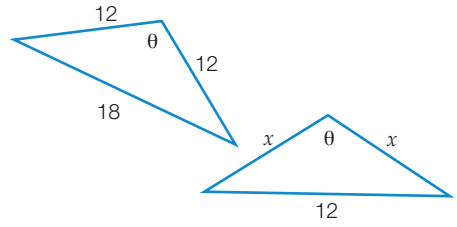
WE2

2 Given that these pairs of triangles are similar, find x.

(a)



(b)



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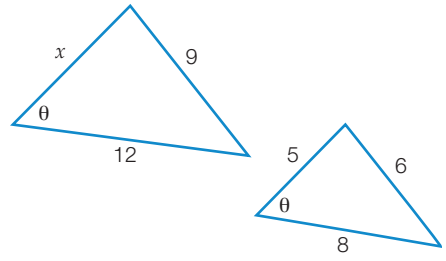
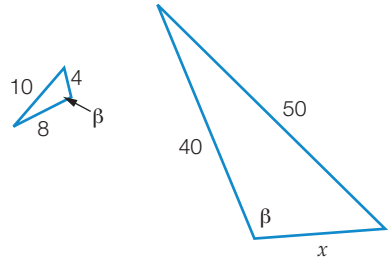
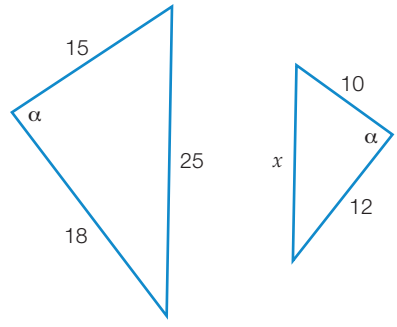


(d)



(e)



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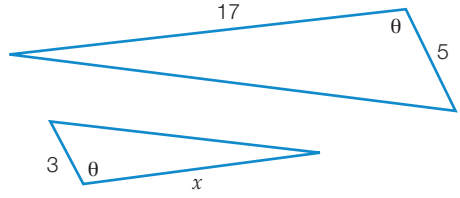
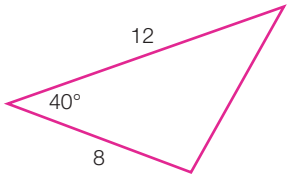
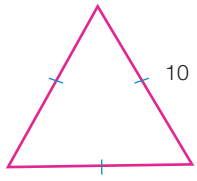
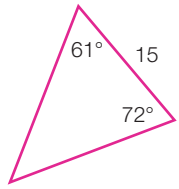
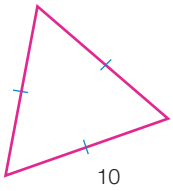


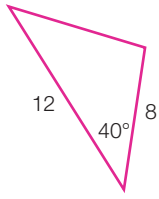
3 Match the pairs of congruent triangles given below.

(a)


(b)


(c)


(d)


(e)


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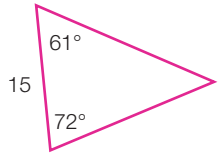
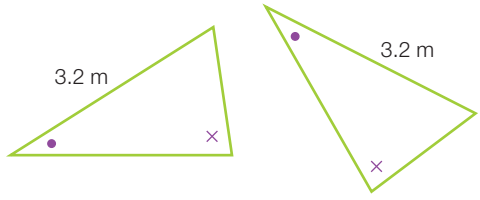


4 Which one of the following conditions does *not* ensure congruence for two triangles?

A SSS **B** SAS **C** AAS **D** AAA

5 Which condition verifies that these two triangles are congruent?

A SSS **B** SAS
C AAS **D** RHS



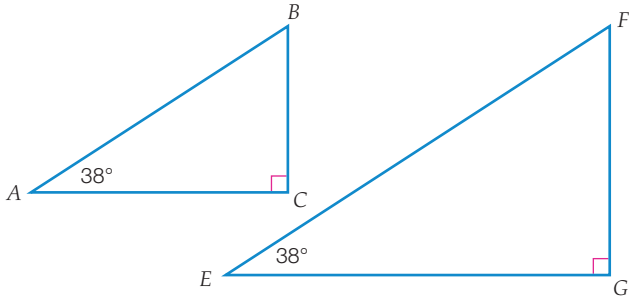
6 Given $\Delta ABC \parallel\!| \Delta EFG$:

(a) the matching side to AB is:

A FG **B** EF
C BC **D** AC

(b) the matching side to FG is:

A BC **B** AC
C EF **D** EG



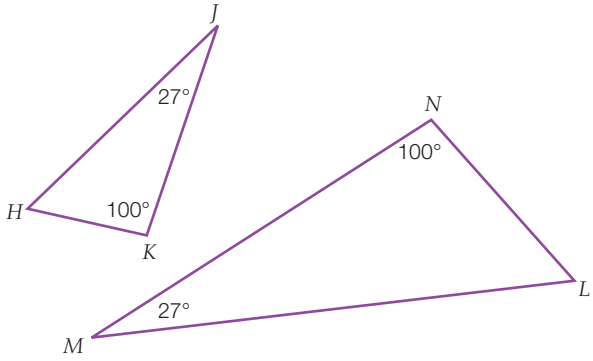
7 Given $\Delta HJK \parallel\!| \Delta LMN$:

(a) the matching side to LM is:

A HJ **B** MN
C JK **D** LN

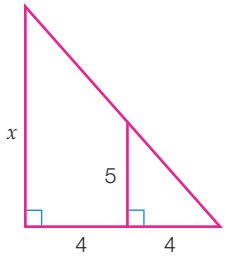
(b) the matching side to HJ is:

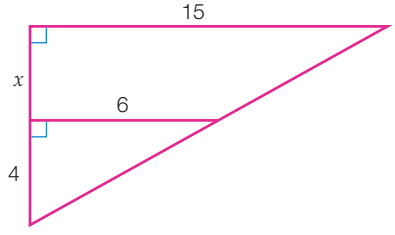
A JK **B** LM
C HK **D** NL



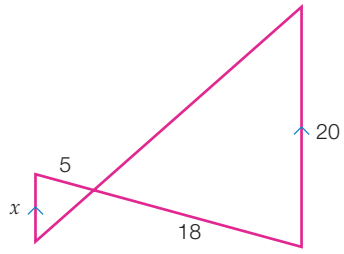
Understanding

8 In the following diagrams, first determine the similar triangles and then find the value of x.

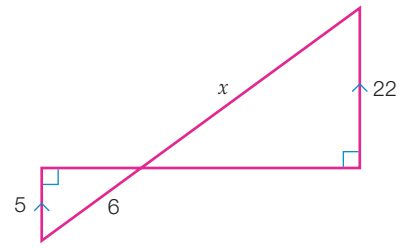
(a)


(b)


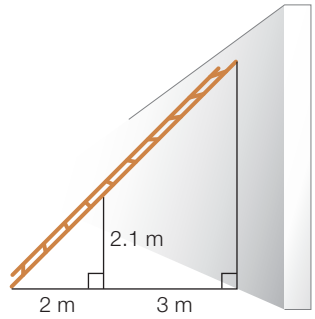
(c)



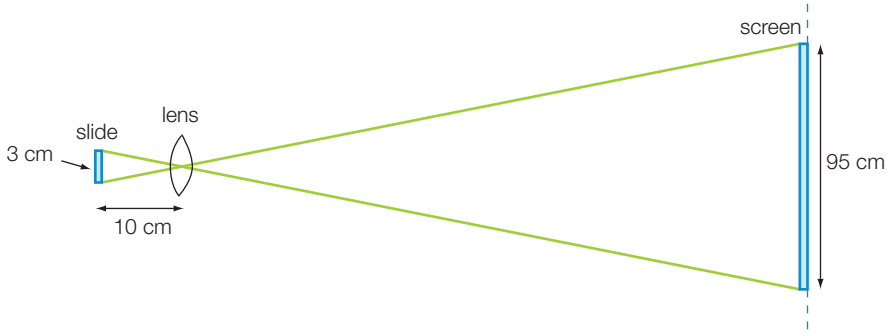
(d)



9 Dirk is climbing up a long ladder leaning against a wall. His friend Sean is holding up the ladder to prevent it wobbling. Sean is 3 m from the wall and 2 m from the base of the ladder. With his arms stretched above his head he holds the ladder at a point 2.1 m above the ground. How far up the wall does the ladder reach?



10 A light source projects the image of a photographic slide onto a screen, as shown in the diagram. The slide is 3 cm high and positioned 10 cm behind a lens. Calculate the distance between the lens and the screen if the image on the screen is 95 cm high. Give your answer correct to two decimal places.



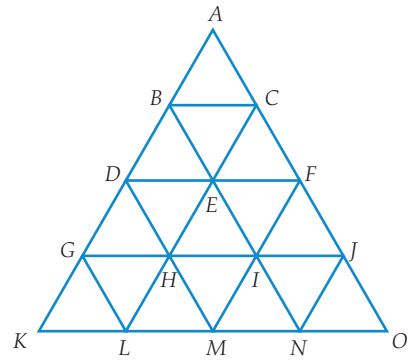
11 Ziva plays tennis at night under floodlights. When she stands 4.5 m from the base of a floodlight, she casts a shadow 1.4 m long. If Ziva is 1.7 m tall, how high is the floodlight?

Reasoning

12 The small triangles in the following diagram are all equilateral.

(a) Find six triangles congruent to ΔDFM.

(b) Find two pairs of congruent triangles larger than ΔDFM.



13 The Akela (leader) of the local scout group was explaining to the scouts how to use a stick to estimate the height of a tree.

Step 1: Have someone hold the stick vertically on the ground.

Step 2: Lie on the ground in a position so that when looking at the tree through the stick the top of the tree and the top of the stick are in line.

Step 3: Unfortunately Akela has forgotten step **3**.

(a) Draw a diagram of the situation and using your knowledge of similar triangles complete the instructions Akela needs to give.

(b) List any practical problems you can see with this method.

Open-ended

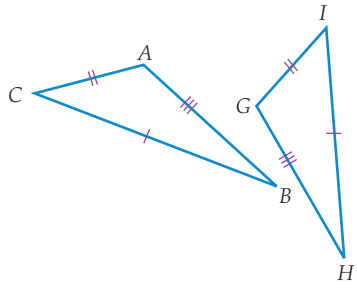
14 Explain why two triangles with three equal angles may not necessarily be congruent. Show this by drawing diagrams.

15 $\Delta ABC \,|||\, \Delta DEF$. The dilation factor is $\frac{3}{2}$. Draw a diagram to show both triangles and write possible values for the sides of both triangles if ΔABC is the smaller triangle and BC is equal to 4 cm.

8.2 Proving congruence and similarity

Proofs involving congruence

It is often necessary to **prove** that two triangles are congruent to complete proofs of other geometric relationships such as the special properties of parallelograms. We use one of the four tests for congruence: SSS, AAS, SAS and RHS.



When showing that two shapes are congruent, write the matching vertices in the same order.

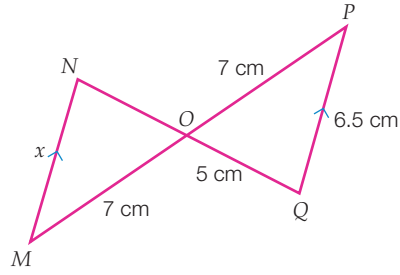
For example, in the congruent triangles shown here, the matching vertices are A and G, B and H, and C and I.

So, we write that $\Delta ABC \equiv \Delta GHI$.

This allows us to easily identify matching sides and angles because they are in the same place in the statement.

Worked Example 3 WE3

Prove that the triangles ΔMNO and ΔPQO are congruent, and hence find the value of the given variable.



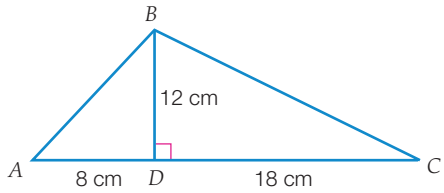
Thinking	Working
1 Name the two triangles.	$\Delta MNO \equiv \Delta PQO$
2 Write three sets of equalities, with reasons, that apply to the two triangles.	$\angle MNO = \angle PQO$ (alternate angles, $MN \parallel PQ$) $\angle MON = \angle POQ$ (vertically opposite angles) $MO = PO$ (given)
3 Write down the pair of congruent triangles and the test used to determine this.	$\Delta MNO \equiv \Delta PQO$ (AAS)
4 Identify the appropriate pair of matching sides.	$MN = PQ$ (matching sides in congruent triangles)
5 State the value of x.	$x = 6.5$ cm

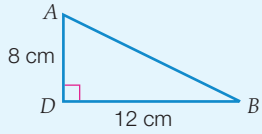
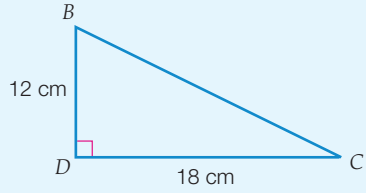
Proving similarity

We use the tests for similarity to prove that triangles are similar.

Worked Example 4 WE4

Prove that the triangles ΔABD and ΔBCD are similar.



Thinking	Working
1 Redraw the two triangles with the same orientation and mark any given information on the diagrams.	 
2 Check whether any of the matching sides given are proportional.	$\frac{BD}{CD} = \frac{12}{18} = \frac{2}{3}$ $\frac{AD}{BD} = \frac{8}{12} = \frac{2}{3}$
3 Write three statements or equations that display similarity.	BD and CD are in the ratio $2:3$. AD and BD are in the ratio $2:3$. $\angle ADB = \angle BDC = 90°$
4 State whether similarity exists and, if so, by which test.	$\Delta ABD \parallel\!| \Delta BCD$ because two sides of one triangle are in the same ratio as two matching sides of the other triangle and the included angles are equal (SAS).

Note: It can also be proved that $\angle ABC$ is also a right angle, so ΔABC is similar to both ΔABD and ΔBCD. (AAA)

8.2 Proving congruence and similarity

Navigator

Answers page 762

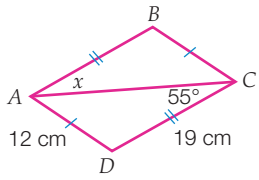
Q1, Q2, Q3, Q4, Q5 (a), Q6 (a), Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7 (a), Q8	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8

Fluency

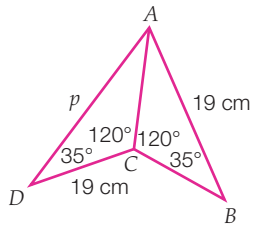
WE3

1 Prove that the following pairs of triangles are congruent and hence find the value of the given variable.

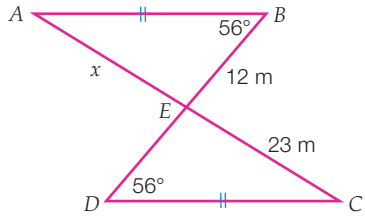
(a)



(b)



(c)



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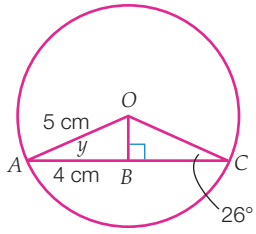

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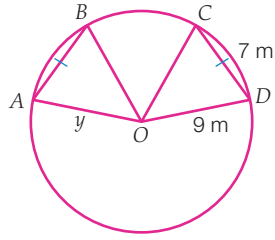
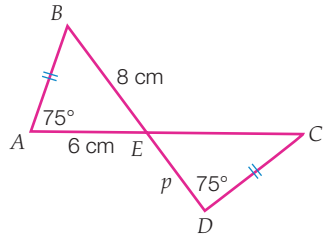


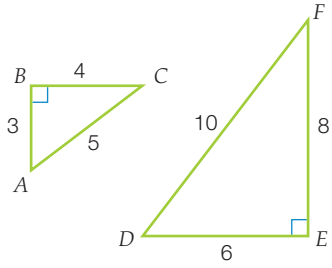
(f)



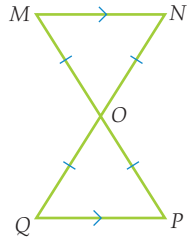
WE4

2 Prove that the following pairs of triangles are similar.

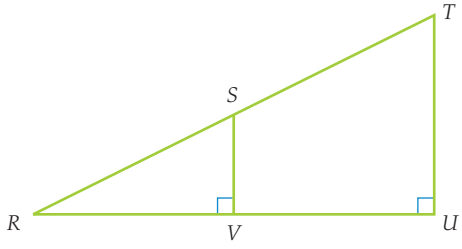
(a)



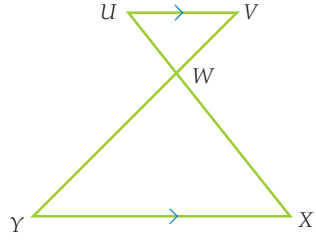
(b)



(c)



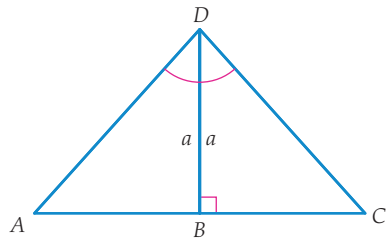
(d)



3 DB is the perpendicular bisector of AC and bisects $\angle ADC$. Which test for congruence can be used to prove that $\Delta ABD \equiv \Delta CBD$?

A SSS **B** SAS

C AAS **D** RHS



4 Which of the following is not enough to determine similarity?

A Two angles of one triangle are equal to two angles of the other triangle.

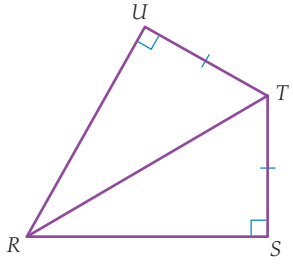
B Two sides of one triangle are proportional to two sides of another triangle.

C The hypotenuse and second side of a right-angled triangle are proportional to the hypotenuse and second side of another right-angled triangle.

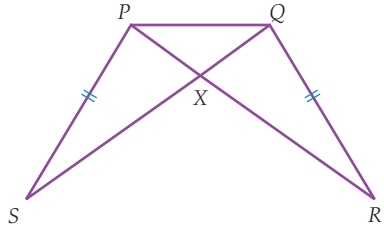
D Two sides of one triangle are proportional to two sides of another triangle and the included angles are equal.

Understanding

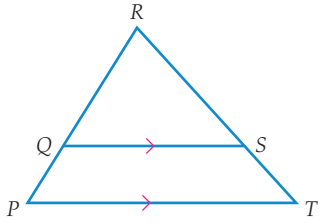
5 **(a)** Prove that $\Delta RUT \equiv \Delta RST$



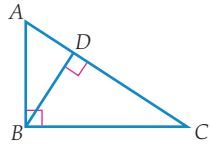
(b) $SP = RQ$ and $\angle SPQ = \angle RQP$. Prove that $\Delta SPQ \equiv \Delta RQP$.



6 **(a)** Prove that $\Delta RQS \parallel\!| \Delta RPT$.



(b) Prove that $\Delta ABC \parallel\!| \Delta BDC \parallel\!| \Delta ADB$.



Reasoning

7 **(a)** Explain why or why not a pair of shapes is similar if they are any two:

(i) rectangles **(ii)** squares

(iii) isosceles triangles **(iv)** equilateral triangles

(v) isosceles right-angled triangles **(vi)** circles.

(b) For each of the similar shapes in part **(a)**, state the conditions necessary for the pairs of shapes to be congruent.

Open-ended

8 Draw four pairs of congruent triangles and label them with enough information to show congruence. Each pair should represent a different test. Write the name of the congruence test that applies to each pair.

8.3 Proofs using congruent triangles

A **definition** is the minimum amount of information needed to identify a particular figure.

A **property** is a fact about a particular figure that can be proved from the definition.

Definitions of triangles

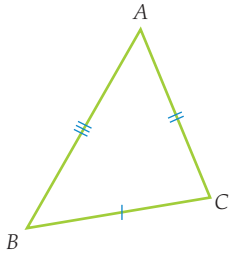
1. A scalene triangle is a triangle with no two sides equal in length.
2. An isosceles triangle is a triangle with only two sides equal in length.
3. An equilateral triangle is a triangle with all sides equal in length.

Properties of triangles

Scalene triangle

A scalene triangle has no particular property as all angles are of different magnitude.

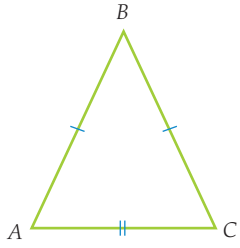
$\angle BAC \neq \angle ABC \neq \angle BCA$



Isosceles triangle

An isosceles triangle has two angles of equal magnitude.

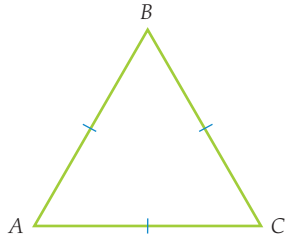
$\angle BAC = \angle BCA \neq \angle ABC$



Equilateral triangle

An equilateral triangle has all angles of equal magnitude of 60°.

$\angle BAC = \angle ABC = \angle BCA = 60°$



Proving properties of triangles

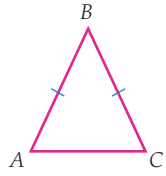
We can use congruent triangles to prove the properties of isosceles and equilateral triangles.

Worked Example 5 WE5

In ΔABC, $BA = BC$.

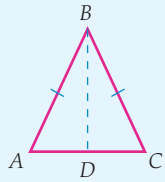
(a) Prove that $\angle BAC = \angle BCA$.

(b) What property of isosceles triangles have you just proved?



Hint

Construct a line from B to the midpoint of AC.

Thinking	Working
(a) 1 Create two triangles that we can prove are congruent by drawing a line BD from B to the midpoint of AC. State this construction.	(a) Line BD meets the midpoint of AC. 
2 Name the two triangles.	ΔABD and ΔCBD
3 Write the three sets of equalities, with reasons, that apply to the two triangles.	$BA = BC$ (given) $BD = BD$ (common side) $AD = CD$ (by construction)
4 Write down the pair of congruent triangles and the test used to determine this.	$\therefore \Delta ABD \equiv \Delta CBD$ (SSS)
5 Identify the appropriate pair of matching angles.	$\angle BAD = \angle BCD$ (matching angles in congruent triangles)
6 State the result you have proved.	$\therefore \angle BAC = \angle BCA$
(b) Write the answer in words.	(b) The result proved that angles opposite equal sides are equal in isosceles triangles.

8.3 Proofs using congruent triangles

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q13, Q14, Q15, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q13, Q14, Q15, Q16	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q15, Q16

Answers page 763

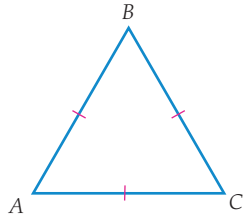
Fluency

WE5

1 In ΔABC, $AB = BC = CA$

(a) Prove that $\angle ABC = \angle BCA = \angle CAB$.

(b) What property of an equilateral triangle have you just proved?



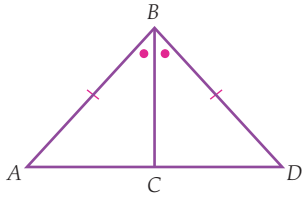
Hint

Construct a line from B to the midpoint of AC and a line from C to the midpoint of AB.

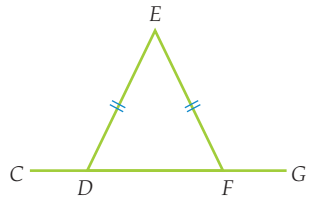
2 Use the triangles shown to prove the following properties of isosceles triangles.

(a) The bisector of the angle between the two equal sides of an isosceles triangle bisects the third side.

(b) The bisector of the angle between the two equal sides of an isosceles triangle is perpendicular to the third side.

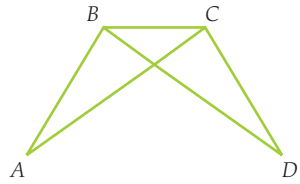


3 ΔDEF is isosceles. Prove that $\angle CDE = \angle GFE$.



4 The diagram shows two triangles, ΔABC and ΔDCB, with $AB = DC$ and $\angle ABC = \angle DCB$.

Which of the following tests can be used to prove that $\Delta ABC \equiv \Delta DCB$?

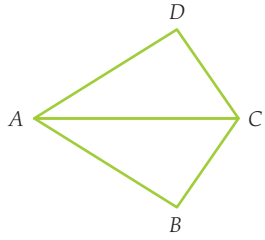


A SAS **B** AAS **C** SSS

D There is insufficient information to determine whether the triangles are congruent.

5 In the diagram opposite, AC bisects $\angle BAD$ and $\angle BCD$.

Which of the following tests can be used to prove that $\Delta ABC \equiv \Delta ADC$?

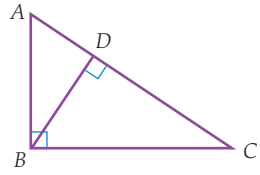


A SAS **B** AAS **C** SSS

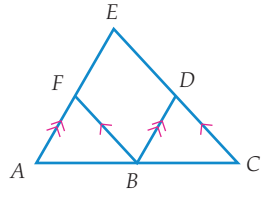
D There is insufficient information to determine whether the triangles are congruent.

Understanding

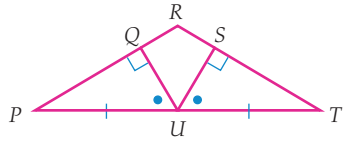
6 For the triangle shown, show that $AC^2 = DC^2 + 2BD^2 + AD^2$.



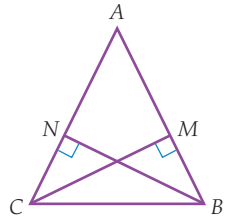
7 B is the midpoint of AC. Show that $\Delta ABF \equiv \Delta BCD$.



8 Prove that ΔPRT is isosceles.

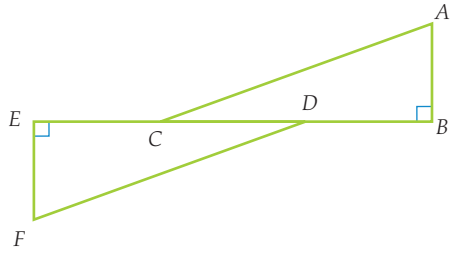


9 In ΔABC, $AB = AC$. Prove that $AM = AN$.

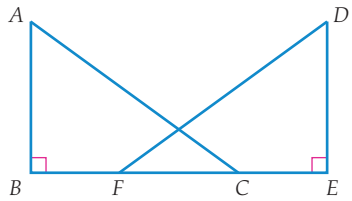


Reasoning

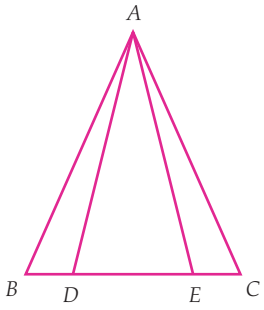
10 If $\angle BAC = \angle DFE$, $\angle ABC = \angle DEF$ and $BD = EC$, prove that $\Delta ABC \equiv \Delta FED$.



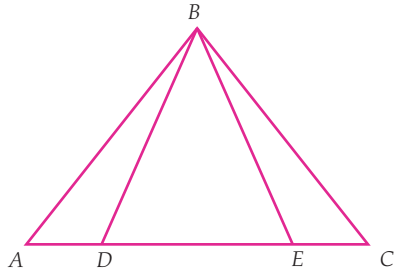
11 If $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BF = EC$, prove that $\Delta ABC \equiv \Delta DEF$.



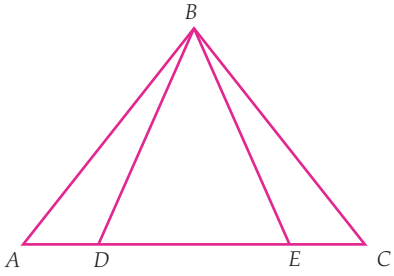
12 **(a)** In the diagram, $\angle ABD = \angle ACE$. Also, $AB = AC$ and $BD = CE$. Prove that $\Delta ABE \equiv \Delta ACD$.



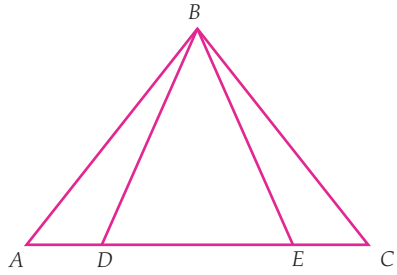
(b) Given $\angle BAD = \angle BCE$ and $\angle ABD = \angle CBE$, prove that $\Delta ABE \equiv \Delta CBD$.



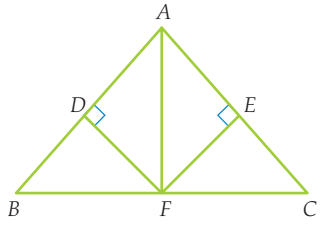
(c) Given $AE = CD$ and $\angle BAD = \angle BCE$, prove that $\Delta ABD \equiv \Delta CBE$.



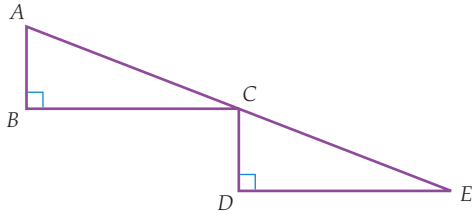
(d) Given $AB = CB$ and $AE = CD$, prove that $\Delta ABD \equiv \Delta CBE$.



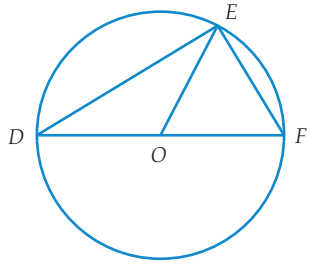
13 In the diagram, $FD = FE$, $FD \perp AB$ and $FE \perp AC$. Prove that $\Delta ADF \equiv \Delta AEF$.



14 ΔABC and ΔCDE are right-angled triangles and $AB = CD$. If C is the midpoint of AE, prove that $\Delta ABC \equiv \Delta CDE$.

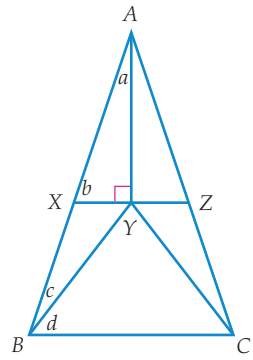


15 When you draw a line from each end of a diameter to a point on the circumference of a circle, the angle formed is called an angle in a semicircle. Prove, using isosceles triangles, that the angle in a semicircle is a right angle.



Open-ended

16 ABC is an isosceles triangle with $AB = AC$. X is a point on AB and Z is a point on AC such that XZ is parallel to BC. Y is the midpoint of XZ and $AY = YC$. Let $\angle BAY = a$.



(a) Prove $AX = AZ$.

(b) Prove $\Delta AXY \equiv \Delta AZY$.

(c) Prove $\Delta BXY \equiv \Delta CZY$.

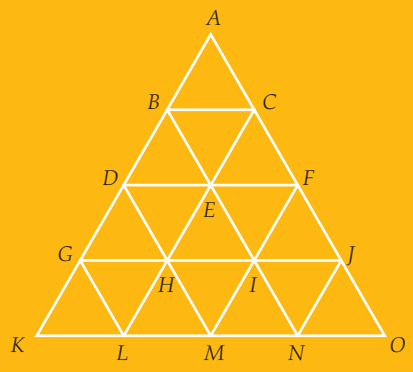
(d) Find b, c and d in terms of a.

17 Draw a diagram of your own that contains two congruent triangles. Prove that they are congruent using one of the congruency tests.

Outside the Square — Problem solving

How many triangles?

How many triangles can you find in this diagram?



Strategy options

- Solve a simpler problem.
- Have I seen a similar problem?
- Draw a diagram.
- Look for a pattern.

Equipment required: 1 brain, 1 TI-Nspire CAS or ClassPad CAS

Similarity, congruence and geometrical conjecture

Exploring similarity and congruence

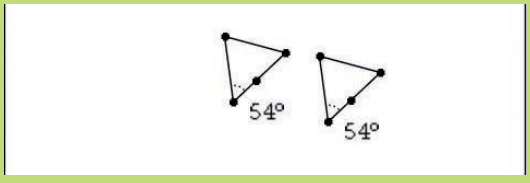
Using the TI-Nspire CAS

Congruent triangles

Open a new Geometry page and press (menu) > Shapes > Triangle > (enter) and a point appears on the screen. Press (enter) where you want to start your triangle and then move to the other vertices of the triangle, pressing (enter) at each one. Press (esc) > (menu) > Transformation > Translation > (enter) and move the cursor until the triangle is selected.

Now press (enter) twice and move the cursor. A copy of the triangle will be translated to a new position on the screen. Press (enter) when you reach where your translation is to be placed.

How can we check that these triangles are the same? We will check that the angles and the side lengths are the same. To check the size of an angle press (menu) > Measurement > Angle > (enter) and press (enter) at each of the vertices, in order, that make up the angle in which you are interested. When you press (enter) at the third vertex the size of the angle will appear on the screen. If you repeat the process on the same angle for the translated triangle you can see the two angles are the same size.



Using the ClassPad

Congruent triangles

From the menu, tap

. Tap on the triangle dropdown arrow and tap on triangle.

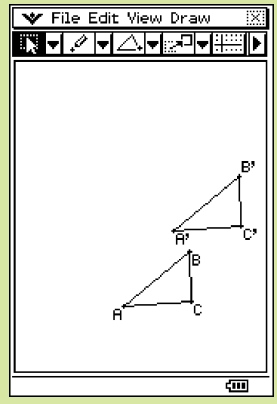
or Draw > Special Shape > Triangle. Tap on the screen. A triangle *ABC* will appear. To resize the triangle, tap on point *A*. Lift stylus then tap *A* again and drag it towards the centre of the screen. Repeat with points *B* and *C* until you have a smaller triangle. Remember to tap the screen in between tapping on the vertices.

The transformations of translations, rotations and reflections all produce congruent shapes. We will use translations here, but you can use a reflection or rotation if you wish. Tap on the sides of the triangle.

Tap on

or Draw > Construct > Translation. A vector will appear. Insert suitable values into the vector $\begin{bmatrix} p \\ q \end{bmatrix}$ e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Tap OK and a congruent triangle will appear on your screen.

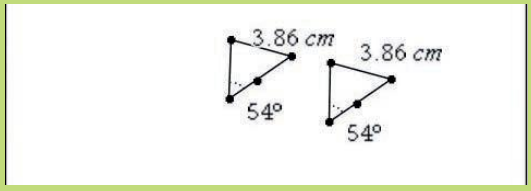
How can we check that these triangles are the same? We will check that the angles and the side lengths are the same.



Using the TI-Nspire CAS

You could then repeat the whole exercise for the other pairs of angles.

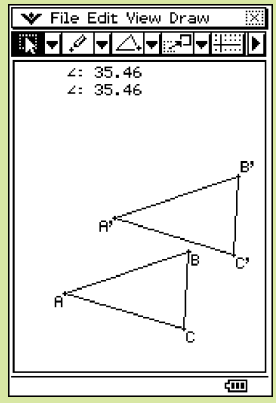
Now press (menu) > Measurement > Length > (enter) and press (enter) at each of the ends of one of the side lengths. You will see the side length appear, slightly shaded. To fix in on the screen press (enter) again. Repeat the process on the translated triangle. Press (esc).



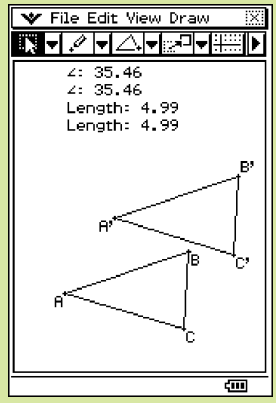
If you now move the cursor to one of the vertices of the original triangle and press until the hand closes, you can change the size of your triangle. You should notice that your translated shape changes in exactly the same way as the original.

Using the ClassPad

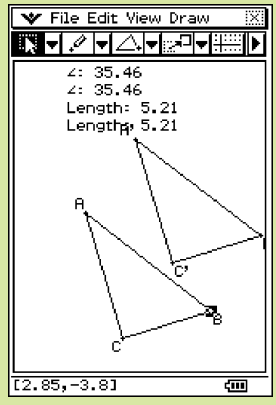
Tap on the two sides that form the angle in which you are interested. Then tap Draw > Measurement > Angle and the angle measurement will appear near the top of the screen. Repeat for the same angle in the image triangle and its angle size will appear. They should be the same.



Tap on the screen. Now tap on a line on the original triangle and tap Draw > Measurement > Length and the length of the line will appear near the top of the screen. Repeat for the matching line on the image triangle.



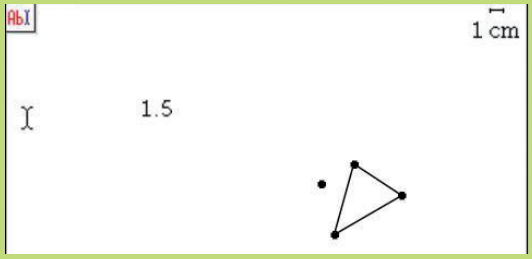
Now select one of the vertices of the original triangle and move it around the screen. The two pairs of measurements should remain the same as each other.



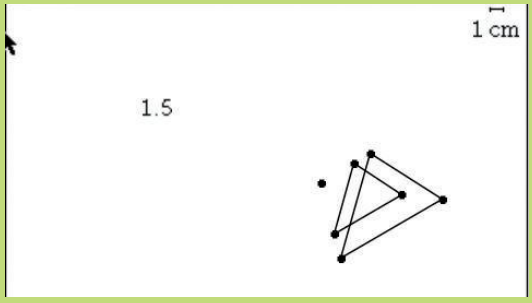
Using the TI-Nspire CAS

Similar triangles

Create a new triangle, as before. Then press (menu) > Points & Lines > Point > (enter), move the cursor to the desired place for the centre of dilation, and press (enter). Now press (menu) > Actions > Text > (enter), move the cursor away from the triangle and press (enter). In the text box created, write the dilation factor, say 1.5, and press (enter) to lock it onto the screen.



Now press (menu) > Transformation > Dilation > (enter) then move the cursor until the triangle is selected and press (enter), then move to the point and press (enter), and finally move to the dilation value and press (enter). You will now have two triangles on your screen.



Measure the lengths of the matching sides of the original and the image, and compare the ratios of these sides. The ratios should all be the same. Measure all the angles of the triangle and its image to check that matching angles are equal. Select a vertex point and change the size of your triangle. Compare side lengths and angle sizes again for the new object and image.

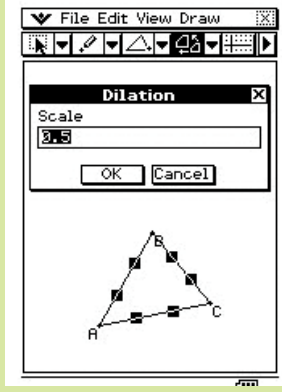
Using the ClassPad

Similar triangles

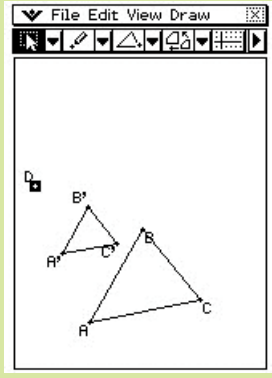
These can be formed using a dilation transformation. Create a new triangle as before and resize it if necessary. Select the three sides on the triangle.

Tap on

or Draw > Construct > Dilation. Tap on the screen and the dialog box will appear.



Enter the required dilation. Unless your original triangle is very small, a dilation < 1 is advisable. Tap OK and a similar triangle will appear on the screen. A point *D* will also appear. This is the centre of dilation. Measure the lengths of the matching sides of the original and the image, and compare the ratios of these sides. The ratios should all be the same. Measure all the angles of the triangle and its image. Matching angles should be equal. Tap on the screen outside the triangles. Select a vertex point and change the size of your triangle. Compare side lengths and angle sizes again for the new object and image.



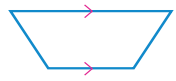
Geometrical conjecture

What is the difference between a theorem and a conjecture? A theorem has been proved to be always true and we can then use it to prove other geometric relationships. To establish a theorem, we need to first write a conjecture. If we can prove that a conjecture is always true, we can then use it as a theorem. Properties of geometric shapes are examples of theorems. If we write a conjecture and show by diagrams that it is true for all possible situations, we can then attempt a formal proof. If we can draw a diagram that is not true for our conjecture, then we know that the conjecture cannot be proved formally.

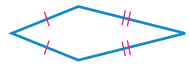
Consider the diagonals of a quadrilateral. Do they bisect each other? Do they intersect at right angles?
Does the answer to these questions depend on the type of quadrilateral being considered?

Below is a list of quadrilaterals with their definitions.

A trapezium is a quadrilateral with one pair of opposite sides parallel.

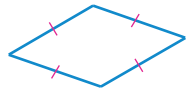


A kite is a quadrilateral with two pairs of adjacent sides equal in length.



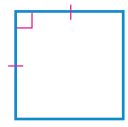
A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

A rhombus is a quadrilateral with all sides equal.



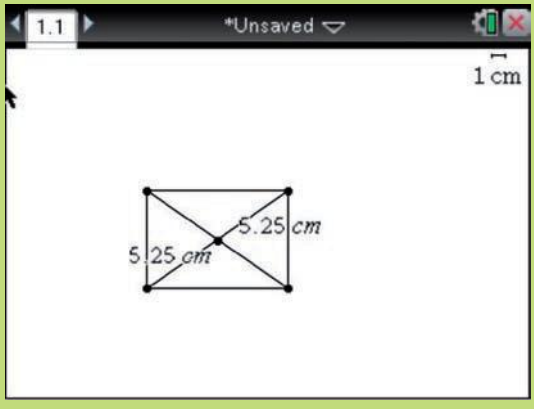
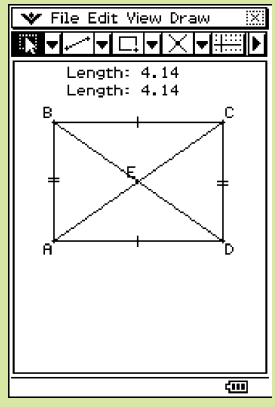
A rectangle is a quadrilateral in which all angles are right angles.

A square is a rectangle with two adjacent sides equal.



Use the list to write as many conjectures as you can about the properties each of the shapes might have. Then use your Geometry package to test these conjectures.

As an example, the screenshots below show how you might start to show that the diagonals of a rectangle bisect each other. They are not complete, but are provided to give you a clue as to what you might need to do.

Using the TI-Nspire CAS	Using the ClassPad
	

Proofs using similar triangles

8.4

The tests for similar and congruent triangles can be used to prove general results known as **theorems** about triangles and quadrilaterals. Once these theorems have been proved they are known as properties of triangles and quadrilaterals, and may be used to prove other relationships in these and other geometric figures, and to solve geometric problems.

Procedure for proving theorems

Theorems are stated in general terms. For instance, this means that a triangle theorem applies to all triangles. We prove theorems by using a general example that contains some given information. We draw a diagram to display the given information and make a statement of the relationship we wish to prove. We use congruence or similarity in our proof. We can then generalise the proof as a property of the figure. This generalisation becomes a theorem, which we can then use for other geometric proofs.

Midpoint theorem

The interval joining the midpoints of two sides of a triangle is parallel to the third side and is half its length.

This theorem is often used to prove relationships. Its proof is provided below.

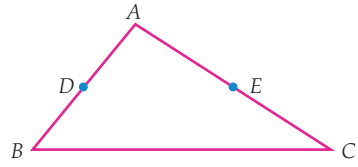
Worked Example 6 WE6

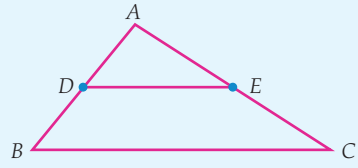
In ΔABC, D is the midpoint of AB and E is the midpoint of AC.

Draw DE and prove that:

(a) DE is parallel to BC.

(b) $DE = \frac{1}{2}BC$.



Thinking	Working
(a) **1** Copy the diagram and join D to E.	(a) 
2 Name the triangles that you will be considering.	ΔADE and ΔABC
3 Write three sets of properties that will allow you to use a similarity test.	$\angle DAE = \angle BAC$ (common angles) $AD = DB$ (D is the midpoint of AB) $\therefore AB = 2AD$ $AE = EC$ (E is the midpoint of AC) $\therefore AC = 2AE$ $\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$

4	Write down a pair of similar triangles and the test used.	$\therefore \Delta ADE \parallel\!| \Delta ABC$ (two sides are proportional and their included angles are equal)
5	Identify pairs of equal angles.	$\angle ADE = \angle ABC$ (matching angles of similar triangles) $\angle AED = \angle ACB$ (matching angles of similar triangles)
6	Link these equal angles to parallel lines.	$\angle ADE$ and $\angle ABC$ are a pair of equal matching angles formed when the transversal AB cuts the lines DE and BC. $\therefore DE \parallel BC$ (Corresponding angles are equal so the lines are parallel.) $\therefore$ The line joining the midpoints of two sides of a triangle (DE) is parallel to the third side (BC).
(b)	Use the property of the similar triangles from **(a)** to obtain the result.	(b) As the triangles are similar, the matching sides are proportional. $\therefore \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC} = \frac{1}{2}$ $\therefore \frac{DE}{BC} = \frac{1}{2}$ $\therefore DE = \frac{1}{2} BC$ $\therefore$ The line joining the midpoints of two sides of a triangle (DE) is half the length of the third side (BC).

When a point is placed on a line interval such that the distance from one end to the point is the same as the distance from the other end to the point we say that the point **bisects** the line.

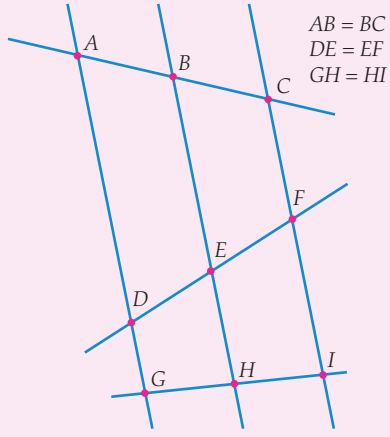
Converse to the midpoint theorem (the inverse theorem)

A line drawn through the midpoint of one side of a triangle and parallel to a second side bisects the third side.

This gives us another theorem.

Intercept theorem

When a set of parallel lines make equal intercepts on one transversal, it makes equal intercepts on any other transversal.



8.4 Proofs using similar triangles

Navigator

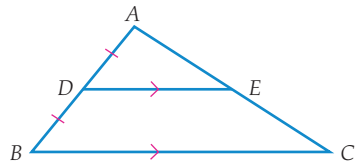
Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q9, Q10, Q12	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12

Answers page 765

Fluency

WE 6

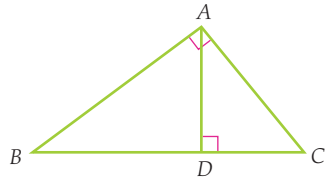
1 The line through the midpoint of a side of a triangle parallel to another side bisects the third side. In triangle ΔABC, D is the midpoint of AB and DE is parallel to BC, meeting AC at E. Prove that:



(a) $\Delta ABC \parallel\!| \Delta ADE$

(b) $AE = EC$.

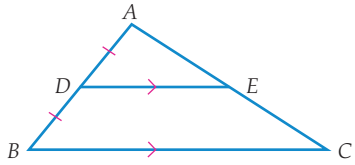
2 In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to each another. Triangle ΔABC is right-angled at A and $AD \perp BC$. Prove that triangles ΔADB, ΔDCA and ΔCAB are similar.



3 In a right-angled triangle, the square on the perpendicular to the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse. Using the diagram for Question **2**, prove that $AD^2 = BD \times DC$.

4 In a right-angled triangle, the square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side. Using the diagram for Question **2**, prove that $BA^2 = BD \times BC$ and $CA^2 = CD \times CB$.

5 Use the result from Question **4** to prove Pythagoras' theorem: $BC^2 = BA^2 + CA^2$



6 Which of the following is not sufficient to prove that two triangles are similar?

A The two triangles have two angles the same.

B The three sides of one triangle are proportional to the three sides of the other triangle.

C Two sides of one triangle are proportional to two sides of another triangle.

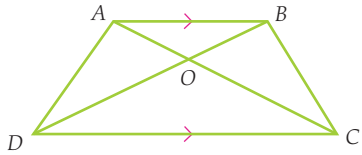
D Two sides of one triangle are proportional to two sides of another triangle and the included angles are equal.

Understanding

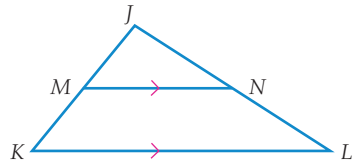
7 In the trapezium $ABCD$, AB is parallel to DC and the diagonals intersect at O.

(a) Show that $OA:OC = OB:OD$.

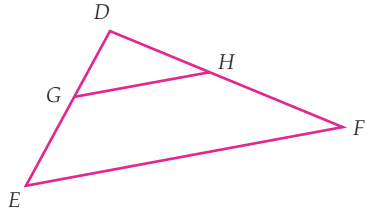
(b) If $DC = 2AB$, show that O is a point of trisection of both diagonals. (That is, show that $AO:OC = BO:OD = 1:2$ and hence that $AO:AC = BO:BD = 1:3$.)



8 A straight line drawn parallel to one side of a triangle cuts the other two sides proportionally. In triangle ΔJKL, M is a point on JK and MN is parallel to KL, meeting JL at N. Prove that $\dfrac{JM}{MK} = \dfrac{JN}{NL}$.



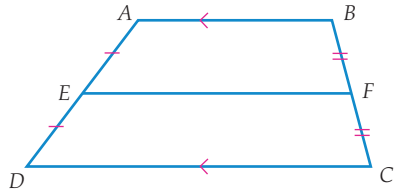
9 Conversely, if a line cuts two sides of a triangle proportionally, it is parallel to the third side. In triangle ΔDEF, $\dfrac{DG}{GE} = \dfrac{DH}{HF}$. Prove that $GH \parallel EF$.



Reasoning

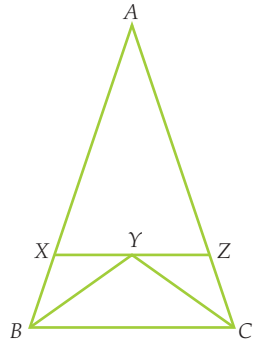
Hint
Extend DA and CB and let them intersect at G.

10 The straight line that joins the midpoints of the slanted sides of a trapezium is parallel to the parallel sides. $ABCD$ is a trapezium in which $AB \parallel DC$. E is the midpoint of AD and F is the midpoint of BC. Prove that $EF \parallel AB$.



Hint
Prove $\Delta BXY \equiv \Delta CZY$ and let them intersect at G.

11 ΔABC is an isosceles triangle such that $AB = AC$. X is a point on AB and Z is a point on AC such that XZ is parallel to BC. Y is the midpoint of XZ and $BX = XY$. Prove that $\Delta BXY \parallel\!| \Delta BYC$.



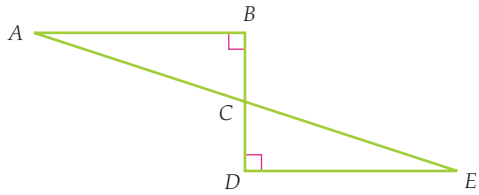
Open-ended

12 Draw a figure different from any given in Exercise 8.4 that contains at least two similar triangles. Prove that your triangles are similar.

Half-time 8

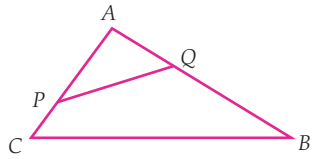
1 **(a)** Given that $BC = DC$, prove that $\Delta ABC \equiv \Delta EDC$.

(b) Hence, prove that $AB = ED$.



Ex. 8.2

2 Given that $\angle ABC = \angle APQ$, prove that $\Delta ABC \,|||\, \Delta APQ$.

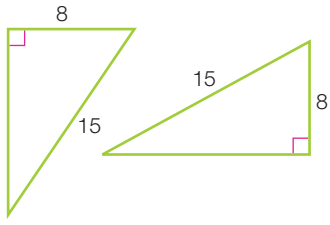


Ex. 8.4

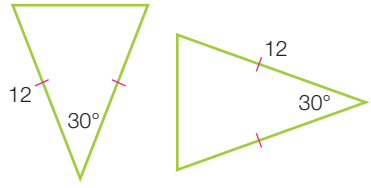
3 Give reasons why each of the following pairs of triangles are congruent.

Ex. 8.1

(a)



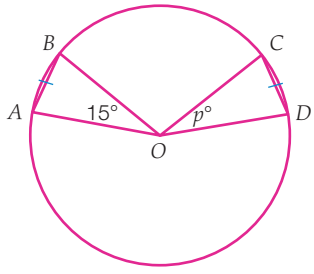
(b)



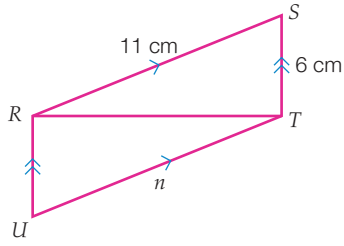
4 For each of the following, prove that the two triangles are congruent and hence find the value of the pronumeral.

Ex. 8.2

(a)



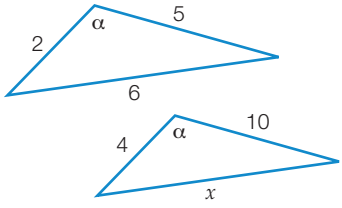
(b)



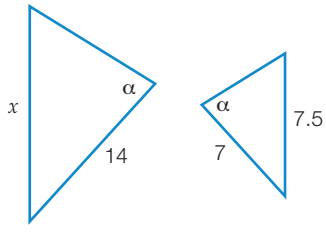
5 Given that these pairs of triangles are similar, find x.

Ex. 8.2

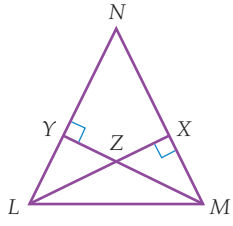
(a)



(b)



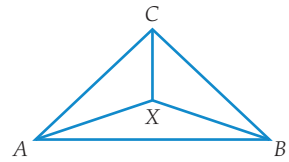
6 In triangle ΔLMN, LX is perpendicular to MN and MY is perpendicular to LN, as shown in the diagram. The point of intersection of LX and YM is Z. Prove that $\Delta LYZ \,|||\, \Delta MXZ$, giving reasons.



Ex. 8.4

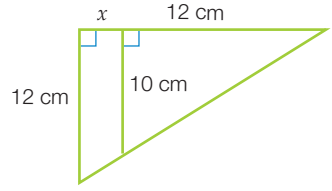
Ex. 8.3

7 Given that $AC = BC$ and $AX = BX$, prove that $\Delta ACX \equiv \Delta BCX$.



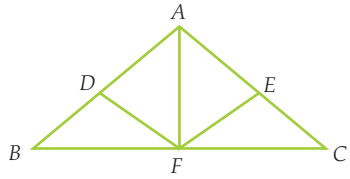
Ex. 8.1

8 Find the value of x in the triangle.



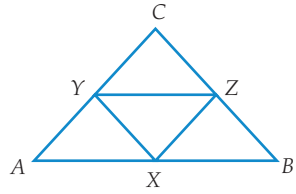
Ex. 8.2

9 In the diagram, $FD = FE$. Also, $FD \perp AB$ and $FE \perp AC$. Prove that $\Delta ADF \equiv \Delta AEF$.



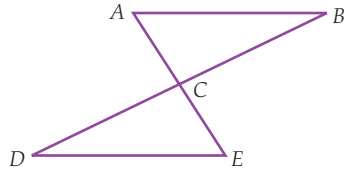
Ex. 8.3

10 In the diagram, ΔABC is isosceles with $AC = BC$. X, Y and Z are the midpoints of AB, AC and BC respectively. Prove that $XY = XZ$.



Ex. 8.2

11 Given $AB \parallel DE$ and that C is the midpoint of BD, prove that $\Delta ABC \equiv \Delta DEC$.

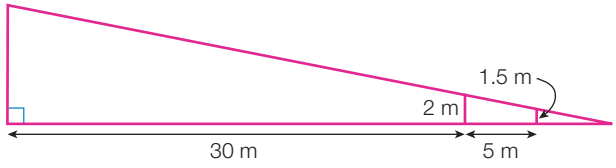


Ex. 8.1

12 Draw two triangles that are not congruent using two matching sides and an angle that is not the included angle.

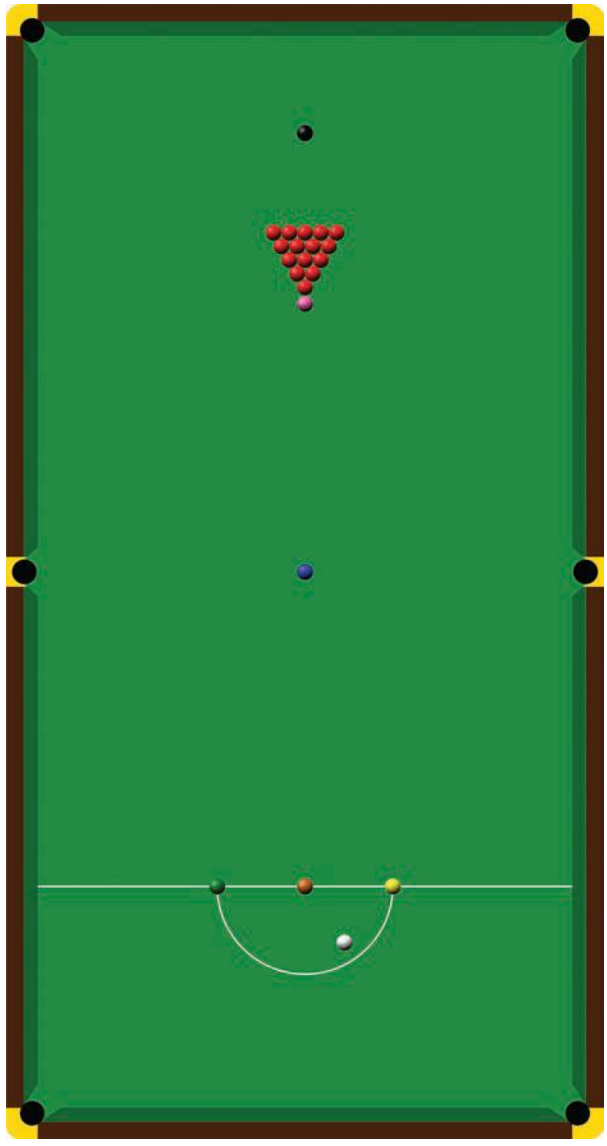
Ex. 8.4

13 A student 1.5 m tall, standing 5 m in front of a 2 m brick wall, can just see the top of a tree growing 30 m behind the wall. How tall is the tree?



Investigation

Snooker

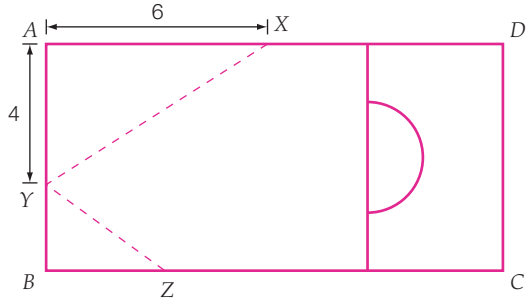


Equipment required: 1 or 2 brains, centimetre graph paper

The Big Question

A player, Neil, is at a point X (6 feet from table end AB) and is aiming for point Y, which is 4 feet from A, along AB. Neil wants to detemine what distance from B the ball would next hit the cushion at point Z.

(We use feet as the unit of measurement in this investigation because that is the way billiards and snooker tables are designed.)



Engage

The game of billiards originated in France in the early 1500s. A railing was added to the original shuffleboard and then cushion was added in 1835 to make the modern table. The object of the game was to use the white ball to hit both the red ball and the spot ball with the same shot. This was called a cannon. The modern game was developed later by the British and uses a 12 foot by 6 foot table (3.66 m × 1.83 m), which has the usual six pockets, one in each corner, and one in the middle of each of the longer sides.

Snooker was invented by British Army officers in India during the latter half of the 19th century and is now a very popular sport with top professional players earning huge amounts of money. This game consists of one white cue ball, 15 red balls (each worth one point), and six balls of different colours; yellow (2 points), green (3), brown (4), blue (5), pink (6) and black (7). The aim of the game is to pot red and coloured balls to earn more points than the opposition player to win a game. A player wins a match when he or she wins a certain number of frames.

Explore

1. The angle at which a ball bounces off the cushion is equal to the angle at which it approaches the cushion. Copy and complete the following: $\angle XYA = \angle$_____.
2. Draw the two triangles from Question **1** with the same orientation.
3. Determine the distance BZ.
4. Locate the position of the point N, where the ball will next strike the cushion of the table. Draw this position on your diagram.

Strategy options

- Draw a diagram.
- Break problem into manageable parts.

Explain

5. Where would the ball strike the cushion BC if Neil aimed at a point W, 2 feet from the corner A? (Neil is at point X.) Draw a diagram to illustrate this situation.
6. Neil is at point X and aiming for a point T, 4 feet from the corner B along the cushion BC, by bouncing off the cushion AB at a point V first. Find the position of V.
7. Is it possible for Neil to hit a ball from point X so that it bounces off all four cushions once before striking a cushion a second time? If it is possible, describe the path of the ball.

Elaborate

8. The path the ball follows after being hit depends on the direction the ball is hit.
   - **(a)** Does the path of the ball follow any pattern?
   - **(b)** How is this pattern related to the original direction of the hit?

Evaluate

9. Did your methods change as you worked through the investigation?
10. Did you discover any shortcuts or rules that you could use to obtain your solutions?

Extend

11. What would happen if the table were a different shape, such as a square or an equilateral triangle? Try Neil's problems on a table with a different shape to the rectangular one above.
12. What would happen if a table had different corner angles? Is it possible to solve Neil's problems?

Geometric properties of special quadrilaterals

8.5

As was stated earlier:

- A definition is the minimum amount of information needed to define a particular figure.
- A property is a fact that can be proven to be always true about a particular figure after it has been defined.

Definitions of the six special quadrilaterals are listed below. We can use these definitions to prove properties of these shapes.

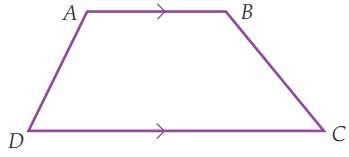
Definitions of special quadrilaterals

1. A trapezium is a quadrilateral with one pair of opposite sides parallel.
2. A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
3. A kite is a quadrilateral with two pairs of adjacent sides equal in length.
4. A rectangle is a quadrilateral in which all angles are right angles.
5. A rhombus is a quadrilateral with all sides equal in length.
6. A square is a rectangle with two adjacent sides equal in length or a rhombus with all angles equal to 90°.

Properties of special quadrilaterals

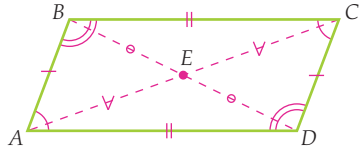
Trapezium

- A trapezium has no particular properties.



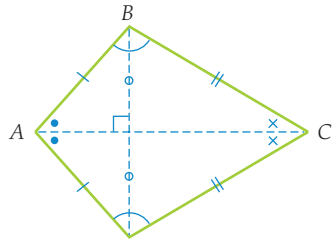
Parallelogram

- Both pairs of opposite angles are equal in magnitude: $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$.
- Both pairs of opposite sides are equal $AB = DC$ and $AD = BC$.
- Diagonals bisect each other: $AE = EC$ and $BE = ED$.

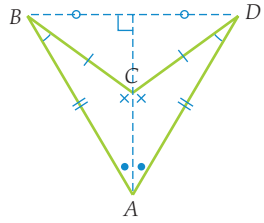


Kite

- One pair of opposite angles are equal in magnitude: $\angle ABC = \angle ADC$.
- A kite has an axis of symmetry that bisects the non-equal vertex angles through which it passes.
- The axis of symmetry of the kite bisects the other diagonal at right angles.



convex kite

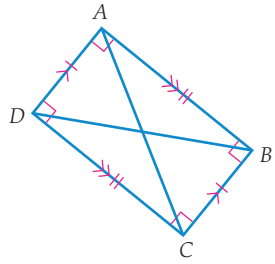


concave kite

Rectangle

All of the properties of a parallelogram, plus:

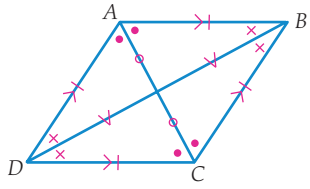
- Diagonals are equal in length, $AC = BD$.
- The lines joining the midpoints of opposite sides are lines of symmetry.



Rhombus

All of the properties of a parallelogram, plus:

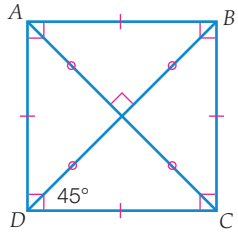
- Diagonals bisect each other at right angles: $AC \perp BD$.
- Diagonals bisect the vertex angles: $\angle DAC = \angle BAC = \angle DCA = \angle BCA$, $\angle ADB = \angle CDB = \angle ABD = \angle CBD$.
- Both diagonals are lines of symmetry.



Square

All of the properties of a rectangle and a rhombus, plus:

- Diagonals make an angle of 45° with each side.
- Each diagonal has length equal to $\sqrt{2}$ times the side length.



A couple of terms that we use in this work need to be defined:

- When two lines intersect at right angles we say they are **perpendicular**.
- When two lines intersect at right angles and one bisects the other (cuts it in half), we say that line is a **perpendicular bisector** of the second line.

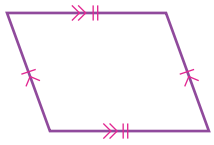
Proving properties of special quadrilaterals

You should already be familiar with many of the properties of special quadrilaterals. We are going to prove some of the properties of these special quadrilaterals.

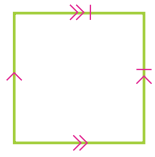
Worked Example 7 WE7

For the following shape, write down the information shown, then use the definitions for quadrilaterals to determine the type of shape.

(a)



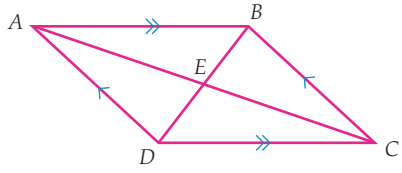
(b)



Thinking	Working
(a) 1 Write down any information displayed or given.	(a) Both pairs of opposite sides are equal and parallel.
2 Use each piece of information to list possible shapes, then state the type of shape and give reasons.	The shape is a parallelogram because both pairs of opposite sides are parallel.
(b) 1 Write down any information displayed or given.	(b) Opposite sides are parallel. Two adjacent sides are equal in length.
2 Use each piece of information to list possible shapes, then state the type of shape and give reasons.	Opposite sides parallel means that the shape is a parallelogram. Two adjacent sides equal in length means that the parallelogram is a rhombus.

Worked Example 8 WE8

ABCD is a parallelogram.



(a) By drawing the diagonal *AC*, prove that $\Delta ADC \equiv \Delta CBA$.

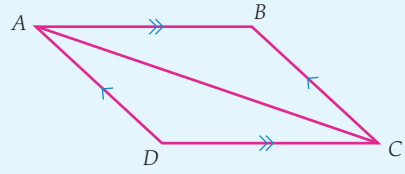
(b) Use these triangles to prove that $\angle ADC = \angle ABC$ and $\angle DAB = \angle BCD$.

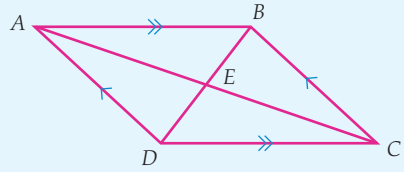
(c) What special property of a parallelogram does this prove?

(d) By now drawing the diagonal *BD*, prove that $\Delta CDE \equiv \Delta ABE$.

(e) Use these triangles to prove that $AE = CE$ and $DE = BE$.

(f) What special property of a parallelogram does this prove?

Thinking	Working
(a) 1 Copy the diagram and complete the construction. Draw on any given side markings. (You can leave out the diagonal *BD*.)	(a) 
2 Name the two triangles to be considered.	ΔADC and ΔCBA

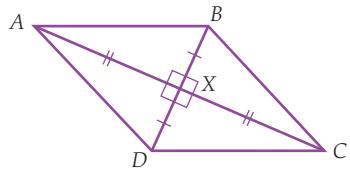
	3 Write three sets of equalities, with reasons, that apply to the two triangles.	$\angle DCA = \angle BAC$ (alternate angles, $AB \parallel DC$) $\angle DAC = \angle BCA$ (alternate angles, $AD \parallel BC$) AC is a common side
	4 Write down the pair of congruent triangles and the test used to determine this.	$\Delta ADC \equiv \Delta CBA$ (AAS)
(b)	**1** Identify the appropriate pair of matching angles.	(b) $\angle ADC = \angle ABC$ (matching angles in congruent triangles)
	2 Show that the other pair of angles is equal.	$\angle DAB = \angle DAC + \angle BAC$ $\angle BCD = \angle BCA + \angle DCA$ But $\angle DAC = \angle BCA$ and $\angle DCA = \angle BAC$ (from part (a) step 3) So $\angle DAB = \angle BCD$
(c)	Write the answer in words.	(c) The property proved is that the opposite angles of a parallelogram are equal.
(d)	**1** Name the two triangles to be considered and complete the construction on the diagram. Label point E.	(d) ΔABE and ΔCDE 
	2 Write three sets of equalities, with reasons, that apply to the two triangles.	$AB = CD$ ($\Delta ADC \equiv \Delta CBA$) $\angle DCE = \angle BAE$ (alternate angles, $AB \parallel DC$) $\angle CDE = \angle ABE$ (alternate angles, $AB \parallel DC$)
	3 Write down the pair of congruent triangles and the test used to determine this.	$\Delta ABE \equiv \Delta CDE$ (AAS)
(e)	Identify the appropriate pair of matching sides.	(e) $AE = CE$ and $DE = BE$ (matching sides in congruent triangles)
(f)	Write the answer in words.	(f) The property proved is that the diagonals of a parallelogram bisect each other.

Sometimes we are told certain properties of a shape. We can use this information to determine the type of shape by finding the appropriate definition.

Worked Example 9

WE9

The diagonals of a quadrilateral bisect each other at right angles. Prove that this quadrilateral is a rhombus.



Thinking	Working
1 Use the definitions for quadrilaterals to determine the information needed to prove the shape is a rhombus.	Two adjacent sides must be proved to be equal in length and the opposite sides must be proved to be parallel.
2 Recognise that congruent triangles need to be found and identify the triangles to be considered.	Consider ΔAXD and ΔAXB
3 List three equalities, with reasons, that apply to the triangles.	$\angle AXD = \angle AXB = 90°$ (given) AX is common $DX = BX$ (given)
4 Write down the congruent triangles and the test used to determine this.	$\Delta AXD \equiv \Delta AXB$ (SAS)
5 Identify the appropriate pairs of matching sides.	$AD = AB$ (matching sides in congruent triangles)
6 Consider another pair of triangles to prove congruent.	Consider ΔAXD and ΔCXB
7 List three equalities, with reasons, that apply to the triangles.	$\angle AXD = \angle CXB = 90°$ (given) $AX = CX$ (given) $DX = BX$ (given)
8 Write down the congruent triangles and the test used to determine this.	$\Delta AXD \equiv \Delta CXB$ (SAS)
9 Identify equal angles and use equal alternate angles to prove the lines are parallel.	$\angle DAX = \angle BCX$ (matching angles in congruent triangles) $AD \parallel BC$ (alternate angles $\angle DAX$ and $\angle BCX$ are equal)
10 Recognise that repeating steps **6–9** with the other two triangles, we can prove the other two sides are parallel.	Similarly, $AB \parallel CD$
11 Summarise the information needed to answer the question.	$AB \parallel CD$ and $AD \parallel BC$ (opposite sides parallel) $AD = AB$ (two adjacent sides equal) So the shape is a rhombus.

8.5 Geometric properties of special quadrilaterals

Navigator

Answers page 767

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11, Q12, Q13, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q19	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19

Fluency

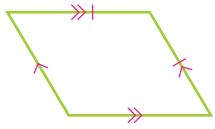
WE7

1 For each of the following shapes:

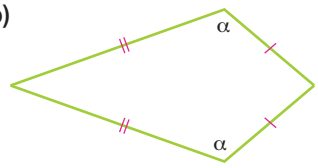
(i) write down the information shown

(ii) use the definitions for quadrilaterals to determine the type of shape.

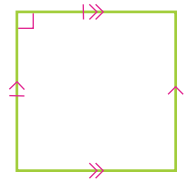
(a)



(b)

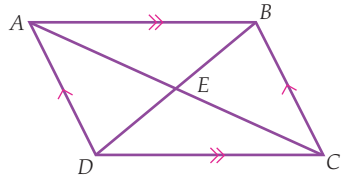


(c)



WE8

2 *ABCD* is a parallelogram Use the diagram shown to prove the following properties of parallelograms.



(a) By drawing the diagonal *BD*, prove that $\Delta DAB \equiv \Delta DCB$.

(b) Use these triangles to prove that $\angle DAB = \angle DCB$ and $\angle ADC = \angle ABC$.

(c) What special property of a parallelogram does this prove?

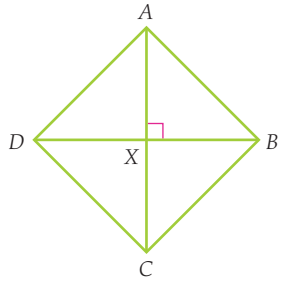
(d) By now drawing the diagonal *AC*, prove that $\Delta DAE \equiv \Delta BCE$.

(e) Use these triangles to prove that $AE = CE$ and $DE = BE$.

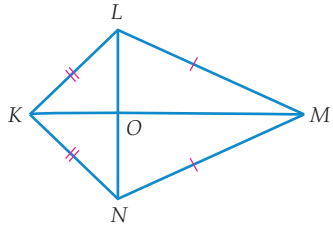
(f) What special property of a parallelogram does this prove?

WE9

3 The diagonals of a quadrilateral are equal in length and bisect each other at right angles. Prove that this quadrilateral is a square.



4 Use the diagram shown to prove the following properties of a kite.



(a) A kite has one pair of opposite equal angles.

(b) A kite has diagonals that cross at right angles.

5 When proving that a quadrilateral is a trapezium but not a parallelogram, it is necessary to show:

A only one pair of parallel sides

B one pair of parallel sides and one pair of non-parallel sides

C one pair of parallel sides and one pair of equal sides

D two pairs of parallel sides.

6 A quadrilateral must be a parallelogram if one pair of opposite sides is:

A only equal in length

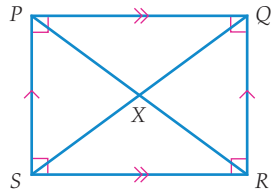
B only parallel

C equal in length and parallel

D parallel and the other pair of opposite sides is equal in length.

7 **(a)** For the diagram shown, prove $PQ = RS$.

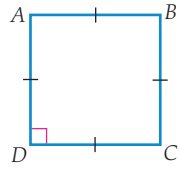
(b) What special property of a rectangle have you just proved?



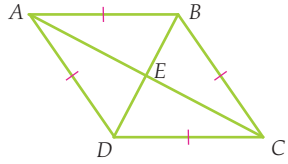
Understanding

8 Prove the following using the diagram shown:

(a) If a quadrilateral has all sides equal and one angle is a right angle, then it is a square.

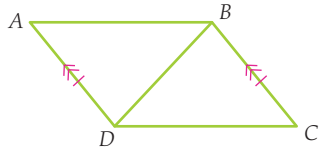


(b) If all sides of a quadrilateral are equal, then the quadrilateral is a rhombus.

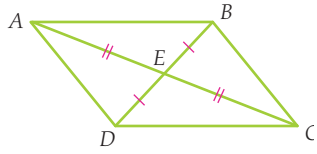


9 Prove the following using the diagrams.

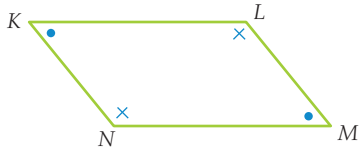
(a) If a pair of opposite sides in a quadrilateral is parallel and equal, then the quadrilateral is a parallelogram.



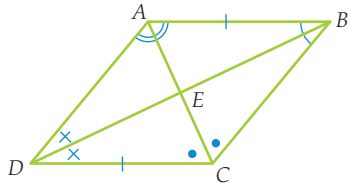
(b) If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



(c) If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.



(d) If both diagonals of a quadrilateral bisect the vertex angles through which they pass, then the quadrilateral is a rhombus.



10 To prove that a parallelogram must be a rectangle it is necessary to prove that its diagonals:

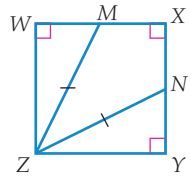
A are equal in length

B bisect each other

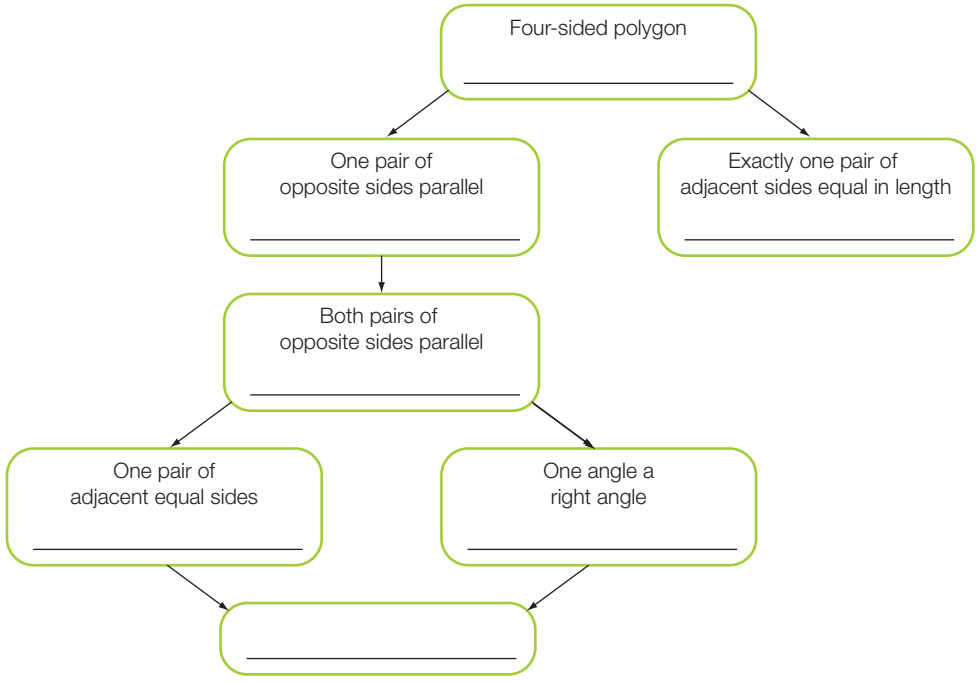
C bisect the angles of the parallelogram

D are perpendicular to each other.

11 *WXYZ* is a square. Show that $\Delta WMZ \equiv \Delta YNZ$.

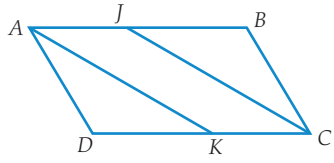


12 Copy and complete the following flowchart to show the relationships between different quadrilaterals.



Reasoning

13 *ABCD* is a parallelogram and $AJ = CK$.
Show that $\Delta AKD \equiv \Delta CJB$.



14 Explain why a square is a rhombus, but a rhombus is not necessarily a square.

15 Explain why a square is a rectangle, but a rectangle is not necessarily a square.

Open-ended

16 Adam and Alvin were discussing how many different quadrilateral shapes they could use to make biscuits. Choose three different quadrilateral shapes that they could use. The shapes need to have one of the following properties in common:

- at least one pair of parallel sides
- at least one pair of equal sides
- at least one pair of equal angles.

Draw the shapes and name them, and then identify the property you have used.

17 (a) The statement 'All squares have equal sides' is true. The converse statement is 'All shapes with four equal sides are squares'. Is this true? Explain why or why not.

(b) Write two or more true statements about special quadrilaterals and their converse statements. Examine whether or not the converse statements are true.

18 (a) Draw any three rhombuses and mark in the diagonals.

(b) Measure the angles at the centre where the diagonals intersect.

(c) What property of rhombuses does this seem to show?

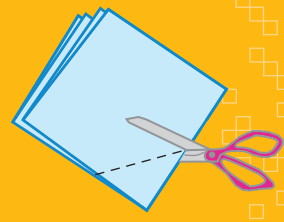
(d) Is this sufficient proof of this property?

19 Gary, Matt and George are arguing about the geometric figure that could have the property that the diagonals bisect at right angles. Matt says there is only one such figure, Gary says there are two and George says there are three or more. Who is right and what shape could the figure(s) be?

Outside the Square Problem solving

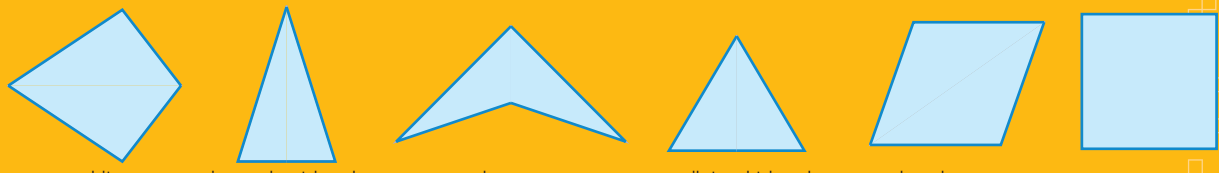
Folding and cutting

Equipment required: some loose sheets of paper and scissors



1 You are going to fold a piece of paper twice and then cut across the dotted corner. Can you predict the shape of the resulting hole?

The shape of the resulting hole will depend on the angle at which you cut and also the angle of the second fold. See if you can produce the following shaped holes.

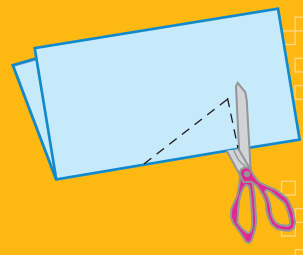


kite | isosceles triangle | chevron | equilateral triangle | rhombus | square

Can you find the maximum number of sides of the hole such a single cut produces for a piece of paper folded twice?

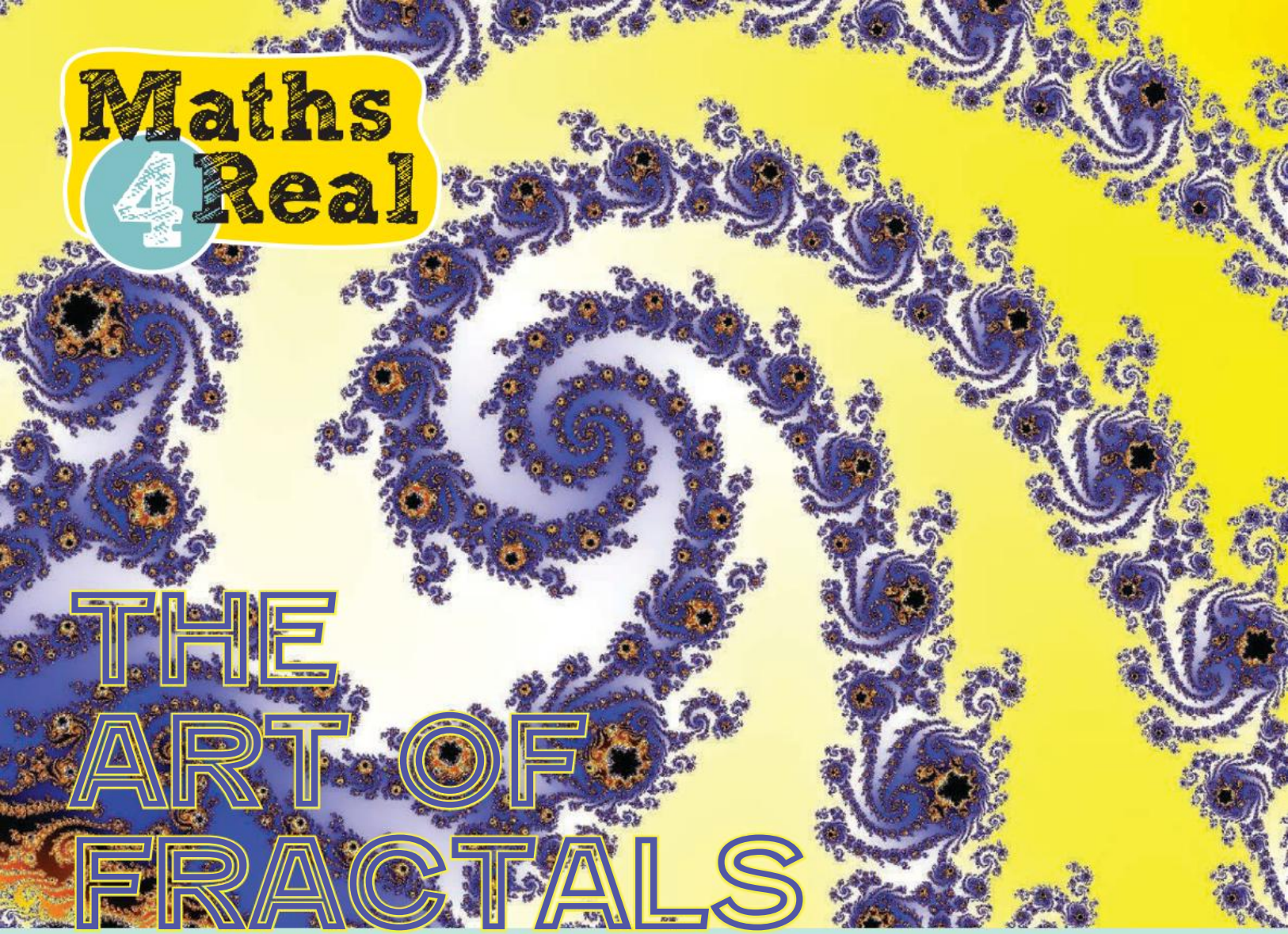
2 Now try folding the sheet of paper just once, but using two cuts to produce a hole when unfolded. What shape will your hole be?

Can you produce an equilateral triangle, an isosceles triangle and a scalene triangle?



Strategy options

- Act it out.
- Make a model.



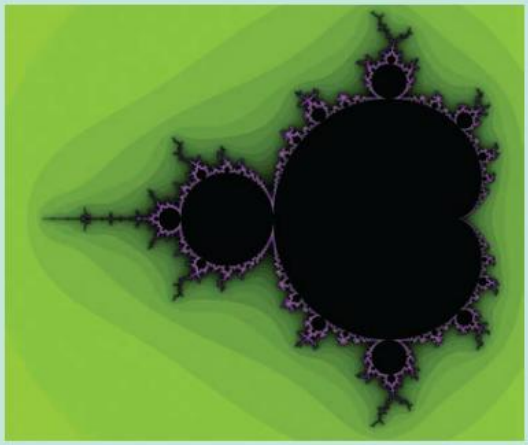
Maths 4 Real

THE ART OF FRACTALS

A *fractal* is a self-replicating pattern. This means that smaller parts of the whole image look like the whole image itself. Fractal-like patterns occur in nature and fractals are sometimes described as the geometry of nature.

Our knowledge of fractals began with the work of the French mathematician Gaston Julia, early in the twentieth century. However, he was working before computers were invented, and to graph the results by hand would have been too time consuming. The illustration at the top of this page is a graph of a *Julia* set.

The word *fractal* was first used by French mathematician Benoit Mandelbrot, who rediscovered the work of Julia in the 1970s. With the advent of the computer age, fractal images were now a reality. However, the first images produced by Mandelbrot were very poor compared to those now produced by modern computers. With the use of suitable software, anybody is now able to produce these amazing patterns on a home computer.



One of the best known fractal images is called a *Mandelbrot set* as shown here. In this image, each point on the number plane is substituted as a variable into a complex equation. The result is then substituted back into the equation and the process repeated (or iterated) many times. The mathematics behind the image is too difficult for us to consider at this level, but we can admire the beauty of the image produced.

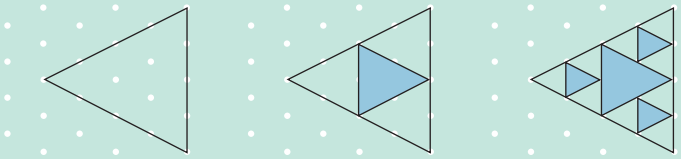
Many fractals can be produced by following simple mathematical procedures. In this task, we will first look at a particular fractal called the *Sierpinski Gasket* or *Sierpinski Triangle*. The first iteration is the first stage in drawing the whole, the second iteration is the next stage in the process, and so on.

The Sierpinski Gasket is drawn using isometric paper. Consider the following:

Draw a triangle of side length 2 units, connect the midpoints of each side, then colour in the middle triangle. This construction is shown in the three diagrams below.

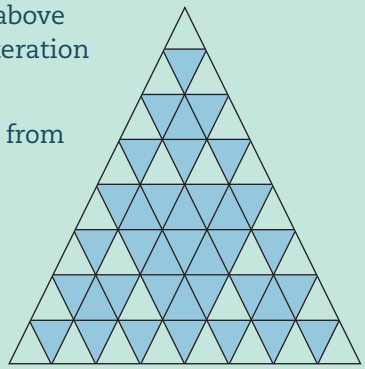
The final diagram in this sequence is the *first iteration* of the Sierpinski Gasket.

For the second iteration, start with an equilateral triangle of side length 4 units. When we join the midpoints and colour in the middle triangle, we are left with three triangles on which we can repeat the first iteration. The following diagrams show this construction.



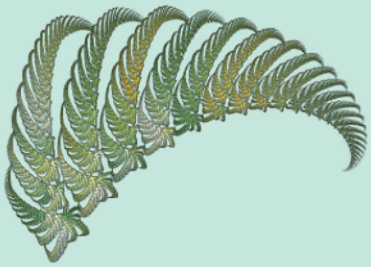
The final diagram in the above sequence is the second iteration of the Sierpinski Gasket.

The third iteration comes from beginning with an 8-unit equilateral triangle, and so on. If you complete it successfully, it should look like this.



1 Draw the fourth iteration of the Sierpinksi Gasket.

As mentioned at the start of this task, fractals can be used to model things which occur in nature. Consider the fractal image of a fern leaf shown here.



This is an example of a type of fractal known as an *L-System*. An L-System is based on a set of rules that model growth that occurs in nature.

Consider the following example in which the rules are:

$A \to DB$; $B \to C$; $C \to D$; $D \to E$; $E \to A$

This produces the following pattern, which models the branching growth of many simple plants.

Stage 0									A							
Stage 1					D							B				
Stage 2					E							C				
Stage 3					A							D				
Stage 4				D				B					E			
Stage 5				E				C					A			
Stage 6				A				D				D				B	
Stage 7			D			B			E				E				C	
Stage 8			E			C			A				A				D	
Stage 9			E			D			D			B					E	
Stage 10		D			B			E				C					A	
Stage 11		E			C			A				D				D		B

2 Consider the L-System that follows the rules $A \to B$ and $B \to AB$ and starts with an A.

Stage 0 is A

Stage 1 is B

Stage 2 is AB

Stage 3 is BAB

(a) Using a grid like the one shown above, write the letter outcome for each of the next four stages (i.e. Stages 4–7).

(b) Can you see a number pattern involving the number of letters in each stage?

(c) Using the pattern, or otherwise, find the number of letters in each of Stages 8, 9 and 10.

The *von Koch snowflake*, which is investigated in Chapter 12, is another type of fractal. It has an infinite perimeter but a finite area. For many mathematicians this is one of the best things about fractals—they give us some contact with infinity.

Research

1 Find out what is meant by the term *fractional dimensionality* and what it has to do with fractals.

2 Produce a colourful fractal image using a computer program such as FRACTINT.

Technology Exploration GeoGebra

Equipment required: 1 brain, 1 computer with GeoGebra

Versions of this Exploration for other technologies are available in Pearson Reader.

Where is the centre of a triangle?

Open the GeoGebra program. You will see seven menu options (File, Edit etc.) at the top of the screen. Below these are eleven icons called tools. We will label these by number counting from the left. By clicking on the small arrow in the bottom right-hand corner of the tool icon a dropdown list of more tools appears. The arrow turns red when you hover the cursor over it. If you hover over a tool, the tool's name and how to use it will appear on the screen.

1 From the Options menu, select 'Labelling' and 'New Points Only'.

2 From the View Menu, select 'Grid' and deselect 'Axis' or right click on the drawing pane and deselect 'Axis'.

3 If at any stage you make a mistake, you can use the 'Undo' tool in the top right-hand corner of the 'Graphics View'.

4 If a larger font is required, click on the Options menu and select 'Font Size'. Choose an appropriate size from the list provided.

Concurrence

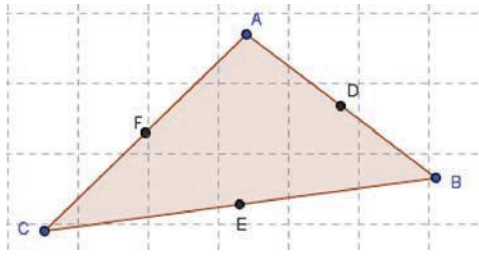
Any two non-parallel lines that lie in the same plane will intersect. If a third line passes through the same point of intersection, we say that the lines are concurrent and the point of intersection is called the point of concurrence.

The centroid of a triangle

A line drawn from a vertex of a triangle to the midpoint of the opposite side is called a median. We will construct the medians of a triangle and investigate their concurrency.

5 From the fifth icon select the 'Polygon' tool . Create a triangle by clicking any three points in the Graphics View and then clicking once more on point *A*.

6 From the second icon select the 'Midpoint or Centre' tool . Click on interval *AB* and the midpoint *D* will appear. Repeat for sides *BC* and *CA* (in that order). Your diagram should look like this.



7 From the third icon select the 'Interval between Two Points' tool . Create interval *AE* by clicking on *A* and *E*. Repeat for *D* and *C*, and *F* and *B*.

8 Notice where the intervals intersect! From the second icon select the 'Intersect Two Objects' tool . Click first on interval *AE* and then interval *FB* or *DC* to create the point of intersection *G*.

9 From the first icon, select the 'Move' tool . Click on one of *A*, *B* or *C* and drag the point about. What do you notice?

10 Complete these sentences:

(a) Lines drawn from each vertex of a triangle to the _____ of the opposite side are concurrent.

(b) The lines are called _____ and the point of concurrence is called the centroid of the triangle.

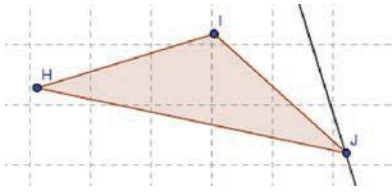
The orthocentre of a triangle

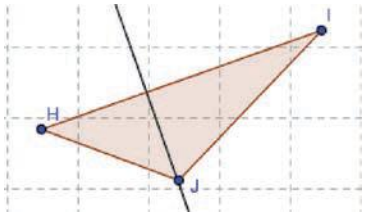
A line that is perpendicular to a side of a triangle that passes through the opposite vertex is called an altitude. We will construct the altitudes of a triangle and investigate their concurrency.

11 From the last icon, select the 'Move Graphics View' tool . Move the Graphics View until you get a clear space to work in.

12 From the fifth icon select the 'Polygon' tool . Create triangle *HIJ* by clicking any three points in the Graphics View and then clicking once more on point *H*.

13 From the fourth icon select the 'Perpendicular Line' tool . Click on interval *HI* and then on point *J*. The line may lie inside or outside of the triangle as shown.





14 Complete the other two altitudes by clicking on interval *HJ*, followed by point *I*, and then click on interval *IJ*, followed by point *H*.

15 Notice where the altitudes intersect! From the second icon select the 'Intersect Two Objects' tool . Click first on the line through *I* and then on either of the lines through *J* or *H* to create the point of intersection *K*.

16 From the first icon, select the 'move' tool . Click on any of *H*, *I* or *J* and drag the point about.

17 What do you notice?

18 Complete these sentences:

(a) Lines _____ to the side of a triangle that pass through the opposite vertex are concurrent.

(b) The lines are called _____ and the point of concurrence is called the orthocentre of the triangle.

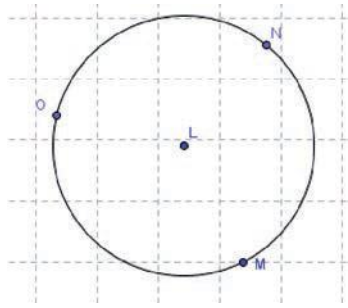
The incentre of a triangle

We will construct a triangle from the tangents to a circle and investigate the links to the angle bisectors of this triangle.

19 From the last icon, select the 'Move Graphics View' too . Move the Graphics View until you get a clear space to work in.

20 From the sixth icon select the 'Circle with Centre through Point' tool . Click and drag to draw a circle that has a radius of between 2 and 3 grid squares.

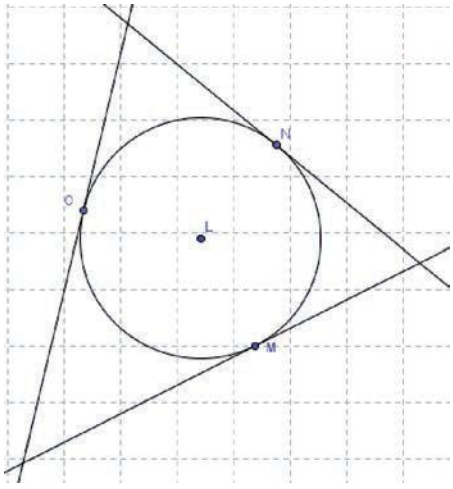
21 From the second icon select the 'New Point' tool, . Add two new points to the circumference of the circle by clicking on it as shown.



22 Look to the 'Algebra View' on the left-hand side of the screen and locate the equation that looks similar to this one. $p: (x - 31.42)^2 + (y + 0.1)^2 = 4.53$. This is the equation of the circle you have just created. The letter at the front tells you the name of the circle (this circle is *p*).

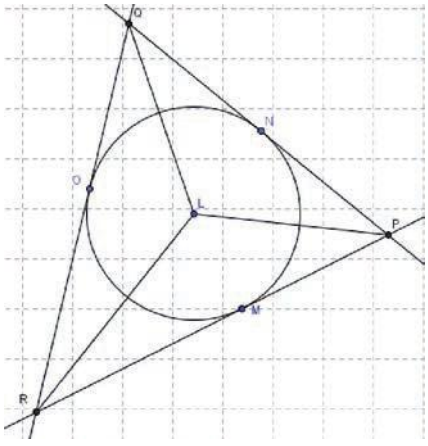
23 In the 'Input' bar at the bottom of the screen, type "tangent[M,p]" as shown Input: tangent[M,p] and press Enter. This will draw a tangent to the circle *p* at the point *M*. (If your circle has a different name to *p*, then use the correct name.)

24 Draw a tangent at N by typing "tangent[N,p]" and one at O by typing "tangent[O,p]". Your diagram will look similar to this.



25 From the second icon select the 'Intersect Two Objects' tool . Locate the points of intersection of each pair of lines, in the following order: The tangents to M and N, the tangents to N and O, the tangents to O and M. You may have to move the Graphics View to see these points of intersection.

26 From the third icon select the 'Interval between Two Points' tool . Create intervals between the centre and each vertex as shown.



27 From the eighth icon select the 'Angle' tool. Measure the size of $\angle OQL$ by clicking on the three points in a clockwise direction. Repeat for $\angle LQN$. (The 'Move' tool can be used to drag the numbers about so you can see them better.)

28 What do you notice about the sizes of these angles?

29 Repeat steps 28 and 29 for $\angle NPL$ and $\angle LPM$, and again for $\angle MRL$ and $\angle LRO$.

30 From the first icon, select the 'Move' tool . Click on any of M, N or O and drag the point about.

31 What do you notice about the sizes of each pair of angles?

32 What word do we give to the process of cutting a line or angle in half?

We have shown that the lines drawn from the centre of the circle inside the triangle bisect the angles. However, we can also use this result in reverse: lines that bisect the angles of a triangle are concurrent.

33 Complete these sentences:

(a) The lines that bisect the angles of a triangle are _____ and the point of concurrence is called the incentre of the triangle because a circle can be drawn so that it touches all three sides.

(b) The circle is called the incircle and the sides of the triangles are _____ to the circle.

The circumcentre of a triangle

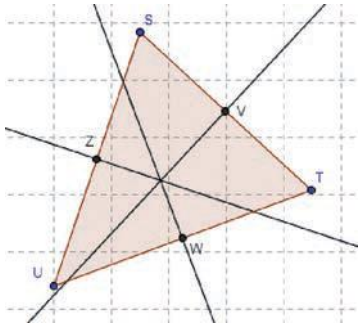
We will construct the perpendicular bisectors of the sides of the triangle and investigate their concurrency.

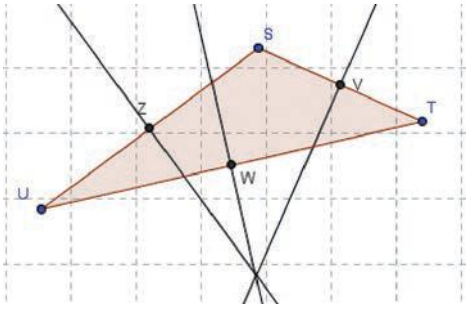
34 From the last icon, select the 'Move Graphics View' tool . Move the graphics view until you get a clear space to work in.

35 From the fifth icon select the 'Polygon' tool . Create triangle STU by clicking any three points in the Graphics View and then clicking once more on point S.

36 From the second icon select the 'Midpoint or Centre' tool . Click on interval ST and the midpoint V will appear. Repeat for sides TU and US (in that order) to get midpoints W and Z.

37 From the fourth icon select the 'Perpendicular Line' tool . Click on a midpoint and then on the interval it lies on. Repeat for the other two midpoints. Your diagram should resemble the following.





38 From the second icon select the 'Intersect Two Objects' tool . Click on one and then another of the perpendicular lines to create the point of intersection A_1.

39 From the first icon, select the 'Move' tool . Click on one of S, T or U and drag the point about.

40 What do you notice about the concurrency of the perpendicular bisectors?

41 From the sixth icon select the 'Circle with Centre through Point' tool . Click on A_1 and then the point S.

42 What do you notice about the points U and T?

43 Copy and complete these sentences:

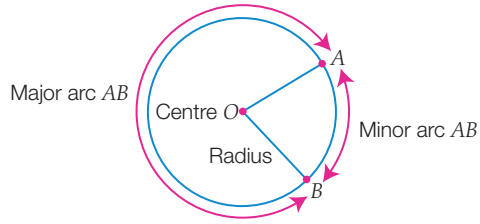
(a) The perpendicular bisectors of each side of a triangle are _____.

(b) The point of concurrence is called the circumcentre of the triangle because a circle can be drawn through all three vertices using the _____ as the centre of the circle. The circle is called the circumcircle.

8.6 Angles in circles 10A

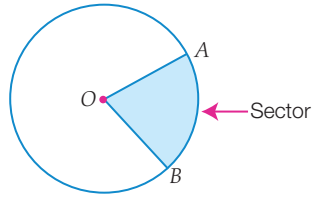
Terms related to circles

1 A circle is a plane curve consisting of all the points at a given distance (called the radius) from a fixed point in the plane (called the centre).

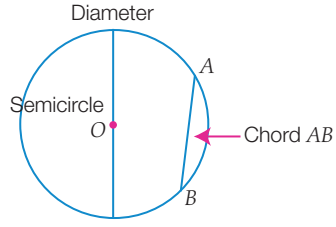


2 An **arc** of a circle is a portion of the circle bounded by two points on the circle. The shorter arc is the **minor arc** and the longer arc is the **major arc**. An arc that goes completely around the circle is called the circumference.

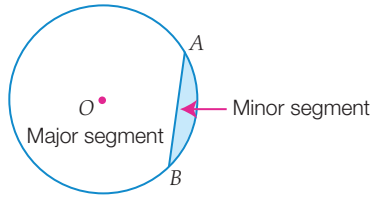
3 A **sector of a circle** is the plane figure bounded by two radii of the circle and one of the arcs that they form.



4 A **chord** of a circle is an interval joining any two points on the circle. A chord that passes through the centre of a circle is called a diameter. The length of the diameter is twice the radius. A diameter divides the circle into two semicircles.

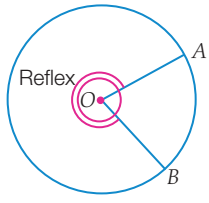


5 A **segment** of a circle is the plane figure bounded by a chord and an arc. A **minor segment** is bounded by a chord and a minor arc. A **major segment** is bounded by a chord and a major arc.



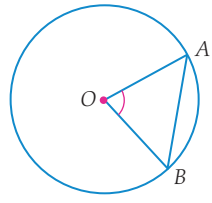
6 The word **subtend** means 'to be opposite to' or 'measure off'. The side of a triangle subtends the opposite angle, and an arc of a circle subtends an angle at the centre of a circle. We also say that the angle at the centre of a circle is **subtended** by the arc or the chord of the circle.

The major arc AB subtends the reflex angle AOB at the centre, O, of the circle.

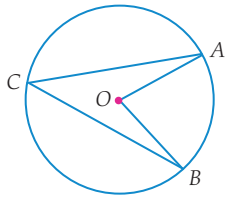


The minor arc AB subtends the acute or obtuse angle AOB at the centre, O, of the circle.

If you join AB, then the chord AB subtends angle AOB at the centre, O.

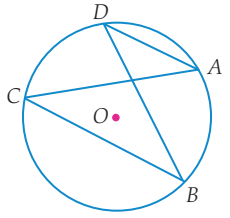


7 Angles may be subtended at the centre of the circle or at the circumference. We say that they stand on the same arc or they are subtended by the same chord if the arms of the angle start from and end at the same points on the circumference of the circle.



$\angle ACB$ is called the **angle at the circumference** standing on the arc AB. $\angle AOB$ is the **angle at the centre** standing on the arc AB. If the chord AB had been drawn, we would say that these angles were standing on the chord AB or they were subtended by the chord AB.

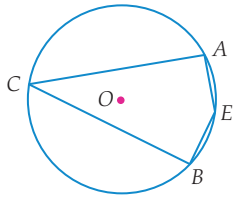
8 Angles at the circumference of a circle drawn from the same two points on the circumference are said to be standing on the same arc. If they are on the same side of the chord they are called **angles in the same segment**.



$\angle BCA$ and $\angle BDA$ are angles at the circumference standing on the minor arc AB.

They are in the same segment because they are on the same side of the chord AB.

Angles CBD and CAD are angles at the circumference standing on the minor arc CD.



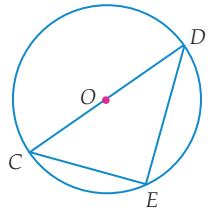
$\angle BCA$ is an angle at the circumference standing on the minor arc AB.

$\angle BEA$ is an angle at the circumference standing on the major arc AB.

They are in opposite segments as they are on opposite sides of the chord AB.

9 The **angle in a semicircle** is formed when the ends of a diameter are joined to a point on the circumference.

$\angle DEC$ is an angle in a semicircle.



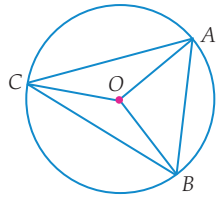
What is a theorem?

A 'theorem' is a mathematical statement that can be shown to be true. A 'proof' is a step-by-step argument that demonstrates the truth of a mathematical theorem.

Angles at the centre and circumference

Draw a circle centre O with three points A, B and C on the circumference. Join AB, BC, CA, OA, OB and OC.

Measure each of the following pairs of angles: $\angle AOB$ and $\angle ACB$; $\angle BOC$ and $\angle BAC$; $\angle COA$ and $\angle CBA$. It appears that the first angle of each pair is double the second angle. Check to see for yourself.



Theorem 1

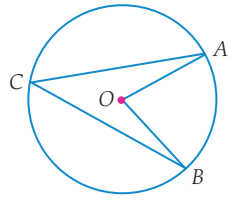
The angle at the centre is twice the angle at the circumference standing on the same arc.

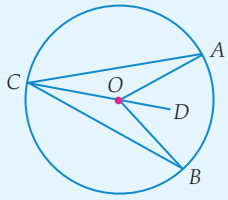
Worked Example 10

WE10

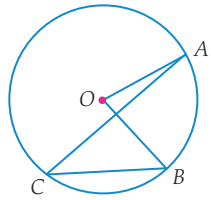
A, B and C are three points on the circumference of a circle with centre O.

Prove that $\angle AOB = 2\angle ACB$.



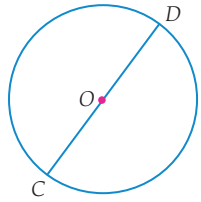
Thinking	Working	
1 Copy the diagram and join CO, extending it to D.		
2 Identify isosceles triangles and list pairs of equal angles.	$OC = OA$	(radii of a circle)
	$\therefore \Delta OCA$ is isosceles	
	$\therefore \angle OCA = \angle OAC$	(equal angles in isosceles triangle)
	$OC = OB$	(radii of a circle)
	$\therefore \Delta OCB$ is isosceles	
	$\therefore \angle OCB = \angle OBC$	(base angles of isosceles trianges)
3 Use this information to obtain $\angle AOD$ and $\angle BOD$.	$\angle AOD = \angle ACO + \angle CAO$	(exterior angle of ΔAOC equals sum of interior opposite angles)
	$\therefore \angle AOD = 2\angle ACO$	($\angle OCA = \angle OAC$)
	$\angle BOD = \angle BCO + \angle CBO$	(exterior angle of ΔBOC equals sum of interior opposite angles)
	$\therefore \angle BOD = 2\angle BCO$	($\angle OCB = \angle OBC$)
4 Find expressions for $\angle ACB$ and $\angle AOB$ and obtain the result to be proved.	Now $\angle ACB = \angle ACO + \angle BCO$	
	and $\angle AOB = \angle AOD + \angle BOD$	
	$\therefore \angle AOB = 2\angle ACO + 2\angle BCO$	(from above)
	$\angle AOB = 2(\angle ACO + \angle BCO)$	
	$\therefore \angle AOB = 2\angle ACB$	

It is important to recognise that the point C may be placed anywhere on the circumference of the circle and the lines drawn within the circle may not necessarily form a quadrilateral. For example, they may look like the diagram shown at right.

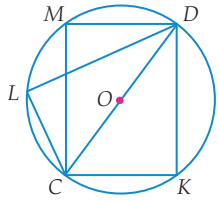


Angle in a semicircle

Draw a circle with centre O and diameter CD.



Join C and D to points K, L and M on the circumference. Measuring $\angle CKD$, $\angle CLD$ and $\angle CMD$, we find that all the angles measured are 90°.



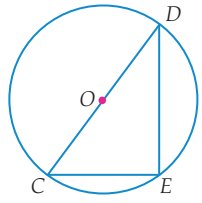
Theorem 2

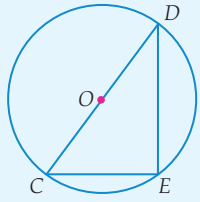
The angle in a semicircle is a right angle.

Worked Example 11

WE 11

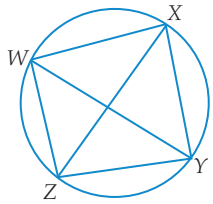
CD is a diameter of a circle with centre O. E is a point on the circumference. Prove that $\angle CED$ is a right angle.



Thinking	Working
1 Copy the diagram.	
2 Look for information so that you can use the angle at the centre.	$\angle COD = 2\angle CED$ (angle at the centre is twice the angle at the circumference standing on the same arc CD) But $\angle COD = 180^\circ$ (COD is a straight line) $\therefore \angle CED = 90^\circ$
3 State the result proved.	$\angle CED$ is a right angle. $\therefore$ the angle in a semicircle is a right angle.

Angles at the circumference on the same arc

Draw a circle with the points *W, X, Y, Z* in order on the circumference.



Join each point to the others, giving *WX, WY, WZ, XY, XZ, YZ*.

Measure $\angle WXZ$ and $\angle WYX$; $\angle WZX$ and $\angle WYX$; $\angle XWY$ and $\angle XZY$; $\angle YXZ$ and $\angle YWZ$ etc. *What do you notice?* It appears as though the pairs of angles are equal.

Theorem 3

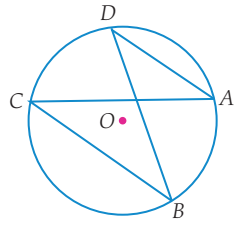
Angles in the same segment subtended by the same arc at the circumference are equal.

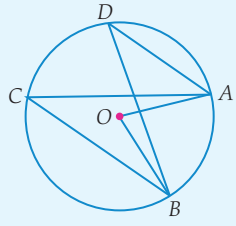
Worked Example 12

WE12

A, B, C and *D* are four points on the circumference of a circle with centre *O*.

Prove that $\angle ACB = \angle ADB$.

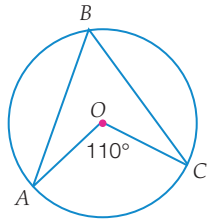


Thinking	Working
1 Copy the diagram and join *AO* and *BO*.	
2 Look for information so that you can use the angle at the centre.	$\angle AOB = 2\angle ADB$ (angle at the centre is twice the angle at the circumference standing on the same arc) Similarly: $\angle AOB = 2\angle ACB$
3 State the result proved.	$\angle ACB = \angle ADB$ (both half of $\angle AOB$)

Worked Example 13

WE13

Find the size of $\angle ABC$, giving reasons for your answer.

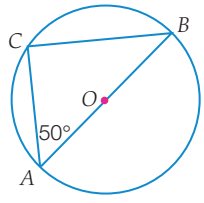


Thinking	Working
1 Write down a result using angles at the centre.	$\angle AOC = 2\angle ABC$ (angle at the centre is twice the angle at the circumference standing on the same arc)
2 Consider any numerical values given in the diagram.	$\angle AOC = 110^\circ$ $\therefore 110^\circ = 2\angle ABC$
3 Calculate the size of the required angle.	$\therefore \angle ABC = 55^\circ$

Worked Example 14

WE14

AB is a diameter of the circle with centre O. $\angle CAB = 50°$. Find the size of $\angle ABC$, giving reasons for your answer.

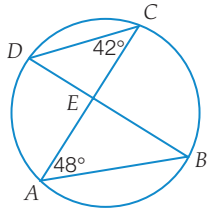


Thinking	Working
1 Write down a result using diameters of a circle.	$\angle ACB = 90^\circ$ (angle in a semicircle is 90°)
2 Consider any numerical values given in the diagram.	$\angle BAC = 50^\circ$ $\angle ABC + 50^\circ + 90^\circ = 180^\circ$ (angle sum of a triangle is 180°)
3 Calculate the size of the required angle.	$\therefore \angle ABC = 40^\circ$

Worked Example 15

WE15

A, B, C and D are points on the circumference of the circle, AC and DB intersect at E. If $\angle CAB = 48°$ and $\angle DCA = 42°$, find the size of $\angle AEB$, giving reasons for your answer.

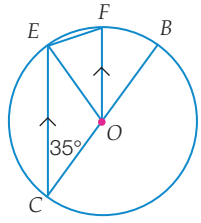


Thinking	Working	
1 Write down a result using the angles of a circle.	$\angle ACD = \angle ABD$	(angles at the circumference on the same arc)
2 Consider any numerical values given in the diagram.	$\therefore \angle ABD = 42^\circ$ $\angle EAB = 48^\circ$	 (given)
3 Calculate the size of the required angle.	$\angle AEB + 42^\circ + 48^\circ = 180^\circ$ $\therefore \angle AEB = 90^\circ$	(angle sum of a triangle is 180°)

Worked Example 16

WE16

BC is a diameter of the circle with centre *O*. *E* and *F* are points on the circumference of the circle such that $OF \parallel CE$. If $\angle OCE = 35°$, find the size of $\angle OFE$, giving reasons for your answer.



Thinking	Working	
1 Write down results using angle properties.	$\angle FOB = \angle ECO$ $\therefore \angle FOB = 35^\circ$	(corresponding angles, $FO \parallel EC$)
	$\angle EOB = 2\angle ECB$	(angle at the centre is twice the angle at the circumference standing on the same arc)
	$\angle EOB = 2 \times 35^\circ = 70^\circ$ $\angle EOF + 35^\circ = 70^\circ$ $\therefore \angle EOF = 35^\circ$	
2 Use the equal radii.	ΔOEF is isosceles	($OE = OF$, radii of the circle)
	$\angle OEF = \angle OFE$	(base angles of an isosceles triangle)
	But $\angle OEF + \angle OFE + 35^\circ = 180^\circ$ $\therefore 2\angle OFE = 145^\circ$	(angle sum of a triangle)
3 Calculate the size of the required angle.	$\angle OFE = 72.5^\circ$	

8.6 Angles in circles

10A

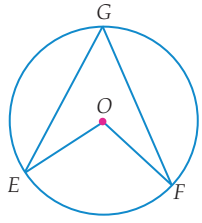
Navigator

Q1, Q2, Q3, Q4, Q5, Q7, Q8, Q9 (a) (c), Q10 (a)–(d), Q11, Q16, Q24, Q26	Q1, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q16, Q17, Q18, Q19, Q20, Q23, Q24, Q25, Q26	Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22, Q23, Q24, Q25, Q26

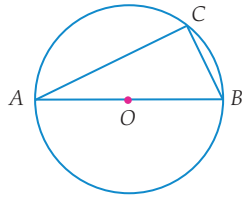
Answers page 769

Fluency

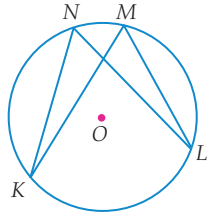
1 E, F and G are three points on the circumference of a circle with centre O. Prove that $\angle EOF = 2\angle EGF$. WE10



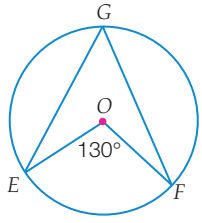
2 AB is a diameter of a circle with centre O. C is a point on the circumference. Prove that $\angle ACB = 90°$. WE11



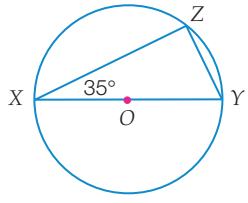
3 K, L, M and N are four points on the circumference of a circle with centre O. Prove that $\angle KML = \angle KNL$. WE12



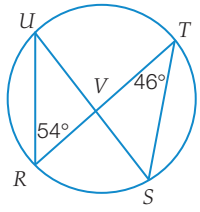
4 $\angle EOF = 130°$. Find the size of $\angle EGF$, giving reasons for your answer. WE13



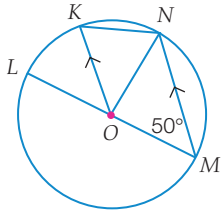
5 XY is a diameter of the circle with centre O. $\angle ZXY = 35°$. Find the size of $\angle ZYX$, giving reasons for your answer. WE14



6 R, S, T and U are points on the circumference of a circle. RT and SU intersect at V. If $\angle URT = 54°$ and $\angle RTS = 46°$, find the size of $\angle TVS$, giving reasons for your answer. WE15

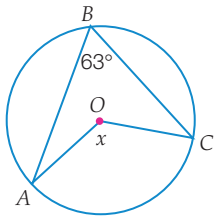


7 LM is a diameter of the circle with centre O. K and N are points on the circumference of the circle such that $OK \parallel MN$. If $\angle OMN = 50°$, find the size of $\angle OKN$, giving reasons for your answer. WE16

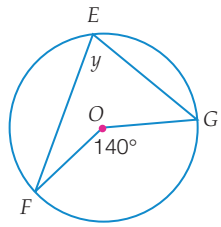


8 Find the value of the variables in each figure. Give reasons for your answer.

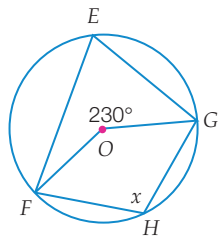
(a)



(b)

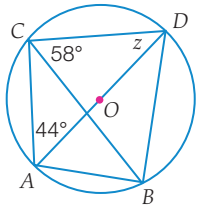


(c)

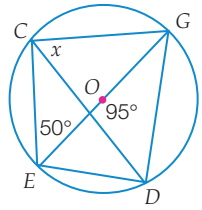


9 Find the value of the variables in each figure. Give reasons for your answer.

(a)



(b)



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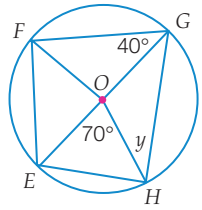
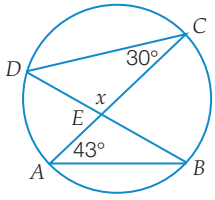


10 Find the value of the variables in each figure. Give reasons for your answer.

(a)



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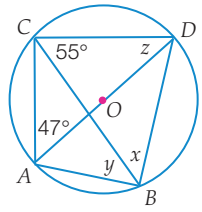

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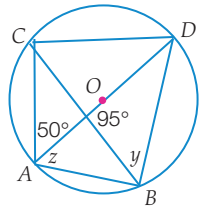

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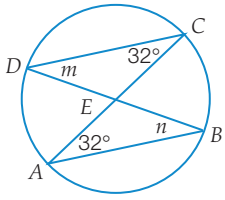

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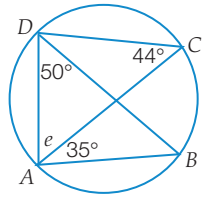
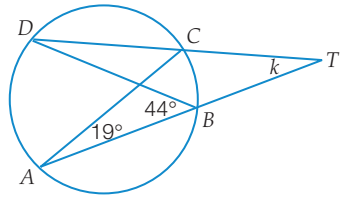


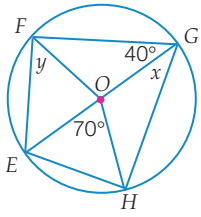
(f)



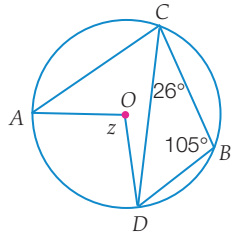
Understanding

11 Find the value of the variables in each figure. Give reasons for your answer.

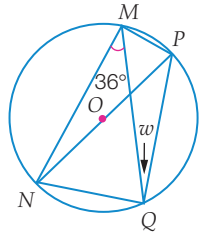
(a) EG is the diameter of the circle



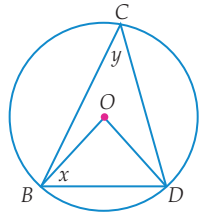
(b) $AC \parallel DB$



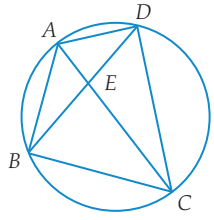
(c) $NM = MQ$, NP is the diameter



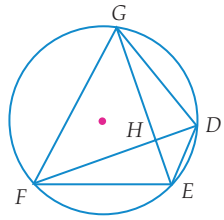
12 O is the centre of the circle. Prove that $x + y = 90°$.



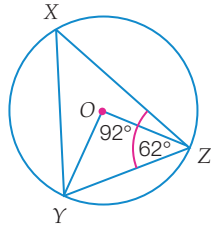
13 AC is the diameter of the circle. If $\angle BDC = 52°$ and $BD = DC$, calculate the size of $\angle ACD$ giving reasons for your answer.



14 If $\angle FGE = 48°$, $\angle GED = 42°$ and $\angle DFE = 20°$, calculate the size of each angle in the quadrilateral $GDEF$, giving reasons for your answers. What can you say about the diagonals of this quadrilateral?

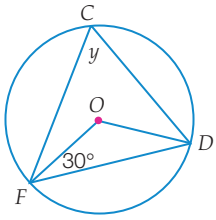


15 If $\angle YOZ = 92°$ and $\angle XZY = 62°$, calculate the size of $\angle OXZ$ giving reasons for your answer.

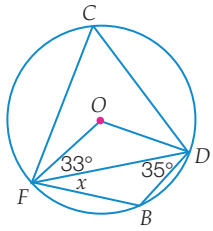


16 Find the value of the variables in each figure. Give reasons for your answer.

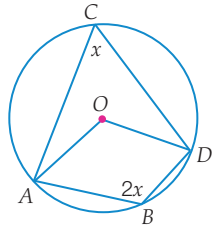
(a)



(b)

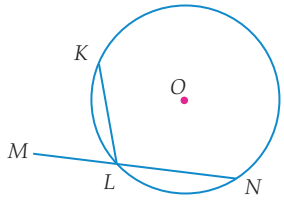


(c)



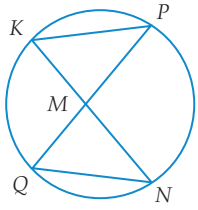
Reasoning

17 $KL = LN$ in the circle with centre O. Prove that $\angle KLM = \angle LON$.

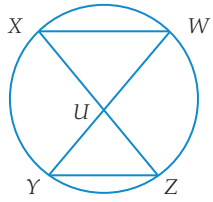


18 **(a)** Prove that $\Delta KPM \,|||\, \Delta QNM$.

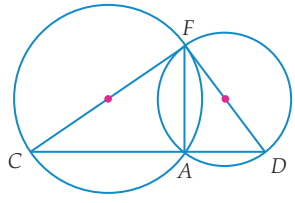
(b) If $KM = MQ$ prove that $MP = MN$.



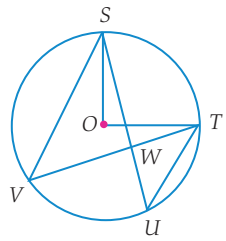
19 XZ and WY intersect at U. If $XU = UW$ prove that $XW \parallel YZ$.



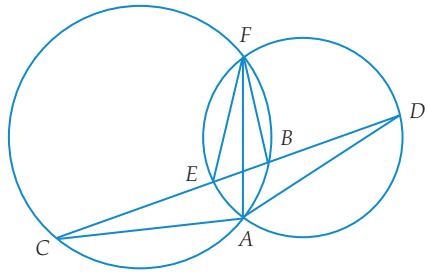
20 CF is a diameter of the circle CFA. FD is a diameter of the circle FDA. Prove that C, A and D are collinear.



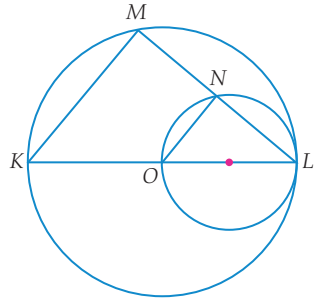
21 In the circle with centre O, $VS \parallel UT$ and $SO \perp OT$. Calculate the size of $\angle TWU$. Give reasons for your answer.



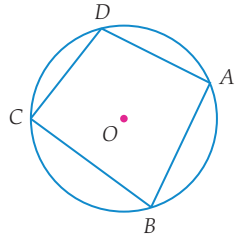
22 If $CA = AD$, prove that AF bisects $\angle EFB$.



23 O is the centre of the circle KML. KL is a diameter. OL is a diameter of the circle ONL. Prove that $KM \parallel ON$.

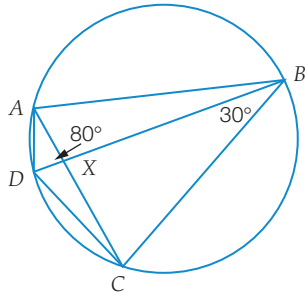


24 A cyclic quadrilateral is a quadrilateral whose vertices lie on the circumference of a circle. $ABCD$ is a cyclic quadrilateral in a circle with centre O. By joining CO and OA, prove that $\angle CDA + \angle CBA = 180°$.



Open-ended

25 $ABCD$ is a cyclic quadrilateral whose diagonals intersect at X. $\angle DBC = 30°$ and $\angle AXD = 80°$.



(a) Find the size of:

(i) $\angle DAC$ **(ii)** $\angle ADB$ **(iii)** $\angle ACB$

(b) If $\angle BAC < 80°$ choose a size for $\angle BAC$ and then find the size of:

(i) $\angle BDC$ **(ii)** $\angle DCA$ **(iii)** $\angle DBA$

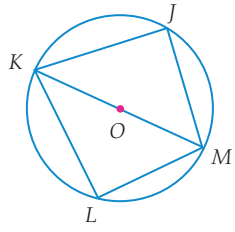
(c) If $\angle BAC = 50°$ write something about:

(i) $\angle ADC$ **(ii)** $\angle ABC$

(d) If $\angle BAC = 10°$ write something about:

(i) $\angle ADB$ **(ii)** $\angle BCD$

26 KM is a diameter of the circle with centre O. J and L are points on the circumference. Claire claims that $JKLM$ is a square; James claims that $JKLM$ is a kite. Who is correct, or is neither correct? Explain your decision.



Investigation

Points on polygons and circles

Equipment required: 1 brain, an equilateral triangle, an isosceles triangle, a scalene triangle, a rectangle and a square, all cut out of cardboard or plastic, a $1 coin and a 50c coin

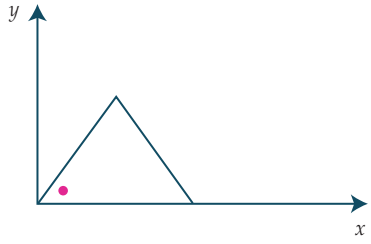
When shapes rotate, the path followed by a point on a vertex follows a path called a *locus*.

The Big Question

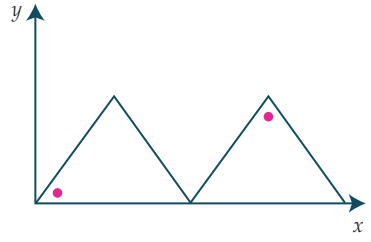
Can we predict the path taken by a point on a polygon or a circle as the shape rotates along a straight line?

Engage

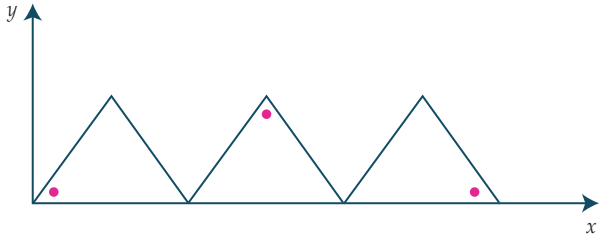
1 Mark a dot on one vertex of the equilateral triangle. Draw up a set of axes and place the equilateral triangle with its base on the x-axis and the vertex with the dot at the origin as shown:



2 Rotate the triangle in a clockwise direction around the vertex on the bottom right-hand side.



3 Rotate the triangle again in a clockwise direction around the vertex on the bottom right-hand side.



4 What path will the vertex with the dot take as the triangle rotates along the x-axis? Can you predict it? Using your triangles, draw the path that the vertex with the dot has taken as the triangle rotated.

5 Now rotate an isosceles triangle starting with the vertex with the dot at the origin and the non-equal side of the triangle along the x-axis. Again, try to predict the path first, then draw the path taken by the vertex with the dot as it is rotated along the x-axis.

6 Repeat this process for a scalene triangle. Always start with the vertex with the dot at the origin and one side of the triangle along the x-axis.

Explore

7 Now rotate a square in the same manner and draw the path taken by one corner as the shape rotates.

8 The shape of a 50c coin is a dodecahedron (12-sided polygon). Predict the path it will produce when rotated in the same manner as described above.

9 Rotate a 50c coin and draw the path you obtain.

Explain

10 Describe the path or locus for each of the shapes you have used so far, noting similarities and differences, and using mathematical terms such as radius and diagonal. You may find it useful to label the sides and the other angles to help you with your description. Use features of the shape to find the maximum height reached for each shape.

11 What is the locus of a point at the centre of **(i)** an equilateral triangle and **(ii)** a square?

Strategy options

- Draw a diagram.
- Make a model.
- Look for a pattern.

Elaborate

12 Rotate a $1 coin and draw the path taken by a point on the circumference. Describe this path.

13 Rotate a rectangle and draw the path taken by a point on a vertex. Describe this path.

14 What would be the locus of a point that is always equidistant from two fixed points?

Evaluate

15 Did you have any difficulty drawing the paths for any of the shapes? What could you do in future to make it easier?

16 Where do you think an understanding of paths like these would be important?

17 Did working with models make it easier for you to predict the correct path in later questions?

Extend

Choose one of the following.

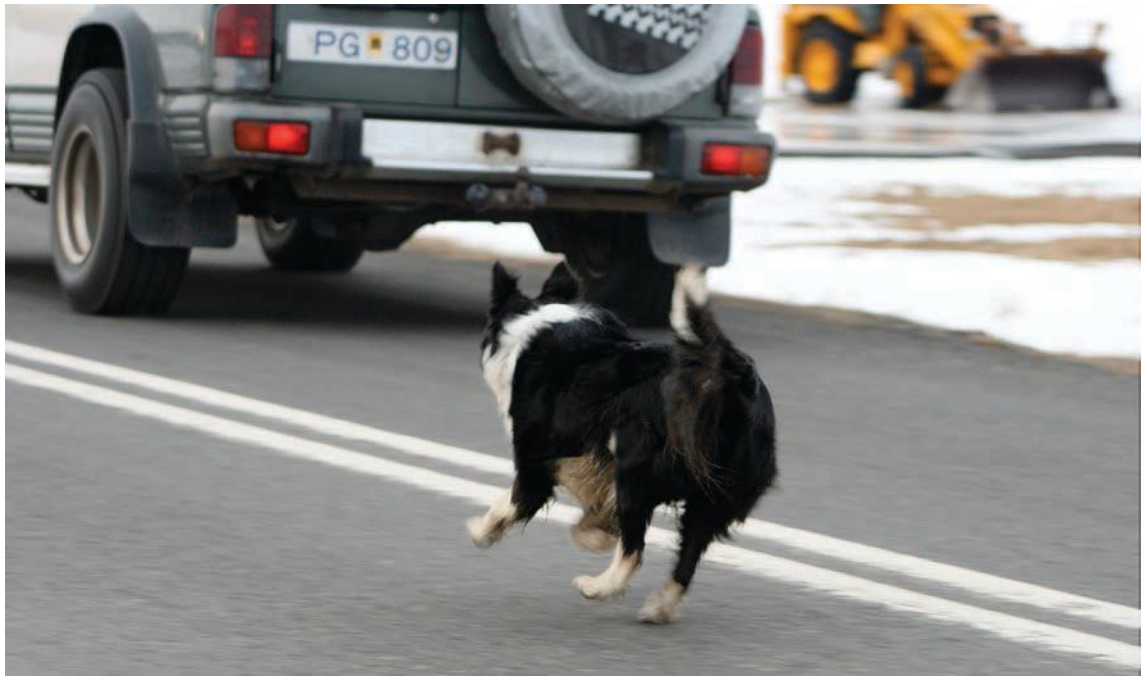
18 Rotate a $1 coin around the circumference of a circle with twice the diameter of the coin and find the locus of a point on the vertex of these shapes.

19 If a goat were tethered to the midpoint of a wall d metres long with a length of rope r metres long, investigate the path the goat could take if it walked at its maximum length of rope from its tether point:

(a) if $r < \frac{d}{2}$ **(b)** if $r = \frac{d}{2}$ **(c)** if $r > \frac{d}{2}$.

20 Investigate the paths formed by *curves of pursuit*.

Imagine a dog on a footpath chasing a car. As the dog heads towards the car, the car moves forward so the dog now has to change direction. As long as the car is travelling faster than the speed of the dog, the dog will not catch the car.



Chords of circles 10A

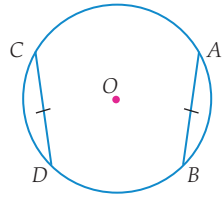
8.7

The following constructions may be completed using pen and paper or dynamic geometry packages. Ideally, both methods should be used so as to understand how the processes differ in each case.

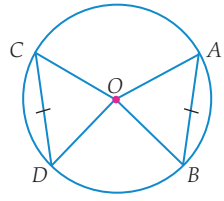
The instructions given are for pen and paper because the steps are different for each software package.

Angles at the centre subtended by chords of equal length

Draw a circle centre O and mark off equal arcs AB and CD. Join AB and CD to form chords of the circle.

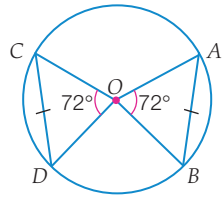


Join AO, BO, CO and DO.



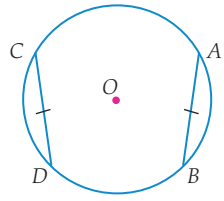
Measure $\angle AOB$ and $\angle COD$. *What can we say about these two angles?*

It appears that $\angle AOB = \angle COD$.



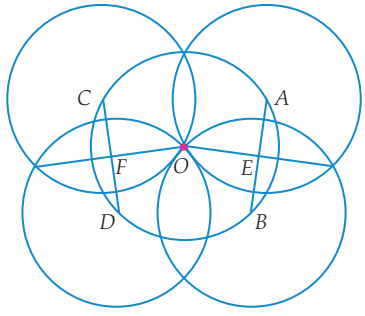
Distance from the centre to chords of equal length

Draw a circle centre O and mark off equal chords AB and CD.



Using compasses, construct the perpendicular from O to each chord, cutting AB at E and CD at F.

Measure OE and OF.
It appears that $OE = OF$
That is, E and F are **equidistant** from O.



Theorem 4

Chords of equal length in a circle subtend equal angles at the centre and are equidistant from the centre.

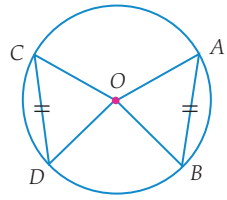
This theorem has two parts and requires two separate proofs, one involving the angles at the centre of the circle and the other involving the perpendicular distance from the centre of the circle to each chord.

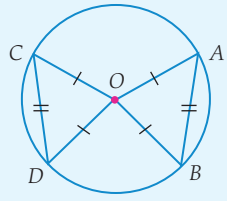
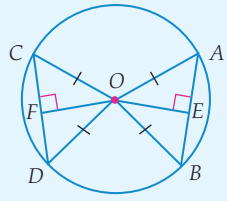
Worked Example 17 WE17

AB and *CD* are equal chords of the circle with centre *O*.

(a) Prove that $\angle AOB = \angle COD$.

(b) *OE* and *OF* are the perpendiculars drawn from *O* to *AB* and *CD*, respectively. Prove that $OE = OF$.

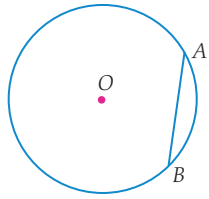


Thinking	Working
(a) 1 Copy the diagram and mark the equal radii.	(a) 
2 Name the triangles that you will be considering.	ΔOAB and ΔOCD
3 List three properties that will allow you to use a congruency test.	$OA = OC$ (radii of circle) $AB = CD$ (given, equal chords) $BO = DO$ (radii of circle)
4 Write down a pair of congruent triangles and the test used.	$\Delta OAB \equiv \Delta OCD$ (SSS)
5 State your answer to the question, giving a reason.	$\angle AOB = \angle COD$ (corresponding angles of congruent triangles)
(b) 1 Copy the diagram, draw *OE* and *OF*, and mark the equal radii.	(b) 
2 Identify results from **(a)** that you need here.	$\angle BAO = \angle DCO$ (corresponding angles in congruent triangles)
3 Name the triangles that you will be considering.	ΔEAO and ΔFCO
4 List three properties that will allow you to use a congruency test.	$OA = OC$ (radii of circle) $\angle EAO = \angle FCO$ (corresponding angles in congruent triangles OAB and OCD) $\angle AEO = \angle CFO$ (both right angles, $OE \perp AB$ and $FO \perp CD$)
5 Write down a pair of congruent triangles and the test used.	$\Delta EAO \equiv \Delta FCO$ (AAS)

6 State your answer to the question, giving a reason.	$OE = OF$	(corresponding sides of congruent triangles)

The perpendicular to a chord

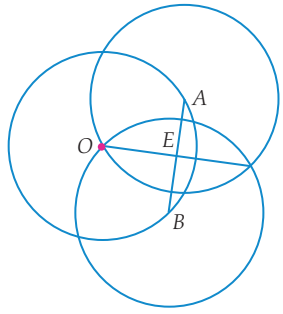
Consider the circle with centre O and a chord AB.



Use compasses to construct the perpendicular from O to the chord, cutting AB at E.

Measure AE and BE. *What can we say about these two intervals?*

It appears that $AE = BE$.



Theorem 5

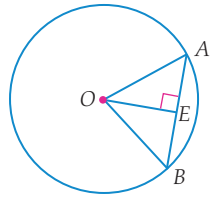
The perpendicular from the centre of a circle to a chord bisects the chord.

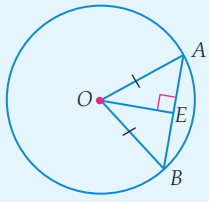
Worked Example 18

WE18

AB is a chord of a circle with centre O. $OE \perp AB$.

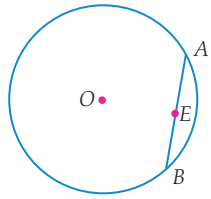
Prove that $AE = EB$.



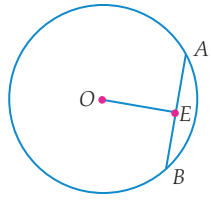
Thinking	Working	
1 Copy the diagram and mark the equal radii.		
2 Name the triangles that you will be considering.	ΔEAO and ΔEBO	
3 List three properties that will allow you to use a congruency test.	$\angle OEA = \angle OEB = 90°$ $OA = OB$ OE is a common side.	($OE \perp AB$) (radii of circle)
4 Write down a pair of congruent triangles and the test used.	$\Delta EAO \equiv \Delta EBO$	(RHS)
5 State your answer to the question, giving a reason.	$AE = EB$	(corresponding sides in congruent triangles)

Midpoint of a chord to the centre

Draw a circle centre O with a chord AB. Construct the midpoint, E, of AB.



Draw OE.
Measure $\angle AEO$ and $\angle BEO$.
It appears that $\angle AEO = \angle BEO = 90°$.



Theorem 6

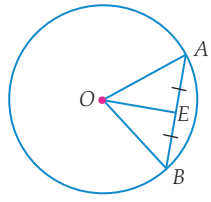
The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

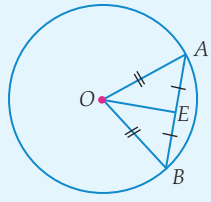
Worked Example 19

WE19

AB is a chord of a circle with centre O. E is the midpoint of AB.

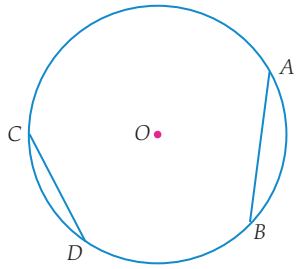
Prove that $OE \perp AB$.



Thinking	Working
1 Copy the diagram and mark the equal radii.	
2 Name the triangles that you will be considering.	ΔEAO and ΔEBO
3 List three properties that will allow you to use a congruency test.	$AE = EB$ (E is midpoint of AB) $OA = OB$ (radii of circle) OE is a common side.
4 Write down a pair of congruent triangles and the test used.	$\Delta EAO \equiv \Delta EBO$ (SSS)
5 State your answer to the question, giving a reason.	$\angle OEA = \angle OEB$ (corresponding angles of congruent triangles) But $\angle OEA = 180° - \angle OEB$ (straight angle) $\therefore \angle OEA = \angle OEB = 90°$ $\therefore OE \perp AB$

Perpendicular bisector of a chord

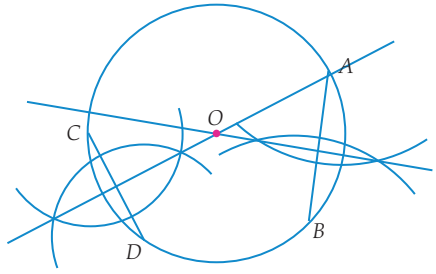
Draw a circle centre O with several chords AB, CD etc., not of equal lengths.



Using compasses, construct a perpendicular bisector of each chord.

What can be said about the perpendicular bisectors?

It appears that they all pass through O.



Theorem 7

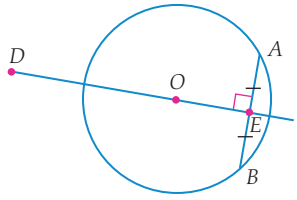
The perpendicular bisector of a chord of a circle passes through the centre of the circle.

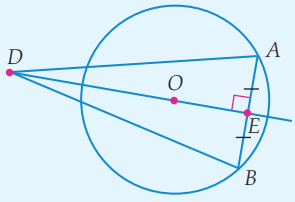
Worked Example 20

WE 20

E is the midpoint of the chord AB in a circle with centre O. DE is the perpendicular bisector of AB. Prove that O lies on DE.

When constructed, DE appears to pass through O. To prove this is true we need to use a different sort of proof.



Thinking	Working	
1 Copy the diagram and join DA and DB.		
2 Write down what you know about DA and DB.	$DA = DB$	(all points on the perpendicular bisector of the interval AB are equidistant from the end points of the interval)
3 Write down what you know about O.	$OA = OB$	(radii of the circle centre O on which A and B lie)

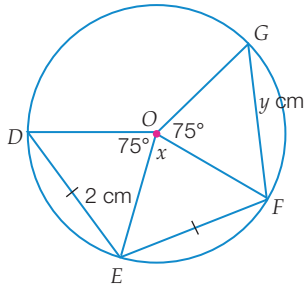
4 Give reasons for your answer.	O is equidistant from A and B (it must lie on the perpendicular bisector of AB). ∴ O lies on DE Hence, the perpendicular bisector of a chord of a circle passes through the centre of the circle.

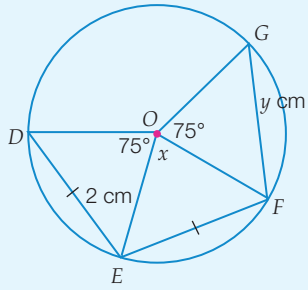
This result could be verified using dynamic geometry software such as GeoGebra. Draw a circle showing its centre. Then draw a chord of the circle. Construct the perpendicular bisector of the chord. Check to see whether the centre of the circle lies on the perpendicular bisector of the chord.

Worked Example 21

WE 21

Calculate the value of the pronumerals. Give reasons for your answers.

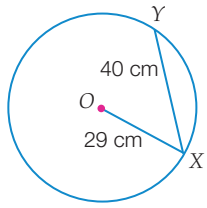


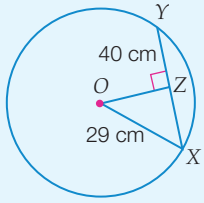
Thinking	Working
1 Copy the diagram.	
2 Write down the information about the equal chords contained in the diagram.	DE = EF (equal chords marked on diagram) ∴ ∠DOE = ∠EOF (equals chords subtend equal angles at the centre of a circle) ∴ $x^\circ = 75^\circ$
3 Write down the information about the equal chords contained in the diagram.	∠DOE = ∠FOG = 75° ∴ DE = GF (equal angles at the centre of a circle are subtended by equal chords) ∴ y = 2 cm

Worked Example 22 WE22

XY is a chord of a circle with centre O.

If $XY = 40$ cm and $OX = 29$ cm, calculate the perpendicular distance of XY from O. Give reasons for your answer.



Thinking	Working
1 Copy the diagram. Let Z be the foot of the perpendicular from O to XY. Draw OZ perpendicular to XY.	
2 Identify any equal lengths.	$XZ = ZY$ (perpendicular from the centre of a circle to a chord bisects the chord) $\therefore XZ = 20$ cm (half of XY)
3 Calculate the length of OZ.	$OX^2 = XZ^2 + OZ^2$ (Pythagoras' theorem) Hence, $OZ^2 = OX^2 - XZ^2$ $\therefore OZ^2 = 29^2 - 20^2$ $= 841 - 400$ $= 441$ $\therefore OZ = 21$ cm
4 Answer the question.	XY is 21 cm from O.

8.7 Chords of circles 10A

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7 (a), (c), Q8 (a), (b), Q10 (a), Q11, Q14, Q16	Q2, Q4, Q5, Q6, Q7 (b), (c), Q8, Q9 (a), (b), Q10 (a), (b), Q11, Q12, Q14, Q15, Q16, Q17	Q3, Q4, Q5, Q6, Q7 (b), (c), Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18

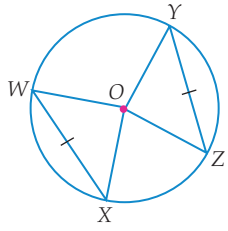
Answers page 772

Fluency

1 WX and YZ are equal chords of a circle with centre O. WE17

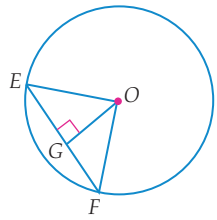
(a) Prove that $\angle WOX = \angle YOZ$.

(b) OP and OQ are the perpendiculars drawn from O to WX and YZ, respectively. Prove that $OP = OQ$.



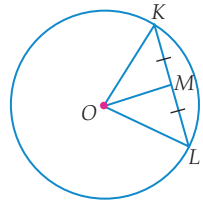
WE18

2 EF is a chord of a circle with centre O. $OG \perp EF$. Prove that $EG = GF$.



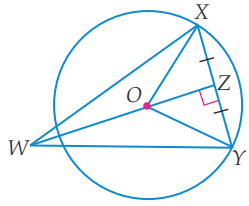
WE19

3 KL is a chord of a circle with centre O. M is the midpoint of KL. Prove that $OM \perp KL$.



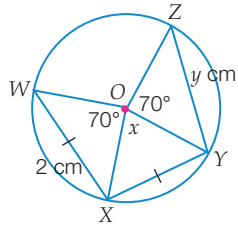
WE20

4 Z is the midpoint of the chord XY in a circle with centre O. WZ is the perpendicular bisector of XY. Prove that O lies on WZ.



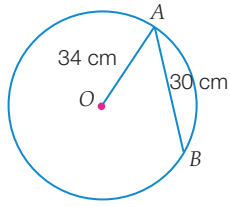
WE21

5 Calculate the value of the pronumerals. Give reasons for your answers.



WE22

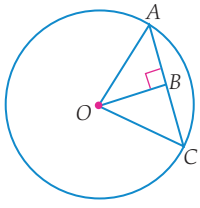
6 AB is a chord of a circle with centre O. If $AB = 30$ cm and $OA = 34$ cm, calculate the perpendicular distance of AB from O. Give reasons for your answer.



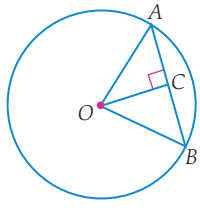
Understanding

7 Calculate the length of AB in each diagram, giving reasons for your answer.

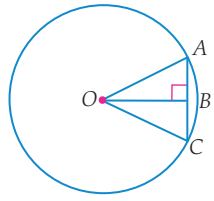
(a) $AC = 10$ cm, $OB \perp AC$



(b) $AC = 10$ cm, $OC \perp AB$

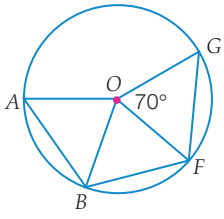


(c) $OA = 17$ cm, $OB = 8$ cm

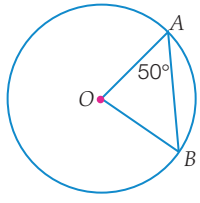


8 Calculate the size of $\angle AOB$ in each figure, giving reasons for your answer.

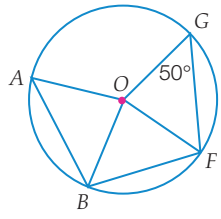
(a) $AB = GF$, $\angle GOF = 70°$



(b) $\angle OAB = 50°$

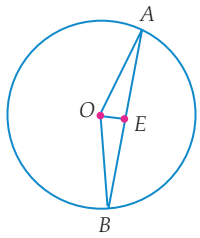


(c) $AB = GF$, $\angle OGF = 50°$

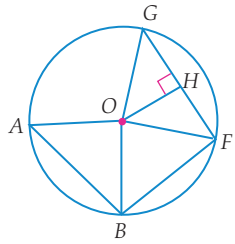


9 Calculate the length of OA in each figure, giving reasons for your answer.

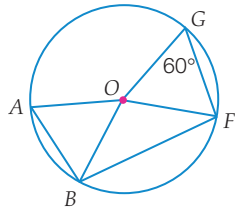
(a) $AB = 48$ cm, $OE = 7$ cm, E is the midpoint of AB.



(b) $GH = HF = 21$ cm, $OH = 20$ cm, $AB = GF$

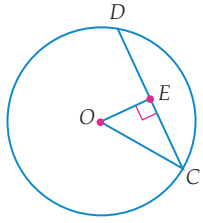


(c) $\angle OGF = 60°$, $AB = GF = 10$ cm

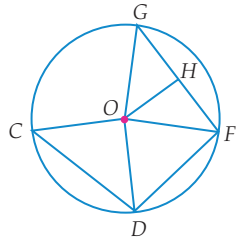


10 Calculate the length of CD in each diagram, giving reasons for your answer.

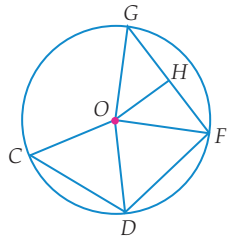
(a) $OE \perp CD$, $OC = 53$ cm, $OE = 28$ cm



(b) $\angle FHO = 90°$, $\angle FOD = \angle DOC$, $GF = FD$, $OF = 73$ cm, $OH = 48$ cm

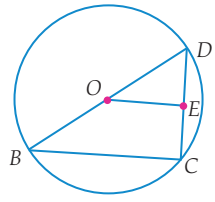


(c) $\angle GOF = \angle COD$, $GH = HF$, $OG = 85$ cm $OH = 77$ cm

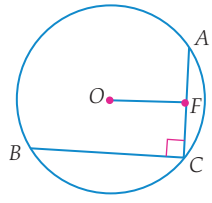


Reasoning

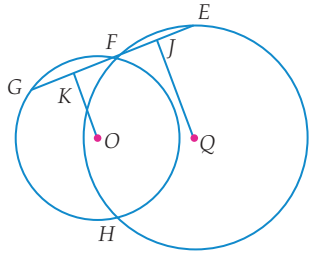
11 BD is the diameter of a circle with centre O. C is a point on the circumference of the circle. E is the midpoint of CD. Prove that $OE \parallel BC$, and hence that $\angle BCD = 90°$.



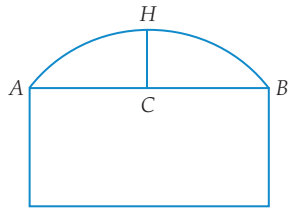
12 O is the centre of the circle ACB. $AF = FC$ and $\angle ACB = 90°$. Mark D, the midpoint of BC, and use it to prove that $BC = 2OF$.



13 The circles GFH, with centre O, and FEH, with centre Q, intersect at F and H. GFE is a straight line. K is the midpoint of GF and J is the midpoint of FE. Prove that $\angle KOQ + \angle OQJ = 180°$.



14 The entrance to a shopping centre is a rectangular doorway with a circular arch above it as shown in the diagram. The doorway *AB* is 4 m wide and the radius of the arch is 2.5 m. Calculate the height of the arch, *CH*. Justify your calculations.



15 *KLM* is an isosceles triangle inscribed in a circle of radius 25 mm. If $KL = KM$ and $LM = 48$ mm, calculate the length of *KL*. Give reasons to justify your calculation. (A triangle inscribed in a circle has its vertices on the circumference of the circle.)

Open-ended

16 The doorways to a shopping centre are being designed. The shape is to be similar to that in the diagram for Question **14**. The rectangular doorways are to be between 4 and 6 m wide and at least 2.5 m high. Each doorway is to have a circular arch on top of the rectangular door opening such that the total height of the door and the arch is between 3 and 4.5 m. Create two designs, giving the width and height of the rectangular doorway, the radius of the arch and the height of the arch above the centre of the doorway.

17 Construct the perpendicular bisectors of the sides *KL* and *LM* of a scalene triangle *KLM*. Explain why their point of intersection, *X*, is the centre of the circle through *L*, *K* and *M*.

18 Circles with centres *E* and *F* intersect at *G* and *H*. Draw this information on a diagram. Use your diagram to explain why **(i)** *EF* bisects *GH* and **(ii)** $CE \perp GH$.

Technology Exploration CAS

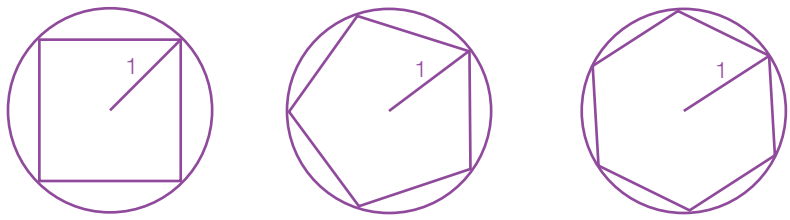
Equipment required: 1 brain, a TI-Nspire CAS or ClassPad CAS

Archimedes and the calculation of π

Archimedes was born c. 290–280 BC in Syracuse, Sicily, and was the best known and most influential of the Greek mathematicians. He was a great inventor. In this investigation, we will find approximations to π using a method similar to that invented by Archimedes. Had he had access to a CAS his investigations would have been much easier!

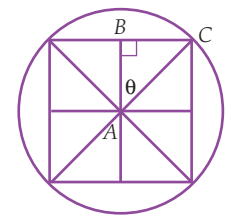
1 Use the formula Area $= \pi r^2$ to find the area of a circle with a radius of one unit. *Note:* A circle where $r = 1$ is called a *unit circle*.

2 To find the area of a unit circle by the method of Archimedes, we start by inscribing (drawing) a square inside the unit circle. The area of the square becomes the first approximation to the area of the unit circle. A square is a regular polygon with $x = 4$ sides. If the square is changed to a regular pentagon ($x = 5$), the area of the pentagon becomes a better approximation to the area of the unit circle. As the number of sides of the polygon is increased, the area of the polygon gets closer and closer to the area of the unit circle.



$x = 4$ $\quad$ $x = 5$ $\quad$ $x = 6$

3 By finding the areas of the regular polygons, Archimedes was able to calculate an accurate approximation for π. To find the area of an inscribed polygon, divide the polygon into equal triangles with a common vertex at the centre. To find the area of one of these triangles:



Let side BC represent the base of ΔABC and let AB represent the perpendicular height of ΔABC.

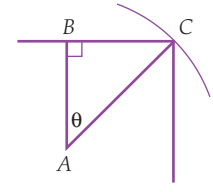
Remember, $AC = 1$ as it is a radius of the unit circle.

$$\sin\theta = \frac{BC}{AC} = \frac{BC}{1} = BC \quad \therefore \text{The base } BC = \sin\theta$$

$$\cos\theta = \frac{AB}{AC} = \frac{AB}{1} = AB \quad \therefore \text{The perpendicular height } AB = \cos\theta$$

The area of the triangle is given by:

$$\text{Area} = \tfrac{1}{2}\ \text{base} \times \text{height} = \tfrac{1}{2}\sin\theta\cos\theta$$



But $\quad \theta = \dfrac{360°}{8} = \dfrac{180°}{4} \qquad \therefore \text{Area} = \tfrac{1}{2}\sin\left(\dfrac{180°}{4}\right)\cos\left(\dfrac{180°}{4}\right)$

As the square contains eight triangles, its area is given by:

$$\text{Area} = 8 \times \tfrac{1}{2}\sin\left(\frac{180°}{4}\right)\cos\left(\frac{180°}{4}\right) = 4\sin\left(\frac{180°}{4}\right)\cos\left(\frac{180°}{4}\right)$$

This gives $\quad$ Area $= 2$, which does not seem to be a very accurate estimate.

4 Show that the area of the inscribed pentagon ($x = 5$) is given by:

$$\text{Area} = 5\sin\left(\frac{180°}{5}\right)\cos\left(\frac{180°}{5}\right)$$

This gives Area ≈ 2.378, a slightly better estimate than that given by the square.

5 Verify that the area of an inscribed polygon with x sides is given by:

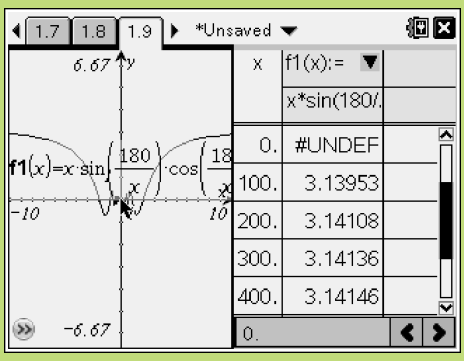
$$\text{Area} = x\sin\left(\frac{180°}{x}\right)\cos\left(\frac{180°}{x}\right)$$

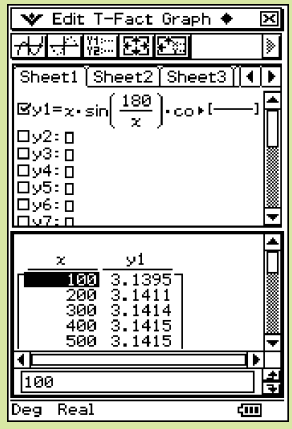
for a value of x of your own choice.

6 We can use a CAS to generate a table of values that shows the number of sides of each polygon (X) and the area of that polygon (f1(x) or y1).

Enter the equation from Question **5** into your CAS. Set up your CAS to start the table with x-values at 0 and to step x up 100 at a time.

Using the TI-Nspire CAS	Using the ClassPad
To make sure your CAS is calculating in degrees, press on > Settings > Graphs & Geometry (or System Info > Document Settings) and check the Angle setting. Press ctrl T (or ctrl T to get the table, and menu > Table > Edit Table Settings (or menu > Function Table > Edit Function Table Settings) to change the table settings.	To make sure your CAS is calculating in degrees look at the bottom of the screen. If you see Rad or Gra tap it to change it to Deg. You need to tap to change the table settings.





By altering the amount that the table steps each time (including using negative step values), find the lowest number of sides on a polygon required to obtain π correct to four decimal places.

Research

Archimedes established that the value of π lay between the lower and upper limits of $\frac{223}{71}$ and $\frac{22}{7}$. Do further research on Archimedes' method for determining π. Write a brief report and describe how Archimedes used polygons that circumscribed the unit circle to find an upper bound to π.

We can find the area of an x-sided circumscribed polygon using:

$$\text{Area} = x\tan\left(\frac{180°}{x}\right)$$

Derive this formula for the area. Use this formula to find an upper bound for π, correct to two and four decimal places.

Chapter review 8

D.I.Y. Summary

Key Words

bisect	included angle	property	similar
congruent	perpendicular	prove	theorem
definition	perpendicular bisector		
10A			
angle at the centre	arc	major segment	segment
angle at the circumference	chord	minor arc	subtend
angle in a semicircle	equidistant	minor segment	subtended
angles in the same segment	major arc	sector of a circle	

Copy and complete the following using the words and phrases from the list where appropriate to write a summary for this chapter. A word or phrase may be used more than once.

1 Triangles are _____ if they are identical in all aspects; i.e. they have exactly the same shape and size.

2 When applying SAS to test for congruence it is important that the angle is the _____.

3 Triangles are _____ if they have exactly the same shape, although their sizes may be different.

4 The minimum amount of information needed to identify a particular figure is called a _____.

5 A _____ is true about a particular figure but is not enough to define that figure.

6 A line at right angles to another line that also cuts that line in half is called a _____.

7 If we divide a line or angle into two equal parts, we _____ the line or angle.

8 If two radii are drawn to different points on the circumference of a circle they divide the circle into a _____ and a _____. **10A**

9 A line joining one point on the circumference of the circle to another is called a _____. **10A**

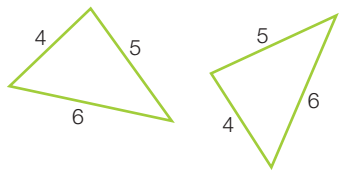
10 An arc will _____ an angle at the circumference. **10A**

11 A chord, that is not a diameter, drawn on a circle divides the circle into a _____ and a _____. **10A**

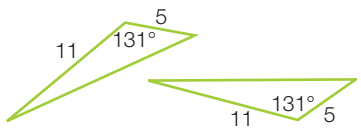
Fluency

1 Give reasons why the following pairs of triangles are congruent. **Ex.8.1**

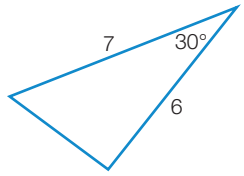
(a)



(b)

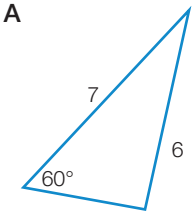


2 The triangle congruent with the triangle shown is:

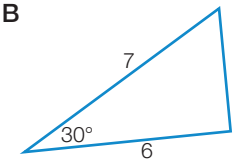


Ex. 8.1

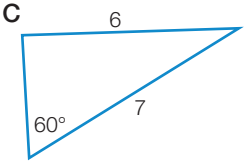
A



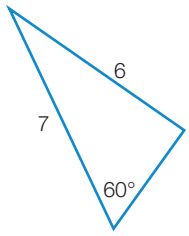
B



C



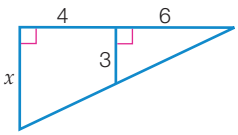
D



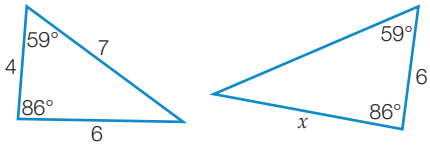
3 Find the value of x in each of the following pairs of triangles.

Ex. 8.1

(a)



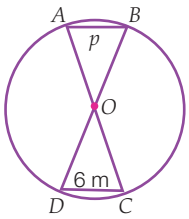
(b)



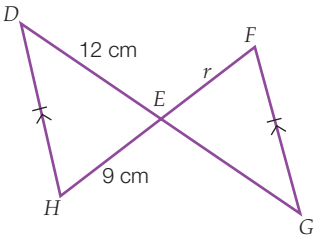
4 Prove that the following pairs of triangles are congruent and hence find the value of the pronumeral.

Ex. 8.2

(a)



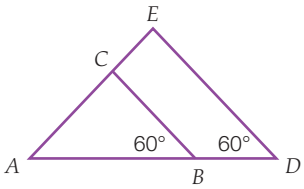
(b)



5 Prove that the following pairs of triangles, ΔABC and ΔADE, are similar.

Ex. 8.2

(a)



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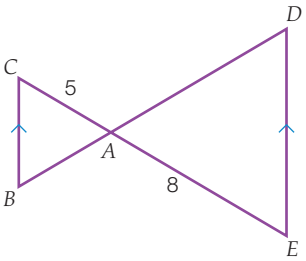
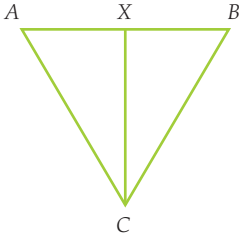
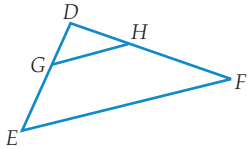


6 Given that CX bisects $\angle ACB$, and $CX \perp AB$, prove that $AX = XB$.



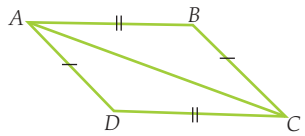
Ex. 8.3

7 In ΔDEF, G is a point on DE and H is a point on DF such that $GH \parallel EF$. Prove that $\dfrac{DG}{GE} = \dfrac{DH}{HF}$.



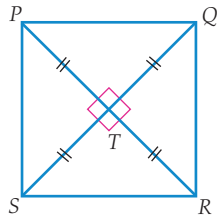
Ex. 8.4

8 If both pairs of opposite sides of a quadrilateral are equal, prove that the quadrilateral is a parallelogram.



Ex 8.5

9 If the diagonals of a quadrilateral are equal and bisect each other at right angles, prove that the quadrilateral is a square.

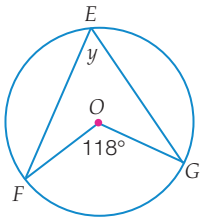


Ex 8.5

10 Find the value of the pronumerals in each figure. Give reasons for your answer.

Ex 8.6 10A

(a)



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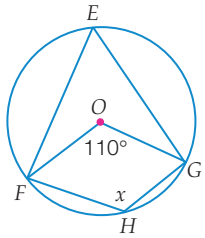

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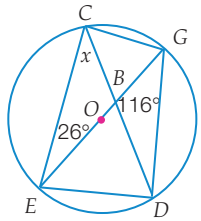

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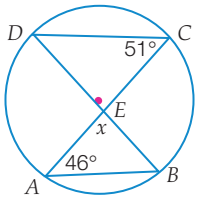
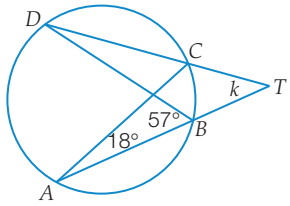


(e)



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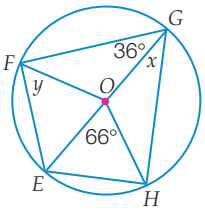
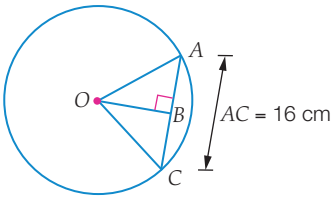


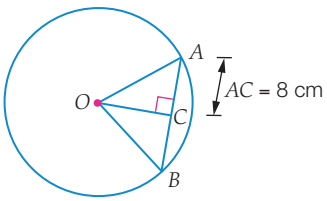
11 Calculate the length of AB in each diagram, giving reasons for your answer.

Ex 8.7 10A

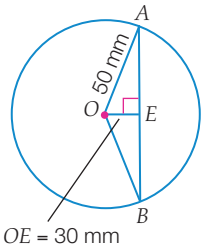
(a)



(b)



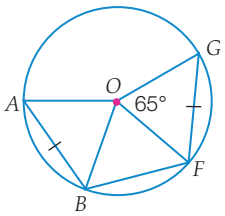
(c)



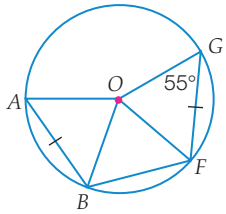
12 Calculate the size of $\angle AOB$ in each figure, giving reasons for your answer.

Ex 8.7 10A

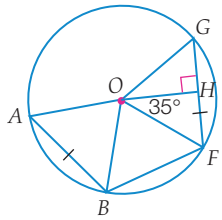
(a)



(b)

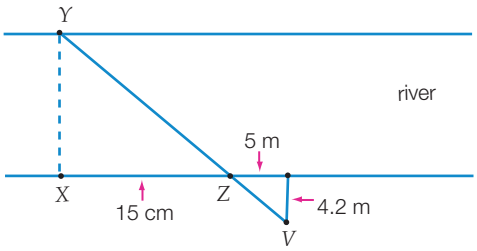


(c)



Understanding

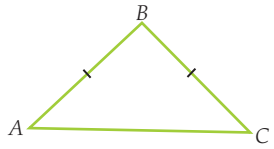
13 Angela is on a walk through the forest when she comes to a river. She wants to know the distance across the river. She draws triangles as shown in the diagram to help her. Find the distance across the river.



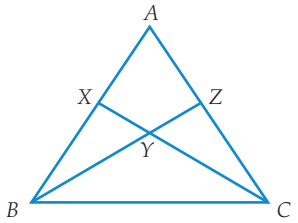
14 In ΔABC, $BA = BC$.

(a) Prove that $\angle BAC = \angle BCA$, by drawing the line BD perpendicular to AC, meeting it at D.

(b) What property of isosceles triangles have you just proved?



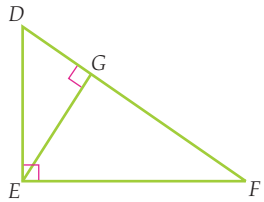
15 $AB = AC$ and $AX = AZ$. If $\angle YBC = \angle YCB$, prove that $XY = ZY$.



16 ΔDEF is right angled at E. $EG \perp DF$.

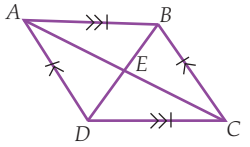
(a) Prove that triangles DEF, DEG and EGF are similar.

(b) Prove that $EG^2 = DG \times GF$.



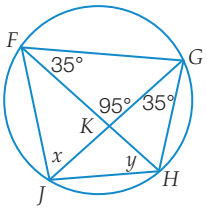
17 **(a)** For the diagram shown, prove $AC \perp BD$.

(b) What special property of a rhombus have you just proved?

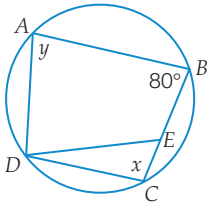


18 Find the value of the variables in each of the following, giving reasons for your answers. 10A

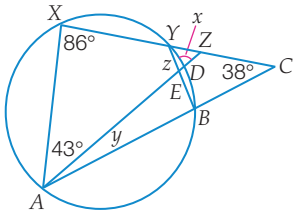
(a) $GH = HJ$



(b) $AB \parallel DC$

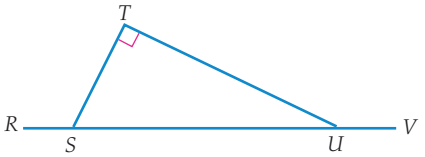


(c) $\angle YDZ = x$

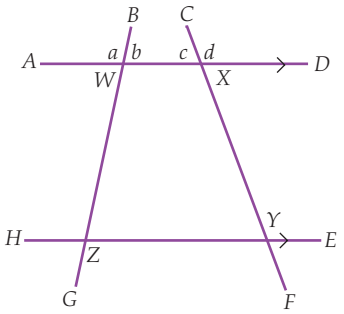


Reasoning

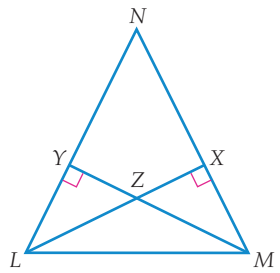
19 Prove that $\angle TSR + \angle TUV = 270°$.



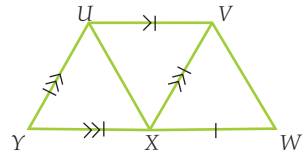
20 Use the diagram to prove that the sum of the interior angles of a trapezium is 360°.



21 In triangle LMN, LX is perpendicular to NM and MY is perpendicular to LN, as shown in the diagram. The point of intersection of LX and MY is Z. Prove that $\Delta LYZ \parallel\!| \Delta MXZ$, giving reasons.

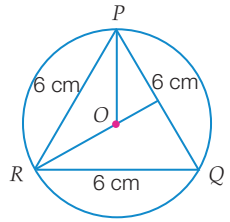


22 $UVXY$ is a rhombus and $YX = XW$. Show that $\Delta UXY \equiv \Delta VWX$.



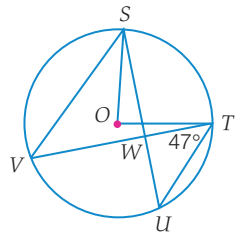
23 PQR is an equilateral triangle inscribed in a circle with centre O. (P, Q, R lie on the circle.) If $PQ = 6$ cm, calculate OP, the radius of the circle, giving reasons for your answer.

10A



24 In the circle with centre O, $SU \perp VT$ and $\angle VTU = 47°$. Find the size of $\angle SOT$, giving reasons for your answer.

10A



9 Non-linear relationships

Maths at sea

Name: David Provis
Job: Oceanographer
Qualifications: Bachelor of Science, PhD in Mathematics

Imagine if your job paid you to study one of your favourite places. That's the case for David Provis, Principal Oceanographer with engineering firm Cardno Lawson Treloar.

David always enjoyed sailing, swimming and surfing and now finds himself observing and measuring the movement of the tides, currents and sand in the marine environment.

He uses the data he collects to perform calculations such as the height and energy of waves, the volume of sand shifted by an ocean current, or how the depth of water affects the readings of his instruments. "Maths helps me understand and describe the laws of the ocean. I have a toolbox of mathematical techniques I can use, such as geometry, trigonometry, and statistics."

The information David produces may be used by engineers when building a marine structure such as a wharf, pipeline or oil rig, or by biologists concerned about the environmental impact of such a structure. Oceanographers may also be involved in studying the effects of climate change on ocean levels, developing tsunami early warning systems, or even in designing the perfect surf break!

Why learn this?

Straight lines are not the only useful type of graph. Quadratics, exponentials and circles are useful in modelling many situations. The path of a ball thrown into the air is parabolic. The ripples made by a stone thrown into still water are circles. The decay of radioactive substances can be modelled by exponential equations. Using graphs to model real-life situations means that future trends can be predicted and relationships between variables, such as global temperatures and the rate of carbon emissions produced worldwide can be established.

After completing this chapter you will be able to:

- identify non-linear graphs
- solve non-linear equations algebraically and graphically using technology
- solve quadratic equations using a variety of techniques
- construct graphs of parabolas, exponentials, circles and hyperbolas
- identify, match and describe algebraic and graphical representations of parabolas, rectangular hyperbolas, exponential functions and circles
- solve problems that can be modelled by quadratic, exponential, circle and hyperbolic relationships
- understand the effect of transforming graphs
- sketch the graphical representation of parabolas, circles, exponential and hyperbolic functions.

9 Recall

Prepare for this chapter by attempting the following questions. if you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 **(a)** If $p = 2m^2 - 3m + 5$, find the value of p when:

(i) $m = 1$ **(ii)** $m = 0$ **(iii)** $m = -1$

Worksheet R9.1

(b) Find the value of $x(x - 3)$ when:

(i) $x = 5$ **(ii)** $x = -3$ **(iii)** $x = 0$

2 Factorise each of these quadratic expressions.

(a) $x^2 - 7x$ **(b)** $2x^2 + 10x$ **(c)** $x^2 - 4$

Worksheet R9.2

3 Expand and simplify each of the the following.

(a) $3x(x + 4)$ **(b)** $(x + 2)(x + 5)$ **(c)** $(x - 7)(x + 3)$

(d) $(x + 8)^2$ **(e)** $(x - 3)(x - 1)$ **(f)** $(x - 4)(x + 4)$

Worksheet R9.3

4 Factorise each of the following expressions.

(a) $x^2 - 25$ **(b)** $9x^2 - 16$ **(c)** $x^2 - 5$

(d) $x^2 - 8x + 16$ **(e)** $x^2 + 14x + 49$ **(f)** $3x^2 - 66x + 363$

Worksheet R9.4

5 Factorise each of the following.

(a) $x^2 + 7x + 10$ **(b)** $x^2 - 5x + 6$ **(c)** $x^2 - 13x - 14$

(d) $x^2 + 5x - 24$ **(e)** $x^2 + 5x - 36$ **(f)** $2x^2 + 14x + 12$

(g) $-x^2 - 3x - 2$ **(h)** $-x^2 + 7x - 10$ **(i)** $-2x^2 + 2x + 24$

6 Write each of the following as a number without an index.

(a) 3^3 **(b)** 4^{-1} **(c)** $5^{-2} + 3$

Worksheet R9.5

7 Solve the following using the Null Factor Law.

(a) $(x - 1)(x + 3) = 0$ **(b)** $(x + 4)(x + 2) = 0$

(c) $(x - 1)^2 = 0$ **(d)** $x(x - 5) = 0$

Worksheet R9.6

8 Solve the following using special product rules to factorise first.

(a) $x^2 - 6x + 9 = 0$ **(b)** $4x^2 - 25 = 0$

(c) $x^2 + 8x + 16 = 0$ **(d)** $121x^2 - 36 = 0$

Worksheet R9.7

9 Factorise each of the following by completing the square.

(a) $x^2 + 4x + 2$ **(b)** $x^2 - 6x + 6$ **(c)** $x^2 - 3x - 5$

Worksheet R9.8

Key Words

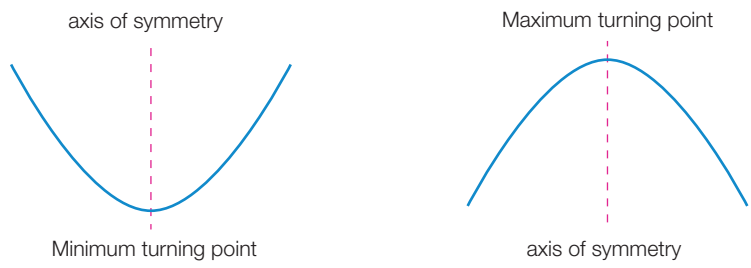
asymptote	exponential	reflection	turning point form
axis of symmetry	Null Factor Law	transformation	vertex
dilation	parabola	translation	
dilation factor	rectangular hyperbola	turning point	

9.1 Identifying non-linear graphs and their equations

Linear equations in the form of $y = mx + b$ graph as straight lines. In this section, we are going to explore some different types of equations and the graphs they produce. It is important to be able to recognise the shape of a graph given a particular equation or identify the form of an equation from the shape of its graph.

Quadratics

Quadratic relationships occur often in everyday life. The path of a ball in flight, the area of land, or the height of a ball thrown vertically in the air, are all modelled by quadratic relationships. A quadratic relationship is recognised by equations in the general form of $y = ax^2 + bx + c$, where x is the independent variable and y is the dependent variable. The highest power of the independent variable is 2 and the highest power of the dependent variable is 1. All powers are integers. The graph of a quadratic equation is called a **parabola**. A parabola has a minimum or maximum **turning point** or **vertex**, which is where the graph changes direction. Every parabola is symmetrical about an **axis of symmetry**, a vertical line with an equation in the form $x = h$, where h is a constant equal to the x-value of the turning point. The shape of a parabola is given below.



Features of the parabola:

- A parabola is a graph of the quadratic relationship $y = ax^2 + bx + c$.
- Every parabola has a vertex or turning point.
- A parabola can have a minimum turning point or a maximum turning point.
- Every parabola has an axis of symmetry.

The graph of a quadratic equation can be plotted using the following steps:

1. Complete a table of values.
2. Draw a set of appropriately sized and scaled axes.
3. Plot the points from the table.
4. Join the points with a smooth curve.
5. Label the graph and the axis of symmetry.

Worked Example 1

WE1

Plot the graphs of the following for values of x between -3 and 3 inclusive:

(a) $y = x^2$ **(b)** $y = 2x^2 - 1$ **(c)** $y = 5 - x^2$.

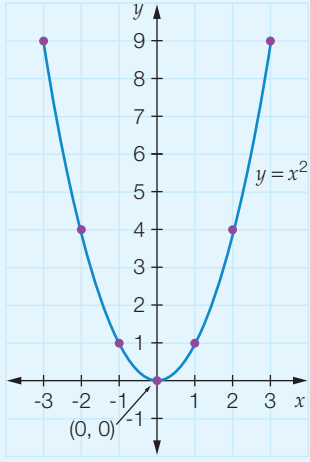
Thinking	Working

(a) 1 Write the equation and draw up a table of values.

(a) $y = x^2$

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the turning point.



3 Write the equation of the axis of symmetry.

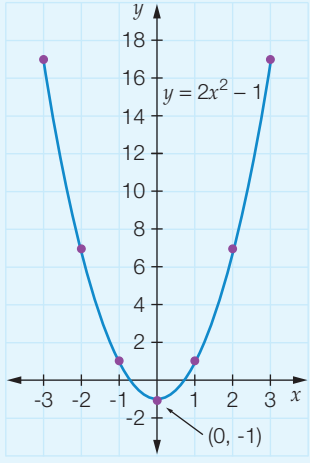
The equation of the axis of symmetry is $x = 0$.

(b) 1 Write the equation and draw up a table of values.

(b) $y = 2x^2 - 1$

x	-3	-2	-1	0	1	2	3
y	17	7	1	-1	1	7	17

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the turning point.



3 Write the equation of the axis of symmetry.

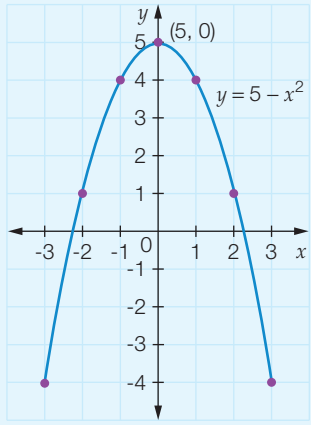
The equation of the axis of symmetry is $x = 0$.

(c) 1 Write the equation and draw up a table of values.

(c) $y = 5 - x^2$

x	-3	-2	-1	0	1	2	3
y	-4	1	4	5	4	1	-4

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the turning point.



3 Write the equation of the axis of symmetry.

The equation of the axis of symmetry is $x = 0$.

Circles

All circular shapes can be modelled by a circle equation that can be derived from Pythagoras' theorem. The equation of a circle with its centre at the origin (0, 0) and with radius r is given by $x^2 + y^2 = r^2$. The power of both the independent and dependent variables is 2.

The evaluation of x- and y-coordinates often involves surds, so we use the centre of the circle and four other points on the circumference to plot a circle. For a circle with its centre at the origin and radius, r, the four points are $(r, 0)$, $(0, r)$, $(-r, 0)$, and $(0, -r)$.

Circle equations written in the form of $(x - h)^2 + (y - k)^2 = r^2$ have the centre at (h, k) and radius r.

The four points plotted on the circumference are $(h + r, k)$, $(h, k + r)$, $(h - r, k)$ and $(h, k - r)$.

Worked Example 2

WE2

Plot the graphs of the following equations.

(a) $x^2 + y^2 = 25$

(b) $(x - 1)^2 + (y + 2)^2 = 36$

Thinking

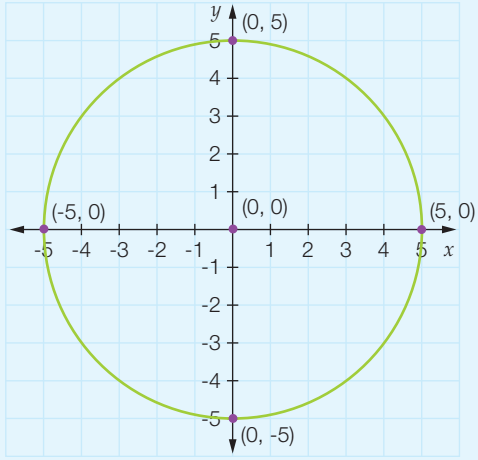
Working

(a) 1 Write the equation, identify the radius (here, $r = 5$) and centre (h, k) (here, the centre is (0, 0)). Draw up a table of values using $(r, 0)$, $(0, r)$, $(-r, 0)$, and $(0, -r)$. If the radius is a surd, use an approximation to one decimal place.

(a) $x^2 + y^2 = 25$

x	-5	0	5
y	0	±5	0

2 Draw a set of labelled Cartesian axes, plot these points and join them with a smooth curve. The scale has to be the same for both axes. Label the graph and the centre (0, 0).

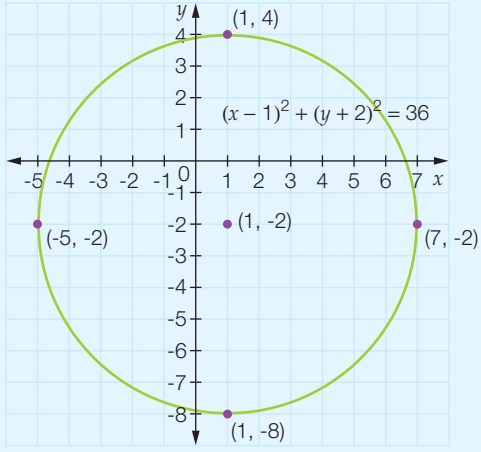


(b) 1 Write the equation and identify the radius (here, $r = 6$) and centre (h, k) (here, $h = 1$, $k = -2$). Draw up a table of values using $(h + r, k)$, $(h, k + r)$, $(h - r, k)$ and $(h, k - r)$. If the radius is a surd, use an approximation to one decimal place.

(b) $(x - 1)^2 + (y + 2)^2 = 36$

x	-5	1	7
y	-2	-8, 4	-2

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale has to be the same for both axes. Label the graph and the centre (h, k).

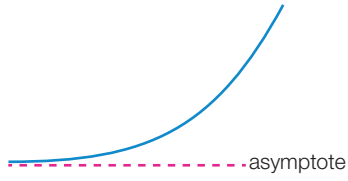


Exponentials

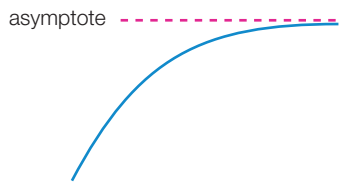
An **exponential** relationship occurs when a quantity grows or decays at a rate proportional to its current value. It can be recognised when the independent variable is the index or exponent of a constant base. All exponential graphs have a limiting value, a value they approach but never reach. This limiting value gives us a horizontal **asymptote**. An asymptote is a line that a curve approaches but never cuts or crosses as the independent variable approaches particular values.

The simplest exponential graph continually increases and models exponential growth. Examples of exponential growth are the growth of money invested with compound interest or the growth of bacteria. These graphs are in the form of $y = a^x$ where a is the base ($a > 1$) and x is the independent variable.

For $y = a^x$, $a > 1$ the graph continually increases as x increases. However, as the x-value decreases (becomes negatively large), the y-value becomes smaller and smaller but never reaches zero. The x-axis ($y = 0$) is the asymptote.

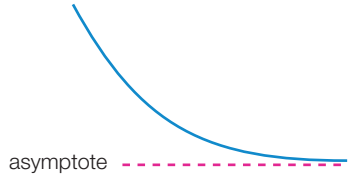


An exponential graph can also increase to an asymptotic value.

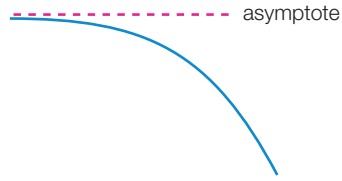


Some exponential graphs decrease to an asymptotic value. Examples of this are radioactive decay or loss of value of an item with time (depreciation). These graphs are in the formof $y = a^{-x}$, $a > 1$.

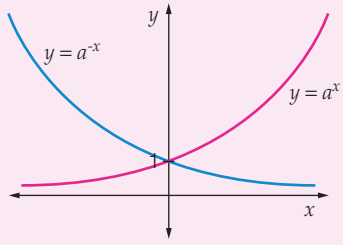
For $y = a^{-x}$, $a > 1$, as the x-value increases, the y-value becomes smaller and smaller but never reaches zero. The x-axis ($y = 0$) is the asymptote for these graphs also.



An exponential graph can also decrease from an asymptotic value.



The graphs of $y = a^x$ and $y = a^{-x}$ pass through the point (0, 1) because $a^0 = 1$, for all values of a.



Worked Example 3

WE3

Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = 2^x$ **(b)** $y = 2^{-x} + 3$

Thinking

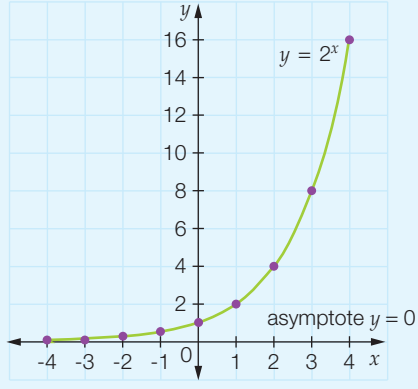
(a) 1 Write the equation, identify the base and draw up a table of values.

Working

(a) $y = 2^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the asymptote.

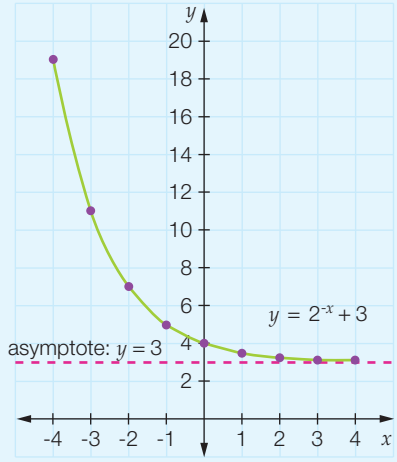


(b) 1 Write the equation, identify the base and draw up a table of values.

(b) $y = 2^{-x} + 3$

x	-4	-3	-2	-1	0	1	2	3	4
y	19	11	7	5	4	$3\frac{1}{2}$	$3\frac{1}{4}$	$3\frac{1}{8}$	$3\frac{1}{16}$

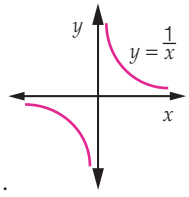
2 Draw a set of labelled Cartesian axes, plot the points, and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the asymptote.



Rectangular hyperbola

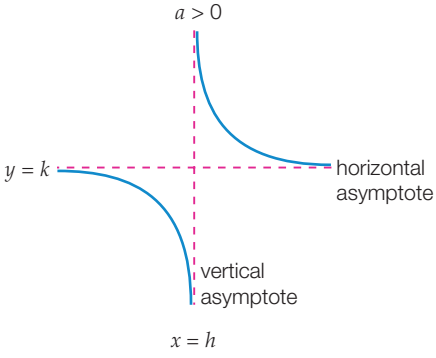
10A

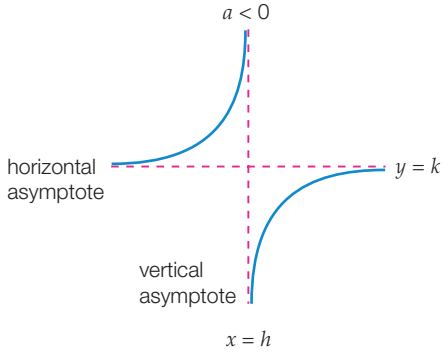
A **rectangular hyperbola** is another non-linear relationship that occurs frequently. In this relationship, as one variable increases, the other decreases. Consider a rectangle with a constant area. As the length increases, the width decreases.



The simplest form of a rectangular hyperbola equation is in the form of $y = \frac{1}{x}$.

Equations of a rectangular hyperbola in the form of $y = \frac{a}{x-h} + k$ look like this:





Graphs of a rectangular hyperbola in the form of $y = \frac{a}{x-h} + k$ have both horizontal and vertical asymptotes. The graph is discontinuous at the vertical asymptote. We say the equation is undefined for x at that value.

Worked Example 4

WE4

Plot the graphs of the following equations for $-4 \le x \le 4$.

(a) $y = \dfrac{4}{x}$

(b) $y = \dfrac{1}{x-1} + 3$

Thinking

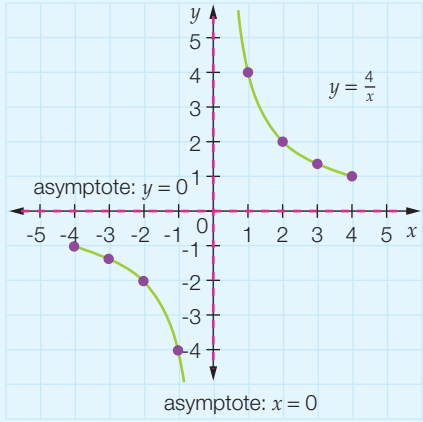
Working

(a) 1 Write the equation, identify the base and draw up a table of values.

(a) $y = \dfrac{4}{x}$

x	-4	-3	-2	-1	0	1	2	3	4
y	-1	$-\frac{4}{3}$	-2	-4	undefined	4	2	$\frac{4}{3}$	1

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the asymptotes.

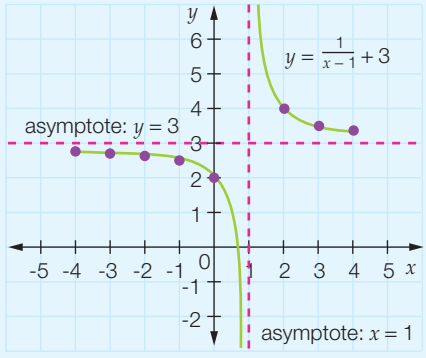


(b) 1 Write the equation, identify the base and draw up a table of values.

(b) $y = \dfrac{1}{x-1} + 3$

x	-4	-3	-2	-1	0	1	2	3	4
y	$2\frac{4}{5}$	$2\frac{3}{4}$	$2\frac{2}{3}$	$2\frac{1}{2}$	2	undefined	4	$3\frac{1}{2}$	$3\frac{1}{3}$

2 Draw a set of labelled Cartesian axes, plot the points and join them with a smooth curve. The scale does not have to be the same for both axes. Label the graph and the asymptotes.



9.1 Identifying non-linear graphs and their equations

Navigator

Q1, Q2, Q3, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q18 (a)–(c), Q19	Q1, Q2, Q3, Q5, Q6, Q8, Q9, Q10, Q12, Q13, Q15, Q17, Q18 (a)–(c), Q19	Q1, Q2, Q3, Q4, Q5, Q6, Q8, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q20

Answers page 777

Equipment required: graph paper for Questions **8** and **9**, calculator for Questions **13** and **16**

Fluency

1 Plot the graphs of each of the following using integer values of x between -3 and 3 inclusive. WE1

(a) $y = x^2 + 1$ **(b)** $y = x^2 - 3$ **(c)** $y = x^2 + 5$

(d) $y = 2x^2$ **(e)** $y = -2x^2$ **(f)** $y = -2x^2 + 1$

2 Plot the graphs of each of the following equations. WE2

(a) $(x + 3)^2 + (y + 2)^2 = 16$ **(b)** $(x - 3)^2 + (y - 2)^2 = 9$

(c) $(x - 1)^2 + (y + 1)^2 = 25$ **(d)** $x^2 + (y + 4)^2 = 36$

(e) $(x + 4)^2 + y^2 = 8$ **(f)** $(x - 4)^2 + (y + 2)^2 = 12$

3 Plot the graphs of the following equations for $-3 \le x \le 3$. WE3

(a) $y = 2^x + 2$ **(b)** $y = 2^x - 1$

(c) $y = 2^x - 3$ **(d)** $y = 3^x - 1$

(e) $y = 3^x - 2$ **(f)** $y = 3^x + 4$

4 Plot the graphs of the following equations for $-4 \le x \le 4$. WE4 10A

(a) $y = \frac{2}{x}$ **(b)** $y = \frac{3}{x}$ **(c)** $y = -\frac{4}{x}$

(d) $y = \frac{1}{x - 2} + 1$ **(e)** $y = \frac{1}{x + 2} - 3$ **(f)** $y = \frac{1}{x + 2} - 1$

Understanding

5 **(a)** Complete a table of values and then plot the graph for each of the following using integer values for x between -4 and 4 inclusive.

(i) $y = (x - 1)(x + 3)$ **(ii)** $y = x(x + 3)$ **(iii)** $y = (4 + x)(2 - x)$

(b) Write down the coordinates of the turning point for each graph and state whether it is a maximum or minimum.

6 The graph of $y = 2^{x-2} + 1$ would be:

A a circle **B** a rectangular hyperbola

C an exponential **D** a parabola

7 A path around an area on a coordinate map is described by the equation $(x + 1)^2 + (y - 3)^2 = 16$.

(a) Make a simple sketch of the path.

(b) Determine whether the following points are within or outside the circle.

(i) (1, 2) **(ii)** (3, 6) **(iii)** (1.5, 2.5)

Hint
Will the temperature ever reach 25°C exactly?

8 A cup of coffee cools exponentially in a room where the temperature is 25°C. The temperature of the coffee was taken every 30 minutes and the results recorded in the table below:

t (min)	0	30	60	90	120	150
T (°C)	100	54	36.2	29.3	26.7	25.6

(a) Use graph paper to plot a graph of t against T (show t on the horizontal axis). Join the points with a smooth curve. Use your graph to answer parts **(b)** to **(e)**.

(b) What is the initial temperature of the coffee?

(c) Use the graph to find the temperature of the coffee after:

(i) 10 min **(ii)** 20 min **(iii)** 40 min **(iv)** 50 min

Give your answer to the nearest degree.

(d) During which 10 min period in the first hour did the coffee cool the fastest? Justify your answer.

(e) When will the temperature be 27.5°C?

(f) Extend the curve. Predict the temperature after 3 hours.

9 The volume of a child's circular swimming pool, with a depth of $\frac{3}{\pi}$ (≈ 0.96 m), depends on the radius of the pool. The volume of the pool is given by $V = \pi r^2 h$, where V is the volume of the pool in cubic metres, r is the radius (in metres) and h is the height (in metres).

(a) Show that $V = 3r^2$.

(b) Explain why this relationship is quadratic.

(c) Complete the table of values for $0 \le r \le 5$.

r	0	1	2	3	4	5
r^2						
V						

(d) Plot a graph of the relationship $V = 3r^2$ on graph paper. (Represent r on the horizontal axis.)

(e) Find the volume of the pool when the radius is:

(i) 2.5 m **(ii)** 3.8 m

(f) Find the radius of the pool if its volume is:

(i) 30 m^3 **(ii)** 65 m^3

(g) What is the difference in volume between a pool of radius 1.5 m and one of radius 3.5 m?

10 **(a)** Write down the equation of a circle with:

(i) centre (3, 1) and radius 7

(ii) centre (-2, -4) and radius 3

(iii) centre (2, -5) and radius 2.

(b) Plot the graph for each of the equations in part **(a)**.

10A **11** The graph of the equation $y = \frac{1}{x-1} + 2$ has asymptotes at:

A $x = 2, y = 1$ **B** $x = 1, y = 2$ **C** $x = -1, y = 2$ **D** $x = 1, y = -2$

12 A point that does not lie on the graph of $(x-3)^2 + (y+2)^2 = 25$ would be:

A (7, 1) **B** (-1, -5) **C** (7, -5) **D** (-7, 1)

13 **(a)** For the relationship $y = 2^x$, copy and complete this table of values for x and y.

x	-2	-1	0	1	2	3	4
y							

(b) Use this table of values to plot a graph of $y = 2^x$. Join the points with a smooth curve.

(c) Use your graph to determine approximate values for:

(i) $2^{1.5}$ **(ii)** $2^{0.8}$ **(iii)** $2^{-0.5}$

(iv) $2^{-1.2}$ **(v)** $\sqrt{2}$ **(vi)** $(\sqrt{2})^3$

(d) Use a calculator to check the answers you obtained in part **(c)**.

(e) Use your graph to find the value of the exponent, correct to one decimal place, when y is:

(i) 5 **(ii)** 12 **(iii)** 15

Reasoning

14 A tablecloth has a length 0.4 m longer than its width.

(a) Write an equation for the area of the tablecloth in terms of its length if the length is represented by the variable l.

(b) Plot a graph of the area against the length.

(c) Use this graph to find the dimensions of the tablecloth if the area of the tablecloth is 3.2 m^2.

15 A town has an initial population of 4000, but the establishment of a new industry in the town is expected to increase the population by 6% per year for future years.

(a) What will be the predicted population 1 year after the industry is established?

(b) Complete a table of values to show the predicted population for the first 5 years after the industry is established. Round the population to the nearest whole number. Use P to represent the population after n years.

(c) Write an equation to model the predicted population over future years.

(d) Sketch a graph of this relationship using your equation.

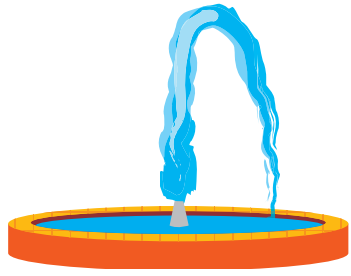
(e) Use your graph to predict when the population will have doubled in size.

(f) Use your equation to confirm your prediction in **(e)** is correct.

(g) Explain why the graph is a first quadrant graph only.

16 The water coming out of a fountain in a pond forms a parabolic shape. At any point p from the place where the water comes out, the height of the water, h, can be found using the equation: $h = -\frac{1}{16}(p-40)^2 + 100$

(h and p are in centimetres).



(a) Plot the graph of the relationship between h and p, with p on the horizontal axis and the values of p between 0 and 80.

(b) How high does the water go up in the air?

(c) How far from the water outlet does the water from the fountain reach?

(d) Assuming that the fountain is in the centre of a pond and reaches to the edge of the pond, what area would be covered by water?

17 The centre of a circular water fountain that has been built is 3 m east and 2 m south of the corner of a building. If the water fountain has a radius of 1 m, find if a tree could be planted 2.4 m east and 2.7 m south of the corner of the building. Justify your answer by using an equation to plot a graph that represents this situation.

Open-ended

18 (a) Write down an equation for a circle that does not have its centre at the origin and plot its graph.

(b) Write down the equation for a parabola that has a maximum turning point and plot the graph.

(c) Write down the equation for an exponential relationship that has an asymptote at $y = -1$.

10A (d) Write down an equation for a rectangular hyperbola that has asymptotes at $x = -2$ and $y = 6$.

19 When Sian threw a ball up in the air and then caught it, the path of the ball traced a parabolic shape.

(a) Sketch what you think might have happened on a set of axes.

(b) Describe the main features of the graph and explain why it has a parabolic shape.

20 Find the equations of at least two circles that pass through the point (1, 2) and have different centres.

Outside the Square — Problem solving

How much is enough?

Bob needs to build a picket fence from his house to the garage. The curve formed by the top of the pickets follows the path of a parabola with equation:

$$y = \frac{3}{2.56}(x^2 - 3.2x + 2.56) + 1.2$$

where y is the height above the ground and x is the distance from the wall of the house. The pickets are 50 mm wide and have a gap of 50 mm between them. The length of the fence is 3.2 m. Each picket is shaped to a point at the top. What length of full width wood is used for the pickets? (A half-width picket is used at each end.)

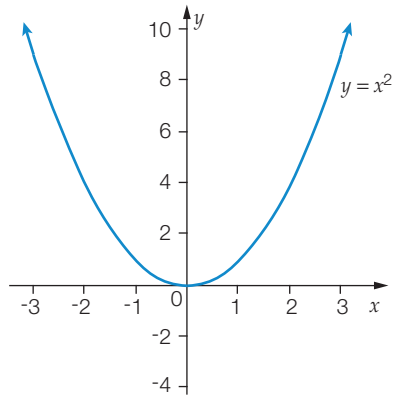
Strategy options

- Draw a diagram.
- Make a model.
- Break problem into manageable parts.

Quadratic transformations

9.2

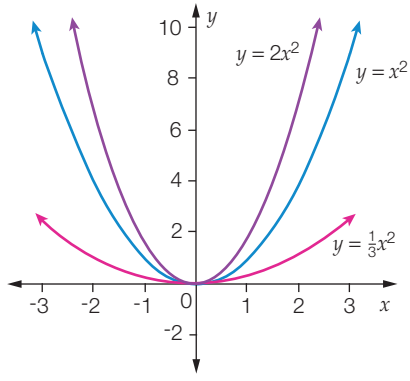
Plotting graphs is time consuming and it is often unnecessary to reproduce an exactly accurate graph. In this section, we will explore the **transformations** of **dilation**, **reflection** and **translation** of quadratics and will sketch simple quadratic graphs instead of plotting them. Sketch graphs are not drawn on graph paper and exact values cannot be read from them.



The basic quadratic relationship is $y = x^2$. The parabola $y = x^2$ has a turning point, or vertex, at (0, 0), the origin. The turning point is a minimum. The x-intercept and y-intercept are both at the point (0, 0).

Dilations

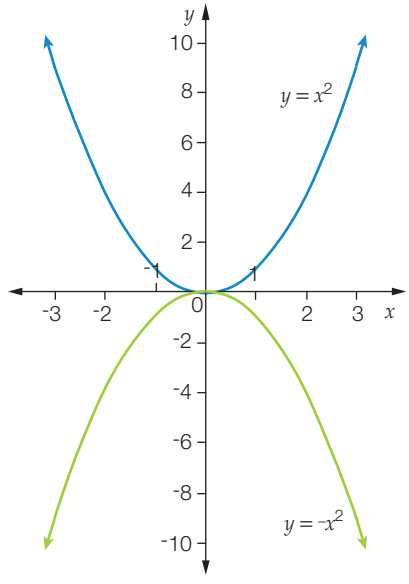
Graphing $y = x^2$, $y = \frac{1}{3}x^2$ and $y = 2x^2$ on the same set of axes shows that the coefficient of x^2 determines the shape of the graph, but the turning point is still the same. This effect is called dilation. The coefficient of x^2 is called the **dilation factor**.



- A factor greater than 1 will make the graph narrower.
- A factor less than 1 but greater than 0 will make the graph wider.

Reflections

The graph of $y = -x^2$ is the same shape as the graph of $y = x^2$ except that it is upside down. We say it is *inverted* as it is a reflection in the x-axis. All graphs that have a negative coefficient for x^2 will have this same basic shape. The turning point of these graphs is a maximum.



Worked Example 5

WE 5

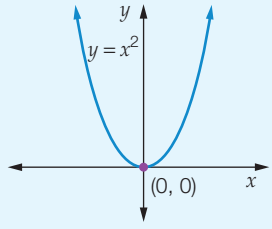
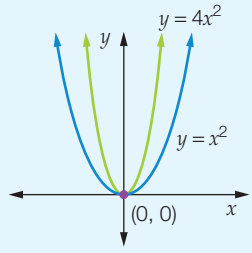
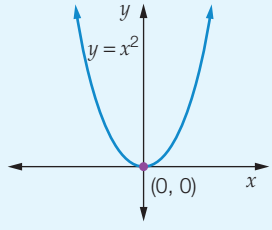
For each of the following equations:

(a) $y = 4x^2$ **(b)** $y = \frac{1}{4}x^2$ **(c)** $y = -3x^2$

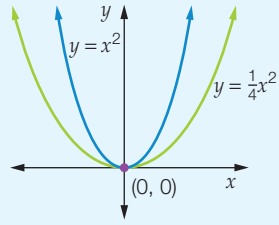
(i) state the dilation factor

(ii) determine whether the graph is narrower or wider than the parabola $y = x^2$

(iii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Thinking	Working
(a) (i) The coefficient of x^2 gives the dilation factor. Write down the dilation factor.	(a) $a = 4$ (i) The dilation factor is 4.
(ii) If the dilation factor is greater than 1, the graph is narrower than $y = x^2$.	(ii) The graph is narrower than $y = x^2$.
(iii) 1 Sketch the graph of $y = x^2$ on a set of Cartesian axes. Label the graph and the turning point of $y = x^2$.	(iii) 
2 Sketch the required graph. Every point has been dilated by the dilation factor (here, $a = 4$). Label the graph and the turning point.	
(b) (i) The coefficient of x^2 gives the dilation factor. Write down the dilation factor.	(b) $a = \frac{1}{4}$ (i) The dilation factor is $\frac{1}{4}$.
(ii) If the dilation factor is greater than 0 but less than 1, the graph is wider than $y = x^2$.	(ii) The graph is wider than $y = x^2$.
(iii) 1 Sketch the graph of $y = x^2$ on a set of Cartesian axes. Label the graph and the turning point of $y = x^2$.	(iii) 

2 Sketch the required graph. Every point has been dilated by the dilation factor (here, $a = \frac{1}{4}$). Label the graph and the turning point.



(c) **(i)** The coefficient of x^2 gives the dilation factor. Write down the dilation factor.

(c) $a = -3$

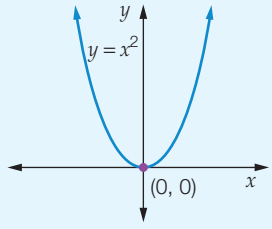
(i) The dilation factor is -3.

(ii) If the dilation factor is less than -1, the graph is narrower than $y = x^2$ and is inverted.

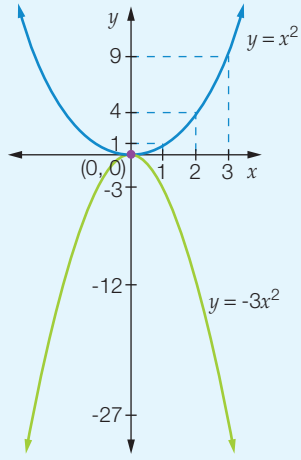
(ii) The graph is narrower than $y = x^2$.

(iii) **1** Sketch the graph of $y = x^2$ on a set of Cartesian axes. Label the graph and the turning point of $y = x^2$.

(iii)

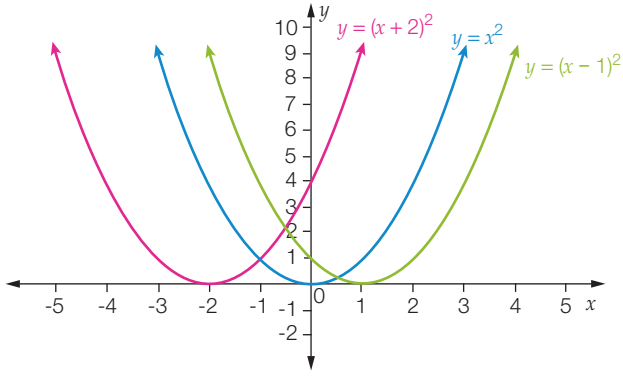


2 Sketch the required graph. Every point has been dilated by the dilation factor. If it is negative, the graph is reflected in the x-axis (here, $a = -3$). Label the graph and the turning point.



Horizontal translations

Graphing $y = x^2$ with $y = (x - 1)^2$ and $y = (x + 2)^2$ shows that when a number is added or subtracted to x in the brackets the basic graph $y = x^2$ is moved left or right. This movement is called a horizontal translation.



If $y = (x - h)^2$, h indicates the size and direction of the horizontal translation. When h is positive, the graph is moved right h units. When h is negative, the graph is moved left h units.

In the graph shown above:

$y = (x - 1)^2$ is the graph of $y = x^2$ translated 1 unit to the right.

$y = (x + 2)^2$ is the graph of $y = x^2$ translated 2 units to the left.

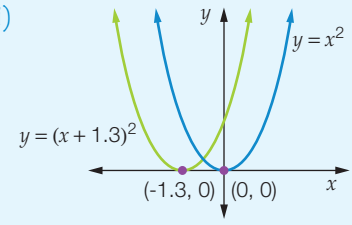
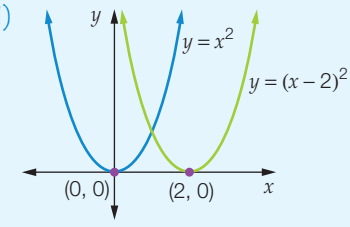
Worked Example 6 WE6

For each of the following equations:

(a) $y = (x + 1.3)^2$ **(b)** $y = (x - 2)^2$

(i) state the translation of the graph of $y = x^2$

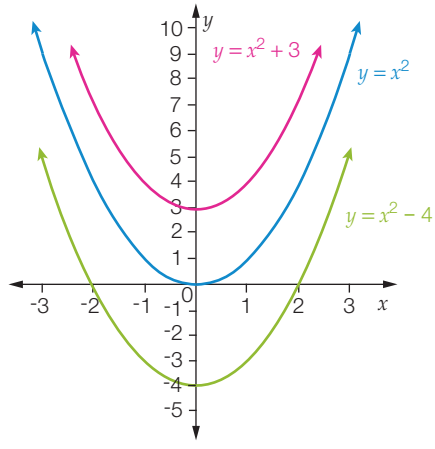
(ii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Thinking	Working
(a) (i) Compare the graph's equation to $y = (x - h)^2$ to identify the value of h (here, $h = -1.3$). If h is negative, $y = x^2$ is moved h units to the left.	(a) (i) $y = x^2$ is translated 1.3 units to the left.
(ii) Sketch the graphs. They should be identical in shape, as every point on the graph of $y = x^2$ has been translated h units horizontally (here, the graph of $y = x^2$ is translated 1.3 units to the left). Label the graphs and the turning points.	(ii) 
(b) (i) Compare the graph's equation to $y = (x - h)^2$ to identify the value of h (here, $h = 2$). If h is positive, $y = x^2$ is moved h units to the right.	(b) (i) $y = x^2$ is translated 2 units to the right.
(ii) Sketch the graphs. They should be identical in shape, as every point on the graph of $y = x^2$ has been translated h units horizontally (here, the graph of $y = x^2$ is translated 2 units to the right). Label the graphs and the turning points.	(ii) 

Vertical translations

Graphing $y = x^2$, $y = x^2 + 3$ and $y = x^2 - 4$ shows that the basic graph $y = x^2$ will move up when adding a number and down when subtracting a number. This movement is called a vertical translation.

If $y = x^2 + k$, the constant term k indicates the size and direction of the vertical translation. When k is positive, the graph is moved up k units. When k is negative, the graph is moved down k units.



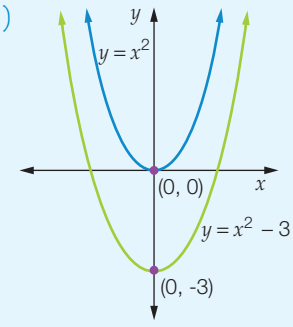
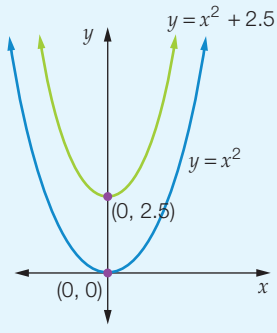
Worked Example 7 WE7

For each of the following equations:

(a) $y = x^2 - 3$ **(b)** $y = x^2 + 2.5$

(i) state the translation of the graph of $y = x^2$

(ii) sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

Thinking	Working
(a) (i) Compare the graph's equation to $y = x^2 + k$ to identify the value of k (here, $k = -3$). If k is negative, $y = x^2$ is moved k units down.	(a) (i) $y = x^2$ is translated 3 units down.
(ii) Sketch the graphs. They should be identical in shape, as every point on the graph of $y = x^2$ has been translated k units vertically (here, the graph of $y = x^2$ is translated 3 units down). Label the graphs and the turning points.	(ii) 
(b) (i) Compare the graph's equation to $y = x^2 + k$ to identify the value of k (here, $k = 2.5$). If h is positive, $y = x^2$ is moved k units up.	(b) (i) $y = x^2$ is translated 2.5 units up.
(ii) Sketch the graphs. They should be identical in shape, as every point on the graph of $y = x^2$ has been translated k units vertically (here, the graph of $y = x^2$ is translated 2.5 units up). Label the graphs and the turning points.	(ii) 

Combined translations

We can combine horizontal and vertical translations.

These equations are in the form of $y = (x - h)^2 + k$, where h is the horizontal translation and k is the vertical translation.

This equation is said to be in turning point form because the turning point has translated from the origin (0, 0) to the point (h, k).

Worked Example 8

WE 8

For each of the following equations, sketch the graph of $y = x^2$ on a set of Cartesian axes and then use transformations to sketch the graph of the given equation on the same set of axes.

(a) $y = (x - 3)^2 - 4$

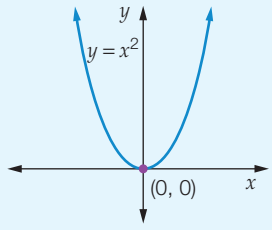
(b) $y = (x + 2)^2 + 3$

Thinking

Working

(a) 1 Sketch the graph of $y = x^2$ on a set of Cartesian axes. Label the graph and the turning point of $y = x^2$.

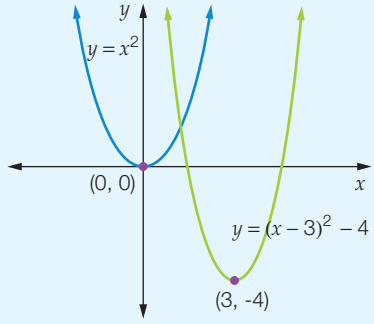
(a)



2 Write down the coordinates of the turning point of the given equation.

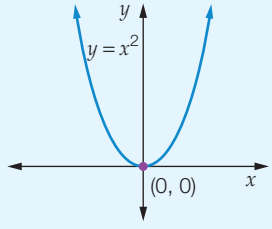
The turning point is at (3, -4).

3 Sketch the graph. It should be identical in shape to the graph of $y = x^2$ but every point has been translated h units horizontally and k units vertically (here, the graph is translated 3 units to the right and 4 units down). Label the graph and the turning point.



(b) 1 Sketch the graph of $y = x^2$ on a set of Cartesian axes. Label the graph and the turning point of $y = x^2$.

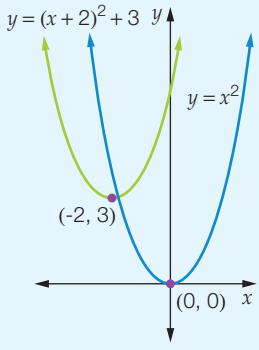
(b)



2 Write down the coordinates of the turning point of the given equation.

The turning point is at (-2, 3).

3 Sketch the graph. It should be identical in shape to the graph of $y = x^2$ but every point has been translated h units horizontally and k units vertically (here, the graph is translated 2 units to the left and 3 units up). Label the graph and the turning point.



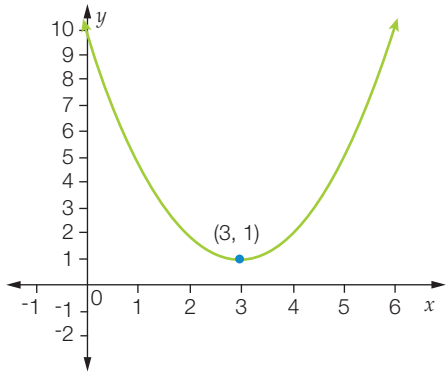
We can use the basic graph of $y = x^2$ to obtain other parabolas by:

- Dilating it by a factor of a, making it narrower ($a > 1$) or wider ($0 < a < 1$).
- Reflecting it in the x-axis (inverting it) ($a < 0$).
- Translating it by moving it h units horizontally, k units vertically or in both directions h units horizontally and k units vertically.

Worked Example 9

WE9

Using the graph shown, find the equation of the curve, assuming there is no dilation.



Thinking	Working
1 As there is no dilation, we can use the equation $y = (x - h)^2 + k$ as $a = 1$.	$y = (x - h)^2 + k$
2 Identify the turning point from the graph to find the values of h and k.	The turning point is at (3, 1) so $h = 3$ and $k = 1$.
3 Write the equation.	$y = (x - 3)^2 + 1$

9.2 Quadratic transformations

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q16, Q17	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17

Answers page 782

Fluency

WE5

1 For each of the following equations:

(a) $y = 2x^2$ **(b)** $y = \frac{x^2}{2}$

(i) state the dilation factor

(ii) determine whether the graph is narrower or wider than the parabola $y = x^2$

(iii) sketch the graph on the same set of axes as the graph of $y = x^2$.

WE6

2 For each of the following equations:

(a) $y = (x - 1.3)^2$ (b) $y = (x + 2)^2$

(i) sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation

(ii) state the translation of the graph of $y = x^2$.

WE7

3 For each of the following equations:

(a) $y = x^2 + 6$ (b) $y = x^2 - 1.2$

(i) sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation

(ii) state the translation of the graph of $y = x^2$.

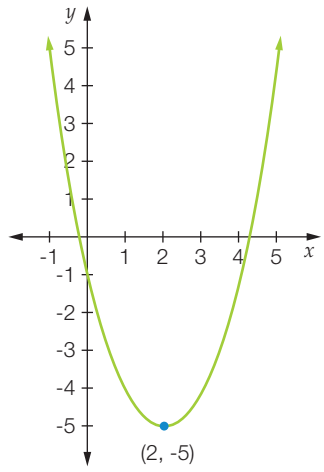
WE8

4 For each of the following equations, sketch the graph of $y = x^2$ on a set of Cartesian axes then use transformations to sketch the graph of the given equation on the same set of axes.

(a) $y = (x + 3)^2 - 4$ (b) $y = (x - 2)^2 - 3$

WE9

5 Using the graph shown on the right, find the equation of the curve, assuming there is no dilation.



6 To sketch the graph of $y = (x + 5)^2 - 2$, the graph of $y = x^2$ will need to be translated:

A 5 units right and 2 units down B 5 units left and 2 units down

C 5 units left and 2 units up D 5 units right and 2 units up

7 On the same set of axes, sketch and label the graphs of:

(a) $y = x^2$ (b) $y = x^2 - 5$ (c) $y = (x + 4)^2$ (d) $y = (x - 1)^2 + 5$

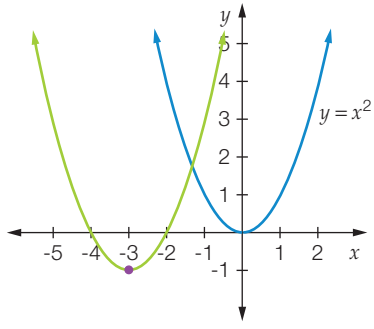
8 On the same set of axes, sketch and label the graphs of:

(a) $y = x^2$ (b) $y = 3x^2$ (c) $y = \frac{x^2}{3}$ (d) $y = -x^2$

9 The equation of the parabola shown on the right is:

A $y = (x - 3)^2 + 1$ B $y = (x + 3)^2 + 1$

C $y = (x + 1)^2 + 3$ D $y = (x + 3)^2 - 1$



Understanding

10 (a) Sketch and label the graph of $y = (x + 2)^2$.

(b) Use the graph from part **(a)** to sketch and label the graph of $y = -(x + 2)^2$ on the same set of axes.

11 **(a)** Sketch the graph of $y = x^2 + 4$.

(b) On the same set of axes, sketch and label the graph formed by dilating the graph in part **(a)** by a factor of 2.

(c) Write the equation for this graph.

12 **(a)** Sketch the graph of $y = (x - 3)^2$.

(b) On the same set of axes, sketch and label the graph formed by dilating the graph in part **(a)** by a factor of 2.

(c) Write the equation for this graph.

Hint
It will have the same turning point as the graph in part **(a)**.

Hint
It will have the same turning point as the graph in part **(a)**.

Reasoning

13 A quadratic graph has a turning point at (-3, 2) and a dilation factor of 1.

(a) Find the equation of this graph. **(b)** Sketch the graph of this equation.

14 A rope tied at both ends to a horizontal bar hangs in the shape of a parabola $y = x^2$. A photo of the rope shows that the bottom of the rope is 2 m left of a person in the photo and 0.5 m above ground level. If the person is standing on a spot marked (0, 0), what would be the equation of the rope?

15 An arch is built in the shape of a parabola to span a roadway. It is modelled by the equation $y = -(x - h)^2 + k$ where the origin (0, 0) is the base of the left hand side of the arch. If the road is 6 m wide and the highest point of the arch is to be 9 m above ground level (assume the ground is flat), find the coordinates of lights if they are to be placed on the arch so they are horizontally 1 m apart and start and end at ground level.

Open-ended

16 **(a)** Write down an equation of a parabola that is a vertical translation of $y = x^2$ and sketch a graph of both $y = x^2$ and your equation on the same set of axes.

(b) Write down an equation of a parabola that is a horizontal translation of $y = x^2$ and sketch a graph of both $y = x^2$ and your equation on the same set of axes.

17 Freya and Syd are asked to sketch the graph of $y = (x + 6)^2 - 5$. Syd gets the wrong answer. Freya's graph is correct. Syd thinks that the + sign in front of the 6 means that the graph could be moved 6 units to the right and the negative sign in front of the 5 means that the graph should be moved 5 units down. Where did Syd go wrong and what explanation would Freya give to him so that he doesn't make the same mistake again.

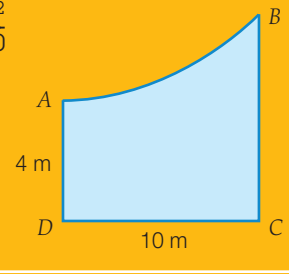
Outside the Square — Problem solving

Finding curved areas

Erica wants to paint a wall of her architect-designed house. It has a shape that is described by the equation:

$$y = \frac{x^2}{50}$$

Strategy options
- Look for a pattern.
- Break problem into manageable parts.



Some wall dimensions are shown on the diagram.

1 How high is the highest point on the wall?

2 Estimate the area by considering the shape to be a trapezium (*ABCD*).

Erica uses her CAS and finds that the actual area of the wall is 46.6 m^2.

3 How close can you get to this answer? Explain the method you used.

Technology Exploration CAS

Equipment required: 1 brain, a TI-Nspire CAS or ClassPad CAS

Investigating quadratics—how varying *b* affects the turning points of the graphs of $y = ax^2 + bx + c$ for given values of *a* and *c*

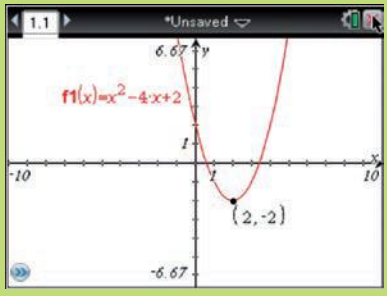
At the end of this investigation you will be able to describe exactly how varying b affects the turning point of the graph $y = ax^2 + bx + c$ for given values of a and c.

Clearly, the position of the vertex and the axis of symmetry of the graph of $y = ax^2 + bx + c$ will change as b varies. To investigate the problem in an orderly way, we will fix the value of a to 1 and c to 2 and vary b from -4 to 4 in steps of 1 unit.

1 Use your CAS to find the coordinates of the turning points of $x^2 + bx + 2$ for integer values of b from -4 to 4.

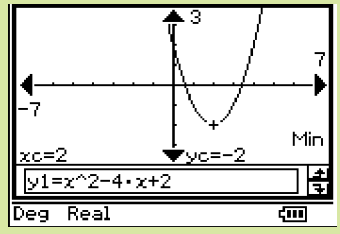
Using the TI-Nspire CAS

Select the Graphs screen and enter $f1(x)$ as $x^2 - 4x + 2$. Find the coordinates of the vertex by pressing (menu) > Analyze Graph > Minimum > (enter) (or (enter)) then move the cursor to just before the minimum value and press (enter) (or (enter)) then move to just to the right of the minimum and press (enter) (or (enter)) again. The minimum will be shown on the screen. Record the coordinate pair in the table.



Using the ClassPad

Select from the Menu and enter y1 as $x^2 - 4x + 2$. View this in the standard window and find the coordinates of the vertex by tapping Analysis > G-Solve > Min. Record the coordinate pair in the table.

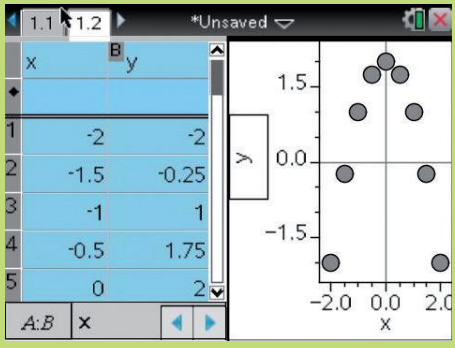
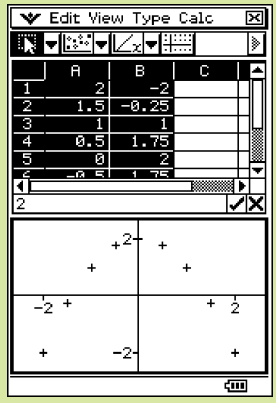


2 Copy the table on the right into your work book. Repeat the process for the other values of b, recording the coordinate pairs for the vertex in the table.

3 Calculate $x = \frac{-b}{2a}$ for each equation. Are these values the same as the x-coordinate of the vertex for the corresponding value of b in the table?

4 Plot these coordinate pairs.

b	*x*-coordinate of vertex	*y*-coordinate of vertex
-4	2	-2
-3		
-2		
-1		
0		
1		
2		
3		
4		

Using the TI-Nspire CAS	Using the ClassPad
Select Lists and spreadsheet and enter the x-coordinate of the vertex into the first column and the y-coordinate into the second column. You will need to name the columns as we are drawing a graph. Then scroll right to the top of A until it is all highlighted then press shift (or CAPS) ▸ to highlight both columns then press menu > Data > Quick Graph. 	Select Spreadsheet from the Menu and enter the x-coordinate of the vertex into column A and the y-coordinate into column B. Highlight both columns and tap Graph > Scatter. A scatter graph will appear. 

5 What kind of relationship would pass through this set of points? Try to guess the equation for the relationship. Write your guess in your work book. Your CAS can find this relationship for you.

Using the TI-Nspire CAS	Using the ClassPad
With the graph highlighted, press menu > Analyze > Regression and choose the type of relationship you think is correct. If you are correct the graph will pass through the centre of each point. As an added bonus you will be given the rule for the relationship.	With the columns still highlighted, tap Calc > Regressions. Test any of the regressions listed. A list of coefficients of the powers of x will appear and a graph will be drawn. Find the one where the graph goes through all given points and $a \neq 0$. You will see that $r^2 = 1$, which means it is a perfect fit. This will give you the required equation. Is this the same as your guess?

6 Repeat steps **1**–**3** using:

(a) $y = 2x^2 + bx - 5$ **(b)** $y = -x^2 + bx + 4$

Can you see a pattern connecting the original equation and the equation of the path formed by the turning points as the value of b changes?

7 Find the x-coordinate of the vertex of $y = x^2 + bx + 2$. Substitute this x-coordinate into $y = x^2 + bx + 2$ to find the y-coordinate of the vertex. Verify that these coordinates are satisfied by the relationship in Question **1**.

Taking it further

8 Find the turning point of the equation $y = ax^2 + bx + c$. Verify that these coordinates satisfy the equation $y = -ax^2 + c$ to show that this is the path (locus) traced out by the vertices as b varies and a and c are held constant.

9 Investigate the effect on the graph of $y = ax^2 + bx + c$ when:

- c is varied and a, b are held constant
- a is varied and b, c are held constant.

Write a brief report on your investigation and state your conclusions.

9.3 Solving quadratic equations using the Null Factor Law

We often need to find solutions for quadratic equations when sketching graphs and solving problems that involve quadratic relationships. The solution to a quadratic equation can be obtained by plotting a quadratic graph and finding the x-intercepts, but this method is time consuming and inaccurate if the solutions involve fractions or surds. We use algebraic techniques to solve quadratic equations exactly. It should be remembered that not all quadratics cross or touch the x-axis so not all quadratic equations will have solutions.

To solve quadratic equations, we factorise the equations and use the **Null Factor Law**. This law is based on the fact that if the product of two or more expressions is equal to zero, then at least one of the factors must equal zero.

Null Factor Law

If $ab = 0$, then $a = 0$, $b = 0$ or $a = b = 0$.

Worked Example 10 WE10

Solve the following equations for x using the Null Factor Law.

(a) $x(x - 2) = 0$ **(b)** $(x + 1)(x - 6) = 0$

Thinking	Working
(a) 1 As the equation is already factorised use the Null Factor Law to equate each factor to zero.	(a) $x(x - 2) = 0$ $x = 0$ or $x - 2 = 0$
2 Solve each individual equation and state your solutions.	$x = 0$ or $x = 2$
(b) 1 As the equation is already factorised use the Null Factor Law to equate each factor to zero.	(b) $(x + 1)(x - 6) = 0$ $x + 1 = 0$ or $x - 6 = 0$
2 Solve each individual equation and state your solutions.	$x = -1$ or $x = 6$

Worked Example 11

WE11

Solve each equation for x:

(a) $4x^2 = 0$ **(b)** $(x + 1)^2 = 0$

Thinking	Working
(a) 1 Divide both sides by the coefficient of x^2.	(a) $4x^2 = 0$ $x^2 = 0$
2 As the equation is the product of two identical linear factors, equate the linear factor to zero to find the solution.	$x = 0$
(b) 1 As the equation is the product of two identical linear factors, equate the linear factor to zero.	(b) $(x + 1)^2 = 0$ $x + 1 = 0$
2 Solve the equation and state your solution.	$x = -1$

Before we can use the Null Factor Law, we must factorise the quadratic equation.

Algebraic techniques used to factorise quadratic equations in this section include:

- taking out all common factors, including a negative sign if appropriate
- using the Difference of Two Squares rule (two terms in the quadratic expression)
- using the Perfect Square rules
- factorising monic trinomials.

Once we have factorised the quadratic equation, we can apply the Null Factor Law to obtain our solutions. Some quadratics cannot be factorised. These will have no solutions. If the factorised quadratic is a perfect square, there will only be one solution. Otherwise there will be two solutions. The first step is always to look for a common factor.

If the quadratic equation contains a two-term quadratic expression, we first look for a common factor.

Worked Example 12

WE12

Solve each equation for x:

(a) $3x^2 - 6x = 0$ **(b)** $11x^2 = x$

Thinking	Working
(a) 1 Factorise the left-hand side by finding the highest common factor.	(a) $3x^2 - 6x = 0$ $3x(x - 2) = 0$
2 Apply the Null Factor Law.	$3x = 0$ or $x - 2 = 0$
3 Solve each equation.	$x = 0, x = 2$
4 State your solutions.	$x = 0$ or 2

(b)	1	Rearrange the equation to make one side equal to zero (usually the RHS).	(b) $11x^2 = x$ $11x^2 - x = 0$
	2	Factorise the other side of the equation by finding the HCF.	$x(11x - 1) = 0$
	3	Apply the Null Factor Law.	$x = 0$ or $11x - 1 = 0$
	4	Solve each equation.	$x = 0$ or $11x = 1$ $x = 0$ or $x = \frac{1}{11}$
	5	State your solutions.	$x = 0$ or $\frac{1}{11}$

If there is no common factor, we may be able to factorise using the Difference of Two Squares rule: $a^2 - b^2 = (a + b)(a - b)$.

Worked Example 13

WE13

Solve the following equations for x.

(a) $x^2 - 49 = 0$ (b) $4x^2 = 25$ (c) $5x^2 = 3$

		Thinking	Working
(a)	1	Use the Difference of Two Squares rule $a^2 - b^2 = (a + b)(a - b)$.	(a) $x^2 - 49 = 0$ $(x + 7)(x - 7) = 0$
	2	Use the Null Factor Law and equate each factor to zero.	$x + 7 = 0$ or $x - 7 = 0$
	3	Solve each equation.	$x = -7$ or $x = 7$
	4	State your solutions.	$x = \pm 7$
(b)	1	Rearrange the equation so that the RHS = 0.	(b) $4x^2 = 25$ $4x^2 - 25 = 0$
	2	Factorise using the Difference of Two Squares rule $a^2 - b^2 = (a + b)(a - b)$.	$(2x)^2 - (5)^2 = 0$ $(2x + 5)(2x - 5) = 0$
	3	Use the Null Factor Law and equate each factor to zero.	$2x + 5 = 0$ or $2x - 5 = 0$
	4	Solve each equation.	$x = -\frac{5}{2}$ or $x = \frac{5}{2}$
	5	State your solutions.	$x = \pm\frac{5}{2}$
(c)	1	Rearrange the equation so that the RHS = 0.	(c) $5x^2 = 3$ $5x^2 - 3 = 0$
	2	Divide each term by the coefficient of x.	$x^2 - \frac{3}{5} = 0$
	3	Factorise using the Difference of Two Squares rule $a^2 - b^2 = (a + b)(a - b)$.	$(x + \sqrt{\frac{3}{5}})(x - \sqrt{\frac{3}{5}}) = 0$

4	Use the Null Factor Law and equate each factor to zero.	$x + \sqrt{\frac{3}{5}} = 0$ or $x - \sqrt{\frac{3}{5}} = 0$
5	Solve each equation.	$x = -\sqrt{\frac{3}{5}}$ or $x = \sqrt{\frac{3}{5}}$
6	State your solutions.	$x = \pm\sqrt{\frac{3}{5}}$

When the quadratic equation contains a three term quadratic expression, we look for a perfect square after checking for common factors.

Worked Example 14

WE14

Solve the following equations for x.

(a) $x^2 + 10x + 25 = 0$ **(b)** $2x^2 - 36x = -162$

Thinking		Working
(a) 1	Recognise this quadratic as a perfect square in the form of $a^2 + 2ab + b^2$.	(a) $x^2 + 10x + 25 = 0$ $(x + 5)(x + 5) = 0$
2	Factorise using the Perfect Square rule $a^2 + 2ab + b^2 = (a + b)^2$.	$(x + 5)^2 = 0$
3	We have the product of two identical factors so there is only one linear equation to solve.	$x + 5 = 0$
4	Solve the equation.	$x = -5$
(b) 1	Rearrange to make one side equal to zero.	(b) $2x^2 - 36x = -162$ $2x^2 - 36x + 162 = 0$
2	Factorise the left-hand side by: • finding the common factor • using the Perfect Square rule $a^2 - 2ab + b^2 = (a - b)^2$.	$2(x^2 - 18x + 81) = 0$ $2(x - 9)^2 = 0$
3	Divide both sides by the common factor.	$(x - 9)^2 = 0$
4	We have the product of two identical factors so there is only one linear equation to solve.	$x - 9 = 0$
5	Solve the equation.	$x = 9$

If the trinomial expression is not a perfect square, we can use an appropriate algebraic technique to factorise the monic trinomial remembering to take out a common factor first. Taking out a negative coefficient of x^2 often makes it simpler to factorise. We can divide both sides by this coefficient before we factorise the trinomial.

Worked Example 15 WE15

Solve the following quadratic equations for x.

(a) $x^2 - 6x + 8 = 0$ **(b)** $2x^2 + 8x = 24$ **(c)** $-x^2 - 4x - 3 = 0$

Thinking		Working
(a) 1	Factorise the trinomial. (Here, we find the two numbers that: × to give 8, + to give -6 } The numbers are -2 and -4)	(a) $x^2 - 6x + 8 = 0$ $x^2 - 2x - 4x + 8 = 0$ $x(x-2) - 4(x-2) = 0$
2	Write down the factorised form.	$(x-2)(x-4) = 0$
3	Apply the Null Factor Law.	$x - 2 = 0$ or $x - 4 = 0$
4	Solve each equation and state the solution.	$x = 2$ or $x = 4$
(b) 1	Rearrange the equation to make one side equal zero.	(b) $2x^2 + 8x = 24$ $2x^2 + 8x - 24 = 0$
2	Look for a common factor. (Here, it is 2.)	$2(x^2 + 4x - 12) = 0$
3	Divide both sides by the common factor.	$x^2 + 4x - 12 = 0$
4	Factorise the trinomial. (Here, we find the two numbers that: × to give -12, + to give +4 } The numbers are +6 and -2)	$x^2 + 6x - 2x - 12 = 0$ $x(x+6) - 2(x+6) = 0$
5	Write down the factorised form.	$(x+6)(x-2) = 0$
6	Apply the Null Factor Law.	$x + 6 = 0$ or $x - 2 = 0$
7	Solve each equation and state the solution.	$x = -6$ or $x = 2$
(c) 1	Write down the equation.	(c) $-x^2 - 4x - 3 = 0$
2	Look for a common factor. (Here, it is -1.)	$-1(x^2 + 4x + 3) = 0$
3	Divide both sides by the common factor.	$x^2 + 4x + 3 = 0$
4	Factorise the trinomial. (Here, we find the two numbers that: × to give +3, + to give +4 } The numbers are +3 and +1)	$x^2 + x + 3x + 3 = 0$ $x(x+1) + 3(x+1) = 0$
5	Write down the factorised form.	$(x+1)(x+3) = 0$
6	Use the Null Factor Law.	$x + 1 = 0$ or $x + 3 = 0$
7	Solve each equation and state the solution.	$x = -1$ or $x = -3$

These methods can be used to solve quadratic equations:

1 Equate one side (usually RHS) to zero.

2 Factorise the expression by looking for:

- a common factor
- a Difference of Two Squares $a^2 - b^2 = (a + b)(a - b)$
- a Perfect Square $a^2 \pm 2ab + b^2 = (a \pm b)^2$
- a monic trinomial where the middle term can be split so that the coefficients of the two middle terms multiply to give the constant term.

You may need to take out the coefficient of x^2 as a common factor and divide both sides by the common factor before factorising.

9.3 Solving quadratics using the Null Factor Law

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Fluency

1 Solve the following equations for x using the Null Factor Law. WE10

(a) $x(x - 11) = 0$ (b) $x(x + 6) = 0$ (c) $x(x + 4) = 0$

(d) $2x(x - 1) = 0$ (e) $3x(x - 1) = 0$ (f) $-4x(x - 6) = 0$

(g) $(x - 3)(x + 7) = 0$ (h) $(x - 4)(x + 1) = 0$ (i) $(x - 2)(x - 6) = 0$

(j) $(4 - x)(x - 5) = 0$ (k) $(7 - x)(x + 2) = 0$ (l) $(3 - x)(x + 10) = 0$

2 Solve the following quadratic equations for x. WE11

(a) $-6x^2 = 0$ (b) $3x^2 = 0$ (c) $2x^2 = 0$

(d) $(x + 7)^2 = 0$ (e) $(x + 1)^2 = 0$ (f) $(x - 9)^2 = 0$

3 Solve the following equations for x. WE12

(a) $x^2 - 4x = 0$ (b) $x^2 + x = 0$ (c) $x^2 - 6x = 0$

(d) $3x^2 = 15x$ (e) $4x^2 = 2x$ (f) $4x^2 = -12x$

4 Solve the following quadratic equations for x. WE13

(a) $x^2 - 25 = 0$ (b) $x^2 - 81 = 0$ (c) $x^2 - 4 = 0$

(d) $16x^2 - 25 = 0$ (e) $9x^2 - 49 = 0$ (f) $100x^2 - 81 = 0$

(g) $3x^2 - 75 = 0$ (h) $4x^2 - 4 = 0$ (i) $7x^2 - 28 = 0$

(j) $49 - x^2 = 0$ (k) $81 - x^2 = 0$ (l) $121 - x^2 = 0$

(m) $x^2 - 13 = 0$ (n) $x^2 - 24 = 0$ (o) $x^2 - 32 = 0$

WE14

5 Solve the following quadratic equations for x.

(a) $x^2 - 14x + 49 = 0$ (b) $x^2 + 8x + 16 = 0$ (c) $x^2 + 16x + 64 = 0$

(d) $x^2 - 4x + 4 = 0$ (e) $x^2 + 2x + 1 = 0$ (f) $x^2 - 6x + 9 = 0$

(g) $2x^2 - 20x + 50 = 0$ (h) $5x^2 - 10x + 5 = 0$ (i) $6x^2 + 24x + 24 = 0$

(j) $3x^2 - 18x = -27$ (k) $5x^2 - 50x = -125$ (l) $24x - 3x^2 = 48$

WE15

6 Solve the following quadratic equations for x.

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 + 9x + 18 = 0$ (c) $x^2 - 11x + 28 = 0$

(d) $x^2 - 10x + 9 = 0$ (e) $x^2 - x - 30 = 0$ (f) $x^2 + 2x - 8 = 0$

(g) $2x^2 - 24x + 64 = 0$ (h) $2x^2 + 4x - 30 = 0$ (i) $3x^2 - 18x - 48 = 0$

(j) $5x^2 + 20x - 60 = 0$ (k) $4x^2 - 4x - 8 = 0$ (l) $6x^2 - 12x - 18 = 0$

7 Solve the following equations. Rewrite each equation so that the right-hand side equals zero.

(a) $x^2 + 7x = -12$ (b) $x^2 + 8x = -12$ (c) $x^2 - 2x = 63$

(d) $x^2 + 5x = 24$ (e) $-x^2 - 5x = 6$ (f) $-x^2 + 2x = -15$

(g) $x^2 + 5x - 20 = 30$ (h) $x^2 - 6x - 13 = 14$ (i) $x^2 + 66 = 17x$

(j) $x^2 + 5 = 6x$ (k) $-x^2 + 16 = 6x$ (l) $-x^2 + 21 = 4x$

Understanding

8 (a) The solution/s to $x(x - 2) = 0$ is/are:

A $x = 2$ B $x = -2$ C $x = 0, x = -2$ D $x = 0, x = 2$

(b) The solution/s to $(x + 3)(x - 4) = 0$ is/are:

A $x = -3, x = 4$ B $x = 3, x = -4$ C $x = -3, x = -4$ D $x = 3, x = 4$

(c) The solution/s to $x^2 - 12x + 36 = 0$ is/are:

A $x = 0, x = -6$ B $x = -6$ C $x = 6, x = 0$ D $x = 6$

(d) The solution/s to $x^2 - 64 = 0$ is/are:

A $x = 8, x = -8$ B $x = 8$ C $x = 0, x = 8$ D $x = 0, x = -8$

9 Solve the following equations.

(a) $x^2 + 3x = 10$ (b) $x^2 - 2x = 3$ (c) $x^2 - 3x - 18 = 10$

(d) $x^2 + 8x + 12 = 5$ (e) $3x^2 - 28 = 20$ (f) $2x^2 - 15 = 3$

(g) $(x + 4)(x + 3) = 6$ (h) $(x - 5)(x + 6) = 12$ (i) $(x - 6)^2 = 36$

10 (a) The solution/s to $x^2 - 9x + 18 = 0$ is/are:

A $x = 3, x = 6$ B $x = -9$ C $x = 3, x = -6$ D $x = -3, x = -6$

(b) The solution/s to $x^2 + 6x + 5 = 0$ is/are:

A $x = 1, x = 5$ B $x = -1, x = 5$ C $x = 1, x = -5$ D $x = -1, x = -5$

(c) The solution/s to $3x^2 - 12x - 63 = 0$ is/are:

A $x = 3, x = 7$ B $x = -3, x = 7$ C $x = 3$ D $x = 3, x = -7$

(d) The solution/s to $-x^2 + x + 12 = 0$ is/are:

A $x = 3, x = 4$ B $x = -3, x = 4$ C $x = 3, x = -4$ D $x = -3, x = -4$

(e) The solution/s to $x^2 + 3x = 4$ is/are:

A $x = 1, x = 3$ B $x = 1, x = -4$ C $x = 1, x = 4$ D $x = -1, x = 4$

(f) The solution/s to $x^2 - 4x = -4$ is/are:

A $x = 2, x = -2$ B $x = 4, x = -1$ C $x = 2$ D $x = 4, x = 1$

11 The height, h centimetres, of a ball above the ground when it is being bounced is given by $h = t(50 - t)$, where t is the time in seconds. Find the two times (in seconds) when the ball is on the ground, that is, when its height (h) is 0 cm.

12 A graph has the equation $y = (x + 1)(x - 4)$. Find the x-intercepts of this graph.

13 Determine the x-intercepts of the following quadratic relationships.

(a) $y = x^2 - 144$

(b) $y = 6x^2 + 24x$

(c) $y = x^2 - 12x + 36$

Reasoning

14 The temperature, T degrees Celsius, is measured during the day and found to follow the quadratic relationship $T = -t^2 + 10t + 5$, where t represents the time in hours after 8 a.m.

(a) Find the temperature when t is equal to 0.

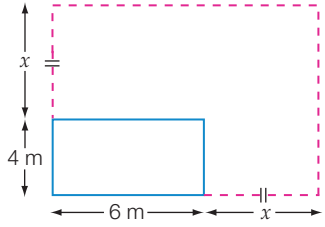
(b) By substituting for T, find the values of t when the temperature is 21°C.

(c) At what times is the temperature 21°C?

15 At the zoo, a wombat enclosure is to be enlarged to accommodate recent arrivals.

(a) Write an expression for the area of the new enclosure if the same length is to be added to two sides, as in the diagram.

(b) The new enclosure is to have double the area of the existing one. Find the dimensions of the new wombat enclosure.



16 Gerald hit his golf ball down the fairway. The path the ball followed was parabolic in shape with the relationship:

$$h = 2d - \frac{1}{30}d^2$$

where h is the height above the ground of the golf ball when it is a horizontal distance d from him. Both h and d are in metres. How far from Gerald did the golf ball land?

17 To determine the number of diagonals (D) in a polygon containing n sides, the following quadratic equation can be used:

$$D = \frac{n}{2}(n - 3)$$

(a) Show that a triangle contains no diagonals.

(b) Show that the given quadratic equation is true for each of these polygons—quadrilateral, pentagon, hexagon—by determining the number of diagonals algebraically and then diagrammatically.

(c) Calculate the number of diagonals in:

(i) an octagon **(ii)** a decagon **(iii)** a dodecagon.

(d) How many sides has a polygon that has:

(i) 170 diagonals? **(ii)** 299 diagonals?

(e) A similar equation can be used to determine the number of consecutive integers, beginning at 1, required to sum to a particular value.

$$S = \frac{n}{2}(n + 1)$$

Use the equation to find the number of integers, beginning at 1, required to sum to:

(i) 105 **(ii)** 903

Open-ended

18 **(a)** Choose pairs of values for x. For each pair, write equations for which these are the solution.

(b) Choose a single value of x. Write down a quadratic equation for which this x value is the only solution.

19 Decide on two values of x that will be the x-intercepts of a quadratic graph. Write down at least two possible equations for the graph.

20 Write three different quadratic equations that have two solutions lying between $x = -3$ and $x = 3$. At least one of the equations should produce a solution containing a surd.

Half-time 9

1 Solve each of the following. Ex. 9.3

(a) $x^2 + 4x = -4$ **(b)** $x^2 - 5x = 6$ **(c)** $x^2 + 8 = 9x$

(d) $x^2 - 4x = 5$ **(e)** $2x^2 + 6x = -4$ **(f)** $3x^2 + 9x = 30$

2 Find the coordinates of the centre and the radius of the circle with equation $(x + 1)^2 + (y - 2)^2 = 4$. Ex. 9.1

3 Sketch each parabola, stating the changes that should be made to the graph $y = x^2$. Ex. 9.2

(a) $y = 2(x + 3)^2 - 4$ **(b)** $y = \frac{1}{2}(x - 1)^2 + 3$ **(c)** $y = (x - 2)^2 - 4$

4 The graph $y = x^2$ is moved 3 units right and 2 units up. What is the equation of the new graph? Ex. 9.2

5 Solve each of the following. Ex. 9.3

(a) $x^2 - 64 = 0$ **(b)** $(x + 1)^2 = 0$ **(c)** $(x - 2)^2 = 4$

6 Plot the graph of: Ex. 9.1

(a) $y = -2^{-x}$ for values of x from -3 to 3 inclusive

(b) $(x - 2)^2 + (y - 3)^2 = 4$, showing the four points $(h + r, k)$, $(h, k + r)$, $(h - r, k)$, $(h, k - r)$ on the circumference of the circle with centre (h, k) and radius r.

7 Solve these quadratic equations. Ex. 9.3

(a) $x^2 + 6x + 5 = 0$ **(b)** $x^2 - 9x = 0$ **(c)** $3x^2 + 9x = 30$

(d) $x^2 + x - 6 = 0$ **(e)** $x^2 - 4x + 3 = 0$ **(f)** $x^2 - 6x + 9 = 0$

8 Consider the equation $y = -x^2 + 1$. Ex. 9.2

(a) Set up a table of values using integer values of x from -3 to 3 inclusive.

(b) Plot the points on a number plane and join them with a smooth curve.

(c) Find the coordinates of the turning point from your graph.

(d) Find the equation of the axis of symmetry.

9 **(a)** Plot the graph of $y = 2^{-x} - 1$ for values of x from -3 to 3 inclusive. Ex. 9.1

(b) What is the equation of the asymptote for the graph?

(c) What are the coordinates of the y-intercept?

10 For the non-linear equation $y = \frac{1}{x - 3} + 4$: Ex. 9.1 10A

(a) What name is given to a graph of this type?

(b) Write down the equations of the x- and y-asymptotes.

9.4 Solving quadratics by completing the square

In the previous section, it was necessary to factorise the quadratic equations before solving using the Null Factor Law. If a quadratic cannot be factorised easily then you could complete the square and then try to factorise using the Difference of Two Squares. Remember that not all quadratics will factorise.

The process of completing the square involves producing a perfect square. This process was studied in Section 2.4.

Once the expression has been factorised by completing the square then we can use the Null Factor Law to solve.

Worked Example 16 WE16

For each of the following expressions:

(a) $x^2 - 6x + 3$ **(b)** $x^2 + 5x - 2$

(i) Complete the square and factorise.

(ii) Equate the expression to zero and use your answer for part **(i)** to solve the quadratic equation formed.

Thinking	Working
(a) (i) 1 Write the expression with descending powers of the variable. Use the first two terms to create a perfect square. The term that is added and subtracted is the square of half the coefficient of the second term.	(a) (i) $x^2 - 6x + 3$ $= [x^2 - 6x + \left(\frac{6}{2}\right)^2] - \left(\frac{6}{2}\right)^2 + 3$ $= (x^2 - 6x + 3^2) - 9 + 3$
2 Factorise the perfect square and simplify the constant term.	$= (x - 3)^2 - 6$
3 Rewrite the expression with the constant term as a square.	$= (x - 3)^2 - (\sqrt{6})^2$
4 Factorise the expression using the Difference of Two Squares rule.	$= (x - 3 + \sqrt{6})(x - 3 - \sqrt{6})$
(ii) 1 Equate the original expression to zero.	(ii) $x^2 - 6x + 3 = 0$

	2	Use the factorised expression obtained in part **(i)** to rewrite the equation.	$(x-3+\sqrt{6})(x-3-\sqrt{6})=0$
	3	Use the Null Factor Law.	$(x-3+\sqrt{6})=0$ or $(x-3-\sqrt{6})=0$
	4	Solve for x.	$x=3-\sqrt{6}$ or $x=3+\sqrt{6}$
	5	State the solution.	$x=3\pm\sqrt{6}$
(b) (i)	1	Write the expression. Use the first two terms to create a perfect square. The term that is added and subtracted is the square of half the coefficient of the x term.	(b) (i) x^2+5x-2 $=[x^2+5x+\left(\frac{5}{2}\right)^2]-\left(\frac{5}{2}\right)^2-2$ $=(x+\frac{5}{2})^2-\frac{25}{4}-2$
	2	Factorise the perfect square and simplify the constant term.	$=(x+\frac{5}{2})^2-\frac{33}{4}$
	3	Rewrite the expression with the constant term as a square.	$=(x+\frac{5}{2})^2-\left(\sqrt{\frac{33}{4}}\right)^2$
	4	Factorise the expression using the Difference of Two Squares rule.	$=(x+\frac{5}{2}-\sqrt{\frac{33}{4}})(x+\frac{5}{2}+\sqrt{\frac{33}{4}})$
(ii)	1	Equate the original expression to zero.	(ii) $x^2+5x-2=0$
	2	Use the factorised expression obtained in part **(i)** to rewrite the equation.	$(x+\frac{5}{2}-\sqrt{\frac{33}{4}})(x+\frac{5}{2}+\sqrt{\frac{33}{4}})=0$
	3	Use the Null Factor Law.	$(x+\frac{5}{2}-\sqrt{\frac{33}{4}})=0$ or $(x+\frac{5}{2}+\sqrt{\frac{33}{4}})=0$
	4	Solve for x and simplify your answer.	$x=-\frac{5}{2}+\sqrt{\frac{33}{4}}$ or $x=-\frac{5}{2}-\sqrt{\frac{33}{4}}$
	5	State the solution.	$x=\frac{-5}{2}\pm\sqrt{\frac{33}{4}}$ or $x=\frac{-5\pm\sqrt{33}}{2}$

Worked Example 17

WE17

Solve $x^2+6x-2=0$ using the method of completing the square. Leave your answer in exact surd form.

Thinking	Working
1 Write down the equation with descending powers of the variable.	$x^2+6x-2=0$

2 Factorise the non-zero expression by completing the square. Use the first two terms to create a perfect square. The term that is subtracted is the square of half the coefficient of the x term.	$[x^2 + 6x + \left(\frac{6}{2}\right)^2] - \left(\frac{6}{2}\right)^2 - 2 = 0$ $(x^2 + 6x + 9) - 9 - 2 = 0$
3 Factorise the perfect square and simplify the constant term.	$(x+3)^2 - 11 = 0$
4 Factorise the expression using the Difference of Two Squares rule.	$(x+3)^2 - (\sqrt{11})^2 = 0$ $(x+3-\sqrt{11})(x+3+\sqrt{11}) = 0$
5 Use the Null Factor Law.	$x+3-\sqrt{11} = 0$ or $x+3+\sqrt{11} = 0$
6 Solve the individual equations.	$x = -3+\sqrt{11}$ or $x = -3-\sqrt{11}$
7 State the solution.	$x = -3 \pm \sqrt{11}$

If there is a common factor in the quadratic equation, we can take it out and divide both sides by it first to form a simpler equivalent equation before we complete the square. For example:

$$2x^2 + 4x - 6 = 0$$
$$2(x^2 + 2x - 3) = 0$$
$$x^2 + 2x - 3 = 0$$

9.4 Solving quadratics by completing the square

Navigator

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Fluency

WE16

1 For each of the following expressions:

(a) $x^2 + 2x - 5$ **(b)** $x^2 + 4x - 7$ **(c)** $x^2 + 8x + 5$

(d) $x^2 + 3x - 1$ **(e)** $x^2 + 5x + 3$ **(f)** $x^2 + 3x - 5$

(i) Complete the square and factorise.

(ii) Equate the expression to zero and use your answer for part **(i)** to solve the quadratic equation formed.

WE17

2 Solve these quadratic equations using the method of completing the square. Leave your answers in exact surd form.

(a) $x^2 - 8x + 6 = 0$ **(b)** $x^2 + 6x + 4 = 0$ **(c)** $x^2 - 3x + 1 = 0$

(d) $x^2 - 5x - 2 = 0$ **(e)** $x^2 + 4x - 3 = 0$ **(f)** $x^2 - 20x - 12 = 0$

(g) $2x^2 - 8x - 10 = 0$ **(h)** $3x^2 + 12x + 3 = 0$ **(i)** $4x^2 + 16x - 20 = 0$

3 (a) The solution to $2x^2 + 4x - 4 = 0$ is:

A $x = 1 \pm \sqrt{3}$ B $x = -2$ or 4 C $x = -1 \pm \sqrt{3}$ D $x = -4$ or 2

(b) The solution to $x^2 - 4x - 3 = 0$ is:

A $x = 4$ or -1 B $x = 2 \pm \sqrt{7}$ D $x = -2 \pm \sqrt{7}$ E $x = -4$ or 1

(c) The solution to $x^2 - 8x + 14 = 0$ is:

A $x = -4$ or $-\sqrt{2}$ B $x = 4\sqrt{2}$ or $-4\sqrt{2}$

C $x = -4 - \sqrt{2}$ or $-4 + \sqrt{2}$ D $x = 4 + \sqrt{2}$ or $4 - \sqrt{2}$

Understanding

4 Solve the following quadratic equations.

(a) $x^2 - 4x = 2$ (b) $x^2 = 6x - 2$ (c) $3 + 4x = x^2$

(d) $x^2 = 5 + 3x$ (e) $x^2 - 5x = -2$ (f) $x^2 = 5 - 7x$

5 (a) Explain why the quadratic equation $x^2 + 4x + 7$ cannot be factorised.

(b) Explain why the quadratic equation $x^2 + 4x + 7 = 0$ has no solution.

(c) As $x^2 + 4x + 7 = 0$ has no solution, what does this tell you about the graph of $y = x^2 + 4x + 7$?

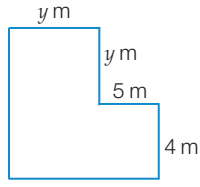
Reasoning

6 An envelope has an area of 80 cm^2. If the length is 4 cm longer than the width, find the dimensions of the envelope:

(a) exactly

(b) rounded to one decimal place.

7 An area of lawn as shown in the diagram is to be mown. Find the value y represents if the total area mown is 44 m^2.



8 If the dimensions of a rectangular park are $(x + 4)$ m and $(x - 5)$ m:

(a) find the exact value of x if the area of the park is 180 m^2

(b) find the value of x rounded to two decimal places.

9 A photo of length l cm and width $(l - 2)$ cm is surrounded by a 2 cm rectangular frame. Altogether, the photo and the frame occupy an area of 215 cm^2.

(a) Write the dimensions of the photo and frame in terms of l.

(b) Write an equation in terms of l for the area of the photo and frame.

(c) Find the exact dimensions of the photo.

(d) Find the dimensions of the photo and frame correct to one decimal place.

Open-ended

10 Write a quadratic equation that gives solutions containing surds. Solve your equation.

11 Write a quadratic equation that has no solution and demonstrate how you know this.

12 By using the process of completing the square, the solution $x = -a \pm \sqrt{3}$ was obtained. By choosing a value for a between 0 and 5, write a possible quadratic equation that could have this solution.

9.5 Sketching parabolas

The graph of a quadratic equation in the form of $ax^2 + bx + c = 0$ can be plotted using a table of values. However, a sketch graph shows us the general shape of the graph and its important features without having to complete a table of values. A sketch graph is not drawn on graph paper but needs to be neat and show the main features as described below.

In this section, we will sketch graphs of $y = ax^2 + bx + c$, where $a = \pm 1$. To make a sketch of a quadratic graph, you will need to identify the following:

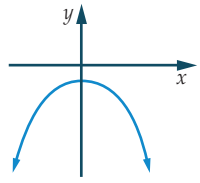
1 The shape of the graph

- A quadratic equation with a positive coefficient of x^2 will produce a graph that is a positive parabola.

- A quadratic equation with a negative coefficient of x^2 will produce a graph that is a negative parabola.

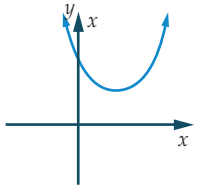
2 The y-intercept

This is where the graph crosses the y-axis and so $x = 0$. It is found by substituting $x = 0$ into the quadratic equation. There is always a y-intercept.

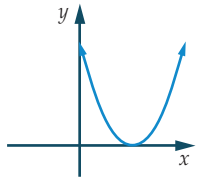


3 The x-intercept(s)

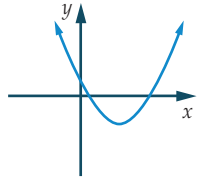
This is where the graph crosses the x-axis and so $y = 0$. There may be no x-intercepts, one x-intercept or two x-intercepts.



no x-intercepts



one x-intercept



two x-intercepts

x-intercepts are found by equating the quadratic to zero and solving.

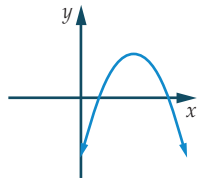
- If there is no solution, there are no x-intercepts.
- If there is one solution, the parabola touches the x-axis. There is only one x-intercept.
- If there are two solutions, there are two distinct x-intercepts. These may involve surds.

4 The turning point

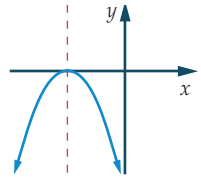
- The turning point can be found by completing the square to transform the equation into **turning point form**.

The graph of $y = (x - h)^2 + k$ has a turning point at (h, k).

- When a graph has two x-intercepts the x-coordinate of the turning point is halfway between the two x-intercepts. The corresponding y-coordinate can be found by substituting the x-value into the quadratic equation.



- When a graph has only one x-intercept, the turning point is the x-intercept.



The axis of symmetry is a vertical line that passes through the turning point, dividing the graph into mirror image parts.

Worked Example 18 WE18

Sketch the graph of the quadratic relationship $y = x^2 + 2x - 8$ by finding:

(a) the shape of the graph

(b) the y-intercept

(c) the x-intercepts

(d) the turning point

(e) the axis of symmetry.

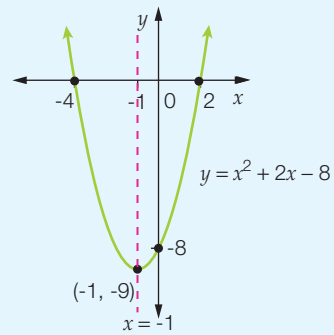
Thinking	Working
(a) Determine the shape of the graph by identifying whether the coefficient of x^2 is positive or negative.	(a) This is a ∪ shaped parabola as the coefficient of x^2 is positive.
(b) Find the y-intercept. Substitute 0 for x in the equation and solve.	(b) When $x = 0$, $y = 0^2 + 2 \times 0 - 8$ $y = 0 + 0 - 8$ $y = -8$ The y-intercept is $(0, -8)$.
(c) Find the x-intercepts. Substitute 0 for y in the equation. Swap sides if it makes the equation easier to solve. Factorise. Use the Null Factor Law. Solve the equations.	(c) When $y = 0$, $0 = x^2 + 2x - 8$ $x^2 + 2x - 8 = 0$ $(x + 4)(x - 2) = 0$ $x + 4 = 0$ or $x - 2 = 0$ $x = -4$ or $x = 2$ The x-intercepts are $(-4, 0)$ and $(2, 0)$.
(d) Find the x-coordinate of the turning point by finding the midpoints of the x-intercepts. Find the y-coordinate of the turning point by substituting the x-coordinate into the equation. Write the coordinates of the turning point. Alternatively, complete the square to obtain turning point form.	(d) For the turning point, $x = \frac{-4 + 2}{2}$ $= -1$ When $x = -1$ $y = (-1)^2 + 2 \times -1 - 8$ $y = 1 - 2 - 8$ $y = -9$ The turning point is at $(-1, -9)$. $y = x^2 + 2x - 8$ $y = x^2 + 2x + 1 - 8 - 1$ $y = (x + 1)^2 - 9$ The turning point is at $(-1, -9)$.

(e) Draw and label axes.

Label the intercepts and turning point. (*Note:* No other values need to be shown on the axes.)

Draw a smooth symmetrical curve for the parabola. Arrows show that the curve continues. Label the graph. Mark in the axis of symmetry with a dotted line and label it.

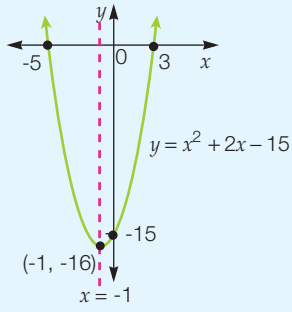
(e)



Worked Example 19

WE19

Sketch the parabola $y = x^2 + 2x - 15$, showing the x- and y-intercepts, turning point and the axis of symmetry. Find the coordinates of the turning point by transforming the equation into turning point form.

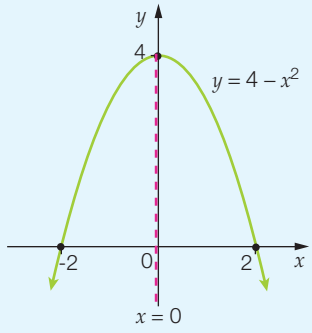
Thinking	Working
1 State the equation.	$y = x^2 + 2x - 15$
2 Determine the shape of the graph by identifying whether the coefficient of x^2 is positive or negative.	This is a ∪ shaped parabola as the coefficient of x^2 is positive.
3 Find the y-intercept by substituting $x = 0$ into the equation.	When $x = 0$, $y = 0 + 0 - 15$ $= -15$ The y-intercept is (0, -15).
4 Find the x-intercepts by substituting $y = 0$ into the equation.	When $y = 0$, $0 = x^2 + 2x - 15$ $x^2 + 2x - 15 = 0$ $(x + 5)(x - 3) = 0$
Factorise.	$x + 5 = 0$ or $x - 3 = 0$
Use the Null Factor Law.	$x = -5$ or $x = 3$
Solve the individual equations.	The x-intercepts are (-5, 0) and (3, 0).
5 Find the turning point.	
Transform the equation into turning point form by completing the square on the first two terms.	$y = x^2 + 2x - 15$ $y = (x^2 + 2x + 1) - 1 - 15$ $y = (x + 1)^2 - 16$
Write the coordinates of the turning point.	The turning point is at (-1, -16).
6 Rule and label the x- and y-axes.	
7 Mark in the known points.	
8 Draw a smooth curve through them to obtain a parabola. (Check that $y = x^2$ has been moved 1 unit left and 16 units down.)	
9 Mark in the axis of symmetry with a dotted line and label it.	

This method is useful when there are no x-intercepts. For example, $y = (x + 1)^2 + 2$.

This graph has no x-intercepts but has a turning point at (-1, 2).

Worked Example 20 WE20

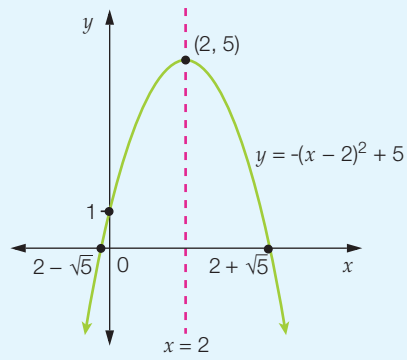
Sketch the graph of the equation $y = 4 - x^2$ showing all the key features.

Thinking	Working
1 State the equation.	$y = 4 - x^2$
2 Determine the shape of the graph by identifying whether the coefficient of x^2 is positive or negative.	This is a ∩ shaped parabola as the coefficient of x^2 is negative.
3 Find the y-intercept by substituting $x = 0$ into the equation.	When $x = 0$, $y = 4 - 0^2$ $y = 4$ The y-intercept is (0, 4).
4 Find the x-intercepts by substituting $y = 0$ into the equation.	When $y = 0$, $0 = 4 - x^2$ $4 - x^2 = 0$
Factorise using Difference of Two Squares.	$(2 + x)(2 - x) = 0$
Solve using the Null Factor Law.	$2 + x = 0$ or $2 - x = 0$ $x = \pm 2$ The x-intercepts are (-2, 0) and (2, 0).
5 Find the x-coordinate of the turning point by finding the midpoint of the x-intercepts.	For the turning point: $x = \frac{-2 + 2}{2}$ $= 0$
Find the corresponding y-value.	When $x = 0$, $y = 4 - 0$ $y = 4$
Write the coordinates of the turning point.	The turning point is at (0, 4).
6 Sketch the graph. (Notice that the turning point is at the same point as the y-intercept for this graph.)	
7 Mark in the axis of symmetry with a dotted line and label it.	

Worked Example 21

WE 21

Sketch the graph of the equation $y = -(x - 2)^2 + 5$, showing all the key features.

Thinking	Working
1 Write down the equation.	$y = -(x - 2)^2 + 5$
2 Compare the equation to $y = (x - h)^2 + k$ to find (h, k), the turning point. $y = -1(x - 2)^2 + 5$ inverted (no dilation); 2; 5	The graph of $y = x^2$ needs to be inverted and translated 2 units right and 5 units up. $\therefore h = 2$ and $k = 5$ The turning point is at $(2, 5)$.
3 Find the y-intercept by substituting $x = 0$ into the equation.	When $x = 0$, $y = -(0 - 2)^2 + 5$ $= -(4) + 5$ $= 1$ The y-intercept is $(0, 1)$.
4 Find the x-intercepts by substituting $y = 0$ into the equation. Solve the equation.	When $y = 0$, $0 = -(x - 2)^2 + 5$ $(x - 2)^2 - 5 = 0$ $(x - 2 + \sqrt{5})(x - 2 - \sqrt{5}) = 0$ $x = 2 - \sqrt{5}$ or $x = 2 + \sqrt{5}$ $x = 2 \pm \sqrt{5}$ The x-intercepts are $(2 - \sqrt{5}, 0)$ and $(2 + \sqrt{5}, 0)$.
5 Use a calculator to give approximate values for x. This may help to sketch the parabola.	$x \approx 4.2$ and -0.2
6 Rule and label the x- and y-axes. **7** Mark in the known points using exact values. **8** Draw a smooth curve through them to produce a parabola. (Check that your result is consistent with inverting $y = x^2$ and moving it 2 units right and 5 units up.) **9** Mark in the axis of symmetry with a dotted line and label it.	

Key features of a parabola

When sketching a parabola, the key features to show are:

- the y-intercept
- the x-intercepts, if they exist
- the turning point, or vertex
- the axis of symmetry.

9.5 Sketching parabolas

Navigator

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4 Column 1, Q5 Column 1, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q16, Q17	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4 Column 2, Q5 Column 2, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q16, Q17	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4 Column 3, Q5 Column 3, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q18

Answers page 786

Fluency

1 Sketch the graph of each of the following quadratic relationships by finding **(i)** the shape of the graph, **(ii)** the y-intercept, **(iii)** the x-intercepts, **(iv)** the turning point and **(v)** the axis of symmetry. WE18

(a) $y = x^2 - 8x + 15$ **(b)** $y = x^2 - 6x + 8$ **(c)** $y = x^2 - 10x + 21$

(d) $y = -x^2 + 4x + 12$ **(e)** $y = -x^2 + 2x + 3$ **(f)** $y = -x^2 + 2x + 8$

(g) $y = x^2 - 8x + 12$ **(h)** $y = x^2 + 2x - 3$ **(i)** $y = x^2 + 4x - 5$

2 Sketch the following parabolas showing the x- and y-intercepts, the turning point and the axis of symmetry. Find the coordinates of the turning point by transforming the equations into turning point form. WE19

(a) $y = x^2 + 2x - 8$ **(b)** $y = x^2 + 8x + 15$ **(c)** $y = x^2 + 3x + 2$

(d) $y = x^2 - 4$ **(e)** $y = 9 - x^2$ **(f)** $y = x^2 - 7$

(g) $y = x^2 - 2x$ **(h)** $y = x^2 + 3x$ **(i)** $y = x^2 + 2x$

3 Sketch the graphs of the following equations showing all the key features. WE20

(a) $y = x^2 - 25$ **(b)** $y = x^2 - 9$ **(c)** $y = x^2 - 16$

(d) $y = 81 - x^2$ **(e)** $y = 4 - x^2$ **(f)** $y = 1 - x^2$

(g) $y = x^2 - 4x$ **(h)** $y = x^2 + 6x$ **(i)** $y = 3x^2 + 6x$

4 Sketch the graphs of the following equations showing all the key features. WE21

(a) $y = (x + 2)^2 - 9$ **(b)** $y = (x + 5)^2 - 1$ **(c)** $y = -(x + 1)^2 + 5$

(d) $y = (x - 3)^2 + 2$ **(e)** $y = (x - 2)^2 + 4$ **(f)** $y = (x - 4)^2 + 2$

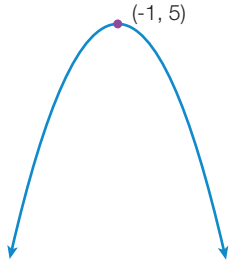
5 Sketch the graphs of the following equations showing all the key features.

(a) $y = x^2 + 10x + 16$ **(b)** $y = x^2 + 8x + 7$ **(c)** $y = -x^2 - 6x - 5$

(d) $y = -x^2 + 4x - 3$ **(e)** $y = x^2 + 6x + 9$ **(f)** $y = x^2 + 10x + 25$

(g) $y = -x^2 + 3x$ **(h)** $y = -x^2 - 5x$ **(i)** $y = -x^2 + 7x - 12$

6 When expressed graphically, which of the following equations would have a shape like that of the parabola shown on the right?



A $y = x^2 + 2x - 5$ **B** $y = -x^2 - 2x + 4$

C $y = 1 - x^2$ **D** $y = 6 - 2x - x^2$

7 **(a)** The coordinates of the y-intercept of $y = x^2 + 4x + 3$ are:

A (0, 8) **B** (-1, 0) **C** (3, 0) **D** (0, 3)

(b) The coordinates of the x-intercepts of $y = x^2 - 9$ are:

A (0, -3), (0, 3) **B** (-3, 0), (3, 0) **C** (-9, 0), (9, 0) **D** (0, -9), (0, 9)

8 (a) The x-intercepts of the parabola $y = x^2 + 5x - 6$ are:

A -2 and 3 B -3 and 2 C -6 and 1 D -1 and 6

(b) If the x-intercepts of a parabola are -1 and 5, the turning point could be:

A (3, 7) B (2, 4) C (1, -6) D (-2, 2)

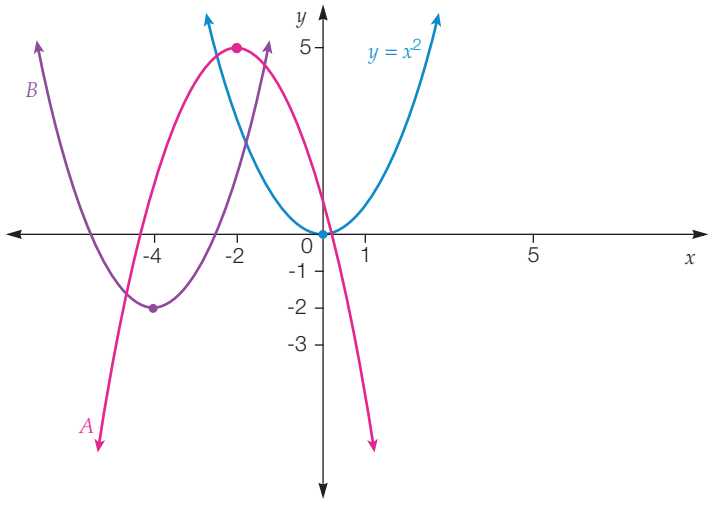
Understanding

9 A ball is thrown into the air and follows a path given by the relationship $h = -t^2 + 10t$, where h is the height of the ball in metres after t seconds of flight. Sketch the graph of the relationship. Use the sketch graph to find:

(a) the total time the ball was in flight

(b) the maximum height the ball reached.

10



(a) The equation of parabola A is:

A $y = (x - 2)^2 + 5$ B $y = -(x + 2)^2 + 5$

C $y = -(x - 2)^2 + 5$ D $y = -(x + 2)^2 - 5$

(b) The equation of parabola B is:

A $y = (x - 4)^2 - 2$ B $y = (x + 4)^2 - 2$

C $y = (x - 4)^2 + 2$ D $y = (x + 4)^2 + 2$

11 (a) In turning point form, the quadratic equation $y = x^2 - 8x + 18$ can be written as:

A $y = (x - 4)^2 + 34$ B $y = (x + 4)^2 + 2$

C $y = (x - 4)^2 - 2$ D $y = (x - 4)^2 + 2$

(b) Hence, write down the coordinates of the turning point for the parabola $y = x^2 - 8x + 18$.

12 (a) Sketch the graph of the equation $y = x^2 - 6x + 5$ showing all its key features.

(b) Use the graph to find the solution to the equation $x^2 - 6x - 7 = 0$.

Hint

What can you add to -7 to equal 5?

13 Over a period of 8 hours, the temperature of a room follows the relationship $T = h^2 - 6h + 15$, where T is the temperature in degrees Celsius, h hours after commencing the experiment.

(a) Express the equation in turning point form and hence sketch the graph of this quadratic relationship.

(b) What is the initial temperature?

(c) After 2 hours, is the temperature of the room increasing or decreasing?

(d) After 6 hours, is the temperature of the room increasing or decreasing?

(e) Find the minimum temperature.

(f) When did the minimum temperature occur?

(g) Would it be realistic to use the equation over a 24-hour period? Explain why or why not.

Reasoning

14 Determine the equation of a graph that has x-intercepts of -3 and 5 and a coefficient of x^2 equal to 1.

15 A rectangular area of land is to be fenced off using 44 m of continuous fencing.

(a) Write an equation for the area enclosed by the fencing.
(Let x = length of one side of the rectangle.)

(b) Sketch a graph of this quadratic relationship showing all key features.

(c) What is the smallest value x can be?

(d) What is the largest value x can be?

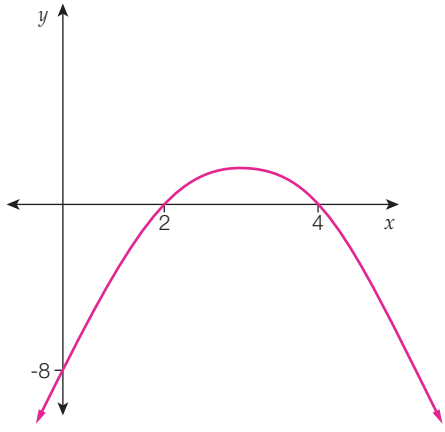
(e) What is the maximum area that can be enclosed by this fencing?

(f) What should the dimensions of the rectangle be for the enclosed area to be as big as possible?

Open-ended

16 Explain with examples how you could determine if x-intercepts will exist for a quadratic graph.

17 Archie was graphing $y = x^2 - 2x - 8$. He drew the graph shown below.



(a) Explain why he is wrong and how he could have avoided these mistakes.

(b) Draw the correct graph.

18 Write down the equations of three parabolas that have $x = 5$ as an axis of symmetry. For each parabola, find the turning point.

Maths 4 Real

What goes up must come down

Isaac Newton is attributed with the famous saying: "What goes up must come down". This oft-quoted expression can be used for the football team presently at the top of the ladder but more appropriately refers to the force of gravity that causes objects that are thrown up in the air to come down.

The movement of objects that have been lauched into the air is called 'projectile motion'. Because of gravity, the object that has been thrown, the 'projectile', takes a curved path through the air. The shape of the curve is a parabola, which can be described by a quadratic equation.

Daredevil motorcross rider Robbie Knievel needs to have a good understanding of projectile motion because he is the projectile and he wants to land in the right place. Suppose Robbie launches his bike upwards off a tower 58.8 m high, with a vertical velocity of 19.6 m/s, the height (h metres) of the bike above the ground at time (t seconds) after it has been launched is given by:

$$h = -4.9t^2 + 19.6t + 58.8$$

The height at ground level will be 0 m.

1 When is the bike at ground level?

2 There are two theoretically possible answers to Question 1. Why is one of them not applicable in this situation?

3 Draw a graph of the equation:

$$h = -4.9t^2 + 19.6t + 58.8$$

Label the part of the graph that shows the ground, the height of the tower, the highest point reached by the bike, and the point at which it lands.

The equation $h = -4.9t^2 + 19.6t + 58.8$ has the form of all quadratic equations: $y = ax^2 + bx + c$. Here, $a = -4.9$, which is half of the acceleration due to the force of gravity, 9.8 m/s^2 and negative as he initially travels upwards.

4 What factors affecting the path of the bike do b and c represent?

5 **(a)** What general equation could you use to describe this motion? (Remember that the value for the effect of gravity is the same, $a = -4.9$.)

(b) What is the maximum height that is attained and at what time after launch is this reached?

When an object is in the air, we may need to know how long it is above a certain height.

6 An object is launched from ground level directly upward at 39.2 m/s. For how long is the object at or above a height of 34.3 m? Form an equation and solve it for $h = 34.3$. Use the solutions to answer the question.

Compare motion

Sometimes two objects can be launched different ways at the same time and we want to determine when they will land. An example of this might be if identical objects are thrown from a building 53 m high. The first is simply dropped (there is no launching velocity) and the other is thrown down with a vertical velocity of 16 m/s. Which one will hit the ground first and by what period of time?

The equation for the object that is dropped would be:

$$h_1 = -4.9t^2 + 53$$

The equation for the second object that is thrown down at 16 m/s would be:

$$h_2 = -4.9t^2 - 16t + 53$$

These two equations could be graphed using technology and the difference in their t-values when their height is zero could be found. Alternatively, you could find the time when each one lands on the ground and find the difference between the two times.

7 Using either of the above methods, determine the difference in the times taken to reach the ground for the two objects described above.

Technology Exploration Excel

Equipment required: 1 brain, a computer with Excel

Versions of this Exploration for other technologies are available in Pearson Reader.

Solving quadratics using iteration

Iteration is a process that involves repeating a series of steps, using the 'answer' from one step to get an improved 'answer' at the next step. As a process it is ideally suited to use in a spreadsheet such as Excel. We will use quadratics to demonstrate the process.

If we wish to find the x-intercepts for $y = x^2 - 4x + 2$ then we need to find the solutions to the quadratic equation $x^2 - 4x + 2 = 0$. By completing the square, we obtain the solutions $x = 2 \pm \sqrt{2}$.

Rounded to eight decimal places the solutions are

$x = 3.41421356$ and $x = 0.58578644$.

We will use these solutions to help us verify the iteration process.

1 Transform the equation so that the term containing x is on the LHS.

$x^2 - 4x + 2 = 0.$

$4x = x^2 + 2$

2 Divide both sides by the coefficient of x.

$x = \dfrac{x^2 + 2}{4}$

Set up a spreadsheet as shown.

B2 =(A2^2+2)/4

	A	B	C	D	E
1	Value	Expression	Difference		
2	1	0.75	0.25		
3	0.75				
4					

Look at the formula bar to see how to enter the expression. Also note that the value of B2 has been repeated in A3. To do this, in A3, type =B2. The formula for cell C2 is just =A2-B2.

3 The difference column has been included so we know when we have found a solution. What value will you expect in the difference column when a solution has been found?

We are going to copy the cells down but we need to even up the rows first. Highlight cells B2 and C2 and put the cursor in the bottom right-hand corner where it will become a small black cross. Drag down to cover B3 and C3.

Highlight A3, B3 and C3 and repeat this process dragging down to say row 25.

4 Have you reached a solution yet? Your spreadsheet should look something like the one shown at right.

A3

	A	B	C
1	Value	Expression	Difference
2	1	0.75	0.25
3	0.75	0.640625	0.109375
4	0.640625	0.6026001	0.0380249
5	0.6026	0.5907817	0.0118184
6	0.590782	0.5872558	0.003526
7	0.587256	0.5862173	0.0010384
8	0.586217	0.5859127	0.0003046
9	0.585913	0.5858234	8.927E-05
10	0.585823	0.5857973	2.615E-05
11	0.585797	0.5857896	7.66E-06
12	0.58579	0.5857874	2.243E-06
13	0.585787	0.5857867	6.571E-07
14	0.585787	0.5857865	1.925E-07
15	0.585787	0.5857865	5.637E-08
16	0.585786	0.5857864	1.651E-08
17	0.585786	0.5857864	4.836E-09
18	0.585786	0.5857864	1.416E-09
19	0.585786	0.5857864	4.148E-10
20	0.585786	0.5857864	1.215E-10
21	0.585786	0.5857864	3.559E-11
22	0.585786	0.5857864	1.042E-11
23	0.585786	0.5857864	3.053E-12
24	0.585786	0.5857864	8.942E-13
25			

You should recognise the number in the Value column and the Expression column as being one of the solutions found by completing the square. But what about the value in the Difference column? Didn't we expect this to be 0 when we had a solution?

5 Explain what 8.942E-13 means.

We can see that the iteration has got closer and closer to the solution at each step. This iteration is said to converge.

6 Replace the 1 in A2 with 5 and see what happens. This iteration is said to diverge, and does not lead us to a solution.

We now need to find the other solution. However, using this transformation, we cannot find it no matter what value we enter. We either converge to the solution 0.585 786 or diverge. We can explore other ways of transposing the original equation.

Taking it further

7 Other expressions that are equivalent to x are:

$\dfrac{2}{4-x}, \dfrac{4x-2}{x}, \pm\sqrt{4x-2}.$

(a) Show how each can be derived.

(b) Try these expressions in your spreadsheet.

(c) Do any of them give the other solution?

(d) Do any of them give both solutions?

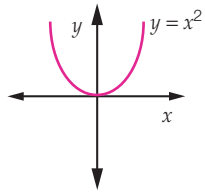
Sketch graphs of other non-linear relationships

9.6

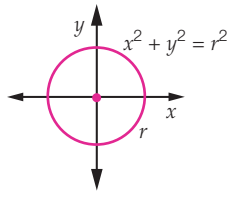
The equations of all relationships are transformed in the same way as quadratics by translations and dilations. The graphs can then be sketched easily if the basic shape and key features are recognised.

In reverse, if a graph is sketched, its equation can be found by recognising the basic shape and then writing the basic equation with the noted translations and/or dilations.

Quadratics

Basic equation $y = x^2$	Basic shape is a parabola. 
Translated equation $y = (x - h)^2 + k$	Basic shape is moved h units to right and k units up.
Dilated equation $y = ax^2$	If $a < -1$ or $a > 1$, then the parabola is narrower than $y = x^2$. If $-1 < a < 1$, then the parabola is wider than $y = x^2$. If the equation has a negative coefficient, $a < 0$, then the basic shape of the parabola is reflected in the x-axis.

Circles

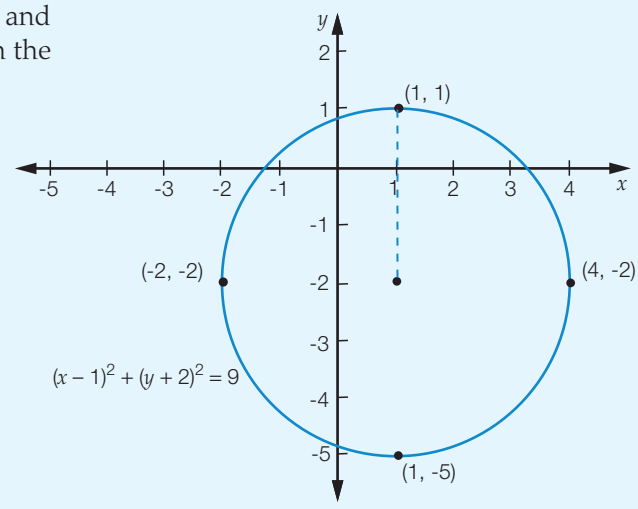
Basic equation $x^2 + y^2 = r^2$	Basic shape is a circle. The centre is (0, 0) and radius is r. 
Translated equation $(x - h)^2 + (y - k)^2 = r^2$	Basic shape is moved h units to right and k units up. The centre is (h, k) and the radius is r.
Dilated equation $ax^2 + ay^2 = r^2$	The radius of the circle is changed. Note that the dilation must be the same for x and y. The centre is (0, 0) and the radius is $\frac{r}{\sqrt{a}}$.

x- and y-intercepts can be difficult to find by hand. A CAS can help to find these values.

Dilation of a circle must happen in both directions at the same time or it doesn't remain a circle. A circle is dilated by changing the radius.

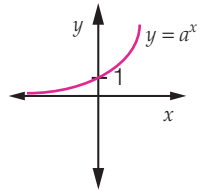
Worked Example 22 WE22

Sketch the graph of $(x-1)^2+(y+2)^2=9$ clearly showing the centre of the circle and the radius.

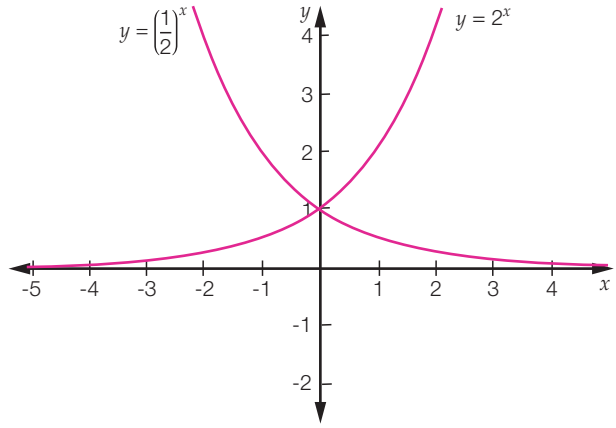
Thinking	Working
1 Determine the translations by comparing to identify $(x-h)^2+(y-k)^2=r^2$ and hence the new centre of the circle.	Coordinates of the centre are (1, -2).
2 Determine the radius of the circle by comparing the equation to $(x-h)^2+(y-k)^2=r^2$. (Here, $r^2=9$.)	$r=\sqrt{9}$ $=3$ $\therefore$ radius is 3 units
3 Sketch the graph using the centre and radius to determine four points on the circumference $(h-r, k)$, $(h, k+r)$, $(h+r, k)$ and $(h, k-r)$	

Exponential graphs

The graph of an exponential with equation $y=a^x$ has an asymptote of $y=0$ and has a y-intercept of (0, 1). The value of a determines the rate of increase of the graph. If a graph is growing exponentially by doubling then $a=2$. If it is trebling then $a=3$. By using these examples it is possible to recognise that the bigger a is, the steeper the graph will be. If $0<a<1$ then the graph will be reflected in the y-axis and so what will be obtained is a graph that is getting closer and closer to $y=0$ as x increases.

Basic equation $y=a^x$, $a>1$	Basic shape is an exponential graph. The graph is continually increasing with an asymptote at $y=0$ and a y-intercept of (0, 1).  If $0<a<1$, the graph is reflected in the y-axis. Note that $\left(\frac{1}{a}\right)=a^{-1}$ so $\left(\frac{1}{a}\right)^x=a^{-x}$ and the graph of $y=a^{-x}$ is the same as $y=\left(\frac{1}{a}\right)^x$.
Translated equation $y=a^{(x-h)}+k$	Basic shape is moved h units to the right and k units up. The asymptote is $y=k$ and the y-intercept is $(0, a^{-h}+k)$. The point (0, 1) has moved to the point $(h, k+1)$.

$y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ look like this:



Note: If you use index laws $y = \left(\frac{1}{2}\right)^x$ is the same as $y = (2^{-1})^x$, which is the same as $y = 2^{-x}$. It is the negative in front of the x that causes it to be reflected in the y-axis.

The graph of $y = 2^x$ translated 3 units to the right and 4 units down gives us $y = 2^{x-3} - 4$. The translation down changes the asymptote from $y = 0$ to $y = -4$. The translation also affects the y-intercept. The original y-intercept of (0, 1) has moved 3 units right and 4 units down and so is now (3, -3). To calculate the new y-intercept, let $x = 0$ and solve the equation.

i.e. $y = 2^{-3} - 4$

$$= \left(\frac{1}{2}\right)^3 - 4$$

$$= \frac{1}{8} - 4$$

$$= -3\frac{7}{8}$$

This gives the y-intercept as the point $(0, -3\frac{7}{8})$.

For the x-intercept, let $y = 0$ and solve. This can be a little difficult by hand but can be calculated using CAS. All exponential graphs will have a y-intercept but not all exponential graphs will have an x-intercept.

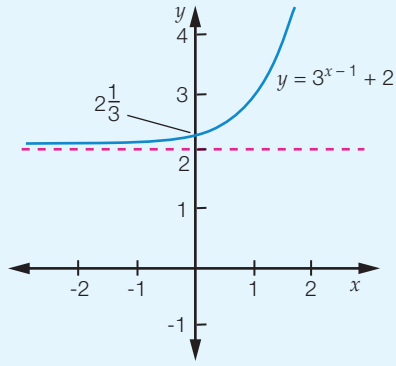
Worked Example 23

WE23

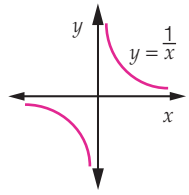
Sketch the graph of $y = 3^{x-1} + 2$ clearly showing the asymptote and other key features.

Thinking	Working
1 State the basic graph and identify the translations that have been undertaken.	$y = 3^{x-1} + 2$ is the graph $y = 3^x$ translated 1 unit right and 2 units up.
2 Determine the asymptote.	The asymptote is $y = 2$.
3 Find the y-intercept by letting $x = 0$.	When $x = 0$, $y = 3^{-1} + 2$ $= 2\frac{1}{3}$ The y-intercept is $(0, 2\frac{1}{3})$.

4 Sketch the graph clearly showing the y-intercept and the asymptote with its equation.



10A Rectangular hyperbolas

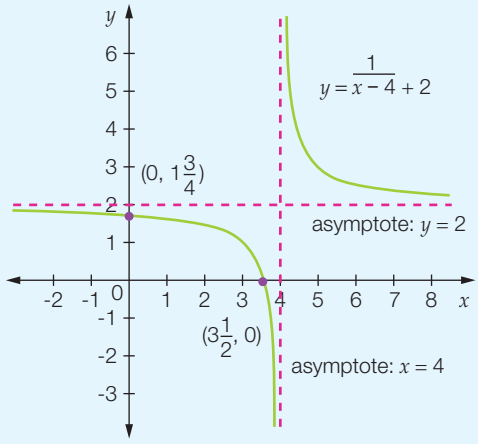
Basic equation $y = \frac{1}{x}$	The basic shape is a hyperbola. It has two asymptotes, the x- and y-axes. There are no x-and y-intercepts. 
Translated equation $y = \frac{1}{x-h} + k$	Basic shape is moved h units to the right and k units up. There are asymptotes at $x = h$ and $y = k$ and graph is discontinuous for $x = h$.

10A Worked Example 24 WE24

Sketch the graph of $y = \frac{1}{x-4} + 2$ clearly showing the key features (asymptotes, and x- and y-intercepts).

Thinking	Working
1 Write the equation.	$y = \frac{1}{x-4} + 2$
2 Identify the asymptotes.	The asymptotes are $x = 4$ and $y = 2$.
3 Find the y-intercept by letting $x = 0$.	When $x = 0$, $y = -\frac{1}{4} + 2$ $= 1\frac{3}{4}$ The y-intercept is $(0, 1\frac{3}{4})$.
4 Find the x-intercept by letting $y = 0$.	When $y = 0$, $0 = \frac{1}{x-4} + 2$ $-2 = \frac{1}{x-4}$ $-2x + 8 = 1$ $-2x = -7$ $x = 3\frac{1}{2}$ The x-intercept is $(3\frac{1}{2}, 0)$.

5 Mark the key features (asymptotes and intercepts) on the Cartesian axes and sketch the graph with a smooth curve.



9.6 Sketch graphs of other non-linear relationships

Navigator

Q1, Q3, Q4, Q5, Q7, Q11	Q1, Q3, Q4, Q7, Q9, Q11	Q1, Q2, Q3, Q4, Q6, Q7, Q8, Q9, Q10, Q11

Answers page 789

Fluency

1 Sketch the graph of each of the following clearly showing the centre of the circle and the radius. WE22

(a) $(x-2)^2 + (y+1)^2 = 16$ (b) $(x+3)^2 + (y-4)^2 = 25$

(c) $(x-2)^2 + (y-2)^2 = 4$ (d) $(x+4)^2 + (y+1)^2 = 16$

2 Sketch the graph of the following clearly showing the asymptote and other key features for each graph. WE23

(a) $y = 2^{x-2} + 1$ (b) $y = 2^{x+1} + 3$

(c) $y = 2^x - 3$ (d) $y = 3^{x-4} - 1$

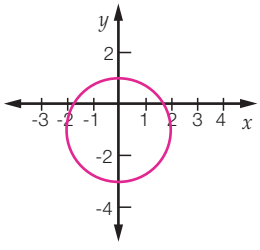
3 Sketch the graph of the following clearly showing the key features (asymptotes, and x- and y-intercepts) for each graph. WE24 10A

(a) $y = \frac{1}{x+3} - 2$ (b) $y = \frac{1}{x-5} + 1$ (c) $y = \frac{1}{x+2} + 3$

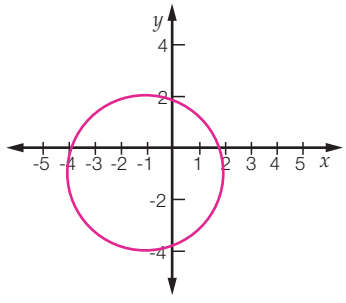
Understanding

4 What is the equation of each of the following graphs?

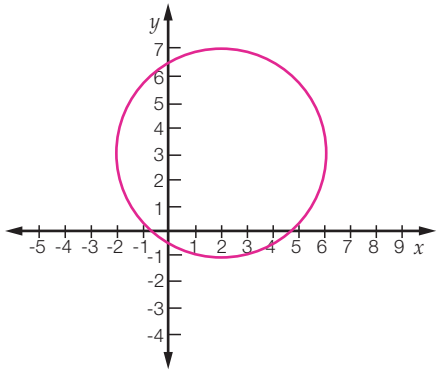
(a)



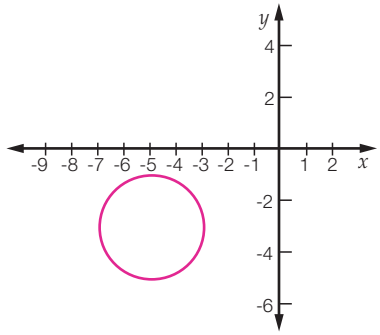
(b)



(c)



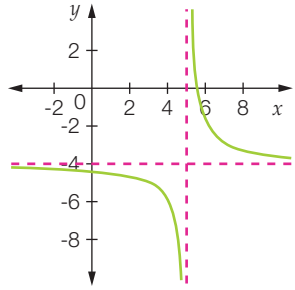
(d)



5 (a) Write the equation formed if the graph of $y = \frac{1}{x}$ is translated 3 units left and 4 units up.

(b) Graph this new equation showing any relevant asymptotes.

10A 6 The equation for this graph would be:



A $y = \frac{1}{x+4} + 5$ B $y = \frac{1}{x-4} - 5$

C $y = \frac{1}{x-5} - 4$ D $y = \frac{1}{x+5} - 4$

7 (a) Write the new equation formed if a circle with equation $(x-3)^2 + (y+4)^2 = 25$ is translated 2 units left and 3 units down.

(b) Write the new equation formed if a circle with equation $(x-3)^2 + (y+4)^2 = 25$ has its radius increased by 3 units.

Reasoning

10A 8 (a) Show that $\frac{1}{x-1} + 3 = \frac{3x-2}{x-1}$.

(b) Hence, sketch the graph of $y = \frac{3x-2}{x-1}$.

9 A circular rose garden is to be planted in a rectangular lawn in front of a house. The lawn is 15 m from the front boundary to the house and 25 m along the front boundary. A diagram of the lawn and the rose garden is to be drawn before the garden is prepared for planting. The left hand side of the front boundary is used as the origin.

(a) The landscape gardener who is creating this garden initially places the centre of the garden in the centre of the lawn. Write down the coordinates of the centre of the circle.

(b) If the rose garden is to have a radius of 3 m, write the equation that would represent the border of the garden.

(c) Sketch a graph of this garden.

(d) If the garden is moved 0.5 m closer to the front boundary and 6.5 m further to the left:

(i) write the new equation (ii) sketch the new graph.

(e) If the radius is now reduced by 1 m after the move:

(i) write the new equation (ii) sketch the new graph.

Open-ended

10A 10 (a) Write the equation of a rectangular hyperbola that has been translated from the origin.

(b) Identify the asymptotes for a graph of your equation.

(c) Sketch the graph.

11 A circle whose centre is not at the origin passes through the point (1, 1). What could be the equation of this circle?

Chapter review 9

D.I.Y. Summary

Key Words

asymptote	exponential	reflection	turning point form
axis of symmetry	Null Factor Law	transformation	vertex
dilation	parabola	translation	
dilation factor	rectangular hyperbola	turning point	

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 The basic graph of $y = x^2$ can be used to obtain other parabolas by using a combination of any of these _____s: _____s, _____s or _____s.

2 A _____ is the type of graph produced by a quadratic relationship.

3 To graph a quadratic it is important to show the x- and y-intercepts and the _____.

4 To change a quadratic that is in the form $y = ax^2 + bx + c$ ($a = 1$) into _____ , it is necessary to complete the square.

5 An _____ is a line on a cartesian plane that a curve approaches but does not cut or touch for certain values of the independent variable.

6 An _____ equation has a horizontal asymptote and may be increasing or decreasing at a rapid rate.

Fluency

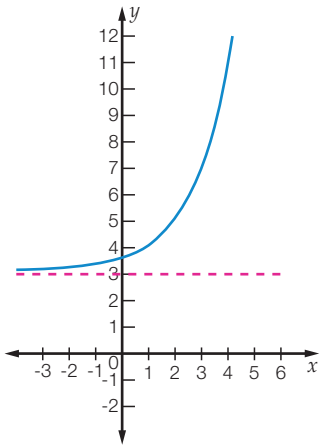
1 For each graph below, identify the type of graph and give the:

Ex. 9.1

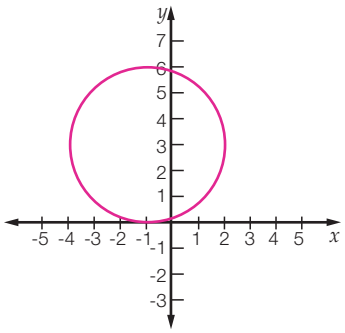
(i) equations of the asymptote if it is an exponential.

(ii) coordinates of the centre and the radius if it is a circle.

(a)



(b)



2 Plot the graph of $y = x^2 - 2x - 8$ for $-3 \le x \le 5$. **Ex. 9.1**

(a) Write down the coordinates of the x- and y-intercepts.

(b) Write down the coordinates of the turning point. Does the parabola have a maximum or a minimum turning point?

(c) Write down the equation of the axis of symmetry.

(d) From the graph, find the value of y when $x = 3$.

(e) What are the values of x when $y = 7$?

(f) Hence, state the values of x when $x^2 - 2x - 8 = 7$.

(g) Write down the values of x when $x^2 - 2x - 8 = 0$.

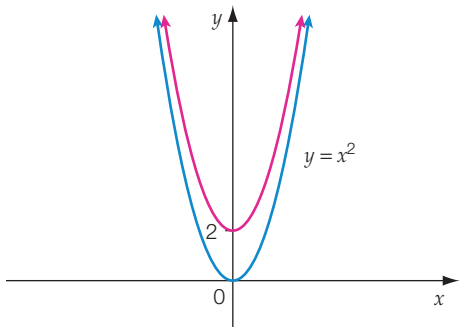
3 (a) The equation of the parabola shown could be: **Ex. 9.2**

A $y = x^2 + 2$

B $y = x^2 - 2$

C $y = (x - 2)^2$

D $y = (x + 2)^2$



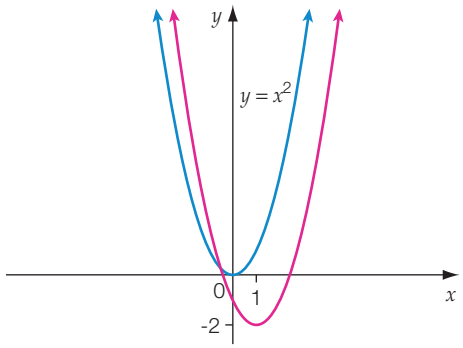
(b) The equation of the parabola shown could be:

A $y = (x - 1)^2 - 2$

B $y = (x - 1)^2 + 2$

C $y = (x + 1)^2 + 2$

D $y = (x + 1)^2 - 2$



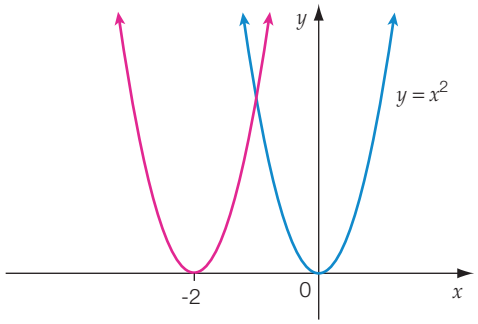
(c) The equation of the parabola shown could be:

A $y = (x - 2)^2$

B $y = (x + 2)^2$

C $y = x^2 - 2$

D $y = x^2 + 2$



4 Solve these quadratic equations using the Null Factor Law. **Ex. 9.3**

(a) $x^2 - 6x = 0$ (b) $4x^2 - 81 = 0$ (c) $x^2 - 5x - 14 = 0$

(d) $x^2 + 6x + 9 = 0$ (e) $x^2 = 8x$ (f) $x^2 + 36 = 12x$

(g) $2x(x + 4) = 0$ (h) $(x - 7)(x + 1) = 0$ (i) $x^2 - 16 = 0$

5 Solve these quadratic equations. **Ex. 9.3**

(a) $x^2 + 12x + 35 = 0$ (b) $x^2 + x - 6 = 0$ (c) $-x^2 - 8x + 9 = 0$

(d) $-x^2 + 9x - 8 = 0$ (e) $3x^2 + 9x - 12 = 0$ (f) $2x^2 - 8x + 6 = 0$

6 Solve these quadratic equations. **Ex. 9.3**

(a) $x^2 - 4x = 5$ (b) $x^2 - 6 = 5x$ (c) $x^2 + 12 = -7x$

(d) $3x^2 = 27$ (e) $(x - 3)(x + 7) = 11$ (f) $(x - 3)(x + 2) = 6$

7 Solve each of the following by completing the square. Ex 9.4

(a) $x^2 - 4x + 2 = 0$ (b) $x^2 + 8x + 9 = 0$ (c) $-x^2 - 2x + 4 = 0$

8 For each of the following: Ex 9.5

(i) state the changes that have been made to the graph of $y = x^2$

(ii) state the coordinates of the turning point

(iii) find the x- and y-intercepts

(iv) sketch a graph of each equation showing the turning point and intercepts.

(a) $y = x^2 - 4$ (b) $y = -2(x + 1)^2 - 4$ (c) $y = \frac{1}{3}(x - 7)^2$

9 For each of the following equations: Ex 9.5

(i) find the y-intercept (ii) find the x-intercepts

(iii) find the coordinates of the turning point (iv) sketch the graph.

(a) $y = x^2 - 4x - 12$ (b) $y = -(x + 1)^2 + 4$

(c) $y = 2(x - 3)^2 - 6$ (d) $y = -x^2 + 3x + 4$

10 Sketch the graph of each of the following. Ex 9.6

(a) $(x + 2)^2 + (y - 3)^2 = 16$ (b) $x^2 + (y - 3)^2 = 4$

(c) $y = 2^{x-2} + 1$ (d) $y = 2^{x+2} - 4$

11 Sketch the graph of each of the following. Ex 9.6 10A

(a) $y = \dfrac{1}{x - 4} + 1$ (b) $y = \dfrac{1}{x + 2} - 1$

Understanding

12 A softball thrown by an outfielder to the pitcher followed a path where the height h, in metres, of the softball after t seconds, is described by the relationship:

$h = -t^2 + 6t + 1$

(a) Complete a table of values for this relationship.

(b) Plot the graph of this relationship on graph paper.

(c) What was the height of the softball after 2 seconds?

(d) Use your graph to find when the ball was first at a height of 7 m above the ground?

(e) Use your graph to find during which time interval the ball was above a height of 9 m?

(f) What was the maximum height of the softball during its flight?

(g) How long was the ball in flight if it was caught at a height of 1.6 m above the ground on its downward path?

13 A train ticket is 5 cm longer than it is wide.

(a) Draw a diagram of the ticket, labelling the width x.

(b) Write down the length of the ticket in terms of x.

(c) Write an expression in terms of x for the area of the train ticket.

(d) If each ticket requires 24 cm^2 of paper, write an equation that represents this area.

(e) Solve this equation to find x.

(f) Hence, state the dimensions of the train ticket.

(g) Did you use both of the values for x that you found using algebra to find the dimensions of the ticket? Explain your answer.

Reasoning

14 A fountain sends a jet of water on a parabolic path, which can be described by the equation $h = -2x^2 + 0.8x$, where h is the height of the water, in metres, at a horizontal distance x, in metres, from the water outlet.

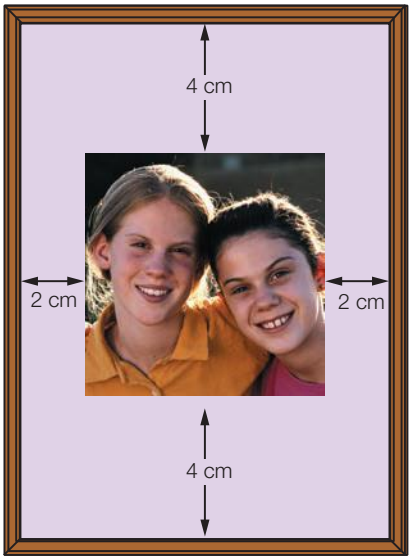
(a) Where is the jet of water again at the height that it left the outlet?

(b) If it hits the base of a pond at a horizontal distance of 0.8 m from the outlet, how far below the outlet does the water land?

(c) What is the maximum height above the base of the pond that the jet of water reaches?

(d) Sketch the graph of the path of the water.

15 A square photograph is to be attached to a coloured mounting and then framed.



(a) Find the length of a side of the photo if the frame encloses a total area of 140 cm^2.

(b) Hence find the dimensions of the photo frame.

(c) Calculate the area of the photograph.

(d) Calculate the area of coloured mounting that can be seen.

(e) What area of coloured mounting would be seen if the same photograph was to have a 2 cm coloured mounting all around?

16 The sum of two numbers is 20.

(a) The two numbers are x and y. Write an equation to represent the sum.

(b) Rearrange the equation so that x is the subject (i.e. in the form $y =$).

(c) Write an expression for the product of the two numbers using x.

(d) Graph the expression and use it to find the maximum product and the numbers that obtain it.

17 A tennis ball is struck in the centre of a tennis court 23.78 m long, 1 m behind the base line. It follows a parabolic path parallel to the sidelines and its height is determined by the equation $h = -0.0045x^2 + 0.135x$, where x is the distance from a fence, which is 6.11 m from the base line.

(a) At what height was it struck?

(b) Show that the ball clears the net, which is 0.913 m high in the centre.

(c) Determine whether the ball lands inside or outside the tennis court.

18 The graph of $y = x^2$ is translated so that it passes through two points (3, 8) and (9, 8).

(a) Write down the x-coordinate of the turning point of the graph.

(b) Find the equation of the curve.

(c) Sketch this graph.

19 A pattern is drawn with four circles with equations:

A $(x - 3)^2 + (y - 2)^2 = 16$

B $(x + 1)^2 + (y - 3)^2 = 4$

C $(x + 5)^2 + (y + 4)^2 = 9$

D $(x + 2)^2 + (y + 1)^2 = 16$

(a) Which two circles are the same size?

(b) By sketching the graphs find which circle intersects with the other three circles.

Mixed review C

Fluency

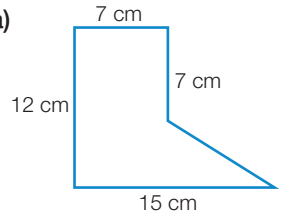
1 An investment of \$5000 is made at 3.5% p.a. for 6 years. Find the difference in interest earned if the interest is compounded monthly compared to annually. Ex.1.2

2 Factorise the following quadratic trinomials. Ex.2.3

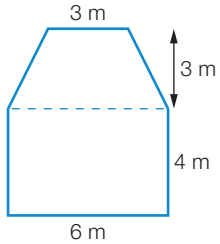
(a) $a^2 - 5a - 36$ (b) $a^2 - 8a + 12$ (c) $4a^2 + 20a + 25$

3 Find the area of each of the following shapes. Ex.3.1

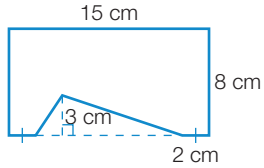
(a)



(b)



(c)

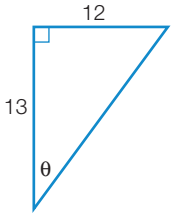


4 Solve the following equations. Ex.4.1

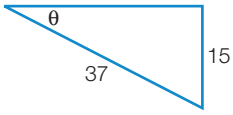
(a) $\frac{3a+4}{7} = -2$ (b) $\frac{4a+3}{2} = \frac{2a}{5}$ (c) $\frac{3a-2}{3} = \frac{4a-5}{2}$

5 Find the value of the pronumeral in each, correct to one decimal place. Ex.5.3

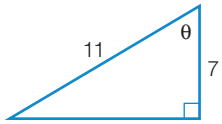
(a)



(b)



(c)



6 Find the length of the third side of a scalene triangle with two sides of length 6 cm and 7 cm and an included angle of 43°. Round your answer to two decimal places. Ex.6.4 10A

7 Find the range and the interquartile range for each of the following grouped data frequency tables. Give your answers correct to two decimal places. Ex.7.2

(a)

Class interval	Frequency
20–<25	8
25–<30	6
30–<35	3
35–<40	2
40–<45	5
45–<50	5

(b)

Class interval	Frequency
110–<115	12
115–<120	15
120–<125	17
125–<130	19
130–<135	13
135–<140	11

8 Prove that the diagonals of a kite intersect at right angles. Ex.8.5

9 Consider the equation $y = 2x^2 + x$. Ex. 9.1

(a) Set up a table of values with x-values from -3 to 3.

(b) Plot the points on a coordinate plane.

(c) Find the turning point of the graph.

(d) Determine the x- and y-intercepts for the graph.

10 What are the asymptotes of the graph of $y = \frac{1}{x+1} - 3$? Ex. 9.1

11 Solve each of the following equations: Ex. 9.3

(a) $x^2 + 13x + 42 = 0$ (b) $x^2 - 100 = 0$ (c) $x^2 - 14x + 49 = 0$

Understanding

12 Sketch the graph of each of the following quadratics, showing all important features.

(a) $y = x^2 + 6x - 4$ (b) $y = x^2 - 4x + 4$ (c) $y = -2x^2 + 3x$

13 Solve each of the following for x.

(a) $2x^2 - 50 = 0$ (b) $4x^2 + 2x = 0$ (c) $x^2 - 5x + 6 = 0$

14 Write down the turning point of each of the following graphs.

(a) $y = (x - 2)^2 + 3$ (b) $y = (x + 4)^2 - 1$ (c) $y = (x - 7)^2 - 6$

15 Write the equation formed when $x^2 + y^2 = 25$ is translated 4 units right and 2 units up.

16 A point that does not lie on the graph of $(x - 3)^2 + (y - 2)^2 = 20$ would be:

A (7, 0) B (5, 6) C (6, 5) D (-1, 4)

17 The graph of the equation $y = 2^{x-1} + 1$ has an asymptote at:

A $x = 1$ B $x = -1$ C $y = 1$ D $y = -1$

Reasoning

18 A mobile phone screen is 2.5 cm longer than it is wide.

(a) Write an equation for the area of the screen in terms of its width, in cm^2.

(b) Plot a graph of the area against the width.

(c) Use this graph to find the dimensions of the screen if the area of the screen is 37.5 cm^2.

19 A graph has x-intercepts at $x = 4$ and $x = 6$.

(a) Determine the x-coordinate of the turning point.

(b) If the x^2 coefficient is 1, determine the equation of the graph.

Challenge C

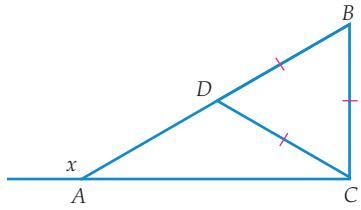
1 The mean of 19 consecutive integers is 99. The largest of these integers is:

A 118 **B** 117 **C** 108 **D** 107

2 Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is:

A 34 **B** 35 **C** 36 **D** 37

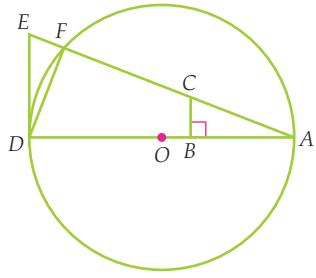
3 Find the value of x if the large triangle has been broken into two triangles of equal area. The large triangle is not marked as being right angled.



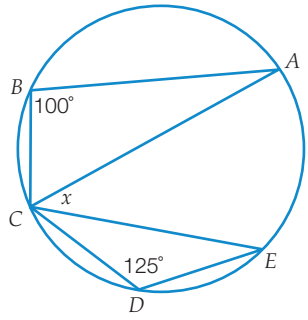
4 In the diagram, $ABC = 90°$, $CB \parallel ED$, $AB = DF$, $AD = 24$, $AE = 25$ and O is the centre of the circle. The perimeter of $CBDF$ is:

A 40 units **B** 42 units

C 43 units **D** 44 units



5 Find x in the following diagram.



6 **(a)** Determine the values of x for which $(x + 3)(x - 6) = -14$.

(b) Determine the values of x for which $(x^2 - 3x)^2 = 4 - 3(3x - x^2)$.

7 **(a)** What is the sum of the two numbers that satisfy the equation $x^2 - 6x - 7 = 0$?

(b) What is the product of the two numbers that satisfy the equation $5x^2 - 20 = 0$?

(c) Find the average of the numbers that satisfy the equation $x^3 - 6x^2 + 5x = 0$.

8 The equation of a parabola is written in the form $y = (x - 2)(x - 8) + 7$.

(a) If $(c, 7)$ lies on the parabola, find all possible values of c.

(b) Find the coordinates of the vertex of the parabola.

9 The circle $x^2 + y^2 = 9$ and the parabola $y = x^2 - 3$ intersect at three points.

(a) Find the y-coordinates of the points of intersection.

(b) Hence, find the exact values of the coordinates of the points of intersection.

10 Polynomials 10A

The mathematics of DNA

Name: Dr Tony Papenfuss
Job: Laboratory Head, Bioinformatics
Qualifications: PhD in Applied Mathematics (astrophysics)

Bioinformatics is the analysis of biological data using mathematics and computer science. It is a field that has exploded in the last decade as recent developments in our ability to isolate and sequence DNA means that genetic data can be generated quickly.

Dr Tony Papenfuss is a Laboratory Head in Bioinformatics at The Walter & Eliza Hall Institute of Medical Research. He works with a team of researchers to analyse sequences of DNA in an attempt to discover more about cancer, malaria and other diseases. The department has grown considerably over the last few years.

"A strong basis in mathematics is crucial in this role, as well as a keen interest in biology and an interest in working on real problems. I enjoy the collaborative aspect of the job, working closely with biologists, clinicians and with my own team to find solutions to these problems."

The data analysis is usually done on large computers. The mathematics can be simple but powerful at the same time. "I would advise anyone looking to move into this field to continue with maths—it's important in lots of different areas of scientific research."

Tony has recently been working on the Tasmanian devil facial tumour disease, and his work has contributed to some important discoveries about what type of cancer it is.

Why learn this?

Graphs convey a great deal of information in a clear and precise way. Graphs of polynomials can be used to model relationships between variables if the patterns are recognised. Therefore, engineers, statisticians, surveyors and research scientists use them extensively to understand and interpret the world around them.

After completing this chapter you will be able to:

- recognise a polynomial
- recognise graphs in the form of $y = ax^n$
- recognise graphs in the form of $y = a(x - h)^n + k$ as translations of graphs in the form of $y = ax^n$
- translate graphs in the form of $y = a(x - h)^n + k$
- perform the four operations (+, –, ×, ÷) on polynomials
- solve non-monic quadratic equations using a variety of techniques
- use the remainder theorem to find remainders after polynomial division
- use the factor theorem to factorise a polynomial
- sketch graphs of polynomials.

10 Recall

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

1 Plot the points in each of the following tables of values and join them with a straight line or a smooth curve to obtain the graph of each relationship.

Worksheet R10.1

(a)

x	-3	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4	6

(b)

x	-3	-2	-1	0	1	2	3
y	11	6	3	2	3	6	11

2 State whether the relationship is linear, quadratic or neither.

Worksheet R10.2

(a) $y = x^3 + 2x^2 + 3$ **(b)** $b = 2c + 3$ **(c)** $z = 4 - 3w^2$

3 **(a)** Complete this table of values for $y = x^2 + 2x - 8$.

Worksheet R10.3

x	-5	-4	-3	-2	-1	0	1	2	3
y									

(b) On graph paper, plot the graph of the parabola $y = x^2 + 2x - 8$.

(c) From the graph, write the:

(i) y-intercept

(ii) x-intercepts

(iii) coordinates of the turning point.

(d) Decide whether the graph has a maximum or a minimum turning point.

(e) What is the equation of the axis of symmetry?

4 Factorise each quadratic expression:

Worksheet R10.4

(a) $x^2 - 5x - 6$ **(b)** $4x^2 - 81$ **(c)** $3x^2 - 3x - 36$

(d) $x^2 - 14x + 49$ **(e)** $-x^2 + 8x + 20$ **(f)** $x^2 - 6x + 1$

5 Solve these equations using the Null Factor Law.

Worksheet R10.5

(a) $(x + 2)(x - 1) = 0$ **(b)** $(x + 9)(x - 9) = 0$ **(c)** $3x(2x - 5) = 0$

Key Words

cubic	factor theorem	point of inflection	remainder
degree	leading coefficient	polynomial	remainder theorem
dividend	leading term	quartic	
divisor	non-monic quadratic	quotient	

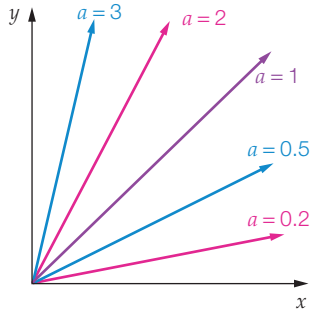
10.1 Equations of the form $y = ax^n$

The straight line with equation $y = 2x$, the parabola with the equation $y = x^2$, the **cubic** function given by $y = x^3$ and the **quartic** function given by $y = 5x^4$ belong to a class of equations described by $y = ax^n$, where a is a constant and n is an integer. We will investigate the properties of graphs of $y = ax^n$ for different non-zero values of a and for $n = 1, 2, 3$ and 4.

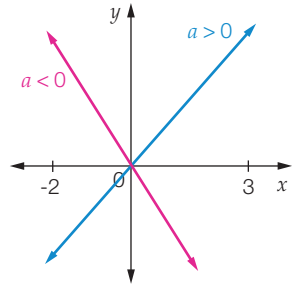
Linear equations: $y = ax$

When $n = 1$, $y = ax^n$ defines a linear equation $y = ax$. The straight line passes through the origin (0, 0) and has gradient a.

- If $a > 0$, the gradient is positive.



- If $a < 0$, the gradient is negative.



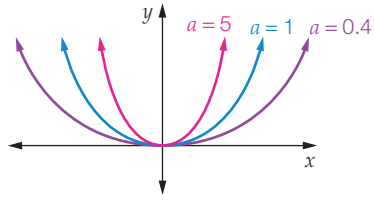
The coefficient of x is the dilation factor. The larger the size of a (ignoring the sign), the steeper the line. This means that every y-value of the graph of $y = x$ is multiplied by a to find the y-values of $y = ax$.

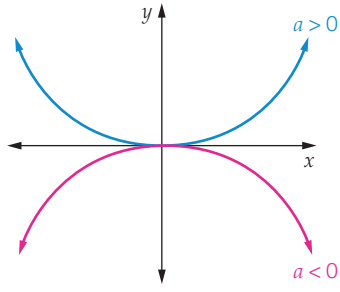
Quadratic equations: $y = ax^2$

For $n = 2$, we have the quadratic function $y = ax^2$. Its graph is a parabola that passes through the origin and is symmetrical about the y-axis. The axis of symmetry is the line $x = 0$.

The coefficient of x^2, a, is the dilation factor.

- If $a > 0$, the graph has a minimum turning point at (0, 0).
- If $a > 1$, then the graph becomes narrower than the graph of $y = x^2$.
- If $0 < a < 1$, then the graph becomes wider than the graph of $y = x^2$.
- If $a < 0$, the graph is inverted. It is a reflection in the x-axis of the graph for $a > 0$ and it now has a maximum turning point at (0, 0).

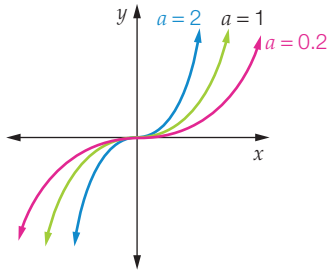




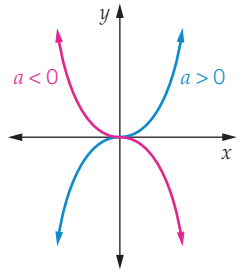
Cubic equations: $y = ax^3$

When $n = 3$, $y = ax^n$ becomes $y = ax^3$. Its graph is a cubic that passes through the origin. It does not have a turning point (maximum or minimum) where the graph changes direction but it does have a special point at the origin called a **point of inflection**. It is not symmetrical about the x-axis or the y-axis but the graph on one side of the origin is reflected in both the x- and y-axes to produce the other half of the graph.

- If $a > 1$, the graph becomes steeper (narrower) than the graph of $y = x^3$.
- If $0 < a < 1$, the graph becomes flatter (wider) than the graph of $y = x^3$.



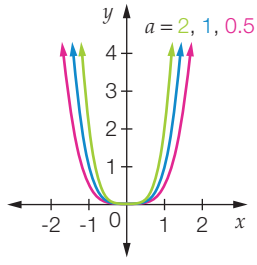
- If $a < 0$, the graph is a reflection in the x-axis of the graph for $a > 0$.



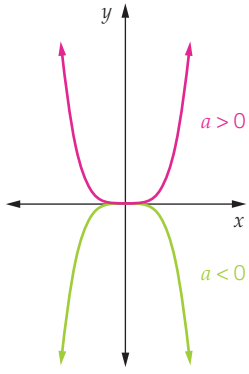
Quartic equations: $y = ax^4$

When $n = 4$, $y = ax^n$ gives the equation $y = ax^4$. Its graph is a quartic that passes through the origin and is symmetrical about the y-axis. A quartic graph is steeper than a quadratic graph with the same dilation factor for $x > 1$ and $x < -1$. It is squarer than a quadratic graph with the same dilation factor for $-1 < x < 1$.

- If $a > 0$, the graph has a minimum turning point at (0, 0).
- If $a > 1$, the graph becomes narrower than the graph of $y = x^4$.
- If $0 < a < 1$, the graph becomes wider than the graph of $y = x^4$.



- If $a < 0$, the graph is a reflection in the x-axis of the graph for $a > 0$ and it now has a maximum turning point at (0, 0).



Worked Example 1

WE1

Construct a table of values for $x = -3$ to 3 and then plot the graph of the following equations:

(a) $y = x^2$ **(b)** $y = x^3$ **(c)** $y = x^4$

Thinking

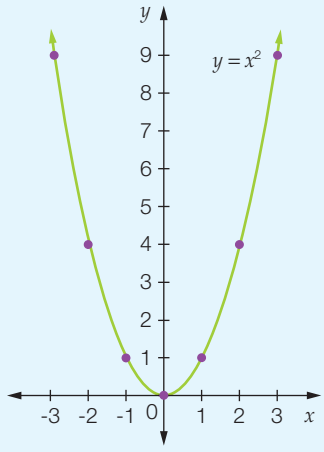
Working

(a) 1 Construct the table of values.

(a)

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

2 Draw the axes using a suitable scale. Plot the points and join them with a smooth curve.

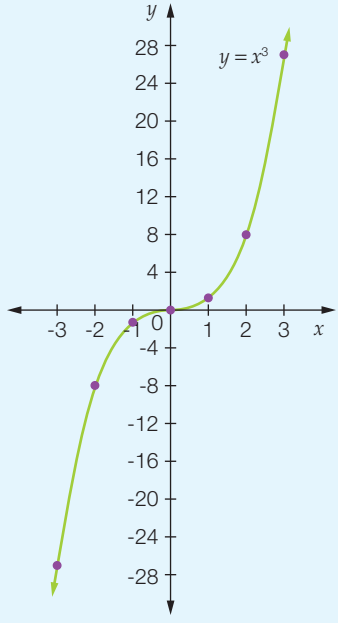


(b) 1 Construct the table of values.

(b)

x	-3	-2	-1	0	1	2	3
y	-27	-8	-1	0	2	8	27

2 Draw the axes using a suitable scale. Plot the points and join them with a smooth curve.

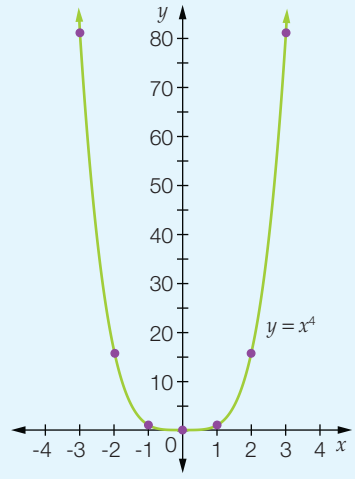


(c) 1 Construct the table of values.

(c)

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

2 Draw the axes using a suitable scale. Plot the points and join them with a smooth curve.



These graphs could also be drawn using a computer drawing program such as GeoGebra or the graphing function on your CAS.

The graph of $y = ax^n$ for $n = 1, 2, 3$ and 4 passes through the origin and has an x-intercept and a y-intercept of 0.

- For $n = 1$, the graph is linear.
- For $n = 2$, the graph is a parabola.
- For $n = 3$, the graph is a cubic.
- For $n = 4$, the graph is a quartic.
- The dilation factor (value of a) narrows the graph for $a > 1$ and widens the graph for $0 < a < 1$.
- The graph of $y = -ax^n$ is the reflection of $y = ax^n$ in the x-axis, $a > 0$.

10.1 Equations of the form $y = ax^n$

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Answers page 795

Fluency

1 Construct a table of values for $x = -3$ to 3 and then plot the following equations.

WE1

(a) $y = 2x^2$ (b) $y = 2x^3$ (c) $y = 2x^4$

2 Match each equation to its graph.

U $y = -\frac{2}{3}x$

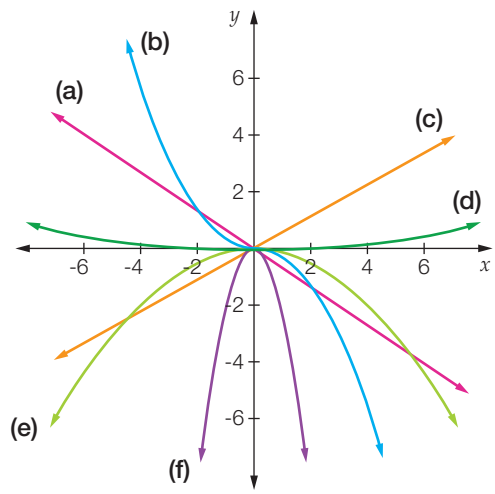
V $y = -6x^2$

W $y = \frac{x}{2}$

X $y = -\frac{1}{10}x^2$

Y $y = 0.01x^2$

Z $y = \left(-\frac{1}{2}x\right)^3$



3 (a) Plot the graphs of each linear equation on the same number plane. Use x-values from -3 to 3.

(i) $y = x$ (ii) $y = 2x$ (iii) $y = 4x$

(b) What can you conclude about the steepness of the line in relation to the coefficient of x?

(c) Does your answer to part **(b)** apply to the equations $y = -x$, $y = -2x$ and $y = -4x$?

4 (a) Plot the graphs of each quadratic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = 2x^2$ (ii) $y = 3x^2$ (iii) $y = 4x^2$

(b) Describe what happens to the shape of the parabola as the coefficient of x^2 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -2x^2$, $y = -3x^2$ and $y = -4x^2$?

Warning
Is -4 > -2?

5 (a) Plot the graphs of each cubic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = x^3$ (ii) $y = 2x^3$ (iii) $y = 3x^3$

(b) Describe what happens to the shape of the cubic graph as the coefficient of x^3 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -x^3$, $y = -2x^3$ and $y = -3x^3$?

6 (a) Plot the graph of each quartic equation on the same number plane. Use x-values from -3 to 3.

(i) $y = 0.5x^4$ (ii) $y = x^4$ (iii) $y = 3x^4$

(b) Describe what happens to the shape of the quartic graph as the coefficient of x^4 gets larger.

(c) Does your answer to part **(b)** apply to the equations $y = -x^4$, $y = -3x^4$, $y = -0.5x^4$?

Understanding

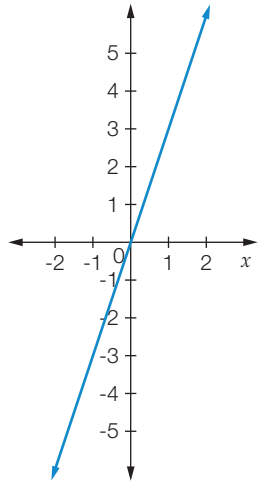
7 The straight line shown is reflected in the x-axis and then reflected in the y-axis. The equation describing the new graph is:

A $y = \frac{1}{3}x$

B $y = 3x$

C $y = -\frac{1}{3}x$

D $y = -3x$



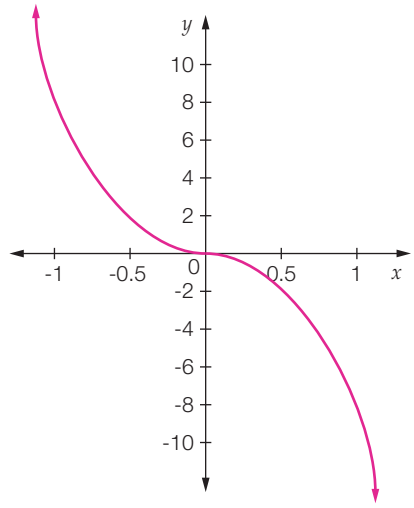
8 A possible equation to describe the graph shown is:

A $y = -8x^2$

B $y = x^3$

C $y = -x^2$

D $y = -8x^3$



Reasoning

9 The shape of a skate ramp is described by the equation $y = \frac{1}{3}x^2$, where x and y are the horizontal and vertical distances respectively, in metres, from the bottom of the ramp.

(a) Draw the graph of the equation using values of x from 0 to 3.

(b) One end of a taut length of rope is attached to the bottom of the ramp and the other is attached to a point on the ramp with x-coordinate 1.5. What is the equation of the line describing the position of the rope?

10 Tim and Matthew set off from their campsite in different directions. Tim's distance, d km, from the campsite at t minutes is given by the equation $d = 0.1t$, and Matthew's distance from the campsite is given by $d = -0.1t$.

(a) Represent each equation on the same number plane using values of t from 0 to 10.

(b) How far apart are the boys after 10 minutes?

(c) What is the equation of the line describing the path Tim could have taken had he walked at a constant speed and travelled 2.4 km in 20 minutes?

Hint
What does a negative distance represent?

11 The shape of a flat weathervane is defined by the two equations $y = 4x$ and $y = 2x^2$ for x-values from 0 to 2, where x and y represent length in metres and the origin is taken to be the base of the weathervane.

(a) Sketch the two graphs on the same number plane using x-values from 0 to 2.

(b) If the area under the curve $y = 2x^2$ from $x = 0$ to $x = 2$ and bounded by the x-axis is $5\frac{1}{3}$ square metres, what is the area covered by one face of the weathervane?

(c) The wind changes direction so that the weathervane swings around 180° about the origin. What are the equations describing the shape of the weathervane now? State the values of x that apply.

12 **(a)** On the same set of axes draw the curves $y = x^3$ and $y = \frac{1}{2}x^4$ for $x = -2$ to 2.

(b) Shade in the region between these two curves between $x = 0$ and $x = 2$.

(c) Reflect this region in the y-axis and draw it on your graph.

(d) What is the equation of the top boundary of this new region?

Open-ended

13 **(a)** Find two consecutive whole number values for c so that the gradient of the straight line described by the equation $y = -\left(\frac{c-9}{1-c}\right)x$ is positive.

(b) What can be concluded about the gradient when $c = 1$? What does this tell you about the line?

14 Consider the graphs of $y = ax^2$ and $y = 2a - ax^2$. Draw the graphs on the same number plane for $a = 1, 2, 3, 4$. What can you discover above the points of intersection of the two graphs? Repeat for negative values of a and comment on your answer.

Outside the Square — Problem solving

Different paths

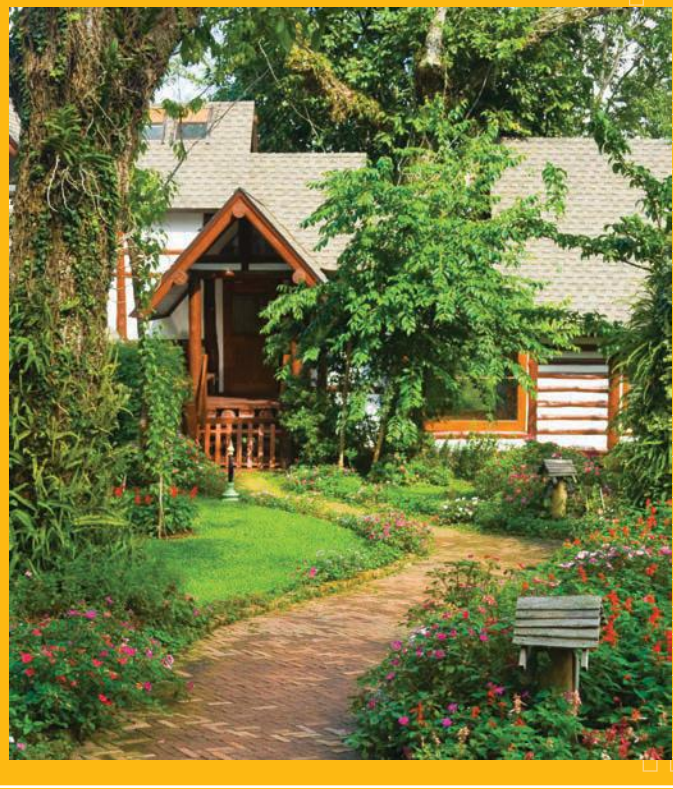
Mark wants to build a path from his front door to the gate. He draws up a grid and determines that his front door is at grid reference (0, 0) and the gate is at (5, 12). He decides that as he has been learning about graphs of the type $y = ax^n$ he will see how many different graphs he can get that fit these two points.

1 Find the equations of the curves that he would need for $n = 1, 2, 3$ and 4.

2 For each value of n, determine at least one other grid reference that he should use when making the path.

Strategy options

- Draw a diagram.
- Test all possible combinations.
- Look for a pattern.

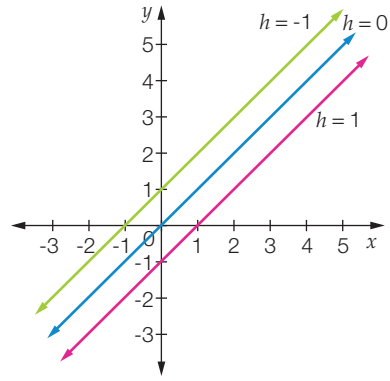


Transformations of $y = ax^n$

10.2

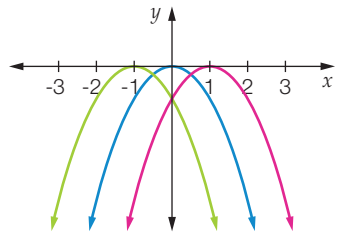
Graphs of $y = a(x - h)^n$

The graph of $y = a(x - h)^n$ is a translation of the graph of $y = ax^n$ by h units horizontally. If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.

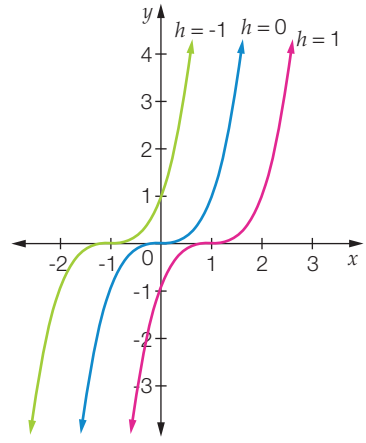


When we substitute the values $a = 1$, $n = 1$ and $h = -1, 0, 1$ into $y = a(x - h)^n$ we obtain the equations $y = x - 1$, $y = x$ and $y = x + 1$. The graphs of these equations are straight lines with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively. The value of h represents the x-coordinate of the x-intercept and has the effect of translating the graph of $y = ax$ a distance of h units in the positive direction of the x-axis.

If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.



When we substitute the values $a = -3$, $n = 2$ and $h = -1, 0, 1$ into $y = a(x - h)^n$ we obtain the equations $y = -3x^2$, $y = -3(x - 1)^2$ and $y = -3(x + 1)^2$. The graphs of the equations are parabolas, symmetrical about the line $x = h$ and the y-value of the y-intercept is $a(-h)^2$, found by substituting $x = 0$ in $y = a(x - h)^2$. The value of h represents the x-coordinate of the x-intercept and has the effect of translating the graph of $y = ax^2$ a distance of h units in the direction of the x-axis. If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.



When we substitute the values $a = 1$, $n = 3$ and $h = -1, 0, 1$ into $y = a(x - h)^n$, we obtain the cubic equations $y = (x + 1)^3$, $y = x^3$ and $y = (x - 1)^3$. The y-value of the y-intercept is $a(-h)^3$, found by substituting $x = 0$ in $y = a(x - h)^3$. This type of cubic has only one x-value of the x-intercept. The value of h represents the x-intercept and has the effect of translating the graph of $y = ax^3$ a distance of h units in the direction of the x-axis. If $h > 0$ the translation is towards the right, and if $h < 0$ the translation is towards the left.

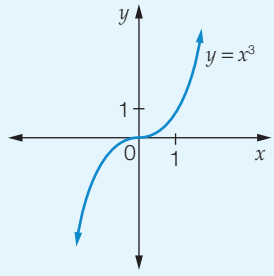
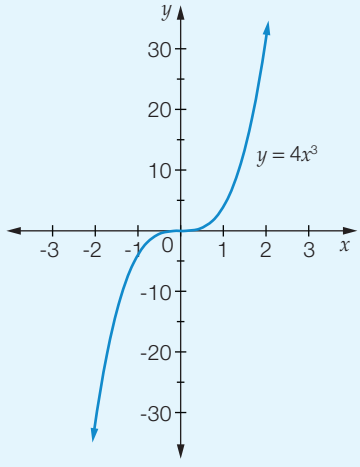
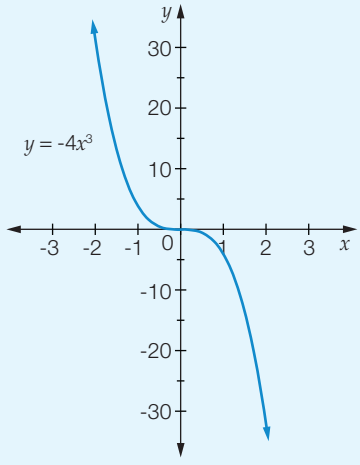
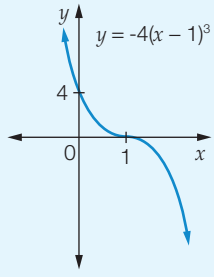
The usual order for graphing transformations is:

1. dilations
2. reflections
3. translations

Worked Example 2

WE2

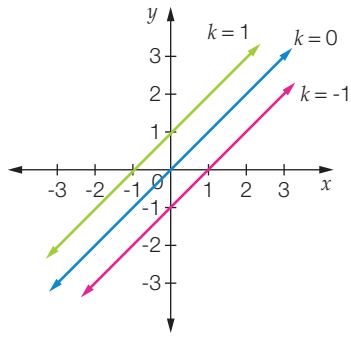
Sketch the graph of $y = -4(x - 1)^3$ by transforming the basic graph of $y = x^3$.

Thinking	Working
1 Sketch the graph of $y = x^3$.	
2 Dilate the graph. Locate the y-intercept. (Here, the dilation factor is 4 so we make it narrower to give the graph $y = 4x^3$. and substitute $x = 0$ to find the y-intercept at the point (0, 0).)	
3 Reflect the graph. Locate the y-intercept. (Here, we reflect the graph in the x-axis to obtain the graph $y = -4x^3$ and substitute $x = 0$ to find the y-intercept at the point (0, 0).)	
4 Translate the graph horizontally. Locate the y-intercept. (Here, we translate 1 unit to the right to obtain the graph $y = -4(x - 1)^3$ and substitute $x = 0$ to find the y-intercept at the point (0, 4).)	

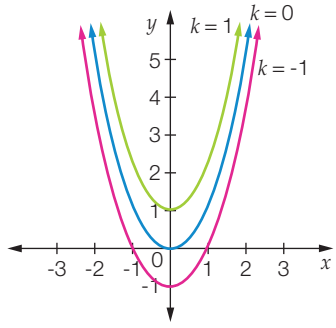
Graphs of $y = ax^n + k$

The graph of $y = ax^n + k$ is a translation of the graph of $y = ax^n$ by k units vertically. If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

When we substitute the values $a = 1$, $n = 1$ and $k = -1, 0, 1$ into $y = ax^n + k$ we obtain the equations $y = x - 1$, $y = x$ and $y = x + 1$. The graphs are straight lines with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively. The value of k represents the y-coordinate of the y-intercept and has the effect of translating the graph of $y = ax$ a distance of k units in the positive direction of the y-axis. If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

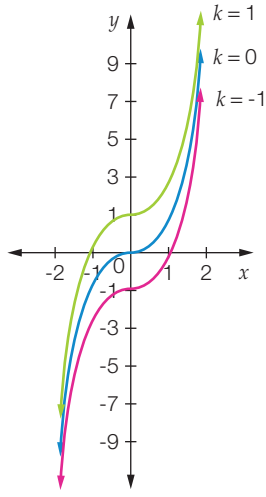


When we substitute the values $a = 1$, $n = 2$ and $k = -1, 0, 1$ into $y = ax^n + k$ we obtain the equations $y = x^2 - 1$, $y = x^2$, $y = x^2 + 1$. The graph is a parabola, symmetrical about the line $x = h$. The graph may have zero, one or two x-intercepts. The value of k represents the y-coordinate of the y-intercept and is found by substituting $x = 0$ in $y = ax^n + k$. It has the effect of translating the graph of $y = ax^2$ a distance of k units in the positive direction of the y-axis. If $k > 0$ the translation is up, and if $k < 0$ the translation is down.



When we substitute the values $a = 1$, $n = 3$ and $k = -1, 0, 1$ into $y = ax^n + k$ we obtain the equations $y = x^3 - 1$, $y = x^3$ and $y = x^3 + 1$. The graphs are straight lines with the x-values of the x-intercepts being 1, 0 and -1, and the y-values of the y-intercepts being -1, 0 and 1 respectively. The value of k represents the y-coordinate of the y-intercept and is found by substituting $x = 0$ in $y = ax^3 + k$, and has the effect of translating the graph of $y = ax^3$ a distance of k units in the positive direction of the y-axis. If $k > 0$ the translation is up, and if $k < 0$ the translation is down.

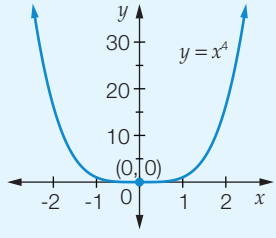
This type of cubic has only one x-intercept.



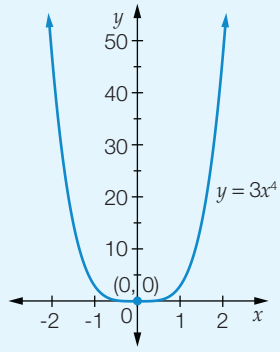
Worked Example 3

WE3

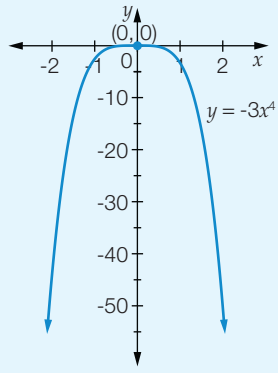
Sketch the graph of $y = -3x^4 + 2$ by transforming the basic graph of $y = x^4$.

Thinking	Working
1 Sketch the graph of $y = x^4$.	

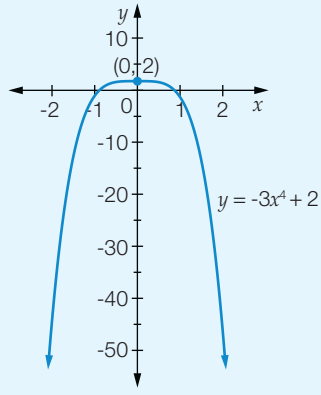
2 Dilate the graph. Locate the y-intercept. (Here, the dilation factor is 3 so we make it narrower to give the graph $y = 3x^4$ and substitute $x = 0$ to find the y-intercept at the point (0, 0).)



3 Reflect the graph. Locate the y-intercept. (Here, we reflect the graph in the x-axis to obtain the graph $x = -3x^4$ and then substitute $x = 0$ to find the y-intercept at the point (0, 0).)



4 Translate the graph vertically. Locate the y-intercept. (Here, we translate 2 units up to obtain the graph $y = -3x^4 + 2$ and substitute $x = 0$ to find the y-intercept at the point (0, 2).)



Graphs of $y = a(x - h)^n + k$

Transformations are applied in the order of dilations, reflections and translations.

The graph of $y = a(x - h)^n + k$ is a translation of the graph of $y = ax^n$ by h units horizontally and k units vertically.

- If $h > 0$, the translation is towards the right and if $h < 0$, the translation is towards the left.
- If $k > 0$, the translation is up and if $k < 0$, the translation is down.

10.2 Transformations of $y = ax^n$

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14

Answers page 797

Fluency

1 Sketch the graph of $y = -3(x + 2)^3$ by transforming the graph of $y = x^3$. WE2

2 Sketch the graph of $y = -x^2 + 1$ by transforming the graph of $y = x^2$. WE3

3 Sketch the graph of $y = 4x - 2$ by transforming the graph of $y = x$.

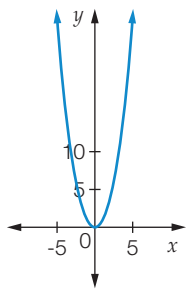
4 Sketch the graph of $y = 3(x - 1)^4$ by transforming the graph of $y = x^4$.

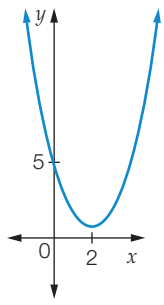
5 Sketch the graph of $y = \frac{1}{2}(x - 3)^2$ by transforming the graph of $y = x^2$.

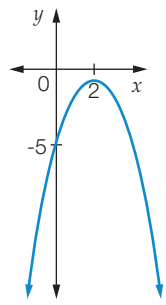
6 Sketch the graph of $y = -\frac{1}{10}(x + 2)^4$ by transforming the graph of $y = x^4$.

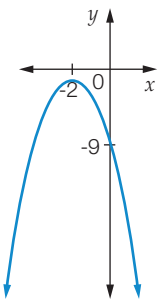
Understanding

7 The graph of the equation $y = x^2$ is transformed to the graph of the equation $y = -2(x + 2)^2 - 1$. The graph of the transformed equation is:

A


B


C


D


8 To obtain the graph of $y = -3(x + 2)^3 + 1$ from $y = x^3$:

A Dilate by a factor of 3 in the y-direction, reflect in the x-axis, translate 2 units right and translate one unit up.

B Dilate by a factor of 3 in the y-direction, reflect in the x-axis and translate 2 units left.

C Dilate by a factor of 3 in the y-direction, reflect in the x-axis, translate 2 units left and translate one unit up.

D Dilate by a factor of 3 in the y-direction, translate 1 unit left and translate 2 units down.

9 The y-intercept of the graph of the equation $y = (16x - 3)^3$ is:

A (0, 54) B (0, -24) C (0, -27) D (0, 48)

Reasoning

10 A quadratic in the form $y = a(x - h)^2 + k$ has a turning point at (1, 2) and passes through the point (2, -1).

(a) State the values of h and k.

(b) Will the value of a be positive or negative? Give a reason for your answer.

(c) Find the value of a.

11 The shape and position of a snake's body at time t seconds is described by the equation $y = \frac{(-1)^t x^3}{10} + 100t$ where x and y are in centimetres and $-10 \le x \le 10$.

(a) What is the equation describing the snake's initial shape? (Initial means $t = 0$.)

(b) Sketch the graph of the equation in part **(a)** using the basic graph $y = x^3$.

(c) On the same axes sketch the snake's shape and position:

(i) 1 second later **(ii)** 2 seconds later.

(d) Compare the position of the snake after 1 second with its initial position. State the y-value about which the snake's new position is a reflection of its initial position.

12 The cross-section of the entrance to a tunnel has the shape of the parabola $y = 5 - 0.2(x - 5)^2$, where x and y are given in metres and $0 \le x \le 10$. A second tunnel with the same basic shape is 5 m higher and has its centre 15 m to the right of the centre of the first tunnel.

(a) What is the equation describing the second tunnel's entrance?

(b) Sketch the graphs of the two equations on the same number plane.

(c) Use Pythagoras' theorem to calculate the distance, to one decimal place, between the tops of the entrances to the tunnels.

13 The shape of the front leg of an antique chair is described by the equation $y = (x - 2)^3 + 8$, where x and y are given in centimetres. The second front leg is a reflection of the first leg in the line $x = 12$.

(a) Sketch the graph of the equation describing the shape of the first leg for $0 \le x \le 6$.

(b) State the equation describing the shape of the second leg.

(c) Sketch the graph of the second leg on the same number plane as your answer to part **(a)**.

(d) What is the minimum distance between the two legs?

Open-ended

14 Starting with the equation $y = x^3$, write two cubic equations so that each equation has a reflection in the x-axis, a dilation factor between 1 and 10, a horizontal translation to the left and a vertical translation downwards.

Polynomials

10.3

A **polynomial** is an expression that contains only one variable and each term in the expression contains the variable raised to a different positive integer power. The terms are arranged in descending powers of the variable. Each term has a numerical coefficient. The coefficient of every term may be zero, except for the coefficient, a_n, of the term with the highest power, x^n. If the coefficient of a term is zero, its value is zero, so the term is not written.

For example, $6x^5$ is a polynomial where all the coefficients of the powers of x are zero except for the coefficient of x^5, which is 6.

Consider the expression $5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$.

- This is a polynomial in x as all the powers are positive integers.
- We write $P(x)$ to represent a polynomial in x.

  e.g. $P(x) = 5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$.

We can also use other capital letters such as $Q(x)$, $R(x)$ to represent polynomials.

To write a general equation of a polynomial where the highest power of the variable, usually x, is n, we need a method of identifying the coefficients of each term. As the first term, called the **leading term**, has an index of n, we write a_n as the coefficient of the leading term and it is called the **leading coefficient**.

In the polynomial $5x^6 - 8x^5 + 7x^4 + 2x^2 + 3x + 9$:

- there is no x^3 term. This means the coefficient of $x^3 = 0$.
- the power of the last term is zero. As $x^0 = 1$, the last term is a constant.

$5x^6$	$-8x^5$	$+7x^4$	$+0x^3$	$+2x^2$	$-3x$	$+9$
$a_n = a_6$	$a_{n-1} = a_5$	$a_{n-2} = a_4$	$a_{n-3} = a_3$	$a_{n-4} = a_2$	$a_{n-5} = a_1$	$a_{n-6} = a_0$
$= 5$	$= -8$	$= 7$	$= 0$	$= 2$	$= -3$	$= 9$

In a polynomial where x^n is the highest power of x, there are $(n + 1)$ terms. However, as many of the coefficients may be zero, fewer that $(n + 1)$ terms may appear in the expression. The **degree** of the polynomial is the highest power of x that appears in the expression. In a polynomial of $(n + 1)$ terms, the polynomial is of degree n.

- A polynomial can be written as $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0$, where $a^n \neq 0$, n is a positive integer and $a_0, a_1, a_2, \ldots a_n$ are the coefficients of the powers of x.
- The degree of the polynomial is the highest power of x. For $P(x)$ given above, the polynomial is of degree n.
- The leading coefficient is a_n and the constant term is a_0 (the term without an x).
- The coefficient of a monic polynomial is equal to 1, $a_n = 1$.

Worked Example 4 WE4

State the degree, leading coefficient and constant term of each polynomial.

(a) $5x^4 + 3x^3 - x^2 + x - 2$ **(b)** $12 + 6x - x^3$

Thinking		Working
(a) 1	Write the polynomial in decreasing order of powers.	(a) $5x^4 + 3x^3 - x^2 + x - 2$
2	Identify the leading term, the first term in your ordered polynomial.	The leading term is $5x^4$.
3	Write down the degree, the power of the leading term.	The degree is 4.
4	Write down the leading coefficient, the coefficient of the leading term.	The leading coefficient is 5.
5	Write down the constant term.	The constant term is -2.
(b) 1	Write the polynomial in decreasing order of powers.	(b) $-x^3 + 6x + 12$
2	Identify the leading term, the first term in your ordered polynomial.	The leading term is $-x^3$.
3	Write down the degree, the power of the leading term.	The degree is 3.
4	Write down the leading coefficient, the coefficient of the leading term.	The leading coefficient -1.
5	Write down the constant term.	The constant term is 12.

Substitution into a polynomial

If we substitute $x = 2$ into $P(x) = 5x^6 - 8x^5 + 7x^4 + 2x^2 - 3x + 9$, we write:

$$P(2) = 5(2)^6 - 8(2)^5 + 7(2)^4 + 2(2)^2 - 3(2) + 9$$
$$= 187$$

In general, if we substitute $x = a$ into $P(x)$ we write this as $P(a)$.

$P(a)$ is the value of $P(x)$ when $x = a$.

Addition and subtraction of polynomials

Polynomials can be added and subtracted by combining like terms (terms of the same degree).

Worked Example 5 WE5

$P(x) = 2x^3 - x^2 + 4x - 7$ and $Q(x) = x^3 + 3x^2 - x + 1$

(a) Evaluate $P(2)$ and $Q(-2)$. **(b)** Write the expression for $P(x) + 2Q(x)$.

Thinking		Working
(a) 1	Write the polynomial for $P(x)$.	(a) $P(x) = 2x^3 - x^2 + 4x - 7$

2	Substitute $x = 2$ and simplify.	$P(2) = 2 \times 2^3 - 2^2 + 4 \times 2 - 7$ $P(2) = 16 - 4 + 8 - 7$ $= 13$
3	Write the polynomial for $Q(x)$.	$Q(x) = x^3 + 3x^2 - x + 1$
4	Substitute $x = -2$ and simplify.	$Q(-2) = (-2)^3 + 3 \times (-2)^2 - (-2) + 1$ $Q(-2) = -8 + 12 + 2 + 1$ $= 7$
(b) 1	Write down the rule.	(b) $P(x) + 2Q(x)$
2	Expand and simplify by collecting like terms.	$= 2x^3 - x^2 + 4x - 7 + 2(x^3 + 3x^2 - x + 1)$ $= 2x^3 - x^2 + 4x - 7 + 2x^3 + 6x^2 - 2x + 2$ $= 4x^3 + 5x^2 + 2x - 5$

Multiplication by a linear expression

A linear expression is a polynomial of degree 1; that is, it is of the form $a_1x + a_0$.

Its product with a polynomial of degree n is a polynomial of degree $n + 1$.

Worked Example 6 WE6

If $P(x) = 3x - 2$ and $Q(x) = 4x^2 + x - 5$, write $P(x).Q(x)$ as a polynomial.

Thinking	Working
1 Write the polynomial product in brackets.	$P(x).Q(x) = (3x - 2)(4x^2 + x - 5)$
2 Expand the brackets by multiplying each term in the second bracket by each term in the first bracket.	$= 3x(4x^2 + x - 5) - 2(4x^2 + x - 5)$ $= 12x^3 + 3x^2 - 15x - 8x^2 - 2x + 10$
3 Simplify by adding like terms.	$= 12x^3 - 5x^2 - 17x + 10$

Multiplication by a non-linear expression

The product will be a polynomial whose degree is the sum of the degrees of the two expressions being multiplied.

Worked Example 7 WE7

If $P(x) = x^2 - 2x - 3$ and $Q(x) = 4x^3 + x^2 - 6$, write $P(x).Q(x)$ as a polynomial.

Thinking	Working
1 Write the polynomial product in brackets.	$P(x).Q(x) = (x^2 - 2x - 3)(4x^3 + x^2 - 6)$
2 Expand the brackets by multiplying each term in the second bracket by each term in the first bracket.	$= x^2(4x^3 + x^2 - 6) - 2x(4x^3 + x^2 - 6)$ $- 3(4x^3 + x^2 - 6)$ $= 4x^5 + x^4 - 6x^2 - 8x^4 - 2x^3 + 12x - 12x^3$ $- 3x^2 + 18$
3 Simplify by adding like terms.	$= 4x^5 - 7x^4 - 14x^3 - 9x^2 + 12x + 18$

Numerical long division

When the improper fraction $\frac{20}{7}$ is written as $2\frac{6}{7}$ it means that 7 divides into 20 twice and leaves a remainder of 6.

Before performing division of polynomials we need to be familiar with numerical long division without using a calculator. Now $25 \div 4 = 6\frac{1}{4}$, which means $25 = 6 \times 4 + 1$. Here, 25 is the **dividend**, 4 is the **divisor**, 6 is the **quotient** and 1 is the **remainder**.

Now consider $625 \div 4$. This can be evaluated with short division. Division by integers up to 12 can be done with short division and our tables. We can often cancel factors to reduce the divisor to an integer from 1 to 12.

If the divisor is greater than 12 and has no common factors with the dividend, we need a method called long division. The process is the same as short division. We are just writing down all the steps instead of doing them mentally.

Worked Example 8 WE8

Evaluate each expression using long division. Do not use a calculator.

(a) $3689 \div 17$ **(b)** $16\,319 \div 23$

Thinking

(a)

1 Write the divisor divided into the dividend in new notation. (Here, 17 is divided into 3689.)

2 Divide the divisor into the first number formed by the leading digits of the dividend that will give a value greater than zero but less than or equal to 9. Write this answer above the last digit used in the dividend. This is the first digit in the quotient. (Here we divide 17 into 36 to get an answer of 2. We write a 2 above the 6. This is just dividing 17 into 3600 to get an answer of 200.)

3 Multiply this answer by the divisor and write this product under the digits used in step **1**. (Here, we multiply 2 by 17 to get 34 and write 34 under 36.)

4 We subtract these numbers to find a remainder. (Here, we subtract 34 from 36 to get an answer of 2. We are finding a remainder of 200 when 3600 is divided by 17 in steps **2** and **3**.)

5 Bring down the next digit and write it after the remainder found in step **4**. (Here, we bring down the 8 to make 28.)

6 Divide the divisor into the number you have formed at step 5 and write the answer after the first digit of your quotient. ($28 \div 17 = 1$. Now we are dividing 17 into 280 to get an answer of 10.)

7 Multiply this answer by the divisor and write this product under the digits you divided into in step **6**. (Here, $1 \times 17 = 17$ and write the 17 under 28.)

Working

(a)

```
     217
17)3689
  - 34
    28
  - 17
    119
  - 119
      0
```

8 Subtract the numbers to find a remainder (28 – 17 = 11. We are finding a remainder of 110 when 280 is divided by 17.)

9 Bring down the next digit and write it after the remainder in step **8**. (Here, we bring down 9 to give 119.)

10 Divide the divisor into the number you have formed in step **9** and write the answer after the second digit in your quotient. You may have to trial a few numbers before you find the correct number. (Divide 17 into 119 to get an answer of 7.)

11 Multiply the divisor by this answer and write this product under the digits you divided into in step **10**. (17 × 7 = 119)

12 Subtract the numbers to find a remainder. (119 – 119 = 0) As there are no other digits to bring down, this means that there is no remainder in this division. If there were more digits to bring down we would continue the process until there were no more digits to be brought down. If there is no remainder, the divisor is a factor of the dividend. (Hence, 17 is a factor of 3689.)

13 Write your answer. 3689 ÷ 17 = 217

(b) **1** Write the divisor divided into the dividend in new notation. (Here, 23 is divided into 16 319.)

2 Divide the divisor into the first number formed by the leading digits of the dividend that will give an answer and write this answer above the last digit used in the dividend. This is the first digit in the quotient. (Here, we divide 23 into 163 to get an answer of 7. We write a 7 above the 3. This is just dividing 23 into 16 300 to get an answer of 700.)

3 Multiply this answer by the divisor and write this product under the digits used in step **1**. (Here, we multiply 7 by 23 to get 161 and write 161 under 163.)

4 We subtract these numbers to find a remainder. (Here, we subtract 161 from the 163 to get an answer of 2. We are finding the remainder of 200 when 16 300 is divided by 23 in steps **2** and **3**.)

5 Bring down the next digit and write it after the remainder found in step **4**. (Here, we bring down the 1 to make 21.)

6 Divide the divisor into the number you have formed at step **5** and write the answer after the first digit of your quotient. (21 ÷ 23 = 0.) If the answer is 0, write a 0 after the first digit in the quotient and bring down the next digit and write it after the remainder found in step **4**. (Here, we bring down 9 to make 219.)

(b)

```
      709
23)16319
  – 161
     219
   – 207
      12
```

7 Divide the divisor into the number you have formed in step **6** and write the answer after the third digit in your quotient. You may have to trial a few numbers before you find the correct number. (Divide 23 into 219 to get an answer of 9.)

8 Multiply the divisor by this answer and write this product under the digits you divided into in step **7**. ($23 \times 9 = 207$)

9 Subtract the numbers to find a remainder. ($219 - 207 = 12$) As there are no other digits to bring down, this means that the remainder in this division has been found. (Hence, the remainder is 12.)

10 Write your answer. $16\,319 \div 23 = 709\frac{12}{23}$

Polynomial division

We can use the same idea for polynomials. When a polynomial of degree n, $P(x)$, is divided by a linear expression $D(x)$, the result is a polynomial $Q(x)$ of degree $n - 1$ with a remainder $R(x)$.

$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ or $P(x) = D(x) \times Q(x) + R(x)$, where $P(x)$ is the dividend, $D(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder. If $D(x)$ is linear, $R(x)$ is an integer. To find $Q(x)$ and $R(x)$ we use a method similar to long division of whole numbers.

Worked Example 9

WE 9

Use long division to find the quotient and remainder for $\frac{x^2 + 6x + 10}{x + 2}$.

Thinking	Working
1 Write the expression in the required form.	$x + 2\overline{)\,x^2 + 6x + 10}$
2 Divide the first term of the dividend by the leading term of the divisor and place the result above the second term in the dividend aligning the powers of x. (Here, the dividend is $x^2 + 6x + 10$ and the divisor is $x + 2$.)	$\begin{array}{r} x \\ x + 2\overline{)\,x^2 + 6x + 10} \end{array}$
3 Multiply each term of the divisor by the first term in the quotient. Write the result underneath.	$\begin{array}{r} x \\ x + 2\overline{)\,x^2 + 6x + 10} \\ x^2 + 2x \end{array}$
4 Insert brackets and subtract the like terms.	$\begin{array}{r} x \\ x + 2\overline{)\,x^2 + 6x + 10} \\ -\,(x^2 + 2x) \\ \hline 0 + 4x \end{array}$

5 Add the next term from the dividend to the result of the subtraction.

$$\begin{array}{r} x \\ x+2\overline{)\,x^2+6x+10} \\ \underline{x^2+2x\quad\;\downarrow} \\ 0+4x+10 \end{array}$$

6 Divide the first term of the new dividend by the first term (x) of the divisor and place the result above the original dividend.

$$\begin{array}{rl} x+4 & \\ x+2\overline{)\,x^2+6x+10} & \text{original dividend} \\ \underline{x^2+2x\quad\;\downarrow} & \\ 4x+10 & \text{new dividend} \end{array}$$

7 Multiply each term of the divisor by the second term in the quotient. Write the result underneath.

$$\begin{array}{r} x+4 \\ x+2\overline{)\,x^2+6x+10} \\ \underline{x^2+2x\qquad} \\ 4x+10 \\ 4x+8 \end{array}$$

8 Insert brackets and subtract the like terms.

$$\begin{array}{rl} x+4 & \text{quotient} \\ x+2\overline{)\,x^2+6x+10} & \\ \underline{x^2+2x\qquad} & \\ 4x+10 & \\ \underline{-(4x+8)} & \\ 0+2 & \text{remainder} \end{array}$$

9 Stop when the degree of the remainder is less than the degree of the divisor. State the result. We are dividing by a linear expression so the remainder will be a constant.

$$\frac{x^2+6x+10}{x+2} = x+4+\frac{2}{x+2}$$

We can also write this result as $x^2 + 6x + 10 = (x + 2)(x + 4) + 2$.

You could also consider the long division process in Worked Example 9 in the following form:

$$\begin{aligned} & x^2 + 6x + 10 \\ = {} & x^2 + 2x + 4x + 10 \\ = {} & x(x + 2) + 4x + 8 + 2 \\ = {} & x(x + 2) + 4(x + 2) + 2 \\ = {} & (x + 2)(x + 4) + 2 \end{aligned}$$

To make $(x + 2)$ a factor of the next two terms we use $4x + 8 = 4(x + 2)$.

All the powers of x in the dividend (up to the degree of the polynomial) should be shown in the calculations. This may require the insertion of coefficients that are zero.

The remainder must be zero or have a degree less than the degree of the divisor.

Worked Example 10

WE 10

Use long division to find the quotient and remainder for $\dfrac{8x^3 + 4x - 1}{2x + 3}$.

Thinking	Working
1 Write the expression in the required form inserting a zero coefficient for any powers of x that are missing.	$2x+3\overline{)\,8x^3 + 0x^2 + 4x - 1}$
2 Divide the first term of the dividend by the leading term of the divisor. Multiply each term of the divisor by this term in the quotient. Insert brackets and subtract the like terms.	$4x^2$ $2x+3\overline{)\,8x^3 + 0x^2 + 4x - 1}$ $-(8x^3 + 12x^2)$ $0 - 12x^2$
3 Bring down the next term from the dividend. Repeat step **2**.	$4x^2 - 6x$ $2x+3\overline{)\,8x^3 + 0x^2 + 4x - 1}$ $-(8x^3 + 12x^2)$ $0 - 12x^2 + 4x$ $-(-12x^2 - 18x)$ $0 + 22x$
4 Bring down the next term from the dividend. Repeat step **2**. Stop when the degree of the remainder is less than the degree of the divisor.	$4x^2 - 6x + 11$ $2x+3\overline{)\,8x^3 + 0x^2 + 4x - 1}$ $8x^3 - 12x^2$ $0 - 12x^2 + 4x$ $-(-12x^2 - 18x)$ $0 + 22x - 1$ $-(22x + 33)$ $0 - 34$
5 State the result.	$\dfrac{8x^3 + 4x - 1}{2x + 3} = 4x^2 - 6x + 11 - \dfrac{34}{2x + 3}$

The result can also be written as $8x^3 + 4x - 1 = (2x + 3)(4x^2 - 6x + 11) - 34$.

- When adding and subtracting polynomials, only combine like terms.
- When expanding the product of binomial expressions, work through the expression in order, being very careful with negative signs.
- When dividing polynomials, make sure that all the powers less than the degree of the dividend appear in your expression for the dividend. This may mean the introduction of zero coefficients.

10.3 Polynomials

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 Column 1, Q7, Q8, Q9 Column 1, Q10, Q11, Q12, Q13, Q14, Q16, Q18	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9 Column 2, Q10, Q11, Q12, Q13, Q14, Q15, Q17, Q19	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9 Column 2, Q10, Q11, Q12, Q14, Q15, Q16, Q17, Q19, Q20

Answers page 798

Fluency

1 State the degree, leading coefficient and constant term of each polynomial. WE4

(a) $2x^3 + 5x^2 - x + 2$ (b) $5 - 4x^4 - 3x^3 + 7x^2$

(c) x (d) $-6 - x - x^2$

2 If $P(x) = 3x^3 - 2x^2 - x + 2$ and $Q(x) = -x^3 - 4x^2 - 5x + 6$, evaluate: WE5

(a) $P(3)$ (b) $Q(4)$ (c) $P(-2)$ (d) $Q(-1)$

3 Write each expression as a polynomial if $P(x) = 5x + 1$, $Q(x) = x^2 + 2x - 3$ and $R(x) = 2x^3 - x^2 - x + 6$. WE6

(a) $P(x).\ Q(x)$ (b) $P(x).\ R(x)$

4 If $P(x) = x^2 - 3x + 4$ and $Q(x) = 2x^3 + 3x^2 - 5$, write $P(x).\ Q(x)$ as a polynomial. WE7

5 Evaluate each expression by performing long division. Do not use a calculator. WE8

(a) $3715 \div 25$ (b) $4061 \div 12$ (c) $5924 \div 17$

(d) $2018 \div 19$ (e) $14\,797 \div 29$ (f) $10\,946 \div 123$

6 Use long division to find the quotient and remainder for: WE9

(a) $\dfrac{x^2 + 6x + 10}{x + 3}$ (b) $\dfrac{x^2 + 3x - 1}{x - 2}$

7 Use long division to find the quotient and remainder for: WE10

(a) $\dfrac{x^3 + 6x + 5}{x + 2}$ (b) $\dfrac{x^4 - 3x^2 + 1}{x - 3}$

Understanding

8 Write $-3A(x)\ B(x) + C(x)$ as a polynomial if $A(x) = -x + 4$, $B(x) = 4x^2 - 2x + 1$ and $C(x) = -x^2 + 3x - 8$.

9 Find the quotient and remainder for $\dfrac{P(x)}{Q(x)}$ if:

(a) $P(x) = x^2 + 7x + 13$, $Q(x) = x + 3$ (b) $P(x) = 4x^2 + 4$, $Q(x) = 2x - 1$

(c) $P(x) = 2x^2 - 3x + 1$, $Q(x) = x + 1$ (d) $P(x) = 3x^3 + 2x - 4$, $Q(x) = x - 2$

10 If $P(x) = -5x^3 - 8x^2 - 3x + 2$ and $Q(x) = x^3 + 4x^2 - x + 1$ and $R(x) = 2P(x) - 2Q(x)$, the leading coefficient of $R(x)$ is:

A -12 B -5 C 1 D 8

11 Show by long division that the remainder is zero when $4x^2 + x - 3$ is divided by $4x - 3$.

12 The remainder when $x^3 + x^2 + x + 1$ is divided by $x - 1$ is:

A -5 B -3 C 2 D 4

13 Calculate $\frac{A(x)}{B(x)}$ when $A(x) = x^4 - 6x^3 + 5$ and $B(x) = x - 5$.

14 $P(x) = 3x^3 - 2x^2 - x + 2$ and $Q(x) = -x^3 - 4x^2 - 5x + 6$.

(a) If $R(x) = 5P(x) - Q(x)$, write the polynomial $R(x)$ and evaluate $R(2)$.

(b) If $R(x) = 2P(x) + 3Q(x)$, write the polynomial $R(x)$ and evaluate $R(-3)$.

(c) If $6P(x) = 2R(x) - 10Q(x)$, write the polynomial $R(x)$ and evaluate $R(-1)$.

Reasoning

15 **(a)** Calculate $\frac{A(x)}{B(x)}$ when $A(x) = 2x^5 - 3x^4 - 6x^2 - 3x + 18$ and $B(x) = 2x - 3$.

(b) What is the value of $\frac{A(x)}{B(x)}$ for $x = 2$?

(c) Evaluate $A(2)$ and $B(2)$ and find $\frac{A(2)}{B(2)}$. Compare your answer to your answer in **(b)**.

16 Alana has $P(x)$ dollars, Bruno has $Q(x)$ dollars and Cien has $\frac{P(x)}{Q(x)}$ dollars, where $P(x) = 12x^4 + 16x^3 + 3x^2 + 28x + 32$ and $Q(x) = 3x + 4$.

(a) Write the polynomial that describes the amount of money Cien has.

(b) **(i)** Dylan has x dollars. What is the ratio of Cien's amount of money to Dylan's?

(ii) Use your answer to part **(i)** to explain why x cannot be zero.

17 A sheet of pastry is $(2x^2 - 9x)$ cm long and $(4x + 3)$ cm wide.

(a) Write an expression for the perimeter of the sheet of pastry.

(b) Write an expression for the area of the sheet of pastry.

(c) **(i)** The sheet of pastry is to be divided into small rectangles of side lengths x cm and $(x - 1)$ cm. Write an expression for the largest number of small rectangles that could be cut out of the pastry if any unused pastry is rerolled to the same thickness to maximise the number of squares that could be formed.

(ii) If all the pastry is used (there is no remainder), what is the smallest possible value of x and how many rectangles can be formed?

Open-ended

18 Write a non-monic polynomial of degree 5 that contains four terms and has a negative constant term.

19 Find a polynomial $P(x)$ of degree 3 so that $\frac{P(x)}{x-3}$ leaves no remainder.

20 $P_n(x)$ is a polynomial of degree n. It has 1 for all its coefficients so that, for example, $P_2(x) = x^2 + x + 1$ and $P_3(x) = x^3 + x^2 + x + 1$.

(a) Write an expression for $\frac{P_n(x)}{x+1}$ for $n = 1, 2, 3, 4, 5$.

(b) By considering your results in part **(a)**, show that if n is odd, $\frac{P_n(x)}{x+1}$ is a polynomial of degree $n - 1$ consisting of even powers of x.

(c) By considering your results in part **(a)**, show that if n is even, $\frac{P_n(x)}{x+1}$ is a polynomial of degree $n - 1$ consisting of odd powers of x plus a remainder of $\frac{1}{x+1}$.

Factorising non-monic quadratic trinomials

10.4

Up to now, the only quadratic trinomials you have factorised have had a coefficient of 1 for the squared term unless there was a common factor that could be taken out of the expression. We are now going to factorise **non-monic quadratic** trinomials in the form of $ax^2 + bx + c$ for all values of a.

Consider $(2x - 5)(3x + 7)$. We can expand this:

$$\begin{aligned} &(2x - 5)(3x + 7) \\ &= 6x^2 + 14x - 15x - 35 \\ &= 6x^2 - x - 35 \end{aligned}$$

Reversing the process:

$$\begin{aligned} &6x^2 - x - 35 \\ &= 6x^2 + 14x - 15x - 35 \\ &= 2x(3x + 7) - 5(3x + 7) \\ &= (2x - 5)(3x + 7) \end{aligned}$$

We use grouping in pairs to factorise by splitting up the middle term, but choosing the correct pair of like terms can be time consuming if we use a trial and error approach.

To factorise $6x^2 - x - 35$ we need to follow the same process as we used for factorising monic quadratics, but we now need to consider the coefficient of the squared term as well as the constant term.

Method 1: Split the middle term

Multiply the coefficient of the squared term a by the constant c:

For $\quad ax^2 + bx + c$
$\quad\quad = 6x^2 - x - 35$

$a = 6$, $b = -1$, $c = -35$

$$\begin{aligned} ac &= 6 \times -35 \\ &= -210 \end{aligned}$$

List factor pairs of -210:

5, -42; 10, -21; 15, -14; 30, -7; …

This can be a lengthy process, but we can simplify it by using some simple strategies.

For a positive constant term:

- pairs will have the same sign as the middle term
- if the magnitude of the coefficient of the middle term is small the factors will be close to each other
- if the magnitude of the coefficient of the middle term is large the factors will be far apart.

For a negative constant term:

- pairs will be opposite in sign

- if the magnitude of the coefficient of the middle term is small, choose factors that are close in magnitude but opposite in sign.
- if the magnitude of the coefficient of the middle term is large, choose factors that are far apart in magnitude but opposite in sign.

As b is small and negative (-1), we will choose factor pairs close to each other but opposite in sign (14 and -15).

We can now write $6x^2 - x - 35 = 6x^2 + 14x - 15x - 35$ and factorise by grouping in pairs as shown on the previous page.

Worked Example 11

WE11

Factorise the following quadratic expressions by splitting the middle term.

(a) $3x^2 - 11x - 4$ **(b)** $5x^2 - 43x + 24$ **(c)** $12x^2 + 11x - 36$

Thinking		Working	
(a) 1	Multiply the coefficient of x^2 and the constant term ($3 \times -4 = -12$) and choose a factor pair that adds to give the coefficient of the middle term ($-12 + 1 = -11$). Rewrite the expression using these factors as coefficients of two middle terms to replace the middle term.	(a) $3x^2 - 11x - 4$ $= 3x^2 - 12x + x - 4$	× to give -12 + to give -11 } -12, 1
2	Factorise using grouping in pairs.	$= 3x(x - 4) + 1(x - 4)$	
3	Take out the common factor.	$= (x - 4)(3x + 1)$	
(b) 1	Multiply the coefficient of x^2 and the constant term ($5 \times 24 = 120$) and choose a factor pair that adds to give the coefficient of the middle term ($-40 + -3 = -43$). Rewrite the expression using these factors as coefficients of two middle terms to replace the middle term.	(b) $5x^2 - 43x + 24$ $= 5x^2 - 40x - 3x + 24$	× to give 120 + to give -43 } -40, -3
2	Factorise using grouping in pairs.	$= 5x(x - 8) - 3(x - 8)$	
3	Take out the common factor.	$= (x - 8)(5x - 3)$	

(c) **1**	Multiply the coefficient of x^2 and the constant term ($12 \times -36 = -432$) and choose a factor pair that adds to give the coefficient of the middle term ($27 + -16 = 11$). Rewrite the expression using these factors as coefficients of two middle terms to replace the middle term.	(c) $12x^2 + 11x - 36$ $= 12x^2 + 27x - 16x - 36$	$\times$ to give -432 + to give 11 } 27, -16
2	Factorise using grouping in pairs.	$= 3x(4x + 9) - 4(4x + 9)$	
3	Take out the common factor.	$= (4x + 9)(3x - 4)$	

Method 2: The cross method

Identify factor pairs for the coefficient of the squared term:

For $6x^2 - x - 35$, 6 has factor pairs of 6, 1 and 3, 2 (we will not consider the negative pairs).

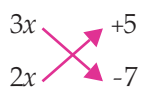
Choose a factor pair that gives the squared term ($3x \times 2x = 6x^2$).

Identify factor pairs for the constant term:

For $6x^2 - x - 35$, -35 has factor pairs of 5, -7; 1, -35; 7, -5 and -1, 35.

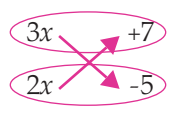
Use the cross method to trial factor pairs:

For $6x^2 - x - 35$, the coefficient of the middle term is -1 so we will try factors that are close to each other in magnitude.



Middle term: $-21x + 10x = -11x$

(not correct in magnitude or sign)



Middle term: $-15x + 14x = -x$

(correct in magnitude and sign)

The circled expressions give us our factors.

$6x^2 - x - 35 = (3x + 7)(2x - 5)$

Worked Example 12

WE 12

Factorise the following quadratic expressions using the cross method.

(a) $3x^2 - 11x - 4$ **(b)** $5x^2 - 43x + 24$ **(c)** $12x^2 + 11x - 36$

Thinking	Working
(a) 1 Choose a factor pair that multiply to give the squared term. ($3x \times x = 3x^2$). Choose a factor pair that multiply to give the constant term ($-4 \times 1 = -4$). Use the cross method to find the correct factors.	(a) $3x^2 - 11x - 4$ $3x$ -4 x $+1$ $3x - 4x = -x$ (incorrect middle term) $3x$ $+1$ x -4 $-12x + x = -11x$ (correct middle term)
2 Write down the factorisation.	$3x^2 - 11x - 4$ $= (3x + 1)(x - 4)$
(b) 1 Choose a factor pair that multiply to give the squared term ($5x \times x = 5x^2$). Choose a factor pair that multiply to give the constant term ($-8 \times -3 = 24$). If the middle term is negative we use negative factor pairs, and if the middle term is quite large in magnitude we use factors that are not too close. Use the cross method to find the correct factors.	(b) $5x^2 - 43x + 24$ $5x$ -8 x -3 $-15x - 8x = -23x$ (incorrect middle term) $5x$ -3 x -8 $-40x - 3x = -43x$ (correct middle term)
2 Write down the factorisation.	$5x^2 - 43x + 24$ $= (5x - 3)(x - 8)$
(c) 1 Choose a factor pair that multiply to give the squared term ($4x \times 3x = 12x^2$). Choose a factor pair that multiply to give the constant term ($-9 \times 4 = -36$). Use the cross method to find the correct factors. If the pairs chosen do not give the correct magnitude, try another pair. If the magnitude is correct but the sign is incorrect, swap the signs on the factors.	(c) $12x^2 + 11x - 36$ $3x$ $+4$ $4x$ -9 $-27x + 16x = -11x$ (incorrect middle term) $3x$ -4 $4x$ $+9$ $27x - 16x = +11x$ (correct middle term)
2 Write down the factorisation.	$12x^2 + 11x - 36$ $= (4x + 9)(3x - 4)$

Note that in Worked Example 12(c) we did not try $(4x - 4)(3x + 9)$ as there would have been a common factor in both brackets. We also use our knowledge of odds and evens to reject unsuitable factor pairs.

Completing the square with non-monic quadratic expressions

As we have seen previously, not all quadratic trinomials factorise with binomial products and some do not factorise at all. We can use 'completing the square' with non-monic quadratics to see if the expression can be factorised.

To complete the square, the coefficient of the quadratic must be 1, so we will need to take the coefficient of x out as a common factor before we can complete the square.

Worked Example 13 WE13

Factorise the following expressions if possible by completing the square

(a) $3x^2 - 6x - 5$ **(b)** $2x^2 - 5x - 2$ **(c)** $2x^2 - 4x + 6$

Thinking	Working
(a) 1 Take out the coefficient of the squared term as a factor of the expression. (Write 3 outside the brackets and divide every term inside the brackets by 3.)	(a) $3x^2 - 6x - 5$ $= 3(x^2 - 2x - \frac{5}{3})$
2 Complete the square inside the brackets and simplify (add and subtract 1).	$= 3(x^2 - 2x + 1 - 1 - \frac{5}{3})$
3 Factorise inside the brackets using the difference of two squares and surds.	$= 3[(x-1)^2 - \frac{8}{3}]$ $= 3(x - 1 - \sqrt{\frac{8}{3}})(x - 1 + \sqrt{\frac{8}{3}})$
(b) 1 Take out the coefficient of the squared term as a factor of the expression. (Write 2 outside the brackets and divide every term inside the brackets by 2.)	(b) $2x^2 - 5x - 2$ $= 2(x^2 - \frac{5}{2}x - 1)$
2 Complete the square inside the brackets and simplify (add and subtract $\frac{25}{16}$).	$= 2[(x^2 - \frac{5}{2}x + \frac{25}{16}) - \frac{25}{16} - 1]$
3 Factorise inside the brackets using the difference of two squares and surds.	$= 2[(x - \frac{5}{4})^2 - \frac{41}{16}]$ $= 2(x - \frac{5}{4} + \frac{\sqrt{41}}{4})(x - \frac{5}{4} - \frac{\sqrt{41}}{4})$
(c) 1 Take out the coefficient of the squared term as a factor of the expression. (Write 2 outside the brackets and divide every term inside the brackets by 2.)	(c) $2x^2 - 4x + 6$ $= 2(x^2 - 2x + 3)$
2 Complete the square inside the brackets and simplify (add and subtract 1).	$= 2[(x^2 - 2x + 1) - 1 + 3]$
3 As we do not have a difference of two squares we cannot factorise further.	$= 2[(x-1)^2 + 2]$ The expression cannot be factorised further.

10.4 Factorising non-monic quadratic trinomials

Navigator

Answers page 798

Q1 Column 1, Q2 Column 1, Q3 Column 1, Q4, Q6, Q7	Q1 Column 2, Q2 Column 2, Q3 Column 2, Q4, Q5, Q6, Q7	Q1 Column 3, Q2 Column 3, Q3 Column 3, Q4, Q5, Q6, Q7

Fluency

WE11

1 Factorise these expressions by splitting the middle term.

(a) $2x^2 + 5x + 2$ **(b)** $2y^2 + 9y + 7$ **(c)** $5b^2 + 8b + 3$

(d) $3k^2 - 11k + 6$ **(e)** $11a^2 - a - 12$ **(f)** $5x^2 - 11x - 12$

(g) $2m^2 + 3m - 9$ **(h)** $7x^2 - 23x + 18$ **(i)** $11c^2 - 25c + 6$

(j) $5y^2 + 19y + 12$ **(k)** $3p^2 + 8p - 16$ **(l)** $2x^2 + 3x - 20$

(m) $3x^2 + 8x + 5$ **(n)** $7y^2 - 18y + 8$ **(o)** $3t^2 + t - 4$

(p) $7p^2 + 22p + 16$ **(q)** $3q + 2q^2 - 5$ **(r)** $4x + 5x^2 - 12$

(s) $8 - 18r + 7r^2$ **(t)** $21 - 16c + 3c^2$ **(u)** $2 - 10t + 12t^2$

WE12

2 Factorise these expressions using the cross method.

(a) $4x^2 + 8x + 3$ **(b)** $10y^2 + 13y + 4$ **(c)** $9c^2 - 19c + 2$

(d) $6w^2 + 17w + 12$ **(e)** $15y^2 + 7y - 30$ **(f)** $8x^2 - 9x - 14$

(g) $4a^2 + 12a - 27$ **(h)** $24k^2 - 2k - 15$ **(i)** $20x^2 - 44x + 21$

(j) $12b^2 + 32b + 5$ **(k)** $6g^2 + 7g + 2$ **(l)** $8a^2 + 22a + 5$

(m) $4x^2 - 11x + 6$ **(n)** $10m^2 + 17m + 7$ **(o)** $12p^2 - 7p - 10$

(p) $9d^2 + 6d - 8$ **(q)** $6x^2 - x - 22$ **(r)** $18n^2 + 31n - 35$

(s) $24y^2 - 17y + 3$ **(t)** $15x^2 + 46x + 16$ **(u)** $12x^2 - 11x - 15$

WE13

3 Factorise the following expressions, if possible, by completing the square.

(a) $2x^2 - 6x - 1$ **(b)** $3x^2 - 12x - 7$ **(c)** $2x^2 - 8x - 3$

(d) $3x^2 - 9x + 4$ **(e)** $5x^2 - 10x + 3$ **(f)** $2x^2 - 7x + 2$

(g) $2x^2 - 4x + 5$ **(h)** $2x^2 - 8x + 9$ **(i)** $3x^2 - 6x + 5$

Understanding

4 The floor (in m^2) of a rectangular room of height x m is given by the expression $6x^2 - 7x - 3$.

(a) Factorise the expression.

(b) Assuming that the two factors represent the length and the width of the room and that the length is 11.5 m, find the height of the room.

(c) What is the width of the room?

Reasoning

5 The number of bird nests in a park depends on the number of trees it has. In Byrd Park there are $12t^2 - 12t + 10$ bird nests, where t is $\frac{1}{100}$ of the number of trees. (The number of trees is estimated to the nearest multiple of 25.)

Plume Park has $4t^2 + 6t + 1$ bird nests.

(a) Write an expression in terms of t for the difference in the number of bird nests in the two parks. (Assume that the number of bird nests in Byrd Park is greater than the number of bird nests in Plume Park for all values of t.)

(b) **(i)** Simplify the expression in part **(a)**.

(ii) Factorise the expression in **(i)**.

(c) Use your answer from **(b) (ii)** to show that the parks will have equal numbers of birds when there are 75 or 150 trees.

6 Paul and Selina each have \$$x$. Later, each person has more money and the product of their money is $12x^2 - 8x - 15$.

(a) Factorise the expression.

(b) Find how much money each person had initially if Paul now has \$65 and has more money than Selina.

(c) How much money does Selina have now?

Open-ended

7 **(a)** Find a non-monic quadratic trinomial that will factorise to give binomial products. Factorise it.

(b) Find a non-monic quadratic trinomial that will factorise but not as binomial products. Factorise it.

(c) Find a non-monic quadratic trinomial that will not factorise. Explain why it cannot be factorised.

Half-time 10

Ex 10.1

1 Graph each equation by constructing a table of values for $x = -3$ to 3.

(a) $y = 4x^2$ **(b)** $y = \frac{1}{4}x^3$

Ex 10.3

2 Find the quotient and remainder for $\frac{3x^3 + 4x^2 - 20x - 5}{3x + 1}$.

Ex 10.3

3 If $P(x) = 3x - 4$ and $Q(x) = 3x^2 - 2x + 1$, write an expression for:

(a) **(i)** $P(4)$ **(ii)** $Q(-2)$ **(iii)** $P(-1).\ Q(-1)$

(b) $2P(x) + 3Q(x)$

(c) $P(x).Q(x)$

(d) $Q(x) - xP(x)$

Ex 10.2

4 **(a)** Sketch the graph of $y = 2x^2 + 3$ by starting with $y = x^2$.

(b) Sketch the graph of $y = -3(x - 2)^3 + 1$ by starting with $y = x^3$.

(c) Sketch the graph of $y = 2(x + 1)^4$ by starting with the graph of $y = x^4$.

Ex 10.3

5 Use polynomial division to find the remainder produced by each division.

(a) $\frac{9x^3 + 6x^2 - 7x - 3}{x + 1}$ **(b)** $\frac{x^4 - x^3 + x^2 + x - 1}{3x - 1}$

Ex 10.4

6 Factorise $2x^2 + 8x + 3$ by completing the square.

Ex 10.2

7 The vertical cross-section of a bottle is described by the equations $y = (x - 4)^3 + 1$ and $y = -(x - 14)^3 + 1$.

(a) Graph both equations on the same axes using x-values from 1 to 17.

(b) State the equation of the axis of symmetry of the bottle.

Ex 10.3

8 Write $x^3 + x - 5 = (x - 1)(x^2 + x + 2) - 3$ as a division of one polynomial divided by another.

Ex 10.4

9 Factorise the following quadratic expressions.

(a) $6x^2 + x - 35$ **(b)** $10x^2 - 17x + 3$ **(c)** $12x^2 + 19x + 4$

Ex 10.3

10 By performing a division, show that:

(a) $x - 1$ is a factor of $x^3 - 6x^2 - 9x + 14$

(b) $x - 2$ is a factor of $x^4 - 2x^3 - 3x^2 + 8x - 4$

Solution of non-monic quadratics

10.5

In Chapter 9, we solved monic quadratic equations. Now that we can factorise non-monic equations we use these techniques to solve non-monic quadratic equations. We write the equation as $ax^2 + bx + c = 0$ where $a \neq 0$ or 1. We factorise the expression $ax^2 + bx + c = 0$, $a \neq 0$ or 1, by splitting the middle term, using the cross method or completing the square. Once the expression is factorised we can use the Null Factor Law to solve the equation.

Worked Example 14

WE14

Solve the quadratic equation $3x^2 - 11x - 4 = 0$. Factorise the expression first by splitting the middle term.

Thinking	Working
1 Write down the quadratic equation.	$3x^2 - 11x - 4 = 0$
2 Identify a, b and c.	$a = 3$, $b = -11$, $c = -4$
3 Form the product ac.	$ac = 3 \times (-4) = -12$
4 Find two numbers that multiply to give ac and add to give b.	$\left.\begin{array}{l} \times \text{ to give } -12 \\ + \text{ to give } -11 \end{array}\right\} -12, 1$
5 Rewrite the LHS of the equation using these numbers as coefficients of two middle terms.	$3x^2 - 12x + x - 4$
6 Factorise the LHS by grouping in pairs.	$3x(x - 4) + 1(x - 4) = 0$ $(x - 4)(3x + 1) = 0$
7 Use the Null Factor Law to solve the equation.	$x - 4 = 0$ or $3x + 1 = 0$ $x = 4$ or $x = -\frac{1}{3}$

Worked Example 15

WE15

Solve $6x^2 - 17x + 12 = 0$. Factorise the expression first using the cross method.

Thinking	Working
1 Write down the quadratic equation.	$6x^2 - 17x + 12 = 0$
2 Identify a, b and c.	$a = 6$, $b = -17$, $c = 12$

3 Find factor pairs for the first term. (Here, we use $3x$ and $2x$). Try factor pairs for the last term. (Here, we try -4 and -3.) We do not try -3 and -4, as this arrangement contains a common factor. We do not try positive factor pairs, as the middle term is negative.

$$\begin{array}{ll} 3x & -4 \\ 2x & -3 \end{array}$$

$-9x - 8x = -17x$

4 Write down the equation in factorised form.

$(3x - 4)(2x - 3) = 0$

5 Use the Null Factor Law to solve the equation.

$3x - 4 = 0$ or $2x - 3 = 0$

$3x = 4$ or $2x = 3$

$x = \frac{4}{3}$ or $x = \frac{3}{2}$

Worked Example 16

WE16

Solve $2x^2 - 7x + 2 = 0$. Factorise the expression first, by completing the square on x.

Thinking	Working
1 Write down the quadratic equation.	$2x^2 - 7x + 2 = 0$
2 Take out the coefficient of x^2 as a factor of each term.	$2\left(x^2 - \frac{7}{2}x + 1\right) = 0$
3 Divide both sides by this factor.	$x^2 - \frac{7}{2}x + 1 = 0$
4 Complete the square. (Here, we add and subtract $\left(\frac{7}{4}\right)^2$.)	$x^2 - \frac{7}{2}x + \left(\frac{7}{4}\right)^2 + 1 - \left(\frac{49}{16}\right) = 0$
5 Use the Perfect Square rule to factorise the first three terms and simplify the remaining terms.	$\left(x - \frac{7}{4}\right)^2 - \frac{33}{16} = 0$
6 Factorise further by using the Difference of Two Squares rule and surds. (Here, $\frac{33}{16} = \left(\frac{\sqrt{33}}{4}\right)^2$.)	$\left(x - \frac{7}{4} + \frac{\sqrt{33}}{4}\right)\left(x - \frac{7}{4} - \frac{\sqrt{33}}{4}\right) = 0$
7 Use the Null Factor Law to solve the equation.	$x - \frac{7}{4} + \frac{\sqrt{33}}{4} = 0$ or $x - \frac{7}{4} - \frac{\sqrt{33}}{4} = 0$ $x = \frac{7}{4} - \frac{\sqrt{33}}{4}$ or $x = \frac{7}{4} + \frac{\sqrt{33}}{4}$ $x = \frac{7 \pm \sqrt{33}}{4}$

The quadratic formula

When solving a quadratic equation we can use the quadratic formula to factorise the quadratic expression. The quadratic formula is developed by completing the square in the quadratic equation $ax^2 + bx + c = 0$.

Deriving the quadratic formula:

1 Write down the equation.	$ax^2 + bx + c = 0$
2 Take out a as a common factor.	$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$
3 Divide both sides by a.	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
4 Complete the square and balance the equation by adding and subtracting $\left(\frac{1}{2} \times \frac{b}{a}\right)^2$.	$\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$
5 Identify the first three terms as a perfect square and factorise them.	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
6 Write the other two terms with a common denominator.	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = 0$
7 Combine these terms with a common denominator. Be careful to change the sign in the numerator.	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$
8 Express as the difference of two squares.	$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$
9 Factorise as the difference of two squares.	$\left(x + \frac{b}{2a} + \sqrt{\frac{b^2 - 4ac}{4a^2}}\right)\left(x + \frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{4a^2}}\right) = 0$
	$\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right) = 0$
10 Use the Null Factor Law.	$x + \frac{b + \sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b - \sqrt{b^2 - 4ac}}{2a} = 0$
11 Solve each equation.	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
12 Write the solution in a shorthand form.	The solution to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked Example 17

WE17

Use the quadratic formula to solve $3x^2 - 4x - 5 = 0$. Give your answers in exact form and rounded to two decimal places.

Thinking	Working
1 Write down the quadratic expression.	$3x^2 - 4x - 5 = 0.$
2 Identify a, b and c.	$a = 3,\ b = -4,\ c = -5$
3 Write down the quadratic formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4 Substitute the values for a, b and c.	$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-5)}}{2 \times 3}$
5 Simplify.	$x = \dfrac{4 \pm \sqrt{16 + 60}}{6}$
6 State your answer in exact form.	$= \dfrac{4 \pm \sqrt{76}}{6}$
7 Use your calculator to evaluate.	$x = 2.12$ or $x = -0.79$

In Chapter 12, you will learn to simplify surds so $\dfrac{4 \pm \sqrt{76}}{6}$ can be written as $\dfrac{2 \pm \sqrt{19}}{3}$.

Solving non-monic quadratic equations: $ax^2 + bx + c = 0$

Factorise non-monic quadratic expressions by:

- splitting the middle term
- using the cross method
- completing the square.

Solve non-monic quadratic equations by:

- using the Null Factor Law
- using the quadratic formula. The solution to $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

10.5 Solution of non-monic quadratics

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q13, Q14, Q16, Q17, Q20	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19 (a) (b), Q20	Q1, Q2, Q3 (c) (d), Q4 (b) (c), Q5, Q6, Q7, Q8, Q9, Q11, Q12, Q13, Q14, Q15, Q16, Q17, Q18, Q19, Q20, Q21, Q22

Answers page 800

Equipment required: calculator for Question **4**

Fluency

1 Solve each quadratic equation. Factorise the expression first by splitting the middle term: WE14

(a) $3x^2 - 4x - 4 = 0$ **(b)** $3x^2 + 8x + 4 = 0$

(c) $5x^2 - 11x + 6 = 0$ **(d)** $4x^2 + 3x - 10 = 0$

2 Solve each equation. Factorise the expression first using the cross method. WE15

(a) $5x^2 - 17x + 12 = 0$ **(b)** $3x^2 + 8x + 5 = 0$

(c) $6x^2 - x - 2 = 0$ **(d)** $8x^2 + 6x - 9 = 0$

3 Solve each equation. Factorise the expression first by completing the square on x. WE16

(a) $2x^2 - 5x + 2 = 0$ **(b)** $3x^2 + 10x + 1 = 0$

(c) $5x^2 - 2x - 6 = 0$ **(d)** $6x^2 + 9x - 2 = 0$

4 Use the quadratic formula to solve each equation. Give your answers in **(i)** exact form and **(ii)** rounded to two decimal places. WE17

(a) $3x^2 - 5x - 4 = 0$ **(b)** $5x^2 + 10x + 1 = 0$

(c) $2x^2 - 11x + 6 = 0$ **(d)** $x^2 + 5x - 3 = 0$

5 For each of these quadratic equations:

(a) $x^2 + 5x + 2 = 0$ **(b)** $2x^2 - 3x + 1 = 0$

(c) $-3x^2 - x + 3 = 0$ **(d)** $5x^2 + 4x = 0$

(i) list the values that a, b and c represent if the equation is of the form $ax^2 + bx + c = 0$

(ii) find the solution to each equation using the quadratic formula.

6 The solutions to the equation $10x^2 - 23x + 12 = 0$ are:

A $x = 3$ or $x = 4$ **B** $x = \frac{2}{3}$ or $x = \frac{5}{4}$

C $x = 1.5$ or $x = 0.8$ **D** $x = -1.5$ or $x = -0.8$

Understanding

7 Solve each quadratic equation using an appropriate method.

(a) $3x^2 - 5x + 2 = 0$ **(b)** $5x^2 + 10x + 2 = 0$

(c) $4x^2 - 9 = 0$ **(d)** $x^2 + 7x - 9 = 0$

(e) $6x^2 - 7x - 10 = 0$ **(f)** $4x^2 + 4x + 1 = 0$

(g) $7x^2 - 6 = 0$ **(h)** $8x^2 + 6x - 4 = 0$

(i) $36x^2 - 60x + 25 = 0$ **(j)** $3x^2 - 11x + 7 = 0$

8 Which of the following equations has the solutions $x = 1.5$ or $x = 2.5$?

A $x^2 - 16x + 15 = 0$

B $4x^2 - 16x + 15 = 0$

C $x^2 - 8x + 15 = 0$

D $4x^2 + 16x + 15 = 0$

9 The solutions to $x^2 + px + q = 0$ are:

A $x = \frac{-p \pm \sqrt{q}}{2}$

B $x = \frac{p \pm \sqrt{p^2 + 4q}}{2}$

C $x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$

D $x = \frac{-2p \pm \sqrt{p^2 - 4q}}{2}$

10 Rick's driving speed was measured over a 10-minute period and the following relationship was found to exist: $s = -4t^2 + 31t$, where s is the speed in kilometres per hour after t minutes. When was Rick travelling at 60 km/h?

11 The temperature inside a tent was measured over a period of time and the following quadratic relationship was found to exist: $T = -2h^2 + 11h + 21$, where T is the temperature in degrees Celsius after h hours.

(a) When was the temperature 0°C?

(b) When was the temperature 26°C?

12 A tourist, high above the ground enjoying the sights from a hot-air balloon, unfortunately drops a camera and watches it fall to the ground. The height of the camera above the ground, h (in metres), t seconds after it has been dropped can be represented by the relationship:

$h = -5t^2 + 192$

(a) At what height above the ground is the camera dropped?

(b) How long does it take for the camera to fall to the ground? Round your answer to one decimal place.

Reasoning

13 Use the quadratic formula to show that none of the following equations has a solution. Write the part of the quadratic formula that indicates that no solutions are possible.

(a) $2x^2 - 3x + 2 = 0$

(b) $5x^2 + 5x + 2 = 0$

(c) $4x^2 - 6x + 9 = 0$

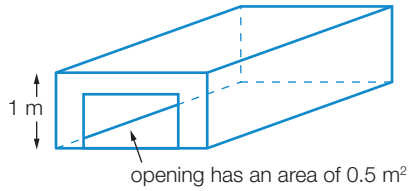
(d) $2x^2 + 7x + 9 = 0$

14 A policeman on a motorbike is following a car along a highway. After a short time, the driver of the car notices the policeman and starts to slow down, finally stopping on the side of the road. The speed of the car during this time can be represented by the quadratic function $S = -3t^2 + 17t + 70$, where S is the speed of the car in kilometres per hour, t minutes after the policeman started following.

(a) How long was the car under surveillance until it stopped?

(b) During this time, did the driver break the speed limit of 100 km/h?

15 Yeti's owners are going to build a dog kennel for him. It will be 1 m high and twice as long as it is wide, and it will have an opening at one end. When they are finished, they are going to paint the outside of it, including the base of the kennel, to keep it waterproof.



(a) Write an algebraic expression for the area of the kennel that is to be painted.

(b) If they buy a 4 L tin of paint, which will cover 31.4 m^2, and they will give the kennel two coats of paint, find the dimensions of the largest kennel that could be constructed. Round your answer to one decimal place.

16 Sally is a metal worker. She has several rectangular pieces of sheetmetal with dimensions of 30 cm by 25 cm and she wants to make three simple trays by cutting out the same size square from each corner of the material and then folding up the edges to complete the tray. When completed, each tray will have a different base area: 400 cm^2, 350 cm^2 and 300 cm^2.

(a) Draw a diagram of the net for each rectangular tray with the corners cut out. Draw broken lines to indicate where the trays will be folded.

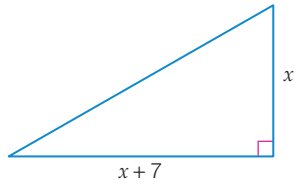
(b) Write an algebraic expression for the length of the base of each tray.

(c) Write an algebraic expression for the width of the base of each tray.

(d) Calculate the length of the square corner cut out in each tray (correct to one decimal place).

(e) What are the dimensions of each tray?

17 A right-angled triangle with shorter sides x cm and $(x + 7)$ cm has an area of 60 cm^2. Find the perimeter of the triangle.



18 The sum of the first n terms of the sequence 2, 5, 8, 11 … is given by the formula

$S = \frac{n}{2}[4 + 3(n - 1)]$ where n is the number of terms in the sequence. Use the formula to find the number of terms needed to make a sum of 805.

19 Solve the following quadratic equations by first multiplying by the lowest common denominator. Note that the denominator can never be zero.

(a) $2x - 1 = \frac{21}{x}$ **(b)** $x = \frac{9x - 14}{x}$ **(c)** $x = \frac{x - 9}{x - 5}$

(d) $\frac{3}{x - 1} = \frac{x}{x + 1}$ **(e)** $\frac{5}{x + 2} + \frac{3}{x} = 1$ **(f)** $\frac{x - 1}{x} + 5 = \frac{3}{x^2}$

Open-ended

20 Make up three non-monic quadratic equations that have as their solutions $x = 2.5$ or $x = -3$.

21 By using the process of completing the square, the solution $x = \frac{-a \pm \sqrt{5}}{2}$ was obtained.

By choosing a value of a between -3 and 3, make up at least three possible quadratic equations that would have this solution.

22 By choosing values for a, b and c in the general quadratic equation $ax^2 + bx + c = 0$, make up an equation that has:

(a) no solution **(b)** one solution **(c)** two solutions.

Write the solutions to your equations where they exist.

Outside the Square Problem solving

Projectiles

When an object is projected vertically upward with an initial velocity v m/s, its height h m above the ground after t secs is approximately given by the equation $h = vt - 5t^2$.

(a) A rocket is fired vertically upward at an initial velocity of 25 m/s. At what times is the height of the rocket above ground going to be zero?

(b) The equation $h = vt - 5t^2$ can also be expressed as $5t^2 - vt + h = 0$.

(i) When the height of the rocket is 30 m and $v = 25$, show that the expression becomes $t^2 - 5t + 6 = 0$

(ii) By factorising $t^2 - 5t + 6$, find at what times the rocket is at a height of 30 m.

(c) The rocket is now projected with an initial velocity of 35 m/s. How long does it take for the rocket to be at a height of 30 m on its way down?

(d) The rocket is again projected, but with an initial velocity of 10 m/s.

(i) Show that a perfect square is obtained when the left side of the equation $5t^2 - vt + h = 0$ is factorised using $h = 5$.

(ii) For how long is the rocket in flight?

(iii) What can you conclude about the maximum height the rocket can reach?

Strategy options

- Draw a diagram.
- Look for a pattern.

Technology Exploration Excel

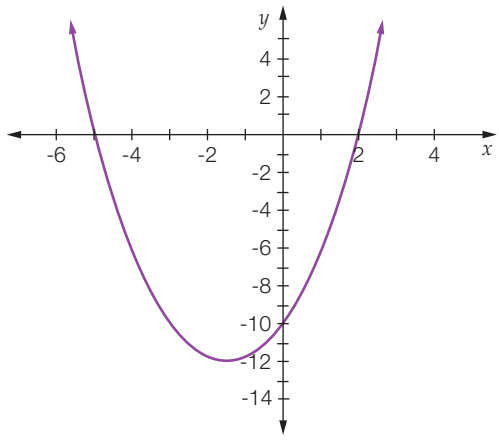
Equipment required: 1 brain, a computer with Excel

Versions of this Exploration for other technologies are available in Pearson Reader.

Factorising using 'zero-opposites'

If we factorise the expression $x^2 + 3x - 10$ we get $(x + 5)(x - 2)$.

The graph of $y = x^2 + 3x - 10$ is shown below.



From the graph we can see that the x-intercepts occur at $x = -5$ and $x = 2$ and that this is where $y = 0$. We call these values the solutions to the equation.

1 Explain the connection between the factors and the solutions.

We can use this connection to find factors for other quadratic expressions. We will use a spreadsheet to check the value of the expression for many values of x. A value of 0 indicates a solution. But we need to be careful how we write the factor!

The spreadsheet shown at right shows two solutions for the original expression above.

2 (a) Identify the two solutions.

(b) Write the factors of $x^2 + 3x - 10$.

3 (a) Reproduce the spreadsheet shown using the following instructions.

In A2 a starting value has been chosen. A3 has the formula =A2+1. B2 has the formula =A2^2+3*A2 – 10. (You should recall that ^ is the way to indicate 'to the power of' in Excel.) The formulas are then copied by highlighting and then moving the cursor to the bottom right corner. It will change into a small black cross that you drag down the page and release when appropriate.

(b) By changing the expression in your spreadsheet factorise the following:

(i) $x^2 - 32x - 144$

(ii) $x^2 - 15x + 56$

(iii) $x^2 - 17x - 84$

C6 f_x

Book1

	A	B
1	x-value	value of expression:x^2 +3x-10
2	-10	60
3	-9	44
4	-8	30
5	-7	18
6	-6	8
7	-5	0
8	-4	-6
9	-3	-10
10	-2	-12
11	-1	-12
12	0	-10
13	1	-6
14	2	0
15	3	8
16	4	18
17	5	30
18	6	44
19	7	60
20	8	78
21	9	98
22	10	120

4 Use a similar technique to find the factors for $x^2 - x - 20$. You should use x-values between -10 and 10 for this expression.

Taking it further

How can we use this technique for expressions whose factors are not whole numbers?

Set up your spreadsheet for $8x^2 - 2x - 3$ with x-values from -3 to +3. The solutions will occur between the values where the value of the expression changes sign. In this case, there is a solution between $x = -1$ and $x = 0$ and another between $x = 0$ and $x = 1$.

B13

	A	B	C	D
1	x-value	value of expression:	8x^2-2x-3	
2	-3	75		
3	-2	33		
4	-1	7		
5	0	-3		
6	1	3		
7	2	25		
8	3	63		
9				
10				

We can go back to cell A2 and change it to -1 and make A3 =A2+0.5, for instance. If we fill this down we get the screen shown at right. We should be able to read off the first solution now.

E2

	A	B	C
1	x-value	value of expression:	8x^2-2x-3
2	-1	7	
3	-0.5	0	
4	0	-3	
5	0.5	-2	
6	1	3	
7	1.5	12	
8	2	25	
9			
10			

5 What is the first factor for $8x^2 - 2x - 3$?

So, how do we find the second factor? We should be able to see that it occurs between $x = 0.5$ and $x = 1$, so we can change the increment.

As the value of the expression gets closer to zero, we can make the change in x even smaller if necessary (e.g. 0.01). We continue in this way until we obtain a value of zero for the expression.

6 What is the second factor for $8x^2 - 2x - 3$?

7 Use a similar technique to find the factors for:

(a) $10x^2 + 21x - 10$ **(b)** $100x^2 - 79x + 3$

Remainder and factor theorems

10.6

Consider the polynomial $P(x) = x^2 + x + 2$. It can also be written in the form $P(x) = x^2 + x - 6 + 8$.

$$\begin{aligned} P(x) &= x^2 + x + 2 \\ &= x^2 + x - 6 + 8 \\ &= (x+3)(x-2) + 8 \end{aligned}$$

So $\dfrac{P(x)}{x-2} = (x+3) + \dfrac{8}{x-2}$ and $\dfrac{P(x)}{x+3} = (x-2) + \dfrac{8}{x+3}$.

When $P(x)$ is divided by $(x-2)$ or $(x+3)$ the remainder is 8.

If we substitute $x = 2$ into $P(x)$, we obtain 8.

$$\begin{aligned} P(2) &= 4 + 2 + 2 \\ &= 8 \end{aligned}$$

If $P(x)$ is divided by $(x-2)$ then $P(2) = 8$ is the remainder.

If we substitute $x = -3$ into $P(x)$, we obtain 8.

$$\begin{aligned} P(-3) &= 9 - 3 + 2 \\ &= 8 \end{aligned}$$

If $P(x)$ is divided by $(x+3)$ then $P(-3) = 8$ is the remainder.

Worked Example 18 WE18

If $P(x) = x^3 + 3x^2 - 4x + 5$ and $Q(x) = x - 2$:

(a) find $\dfrac{P(x)}{Q(x)}$ **(b)** evaluate $P(2)$ **(c)** comment on the result.

Thinking	Working
(a) Set up and perform the long division.	(a) $$\begin{array}{r} x^2 + 5x + 6 \\ x-2\overline{\big)\, x^3 + 3x^2 - 4x + 5} \\ -(x^3 - 2x^2) \\ \hline 5x^2 - 4x \\ -(5x^2 - 10x) \\ \hline 6x + 5 \\ -(6x - 12) \\ \hline 17 \end{array}$$
(b) Evaluate $P(2)$.	(b) $P(2) = 8 + 12 - 8 + 5 = 17$
(c) Compare the answers.	(c) The remainder when $P(x)$ is divided by $x - 2$ is the same as the value of $P(2)$.

Remainder theorem

The **remainder theorem** can be used to find the remainder when a polynomial $P(x)$ is divided by a linear factor $(ax - b)$.

- If the polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$.
- If the polynomial $P(x)$ is divided by $(ax - b)$, then the remainder is $P\left(\frac{b}{a}\right)$.

Worked Example 19 WE19

Use the remainder theorem to find the remainder for:

(a) $\dfrac{2x^2 - x + 1}{x - 2}$ (b) $\dfrac{-4x^3 + x^2 - x + 5}{2x + 3}$

Thinking	Working
(a) 1 Write down $P(x)$.	(a) $P(x) = 2x^2 - x + 1$
2 Write down the value of a.	$x - 2 = 0$ $x = 2$, hence $a = 2$
3 Evaluate $P(a)$. (Here, $a = 2$).	$P(2) = 8 - 2 + 1 = 7$
4 Write down the remainder.	The remainder is 7.
(b) 1 Write down $P(x)$.	(b) $P(x) = -4x^3 + x^2 - x + 5$
2 Write down the value of $\frac{b}{a}$.	$2x + 3 = 0$ $x = \frac{-3}{2}$ $\therefore \frac{b}{a} = \frac{-3}{2}$
3 Evaluate $P\left(\frac{b}{a}\right)$. (Here, we evaluate $P\left(-\frac{3}{2}\right)$.)	$P\left(\frac{-3}{2}\right) = \frac{27}{2} + \frac{9}{4} + \frac{3}{2} + 5$ $= 22\frac{1}{4}$
4 Write down the remainder.	The remainder is $22\frac{1}{4}$.

Factor theorem

The **factor theorem** can be used to find factors of a polynomial $P(x)$ in the form of $(ax - b)$.

If the remainder is 0, then the expression that you are dividing by is a factor of the polynomial. For the polynomial $P(x)$:

If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

If $P\left(\frac{b}{a}\right) = 0$, for the polynomial $P(x)$, then $(ax - b)$ is a factor of $P(x)$.

Worked Example 20

WE 20

Use the factor theorem to fully factorise $x^3 + 6x^2 + 11x + 6$.

Thinking	Working
1 Find a value of x that makes $P(x) = 0$. Try the factors of the constant term of $P(x)$.	$P(x) = x^3 + 6x^2 + 11x + 6$ Try $x = 1$ $P(1) = 1 + 6 + 11 + 6 = 24$ $\therefore$ $(x - 1)$ is not a factor
As all the terms have a positive coefficient, a negative value is needed.	Try $x = -1$ $P(-1) = -1 + 6 - 11 + 6 = 0$ Hence, $(x + 1)$ is a factor.
2 Divide $P(x)$ by the first factor found.	$x^2 + 5x + 6$ $x + 1 \overline{)\, x^3 + 6x^2 + 11x + 6}$ $-(x^3 + x^2)$ $5x^2 + 11x$ $-(5x^2 + 5x)$ $6x + 6$ $-(6x + 6)$ 0
3 Write $P(x)$ as the product of the factors.	$P(x) = (x + 1)(x^2 + 5x + 6)$
4 Factorise the quadratic factor.	$x^2 + 5x + 6 = (x + 2)(x + 3)$
5 Write $P(x)$ as the product of all its factors.	$P(x) = (x + 1)(x + 2)(x + 3)$

Worked Example 21

WE 21

Use the factor theorem to fully factorise $x^4 + x^3 - 7x^2 - x + 6$.

Thinking	Working
1 Find a value of x that makes $P(x) = 0$.	$P(x) = x^4 + x^3 - 7x^2 - x + 6$
Try the factors of the constant term of $P(x)$.	Try $x = 1$ $P(1) = 1 + 1 - 7 - 1 + 6 = 0$ Hence, $(x - 1)$ is a factor.
2 Divide $P(x)$ by the first factor found.	$x^3 + 2x^2 - 5x - 6$ $x - 1 \overline{)\, x^4 + x^3 - 7x^2 - x + 6}$ $-(x^4 - x^3)$ $2x^3 - 7x^2$ $-(2x^3 - 2x^2)$ $-5x^2 - x$ $-(-5x^2 + 5x)$ $-6x + 6$ $-(-6x + 6)$ 0

3 Write $P(x)$ as the product of the factors.	$P(x) = (x-1)(x^3 + 2x^2 - 5x - 6)$
4 Define the non-linear factor $M(x)$.	$P(x) = (x-1)M(x)$
5 Find a value of x so that $M(x) = 0$. Try the factors of the constant term of $M(x)$.	Try $x = -1$ (we have used $x = 1$) $M(x) = x^3 + 2x^2 - 5x - 6$ $M(-1) = -1 + 2 + 5 - 6 = 0$ Hence, $(x + 1)$ is a factor.
6 We now divide $M(x)$ by $(x + 1)$ to obtain the quadratic factor and then factorise.	$M(x) = (x+1)(x^2 + x - 6)$ $= (x+1)(x-2)(x+3)$
7 Write $P(x)$ as the product of its factors.	$P(x) = (x-1)(x+1)(x-2)(x+3)$

There are other methods that can be used to find the factors. One method we could use is to try all integer factors of 6: $\pm 1, \pm 2, \pm 3$. We would find $P(1) = 0$, $P(-1) = 0$, $P(2) = 0$, $P(-3) = 0$ so the factors of $P(x)$ would be $(x - 1)$, $(x + 1)$, $(x - 2)$ and $(x + 3)$.

10.6 Remainder and factor theorems

Navigator

Answers page 801

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q12, Q13, Q14, Q15

Fluency

WE18

1 If $P(x) = x^3 + 4x^2 - 5x + 3$ and $Q(x) = x - 2$, find $\dfrac{P(x)}{Q(x)}$, evaluate $P(2)$ and comment on the result.

WE19

2 Use the remainder theorem to find the remainder for:

(a) $\dfrac{x^2 - 2x + 1}{x - 2}$ **(b)** $\dfrac{-3x^3 + 2x^2 - x + 2}{2x + 6}$

WE20

3 Use the factor theorem to fully factorise $x^3 - 6x^2 + 11x - 6$.

WE21

4 Use the factor theorem to fully factorise $x^4 - x^3 - 7x^2 + x + 6$.

5 Use the remainder theorem to find the remainder if:

(a) $4x^2 + 3x - 1$ is divided by $x - 3$

(b) $5x^3 - 2x^2 + 6x - 10$ is divided by $x + 1$

(c) $x^4 + x^3 - 3x^2 - x - 1$ is divided by $2x + 1$

(d) $x^5 - 1$ is divided by $2x - 1$

(e) $(6x - 8)(4x + 5)$ is divided by $x - 5$

(f) $49x^2 - 144$ is divided by $7x + 2$

6 Use the factor theorem to factorise the following polynomials.

(a) $x^3 + x^2 - 10x + 8$

(b) $x^3 - 5x^2 + 3x + 9$

(c) $x^3 - 6x^2 + 3x + 10$

(d) $x^3 + 3x^2 - 16x + 12$

(e) $x^3 + 6x^2 + 12x + 8$

(f) $x^3 - 3x^2 - 9x - 5$

Understanding

7 Calculate the remainder for $\frac{P(x)}{Q(x)}$ given:

(a) $P(x) = 4x^3 + 2x^2 - x + 1$, $Q(x) = 5x + 15$

(b) $P(x) = -x^4 - 3x^2 + 2x + 1$, $Q(x) = -x - 1$

(c) $P(x) = -x^3 - 9x^2 + 10x - 12$, $Q(x) = -2x$

(d) $P(x) = x^5 - x^4 - x^3 + 2x^2 - x - 1$, $Q(x) = 4x + 8$

8 Use the factor theorem to factorise the following polynomials.

(a) $x^4 + 6x^3 + 7x^2 - 6x - 8$

(b) $x^4 - 7x^3 + 14x^2 - 8x$

(c) $x^4 - x^3 - 7x^2 + 13x - 6$

(d) $x^4 + 2x^3 - 12x^2 + 14x - 5$

(e) $x^4 - 4x^3 + 3x^2 + 4x - 4$

(f) $x^4 + 2x^3 - 13x^2 - 14x + 24$

9 When $2x^3 - 4x^2 + x - 1$ is divided by $x + \frac{1}{2}$, the remainder is:

A $1\frac{1}{2}$ B $-2\frac{3}{4}$ C $1\frac{3}{4}$ D $-3\frac{1}{2}$

Reasoning

10 If $\frac{x^3 - 4x^2 - 4x + 16}{x - 4} = ax^2 + bx + c$ the values of a, b and c are:

A 1, -1, -4 B -3, 4, 0 C 4, 3, -1 D 1, 0, -4

11 If $x + 2$ is a factor of $6x^3 + 19x^2 + 11x + a$, find the value of a.

12 Use the factor theorem to fully factorise each polynomial. Then solve the polynomial equation.

(a) $x^3 - 6x^2 + 11x - 6 = 0$

(b) $x^3 + x^2 - 4x - 4 = 0$

(c) $x^3 + 9x^2 + 26x + 24 = 0$

(d) $x^4 + x^3 - 7x^2 - x + 6 = 0$

(e) $x^4 - 5x^2 + 4 = 0$

(f) $x^4 - x^3 - 6x^2 + 6x = 0$

13 A toy top is moving along the floor as it spins. Its distance, d cm, from a fixed point at time t seconds, is given by the equation $d = t^4 - 6t^3 + 13t^2 - 12t + 4$.

(a) Factorise the polynomial.

(b) How far is the top from the fixed point initially?

(c) At what times is the top at the fixed point?

(d) By inspection of your answer to part **(a)**, or by sketching the graph, describe the motion of the top after 2 seconds.

Open-ended

14 Give two monic polynomials of degree 3 that leave a remainder of -2 when they are divided by $x - 3$.

15 A polynomial equation of degree 4 has solutions $x = 0$, $x = -1$, $x = 2$, $x = 3$. Write down two possible equations.

Technology Exploration CAS

Equipment required: 1 brain, a TI-Nspire CAS or a ClassPad CAS

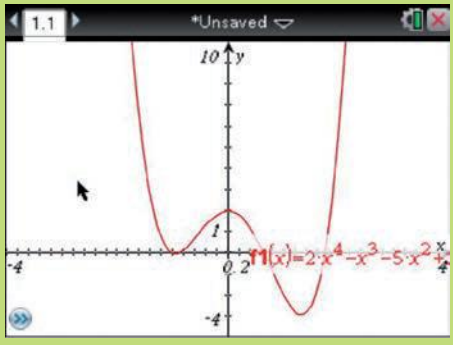
Sketching polynomials with a CAS

We often need information from a polynomial graph. Sketching a graph by hand is not very accurate and time-consuming. The exact values for the x-intercepts are often difficult to find and turning points and points of inflection require mathematical techniques beyond this course. Being able to use a CAS efficiently to sketch a polynomial and find relevant information enables you to use polynomials to interpret situations and solve problems using polynomials more effectively.

1 Use a CAS to sketch $y = 2x^4 - x^3 - 5x^2 + 2x + 2$. Identify the x- and y-intercepts and any turning points and/or points of inflection.

Using the TI-Nspire CAS

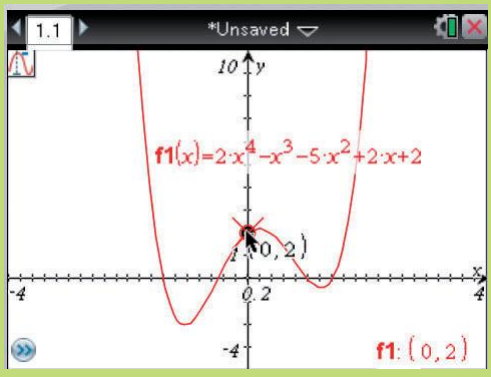
Open a new Graphs screen from the main menu. Enter $f1(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$. Press (menu) > Window/Zoom > Window Settings and change the axis values to those shown on the graph below. Press (enter) to make the graph appear on the screen. Select OK.



We will now find the x- and y-intercepts.

y-intercept:

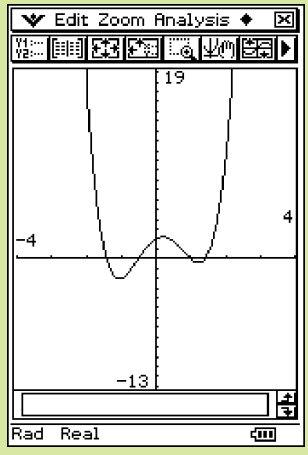
Press (menu) > Trace > Graph Trace > (enter) and the y-intercept will appear on the screen.



The y-intercept is (0, 2).

Using the ClassPad

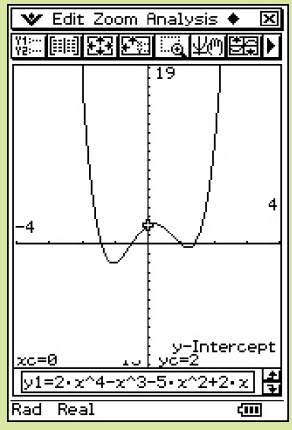
Tap on Graph and Tab . Enter y1 as $2x^4 - x^3 - 5x^2 + 2x + 2$ and tap (EXE). Then tap and Resize (from the bottom of the screen) to get a full screen graph. Reset the window values so that all the key features of the graph are visible. To do this tap and change the axis values to those shown on the screen. Tap OK.

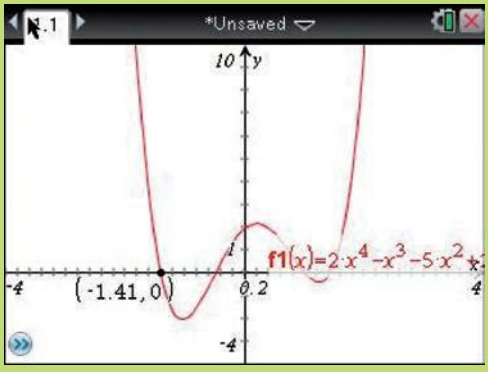
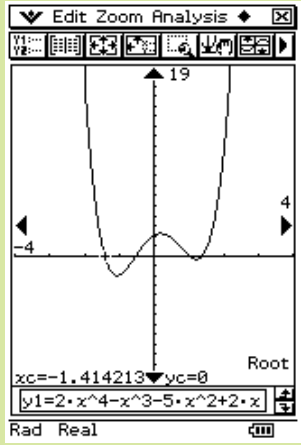


We will now find the x- and y-intercepts.

y-intercept:

Tap on Analysis > G-Solve > y-Intercept and the y-intercept will be displayed. It is (0, 2)

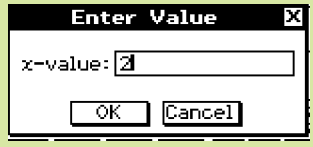


Using the TI-Nspire CAS	Using the ClassPad
x-intercepts: Press (menu) > Analyse Graph > Zero > (enter). Move the pointer to a point on the graph on the LHS of the first x-intercept and press (enter). You now need to identify the upper bound. Move the pointer to the RHS of the first x-intercept and press (enter). The coordinates of the first x-intercept will appear on your screen: (-1.41, 0).  Repeat to find the other three x-intercepts: (-0.5, 0), (1, 0) and (1.41, 0).	x-intercepts: Tap on Analysis > G-Solve > Root. The cursor will blink on the first x-intercept and the coordinates of this point will be displayed (-1.414213, 0). Use the right arrow to jump to the next x-intercept (-0.5, 0). Repeat this action two more times to find (1, 0) and (1.414213, 0). 
Maximum and minimum points: Press (menu) > Analyse Graph > Maximum (enter). A hand will appear on the screen asking for a lower bound. Choose a point on the graph on the LHS of the maximum and press (enter). You will then be asked for an upper bound. Move the hand to the RHS of the maximum and press (enter). The coordinates of the maximum will appear on your screen (0.195, 2.2). Press (menu) > Analyse Graph > Minimum (enter). A hand will appear on the screen asking for a lower bound. Choose a point on the graph on the LHS of a minimum and press (enter). You will then be asked for an upper bound. Move the hand to the RHS of the minimum and press (enter). The coordinates of a minimum will appear on your screen (-1.05, -2.02). Repeat for the second minimum value (1.23, -0.388).	Maximum and minimum points: Tap on Analysis > G-Solve > Max. The coordinates of the maximum value (0.1945363, 2.195353) will be displayed. Tap on Analysis > G-Solve > Min. The coordinates of the first minimum value (-1.046979, -2.023965) will be displayed. Press the right arrow on the central pad and the cursor will jump to the second minimum (1.2274427, -0.387695).

You now have enough information to sketch your graph by hand.

You can use your graph to find values for x and solve the equation for various values of y.

2 If $y = P(x)$, find $P(2)$ and $P(-3)$.

Using the TI-Nspire CAS	Using the ClassPad
Evaluating polynomials: Using the calculator screen, enter $f1(2)$ and press (enter). 10 will appear on the RHS of the screen. Repeat using $f1(-3)$ and you will get 140. $P(2) = 10$ and $P(-3) = 140$	Evaluating polynomials: Tap on Analysis > G-Solve > y-Calc. A dialog box will open. Enter 2 as the x-value and tap OK. The coordinate (2, 10) will be displayed. $P(2) = 10$ $P(-3) = 140$ 

3 Solve $2x^4 - x^3 - 5x^2 + 2x + 2 = 2$. Write your answer correct to two decimal places.

Using the TI-Nspire CAS	Using the ClassPad
Solving polynomials: Using the calculator screen, press (menu) > Algebra > solve. Press (enter). Enter $f1(x) = 2$, x inside the brackets and press (enter). Four solutions will be displayed. The solutions are: $x = -1.54, 0, 0.39, 1.65$ (2 d.p.)	Solving polynomials: Tap on Analysis > G-Solve > x-Calc A dialog box will open. Enter 2 as the x-value and tap OK. The coordinate (-1.541936, 2) will be displayed. Press the right arrow on the central pad and the cursor will jump to the next solution (0, 2). Repeat this two more times to obtain (0.3934009, 2) and (1.6485353, 2). The solutions are: $x = -1.54, 0, 0.39, 1.65$ (2 d.p.)

4 (a) Use a CAS to sketch $y = 6x^4 + 7x^3 - 9x^2 - 7x + 3$. Identify x- and y-intercepts and any turning points and/or points of inflection.

(b) If $y = Q(x)$, find $Q(2)$and $Q(-3)$.

(c) Solve $6x^4 + 7x^3 - 9x^2 - 7x + 3 = 2$. Write your answer correct to two decimal places.

Sketching polynomials

10.7

Before we can sketch a polynomial we need to identify its key features.

Step 1 Identify the shape of $y = a(x - h)^2 + k$ using transformations of $y = ax^n$.

Step 2 Find the y-intercept. All polynomial graphs have a y-intercept. Find the y-value of the y-intercept by substituting $x = 0$ into the polynomial.

Step 3 Find the x-intercepts. When a polynomial $P(x)$ is written as the product of its factors, the solution of $P(x) = 0$ can be found by equating each factor equal to zero and solving for x. The solutions are the x-values of the x-intercepts of the graph of $y = P(x)$.

- If the degree of $P(x)$ is 1, the linear polynomial $P(x) = mx + b$ has one solution for $P(x) = 0$. There is always exactly one x-intercept.
- If the degree of $P(x)$ is 2, the quadratic polynomial $P(x) = ax^2 + bx + c$ has at most two solutions for $P(x) = 0$. There may be zero, one or two x-intercepts. The x-values of the x-intercepts may not be rational.
- If the degree of $P(x)$ is 3, the cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ has at most three solutions for $P(x) = 0$. There will always be at least one x-intercept. The x-values of the x-intercepts may not be rational.
- The polynomial $P(x)$ of degree n has at most n solutions for $P(x) = 0$. If n is odd, there will always be at least one x-intercept. If n is even, there may be no x-intercepts.

  We can solve any polynomial equation, if solutions exist, by using the factor theorem to factorise and the Null Factor Law to find any x-intercepts of the graph of the polynomial.

Step 4 Find the turning points or points of inflection where the graph changes direction giving a maximum or minimum or a special point (h, k) where a reflection in both the lines $x = h$ and $y = k$ of one half of the graph will give the other half.

Quadratics

The general equation of a quadratic is $P(x) = ax^2 + bx + c$ where a, b and c are constants.

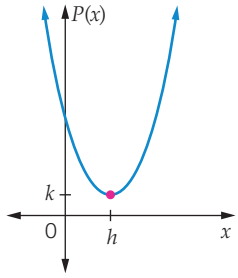
The shape of a quadratic is called a parabola.

Features of the parabola:

- Every parabola is symmetrical about the axis of symmetry which is a vertical line with equation in the form $x = d$, where d is a constant.
- Every parabola has a turning point where the graph changes direction.
- A parabola can have a minimum or maximum turning point. A quadratic equation written in turning point form $P(x) = a(x - h)^2 + k$ has a turning point at (h, k).
- The y-value of the y-intercept can be found by substituting $x = 0$ into the quadratic equation.
- The x-intercepts, if they exist, can be found by solving $P(x) = 0$. We can solve quadratic equations either by factorising and using the Null Factor Law or by substituting into the quadratic formula. These methods are preferable to the factor theorem for quadratics.

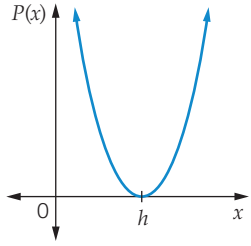
The graphs of a quadratic fit one of the following three types for $a > 0$. If $a < 0$, the graphs will be reflected in the x-axis.

$P(x) = a(x-h)^2 + k, k > 0$



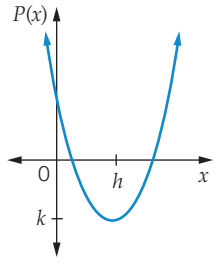
- no solution for $P(x)$
- no x-intercept

$P(x) = a(x-h)^2, k = 0$



- 1 solution for $P(x)$
- 1 x-intercept

$P(x) = a(x-h)^2 + k, k < 0$



- 2 solutions for $P(x)$
- 2 x-intercepts

Worked Example 22

WE22

Sketch the graph of each quadratic polynomial showing the x- and y-intercepts and the coordinates of the turning point.

(a) $P(x) = 2x^2 + 2x + 1$ **(b)** $P(x) = 3x^2 + 6x + 3$ **(c)** $P(x) = -6x^2 + x + 2$

Thinking	Working
(a) 1 Write down the quadratic equation.	(a) $P(x) = 2x^2 + 2x + 1$
2 Find the y-intercept by letting $x = 0$.	$x = 0$, $P(0) = 2(0)^2 + 2(0) + 1$ $= 1$ The y-intercept is $(0, 1)$.
3 Find the x-intercept by letting $P(x) = 0$. Solve the quadratic equation if possible. If there are no obvious factors, use the quadratic formula.	There are no obvious factors so use the quadratic formula. $a = 2, b = 2, c = 1$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-2 \pm \sqrt{4 - 8}}{2 \times 2}$ $= \frac{-2 \pm \sqrt{-4}}{4}$ This gives no solution so the graph has no x-intercepts.
4 Identify whether the graph has a maximum or minimum turning point.	Since a is positive, the graph has a minimum turning point.

5 Identify the turning point using $x = -\frac{b}{2a}$.

$x = -\frac{b}{2a}$

$= -\frac{2}{4}$

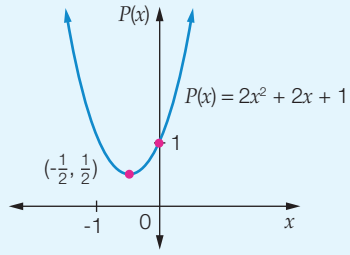
$= -\frac{1}{2}$

when $x = -\frac{1}{2}$, $y = \frac{2}{4} - \frac{2}{2} + 1$

$= \frac{1}{2}$

The turning point is at $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

6 Sketch the graph and label it showing the turning point and y-intercept.



(b) 1 Write down the quadratic equation.

(b) $P(x) = 3x^2 + 6x + 3 = 0$

2 Find the y-intercept by letting $x = 0$.

$x = 0$, $P(0) = 3(0)^2 + 6(0) + 3$
$= 3$

The y-intercept is $(0, 3)$.

3 Find the x-intercept by letting $P(x) = 0$

Identify the common factor and a perfect square. Solve the quadratic equation.

For $P(x) = 0$, $3x^2 + 6x + 3 = 0$
$3(x^2 + 2x + 1) = 0$
$3(x + 1)^2 = 0$
$x = -1$

The x-intercept is $(-1, 0)$.

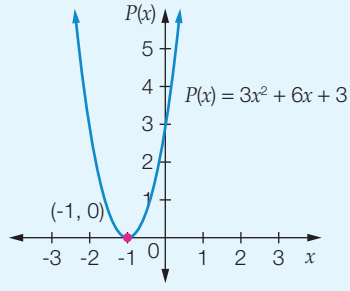
4 Identify whether the graph has a maximum or minimum turning point.

As a is positive, the graph has a minimum turning point.

5 Identify the turning point using symmetry as there is only one x-intercept.

The turning point is at $(-1, 0)$.

6 Sketch the graph and label it showing the turning point and y-intercept.



(c) **1** Write down the quadratic equation.

(c) $P(x) = -6x^2 + x + 2$

2 Find the y-intercept by letting $x = 0$.

$x = 0,\ P(0) = -6(0)^2 + 0 + 2$
$= 2$

The y-intercept is (0, 2).

3 Find the x-intercept by letting $P(x) = 0$.

Factorise the trinomial and use the Null Factor Law to solve the quadratic equation.

$P(x) = 0$
$-6x^2 + x + 2 = 0$
$-(6x^2 - x - 2) = 0$
$-(3x - 2)(2x + 1) = 0$
$x = \frac{2}{3}$ or $x = -\frac{1}{2}$

The x-intercepts are $\left(\frac{2}{3}, 0\right)$ and $\left(-\frac{1}{2}, 0\right)$.

4 Identify whether the graph has a maximum or minimum turning point.

As a is negative, the graph has a maximum turning point.

5 Identify the turning point using $x = -\frac{b}{2a}$ or use symmetry as the x-coordinate is halfway between the two x-coordinates of the x-intercepts.

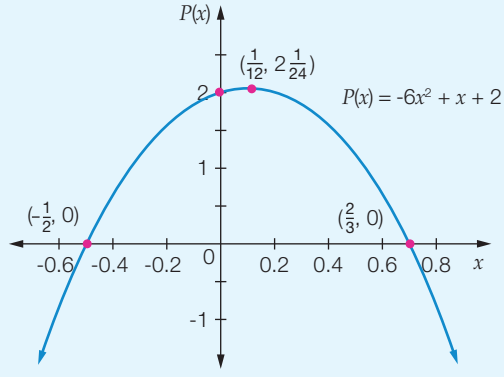
(Here, $x = \frac{1}{2} \times \left(-\frac{1}{2} + \frac{2}{3}\right) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$)

$x = -\frac{b}{2a}$
$= -\frac{1}{2 \times -6}$
$= \frac{1}{12}$

$y = -6\left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right) + 2$
$= -6 \times \frac{1}{144} + \frac{1}{12} + 2$
$= 2\frac{1}{24}$

The turning point is at $\left(\frac{1}{12}, 2\frac{1}{24}\right)$.

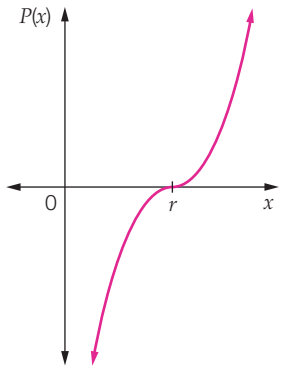
6 Sketch the graph and label it showing the turning point, the y-intercept and the x-intercepts.



Cubics

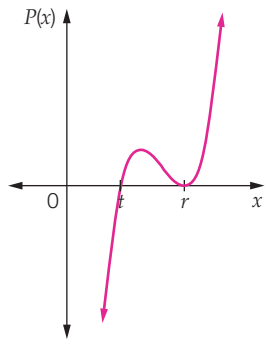
A cubic polynomial expressed in the form $P(x) = a(x - r)(x - s)(x - t)$ is easy to sketch as the x-intercepts are given by $x = r$, $x = s$ and $x = t$. Depending on the values of r, s and t there will be one, two or three solutions to the equation $P(x) = 0$.

$P(x) = a(x - r)^3$



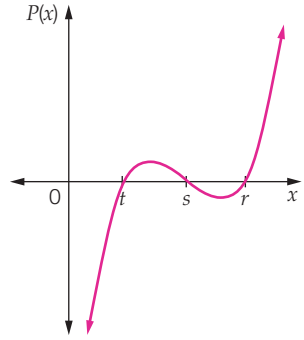
- r, s and t all have the same value
- 1 solution for $P(x)$
- cuts the x-axis once at $(r, 0)$

$P(x) = a(x - r)^2(x - t)$



- r and s have the same value
- 2 solutions for $P(x)$
- cuts the x-axis once at $(t, 0)$ and touches the x-axis once at $(r, 0)$

$P(x) = a(x - r)(x - s)(x - t)$



- r, s and t all have distinc valuest
- 3 solutions for $P(x)$
- cuts the x-axis 3 times at $(t, 0)$, $(s, 0)$, $(r, 0)$

Hence, to sketch cubics, it will be easiest if you can factorise the cubic polynomial. Notice that the ends of a cubic graph point in opposite directions.

Worked Example 23

WE 23

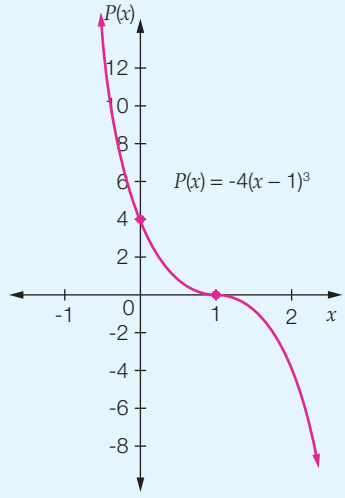
Sketch the graph of each cubic polynomial showing all x- and y-intercepts. Use a CAS to find the coordinates of the turning points or points of inflection, if necessary.

(a) $P(x) = -4(x - 1)^3$ **(b)** $P(x) = (x + 2)(x - 1)^2$ **(c)** $P(x) = x^3 - 3x^2 - x + 3$

Thinking	Working
(a) 1 Write down the cubic equation.	(a) $P(x) = -4(x - 1)^3$
2 Find the y-intercept.	$x = 0$, $P(0) = -4 \times (-1)^3$ $= 4$ The y-intercept is $(0, 4)$.
3 Find the x-intercept(s) by solving the cubic equation.	$P(x) = 0$ $-4(x - 1)^3 = 0$ $x - 1 = 0$ $x = 1$ is the only solution The curve cuts the x-axis at $(1, 0)$.

4 Sketch the graph and label it showing the x- and y-intercepts.

This graph could also have been obtained by sketching the graph of $y = x^3$, applying a dilation factor of 4, reflecting it in the x-axis and translating it 1 unit to the right.



(1, 0) is a point of inflection.

(b) 1 Write down the cubic equation.

(b) $P(x) = P(x) = (x + 2)(x - 1)^2$

2 Find the y-intercept.

$x = 0$, $P(0) = 2 \times (-1)^2$
$= 2$

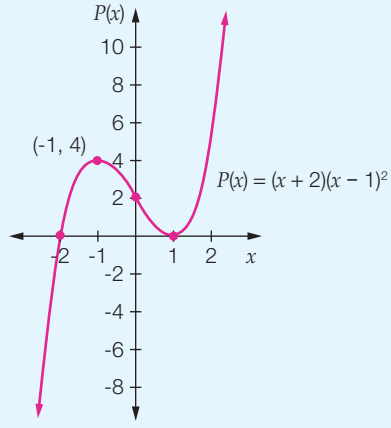
The y-intercept is (0, 2).

3 Find the x-intercept(s) by solving the cubic equation $P(x) = 0$. Any repeated factor indicates where the graph will touch the x-axis only. (Here, a repeated factor of $(x - 1)^2$ indicates that $P(x)$ touches the x-axis at $x = 1$.)

$P(x) = 0$
$(x + 2) = 0$ or $(x - 1)^2 = 0$
$x = -2$, $x = 1$

The curve cuts the x-axis at (-2, 0) and touches it at (1, 0).

4 Sketch the graph and label it showing the x- and y-intercepts. The maximum point can be found with a CAS.



(1, 0) is a minimum turning point.

The maximum turning point at (-1, 4).

A CAS can be used to find this point.

(c) 1 Write down the cubic equation.

(c) $P(x) = x^3 - 3x^2 - x + 3$

2 Find the y-intercept.

$x = 0$, $P(0) = 3$

The y -intercept is (0, 3).

3 Find the x-intercept(s) by solving the cubic equation. Use the factor theorem. If $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

$x^3 - 3x^2 - x + 3 = 0$

$P(-1) = -1 - 3 + 1 + 3$
$= 0$

$(x + 1)$ is a factor.

$P(3) = 27 - 27 - 3 + 3$
$= 0$

$(x - 3)$ is a factor.

$P(1) = 1 - 3 - 1 + 3$
$= 0$

$(x - 1)$ is a factor.

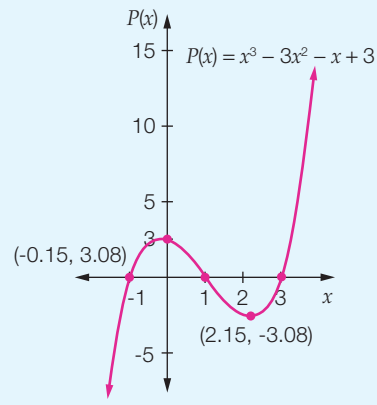
$(x + 1)(x - 1)(x - 3) = 0$

$x = -1, x = 1, x = 3$

The curve cuts the x-axis at (-1, 0), (1, 0) and (3, 0).

4 Sketch the graph and label it showing the x- and y-intercepts.

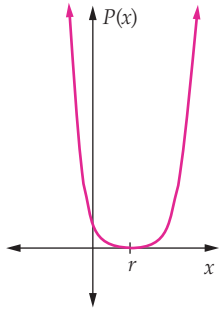
The maximum and minimum points can be found with a CAS. Round to two decimal places if necessary.



Quartics

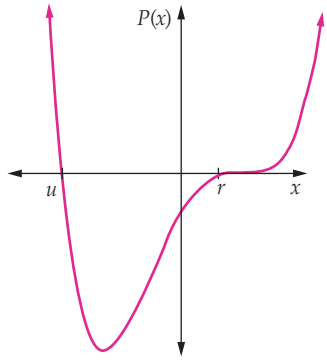
Some quartic polynomials of the form $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ can be written in the form $P(x) = a(x - r)(x - s)(x - t)(x - u)$. In this case, there can be up to four solutions for $P(x) = 0$. Four possibilities are shown for $a > 0$. If $a < 0$ then the graphs are reflected in the x-axis and both ends would be pointing downwards.

$P(x) = (x - r)(x - r)(x - r)(x - r)$
$= (x - r)^4$



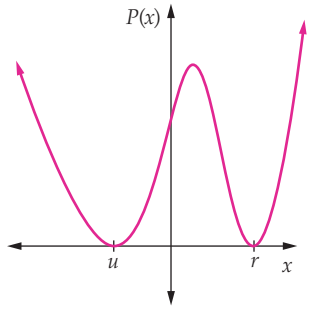
- r, s, t and u all have the same value
- 1 solution for $P(x)$
- touches the x-axis at $(r, 0)$

$P(x) = (x - r)(x - r)(x - r)(x - u)$
$= (x - r)^3(x - u)$



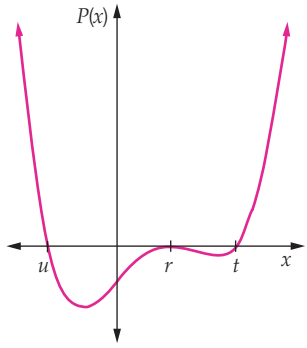
- r, s and t all have the same value
- 2 solutions for $P(x)$
- cuts the x-axis at $(u, 0)$ and touches the x-axis at $(r, 0)$

$P(x) = (x - r)(x - r)(x - u)(x - u)$
$= (x - r)^2(x - u)^2$



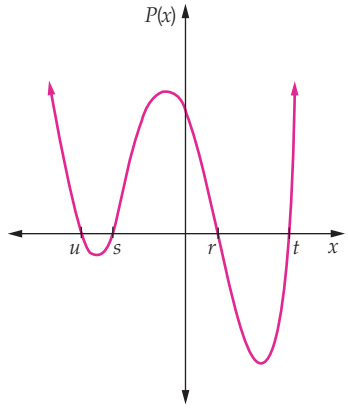
- r and s have the same value, t and u have the same value
- 2 solutions for $P(x)$
- touches the x-axis at $(r, 0)$ and $(u, 0)$

$$P(x) = (x - r)(x - r)(x - t)(x - u)$$
$$= (x - r)^2(x - t)(x - u)$$



- r and s have the same value
- 3 solutions for $P(x)$
- Touches x-axis at $(r, 0)$ and cuts it at $(t, 0)$ and $(u, 0)$

$$y = (x - r)(x - s)(x - t)(x - u)$$



- r, s, t and u all have different values
- 4 solutions for $P(x)$
- Cuts the x-axis at $(r, 0)$, $(s, 0)$, $(t, 0)$ and $(u, 0)$

If a factor occurs only once, it corresponds to a point where the curve cuts the x-axis.

If a factor is repeated an even number of times, it corresponds to a point where the curve touches the x-axis, creating a turning point, either a maximum or a minimum.

If a factor is repeated an odd number of times, it corresponds to a point of inflection, where the graph cuts the x-axis and flattens out slightly.

A quartic can have one, two or three turning points. If it has two turning points, it must also have a point of inflection as the ends of the graph have to point in the same direction.

Like parabolas, some quartics also have no solutions. These quartics cannot be factorised.

Worked Example 24

WE24

Sketch the graph of $P(x) = x^4 - 9x^2 + 4x + 12$ showing all x- and y-intercepts. Use a CAS to find the turning points or points of inflection, if necessary.

Thinking	Working
1 Write down the polynomial.	$P(x) = x^4 - 9x^2 + 4x + 12$
2 Find the y-intercept.	$P(0) = (0)^4 - 9(0)^2 + 4(0) + 12$ $= 12$ The y-intercept is $(0, 12)$.
3 Find the x-intercept(s). Factorise $P(x)$ and then solve $P(x) = 0$. Use the factor theorem to find a factor.	Factorise using the factor theorem. $P(-1) = 1 - 9 - 4 + 12$ $= 0$ $(x + 1)$ is a factor.

Perform a polynomial division.

$$
\begin{array}{r}
x^3 - x^2 - 8x - 12 \\
x+1\overline{)\,x^4 + 0x^3 - 9x^2 + 4x + 12} \\
\underline{x^4 + x^3} \\
-x^3 - 9x^2 \\
\underline{-x^3 - x^2} \\
-8x^2 + 4x \\
\underline{-8x^2 - 8x} \\
12x + 12 \\
\underline{12x + 12} \\
0
\end{array}
$$

Write as a product of factors.

$P(x) = (x+1)(x^3 - x^2 - 8x + 12)$

Use the factor theorem to find another factor.

$M(x) = x^3 - x^2 - 8x + 12$

$M(2) = 8 - 4 - 16 + 12$
$= 0$

Perform another division.

$(x - 2)$ is a factor.

$$
\begin{array}{r}
x^2 + x - 6 \\
x-2\overline{)\,x^3 - x^2 - 8x + 12} \\
\underline{x^3 - 2x^2} \\
x^2 - 8x \\
\underline{x^2 - 2x} \\
-6x + 12 \\
\underline{-6x + 12} \\
0
\end{array}
$$

Write as a product of factors.

Factorise the quadratic.

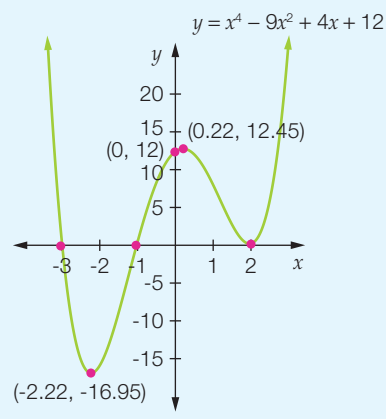
$P(x) = (x+1)(x-2)(x^2 + x - 6)$
$P(x) = (x+1)(x-2)(x-2)(x+3)$
$P(x) = (x+1)(x+3)(x-2)^2$

For $P(x) = 0$, $x = -1$, $x = -3$, $x = 2$ (repeated solution)

Write down the x-intercepts.

The curve cuts the x-axis at $(-3, 0)$ and $(-1, 0)$ and touches it at $(2, 0)$ (repeated factor).

4 Sketch the graph showing the x- and y-intercepts. Use a CAS to find the turning points. Round to two decimal places if necessary.



The curve cuts the x-axis twice, touches it once and has 3 turning points.

- If possible, factorise polynomials before sketching.
- Find the y-intercept.
- Find the x-intercept(s) by solving the equation $P(x) = 0$. Use the factor theorem and the quadratic formula if necessary.
- Identify whether the polynomial cuts or touches the x-axis. A repeated factor with an even power indicates that the graph touches the x-axis. A repeated factor with an odd power indicates a point of inflection on the x-axis.
- Use a CAS to find maximum and minimum turning points and points of inflection.

10.7 Sketching polynomials

Navigator

Answers page 801

Q1, Q2, Q3, Q4 Column 1, Q5 (a)–(d), Q6 (a)–(d), Q7, Q8, Q9, Q11, Q12, Q16	Q1, Q2, Q3, Q4 Column 2, Q5 (b)–(e), Q6 (b)–(e), Q7, Q8, Q9, Q10, Q11, Q12, Q14, Q15, Q16	Q1, Q2, Q3, Q4 Column 3, Q5 (c)–(f), Q6 (c)–(f), Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15, Q16, Q17

Equipment required: a CAS

Fluency

WE22

1 Sketch the graph of each quadratic polynomial showing all x- and y-intercepts. Use a CAS to find the coordinates of the turning points, if necessary.

(a) $P(x) = 2x^2 + 6x + 5$ **(b)** $P(x) = 3x^2 + 12x + 12$ **(c)** $P(x) = -5x^2 - 6x + 8$

WE23

2 Sketch the graph of each cubic polynomial showing all x- and y-intercepts. Use a CAS to find the coordinates of the turning points or point of inflection, if necessary.

(a) $P(x) = -2(x + 1)^3$ **(b)** $P(x) = (x - 2)(x + 1)^2$ **(c)** $P(x) = x^3 + 3x^2 - x - 3$

WE24

3 Sketch the graph of $P(x) = x^4 - 9x^2 - 4x + 12$ showing all x- and y-intercepts. Use a CAS to find the coordinates of the turning points or points of inflection, if necessary.

4 Sketch the graph of each polynomial.

(a) $P(x) = x^2 - 5x - 14$ **(b)** $P(x) = x^2 - 6x + 8$ **(c)** $P(x) = x^2 - 7x - 12$

(d) $P(x) = x^2 - 12x + 36$ **(e)** $P(x) = x^2 - 15x + 56$ **(f)** $P(x) = x^2 - x - 90$

5 Sketch the graph of each polynomial.

(a) $P(x) = x^3 + 7x^2 + 7x - 15$ **(b)** $P(x) = -x^3 + 9x^2 - 27x + 27$

(c) $P(x) = x^3 - 12x + 16$ **(d)** $P(x) = x^3 + x^2 - 12x$

(e) $P(x) = x^3 - 2x^2 - 11x + 12$ **(f)** $P(x) = x^3 + 5x^2 - x - 5$

6 Sketch the graph of each polynomial.

(a) $P(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$ **(b)** $P(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

(c) $P(x) = x^4 - 2x^3 + x^2$ **(d)** $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$

(e) $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ **(f)** $P(x) = x^4 + 3x^3 - 5x^2 - 3x + 4$

Understanding

7 The equation $x^3 - 7x^2 + 11x - 5 = 0$ has:

A no solutions
B only one solution at $x = 1$
C three solutions
D two solutions

8 The zeroes of the polynomial $P(x) = x(x - 8)(x - 2)(x - 8)$ are:

A 0 and 8
B 0 and 2
C 2 and 8
D 0, 2 and 8

9 $P(x) = 2x^3 + 5x^2 + x - 20$ and $Q(x) = 2x^3 + 4x^2 - x + 15$. Find the solutions to $P(x) = Q(x)$.

10 $P(x) = 5x^4 + 4x^3 - 3x^2 - 5x + 5$ and $Q(x) = 3x^4 + 2x^3 - x^2 - 2x + 2$. Find the solutions to $2P(x) = 3Q(x)$.

Reasoning

11 A flying fish, initially under water, suddenly leaps above the surface before re-entering the water. Its motion is given by $P(t) = -t^3 + 3t^2 - 2t$, where $P(t)$ is the vertical distance of the fish from sea level at time t seconds, with $0 \le t \le 2$.

(a) Factorise the polynomial and sketch the graph for values of t from 0 to 2.

(b) For how long was the fish swimming until it broke the sea surface? Approximately what was its greatest depth?

(c) For how long was the fish above the surface and approximately what was its maximum height?

12 A plumber inserts a long flexible rod in a length of water pipe to test for blockages. The pipe is described by the equation $y = x^3 - 6x^2 + 9x$, where x is the horizontal distance and y is the vertical distance from the rod's insertion point (0, 0) and the units are in metres.

(a) Factorise the expression and sketch the graph for values of x from 0 to 4.

(b) There are blockages located at the turning points of the graph. What are the coordinates of the blockages?

(c) To avoid the first blockage, the plumber decides to install a new system of pipes starting at (0, 0) and ending at the second blockage. The pipes take the shape of a parabola. Give one equation for this parabola.

(d) While installing the new pipes, the plumber is informed that there is an underground power cable located at (2.5, 1.2), which he must avoid. Can he still use the parabola pipe layout?

13 A wave pattern on a cathode ray oscilloscope is described by the polynomial $P(x) = x^4 + 5x^3 - x^2 - 5x$. A negative pulse is that section of the wave that is below the x-axis and between the x-intercepts. Similarly, a positive pulse is that section of the wave that is above the x-axis and between the x-intercepts.

(a) Factorise the polynomial and sketch its graph.

(b) Describe the location of each type of pulse.

(c) The wave pattern is a model for part of the cross-section of an island that consists of two lagoons and a small rocky outcrop with mountains on either side. The x-axis represents sea level, and each unit on the axis is 100 metres. Each unit on the y-axis is in metres.

(i) How wide and approximately how deep is the larger lagoon?

(ii) A research observation tower planned for the rocky outcrop requires a minimum length of land of 110 m. Can it be built?

14 A dam is located at the bottom of a hill. If $y = 0$ is taken as ground level, the cross-section (west to east) of both shapes can be described by the polynomial $x^3 - 5x^2 - 2x + 24$ over $-2 \leq x \leq 5$, where each unit of x represents 100 m and each unit of y represents 1 m.

(a) Factorise the polynomial.

(b) Sketch the graph of the polynomial for $-2 \leq x \leq 5$.

(c) What is the horizontal distance between the western base of the hill and the eastern bank of the dam?

15 A section of a roller-coaster track is described by the equation $y = 0.001(x - r)(x - s)^2$, where x is the horizontal distance in metres from Daredevil Corner and y is the height of the roller-coaster in metres. The graph has intercepts at $x = 0$ and $x = 60$ and the y-intercept is zero. The turning points occur at $x = 60$ and $x = 20$.

(a) Use this information to write the equation.

(b) Sketch the graph of the equation for $0 \leq x \leq 80$.

(c) The top of Heartstop Hill is a horizontal distance of 80 m from Daredevil Corner. The length of the track from Daredevil Corner to Heartstop Hill is six times the height of the roller-coaster at Heartstop Hill. The roller-coaster on average moves on the track at 32 km/h. How many minutes does it take to travel from Daredevil Corner to Heartstop Hill?

Open-ended

16 Find two monic quartic polynomials that have three zeroes, and sketch their graphs.

17 A quartic polynomial has factors of $(x + 1)$ and $(x - 3)$. Using at least these factors, write down two quartic polynomials that:

(a) have exactly two zeroes

(b) have three zeroes

(c) have four zeroes

(d) are non-monic with only one turning point.

Chapter review 10

D.I.Y. Summary

Key Words

cubic	factor theorem	points of inflection	remainder
degree	leading coefficient	polynomial	remainder theorem
dividend	leading term	quartic	
divisor	non-monic quadratic	quotient	

1 The highest power of a polynomial gives the _____ of the polynomial.

2 The first term of a polynomial written with powers in descending order is called the _____.

3 A linear factor is divided into a _____ to find a _____ and a _____.

4 A polynomial can be factorised by using the _____.

Fluency

1 Graph each equation by constructing a table of values for $x = -3$ to 3. Ex.10.1

(a) $y = 3x^2$ (b) $y = \frac{1}{2}x^3$ (c) $y = -x^4$

2 (a) Sketch the graph of $y = 2x^2 - 3$ starting with $y = x^2$. Ex.10.2

(b) Sketch the graph of $y = -2(x + 2)^3 + 1$ starting with $y = x^3$.

(c) Sketch the graph of $y = -3(x + 2)(x - 4)(x + 1)$.

3 (a) If $P(x) = 4x - 3$ and $Q(x) = 3x^2 - x + 2$, evaluate: Ex.10.3

(i) $P(4)$ (ii) $Q(-2)$ (iii) $P(-1).Q(-1)$

(b) Write an expression for:

(i) $2P(x) + 3Q(x)$ (ii) $P(x) \times Q(x)$ (iii) $Q(x) - P(x)$

4 Find the quotient and remainder for $\frac{4x^3 + 6x^2 + 26x + 13}{4x + 2}$. Ex.10.3

5 Factorise the following quadratic expressions. Ex.10.4

(a) $12x^2 + x - 20$ (b) $12x^2 - 4x - 5$ (c) $6x^2 + 17x - 14$

(d) $3x^2 + 5x + 1$ (e) $4x^2 - 7x - 5$ (f) $2x^2 + 6x + 3$

6 Solve each quadratic equation using an appropriate method. Ex.10.5

(a) $2x^2 - 18 = 0$ (b) $2x^2 - 9x = 5$ (c) $(2x - 5)^2 = 3$

(d) $2x^2 + 5x + 1 = 0$ (e) $3x^2 - 6x + 2 = 0$ (f) $13x^2 + 7x - 3 = 0$

7 Use the remainder theorem to find the remainder produced by each division. Ex.10.6

(a) $\frac{10x^3 + 7x^2 - 4x - 3}{x + 1}$ (b) $\frac{x^4 - x^3 + x^2 + x - 1}{2x - 1}$

8 Use the factor theorem to factorise:

(a) $x^3 + 6x^2 - 9x - 14$ (b) $x^4 - 5x^2 + 4$

9 Sketch the graph of each polynomial.

(a) $P(x) = x^3 - 8x^2 - x + 8$ (b) $P(x) = x^4 + 2x^3 - 4x^2 - 2x + 3$

Ex. 10.6

Ex. 10.7

Understanding

10 An equation $y = x^4$ is dilated by a factor of $\frac{1}{2}$ in the y-direction, reflected in the x-axis and translated 2 units left and 3 units up.

(a) Write the new equation.

(b) Sketch the graph of the equation. Use a CAS to find the x-intercepts correct to two decimal places.

11 Find the remainder using long division when $P(x) = x^3 - 3x^2 - 6x + 1$ is divided by $x + 2$. Confirm your answer by finding the remainder using the remainder theorem.

Reasoning

12 Solve $\frac{3(x+1)}{5x} - \frac{1}{2x} = 4x$.

13 Substitute $m = x^2$ into $x^4 - x^2 - 12 = 0$ to solve the equation for x.

14 The sum of a sequence is given by $S = \frac{n}{2}[2a + (n-1)d]$ where n stands for the number of terms in the sequence. If $a = 5$ and $d = 4$, find the number of terms in the sequence needed to make a sum of 860.

15 A bowl has a shape that can be described by the parabola $y = \frac{1}{10}x^2$ for $-10 \le x \le 10$.

The position of a length of uncooked spaghetti that just touches the rim of the bowl is described by the line $y = x$. The coordinate (0, 0) is the middle of the bottom of the bowl and all units are in centimetres.

(a) Graph both equations on the same axes.

(b) What is the vertical distance, above the centre of the bowl, of the end of the spaghetti?

16 The vertical cross-section of a bottle is described by the equations $y = (x - 5)^3 + 2$ and $y = -(x - 15)^3 + 2$.

(a) Graph both equations on the same axes using x-values from 2 to 18.

(b) State the equation of the axis of symmetry of the bottle.

17 A capsule less dense than water is released under water and moves upwards subject to ocean currents. Its height, h metres, from sea level at time, t seconds, is given by $h = t^3 - 9t^2 + 26t - 24$.

(a) Factorise the equation and sketch the graph for $0 \le t \le 4$.

(b) Describe the motion of the capsule in this time period.

Probability

11

You want how much?

Name: John Roumeliotis
Job: Global Head of Credit Model Development
Qualifications: PhD in applied mathematics

When a person applies for a loan or credit card, how does the bank decide how much to lend? John Roumeliotis leads the credit-risk team at NAB, which develops analytical statistical models that assess credit risk.

Large banks can have hundreds of billions of dollars of loans—some of which will default; that is, people may not be able to pay back their credit cards or companies may become bankrupt. John's team uses data gathered from different sources to determine the likelihood of a loan or set of loans defaulting.

As part of his role, John has to explain his team's work to other people: "The trick is to take something that might be very quantitative and mathematically or statistically intensive and try to explain it in a way that makes sense and is meaningful to others. I enjoy trying to get people to understand mathematics. My team needs good communication skills to communicate their results and the methods that they used. They also need to work collaboratively, share ideas and listen to others. Some of the mathematics is high-level, but many of the skills are learnt in high school. The first thing I ask my team is "Are you comfortable with data?" I expect the team to be innovative and insightful. We always need to refine our approach."

John also appreciates the usefulness of his mathematical skills: "The skill set I have is adaptable. Skills in mathematics give people an ability to do all sorts of things. We have mathematically trained people working in all parts of our organisation."

Why learn this?

Probability provides a way of making judgements about future events. For example, retailers are using probability when they estimate how many staff will be required to work on a particular day and traffic engineers establish how long the traffic lights should stay green in a particular direction by predicting future traffic flows. Also, a good knowledge of probability can enhance our ability to play many strategy games.

After completing this chapter you will be able to:

- describe the results of two- and three-step chance experiments both with and without replacement
- assign probabilities to outcomes and determine probabilities of events
- distinguish between independent and dependent events
- identify mutually exclusive events
- use the language of conditional statements to identify common mistakes interpreting such language.

Recall 11

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

Worksheet R11.1

1 Express in simplest form.

(a) $\frac{3}{15}$ **(b)** $\frac{16}{52}$

Worksheet R11.2

2 Evaluate the following.

(a) $\frac{1}{8} + \frac{3}{4}$ **(b)** $\frac{2}{5} + \frac{1}{3}$ **(c)** $1 - \frac{3}{5}$

(d) $\frac{2}{5} \times \frac{3}{4}$ **(e)** 0.8×0.5 **(f)** 0.05×0.2

Worksheet R11.3

3 **(a)** Two students are to be chosen from their coeducational class. Complete a list of the gender pairings possible, in order first student, second student.

(b) What are the two possible outcomes when a coin is tossed?

Worksheet R11.4

4 **(a)** A container has 25 numbered tokens; 6 are red, 10 are blue and the rest are yellow. A token is chosen at random from the container.

(i) How many different tokens can be chosen?

(ii) How many possible outcomes are there?

(iii) How many of these outcomes result in a red token being chosen?

(iv) How many of these outcomes result in a yellow token being chosen?

(v) How many of these outcomes result in a token that is not blue being chosen?

(b) A fair six-sided *die* (singular of *dice*) is rolled.

(i) List all the possible outcomes.

(ii) How many of these outcomes are even?

Key Words

Addition Law for Mutually Exclusive Events	independent events	sample space
complementary events	long-run proportion	tree diagram
conditional statement	mutually exclusive	two-way table
dependent events	probability	universal set
event	probability tree diagram	

Probability review

11.1

An understanding of **probability** helps increase your chances of making good decisions in a variety of situations. Will I need an umbrella today? Should I take this shot at goal or pass it to a team member who is in a better position? Do I need to insure my property? Probability is the study of the chance, or likelihood, of a particular **event** occurring. The probability of an event lies between 0 and 1 with 0 being impossible and 1 being certain.

$0 \leq \Pr(\text{event}) \leq 1$

Estimating probability

An estimate of the probability of an outcome can sometimes be found by examining past results. The long-term pattern of results can be used to indicate a similar event occurring in the future. This process is known as estimating probability and assumes that conditions will remain the same for future events. The greater the number of trials, the more accurate this estimate will be.

$$\text{An estimate of Pr(success)} = \frac{\text{number of past successes}}{\text{total number of past trials}}$$

For example, Greg is playing tennis and has served 15 aces in the last 80 points. An estimate of the probability of Greg serving an ace is:

$$\Pr(\text{Greg serving an ace}) = \frac{15}{80} = \frac{3}{16}$$

If we assume this will always be the proportion of aces Greg will serve then we can use $\frac{3}{16}$ as the **long-run proportion**. However, it is important to understand that this is an estimate.

Theoretical probability

In situations where we can list all the possible equally likely outcomes we can deduce probabilities. However, it is important that the outcomes can reasonably be assumed to be equally likely. We need to examine these assumptions carefully at the beginning of each question. For example, when we are flipping a coin it would be reasonable to assume that heads and tails are equally likely, but if two different footballers were lined up to take a shot at goal from 40 m out on a 45° angle it probably would not be reasonable to assume that they had the same probability of scoring a goal.

Consider the situation where a \$1 coin and a \$2 coin are both tossed. We know there are four possible outcomes and we can list them in what is called a **sample space**.

HH, HT, TH, TT

The $\Pr(2 \text{ heads}) = \frac{1}{4}$, $\Pr(0 \text{ heads}) = \frac{1}{4}$ and $\Pr(1 \text{ head}) = \frac{2}{4} = \frac{1}{2}$.

Note: The total probability is 1 because one of these events is certain to occur.

$$\Pr(\text{success}) = \frac{\text{number of equally likely successful outcomes}}{\text{total number of equally likely possible outcomes}}$$

Complementary events

When we roll a fair six-sided die there are 6 outcomes possible: 1, 2, 3, 4, 5 and 6.

It would seem reasonable to assume that these six events are all equally likely, so Pr(1) = Pr(2) = Pr(3) = Pr(4) = Pr(5) = Pr(6) = $\frac{1}{6}$. The sum of the probabilities of these six outcomes is 1, which we should expect as nothing else can (reasonably) happen when we roll the die. However, sometimes we like to break the outcomes up into just two events, for instance Pr(1) and Pr(not a 1). Pr(1) = $\frac{1}{6}$ and Pr(not a 1) = Pr(2) + Pr(3) + Pr(4) + Pr(5) + Pr(6) = $\frac{5}{6}$. Here we can see that Pr(1) + Pr(not a 1) = $\frac{1}{6} + \frac{5}{6}$. We say that Pr(1) and Pr(not a 1) are a pair of **complementary events**. One of the reasons we like to do this is that Pr(not a 1) = 1 – Pr(1), which can greatly simplify some of the calculations we will make. Further to this we may define 'rolling a 1' as a success and 'not rolling' a 1 as a failure. In this case we would have Pr(success) + Pr(failure) = 1.

Worked Example 1 WE1

Over the past 6 months Jake has been late to school on 12 occasions in 100 school days.

(a) What is the estimated probability he will be late next Wednesday?

(b) What is the estimated Pr(Jake being on time next Friday)?

Thinking	Working
(a) **1** Find the long-term pattern of results.	(a) Jake has been late 12 out of 100 days.
2 Express the trend using the formula: Pr(success) $= \frac{\text{number of equally likely past successes}}{\text{total number of equally likely past trials}}$	$\text{Pr(late)} = \frac{12}{100}$
3 Cancel to express the fraction in simplest form (if necessary).	$= \frac{3}{25}$
(b) As the events are complementary, use the formula: Pr(success) + Pr(failure) = 1. In this case, success = 'being late' and failure = 'being on time'.	(b) Pr(late) + Pr(on time) = 1 Pr(on time) = 1 – Pr(late) $= 1 - \frac{3}{25}$ $= \frac{22}{25}$

In the above worked example, the assumption made was that 'being late' and 'being on time' covered all the possibilities. Do you think this was reasonable?

Multiple events

If we flip a coin once there are two equally likely outcomes:

- a head H
- a tail T

If we flip a coin twice, there are four equally likely outcomes:

- two heads HH
- a head and a tail HT
- a tail and a head TH
- two tails TT

If we flip a coin three times, there are eight equally likely outcomes:

- three heads HHH
- a head, a head and a tail HHT
- a head, a tail and a head HTH
- a head, a tail and a tail HTT
- a tail, a tail and a head TTH
- a tail, a head and a tail THT
- a tail, a head and a head THH
- three tails TTT

By listing all the outcomes we can calculate the probability of any desired outcome. For example, the probability of flipping a head only once in three flips can be found by counting the successful outcomes and dividing by the total number of outcomes. This is because we regard all of the outcomes as equally likely. As there are 3 successful outcomes, HTT, THT and TTH out of 8 possible outcomes:

Pr(only 1 head) = $\frac{3}{8}$

Worked Example 2

WE2

Robin loves to play football with his mates after school but he also loves to play his guitar. He can't decide what to do so he flips a coin. A head means he will play football and a tail means he will play his guitar. Find the probability that he:

(a) plays football 3 afternoons in a row

(b) plays football at least twice in 3 afternoons

(c) plays his guitar 2 afternoons in a row

(d) alternates his activities for the 3 afternoons

(e) doesn't play football in 3 afternoons.

Thinking	Working
Make a list of all the possible outcomes and count them. Can we assume that they are equally likely?	HHH, HHT, HTH, HTT, THH, TTH, THT, TTT are all equally likely as long as the coin is fair.
(a) How many successful outcomes are listed? Use this to calculate the probability required.	(a) Pr(football 3 afternoons in a row) $= \text{Pr(HHH)} = \frac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}} = \frac{1}{8}$
(b) How many successful outcomes are listed? Use this to calculate the probability required.	(b) Pr(football at least twice in 3 afternoons) $= \text{Pr(HHT or HTH or THH or HHH)} = \frac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}} = \frac{4}{8} = \frac{1}{2}$

(c) How many successful outcomes are listed? Use this to calculate the probability required.	(c) Pr(plays guitar 2 afternoons in a row) = Pr (TTH or HTT or TTT) $= \frac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}}$ $= \frac{3}{8}$
(d) How many successful outcomes are listed? Use this to calculate the probability required.	(d) Pr(alternates activities for the 3 afternoons) = Pr(HTH or THT) $= \frac{\text{number of successful outcomes}}{\text{number of equally likely outcomes}}$ $= \frac{2}{8}$ $= \frac{1}{4}$
(e) Use complementary events to find the probability. Pr(failure) = 1 – Pr(success)	(e) Pr(doesn't play football in 3 afternoons) = 1 – Pr(plays football 3 afternoons in a row) $= 1 - \frac{1}{8}$ $= \frac{7}{8}$

Sometimes we have multiple events where the number of equally likely outcomes for one event is different from the second event. Consider flipping a coin and rolling a die. There are 2 equally likely outcomes for flipping a coin: H or T. There are 6 equally likely outcomes for rolling a die: 1, 2, 3, 4, 5, 6.

If we flip a coin and roll a die we can list all equally likely outcomes:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6

Worked Example 3

WE 3

Jemma has a dilemma. She has 6 favourite outfits: 1 black, 1 red, 1 blue, 1 green, 1 purple and 1 multi-coloured. She has 2 favourite pairs of shoes; 1 has a pattern in the leather and 1 is plain. She can never decide what to wear so she rolls a die to choose her outfit and flips a coin to choose her shoes. On the die: 1 is black, 2 is red, 3 is blue, 4 is green, 5 is purple, 6 is multi-coloured. On the coin: heads is shoes with pattern, tails is plain. Find the probability that she will wear:

(a) the black outfit with the plain shoes

(b) the green, purple or red outfit

(c) the multi-coloured outfit and the patterned shoes

(d) the red or green outfit with plain shoes

(e) the red outfit

(f) the plain pair of shoes.

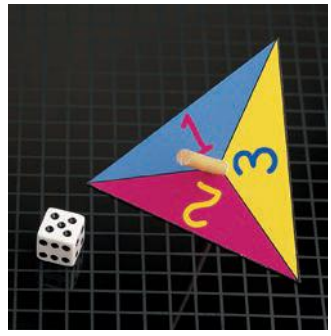
Thinking	Working
Make a list of all the possible outcomes and count them. Can we assume that they are equally likely?	H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6 There are 12 equally likely outcomes.
(a) How many successful outcomes are listed? Use this to calculate the probability required.	(a) Pr(black outfit, plain shoes) $= \Pr(T1)$ $= \frac{1}{12}$
(b) How many successful outcomes are listed? Use this to calculate the probability required.	(b) Pr(green or purple or red outfit) $= \Pr(H2, H4, H5, T2, T4, T5)$ $= \frac{6}{12}$ $= \frac{1}{2}$
(c) How many successful outcomes are listed? Use this to calculate the probability required.	(c) Pr(multi-coloured outfit and shoes with pattern) $= \Pr(H6)$ $= \frac{1}{12}$
(d) How many successful outcomes are listed? Use this to calculate the probability required.	(d) Pr(red or green outfit with plain shoes) $= \Pr(T2, T4)$ $= \frac{2}{12}$ $= \frac{1}{6}$
(e) How many successful outcomes are listed? Use this to calculate the probability required.	(e) Pr(red outfit) $= \Pr(T2, H2)$ $= \frac{2}{12}$ $= \frac{1}{6}$
(f) How many successful outcomes are listed? Use this to calculate the probability required.	(f) Pr(plain shoes) $= \Pr(T1, T2, T3, T4, T5, T6)$ $= \frac{6}{12}$ $= \frac{1}{2}$

$$\Pr(\text{successful event}) = \frac{\text{number of equally likely successful outcomes}}{\text{total number of equally likely possible outcomes}}$$

$0 \le \Pr \le 1$

$\Pr(A) + \Pr(\text{not } A) = 1$

Two-way tables and tree diagrams



Duncan spins a spinner and throws a die. What is the probability of obtaining a 2 with the spinner and a 5 with the die?

We need to find the total number of possible outcomes. We can do this in at least two ways: a two-way table or a tree diagram.

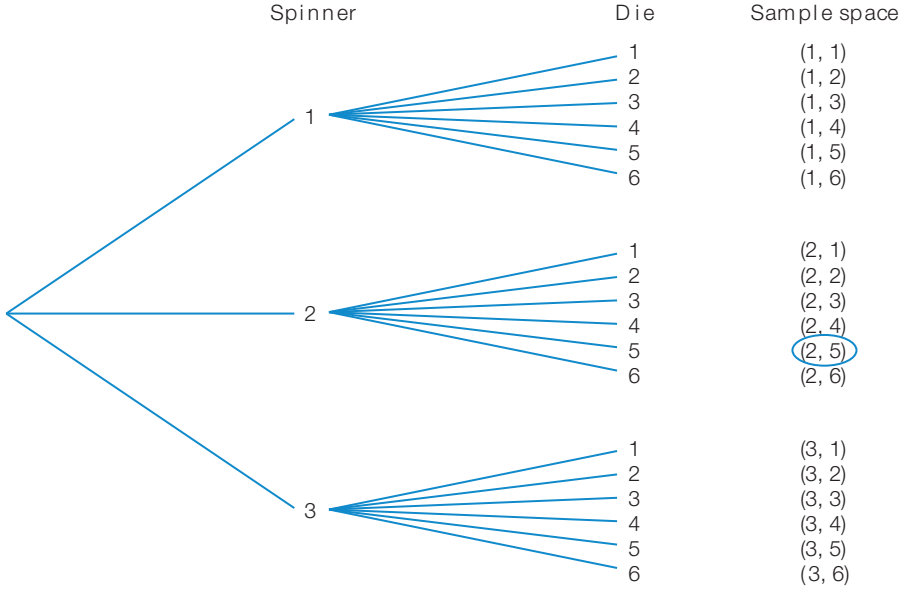
Possible outcomes

		Die					
		1	**2**	**3**	**4**	**5**	**6**
Spinner	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)

From this **two-way table**, we can see that the total number of equally likely possible outcomes is 18. We consider these outcomes to be equally likely because the die has an equal chance of showing any of its six faces and the spinner has an equal chance of landing in any of its three compartments, as they are all the same size.

So $\Pr(2 \text{ on the spinner, } 5 \text{ on the die}) = \frac{1}{18}$

We can obtain the same result using a **tree diagram**.



The full list of outcomes is called the sample space.

What is the probability of obtaining a total of 4 using the spinner and die? A two-way table can be used to present all possible outcomes. The numbers have been added together. So, the outcome (1, 1) becomes 2, as 1 + 1 = 2. As we assumed the outcomes in the first table were equally likely then so too are these. Each of the 18 points in the table is equally likely to occur.

		Die					
		1	**2**	**3**	**4**	**5**	**6**
Spinner	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9

$$\Pr(\text{total of } 4) = \frac{3}{18} = \frac{1}{6}$$

Worked Example 4 WE4

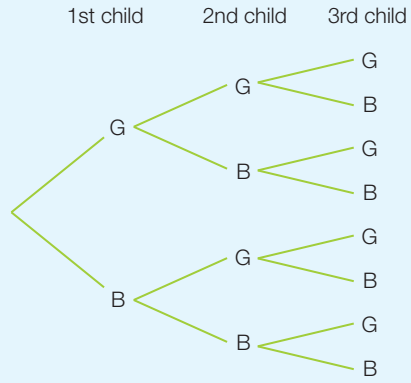
Libby and Chris wish to start a family. They plan to have three children. The chance of any particular child being a boy or a girl is considered to be equally likely.

(a) Represent all the possible outcomes using a tree diagram.

(b) List the sample space.

(c) Find the probability that they have:

- **(i)** a girl, a boy, then another girl
- **(ii)** 2 girls and a boy (in any order)
- **(iii)** 3 boys
- **(iv)** 3 children of the same gender
- **(v)** at least 2 girls

Thinking	Working
(a) Draw the tree diagram.	(a) 
(b) Follow each path on the tree diagram to list the sample space. Comment on why these can be considered equally likely events.	(b) (G, G, G), (G, G, B), (G, B, G), (G, B, B), (B, G, G), (B, G, B), (B, B, G), (B, B, B) There are 8 equally likely outcomes. These outcomes are equally likely because each child is considered equally likely to be a boy or a girl.
(c) **(i)** How many successful outcomes are listed? (Here, there is only 1 successful outcome out of 8 possible outcomes: (G, B, G).) Use this to calculate the probability required.	(c) (i) Pr(G, B, G) $= \frac{1}{8}$
(ii) How many successful outcomes are listed? (Here, there are 3 successful outcomes out of 8 possible outcomes: (G, G, B), (G, B, G), (B, G, G).) Use this to calculate the probability required.	(ii) Pr(2 girls and 1 boy) $= \frac{3}{8}$

(iii) How many successful outcomes are listed?

(Here, there is only 1 successful outcome out of 8 possible outcomes: (B, B, B).)

Use this to calculate the probability required.

(iii) Pr(3 boys)

$= \frac{1}{8}$

(iv) How many successful outcomes are listed?

(Here, there are 2 successful outcomes out of 8 possible outcomes: (G, G, G), (B, B, B).)

Use this to calculate the probability required.

(iv) Pr(3 children of the same gender)

$= \frac{2}{8}$

$= \frac{1}{4}$

(v) How many successful outcomes are listed?

(Here, there are 4 successful outcomes out of 8 possible outcomes: (G, G, G), (G, G, B), (G, B, G), (B, G, G).)

Use this to calculate the probability required.

(v) Pr(at least 2 girls)

$= \frac{4}{8}$

$= \frac{1}{2}$

11.1 Probability review

Navigator

Answers page 806

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q13, Q14, Q16, Q18, Q20, Q22	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q15, Q17, Q18, Q21, Q23	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q12, Q13, Q15, Q17, Q18, Q19, Q21, Q22, Q23

Fluency

WE1

1 Ben's car has been hard to start on 8 days over the last 4 weeks.

(a) Estimate Pr(Ben's car will be hard to start tomorrow).

(b) Estimate Pr(Ben's car will be easy to start in 3 days' time).

(c) What assumptions have you made to calculate these answers?

WE2

2 Fiona is bushwalking through a national park. She finds that the path keeps dividing and she has to decide whether to take the left path or the path to the right. If she takes 3 left turns she will arrive at a beach. If she takes 3 right turns, she will reach a waterfall. If she turns left twice and right once, not necessarily in that order, she will get to a lookout. If she turns left once and right twice, not necessarily in that order, she will reach a picnic area. She can't decide where to go so she flips a coin. Heads will mean she turns right and tails will mean that she turns left. Find the probability that she:

(a) gets to the beach

(b) arrives at the picnic area or waterfall

(c) reaches the lookout

(d) alternates the direction she takes

(e) doesn't arrive at the beach.

3 Mario likes to eat at a local restaurant but he has difficulty choosing his meal as he likes 6 main courses equally and there are 2 desserts he loves. He decides to roll a die to choose his main course and flip a coin to decide his dessert. He assigns the meals this way: chicken parmigiana as 1, chilli prawns as 2, spaghetti bolognese as 3, grilled barramundi as 4, pepper steak as 5 and spicy calamari as 6. He assigns his desserts this way: sticky date pudding as H and tiramisu as T. Find the probability that he chooses: WE3

(a) chicken parmigiana and tiramisu

(b) a seafood dish—chilli prawns, grilled barramundi or spicy calamari

(c) pepper steak and a sticky date pudding

(d) grilled barramundi or spaghetti bolognese as a main course with a tiramisu desert

(e) chilli prawns

(f) sticky date pudding.

4 A coin is tossed three times. WE4

(a) Represent all the possible outcomes using a tree diagram.

(b) List the sample space.

(c) Find the probability of tossing:

(i) a head, a tail then a head

(ii) a head, a head then a tail

(iii) 3 tails

(iv) all heads or all tails

(v) a least two heads.

5 Nikola and Brad are in a class containing 14 girls and 10 boys. To select a class representative, a person's name is chosen at random from the class roll. Find the probability that the class representative is:

(a) a girl

(b) not a girl

(c) Nikola

(d) Brad

(e) Nikola or Brad

(f) neither Nikola nor Brad.

6 Pedro has two drawers containing loose socks. There are a black, a white and a green sock in each. He is half asleep and grabs one sock, at random, from each drawer.

(a) Represent all the possible outcomes using a tree diagram.

(b) Why is it reasonable to regard these as equally likely outcomes?

(c) Find the probability of Pedro obtaining:

(i) a pair of black socks

(ii) one green and one white sock

(iii) a matching pair of socks

(iv) two socks that do not match.

7 Jacob and Ted are 2 members of a 12 member cricket team. The captain is chosen randomly each week. Find the probability that the captain is:

(a) Ted

(b) not Jacob

(c) Jacob or Ted

(d) not Jacob or Ted.

8 The local supermarket experiences a flood in its store and the labels are washed off 12 cans of peaches, 5 cans of apples and 3 cans of beans. They place them all for sale for 50 cents a can, but you will not know what is inside until it is opened.

(a) If you buy only 1 can the probability of it being apple is:

A $\frac{1}{5}$ B $\frac{1}{4}$ C $\frac{1}{3}$ D $\frac{5}{8}$

(b) If you buy 1 can the probability of it containing fruit (not beans) is:

A $\frac{3}{20}$ B $\frac{5}{20}$ C $\frac{12}{20}$ D $\frac{17}{20}$

(c) If you want to be sure of purchasing at least 1 can of peaches, the minimum number of cans you must purchase is:

A 1 **B** 3 **C** 9 **D** 12

9 A spinner evenly divided and showing the numbers 1, 2, 3, 4 is spun once, and an eight-sided die showing 1 to 8 is rolled once.

(a) The probability that the total of the two numbers is 7 is:

A $\frac{3}{11}$ **B** $\frac{3}{32}$ **C** $\frac{1}{8}$ **D** $\frac{5}{32}$

(b) The probability that the total of the two numbers is 11 is:

A $\frac{1}{32}$ **B** $\frac{1}{16}$ **C** $\frac{1}{8}$ **D** $\frac{3}{32}$

(c) The probability that the total of the two numbers is a multiple of 3 is:

A $\frac{5}{16}$ **B** $\frac{11}{32}$ **C** $\frac{3}{8}$ **D** $\frac{15}{32}$

10 Suresh and Julian regularly play tennis against one another. Over the past 2 years, Suresh has won 16 times and Julian has won 20 times. What is the estimated probability that Suresh will win their next match?

11 Of the 125 students in Year 10, 15 ride to school every day and 30 ride sometimes. A student is chosen at random from the Year 10 students.

(a) What is the probability that he or she always rides to school?

A $\frac{3}{25}$ **B** $\frac{9}{25}$ **C** $\frac{17}{25}$ **D** $\frac{22}{25}$

(b) What is the probability that he or she never rides to school?

A $\frac{3}{25}$ **B** $\frac{9}{25}$ **C** $\frac{16}{25}$ **D** $\frac{22}{25}$

12 If the probability of it raining tomorrow is $\frac{2}{9}$, what is the probability of it not raining?

Understanding

13 In a game of chance, 45 balls numbered from 1 to 45 are placed in a barrel. If one ball is drawn at random, find the probability that its number is:

(a) 2 **(b)** less than 10

(c) even **(d)** odd

(e) greater than 50 **(f)** between 10 and 20 inclusive.

14 In the game of Monopoly, players advance clockwise around the 40 squares of the board according to the total of the numbers shown on the throw of two dice.

(a) Use a two-way table to represent all the possible totals obtained from the throw of two dice.

(b) How many possible outcomes are there?

(c) Why is it reasonable to regard these as equally likely outcomes?

(d) If your token is on Whitechapel Road, what is the probability that your next throw will result in your landing on Euston Road?

(e) Suppose you do land on Euston Road. Find the probability of advancing to Bow Street or further on your next throw.

(f) One way of getting out of jail in the game of Monopoly is to throw a 'double' with the two dice (i.e. to throw the dice and obtain the same result on both). Supposing you are in jail, find the probability of getting out of jail with your next throw of the dice.

15 Skye has the mobile telephone numbers of 40 'friends' saved on her mobile telephone. If we were trying to estimate the probability of Skye messaging one of these friends would it be reasonable to assume they were equally likely outcomes? Explain your thoughts in a sentence.

16 Val and Peter have a bag of sweets each. Val's bag of sweets contains 3 licorice blocks, 2 jelly babies and 1 chocolate caramel. Peter has 2 licorice blocks and 2 jelly babies in his bag. Both Val and Peter offer a sweet to Narelle. If Narelle randomly selects a sweet from each bag, find the probability that she obtains:

(a) 2 licorice blocks

(b) 2 jelly babies

(c) no licorice blocks

(d) no jelly babies

(e) 1 licorice block

(f) at least 1 jelly baby.

17 In a round-robin competition in tenpin bowling, Shelley plays 4 rounds. She estimates that she has an equal chance of winning or losing each round.

(a) Draw a tree diagram showing all the possible outcomes. (W = a win; L = a loss)

(b) List the sample space. How many possible outcomes are there?

(c) Find the probability that Shelley:

(i) wins all 4 rounds

(ii) wins only 2 of the 4 rounds

(iii) wins at least 3 rounds

(iv) wins the second round

(v) doesn't win the third round

(vi) doesn't win neither the first nor the fourth round.

18 There are 12 houses in the court where Zoran lives. Of the 20 cars owned between the people who live in the court, 7 are Holdens, 5 are Fords and the rest are other makes. The owner of one of the Holdens moves out and a new person, who owns a car, moves into the court. Is it reasonable to say that the probability this new person owns a Holden is $\frac{7}{20}$? Explain your thoughts in a sentence.

19 Two fair six-sided dice are rolled. One is numbered 1, 1, 2, 2, 3, 3 and the other is numbered 2, 4, 6, 8, 10, 12.

(a) Draw up a two-way table to show the sample space.

(b) Draw a two-way table to show the sum of the two numbers shown.

(c) Find the following probabilities.

(i) Pr(sum is even)

(ii) Pr(sum is multiple of 5)

(iii) Pr(sum is greater than 10)

(iv) Pr(sum is not less than 11)

(d) Why is it harder to answer the questions in part **(c)** for this two-way table than for a diagram with consecutive numbers on each side?

(e) Draw up another two-way table this time showing the product of the two numbers.

(f) Draw up a table that shows the probability for each possible product.

Reasoning

20 Explain the following observations.

(a) In the long run, the observed probability of Aiden wearing a tie to work is $\frac{19}{20}$, but on three consecutive days he is seen not wearing a tie.

(b) Lesley has kept track of the lotto numbers and has noticed that 16 has not occurred for 12 months. Thinking the probability of it coming out soon must be high, she includes 16 in her numbers for the next 2 months, but it still doesn't come out.

Open-ended

21 You are given a 12-sided die, numbered 1 to 12. Write an event that would have the following probabilities.

(a) $\Pr(A) = \frac{1}{6}$

(b) $\Pr(B) = \frac{1}{3}$

(c) $\Pr(C) = \frac{5}{12}$

Hint

Assume that there is an equal probability of having a boy as having a girl.

22 The long-term probability of Maria's team beating Abbey's team at netball is $\frac{4}{5}$. List three possible sets of long-term results that would produce this figure.

23 Sharanpreet calculates the probability of a family having at least one boy in a family of three children is $\frac{3}{8}$. What is the correct answer and what mistake has Sharanpreet made?

Outside the Square

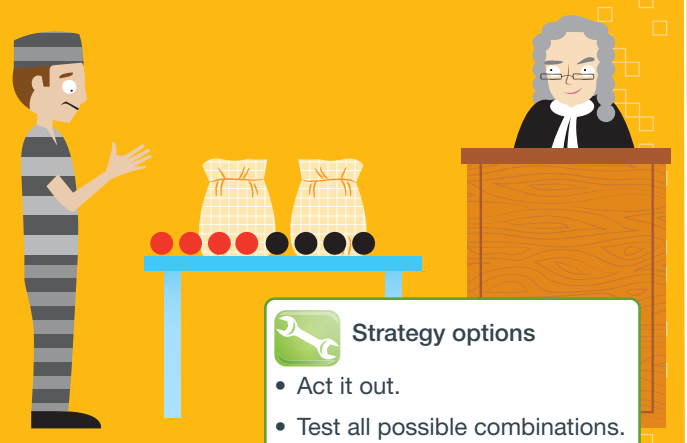
Problem solving

A chance of freedom

A prisoner is brought before a judge who says: "I have decided to give you a chance of freedom. Here are four red balls and four black balls. Divide the eight balls into two groups, any way you wish, and then place each group in a bag. I will hide the two bags behind my back and you will choose a bag from either my right or my left hand. You can then choose a ball at random from the bag. If it is red you can go free. If it is black your sentence will double."

(a) How can the prisoner divide the balls to give him the best chance of freedom?

(b) What division would give him the worst chance?



Strategy options

- Act it out.
- Test all possible combinations.
- Solve a simpler problem.

Technology Exploration CAS

Equipment required: 2 brains, 2 TI-Nspire CAS or ClassPad CAS

Investigating the reliability of lie detectors

A lie detector, or *polygraph*, is sometimes used by detectives and police officers to determine the truth of a suspect's responses to questions. Despite the widespread use of lie detectors by police in the USA, the validity of their use has not been agreed upon by psychologists, many of whom believe that they give accurate results about 80% of the time at best. This investigation uses two CAS to simulate the use of a lie detector.

Assumptions:

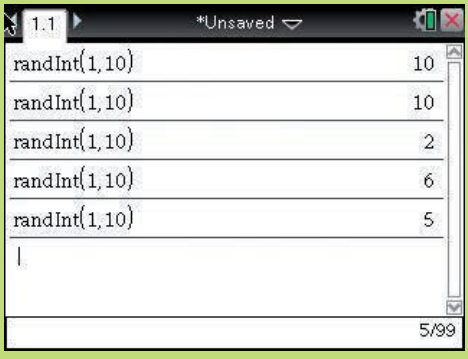
- The lie detector is accurate 80% of the time (therefore incorrect 20% of the time).
- At any time 90% of people tell the truth (therefore 10% lie).

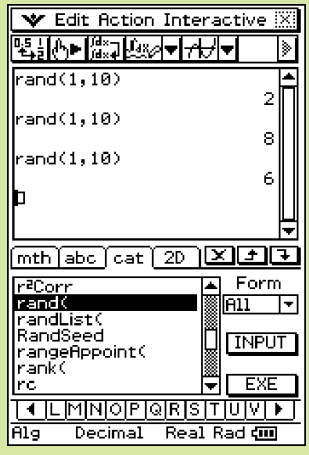
Working in pairs, one person uses a CAS to play the role of the detective who operates the lie detector. The other person uses a CAS to play the role of the suspect undergoing the lie detector test.

Setting up the 'suspect' CAS

We can use the first CAS (the suspect) to generate random counting numbers from 1 to 10. If the 'suspect' CAS generates a 5, the suspect is lying. Any other number indicates that the suspect is telling the truth. The table below shows how to generate random counting numbers on the TI-Nspire and ClassPad.

Using the TI-Nspire CAS	Using the ClassPad
The command randInt is found in the catalog (/). The syntax is randInt(lower bound, upper bound) and you get a new value each time you press (enter/).	The command rand is found in cat. You need to tap INPUT to insert the command. The syntax is rand(1,10) and you get a new value each time you press EXE.





Setting up the 'lie detector' CAS

We use the second CAS to generate random counting numbers from 11 to 20 inclusive. Try to adapt the commands given for the 'suspect' so that the 'lie detector' CAS generates the required numbers. If the lie detector CAS generates an 11 or 12, the result of the test is false. If the CAS generates a number from 13 to 20 inclusive, the result is true.

Recording the simulation

Copy the table below and record your results using tally marks.

	Suspect is lying CAS 1 shows 5	Suspect is telling the truth CAS 1 does not show 5
Lie detector result is correct CAS 2 shows 13 to 20	*A*	*B*
Lie detector result is incorrect CAS 2 shows 11 to 12	*C*	*D*

Performing the simulation

The suspect presses enter (or ≈ enter) or EXE (remember, a 5 means the suspect has just told a lie). The detective then presses enter (or ≈ enter) or EXE on the lie detector CAS to determine the truth or falsity of the suspect's statement.

Working in pairs, simulate 100 tests. Your class teacher will aggregate the results of your class into a summary table. We are aiming to simulate 1000 tests across the class. Use the aggregated results in the summary table to answer the following questions.

1 Calculate the probability that:

(a) a person selected at random is telling the truth

(b) a person selected at random is lying

(c) a lie detector test gives a true result

(d) a lie detector test gives a false result

(e) a randomly selected person who is telling the truth is found to be telling the truth by the lie detector test

(f) a randomly selected person who is telling the truth is found to be telling a lie by the lie detector test.

2 Which two outcomes in the table are undesirable? What is the probability of each?

3 Assume that 1000 people undertake a lie detector test. Use a tree diagram to determine mathematically the probability of a randomly selected person being in each section of your results table, and the number of people in each section.

For example: Section *B* of table = $0.8 \times 0.9 = 0.72$; $0.72 \times 1000 = 720$ people.

4 Use your answers to Question **3** to answer Question **1** again. Compare the theoretical and experimental probabilities.

5 Write a brief report on this activity and address the following questions. What assumptions have been made? How valid are they? Do you think lie detectors should be used? Why? Can you think of other 'non-mathematical' reasons why we should or shouldn't use lie detectors?

Venn diagrams and two-way tables

11.2

Venn diagrams and two-way tables are useful methods to help us visualise probability questions and list the sample space.

Venn diagrams

A Venn diagram consists of a rectangle with one or more circles inside it. The rectangle represents the **universal set**, which is every element under consideration for the particular situation being examined. On the Venn diagram it is represented by ξ. The circles represent particular events and are represented by capital letters.

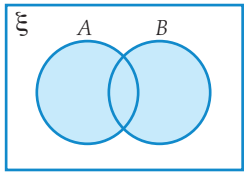
For example we might have:

ξ = 'students in year 10'
A = 'students who travel by bus'
B = 'students with red hair'

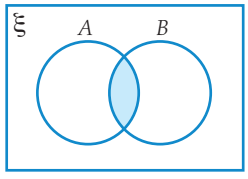
There are three situations for which Venn diagrams are really useful. These are when we are dealing with the words 'and', 'or' and 'not'. Using the example above we would have:

A or B	students who travel by bus or who have red hair (this includes those with both characteristics)
A and B	students who travel by bus and have red hair
not A	students who do not travel by bus

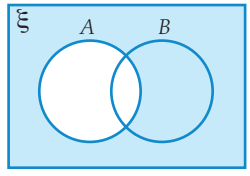
These can be shown on Venn diagrams as follows:



A or B



A and B



not A

Two-way tables

We can use a two-way table to show similar information.

	B	**not *B***
A	A and B	A and not B
not *A*	not A and B	not A and not B

Together, the sections of the table make up the universal set. In addition to this, no section has any overlap with any other section.

Worked Example 5

WE 5

A single card is drawn from a set of 25 cards numbered from 1 to 25.

Let A = 'odd numbers' and B = 'multiples of 3'

(a) Represent this on a Venn diagram.

(b) Use the Venn diagram to find:

(i) Pr(A)

(ii) Pr(A and B)

(iii) Pr(not B)

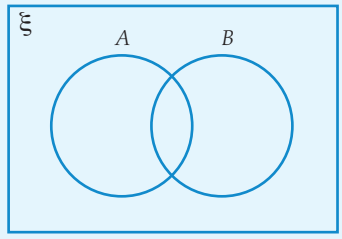
(c) Draw a two-way table to display the probability of possible events.

Thinking

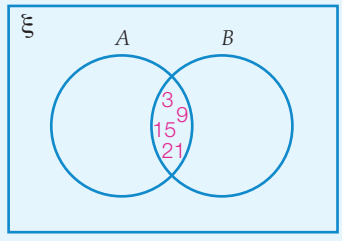
Working

(a) **1** Draw a Venn diagram that contains two overlapping circles labelled A and B.

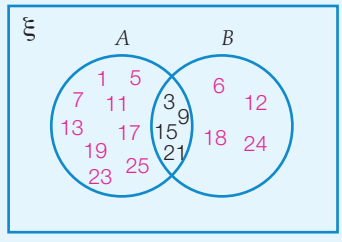
(a)



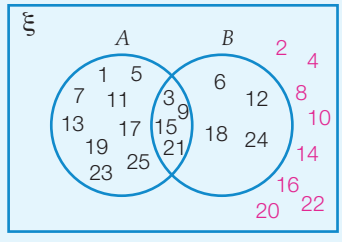
2 Place the elements that are common to both sets in the overlap of the circles.



3 Put the remainder of elements that are not included in the overlap into the correct circle.



4 Place elements that do not belong inside the circles but are part of the universal set into the Venn diagram.



(b) **(i)** Use the formula:
$\Pr(A) = \dfrac{\text{number of elements in } A}{\text{number of elements in universal set}}$

(b) (i) $\Pr(A) = \dfrac{13}{25}$

(ii) Use the formula:
$\Pr(A \text{ and } B) = \dfrac{\text{number of elements in } A \text{ and } B}{\text{number of elements in universal set}}$

(ii) $\Pr(A \text{ and } B) = \dfrac{4}{25}$

(iii) Use the formula:
$\Pr(\text{not } B') = 1 - \Pr(B) = 1 - \dfrac{\text{number of elements in } B}{\text{number of elements in universal set}}$

(iii) $\Pr(\text{not } B) = 1 - \Pr(B)$
$= 1 - \dfrac{8}{25}$
$= \dfrac{17}{25}$

(c) **1** Draw a two-way table and label the possible outcomes. Enter the known probabilities from **(b)**.

(c)

	B	not B	
A	$\frac{4}{25}$		$\frac{13}{25}$
not A			
		$\frac{17}{25}$	

2 Calculate any missing probabilities.

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A			
		$\frac{17}{25}$	

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A			$\frac{12}{25}$
		$\frac{17}{25}$	

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A		$\frac{8}{25}$	$\frac{12}{25}$
	$\frac{8}{25}$	$\frac{17}{25}$	

	B	not B	
A	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{13}{25}$
not A	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{12}{25}$
	$\frac{8}{25}$	$\frac{17}{25}$	

11.2 Venn diagrams and two-way tables

Navigator

Answers page 808

Q1, Q2, Q3, Q5, Q8, Q9, Q11, Q14	Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q9, Q10, Q12, Q13	Q1, Q2, Q3, Q5, Q6, Q7, Q9, Q10, Q12, Q13

Fluency

WE5

1 In a non-leap year there are 28 days in February. A date in February is to be chosen to celebrate the 100th birthday of a city.

Let A = 'even numbered dates' and B = 'dates that are multiples of 5'.

(a) Represent this information on a Venn diagram.

(b) Use the Venn diagram to find:

(i) Pr(A)

(ii) Pr(A and B)

(iii) Pr(not B)

(c) Draw a two-way table to display the probability of possible events.

2 Use the two-way table shown to find:

(a) Pr(A and B)

(b) Pr(not A and not B)

(c) Pr(not A and B)

(d) Pr(A and not B)

(e) Pr(A)

(f) Pr(B)

	B	not B
A	$\frac{3}{16}$	$\frac{5}{16}$
not A	$\frac{1}{16}$	$\frac{7}{16}$

3 In a class of 20, 8 students learn the piano, 6 take singing lessons and 10 students neither learn the piano nor have singing lessons.

(a) Represent this information in a two-way table.

(b) Use the two-way table to find the number of people who learn piano and take singing lessons.

(c) Use the two-way table to find the probability of:

(i) taking singing lessons but not piano

(ii) taking piano lessons but not singing.

Understanding

4 In a group of 30 people there are 15 who like cats, 20 who like dogs and 6 who dislike both cats and dogs. One person is chosen from the group at random.

(a) Represent this information on a Venn diagram.

(b) Use the Venn diagram to find the probability that the person chosen at random is:

(i) a cat lover only

(ii) a dog lover only

(iii) a person who likes both.

5 Complete these two-way tables.

(a) $\Pr(A) = \frac{7}{16}$

	B	not *B*
A		$\frac{5}{16}$
not *A*	$\frac{1}{16}$	

(b) $\Pr(D) = 0.7$

	C	not *C*
D	0.2	
not *D*		0.1

6 Given the following Venn diagram, identify the numbers represented by each of the following:

(a) *A* or *B*

(b) *A* or *C*

(c) *B* or *C*

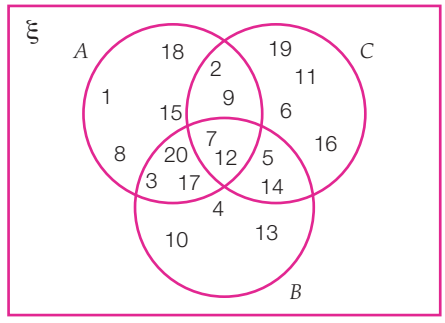
(d) *A* and *B*

(e) *A* and *C*

(f) *B* and *C*

(g) *A* and *B* and *C*

(h) not *A* and not *B*



7 A coin is tossed, a six-sided die is thrown and a ball is selected from a bag containing 4 red and 3 green balls. *M* is the set of outcomes that contains 'head is tossed', *N* is the set of outcomes that contains 'a 2 is not thrown' and *P* is the set of outcomes that contains 'a red ball is selected'.

(a) Draw a Venn diagram to show the sample space.

(b) Write down the elements of not *M*, not *N* and not *P*.

(c) Is the probability of H1R the same as H1G? Explain your answer.

8 A poll of 2000 students revealed that 554 respondents only watch TV and do not browse the internet while they watch. 1200 respondents only use the internet and the TV is not on while they're browsing. 10 students claim to neither watch TV nor use the internet.

(a) Draw a two-way table to represent this situation.

(b) What is the probability of a student in this survey watching TV and browsing the internet at the same time?

Reasoning

9 A survey of 1200 college students revealed that $\frac{3}{8}$ have a profile on MySpace, $\frac{3}{4}$ have a profile on Facebook, and $\frac{1}{4}$ have a profile on both sites.

(a) Represent the information in a two-way table.

(b) If a student is chosen at random find the probability that the student has:

(i) no profile on either

(ii) a profile on either, but not on both.

10 Mrs Lauder is creating the timetable for Year 12 students. The request forms have been entered online:

- 163 want to study mathematics, 95 physics, 124 chemistry
- 61 requested maths and physics
- 67 requested maths and chemistry
- 49 requested physics and chemistry
- 27 want to study all three subjects
- 17 want to study none of these subjects

(a) Draw a Venn diagram to illustrate the situation.

(b) How many students requested only mathematics?

(c) How many students requested only physics?

(d) How many students requested only chemistry?

(e) What was the total number of requests?

(f) What is the probability of a Year 12 student at Mrs Lauder's school selecting mathematics as one of their subjects? Will the answer be the same at every school?

11 In a scout troop there are 15 boys and 12 girls. 10 of the boys and 8 of the girls say they are interested in tennis.

(a) Show this information on a Venn diagram.

(b) A student is selected at random. Use the diagram to find the probability that the student selected is:

(i) a girl who is interested in tennis

(ii) a boy who is not interested in tennis

(iii) someone who is interested in tennis

(iv) someone who is not interested in tennis

12 Human blood can contain up to three different antigens; A, B and Rh. The following table outlines the antigens present in each blood type. (The actual situation is slightly more complicated for the O group.) The values given in the table refer to Australia; countries with different ethnic make-up would have different values.

Blood Group	A	B	Rh	% in Australia
A Positive	Yes	No	Yes	31
A Negative	Yes	No	No	7
B Positive	No	Yes	Yes	8
B Negative	No	Yes	No	2
AB Positive	Yes	Yes	Yes	2
AB Negative	Yes	Yes	No	1
O Positive	No	No	Yes	40
O Negative	No	No	No	9

The antigens present in the blood of patients at a clinic were recorded for 1 week:

- 39 patients had the A antigen
- 2 had the A and B antigens
- 12 had the B antigen
- 9 had the B and Rh antigens
- 67 had the Rh antigen
- 30 had the A and Rh antigens
- 2 had all three antigens
- 5 had no antigens.

(a) Draw a Venn diagram to show this situation.

(b) How many patients were there?

(c) What was the Pr(exactly one antigen)?

(d) What was the Pr(exactly two antigens)?

(e) Use the data to calculate the probability of each of the eight blood groups.

(f) Compare the answers from part **(e)** with those expected from long-term data as shown in the table.

Open-ended

13 From a class of 25 students 14 like watching the news on TV and 16 said they use the internet to find out the news. Mert says there must be something wrong with the statistics because there are 30 different opinions from only 25 students. Sasha says Mert hasn't taken into account students who do both. Neither of them is correct. What is the correct interpretation of the statistics and what mistakes have Mert and Sasha made?

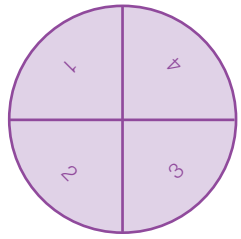
14 What is wrong with the following question?

In a survey of 200 people with a driving licence, 18% have never received a speeding fine, 44 people have received a parking ticket but have never received a speeding fine, and 78% have received both a speeding fine and a parking ticket. How many people have received a speeding fine but have never received a parking ticket?

11.3 Mutually exclusive events

If two events cannot occur at the same time they are called **mutually exclusive** events.

The spinner at right is spun.



Consider two events.

Let A = obtaining an even number

= obtaining a 2 or a 4

B = obtaining a 3

These events are said to be mutually exclusive as it is impossible for both of them to occur at the same time.

$\Pr(A \text{ and } B) = 0$

Note that there is no overlap between these two events. Also, since the spinner is evenly divided into four pieces, each number has an equal chance of occurring.

$$\Pr(A) = \frac{\text{number of successful equally likely outcomes}}{\text{total number of equally likely outcomes}}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

$$\Pr(B) = \frac{\text{number of successful equally likely outcomes}}{\text{total number of equally likely outcomes}}$$

$$= \frac{1}{4}$$

$$\Pr(A \text{ or } B) = \frac{\text{number of successful equally likely outcomes}}{\text{total number of equally likely outcomes}}$$

$$= \frac{3}{4}$$

Notice that $\Pr(A) + \Pr(B) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

So $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$. This is called the **Addition Law for Mutually Exclusive Events**. It can only be used when one event *or* the other event may occur, but not both, i.e. when the events are mutually exclusive.

Worked Example 6

WE 6

A six-sided die is rolled. What is the probability that the number obtained is:

(a) a 2 or a 5?

(b) neither odd nor a 4?

Thinking	Working
(a) Use the Addition Law for Mutually Exclusive Events as both events cannot occur at the same time. $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$ (Here, $\Pr(A) = \Pr(2) = \frac{1}{6}$ and $\Pr(B) = \Pr(5) = \frac{1}{6}$.)	(a) $\Pr(2 \text{ or } 5)$ $= \Pr(2) + \Pr(5)$ $= \frac{1}{6} + \frac{1}{6}$ $= \frac{2}{6}$ $= \frac{1}{3}$
(b) 1 Use complementary probabilities. $\Pr(\text{not } A) = 1 - \Pr(A)$ **2** Use the Addition Law for Mutually Exclusive Events as both events cannot occur at the same time. $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$ (Here, $\Pr(A) = \Pr(\text{odd}) = \frac{3}{6} = \frac{1}{2}$ and $\Pr(B) = \Pr(4) = \frac{1}{6}$.)	(b) $\Pr(\text{neither odd nor a } 4)$ $= \Pr[\text{not}(\text{odd or } 4)]$ $= 1 - \Pr(\text{odd or } 4)$ $= 1 - [\Pr(\text{odd}) + \Pr(4)]$ $= 1 - \left(\frac{1}{2} + \frac{1}{6}\right)$ $= 1 - \frac{2}{3}$ $= \frac{1}{3}$

Can you think of an alternative, perhaps simpler, way we could have worked out the answer to part **(b)** above? (*Hint:* Try listing the sample space.)

Addition Law for Mutually Exclusive Events

If A and B are mutually exclusive events (i.e. they cannot occur at the same time), then:

$$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$$

11.3 Mutually exclusive events

Navigator

Q1, Q2, Q3, Q4, Q5, Q6 Column 1, Q7, Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q10	Q1, Q2, Q3, Q4, Q5, Q6 Column 2, Q7, Q8, Q9, Q10

Answers page 809

Fluency

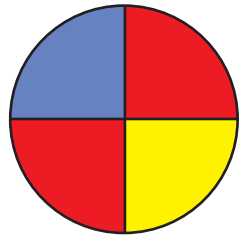
1 A fair eight-sided die, numbered 1 to 8, is rolled. What is the probability that the number obtained is:

WE 6

(a) a 3 or a 6?

(b) neither 5 nor even?

2 A spinner is coloured as shown.



(a) What is the probability of spinning blue?

(b) What is the probability of spinning red?

(c) What is the probability of spinning blue or red?

(d) What is the probability of spinning blue or yellow?

3 Are any of the following pairs of events mutually exclusive?

(a) Obtaining a head on the toss of a coin and a tail on the toss of a coin.

(b) Choosing an odd-numbered horse in a race and horse number 7 in another race.

(c) Winning first prize in a raffle with a blue ticket and first prize in a raffle with a yellow ticket.

4 (a) A basket contains 3 red, 2 white and 5 black scarves. The probability that a scarf drawn at random will be red or black is:

A $\frac{1}{5}$ B $\frac{2}{3}$ C $\frac{7}{10}$ D $\frac{4}{5}$

(b) A jar contains 6 red lollies, 3 green lollies and 2 yellow lollies. The probability that a lolly drawn at random will be green or yellow is:

A $\frac{5}{11}$ B $\frac{6}{11}$ C $\frac{3}{11}$ D $\frac{5}{6}$

(c) A drawer contains 6 blue socks, 4 black socks and 2 brown socks. The probability that a sock drawn at random will *not* be black is:

A $\frac{1}{4}$ B $\frac{1}{3}$ C $\frac{1}{2}$ D $\frac{2}{3}$

Understanding

5 These probabilities apply to different modes of transport students use to travel to and from a particular school. The modes are all mutually exclusive.

$\Pr(\text{car}) = \frac{7}{20}$ $\Pr(\text{train}) = \frac{9}{20}$

$\Pr(\text{bus}) = \frac{1}{20}$ $\Pr(\text{walking}) = \frac{3}{20}$

A student is asked how he travels to school. Find the probability that:

(a) he travels by car or train

(b) he walks or catches a bus

(c) he does not walk nor catch a bus

(d) he does not travel by car nor by bus

(e) he travels by car, train or bus.

6 Twenty tokens numbered in order from 1 to 20 are placed in a bag and one is randomly selected. Find the probability that the number on the token is:

(a) 19 or 20

(b) a multiple of 3

(c) a multiple of 3 or a multiple of 7

(d) even or the number 15

(e) less than 10 or greater than 18

(f) neither 1 nor 2

(g) neither odd nor a multiple of 4

(h) 5, 6 or 7.

7 A pack of playing cards contains 52 cards, half red and half black. The red cards are divided into two suits: hearts and diamonds. The black cards are divided into two suits: clubs and spades. Each suit is made up of the cards ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen and king.

A card is drawn at random from the pack. Find the probability that the card is:

(a) red

(b) a heart

(c) an ace

(d) the king of hearts

(e) red or the ace of spades

(f) a diamond or the 4 of hearts

(g) the 5 of clubs or the queen of spades

(h) a king or an ace

(i) any king or any ace or the 10 of hearts

(j) the jack of diamonds, any 2, or the 7 of clubs

(k) neither a 6 nor a 9 of any suit

(l) neither a spade nor red.

Reasoning

8 A normal six-sided die is rolled. The following events have been defined:

A = rolling an even number

B = rolling a 2 or a 6

C = rolling a 1 or a 3

D = rolling a multiple of 3

(a) List all possible pairs of events (order does not matter).

(b) Are any of these pairs mutually exclusive?

(c) Find the probability of each event.

(d) Find the probability that either event occurs for each of the pairs listed in **(a)**.

(e) Find the probability that both of the events occur for each of the pairs listed in **(a)**.

(f) Find the sum of the probabilities of each event in all the pairs listed in **(a)**.

(g) Compare your answers to parts **(d)**, **(e)** and **(f)**. What do you find? Write an equation to summarise your findings.

9 A class conducted a traffic survey. The types of vehicles they recorded were:

sedan	station wagon	4WD	hatchback
truck	van	bus	2 door

(a) What difficulties can you see arising from the data collection?

(b) How could these difficulties have been avoided?

Open-ended

10 Consider a fair 20-sided die numbered 1 to 20. Write down at least eight pairs of mutually exclusive events related to this die.

Half-time 11

Ex 11.2

1 (a) Complete the following two-way table which relates to the number of surveyed students who own either, or both, an iPhone or an iPad.

	iPhone	not iPhone	
iPad	60		75
not iPad			
	210		400

(b) Now use the completed two-way table to state the probability of a randomly chosen student owning:

(i) an iPhone but not an iPad (ii) an iPad but not an iPhone.

Ex 11.1

2 Ten balls numbered from 1 to 10 are placed in a bag and one is drawn at random. Find the probability that the number on the ball is:

(a) 1 or 2

(b) an even number

(c) a 6 or an odd number

(d) neither an odd number nor a multiple of 4.

Ex 11.1

3 The probability of it raining on a given day of the weekend is 0.3. What is the probability that it rains on both Saturday and Sunday?

Ex 11.1

4 In a chess tournament, each contestant must play three others. Suzanne estimates that she has an equal chance of winning or losing each game.

(a) Draw a tree diagram and list the sample space for the possible outcomes of Suzanne's three games.

(b) How many possible outcomes are there?

(c) Find the probability of Suzanne:

(i) winning all three games (ii) winning only one game

(iii) winning the first game (iv) not winning the third game

(v) winning at least two games (vi) not winning either of the first two games.

Ex 11.3

5 A fair eight-sided die is numbered 1 to 8. The die is rolled once. The following events are defined:

A = 'rolling an even number'
B = 'rolling a prime number'
C = 'rolling a multiple of 5'

(a) Given that order does not matter list the pairs of events that are possible.

(b) Find the probability of each individual event.

(c) State which, if any, of the pairs of events are mutually exclusive.

Technology Exploration CAS

Equipment required: 2 brains, 1 TI-Nspire CAS or ClassPad CAS

Investigating the 'one son' policy

China is the most populous country in the world, with approximately 1.3 billion people. In an effort to reduce the population, the Chinese government has encouraged a policy of one child per family. Eventually such a policy, if strictly adhered to, would reduce the population. Can you explain why?

Some Chinese have advocated a 'one son' policy. Under this policy a family would be allowed to continue to have additional children until a son is born. Possible families under this policy, excluding parents, would be: one boy only, one girl and one boy (in that order), two girls and one boy (in that order) etc. Note that if the first born is a son, then no additional children are allowed. Your task is to assess the implications of adopting the 'one son' policy.

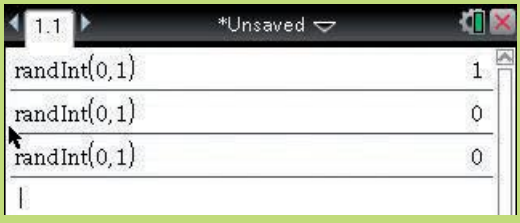
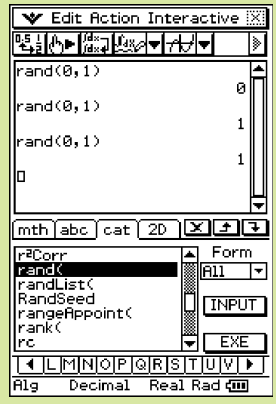
1 Write down what you think would be the ratio of the number of births of girls to births of boys under this policy.

2 What do you think would be the mean number of children in a family? Discuss your answers with other students and your teacher. Write down your estimate.

Modelling the possible families

We can model the 'one son' policy using the calculator's random number generator.

We want to make the CAS randomly generate 1s and 0s. To do this, follow the steps for your CAS.

Using the TI-Nspire CAS	Using the ClassPad
In the Calculator application, enter the Catalog by pressing (or), press R (or R) to get to the start of the Rs, then scroll down to randInt and press enter (or). You then enter the beginning value and the end value, in this case, 0 and 1. Each time you press enter (or) you get a new value. 	It is better to have your calculator set to Decimal mode rather than Standard for generating random numbers. Just tap the word Standard at the bottom of the screen and it will change to Decimal. At the Main screen, tap Keyboard cat then scroll across the alphabet and tap R. Highlight rand(and tap INPUT. To get 0 or 1 as the only options you need to have rand(0,1) and each time you tap EXE you get a new value. 

3 Work in pairs to complete the table on the following page. Each pair should aim to simulate the results for 50 families. Press enter (or) or EXE to generate the random numbers. If, for example, you obtained the output 0 0 1, then put a tally mark next to 001 (i.e. girl, girl, boy).

Generate random numbers until you get a 1: this represents the children in the first family. Continue until the next 1: this represents the children in the next family. For example, the numbers below represent 10 families:

(0 1) (0 0 1) (0 0 0 1) (1) (0 1) (0 0 1) (0 1) (1) (1) (0 1)

- 3 with 1 child only (1)
- 2 with 3 children (001)
- 4 with 2 children (01)
- 1 with 4 children (0001)

When all pairs of students have finished the table, combine 10 sets of results into a summary table to represent 500 families. Use the combined results to answer the questions below on the next page.

Family type	Tally	Total number of families	Total number of boys	Total number of girls	Total number of children
1					
01					
001					
0001					
00001					
0000001					
00000001					
000000001					
0000000001					
_____					
_____					
	Totals:				

4 What was the ratio of births of girls to births of boys?

5 How did this ratio compare with your estimation in Question **1**?

6 Calculate the ratio of 'boy births' to 'total births'.

Finding the mean number of children per family

Suppose the following results were obtained by your class:

Number of children	1	2	3	4	5	6	7	8	9	10
Number of families	241	130	64	32	18	7	4	3	0	1

To calculate the mean number of children per family:

$$\text{Mean number of children per family} = \frac{\text{Total number of children}}{\text{Total number of families}}$$

$$= \frac{1 \times 241 + 2 \times 130 + 3 \times 64 + 4 \times 32 + 5 \times 18 + 6 \times 7 + 7 \times 4 + 8 \times 3 + 9 \times 0 + 10 \times 1}{241 + 130 + 64 + 32 + 18 + 7 + 4 + 3 + 0 + 1}$$

$= 2.03$ children/family

7 Find the mean number of children per family using the results obtained by the class. Compare this answer to your estimate in Question **2**.

Examining assumptions

8 When we use mathematics as a model for a real-world problem, we need to make assumptions. The model above assumes that the probability of giving birth to a boy is the same as the probability of giving birth to a girl. Is this assumption valid? If not, is our model still useful?

9 If we were to model one million births, predict the mean number of families with 1, 2, 3, …, 10 children.

10 What other assumptions have been made by this model?

11 If the 'one son' policy were adopted, what would be the effect on the population?

12 If you were an adviser to the Chinese government, would you recommend the policy be adopted or rejected? Why?

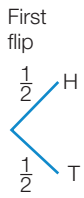
Probability tree diagrams

11.4

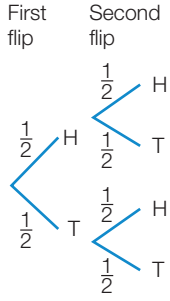
Let us consider the case of flipping a fair coin three times. There are eight equally likely outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

So Pr(HHH) = Pr(HHT) = Pr(HTH) = Pr(HTT) = Pr(THH) = Pr(THT) = Pr(TTH) = Pr(TTT) = $\frac{1}{8}$ and Pr(HHH) + Pr(HHT) + Pr(HTH) + Pr(HTT) + Pr(THH) + Pr (THT) + Pr(TTH) + Pr(TTT) = 1, as these are the only possible outcomes.

We can represent these outcomes with a **probability tree diagram**. We start with a dot and branch this to represent the possible outcomes from the first flip of the coin. In this case, there are two (H and T) so there are two branches. The outcomes are equally likely as the coin is fair, so we can add the probability of $\frac{1}{2}$ to the outcome on each branch.

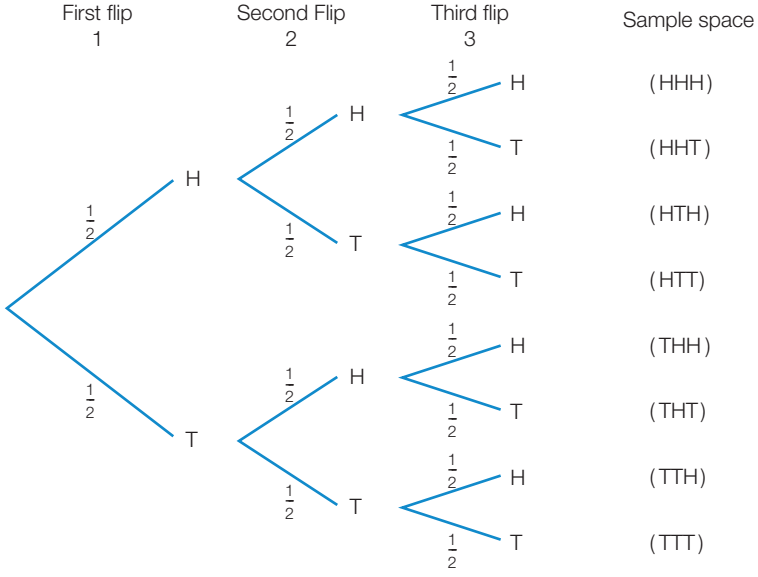


When we flip the coin the second time there are again two possible outcomes (H and T) but this time we have two separate starting positions, Again, Pr(H) = Pr(T) = $\frac{1}{2}$ so we get the probability tree diagram shown at right.



Flipping the coin for the third time, we once again have 2 possible outcomes (H and T) for each of the previous 4 outcomes.

As there are 8 possible outcomes, we can write these in the sample space.



We can then find the probabilities for each outcome as each event is equally likely (Pr(H) = Pr(T) = $\frac{1}{2}$).

Pr(HHH) = $\frac{1}{8}$, Pr(HHT) = $\frac{1}{8}$, Pr(HTH) = $\frac{1}{8}$, Pr(HTT) = $\frac{1}{8}$, Pr(THH) = $\frac{1}{8}$, Pr (THT) = $\frac{1}{8}$, Pr(TTH) = $\frac{1}{8}$ and Pr(TTT) = $\frac{1}{8}$.

The probability of obtaining 2 heads equals $\frac{3}{8}$ as there are 3 outcomes out of 8 that are successful.

Calculating probabilities using the probability tree

If we follow along the branches from the start to each possible outcome, each with $\frac{1}{2}$ written on it and multiplying the probabilities on each branch, we see that:

$$\Pr(HHH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(HHT) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(HTH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(HTT) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$\Pr(THH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(THT) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(TTH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \qquad \Pr(TTT) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

These probabilities correspond to the probabilities we previously calculated.

The probability of obtaining 2 heads is the sum of three probabilities:

$$\Pr(HHT) + \Pr(HTH) + \Pr(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

In this example, the outcomes of all events in the sample space are equally likely. In many circumstances, however, the outcomes of events in the sample space are not equally likely.

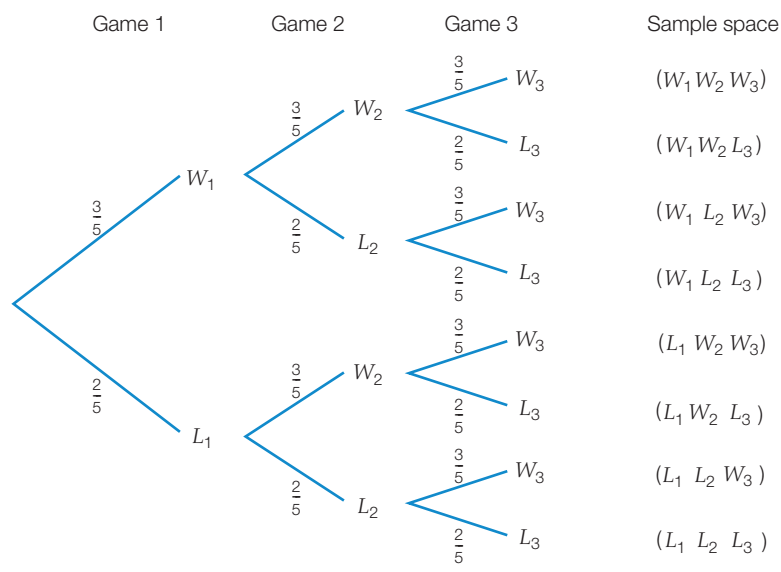
Now consider the following situation.

Neale and Mario play three games of golf. Neale's chance of winning any one game is $\frac{3}{5}$. We are assuming this probability remains constant over the three games and no games are drawn. What is the probability that Neale wins two games out of three?

Let W represent the event 'Neale wins the golf game'.

Let L represent the event 'Neale loses the golf game'.

$\Pr(W) = \frac{3}{5}$ and $\Pr(L) = \frac{2}{5}$. The probabilities of the branches from one point must be add to 1. In this case $\frac{3}{5} + \frac{2}{5} = 1$.



There are three possible situations where Neale wins two games:

- $(W_1\ W_2\ L_3)$ Neale wins 1st game, *and* wins 2nd game, *and* loses 3rd game.
- $(W_1\ L_2\ W_3)$ Neale wins 1st game, *and* loses 2nd game, *and* wins 3rd game.
- $(L_1\ W_2\ W_3)$ Neale loses 1st game, *and* wins 2nd game, *and* wins 3rd game.

Here, we cannot just use 3 possible outcomes in 8 total outcomes to calculate the required probability because the events in the sample space are not equally likely.

$$\begin{aligned}\Pr(W_1W_2) &= \Pr(W_1 \text{ and } W_2)\\ &= \frac{3}{5}\times\frac{3}{5}\\ &= \frac{9}{25}\end{aligned}$$

$$\begin{aligned}\text{So } \Pr(W_1W_2L_3) &= \Pr(W_1 \text{ and } W_2 \text{ and } L_3)\\ &= \frac{9}{25}\times\frac{2}{5}\\ &= \frac{18}{125}\end{aligned}$$

$$\begin{aligned}\text{Similarly } \Pr(W_1L_2\ W_3) &= \Pr(W_1 \text{ and } L_2 \text{ and } W_3)\\ &= \frac{3}{5}\times\frac{2}{5}\times\frac{3}{5}\\ &= \frac{18}{125}\end{aligned}$$

$$\begin{aligned}\text{and } \Pr(L_1W_2W_3) &= \Pr(L_1 \text{ and } W_2 \text{ and } W_3)\\ &= \frac{2}{5}\times\frac{3}{5}\times\frac{3}{5}\\ &= \frac{18}{125}\end{aligned}$$

If we follow along the branches from the start to each possible outcome, multiplying the probabilities on each branch, we see that we calculate the same probabilities.

$$\Pr(W_1\ W_2\ L_3) = \frac{3}{5}\times\frac{3}{5}\times\frac{2}{5} = \frac{18}{125}$$

$$\Pr(W_1\ L_2\ W_3) = \frac{3}{5}\times\frac{2}{5}\times\frac{3}{5} = \frac{18}{125}$$

$$\Pr(L_1\ W_2\ W_3) = \frac{2}{5}\times\frac{3}{5}\times\frac{3}{5} = \frac{18}{125}$$

These events are mutually exclusive, as they cannot occur at the same time, so you can add probabilities to find the probability that Neale wins two games.

$$\begin{aligned}\Pr(\text{Neale wins 2 games}) &= \Pr(W_1\ W_2\ L_3 \text{ or } W_1\ L_2\ W_3 \text{ or } L_1\ W_2\ W_3)\\ &= \Pr(W_1\ W_2\ L_3) + \Pr(W_1\ L_2\ W_3) + \Pr(L_1\ W_2\ W_3)\\ &= \frac{18}{125} + \frac{18}{125} + \frac{18}{125}\\ &= \frac{54}{125}\end{aligned}$$

Remember 'or' means + if the outcomes are mutually exclusive.

The probability that Neale wins two golf games out of three against Mario is $\frac{54}{125}$ or 0.432.

Worked Example 7 WE7

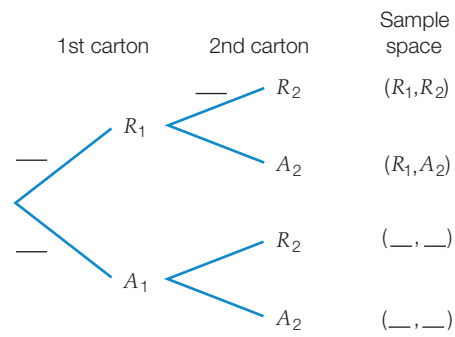
At supermarkets, cartons of eggs are inspected for breakages by the checkout assistant before payment is made. The probability that a carton of eggs will be rejected is found to be 0.1. Two cartons of eggs are brought up to a checkout.

(a) Copy and complete the following tree diagram. Evaluate the probability of each outcome.

Let R = carton of eggs is rejected and A = carton of eggs is not rejected (is accepted)

(b) Find the probability that:

- **(i)** both cartons will be rejected
- **(ii)** only one carton will be rejected
- **(iii)** neither carton will be rejected
- **(iv)** at least one carton will be rejected.



Thinking

(a) 1 Use $\Pr(R) + \Pr(A) = 1$ to find $\Pr(A)$.

2 Copy and complete the tree diagram.

Assign the probability of each branch, list the sample space and then calculate the probability of each outcome.

$\Pr(R_1R_2)$ means $\Pr(R_1 \text{ and } R_2)$. As we have an 'and' situation, multiply to find the probabilities.

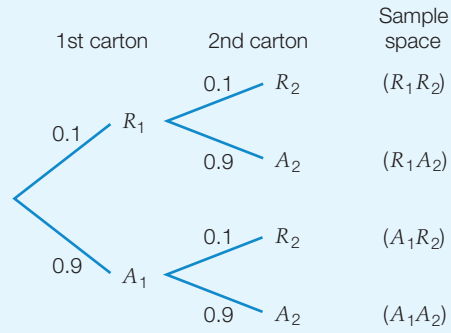
(b) (i) Use the sample space to find the required outcomes.

(ii) 1 Use the sample space to find the required outcomes.

2 These outcomes are mutually exclusive, so use the Addition Law for Mutually Exclusive Events.

Working

(a) $\Pr(R) = 0.1$, so $\Pr(A) = 0.9$



$\Pr(R_1R_2) = 0.1 \times 0.1$
$= 0.01$

$\Pr(R_1A_2) = 0.1 \times 0.9$
$= 0.09$

$\Pr(A_1R_2) = 0.9 \times 0.1$
$= 0.09$

$\Pr(A_1A_2) = 0.9 \times 0.9$
$= 0.81$

(b) (i) Pr(both cartons will be rejected)
$= \Pr(R_1R_2)$
$= 0.01$

(ii) Pr(only one carton will be rejected)
$= \Pr(R_1A_2 \text{ or } A_1R_2)$

$= \Pr(R_1A_2) + \Pr(A_1R_2)$
$= 0.09 + 0.09$
$= 0.18$

(iii) Use the sample space to find the required outcomes.	(iii) Pr(neither carton will be rejected) $= \Pr(A_1A_2)$ $= 0.81$
(iv) **1** 'At least one' means one or two.	(iv) Pr(at least one carton will be rejected) = Pr(one carton rejected or both cartons rejected)
2 These situations are mutually exclusive, so use the Addition Law for Mutually Exclusive Events.	= Pr(one carton rejected) + Pr(both cartons rejected)
3 Use your answers from parts **(a)** and **(b)**.	$= 0.18 + 0.01$ $= 0.19$

In the next example the probabilities change because we are sampling without replacement.

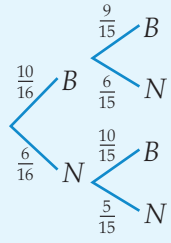
Worked Example 8

WE8

The lucky dip at the school fete contains 16 different gifts, 10 for the 4–7 age group and 6 for the 8–11 age group. Unfortunately, the 16 gifts have been mixed together. Since you cannot tell what is inside without opening the gift, the lucky dips are offered as a special two for the price of one. A 5-year-old takes his two gifts without replacement. Let *B* represent a gift is chosen in the 4–7 age group and *N* represent a gift in the 4–7 age group is not chosen.

(a) Find Pr(both gifts are in the 4–7 age group).

(b) Find Pr(the 5-year-old chooses at least one suitable gift).

Thinking	Working
1 Draw a tree diagram and assign a probability to each branch.	
2 List the sample space.	*BB, BN, NB, NN*
(a) **1** Use the sample space to find the required outcomes.	(a) *BB*
2 This is an 'and' situation, so we multiply.	$\Pr(BB) = \frac{10}{16} \times \frac{9}{15} = \frac{3}{8}$
(b) Use a complementary event to find the probability. (Here, there are three ways of choosing at least one gift in the 4–7 age group (*BN*, *NB*, *BB*) and only one way of not (*NN*), so it is more efficient to calculate 1 – Pr(*NN*).	(b)
1 Find Pr(*NN*).	$\Pr(NN) = \frac{6}{16} \times \frac{5}{15} = \frac{1}{8}$

Pr(the 5-year-old chooses at least one suitable gift)

$= 1 - \frac{1}{8}$

2 Find 1 – Pr(NN). $= \frac{7}{8}$

We can use a tree diagram and multiply along its branches to obtain probabilities whether the outcomes are equally likely or not.

11.4 Probability tree diagrams

Navigator

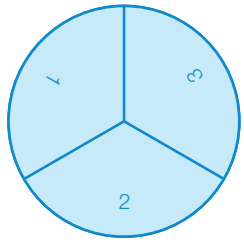
Answers page 810

Q1, Q2, Q3, Q4, Q6, Q7, Q9, Q10, Q12, Q14, Q17	Q1, Q2, Q3, Q4, Q7, Q8, Q10, Q13, Q15, Q16	Q1, Q2, Q4, Q5, Q7, Q9, Q11, Q12, Q16, Q17

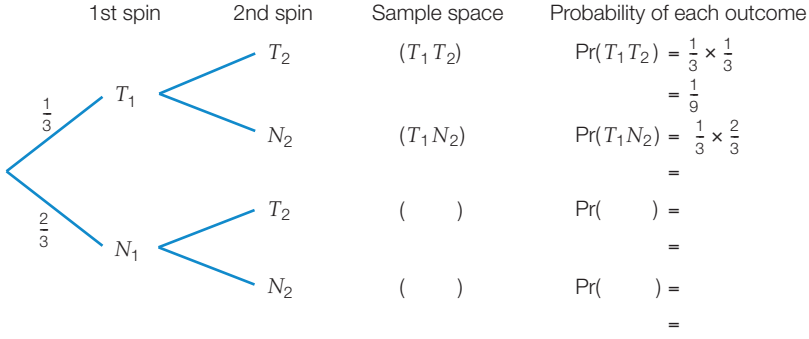
Fluency

WE7

1 A spinner numbered 1 to 3 is spun twice. Let T represent 'the number 2 is spun' and N represent 'the number 2 is not spun'.



(a) Copy and complete the following tree diagram. Evaluate the probability of each outcome.



(b) Find the probability of:

(i) spinning 2 twice

(ii) spinning 2 once only

(iii) *not* spinning a 2

(iv) spinning 2 once at least.

WE8

2 Two balls are drawn without replacement from a bag containing 4 green marbles and 7 blue marbles. Draw a tree diagram to represent the situation and find:

(a) Pr(both marbles being blue)

(b) Pr(getting at least one blue marble).

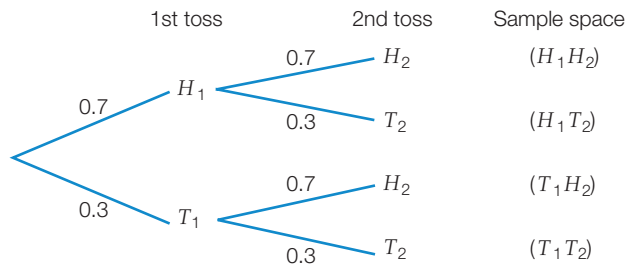
3 The spinner described in Question **1** is now spun three times.

(a) Draw a probability tree diagram for this situation.

(b) What is the probability of spinning a 2 three times?

(c) What is the probability of *not* obtaining a 2 on any of the three spins?

4 A biased coin is tossed twice. Let H represent 'obtaining a head with the coin' and T represent 'obtaining a tail with the coin'. The following probability tree diagram can be drawn.



(a) The probability of obtaining two heads is:

A $0.7 + 0.7$ **B** 0.7×0.7 **C** 0.25 **D** 0.7×0.3

(b) The probability of obtaining two tails is:

A 0.6 **B** 0.3 **C** 0.21 **D** 0.09

(c) The probability of obtaining a tail first, and then a head is:

A 0.25 **B** 0.21 **C** 0.49 **D** 1.0

(d) The probability of obtaining only one tail is:

A 0.21 **B** 0.3 **C** 0.42 **D** 0.63

(e) The probability of obtaining at least one tail is:

A 0.75 **B** 0.51 **C** 0.42 **D** 0.39

5 Two cards are drawn, without replacement, from a set of 16 cards numbered from 1 to 16.

Let M = the event a multiple of four is drawn and N = a multiple of four is not drawn.

(a) Draw a tree diagram and assign a probability to each branch.

(b) List the sample space.

(c) Find Pr(both cards are a multiple of 4).

(d) Find Pr(only one card is a multiple of 4).

(e) Find Pr(neither card is a multiple of 4).

Understanding

6 For afternoon tea, patrons of a cafe can select from tea, coffee or hot chocolate, and either scones or carrot cake. The probability of a person selecting tea is 0.45, whereas the probability of a person selecting coffee is 0.35. The probability of a person selecting scones is 0.6.

(a) What is the probability of a person selecting hot chocolate?

(b) What is the probability of a person selecting carrot cake?

(c) Draw a probability tree diagram to represent this situation.

(d) Find the probability of a person:

(i) selecting tea and scones

(ii) selecting hot chocolate and carrot cake

(iii) selecting tea and scones or hot chocolate and carrot cake

(iv) *not* selecting coffee and scones.

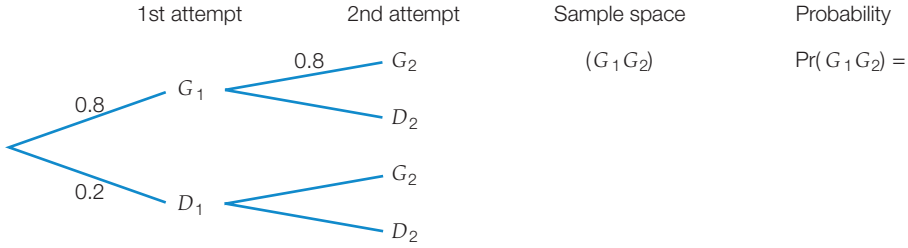
7 For each of the following:

(i) copy and complete the probability tree diagram

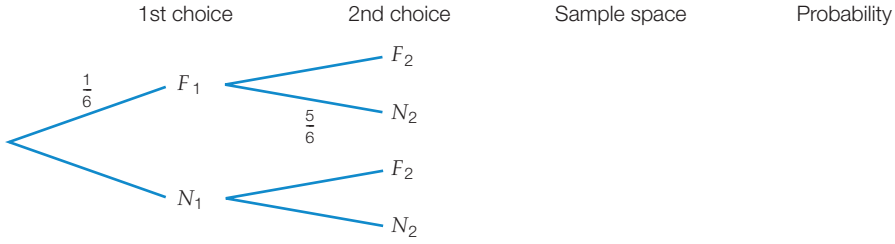
(ii) write down the probability of each possible outcome

(iii) find the sum of the probabilities of all outcomes.

(a) Let G = scores a goal and D = does not score a goal.



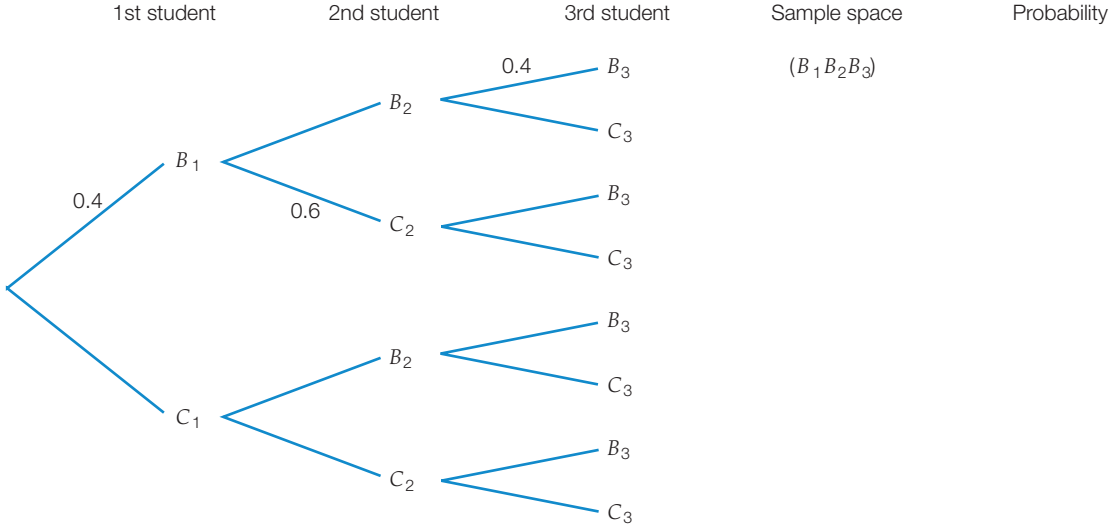
(b) Let F = a 4 is chosen from the numbers 1 to 6 and N = a 4 is not chosen.



(c) P = Patrick is chosen from a group of boys.

A = Anne is chosen from a group of girls.

(d) Let B = student travels to school by bus and C = student does not travel by bus.



8 In a game of darts, each player has a turn to throw three darts one at a time. Seline estimates her chances of scoring a bullseye with any one throw of a dart to be 0.2. Find the probability that:

(a) Seline scores three bullseyes in her turn

(b) she scores two bullseyes only

(c) she scores at least one bullseye.

9 Two fair six-sided dice are rolled. Let T represent the event 'a 3 is rolled'.

(a) What would not T represent?

(b) Draw a probability tree diagram to illustrate this situation.

(c) Find the probability of:

(i) rolling two 3s **(ii)** not rolling any 3s **(iii)** rolling one 3 only

(iv) rolling at least one 3 **(v)** rolling no more than one 3.

10 A card is selected from a pack of 52 playing cards and then replaced. Another card is selected. Find the probability that:

(a) both cards are aces

(b) the first card is a club and the second card is red

(c) both cards are of the same colour

(d) one card is red and the other is black.

11 Wesley's lucky number is 5. At the races, he decides to bet on horse number 5 in the first three events. In race number 1, there is a field of 12 horses. In race number 2, there is a field of 9 horses. In race number 3, there is a field of 10 horses.

(a) If each horse in an event has an equal chance of winning, draw a probability tree diagram to represent Wesley's chances of winning or losing on each race.

(b) Find the probability that:

(i) Wesley wins all three racing events

(ii) only one of his chosen horses wins

(iii) at least one of his chosen horses wins.

Reasoning

12 A mathematics quick quiz consists of five multiple choice questions, each with five choices for the solution. Draw up a table to show the probability of obtaining each of the totals of correct answers from 0 to 5 if you randomly guess the answers.

13 A fair six-sided die is rolled until a 6 is obtained.

(a) Copy and complete the following table, which shows the probability that this will take the indicated number of rolls. Round your answer to three decimal places.

Number of rolls	1	2	3	4	5	6
Probability						

(b) Add the probabilities you found in part **(a)**. What sum do you get?

(c) Explain what this answer means.

14 A party game for 29 people starts by assigning everyone a number from 1 to 29. In the first round, 7 numbers are drawn at random and those people eliminated. In the second round 6 people are eliminated, the third round eliminates 5 people, then 4, 3, 2 and finally 1, to leave the winner.

(a) Find Pr(being eliminated in round 1).

(b) Assuming you are not eliminated in the first round, find Pr(being eliminated in round 2).

(c) Assuming you are not eliminated in the previous rounds find the probability of being eliminated in each round.

(d) Add the results for the probability of being eliminated in each round. Explain the result.

Open-ended

15 A set of volleyball is won when one side reaches 25 points. If we knew nothing about the ability of the teams we would assign the chance of winning as 0.5 each. If, however, one side was leading 15–0 we may rethink the original assignment of 0.5 each. Estimate the new probability for each team to win the set? Justify your answer.

16 Draw up a probability tree diagram that finishes with at least 9 branches where one of the final branches has a probability of $\frac{1}{9}$ and another has a probability of $\frac{1}{8}$. Explain what your tree diagram represents.

17 The estimated probability of a 25-year-old male playing cricket is 0.2 and a 25-year-old male playing golf is 0.18. Armapreet calculates that the probability of a man playing cricket or golf must be 0.38. Is she right? If not, what mistake has she made?

Outside the Square — Problem solving

The origins of probability theory

The origins of modern probability theory involve a problem known as the 'problem of the points' or 'division of the stakes'. It involves a two-person game where each has made an equal contribution to the prize pot and the agreement is that the first person to get to a certain number of points will be the winner and collect the entire prize pot. The problem arises when the game is interrupted, and cannot be concluded. How should the prize pot be divided?

Let's say the the winner is the first to 10 points.

1 How would you divide the pot if the score was 7 to 5?

2 One of the early solutions suggested that the pot should be divided in proportion to the number of games won. Using this solution, the pot above would be divided $\frac{7}{12}:\frac{5}{12}$. What do you think of this solution? What would happen if the score was 1 to 0? Pascal and Fermat thought the crucial issue was the possible ways the game could have continued rather than what had already happened.

3 Think of a game where the score is 8 to 7. Write all the ways the game could continue and use this to decide how to divide the pot.

4 What if the score was 2 to 5? What is the problem with this sort of score?

5 What do you think would be the easiest scores to deal with if the game was interrupted?

Strategy options

- Act it out.
- Draw a diagram.

Extension

It is considered that Pascal and Fermat solved this problem. Conduct some internet research to discover how they resolved the issue.

Conditional statements

11.5

Let's assume we have rolled two fair six-sided dice, one green and one red. The following array shows the possible outcomes.

		Green die					
		1	2	3	4	5	6
Red die	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

There are 36 equally likely outcomes. They are equally likely as we assume each die is fair. So, for example, Pr(1, 1) = $\frac{1}{36}$. If we know that the number on the green die is 1, then Pr(1, 1) would be $\frac{1}{6}$ because only the first column of results is possible from our array. In short, the universal set has been restricted in this case, so we effectively only consider the outcomes that have a 1 on the green die.

We would write: Pr[(1, 1) given green die is 1] = $\frac{1}{6}$

Similarly, we can calculate Pr(red die is even given green die is a multiple of 3). We use the information after the word 'given' to restrict the universal set—the value that becomes the denominator for our probability. This means we are dealing with a **conditional statement**.

The number of outcomes in the restricted set becomes the denominator for the probability fraction.

We can shade our array to show this.

		Green die					
		1	2	3	4	5	6
Red die	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

		Green die	
		3	6
Red die	1	(3, 1)	(6, 1)
	2	(3, 2)	(6, 2)
	3	(3, 3)	(6, 3)
	4	(3, 4)	(6, 4)
	5	(3, 5)	(6, 5)
	6	(3, 6)	(6, 6)

The green shading shows the condition we were given—the green die is a multiple of 3 (which is a 3 or a 6). There are 12 possible outcomes in the restricted set, shaded green, so the denominator for our probability fraction is 12. We now need to count the number of outcomes that have an even number on the red die, shaded red. There are 6 possible outcomes.

Pr(red die is even given green die is a multiple of 3) = $\frac{6}{12} = \frac{1}{2}$.

The order of the statements is important. To demonstrate this, we will find Pr(green die is a multiple of three given the red die is even).

		Green die					
		1	2	3	4	5	6
Red die	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

		Green die					
		1	2	3	4	5	6
Red die	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
	6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

There are 18 possible outcomes in the restricted set, shaded red, so the denominator for our probability fraction is 18. There are 6 possible outcomes that have a 3 or 6 on the green die, which are shaded green.

Pr(green die is a multiple of three given the red die is even) $= \frac{6}{18} = \frac{1}{3}$

In these examples we have shown the events using a two-way table. However, we could also use a Venn diagram to illustrate this type of information.

Using Venn diagrams

Worked Example 9

WE 9

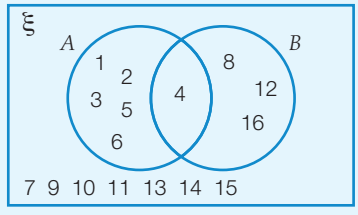
A 16-sided die is rolled, with faces numbered 1–16. The following events are defined.

A = 'numbers less than 7' and B = 'multiples of 4'

(a) Draw a Venn diagram to show this information.

(b) Calculate Pr(a number less than 7 given it is a multiple of 4).

(c) Calculate Pr(a multiple of 4 given it is less than 7).

Thinking	Working
(a) Draw the Venn diagram. Place the element(s) in the overlap (intersection) of A and B first as it makes it easier to draw the diagram.	(a) 
(b) **1** Identify the denominator for the probability.	(b) The denominator is the number of elements in B (a multiple of 4): 4
2 Identify the numerator for the probability.	Find how many of the elements of B are also in A (a number less than 7): 1
3 State the probability.	Pr(a number less than 7 given it is a multiple of 4) $= \frac{1}{4}$
(c) **1** Identify the denominator for the probability.	(c) The denominator is the number of elements in A (a number less than 7): 6
2 Identify the numerator for the probability.	Find how many of the elements of A are also in B (a multiple of 4): 1

3 State the probability.	Pr(a multiple of 4 given it is a number less than 7) $= \frac{1}{6}$

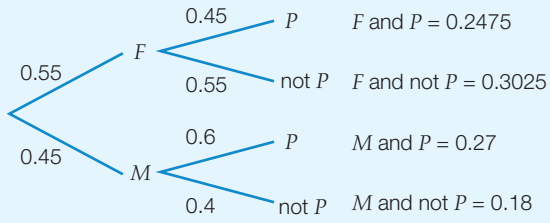
Using tree diagrams

So far, we have used the expression 'given that' as the conditional statement. However, other expressions can also be used. These include expressions such as 'knowing that', 'when' and 'if … then'. We can also use tree diagrams to help us with these probability calculations.

Worked Example 10

WE10

In a particular electorate, 55% of the voting population is female. It is also known that 60% of the males support the Progressive Conservatives Party, whereas only 45% of the females support this party. Find the probability that a randomly selected voter from that electorate is male if you know that they support the Progressive Conservative Party. Round your answer to four decimal places.

Thinking	Working
1 Draw a tree diagram to show the information. Define the variables to be used to represent the events.	F = 'female' M = 'male' P = 'supporters of Progressive Conservative Party' 
2 State the outcome we are interested in finding.	Pr(M given P)
3 Find the value for the denominator. (In this case it will be a probability value, but as long as the numerator is a probability value, this will not affect our calculation.)	Pr(support Progressive Conservative Party) $= \Pr(FP) + \Pr(MP)$ $= 0.2475 + 0.27$ $= 0.5175$
4 Find the value for the numerator. (This will be those supporters of the Progressive Conservative Party who are male.)	Pr(male and support Progressive Conservative Party) $= \Pr(MP)$ $= 0.27$
5 Find the required probability.	Pr(male given they support Progressive Conservative Party) $= \frac{0.27}{0.5175}$ $= 0.5217$

11.5 Conditional statements

Navigator

Answers page 811

Q1, Q2, Q3, Q4, Q5, Q6, Q9 (a)–(c), Q11	Q1, Q2, Q4, Q5, Q7, Q9 (a)–(c), Q11	Q1, Q2, Q4, Q5, Q7, Q8, Q9, Q10, Q11

Fluency

WE9

1 **(a)** A 16-sided die is rolled, with faces numbered 1–16. The following events are defined.

A = 'numbers less than 9' and B = 'multiples of 3'

(i) Draw a Venn diagram to show this information.

(ii) Calculate Pr(number less than 9 given it is a multiple of 3).

(iii) Calculate Pr(multiple of 3 given it is a number less than 9).

(b) A 16-sided die, with faces numbered 1–16, is rolled and the following events are defined.

A = 'numbers greater than 5' and B = 'prime numbers'

(i) Draw a Venn diagram to show this information.

(ii) Calculate Pr(number greater than 5 given it is a prime number).

(iii) Calculate Pr(prime number given it is greater than 5).

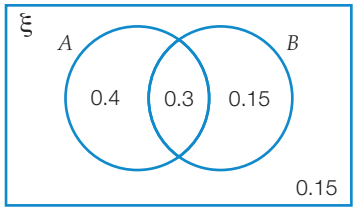
WE10

2 In a particular electorate, 65% of the voting population is female. It is also known that 55% of the males support the Progressive Conservatives Party, whereas only 40% of the females support this party. Find the probability that a randomly selected voter from that electorate is female if you know that they support the Progressive Conservative Party. Round your answer to four decimal places.

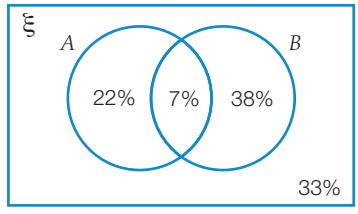
3 Use the information in the Venn diagram to find:

(i) Pr(A given B) **(ii)** Pr(B given A)

(a)



(b)



4 A coin is flipped three times.

(a) The probability that the third flip results in a head, given the previous two flips were both heads, is:

A $\frac{1}{8}$ **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** 1

(b) The probability that the third flip results in a head, given at least two of the three flips are heads, is:

A $\frac{3}{8}$ **B** $\frac{1}{2}$ **C** $\frac{2}{3}$ **D** $\frac{3}{4}$

5 You have a reduced pack of playing cards consisting only of the 26 red cards. A card is selected at random from the pack. Find the probability that the card is:

(a) an ace given it is a heart **(b)** a heart given it is a picture card

(c) the five of diamonds given it is a diamond **(d)** the five of diamonds given it is a five.

Understanding

6 A new disease is known to infect 2.5% of the population. A test for the disease has been developed, but it is not completely accurate. It correctly identifies the disease in infected patients 95% of the time, but also (incorrectly) identifies the disease in 7% of the uninfected population.

(a) Draw a tree diagram to show this information.

(b) Find Pr(have disease given test positive), rounding your answer to four decimal places.

(c) Find Pr(don't have disease given test negative), rounding your answer to four decimal places.

7 A sportswear company obtains its shoes from two suppliers, A and B. It is known that 5% of the shoes from supplier A are defective, whereas the figure is 9% for supplier B. Artemis obtains 65% of its shoes from supplier A. Where necessary, round your answers to four decimal places.

(a) Draw a tree diagram to show this information.

(b) Find Pr(A given the shoe is defective).

(c) Find Pr(B given the shoe is not defective).

8 A university student is studying both Economics and Voodoo. The probability that she will pass Economics is 0.95 and that she will pass Voodoo is 0.8. There is a 0.75 chance she will pass both. Find:

(a) Pr(passes Economics given passes Voodoo).

(b) Pr(passes Voodoo given passes Economics).

(c) Pr(passes Economics given passes only one of the subjects).

Reasoning

9 Abbud plays a lot of table-tennis. He has a 70% chance of winning any particular game. Find the probability that Abbud wins a match decided by the:

(a) best of three games

(b) best of three games, given that he won the first game

(c) best of three games, given that it lasts three games

(d) best of five games

(e) best of five games, given that he won the first two games.

10 A new cola is produced which the maker claims has the same taste as the market leading cola. A test is set up whereby people are offered a drink and they say whether it is the market leader or the new cola. The new cola is correctly identified 40% of the time and the market leader is correctly identified 75% of the time. If a person has an equal chance of being given either cola and they say the cup contains the market leader, what is the probability that this is true?

Open-ended

11 A tired teacher wrote this question:

Given Pr(A) = 0.6 and Pr(B) = 0.4, find Pr(A given B).

(a) What is wrong with the question?

(b) Rewrite the question so it is correct, and then find the answer.

Maths 4 Real

MENDEL AND THE BIRTH OF MODERN GENETICS

Your genetic code

Inside every one of the cells of your body lies your genetic code—the unique combination of DNA that contains the instructions for making the individual person that you are. A length of DNA that contains the instructions for a particular characteristic (such as eye colour or hair colour) is called a gene. Your genes are organised into 46 packages called chromosomes. These chromosomes form 23 pairs. One chromosome from each pair is from your biological mother, and the other is from your biological father. Which chromosome from a pair that a mother or father passes onto their offspring is a random and independent event.

Will it be a boy or a girl?

Although it may appear that certain families have a greater chance of having boys than girls, or girls than boys, the gender of a newborn baby is an independent event. Regardless of how many boys or girls were born previously, the probability of the baby being female is 50%, and the probability for a male is also 50%. The gender of a baby is determined by which copy of chromosome 23 it receives from its father. If it receives an 'X' chromosome, it will be a girl. A 'Y' chromosome means that the baby will be a boy. Which chromosome is passed on is a completely random process.

1 Determine the probability of a mother having:

(a) two boys in a row

(b) three girls in a row

(c) a girl, then a boy

(d) one of each gender, in either order.

Dominant and recessive genes

Although you have two copies of every gene (one from your mother and one from your father), usually, the physical characteristic of only one gene is actually seen, or 'expressed', in the body. This is the 'dominant' gene. The other gene is called 'recessive'.

The modern science of genetics began with an Austrian monk named Gregor Mendel. Mendel studied the effects of cross-breeding in pea plants. He was able to predict the probability of a cross-bred pea showing certain characteristics. As a simple example we can look at seed colour.

The following diagrams are similar to two-way tables. They show the different gene combinations that can result from each parent passing on one of their genes to their offspring.

Take two parent plants: one is pure-bred green GG (both genes for colour are green) and the other is pure-bred yellow YY (both genes for colour are yellow). The table below shows that all offspring are YG (one yellow gene and one green gene). Y is dominant over G so all the offspring appear yellow.

Parent

		G	G
Parent	Y	YG	YG
	Y	YG	YG

Possible offspring

If, however, both parents are YG (hybrids) then the offspring will be 25% YY 50% YG and 25% GG. The GG will be green even though its parents appeared yellow!

Parent

		Y	G
Parent	Y	YY	YG
	G	YG	GG

Possible offspring

2 What will be the probabilities for the seed colour of the offspring of a YY crossed with a YG, where Y (yellow) is dominant over G (green)?

Genetic diseases

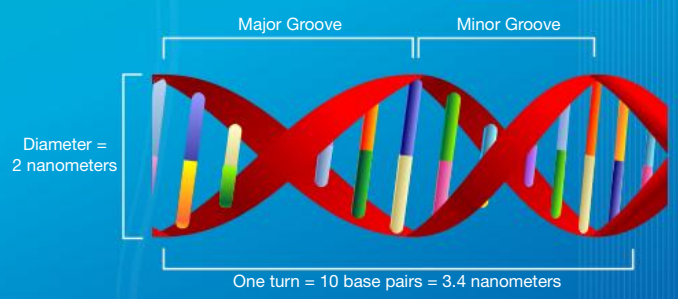
Genetics has very important implications for potentially fatal human illnesses. Let us assume there is a disease, such as cystic fibrosis, where there is a dominant gene A and a recessive gene a. It is the recessive gene, a, that is responsible for the disease. You suffer the disease if you are aa. You carry it but don't suffer it if you are Aa.

If two carriers have a child there is a 1 in 4 (25%) chance of the child having the illness (aa), a 1 in 2 (50%) chance of the child being a carrier (Aa) and a 25% chance of the child not being a carrier (AA).

	A	a
A	AA	Aa
a	Aa	aa

If a carrier and a sufferer have a child the situation gets worse. There is a 50% chance of the child having the illness and a 50% chance of being a carrier.

	A	a
a	Aa	aa
a	Aa	aa



3 Blood groups are another simple example of genetics. Calculate the probabilities of each possible outcome from parents having the following characteristics. Note: Group A is dominant over Group O.

(a)

	O	O
A		
O		

(b)

	A	O
A		
O		

(c)

	O	O
A		
A		

Taking it further

4 Research one or more of the following topics and present your findings in a suitable manner.

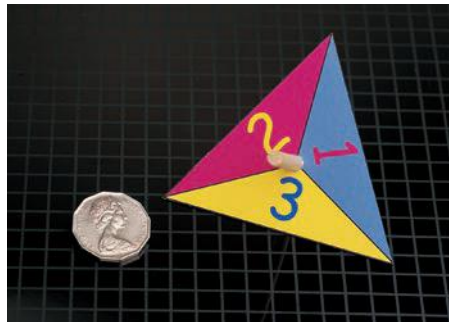
(a) The life of Mendel and his work on genetics.

(b) There is a third blood group, Group B, which is co-dominant with Group A. Find out what this means.

(c) Genetic fingerprinting is used to help solve many crimes. Find out how it works and whether it is related at all to fingerprint analysis, which has been used for many years now.

11.6 Dependent and independent events

Now let us consider the situation where we have a fair coin and a spinner equally divided into three sections, labelled 1, 2 and 3.



We can find the sample space by looking at the following array:

		Coin	
		H	T
Spinner	1	(H, 1)	(T, 1)
	2	(H, 2)	(T, 2)
	3	(H, 3)	(T, 3)

We consider these to be equally likely outcomes as we assume the coin is fair and we know the spinner is equally divided. From this array we can see that:

$$\begin{aligned}\Pr(\text{H}) &= \Pr(\text{H}, 1) + \Pr(\text{H}, 2) + \Pr(\text{H}, 3)\\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\\ &= \frac{3}{6}\\ &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\Pr(1) &= \Pr(\text{H}, 1) + \Pr(\text{T}, 1)\\ &= \frac{1}{6} + \frac{1}{6}\\ &= \frac{2}{6}\\ &= \frac{1}{3}\end{aligned}$$

Now, what is the probability of getting a head *and* a 1? This is just $\Pr(\text{H}, 1) = \frac{1}{6}$.

We can see that $\Pr(\text{H and } 1) = \Pr(\text{H}) \times \Pr(1)$ as $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

When $\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$ we say we are dealing with **independent events**. Knowing anything about one event has no impact on what happens with the other. The outcome of the flip of the coin has no influence on what happens to the spinner.

Two events A and B are said to be independent if the result of one event has no effect on the result of the other event.

When A and B are independent:

$$\Pr(A \text{ and } B) = \Pr(A) \times \Pr(B)$$

Dependent events are influenced by the outcome of another event; sampling without replacement is an example of a dependent event.

Consider a situation where two balls are drawn successively without replacement from a box that contains 5 white and 3 green balls.

For the first ball drawn Pr(white) = $\frac{5}{8}$ and Pr(green) = $\frac{3}{8}$.

However, for the second draw the probabilities depend upon which colour ball was drawn first.

- If the first ball drawn was white, then Pr(white) = $\frac{4}{7}$ and Pr(green) = $\frac{3}{7}$.
- If the first ball drawn was green, then Pr(white) = $\frac{5}{7}$ and Pr(green) = $\frac{2}{7}$.

In this case the probability of the second event is dependent on the first event. Other examples of dependent events might be the possibility of rain on two successive days, because if it rains one day it is more likely to rain again the next day than if it had been fine.

Worked Example 11

WE 11

Two representatives, one girl and one boy, are chosen at random from a group of 6 girls and 4 boys. If Linda and Adam are in this group, what is the probability that both Linda and Adam will be chosen?

Thinking	Working
1 Assign letters to stand for each of the basic events.	Let L = Linda chosen and A = Adam chosen
2 Are these events independent or does one result affect the other?	L and A are independent events.
3 Both must occur. Use Pr(A and B) = Pr(A) × Pr(B).	Pr(L and A) = Pr(L) × Pr(A)
4 Calculate the probability of each event happening in isolation and then calculate the probability of both.	$\Pr(L) = \frac{1}{6}$ $\Pr(A) = \frac{1}{4}$ $\Pr(L \text{ and } A) = \frac{1}{6} \times \frac{1}{4}$ $= \frac{1}{24}$
5 State the answer.	The probability that both Linda and Adam will be chosen is $\frac{1}{24}$ (i.e. 1 chance in 24).

What we are doing here is formalising the sort of calculation we may have previously completed using a tree diagram. If we can state the individual probabilities for a chain of events we can then multiply those probabilities to find the probability of all of these events occurring. We can now use multiplication of probabilities instead of drawing a tree diagram or writing out the sample space.

Worked Example 12 WE12

Two cards are drawn out of a 52-card deck of playing cards without replacement.

(a) Find the Pr(drawing 2 kings).

(b) Find the Pr(both cards are from the same suit).

Thinking	Working
(a) 1 Find the probability of the given event. (Here, Pr(K on first draw).)	(a) Pr(king on 1st draw) $= \frac{4}{52} = \frac{1}{13}$
2 Find the probability of the second event knowing the first has occurred. (Here, Pr(K on second draw after K on first draw).)	Pr(king on 2nd draw) $= \frac{3}{51} = \frac{1}{17}$
3 Find the probability of the stated event. (Here, Pr(KK), with no replacement.)	Pr(both cards being a king) $= \frac{1}{13} \times \frac{1}{17}$ $= \frac{1}{221}$
(b) 1 Find the probability of each successful outcome. (Here, Pr(CC), Pr(SS), Pr(HH), Pr(DD).)	(b) Pr(club/club) $= \frac{13}{52} \times \frac{12}{51} = \frac{3}{51} = \frac{1}{17}$
	Pr(spade/spade) $= \frac{13}{52} \times \frac{12}{51} = \frac{3}{51} = \frac{1}{17}$
	Pr(heart/heart) $= \frac{13}{52} \times \frac{12}{51} = \frac{3}{51} = \frac{1}{17}$
	Pr(diamond/diamond) $= \frac{13}{52} \times \frac{12}{51} = \frac{3}{51}$ $= \frac{1}{17}$
2 As the events are mutually exclusive we add the individual probabilities to find the probability of all successful outcomes.	Pr(both cards are from the same suit) $= \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17}$ $= \frac{4}{17}$

11.6 Dependent and independent events

Navigator

Q1, Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q9, Q15, Q18, Q19, Q20	Q1, Q2, Q3, Q5, Q7, Q8, Q10, Q12, Q13, Q14, Q15, Q17, Q18, Q20	Q1, Q2, Q4, Q6, Q7, Q9, Q10, Q11, Q13, Q14, Q16, Q17, Q19, Q20

Answers page 812

Fluency

1 Katrina finds that on average her train is late 2 mornings out of 5, whereas Frank's bus is late 1 morning in every 4. Find the probability that on any given day both Katrina and Frank are late. WE11

2 A box containing 40 apples, 10 of each of 4 different varieties, has 5 rotten ones. If 2 apples are taken out without replacement, calculate: WE12

(a) Pr(2 rotten apples)

(b) Pr(2 apples of the same variety)

3 Phillip has 2 pairs of blue and 1 pair of black jeans hanging in his wardrobe. In a drawer, he has 1 red, 3 black and 5 white T-shirts. If he randomly chooses a pair of jeans and a T-shirt to wear, find the probability that he chooses:

(a) a white T-shirt and blue jeans

(b) a black T-shirt and black jeans

(c) black jeans and a T-shirt that is not black

(d) blue jeans and a T-shirt that is not red.

4 In a group of 20 people, 2 have red hair, 6 are blond and the rest have dark hair. Of the group, 15 people are right-handed. If a person is chosen at random from the group, find the probability that the person:

(a) has red hair and is right-handed

(b) is blond and is left-handed

(c) is right-handed and does not have red hair.

5 Label each of these pairs of events as either independent (I) or dependent (D).

(a) Tossing a coin and rolling a die.

(b) Drawing two cards from a deck without replacing the first before drawing the second.

(c) Drawing two cards from a deck replacing the first before drawing the second.

(d) Raining in two cities 100 km apart on the same day.

(e) Hours of study and test result.

(f) The number of hours of practice a learner driver does before getting their license and the chance of having an accident in the first month of having their license.

6 An urn contains 3 red beads, 4 blue beads and 8 green beads. If two beads are drawn without replacement, find:

(a) Pr(RR) (b) Pr(BB) (c) Pr(GG) (d) Pr(RB)

(e) Pr(BG) (f) Pr(GB) (g) Pr(same colour) (h) Pr(different colours)

7 **(a)** If a 20-cent coin is tossed in the air twice, the probability of obtaining two heads is:

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$

(b) If a six-sided die is rolled twice, the probability of obtaining a double 6 is:

A $\frac{1}{3}$ **B** $\frac{1}{6}$ **C** $\frac{1}{12}$ **D** $\frac{1}{36}$

(c) If a 20-cent coin is tossed and a six-sided die is rolled, the probability of getting a head and a multiple of 3 is:

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$

Understanding

8 Joshua and Tania are competing in different sections of a chess tournament. Joshua estimates that his chance of winning against his opponent is 3 in 5, whereas Tania estimates that her chance of winning against her opponent is 5 in 7. Find the probability that:

(a) both Joshua and Tania win their chess game

(b) they both lose their chess game

(c) Joshua loses his game but Tania wins hers.

9 A drawer contains 6 black socks, 8 red socks and 4 white socks. If two socks are drawn without replacement, find:

(a) Pr(2 black socks)

(b) Pr(2 red socks)

(c) Pr(2 white socks)

(d) Pr(a pair of socks of matching colour)

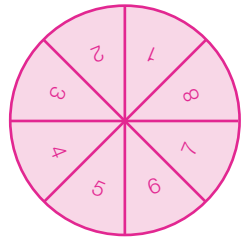
(e) Pr(two odd socks)

(f) Pr(one black sock)

(g) Pr(at least one black sock)

(h) Pr(no white socks)

10 A spinner numbered 1 to 8 is spun twice. Find the probability that:



(a) a 5 is obtained on the first spin

(b) a 5, then a 7, is spun

(c) a 5 is obtained on the first spin and a number less than 4 is obtained on the second

(d) an even number is obtained on the first spin and an odd number is obtained on the second.

11 If the probability that person A will be alive in 20 years is 0.7 and the probability that person B will be alive in 20 years is 0.5, find:

(a) the probability that they will both be alive in 20 years

(b) the probability that only one is alive in 20 years.

12 Natalie and her team play in a weekly squash competition. The probability that her team wins in any given week is 0.7. The outcome is not affected by any previous wins or losses. Find the probability that:

(a) Natalie's team does not win in any given week

(b) her team wins 2 weeks running

(c) her team loses 3 weeks running

(d) her team loses 4 weeks running.

13 A bag contains 6 white marbles, 5 black marbles and 4 green marbles. In each draw, a marble is drawn from the bag and not replaced. In three draws, find:

(a) Pr(WBG in that order)

(b) Pr(WBG in any order).

Reasoning

14 Two normal unbiased six-sided dice are rolled. These events are defined:

A = rolling a 6 with the first die

B = rolling a 5 with the second die

C = rolling a total of 8

D = rolling a total greater than 10

(a) Write the sample space for this experiment.

(b) Find the probability for each event.

(c) How many pairs of events are possible when order is not important? List them.

(d) How can you check which pairs are independent?

(e) Which, if any, of these pairs of events are independent?

15 You know that two events, A and B, are independent and that $\Pr(A) = 0.3$ and $\Pr(A \text{ and } B) = 0.15$.

(a) Find $\Pr(B)$.

(b) Now find $\Pr(A \text{ or } B)$. Explain how you obtained this value.

16 A secondary school has 900 students, of whom 475 are female and 870 study mathematics. There are 470 female mathematics students. If a student is picked at random, show that the events 'picking a female student' and 'picking a maths student' are not independent.

17 For two events, A and B, we are given $\Pr(A) = 0.35$ and $\Pr(B) = 0.45$. Find $\Pr(A \text{ and } B)$ if:

(a) A and B are mutually exclusive

(b) A and B are independent.

Open-ended

18 You have an unbiased ten-sided die, numbered 1 to 10. Write at least five questions related to independent events that deal with this die. Include the answers for your questions.

19 **(a)** Give two examples of pairs of dependent events.

(b) Give two examples of pairs of independent events.

20 A punter placed a $5000 bet on his local team to win $200 000 if the local side, who had only won one game in the last two seasons, finished on top of the ladder at the end of the season. After laying the bet the punter donated $100 000 to the local club to pay for three new star players. At the end of the season the bookmaker refused to pay because the punter had influenced the probability of the team finishing on top. What do you think?

Outside the Square — Problem solving

Winning the lottery

In a particular raffle you purchase a ticket that has a unique six-digit number from 000001 to 999999 printed on it. The winning number is drawn at random and the ticket with that number printed on it wins the first prize. In this case the prize is $100 000. Any ticket that has the last five of the winning digits in order in the last five positions in their number wins second prize ($20 000). Any ticket with the last four digits in order in the last four positions wins third prize ($10 000), and any ticket with the last three digits in order in the last three positions wins fourth prize ($2000). Each ticket can claim only one prize.

1 What is the probability that you win:

(a) $100 000? **(b)** $20 000?

(c) $10 000? **(d)** $2000?

2 What is the total payout on the lottery, assuming every prize is claimed?

3 How much do the tickets have to sell for so that the organisers make a profit?

Strategy options

- Break problem into manageable parts.
- Have I seen a similar problem?

Investigation

Biometrics and security

Equipment required: a calculator

How secure are we? In today's society we are becoming increasingly aware that we need to protect ourselves, our money, our possessions, even our ideas from theft or danger. Keyed locks, while still useful at times, are impractical for many modern applications and too easily opened by the skilful burglar. We need security for diverse reasons, at airports, on computers, at ATMs, on credit cards, doors, safes and passports.

The Big Question

Does fingerprinting provide more security than PIN numbers or combination locks?

Engage

1 Combination locks are a cheap and more secure alternative to keyed locks. Students use them on their lockers and they are built into the fasteners of many briefcases and suitcases. They usually need the selection of 3 digits.

(a) Assuming a 3-digit combination lock is used, what is the probability of correctly guessing:

(i) the first digit?

(ii) the second digit?

(iii) the third digit?

(iv) the all 3 digits?

Cards such as credit and debit cards are issued with a 4-digit PIN. Mobile phones can also be secured with a 4-digit PIN.

(b) What is the probability of correctly guessing a 4-digit PIN?

Personal computers and computer networks can be protected by using a password. A standard password must conform to certain rules.

Rule 1 The password must contain at least 7 characters.

Rule 2 It must contain three out of the following four character groups:

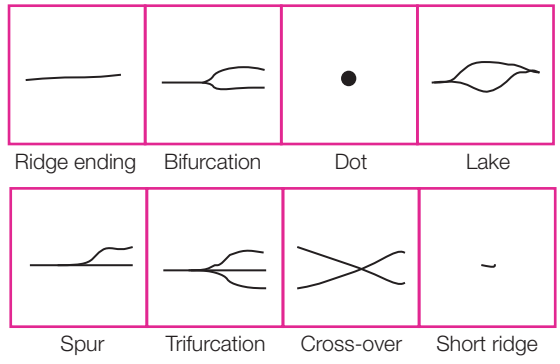
- English upper case (A–Z). There are 26 of these.
- English lower case (a–z). There are 26 of these.
- Numerics (0–9). There are 10 of these.
- Non-alphabetic (!,@,#,…). There are 32 of these on a standard keyboard.

(c) Assuming that a 7-digit password is chosen consisting of 1 upper case letter, 4 different lower case letters, a numeric and a non-alphabetic selected in that order, calculate the probability that this password would be guessed correctly.

Explore

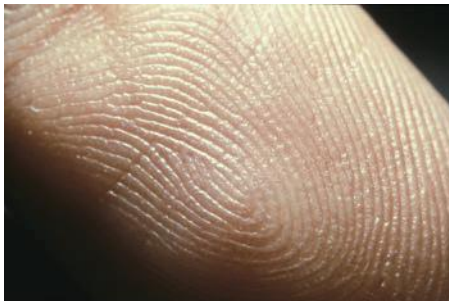
People forget their passwords, and PINs can be stolen and copied. To ensure security today, there is increasing use of biometrics to safeguard our security. Biometrics is the science of verifying a person's identity by measuring personal characteristics. Fingerscan units are the most widely used and cost-effective biometric devices available. This is an automated way of using fingerprints as a method of biometric identification and verification.

Fingerprint experts use a number of 'ridge characteristics' to describe fingerprints. Some of these are shown below.



In over 140 years of fingerprint comparisons worldwide, no two fingerprints have ever been found to be alike, not even those of identical twins. Fingerprint experts

compare the pattern of ridges and furrows on the fingertips (ridge characteristics), as well as the types of minutiae points (ridge characteristics that occur when a ridge splits into two, or ends such as arch, tented arch, left loop, right loop, double loop, whorl, …) of a specimen print with a database of prints on file.



A simple method of fingerprint verification is to calculate the probability that 2 people will have the same ridge characteristics in the same location.

2 If the probability of having one identical ridge characteristic is assumed to be equal to $\frac{1}{10}$, find the probability that 2 people will have:

(a) 2 identical ridge characteristics in the same location

(b) 10 identical characteristics in the same location

(c) N identical characteristics in the same location.

3 Using types of minutiae points of a fingerprint to obtain probabilities to identify individuals was first suggested by Galton in 1892.

$$\Pr(\text{fingerprint configuration}) = \frac{1}{16} \times \frac{1}{256} \times \left(\frac{1}{2}\right)^{24}$$

(a) Evaluate this probability writing your answer in scientific notation.

In 1930, Pearson argued that this probability was more accurately found using the following calculation.

$$\Pr(\text{fingerprint configuration}) = \frac{1}{16} \times \frac{1}{256} \times \left(\frac{1}{36}\right)^{24}$$

(b) Calculate this probability writing your answer in scientific notation.

Stoney et al in 1985 came up with a quite different and even more accurate formula.

Pr(fingerprint configuration) =

$\frac{N}{5} \times 0.6 \times (0.5 \times 10^{-3})^N$, where N is the number of minutiae on an average fingerprint.

(c) Evaluate this probability when $N = 12$ writing your answer in scientific notation.

(d) Evaluate this probability if $N = 36$ writing your answer in scientific notation.

Strategy options

- Look for a pattern.
- Make a table.

Explain

4 **(a)** Using Galton's fingerprint probability, would a 7-character password as described in **1(c)** be more secure than a fingerprint verification? Justify your answer.

(b) Would your answer be the same if Pearson's fingerprint probability was used instead? Justify your answer.

Elaborate

5 How many particular characteristics using the formula in **2 (a)** would need to be compared to be more secure than a computer password as described in **1(c)**?

Evaluate

6 **(a)** Why is fingerprinting being used more often for verification (e.g. access control, time and attendance at work, alarm activation/deactivation, safety deposits, club membership privileges, border control, parent entry to preschool, access to self serve facilities).

(b) If fingerprinting is more secure than PINs or Passwords, why aren't fingerscans used more extensively for verification?

(c) Do you think that fingerprint technology will ever replace PINs and passwords? Justify your answer.

Extend

Some other methods of biometric authentication are:

- Face recognition
- Retina scan
- Signature
- Hand geometry biometric
- Iris scan
- Voice analysis

7 Investigate one of the above methods. See if you can find data to support the argument that the probability that no two people would give the same biometric data is very low.

Chapter review 11

D.I.Y. Summary

Key Words

Addition Law for Mutually Exclusive Events	independent events	sample space
complementary events	long-run proportion	tree diagram
conditional statement	mutually exclusive	two-way table
dependent events	probability	universal set
event	probability tree diagram	

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 Every element under consideration in a particular question is called the _____.

2 If two events, A and B, are _____, then $\Pr(A) \times \Pr(B) = \Pr(A \text{ and } B)$.

3 The event 'having brown hair' and 'not having brown hair' are an example of _____s.

4 When we put the probabilities on each of the branches of a tree diagram we then have a _____.

5 Events that cannot occur at the same time are said to be _____.

6 A _____ or a _____ can be used to help find the sample space.

7 Spinning a spinner that has been equally divided into four pieces labelled 1, 2, 3 and 4 then calculating $\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) = 1$ is an example of the _____.

8 Words and phrases such as 'given', 'knowing that' and 'if … then' are indications we are dealing with a _____.

Fluency

1 A spinner numbered 1 to 5 is spun. Find: Ex. 11.1

(a) Pr(spinning a 2)
(b) Pr(not spinning a 2)
(c) Pr(spinning a number greater than 2)
(d) Pr(spinning an even number)
(e) Pr(spinning an odd number)
(f) Pr(spinning a number less than 5).

2 A box contains 15 blocks. Seven are red, five are white and the rest are blue. A block is chosen at random from the box. Find the probability that the block chosen is: Ex. 11.1

(a) blue
(b) not red
(c) red or blue.

3 A fair six-sided die is rolled. What is the probability of obtaining an even number or a 5? Ex. 11.1

4 A spinner numbered 1 to 4 is spun twice. Ex. 11.1

(a) Represent all the possible outcomes using a two-way table.

(b) Find:

(i) Pr(spinning a pair of 3s)
(ii) Pr(spinning only one 3)
(iii) Pr(not spinning any 3s)
(iv) Pr(the sum of the two numbers obtained is 5).

5 A 50-cent coin is tossed in the air twice. Find the probability of obtaining two tails. **Ex. 11.1**

6 This two-way table shows the number of elements in each of three sets. **Ex. 11.2**

	A and *B*	*A* and not *B*	not *A* and *B*	not *A* and not *B*	
C	1	4	3	2	**10**
not *C*	2	1	2	3	**8**
	3	**5**	**5**	**5**	**18**

Write down the number of elements in:

(a) the universal set

(b) A

(c) A or B

(d) B and C

(e) not B

(f) not A and not B and not C.

7 A cutlery rack contains 6 knives, 5 forks and 2 spoons. **Ex. 11.3**

(a) The probability that a piece of cutlery drawn at random will be a knife or a spoon is:

A $\frac{2}{13}$ **B** $\frac{2}{5}$ **C** $\frac{6}{13}$ **D** $\frac{8}{13}$

(b) The probability that a piece of cutlery drawn at random will not be a knife is:

A $\frac{6}{13}$ **B** $\frac{7}{13}$ **C** $\frac{6}{7}$ **D** $\frac{7}{6}$

8 Two cards are drawn, without replacement, from a set of 20 cards numbered 1 to 20. **Ex. 11.4**

Let P = 'a prime number is drawn' and N = 'a prime number is not drawn'

(a) Draw a tree diagram to show this information.

(b) Find Pr(both cards show a prime number).

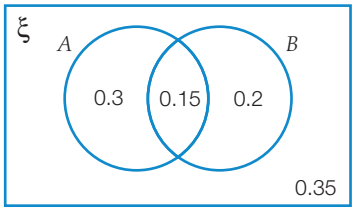
(c) Find Pr(no cards show a prime number).

(d) Find Pr(at least one card shows a prime number).

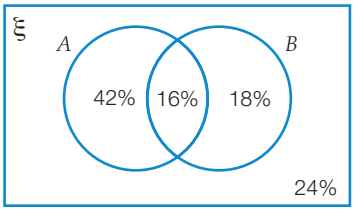
9 Use the information in the Venn diagram to find the following probabilities. Where necessary, give your answers correct to four decimal places. **Ex. 11.5**

(i) Pr(A given B) **(ii)** Pr(B given A)

(a)



(b)



10 John and Petula are in the same class at school and both have trouble getting to school on time. John is late 2 days out of 9, whereas Petula is late 3 days out of 11. What is the probability that they will both be late on any particular day? **Ex. 11.6**

Understanding

11 A fair seven-sided spinner, numbered 1 to 7, is spun. Find the probability of:

(a) obtaining a 5

(b) obtaining an even number

(c) obtaining a number less than 4

(d) not obtaining a 6

(e) obtaining a multiple of 3

(f) not obtaining a prime number.

12 In clay pigeon shooting, a clay block in the shape of a pigeon is used as a target for shooting. Jackie estimates her chance of hitting the moving clay target with one shot to be 0.75.

(a) What is the probability of her not hitting the target with one shot?

(b) Draw a probability tree diagram to represent the possible outcomes of shooting at three clay targets.

(c) Find the probability that Jackie hits:

(i) all three clay targets **(ii)** only one clay target **(iii)** at least one clay target.

13 In a survey of 60 people, it was found that 40 people liked lemonade, whereas 30 people liked cola. Two people said they disliked both.

(a) Present this information in a two-way table.

(b) If one person is chosen at random, find the probability that this person:

(i) likes lemonade **(ii)** likes lemonade only

(iii) likes cola **(iv)** likes cola only

(v) likes both lemonade and cola **(vi)** does not like lemonade

(vii) likes lemonade or cola **(viii)** likes neither cola nor lemonade.

14 These probabilities apply to eye colour for students at a particular school. The colours are all mutually exclusive.

$\Pr(\text{green}) = \frac{3}{16}$ $\Pr(\text{blue}) = \frac{1}{4}$ $\Pr(\text{hazel}) = \frac{3}{8}$ $\Pr(\text{brown}) = \frac{3}{16}$

A student is chosen at random. What is the probability that he or she has:

(a) blue or green eyes **(b)** neither blue nor brown eyes

(c) blue, hazel or brown eyes **(d)** neither hazel nor blue eyes?

15 Nerida and Ngaio play each other at badminton every week. The probability that Nerida wins any particular game is 0.65. Find, rounded to three decimal places where necessary, the probability that:

(a) Ngaio wins in any particular week

(b) Nerida wins for 3 weeks in a row

(c) Ngaio wins for 4 weeks in a row

(d) over a 6-week period the winning sequence is Nerida, Ngaio, Nerida, Ngaio, Nerida, Ngaio.

16 You toss three coins: a 10-cent, a 20-cent and a 50-cent. You note the results in that order.

(a) Find the probability of obtaining:

(i) three heads **(ii)** two heads

(iii) one head **(iv)** no heads.

(b) What do the answers in part **(a)** add up to? Is this an expected result? Explain your reasoning.

(c) Without doing any further calculations, explain how you can find the probability of each number of tails.

(d) Find the probability of getting:

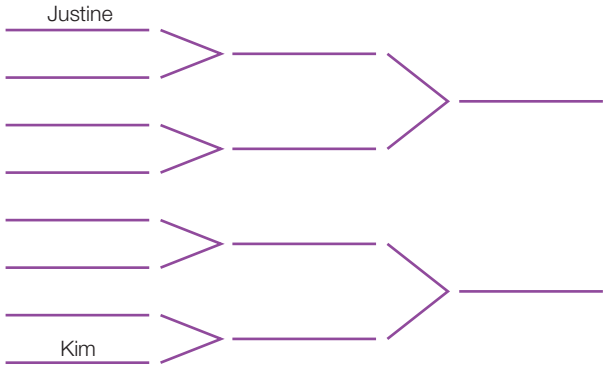
(i) HHT **(ii)** HTH **(iii)** THH.

(e) Write a statement about what you have found in part **(d)**.

Reasoning

17 Justine and Kim are entered for a tennis tournament that begins with eight players. The draw profile looks like the following (because they are so good Justine and Kim are never scheduled to play until the final).

Justine and Kim are very confident players and they think they have a probability of $\frac{2}{3}$ of beating any other player except for each other. If they play each other, Kim has a $\frac{3}{5}$ chance of winning. Find the following probabilities.



(a) The two girls play each other in the final.

(b) Kim wins the tournament (not necessarily against Justine).

(c) Justine wins the tournament (not necessarily against Kim).

18 A weather forecast predicting the chance of rain in the next 24 hours in some capital cities is recorded below.

City	Brisbane	Sydney	Canberra	Melbourne	Hobart	Adelaide	Perth	Darwin
Chance of rain	0.15	0.4	0.3	0.1	0.2	0.4	0.4	0.1

Assuming the events to be independent, find:

(a) Pr(rain in Melbourne and Brisbane)

(b) Pr(rain in Sydney and Perth)

(c) Pr(rain in Adelaide, Hobart and Perth)

(d) Pr(rain in Sydney and Canberra).

Long-term statistics are kept and found that the probability of rain in both Sydney and Canberra on the same day is actually 0.2.

(e) Explain the difference between your answer to **(d)** and the long-term probability.

Surds and logarithms

10A

12

Can maths help us find alien life?

Name: Tanya Hill
Job: Astronomer
Qualifications: Bachelor of Science, PhD in Astronomy

Astronomers such as Dr Tanya Hill, from the Planetarium at Scienceworks, are using the power of mathematical modelling to test their theories about where planets capable of supporting life could be formed.

Mathematical modelling involves developing a formula or equation that fits a set of observed data. This formula can then be tested by computer simulation to see whether it produces results that are the same as those observed.

At school, Tanya "loved the certainty of maths—knowing that you had a clear-cut answer you could work towards. I was drawn to physics and later on astronomy, because it gave you a reason for doing the maths."

In her day-to-day work at the Planetarium, Tanya gives public presentations on different areas of astronomy, including recent developments and discoveries. Part of this involves converting the units used to measure astronomical distance (parsecs and light years) into ideas that help people understand the enormous scale of the universe.

Why learn this?

Irrational numbers expressed in decimal form have an infinite number of decimal places. Some irrational numbers, however, can be written in exact form as surds. For example, the surd $\sqrt{7}$ has the decimal representation 2.645 751 311 ... For some purposes we could round this to 2.6, but mathematicians often require exact measurements and surds allow us this level of precision. Many measurement scales use logarithms. The Richter scale, which is used to measure earthquake intensity, is an example.

After completing this chapter you will be able to:

- add, subtract and multiply simple surd expressions
- carry out simple calculations and simplifications involving surds
- rationalise the denominator of a surd expression
- apply the index laws to numerical and algebraic expressions, evaluating and simplifying where necessary
- understand the relationship between exponential and logarithmic expressions
- apply the logarithmic laws to numerical and algebraic expressions.

Recall 12

Prepare for this chapter by attempting the following questions. If you have difficulty with a question, go to Pearson Places and download the Recall Worksheet from Pearson Reader.

Worksheet R12.1

1 Write in mathematical symbols:

(a) the square root of 3

(b) the cube root of 5

(c) the number that when squared equals 7

(d) the number that when cubed equals 6

(e) the number that when raised to the fourth power equals 10.

Worksheet R12.2

2 Evaluate each of the following. Do not use a calculator.

(a) $\sqrt{4}$ **(b)** $\sqrt{36}$ **(c)** $\sqrt{9}$ **(d)** $\sqrt{1}$ **(e)** $\sqrt[3]{8}$

(f) $\sqrt[3]{64}$ **(g)** $\sqrt[3]{-27}$ **(h)** $\sqrt[3]{1}$ **(i)** $\sqrt[4]{16}$ **(j)** $\sqrt[5]{-1}$

Worksheet R12.3

3 Use a calculator to evaluate each of the following, correct to two decimal places where necessary.

(a) $\sqrt{11}$ **(b)** $-\sqrt{7}$ **(c)** $\sqrt[3]{15}$ **(d)** 34^2 **(e)** 3^9

Worksheet R12.4

4 Expand each of the following.

(a) $3(x + 4)$ **(b)** $-4(2y - 3)$

Worksheet R12.5

5 Simplify each of the following.

(a) $m^3 \times m^2$ **(b)** $3k^2p^4 \times 4kp^5$ **(c)** $\dfrac{x^3 \times x^6}{x^2}$

(d) $3x^0$ **(e)** $(m^3n)^4$ **(f)** $(x^{-3}y^2)^{-2}$

Worksheet R12.6

6 Expand and simplify each of the following.

(a) $(x - y)(x + y)$ **(b)** $(c - 9)(c + 9)$ **(c)** $(2p + 3)(2p - 3)$

Key Words

conjugate surds	irrational numbers	rationalising the denominator
Distributive Law	logarithm	real numbers
entire surd	rational numbers	surds

Rational and irrational numbers

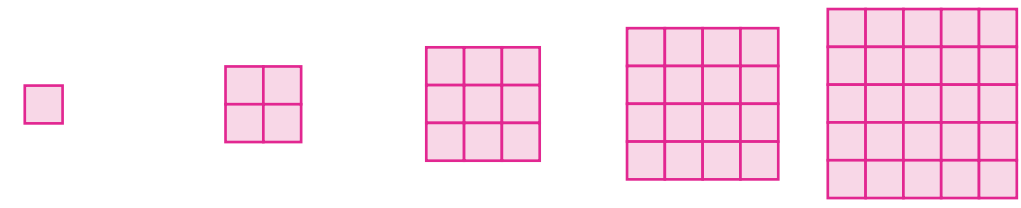
12.1

Real numbers

Real numbers are numbers that can be shown on a continuous number line. They consist of all **rational numbers** combined with all **irrational numbers**.

- A rational number can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Rational numbers include whole numbers, fractions and repeating decimals.
- An irrational number cannot be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Irrational numbers include all surds and other special numbers such as π. When represented in decimal form an irrational number has an infinite number of non-repeating decimal places.

Surds are numbers that can only be expressed in exact form using the radical or root sign ($\sqrt{\ }$). However, just because a number is written using a root sign does not, of itself, make the number a surd. For instance, $\sqrt{\text{perfect square}}$ will not be a surd. The first five perfect squares are shown below.

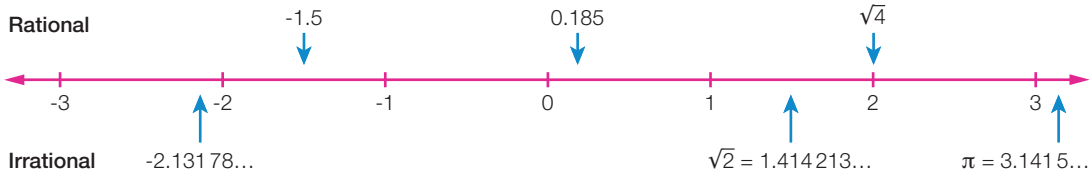


$\sqrt{1} = 1$ $\quad$ $\sqrt{4} = 2$ $\quad$ $\sqrt{9} = 3$ $\quad$ $\sqrt{16} = 4$ $\quad$ $\sqrt{25} = 5$

The square root of a whole number that is not a perfect square is a surd. So, $\sqrt{5}$ is a surd because 5 is not a perfect square. We cannot write $\sqrt{5}$ in fraction form, and if we enter it on our calculator we get 2.236 067 ..., a decimal with an infinite number of non-repeating decimal places.

In mathematics we do not just deal with square roots. We also need to be able to tell if a cube root ($\sqrt[3]{\ }$), or a fourth root ($\sqrt[4]{\ }$), or a fifth root ($\sqrt[5]{\ }$) and so on, is a surd. In each case, if the value inside the radical sign is a perfect cube, or perfect fourth power, or perfect fifth power, as the case may be, then the expression is not a surd. As examples, $\sqrt[3]{8}$ is not a surd because 8 can be expressed as 2^3, so is a perfect cube, but $\sqrt[3]{9}$ is a surd because 9 is not a perfect cube.

A calculator may be needed to identify large perfect squares, perfect cubes and higher order powers.



Real numbers

Rational numbers

- Integers

  $8 = \frac{8}{1}$, $\sqrt{4} = 2 = \frac{2}{1}$, $-15 = -\frac{15}{1}$

- Terminating decimals

  $0.8 = \frac{4}{5}$, $1.23 = \frac{123}{100}$, $0.0006 = \frac{6}{10\,000}$

- Recurring decimals

  $0.3\ldots = \frac{1}{3}$, $0.636\,363\ldots = \frac{7}{11}$

Irrational numbers

- Surds

  $-\sqrt{3}$, $\sqrt[3]{7}$, $4\sqrt{19}$

- Other infinite non-recurring decimals

  $\pi = 3.141\,592\ldots$

Worked Example 1 WE1

For each of the following numbers, state whether it is rational or irrational.

(a) $\sqrt{12}$ **(b)** $\sqrt{361}$

Thinking	Working
(a) Identify whether the number under the radical sign can be written as a perfect square. If it can, the expression is rational. If not, it is irrational.	(a) 12 is not a perfect square, so $\sqrt{12}$ is irrational.
(b) Identify whether the number under the radical sign can be written as a perfect square. If it can, the expression is rational. If not, it is irrational.	(b) $361 = 19^2$, so $\sqrt{361}$ is rational.

Worked Example 2 WE2

Evaluate the following. Do not use a calculator.

(a) $-7\sqrt{81}$ **(b)** $3\sqrt{100} - 2\sqrt{16} + \sqrt{9}$

Thinking	Working
(a) 1 Identify and simplify the $\sqrt{\text{perfect square}}$ ($\sqrt{81} = 9$).	(a) $-7\sqrt{81}$
2 Simplify.	$= -7 \times 9$ $= -63$
(b) 1 Identify and simplify the $\sqrt{\text{perfect squares}}$.	(b) $3\sqrt{100} - 2\sqrt{16} + \sqrt{9}$ $= 3 \times 10 - 2 \times 4 + 3$
2 Simplify.	$= 30 - 8 + 3$ $= 25$

In some situations, such as finding the side length of a right-angled triangle, an approximate value for a surd can be useful. We can use a calculator to obtain approximate values for surds.

Worked Example 3 WE3

Use a calculator to evaluate each of the following correct to three decimal places.

(a) $\sqrt{3} + \sqrt{5}$ **(b)** $-7 \times \sqrt[3]{9}$

Thinking	Working
(a) Use your calculator.	(a) $\sqrt{3} + \sqrt{5} \approx 3.968\,118\,785$ Correct to three decimal places this is 3.968.
(b) Use your calculator.	(b) $-7 \times \sqrt[3]{9} \approx -14.560\,586\,76$ Correct to three decimal places this is -14.561.

Rational numbers can be expressed as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

Irrational numbers cannot be expressed as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$.

12.1 Rational and irrational numbers

Navigator

Q1 (a)–(f), Q2 (a)–(c), (g)–(i), Q3, Q4, Q5, Q6, Q7 (a) (d) (g), Q8 (a) (c), Q9, Q12	Q1 Q2, Q3, Q4, Q5, Q7, Q8 (b) (c), Q9, Q10 (b) (c), Q13	Q1 (g)–(l), Q2 (d)–(f), (j)–(l), Q3, Q5, Q6, Q7, Q8 (c) (d), Q10, Q11, Q13

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Equipment required: calculator for Questions **1**, **3**, **4** and **7**

Fluency

1 For each of the following numbers, state whether it is rational or irrational. WE1

(a) $\sqrt{5}$ **(b)** $\sqrt{36}$ **(c)** $-\frac{2}{7}$ **(d)** $\sqrt{10}$

(e) 4.6 **(f)** $\sqrt[3]{4}$ **(g)** $-\sqrt[3]{2}$ **(h)** $\sqrt[4]{1}$

(i) 2π **(j)** $\sqrt{20}$ **(k)** $-\sqrt{3}$ **(l)** 8.621

2 Evaluate the following. Do not use a calculator. WE2

(a) $\sqrt{64}$ **(b)** $-\sqrt{25}$ **(c)** $-\sqrt{36}$

(d) $3\sqrt{9}$ **(e)** $7\sqrt{36}$ **(f)** $-4\sqrt{49}$

(g) $\sqrt{25 + 144}$ **(h)** $\sqrt{9} + \sqrt{16}$ **(i)** $7\sqrt{49} + 4\sqrt{25}$

(j) $3\sqrt{16} + 2\sqrt{81}$ **(k)** $5\sqrt{36} - 7\sqrt{4}$ **(l)** $9\sqrt{100} - 8\sqrt{16} + 5\sqrt{9}$

3 Use a calculator to find the approximate value of each of the following surds, correct to three decimal places where necessary. WE3

(a) $\sqrt{3} + \sqrt{2}$ **(b)** $\sqrt{8} - \sqrt{5}$ **(c)** $\sqrt{7} - \sqrt{13}$

(d) $\sqrt{35}$ **(e)** $-\sqrt{3}$ **(f)** $\sqrt[3]{17}$

(g) $\sqrt[3]{10}$ **(h)** $\sqrt{6}$ **(i)** $-\sqrt[3]{72}$

4 (a) Which one of the following is a surd?

A $\sqrt{9}$ B $-\sqrt{1}$ C $\sqrt{8}$ D $\sqrt[3]{8}$

(b) Which one of the following is *not* a surd?

A $\sqrt[3]{7}$ B $\sqrt{11}$ C $\sqrt[3]{16}$ D $\sqrt{25}$

(c) Which one of the following is a rational number?

A $\sqrt{3}$ B $\sqrt[3]{2}$ C $\sqrt[3]{4}$ D $\sqrt{4}$

(d) Which one of the following is *not* a rational number?

A $\frac{1}{3}$ B 1.45 C 2.717 171 … D 3.721 452 …

5 (a) $4\sqrt{3}$ is the same as:

A $4+\sqrt{3}$ B $4-\sqrt{3}$ C $4\times\sqrt{3}$ D $\sqrt[4]{3}$

(b) $8\sqrt{16}$ is the same as:

A 8×4 B $8\div 4$ C $8+4$ D 8×16

6 Do not use a calculator for this question.

(a) $\sqrt{7}$ is between $\sqrt{4}$ and $\sqrt{9}$, so $\sqrt{7}$ is between:

A 1 and 2 B 2 and 3 C 3 and 4 D 4 and 5

(b) $\sqrt{31}$ is between:

A 2 and 3 B 3 and 4 C 4 and 5 D 5 and 6

(c) $\sqrt[3]{11}$ is between:

A 1 and 2 B 2 and 3 C 3 and 4 D 4 and 5

Understanding

7 Use a calculator to find the approximate value of each of the following, correct to three decimal places.

(a) $\sqrt{3+2}$ (b) $\sqrt{8-5}$ (c) $2\sqrt{6}+\sqrt{10}$

(d) $3\sqrt{2}+\sqrt{8}$ (e) $4\sqrt{5}-3\sqrt{6}$ (f) $6\sqrt{3}-6\sqrt{7}$

(g) $\sqrt[3]{-14}$ (h) $-\sqrt{43}$ (i) $2\sqrt{3}$

(j) $6\sqrt{5}$ (k) $-3\sqrt{11}$ (l) $5\times 6\sqrt{5}$

Reasoning

8 Find the exact side length of a square with area:

(a) 25 m^2 (b) 36 cm^2 (c) 46 mm^2 (d) 17 m^2

9 The area of a square vegetable garden is 60 m^2.

(a) What is the exact length of each side of the vegetable garden?

(b) If a fence of chicken wire is to be constructed around the vegetable garden, what length (to the nearest metre) of chicken wire is needed?

(c) Why do you think the measurement was to the nearest metre?

10 Find the exact side length of a cube with volume:

(a) $8\ m^3$ **(b)** $27\ m^3$ **(c)** $35\ cm^3$ **(d)** $15\ cm^3$

11 Two identical dice have a combined volume of $14\ cm^3$.

(a) Find the exact side length of each die.

(b) What is the side length of each die to the nearest mm?

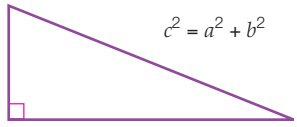
Open-ended

12 **(a)** Write three irrational numbers greater than zero.

(b) Write three irrational numbers less than zero.

(c) Draw a number line and mark the approximate places of these numbers.

13 Pythagoras' theorem is used to determine the side lengths of right-angled triangles. Explain how to cut a piece of string exactly $\sqrt{2}$ m long using a pencil, a metre ruler and a piece of string.



Outside the Square — Problem solving

Surd shells

Equipment required: graph paper

A spiral can be created using straight lines to form a spiral made with ever increasing right-angled triangles. A very special spiral is created when starting with an isosceles right-angled triangle with equal sides of length 1 cm. Each subsequent triangle has a height of 1 cm and a base that is the length of the hypotenuse of the previous triangle.

(a) What is the perimeter of the tenth right-angled triangle?

(b) Calculate the exact perimeter of the entire spiral.

(c) Does the entire spiral have a rational or an irrational perimeter?

(d) Draw the spiral on graph paper. Measure the length of the perimeter of the entire spiral to check your answer to **(b)**.

(e) The first triangle has a rational area of $\frac{1}{2}\ cm^2$.

(i) Which is the next triangle to have a rational area?

(ii) Find this area and compare it to the area of the first triangle.

(iii) Find the next triangle to have a rational area.

(iv) Find this area and compare it to the area of the first triangle.

Strategy options

- Look for a pattern.
- Make a table.

12.2 Multiplying and dividing surds

As we have seen, surds can be cube roots ($\sqrt[3]{5}$), fourth roots ($\sqrt[4]{9}$), as well as square roots ($\sqrt{3}$). In this section, we will concentrate on square roots only. Remember, the square roots of perfect squares, such as 16, 36 and 81 are rational numbers, not surds.

Multiplying surds

When dealing with perfect squares we can show:

$$\begin{aligned} &\sqrt{4} \times \sqrt{9} \\ &= 2 \times 3 \\ &= 6 \\ &= \sqrt{36} \\ &= \sqrt{4 \times 9} \end{aligned}$$

The result we are interested in here is $\sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9}$.

We can then do the same with surds.

So, $\sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6}$.

We can check this with our calculator

$\sqrt{3} \times \sqrt{2} = 2.449\,489\,743\ldots$ and $\sqrt{6} = 2.449\,489\,743\ldots$

In general, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$, where a and b are positive numbers.

Surds should always be written in their simplest form. Surds can often be simplified by expressing the number under the root sign as a product of factors, one of which is a perfect square. Use the largest perfect square that is a factor.

$\sqrt{2} \times \sqrt{6} = \sqrt{12}$

The factors of 12 are: 1, 2, 3, 4, 6, 12.

4 is a perfect square, therefore $\sqrt{12}$ can be simplified further.

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \times 3} \\ &= \sqrt{4} \times \sqrt{3} \\ &= 2 \times \sqrt{3} \\ &= 2\sqrt{3} \end{aligned}$$

$2\sqrt{3}$ is considered to be simpler than $\sqrt{12}$ as it contains a smaller surd.

Worked Example 4 WE4

Simplify:

(a) $\sqrt{2} \times \sqrt{12}$

(b) $4\sqrt{3} \times 5\sqrt{7}$

Thinking	Working
(a) 1 Use $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b} = \sqrt{ab}$ to combine the surds.	(a) $\sqrt{2} \times \sqrt{12}$ $= \sqrt{2 \times 12}$ $= \sqrt{24}$
2 Simplify if the number under the square root sign has a factor that is a perfect square. (Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24)	$= \sqrt{4 \times 6}$ $= \sqrt{4} \times \sqrt{6}$ $= 2\sqrt{6}$
(b) 1 Multiply whole numbers together and surds together.	(b) $4\sqrt{3} \times 5\sqrt{7}$ $= 4 \times 5 \times \sqrt{3} \times \sqrt{7}$
2 Simplify. (Factors of 21: 1, 3, 7, 21. 21 does not have any factors that are perfect squares, therefore it cannot be simplified further.)	$= 20\sqrt{21}$

Dividing surds

When dealing with perfect squares we can show:

$$\sqrt{36} \div \sqrt{9} = \frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$$

and

$$\sqrt{\frac{36}{9}} = \sqrt{4} = 2$$

So, $\dfrac{\sqrt{36}}{\sqrt{9}} = \sqrt{\dfrac{36}{9}}$

We operate in the same way with surds.

$$\frac{\sqrt{14}}{\sqrt{2}} = \sqrt{\frac{14}{2}} = \sqrt{7}$$

We can check this using our calculator.

$\dfrac{\sqrt{14}}{\sqrt{2}} = 2.645\,751\,311\ldots$ and $\sqrt{7} = 2.645\,751\,311\ldots$

In general, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$, where a and b are positive numbers and $b \neq 0$.

Worked Example 5

WE5

Simplify:

(a) $\sqrt{21} \div \sqrt{7}$

(b) $\dfrac{12\sqrt{40}}{3\sqrt{5}}$

Thinking	Working
(a) Use $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	(a) $\sqrt{21} \div \sqrt{7}$ $= \frac{\sqrt{21}}{\sqrt{7}}$ $= \sqrt{\frac{21}{7}}$ $= \sqrt{3}$
(b) 1 Divide the whole numbers and divide the surds. **2** Simplify. (Factors of 8: 1, 2, 4, 8)	(b) $\frac{12\sqrt{40}}{3\sqrt{5}}$ $= \frac{12}{3} \times \sqrt{\frac{40}{5}}$ $= 4\sqrt{8}$ $= 4 \times \sqrt{4 \times 2}$ $= 4 \times \sqrt{4} \times \sqrt{2}$ $= 4 \times 2 \times \sqrt{2}$ $= 8\sqrt{2}$

Sometimes it is necessary to convert a surd into an **entire surd**, which is where the entire surd is under the square root sign.

$2\sqrt{6}$
$= 2 \times \sqrt{6}$
$= \sqrt{4} \times \sqrt{6}$
$= \sqrt{24}$

$\sqrt{24}$ is an entire surd

12.2 Multiplying and dividing surds

Navigator

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Q1 (a) (d) (g) (j) (m) (p), Q2 (a) (d) (g) (j) (m) (p) (s) (v), Q3 (a)–(f), Q4 (a)–(f), Q5 (a)–(c), Q6 (a) (d) (g) (j), Q7 (a)–(f), Q8 (a) (b), Q10, Q12, Q13	Q1 (b) (e) (h) (k) (n) (q), Q2 (b) (e) (h) (k) (n) (q) (t) (w), Q3 (a)–(h), Q4 (d)–(l), Q5 (b) (c) (d), Q6 (b) (e) (h) (k), Q7 every 2nd, Q8 (a) (c), Q10, Q13	Q1 (c) (f) (i) (l) (o) (r), Q2 (c) (f) (i) (l) (o) (r) (u) (x), Q3 (g)–(l), Q4 (g)–(l), Q5 (c) (d) (e), Q6 (c) (f) (i) (l), Q7 (g)–(l), Q8 (c) (d), Q9, Q11, Q13

Fluency

WE4

1 Simplify each of the following.

(a) $\sqrt{3} \times \sqrt{2}$ **(b)** $\sqrt{11} \times \sqrt{5}$ **(c)** $\sqrt{7} \times \sqrt{10}$

(d) $\sqrt{7} \times \sqrt{7}$ **(e)** $\sqrt{4} \times \sqrt{4}$ **(f)** $\sqrt{16} \times \sqrt{4}$

(g) $\sqrt{36} \times \sqrt{2}$ **(h)** $\sqrt{25} \times \sqrt{7}$ **(i)** $\sqrt{2} \times \sqrt{9}$

(j) $2\sqrt{3} \times 5\sqrt{2}$ **(k)** $\sqrt{11} \times 4\sqrt{3}$ **(l)** $5 \times 8\sqrt{2}$

(m) $7\sqrt{5} \times \sqrt{5}$ **(n)** $\sqrt{6} \times 4\sqrt{6}$ **(o)** $5\sqrt{2} \times 4\sqrt{2}$

(p) $3\sqrt{7} \times 2\sqrt{7}$ **(q)** $5\sqrt{3} \times 3\sqrt{2}$ **(r)** $8\sqrt{6} \times 3\sqrt{12}$

WE5

2 Simplify each of the following.

(a) $\sqrt{\frac{27}{3}}$ (b) $\sqrt{\frac{32}{2}}$ (c) $\frac{\sqrt{125}}{\sqrt{5}}$

(d) $\frac{\sqrt{60}}{\sqrt{15}}$ (e) $\sqrt{72} \div \sqrt{2}$ (f) $\sqrt{45} \div \sqrt{5}$

(g) $\sqrt{7} \div \sqrt{7}$ (h) $\sqrt{\frac{64}{16}}$ (i) $\sqrt{25} \div \sqrt{9}$

(j) $\sqrt{\frac{100}{49}}$ (k) $\frac{\sqrt{4}}{\sqrt{81}}$ (l) $\sqrt{\frac{36}{121}}$

(m) $\sqrt{4} \div \sqrt{64}$ (n) $\sqrt{3} \div \sqrt{27}$ (o) $\frac{2\sqrt{36}}{3}$

(p) $\frac{5\sqrt{49}}{7}$ (q) $\frac{5\sqrt{9}}{3\sqrt{25}}$ (r) $\frac{6\sqrt{9}}{9\sqrt{36}}$

(s) $\frac{18\sqrt{75}}{3\sqrt{3}}$ (t) $5\sqrt{20} \div \sqrt{5}$ (u) $18\sqrt{48} \div 3\sqrt{8}$

(v) $\frac{5\sqrt{30}}{25\sqrt{6}}$ (w) $6\sqrt{14} \div 15\sqrt{2}$ (x) $12\sqrt{30} \div 18\sqrt{10}$

3 Simplify each of the following.

(a) $\sqrt{27}$ (b) $\sqrt{108}$ (c) $\sqrt{162}$ (d) $\sqrt{96}$

(e) $\sqrt{99}$ (f) $\sqrt{500}$ (g) $\sqrt{486}$ (h) $\sqrt{405}$

(i) $5\sqrt{8}$ (j) $-4\sqrt{24}$ (k) $-5\sqrt{63}$ (l) $4\sqrt{243}$

4 Express each of the following as an entire surd.

(a) $2\sqrt{3}$ (b) $3\sqrt{7}$ (c) $6\sqrt{2}$ (d) $3\sqrt{3}$

(e) $-10\sqrt{6}$ (f) $-2\sqrt{10}$ (g) $-8\sqrt{3}$ (h) $5\sqrt{7}$

(i) $8\sqrt{2}$ (j) $7\sqrt{5}$ (k) $-6\sqrt{3}$ (l) $-5\sqrt{11}$

Hint
Leave the negative sign outside the square root sign.

5 (a) $\sqrt{32}$ written in simplest form is:

A $2\sqrt{4}$ B $2\sqrt{8}$ C $4\sqrt{2}$ D $4\sqrt{8}$

(b) $\sqrt{3} \times \sqrt{7}$ simplifies to:

A $\sqrt{10}$ B $\sqrt{21}$ C $\sqrt{37}$ D $3\sqrt{7}$

(c) $\frac{\sqrt{49}}{\sqrt{4}}$ simplifies to:

A $\frac{2}{7}$ B $\frac{7}{4}$ C $\frac{7}{2}$ D 14

(d) $3\sqrt{5} \times 2\sqrt{6}$ simplifies to:

A $5\sqrt{11}$ B $6\sqrt{11}$ C $5\sqrt{30}$ D $6\sqrt{30}$

(e) $\frac{15\sqrt{6}}{3\sqrt{2}}$ simplifies to:

A $\sqrt{15}$ B $5\sqrt{3}$ C $5\sqrt{4}$ D $12\sqrt{3}$

Understanding

6 Simplify each of the following.

(a) $3\sqrt{6} \times 5\sqrt{3}$ (b) $4\sqrt{2} \times 2\sqrt{14}$ (c) $-2\sqrt{6} \times 7\sqrt{10}$

(d) $-4\sqrt{8} \times 2\sqrt{10}$ (e) $-5\sqrt{3} \times \sqrt{15}$ (f) $10\sqrt{2} \times 3\sqrt{8}$

(g) $2\sqrt{12} \times 4\sqrt{3}$ (h) $2\sqrt{18} \times 3\sqrt{12}$ (i) $4\sqrt{27} \times 7\sqrt{28}$

(j) $-2\sqrt{20} \times -5\sqrt{75}$ (k) $-3\sqrt{72} \times -4\sqrt{63}$ (l) $\sqrt{54} \times \sqrt{162}$

7 Simplify each of the following.

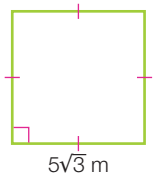
(a) $\sqrt{\frac{24}{3}}$ (b) $\sqrt{\frac{54}{2}}$ (c) $3\sqrt{\frac{90}{2}}$ (d) $\frac{-6\sqrt{120}}{\sqrt{5}}$

(e) $\frac{\sqrt{27}}{\sqrt{4}}$ (f) $\frac{\sqrt{32}}{\sqrt{25}}$ (g) $\frac{9\sqrt{8}}{\sqrt{9}}$ (h) $\frac{15\sqrt{28}}{2\sqrt{25}}$

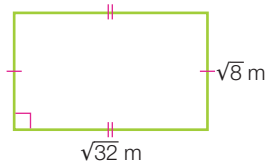
(i) $\sqrt{\frac{125}{36}}$ (j) $-3\sqrt{\frac{44}{81}}$ (k) $-2\sqrt{\frac{27}{100}}$ (l) $\frac{4}{5}\sqrt{\frac{147}{64}}$

8 Calculate the area of the following shapes, writing the answer as a surd in its simplest form.

(a)



(b)



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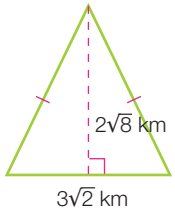
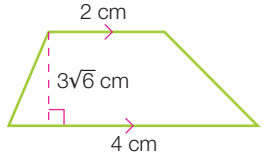


(d)



Area formulas:

Square: $A = l^2$

Rectangle: $A = lw$

Triangle: $A = \frac{bh}{2}$

Trapezium: $A = \frac{(a+b)h}{2}$

9 Given $\sqrt{a} \times \sqrt{b} = 3\sqrt{c}$, what are possible values of b and c if $a = 2$?

Reasoning

10 **(a)** Chris calculated the side of a triangle and found it was $\sqrt{41}$ mm. His friends' calculations gave lengths of $3\sqrt{5}$ mm and $2\sqrt{10}$ mm. If the actual length was 6 mm, which measurement was closest to this?

(b) If the area of a circular swimming pool is 45π m^2, find its *exact* circumference.

Hint

Exact means write in surd form if necessary.

11 Adrianna has $\sqrt{50}$ m of tape that is to go across the two diagonals of rectangular windows that have side lengths 40 cm and 35 cm.

(a) What is the length of the diagonal?

(b) How many windows can she tape? (She may have some tape left over.)

Open-ended

12 Find at least three surd calculations that simplify to $2\sqrt{6}$.

13 **(a)** Andrew multiplied two surds and got the answer 5. What could the two surds have been?

(b) Fiona divided two surds and got the answer 4. What could the two surds have been?

Adding and subtracting surds

12.3

When dealing with algebraic expressions, $3x + 9x$ can be simplified to $12x$. This is because both terms have the same pronumeral. The expression $8g - 3y$ cannot be simplified as the pronumerals are not like terms.

The same rule is true when working with surds.

Let's consider $3\sqrt{2} + 5\sqrt{2}$.

Notice that the surd $\sqrt{2}$ is contained in each term. This expression can be factorised by taking out the common factor of $\sqrt{2}$.

$$\begin{aligned} & 3\sqrt{2} + 5\sqrt{2} \\ = \; & 3 \times \sqrt{2} + 5 \times \sqrt{2} \\ = \; & \sqrt{2}(3 + 5) && \text{(take out common factor)} \\ = \; & \sqrt{2} \times 8 && \text{(simplify)} \\ = \; & 8\sqrt{2} \end{aligned}$$

So, $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$

Numbers with the same surd are called *like* surds. Only terms that contain like surds can be added or subtracted.

What about $4\sqrt{3} + 7\sqrt{2}$? We cannot simplify this expression as the terms do not contain the same surd. (We can evaluate the expression approximately using a calculator.)

Worked Example 6

WE 6

Simplify:

(a) $10\sqrt{3} - 4\sqrt{3}$ **(b)** $3\sqrt{5} + 2\sqrt{6} + \sqrt{5}$

Thinking	Working
(a) Add or subtract terms containing like surds.	(a) $10\sqrt{3} - 4\sqrt{3}$ $= 6\sqrt{3}$
(b) Add or subtract terms containing like surds. Different surds cannot be simplified.	(b) $3\sqrt{5} + 2\sqrt{6} + \sqrt{5}$ $= 4\sqrt{5} + 2\sqrt{6}$

To see whether there are like terms, it may be necessary to express each term in simplest form to reveal the like surds.

Worked Example 7 WE7

Simplify:

(a) $\sqrt{18}+\sqrt{8}$ **(b)** $2\sqrt{12}-\sqrt{20}-\sqrt{27}+2\sqrt{45}$

Thinking	Working
(a) 1 Simplify each term.	(a) $\sqrt{18}+\sqrt{8}$ $=\sqrt{9\times 2}+\sqrt{4\times 2}$ $=3\sqrt{2}+2\sqrt{2}$
2 Add like terms.	$=5\sqrt{2}$
(b) 1 Simplify each term.	(b) $2\sqrt{12}-\sqrt{20}-\sqrt{27}+2\sqrt{45}$ $=2\sqrt{4\times 3}-\sqrt{4\times 5}-\sqrt{9\times 3}+2\sqrt{9\times 5}$ $=2\times 2\sqrt{3}-2\sqrt{5}-3\sqrt{3}+2\times 3\sqrt{5}$ $=4\sqrt{3}-2\sqrt{5}-3\sqrt{3}+6\sqrt{5}$
2 Group like terms together.	$=4\sqrt{3}-3\sqrt{3}-2\sqrt{5}+6\sqrt{5}$
3 Add or subtract like terms. ($\sqrt{3}+4\sqrt{5}$ cannot be simplified further.)	$=\sqrt{3}+4\sqrt{5}$

12.3 Adding and subtracting surds

Navigator

Answers page 814

Q1 (a)–(c), (g)–(i), Q2 (a)–(c), (g)–(i), Q3, Q5 (a) (b), Q7, Q9, Q10, Q11, Q12	Q1 Column 1, Q2 Column 1, Q4, Q5 (b) (c), Q6 (a), Q7, Q9, Q11, Q12	Q1 Column 2, Q2 Column 2, Q4, Q5 (c)–(d), Q6, Q8, Q9, Q11, Q12

Fluency

WE6

1 Simplify each of the following.

(a) $3\sqrt{2}+7\sqrt{2}$ **(b)** $8\sqrt{5}-5\sqrt{5}$

(c) $11\sqrt{3}-6\sqrt{3}$ **(d)** $\sqrt{7}+7\sqrt{7}$

(e) $5\sqrt{3}-9\sqrt{3}$ **(f)** $12\sqrt{11}-35\sqrt{11}$

(g) $5\sqrt{2}-3\sqrt{2}+9\sqrt{2}$ **(h)** $8\sqrt{3}-9\sqrt{3}+\sqrt{3}$

(i) $7\sqrt{5}-13\sqrt{5}+4\sqrt{5}-\sqrt{5}$ **(j)** $10\sqrt{6}+3\sqrt{3}-5\sqrt{6}-8\sqrt{3}$

(k) $5\sqrt{2}-7\sqrt{5}+2\sqrt{5}+2\sqrt{2}$ **(l)** $7\sqrt{3}+6\sqrt{11}-8\sqrt{3}-10\sqrt{11}$

WE7

2 Simplify each of the following.

(a) $\sqrt{2}+\sqrt{18}$
(b) $\sqrt{28}-\sqrt{7}$
(c) $\sqrt{125}-2\sqrt{5}$
(d) $7\sqrt{18}+3\sqrt{50}$
(e) $5\sqrt{63}-7\sqrt{28}$
(f) $4\sqrt{90}-4\sqrt{160}$
(g) $\sqrt{40}+\sqrt{90}+\sqrt{160}$
(h) $2\sqrt{32}+7\sqrt{2}+4\sqrt{18}$
(i) $\sqrt{150}+5\sqrt{54}-6\sqrt{24}$
(j) $9\sqrt{48}-\sqrt{27}+2\sqrt{192}$
(k) $6\sqrt{80}-3\sqrt{180}-4\sqrt{405}$
(l) $\sqrt{8}+\sqrt{162}+\sqrt{18}+\sqrt{50}$
(m) $\sqrt{28}-\sqrt{112}+\sqrt{567}-\sqrt{175}$
(n) $2\sqrt{300}-6\sqrt{27}-3\sqrt{108}+4\sqrt{48}$

3 (a) $7\sqrt{3}-2\sqrt{3}$ simplifies to:

A 5 B $5\sqrt{3}$ C $5\sqrt{6}$ D $9\sqrt{3}$

(b) $5\sqrt{5}-3\sqrt{5}$ simplifies to:

A $2\sqrt{2}$ B $2\sqrt{5}$ C $2\sqrt{7}$ D $8\sqrt{12}$

(c) $5\sqrt{3}+7\sqrt{3}+3\sqrt{2}$ simplifies to:

A $15\sqrt{3}$ B $12\sqrt{6}+3\sqrt{2}$ C $15\sqrt{8}$ D $12\sqrt{3}+3\sqrt{2}$

4 (a) $\sqrt{3}+\sqrt{12}$ simplifies to:

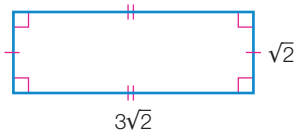
A $\sqrt{15}$ B $2\sqrt{3}$ C $3\sqrt{3}$ D $5\sqrt{3}$

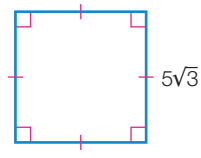
(b) $5\sqrt{2}-2\sqrt{8}$ simplifies to:

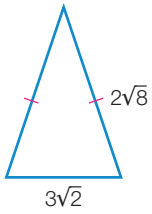
A $3\sqrt{-6}$ B $-3\sqrt{2}$ C 1 D $\sqrt{2}$

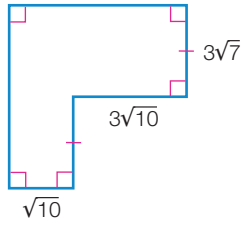
Understanding

5 Calculate the perimeter of the following shapes, writing your answer as a surd in its simplest form.

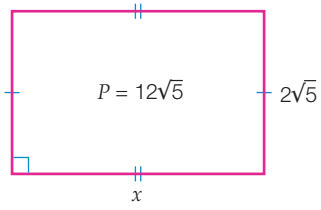
(a)


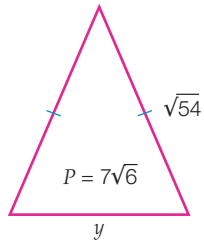
(b)


(c)


(d)


6 Find the value of the variable in each of the following shapes.

(a)


(b)


7 Find the exact area of a circle if its radius is $\sqrt{5}$ cm.

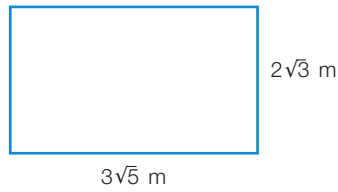
8 Find the exact area of a circular coin with diameter $\sqrt{3}$ cm.

Reasoning

9 (a) Write down the perimeter of the rectangle shown.

(b) What is the exact area of this rectangle?

(c) If this rectangle represents the available floor space of a bedroom, can the floor be covered by a piece of carpet with area 22 m^2?



Open-ended

10 Make up at least three questions, using addition and/or subtraction, where $3\sqrt{5}$ is the answer.

11 A rectangle has an area of $\sqrt{96}$ cm^2. What might its perimeter be?

12 Ruby's answers to a surds spot quiz are shown.

(i) Find the perimeter of a triangle with side lengths $2\sqrt{3}$, $2\sqrt{3}$, $3\sqrt{3}$.

$$\begin{aligned} \text{Perimeter} &= 2\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} \\ &= 7\sqrt{9} \\ &= 7 \times 3 \\ \therefore P &= 21 \end{aligned}$$

(ii) The perimeter of a rectangle is $8\sqrt{6}$. If the length of one pair of sides is $\sqrt{6}$, find the length of the other pair.

$$\begin{aligned} \text{Length of other pair of sides} &= \frac{8\sqrt{6} - 2\sqrt{6}}{2} \\ &= \frac{6}{2} \\ \therefore l &= 3 \end{aligned}$$

(a) How would you explain to Ruby the mistake she made in each question?

(b) Write the correct answer for each question.

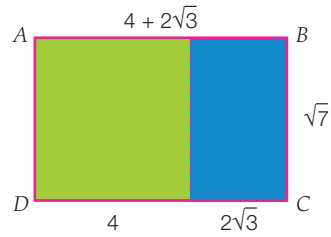
(c) Draw a diagram that could accompany each question.

Surds and the Distributive Law

12.4

Look at the rectangle on the right.

The side lengths AB and BC include surd terms. The two terms are not like, so cannot be simplified to one term. To find the area of this rectangle, the length and width have to be multiplied:



$A = l \times w$

$= \sqrt{7} \times (4 + 2\sqrt{3})$

$= \sqrt{7}(4 + 2\sqrt{3})$

To simplify surd expressions involving brackets we can use the **Distributive Law**:

$a(b + c) = ab + ac$ or $(a + b)(c + d) = ac + ad + bc + bd$

To calculate the exact area of the above rectangle, the term outside the bracket is multiplied by each term inside the bracket.

$A = \sqrt{7}(4 + 2\sqrt{3})$

$= \sqrt{7} \times 4 + \sqrt{7} \times 2\sqrt{3}$

$= 4\sqrt{7} + 2\sqrt{21}$

Worked Example 8

WE8

Expand and simplify:

(a) $2\sqrt{3}(5 - \sqrt{3})$ **(b)** $(2\sqrt{2} + 1)(\sqrt{6} - 4\sqrt{3})$

Thinking	Working
(a) 1 Multiply each term inside the bracket by the term outside the bracket. $2\sqrt{3}(5 - \sqrt{3})$	(a) $2\sqrt{3}(5 - \sqrt{3})$ $= 10\sqrt{3} - 2\sqrt{9}$
2 Simplify any surds if necessary. $(\sqrt{9} = 3)$	$= 10\sqrt{3} - 2 \times 3$
3 Simplify further.	$= 10\sqrt{3} - 6$

(b) 1 Multiply each term inside the second bracket by each term inside the first bracket.

$(2\sqrt{2}+1)(\sqrt{6}-4\sqrt{3})$

(b) $(2\sqrt{2}+1)(\sqrt{6}-4\sqrt{3})$
$= 2\sqrt{12}-8\sqrt{6}+\sqrt{6}-4\sqrt{3}$

2 Simplify any surds if necessary.
$(\sqrt{12} = \sqrt{4\times 3} = 2\sqrt{3})$

$= 2\times 2\sqrt{3}-8\sqrt{6}+\sqrt{6}-4\sqrt{3}$
$= 4\sqrt{3}-8\sqrt{6}+\sqrt{6}-4\sqrt{3}$

3 Add or subtract like terms.
($\sqrt{6}$ cannot be simplified further.)

$= -7\sqrt{6}$

12.4 Surds and the Distributive Law

Navigator

Answers page 815

Q1 Column 1, Q2, Q3, Q4, Q5, Q6, Q8, Q10	Q1 Column 2, Q2, Q3, Q5, Q6, Q8, Q9, Q10	Q1 Column 2, Q2, Q3, Q4, Q6, Q7, Q9, Q11

Equipment required: calculator for Questions **5**, **6**, **7**, **8** and **9**

Fluency

WE8

1 Expand and simplify each of the following.

(a) $3(\sqrt{7}+2)$
(b) $-4(\sqrt{11}+\sqrt{3})$
(c) $9(2\sqrt{5}-\sqrt{2})$
(d) $-\sqrt{7}(5\sqrt{2}-3\sqrt{5})$
(e) $4\sqrt{2}(2\sqrt{5}+6\sqrt{3})$
(f) $3\sqrt{5}(\sqrt{2}-7\sqrt{3}+3\sqrt{7})$
(g) $\sqrt{3}(\sqrt{3}+2)$
(h) $5\sqrt{7}(3\sqrt{2}-2\sqrt{7})$
(i) $-3\sqrt{5}(2-8\sqrt{3}+\sqrt{5})$
(j) $3\sqrt{2}(\sqrt{14}-\sqrt{8})$
(k) $4\sqrt{3}(\sqrt{3}-\sqrt{6}+5)$
(l) $-4\sqrt{5}(1-2\sqrt{5}+\sqrt{15})$
(m) $(\sqrt{5}+4)(\sqrt{2}+3)$
(n) $(\sqrt{3}-\sqrt{5})(\sqrt{3}+\sqrt{7})$
(o) $(\sqrt{2}+\sqrt{3})^2$
(p) $(4\sqrt{5}-2\sqrt{2})^2$
(q) $(2\sqrt{7}+\sqrt{3})(2\sqrt{7}-\sqrt{3})$
(r) $(3\sqrt{5}-7\sqrt{2})(3\sqrt{5}+7\sqrt{2})$

2 (a) $5(2\sqrt{5}-\sqrt{2})$ is the same as:

A $10\sqrt{5}-\sqrt{10}$ **B** $10\sqrt{5}-5\sqrt{2}$ **C** $10\sqrt{5}-\sqrt{2}$ **D** $10\sqrt{3}$

(b) $4\sqrt{6}(\sqrt{2}-\sqrt{3})$ is the same as:

A $4\sqrt{8}-4\sqrt{18}$ **B** $8\sqrt{6}-12\sqrt{2}$ **C** $4\sqrt{8}-12\sqrt{2}$ **D** $8\sqrt{3}-12\sqrt{2}$

(c) $(2-\sqrt{5})(\sqrt{5}-1)$ expands to:

A $3\sqrt{5}+7$ **B** $3\sqrt{5}$ **C** $3\sqrt{5}-7$ **D** $3\sqrt{5}-3$

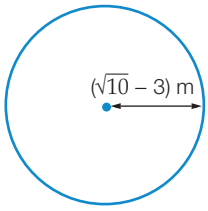
(d) $(\sqrt{7}+\sqrt{2})^2$ expands and simplifies to:

A $9-2\sqrt{14}$ **B** 9 **C** $9+2\sqrt{14}$ **D** $11\sqrt{14}$

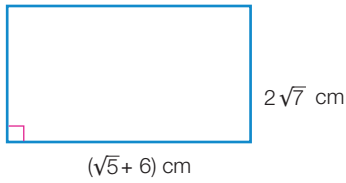
Understanding

3 Find the exact area of each shape.

(a)

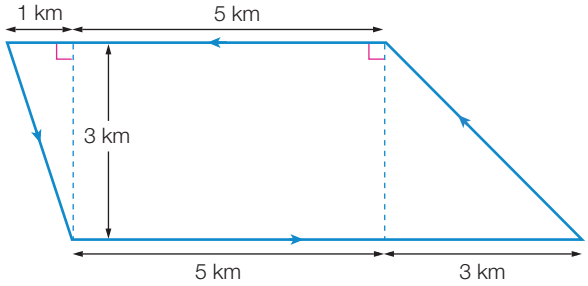


(b)



4 Find the exact area of a compact disc if it has a radius of $(4 + \sqrt{5})$ cm.

5 On a walk around the city, Luke followed the path shown in the diagram, finishing back where he started.



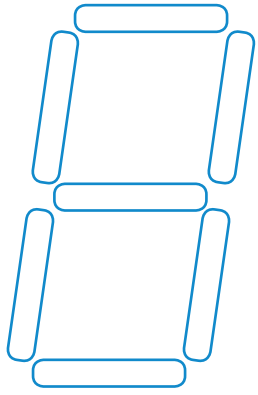
Hint
What shape can you see?

(a) Find the area enclosed by his path.

(b) Find the exact distance he walked.

(c) State the distance he walked, to the nearest metre.

6 On a scoreboard, tube lighting is set up so that any digit from 0 to 9 can be displayed as shown in the diagram at right.



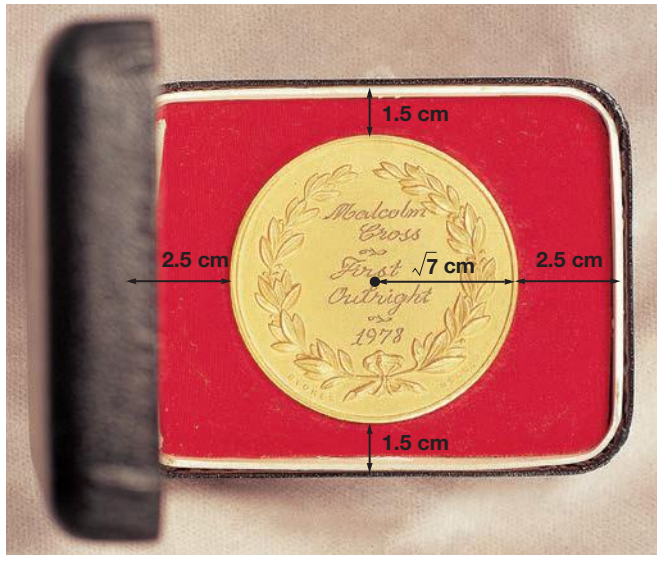
(a) If each individual section of tubing is $(3\sqrt{2} - 2)$ m long, what exact length of tubing would be lit to produce the digit:

(i) 0 **(ii)** 4

(iii) 5 **(iv)** 8?

(b) Use your calculator to express each length found correct to two decimal places.

7 A sports medal is to be placed into a presentation box as shown. Give exact answers for parts **(a)** to **(d)**.



(a) What area of the base of the box does the medal take up?

(b) What are the dimensions (length and width) of the base of the box?

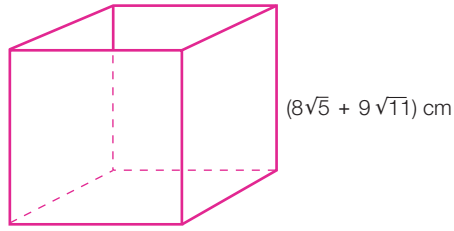
(c) What is the area of the base of the box?

(d) To stop the medal moving freely in the box, a thick layer of velvet will surround it. What area of velvet will be seen around the medal?

(e) Express your answer to part **(d)** correct to two decimal places.

Reasoning

8 Zev has a toy box in the shape of a cube (open at the top). He wants to cover its surface (both inside and out) with adhesive-backed plastic and needs to know how much he should buy.



(a) What is the length of a side of the toy box, correct to one decimal place?

(b) Use your answer to part **(a)** to find the area to be covered with plastic. Assume that the box has walls of negligible thickness.

(c) Find the area to be covered with plastic using the value $8\sqrt{5} + 9\sqrt{11}$ cm and keeping your answer as a surd expression.

(d) Use a calculator to express your answer to part **(c)** correct to one decimal place.

(e) Compare your answers to parts **(b)** and **(d)**. Which answer should Zev use when purchasing the plastic?

9 Square biscuits are to be cut out of a length of biscuit dough. Each biscuit has a line of chocolate icing running along the 6 cm diagonal of the square.

(a) Use Pythagoras' theorem to find the exact side length of each biscuit.

(b) What exact length of biscuit dough would be needed to cut out a row of 50 biscuits? (Keep your answer in surd form.)

(c) Use a calculator to express your answer to part **(b)** correct to one decimal place.

(d) Use a calculator to express the side length of each biscuit correct to one decimal place.

(e) Use your answer to part **(d)** to calculate the length of biscuit dough needed to cut out a row of 50 biscuits.

(f) Compare your answers to parts **(c)** and **(e)**. If the biscuits are cut out by a machine that can be calibrated to $\frac{1}{100}$th of a centimetre, which answer should be used?

Open-ended

10 The area of a triangle is $\sqrt{48}$ cm^2. Draw at least two diagrams with different dimensions that could be the triangle.

11 The perimeter of a rectangle is given by $2(m\sqrt{n} + p)$.

(a) What might be the dimensions of the rectangle if the perimeter is greater than 20 m? Draw a diagram, showing lengths $m\sqrt{n}$ and p.

(b) Write an equation for the rectangle's area, using side lengths $m\sqrt{n}$ and p.

(c) State values of m, n and p so that the area of the rectangle is less than 10 m^2.

Outside the Square

Problem solving

Connecting squares

The diagram below shows increasingly larger squares connecting to the smaller square along one edge.

(a) If the side lengths of the smallest squares are 1 cm, what is the area of the entire rectangle?

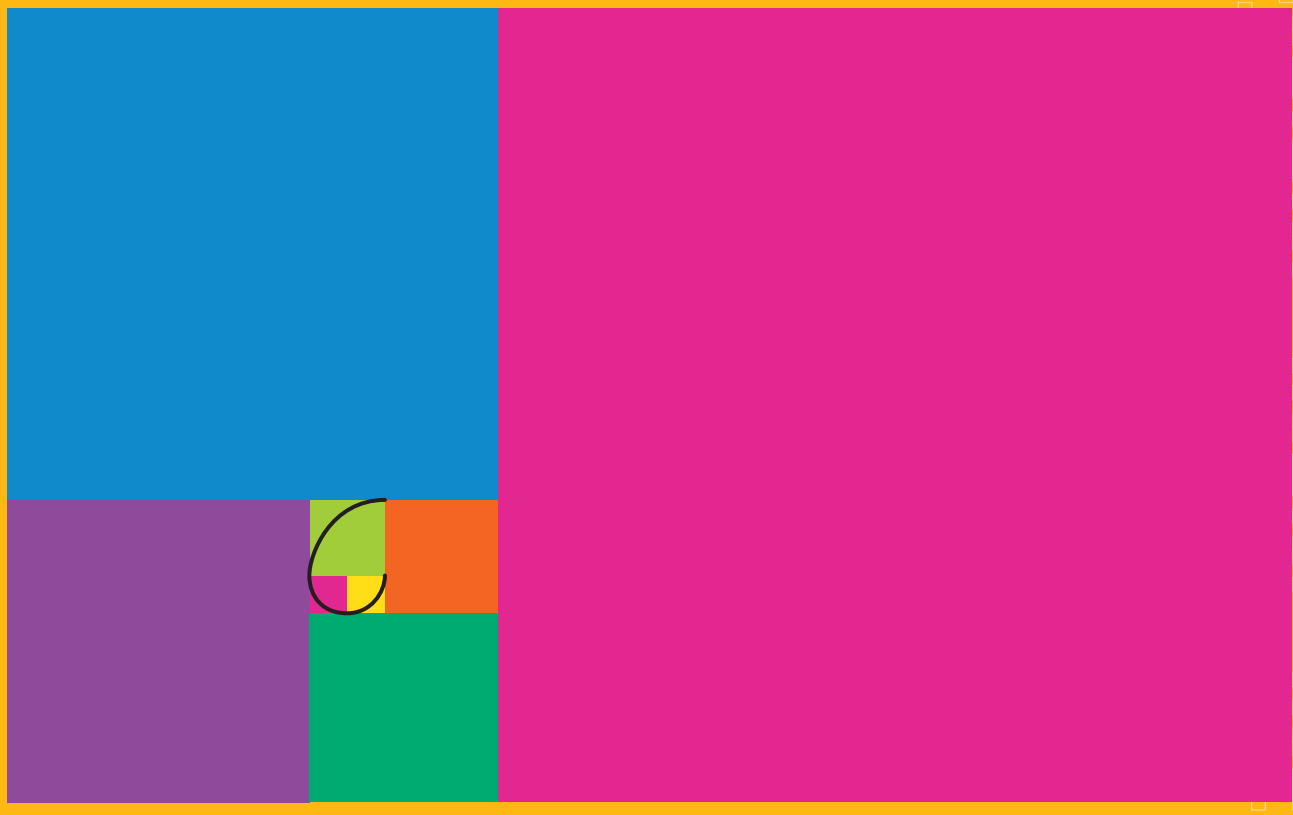
(b) This diagram is a spiral pattern that started with the yellow square. Complete the spiral pattern that has been started. If the pattern was continued, what would be the dimensions and area of the next square?

(c) Write down the side lengths of each square in ascending order. Have you seen this number pattern before? If so, what is its name?

(d) Explain the pattern and state the side lengths of the next five squares.

Strategy options

- Look for a pattern.
- Break problem into manageable parts.



12.5 Rationalising the denominator

Fractions are easier to work with if the denominator is a rational number. Consider the number $\frac{1}{\sqrt{3}}$. We can change the denominator of $\sqrt{3}$ to a rational number if we multiply it by itself. The numerator must also be multiplied by $\sqrt{3}$ to keep the fraction equivalent. $\left(\frac{\sqrt{3}}{\sqrt{3}} = 1\right)$

$$\begin{aligned}\frac{1}{\sqrt{3}} &= \frac{1}{\sqrt{3}} \times 1\\ &= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

So $\frac{1}{\sqrt{3}}$ can be written as $\frac{\sqrt{3}}{3}$, which has a rational denominator.

This process is called **rationalising the denominator**.

Rationalising the denominator of a fraction changes the denominator to a rational number, but does not change the value of the fraction.

Worked Example 9 WE9

Rationalise the denominator of:

(a) $\frac{3\sqrt{5}}{\sqrt{6}}$ **(b)** $\frac{2\sqrt{3}}{\sqrt{20}}$

Thinking	Working
(a) 1 Identify the surd value that will rationalise the denominator ($\sqrt{6}$) and multiply the numerator and denominator by that value.	(a) $\frac{3\sqrt{5}}{\sqrt{6}}$ $= \frac{3\sqrt{5}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$
2 Simplify the surd calculation.	$= \frac{3\sqrt{30}}{6}$
3 Cancel, if possible, to write your answer in the simplest form.	$= \frac{\sqrt{30}}{2}$

(b) 1	Simplify the denominator.	(b) $\frac{2\sqrt{3}}{\sqrt{20}}$
2	Identify the surd value that will rationalise the denominator $(\sqrt{5})$ and multiply the numerator and denominator by that value.	$= \frac{2\sqrt{3}}{2\sqrt{5}}$ $= \frac{2\sqrt{3}}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$
3	Simplify the surd calculation.	$= \frac{2\sqrt{15}}{2 \times 5}$
4	Cancel, if possible, to write your answer in the simplest form.	$= \frac{\sqrt{15}}{5}$

Using the conjugate to rationalise denominators

We can use the Distributive Law to show the following:

$$\begin{aligned}(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) &= \sqrt{a}\times\sqrt{a}-\sqrt{a}\times\sqrt{b}+\sqrt{b}\times\sqrt{a}-\sqrt{b}\times\sqrt{b}\\ &= \sqrt{a^2}-\sqrt{ab}+\sqrt{ab}-\sqrt{b^2}\\ &= a-b\end{aligned}$$

This simplifies to the Difference of Two Squares rule:

$$\begin{aligned}(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) &= (\sqrt{a})^2-(\sqrt{b})^2\\ &= a-b\end{aligned}$$

The two expressions $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are called **conjugate surds**, each being the conjugate of the other. Conjugate surds are useful because their product does not contain a surd term.

For example, the conjugate of $(3-\sqrt{2})$ is $(3+\sqrt{2})$. When these two numbers are multiplied, the result is a rational number:

$$\begin{aligned}&(3-\sqrt{2})(3+\sqrt{2})\\ &= 9+3\sqrt{2}-3\sqrt{2}-2\\ &= 9-2\\ &= 7\end{aligned}$$

or

$$\begin{aligned}&(3-\sqrt{2})(3+\sqrt{2})\\ &= 3^2-(\sqrt{2})^2\\ &= 9-2\\ &= 7\end{aligned}$$

$(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ are conjugate surds.

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$

Worked Example 10

WE10

Rationalise the denominator of each of the following.

(a) $\frac{3}{3-\sqrt{2}}$

(b) $\frac{\sqrt{2}}{\sqrt{11}+3}$

Thinking	Working
(a) 1 Multiply both the numerator and denominator by the conjugate of the denominator $(3+\sqrt{2})$.	(a) $\frac{3}{3-\sqrt{2}}$ $= \frac{3}{(3-\sqrt{2})} \times \frac{(3+\sqrt{2})}{(3+\sqrt{2})}$
2 Use the difference of two squares to simplify the denominator.	$= \frac{3(3+\sqrt{2})}{9-2}$
3 Expand the numerator and write the final answer in the simplest way.	$= \frac{9+3\sqrt{2}}{7}$
(b) 1 Multiply both the numerator and denominator by the conjugate of the denominator $(\sqrt{11}-3)$.	(b) $\frac{\sqrt{2}}{\sqrt{11}+3}$ $= \frac{\sqrt{2}}{(\sqrt{11}+3)} \times \frac{(\sqrt{11}-3)}{(\sqrt{11}-3)}$
2 Use the difference of two squares to simplify the denominator.	$= \frac{\sqrt{2}(\sqrt{11}-3)}{11-9}$
3 Expand the numerator and write the final answer in the simplest way.	$= \frac{\sqrt{22}-3\sqrt{2}}{2}$

The denominator can be rationalised in two ways:

- Multiply the numerator and denominator by the smallest surd that will give a rational denominator (if the denominator is a surd only).
- Multiply the numerator and denominator by the conjugate of the denominator (if the denominator has rational and irrational elements).

12.5 Rationalising the denominator

Navigator

Answers page 815

Q1 (a)–(d), Q2, Q3, Q4, Q5, Q6, Q9, Q12	Q1 (c)–(f), Q2, Q3, Q4, Q5, Q6, Q7, Q11, Q12	Q1 (e)–(h), Q2, Q3, Q4, Q5, Q6, Q7, Q8, Q10, Q11

Fluency

WE9

1 Express each of the following in simplest form with a rational denominator.

(a) $\frac{1}{\sqrt{7}}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{-12}{\sqrt{6}}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$

(e) $\frac{5\sqrt{11}}{\sqrt{5}}$ (f) $3\sqrt{\frac{5}{6}}$ (g) $-\frac{3}{4\sqrt{7}}$ (h) $\frac{6\sqrt{5}}{7\sqrt{3}}$

WE10

2 Express each of the following in simplest form with a rational denominator.

(a) $\dfrac{1}{3-\sqrt{7}}$ (b) $\dfrac{4}{2+\sqrt{3}}$ (c) $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{2}}$ (d) $\dfrac{2}{2\sqrt{7}+5}$

(e) $\dfrac{6+\sqrt{3}}{2\sqrt{3}-3}$ (f) $\dfrac{\sqrt{5}+1}{7-3\sqrt{5}}$ (g) $\dfrac{\sqrt{6}-1}{3\sqrt{2}+\sqrt{7}}$ (h) $\dfrac{\sqrt{2}-3}{2\sqrt{3}+\sqrt{6}}$

3 (a) $\dfrac{5}{\sqrt{2}}$ expressed with a rational denominator is:

A $\dfrac{\sqrt{10}}{2}$ B $\dfrac{2\sqrt{5}}{5}$ C $\dfrac{5\sqrt{2}}{\sqrt{2}}$ D $\dfrac{5\sqrt{2}}{2}$

(b) $\dfrac{4}{3\sqrt{3}}$ expressed with a rational denominator is:

A $\dfrac{4\sqrt{3}}{9}$ B $\dfrac{\sqrt{12}}{9}$ C $\dfrac{4\sqrt{3}}{3}$ D $\dfrac{4\sqrt{3}}{27}$

(c) $\dfrac{2\sqrt{2}}{3\sqrt{7}}$ expressed with a rational denominator is:

A $\dfrac{2\sqrt{6}}{21}$ B $\dfrac{2\sqrt{14}}{21}$ C $\dfrac{14\sqrt{2}}{3}$ D $\dfrac{2\sqrt{14}}{3}$

4 (a) The conjugate of $2\sqrt{3}-5$ is:

A $2\sqrt{3}-5$ B $2\sqrt{3}+5$ C $\sqrt{3}+5$ D $5\sqrt{3}+2$

(b) To rationalise the denominator of $\dfrac{4+\sqrt{7}}{2-3\sqrt{3}}$ it is necessary to multiply by:

A $\dfrac{4-\sqrt{7}}{4-\sqrt{7}}$ B $\dfrac{1}{2+3\sqrt{3}}$ C $2+3\sqrt{3}$ D $\dfrac{2+3\sqrt{3}}{2+3\sqrt{3}}$

Understanding

5 Simplify each of the following. (Rationalise each denominator first.)

(a) $\dfrac{1}{\sqrt{3}+1}+\dfrac{1}{3-\sqrt{5}}$ (b) $\dfrac{1}{\sqrt{7}+1}+\dfrac{1}{3-\sqrt{6}}$

(c) $\dfrac{1}{\sqrt{6}+\sqrt{3}}-\dfrac{1}{4-\sqrt{11}}$ (d) $\dfrac{1}{\sqrt{5}+\sqrt{3}}-\dfrac{1}{\sqrt{7}-2}$

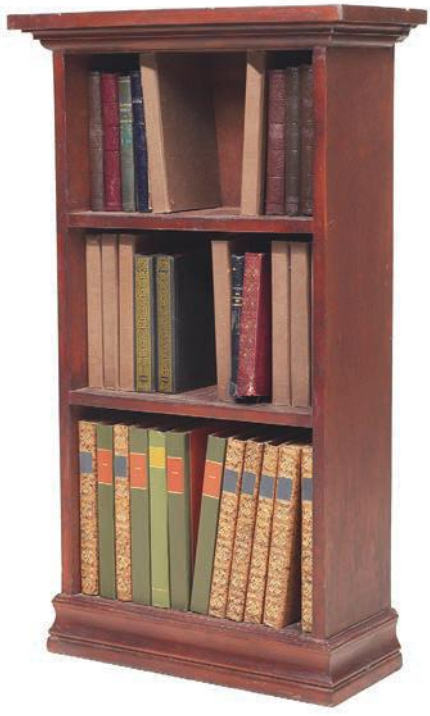
Reasoning

6 A piece of wood with length $5\sqrt{6}$ m is cut to make shelves for a bookcase with length $2\sqrt{2}$ m.

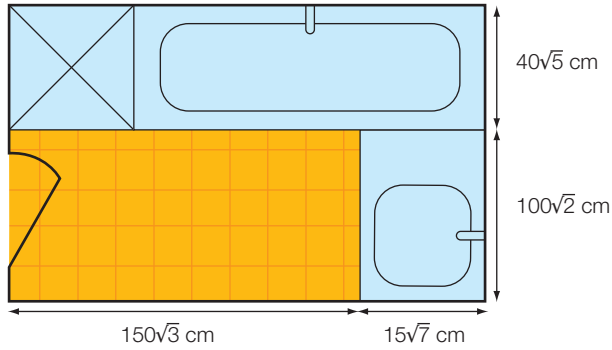
(a) Show $\dfrac{5\sqrt{6}}{2\sqrt{2}}$ as a fraction with a rational denominator.

(b) How many shelves can be cut from the wood?

(c) What is the exact length of wood left over?



7 The floor plan of a bathroom to be tiled is shown.



(a) Calculate the area to be tiled shaded orange.

(b) The square tiles chosen have side lengths of $10\sqrt{3}$ cm. How many tiles are required for the bathroom floor? State the answer as a simplified surd with a rational denominator.

(c) How many tiles would you order?

(d) Find the exact area of the entire bathroom in square centimetres.

8 If $\frac{3+2\sqrt{3}}{4-\sqrt{2}}a = \frac{4+\sqrt{2}}{5-6\sqrt{3}}$, find the value of a, expressing your answer with a rational denominator.

Open-ended

9 Find at least three fractions that are equivalent to $\frac{4\sqrt{2}}{\sqrt{3}}$, one of which must have a rational denominator.

10 For each of the following, find two fractions that the stated number is between, with a rational denominator formed from the original fraction.

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{3}{\sqrt{7}}$ (c) $\frac{2\sqrt{3}}{\sqrt{5}}$ (d) $\frac{2}{3\sqrt{2}}$

11 Find at least two expressions that $2+\sqrt{3}$ can be multiplied by to give a rational answer.

12 Sura is trying to rationalise the denominator of the fraction $\frac{\sqrt{2}-3}{2\sqrt{3}+\sqrt{6}}$. She is struggling to understand why the denominator can be rationalised by multiplying both the numerator and denominator by the denominator's conjugate $2\sqrt{3}-\sqrt{6}$. How could you explain this to Sura?

$$\frac{\sqrt{2}-3}{(2\sqrt{3}+\sqrt{6})} \times \frac{(2\sqrt{3}-\sqrt{6})}{(2\sqrt{3}-\sqrt{6})}$$

Investigation

Phi (Φ) and the golden rectangle

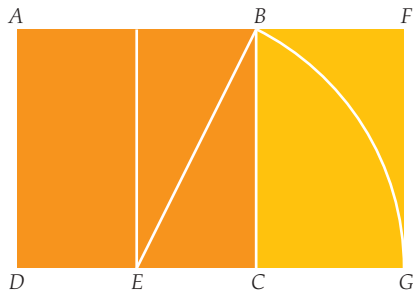
Equipment required: 1 brain, pair of compasses, ruler

The Big Question

What are phi, the golden ratio and the golden rectangle and how do they interrelate?

Engage

Square *ABCD* has side lengths of 2 units. A straight line joins point *B* with point *E*, which is halfway between *C* and *D*. The rectangle *AFGD* is formed using the radius of *BE*, as shown.



1 **(a)** What is the length *BE*, as a surd?

(b) What is the length *EG*, expressed as a surd?

(c) What is the length *DG*, as a surd expression?

(d) Find the ratio of length and width $\left(\frac{L}{W}\right)$ of rectangle *AFGD* as an expression using surds and as a decimal number rounded to nine decimal places.

The ratio calculated in **(d)** is known as the golden ratio or phi (Φ), and is an irrational number. A rectangle with this ratio is called the golden rectangle.

2 **(a)** What is the length *CG* as a surd?

(b) Repeat **1(d)** for rectnagle *BFGC*.

(c) Is your answer to **2(b)** the same as the answer to **1(d)**?

Explore

3 **(a)** Make a golden rectangle, with a ruler and pair of compasses, starting with a 10 cm square.

(b) What is the length of this rectangle? Write the length as a simplified surd. Record the length, width and ratio $\frac{L}{W}$ in a table.

(c) Cut the rectangle into a square and a rectangle.

Prove that the smaller rectangle is also a golden rectangle by folding the rectangle into a square and a rectangle. Record this rectangle's length, width and ratio $\frac{L}{W}$ into the table created in **(b)**.

(d) Continue to divide the rectangles into the square and the rectangle, writing the results in the table. What is the smallest golden rectangle you are able to make?

> **Strategy options**
> - Make a model.
> - Make a table.

Explain

4 **(a)** What did you discover about the ratio $\frac{L}{W}$ of the rectangles created?

(b) What is the exact value of phi (Φ)?

Elaborate

5 "A golden rectangle is …". How would you finish this sentence? Use the words phi and golden ratio within your sentence.

Evaluate

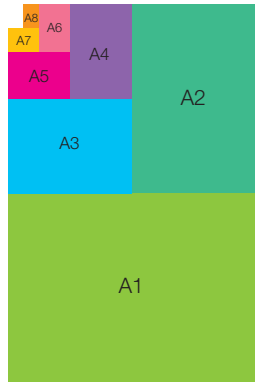
6 **(a)** Describe how to create a golden rectangle using a step-by-step method.

(b) Were you surprised by the findings? Explain.

(c) Did you have any difficulties making the golden rectangle or dividing the rectangle into smaller golden rectangles? If so, what were they and how were these difficulties overcome?

Extend

7 The metric paper system, such as A4 and A3 paper sizes, also involves a special ratio. Investigate the metric paper system, its uniqueness and its benefits.



Half-time 12

Ex. 12.2

1 Simplify each of the following.

(a) $2\sqrt{3} \times \sqrt{3}$ (b) $4\sqrt{11} \times 2\sqrt{11}$ (c) $\sqrt{\frac{81}{16}}$

Ex. 12.4

2 What is the area of a rectangular patio with dimensions of $(5 + \sqrt{3})$ m and $\sqrt{6}$ m?

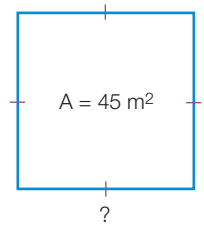
Ex. 12.5

3 Express each of the following in simplest form with a rational denominator.

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{5}{2\sqrt{3}}$ (c) $\frac{3}{4 - \sqrt{11}}$

Ex. 12.1

4 What is the exact side length of a square with an area of 45 m^2?



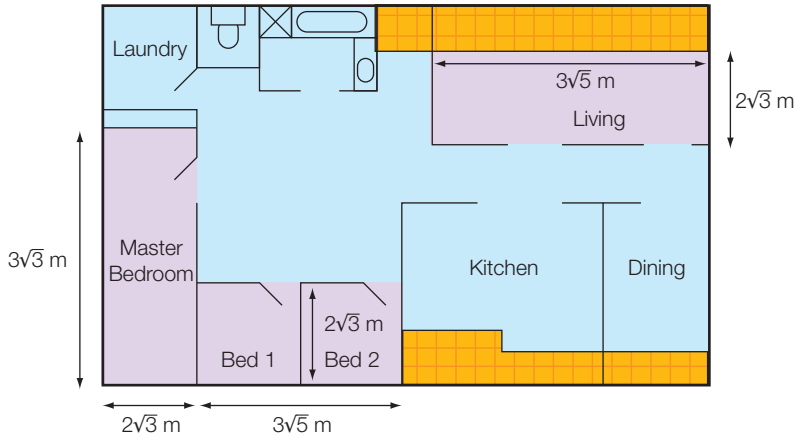
Ex. 12.1

5 For each of the numbers below, state whether it is rational (R) or irrational (I).

(a) 3.656565… (b) 0.924 (c) $\sqrt[3]{9}$ (d) $-\sqrt{25}$

Ex. 12.4

6 Calculate the exact total area to be carpeted in the bedrooms and the living room in the house below:



Ex. 12.2

7 Simplify each of the following.

(a) $\sqrt{72}$ (b) $3\sqrt{20} \times 4\sqrt{45}$ (c) $\frac{3\sqrt{8}}{\sqrt{200}}$

Ex. 12.3

8 Evaluate the following without the use of a calculator.

(a) $-5\sqrt{9}$ (b) $\sqrt{81} + \sqrt{64}$ (c) $4\sqrt{36} - \sqrt{16} + 2\sqrt{49}$

Summary of index laws

12.6

The number 2^3 has a base of 2 and a power or index of 3. This means that 2 has been multiplied by itself three times. The same applies when dealing with pronumerals. a^4 means that a has been multiplied by a four times.

$a^4 = a \times a \times a \times a$

There are laws to make simplifying expressions involving indices easier. A summary of the work we have covered is as follows:

$a^m \times a^n = a^{m+n}$	When multiplying numbers that are in index form and have the same base, keep the base and add the indices.
$\frac{a^m}{a^n} = a^{m-n}$	When dividing numbers that are in index form and have the same base, keep the base and subtract the indices.
$a^0 = 1$	Any number to the power of zero equals one, $a \neq 0$.
$(a^m)^n = a^{mn}$	When raising a number in index form to a power, keep the base and multiply the indices.
$(ab)^m = a^m b^m$	When a product in brackets is raised to a power m, each factor in the brackets is raised to the power m.
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	When a quotient in brackets is raised to a power m, both the numerator and the denominator are raised to the power m.
$a^{-m} = \frac{1}{a^m}$	A number raised to a negative power is the same as the reciprocal of the number raised to the positive of that power.

More difficult expressions often require the use of more than one index law to simplify them fully.

Worked Example 11

WE11

Simplify: $(a^4)^3b^7 \times (a^3b^{-4})^2$

Thinking	Working
1 Expand the brackets using the appropriate index laws.	$(a^4)^3b^7 \times (a^3b^{-4})^2$ $= a^{12}b^7 \times a^6b^{-8}$
2 To multiply, keep the bases and add the indices.	$= a^{18}b^{-1}$
3 Express the answer with positive indices.	$= \frac{a^{18}}{b}$

Worked Example 12 WE12

Simplify: $\left(\frac{m^2n}{m^5n^3}\right)^4 \div \frac{n^{-5}}{m^{12}}$

Thinking	Working
1 Change division to multiplication by taking the reciprocal of the second fraction.	$\left(\frac{m^2n}{m^5n^3}\right)^4 \div \frac{n^{-5}}{m^{12}}$ $= \left(\frac{m^2n^1}{m^5n^3}\right)^4 \times \frac{m^{12}}{n^{-5}}$
2 Expand brackets by multiplying indices.	$= \frac{m^8n^4}{m^{20}n^{12}} \times \frac{m^{12}}{n^{-5}}$
3 Multiply the numerators and denominators by adding the indices of like bases. ($m^8 \times m^{12} = m^{20}$; $n^{12} \times n^{-5} = n^7$)	$= \frac{m^{20}n^4}{m^{20}n^7}$
4 Divide the numerator by the denominator by subtracting the indices of like bases. ($m^{20} \div m^{20} = m^0$; $n^4 \times n^7 = n^{-3}$)	$= m^0n^{-3}$
5 Perform any further simplification. ($m^0 = 1$.)	$= n^{-3}$
6 Express the answer with a positive index.	$= \frac{1}{n^3}$

When performing calculations using indices, remember to follow these steps:

1. Cancel any numerical common factors (unless they are inside brackets).
2. Expand brackets by multiplying indices.
3. Simplify terms with the same base using the index laws. With fractions:
   - multiply each numerator together and each denominator together by adding indices
   - divide terms with the same base by subtracting indices.
4. Apply any other index laws if necessary.
5. Express your answer with positive indices.

12.6 Summary of index laws

Navigator

Answers page 816

Q1 (a)–(e) (j) (l), Q2 (a) (c) (e) (j), Q3, Q4 (a) (b) (f) (h), Q5 (a) (c) (g) (i), Q6, Q7 (a) (d) (h), Q8, Q9, Q11	Q1 (c) (d) (f) (g) (m) (n), Q2 (b) (d) (f) (j) (k), Q3, Q4 (c) (e) (h) (i), Q5 (b) (e) (f) (j) (k), Q7 (b) (d) (f) (h), Q8, Q10, Q11	Q1 (d) (h) (i) (k) (n) (o), Q2 (g)–(l), Q3, Q4 (d) (g) (i)–(l), Q5 (d) (h)–(l), Q7 (c) (e) (g)–(i), Q8, Q10, Q11

Fluency

1 Simplify each of the following. WE11

(a) $4x^7 \times 3x^{-2}$ (b) $a^3b^{-5} \times a^2b^2$ (c) $\dfrac{p^4q^6}{p^3q^9}$

(d) $\dfrac{(x^5)^{-2} \times x^8}{x^{-4}}$ (e) $(a^4)^3b^5 \times (a^2b)^4$ (f) $2(m^3)^2n^{-3} \times 3(mn)^4$

(g) $\dfrac{(p^2q^3)^{-1} \times p^5q^{-2}}{p^3q^2}$ (h) $\dfrac{(c^{-3}d^{-4})^2 \times c^2d^6}{c^5d^3}$ (i) $\dfrac{4(a^2b)^3 \times 3a^{-5}b^{-2}}{6ab}$

(j) $\dfrac{a^5b^{-2} \times a^2b^3}{a^4b^7 \times ab^{-4}}$ (k) $\dfrac{m^4n^{-3} \times m^5n^2}{mn^6 \times m^3n^{-2}}$ (l) $\dfrac{x^3y^{-6}}{x^{-4}y^7} \times \dfrac{x^{-2}y^4}{x^5y^{-9}}$

(m) $\dfrac{p^4q^{-2}}{p^2q^{-4}} \times \dfrac{pq^5}{p^3q^6}$ (n) $\dfrac{3k^2h^4}{4k^{-3}h^2} \times \dfrac{2k^{-1}h}{5kh^4}$ (o) $\dfrac{4a^4b}{9a^{-5}b^3} \times \dfrac{3a^{-2}b^5}{2ab^{-1}}$

2 Simplify each of the following. WE12

(a) $\dfrac{k^9p^6}{k^4p^2} \div \dfrac{k^3p^2}{k^5p^4}$ (b) $\dfrac{x^6y^4}{x^3y^2} \div \dfrac{xy^5}{x^3y^8}$ (c) $\dfrac{6m^8n^6}{2m^4n^3} \div \dfrac{5m^3n^2}{2m^5n^4}$

(d) $\dfrac{2p^5q^3}{p^2q^{-4}} \div \dfrac{p^7}{q^4}$ (e) $\dfrac{7x^3y^5}{xy^{-3}} \div \dfrac{x^6}{y^2}$ (f) $\dfrac{m^{-4}n^{-2}}{m^5n^3} \div \dfrac{m^2n}{m^5n^4}$

(g) $\dfrac{k^{-5}p^2}{k^3p} \div \dfrac{k^5p^2}{k^6p^{-3}}$ (h) $\dfrac{2a^5b}{a^{-4}b^{-3}} \div \dfrac{3a^2b^6}{a^{-2}b}$ (i) $\dfrac{5f^4g^6}{f^{-3}g^{-1}} \div \dfrac{2f^2g^4}{f^{-4}g}$

(j) $\left(\dfrac{x^5}{y^2}\right)^2 \div \left(\dfrac{x^2}{y^2}\right)^4$ (k) $\left(\dfrac{4m^3n^4}{3mn}\right)^2 \div \dfrac{(2m^2n^3)^3}{3m^5n^3}$ (l) $\left(\dfrac{5p^2q^4}{2pq^3}\right)^2 \div \dfrac{(5p^3q^2)^3}{4p^7q^6}$

3 (a) $\dfrac{k^3 \times k^4}{k^9}$ is the same as:

A k^2 B $\dfrac{1}{k^2}$ C k^3 D $\dfrac{1}{k^3}$

(b) $(2x^3y^{-2})^3$ is the same as:

A $\dfrac{8x^9}{y^6}$ B $2x^9y^6$ C $8x^3y^6$ D $\dfrac{8}{x^9y^6}$

(c) $4(3a)^0$ is the same as:

A 0 B 1 C 4 D $12a$

4 Simplify each of the following.

(a) $\dfrac{(x^4y^{-2})^3 \times (xy^3)^2}{x^5y^{-4}}$ (b) $\dfrac{(a^3b^{-4})^5 \times (a^2b)^3}{a^9b^{-6}}$

(c) $\dfrac{(2k^{-3}p^2)^4 \times (k^2p^4)^{-2}}{(2kp)^2}$ (d) $\dfrac{(3m^4n^2)^4 \times (m^{-3}n^2)^{-2}}{(3mn)^2}$

(e) $\dfrac{(x^5y^{-3})^3}{(x^{-2}y)^2} \times \dfrac{(x^{-2}y^{-3})^2}{(x^4y^2)^{-3}}$ (f) $\left(\dfrac{m^3}{n^2}\right)^4 \times \left(\dfrac{m^{-2}}{n^4}\right)^3$

(g) $\left(\dfrac{p^2}{q^4}\right)^3 \times \left(\dfrac{p^{-3}}{q^5}\right)^2$ (h) $\left(\dfrac{f^{-3}}{g^2}\right)^{-4} \div \left(\dfrac{f^2}{g}\right)^3$

(i) $\left(\dfrac{a^{-2}b^2}{ab}\right)^3 \times \left(\dfrac{a^4b^{-3}}{a^{-1}b^2}\right)^2$ (j) $\left(\dfrac{m^{-3}n}{m^2n^2}\right)^2 \times \left(\dfrac{m^5n^{-2}}{m^{-3}n}\right)^3$

(k) $\dfrac{4x^3(y^{-2})^4}{(xy)^2} \times \left(\dfrac{x^{-4}y}{x^2y^{-3}}\right)^{-2}$ (l) $\left(\dfrac{3c^4d^3}{2cd}\right)^2 \div \dfrac{3cd}{(2c^{-2}d^3)^4}$

Understanding

5 Simplify each of these, expressing your answers with positive indices.

(a) $x^{-4} \times x^{6}$ (b) $a^{3}b^{-2} \times b^{-3}$ (c) $p^{-4}q^{3} \times p^{5}q^{-2}$

(d) $d^{7}f^{-2} \times d^{-3}f^{4}$ (e) $x^{5}y^{2} \times y^{-4}x^{-5}$ (f) $a^{3}b \times b^{-1}a^{-6}$

(g) $(m^{3})^{-2}n^{3} \times m^{4}(n^{-1})^{2}$ (h) $(p^{4})^{-3}q^{6} \times p^{5}(q^{-2})^{2}$ (i) $(a^{-2}b^{3})^{2} \times (a^{4}b^{-1})^{-3}$

(j) $(x^{4}y^{-3})^{2} \times (x^{-5}y)^{-1}$ (k) $4(p^{-3})^{5}q^{6} \times 3(pq)^{-2}$ (l) $(2x^{2}y)^{-3} \times (3x^{-4}y^{5})^{2}$

6 (a) Using base 5 index notation, write the numbers 125, 25, 5 and 1.

(b) Using base 5 index notation, write the next five numbers in the sequence.

(c) Write down the whole sequence without using index notation.

Reasoning

7 Simplify each of these, expressing your answers with positive indices.

(a) $\dfrac{m^{7}n^{-3}}{m^{-2}n^{-1}}$ (b) $\dfrac{a^{-5}b^{3}}{a^{2}b^{-4}} \times \dfrac{a^{3}b^{2}}{a^{-3}b^{6}}$ (c) $\dfrac{x^{7}y^{-5}}{x^{-4}y^{3}} \times \dfrac{x^{-4}y^{3}}{x^{2}y^{-1}}$

(d) $\dfrac{x^{-4}y^{7}}{x^{-2}y^{-5}} \div \dfrac{x^{3}y^{-6}}{x^{-1}y}$ (e) $\dfrac{(3a^{-4}b^{-2})^{2}}{(a^{-2}b)^{3}}$ (f) $\dfrac{(2d^{-3}f^{-1})^{3}}{(d^{-2}f^{2})^{2}}$

(g) $\dfrac{5(a^{-4})^{3}b^{-6}}{(a^{2}b^{-1})^{4}} \times \dfrac{2a^{7}(b^{-3})^{2}}{a^{-4}b^{-9}}$ (h) $\left(\dfrac{m^{5}n^{-2}}{m^{-3}n}\right)^{-1}$ (i) $\left(\dfrac{x^{3}}{y^{2}}\right)^{-4} \div \dfrac{x^{-10}}{(y^{-4})^{2}}$

8 (a) If I divide a number by its square, what is the result?

(b) I divide a room into squares that have a side length of a units. If the room has a width of $16a^{2}$ units and a length of $12a$ units, how many squares would I require to carpet the room?

Open-ended

9 (a) Write two expressions involving powers and multiplication that when simplified equal $36m^{4}n^{8}$.

(b) Write two expressions involving powers and division that can be simplified to $\dfrac{4m^{4}}{5n^{6}}$.

10 For the expression a^{-b}, write three different sets of values for a and b, so that its value is less than $\frac{1}{4}$. Mark each value on a number line.

11 Raphael and Murray were arguing over a question involving indices. Murray believes that $a^{-2} \times a^{-4}$ simplifies to $\dfrac{1}{a^{8}}$. Raphael disagrees.

(a) How did Murray calculate his answer? Is he correct?

(b) Why does Raphael disagree? What might Raphael's answer be, assuming it is correct?

(c) How would you explain the correct answer to the students?

Fractional indices

12.7

The index laws can also be used when the index or power is a fraction.

Consider $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$. Using the index law $a^m \times a^n = a^{m+n}$, we would keep the base 3 and add the powers.

$$\begin{aligned} 3^{\frac{1}{2}} \times 3^{\frac{1}{2}} &= 3^{\frac{1}{2}+\frac{1}{2}} \\ &= 3^1 \\ &= 3 \end{aligned}$$

However, what does $3^{\frac{1}{2}}$ mean? What number when multiplied by itself gives 3?

$$\sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$$

So $3^{\frac{1}{2}}$ is the same as $\sqrt{3}$.

$$a^{\frac{1}{2}} = \sqrt{a}\text{, where } a \text{ is positive or zero.}$$

Similarly, consider $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$.

This equals $5^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$ or $5^{\frac{3}{3}}$ or 5. What number multiplied three times by itself gives 5?

$$\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5} = \sqrt[3]{125} = 5$$

So $5^{\frac{1}{3}}$ is the same as $\sqrt[3]{5}$.

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

A number raised to a fractional power $\frac{1}{n}$ is the nth root of the number.

$$a^{\frac{1}{2}} = \sqrt{a}\text{, where } a > 0$$

$$a^{\frac{1}{3}} = \sqrt[3]{a}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a}\text{, where } a > 0$$

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

Numbers raised to a fractional power obey all the index laws.

Worked Example 13 WE13

Simplify:

(a) $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$ (b) $\sqrt{x^6}$ (c) $\dfrac{(m^3)^{\frac{1}{3}} \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$

Thinking	Working
(a) Multiply by keeping the base and adding the indices. To add fractions a common denominator is needed. (LCM of 2 and 4 is 4.)	(a) $a^{\frac{1}{2}} \times a^{\frac{1}{4}}$ $= a^{\frac{1}{2}+\frac{1}{4}}$ $= a^{\frac{(2+1)}{4}}$ $= a^{\frac{3}{4}}$
(b) 1 Change to index form by using $\sqrt{a} = a^{\frac{1}{2}}$. 2 Expand the brackets by using $(a^m)^n = a^{mn}$.	(b) $\sqrt{x^6}$ $= (x^6)^{\frac{1}{2}}$ $= x^{6 \times \frac{1}{2}}$ $= x^3$
(c) 1 Expand the brackets by using $(a^m)^n = a^{mn}$. 2 Simplify the numerator by keeping the base and adding the indices. 3 Divide by keeping the base and subtracting the indices.	(c) $\dfrac{(m^3)^{\frac{1}{3}} \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$ $= \dfrac{m^1 \times m^{-\frac{1}{3}}}{m^{\frac{1}{3}}}$ $= \dfrac{m^{1-\frac{1}{3}}}{m^{\frac{1}{3}}}$ $= \dfrac{m^{\frac{2}{3}}}{m^{\frac{1}{3}}}$ $= m^{\frac{2}{3}-\frac{1}{3}}$ $= m^{\frac{1}{3}}$

Fractional powers on a calculator

Consider calculating $625^{\frac{1}{4}}$. Now $625^{\frac{1}{4}}$ is the same as $\sqrt[4]{625}$.

On your calculator look for $\sqrt[x]{\ }$ or $\sqrt[x]{y}$ or $\sqrt[y]{x}$ or $x^{1/y}$.

You may need to press the second function 2nd or Shift or inverse INV key before this key. The root, which is the small number in the radical sign ($\sqrt{\ }$), is usually entered first.

For $625^{\frac{1}{4}}$, press: 4 2nd $\sqrt[x]{\ }$ 6 2 5 = . The screen should show 5.

(Your calculator may require a different sequence, such as 6 2 5 INV $x^{1/y}$ 4 = . Learn how your calculator works.)

Worked Example 14 WE14

Evaluate:

(a) $7776^{\frac{1}{5}}$ **(b)** $289^{\frac{1}{2}}$ **(c)** $1331^{\frac{1}{3}}$

Thinking	Working
(a) Use your calculator to evaluate. (Press: 5 2nd $\sqrt[x]{\ }$ 7 7 7 6 =)	(a) $7776^{\frac{1}{5}}$ $= \sqrt[5]{7776}$ $= 6$
(b) Use your calculator to evaluate. (Press: $\sqrt{\ }$ 2 8 9 = or 2 2nd $\sqrt[x]{\ }$ 2 8 9 =)	(b) $289^{\frac{1}{2}}$ $= \sqrt{289}$ $= 17$
(c) Use your calculator to evaluate. (Press: $\sqrt[3]{\ }$ 1 3 3 1 = or 3 2nd $\sqrt[x]{\ }$ 1 3 3 1 =)	(c) $1331^{\frac{1}{3}}$ $= \sqrt[3]{1331}$ $= 11$

If you do not want to worry about working out which root it is, you can use the following alternative calculator key sequence.

For $7776^{\frac{1}{5}}$, press 7 7 7 6 y^x (1 ÷ 5) = and the screen display will be 6 as in part **(a)** above.

12.7 Fractional indices

Navigator

Answers page 817

Q1 Column 1, Q2 Columns 1–2, Q3, Q4, Q5 (a)–(d), Q7 (a)–(d), Q9, Q10, Q11, Q12	Q1 Column 2, Q2 Columns 2–3, Q3, Q4, Q5 Column 1, Q6, Q7 (c)–(f), Q9, Q10, Q11, Q12	Q1 Column 3, Q2 Columns 3–4, Q3, Q5 Column 2, Q6, Q7 (e)–(h), Q8, Q9, Q10, Q11, Q13

Equipment required: calculator for Questions **2**, **3**, **4**, **7**, **10**, **11** and **12**

Fluency

WE13

1 Simplify each of the following.

(a) $x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ **(b)** $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$ **(c)** $k^{\frac{1}{3}} \times k^{\frac{1}{5}}$

(d) $\dfrac{m^{\frac{1}{2}}}{m^{\frac{1}{4}}}$ **(e)** $\dfrac{p^{\frac{1}{2}}}{p^{\frac{1}{3}}}$ **(f)** $\left(y^{\frac{1}{2}}\right)^{\frac{1}{2}}$

(g) $(25p^4)^{\frac{1}{2}}$ **(h)** $(49x^6)^{\frac{1}{2}}$ **(i)** $(8ab^3)^{\frac{1}{3}}$

(j) $\sqrt{x^4y}$ **(k)** $\sqrt{kn^8}$ **(l)** $\left(\dfrac{p^4}{q^{10}}\right)^{\frac{1}{2}}$

WE14

2 Evaluate each of the following. Express your answer correct to two decimal places where appropriate.

(a) $196^{\frac{1}{2}}$ **(b)** $529^{\frac{1}{2}}$ **(c)** $6859^{\frac{1}{3}}$ **(d)** $512^{\frac{1}{3}}$

(e) $1296^{\frac{1}{4}}$ **(f)** $4096^{\frac{1}{4}}$ **(g)** $3125^{\frac{1}{5}}$ **(h)** $1024^{\frac{1}{5}}$

(i) $500^{\frac{1}{3}}$ **(j)** $3000^{\frac{1}{3}}$ **(k)** $98^{\frac{1}{6}}$ **(l)** $145^{\frac{1}{6}}$

3 Using your calculator to assist you, place these numbers in ascending order.

$2^{\frac{1}{2}}, 3^{\frac{1}{3}}, 4^{-1}, 6^{-\frac{1}{5}}$

4 **(a)** 3^5 is closest to:

A 1.71 **B** 15 **C** 125 **D** 243

(b) 4^{-3} is closest to:

A 0.012 **B** 0.016 **C** 64 **D** 81

(c) $20^{\frac{1}{4}}$ is closest to:

A 0.000 006 **B** 2.1 **C** 5 **D** 160 000

(d) $75^{\frac{1}{3}}$ is closest to:

A -25 **B** 0.24 **C** 4.22 **D** 421 875

5 Simplify each of the following expressing your answers with positive indices.

(a) $\dfrac{\left(a^{\frac{1}{3}}\right)^{6} \times a}{a^{2}}$

(b) $\dfrac{\left(d^{\frac{1}{2}}\right)^{8} \times d^{2}}{d^{3}}$

(c) $(x^{4}y^{6})^{\frac{1}{2}} \times (x^{-3}y^{-12})^{\frac{1}{3}}$

(d) $(m^{6}n^{3})^{\frac{1}{3}} \times (m^{-8}n^{-10})^{\frac{1}{2}}$

(e) $\dfrac{(pq^{3})^{\frac{1}{2}} \times p^{\frac{1}{2}}q^{\frac{1}{2}}}{pq^{4}}$

(f) $\dfrac{(a^{5}b)^{\frac{1}{2}} \times a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{3}b^{5}}$

(g) $\left(\dfrac{m^{2}n}{n^{-\frac{1}{2}}}\right)^{-\frac{1}{3}} \times \dfrac{m\left(n^{\frac{1}{3}}\right)^{2}}{2n^{\frac{1}{3}}}$

(h) $\left(\dfrac{x^{3}y^{2}}{y^{\frac{1}{2}}}\right)^{-\frac{1}{3}} \times \dfrac{(x^{2})^{\frac{1}{3}}y}{5y^{-\frac{2}{3}}}$

6 **(a)** Find the expression that is *not* the same as $a^{\frac{3}{2}}b^{\frac{1}{2}}$.

A $\sqrt{a^{3}b}$ **B** $(\sqrt{ab})^{3}$ **C** $(a^{3}b)^{\frac{1}{2}}$ **D** $(\sqrt{a})^{3}\sqrt{b}$

(b) Simplifying $\left(\dfrac{a^{\frac{1}{2}}b^{3}}{b^{2}}\right)^{2} \div \dfrac{b^{\frac{3}{2}}a}{b^{4}}$ gives:

A $b^{\frac{9}{2}}$ **B** $b^{\frac{13}{2}}$ **C** $\dfrac{b^{\frac{7}{2}}}{a^{\frac{1}{2}}}$ **D** $\dfrac{b^{7}}{a^{\frac{1}{2}}}$

7 Evaluate each of the following correct to three decimal places.

(a) $8^{-\frac{1}{3}}$ **(b)** $12^{-\frac{1}{3}}$ **(c)** $37^{-\frac{1}{2}}$ **(d)** $26^{-\frac{1}{2}}$

(e) $54^{-\frac{1}{4}}$ **(f)** $72^{-\frac{1}{4}}$ **(g)** $100^{-\frac{1}{5}}$ **(h)** $167^{-\frac{1}{5}}$

Understanding

8 By simplifying first, find the value of x in each of the following equations.

(a) $\dfrac{72x^{2}}{\left(x \times x^{\frac{3}{2}}\right)^{2}} = 24$

(b) $\dfrac{\left(x^{\frac{1}{2}}\right)^{5}}{x^{-2}} \times \dfrac{5x^{-1}}{x^{2}} = 40$

(c) $\dfrac{7x^{-3}}{x^{3}}\left(\dfrac{x^{\frac{3}{2}}}{x}\right)^{6} = 56$

9 Simplify the following.

(a) $\sqrt{x^{6}}$ **(b)** $\sqrt[3]{a^{9}}$ **(c)** $\sqrt[4]{16m^{8}n^{4}}$

Reasoning

10 Find the value of x in each of the following equations.

(a) $x^{-2} = \frac{1}{4}$ **(b)** $3^{x} = 27$ **(c)** $3^{x} = \frac{1}{9}$ **(d)** $2 \times 5^{x} = 50$

11 **(a)** Using your calculator, find $16\,807^{\frac{1}{5}}$.

(b) Hence, find the value of x and y in the expression $x^y = 16\,807$.

(c) Using your calculator, find $64^{\frac{4}{3}}$.

(d) Hence, find the value of d and e in the expression $d^e = 64$.

Open-ended

12 Using your calculator, find at least three ways of writing 343 as a power. When the base number is not exact, use three decimal places. For example, $343 = (18.520)^2$.

13 Write the expression $n^{\frac{3}{4}} m^{\frac{3}{5}}$ in at least three different ways.

Outside the Square

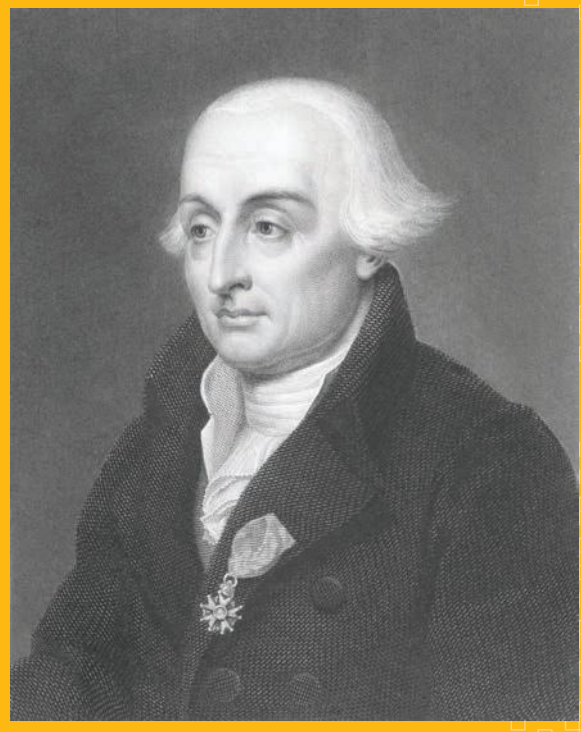
Problem solving

Lagrange's four-square theorem

Diophantus was a Greek mathematician who lived in the 3rd century. He wrote a famous book called *Arithmetica*. In this book he wrote the following conjecture (a statement that has not been proven).

> Every whole number is either a square itself or the sum of two, three or four squares.

This conjecture was subsequently proven in 1770 by Joseph-Louis Lagrange.



(a) Write a square number in each box so that each column and row adds to the total at the end.

			1^2	39
		1^2		27
	1^2			34
1^2				62
36	30	66	54	

(b) Can any of the totals in the grid be written using fewer square numbers?

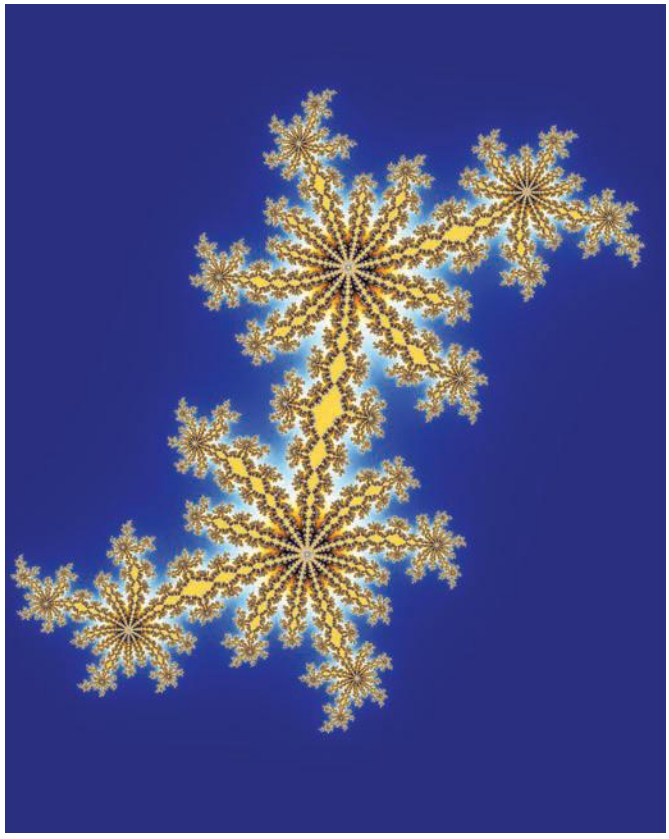
(c) Starting at 1, write each whole number as a sum of four or fewer square numbers. How far can you get in 3 minutes?

Strategy options

- Make a table.
- Break problem into manageable parts.

Investigation

The von Koch snowflake



Equipment required: 1 brain, isometric paper and a calculator

The Big Question

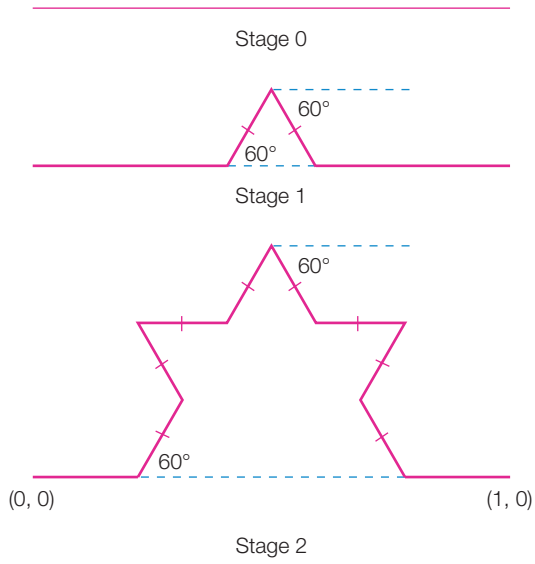
How can the perimeter and area of a von Koch snowflake, a fractal that continues indefinitely, be described?

Strategy options

- Look for a pattern.
- Make a table.

Engage

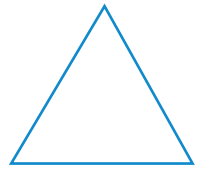
Swedish mathematician Helge von Koch produced the von Koch curve in 1904. It starts with a line and at each stage the middle third of the line is removed and replaced with two lines of the same length as the removed third.



A geometric pattern that repeats the same process at each stage indefinitely is known as a fractal.

1 Follow the steps below to construct some of the early stages of what is known as the *von Koch snowflake*.

(a) Begin with an equilateral triangle with 15 cm sides drawn on isometric grid paper. We will call this 'stage 0'.

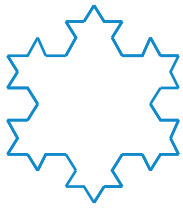


Stage 0

(b) Divide each side into equal thirds and construct a new equilateral triangle on the middle third of each side. Erase the internal lines to complete the next stage.

Stage 1

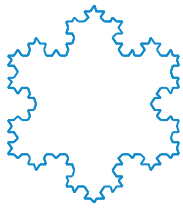
(c) Repeat **1(b)** on the stage 1 shape.



Stage 2

(d) Continue this process—through as many stages as possible.

(e) Describe the von Koch snowflake and the process of drawing the next stage.



Stage 3

Explore

2 **(a)** If the perimeter of the initial triangle is 3 units, calculate the perimeter of the von Koch snowflake at stages 0, 1 and 2. Record your results in a table.

(b) If the area of the initial triangle is 1 unit^2, calculate the area of the von Koch snowflake at stages 0, 1 and 2. Record your results in the table created in **2(a)**.

(c) Describe any trends in the perimeter and area at these stages. Estimate the perimeter and area for stage 3.

(d) At each stage, the middle third of each edge is removed and replaced with two edges of the same length or in other words, a length 1 unit long or $\frac{3}{3}$ becomes $\frac{4}{3}$ long with each stage. The perimeter at any stage (P_n) is equal to the initial length (P_0) multiplied by $\frac{4}{3}$ to the power of n. Write this formula.

(e) Use the formula to find the perimeter for $n = 0, 1, 2$ and 3.

Explain

3 **(a)** Were the perimeters you calculated in **2(a)** the same as the perimeters calculated using the formula in **2(e)**?

(b) Describe the pattern for finding the perimeter of the von Koch snowflake at consecutive stages.

(c) Can a similar formula be written to describe the area at each stage of the von Koch snowflake? If yes, what is it? If no, why not?

Elaborate

4 **(a)** Does the formula for calculating the perimeter of the von Koch snowflake suggest that the perimeter is finite or infinite? Give reasons for your answer.

(b) Describe the area of the von Koch snowflake. Does a von Koch snowflake have a finite or infinite area?

(c) Summarise your investigation of the von Koch snowflake, describing what it is, and then commenting on the patterns found with the perimeter and area.

Evaluate

5 **(a)** Describe the method you used to calculate the perimeter and area of the von Koch triangle at each stage.

(b) Did you find any shortcuts to calculate the perimeter or area? If yes, what were they?

(c) Describe any difficulties you had with this investigation and the methods used to overcome them.

Extend

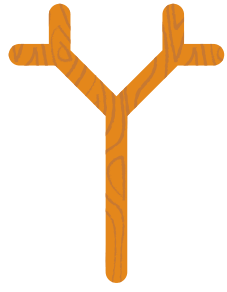
6 The von Koch snowflake is an example of a fractal. The Sierpinski triangle, also known as the Sierpinski Gasket, is another example of a fractal.

Stage 0 Stage 1 Stage 2

(a) Explain its pattern

(b) Draw the Sierpinski triangle at stage 3.

7 Stage 2 of another fractal pattern is shown. Each new set of branches are half the length of the previous set.



(a) Copy and complete the pattern up to stage 6.

(b) Describe the appearance of this fractal pattern.

(c) Where do we see fractal patterns in nature?

Logarithms

12.8

Logarithm is another word for power or index. We know that $10^2 = 100$, so we say the logarithm of 100 to the base 10 is 2.

logarithm

$$100 = 10^2$$

numerical value base

In this case we used 10 as the base, but any number can be used. For example, $2^3 = 8$, so the logarithm of 8 to the base 2 is 3. We write this as $\log_2 8 = 3$.

If $y = \log_b x$ then $b^y = x$, where b is the base, y is the power or logarithm and x is the numerical value.

Any number can be used as the base number, however, the most frequently used values are 2 and 10. The **log** button on your calculator will give you the base 10 logarithm for any number keyed in. We know that an index can be negative, so we can deal with negative logarithms in the same way.

Worked Example 15

WE15

(a) Write $2^6 = 64$ in its equivalent logarithmic form.

(b) Write $2^{-4} = \frac{1}{16}$ in its equivalent logarithmic form.

Thinking	Working
(a) 1 Identify the logarithm (power), the base and the numerical value.	(a) 6 is the logarithm 2 is the base 64 is the numerical value
2 Write the logarithmic form.	$6 = \log_2 64$
(b) 1 Identify the logarithm (power), the base and the numerical value.	(b) -4 is the logarithm 2 is the base $\frac{1}{16}$ is the numerical value
2 Write the logarithmic form.	$-4 = \log_2\left(\frac{1}{16}\right)$

Worked Example 16 WE16

Write $\log_{10}(10\,000) = 4$ in its equivalent index form.

Thinking	Working
1 Identify the logarithm (power), the base and the numerical value.	4 is the logarithm 10 is the base 10 000 is the numerical value
2 Write the index form.	$10^4 = 10\,000$

12.8 Logarithms

Navigator

Answers page 817

Q1 Column 1, Q2 Column 1, Q3, Q4 Column 1, Q5 Column 1, Q6, Q9, Q11	Q1 Column 2, Q2 Column 2, Q3, Q4 Column 2, Q5 Column 2, Q6, Q7, Q9, Q11	Q1 Column 3, Q2 Column 3, Q3, Q4 Column 3, Q5 Column 3, Q6, Q8, Q9, Q10

Equipment required: calculator for Questions **4–5**

Fluency

WE15

1 Write each of the following in its equivalent logarithmic form.

(a) $2^5 = 32$ **(b)** $2^8 = 256$ **(c)** $2^{12} = 4096$

(d) $3^3 = 27$ **(e)** $3^5 = 243$ **(f)** $3^8 = 6561$

(g) $10^2 = 100$ **(h)** $10^6 = 1\,000\,000$ **(i)** $10^8 = 100\,000\,000$

(j) $2^{-2} = \frac{1}{4}$ **(k)** $2^{-3} = \frac{1}{8}$ **(l)** $2^{-4} = \frac{1}{16}$

WE16

2 Write each of the following in its equivalent index form.

(a) $\log_{10}100 = 2$ **(b)** $\log_{10}1000 = 3$ **(c)** $\log_{10}100\,000 = 5$

(d) $\log_2 16 = 4$ **(e)** $\log_2 128 = 7$ **(f)** $\log_2 512 = 9$

(g) $\log_{10}\left(\frac{1}{1000}\right) = -3$ **(h)** $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **(i)** $\log_{10}\left(\frac{1}{1\,000\,000}\right) = -6$

(j) $\log_2\left(\frac{1}{8}\right) = -3$ **(k)** $\log_2\left(\frac{1}{32}\right) = -5$ **(l)** $\log_2\left(\frac{1}{128}\right) = -7$

3 **(a)** If $\log_{10}x = -2$ then x is equal to:

A -200 **B** $-\frac{1}{100}$ **C** $\frac{1}{100}$ **D** 200

(b) Writing $2^x = 2048$ in logarithmic form gives:

A $\log_2 x = 2048$ **B** $\log_x 2 = 2048$ **C** $\log_x 2048 = 2$ **D** $\log_2 2048 = x$

Understanding

4 Solve the following for x:

(a) $3^x = 81$ **(b)** $3^x = 6561$ **(c)** $3^x = 19\,683$

(d) $5^x = \frac{1}{25}$ **(e)** $5^x = \frac{1}{125}$ **(f)** $5^x = \frac{1}{15\,625}$

(g) $6^{-x} = 216$ **(h)** $6^{-x} = 7776$ **(i)** $6^{-x} = \frac{1}{1296}$

Hint

Write the number on the RHS as an index number, using the same base as the LHS.

5 Solve the following for x:

(a) $\log_2 x = 4$ **(b)** $\log_2 x = 6$ **(c)** $\log_2 x = 7$

(d) $\log_3\left(\frac{1}{81}\right) = x$ **(e)** $\log_3\left(\frac{1}{243}\right) = x$ **(f)** $\log_3\left(\frac{1}{6561}\right) = x$

(g) $\log_x 625 = 4$ **(h)** $\log_x 343 = 3$ **(i)** $\log_x\left(\frac{1}{3125}\right) = -5$

Reasoning

6 **(a)** Solve for x in each of the following:

(i) $\log_{10} x = 1$ **(ii)** $\log_5 x = 1$ **(iii)** $\log_7 x = 1$

(b) Write a general rule based on the above, using a as your base.

7 **(a)** Solve for x in each of the following:

(i) $\log_{10} x = 0$ **(ii)** $\log_5 x = 0$ **(iii)** $\log_7 x = 0$

(b) Write a general rule to explain what you have just found out.

8 In the equation $\sqrt{a} = a^{\frac{1}{2}}$, why does a have to be greater than or equal to 0?

9 **(a)** Check that both of the following logarithmic equations are correct:

$\log_2 64 = 6$ and $\log_4 64 = 3$

(b) Now, given $\log_2(4096) = 12$, find the value of y in $\log_4(4096) = y$.

(c) Explain how you got your answer for **(b)**.

Open-ended

10 Given $\log_3(531\,441) = 12$, write two other related log statements that are true.

11 Lee wrote:

$-(6)^2 = -36$

because $-\log_6(-36) = 2$

Explain why this is incorrect and how the question should be tackled.

Maths 4 Real

MEASURING EARTHQUAKES

Earthquakes occur when forces between the tectonic plates that make up the Earth's crust cause energy to be released abruptly. The strength of the Earthquake and the location of its epicentre (the point on the crust where the energy is released) determine the effects the earthquake has on human life. The strength of an earthquake is commonly described as a magnitude between 0 and 10 on the Richter scale. The Richter scale increases by powers of 10, which means an increase of 1 on the Richter scale represents an earthquake 10 times stronger. An earthquake with a recorded magnitude of 5 on the Richter scale is 10 times stronger than an earthquake with a magnitude of 4. An earthquake with a magnitude of 6 on the Richter scale is 10×10, or 100 times stronger than an earthquake measuring 4.

1 **(a)** Complete the Richter scale chart below to show the comparative strength of earthquakes.

Richter scale	0	1	2	3	4	5	6	7	8	9
Comparative strength	1	10^1	10^2							

(b) **(i)** How many times stronger is an earthquake with magnitude 8.0 compared to an earthquake with magnitude 5.0?

(ii) How many times stronger is an earthquake with magnitude 6.5 compared to an earthquake with magnitude 7.5?

(c) Draw a graph showing the comparative strength of earthquakes on the Richter scale.

(d) What type of graph is this?

(e) Write an equation to describe this graph.

(f) Use the equation to find the comparative strength of an earthquake with magnitude 5.2. What does this figure mean?

2 The Earth has seven major tectonic plates that can cause earthquakes when sliding past each other. New Zealand, Indonesia, Japan, China, the west coast of the USA and South America all lie directly above where two tectonic plates meet and therefore are more susceptible to frequent and severe earthquakes.

(a) Show the position of the following deadly earthquakes on the graph drawn in **1(c)**.

- Sichuan earthquake, China, 2008, magnitude of 7.9, killing 88 000 people
- The Great Kanjō Earthquake, Japan, 1923, magnitude of 8.3, killing 140 000 people
- The Great Hanshin Earthquake, Kobe, Japan, 1995, magnitude of 7.2, killing 6433 people
- Java earthquake, Indonesia, May 2006, magnitude of 6.4, killing 5749 people
- Christchurch, New Zealand, February 2011, magnitude of 6.3, killing 181 people

One day after the Sichuan earthquake in China, another earthquake hit China with a magnitude of 6.1. The difference in strength between the two earthquakes can be found by calculating the ratio of the strengths of the two earthquakes using the following method:

$$\frac{\text{Magnitude}_1}{\text{Magnitude}_2} = \frac{10^{7.9}}{10^{6.1}} = 10^{1.8} = 63$$

The Sichuan earthquake was 63 times stronger than the earthquake the following day.

(b) Calculate the differences in strength between the following earthquakes listed in **3(a)**.

(i) The Great Kanjō Earthquake and The Great Hanshin Earthquake

(ii) The 2006 Java earthquakes and the 2011 Christchurch earthquake.

3 In 1906 an earthquake hit San Francisco, leaving 3000+ people dead and starting fires that almost flattened the whole city. This earthquake was 8 times stronger than the 1989 San Francisco earthquake that measured 6.9 on the Richter scale. The information above can be shown as:

$$\frac{\text{Magnitude}_1}{\text{Magnitude}_2} = \frac{10^y}{10^{6.9}} = 10^x = 8$$

$10^x = 8$ can be written as $\log_{10} 8 = x$

This can be solved on a calculator:

$\log_{10} 8 = 0.9$ (1 d.p.)

$$\frac{\text{Magnitude}_1}{\text{Magnitude}_2} = \frac{10^y}{10^{6.9}} = 10^{0.9} = 8$$

Therefore, the magnitude of the 1906 San Francisco earthquake had a magnitude of 7.8. Between 12 September 2007 and 13 September 2007, three earthquakes hit Indonesia, with the first registering a magnitude of 8.5. This was 4 times stronger than the second earthquake 12 hours later.

(a) What was the magnitude of the second earthquake?

(b) The first earthquake was 25 times stronger than the third earthquake. What was the magnitude of the third earthquake?

4 An earthquake in Taiwan in 2006 had a magnitude of 7.0, the same as recorded at Haiti in the 2010 earthquake. Over 230 000 people were killed in Haiti, whereas only two were killed in the Taiwan earthquake. Give reasons for the differences in devastation caused by the two similar strength earthquakes.

Research

5 The following measures also involve exponentials:

- loudness of sound (decibels)
- brightness of light
- acidity levels (pH readings)

Investigate how these scales are exponential and how they produce accurate and comparative readings.

12.9 Laws of logarithms

Just as there are laws that help us to simplify indicial expressions so there are laws to assist us with logarithmic expressions.

The proofs of each law are provided below.

Prove: $\log_b(xy) = \log_b x + \log_b y$

Proof: Let $\log_b x = m$ and $\log_b y = n$

$b^m = x$ and $b^n = y$

$$\begin{aligned} xy &= b^m \times b^n \\ &= b^{m+n} \end{aligned}$$

$\log_b(xy) = m + n$

$\therefore \log_b(xy) = \log_b x + \log_b y$

Worked Example 17

WE17

Simplify $\log_2 8 + \log_2 16$, expressing your answer in logarithmic form.

Thinking	Working
1 Write the rule for simplifying logarithmic expressions that are added.	$\log_b x + \log_b y = \log_b xy$ $\log_2 8 + \log_2 16$
2 Substitute the known values on the right-hand side.	$= \log_2(8 \times 16)$
3 Simplify the expression.	$= \log_2 128$

Prove: $\log_b \frac{x}{y} = \log_b x - \log_b y$

Proof: Let $\log_b x = m$ and $\log_b(y) = n$

$b^m = x$ and $b^n = y$

$$\frac{x}{y} = \frac{b^m}{b^n}$$

$$= b^{m-n}$$

$$\log_b\left(\frac{x}{y}\right) = m - n$$

$$\therefore \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

Worked Example 18

WE18

Simplify $\log_2 12 - \log_2 4$, expressing your answer in logarithmic form.

Thinking	Working
1 Write the rule for simplifying logarithmic expressions that are subtracted.	$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$ $\log_2 12 - \log_2 4$
2 Substitute the known values on the right-hand side.	$= \log_2\left(\frac{12}{4}\right)$
3 Simplify the expression.	$= \log_2 3$

Prove: $\log_b x^n = n\log_b x$

Proof: Let $\log_b x = m$

So $x = b^m$

$x^n = (b^{mn}) = b^{mn}$

$\log_b x^n = mn$

$= nm$

$= n\log_b x$

$\therefore \log_b x^n = n\log_b x$

Worked Example 19

WE19

Simplify $\log_2 5^2$, expressing your answer in logarithmic form without a power.

Thinking	Working
1 Write the rule for simplifying logarithmic expressions that involve a power.	$\log_b x^n = n\log_b x$
2 Simplify the expression by substituting the known values.	$\log_2 5^2 = 2\log_2 5$

There are two additional results that help us simplify logarithmic expressions.

$b^1 = b$ so $\log_b b = 1$

$b^0 = 1$ so $\log_b 1 = 0$

To help simplify logarithmic expressions we use:

$\log_b(xy) = \log_b x + \log_b y$

$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

$\log_b x^n = n\log_b x$

$\log_b b = 1$

$\log_b 1 = 0$

Worked Example 20 WE20

Simplify the following, writing your answer as a basic numeral if possible.

(a) $\log_2 16$ **(b)** $\log_{10}1000 - \log_{10}10^2$

Thinking	Working
(a) 1 Rewrite the number inside the brackets using index notation.	(a) $\log_2 16$ $= \log_2 2^4$
2 Simplify using the appropriate logarithm law.	$= 4\log_2 2$
3 Evaluate using $\log_b b = 1$.	$= 4 \times 1$ $= 4$
(b) 1 Rewrite without the power in the logarithm.	(b) $\log_{10}1000 - \log_{10}10^2$ $= \log_{10}10^3 - \log_{10}10^2$
2 Simplify.	$= 3\log_{10}10 - 2\log_{10}10$ $= 3 - 2$ $= 1$

12.9 Laws of logarithms

Navigator

Answers page 817

Q1, Q2, Q3, Q4, Q6 (a)–(b), Q7 (a)–(c), Q8, Q10	Q1, Q2, Q3, Q4, Q5, Q6 (a)–(d), Q7, Q8, Q10, Q11	Q1, Q2, Q3, Q4, Q5, Q6 (c)–(f), Q7, Q9, Q10, Q11

Fluency

WE17

1 Simplify the following, expressing your answer in logarithmic form.

(a) $\log_2 4 + \log_2 8$ **(b)** $\log_2 16 + \log_2 4$

(c) $\log_2 8 + \log_2 32$ **(d)** $\log_{10}5 + \log_{10}6$

(e) $\log_{10}7 + \log_{10}3$ **(f)** $\log_{10}4 + \log_{10}9$

WE18

2 Simplify the following, expressing your answer in logarithmic form.

(a) $\log_2 16 - \log_2 8$ **(b)** $\log_2 20 - \log_2 4$

(c) $\log_2 25 - \log_2 5$ **(d)** $\log_{10}70 - \log_{10}5$

(e) $\log_{10}35 - \log_{10}7$ **(f)** $\log_{10}36 - \log_{10}6$

WE19

3 Simplify the following, expressing your answer in logarithmic form without a power.

(a) $\log_2 5^4$ **(b)** $\log_2 6^3$ **(c)** $\log_2 7^4$

(d) $\log_{10}7^3$ **(e)** $\log_{10}8^4$ **(f)** $\log_{10}11^5$

WE20

4 Simplify the following, expressing your answer as a basic numeral.

(a) $\log_2 32$ **(b)** $\log_2 128$ **(c)** $\log_2 256$

(d) $\log_{10}10\,000 - \log_{10}100$ **(e)** $\log_3 81 - \log_3 27$ **(f)** $\log_4 64 - \log_4 16$

5 (a) $\dfrac{\log_2 32}{\log_2 4}$ is equal to:

A $\dfrac{5}{2}$ B 3 C $\log_2(8)$ D $\log_2(28)$

(b) $\log_a(mn)$ can be written as:

A $\log_a m \times \log_a n$ B $\log_a m^n$

C $\log_a m + \log_a n$ D $\log_a m - \log_a n$

Understanding

6 Simplify:

(a) $\log_{10}4 + \log_{10}250$ (b) $\log_2 64 + \log_2 32$

(c) $\log_2 4 + \log_2 8 - \log_2 16$ (d) $\log_3 27 + \log_3 81 - \log_3 243$

(e) $2\log_2 2 - \log_2\left(\dfrac{1}{8}\right) + \log_2 8$ (f) $\log_3 3 + 2\log_3\left(\dfrac{1}{9}\right) - 3\log_3 9$

7 Simplify:

(a) $\dfrac{\log_5 x^2}{\log_5 x}$ (b) $\dfrac{\log_3 x^5}{\log_3 x^6}$ (c) $\dfrac{\log_7 x^9}{\log_7 x^3}$

(d) $\dfrac{\log_2(x+4)^2}{\log_2(x+4)}$ (e) $\dfrac{3\log_5(2x-1)}{\log_5(2x-1)^2}$ (f) $\dfrac{\log_7(3x+4)^4}{3\log_7(3x+4)}$

Reasoning

8 Simplify:

(a) $\dfrac{\log_x 2}{\log_x 16}$ (b) $\dfrac{\log_x 25}{\log_x 5}$ (c) $\dfrac{\log_x 49}{\log_x 7^3}$

9 (a) If $\log_a y = \log_a 4 + 3\log_a x$, express y in terms of x.

(b) If $\log_a y = \log_a 11 + 5\log_a x$, express y in terms of x.

10 If $5\log_2 y = \log_2(x-1) + 2$, show that $y^5 = 4(x-1)$.

Open-ended

11 One of the questions on a test was:

Simplify $\dfrac{\log_2 x}{\log_2 \sqrt{x}}$

Danielle wrote: $\dfrac{\log_2 x}{\log_2 \sqrt{x}} = \dfrac{\log_2 x}{\log_2 x^{\frac{1}{2}}} = \dfrac{1}{2} \times \dfrac{\log_2 x}{\log_2 x} = \dfrac{1}{2}$

Is Danielle's answer correct? If not, what is the correct answer and how would you explain her mistake to Danielle?

Chapter review 12

D.I.Y. Summary

Key Words

conjugate surds	irrational numbers	rationalising the denominator
Distributive Law	logarithm	real numbers
entire surd	rational numbers	surds

Copy and complete the following using the words and phrases from the list, where appropriate, to write a summary for this chapter. A word or phrase may be used more than once.

1 The real number system contains _____ and _____.

2 $\sqrt{2}$ and π are examples of _____.

3 _____ are numbers that contain a radical or root sign and cannot be written as an exact number.

4 To simplify surd expressions involving brackets, we can use the _____.

5 When _____ of a fraction it is necessary to multiply both the numerator and the denominator by a surd or _____ in order to make the denominator rational.

6 The _____ $\sqrt{75}$ can be simplified to $3\sqrt{5}$.

7 When we write statements such as $3^4 = 81$ is equivalent to $\log_3(81) = 4$, we refer to 4 as the _____.

8 (a) Three examples of surds are _____.

(b) The conjugate of $\sqrt{3} + 1$ is _____.

(c) Write down the rules that are applied when multiplying numbers with an exponent, when dividing numbers with an exponent, when the exponent is 0, and when raising a number with an exponent to a power.

Fluency

1 Use a calculator to evaluate the following correct to three decimal places. Ex. 12.1

(a) $\sqrt{17}$ (b) $\sqrt[3]{25}$ (c) $5 \times 4\sqrt{3}$ (d) $3\sqrt{7} - 4\sqrt{2}$

2 Simplify each of the following. Ex. 12.2

(a) $4\sqrt{5} \times 3\sqrt{6}$ (b) $5\sqrt{7} \times 2\sqrt{7}$ (c) $\sqrt{\frac{64}{25}}$ (d) $\frac{18\sqrt{6}}{2\sqrt{3}}$

(e) $\sqrt{40}$ (f) $2\sqrt{125}$ (g) $3\sqrt{2} \times 5\sqrt{10}$ (h) $4\sqrt{\frac{75}{16}}$

3 Express each of the following as an entire surd. Ex. 12.2

(a) $6\sqrt{2}$ (b) $-3\sqrt{7}$

4 Simplify each of the following. Ex. 12.3

(a) $6\sqrt{2} - 8\sqrt{2} + 3\sqrt{2}$ (b) $5\sqrt{7} - 2\sqrt{3} - \sqrt{7} + \sqrt{3}$

(c) $4\sqrt{12} + 7\sqrt{27}$ (d) $6\sqrt{50} - 2\sqrt{24} + 4\sqrt{32} + \sqrt{54}$

5 Expand and simplify each of the following. **Ex.12.4**

(a) $\sqrt{5}(4\sqrt{3}-2\sqrt{5}+\sqrt{2})$ **(b)** $-6\sqrt{2}(3\sqrt{7}-\sqrt{2}+1)$

(c) $(2\sqrt{5}+9)(\sqrt{5}-2)$ **(d)** $(3\sqrt{7}+4\sqrt{2})^2$

6 Express each of the following in simplest form with a rational denominator. **Ex.12.5**

(a) $\dfrac{5}{\sqrt{2}}$ **(b)** $\dfrac{4\sqrt{3}}{\sqrt{6}}$ **(c)** $\dfrac{7}{2\sqrt{5}}$

7 Express each of the following in simplest form with a rational denominator. **Ex.12.5**

(a) $\dfrac{7}{3-\sqrt{5}}$ **(b)** $\dfrac{\sqrt{2}-2}{2\sqrt{3}+\sqrt{7}}$

8 Simplify each of the following. **Ex.12.6**

(a) $(3ab^5)^2$ **(b)** $\dfrac{(cd^2)^8}{(c^3d)^2}$ **(c)** $\left(\dfrac{t^2}{u^3}\right)^4\times\left(\dfrac{u^2}{t}\right)^7$

(d) $\dfrac{8x^5y^6}{6x^3y^2}\times\dfrac{3x^4y^7}{2x^5y}$ **(e)** $\dfrac{(m^4n)^3\times(m^2n^5)^2}{(mn)^4}$ **(f)** $\left(\dfrac{a^4b^2}{ab^3}\right)^2\div\left(\dfrac{a^3b}{a^5b^2}\right)^3$

9 Simplify, expressing your answers with positive indices. **Ex.12.6**

(a) $3a^{-4}\times 2a^{-2}$ **(b)** $(4xy^2)^{-2}\times(3x^2y)^3$ **(c)** $\dfrac{(m^{-3}n^2)^4}{2(mn^3)^{-2}}\times\left(\dfrac{m^2n^{-1}}{m^{-3}n^3}\right)^2$

10 Simplify each of the following. **Ex.12.7**

(a) $(64m^3n)^{\frac{1}{3}}$ **(b)** $\sqrt{p^6q}$

(c) $\dfrac{(a^6b^3)^{\frac{1}{2}}\times a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^3b^2}$ **(d)** $(a^2b^{-3})^{\frac{1}{2}}\times(a^{-4}b)^{\frac{5}{2}}$

(e) $\left(\dfrac{m^{\frac{1}{3}}n^{\frac{2}{3}}}{mn^{\frac{1}{2}}}\right)^2\times\dfrac{2m^3n^5}{\left(m^{\frac{1}{2}}n\right)^4}$ **(f)** $\left(\dfrac{3^2m^{-2}n^2}{mn}\right)^3\div\left(\dfrac{m^{\frac{1}{3}}n^{\frac{2}{3}}}{n^2m^{-3}}\right)^3$

11 Express each of the following in index form. **Ex.12.7**

(a) $(\sqrt{k})^5$ **(b)** $\sqrt[4]{x^3}$ **(c)** $\dfrac{1}{\sqrt[3]{m^2}}$

12 **(a)** $2\sqrt{6}(\sqrt{10}+3\sqrt{3})$ can be simplified to: **Ex.12.4**

A $2\sqrt{60}+5\sqrt{18}$ **B** $8\sqrt{15}+18\sqrt{2}$ **C** $4\sqrt{15}+18\sqrt{2}$ **D** $30\sqrt{2}+5\sqrt{18}$

(b) $8\times\sqrt{\dfrac{81}{64}}$ can be simplified to: **Ex.12.1**

A $\sqrt{9}$ **B** $24\times\sqrt{\dfrac{1}{8}}$ **C** $8\times\sqrt{\dfrac{9}{8}}$ **D** 9

13 Write each of the following in its equivalent logarithmic form. **Ex.12.8**

(a) $4^7=16\,384$ **(b)** $2^{10}=1024$ **(c)** $10^{-3}=\dfrac{1}{1000}$

14 Write each of the following in its equivalent index form.

Ex. 12.8

(a) $\log_2(16\,384) = 14$ **(b)** $\log_3 243 = 5$ **(c)** $\log_{10}\left(\frac{1}{100\,000}\right) = -5$

15 Simplify the following, expressing your answer as a basic numeral where possible.

Ex. 12.9

(a) $\log_2 8 + \log_2 64$ **(b)** $\log_2 16 - \log_2 64$

(c) $\log_3 5 + \log_3 10$ **(d)** $\log_5 110 - \log_5 11$

Understanding

16 A DVD has an area of 232π cm^2.

(a) What is its exact radius? (Leave your answer in surd form.)

(b) State the radius of the disc correct to two decimal places.

17 A gift box is to have its four sides covered by adhesive paper. It has the dimensions of length $8\sqrt{10}$ cm, width 18 cm and height $(5\sqrt{6} + 3\sqrt{7})$ cm.

(a) What are the approximate length and height of the box, correct to one decimal place?

(b) Using your answers to part **(a)**, calculate the area to be covered by adhesive paper, correct to one decimal place.

(c) Using the surd expression, find the area to be covered by the paper. Express your answer as a surd.

(d) Use a calculator to express your answer to part **(c)** correct to one decimal place.

(e) Compare your answers to parts **(b)** and **(d)**.

18 Solve for x:

(a) $4^x = 1024$ **(b)** $3^x = 2187$ **(c)** $2^x = \frac{1}{64}$

(d) $\log_2(x) = -4$ **(e)** $\log_3(6561) = x$ **(f)** $\log_5\left(\frac{1}{625}\right) = x$

19 Simplify each of the following, expressing your answer as a basic numeral where possible.

(a) $\log_{10} 2 + \log_{10} 500$ **(b)** $\log_3 81 - \log_3 27 + \log_3 9$

(c) $\dfrac{\log_5 x^4}{2\log_5 x}$ **(d)** $\dfrac{\log_8 (5x-7)^3}{4\log_8 (5x-7)}$

Reasoning

20 **(a)** Find the exact side length of a square with an area of 50 cm^2.

(b) Find the exact side length of an equilateral triangle with an area of 40 cm^2.

(c) Find the exact circumference of a circular swimming pool with an area of 75π m^2.

21 If $\left(\frac{4 - 3\sqrt{5}}{5 + 2\sqrt{2}}\right)a = \frac{5 - 2\sqrt{2}}{4 + \sqrt{5}}$, find the value of a, expressing your answer with a rational denominator.

22 **(a)** If $\log_a y = \log_a 7 + 3\log_a x$, express y in terms of x.

(b) If $\log_a y = \log_a 5 - 2\log_a x$, express y in terms of x.

Mixed review D

Fluency

1 An investment of \$5000 is made at 3.5% p.a. for 6 years. Find the difference in interest earned, to the nearest cent, if interest is compounded monthly compared to annually. Ex.1.2

2 Find the area and perimeter, correct to one decimal place, of a semicircle with a diameter of 9.6 cm. Ex.3.1

3 Write the following in simplest form. Ex.12.2 10A

(a) $\sqrt{3} \times \sqrt{8}$ (b) $\sqrt{3} \times 3\sqrt{12}$ (c) $\sqrt{2} \times \sqrt{18}$

(d) $2\sqrt{5} \times 4\sqrt{10}$ (e) $2\sqrt{6} \times 3\sqrt{10}$ (f) $3\sqrt{6} \times 7\sqrt{12}$

4 Simplify the following. Ex.12.2 10A

(a) $\sqrt{80}$ (b) $3\sqrt{240}$ (c) $7\sqrt{525}$

5 Solve the following equations. Ex.4.3

(a) $\frac{3a+4}{7} = -2$ (b) $\frac{4a+3}{2} = \frac{2a}{5}$ (c) $\frac{3a-2}{3} = \frac{4a-5}{2}$

6 Express each of the following in simplest form with a rational denominator. Ex.12.5 10A

(a) $\frac{4\sqrt{3}}{\sqrt{5}}$ (b) $\frac{2}{5-\sqrt{3}}$ (c) $\frac{\sqrt{2}}{\sqrt{3}+\sqrt{5}}$

7 Factorise each of the following expressions. Ex.2.2

(a) $k(k-3)-(k-3)$ (b) $3t(t+2)-5(t+2)$

8 Simplify the expressions below: Ex.12.6 10A

(a) $\frac{4a^4b}{9a^{-5}b^3} \times \frac{3a^{-2}b^5}{2ab^{-1}}$ (b) $\frac{(a^4b^2)^2}{(a^2b^3)^2} \times \frac{(a^{-3}b)^3}{(a^5b^2)^{-3}}$ (c) $\frac{k^{-8}}{n^4} \div \frac{k^9}{n}$

9 Find the value of each pronumeral, correct to two decimal places. Ex.5.3

(a) $\cos a = 0.1314$ (b) $\tan b = 2.621$ (c) $\sin c = 0.5231$

10 Find the gradient of the line joining: Ex.4.2

(a) points (1, 7) and (4, 14) (b) points (3, 6) and (7, -2).

11 Find the mean for each of the following grouped data frequency tables. Give your answers correct to two decimal places. Ex.7.7 10A

(a)

Class interval	Frequency
20–<25	8
25–<30	6
30–<35	3
35–<40	2
40–<45	5
45–<50	5

(b)

Class interval	Frequency
111–115	12
116–120	15
121–125	17
126–130	19
131–135	13
136–140	11

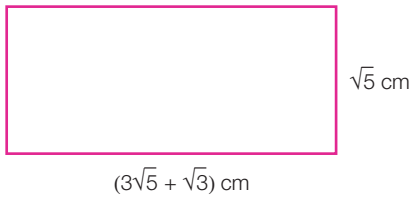
12 Sketch the graphs of each of the following, clearly showing the turning point: Ex. 8.9 10A

(a) $y = (x + 3)(x - 2)$ **(b)** $y = x^2 + 2x + 5$ **(c)** $y = 2x^2 - 4x$

13 For the rectangle shown: Ex. 12.3 10A

(a) State its perimeter in simplest form.

(b) State its area in simplest form.



14 Write a definition for each of the following quadrilaterals. Ex. 8.5

(a) rectangle **(b)** rhombus **(c)** kite

15 Two dice, both numbered 1 to 6, are rolled at the same time. Find the probability that the sum of the two faces showing will be: Ex. 11.2

(a) 4 **(b)** 6 **(c)** greater than 9.

16 Without using a calculator, determine which is the larger of the following pairs, placing Ex. 12.6 10A
<, > or = between the numbers below.

(a) 3^{-5} _____ 5^{-3} **(b)** $9^{\frac{1}{2}}$ _____ 9^{-1} **(c)** $8^{-\frac{1}{3}}$ _____ $\dfrac{2^{-3}}{2^{-2}}$

17 On a Venn diagram show the region described by not A and B and not C. Ex. 11.2

18 Describe what is needed for an equation to be a quadratic equation. Ex. 2.3

Understanding

19 Leonard and Faye want to develop a vegetable garden in their backyard. They have 20 m of chicken wire and enough posts to erect a fence around the four sides of the rectangular garden.

(a) If the length of the rectangular garden is x m, find an expression (in terms of x) for the width of the graden.

(b) Show that the area of the garden, A m^2, is given by the quadratic expression $A = -x^2 + 10x$.

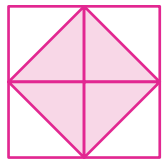
(c) Sketch the graph of the area of the vegetable garden, A m^2, against the length of the garden, x m. (Show length on the horizontal axis.)

(d) What should be the dimensions of the garden if the area is to be as large as possible?

(e) What is the area of the largest vegetable garden that Leonard and Faye can make?

Reasoning

20 A square with side lengths $\sqrt{2}$ has been drawn using four squares with side lengths of 1 cm, as shown. What is the perimeter of a square that is made by joining the diagonals of four squares with side lengths of 2 cm? 10A



21 **(a)** Draw an isosceles triangle with base length 10 cm and height 10 cm. 10A

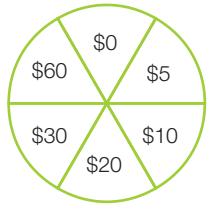
(b) Show that the length of the equal sides in the triangle in part **(a)** is an irrational number.

Challenge D

1 In a dice game, a player rolls two dice. His score is the larger of the two numbers on the dice. For example, if he rolls 3 and 5, his score is 5, and if he rolls 4 and 4, his score is 4. What is the probability that his score is 3 or less?

A $\frac{5}{36}$ **B** $\frac{1}{6}$ **C** $\frac{7}{36}$ **D** $\frac{1}{4}$

2 A prize wheel is divided into 6 equal sectors. Each time the wheel is spun it stops on one of the sectors, resulting in a prize of the amount shown.



(a) On the first spin \$5 is won. What is the probability of winning \$5 on the second spin?

(b) The wheel is spun a total of 50 times and the results recorded in the table.

Prize	Frequency	Relative frequency
\$0	8	0.16
\$5		0.14
\$10	9	
\$20	10	0.20
\$30	8	0.16
\$60	8	0.16
Total	**50**	**1**

(i) Copy and complete the table by writing in the two missing pieces of information.

(ii) How much prize money was won?

(c) To play this game you have to pay \$20 per throw.

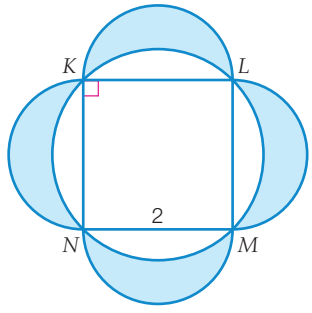
(i) If each outcome was equally likely and you spun the wheel 48 times, would you expect to make a profit?

(ii) If you spun the wheel 50 times and the results were as shown in the table in part **(b)**, would you make a profit?

3 A pyramid has a square base with an area of 1440 cm^2. Each of the pyramid's triangular faces is identical and has an area of 840 cm^2. The height of the pyramid is:

A $20\sqrt{3}$ cm **B** 40 cm **C** $30\sqrt{2}$ cm **D** $4\sqrt{145}$ cm

4 The square *KLMN*, with side length 2 units, is inscribed in a circle as shown. Using each side of the square as a diameter, semicircular arcs are drawn. The total area of the shaded region outside the circle and inside the semicircles is:



A 4 units^2 **B** π units^2

C 2π units^2 **D** $(2\pi - 4)$ units^2

5 Find the value of x in each of the following.

(a) $x^{-1} = 3^{-1} + 4^{-1}$ **(b)** $x + 27^{\frac{1}{3}} = 125^{\frac{1}{3}}$ **(c)** $2^x = 2(16^{12}) + 2(8^{16})$

(d) $2^{2x-4} = 8$ **(e)** $16^x = 2^{x+5} - 2^{x+4}$

Answers

Worked solutions and answers to ***all*** activities appear in the Teacher Companion.

Chapter 1

Recall 1 (p. 3)

1 (a) 0.17 (b) 0.06 (c) 0.192
(d) 0.048 (e) 0.0201 (f) 0.0025

2 (a) \$3000 (b) \$600 (c) \$33 250 (d) \$3375

3 (a) (i) 80 (ii) 60 (iii) 50 (iv) 120
(b) (i) 30 (ii) 42 (iii) 54 (iv) 15

4 (a) 49 (b) 8 (c) 1 000 000
(d) 0.36 (e) $\frac{1}{32}$ (f) 1.4641

5 (a) 0.5 (b) 0.25 (c) 8.75
(d) 0.5 (e) 0.25 (f) 0.4

Exercise 1.1 (p. 8)

1 (a) (i) \$3200 (ii) \$43 200
(b) (i) \$2700 (ii) \$32 700
(c) (i) \$10 500 (ii) \$40 500
(d) (i) \$24 000 (ii) \$64 000
(e) (i) \$281 250 (ii) \$431 250
(f) (i) \$468 000 (ii) \$708 000
(g) (i) \$22 275 (ii) \$322 275
(h) (i) \$27 912.50 (ii) \$377 912.50
(i) (i) \$6175 (ii) \$19 175
(j) (i) \$6794 (ii) \$22 594
(k) (i) \$28 125 (ii) \$73 125
(l) (i) \$13 860 (ii) \$49 860
(m) (i) \$3375 (ii) \$13 375
(n) (i) \$10 200 (ii) \$26 200

2 (a) \$4000 (b) \$8000 (c) 14% p.a.
(d) 11% p.a. (e) 10 years (f) 12 years

3 (a) (i) \$43 200 (ii) \$3200
(b) (i) \$32 700 (ii) \$2700
(c) (i) \$42 076.55 (ii) \$12 076.55
(d) (i) \$71 834.25 (ii) \$31 834.25
(e) (i) \$914 750.94 (ii) \$764 750.94
(f) (i) \$1 587 447.88 (ii) \$1 347 447.88
(g) (i) \$322 348.70 (ii) \$22 348.70
(h) (i) \$378 011.55 (ii) \$28 011.55
(i) (i) \$20 465.10 (ii) \$7465.10
(j) (i) \$23 770.16 (ii) \$7970.16
(k) (i) \$82 809.23 (ii) \$37 809.23
(l) (i) \$52 229.19 (ii) \$16 229.19
(m) (i) \$13 695.26 (ii) \$3695.26
(n) (i) \$28 828.16 (ii) \$12 828.16

4 Lily (a) (i) \$5830.96 (ii) \$5250
(b) Compound interest is \$580.96 higher.
Jarrah (a) (i) \$39 939.82 (ii) \$29 925
(b) Compound interest is \$10 014.82 higher.

5 (a) \$9419.20 (b) \$9240

6 C 7 1.2% p.a. 8 \$129.17

9

Amount (*A*)	Principal (*P*)	Interest rate p.a. (*R*)	Time in years (*T*)
\$1359.81	\$500	8%	13
\$43 742.08	\$24 600	12.2%	5
\$1 048 209.60	\$1 000 200	4.8%	1
\$151.71	\$84	3%	20
\$60.50	\$50	10%	2

10 (a) (i) simple interest (ii) 40% p.a., \$12
(b) (i) compound interest (ii) \$994
(c) (i) compound interest (ii) \$25 729
(d) (i) compound interest (ii) \$51 452
(e) (i) simple interest (ii) \$1250
(f) (i) simple interest (ii) 12% p.a.

11 A

12 (a) 100 years (b) 47 years

13 5% p.a.

Open-ended – Sample answers

14 \$12 000 invested at 11.8% p.a. would give a final value of \$14 999 (clearly within the \$100 buffer!).

15 $R \times T = 0.8$, e.g. 16% p.a. for 5 years, 15% p.a. for 5 years and 4 months, 12% p.a. for 6 years and 8 months.

Exercise 1.2 (p. 16)

1 (a) \$20 793.60 (b) \$20 972.74 (c) \$21 068.94

2 (a) \$9528.13 (b) \$9552.42

3 B 4 C

5 (a) \$64 656.50 (b) \$65 731.65 (c) \$66 472.15

6 \$647.54 7 \$13 499.14

8 \$5468.12 and \$5459.91; Martina is better off with the 4.5% p.a. compounded quarterly. The difference is \$8.21.

9 (a) (i) \$3215.09 (ii) \$2879.21
(iii) \$2545.44 (i) gives the best return

(b) No

(c) (i) $I \approx 16\%$ of P (ii) $I \approx 14\%$ of P (iii) $I \approx 13\%$ of P

Open-ended – Sample answers

10 The statement means that if compound interest is calculated more frequently (e.g. every month instead of every quarter), it will result in a greater amount of interest. Quarterly compounding gives interest of $4859.47. (48.6%) Monthly compounding gives interest of $4898.46. (49.0%) Weekly compounding gives interest of $4913.66. (49.1%)

11 Using the suggested example, simple interest is $600 and interest compounded annually is $630.50. The principal gets bigger after each time period.

Exercise 1.3 (p. 23)

1 (a) $5996 (b) $5437 (c) $5456.76 (d) B

2 (a) 10.2% p.a. (b) 3.24% p.a. (c) 2.6% p.a. (d) B

3 (a) 12 years (b) 10 years (c) 4.2 years

4 (a) $5956.42 (b) $956.42

(c) only marginally better, $I = \$956.46$

(d) $956.42 (e) 4.47% p.a.

(f) At least 4 years, as it would be worth $5788.13 at the end of the third year and $6077.53 at the end of the fourth year.

5 2

6 (a) every 4 months (b) 9.3% p.a.

(c) monthly compounding

7 At least every 4 months.

Open-ended – Sample answers

8 Rounding error: $\frac{0.065}{4} = 0.01625$, $P = \$18\,751.14$

9 11% p.a. with annual compounding, 10.5% p.a. compounding half-yearly, 10.2% p.a. compounding daily

Half-time 1 (p. 26)

1 B 2 $8673.33 3 $8889.26 4 200%

5 (a) $50 649.49 (b) $51 824.82 (c) $52 059.90

6 9

7 (a) $10 866 (b) $9000

(c) Compound interest is $1866 more than simple interest.

8 17.27% p.a.

Exercise 1.4 (p. 30)

1 (a) 7% p.a. (b) 9.4% p.a. (c) 13.52% p.a.

(d) 21.73% p.a. (e) 17.88% p.a. (f) 6.45% p.a.

2 (a) (i) $R = 7.3\%$ p.a., $n = 5$ (ii) 7.52% p.a.

(b) (i) $R = 15.5\%$ p.a., $n = 12$ (ii) 16.65% p.a.

(c) (i) $R = 21\%$ p.a., $n = 4$ (ii) 22.71% p.a.

(d) (i) $R = 16\frac{1}{4}\%$ p.a., $n = 3$ (ii) 17.15% p.a.

(e) (i) $R = 20\frac{1}{3}\%$ p.a., $n = 365$ (ii) 22.54% p.a.

(f) (i) $R = 5.25\%$ p.a., $n = 2$ (ii) 5.32% p.a.

3 B

4 (a) Q Bank: 15.87% p.a., D Bank: 16.06% p.a.

(b) D Bank

5 (a) 22.13% p.a. (b) 22.26% p.a.

(c) Personal loan because the effective rate is lower.

6 (a) 25.85% p.a. (b) online

7 (a) 15% p.a. compounded daily—higher rate and more frequent compounding

(b) 10.5% p.a. compounded weekly—rate is the same so more frequent compounding is better

(c) 12% p.a. compounded daily—compounding frequency is the same so higher rate is better

(d) 15% p.a. compounded annually for 2 years—compounded beats flat rate over 2 years

8 Yes. In each case choose the other option because for loans we want the lowest effective rate.

9 Option C has the highest effective rate but all the options look too good to be true.

10 (a) car company loan (b) flat rate

Open-ended – Sample answers

11 1-year loan of $10 000 at 6% p.a. compounded daily. Nominal rate is 6% p.a. The interest is compounded, not simple. However, the same amount of interest will be paid if $10 000 is borrowed for a year at a simple interest rate of 6.18% p.a. Either way nominal isn't the same as simple.

12 42% p.a. compounded half-yearly, 40% p.a. compounded quarterly

Exercise 1.5 (p. 39)

1 (a) (i) $10 500 (ii) $19 500

(b) (i) $28 500 (ii) $19 000

(c) (i) $100 000 (ii) No value

(d) (i) $772 225 (ii) $17 775

(e) (i) $60 900 (ii) $84 100

(f) (i) $107 250 (ii) $192 750

(g) (i) $4928.13 (ii) $42 571.87

(h) (i) $76 159.13 (ii) $76 540.87

2 (a) (i) $20 870.65 (ii) $9 129.35

(b) (i) $25 667.10 (ii) $21 832.90

(c) (i) $35 848.59 (ii) $64 151.41

(d) (i) $290 950.49 (ii) $499 049.51

(e) (i) $93 037.95 (ii) $51 962.05

(f) (i) $207 293.23 (ii) $92 706.77

(g) (i) $42 285.67 (ii) $5214.33

(h) (i) $83 275.25 (ii) $69 424.75

3 (a) (i) $6000 (ii) $2/hour

(iii) $20 (iv) $6130

(b) (i) \$1000 (ii) 50c/hour

(iii) \$500 (iv) \$500

4 \$526.50 5 B 6 \$2240

7 Dep 1st year = \$9000, C dep 1st year = \$9000

Written-down value 1st year = \$36 000

Dep 2nd year = \$9000, C dep 2nd year = \$18 000

Written-down value 2nd year = \$27 000

Dep 3rd year = \$9000, C dep 3rd year = \$27 000

Written-down value 3rd year = \$18 000

8 C

9 \$2923.24. Reducing balance depreciation is less so written-down value after 4 years is higher than under straight-line depreciation.

10 \$485.82

11 Written-down value 1st year = \$36 000, C dep 1st year = \$9000 Dep 1st year = \$9000

Written-down value 2nd year = \$28 800, C dep 2nd year = \$16 200 Dep 2nd year = \$7200

Written-down value 3rd year = \$23 040, C dep 3rd year = \$21 960 Dep 3rd year = \$5760

1st year is the same as it was for linear. After that, the depreciation slows down and depreciation for each year is less than the year before. After the 1st year, written-down value is also better than it was for linear.

12 C

13 (a) D (b) A

14 (a) \$3916.67 (b) \$6083.33

15 \$9943.11 16 \$2415

17 Written-down value after the 1st year is \$3353.75

Written-down value after the 2nd year is \$2768.75

18 (a) (i) \$50 000 (ii) \$36 000 (iii) \$25 800 (iv) \$18 390

(b) after 21 years

Open-ended – Sample answers

19 5% p.a. and 14 years, 10% p.a. and 7 years, 20% p.a. and 3.5 years, 70% p.a. and 1 year. Must have $R \times T = 0.7$

20 Any values that satisfy $0.3 = (1 - R)^T$; e.g. 70% p.a. and 1 year, 45.23% p.a. and 2 years, 21.4% p.a. and 5 years

Exercise 1.6 (p. 46)

1 (a) 20 524 (b) 15 200 (c) 1171

(d) \$875 855 (e) 242 057 (f) 1062

2 (a) 42 (b) \$896 (c) 6218

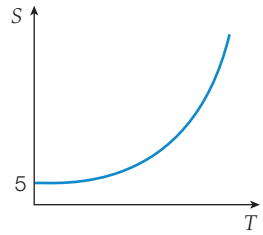
(d) 164 870 (e) 1534 (f) 4 566 189

3 (a)

4	5	6
405	1 215	3 645

(b) $S = 5 \times 3^T$ (c) 388 564

(d)



(e) after 2.5 years

4 (a) C (b) B

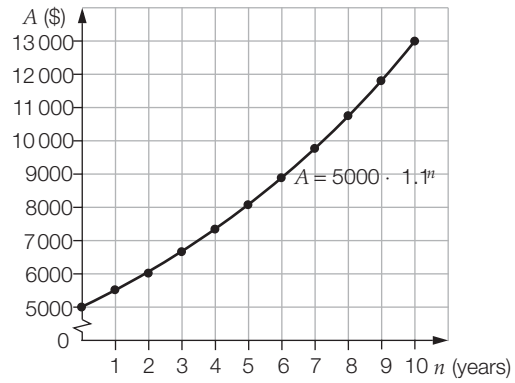
5 (a) \$5500 (b) 110% (c) 1.1 (d) \$6050 (e) 1.1

(f)

Year	Balance (\$)
0	5 000
1	5 500
2	6 050
3	6 655
4	7 320.50
5	8 052.55
6	8 857.51
7	9 743.59
8	10 717.94
9	11 789.74
10	12 968.71

(g) $A = 5000 \times 1.1^n$

(h)



(i) \$6979.82 (j) after 4 years, 11 months

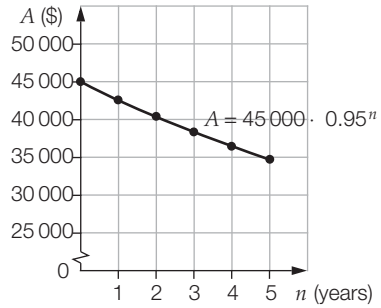
(k) after 7 years, 3 months

6 (a) 95% (b) 0.95 (c) $A = 45\,000 \times 0.95^n$

(d)

Year	Balance (\$)
0	45 000
1	42 750
2	40 612.50
3	38 581.88
4	36 652.78
5	34 820.14

(e)



(f) \$39 584.17

(g) after 4 years, 4 months

7 (a) $I = 10^6 \times 0.5^T$ (b) 95c

(c) When she's 7 years old.

8 (a) $A = 5\left(\frac{2}{3}\right)^n$ (b) 7 g

Open-ended – Sample answers

9 $T = 6$ years, $R = 200\%$ p.a., $T = 3$ years, $R = 800\%$ p.a., $T = 20$ years, $R \approx 39\%$ p.a.

10 There is no error in the report. Janet has assumed a linear model of decay. Radioactive material follows an exponential decay model. We just need to show that

$$0.1 = \left(1 - \frac{0.15}{60}\right)^{850}$$

Chapter review 1 (p. 49)

1 (a) D (b) D

2 (a) \$25 913.02 (b) \$26 221.74 (c) \$26 438.68

3 \$30 000

4 (a) 8.7% p.a. (b) 9.5% p.a.

(c) 4% p.a. (d) 18.7% p.a.

5 (a) \$39 000 (b) \$41 741.30

6 417 7 11.5% p.a. 8 \$59 193 9 B

10 8.1% p.a. 11 B

12 (a) If the prime cost of a photocopier is \$10 800 and it has a scrap value of \$500, then the total possible depreciation is \$10 300. If it has an expected life of 206 000 pages, then the unit cost depreciation is \$0.05 per page. After it copies 125 000, the depreciation is \$6250 and the written-down value is \$4550.

(b) If the prime cost of an item is \$52 500 and it has scrap value of \$750 then the total possible depreciation is \$51 750. If it has an expected life of 5000 days, then the unit cost depreciation is \$10.35 per day. After it does a years' work, the depreciation is \$3777.75 and the written-down value is \$48 722.25.

13 63 680 14 20.4°C 15 4.2 years

16 (a) 11% p.a. compounded quarterly—lower nominal rate and less frequent compounding.

(b) 10% p.a. compounded monthly—same nominal rate but less frequent compounding.

(c) 17% p.a. compounded daily—same compounding period but lower rate.

(d) 12% p.a. flat rate for 2 years—after 1 year, flat rate is cheaper than compounded.

17 77.8% per month

18 10% per month

Chapter 2

Recall 2 (p. 53)

1 (a) 3 (b) 2 (c) 4

2 (a) 4 (b) -6 (c) 1 (d) -12

3 (a) $3a + 5b$ (b) $24x - 8y$

4 (a) $3x + 6$ (b) $5x - 15$ (c) $5x - 8$

5 (a) $-2x - 6$ (b) $-8x + 40$ (c) $-x - 2$

6 (a) $9x^2$ (b) $4y^2$ (c) $49k^2$

7 (a) $14x^2$ (b) $-8p^2$ (c) $2y^2$

8 (a) $x = \frac{y-3}{2}$ (b) $x = 2ac + 4b$ (c) $x = \frac{3s + at^2}{5}$

9 (a) 3, 7 (b) 2, 12 (c) -5, -4

(d) -3, 6 (e) -10, 2 (f) -8, 5

10 (a) $\frac{11}{12}$ (b) $\frac{11}{24}$ (c) $\frac{5a}{3}$

11 (a) $\frac{1}{6}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$

Exercise 2.1 (p. 57)

1 (a) $4x + 12$ (b) $8x + 16$ (c) $12x + 132$

(d) $12x - 2$ (e) $14x - 35$ (f) $5x^2 - 4x$

(g) $-2x - 2$ (h) $-x^2 + 7x$ (i) $-3x^2 + 6x$

(j) $-3x^2 - 18x$ (k) $-2x^2 + 18x$ (l) $-4x^2 - 8x$

(m) $x^2 + 5x + 6$ (n) $2x^2 + 5x + 2$ (o) $5x^2 + 17x + 6$

(p) $2x^2 - 11x - 30$ (q) $10x^2 + 13x + 32$ (r) $12x^2 - 16x - 6$

2 (a) $x^2 + 4x + 3$ (b) $x^2 + 6x + 8$ (c) $x^2 + 9x + 20$

(d) $x^2 + 5x - 24$ (e) $x^2 - 2x - 63$ (f) $x^2 + 4x - 21$

(g) $x^2 - 8x + 7$ (h) $x^2 - 9x + 20$ (i) $x^2 - 12x + 27$

(j) $2x^2 - x - 6$ (k) $4x^2 + 7x - 15$ (l) $7x^2 - 27x - 4$

(m) $6x^2 - 17x + 5$ (n) $-5x^2 + 38x - 21$ (o) $-18x^2 + 27x - 10$

3 (a) $4 - x^2$ (b) $49 - x^2$ (c) $36 - x^2$

(d) $4x^2 - 1$ (e) $9x^2 - 16$ (f) $25 - 9x^2$

(g) $x^2 + 12x + 36$ (h) $x^2 + 22x + 121$ (i) $64 + 16x + x^2$

(j) $x^2 - 6x + 9$ (k) $x^2 - 14x + 49$ (l) $x^2 - 24x + 144$

(m) $4x^2 + 20x + 25$ (n) $9x^2 + 12x + 4$ (o) $49x^2 - 28x + 4$

4 (a) $x^3 + 10x^2 + 24x$ (b) $x^3 + 4x^2 - 5x$

(c) $x^3 - 3x^2 - 4x$ (d) $-x^3 + x^2 + 6x$

(e) $-3x^3 + 9x^2 - 6x$ (f) $-4x^3 + 20x^2 + 96x$

(g) $x^3 + 11x^2 + 38x + 40$ (h) $x^3 - 4x^2 - 7x + 10$

(i) $x^3 - 11x^2 + 36x - 36$ (j) $x^3 + 5x^2 - 9x - 45$

(k) $x^3 + 7x^2 - 4x - 28$ (l) $2x^3 + x^2 - 32x - 16$

(m) $x^3 + 5x^2 - 8x - 48$ (n) $x^3 + 3x^2 - 4$

(o) $x^3 - 7x^2 - 24x + 180$

5 B

6 (a) $8x^2 + 14xy + 15y^2$ (b) $35x^2 + 44xy + 12y^2$

(c) $15x^2 - 44xy + 32y^2$ (d) $4x^2 + 28xy + 49y^2$

(e) $49x^2 + 84xy + 36y^2$ (f) $25x^2 + 80xy + 64y^2$

(g) $16x^2 - 8xy + y^2$ (h) $36x^2 - 60xy + 25x^2$

(i) $81x^2 - 126xy + 49y^2$ (j) $25y^2 - x^2$

(k) $9x^2 - y^2$ (l) $4x^2 - 9y^2$

(m) $6x^3 + 7x^2 - 5x$ (n) $15x^3 - 8x^2 - 12x$

(o) $24x^3 + 78x^2 + 63x$ (p) $-15x^3 + 8x^2 + 16x$

(q) $-30x^3 - 58x^2 - 24x$ (r) $-24x^3 + 76x^2 - 60x$

7 (a) $(x - 4)(3x + 1) = 3x^2 - 11x - 4$

(b) $(2x + 3)(2x + 3) = 4x^2 + 12x + 9$

(c) $\frac{1}{2}(x + 5)(x - 5) = \frac{1}{2}(x^2 - 25)$ or $\frac{1}{2}x^2 - \frac{25}{2}$

8 (a) $x^3 + (a + b)x^2 + abx$

(b) (i) $a + b$ (ii) ab

(iii) $a + b = -4$ and $ab = -12$. Using the expansion from **(a)** this shows that $x(x - 6)(x + 2) = x^3 - 4x^2 - 12x$.

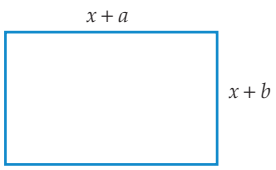
(c) (i) $a = 25, b = -4$ (ii) $x^3 + 21x^2 - 100x$

9 (a) $(x - 2)(x)(x + 2)$

(b) $x^3 - 4x$

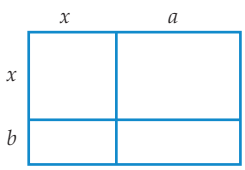
(c) It is easier to have x as the second number as the other two expressions, $(x - 2)$ and $(x + 2)$, form a difference of two squares which is simpler to multiply.

10 (a)



(b) $A = (x + a)(x + b)$ (c) $A = x^2 + ax + bx + ab$

(d)



(e) $x^2 + ax + bx + ab$. This is the same as the answer in part **(c)**.

11 (a) $x + 5$ and $x - 2$

(b) $A = (x + 5)(x - 2) = x^2 + 3x - 10$

(c) x^2 and $5x$

(d) $2x + 10$

(e) $A = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$

(f) The answers are the same.

12 (a) $1296 - x^2$ cm^2 (b) $(36 - x)(36 + x)$ (c) 1260 cm^2

13 (a) $x^2 - 2ax$ (b) $x^2 - 2ax$

(c) They are equivalent.

(d) Rectangular lawn area $= x(x - 2a)$, square lawn area $= (x - a)^2 - a^2$. They will cost equal amounts to turf as we have shown that these expressions are equivalent.

(e) \$19 092

14 (a) $55 - y$ m (b) $x(55 - y)$ m^2

(c) $55x$ m^2 (d) $(x - 30)(55 - y)$ m^2

(e) $55x - xy - 1650 + 30y$ (f) $45x - 1350$ m^2

(g) $10x + 1350$

15 (a) $(x - 2)^2(x + 3)$ (b) 28 cm^3

(c) When $x = 5$, dimensions are 3 cm, 8 cm and 3 cm, so the volume $= 3 \times 8 \times 3 = 72$ cm^3.

(d) $x^3 - x^2 - 8x + 12$; 72 cm^3

(e) height and width 3 cm, length 8 cm

(f) $x = 3$ (g) 6 cm^3

16 If $(x^2 + a^2)^3 = 0$ then $(x^2 + a^2) = 0$ and $x^2 = -a^2$, which can only be true if x and a are both equal to zero. No other values of x and a are possible as you cannot have a square number equal to a negative number.

Open-ended – Sample answers

17 Examples: $a = 2, b = 1, c = 0, d = 1$; $a = 1, b = 0, c = 1, d = 1$

18 (a) $x^2 + y^2$

(b) Examples: $x = 3, y = 4$; $x = 5, y = 12$

19 (a) -4, -5 (b) 4, 5 (c) 2, -3

Exercise 2.2 (p. 63)

1 (a) $4(x + 2)$ (b) $3(6x - 5)$ (c) $3(8x - 3)$

(d) $x(y + 7)$ (e) $3k(4x + 7)$ (f) $4b(8a + 5)$

(g) $7y(x^2 + y)$ (h) $3x(3x + y)$ (i) $8m(6m + 1)$

(j) $-x(6 + x)$ (k) $-x(4 - x)$ (l) $-2a(12a - 5)$

(m) $6ab(1 - 3a)$ (n) $8xy(5x + 2y)$ (o) $2cd(5d - 1)$

2 (a) $(k + 1)(x + 4)$ (b) $(x - 4)(a + 6)$

(c) $(n - 7)(m + 2)$ (d) $(p + 3)(x - 9)$

(e) $(q - 2)(n - 7)$ (f) $(u - 4)(x - 5)$

(g) $(c + 5)(d + 1)$ (h) $(d + 2)(a - 1)$

(i) $(m + 7)(k - 1)$ (j) $(x - 3)(x + 5)$

3 (a) $(y + 5)(x + 3)$ (b) $(b + 2)(a + 7)$

(c) $(n + 4)(m + k)$ (d) $(y + 3)(x - 1)$

(e) $(3 + c)(a - b)$ (f) $(m + 2)(k - 4)$

(g) $(5b - 4)(3a - 2)$ (h) $(2b - 5)(3a - 2)$

(i) $(2n - 3)(4m - 3)$ (j) $(v - a)(x - y)$

(k) $(q - n)(p - r)$ (l) $(h - d)(g - f)$

(m) $(n + t)(m^2 + p)$ (n) $(ab - c)(a + b)$

(o) $(dk - r)(d - 1)$ (p) $(9y - 5)(3x + 2)$

(q) $(4a - 5)(9 - c)$ (r) $(5v - 3)(5w + 1)$

(s) $(8a^2 - 3b)(4ac - 5)$ (t) $(3d - 5f)(7d - 4h)$

(u) $(mr - 5)(4p + mn)$

4 (a) True (b) False (c) False

(d) True (e) False (f) False

5 D 6 B 7 A

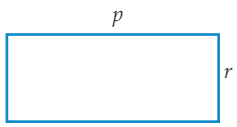
8 $12(2y - 3x)$ cents

9 $8x - x^2 = x(8 - x)$

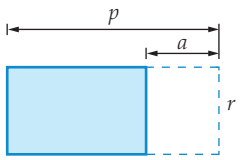
10 (a) $a(p + 5) + b(p + 5)$ (b) $(a + b)(p + 5)$

(c) $3a(p + 5)$

11 (a)

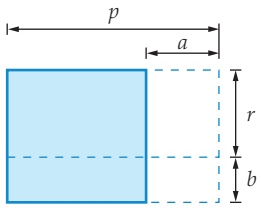


(b)



Area = $pr - ar$

(c)



Area added = $pb - ab$

(d) Area = $pr - ar + pb - ab$

(e) Area = $(r + b)(p - a)$

(f) part (e) is the factorised form of part (d)

12 (a) $\frac{2\pi r^3}{3} + \frac{\pi r^2 h}{3}$ (b) $\frac{\pi r^2}{3}(2r + h)$ (c) 150.82 cm^3

(d) Cone volume is $\frac{\pi r^2(2r)}{3} = \frac{2\pi r^3}{3}$, which equals the half-sphere volume of the ice-cream.

Open-ended – Sample answers

13 (a) $2x^2 + 4xy$ cm^2 (b) $2x(x + 2y)$

(c) $x = 12, y = 3; x = 8, y = 4$

Exercise 2.3 (p. 68)

1 (a) $(x + 2)(x + 4)$ (b) $(x - 3)(x + 6)$

(c) $(x - 1)(x + 7)$ (d) $(x + 5)(x - 2)$

(e) $(x - 4)(x - 3)$ (f) $(x - 4)(x + 3)$

(g) $(x + 4)(x - 2)$ (h) $(x + 1)(x - 3)$

(i) $(x - 8)(x - 2)$

2 (a) $(x + 1)(x + 2)$ (b) $(y + 2)(y + 7)$

(c) $(a - 4)(a - 6)$ (d) $(k + 3)(k + 8)$

(e) $(m - 5)(m + 3)$ (f) $(x - 6)(x + 5)$

(g) $(p - 2)(p - 8)$ (h) $(d - 1)(d - 4)$

(i) $(y - 3)(y - 2)$ (j) $(m - 6)(m + 2)$

(k) $(c - 7)(c - 3)$ (l) $(g + 7)(g - 5)$

(m) $2(y + 3)(y + 4)$ (n) $3(m + 2)(m + 4)$

(o) $5(x + 4)(x - 5)$ (p) $-4(a + 3)(a - 1)$

(q) $-3(k + 3)(k - 7)$ (r) $-5(x + 7)(x - 3)$

3 (a) $(z - 5)(z - 3)$ (b) $(k + 6)(k - 1)$

(c) $(z + 3)(z + 7)$ (d) $(t + 4)(t - 5)$

(e) $(r + 6)(r - 5)$ (f) $(t - 11)(t - 2)$

(g) $(z + 6)(z - 7)$ (h) $(m - 5)(m - 4)$

(i) $(y + 12)(y - 5)$ (j) $(n - 7)(n + 4)$

(k) $(p + 12)(p - 3)$ (l) $(a - 6)(a - 8)$

4 C

5 $(2x + 4)$ cm

6 $(3x - 12)$ mm

7 (a) $x - 30$ (b) 120 cm (c) 190 cm

8 (a) $2w^2 - (6w - 4) = 2w^2 - 6w + 4$

(b) $(2w - 2)(w - 2) = 2(w - 1)(w - 2)$

(c) 1 m

Open-ended – Sample answers

9 $x^2 - x - 2 = (x + 1)(x - 2)$ and $x^2 + x - 42 = (x + 7)(x - 6)$

10 (a) $x + 8$ and $x - 3$ are the factors so possible dimensions could be 12 units by 1 unit or 14 units by 3 units.

(b) Area = $12 \times 1 = 12$ $units^2$ or Area = $14 \times 3 = 42$ $units^2$.

Exercise 2.4 (p. 73)

1 (a) $(p - 2)(p + 2)$ (b) $(d - 10)(d + 10)$

(c) $(3 - k)(3 + k)$ (d) $3(p - 5)(p + 5)$

(e) $5(4 - m)(4 + m)$ (f) $8(k - 2)(k + 2)$

(g) $(2y - 7)(2y + 7)$ (h) $(5 - 4x)(5 + 4x)$

(i) $(6m - 5n)(6m + 5n)$ (j) $(3x - 4)(3x + 4)$

(k) $(2 - 5a)(2 + 5a)$ (l) $(9d - 8)(9d + 8)$

2 (a) $(x - 1)(x + 5)$ (b) $(a - 6)(a + 4)$

(c) $(-k - 1)(k + 9)$ or $-(k + 1)(k + 9)$

(d) $(5 - p)(7 + p)$ (e) $4(2 - y)(y - 1)$

(f) $3(1 - x)(3x - 1)$ (g) $5(x + 3 - y)(x + 3 + y)$

(h) $7(a + 5 - b)(a + 5 + b)$ (i) $3(x - 6)(x - 2)$

3 (a) $(x - \sqrt{2})(x + \sqrt{2})$ (b) $(\sqrt{7} - p)(\sqrt{7} + p)$

(c) $(2k - \sqrt{5})(2k + \sqrt{5})$ (d) $(3p - \sqrt{2})(3p + \sqrt{2})$

(e) $3(a - \sqrt{11})(a + \sqrt{11})$ (f) $2(b - \sqrt{3})(b + \sqrt{3})$

4 (a) $(x - y + 1)(x + y + 3)$ (b) $-(2a + 5)$

(c) $(2k - m + 6)(2k + m)$ (d) $16x$

(e) $(x - 2)(3x + 4)$ (f) $(4x + 3)(6x - 11)$

(g) $(x + 2 - \sqrt{3})(x + 2 + \sqrt{3})$ (h) $(y - 3 - \sqrt{6})(y - 3 + \sqrt{6})$

(i) $(x + 8 - \sqrt{5})(x + 8 + \sqrt{5})$

(j) $(x - 5 - \sqrt{13})(x - 5 + \sqrt{13})$

(k) $(x - 1 - \sqrt{19})(x - 1 + \sqrt{19})$

(l) $(2x + 1 - \sqrt{2})(2x + 1 + \sqrt{2})$

5 (a) $(x + 3)(x + 1)$ (b) $(a - 4)(a + 9)$

(c) $(x - y)(x + y + 7)$ (d) $(3a + b)(3a - b + 2)$

(e) $(x - 4y)(x + 4y + 2)$ (f) $(2p + q)(2p - q - 3)$

6 (a) $(x+3)^2$ (b) $(x+2)^2$ (c) $(x-7)^2$
(d) $(x-4)^2$ (e) $(x+1)^2$ (f) $(x+6)^2$
(g) $(2x+5)^2$ (h) $(3x+4)^2$ (i) $(5x-1)^2$

7 B

8 D

9 A

10 (a) $8(m+5n)(m-5n)$ (b) $6(x-2r)(x+2r)$
(c) $5(a-3c)(a+3c)$ (d) $3(3a-4b)(3a+4b)$
(e) $2(6x-7y)(6x+7y)$ (f) $7(5t-2p)(5t+2p)$
(g) $6(3k-2m)(3k+2m)$ (h) $2(xy-4z)(xy+4z)$
(i) $3(pq-2n)(pq+2n)$

11 (a) $(4c+5d)(4c-5d+2)$ (b) $(k-8)(k+8+m)$
(c) $(9-7a)(9+7a-3d)$ (d) $(4-3y)(4+3y-4w)$
(e) $(5m+3n)(5m-3n-4)$ (f) $(12p-7r)(12p-q+7r)$

12 (a) $x^2-y^2=56$ (b) $(x-y)(x+y)=56$
(c) The difference between consecutive odd numbers is 2.
(d) $4(y+1)=56$ (e) $x=15, y=13$

13 (a) $\frac{1}{2}(a-b)(a+b)$ cm^2 (b) $4b^2$ cm^2
(c) $a=15$ cm, $b=5$ cm

14 (a) $(a-b)(a+b)$
(b) two
(c) $a+b$
(d) $(a^2-b^2)(a^2+b^2)$
(e) $(a-b)(a+b)(a^2+b^2)$
(f) three
(g) $(a+b)(a^2+b^2)$
(h) $(a^4-b^4)(a^4+b^4)$
(i) $(a-b)(a+b)(a^2+b^2)(a^4+b^4)$
(j) $(a+b)(a^2+b^2)(a^4+b^4)$
(k) $(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)$
(l) $(2^{16}-1) = (2^{16}-1^{16}) \div (2-1)$
$= (2+1)(2^2+1^2)(2^4+1^4)(2^8+1^8)$
$= 3\times5\times17\times257$

15 (a) x^2-y^2 m^2 (b) $2(x-y)$ m^2
(c) $(x-y)(x+y+2)$ m^2

16 (a) $a^2+b^2=(m-n)^2+(2\sqrt{mn})^2$
$=m^2-2mn+n^2+4mn$
$=m^2+2mn+n^2$
$=(m+n)^2$
$=c^2$
(b) $t^2=(kr)^2-r^2$
$=(kr-r)(kr+r)$
$=r(k-1)r(k+1)$
$=r^2(k-1)(k+1)$
(c) $k=1$ or -1
(d) k cannot be equal to zero as this creates the impossible equation $a^2=-b^2$, and k cannot be a fraction between -1 and 1 as this creates a similar situation as when k is equal to zero, where a^2 is equal to a negative fraction of b^2.

17 (a) x^2 cm^2
(b) 36 cm^2
(c) x^2-36 cm^2
(d) $(x-6)(x+6)$
(e) Length is $x+6$ cm and width is $x-6$ cm.
(f) The factors in part **(d)** correspond to the length and width of the rectangle.

Open-ended – Sample answers

18 $2x^2+8x-40$ and x^2-8x-9

19 $a^2-b^2+3a+3b=(a+b)(a-b+3)$

Half-time 2 (p. 77)

1 (a) $x^2+8x+12$ (b) $x^2+2x-35$
(c) $3x^2+13x-10$ (d) $2x^2+5x-3$

2 (a) $(x+5)(x-3)$ (b) $(x-2)(x-12)$
(c) $(x+5)(x-8)$

3 (a) $(x+5)(x+7)$ (b) $(x-3)(x-4)$
(c) $(x+7)(x-3)$

4 (a) $(x+7)^2$ (b) $(x-9)^2$ (c) $(3x-5)^2$

5 (a) $7(x+2y)$ (b) $6x(3x+4y)$
(c) $(x-2)(3+y)$ (d) $(x+5)(3+2y)$

6 (a) x^2-9 (b) x^2-64
(c) $9x^2-y^2$ (d) $36x^2-25y^2$

7 (a) x^3+5x^2+6x (b) $3x^2-12$
(c) $63x^3+150x^2-141x-228$ (d) $48x^3-156x^2+36x$

8 (a) $(x-3)(x+3)$ (b) $9(x-2y)(x+2y)$
(c) $2(2x-5y)(2x+5y)$ (d) $3(3-4x)(3+4x)$

9 D 10 C 11 B

Exercise 2.5 (p. 80)

1 (a) $(x+4-y)(x+4+y)$ (b) $(y-2-k)(y-2+k)$
(c) $(y+4-m)(y+4+m)$ (d) $(m-5-2n)(m-5+2n)$
(e) $(x+1-3y)(x+1+3y)$ (f) $(q+3-4p)(q+3+4p)$
(g) $(a-3b-4)(a-3b+4)$ (h) $(x-5y-6)(x-5y+6)$
(i) $(2a-3-4b)(2a-3+4b)$ (j) $(3c+5-2d)(3c+5+2d)$

2 (a) $(x+3-\sqrt{7})(x+3+\sqrt{7})$
(b) $(x-5-\sqrt{21})(x-5+\sqrt{21})$
(c) $(x-4-\sqrt{19})(x-4+\sqrt{19})$
(d) $(x+10-\sqrt{23})(x+10+\sqrt{23})$
(e) $(x-6-\sqrt{15})(x-6+\sqrt{15})$
(f) $(x-2)(x+4)$
(g) $(x-7-\sqrt{51})(x-7+\sqrt{51})$

(h) $(x-2-\sqrt{6})(x-2+\sqrt{6})$

(i) $(x+8-\sqrt{17})(x+8+\sqrt{17})$

(j) $\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$

(k) $\left(x+\frac{7}{2}-\frac{\sqrt{13}}{2}\right)\left(x+\frac{7}{2}+\frac{\sqrt{13}}{2}\right)$

(l) $\left(x-\frac{1}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{1}{2}+\frac{\sqrt{17}}{2}\right)$

(m) $\left(x-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)$

(n) $\left(x+\frac{5}{2}-\frac{\sqrt{33}}{2}\right)\left(x+\frac{5}{2}+\frac{\sqrt{33}}{2}\right)$

(o) $\left(x+\frac{1}{2}-\frac{\sqrt{21}}{2}\right)\left(x+\frac{1}{2}+\frac{\sqrt{21}}{2}\right)$

(p) $\left(x-\frac{9}{2}-\frac{\sqrt{29}}{2}\right)\left(x-\frac{9}{2}+\frac{\sqrt{29}}{2}\right)$

(q) $\left(x-\frac{3}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{3}{2}+\frac{\sqrt{17}}{2}\right)$

(r) $\left(x-\frac{7}{2}-\frac{\sqrt{53}}{2}\right)\left(x-\frac{7}{2}+\frac{\sqrt{53}}{2}\right)$

3 (a) $2(x+1-\sqrt{6})(x+1+\sqrt{6})$

(b) $4(x-2-\sqrt{5})(x-2+\sqrt{5})$

(c) $10(x-4-\sqrt{19})(x-4+\sqrt{19})$

(d) $3(x-1-\sqrt{5})(x-1+\sqrt{5})$

(e) $5(x+2-\sqrt{5})(x+2+\sqrt{5})$

(f) $7(x+3-\sqrt{7})(x+3+\sqrt{7})$

4 D 5 A

6 (a) 14 m (b) $(t-4-\sqrt{2})(t-4+\sqrt{2})$

(c) 2.83 seconds

(d) The seagull is at sea level.

(e) The seagull is below sea level, catching a fish.

7 (a) $-5(x-4)^2+75=-5(x-4-\sqrt{15})(x-4+\sqrt{15})$

(b) $P=-5(4\pm\sqrt{15}-4)^2+75$
$=-5(\pm\sqrt{15})^2+75$
$=-5\times 15+75=0.$

(c) When $x=0$, $P=-5$, so the company initially operated at a loss. When $x=4$, $P=75$, so the company had made a healthy profit after 4 months. When $x=8$, $P=-5$, so the company made a loss again after 8 months.

8 (a) $x^2+14x+35$ (b) $(x+7-\sqrt{14})(x+7+\sqrt{14})$

(c) $8-\sqrt{14}$ km (d) 112.75 km

9 (a) 100 worms

(b) $-(d+2+2\sqrt{26})(d+2-2\sqrt{26})$

(c) When $d=2(\sqrt{26}-1)$, $(d+2-2\sqrt{26})=0$, so no earthworms remain.

(d) 8.20, so the number of earthworms reduced to zero during the 9th day.

(e) (i) $-(n+1+\sqrt{65})(n+1-\sqrt{65})$

(ii) When $n=\sqrt{65}-1$, $(n+1-\sqrt{65})=0$, so no earthworms remain.

(f) $\sqrt{65}-1=7.06$, so the first colony will stay longer in their compost heap.

Open-ended – Sample answers

10 (a) $(x-a)^2+2b-a^2$ (b) $A=2b-a^2$

(c) (i) $2b-a^2=0$ (ii) $a=2, b=2$

Exercise 2.6 (p. 88)

1 (a) $\frac{4x}{5}$ (b) $\frac{k}{7}$ (c) $\frac{5p}{9}$

(d) $2(a-b)$ (e) $\frac{3(x+1)}{2}$ (f) $\frac{7(y-2)}{4}$

(g) $\frac{y(a-4)}{2}$ (h) $\frac{2x(x+5)}{9}$ (i) $\frac{3x(x-4)}{4}$

2 (a) $\frac{x+6}{x-5}$ (b) $\frac{x+2}{x-2}$ (c) $\frac{2x-3}{2x+1}$

(d) $\frac{x-2}{x-7}$ (e) $\frac{b+5}{b+3}$ (f) $\frac{x+7}{x-1}$

3 (a) $\frac{1}{3}$ (b) $\frac{3(5+2x)}{5-2x}$ (c) $\frac{x+4}{x-4}$

(d) $\frac{x-3}{x+3}$ (e) $\frac{x+7}{8}$ (f) $\frac{7}{4}$

(g) $\frac{(x-3)(x+2)}{(x-4)(x-1)}$ (h) $\frac{x+4}{x+1}$

(i) $\frac{(3x-2)(x-3)}{(x-6)(x+2)}$ (j) $\frac{(5x-3)(x-3)}{(x-4)(5x-4)}$

4 (a) $x+8$ (b) $x-9$ (c) $\frac{1}{x-5}$

(d) $\frac{1}{x-7}$ (e) $\frac{2x-1}{2}$ (f) $3(x-2)$

(g) $\frac{2x+3}{2}$ (h) $\frac{3x+5}{2}$ (i) $\frac{x-1}{x-8}$

(j) $\frac{x+4}{x+5}$ (k) $\frac{x+1}{x-2}$ (l) $\frac{x-1}{x+4}$

(m) 1 (n) $\frac{5}{6}$ (o) $\frac{x-5}{6(x+3)}$

(p) $\frac{8(x-7)}{x+5}$ (q) $\frac{x-2}{x+2}$ (r) $\frac{x-3}{x-7}$

5 (a) $\frac{3(x-2)}{x+3}$ (b) $\frac{5(x+5)}{3(x-7)}$ (c) $\frac{x+6}{x+7}$

(d) $\frac{(x+2)(x+1)}{(x+5)(x+8)}$ (e) $\frac{5(x+y+3)}{4}$ (f) $\frac{7(x-1-2y)}{3(3x-2)}$

6 (a) $\frac{5x}{6}$ (b) $\frac{7x}{10}$ (c) $\frac{5x}{12}$

(d) $\frac{x}{2}$ (e) $\frac{11x}{15}$ (f) $\frac{-7x}{12}$

(g) $\frac{8x-3y}{6}$ (h) $\frac{20x-21y}{28}$ (i) $\frac{99x-64y}{72}$

(j) $\frac{5x+22}{6}$ (k) $\frac{3x-2}{8}$ (l) $\frac{4x-16}{9}$

(m) $\frac{3x-8}{12}$ (n) $\frac{13x-34}{20}$ (o) $\frac{13x+14}{36}$

(p) $\frac{29x+2y}{12}$ (q) $\frac{38x-19y}{15}$ (r) $\frac{34x+5y}{12}$

(s) $\frac{9x-8y}{30}$ (t) $\frac{3x-4y}{4}$ (u) $\frac{17x+11y}{6}$

7 C

8 (a) $\frac{3(b-a)}{3b+a}$ (b) $\frac{3(7-5)}{3\times 7+5}=\frac{3}{13}$

9 (a) $(a-b)(a^2+ab+b^2)=a^3+a^2b+ab^2-a^2b-ab^2-b^3=a^3-b^3$

(b) $(a^2-b^2)(a^4+a^2b^2+b^4)$

(c) $\frac{a^6-b^6}{a^2-b^2}=\frac{(a^2-b^2)(a^4-a^2b^2+b^4)}{a^2-b^2}=a^4+a^2b^2+b^4$

10 (a) $4w^2-\pi w^2$ (b) $\frac{4-\pi}{4}$

11 (a) $\frac{3m}{4\pi r^3}$ kg/m^3 (b) Students' answers. (c) $\frac{4}{25}$

Open-ended – Sample answers

12 $\frac{x+1}{2}+\frac{1-x}{6}$

Exercise 2.7 (p. 93)

1 (a) (i) $s=\pm\sqrt{A}$ (ii) $s=\pm 3\sqrt{3}$

(b) (i) $v=\pm\sqrt{\frac{2K}{m}}$ (ii) $v=\pm\sqrt{5}$

(c) (i) $v=\pm\sqrt{\frac{Fr}{m}}$ (ii) $v=\pm\sqrt{10.5}$

(d) (i) $r=\pm\sqrt{\frac{Gm_1m_2}{F}}$ (ii) $r=\pm\sqrt{33.75}$

(e) (i) $r_2=\pm\sqrt{\frac{g_1r_1^2}{g_2}}$ (ii) $r_2=\pm 14\sqrt{2}$

(f) (i) $V=\pm\sqrt{PR}$ (ii) $V=\pm 3\sqrt{7}$

(g) (i) $I=\pm\sqrt{\frac{P}{R}}$ (ii) $I=\pm\sqrt{\frac{20}{9}}$

(h) (i) $v=\pm\sqrt{\frac{2E}{c}}$ (ii) $v=\pm 4\sqrt{2}$

(i) (i) $D=\pm\sqrt{\frac{4A}{\pi}}$ (ii) $D=\pm 6$

2 (a) (i) $A_y=A^2-A_x$ (ii) $A_y=38$

(b) (i) $c=\frac{b^2-(2ax+b)^2}{4a}$ (ii) $c=5$

(c) (i) $T=\frac{9r^2}{l^2}$ (ii) $T=64$

(d) (i) $z=y-x^2$ (ii) $z=-96$

(e) (i) $z=\frac{y}{x^2}$ (ii) $z=0.28$

(f) (i) $b=cx^2-a$ (ii) $b=16$

(g) (i) $y=x^2+z$ (ii) $y=22$

(h) (i) $c=\frac{a+b}{x^2}$ (ii) $c=\frac{1}{3}$

(i) (i) $y=zx^2$ (ii) $y=96$

3 C

4 (a) $r=\sqrt{\frac{A}{\pi}}$

(b) The radius is a distance and must be positive.

(c) 3.57 cm (2 d.p.)

5 (a) $r=\sqrt[3]{\frac{3V}{4\pi}}$

(b) Because it is a cube root and not a square root.

(c) 3.10 cm (2 d.p.)

6 (a) $h=\frac{3V}{\pi r^2}=1.52$ m (2 d.p.)

(b) $r=\sqrt{\frac{3V}{\pi h}}=4.37$ cm (2 d.p.)

7 (a) 41m/s (b) The car is slowing down.

8 (a) $k=1243.57$ (b) 2.1 Hz

Open-ended – Sample answers

9 (a) Line 3 does not have ± symbol.

(b) One possible explanation is that if $x^2=9$, then x can be 3 or -3, since $(3)^2=9$ and $(-3)^2=9$.

10 (a) $a=2b+c^2$ $c=\pm\sqrt{a-2b}$

(b) $d=2e+\sqrt{f}$ $f=d^2-4de+4e^2$

Chapter review 2 (p. 100)

1 (a) $10x+35$ (b) x^2-x-6

(c) $x^2+10x+24$ (d) $8x^2+10x-3$

2 (a) $x^2+12x+36$ (b) $4x^2-12x+9$

(c) $25-x^2$ (d) $9x^2-49$

3 (a) $4x^2-20x-56$ (b) $-3x^3-15x^2+72x$

(c) $x^3-2x^2-29x+30$ (d) $-2x^3-7x^2+10x+24$

4 (a) $7(x+3)$ (b) $4a(3b-2)$ (c) $6x(3x+4)$

5 (a) $(c-4)(d+6)$ (b) $(x+2)(x-3)$

(c) $(a+1)(5a-2)$ (d) $(3y-5)(2y-7)$

6 (a) $(x+5)(x+6)$ (b) $(x+2)(x-12)$

7 (a) $(x+5)^2$ (b) $(x-9)^2$ (c) $(3x-4)^2$ (d) $(2x+1)^2$

8 (a) $(x-3)(x+3)$ (b) $(5-y)(5+y)$

(c) $(2x-y)(2x+y)$ (d) $(7x-4)(7x+4)$

(e) $(6x-9y)(6x+9y) = 9(2x-3y)(2x+3y)$

(f) $(8-3x)(8+3x)$ (g) $(2x-7)(2x-3)$

(h) $-13(2x+3)$ (i) $5(2x-1)$

(j) $(x-\sqrt{3})(x+\sqrt{3})$ (k) $(x+7-\sqrt{5})(x+7+\sqrt{5})$

(l) $(x-4-\sqrt{7})(x-4+\sqrt{7})$

9 (a) $(x+3-\sqrt{11})(x+3+\sqrt{11})$

(b) $(x-5-\sqrt{2})(x-5+\sqrt{2})$

(c) $(x-2-\sqrt{11})(x-2+\sqrt{11})$

(d) $(x+4-\sqrt{3})(x+4+\sqrt{3})$

10 (a) $\frac{2x}{5}$ (b) $\frac{a-3}{3}$ (c) $\frac{5(x-1)}{4}$ (d) $3a$

11 (a) $\frac{x+4}{x+2}$ (b) $\frac{(x+1)(3x+7)}{(x-7)(x-1)}$

(c) $2x-1$ (d) $\frac{2(x+5)}{x+4}$

12 (a) $\frac{7x}{6}$ (b) $\frac{8x-7}{15}$ (c) $\frac{4x-31}{30}$

(d) $\frac{12x+21y}{14}$ (e) $\frac{3x+4y+23}{12}$ (f) $\frac{2x-y-9}{10}$

13 (a) $b = \frac{A}{l}$ (b) $l = \frac{A^2g}{4\pi^2}$

(c) $r = \pm\sqrt{\frac{SA}{4\pi}}$ (d) $x = \pm\sqrt{\frac{2(a+z)}{y}}$

14 $a = 3, b = -2$

15 (a) -5, 6 (b) 6 kg gain (c) 5 kg loss

(d) From April to September John lost 11 kg in weight.

16 (a) $ax+2x+ay+2y$ (b) $(a+2)(x+y)$

(c) 175 cm

17 (a) (i) E (ii) B (iii) D

(iv) F (v) A (vi) C

(b) (i) $(x+8)(x-3)$ (ii) $(5a-b)(5a+b)$

(iii) $(x+3-y)(x+3+y)$ (iv) $(x+6-\sqrt{2})(x+6+\sqrt{2})$

(v) $x(16x-25)$ (vi) $(x-y)(4+x+y)$

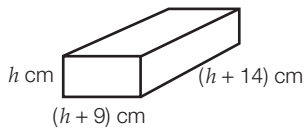
18 (a) $(x+2)(x+8)$ (b) $18(x+2)(x+8)$ (c) \$720

19 $16x^2+24xy+9y^2$

20 (a) $5x-x^2$ m^2 (b) $x(5-x)$

(c) (i) 4 m^2 (ii) 2.25 m^2 (iii) 3.1875 m^2

21 (a)



(b) $h(h+9)(h+14)$

(c) h^3+23h^2+126h (d) 1800 cm^3

(e) 1800 cm^3 (f) Students' own answers.

22 (a) πx^2 (b) $x+2$ (c) $\pi(x+2)^2$

(d) $x-2$ (e) $\pi(x-2)^2$

(f) $\pi(x+2)^2-\pi(x-2)^2$ (g) π

(h) $8\pi x$ (i) 80π (j) 17 mm

23 (a) $(8x+3)(2y+7)$ cm^2 (b) $(8x+3)(8x-3)$ cm^2

(c) $64x^2-9$ cm^2

(d) (i) $y = 4x-2$ (ii) $64x^2+48x+9$

24 (a) $x^2+p^2x+p^2y-y^2$ (b) $(x+y)(x-y+p^2)$

(c) $(x+y)(x-y+p^2) = (x+y)(x-y)$; $x-y$ must be positive, so x must be greater than y.

(d) (i) $2p^2x$ or $2p^2y$ vehicles

(ii) $2px(p-1)$ non-white vehicles

Chapter 3

Recall 3 (p. 105)

1 (a) 625 (b) 29.791 (c) 3.45

2 (a) $A = 16$ (b) $h = 5$ (c) $b = 7$

3 (a) 410 cm (b) 425 000 cm (c) 400 000 cm^2

(d) 0.0578 m^2 (e) 8 250 000 m^2 (f) 0.6 L

(g) 20 mL (h) 30 800 L (i) 0.007 25 m^3

4 (a) 6.11 m (b) 16.11 m (c) 15.27 m^2

5 (a) (i) 15.59 cm (ii) 140.3 cm^2

(b) (i) 22.36 cm (ii) 447.21 cm^2

(c) (i) 4.36 m (ii) 26.18 m^2

(d) (i) 1.5 m (ii) 6 m^2

Exercise 3.1 (p. 108)

1 (a) 1263 mm^2 (b) 583 cm^2 (c) 157.5 cm^2

(d) 349.94 mm^2 (e) 25.63 mm^2 (f) 493.65 cm^2

2 (a) 76.37 cm^2 (b) 43.37 cm^2 (c) 48.29 cm^2

(d) 1809.56 cm^2 (e) 18.27 mm^2 (f) 88.5 m^2

(g) 422.43 mm^2 (h) 13.72 m^2 (i) 69.53 m^2

3 C 4 C

5 (a) 123.61 cm^2 (b) 52.15 cm^2 (c) 47.12 cm^2

(d) 32.47 cm^2 (e) 2194.29 cm^2 (f) 441.06 cm^2

(g) 437.5 cm^2 (h) 294 cm^2

6 triangle: 173 m^2; sector: 1047 m^2

7 (a) 156.25 hours = 22.32 days (b) 75 000 m^2

8 9.78 m^2 9 25.13 m^2

10 (a) 0.05 m^2 (b) 190.42 m^2

11 4×4; 3×6 12 4.51 m

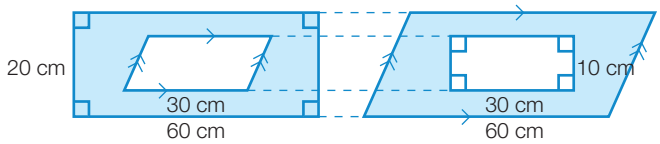
13 A, B, F: 50 cm^2; C, G: 25 cm^2; D, E: 100 cm^2

Open-ended – Sample answers

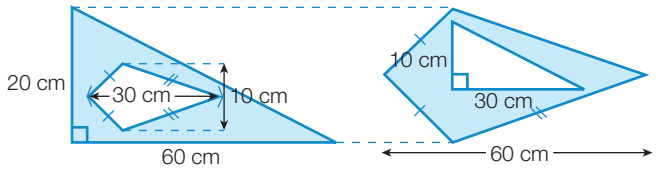
14 **(a)** 26 cm^2 **(b)** various answers

15 **(a)** The formulas for the overall area and the subtracted area are the same. The substituted values are the same hence the calculations are identical.

(b) Possible pair of shapes:



(c) Another possible pair of shapes:



Exercise 3.2 (p. 115)

1 **(a)** 766.14 m^2 **(b)** 82.14 m^2 **(c)** 376 cm^2
(d) 1710 cm^2 **(e)** 108 cm^2 **(f)** 684 cm^2

2 **(a)** 904.78 cm^2 **(b)** 1258.21 mm^2

3 **(a)** D **(b)** D

4 6300 cm^2 **5** 810.27 cm^2 **6** 12.25 m^2

Open-ended – Sample answers

7 **(a)** $2(lw + wh + lh)$

(b) Prism 1: $l = 10$ cm, $w = 5$ cm, $h = 2$ cm, SA = 160 cm^2

Prism 2: $l = 20$ cm, $w = 10$ cm, $h = 4$ cm, SA = 640 cm^2

The surface area quadruples when the dimensions are doubled.

8 **(a)** $r = 2$, $h = 5$, surface area = 28π

(b) $r = 2$, $h = 10$, surface area = 48π

(c) $r = 4$, $h = 5$, surface area = 72π

(d) Doubling the height increases the area of the curved face but not the circular ends. Doubling the radius increases the area of both the curved face and the circular ends. Also, the effect of doubling the radius is magnified, as the radius is then squared in the formula.

Exercise 3.3 (p. 122)

1 **(a)** 450 cm^3 **(b)** 3192 cm^3 **(c)** 5355 mm^3
(d) 585 cm^3 **(e)** 90 053.75 mm^3 **(f)** 552.92 m^3

2 374.36 mL

3 **(a)** 115 000 cm^3 **(b)** 20 250 mm^3 **(c)** 9625 cm^3
(d) 162 m^3 **(e)** 135 000 cm^3 **(f)** 840 cm^3

4 **(a)** B **(b)** D

5 1.31 m^3 **6** 0.07 m^3 **7** 6.64 m^3

8 21 082 L

Open-ended – Sample answers

9 **(a)** $4 \times 5 \times 10$ cm, $2 \times 10 \times 10$ cm, $1 \times 10 \times 20$ cm

(b) $8 \times 2.5 \times 10$ cm, $4 \times 4 \times 12.5$ cm, $0.5 \times 40 \times 10$ cm

(c) $4 \times 5 \times 10$: SA = 220 cm^2

$2 \times 10 \times 10$: SA = 280 cm^2

$1 \times 10 \times 2$: SA = 460 cm^2

10 **(a)** Anita has used the diameter value for the radius and did not convert one of the values to cm.

(b) 628 318.53 cm^3

(c) Point 1: Identify radius or diameter before using circle formulas; point 2: Check for matching units before substituting values into formulas.

Half-time 3 (p. 125)

1 **(a)** 19 101 cm^2 **(b)** 2228 mm^2

2 Surface area = 14π cm^2, volume = 5π cm^3

3 **(a)** 44 m^2 **(b)** 12.96 m^2

4 Volume A = $xy^2\pi$ m^3, Volume B = $4xy^2\pi$ m^3, Volume C = $xy^2\pi$ m^3

B has the largest volume

5 **(a)** 191 038.85 m^3 **(b)** 1323.05 m^3

6 72 206 cartons

Exercise 3.4 (p. 130)

1 **(a)** 576 cm^2 **(b)** 21 m^2

2 **(a)** 1884.96 cm^2 **(b)** 490.09 cm^2

3 **(a)** 20 106.19 cm^2 **(b)** 3848.45 cm^2

4 **(a)** 3402.34 m^2 **(b)** 829.38 cm^2 **(c)** 141 m^2
(d) 1599.56 cm^2 **(e)** 3619.11 cm^2 **(f)** 600 cm^2
(g) 9424.78 cm^2 **(h)** 1413.72 cm^2

5 **(a)** C **(b)** D

6 483.81 cm^2 **7** 414 cm^2 **8** 70 185 m^2

9 **(a)** Surface area of bowling ball = 2827.43 cm^2 (2 d.p.); area of material = 3200 cm^2. The area of material is larger than the surface area of the bowling ball.

(b) The material is not long enough to wrap around the ball, so he will need to cut and sew his existing material to wrap it. $C = \pi d = \pi \times 30 = 94.25$ cm (2 d.p.)

10 **(a)** 31 cm **(b)** 12 cm

Open-ended – Sample answers

11 $r = 5$, $s = 10$ gives a surface area of 75π cm^2 (235.62 cm^2)

$r = 6$, $s = 8$ gives a surface area of 84π cm^2 (263.89 cm^2)

12 The slant height is the hypotenuse of a right-angled triangle. The radius is one of the shorter sides.

13 **(a)** $h = 4$ m, $r = 3$ m, total surface area = 24π m^2

(b) total surface area = $3\pi(3 + \sqrt{73})$ m^2

(c) $h = 12$ m, $r = 5$m, total surface area = 90π m^2

(d) total surface area = $5\pi(5 + \sqrt{601})$ m^2

(e) No

14 (a) $h = 15$ m, $r = 8$m, total surface area = 200π m^2

(b) total surface area = $16\pi(16 + \sqrt{481})$ m^2

(c) $h = 24$ m, $r = 7$ m, total surface area = 224π m^2

(d) total surface area = $14\pi(14 + \sqrt{821})$ m^2

(e) No

Exercise 3.5 (p. 136)

1 (a) 716.28 cm^3 (b) 804.25 cm^3 (c) 66 666.67 cm^3

(d) 25 200 cm^3 (e) 385 mm^3 (f) 2338.27 cm^3

2 (a) 0.52 m^3 (b) 179 594.38 cm^3

3 (a) 150.53 cm^3 (b) 2900 cm^3

(c) 160.22 cm^3 (d) 197.60 cm^3

4 1 098 066 219 000 km^3

5 D 6 A

7 Pyramid is 1287.61 cm^3 larger.

8 157 mL 9 129 mL 10 70

11 (a) 28 (b) 22 338

(c) Calculations for volume involves cubing the radius.

(d) 1253 Earths in Jupiter, 688 Earths in Saturn, 50 Earths in Uranus, 60 Earths in Neptune

12 340 trucks

13 (a) $b = 5$ cm, $h = 9$ cm; volume = 75 cm^3

(b) Volume = 300 cm^3

(c) 4 times greater ($300 \div 75 = 4$)

(d) $b = 6$ cm, $h = 10$ cm; Volume of first pyramid = 120 cm^3

Volume of second pyramid = 480 cm^3

Conclusion: When the base of any square-based pyramid is doubled, the volume is quadrupled.

14 (a) $h = 4$ m, $r = 3$m, volume = 12π m^3

(b) volume = 24π m^3

(c) $h = 12$ m, $r = 5$m, volume = 100π m^3

(d) volume = 200π m^3

(e) When the height of a cone is doubled, the volume of the cone is also doubled.

15 (a) $h = 15$ m, $r = 8$ m, volume = 320π m^3

(b) volume = 1280π m^3

(c) $h = 24$ m, $r = 7$ m, volume = 392π m^3

(d) volume = 1568π m^3

(e) When the radius of a cone is doubled, the volume of the cone is multiplied by 4.

Exercise 3.6 (p. 145)

1 (a) 7.1 g/cm^3 (b) 6.4 g/cm^3 (c) 0.8 g/cm^3

2 (a) 6240 g (b) 486 g (c) 3.96 kg

3 (a) 13.27 cm^3 (b) 694.29 mL (c) 15.29 m^3

4 20 g/L 5 2.4 g/100 g

6 (a) D (b) B

7 7.6 mg/mL

8 (a) 4 cups, 16 cups (b) 200 mL/L

9 (a) 131.175 kg (b) 22 t

10 0.04 g/L

11 (a) wooden log: 0.40 g/cm^3; steel rod: 7.81 g/cm^3; cork: 0.22 g/cm^3; aluminium cylinder: 2.65 g/cm^3; ice block: 0.91 g/cm^3; gold bar: 18.29 g/cm^3

The wooden log, cork and ice block will all float on water.

(b) lead log: 2 852 562 g; lead rod: 726 g; lead cork: 103 g; lead cylinder: 856 g; lead block: 125 g; lead bar: 9931 g

12 (a) Johnstone's abattoir (50 kg/day compared with Arvon River 40 kg/day)

(b) 0.44 mg/L

Open-ended – Sample answers

13 (a) 875 organisms/100 mL (b) no

(c) 600 organisms/50 mL; 2000 organisms/100 mL; 800 organisms/60 mL

Chapter review 3 (p. 148)

1 (a) 925 cm^2 (b) 878.43 cm^2 (c) 5105.09 cm^2

2 B

3 (a) 1550 cm^2 (b) 7.49 m^2

4 C

5 (a) 1080 cm^3 (b) 21 205.75 cm^3 (c) 5040 cm^3

6 (a) 1884.96 cm^2 (b) 11 309.73 cm^2

7 (a) 527.79 cm^3 (b) 268 082.57 mm^3

8 A 9 30 kg/m^3 10 250 mL/L

11 1800 bricks

12 (a) 15 708 L (b) 5109 L

13 1.70 m^3

14 (a) 1885 cm^2, 6807 cm^3 (b) 327 cm^2, 302 cm^3

15 (a) 7527 m^2 (b) 361 m

16 (a) 2000 cm^3 (b) 15 g

17 335 cm^3 18 804 L 19 9839 kg

20 (a) 1950 kL (b) 550 kL

21 21.99 cm

22 (a) 62 832 m^3 (b) \$9 424 778

(c) 12 566 m^2 (d) \$1 256 637

(e) \$1 800 000 (f) \$12 481 415

(g) \$6 300 000 (h) the road over the top by \$6 181 415

23 4.73 g/L

Mixed Review A (p. 152)

1 \$957.04

2 (a) 2786.89 cm^2 (b) 256.31 m^2

3 (a) $3x^2 - 5xy + 2y^2$ (b) $6a^2 - 27ab + 30b^2$
(c) $15t^2 - 33ht - 36h^2$

4 (a) 7% (b) 7% (c) 7.19% (d) 7.23%

5 (a) $(x-2)^2$ (b) $(x+3)(x-3)$
(c) $(x+7)^2$ (d) $25(2x+1)(2x-1)$
(e) $2x(2y^2+y-4)$ (f) $4(x+4)^2$

6 (a) $-\frac{8x}{35}$ (b) $\frac{11x+43}{30}$

7 (a) $-x^2 - 5x + 66$ (b) $-x^2 - 2x + 24$
(c) $2x^2 + 5x - 12$ (d) $7y^2 - 5y - 2$

8 (a) \$5516.35 (b) \$5530.10 (c) \$5533.23

9 (a) 301.59 cm^3 (b) 164.64 cm^3

10 (a) $(x-9)(x+3)$ (b) $(x-5)(x-2)$ (c) $(3x-2)(2x+1)$

11 166.7 g/L

12 (a) $25x^2 - 10x + 1$ (b) $16x - 24x + 9$
(c) $64 - 48x + 9x^2$ (d) $36 - 60x + 25x^2$

13 Option A (\$6248.58 after 3 years) is better than Option B (\$6127.96 after 3 years)

14 (a) \$920 (b) \$1200.61

15 384 cm^2

16 (a) 81 cm^2 (b) 66.21 cm^2 (c) 37.12 cm^2

17 (a) 24.5 m^2 (b) 34.1 m

18 (a) $(x+5)(x+3)$ (b) $x^2 + 8x + 15$
(c) (i) 16 m^2 (ii) 63 m^2
(d) \$504

19 20 mL

Challenge A (p. 154)

1 $1000 - 1 + 3 - 5 + 7 \ldots$
$- 197 + 199 - 201 = 1000 + 50 \times 2 - 201 = 899$

2 $18x + 23y + 52z + 69w = 733$, $x + y + z + w = 20$. One of x and w must be odd, as must one of x and z. Use trial and error or a spreadsheet to obtain $x = 3$, $y = 10$, $z = 2$, $w = 5$. That is, 3 gifts at \$18, 10 gifts at \$23, 2 gifts at \$52 and 5 gifts at \$69.

3 B

4 D

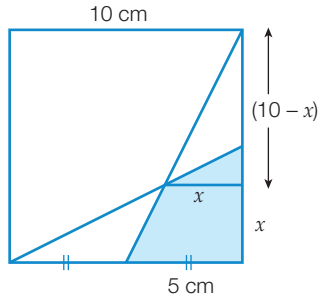
$35x - 35 < 75 < 21x + 84$. $35x < 110$, $x < 3\frac{1}{7}$ and $-9 < 21x$,

$x > -\frac{3}{7}$. $-\frac{3}{7} < x < 3\frac{1}{7}$, $x = 0, 1, 2, 3$.

5 D

6 $\frac{5}{10} = \frac{x}{10-x}$, $50 - 5x = 10x$, $3x = 10$, $x = \frac{10}{3}$

Shaded area $= x^2 + x \times (5 - x) = 5x = \frac{50}{3} = 16\frac{2}{3}$ cm^2



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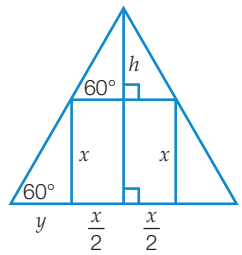


$\frac{x}{y} = \tan 60°$, $\frac{x}{y} = \sqrt{3}$, $y = \frac{x}{\sqrt{3}}$

Base of equilateral triangle $= x + 2y = x + \frac{2x}{\sqrt{3}} = \frac{(3+2\sqrt{3})x}{3}$

$\frac{h}{\frac{x}{2}} = \tan 60°$, $h = \frac{x \tan 60°}{2} = \frac{x\sqrt{3}}{2}$

Base of equilateral triangle $= x + 2y = x + \frac{2x}{\sqrt{3}} = \frac{(2+\sqrt{3})x}{2}$

Height of equilateral triangle $= x + h = x + \frac{\sqrt{3}x}{2} = \frac{(2+\sqrt{3}x)x}{2}$

Area of square $= x^2$

Area of triangle $= \frac{1}{2} \times \frac{(3+2\sqrt{3})x}{3} \times \frac{(2+\sqrt{3})x}{2}$

$= \frac{(12+7\sqrt{3})x^2}{12}$

Square as fraction of triangle $= x^2 \div \frac{(12+7\sqrt{3})x^2}{12}$

$= \frac{12}{12+7\sqrt{3}}$, 49.7%

OR, using similar triangles, $\frac{h}{\frac{x}{2}} = \frac{x}{y}$ so $h = \frac{x^2}{2y}$

Base of triangle $= x + 2y$

Height of triangle $= h + x = \frac{x^2}{2y} + x = \frac{x(x+2y)}{2y}$

Area of triangle $= \frac{1}{2} \times (x+2y) \times \frac{x(x+2y)^2}{2y} = \frac{x(x+2y)^2}{4y}$

Square as fraction of triangle $= x^2 \div \frac{x(x+2y)^2}{4y} = \frac{4xy}{(x+2y)^2}$

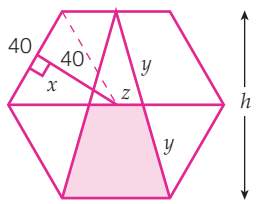
Now $\frac{x}{y} = \tan 60°$ so $y = \frac{x}{\sqrt{3}}$ (or $y = 0.5774x$)

Square as fraction of triangle $= \frac{4xy}{(x+2y)^2} = \frac{4x \times \frac{x}{\sqrt{3}}}{\left(x + \frac{2x}{\sqrt{3}}\right)^2}$

$= \frac{4x^2}{\sqrt{3}x^2\left(1 + \frac{2}{\sqrt{3}}\right)^2} = \frac{4\sqrt{3}}{(\sqrt{3}+2)^2} = \frac{4\sqrt{3}}{(7+4\sqrt{3})}$

$= 0.497$, hence 49.7%

8



$x^2 = 40^2 - 20^2 = 1200$. $x = 20\sqrt{3}$

$h = 2x = 40\sqrt{3}$

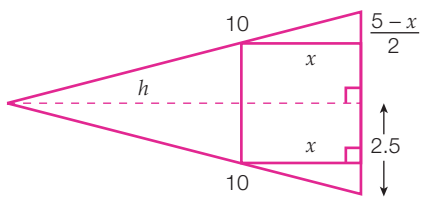
$4y^2 = 20^2 + 1600 \times 3 = 5200$. $y^2 = 1300$. $y = 10\sqrt{13}$

$z^2 = y^2 - x^2 = 1300 - 1200 = 100$. $z = 10$

Shaded area $= \frac{20+40}{2} \times 20\sqrt{3} = 600\sqrt{3}$ cm^2

9 D

10



$h^2 = 10^2 - 2.5^2 = 93.75$, $h = \sqrt{\frac{375}{4}} = \frac{5}{2}\sqrt{15}$

From similar right-angled triangles, $\frac{h}{2.5} = \frac{x}{\frac{5-x}{2}}$

$h = \frac{5x}{5-x}$

$5h - hx = 5x$

$(5+h)x = 5h$

$x = \frac{5h}{5+h}$

$= \frac{5 \times \frac{5\sqrt{15}}{2}}{5 + \frac{5\sqrt{15}}{2}}$

$= \frac{5\sqrt{15}}{\sqrt{15}+2} = \frac{5\sqrt{15}(\sqrt{15}-2)}{15-4}$

$= \frac{5}{11}(15 - 2\sqrt{15})$ cm

11 Radius = 15 cm, Area of base $= 3 \times \frac{5}{6} \times \pi \times 15^2 + 15$

$\times 15 \tan 60° = \frac{1125\pi}{2} + 225\sqrt{3} \approx 2157$ cm^2

Volume $= \left(\frac{1125\pi}{2} + 225\sqrt{3}\right) \div 10\,000 \times 6 \approx 1.294$ m^3

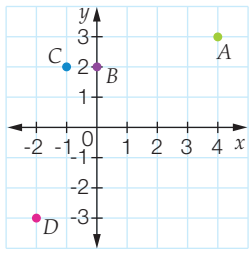
Chapter 4

Recall 4 (p. 156)

1 **(a)** $P = -7$ **(b)** $P = 3$ **(c)** $P = -15$ **(d)** $P = -3$

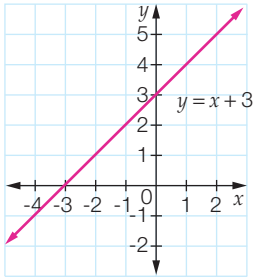
2 **(a)** $a = 7$ **(b)** $b = 24$ **(c)** $c = 10$ **(d)** $d = 7$

3



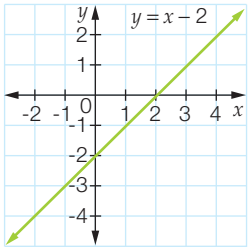
4 **(a)**

x	-2	-1	0	1	2
y	1	2	3	4	5



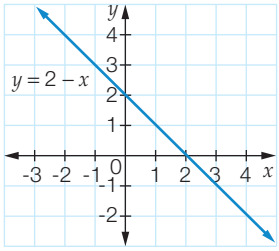
(b)

x	-2	-1	0	1	2
y	-4	-3	-2	-1	0



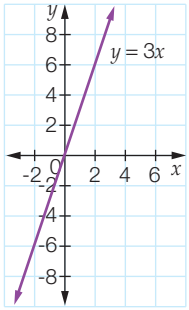
(c)

x	-2	-1	0	1	2
y	4	3	2	1	0

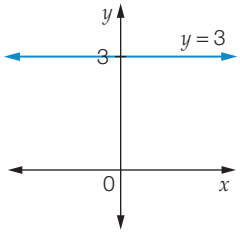


(d)

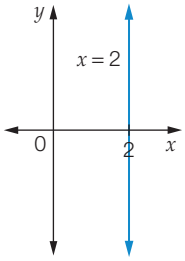
x	-2	-1	0	1	2
y	-6	-3	0	3	6



5 (a)



(b)



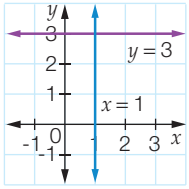
6 (a) (i) 3 (ii) 5

(b) (i) -2 (ii) 3

(c) (i) $-\frac{2}{3}$ (ii) $\frac{4}{3}$

7 (a) $-\frac{5}{3}$ (b) 1

8 (a)



(b) (1, 3)

Exercise 4.1 (p. 163)

1 (a) $x = 3$ (b $x = 8$ (c) $c = -3$ (d) $a = 4$

(e) $x = -6$ (f) $d = -3$ (g) $m = 2$ (h) $n = 1$

(i) $x = -1$ (j) $f = 2$ (k) $e = -2$ (l) $p = \frac{1}{5}$

(m) $x = 0$ (n) $g = 21$ (o) $h = 16$ (p) $k = 3$

(q) $x = 3$ (r) $y = 0$ (s) $x = \frac{5}{4}$ (t) $x = 3$

(u) $x = 7$

2 (a) $a = 1$ (b) $d = -1$ (c) $e = \frac{5}{6}$

(d) $c = 2$ (e) $b = 4$ (f) $f = \frac{13}{6}$ or $2\frac{1}{6}$

(g) $g = 8$ (h) $h = \frac{4}{5}$ (i) $k = -1$

(j) $x = -1$ (k) $x = 1$ (l) $x = -\frac{7}{3}$ or $-2\frac{1}{3}$

(m) $x = \frac{3}{2}$ or $1\frac{1}{2}$ (n) $x = \frac{1}{4}$ (o) $x = \frac{1}{3}$

(p) $m = -\frac{11}{2}$ or $-5\frac{1}{2}$ (q) $d = \frac{17}{4}$ or $4\frac{1}{4}$ (r) $d = 7$

(s) $a = 8$ (t) $a = -\frac{16}{3}$ or $-5\frac{1}{3}$ (u) $g = \frac{8}{7}$ or $1\frac{1}{7}$

(v) $t = 0$ (w) $h = \frac{11}{2}$ or $5\frac{1}{2}$ (x) $j = -\frac{3}{11}$

3 (a) $a = -22$ (b) $b = 8$ (c) $c = 9$

(d) $d = -1$ (e) $x = -2$ (f) $y = 1$

(g) $m = \frac{3}{4}$ (h) $k = \frac{3}{2}$ or $1\frac{1}{2}$ (i) $t = 2$

(j) $a = -25$ (k) $p = 15$ (l) $m = -27$

(m) $p = 5$ (n) $n = 2$ (o) $t = -18$

(p) $m = \frac{13}{22}$ (q) $k = -\frac{7}{4}$ or $-1\frac{3}{4}$ (r) $x = \frac{5}{6}$

4 (a) $x + 6 = 13, x = 7$ (b) $y - 7 = -2, y = 5$

(c) $2x = 17, x = 8\frac{1}{2}$ (d) $\frac{x}{3} = 12, x = 36$

(e) $3x + 2 = 23, x = 7$ (f) $4x - 5 = -3, x = \frac{1}{2}$

5 (a) C (b) D (c) D

6 (a) B (b) B (c) C

7 8, 9, 10, 11 8 40, 42, 44

9 Nikki 7, Sam 14, Minh 21 10 length 100 m, width 50 m

11 (a) $p + 7$ (b) $2(p + 7)$ (c) 17

12 98 m 13 walk 1.5 km, train 9 km

14 15 years old 15 45

16 24 years old 17 43 km/h and 53 km/h

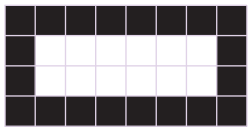
Open-ended – Sample answers

18 Students to check own answers but examples would be:

$\frac{x+8}{6} = -\frac{x}{2}$, $5x - 2 = -10 + x$ and $-4x - 11 = \frac{3x}{2}$

19 **(a)** black tiles $= 2l + 10$ or $2(l + 5)$

(b) **(i)**



(ii) 20 black tiles needed

(iii) $b = 2t + 8$ or $b = 2(t + 4)$

(c) For 5×3 white tiles, $n = 20$

Exercise 4.2 (p. 171)

1 **(a)** $\frac{1}{2}$ **(b)** -2 **(c)** 0

(d) $\frac{3}{0}$, undefined

2 **(a)** $y = 6x - 3$ **(b)** $y = -2x + 7$

(c) $y = 3x + 9$ **(d)** $y = -5x + 8$

3 **(a)** positive **(b)** negative **(c)** zero

(d) undefined

4 **(a)** $\frac{1}{3}$ **(b)** 2 **(c)** -4 **(d)** $-\frac{3}{2}$

(e) $-\frac{5}{4}$ **(f)** -2 **(g)** 3 **(h)** 2

(i) -1 **(j)** $-\frac{2}{3}$ **(k)** 0 **(l)** 0

(m) $-\frac{2}{5}$ **(n)** $\frac{1}{7}$ **(o)** undefined

5 **(a)** C **(b)** B **(c)** D

6 **(a)** **(i)** (-3, 2) and (2, 2) or (5, -4) and (-7, -4)

(ii) (2, 5) and (2, 7)

(iii) (4, 10) and (5, 6)

(iv) (-1, -1) and (6, 0)

(b) (0, 3) and (2, 7) parallel to (2, 1) and (-2, -7)

7 **(a)** 800 m **(b)** 600 m **(c)** 120

(d) 120 m/min **(e)** 440 m **(f)** 560 m

(g) $d = 200 + 120t$

(h) Alex is walking at a constant speed.

8 **(a)** 4800 grains **(b)** 1600 grains

(c) -1600 **(d)** 1600 grains/min

(e) decrease **(f)** $S = 4800 - 1600t$

(g) When $t = 2$, $s = 4800 - 3200$
$= 1600$

(h) $0 \le t \le 3$

9 **(a)** **(i)** $y = 2.5x - 1$ **(ii)** $y = 2.5x - 1$

(b) Yes, either point can be used as (x_1, y_1) to find the equation of the line.

10 $a = 17$

11 $-\frac{9}{35}$

12 Gradient of the line joining the points (-1, -5) and (2, 4) = 3 and the gradient of the line joining the points (2, 4) and (5, 13) = 3. As the two lines have a point in common, (2, 4) and the same gradient, the points are collinear.

Open-ended – Sample answers

13 **(a)** (1, 9), (2, 6), (3, 3) **(b)** (-2, 0), (-1, -1), (1, -3)

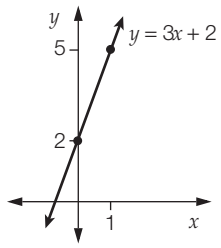
(c) (0, 4), (2, 3), (4, 2) **(d)** (2, -5), (3, -3), (4, -1)

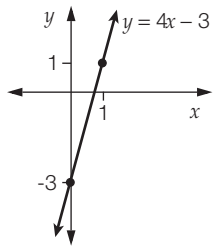
(e) (1, 0), (2, 5), (3, 10)

14 $m = -3$, (1, 4) and (4, -5)

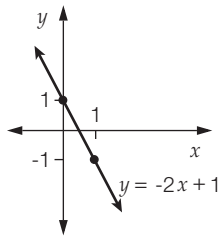
Exercise 4.3 (p. 177)

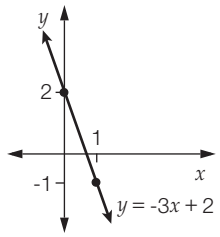
1 **(a)** $b = 2$, $m = 3$ **(b)** $b = -3$, $m = 4$



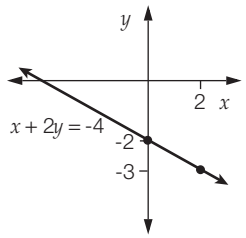


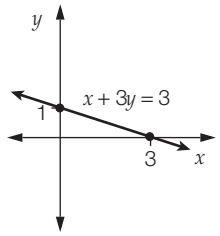
(c) $b = 1$, $m = -2$ **(d)** $b = 2$, $m = -3$



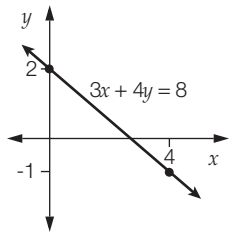


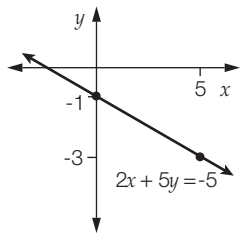
2 **(a)** $b = -2$, $m = -\frac{1}{2}$ **(b)** $b = 1$, $m = -\frac{1}{3}$



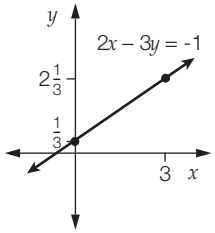


(c) $b = 2$, $m = -\frac{3}{4}$ **(d)** $b = -1$, $m = -\frac{2}{5}$

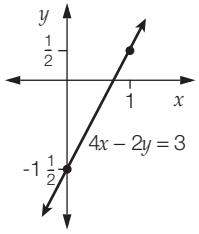




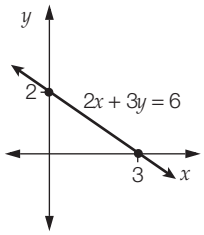
(e) $b = \frac{1}{3}$, $m = \frac{2}{3}$



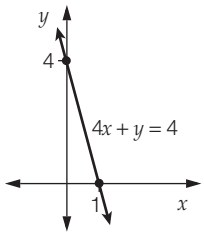
(f) $b = -1\frac{1}{2}$, $m = 2$



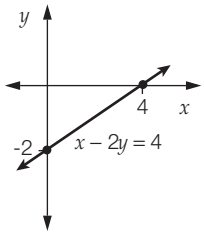
3 (a)



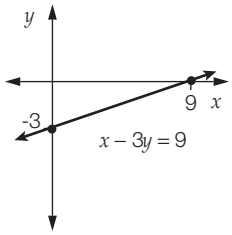
(b)



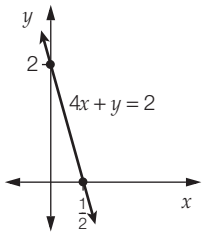
(c)



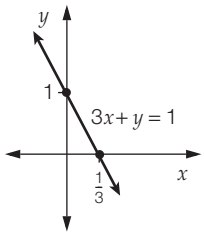
(d)



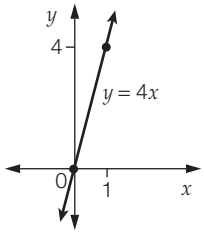
(e)



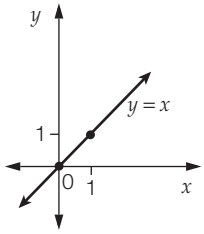
(f)



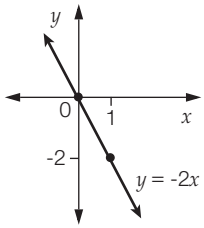
4 (a)



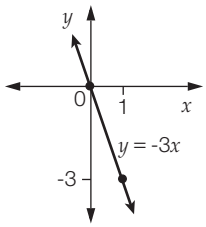
(b)



(c)



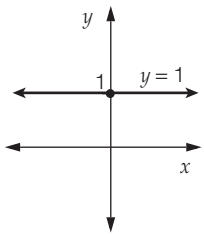
(d)



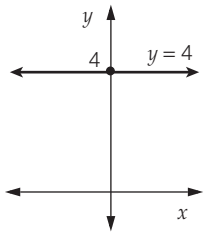
5 (a) D (b) B (c) A

6 (a) C (b) C (c) B

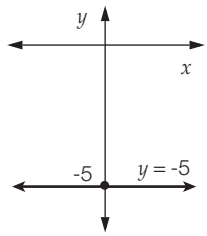
7 (a)



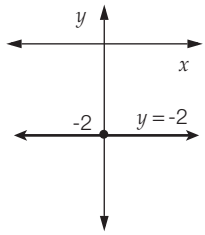
(b)



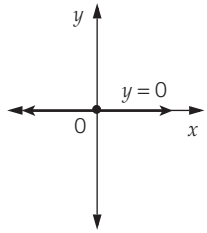
(c)



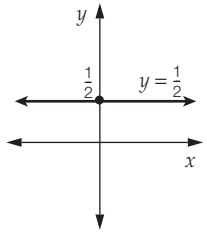
(d)



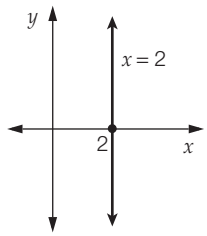
(e)



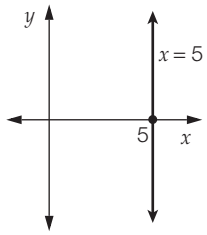
(f)



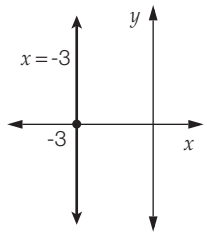
(g)



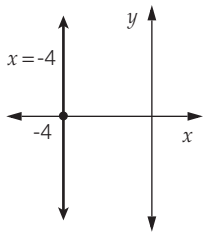
(h)



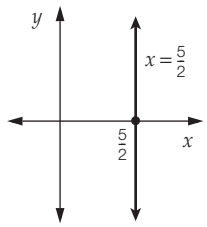
(i)



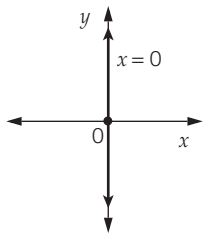
(j)



(k)



(l)



8 (a) D (b) C (c) D

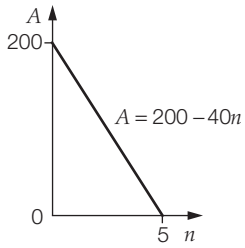
9 (a) 100 km (b) 80 (c) $D = 100 + 80T$

(d) 80 km/h (e) $D = 250 + 100T$ (f) 93 km/h

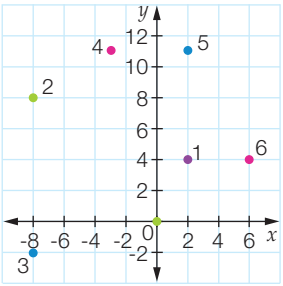
10 (a) \$80 (b) $A = 200 - 40n$

(c) \$200 (d) 5 weeks

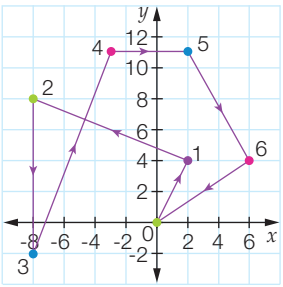
(e)



11 **(a)**



(b)



(c) **(i)** $y = 2x$

(ii) $y = -\frac{2}{5}x + 4\frac{4}{5}$ or $5y + 2x = 24$

(iii) $x = -8$

(iv) $y = \frac{13}{5}x + 18\frac{4}{5}$ or $5y - 13x = 94$

(v) $y = 11$

(vi) $y = -\frac{7}{4}x + 14\frac{1}{2}$ or $4y + 7x = 58$

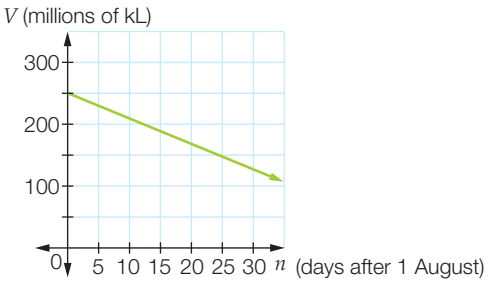
(vii) $y = \frac{2}{3}x$

(d) Yes. It will be on the direct path between the 5th and 6th buoy.

12 **(a)** $V = 225 - 0.75T$ **(b)** 5 min

13 **(a)** $V = 248 - 4n$

(b) Maximum capacity = 248 million kL



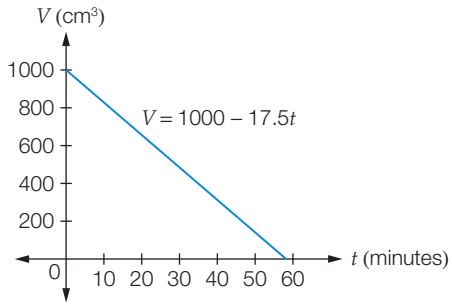
(c) It does not make sense to extend the time and volume into negative values.

14 **(a)** 1000 cm^3 **(b)** $V = 1000 - kt$

(c) 125 cm^3 **(d)** $k = 17.5$

(e) The volume is decreasing. **(f)** $V = 1000 - 17.5t$

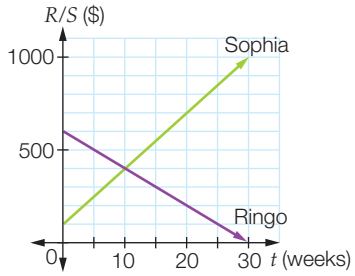
(g)



The ice block has completely melted after 57.15 minutes.

15 **(a)** Ringo: $R = 600 - 20t$, Sophia: $S = 100 + 30t$

(b) 15.7 weeks



Open-ended – Sample answers

16 $y = 2x - 4$, $y = 2x - 1$, $y = 2x + 2$. The coefficient of x in all three equations is the same, indicating that they have the same gradient.

17 (1) The pink line extending from (0, 0).

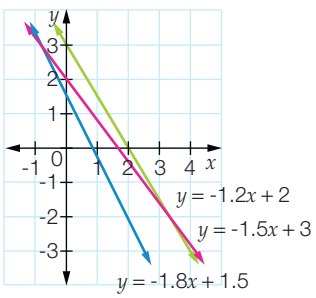
(2) The purple line (gradient is half of the pink one above it).

(3) The blue line (negative gradient shows the reduction in remaining credit).

(4) The black line starts part way up the y-axis representing connection fees.

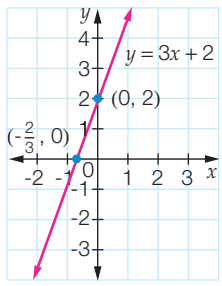
(5) The green line.

18

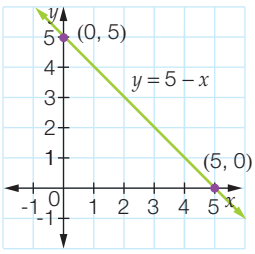


Half-time 4 (p. 181)

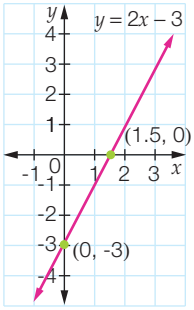
1 **(a)**



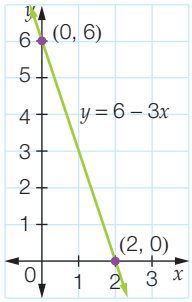
(b)



(c)

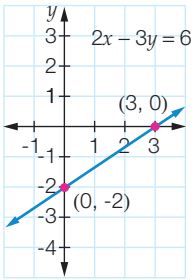


(d)

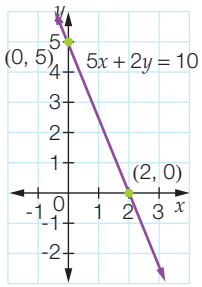


2 (a) $m = 0.5$ (b) $m = -2$ (c) $m = 0.5$ (d) $m = -8$

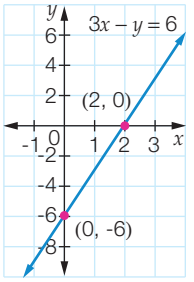
3 (a)



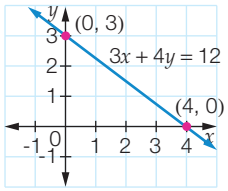
(b)



(c)



(d)



4 Using the points (32, 0) and (212, 100):

$$m = \frac{100 - 0}{212 - 32} = \frac{100}{180} = \frac{5}{9}$$

Hence, $C = \frac{5}{9}F + c$. Substituting in (32, 0):

$$0 = \frac{5}{9} \times 32 + c$$

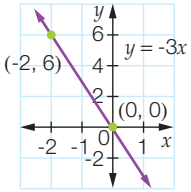
$$0 = \frac{160}{9} + c$$

$$c = -\frac{160}{9}$$

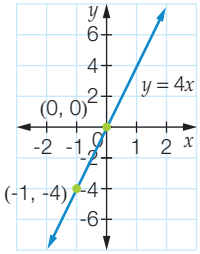
Therefore, $C = \frac{5}{9}F - \frac{160}{9}$

$$C = \frac{5}{9}(F - 32)$$

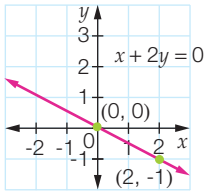
5 (a)



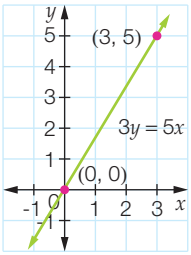
(b)



(c)



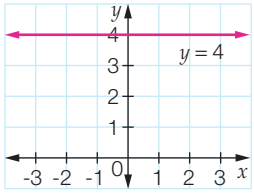
(d)



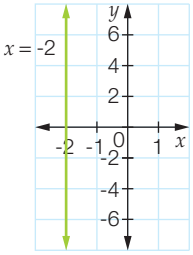
6 (a) $x = 2$ (b) $x = 2$ (c) $x = 19$

(d) $x = 3$ (e) $x = 3$ (f) $x = 4$

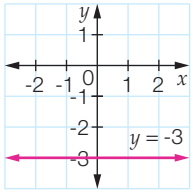
7 (a)



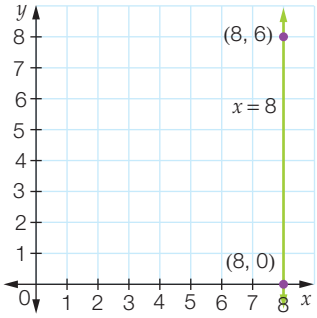
(b)



(c)



(d)



8 (a) $y = 4x + 9$ (b) $y = -x - 4$

Exercise 4.4 (p. 188)

1 (a) The gradient of both lines is 2.5. Hence, they are parallel.

(b) The gradient of both lines is 0.5. Hence, they are parallel.

2 (a) The gradient of $4x + 3y = 7$ is $-\frac{4}{3}$ and the gradient of $8y = 6x - 2$ is $\frac{3}{4}$. $-\frac{4}{3} \times \frac{3}{4} = -1$. Hence, the two lines are perpendicular.

(b) The gradient of $2x - 5y - 1 = 0$ is $\frac{2}{5}$ and the gradient of $4y + 10x = 3$ is $-\frac{5}{2}$. $\frac{2}{5} \times -\frac{5}{2} = -1$. Hence, the two lines are perpendicular.

3 (a) $m_1 = 3$, $m_2 = 3$; lines are parallel.

(b) $m_1 = 2$, $m_2 = -2$; lines are not parallel.

(c) $m_1 = 0.5$, $m_2 = 0.5$; lines are parallel.

(d) $m_1 = -4$, $m_2 = 4$; lines are not parallel.

(e) $m_1 = 5$, $m_2 = 9$; lines are not parallel.

(f) $m_1 = 2$, $m_2 = 2$; lines are parallel.

(g) $m_1 = -3$, $m_2 = -3$; lines are parallel.

(h) $m_1 = -3$, $m_2 = -4$; lines are not parallel.

(i) $m_1 = 4$, $m_2 = 4$; lines are parallel.

(j) $m_1 = -\frac{5}{3}$, $m_2 = \frac{1}{5}$ lines are not parallel.

4 (a) $m_1 = 2$, $m_2 = -\frac{1}{2}$, $2 \times -\frac{1}{2} = -1$; lines are perpendicular.

(b) $m_1 = 4$, $m_2 = -\frac{1}{2}$, $4 \times -\frac{1}{2} = -2$; lines are not perpendicular.

(c) $m_1 = \frac{1}{3}$, $m_2 = -\frac{1}{3}$, $\frac{1}{3} \times -\frac{1}{3} = -\frac{1}{9}$; lines are not perpendicular.

(d) $m_1 = \frac{1}{2}$, $m_2 = -2$, $\frac{1}{2} \times -2 = -1$; lines are perpendicular.

(e) $m_1 = \frac{3}{4}$, $m_2 = -\frac{3}{2}$, $\frac{3}{4} \times -\frac{3}{2} = -\frac{9}{8}$; lines are not perpendicular.

(f) $m_1 = \frac{4}{5}$, $m_2 = -\frac{5}{8}$, $\frac{4}{5} \times -\frac{5}{8} = -\frac{1}{2}$; lines are not perpendicular.

5 C 6 D

7 Gradient of $y = x + 10$ is 1. Gradient between the two points is $\frac{8-2}{4-(-2)} = \frac{6}{6} = 1$. Therefore, the lines are parallel.

8 Gradient of $2x + y - 10 = 0$ is -2. Gradient between the two points is $\frac{-7-5}{4-(-2)} = \frac{-12}{6} = -2$. Therefore, the lines are parallel.

9 Gradient of $y = -\frac{4}{3}x - 2$ is $-\frac{4}{3}$. Gradient between the two points is $\frac{4-1}{0-(-4)} = \frac{3}{4}$. $\frac{3}{4} \times -\frac{4}{3} = -1$. Therefore, the lines are perpendicular.

10 Gradient of $y = -\frac{5}{3}x + 10$ is $-\frac{5}{3}$. Gradient between the two points is $\frac{8-5}{9-4} = \frac{3}{5}$. $-\frac{5}{3} \times \frac{3}{5} = -1$. Therefore, the lines are perpendicular.

11 (a) (i) and (iii)

(b) (ii) and (v)

12 $y = 3x - 9$

13 Parallel line is $y = -2x + 5$, perpendicular line is $y = 0.5x + 5$.

14 $m_{AB} = -\frac{3}{2}$, $m_{BC} = 5$, $m_{CA} = \frac{2}{3}$. $-\frac{3}{2} \times \frac{2}{3} = -1$. Therefore, AB is perpendicular to CA and ABC must be a right-angled triangle.

15 $m_{AB} = 1$, $m_{BC} = -\frac{1}{2}$, $m_{DC} = 1$, $m_{AD} = -\frac{1}{2}$.
$AB \parallel DC$ and $BC \parallel AD$. Therefore $ABCD$ is a parallelogram.

16 $m_{AB} = -\frac{5}{2}$, $m_{BC} = \frac{2}{5}$, $m_{DC} = -\frac{5}{2}$, $m_{AD} = \frac{2}{5}$. $-\frac{5}{2} \times \frac{2}{5} = -1$, therefore $AB \perp BC$, $BC \perp CD$, $CD \perp DA$ and $DA \perp AB$. Therefore $ABCD$ is a quadrilateral with four right angles. $|AB| = |BC| = |CD| = |DA| = 29$. All four sides are of equal length, therefore $ABCD$ is a square.

Open-ended – Sample answers

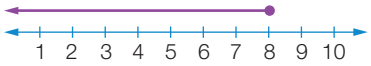
17 (a) $y = 0.5x + 4$, $2y = x + 3$, $2y - x = 10$
(all have gradients of 0.5)

(b) $y = -3x - 1$, $3x + y = 10$, $3x = 5 - y$
(all have gradients of -3)

Exercise 4.5 (p. 194)

1 (a)

(b)



(c)

(d)

(e)

(f)

(g)

(h)

(i)

(j)

(k)

(l)

(m)

(n)

(o)

(p)

(q)

(r)

2 (a) $x > -10$ (b) $x < -5$ (c) $x \le -10$

(d) $x \le 35$ (e) $x \ge -9$ (f) $x > -28$

(g) $x < -\frac{16}{3}$ (h) $x > -\frac{27}{5}$ (i) $x \le -9$

(j) $x \ge 8$ (k) $x \le -2$ (l) $x > -8$

(m) $x > 4$ (n) $x < 11$ (o) $x \le 4$

(p) $x \le -8$ (q) $x > 11$ (r) $x > 10$

3 (a) $y \le -5$ (b) $y \ge -6$ (c) $y \ge -5$

(d) $y > 12$ (e) $y < 21$ (f) $y < 12$

(g) $y \ge -8$ (h) $y \le 2$ (i) $y \ge -4$

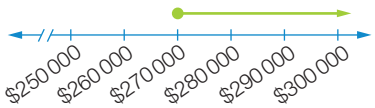
(j) $y \le -\frac{1}{15}$ (k) $y \ge \frac{1}{2}$ (l) $y \le \frac{19}{6}$

(m) $y < \frac{13}{3}$ (n) $y > \frac{23}{6}$ (o) $y < 4$

4 (a) B (b) D (c) A (d) D

5 (a) A (b) D (c) D (d) D

6 (a)



(b) $P \ge 270\,000$

(c) $270\,000 \le P < 320\,000$

7 $x \le 100$ 8 $n \ge 5$ 9 $j > 55$

10 (a) $x \ge \frac{2}{3}$ (b) $x \le -\frac{3}{11}$ (c) $x < 0$

(d) $x > -2$ (e) $x \le 4$ (f) $x \ge 12$

(g) $x > -25$ (h) $x < \frac{34}{7}$ (i) $x \ge 2$

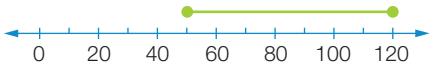
11 (a) $y \ge -1$ (b) $y < 3$ (c) $y \le -\frac{7}{3}$

(d) $y > \frac{1}{3}$ (e) $y \ge 30$ (f) $y \le 36$

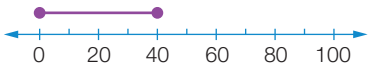
(g) $y > -12$ (h) $y < \frac{6}{7}$ (i) $y < \frac{61}{13}$

12 (a) Let f be the number of cartons of full cream milk.
Let l be the number of cartons of low fat milk.

$50 \le f \le 120$



$0 \le l \le 40$



(b) $l + f \le 120$

13 $20p > 428$, $p > 21.4$. Therefore Tony should sell the calculators for any price higher than $21.40 each.

14 Let n be the number of liquorice straps. $0.4n \le 1.55$, $n \le 3.875$. As n must be an integer $n \le 3$.

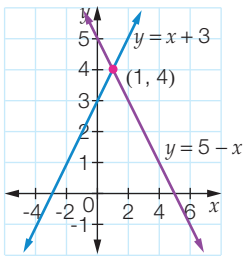
Open-ended – Sample answers

15

16 (a) $x - 10 \ge 3x - 8$ (b) $13x - 4 \ge 9x + 8$

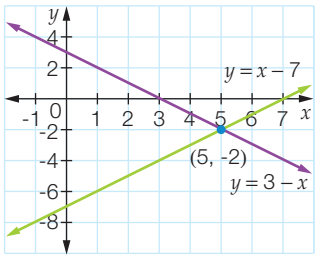
Exercise 4.6 (p.206)

1 (a) (i)



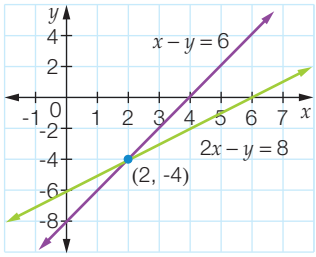
(ii) $x = 1, y = 4$

(b) (i)



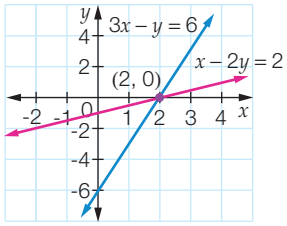
(ii) $x = 5, y = -2$

(c) (i)



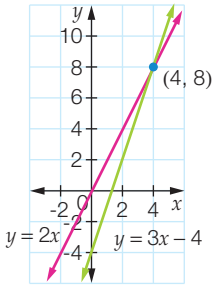
(ii) $x = 2, y = -4$

(d) (i)



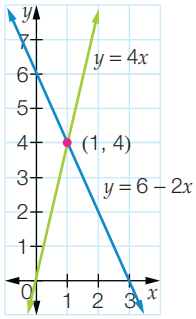
(ii) $x = 2, y = 0$

(e) (i)



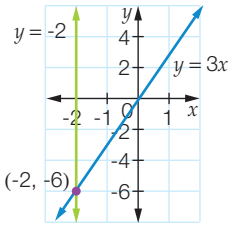
(ii) $x = 4, y = 8$

(f) (i)



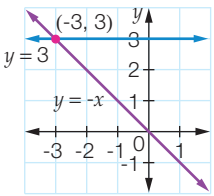
(ii) $x = 1, y = 4$

(g) (i)



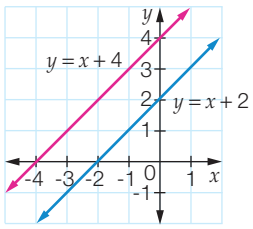
(ii) $x = -2, y = -6$

(h) (i)



(ii) $x = -3, y = 3$

(i) (i)



(ii) no solution

2 (a) (4, 1) (b) (5, -4) (c) (3, 1)
(d) (5, -1) (e) (1, 3) (f) (2, -6)
(g) (-2, 3) (h) (2, 2) (i) (2, 3)

3 (a) (1, -1) (b) (-3, 4) (c) (-1, 1)
(d) (2, -1) (e) (1, -1) (f) (-3, 1)
(g) (3, 1) (h) (-3, 4) (i) (1, 1)
(j) (4, -2) (k) (5, 12) (l) (9, 4)
(m) (-5, -10) (n) (-3, -8) (o) (-6, 5)

4 (a) (2, 4) (b) (-3, -9) (c) (4, -2)
(d) (1, 5) (e) (3, 2) (f) (3, -2)
(g) (-2, 5) (h) (9, -1) (i) (-1, 9)

5 (a) C (b) D

6 (a) D (b) B

7 (a) A (b) C

8 (a) D (b) D

9 a pineapple costs \$2.60, a kiwi fruit costs 35 cents

10 a salad roll costs \$3.60, a pie costs \$2.44

11 $x = 7, y = 5$

12 7 and 31

13 128 pages and 372 pages

14 30 m by 40 m

15 three 5-cent coins, 15 10-cent coins

16 Michelle is 18 years old, Yesmin is 31 years old

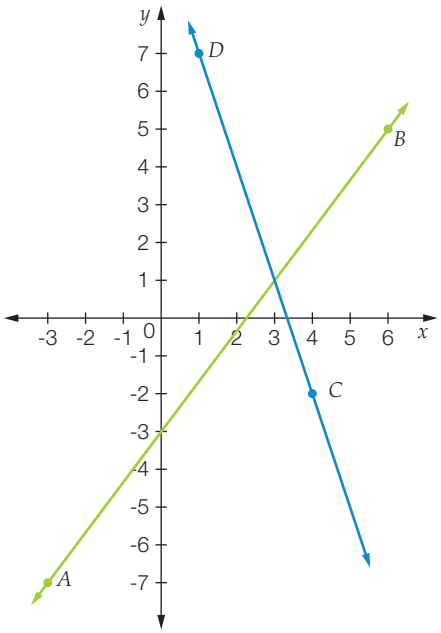
17 (a) Let D = distance in km from Wes' house and t = time taken in hours. Wes: $D = 80t$, Tony: $D = 360 - 100t$.

(b) 2 hours

18 Speed of barge is 12 km/h, speed of current is 3 km/h

19 (a) $A(-3, -7)$, $B(6, 5)$, $C(4, -2)$, $D(1, 7)$

(b)



(c) (3, 1)

(d) $y_{AB} = \frac{4}{3}x - 3$ (1)

$y_{CD} = -3x + 10$ (2)

substitute (1) into (2):

$\frac{4}{3}x - 3 = -3x + 10$

$4x - 9 = -9x + 30$

$13x - 9 = 30$

$13x = 39$

$x = 3$

substitute $x = 3$ into (2): $y = -9 + 10 = 1$

Therefore, the treasure is at (3, 1).

Open-ended – Sample answers

20 $x + y = 7$ and $x - y = -1$, $2x - y = 2$ and $x + 2y = 11$, $5x + 2y = 23$ and $3x - y = 5$

21 3, 8 and 14. Method example: Let x, y and z be the three numbers.

$x + y = 11$ (1)
$x + z = 17$ (2)
$y + z = 22$ (3)

Using elimination, (1) – (2):

$$\begin{array}{rl} & x + y = 11 \\ - & x + z = 17 \\ \hline & y - z = -6 \end{array}$$

$y = z - 6$ (4)

Substitute (4) into (3):

$z - 6 + z = 22$
$z = 14$

Substitute $z = 14$ into (2):

$x + 14 = 17$
$x = 3$

Substitute $x = 3$ into (1):

$3 + y = 11$
$y = 8$

Chapter review 4 (p. 209)

1 **(a)** $c = 8$ **(b)** $x = -12.5$ **(c)** $v = 3$ **(d)** $a = 1$

2 D

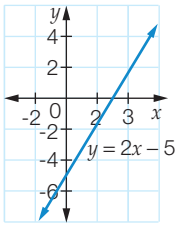
3 **(a)** $m = 5$, y-intercept = (0, 2)

(b) $m = -2$, y-intercept = (0, 0)

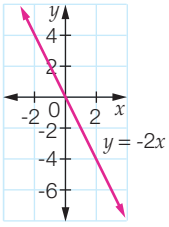
(c) $m = \frac{2}{5}$, y-intercept = (0, -2)

(d) $m = -4$, y-intercept = (0, 8)

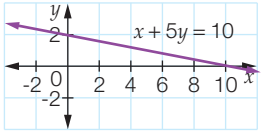
4 **(a)**



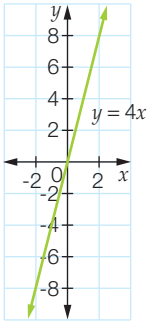
(b)



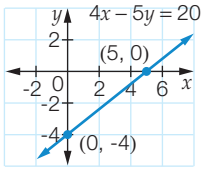
(c)



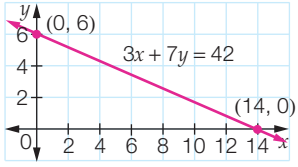
(d)



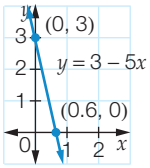
5 **(a)**



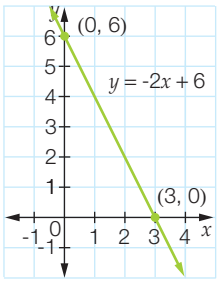
(b)



(c)



(d)



6 **(a)** Both have gradients of -0.5, therefore they are parallel.

(b) With different gradients of 1 and 1.5 they are not parallel.

(c) Both have gradients of -4, therefore they are parallel.

(d) Both have gradients of 4 so they are parallel.

7 **(a)** The lines have gradients of -1 and -2. $-1 \times -2 = 2$. Therefore they are not perpendicular.

(b) The lines have gradients of 1 and -1. $1 \times -1 = -1$. Therefore they are perpendicular.

(c) The lines have gradients of 0.5 and -2. $0.5 \times -2 = -1$. Therefore they are perpendicular.

(d) The lines have gradients of $-\frac{3}{2}$ and $\frac{2}{3}$. Their product is equal to -1 so they are perpendicular.

8 **(a)** $x \geq \frac{7}{3}$ **(b)** $x > -\frac{35}{3}$ **(c)** $x > \frac{5}{6}$

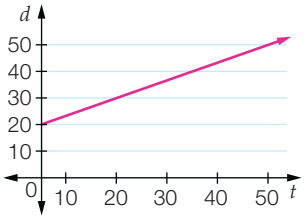
9 (4, 6) **10** (-1, 3)

11 18 wombats, 9 emus **12** 4 hours

13 **(a)** 8 hours **(b)** 1500 L **(c)** 1500 **(d)** $V = 1500T$

14 **(a)** $d = 0.6t + 20$

(b)



(c) Every minute of travel takes Manny 0.6 km further away from Melbourne.

(d) He began 20 km away from Melbourne.

(e) 28.4 km

(f) 46 min and 40 s

15 $y = -x + 9$ **16** $y = 5x - 42$

17 **(a)** $x < 2\frac{4}{5}$ **(b)** $x \leq -2\frac{2}{3}$

18 6 and -5 **19** $a = 9$, $b = -3$

20 **(a)** $570\,000 \leq p \leq 620\,000$ **(b)** $57\,000 \leq d \leq 62\,000$

(c) $p > 670\,000$ **(d)** $p < 670\,000$

21 $d = 165$ km, $s = 82$ km/h

Chapter 5

Recall 5 (p. 213)

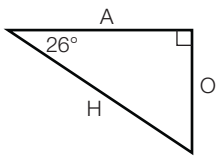
1 (a) $x = 8.17$ (b) $x = 15.7$ (c) $x = 5.20$

(d) $x = 46.377$

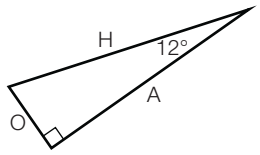
2 (a) 3 cm (b) 15 mm

3 (a) 26 (b) 33.23 (c) 9.95

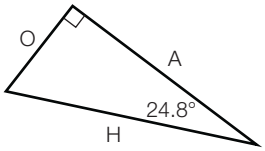
4 (a)



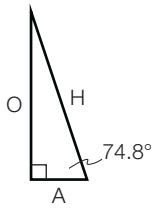
(b)



(c)

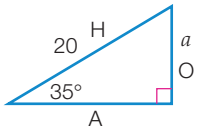


(d)



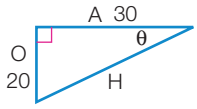
Exercise 5.1 (p. 218)

1 (a)



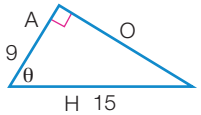
$\sin 35° = \frac{a}{20}$

(b)



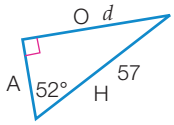
$\tan \theta = \frac{20}{30}$

(c)



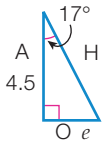
$\cos \theta = \frac{9}{15}$

(d)



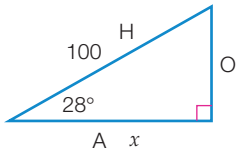
$\sin 52° = \frac{d}{57}$

(e)



$\tan 17° = \frac{e}{4.5}$

(f)



$\cos 28° = \frac{x}{100}$

2 (a) 0.3907 (b) 0.9563 (c) 0.8098

(d) 0.9962 (e) 0.6561 (f) 0.6561

(g) 0.3581 (h) 0.9171 (i) 0.9540

3 (a) 15.50° (b) 35.27° (c) 23.17° (d) 76.80°

(e) 44.99° (f) 8.75°

4 (a) 0.4350 (b) 38.1885 (c) 0.4339 (d) 0.7403

(e) 0.3490 (f) 0.1437

5 (a) C (b) D

6 (a) (i) 1.5 (ii) 1.5 (iii) 1.5

(b) 0.5 (c) 0.5

(d) Yes, because the triangles are similar.

(e) $\propto = \beta = 30°$ (f) yes (g) yes

(h) The triangles are similar so the ratio between the matching sides of the triangles is constant. This means that the ratio of a pair of sides within one triangle is the same as the ratio of the matching pair of sides within the other triangle.

7 (a) (i) 0.5736 (ii) 0.8192 (iii) 0.7002 (iv) 0.7002

(b) The ratio of $\sin 35°$ and $\cos 35°$ is the same as $\tan 35°$.

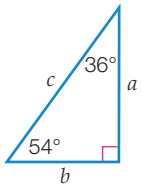
8 1.151

9 B

10 (a) (i) 0.722 (ii) 0.722

(b) Ratios use the same side lengths $\left(\frac{4.6}{6.37}\right)$.

11 (a)



(b) $\sin 54° = \frac{a}{c}$, $\cos 54° = \frac{b}{c}$

(c) $\sin 54° = 0.8090$, $\cos 54° = 0.5878$

(d) $(\sin 54°)^2 + (\cos 54°)^2$
$= 0.8090^2 + 0.5878^2$
$= 0.6545 + 0.3455$
$= 1$

(e) $(\sin 54°)^2 + (\cos 54°)^2$

$= \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$

$= \frac{a^2}{c^2} + \frac{b^2}{c^2}$

$= \frac{a^2 + b^2}{c^2}$

$= \frac{c^2}{c^2}$ (Pythagoras' theorem $a^2 + b^2 = c^2$)

$= 1$

12 An isosceles triangle has two sides equal and two equal angles. In a right-angled isosceles triangle, these two sides are the Opposite and Adjacent of the two base angles, which are each 45°. So:

$\tan \theta = \frac{O}{A}$

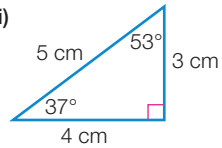
$= \frac{x}{x}$

$= 1$

13 The value of the tan ratio can be greater than 1. The values of the cos and sin ratios can't be greater than 1, as to calculate them you need to divide either the adjacent or opposite side by the hypotenuse, and by definition the hypotenuse will always be larger than the other two sides.

Open-ended – Sample answers

14 **(a)** **(i)** **(ii)**



(iii) $\sin 37° = \frac{3}{5}$ $\quad$ $\sin 53° = \frac{4}{5}$

$= 0.60$ $\quad$ $= 0.80$

$\cos 37° = \frac{4}{5}$ $\quad$ $\cos 53° = \frac{3}{5}$

$= 0.80$ $\quad$ $= 0.60$

$\tan 37° = \frac{3}{4}$ $\quad$ $\tan 53° = \frac{4}{3}$

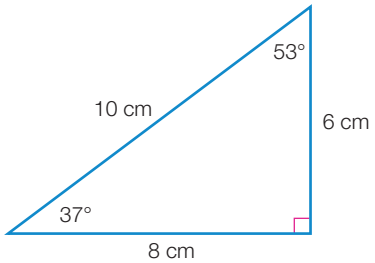
$= 0.75$ $\quad$ $= 1.33$

(iv) $\sin 37° = \cos 53°$

$\cos 37° = \sin 53°$

$\tan 37° = \frac{1}{\tan 53°}$

(b)

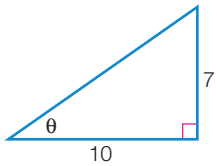


$\sin 37° = 0.60$ $\quad$ $\sin 53° = 0.80$
$\cos 37° = 0.80$ $\quad$ $\cos 53° = 0.60$
$\tan 37° = 0.75$ $\quad$ $\tan 53° = 1.33$

(c) Values of sin, cos and tan are identical as the angles in each triangle are the same.

(d) Measured values agree with the calculator values, rounded to two decimal places.

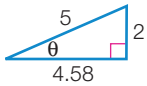
15 **(a)** hypotenuse = 12.2 cm

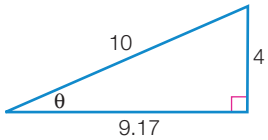


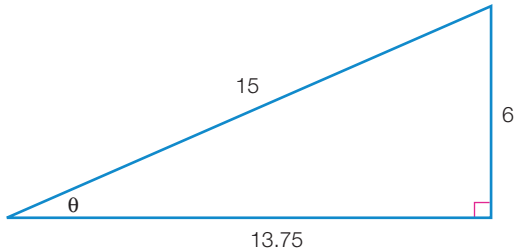
(b) $\sin\theta = \frac{7}{12.2}$ $\quad$ $\cos\theta = \frac{10}{12.2}$

$= 0.5738$ $\quad$ $= 0.8197$

16







17 **(a)** $\theta = 20°$ and $\varphi = 70°$

(b) $\theta = 90 - \varphi$ or $\varphi = 90 - \theta$

Exercise 5.2 (p. 224)

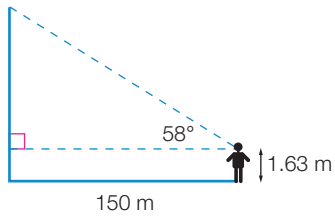
1 **(a)** $a = 167.71$ **(b)** $b = 12.44$ **(c)** $c = 1.05$
(d) $x = 11.53$ **(e)** $x = 11.22$ **(f)** $x = 13.29$
(g) $x = 148.84$ **(h)** $a = 20.58$ **(i)** $b = 42.56$
(j) $d = 32.09$ **(k)** $e = 7.40$ **(l)** $f = 41.08$
(m) $d = 95.83$ **(n)** $c = 244.74$ **(o)** $e = 6.91$

2 193 cm **3** 3.74 m

4 **(a)** B **(b)** B

5 9.89 m

6 **(a)**



(b) 241.7 m

7 15.37 m

8 325.87 m **9** 39.95 cm

10 **(a)** 19.73 m

(b) dropped from 9.86 m along rope, falls 28.7 m

11 **(a)** trapezium 10.74 m^2, rectangle 12 m^2

(b) rectangular room

Open-ended – Sample answers

12 For a pitch angle of 15°:

(a) 1.21 m **(b)** 19.53 m

For a pitch angle of 25°:

(a) 2.1 m **(b)** 21.03 m

13 16 cm, 16.5 cm, 17 cm

14 Taking 'just clear' to be a height of 8.1 m, two possible angles are 40° and 35°.

$\tan 40° = \frac{8.1 - 1.2}{x}$

$x = \frac{6.9}{\tan 40°} = 8.22$ m

$\tan 35° = \frac{8.1 - 1.2}{x}$

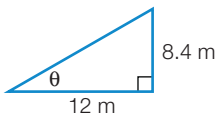
$x = \frac{6.9}{\tan 35°} = 9.9$ m

Exercise 5.3 (p. 229)

1 (a) (i) $a = 81.15°$ (ii) $a = 81°9'10''$
(b) (i) $b = 42.21°$ (ii) $b = 42°12'50''$
(c) (i) $c = 66.58°$ (ii) $c = 66°34'43''$
(d) (i) $k = 56.22°$ (ii) $k = 56°13'13''$
(e) (i) $m = 77.96°$ (ii) $m = 77°57'22''$
(f) (i) $p = 76.29°$ (ii) $p = 76°17'17''$

2 (a) 12°22′25″ (b) 25°42′36″ (c) 47°9′23″
(d) 14°33′5″ (e) 24°11′40″ (f) 54°35′10″

3 (a)



(b) 35°

4 (a) B (b) C

5 0.40° 6 48° 7 53°36′56″ 8 B

9 (a) 33° (b) 14 cm

10 (a) 147° (b) 25.7 m

11 (a) 11.54° to 36.87°
(b) The ladder is too long for the 2 m window. It would be placed 9.8 m away, at an angle of 11.5°.

Open-ended – Sample answers

12 30 cm (22°), 40 cm (30°), 50 cm (39°), 60 cm (49°), 70 cm (61°)

13 Angle of Bilal's shadow with the sun is $\theta = \tan^{-1}\frac{185}{94}$ = 63.06°. For example, take the height of the lamp post as 5 m and the height of the tree as 12 m. Then the length of the lamp post's shadow is $\frac{5}{\tan 63.06°} = 2.54$ m and the length of the tree's shadow is $\frac{12}{\tan 63.06°} = 6.10$ m

Exercise 5.4 (p. 235)

1 14° 2 506 m 3 53°

4 39 m 5 8.55 m

6 (a) 209 m (b) 134 m

7 (a) A (b) B

8 24 m 9 28.84 m 10 4°

11 (a) 177 cm (b) 20.2°

12 (a) 27.7° (b) 12.25 m

Open-ended – Sample answers

13 (a) 46°, 50°, 54°
(b) 12.95 m (46°), 13.79 m (50°), 14.56 m (54°)

14 0.95° (after 20 min), 0.48° (after 44 min)

Half-time 5 (p. 237)

1 6.4° 2 B 3 B

4 θ = 50.36°, height = 23.23 m

5 D 6 320 m

Exercise 5.5 (p. 243)

1 (a) (i) N21°W (ii) 339°T
(b) (i) S73°E (ii) 107°T
(c) (i) N58°E (ii) 058°T
(d) (i) S53°W (ii) 233°T
(e) (i) S51°E (ii) 129°T
(f) (i) N73°W (ii) 287°T

2 115 km

3 (a) True bearing: 152°T Compass bearing: S28°E
(b) True bearing: 147°T Compass bearing: S33°E

4 (a) B (b) C

5 75.74 km 6 260 km

7 (a) 056°T
(b) (i) 288 km (ii) 236°T

8 (a) 2.11 km
(b) (i) 5.87 km (ii) 7.98 km
(c) (i) 1.25 km (ii) 5.78 km

9 (a) 1155 m (b) South 227 m

10 30.39 m

11 (a) (i) 115 m (ii) 164 m
(b) 027°T (c) 1004 m

12 (a) 115°T (b) 3.32 km

13 (a) S23°E (b) 674 km (c) 257 km south
(d) N69°W (e) 721 km

Open-ended – Sample answers

14 If θ = 30°, then:
(a) 150° T, S30° E (b) 330° T, N30° W
(c) 210° T, S30° W

15 Students' own answers.

Exercise 5.6 (p. 251)

1 (a) $x = 16.41, y = 27.27$ (b) $x = 31.40, y = 28.23$
(c) $x = 25.02, y = 3.32$ (d) $x = 28.87, y = 22.91$
(e) $x = 66.75, y = 84.57$ (f) $x = 63.23, y = 78.50$

2 65.80 m

3 (a) $x = 14.4$ (b) $x = 25.87$ (c) $x = 37.14$ (d) $x = 10.51$

4 (a) $a = 34.5°, x = 123.4$ (b) $k = 57.6°, x = 9.5$
(c) $m = 58.5°, x = 69.0$ (d) $a = 31.1°, x = 30.2$

5 A 6 D 7 $24 000

8 31°23′ 9 53 m 10 154 km 11 62°

12 45°38′31″ 13 180.78 m 14 3.80 cm

15 16.83 mm 16 11.92 m

17 (a) 3.46 cm (b) 38 cm^2 (c) 4.53 cm

Open-ended – Sample answers

18 $a = 20$, $\theta = 10°$ so $c = 20.3$; $b = 5$, $\varphi = 20°$ so $d = 14.6$

19 Let $x = 13$ and $y = 11$, then $\theta = 134.5°$.

Chapter review 5 (p. 255)

1 **(a)** 0.5736 **(b)** 1.2482 **(c)** 0.9944 **(d)** 0.5715 **(e)** 0.2823 **(f)** 0.2905

2 **(a)** 22.26 cm **(b)** 11.72 m

3 **(a)** **(i)** 42.27° **(ii)** 42°16′7″
(b) **(i)** 69.29° **(ii)** 69°17′23″
(c) **(i)** 27.45° **(ii)** 27°26′43″

4 62.45 m **5** N64°W

6 $a = 26.7°$, $x = 21.2$

7 418 cm **8** D **9** D

10 17.16 m **11** 19.3° **12** 101 m

13 23.56 m **14** 72 km, N80°25′

15 No, the length of the side of the wedge is correct, but the angle at the point is greater than 40°.

16 84 km

17 Draw a line from the top of the triangle to the midpoint of the base to create a right-angled triangle. Now either use the sine ratio with one side of the original triangle as the hypotenuse and the height as the opposite side, or use the tangent ratio with half of the base as the adjacent side and the height as the opposite side. In both cases the angle will be 60°.

18 34.22 cm

Chapter 6

Recall 6 (p. 258)

1 **(a)** 3.72 **(b)** 46.05 **(c)** 3.77

2 **(a)** 53°8′ **(b)** 67°30′ **(c)** 77°44′

3 **(a)** 8.95 **(b)** 107.57

4 20.32 m

5 **(a)** 10.34 km **(b)** 288.74°T

Exercise 6.1 (p. 262)

1 **(a)** 66° **(b)** 12.81 cm

2 4.05 m^2

3 **(a)** D **(b)** B

4 **(a)** 90° **(b)** 40.6° **(c)** 9.22 mm **(d)** 11.58 mm **(e)** 31.2°

5 8.66 cm

6 **(a)** 147.52 m **(b)** 134.76 m
(c) 17.12 m/min **(d)** 6.66°
(e) gradient of AB is 0.45, gradient of AC is 0.12, so AB is steeper.

7 **(a)** 185.22 cm **(b)** 2°47′7″

8 **(a)** 43.46 m **(b)** 78.41 m **(c)** 93.00 m **(d)** 122.5°T

9 **(a)** 5.37 cm **(b)** 5.13 cm

10 14 289 m

11 Yes. The distance from A to B is 49.75 cm

12 248 m

13 **(a)** 67.71 m **(b)** 75.12 m

14 1941 cm^2

15 Yes. The angle is 75°. **16** 223.07 m

Open-ended – Sample answers

17 The ratio is always 0.07

18 Two values: 20° and 30°

(a) $BF = 4.45$ cm, 6.5 cm

(b) $DF = 12.22$ cm, 11.26 cm
$EF = 9.23$ cm, 7.92 cm
$BC = 9.15$ cm, 10.31 cm

(c) 84.50 cm^2, 81.66 cm^2

(d) As the value of θ increased, the area of $ABCD$ decreased.

(e) 234.97 cm^2, 247.98 cm^2

(f) Less plastic was used when θ was smaller and $ABCD$ had a larger area.

19 **(a)** $h = a\sin\theta$ and $h = b\sin\phi$ so $\frac{a}{b} = \frac{\sin\phi}{\sin\theta}$.

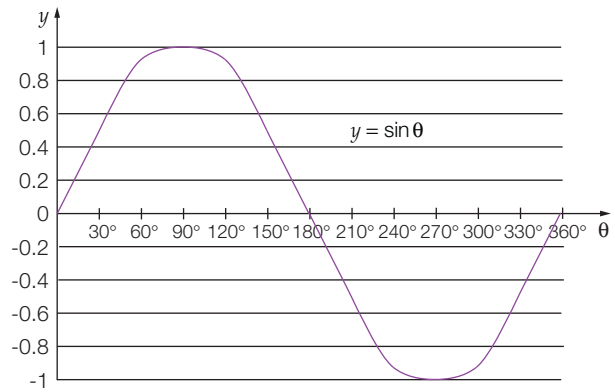
(b) Any of the following values:
$\theta = 21°$, $\phi = 46°$
$\theta = 22°$, $\phi = 49°$
$\theta = 23°$, $\phi = 51°$
$\theta = 24°$, $\phi = 54°$
$\theta = 25°$, $\phi = 58°$

Exercise 6.2 (p. 277)

1 **(a)** 4 **(b)** 3 **(c)** 2 **(d)** 2 **(e)** 3 **(f)** 1 **(g)** 3 **(h)** 2 **(i)** 4 **(j)** 4 **(k)** 2 **(l)** 3

2 **(a)** sin 45° **(b)** -cos 44° **(c)** -tan 69° **(d)** -sin 84° **(e)** cos 39° **(f)** -tan 57° **(g)** -sin 83° **(h)** -cos 83° **(i)** tan 6° **(j)** -sin 18° **(k)** -cos 14° **(l)** tan 27°

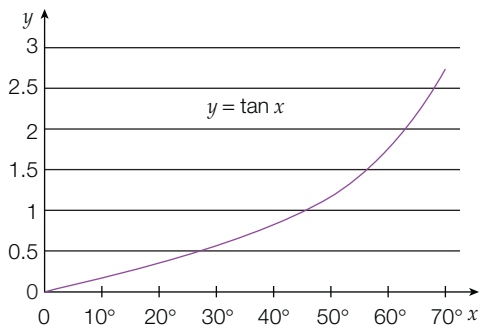
3



4 **(a)** B **(b)** D

5 C

6 (a)



(b) The value of tan x becomes larger at a faster rate and approaches infinity as x approaches 90°.

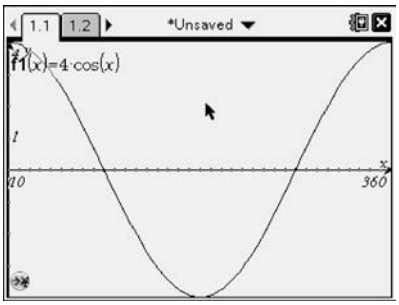
7 For 30°: 150°, 210°, 330°
For 45°: 135°, 225°, 315°
For 60°: 120°, 240°, 300°

8 (a) (i) 360° (ii) 360° (iii) 360°

(b) (i) 1 (ii) 2 (iii) 3

(c) The value of a tells you the amplitude.

(d) A graph like the following should have been sketched.



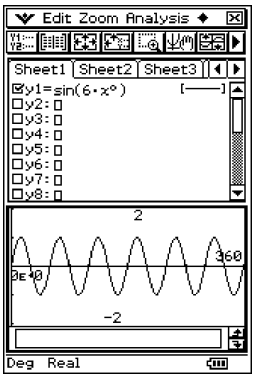
9 (a) (i) 360° (ii) 180° (iii) 120°

(b) (i) 1 (ii) 1 (iii) 1

(c) The value of n changes the period.

(In fact, the period is $\frac{360°}{n}$)

(d) A graph like the following should have been sketched.



10 It will be 156°, the equivalent angle in quadrant 2.

Open-ended – Sample answers

11 Use 24° and 204°

sin	+	–
cos	+	–
tan	+	+

12 (a) If θ is in quadrant 1, -θ is in quadrant 4 (a clockwise rotation of θ). So cos (-θ) and cos θ are positive and of equal magnitude.

If θ is in quadrant 2, -θ is in quadrant 3 so both cos (-θ) and cos (θ) are negative and of equal magnitude.

If θ is in quadrant 3, -θ is in quadrant 2 so both cos (-θ) and cos (θ) are negative and of equal magnitude.

If θ is in quadrant 4, -θ is in quadrant 1 so both cos (-θ) and cos (θ) are positive and of equal magnitude.

(b) Using a similar method as in (a), sin θ is positive in quadrants 1 and 2, and negative in quadrants 3 and 4.

(c) $\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$

Exercise 6.3 (p. 284)

1 (a) $x = 120°, x = 240°$ (b) $x = 30°, x = 150°$

2 $x = (120 \pm 360n)°, n = 0, 1, 2 \ldots$ and
$x = (240 \pm 360n)°, n = 0, 1, 2 \ldots$

3 $x = (14.48 \pm 360n)°, n = 0, 1, 2 \ldots$ and
$x = (165.52 \pm 360n)°, n = 0, 1, 2 \ldots$

4 B

5 (a) 48.59° and 131.41° (b) 158.20° and 338.20°

(c) 53.13° and 306.87° (d) 41.81° and 138.19°

6 9.10 p.m. 7 C 8 C

9 (a) If the denominator becomes equal to 0.

(b) $x = 210°$ and 330°

10 $x = 0°$, 180° and 360°

Open-ended – Sample answers

11 $5 \sin x + 2 = 0; x = 203.58°, 336.45°$
$7 \tan x - 3 = 0; x = 23.20°, 203.20°$
$4 \cos x - 3 = 0; x = 41.41°, 318.59°$

12 (a) Students' own answers.

(b) In (i) it had to be a positive tan equation and in (ii) it had to be a negative sin equation.

Half-time 6 (p. 286)

1 (a) 17.10° (b) 27.20 cm

2 (a) $x = 41.8°, 138.2°$ (b) $x = 21.8°, 201.8°$

3 (a) sin 45° (b) -cos 82° (c) tan 20°

(d) -sin 67° (e) cos 60° (f) -tan 21°

4 (a) 65° (b) 13.6 cm

(c) (i) decrease (ii) 9°

5 C

6 (a) $x = (60 \pm 360n)°$ and $x = (300 \pm 360n)°, n = 1, 2, 3, \ldots$

(b) $x = (19.47 \pm 360n)°$ and $x = (160.53 \pm 360n)°, n = 1, 2, 3, \ldots$

Exercise 6.4 (p. 292)

1 (a) 6.56 (b) 5.89 (c) 48.42

2 (a) 82.16° (b) 40.45° (c) 68.03°

3 (a) 56.23°, 123.77° (b) 45.77°, 134.23°

(c) 68.03°, 111.97°

4 (a) 9.01 (b) 51.50 (c) 42.46

5 (a) 54.90° (b) 110.52° (c) 26.05°

6 39° 7 6.78 units

8 $c^2 = a^2 + b^2 - 2ab\cos\theta$ and $d^2 = 4b^2 + 4a^2 - 8ab\cos\theta$
$d^2 = 4c^2 \Rightarrow d = 2c$

9 113.62°

10 $\angle ABC = \theta - \phi$, so $\dfrac{x}{\sin(\theta - \phi)} = \dfrac{y}{\sin\phi}$

$x = \dfrac{y\sin(\theta - \phi)}{\sin\phi}$

11 D 12 60.99 13 C

14–16 Students' own answers.

Open-ended – Sample answers

17 (a) $c^2 = a^2 + b^2 - 2ab\cos C$
$k^2a^2 = a^2 + a^2 - 2a^2\cos C$
$k^2 = 2 - 2\cos C$
$2 - k^2 = 2\cos C$
$\cos C = \dfrac{2 - k^2}{2}$

(b) $k = -1$, $\cos\theta = 0.5$
$k = 1.5$, $\cos\theta = -0.125$

(c) $-2 \le k \le 2$

18 If $b = 4a$ and $c = 4a$
$\theta = 82.82°$ ($\cos^{-1}\theta = \frac{1}{8}$)

Exercise 6.5 (p. 300)

1 4.92 m 2 10.60 km 3 32.76 m 4 61.47 cm

5 68.40 m

6 Yes, because the angle is 104.478°.

7 Tim 8.22 km, Zoe 6.45 km; so Zoe is closer.

8 46.55° 9 254 m 10 A

11 037°T and 235°T

12 (a) 47.55 m (b) 123.46 m

13 27.41 mm

14 16.33 cm or 21.50 cm

15 B 16 4885 m

Open-ended – Sample answers

17 Students' own answers.

18 $\theta = 45°$, $x = 16$ m, $y = 22.63$ m
$\theta = 60°$, $x = 27.71$ m, $y = 32$ m

Exercise 6.6 (p. 305)

1 (a) 175.96 cm^2 (b) 1269.64 m^2 (c) 330.90 mm^2

2 38°

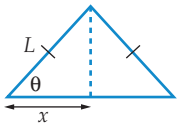
3 (a) 365.01 cm^2 (b) 115.43 m^2 (c) 1718.63 mm^2

4 1292 km^2 5 5.85 mm^2 6 C

7 (a) 36.38 cm^2 or 431.01 cm^2 (b) 278.60 m^2

(c) 583.30 km^2

8 (a) $x = L\cos\theta$



Length of base $= 2L\cos\theta$

Area of triangle $= \frac{1}{2}L \times 2L\cos\theta \times \sin\theta$
$= L^2\cos\theta\sin\theta$

(b) 15.68 km^2

9 (a) Area $= \frac{1}{2}a(b + c)\sin\theta - \frac{1}{2}ab\sin\theta$
$= \frac{1}{2}ab\sin\theta + \frac{1}{2}ac\sin\theta - \frac{1}{2}ab\sin\theta$
$= \frac{1}{2}ac\sin\theta$

(b) 90°, $\frac{1}{2}ac$ mm^2

10 \$1426

11 (a) $0° < \theta < 180°$

(b) The area cannot be negative.

12 22 cm

Open-ended – Sample answers

13 Any triangle where the length of the equal sides is a square number and the included side is 30° (e.g. 36, 36, 30° or 16, 16, 30°).

Chapter review 6 (p. 308)

1 (a) 24.74 cm (b) 118.27 m

2 26.31 m

3 (a) 1 (b) 4 (c) 2 (d) 3

4 (a) -cos 7° (b) -sin 74° (c) -tan 48°

5 B 6 135°, 315°

7 $x = (75.52 \pm 360n)°$ and $x = (284.48 \pm 360n)°$, $n = 0, 1, 2 \ldots$

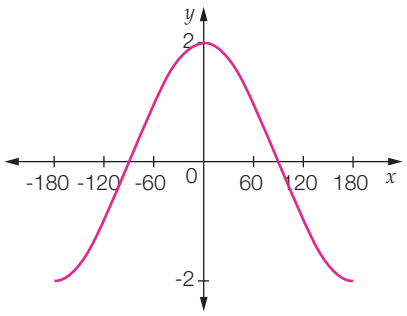
8 C 9 12.81 10 73° 11 207.70 m^2

12 D

13 (a) 180°, 7 (b) 72°, 6 (c) 45°, no amplitude

14 B

15



16 (a) quadrant 1 (b) 320°

17 122° 18 46.03 cm^2 19 39.55 m

20 (a) 62.7° (b) 13.95 cm

21 **(a)** **(i)** 332.86 m **(ii)** 495.02 m

(b) 596.52 m **(c)** 056°T

22 multiples of 90°

23 **(a)** **(i)** Find the third angle α and use the sine formula

$$\frac{x}{\sin\theta} = \frac{a}{\sin\alpha}, \text{ where } \alpha = 180° - \theta - \phi$$

(ii) 13.88

(b) **(i)** Each of the two other angles is $90° - \frac{1}{2}\theta$.

Use the sine formula $\frac{x}{\sin\theta} = \frac{a}{\sin\left(90° - \frac{1}{2}\theta\right)}$.

(ii) 14.51

24 D

25 Students' own answers.

Mixed review B (p. 311)

1 **(a)** \$788.20 **(b)** \$796.82 **(c)** \$801.22

2 4.4%

3 **(a)** $3x^2 + 2x - 8$ **(b)** $6x^2 + 13x - 5$ **(c)** $8x^2 - 26x + 15$

4 **(a)** $(3x - 7)(2x + 1)$ **(b)** $4x(3x - 2)(2x + 3)$

5 **(a)** $(x - 4)(x + 1)$ **(b)** $(x - 5)(x - 9)$

(c) $(x - 4)(x - 3)$

6 **(a)** $\frac{-5}{3x}$ **(b)** $\frac{3x^2 - 5x - 2}{x^2 - 1}$ **(c)** 3

7 **(a)** 75.40 mm^2 **(b)** 382.08 cm^2

8 319.41 cm^2 **9** 720 cm^3 **10** 6.4 g/cm^3

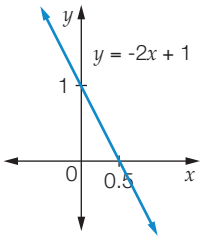
11 **(a)** $x = 9$ **(b)** $x = \frac{7}{3}$ **(c)** $x = 2$ **(d)** $x = 2$

(e) $x = 2$ **(f)** $x = 5$ **(g)** $x = -12$ **(h)** $x = 5$

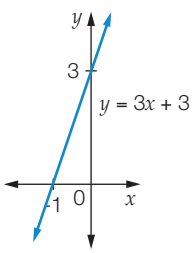
12 **(a)** 1.5 **(b)** 3 **(c)** -2 **(d)** 2

13 **(a)** (-2, -1) **(b)** (1, -1) **(c)** (-8, 0)

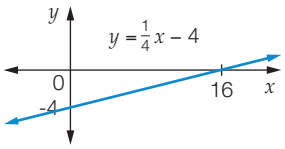
14 **(a)**



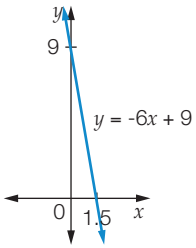
(b)



(c)



(d)



15 **(a)** Yes

$m_1 = -\frac{1}{2}$, $m_2 = 2$, $m_1m_2 = -1$

(b) Yes

$m_1 = 2$, $m_2 = -\frac{1}{2}$, $m_1m_2 = -1$

(c) Yes

$m_1 = -\frac{5}{2}$, $m_2 = \frac{2}{5}$, $m_1m_2 = -1$

(d) No

$m_1 = \frac{4}{3}$, $m_2 = \frac{3}{4}$, $m_1m_2 = -1$

16 **(a)** No

$m_1 = 2$, $m_2 = 1$, $m_1 \neq m_2$,

(b) No

$m_1 = 1$, $m_2 = -1$, $m_1 \neq m_2$

(c) No

$m_1 = -\frac{2}{5}$, $m_2 = \frac{2}{5}$, $m_1 \neq m_2$

(d) Yes

$m_1 = \frac{3}{10}$, $m_2 = \frac{3}{10}$, $m_1 = m_2$

17 **(a)** $x \geq -\frac{3}{2}$ **(b)** $x < -17$

18 **(a)** $x = -1, y = -2$ **(b)** $x = 7, y = -1$

19 **(a)** $h = 20.22$ **(b)** $h = 15.60$ **(c)** $a = 22.85$

20 63.43° **21** 7.79 km **22** 73 km

23 **(a)** 61.9° **(b)** 13.2 cm

24 **(a)** $x = 36.96$ m **(b)** $x = 39.47$ mm

(c) $x = 12.85°$ or $x = 167.15°$

25 **(a)** 141.42 m^2 **(b)** 513.07 mm^2 **(c)** 233.18 cm^2

26 22 cm

27 14 m × 10 m

28 Yes

29 49.62 cm

30 D

31 *AB* has a gradient of $\frac{-2}{3}$, *BC* has a gradient of $\frac{1}{5}$, *CA* has a gradient of -5.

BC and *CA* are perpendicular. Length of *BC* = length of *CA* = $\sqrt{26}$. Therefore, *ABC* is a right-angled isosceles triangle.

Area of *ABC* = 13 $units^2$

32 *AB* has a gradient of $-\frac{2}{3}$, *BC* has a gradient of $\frac{3}{2}$, *CD* has a gradient of $-\frac{2}{3}$, *DA* has a gradient of $\frac{3}{2}$. *AB* and *BC* are perpendicular, *BC* are *CD* are perpendicular, *CD* and *DA* are perpendicular and *DA* and *AB* are perpendicular. Therefore, *ABCD* is a rectangle.

Area of *ABCD* = 39 $units^2$

33 59°, 121°, 115° and 65°

34 $5y = 3x + 21$

35 **(a)** 67.86° **(b)** 756 cm

36 41.71 m

Challenge B (p. 316)

1 B

2 $a = 15 - 4 = 11$, $b = 13 - 3 = 10$. $a - b = 1$

3 D

4 (a) $R(0, 2)$, BR: $2x + 9y - 18 = 0$

(b) $P(3, 0)$, PC: $2x + y - 6 = 0$

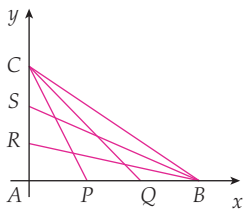
(c) X: $8y - 12 = 0$, $y = 1.5$, $x = 3.6$. $X(2.25, 1.5)$

$Q(6, 0)$, QC: $x + y - 6 = 0$

$S(0, 4)$, SB: $4x + 9y - 36 = 0$

Y: $5y - 12 = 0$. $y = 2.4$, $x = 3.6$. $Y(3.6, 2.4)$

XY: $2x - 3y = 0$. This passes through $A(0, 0)$ so X, Y and A are collinear.



5 (a) -1

(b) CE: $y = -x + 7$ or $x + y - 7 = 0$. $D(0, 7)$

(c) AE: $y - 4 = -3(x - 3)$, $3x + y - 13 = 0$. $A\ 4\frac{1}{3}, 0$. $B(0, 13)$

(d) Shaded area = area ΔODC + area $\Delta BDE = 24.5 + 9$ $= 33.5$ units^2.

6 D $P(z, 5z + 3)$, $Q(3, -2)$. $M\ \frac{z+3}{2}, \frac{5z+1}{2}$. Hence $x = \frac{z+3}{2}$,

$y = \frac{5z+1}{2}$, $z = 2x - 3$, $y = \frac{5(2x-3)+1}{2}$, $y = 5x - 7$

7 B 8 59°

9 $n = 14$, area = 84 cm^2

10 $\cos(\angle MQP) = 0.28$

11 4.58 units

12 13.55 units

13 $AB = 3.2$ m. Length of timber = 16.9 m, $\angle BAX = 22°$, $\angle BYX = 44°$

Chapter 7

Recall 7 (p. 318)

1 (a) mean = 3.2; median = 4; mode = 4; range = 4

(b) mean = 4.7; median = 5.0; mode = 5.0; range = 2

2 (a) discrete (b) continuous (c) nominal

3 (a)

STEM	LEAF	
16	1 3	
16	5 5 5 6 7 7 7	9 9 (Median)
17	0 1 2 2 3	
17	8 9	
18	0 3	

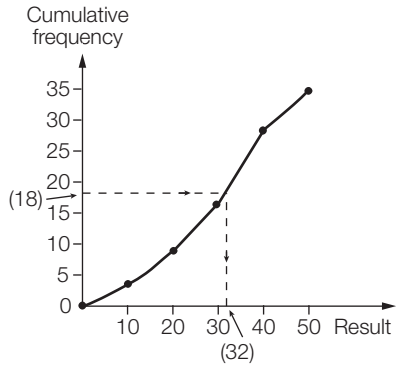
(b) median = 169 cm

4 (a) 6.4875 (b) 6.50–<6.75

Exercise 7.1 (p. 321)

1 (a)

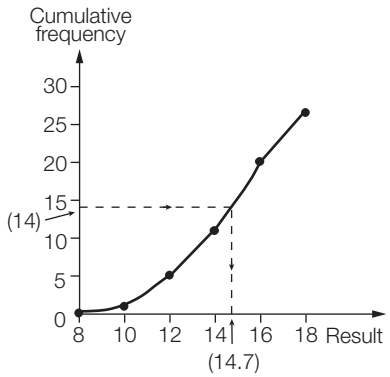
Result	Cumulative frequency
<0	0
<10	4
<20	9
<30	17
<40	28
<50	35



median = 18th term ≈ 32

(b)

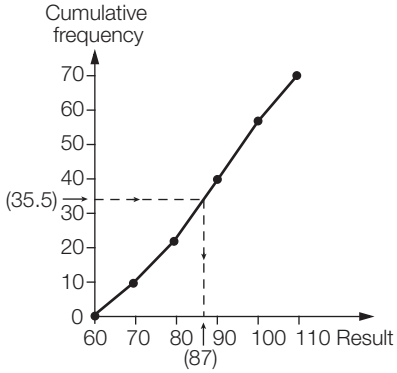
Result	Cumulative frequency
<8	0
<10	1
<12	5
<14	11
<16	20
<18	27



median = 14th term ≈ 14.7

(c)

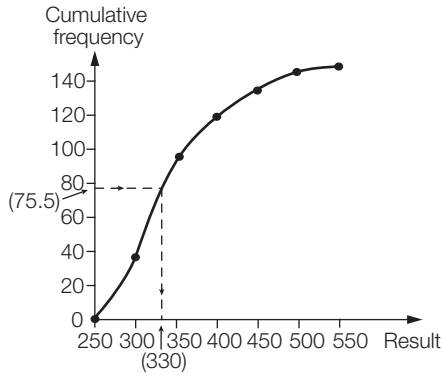
Result	Cumulative frequency
<60	0
<70	9
<80	21
<90	40
<100	57
<110	70



median = 35.5th term ≈ 87

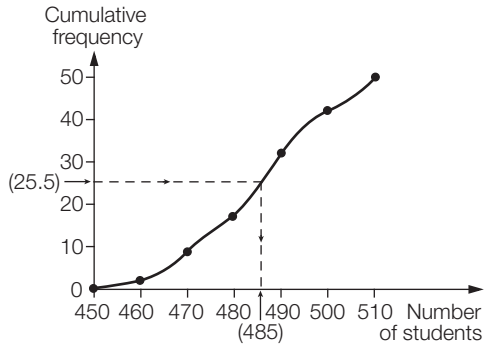
(d)

Result	Cumulative frequency
<250	0
<300	37
<350	95
<400	121
<450	137
<500	146
<550	150



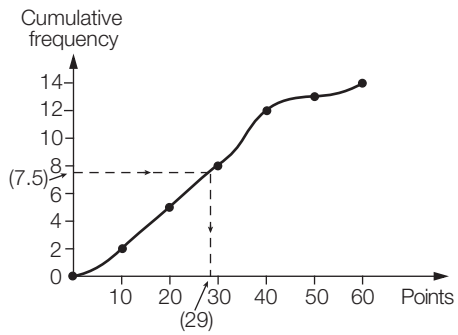
median = 75.5th term ≈ 330

2 (a)



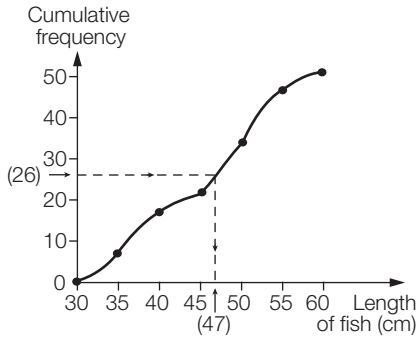
median = 25.5th term ≈ 485 students

(b)



median = 7.5th term ≈ 29 points

3



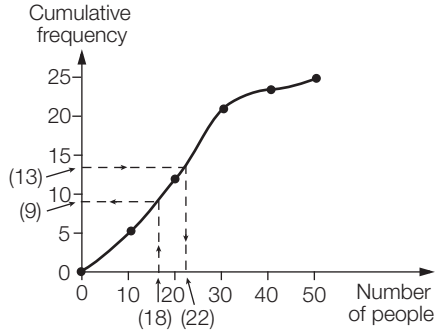
median = 26th term ≈ 47 cm

4 (a) A (b) C (c) B

5 (a) median = 21 people

(b)

No. of people	Cumulative frequency
≤0	0
≤10	5
≤20	12
≤30	21
≤40	23
≤50	25



median = 13th term ≈ 22 people

(c) from raw data 9 trains, from cumulative frequency curve 9 trains

6 (a) 14.7 (b) same

(c) 485.7 students (d) basically the same

(e) If the data is given in frequency table form, the new method is much quicker and seems to give approximately the same results. If the data is given in cumulative frequency graph form, the new method does not seem to have much to offer. The new method assumes that the results are evenly spread throughout the class interval, but this is not necessarily true. However, this is basically the same assumption made by connecting the points on a cumulative frequency graph with straight line segments.

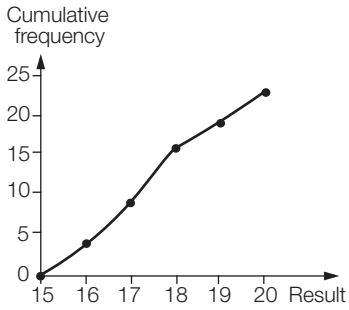
Open-ended – Sample answers

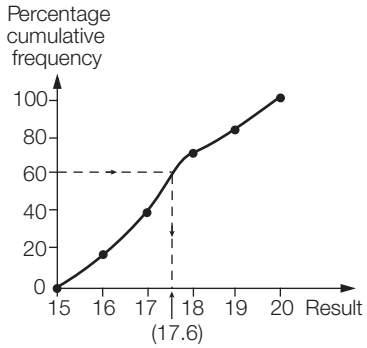
7 Students' own answers.

Exercise 7.2 (p. 332)

1 (a)

Result	Cumulative frequency	Percentage cumulative frequency
<15	0	0
<16	4	17
<17	9	39
<18	16	70
<19	19	83
<20	23	100

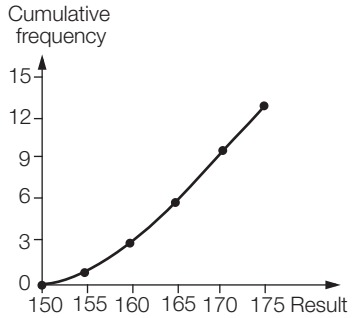


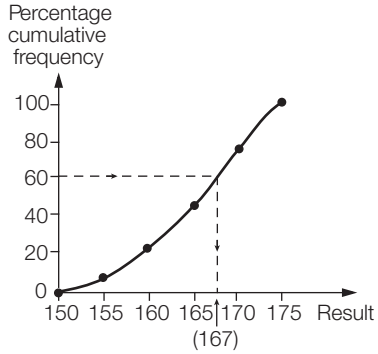


60th percentile is 17.6

(b)

Result	Cumulative frequency	Percentage cumulative frequency
<150	0	0
<155	1	8
<160	3	23
<165	6	46
<170	10	77
<175	13	100



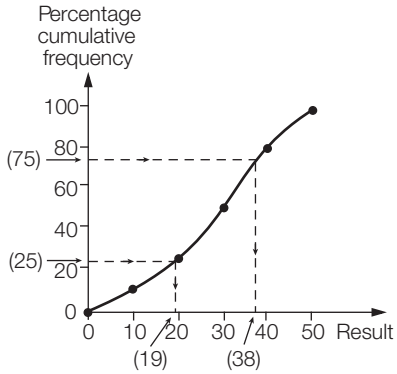


60th percentile is 167

2 (a) (i) Range = 50 – 0 = 50

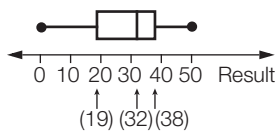
(ii)

Result	Cumulative frequency	Percentage cumulative frequency
<0	0	0
<10	4	11
<20	9	26
<30	17	49
<40	28	80
<50	35	100



IQR = 38 – 19 = 19

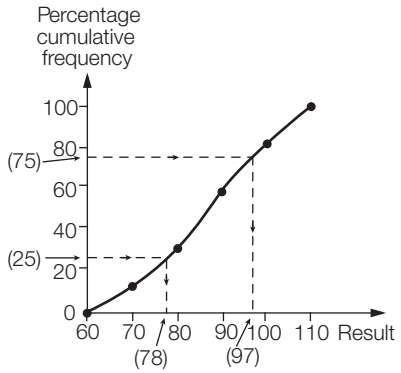
(iii)



(b) (i) Range = 109 – 60 = 49

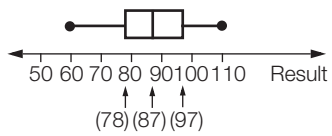
(ii)

Result	Cumulative frequency	Percentage cumulative frequency
<60	0	0
<70	9	13
<80	21	30
<90	40	57
<100	57	81
<110	70	100



IQR = 97 – 78 = 19

(iii)



3 IQR = 14.5, Q_L = 16.5 and Q_U = 31. The score of 54 is an outlier. Even though 8 seems quite a small total it is well within the lower bound for outliers. The high score may have been the result of a low-quality opposition.

4 (a) 65

(b) Q_L = 159.5; Q_U = 179.5; IQR = 20

(c) lower than 129.5 or higher than 209.5

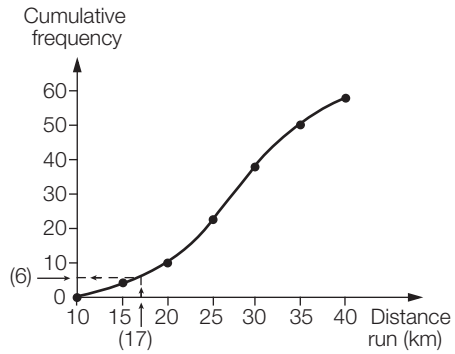
(d) There are no outliers at either end of the data set.

5 (a) C (b) D (c) B

6 D

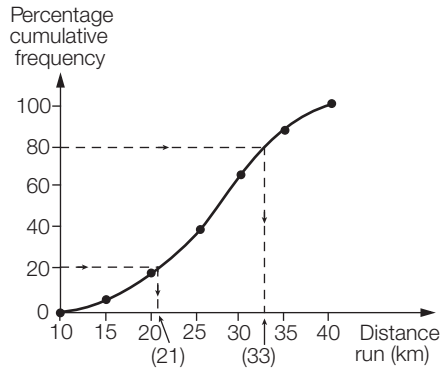
7 (a)

Distance run (km)	Cumulative frequency	Percentage cumulative frequency
<10	0	0
<15	4	7
<20	10	17
<25	23	39
<30	39	66
<35	50	85
<40	59	100



(b) Six distances run were less than 17 km.

(c)

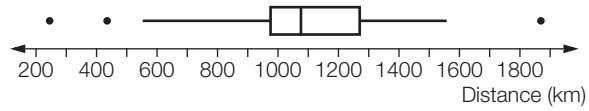


(d) 20th percentile is 21; 0.8 quantile is 33.

8 (a) 1635

(b) Q_L = 972; Q_U = 1275; IQR = 303

(c)



9 (a) Yes, as long as the outlier is more than 1.5 × IQR above Q_U or more than 1.5 × IQR below Q_L.

(b) Students' own answer.

10 (a) (i) less than -5 and greater than 115

(ii) no

(b) (i) less than 32.5 and greater than 52.5

(ii) yes, at both ends

11 (a) (i) As an example: 2 2 2 2 2 2 4 4 5 5
5 5 5 5 5 5 6 6 7 7

(ii) The lowest value is the same as the lower quartile so there needs to be at least six 2s in the list, assuming we are using exactly 20 values. The median does not exist in a discrete fashion so must be the same as either the lower or upper quartile. In this case the upper quartile was chosen. If the lower quartile was chosen, then there would have to be at least eleven 2s.

(b) (i) As an example: 3 3 3 3 3 3 4 5 5 6
6 6 6 6 7 7 7 7 7 7

(ii) In this case there are no whiskers so we need at least six 3s and six 7s. Place the 6s in the middle to ensure that 6 is the median and then fill in the gaps.

Open-ended – Sample answers

12 **(a)** mean = \$46 829.27, median = \$45 000

(b) range = \$50 000, IQR = \$10 104.17 (using the linear approximation method)

(c) Yes, values above \$66 093.76, so at the very least the three highest values are outliers and perhaps one or both of the values in the \$65 000–< \$70 000 class interval.

(d) Removing only the three highest values gives mean = \$45 664.56, median = \$44 791.67 (using the linear approximation method), range = \$35 000, IQR = \$9687.50 (using the linear approximation method).

(e) The mean and range are more affected by the removal of the outliers than are the median and the IQR. This needs to be considered when analysing data sets that contain outliers—the mean and range will need to be treated with some caution.

13 **(a)** Students' own graph.

(b) Students' own graph.

14 **(a)** Example: 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 34

IQR = 11, $Q_L = 5.5$, $Q_U = 16.5$, so any value above 16.5 + 16.5 = 33 will be an outlier.

(b) Example: 36 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69

IQR = 11, $Q_L = 53.5$, $Q_U = 64.5$, so any value below 53.5 – 16.5 = 37 will be an outlier.

Exercise 7.3 (p. 346)

1 **(a)** Set A:

(i) 30.74 **(ii)** 30 **(iii)** 18 **(iv)** 10

Set B:

(i) 30.21 **(ii)** 30 **(iii)** 28 **(iv)** 10

(b) Set A:

(i) 51.22 **(ii)** 51.5 **(iii)** 16 **(iv)** 8

Set B:

(i) 51.22 **(ii)** 51.5 **(iii)** 56 **(iv)** 8

(c) Set A:

(i) 16.82 **(ii)** 15.5 **(iii)** 15 **(iv)** 8

Set B:

(i) 16.64 **(ii)** 14.5 **(iii)** 15 **(iv)** 8

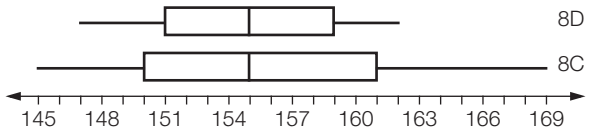
(d) The mean is affected by all changes. The only time the mean did not change in these samples was when the lowest number and the highest number were adjusted by the same amount in opposite directions. The median is very stable and was only affected slightly when data in the middle of the data set was changed. The range fluctuates wildly and is severely affected by any change in the extreme values. The IQR is also quite stable.

2 **(a)** 8C: 155 cm; 8D: 155 cm

(b) 8C: 24; 8D: 15

(c) 8C: 11; 8D: 8

(d)



(e) 8C has a much greater range and IQR than 8D. They have the same median value. The height of students in 8D is much more consistent than for the students in 8C.

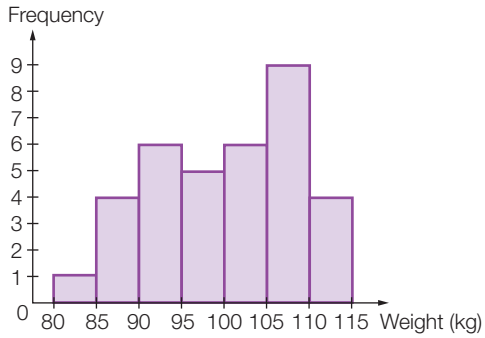
3 **(a)** **(i)** Set A: 45; Set B: 50

(ii) Set A: 35; Set B: 35

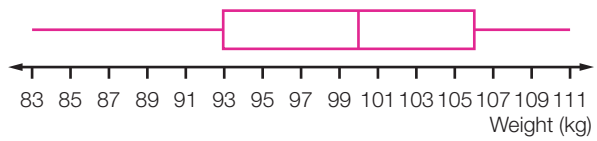
(iii) Set A: 10; Set B: 20

(b) Although they have the same range the two data sets are quite different. Set B has a much bigger IQR, whereas Set A has a much longer right-hand whisker. About 75% of Set A are shorter than the median of Set B.

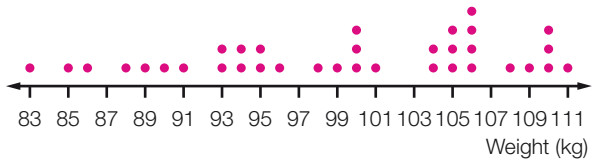
4 **(a)**



(b) min: 83; lower quartile: 93; median: 100; upper quartile: 106; max: 111



(c)



(d) The dot plot shows that the data set is relatively evenly spread out with a slight skew to the right (negative skew). The box plot shows that the median is close to the middle, but slightly to the right. The grouping in the histogram can mask some trends but it is also relatively even with the highest frequency occurring near the right-hand end.

5 **(a)** B **(b)** B

6 The huge range, and relatively big IQR, for Class A means there were some students who did not perform very well on the test as well as some who did extremely well. The fact that some students did well dragged up the mean. Class B was the most consistent class. It depends on what you mean by 'best' as you could argue Class B did best because of its

even results. However, the very highest results are unlikely to be found in Class B. The high IQR for Class C indicates that the middle results were very spread out but the other results were not far outside the middle box.

7 (a) (i) 3 (ii) 1

(b) (i) 21 (ii) 1

(c) The median and IQR give a better feel for the data set as a whole but do not give even a hint that we have two extreme outliers.

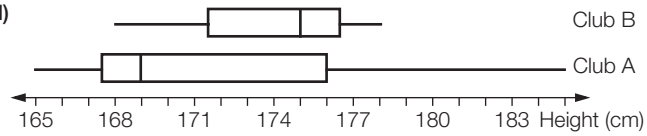
8 (a) Club B seems to be taller.

(b) Mean heights: Club A is 171.75 cm and Club B is 174.1 cm

Median heights: Club A is 169 cm and Club B is 175 cm

(c) Range: Club A is 20 and Club B is 10; IQR: Club A is 8.5 and Club B is 5.

(d)



(e) Yes. Even though Club B does not have the tallest player, it has more consistently tall players, shown by the smaller range and IQR, spread over higher values.

(f) The bunching of the dots for Club B gives us a clue that it will have a small IQR. The parallel dot plot clearly indicates that the range for Club A is going to be much greater.

9 (a) The distances range from 2 to 40 km with a median distance of 12 km. This means about 50% of the students travel up to 12 km to school. The IQR lies between 8 and 26 km.

(b) Students to give their own answers, but it should be greater than 12 km.

(c) Students to give their own answer, but a possible answer would be based on:

Distance (km)	Frequency
0–<4	10
4–<8	30
8–<12	40
12–<16	10
16–<20	15
20–<24	10
24–<28	15
28–<32	10
32–<36	10
36–<40	10

This data set has a mean of 16.5 km.

10 (a) Mean heights: Adelaide 187.77 cm and Essendon 187.83 cm

Mean weights: Adelaide 86.11 kg and Essendon 86.54 kg

(b) Median heights: Adelaide 188 cm and Essendon 187 cm

Median weights: Adelaide 85 kg and Essendon 86 kg

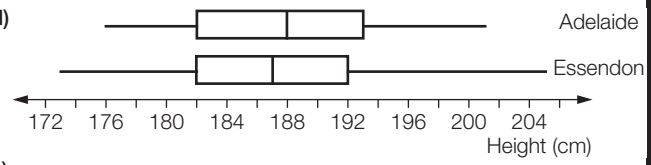
(c) Heights range: Adelaide 25 cm and Essendon 32 cm

Heights IQR: Adelaide 11 cm and Essendon 10 cm

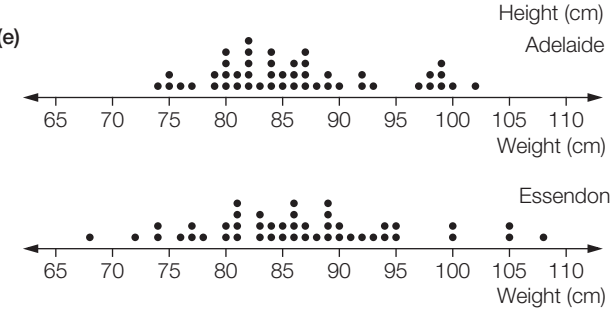
Weights range: Adelaide 28 kg and Essendon 40 kg

Weights IQR: Adelaide 9 kg and Essendon 10 kg

(d)



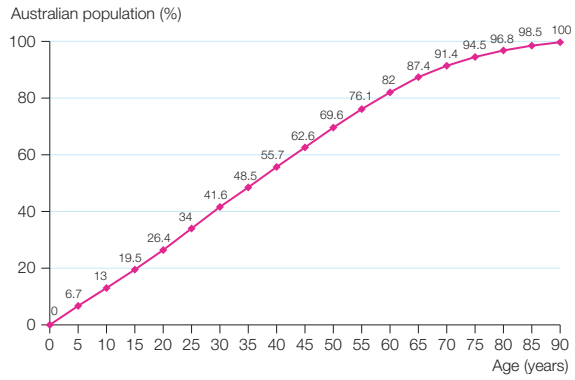
(e)

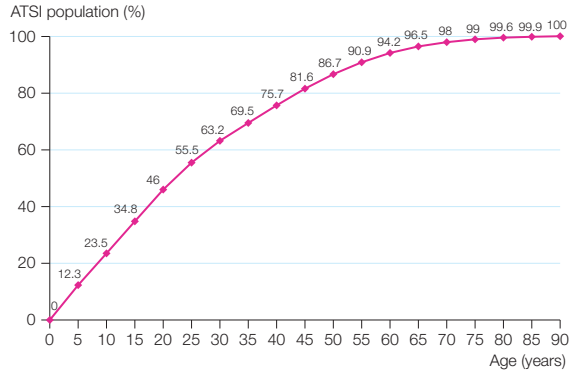


(f) Putting together all of the information collected the final comments would be that Essendon has a greater range for height than Adelaide but essentially the two clubs are extremely close in terms of height.

(g) Sample comment: Essendon has the two lightest players but also the two heaviest. Essendon seems to be spread out more evenly, whereas Adelaide is clumped in a couple of places. The weights are quite even but Essendon may be marginally heavier.

11 (a)

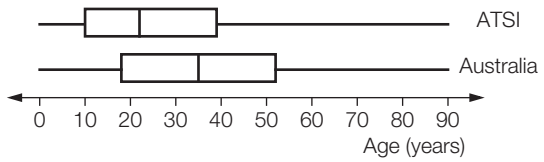




(b) Australia: min = 0; $Q_L = 18$; median = 35; $Q_U = 52$; max = 90

ATSI: min = 0; $Q_L = 10$; median = 22; $Q_U = 39$; max = 90

(c)



(d) The cumulative frequency curves start to tell the story with their very different shapes. The rapid rise at the beginning of the ATSI graph indicates that a great percentage of the population are young. This is confirmed by the box plot, which shows the median age to 22 and the lower quartile to be 10. This means 25% of the ATSI population is less than 10 years of age, compared to an age of 18 for the Australian population as a whole. The medians are different by 13 years. The upper quartile for the ATSI population is barely bigger than the median for the Australian population.

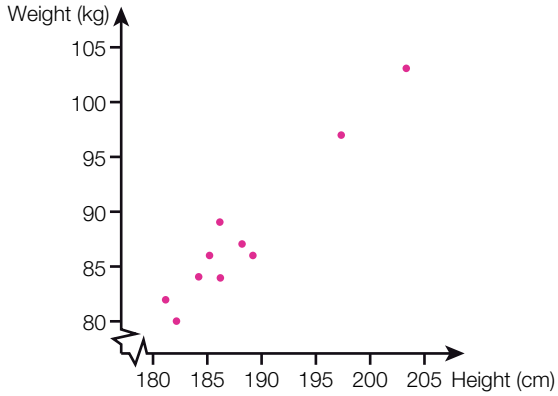
Open-ended – Sample answers

12 Students' own answers.

Exercise 7.4 (p. 356)

1 **(a)** primary **(b)** primary **(c)** secondary **(d)** primary **(e)** secondary

2



As height increases so it seems does weight. However, this relationship is not perfect so we would probably say there is a strong positive relationship between the two variables.

3 **(a)** census **(b)** sample **(c)** census **(d)** census

4 **(a)** Inappropriate—not relevant

(b) Probably appropriate—if not often then may not know much about sport

(c) Inappropriate—is on the way to being appropriate but should not be closed

(d) Inappropriate—not relevant

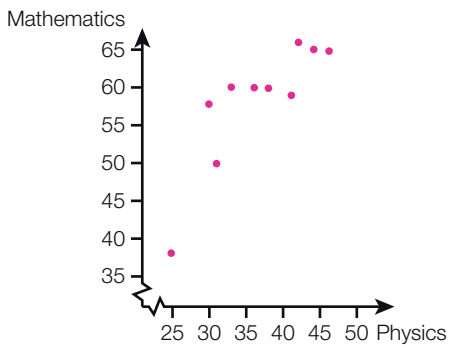
5 **(a)** Biased—not a broad enough range of people

(b) Biased—may be more likely to support environmental causes

(c) Unbiased

(d) An inappropriate method, whether biased or not is unclear.

6 **(a)**



(b) For the lower physics marks it seems that as the mark increased so did the mathematics mark. However, this trend did not continue with the higher physics marks where the mathematics mark seemed to flatten out and not increase at the same rate as the physics mark. Overall, there is a moderate positive relationship.

7 **(a)** This is a very strong negative linear relationship.

(b) This is a strong positive linear relationship.

(c) This looks like a non-linear relationship; it has a curve associated with it. However, it does display a negative relationship.

(d) This looks like there is no relationship between the variables.

(e) This looks like a non-linear relationship. The points go up and down as you move across the horizontal axis.

(f) This looks like a weak negative linear relationship.

8 **(a)** It would be difficult and time consuming to get the opinions of parents compared to students and staff who are on campus.

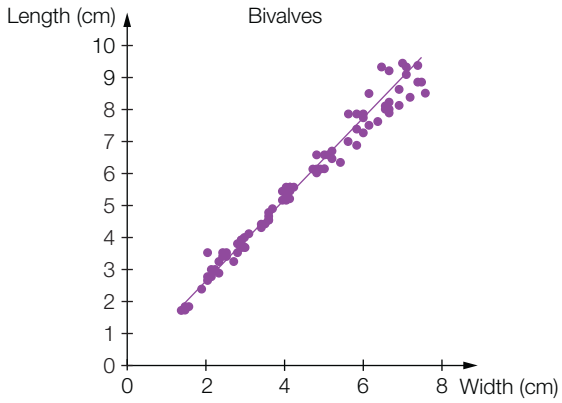
(b) Too many to conduct a census

(c) You need to keep the representation level the same as for students

(d) A stratified random sample will randomly select students from each strata, in this case year levels. It is quick and relatively cheap to conduct, and representative of the structure of the school.

(e) Students will provide their own answers.

9 **(a)**



(b) As the width of the bivalve increases so does the length. These two variables seem to have a very strong positive linear relationship.

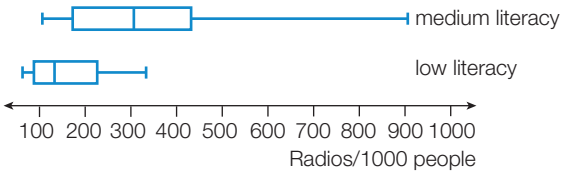
10 (a) secondary

(b) There seems to be a strong positive linear connection between the two variables. Male rates are possibly a little higher than female rates.

(c) No—a different population group entirely.

11 (a) The countries with a higher literacy rate have a higher number of radios.

(b)



(c) The range for medium literacy countries is much higher but the lowest value for the medium countries is higher than the lower quartile for the low countries. The highest literacy value for the low countries is only just higher than the median value for the medium countries. The medium literacy countries have a very long right-hand whisker; it may be that there are one or two outliers that have made the comparisons inaccurate. The numbers of radios are clearly different between these two types of groups of countries. However, it would be incorrect to conclude that a country's literacy rate determines the number of radios.

(d) 907 radios per 1000 people in a medium literacy rate country.

Open-ended – Sample answers

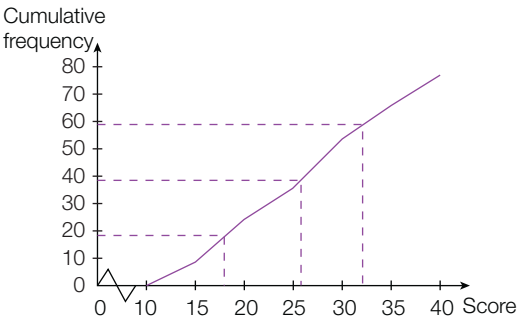
12 Students' own answers.

13 Students' own answers.

Half-time 7 (p. 360)

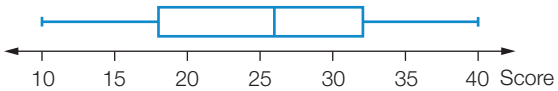
1 (a) Range: 40 – 10 = 30

(b)



From the cumulative frequency graph we get $Q_L = 18$ and $Q_U = 32$, which gives an IQR of 32 – 18 = 14.

(c)

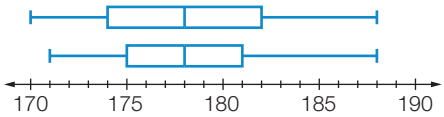


2 (a) Club B appears to be taller but this is mainly based on the fact that it does not have as many players at the lower end of the height scale.

(b) Club A: mean = 178.47, median = 178, range = 18, IQR = 8

Club B: mean = 178.4, median = 178, range = 17, IQR = 6

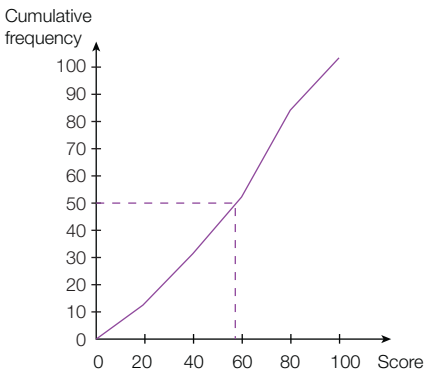
(c)



(d) Club A has a very slightly larger average height, however, they are essentially the same. The IQR in Club A is also slightly larger, which shows a slightly larger spread of heights compared to Club B.

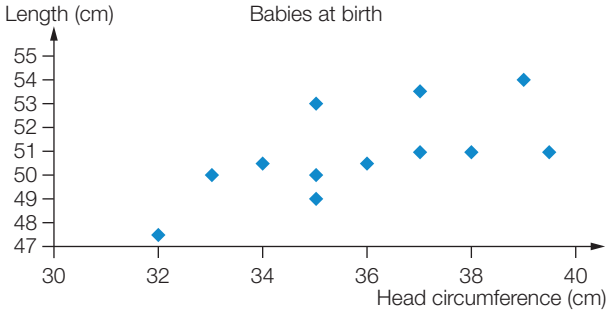
(e) The IQR, as this statistic has the greatest difference and shows how the spread of the data varies between the two clubs.

3



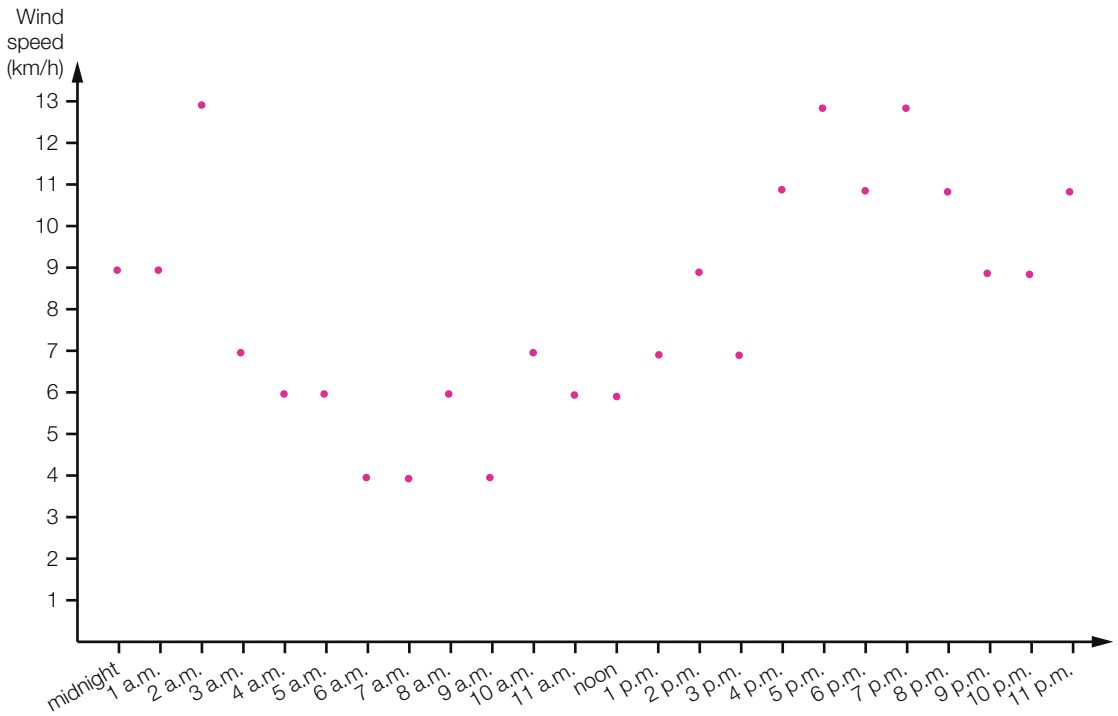
Median = 58

4 There appears to be a moderate positive linear relationship between head circumference and length in babies at birth.



Exercise 7.5 (p. 362)

1 (a)



(b) It is much higher than the readings around it.

(c) The wind speed fluctuates between 11 km/h and 13 km/h.

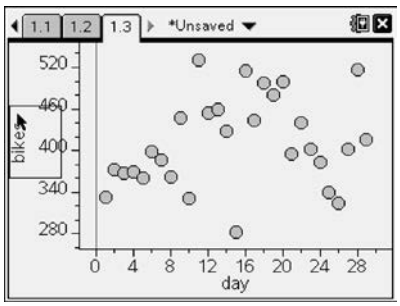
2 (a) Hours spent studying; this is the variable under our control.

(b) Age; this is the variable that is always changing—height does not necessarily change.

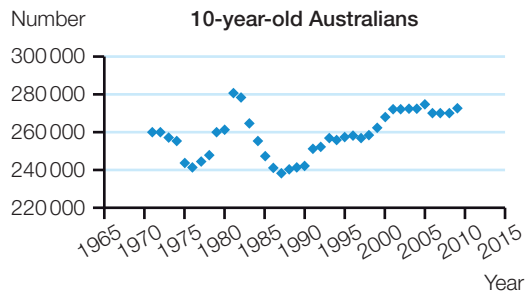
(c) Number of cigarettes; this is what has an impact on age at death.

(d) Number of people; change the number of people and you would expect a change in the amount of rubbish.

3 There does not appear to be a pattern in the number of bicycles. This is perhaps not an unexpected result because the weather could play a significant role in the number of cyclists on any particular day.

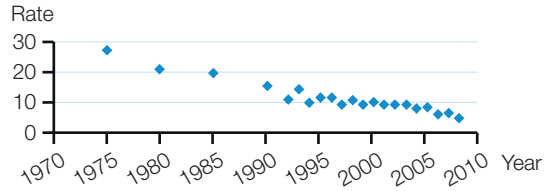


4 (a)



(b) The number of 10-year-old Australians has fluctuated a little over this time period but is now about the same as it was at the beginning of the period. There was a peak in the early 1980s, followed by a trough in the mid-1980s when the lowest number was recorded, but the number has steadily increased since then. The number appears to be becoming more stable.

5 (a)



(b) There is certainly a trend for a considerable decrease in the death rate per 100 000 in South Australia, although the fall has not been consistent. As may have been expected, the most significant reductions were made in the earlier years and there have been some slight ups and downs along the way. It would be interesting to compare the rate with significant road toll initiatives, such as the introduction of the compulsory wearing of seat belts.

6 (a) The population has increased consistently over this period of time. The last 40–50 years have seen the greatest rate of increase.

(b) This inclusion would have minimal impact except perhaps in the very early years shown on the graph. The graph would have started at a higher value, clearly off the axis. Depending on how rapidly the Aboriginal population declined would determine whether or not their inclusion would be detectable.

(c) The population was very small, but the scale needs to be as is to fit today's population into a reasonably sized graph.

(d) In the very early years by counting every individual. Records of arrivals, departures, births and deaths would also have been used.

(e) The very early years may actually be more reliable than the later ones because the later ones have some element of estimation built into them.

(f) There is a census every 5 years and population models are based on the data collected.

7 The number of 90-year-olds is increasing at a fairly rapid rate. The average lifespan is increasing in parallel with improvements in medical care. For younger people, the same trend would probably not be seen.

Open-ended – Sample answers

8 The graph lacks a title and axes labels. There also needs to be a key to identify which colour represents which model. The vertical scale could also be stretched a little to show the differences more clearly.

9 Students' own answers.

10 Students' own answers.

Exercise 7.6 (p. 367)

1 (a) The shaded numbers are approximations, the others are certain.

Position	Sports Identity
1	Greg Norman
2	Andrew Bogut
3	Harry Kewell
4	Mark Webber
5	Chad Reed
8	Geoff Ogilvy
9	Marcos Ambrose
10	Casey Stoner
13	Ricky Ponting
17	Brett Lee
19	Lleyton Hewitt
21	Michael Clarke
41	Michael Hussey
50	Andrew Symonds

It is difficult to be precise about some of these positions because there are large gaps between the positions mentioned in the article.

(b) approximately 9.5% drop

(c) approximately 56.4% drop

(d) $\frac{65}{5} \neq 13$ so it is not just a straight division. The contract must have some higher up–front payments.

(e) \$1.53m is better than 41st but not as good as 21st (\$2.54m). Somewhere around the low 30s would seem appropriate.

2 (a) What a mess! A pie graph assigns percentages of a whole and these percentages clearly do not add up to 100. The explanation given is that it was possible to offer support to more than one candidate. That is fine, but a pie chart should never have been used to represent it. You would not be able to draw a correct graph from this information as you would need to know which one candidate each person supported.

(b) The lottery graph shows that fewer people pick the numbers from 0 to 100, possibly because they don't like starting their choice with one, or even two, zeroes. With few people to claim the prize pool, the winning payoff is larger.

3 (a) \$390 (b) \$341

(c) 14.4%, so the article has not exaggerated the rise in its claim of 14%

(d) \$345.43

(e) old: \$1020.41; new: \$1070.41

(f) 11.1% p.a. increase

(g) Crash repairs are a big part of the cost of insurance. If car crashes are reduced the overall cost of providing insurance cover should also reduce.

4 (a) Rounded to one decimal place:

Guildford 17.3%; Mount Richon 16.9%; Victoria Park 15.7%; Darlington 14.6%; Orelia 13.9%; Maylands 12.5%; South Fremantle 12.1%; Stoneville 11.7%; Fremantle 10.7%; Joondanna 10.2%

(b) Rounded to one decimal place:

North Beach -35.4%; Ascot -32.1%; Mount Lawley -28.3%; Mosman Park -24.5%; Forrestdale -24.0%; Applecross -22.8%; Perth -21.1%; Marmion -21.0%; Kallaroo -17.4%; Swanbourne -17.3%

(c) There is wide variability in prices both in absolute terms and in percentage change terms. It might even be described as volatile. The big losses seem to be associated with some of the more expensive suburbs.

(d) The number of houses in each of these suburbs. This would enable a percentage turnover to be calculated.

(e) House buyers concentrated on lower priced houses so the prices of houses in the more expensive suburbs fell as there was little competition for these properties.

(f) Mode is almost useless as there is such a range of prices possible. It might be that the only two houses sold for the same price were the least expensive houses of all; this is hardly representative. Mean is affected by very low and very high prices.

5 (a) 360

(b) Students to provide their own answer, but it is likely they will say it is not a very big sample.

(c) self-selecting

(d) Yes, the questions are quite different. It is quite possible for a person to hold a view about what should happen in general but have a different view about the same issue as it directly affects them.

(e) The telcos may be talking themselves up. It would be interesting to see some data to support this view. However, it is quite possible, as the survey methodology seems to be quite flawed.

(f) These findings are probably not very reliable. The sample size is relatively small and the selection process would not lead to robust results.

6 (a) It is between two outlets that have both recently sold tickets that won multimillion dollar prizes, so it could be next.

(b) No, this is nonsense.

(c) At the very least you would want to know how many tickets were sold. What percentage of tickets sold are winners? But even this is ridiculous as each ticket, no matter where sold, has the same chance of winning.

(d) If the numbers have been drawn often then they are lucky numbers that will continue to be drawn. If they haven't been drawn for a long time then it is their turn to be drawn. Both arguments are bereft of statistical merit.

7 (a) Mr Casey is the preferred Prime Minister (48%) compared with Mr Smith (46%). Only female voters preferred Mr Smith (47%) compared with Mr Casey (41%). The heading is misleading.

(b) Sample size 1030, $n = 1030^2 = 1\,060\,900$. This is a reasonable sample size, given that the voting population may be less than 10 million.

(c) Telephone survey is the most convenient sampling method.

(d) The figures for female opinions (as in part **(a)**) give some justification for the heading 'Voters back Smith'—but this is not true for male voters.

(e) When you add the votes for either preferred Prime Minister, the total male votes equal 99% and the total female votes equal 88%. The missing 1% and 12% of votes respectively are due to the 'neither' and 'don't know' categories chosen by some persons surveyed.

(f) These categories attempt to identify the grounds for the voters' opinions. We do not know whether these categories are important to the voters.

(g) This poll does indicate that the preferred Prime Minister is Mr Casey (in the general population), but only slightly. The survey also covers the personal qualities of the candidates.

8 (a) The other States and the ACT probably don't have a high enough Indigenous population to make the statistics accurate.

(b) Males 78.7 years and females 82.6 years.

(c) Students to provide their own answer, but would probably include thoughts such as lack of medical facilities in many traditional communities and the impact of certain illnesses and diseases, such as diabetes, on the Indigenous communities.

(d) It is difficult because the ATSI population count is probably not accurate and deaths are not always reported.

(e) Canada, New Zealand and the United States also have significant Indigenous populations and the countries themselves are quite similar to Australia.

(f) Students to provide their own answer, but some of the following may be included. The percentage increase for Indigenous males is 2.54% compared to 1.56% for non-Indigenous males and for females it is 2.00% for the Indigenous population compared to 1.09% for the non-Indigenous population. So the rate of increase is higher but this is due to the lower starting figure. A fair society would want to see these differences become much smaller.

Open-ended – Sample answers

9 Students will provide their own answers.

10 The most interesting question is where the figure for the probability of finding buried treasure came from. The other probability can probably be calculated although it is open to debate what the expression 'top prize at the pokies' means. Each machine would have a different 'top prize' so this makes things more difficult. However, the thrust of the advertisement about the unlikelihood of winning a huge prize on the pokies is to be applauded.

Exercise 7.7 (p. 383)

1 5.72

2 English: mean = 34.33 and standard deviation = 9.22

Mathematics: mean = 34.67 and standard deviation = 7.12

Although the means are effectively the same the standard deviation for mathematics is smaller indicating that the results in that subject are more tightly packed around the mean. In turn, this indicates that the more extreme results will be found in the English marks.

3 mean = 3.69 and standard deviation = 2.52

4 (a) Australia: 241; New Zealand: 597; Brazil: 940; Sri Lanka: 628

(b) Australia: 76.50; New Zealand: 190.74; Brazil: 250.39; Sri Lanka: 165.97

(c) As the range increases so does the standard deviation except for Sri Lanka. Its standard deviation is less than New Zealand's despite its range being slightly higher than New Zealand's. This is probably due to the highest peak in Sri Lanka being quite a bit higher than the next value. It may be an outlier.

5 (a) (i) 4.81 (ii) 8.77 (iii) 6.12 (iv) 9.66

(b) Introducing an outlier at either end has a similar effect as increasing the standard deviation (one of them was further away) and introducing two outliers at either end stretches the standard deviation even further.

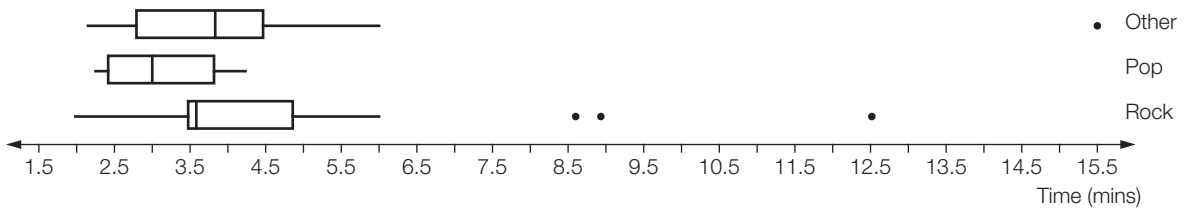
6 B

7 (a) mean = 4:22 and standard deviation = 2:12

(b) mean = 3:06 and standard deviation = 0:41

(c) mean = 4:12 and standard deviation = 2:36

(d)



(e) From the evidence presented you could conclude that pop songs are more formulaic than either of the other categories. The standard deviation was very small, the box has no outliers and the range is very small. The longest pop song is within the third quarter of each of the other box plots. This means at least 25% of each of the other categories are longer than the longest pop song. The number of pop songs was the smallest, which may have had an effect on the data. The Other category has one wild outlier. It is 9.5 minutes longer than the next longest song. On the other hand there were three outliers in the Rock category.

Open-ended – Sample answers

8 (a) Students' own answers.

(b)–(c) The range condition means the data set cannot be mostly one number with just a little tinkering. The first step would be to calculate what the sum of the data set would be, number of values × mean. Then, make a start on the data set so it adds up to this value. Some adjustments may be required to make the standard deviation work.

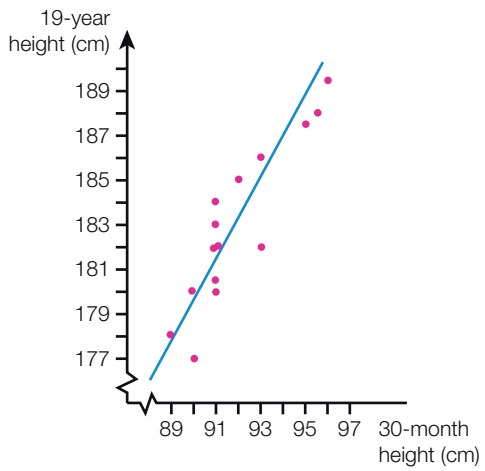
9 The first step is to identify the five-number summary:
min = 12, $Q_L = 13$, median = 16.5, $Q_U = 18$ and max = 20. Construct the data set around these values and then find the mean and the standard deviation.

Exercise 7.8 (p. 389)

1 (a) $y = 12.107x + 4.536$ (b) $y = 15.497x - 14.552$

(c) $y = -4.261x + 46.931$ (d) $y = -4.863xx + 39.999$

2 (a) and (b)



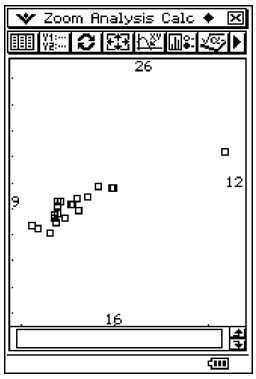
(c) The 30-month height does seem to be a reasonably good predictor of adult height.

(d) approximately 187 cm (will vary depending on the line of best fit)

3 (a) C

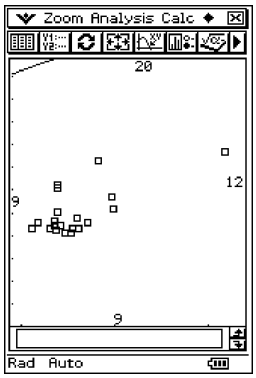
(b) (i) $q = 2.69 + 1.16p$ (ii) A

4 (a)



(b) There does appear to be a strong positive linear connection between these two times. In general, as one value increases, so does the other. It would seem appropriate to fit a straight line for these points.

(c)



(d) There does not appear to be a strong connection between these two times. It would not seem appropriate to fit a straight line to these points.

(e) There is a strong probability that the same person will hold the two sprint records but it will be a different person who holds the other record. The two sprint times are therefore closely linked but no such link is established for a sprint and a middle distance run.

5 (a) The value of r is 0.9987 so it is appropriate to find the equation of a linear line of best fit. It is $F = 4.90m - 0.15$.

(b) This ratio is the gradient of the line of best fit, 4.90.

(c) (i) 19.45 N (ii) 53.75 N (iii) 90.50 N

(d) (i) 244.85 N (ii) 979.85 N (iii) 4899.85 N

6 (a) Solubility = 0.05 × Temperature + 35.05

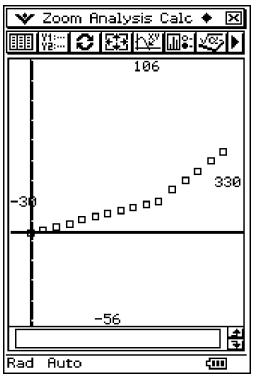
(b) (i) 36.30 units (ii) 38.95 units (iii) 41.05 units

(c) Temperature = 20.02 × Solubility − 701.40

(d) (i) 19.32°C (ii) 69.37°C (iii) 199.50°C

(e) Both of these are extrapolated values; we would need to assume that the connection between the variables remains the same.

7 (a)



The first group ends at the point (200, 20)

(b) $F = 0.100\Delta x - 0.04$, $0 \le \Delta x \le 200$ and
$F = 0.294\Delta x - 37.75$, $200 < \Delta x \le 300$.

(c) (i) 7.5 N (ii) 15.0 N (iii) 28.4 N

(d) $\Delta x = 9.965F + 0.440$, $0 \le F \le 20$ and
$\Delta x = 3.406F + 128.667$, $20 < F < 50.2$

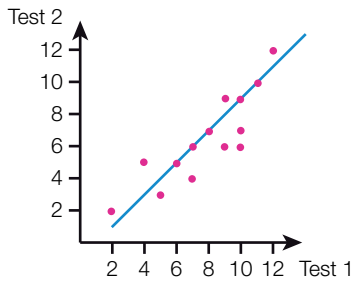
(e) (i) 50 mm (ii) 231 mm (iii) 282 mm

(f) If we substitute 500 mm into the appropriate equation we get 110.4 N which is much greater than the stated value, so we would assume the equation is not valid in that region. Without any other information the best we could do is assume the data is linear between (300, 50.2) and (500, 90). This would give the equation $F = 0.199\Delta x - 9.5$.

Open-ended – Sample answers

8 (a) Emily has just drawn the line of best fit as connecting the first and last points. It should be drawn to represent the general trend of the points with approximately equal numbers of values either side of the line.

(b)

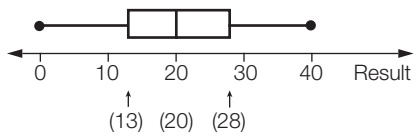


9 Students' own answers.

Chapter review 7 (p. 393)

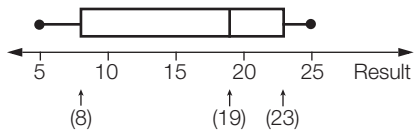
1 (a) (i) median = 20.0 (ii) IQR = 28 − 13 = 15

(iii)



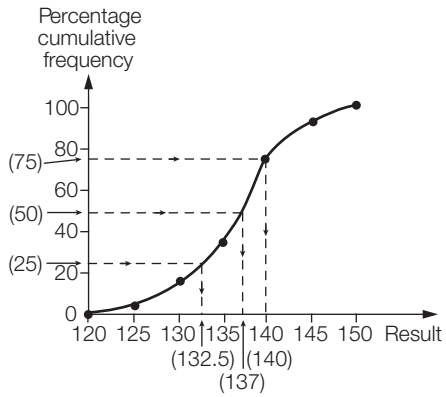
(b) (i) median = 19 (ii) IQR = 23 − 8 = 15

(iii)



2 (a)

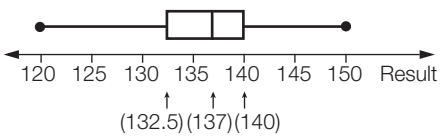
Result	Frequency	Percentage cumulative frequency
<120	0	0
<125	5	3
<130	17	15
<135	32	37
<140	54	74
<145	24	91
<150	13	100



median = 137

(b) IQR = $Q_U - Q_L = 140 - 132.5 = 7.5$

(c)



3 less than 11 cm or greater than 27 cm

4 Team A has been far more consistent than Team B showing lower values for the range and the IQR. However, consistency in scores is not a predictor of success in team-based sports. It looks as if Team B would have lost the game when it scored 4 goals and won the game when it scored 24 goals but other than this it is difficult to make any firm statements. Both teams have the same median number of goals. If pushed you might lean towards saying Team B has been more successful. You definitely need to have access to the raw data.

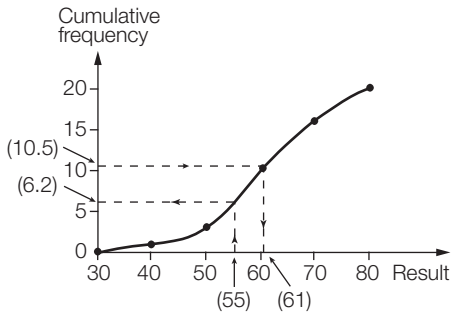
5 (a) primary (b) secondary (c) secondary (d) primary

6 (a) 3.77 (b) 1.08 (c) 5.35

7 (a) $y = 5.536x + 2.536$ (b) $y = -2.080x + 19.428$

8 (a)

Class interval	Frequency	Cumulative frequency
20–<30	0	0
30–<40	1	1
40–<50	2	3
50–<60	7	10
60–<70	6	16
70–<80	4	20



(b) min = 34.5; $Q_1 = 53.55$; median = 60.85; $Q_3 = 67.75$; max = 73.6

(c) From cumulative frequency curve,

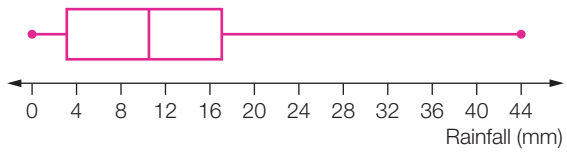
Median = 10.5th term = 61, compared with

Median = $\frac{59.9 + 61.8}{2}$ = 60.85 from the raw data, so they are basically the same.

9 (a) 44 – 0 = 44

(b) min = 0; $Q_1 = 3$; median = 10.5; $Q_3 = 17$; max = 44

(c)



10 (a) (i) less than 8.5 and greater than 20.5

(ii) yes (at least one result greater than 20.5)

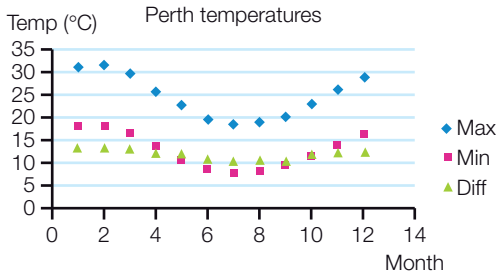
(b) (i) less than 24 and greater than 40

(ii) no (the lowest result is right on the margin)

11 (a)

Jan	Feb	Mar	Apr	May	Jun
13.0	13.3	13.0	12.1	11.9	10.9
Jul	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
10.5	10.7	10.5	11.7	12.1	12.6

(b)



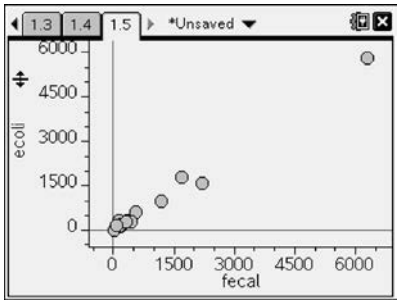
(c) Perth follows a typical temperature pattern for the southern part of Australia—warm to hot in the summer and cool to cold in the winter. As might be expected the difference graph shows the most consistency of the three, but does appear to be remarkably consistent. The smallest differences are seen in the winter months.

(d) No—temperature patterns can be quite local, and temperatures inland vary from coastal temperatures.

(e) The mean minimum for June 2010 was 7.1°C compared to a long-term mean of 8.4°C, whereas the mean maximum was 19.0°C compared to the long-term mean of 19.3°C. June 2010 was slightly more varied in temperature than normal with the minimum being lower and the maximum (slightly) higher. More data would need to be collated before any conclusions could be drawn about changing weather patterns.

12 (a) 3

(b)



(c) $y = 0.91x + 8.02$

13 (a) While there may be good reason for the introduction of City Link toll roads to be unpopular, the survey does not really show an obvious intention for voters to vote against the Government based on this one issue. The percentage results are far too close to tell.

(b) Government + Opposition voters equals 80% (33 + 47%) for Tullamarine. 20% of people surveyed chose not to support the Government or Opposition party, for any number of reasons.

This is important for the overall judgment of the poll because, if these 20% of voters then decided to vote, the result could change dramatically.

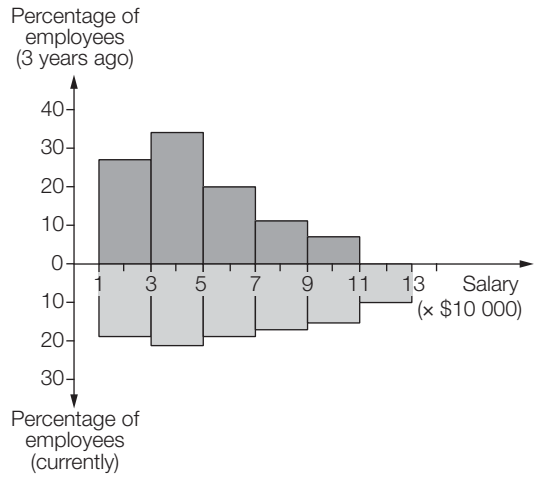
(c) Sample size = 1200 voters, $n^2 = 1\,440\,000$. This is an appropriate size.

(d) It is convenient to do the survey by telephone.

(e) Tullamarine and Essendon residents are largely affected by the new tolls, Oakleigh and Knox residents to a lesser extent. Many people avoid paying the tolls if possible whether they use the City Link roads often or not.

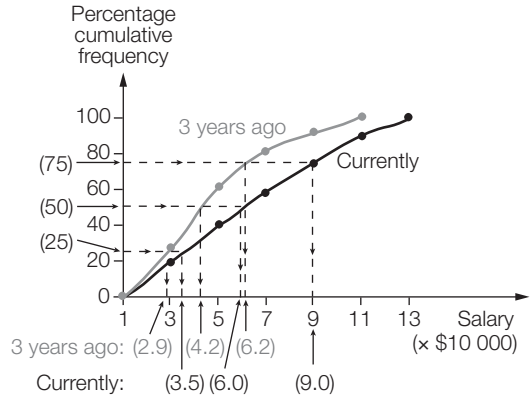
14 (a)

Salary (× $10 000)	Percentage of employees (3 years ago)	Percentage of employees (currently)
1–<3	27	19
3–<5	34	21
5–<7	20	19
7–<9	11	17
9–<11	7	15
11–<13	0	10



(b)

Salary (× $10 000)	Cum. freq. (3 years ago)	Per. cum. freq. (3 years ago)	Cum. freq. (current)	Per. cum. freq. (current)
<1	0	0	0	0
<3	12	27	9	19
<5	27	61	19	40
<7	36	82	28	58
<9	41	93	36	75
<11	44	100	43	90
<13	44	100	48	100



The following values are in $(× 10000).

x (salary × $10 000)	f (3 years ago)	xf	f (currently)	xf
2	12	24	9	18
4	15	60	10	40
6	9	54	9	54
8	5	40	8	64
10	3	30	7	70
12	0	0	5	60
	44	208	48	306

3 years ago

Mean = 4.73

Median = 4.2

IQR $= Q_U - Q_L$
$= 6.2 - 2.9$
$= 3.3$

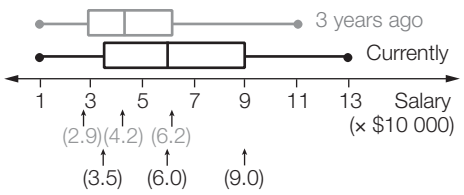
Currently

Mean = 6.38

Median = 6.0

IQR $= Q_U - Q_L$
$= 9.0 - 3.5$
$= 5.5$

(c)



(d) Salaries have increased, on average, by $15 000, and there is a much greater variation in the salaries paid.
mean difference = $16 500
median difference = $18 000

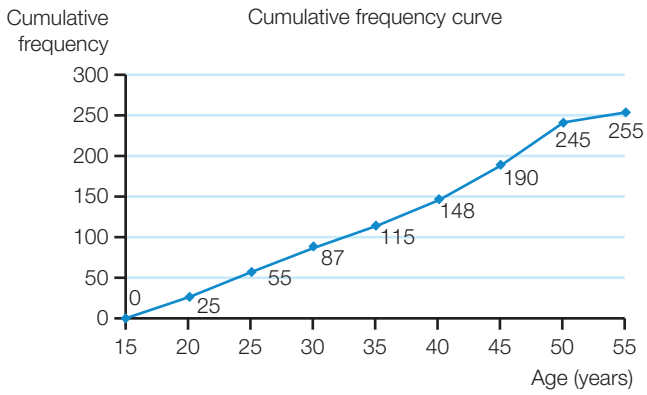
(e) total salary bill 3 years ago = $2 080 000,
total salary bill currently = $3 060 000;
i.e. $980 000 increase

(f) about 47%

15 (a) There are 255 employees. The data set has most people in the range 41–50 but this is balanced by very few in 51–55. Other than this, it is relatively uniform so the mean will be near the median, say about 38.

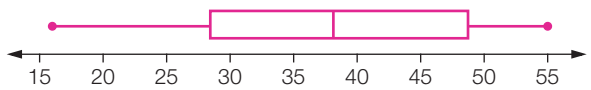
(b) mean = 36.04 and standard deviation = 10.57

(c)



min = 16, $Q_L = 28$, median = 38, $Q_U = 48$, max = 55

(d)



(e) (i) approx 100 (ii) approx 35

Chapter 8

Recall 8 (p. 399)

1 (a) $a = 23°$ (b) $a = 69°$ (c) $x = 15°$

2 (a) $a = 46°$; $b = 46°$; $c = 46°$

3 (a) equilateral (b) right-angled
(c) scalene (d) isosceles
(e) right-angled and isosceles (f) scalene

4 Going from left to right the letters to match the shapes are: E, B, A, D and C.

Exercise 8.1 (p. 403)

1 (a) $XY = PQ$ (both 9 cm), $YZ = QR$ (both 16 cm), $ZX = RP$ (both 8 cm); SSS

(b) $\angle BAC = \angle EFD$ (both 65°), $\angle BCA = \angle FED$ (both 50°), $AC = FE$ (both 10); AAS

(c) $\angle JKL = \angle MPN$ (both 90°), $JL = NM$ (both 5), $KJ = PN$ (both 3); RHS

(d) $XZ = RT$ (both 10), $YZ = ST$ (both 10), $\angle XZY = \angle STR$ (both 40°); SAS

(e) $\angle HGJ = MLK$ (both 56°), $JHG = \angle KML$ (both 44°), $HJ = MK$ (both 6); AAS

(f) $\angle PQR = \angle YXW$ (both 90°), $PR = YW$ (both 8), $RQ = WX$ (both 5); RHS

(g) $\angle CBD = \angle FHG$ (both 28°), $CB = FH$ (both 6), $DB = GH$ (both 10); SAS

(h) $JL = AC$ (both 4), $LK = CB$, $JK = AB$ (both 9); SSS

2 (a) $x = 14$ (b) $x = 8$
(c) $x = 7.5$ (d) $x = 20$
(e) $x = \frac{50}{3} = 16\frac{2}{3}$ (f) $x = \frac{51}{5} = 10\frac{1}{5}$

3 (b) and (d) SSS; (c) and (f) AAS; (a) and (e) SAS

4 D 5 C

6 (a) B (b) A

7 (a) A (b) B

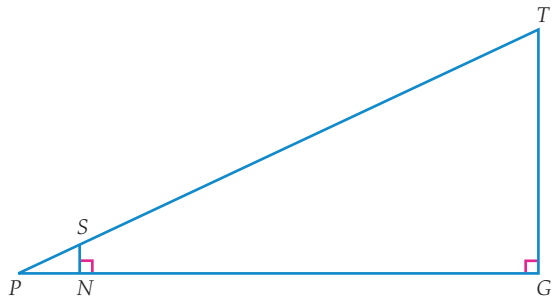
8 (a) $x = 10$ (b) $x = 6$ (c) $x = 5\frac{5}{9}$ (d) $x = 26.4$

9 5.25 m 10 316.67 cm 11 7.16 m

12 (a) ΔDFA ΔBGI, ΔCHJ, ΔDKM, ΔELN, ΔFMO

(b) ΔBKN and ΔCLO; ΔAJG and ΔKNB

13 (a)



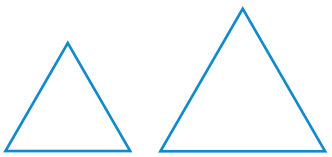
Measure SN, PN and NG

$\frac{TG}{PG} = \frac{SN}{PN}$ so, $TG = \frac{SN \times PG}{PN}$ where $PG = PN + NG$

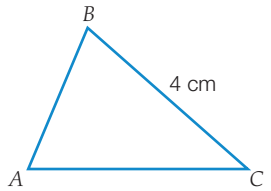
(b) Need to measure accurately; can't really get your head level with the ground.

Open-ended – Sample answers

14 AAA does not guarantee any sides are equal in length. If it did every equilateral triangle would be the same size.



15 Draw any triangle in which $BC = 4$ cm and then measure the other two sides.



In this triangle $AB = 2.6$ cm and $AC = 5$ cm. So, in ΔDEF, $DE = 2.6 \times 1.5 = 3.9$ cm; $AC = 5 \times 1.5 = 7.5$ cm and $EF = 4 \times 1.5 = 6$ cm

Exercise 8.2 (p. 410)

1 (a) $\Delta ABC \equiv \Delta CDA$

$BC = DA$	(given)
$BA = DC$	(given)
$AC = CA$	(same side)
$\Delta ABC \equiv \Delta CDA$	(SSS)
$\angle BAC = \angle DCA$, so $x = 55°$	

(b) $\Delta ACD \equiv \Delta ACB$

$\angle ACD = \angle ACB$	(given)
$\angle ADC = \angle ABC$	(given)
$AC = AC$	(same side)
$\Delta ACD \equiv \Delta ACB$	(AAS)
$AD = AB$, so $p = 19$ cm	

(c) $\Delta ABE \equiv \Delta CDE$

$AB = CD$	(given)
$\angle ABE = \angle CDE$	(given)
$\angle AEB = \angle CED$	(vertically opposite)
$\Delta ABE \equiv \Delta CDE$	(AAS)
$AE = CE$, so $x = 23$ cm	

(d) $\Delta AOB \equiv \Delta COB$

$AO = CO$	(radius of circle)
$\angle OBA = \angle OBC$	(given)
$\angle OAB = \angle OCB$	(isosceles triangle)
$\Delta AOB \equiv \Delta COB$	(AAS)
$\angle OAC = \angle OCB$, so $y = 26°$	

(e) $\Delta ABO \equiv \Delta CDO$

$AB = CD$	(given)
$AO = CO$	(radius of circle)
$BO = DO$	(radius of circle)
$\Delta ABO \equiv \Delta CDO$	(SSS)
$AO = CO$, so $y = 9$ m	

(f) $\Delta ABE \equiv \Delta DCE$

$\angle AEB = \angle DEC$	(vertically opposite)
$\angle BAE = \angle CDE$	(given)
$AB = DC$	(given)
$\Delta ABE \equiv \Delta CDE$	(AAS)
$AE = ED$, so $p = 6$ cm	

2 (a) $\frac{AB}{DE} = \frac{3}{6} = \frac{1}{2}$; $\frac{BC}{EF} = \frac{4}{8} = \frac{1}{2}$; $\frac{AC}{DF} = \frac{5}{10} = \frac{1}{2}$;

$\Delta ABC \parallel\!| \Delta DEF$	(SSS)

(b)

$OM = OP$	(given)
$ON = OQ$	(given)
$\angle MON = \angle POQ$	(vertically opposite)
$\Delta MON \equiv \Delta POQ$	

Since every pair of congruent triangles are also similar we can say $\Delta MON \parallel\!| \Delta POQ$

(c)

$\angle RVS = \angle RUT$	(given)
$\angle SRV = \angle TRU$	(same angle)
$\angle RSV = \angle RTU$	(angle sum of the triangle)
$\Delta RSV \parallel\!| \Delta RTU$	(AAA)

(d)

$\angle YWX = \angle VWU$	(vertically opposite)
$\angle WYX = \angle UVW$	(alternate angles between parallel lines)
$\angle WXY = \angle WUV$	(alternate angles between parallel lines)
$\Delta WYX \parallel\!| \Delta WVU$	(AAA)

3 C **4** B

5 (a)

$\angle RUT = \angle RST$	(given—right angle)
$RT = RT$	(same side—hypotenuse)
$UT = ST$	(given)
$\Delta RUT \equiv \Delta RST$	(RHS)

(b)

$SP = RQ$	(given)
$\angle SPQ = \angle RQP$	(given)
$PQ = PQ$	(same side)
$\Delta SPQ \equiv \Delta RQP$	(SAS)

6 (a)

$\angle QRS = \angle PRT$	(same angle)
$\angle RQS = \angle RPT$	(corresponding angles on parallel lines)
$\angle RSQ = \angle RTP$	(corresponding angles on parallel lines)
$\Delta RQS \parallel\!| \Delta RPT$	(AAA)

(b)

$\angle ABC = \angle BDC$	(given—right angle)
$\angle ACB = \angle BCD$	(same angle)
$\angle BAC = \angle DBC$	(third angle in triangle)
$\Delta ABC \parallel\!| \Delta BDC$	(AAA)

$\angle ADB = \angle ABC$	(given—right angle)
$\angle BAD = \angle CAB$	(same angle)
$\angle ABD = \angle ACB$	(third angle in triangle)
$\Delta ABC \parallel\!| \Delta ADB$	(AAA)
$\therefore$, $BDC \parallel\!| \Delta ADB$	(AAA)

7 (a) **(i)** not necessarily similar, angles same but sides might not be proportional

(ii) all matching sides in same ratio

(iii) not necessarily similar, angles between equal sides might not be equal

(iv) all angles the same (AAA)

(v) all angles the same (AAA)

(vi) radius is only measure so all distances will be in the same ratio

(b) **(i)** not similar **(ii)** side length the same

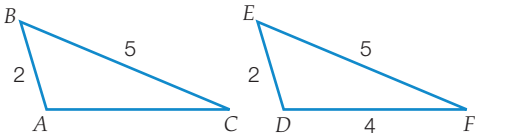
(iii) not similar **(iv)** side length the same

(v) one matching side length the same

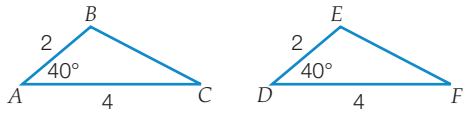
(vi) radius the same

Open-ended – Sample answers

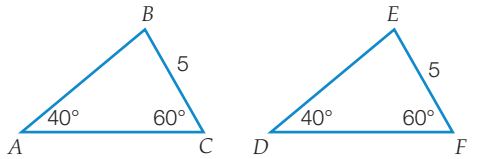
8



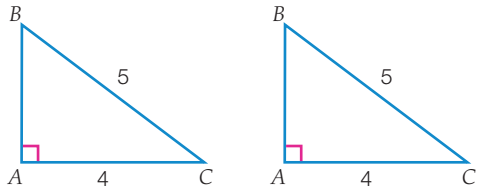
SSS



SAS



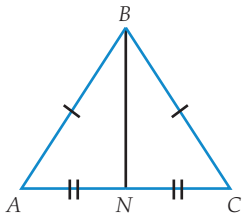
AAS



RHS

Exercise 8.3 (p. 413)

1 (a)

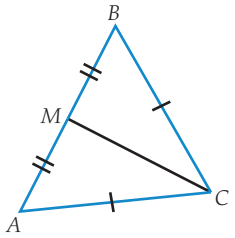


With this construction:

$\Delta ABN \equiv \Delta CBN$

$AN = CN$	(given by construction)
$BN = BN$	(same side)
$AB = CB$	(given)
$\Delta ABN \equiv \Delta CBN$	(SSS)

So, $\angle BAC = \angle BCA$



With this construction:

$\Delta BCM \equiv \Delta ACM$

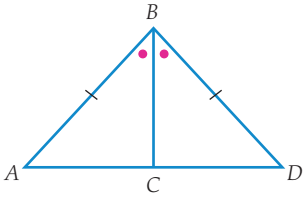
$AM = BM$	(given by construction)
$CM = CM$	(same side)
$CB = CA$	(given)
$\Delta BCM \equiv \Delta ACM$	(SSS)

So, $\angle CBA = \angle CAB$

But, $\angle CAB = \angle BCA$ from the first construction, so $\angle CBA = \angle CAB = \angle BCA$

(b) All angles are the same (60°)

2 (a)

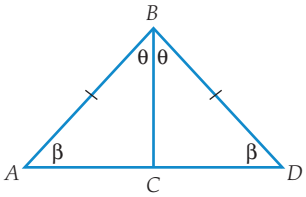


To show: $AC = CD$

$\angle BAC = \angle BDC$	(isosceles triangle)
$\angle ABC = \angle DBC$	(by construction)
$BC = BC$	
So, $\Delta BAC \equiv \Delta BDC$	(AAS)

$\therefore$ so $AC = DC$ (matching sides of congruent triangles)

(b)



To show: $\angle BCA = \angle BCD = 90°$

$2 \times \theta + 2 \times \beta = 180$	(angle sum of triangle)
$\theta + \beta = 90°$	
So, $\angle BCA = 180 - (\theta + \beta) = 90°$	

3

$\angle CDE + \angle EDF = 180°$	(straight angle)
$\angle EFG + \angle EFD = 180°$	(straight angle)
$\angle EDF = \angle EFD$	(equal angles of isosceles triangle)

$\angle CDE = 180 - \angle EDF$

$\angle EFG = 180 - \angle EFD$
$= 180 - \angle EDF$
$= \angle CDE$

4 A **5** B

6

$AC^2 = AB^2 + BC^2$	(Pythagoras)
$BC^2 = BD^2 + DC^2$	(Pythagoras)
So, $AC^2 = AB^2 + BD^2 + DC^2$	(1)
Also, $AB^2 = DB^2 + AD^2$	(Pythagoras)

Substituting into (1):

$AC^2 = DB^2 + AD^2 + BD^2 + DC^2$
$= DC^2 + 2BD^2 + AD^2$

7 $\Delta ABF \equiv \Delta BCD$

$AB = BC$	(given)
$\angle FAB = \angle DBC$	(corresponding angles, $FA \parallel BD$)
$\angle FBA = \angle DCB$	(corresponding angles, $FB \parallel DC$)
$\Delta ABF \equiv \Delta BCD$	(AAS)

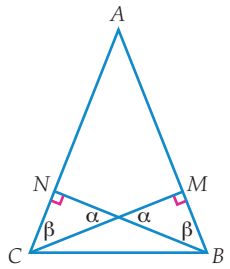
8 To show: $\angle RPT = \angle RTP$

$\angle QPU = \angle STU$ as both these angles are the third angle in a triangle where the other two angles are the same.

So in ΔRPT, $RP = RT$ (opposite equal sides)

$\therefore$ ΔRPT is isosceles.

9



$\angle CNB = \angle BMC$ (90°, given)
$\angle NCB = \angle MBC$ (angles opposite equal sides)
ΔCNB and ΔBMC have a common side, BC

So, $\Delta CNB \equiv \Delta BMC$ (AAS)
So, $NC = MB$ (matching sides)
$AC = AN + NC$
$AB = AM + MB$
$AB = AC$ (given)
So $AN = AM$ (equal sides AC and AB; NC and NB)

10 Let $EC = DB = x$

$\therefore ED = x + DC = BC$

In ΔEFD and ΔBAC

$ED = BC$ (above)

$\angle EFD = \angle BAC$ (given)

$\angle FED = \angle ABC$ (given)

$\therefore \Delta ABC \equiv \Delta FED$ (AAS)

11 Let $BF = EC = x$

$\therefore BC = x + FC = FE$

In ΔABC and ΔDEF

$BC = EF$ (above)

$\angle ABC = \angle DEF$ (given)

$\angle ACB = \angle DFE$ (given)

$\therefore \Delta ABC \equiv \Delta DEF$ (AAS)

12 (a) $\Delta ABE \equiv \Delta ACD$
$\angle ABE = \angle ACE$ (given)
$BE = CD$ ($BC = EC$ and DE is common)
$AB = AC$ (given)
$\Delta ABE \equiv \Delta ACD$ (SAS)

(b) $\Delta ABE \equiv \Delta CBD$
$\angle BAE = \angle BCD$ (given)
$\angle ABE = \angle CBD$ ($\angle ABD = \angle CBE$ and $\angle EBD$ is common)
$AB = BC$ (isosceles triangle)
$\Delta ABE \equiv \Delta CBD$ (AAS)

(c) $\Delta ABD \equiv \Delta CBE$
$AD = CE$ ($AD = AE - DE$ and $CE = CD - DE$; DE common)
$\angle BAD = \angle BCE$ (given)
$AB = CB$ (isosceles triangle)
$\Delta ABD \equiv \Delta CBE$ (SAS)

(d) $\Delta ABD \equiv \Delta CBE$
$AB = CB$ (given)
$AD = CE$ ($AD = AE - DE$ and $CE = CD - DE$; DE common)
$\angle BAD = \angle BCE$ (isosceles triangle)
$\Delta ABD \equiv \Delta CBE$ (SAS)

13 $\Delta ADF \equiv \Delta AEF$
$DF = EF$ (given)
$\angle ADF = \angle AEF$ (both 90° as $FD \perp AB$ and $FE \perp AC$)
$AF = AF$ (common side)
$\Delta ADF \equiv \Delta AEF$ (RHS)

14 $\Delta ABC \equiv \Delta CDE$
$\angle ABC = \angle CDE$ (given—right angle)
$AC = CE$ (given C is the midpoint of AE hypotenuse)
$AB = CD$ (given)
$\Delta ABC \equiv \Delta CDE$ (RHS)

15 $\angle OEF = \angle OFE$ (isosceles triangle; call it α)
$\angle OED = \angle ODE$ (isosceles triangle; call it β)
$\angle EDO + \angle OFE + \angle FED = 180°$ (angle sum of triangle)

So, $\beta + \alpha + (\alpha + \beta) = 180°$
$2(\alpha + \beta) = 180°$
$\alpha + \beta = 90°$

As $\angle DFE = \alpha + \beta = 90°$ the angle in a semicircle is a right angle.

Open-ended – Sample answers

16 (a) $\angle ABC = \angle ACB$ (opposite equal angles)
$\angle AXY = \angle ABC$ (corresponding angles $XZ \parallel BC$)
$\angle AZY = \angle ACB$ (corresponding angles $XZ \parallel BC$)
$\therefore \angle AXY = \angle AZY$
$\therefore AX = AZ$ (sides opposite equal angles)

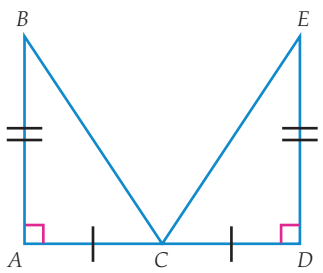
(b) $\Delta AXY \equiv \Delta AZY$
$AX = AZ$ (from **(a)**)
$XY = ZY$ (given)
$\angle AXY = \angle AZX$ (from **(a)**)
$\Delta AXY \equiv \Delta AZY$ (SAS)

(c) $\Delta BXY \equiv \Delta CZY$
$BX = CZ$ ($AB = AC$ and $AX = AZ$)
$XY = ZY$ (given)
$\angle BXY = \angle CZY$ (Each $(180 - b)°$)
$\Delta BXY \equiv \Delta CZY$ (SAS)

(d) $a + b + 90 = 180$ (angle sum in triangle)
$b = 90 - a$

$\angle ACY = c$ and $\angle CAY = a$ but these are equal since ΔACY is isosceles.

So, $c = a$
$c + d = b$ (corresponding angles on parallel lines)
$a + d = 90 - a$
$d = 90 - 2a$

17 

$\Delta ABC \equiv \Delta DEC$

Exercise 8.4 (p. 423)

1 (a) To prove: $\Delta ABC \,|||\, \Delta ADE$

$\angle ADE = \angle ABC$	(corresponding angles, $DE \parallel BC$)
$\angle DAE = \angle BAC$	(same angle)
$\angle AED = \angle ACB$	(corresponding angles, $DE \parallel BC$)
$\Delta ABC \,|||\, \Delta ADE$	(AAA)

(b) To prove: $AE = EC$

$\Delta ABC \,|||\, \Delta ADE$ (from (a))

So, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$

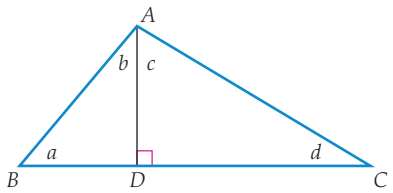
So, $\dfrac{AE}{AC} = \dfrac{1}{2}$

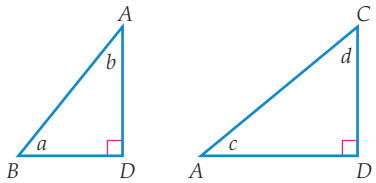
So, $2AE = AC$

So, $AE = EC$

2 To prove: $\Delta ADB \,|||\, \Delta DCA$

The orientation of the two triangles must be as shown below as AD cannot be in the same position in both triangles.





$a + b = 90°$	(rest of angle sum of 180° for triangle)
$c + d = 90°$	(rest of angle sum of 180° for triangle)
$b + c = 90°$	(right angle in original triangle)

$a + b = b + c$, so $a = c$

$b + c = c + d$, so $b = d$

$\Delta ADB \,|||\, \Delta DCA$	(AAA)

To prove: $\Delta DCA \,|||\, \Delta CAB$

$\angle ADC = \angle CAB$	(both 90°)
$\angle ACD = \angle ACB$	(common angle)
$\angle ABD = \angle ABC$	(common angle)
$\Delta DCA \,|||\, \Delta CAB$	(AAA)

3 To prove: $AD^2 = BD \times DC$

$\Delta ADB \,|||\, \Delta DCA$ (from Question 2)

So, $\dfrac{AD}{DC} = \dfrac{BD}{AD}$ (matching sides of similar triangles)

So, $AD \times AD = BD \times DC$

$AD^2 = BD \times DC$

4 To prove: $BA^2 = BD \times BC$ and $CA^2 = CD \times CB$

$\Delta ADB \,|||\, \Delta CAB$ (from Question 2)

$\dfrac{BA}{BC} = \dfrac{BD}{BA}$ (matching sides of similar triangles)

So, $BA^2 = BC \times BD$

To prove: $CA^2 = CD \times CB$

$\Delta ADC \,|||\, \Delta BAC$ (from Question 2)

$\dfrac{CA}{BC} = \dfrac{CD}{AC}$ (matching sides of similar triangles)

$CA^2 = CD \times BC$

5 $AB^2 = BC \times BD$ (from Question 4)

$AC^2 = CD \times BC$ (from Question 4)

$$\begin{aligned} AB^2 + AC^2 &= BC \times BD + CD \times BC \\ &= BC(BD + CD) \\ &= BC \times BC \\ &= BC^2 \end{aligned}$$

6 C

7 (a) To show: $\Delta ABO \,|||\, \Delta DCO$

$\angle ABD = \angle BDC$	(alternate angles, $AB \parallel DC$)
$\angle BAC = \angle ACD$	(alternate angles, $AB \parallel DC$)
$\angle AOB = \angle COD$	(vertically opposite angles)
$\Delta ABO \,|||\, \Delta DCO$	(AAA)

Now, $\dfrac{OA}{OC} = \dfrac{OB}{OD} = \dfrac{AB}{DC}$ (matching sides of similar triangles)

So, $OA : OC = OB : OD$

(b) We have $\dfrac{AB}{CD} = \dfrac{1}{2}$

So, $\dfrac{AO}{OC} = \dfrac{1}{2}$ (matching sides of similar triangles are in same ratio)

So, $2AO = OC$

Similarly, $2OB = OD$

$$\begin{aligned} AC &= AO + OC \\ &= AO + 2AO \\ &= 3AO \end{aligned}$$

So, $AO : AC = 1 : 3$ and similarly $BO : BD = 1 : 3$

8 To prove: $\dfrac{JM}{MK} = \dfrac{JN}{NL}$

We will show $\Delta JMN \,|||\, \Delta JKL$

$\angle MJN = \angle KJL$	(common angle)
$\angle JMN = \angle JKL$	(corresponding angles, $MN \parallel KL$)
$\angle JNM = \angle JLK$	(corresponding angles, $MN \parallel KL$)
$\Delta JMN \,|||\, \angle JKL$	(AAA)

So, $\frac{JK}{JM} = \frac{JL}{JN}$ (matching sides of similar triangles are proportional)

$\frac{JK}{JM} - 1 = \frac{JL}{JN} - 1$

$\frac{JK - JM}{JM} = \frac{JL - JN}{JN}$

$\frac{MK}{JM} = \frac{NL}{JN}$ ($JK = JM + MK$)

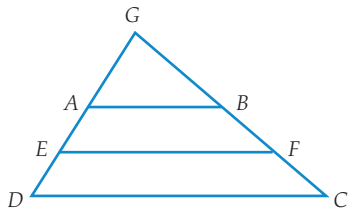
($JL = JN + NL$)

So, $\frac{JM}{MK} = \frac{JN}{NL}$ (reciprocal)

9 To prove: $GH \parallel EF$
We will show $\Delta DGH \,|||\, \Delta DEF$
$\angle GDH = \angle EDF$ (common angle)
$\frac{DG}{GE} = \frac{DH}{HF}$ (given)
$\frac{GE}{DG} = \frac{HF}{DH}$ (reciprocal)
$\frac{GE}{DG} + 1 = \frac{HF}{DH} + 1$
$\frac{GE + DG}{DG} = \frac{HF + DH}{DH}$
$\frac{DE}{DG} = \frac{DF}{DH}$

Corresponding sides are in the same proportion so
$\Delta DGH \,|||\, \Delta DEF$ (SAS)
So, $\angle DGH = \angle DEF$ (matching angles in similar triangles)
And $\angle DHG = \angle DFE$ (matching angles in similar triangles)
So, $GH \parallel EF$ (corresponding angles equal on parallel lines)

10 Following the hint we have this diagram:



In ΔGAB and ΔGDC
$\angle AGB = \angle DGC$ (common angle)
$\angle GAB = \angle GDC$ (corresponding angles, $AB \parallel DC$)
$\therefore \Delta GAB \,|||\, \Delta GDC$ (AAA)
$\therefore$ matching sides are proportional

Let $GA = x$, $AE = y$, $ED = y$

$\therefore GB = mx$, $BC = 2my$, $BF = my$, $FC = my$

$\frac{GA}{GE} = \frac{x}{x + y}$

$\frac{GB}{GF} = \frac{mx}{mx + my} = \frac{x}{x + y}$

now LG is common
$\therefore \Delta GAB \,|||\, \Delta GEF$ (SAS)
$\therefore \angle GAB = \angle GEF$ (matching angles)
However, these angles are equal corresponding angles
$\therefore AB \parallel EF$

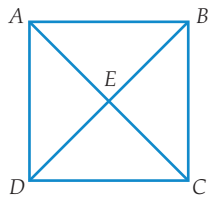
11 To prove: $\Delta BXY \,|||\, \Delta BYC$
Hint: Prove $\Delta BXY \equiv \Delta CZY$

(See Exercise 8.3 Question 16 for the proof of this; it will not be repeated here.)

$BY = CY$ (congruent triangles)
We know $\frac{BX}{BY} = \frac{XY}{YC}$ (as $BX = XY$ and $BY = YC$)
Need to show: $\angle BXY = \angle BYC$
Now, $\angle XBY = \angle XYB$ (isosceles triangle and call it b)
So, $\angle BXY = (180 - 2b)°$
But, $\angle ZYC = b$ (congruent triangle)
And, $\angle XYB + \angle BYC + \angle ZYC = 180°$ (straight angle)
$b + \angle BYC + b = 180°$
$\angle BYC = (180 - 2b)°$
So, $\angle BXY = \angle BYC$
So, $\Delta BXY \,|||\, \Delta BYC$ (SAS)

Open-ended – Sample answers

12



To prove: $\Delta AED \,|||\, \Delta ADC$
The diagonals of a square bisect each other at right angles
So, $\frac{AE}{AD} = \frac{ED}{DC}$ since $AE = ED$ and $AD = DC$
$\angle AED = \angle ADC$ (both right angles)
So, $\Delta AED \,|||\, \Delta ADC$ (SAS)

Half-time 8 (p. 425)

1 **(a)** To prove: $\Delta ABC \equiv \Delta EDC$
$\angle ABC = \angle CDE$ (both right angles)
$\angle BCA = \angle ECD$ (vertically opposite)
So, $\Delta ABC \,|||\, \Delta EDC$ (AAA)
$BC = DC$ (AAS)
$\therefore \Delta ABC \equiv \Delta EDC$

(b) AB and ED are corresponding sides in congruent triangles so must be equal.

2 To prove: $\Delta ABC \,|||\, \Delta APQ$
$\angle ABC = \angle APQ$ (given)
$\angle CAB = \angle QAP$ (common angle)
$\angle AQP = \angle ACB$ (third angle in triangle)
$\Delta ABC \,|||\, \Delta APQ$ (AAA)

3 **(a)** RHS **(b)** SAS

4 **(a)** $OA = OD$ (radii)
$OB = OC$ (radii)
$AB = CD$ (given)
So, $\Delta AOB \equiv \Delta DOC$ (SSS)
$p = 15$

(b) In ΔURT and ΔSTR

$RT = RT$	(common side)
$\angle URT = \angle STR$	(alternate angles, $RU \parallel TS$)
$\angle RTU = \angle SRT$	(alternate angles, $RS \parallel UT$)
$\therefore \Delta URT \equiv \Delta STR$	(AAS)
$\therefore a = 11$ cm	(matching sides in congruent triangles)

5 (a) $x = 12$ (b) $x = 15$

6 To prove: $\Delta LYZ \parallel\!| \Delta MXZ$

$\angle LYZ = \angle MXZ$	(both right angles)
$\angle YZL = \angle XZM$	(vertically opposite)
$\angle YLZ = \angle XMZ$	(third angle in triangle)
$\Delta LYZ \parallel\!| \Delta MXZ$	(AAA)

7 In ΔACX and ΔBCX

$AC = BC$	(given)
$AX = BX$	(given)
$CX = CX$	(common side)
$\therefore \Delta ACX \equiv \Delta BCX$	(SSS)

8 2.4 cm

9 To prove: $\Delta ADF \equiv \Delta AEF$

$\angle ADF = \angle AEF$	(both right angles)
$AF = AF$	(common side)
$DF = EF$	(given)
$\Delta ADF \equiv \Delta AEF$	(RHS)

10 To prove: $XY = XZ$

We will prove $\Delta YAX \equiv \Delta ZBX$

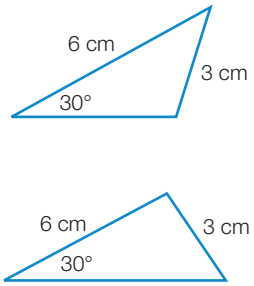
$\angle YAX = \angle ZBX$	(isosceles ΔABC)
$YA = ZB$	(halves of equal sides)
$XA = XB$	(given)
$\Delta YAX \equiv \Delta ZBX$	(SAS)

XY and XZ are corresponding sides in congruent triangles, so $XY = XZ$.

11 To prove: $\Delta ABC \equiv \Delta DEC$

$\angle ACB = \angle ECD$	(vertically opposite)
$\angle CBA = \angle CDE$	(alternate angles, $AB \parallel DE$)
$BC = CD$	(C is the midpoint of DB)
$\therefore \Delta ABC \equiv \Delta DEC$	(AAS)

12 Sample answer:



13 5 m

Exercise 8.5 (p. 434)

1 (a) (i) Opposite sides are parallel and one pair of adjacent sides is equal

(ii) rhombus

(b) (i) Two pairs of adjacent sides equal.

(ii) kite

(c) (i) Two pairs of opposite sides parallel; one pair of adjacent sides equal; one right angle.

(ii) square

2 (a) To prove: $\Delta DAB \equiv \Delta DCB$

$\angle DBA = \angle BDC$	(alternate angles, $AB \parallel CD$)
$\angle BDA = \angle CBD$	(alternate angles, $AD \parallel BC$)
$DB = DB$	(common side)
$\Delta DAB \equiv \Delta DCB$	(AAS)

(b) $\angle DAB = \angle BCD$ (matching angles, congruent triangles from (a))

$\angle ABC$ is supplement of $\angle DAB$ (co-interior angles, $AB \parallel CD$)

$\angle ADC$ is supplement of $\angle DCB$ (co-interior angles, $AD \parallel BC$)

$\therefore \angle ABC = \angle ADC$ (supplements of equal angles)

(c) Opposite angles in a parallelogram are equal

(d) To prove: $\Delta DAE \equiv \Delta BCE$

$AD = BC$	(matching sides in congruent triangles)
$\angle AED = \angle BEC$	(vertically opposite)
$\angle ADE = \angle CBE$	(alternate angles, $AD \parallel BC$)
$\Delta DAE \equiv \Delta BCE$	(AAS)

(e) These are matching sides in congruent triangles, so $AE = CE$ and $DE = BE$

(f) The diagonals of a parallelogram bisect each other

3 $AC = DB$ and bisect each other

$\therefore AX = XC = DX = XB$

$AC \perp BD$ $\therefore$ all angles at centre are 90°

The four triangles are congruent (SAS)

$\therefore$ all hypotenuses are equal (matching sides)

acute angles of each triangle are 45° (isosceles triangle, angle sum of triangle)

$\therefore \angle ABC = \angle BCD = \angle CDA = \angle DAB = 90°$

$\therefore$ quadrilateral is a square (all sides equal, angles are right angles)

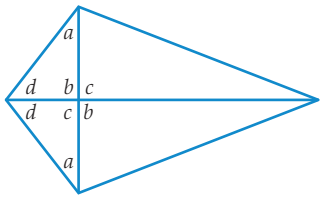
4 (a) To prove: $\angle KLM = \angle KNM$

Will show: $\Delta KLM \equiv \Delta KNM$

$KL = KN$	(given)
$LM = NM$	(given)
$KM = KM$	(common side)
$\Delta KLM \equiv \Delta KNM$	(SSS)

$\angle KLM = \angle KNM$ as these are matching angles in congruent triangles

(b) To prove: $\angle KOL = 90°$



We know the angles marked a are equal as they are in an isosceles triangle.

We know the angles marked d are equal because they are matching angles in congruent triangles.

The angles marked b and c are pairs of vertically opposite angles.

So, $a + b = a + c$	(supplements of d)
So, $b = c$	
But, $b + c = 180°$	(straight angle)
So, $b = c = 90°$	
So, $\angle KOL = 90°$	

5 B **6** C

7 (a) To prove: $PQ = RS$

Will show: $\Delta PQS \equiv \Delta RSQ$

$SQ = SQ$	(common side)
$\angle PSQ = \angle RQS$	(alternate angles, $PS \parallel RQ$)
$\angle PQS = \angle QSR$	(alternate angles, $PQ \parallel RS$)
$\Delta PQS \equiv \Delta RSQ$	(AAS)

So, $PQ = RS$ as they are matching sides on congruent triangles

(b) Opposite sides of a rectangle are equal

8 (a) To show: $\Delta ACD \equiv \Delta ACB$

$AD = BC$	(given)
$DC = AB$	(given)
$AC = AC$	(given)
$\Delta ACD \equiv \Delta ACB$	(SSS)
$\angle BAC = \angle BCA = 45°$	(isosceles triangles)
$\angle DAC = \angle DCA = 45°$	(isosceles triangles)

$\angle BAD = \angle BAC + \angle DAC$
$= 45° + 45°$
$= 90°$

So, $\angle BAD = 90°$, so we have a square

(b) To show: $\Delta ABD \equiv \Delta CBD$

$AB = BC$	(given)
$AD = DC$	(given)
$DB = DB$	(given)
$\Delta ABD \equiv \Delta CBD$	(SSS)

So, $\angle ABD = \angle BDC$ as these are matching angles in congruent triangles. But, these are alternate angles, so the lines must be parallel.

Then, do the same for the other pair of sides

9 (a) Will show: $\Delta ABD \equiv \Delta BDC$

$AD = BC$	(given)
$\angle ADB = \angle DBC$	(alternate angles on parallel lines)
$\angle ABD = \angle BDC$	(alternate angles on parallel lines)
$\Delta ABD \equiv \Delta BDC$	(AAS)
So, $AB = DC$	(matching sides in congruent triangles)
And, $\angle DAB = \angle DCB$	(matching angles in congruent triangles)

So, we have a parallelogram

(b) Will show: $\Delta AED \equiv \Delta CEB$

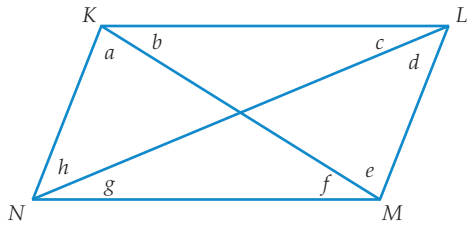
$AE = CE$	(given)
$DE = BE$	(given)
$\angle AED = \angle BEC$	(vertically opposite)
$\Delta AED \equiv \Delta CEB$	(SAS)
So, $AD = BC$	(matching sides in congruent triangles)

In a similar way show $\Delta AEB \equiv \Delta DEC$ and so $AB = DC$

Thus, we have two pairs of opposite sides equal and parallel.

So the quadrilateral is a parallelogram.

(c) Consider the following diagram:



A number of relationships can be established from the diagram

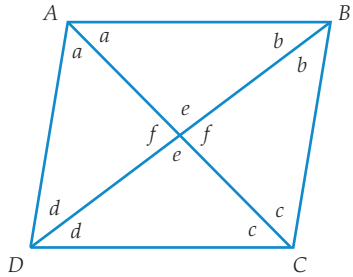
$a + b + c + h = 180°$	(angles in triangle)
$e + f + d + g = 180°$	(angles in triangle)
but $e + f = a + b$	(given)

So, $a + b + c + h = e + f + d + g$
becomes $a + b + c + h = a + b + d + g$
so, $c + h = d + g$

but $c + d = h + g$	(given)

so, $d = h$ and $g = c$
These are alternate angles so $KL \parallel NM$
Then do a similar thing to show $KN \parallel LM$

(d) Consider this diagram:



$2a + b + d = 2c + b + d$ (sum of angles in a triangle = 180°)

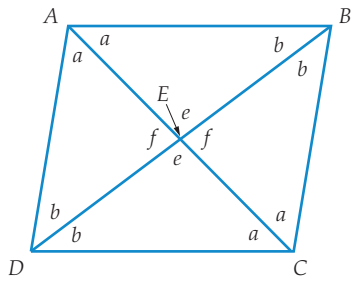
so, $a = c$

$2b + a + c = 2d + a + c$ (sum of angles in a triangle = 180°)

so, $b = d$

similarly $a = c$

Redraw the diagram:



$a + b + e = a + b + f$ (sum of angles in a triangle = 180°)

so, $e = f$

$AD = AB$ (opposite equal angles ΔABD)

$BC = DC$ (opposite equal angles ΔBCD)

$AD = DC$ (opposite equal angles ΔADC)

$\therefore AB = BC = CD = DA$

$\therefore ABCD$ is a rhombus (all sides equal)

10 A

11 To show: $\Delta WMZ \equiv \Delta YNZ$

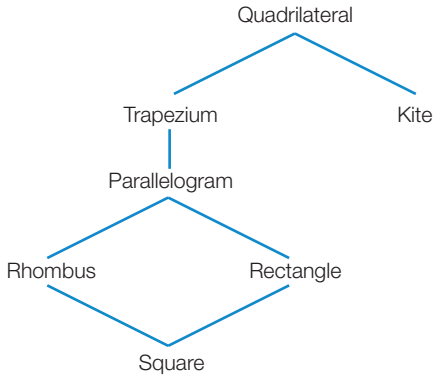
$ZM = NZ$ (given)

$WZ = YZ$ (given)

$\angle ZWM = \angle ZYN$ (both 90°, given)

$\Delta WMZ \equiv \Delta YNZ$ (RHS)

12 The words link in the following way:



13 To show: $\Delta AKD \equiv \Delta CJB$

As $AJ = CK$ and $AB = DC$ (opposite sides of parallelogram) then $JB = DK$

So, $JB = DK$

$AD = BC$ (parallelogram)

$\angle ADK = \angle JBC$ (opposite angles in parallelogram)

$\Delta AKD \equiv \Delta CJB$ (SAS)

14 A square has four right angles and two pairs of opposite sides equal and parallel with all sides the same length, while a rhombus has two pairs of opposite sides equal and parallel with all sides the same length. So, the square has an extra condition.

15 A square has four right angles and two pairs of opposite sides equal and parallel with all sides the same length, whereas a rectangle has four right angles and two pairs of opposite sides equal and parallel. So, the square has an extra condition.

Open-ended – Sample answers

16 For the condition *at least one pair of parallel sides* the shape could be: trapezium, square, rectangle, rhombus or parallelogram

For the condition *at least one pair of equal sides* the shape could be: square, rectangle, parallelogram, rhombus or kite

For the condition *at least one pair of equal angles* the shape could be: square, rectangle, rhombus, parallelogram or kite

17 **(a)** A rhombus has four equal sides but is not a square, because it has no right angle.

(b) Every square has four right angles. Every shape with four right angles is a square. False—it could be a rectangle.

Every kite has two pairs of adjacent sides equal. Every shape with two pairs of adjacent sides equal is a kite. False—it could be a square or a rhombus.

18 **(a)** Students' own diagrams.

(b) Students to draw and measure from their own diagrams.

(c) The diagonals of a rhombus intersect at right angles.

(d) A demonstration involving measurement is never sufficient as a proof.

19 A square, rhombus and kite all have diagonals that bisect at right angles, so George is correct.

Exercise 8.6 (p. 451)

1 Join GO and extend it to H.

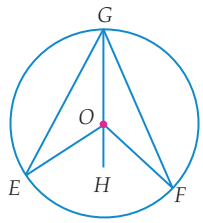
In ΔOGE, $OG = OE$ (radii of a circle)

$\therefore \Delta OGE$ is isosceles

$\therefore \angle OGE = \angle OEG$ (base angles of isosceles triangles)

Hence $\angle HOE = \angle OGE + \angle OEG$ (exterior angle of triangle is sum of interior opposite angles)

$\therefore \angle HOE = 2\angle OGE$



Repeating these steps in ΔOGF we obtain $\angle HOF = 2\angle OGF$.

Now $\angle EOF = \angle HOE + \angle HOF$.

$\therefore \angle EOF = 2\angle OGE + 2\angle OGF$

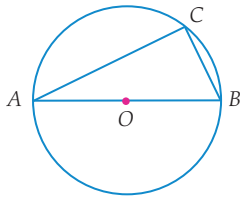
$= 2(\angle OGE + \angle OGF)$

$\therefore \angle EOF = 2\angle EGF$

2 $\angle AOB = 2\angle ACB$ (angle at the centre is twice the angle at the circumference standing on the same arc)

But $\angle AOB = 180°$ (straight angle)

$\therefore \angle ACB = 90°$



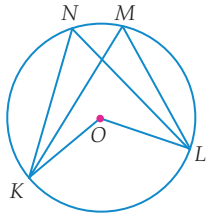
3 Join KO and LO.

$\angle KOL = 2\angle KNL$ (angle at the centre is twice the angle at the circumference standing on the same arc)

Similarly, $\angle KOL = 2\angle KML$

Hence $2\angle KNL = 2\angle KML$

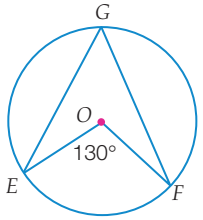
So $\angle KML = \angle KNL$



4 $\angle EOF = 2\angle EGF$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\therefore 130° = 2\angle EGF$

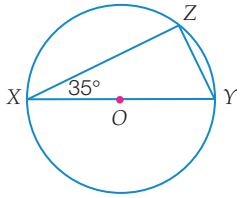
$\angle EGF = 65°$



5 $\angle XZY = 90°$ (angle in a semicircle)

But $\angle ZXY = 35°$

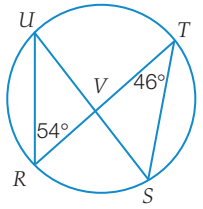
So $\angle ZYX = 55°$ (angle sum of triangle is 180°)



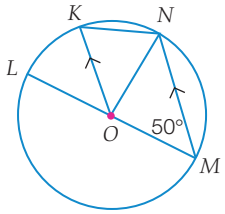
6 $\angle RUS = 46°$ (angles in the same segment)

$\angle UVR = 80°$ (angle sum of ΔUVR)

$\angle TVS = 80°$ (vertically opposite angles)



7 $\angle LOK = 50°$ (corresponding angles, $OK \parallel MN$)



$\angle LON = 2\angle LMN = 100°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$\angle OMN = \angle ONM$ (opposite equal sides (radii) of ΔONM)

$\angle KON = \angle ONM$ (alternate angles, $OK \parallel MN$)

Hence $\angle KON = \angle LON - \angle LOK = 100° - 50° = 50°$

$OK = ON$ (radii of a circle)

Hence ΔOKN is isosceles and $\angle OKN = \angle ONK$.

$50° + \angle OKN + \angle ONK = 180°$ (angle sum of ΔOKN)

$\therefore 2\angle OKN = 130°$

$\therefore \angle OKN = 65°$

8 (a) $x = 126°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $y = 70°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $x = 115°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc)

9 (a) $\angle ACD = 90°$ (angle in a semicircle), $z = 46°$ (angle sum of ΔACD)

(b) $\angle ECG = 90°$ (angle in a semicircle), $\angle CGE = 40°$ (angle sum of ΔECG), $x + 40° = 95°$ (exterior angle of triangle is sum of interior opposite angles), $x = 55°$

(c) $\angle OGH = y$ (isosceles triangle, $OG = OH$), $2y = 70°$ (exterior angle of triangle is sum of interior opposite angles), $y = 35°$

10 (a) $\angle BDC = 43°$ (angles in the same segment), $x = 107°$ (angle sum of ΔEDC)

(b) $x = 47°$ (angles in the same segment), $\angle ABD = 90°$ (angle in a semi circle), $y = 43°$, $z = 43°$ (angles in the same segment)

(c) $y = 50°$ (angles in the same segment), $ABD = 90°$ (angle in a semi circle), $\angle ABC = 40°$, $z + 40° = 95°$ (exterior angle of triangle is sum of interior opposite angles), $z = 55°$

(d) $n = 32°$, $m = 32°$ (angles in the same segment)

(e) $\angle BDC = 35°$ (angles in the same segment), $e = 51°$ (angle sum of ΔADC)

(f) $\angle DCA = 44°$ (angles in the same segment), $k = 25°$ (exterior angle of triangle is sum of interior opposite angles)

11 (a) $x = 35°$ (angle at the centre is twice the angle at the circumference standing on the same arc), $\angle FOE = 80°$ (angle at the centre is twice the angle at the circumference standing on the same arc), $2y + 80° = 180°$ (angle sum of isosceles triangle FOE), $y = 50°$

(b) $\angle CDB = 49°$ (angle sum of ΔBDC), $\angle ACD = 49°$ (alternate angles, $AC \parallel DB$), $z = 98°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $\angle MNQ = \angle MQN = 72°$ (base angles of isosceles ΔMNQ), $\angle NQP = 90°$ (angle in a semi circle), $w = 18°$

12 $\angle BOD = 2y$ (angle at centre is twice angle at circumference standing on the same arc)

$OB = OD$ (radii)

$\angle OBD = \angle ODB = x°$ (angles opposite equal sides of a triangle)

$\therefore x + x + 2y = 180$ (angle sum of a triangle)

$\therefore x + y = 90$

13 ΔBDC is isosceles, so $\angle DBC = \angle DCB = 64°$

$\angle ABC = 90°$ (angle in a semicircle) $\therefore \angle ADE = 38°$

$\angle ABD = 26° \therefore \angle ACB = 38°$

$\angle ACD = 26°$ (angles in the same segment)

14 $\angle DGE = 20°$ (angles in the same segment)

$\angle FDE = 48°$ (angles in the same segment)

$\angle GFD = 42°$ (angles in the same segment)

$\angle GDF = 70°$ (angle sum of triangle)

$\angle GEF = 70°$ (angles in the same segment)

$\angle FGD = \angle FGE + \angle DGE = 68°$

$\angle GDE = \angle GDF + \angle FDE = 118°$

$\angle DEF = \angle GED + \angle GEF = 112°$

$\angle EFG = \angle DFE + \angle GFD = 62°$

$\angle GHF = 90°$ (angle sum of a triangle)

$\therefore$ The diagonals GE and DF are perpendicular.

15 $\angle OZY = \angle OYZ = 44°$ (base angles of isosceles triangle OYZ)

$\angle OZX = 62° - 44° = 18°$

$\angle OXZ = 18°$ (base angles of isosceles triangle OXZ)

16 (a) $\angle ODF = 30°$ (isosceles triangle), $\angle FOD = 120°$ (angle sum of triangle), $y = 60°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $\angle ODF = 33°$ (isosceles triangle), $\angle FOD = 114°$ (angle sum of triangle), reflex $\angle FOD = 246°$ (angles at a point), $\angle FBD = 123°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc), $x = 22°$ (angle sum of triangle)

(c) $\angle AOD = 2x$ (angle at the centre is twice the angle at the circumference standing on the same arc), reflex $\angle AOD = 4x°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc), $6x = 360°$ (angles at a point), $x = 60°$

17 Join KO, LO, NO

Let $\angle LNO = x$

$\therefore \angle OLN = \angle ONL = x$ (angles opposite equal sides of a triangle)

$\therefore \angle LON = (180 - 2x)°$ (angles sum of a triangle)

In ΔOKL and ΔONL, $OK = ON$ (radii)

$KL = NL$ (given)

LO is common

$\therefore \Delta OKL \equiv \Delta ONL$

$\therefore \angle OLK = \angle OLN = x$ (corresponding angles in congruent triangles)

$\therefore \angle KLM = (180 - 2x)°$ (MKL is a straight line)

$\therefore \angle KLM = \angle LON$ (both $(180 - 2x)°$)

18 (a) In Δs KPM and QNM,

$\angle KPQ = \angle QNK$ (angles in the same segment)

$\angle PKN = \angle NQP$ (angles in same segment)

$\angle KMP = \angle QMN$ (vertically opposite angles)

$\therefore \Delta KPM \equiv \Delta QNM$ (AAA)

(b) $KM = MQ$ (given)

$\angle PKM = \angle NQM$ (matching angles in similar triangles)

$\angle MNQ = \angle MPK$ (matching angles in similar triangles)

$\therefore \Delta KMP \equiv \Delta QMN$ (AAS)

$\therefore MP = MN$ (matching sides in congruent triangles)

19 In Δs XWU and YZU,

$\angle XWY = \angle XZY$ (angles in the same segment)

$\angle WXZ = \angle WYZ$ (angles in same segment)

But $\angle XWU = \angle WXU$ (opposite equal sides (radii) ΔXWU)

So $\angle XWU = \angle UYZ$ (both equal to $\angle XWY$)

But these are a pair of equal alternate angles between XW and YZ

$\therefore XW \parallel YZ$

20 $\angle CAF = 90°$ (angle in a semicircle)

$\angle DAF = 90°$ (angle in a semicircle)

$\therefore \angle CAF + \angle DAF = 180°$

$\therefore \angle CAD = 180°$, so C, A and D are collinear

21 $\angle SVT = \angle SUT = 45°$ (angle at centre is twice angle at circumference standing on the same arc)

$\angle UTV = \angle TVS = 45°$ (alternate angles equal, $VS \parallel UT$)

$\therefore \angle TWU = 90°$ (angle sum of ΔTWU)

22 $\angle ACD = \angle ADC$ (base angles of isosceles triangle ACD)

$\angle AFB = \angle ACB$ (angles in the same segment standing on AB)

$\angle AFE = \angle ADE$ (angles in the same segment standing on AE)

$\therefore \angle AFB = \angle AFE$ (from line 1)

$\therefore AF$ bisects $\angle EFB$

23 $\angle KML = 90°$ (angle in a semicircle on diameter KL)

$\angle ONL = 90°$ (angle in a semicircle on diameter OL)

$\therefore \angle KML = \angle ONL = 90°$

$\therefore KM \parallel ON$ (pair of corresponding angles are equal)

24 $\angle COA = 2\angle CBA$ (angle at the centre is twice the angle at the circumference standing on the same arc)

Reflex $\angle COA = 2\angle CDA$ (angle at the centre is twice the angle at the circumference standing on the same arc)

But $\angle COA$ + reflex $\angle COA = 360°$ (angles at a point)

So $2\angle CBA + 2\angle CDA = 360°$

Hence $\angle CBA + \angle CDA = 180°$

Open-ended – Sample answers

25 **(a)** **(i)** $\angle DAC = 30°$ **(ii)** $\angle ADB = 70°$

(iii) $\angle ACB = 70°$

(b) Students' own answer.

(c) **(i)** ΔADC is isosceles because $\angle DAC = \angle DCA = 30°$

(ii) $\angle ABC = 60°$

$\angle ADC = 120°$

(d) **(i)** ΔADB is isosceles because $\angle ADB = \angle ABD = 70°$

(ii) $\angle BCD = 140°$

$\angle ADB = 40°$

26 Neither is correct. It is just a quadrilateral with a pair of opposite angles each equal to 90°.

$\angle KJM = \angle KLM = 90°$ (angle in a semicircle)

Nothing is known about the lengths of the sides. To be a kite, you would need $JM = LM$. To be a square, you would need $JM = LM = LK = KJ$.

Exercise 8.7 (p. 463)

1 **(a)** In Δs WOX and YOZ, $WX = YZ$ (given)

$WO = YO$ (radii)

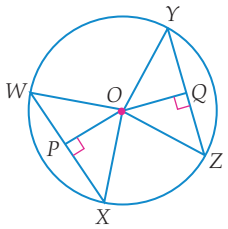
$OX = OZ$ (radii)

$\therefore \Delta WOX \equiv \Delta YOZ$ (SSS)

$\therefore \angle WOX = \angle YOZ$ (matching angles in congruent triangles)

(b) $\angle WXO = \angle YZO$ (matching angles in congruent triangles)

In Δs OXP and OZQ



$\angle OPX = \angle OQZ = 90°$ (given)

$\angle PXO = \angle QZO$ (from above)

$OX = OZ$ (radii)

$\therefore \Delta OXP \equiv \Delta OZQ$ (AAS)

$\therefore OP = OQ$ (matching sides in congruent triangles)

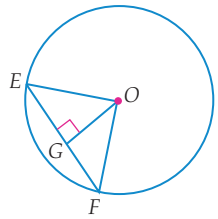
2 In Δs OGE and OGF, $\angle OGE = \angle OGF = 90°$ (given)

$OE = OF$ (radii)

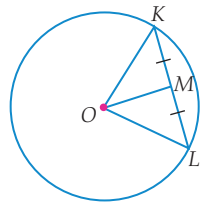
OG is a common side

$\therefore \Delta OGE \equiv \Delta OGF$ (RHS)

$\therefore EG = GF$ (matching sides in congruent triangles)



3 In Δs OMK and OML, $KM = ML$ (given)



$OK = OL$ (radii)

OM is a common side

$\therefore \Delta OKM \equiv \Delta OLM$ (SSS)

$\therefore \angle OMK = \angle OML$ (matching angles in congruent triangles)

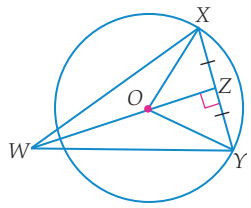
But $\angle OMK + \angle OML = 180°$, so $\angle OMK = \angle OML = 90°$. Hence, $OM \perp KL$.

4 $WX = WY$ as all points on the perpendicular bisector of the interval XY are equidistant from the end points of the interval.

$OX = OY$ (radii of the circle centre O on which X and Y lie)

Because O is equidistant from X and Y, it must lie on the perpendicular bisector of XY.

$\therefore O$ lies on WZ.

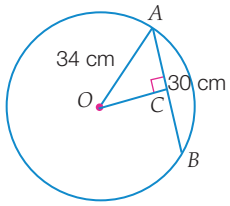


5 $WX = XY$ (given)

Hence, $x = 70$ (equal chords subtend equal angles at the centre)

Because $\angle WOX = \angle YOZ = 70°$, $y = 2$ (equal angles at the centre are subtended by equal chords)

6 Draw OC perpendicular to AB.



$AC = CB = 15$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

$34^2 = 15^2 + OC^2$

$OC^2 = 34^2 - 15^2 = 1156 - 225 = 931$

$OC = \sqrt{931} \approx 30.5$ cm

7 **(a)** $AB = 5$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $AB = 2 \times AC = 20$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(c) $AB = 15$ cm (Pythagoras' theorem)

8 **(a)** $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

(b) $\angle AOB = 80°$ (angle sum of isosceles triangle is 180°)

(c) $\angle OFG = 50°$ (base angles of isosceles triangle), $\angle FOG = 80°$ (angle sum of triangle), $\angle AOB = 80°$ (equal chords subtend equal angles at the centre)

9 **(a)** $AE = EB = 24$ cm (E is midpoint of AB, given)

$\angle OEA = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$OA^2 = 7^2 + 24^2$ (Pythagoras' theorem)

$OA = 25$ cm

(b) $OG^2 = 20^2 + 21^2 = 841$ (Pythagoras' theorem), $OG = 29$, $OA = 29$ cm

(c) ΔOGF is equilateral; $OA = OG = 10$ cm

10 **(a)** $CE^2 = 53^2 - 28^2 = 2025$ (Pythagoras' theorem), $CE = 45$, $CD = 2 \times 45 = 90$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $GH^2 = 73^2 - 48^2 = 3025$ (Pythagoras' theorem), $GH = 55$, $GF = 2 \times 55 = 110$ cm (perpendicular from the centre of a circle to a chord, bisects the chord); $FD = GF = 110$ cm (given), $CD = FD = 110$ cm (equal angles at the centre are subtended by equal chords)

(c) $GH^2 = 85^2 - 77^2 = 1296$ (Pythagoras' theorem), $GH = 36$, $GF = 2 \times 36 = 72$ cm (perpendicular from the centre of a circle to a chord, bisects the chord), $CD = GF = 72$ cm (equal angles at the centre are subtended by equal chords)

11 In Δs DOE and BDC, $DO = OB$ (radii)

$DE = EC$ (E is midpoint of CD)

$\therefore \dfrac{DO}{DB} = \dfrac{DE}{DC} = \dfrac{1}{2}$

$\angle ODE = \angle BDC$ (same angle)

$\Delta DOE \parallel\!| \Delta DCB$ (common angle with the sides containing it proportional)

$\angle DEO = \angle DCB$ (matching angles of similar triangles)

$\therefore OE \parallel BC$ (pair of corresponding angles are equal)

But $\angle DEO = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle BCD = 90°$

12 Join OD

$\angle OFC = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle OFC + \angle FCD = 180°$ ($\angle FCD = 90°$, given)

But these are a pair of supplementary co-interior angles

$\therefore OF \parallel DC$

$\angle ODC = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle ODC + \angle FCD = 180°$ ($\angle FCD = 90°$, given)

But these are a pair of supplementary co-interior angles)

$\therefore OD \parallel FC$

$\therefore OFCD$ is a parallelogram

$\therefore OF = DC$ (opposite sides of a parallelogram are equal)

But $BC = 2DC$ (D is midpoint of BC)

$\therefore BC = 2OF$

13 $\angle OKF = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\angle QJF = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$\therefore \angle OKF + \angle QJF = 180°$

$\angle OKJ$ and $\angle QJK$ are a pair of supplementary co-interior angles contained between OK and QJ.

$\therefore OK \parallel QJ$

$\therefore \angle KOQ + \angle OQJ = 180°$ (co-interior angles, $OK \parallel QJ$)

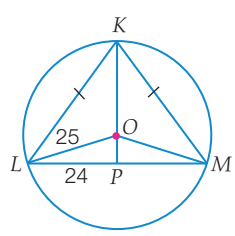
14 Let O be the centre of the circular arch.
By symmetry, the Point C must be in the middle of AB, so $AC = 2$ m.

$OC^2 = 2.5^2 - 2^2$ (Pythagoras' theorem)

$OC = 1.5$ m

$CH = OH - OC = 2.5 - 1.5 = 1$ m

15 Let O be the centre of the circle and P the midpoint of LM.



$\angle OPL = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)

$OP^2 = 25^2 - 24^2$ (Pythagoras' theorem)

$OP = 7$ mm

$KP = 25 + 7 = 32$ mm ($OK = 25$ mm, radius)

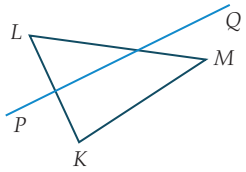
$KL^2 = 32^2 + 24^2$ (Pythagoras' theorem)

$KL = 40$ mm

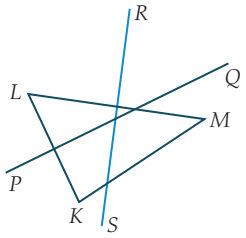
Open-ended – Sample answers

16 Students' own answers.

17 The centres of all the circles passing through K and L lie on the line PQ.

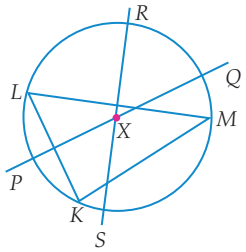


The centres of all the circles passing through L and M lie on the line RS.

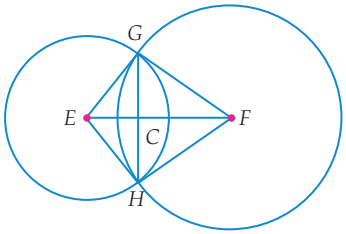


The point of intersection of PQ and RS must be the centre of the only circle that passes through K, L and M.

Label the point of intersection of PQ and RS, X, and draw the circle centre X and radius XL. It will pass through all the points K, L and M.



18 (i)



In Δs EGF and EFH,
$EG = EH$ (radii of circle centre E)
$GF = HF$ (radii of circle centre F)

EF is a common side $\therefore \Delta EGF \equiv \Delta EFH$ (SSS)

$\angle GEF = \angle HEF$ (corresponding angles in congruent triangles)

In Δs EGC and EHC, $EG = EH$ (radii of circle centre E)

$\angle GEC = \angle HEC$ (corresponding angles in congruent triangles)

EC is a common side.

$\therefore \Delta EGC = \Delta EHC$ (SAS)

$GC = CH$ (corresponding sides in congruent triangles)

$\therefore EF$ biscets GH

(i.e. the line of centres bisects the common chord).

(ii) CE is the line joining the centre of the chord GH to the centre E of the circle.

$\therefore CE \perp GH$

(i.e. the line of centres is perpendicular to the common chord).

Chapter review 8 (p. 469)

1 (a) SSS **(b)** SAS

2 B

3 (a) 5 **(b)** 9

4 (a) In ΔAOB and ΔDOC

$\angle AOB = \angle DOC$ (vertically opposite angles)

$AO = DO$ (radii in the same circle)

$BO = CO$ (radii in the same circle)

$\therefore \Delta AOB = \Delta DOC$ (SAS)

So $AB = DC$ (corresponding sides in congruent triangles)

$\therefore p = 6$ m

(b) In ΔDEH and ΔGEF

$\angle DEH = \angle GEF$ (vertically opposite angles)

$DH = GF$ (given)

$\angle EDH = \angle EGF$ (alternate angles, $DH \parallel GF$)

$\therefore \Delta DEH \equiv \Delta GEF$ (AAA)

So $HE = FE$ (matching sides in congruent triangles)

$\therefore r = 9$ cm

5 (a) In ΔABC and ΔADE,

$\angle CAB = \angle EAB$ (common angle)

$\angle ABC = \angle ADE$ (given)

$\therefore \angle ACB = \angle AED$ (third angle in a triangle)

$\therefore \Delta ABC \parallel\!| \Delta ADE$ (AAA)

(b) In ΔABC and ΔADE,

$\angle BAC = \angle DAE$ (vertically opposite angles)

$\angle BCA = \angle AED$ (alternate angles and $BC \parallel DE$)

$\angle CBA = \angle ADE$ (alternate angles and $BC \parallel DE$)

$\therefore \Delta ABC \parallel\!| \Delta ADE$ (AAA)

6 In ΔAXC and ΔBXC,

$\angle AXC = \angle BXC = 90°$ (given that $CX \perp AB$)

$\angle ACX = \angle BCX = \frac{1}{2} \times \angle ACB$ (given that CX bisects $\angle ACB$)

Now $CX = CX$ (common side in ΔAXC and ΔBXC)

$\therefore \Delta AXC \equiv \Delta BXC$ and so $AX = BX$ (matching sides in congruent triangles)

7 In Δs DGH and DEF

$\angle GDH = \angle EDF$ (common angle)

$\angle DHG = \angle DFE$ (corresponding angles equal, $GH \parallel EF$)

$\angle DGH = \angle DEF$ (corresponding angles equal, $GH \parallel EF$)

$\therefore \Delta DGH \equiv \Delta DEF$ (AAA)

$$\frac{DE}{DG} = \frac{DF}{DH}$$

$$\frac{DG + GE}{DG} = \frac{DH + HF}{DH}$$

$$\frac{DG}{DG} + \frac{GE}{DG} = \frac{DH}{DH} + \frac{HF}{DG}$$

$$1 + \frac{GE}{DG} = 1 + \frac{HF}{DH}$$

$$\therefore \frac{GE}{DG} = \frac{HF}{DH}$$

8 In ΔABC and ΔCDA

$AB = CD$ (given)

$BC = DA$ (given)

$AC = CA$ (common side)

$\therefore \Delta ABC \equiv \Delta CDA$ (SSS)

So $\angle BAC = \angle DCA$ (matching angles in congruent triangles)

But $\angle BAC = \angle DCA$ are alternate angles,

$\therefore AB \parallel CD$

And $\angle BCA = \angle DAC$ (matching angles in congruent triangles)

But $\angle BCA = \angle DAC$ are alternate angles $\therefore AD \parallel CB$

The quadrilateral is a parallelogram because both pairs of opposite sides are parallel.

Have proved that if both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.

9 In ΔPTQ, ΔRTQ, ΔRTS and ΔPTS

$PT = RT$ (given)

$ST = QT$ (given)

$\angle PTQ = \angle RTQ = \angle RTS = \angle PTS = 90°$ (given)

$\therefore \Delta PTQ \equiv \Delta RTQ \equiv \Delta RTS \equiv \Delta PTS$ (SAS)

So $PQ = RQ$ (corresponding sides in congruent triangles)

ΔPTQ is isosceles so $\angle TPQ = \angle TQP$

$\angle TPQ + \angle TQP + \angle PTQ = 180°$

$2\angle TQP + 90° = 180°$

$2\angle TQP = 90°$

$\angle TQP = 45°$

But $\angle TQP = \angle TQR = \angle TRQ = \angle TRS = \angle TSR = \angle TSP = \angle TPS = \angle TPQ$ (corresponding angles in congruent triangles)

So $\angle TQR = 45°$

$\angle PQR = \angle TQR + \angle TQP = 90°$

$\angle QRS = \angle TRQ + \angle TRS = 90°$

$\angle SPQ = \angle TPQ + \angle TPS = 90°$

$\angle PQR$ and $\angle QRS$ are co-interior angles and they add to 180°.

$\therefore PQ \parallel SR$

$\angle PQR$ and $\angle SPQ$ are co-interior angles and they add to 180°.

$\therefore PS \parallel QR$

Have shown $PQ = RQ$, $\angle PQR = 90°$, $PQ \parallel SR$ and $PS \parallel QR$.

$PQRS$ is a square as it has two adjacent sides, one right angle and both pairs of opposite sides parallel.

Have proved that if the diagonals of a quadrilateral are equal and bisect each other at right angles, then quadrilateral is a square.

10 **(a)** $y = 59°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

(b) $x = 125°$ (reflex angle at the centre is twice the angle at the circumference standing on the same arc)

(c) $x = 38°$ (vertically opposite angles, angle sum of triangle)

(d) $x = 83°$ (angles in the same segment, angle sum of triangle)

(e) $k = 39°$ (angles in the same segment, exterior angle of a triangle)

(f) $x = 33°$ (angle at the centre is twice the angle at the circumference standing on the same arc)

$y = 90 - 36 = 54°$ (angle in a semicircle, isosceles triangle)

11 **(a)** $AB = 8$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(b) $AB = 16$ cm (perpendicular from the centre of a circle to a chord, bisects the chord)

(c) $AE = 40$ mm (line joining midpoint of chord to centre of circle $\perp$ chord, Pythagoras' theorem), $AB = 80$ mm

12 **(a)** $\angle AOB = 65°$ (equal chords subtend equal angles at the centre)

(b) $\angle OFG = 55°$ (base angles of isosceles triangle), $\angle GOF = 70°$ (angle sum of triangle), $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

(c) $\angle GOH = 35°$ (perpendicular bisector of chord bisects angle at centre), $\angle GOF = 70°$, $\angle AOB = 70°$ (equal chords subtend equal angles at the centre)

13 12.6 m

14 **(a)** In ΔBAD and ΔBCD

$\angle BDA = \angle BDC = 90°$ (by construction)

$BA = BC$ (given)

$BD = BD$ (common side)

$\therefore \Delta BAD \equiv \Delta BDC$ (RHS)

So $\angle BAD = \angle BCD$ (matching angles in congruent triangles)

$\therefore \angle BAC = \angle BCA$

(b) Have proved that the angles opposite the equal sides in an isosceles triangle are also equal.

15 In ΔABZ and ΔACX,

$AB = AC$ (given)

$AZ = AX$ (given)

$\angle BAZ = \angle CAX$ (common angle)

$\therefore \Delta ABZ \equiv \Delta ACX$ (SAS) (and so $BZ = CX$)

Now consider ΔBYC

$\angle YBC = \angle YCB$ (given)

$BY = CY$ (If two angles in a triangle are equal then the sides opposite these angles are equal—the triangle is isosceles.)

Consider ΔBYX and ΔCYZ

$\angle BYX = \angle CYZ$ (vertically opposite angles)

$BX = CZ$ (as $AB = AC$ and $AZ = AX$)

$XY = ZY$ (as $BY = CY$ and $BZ = CX$)

$\therefore \Delta BXY \equiv \Delta CZY$ (SSS)

16 (a) Let $\angle DFE = x$

$\therefore \angle GEF = (90 - x)$ (angle sum of ΔGEF is 180°)

$\therefore \angle GED = x$ ($\angle DEF = 90°$)

In Δs DEF and DEG,

$\angle DFE = \angle DEG = x$

$\angle DEF = \angle DGE = 90°$

$\therefore \Delta DEF \parallel\!| \Delta DGE$ (AAA)

$\frac{EF}{EG} = \frac{DF}{DE} = \frac{DE}{DG}$ (1) (matching sides of similar triangles are proportional)

In Δs DEF and EGF,

$\angle DFE = \angle GFE = x$

$\angle DEF = \angle FGE = 90°$

$\therefore \Delta DEF \parallel\!| \Delta EGF$ (AAA)

(b) $\frac{EF}{GF} = \frac{DF}{EF} = \frac{DE}{EG}$ (2) (matching sides of similar triangles are proportional)

(1) gives $EF = \frac{EG \times DE}{DG}$

(2) gives $EF = \frac{GF \times DE}{EG}$

$\therefore \frac{EG \times DE}{DG} = \frac{GF \times DE}{EG}$

$\therefore EG^2 = DG \times GF$

17 (a) In ΔEAB and ΔECD

$\angle ABE = \angle CDE$ (alternate angles, $AB \parallel CD$)

$\angle AEB = \angle CED$ (vertically opposite)

$AB = CD$ (given)

$\therefore \Delta EAB = \Delta ECD$ (by ASA)

$AE = CE$ (matching sides in congruent triangles)

In ΔEAB and ΔECB

$AE = CE$ (found earlier)

$AB = CB$ (given)

$BE = BE$ (common side)

$\therefore \Delta EAB = \Delta ECB$ (by SSS)

$\therefore \angle AEB = \angle CEB$ (matching angles in congruent triangles)

But $\angle AEB + \angle CEB = 180°$ (angles on a straight line, supplementary)

$\therefore \angle AEB = 90°$ and $\angle CEB = 90°$

$\therefore AC \perp BD$

(b) Have just proved that the diagonals of a rhombus intersect at right angles.

18 (a) $x = 60°$, $y = 50°$ ($\angle$ at centre, $\angle$s in same segment)

(b) $x = 100°$, $y = 80°$ (co-interior angles, opposite angles in cyclic quadrilateral)

(c) $x = 73°$, $y = 13°$, $z = 107°$ (straight angle, opposite angles in cyclic quadrilateral)

19 $\angle TSR + \angle TSU = 180°$ (angles on a straight line)

$\angle TUS + \angle TUV = 180°$ (angles on a straight line)

$\therefore \angle TSR + \angle TSU + \angle TUS + \angle TUV = 180° + 180° = 360°$

$\angle TSU + \angle TUS + \angle UTS = 180°$ (angles in a triangle)

$\angle TSU + \angle TUS + 90° = 180°$

$\angle TSU + \angle TUS = 90°$

$\angle TSR + \angle TSU + \angle TUS + \angle TUV = 360°$ (found earlier)

$\angle TSR + 90° + \angle TUV = 360°$

$\therefore \angle TSR + \angle TUV = 270°$

20 $a + b = 180°$ (angles on a straight line)

$c + d = 180°$ (angles on a straight line)

$\therefore a + b + c + d = 180° + 180° = 360°$

$\angle GWD = \angle AWB = a$ (vertically opposite angles)

$\angle FXA = \angle DXC = d$ (vertically opposite angles)

$\angle BZE = \angle BWD = b$ (corresponding angles, $AD \parallel HE$)

$\angle CYH = \angle CXA = c$ (corresponding angles, $AD \parallel HE$)

$\angle GWD + \angle BZE + \angle FXA + \angle CYH$

$= a + b + c + d$

$= 360°$

So the interior angles of a trapezium add to 360°.

21 To show: $\Delta LYZ \parallel\!| \Delta MXZ$

$\angle LZY = \angle MZX$	(vertically opposite)
$\angle LYZ = \angle MXZ$	(both 90°)
$\angle ZMX = \angle ZLY$	(third angle in a triangle)
$\Delta LYZ \parallel\!| \Delta MXZ$	(AAA)

22 In ΔUXY and ΔVWX

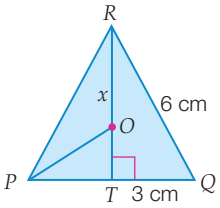
$YX = XW$ (given)

$UY = VX$ (given)

$\angle UYX = \angle VXW$ (corresponding angles, $UY \parallel VX$)

$\therefore \Delta UXY = \Delta VWX$ (SAS)

23 $RT = \sqrt{36-9} = \sqrt{27} = 3\sqrt{3}$ (Pythagoras' theorem in ΔRTQ)



Let radius = x. Hence $OP = x$.

Use Pythagoras' theorem in ΔPOT:

$x^2 = 3^2 + (\sqrt{27} - x)^2$
$x^2 = 9 + 27 - 2\sqrt{27}x + x^2$
$2\sqrt{27}x = 36$
$6\sqrt{3}x = 36$
$x = \dfrac{36}{6\sqrt{3}} = 2\sqrt{3}$

24 $\angle TWU = 90°$ ($SU \perp VT$)

$\angle WUT = 43°$ (angle sum of triangle)

$\angle SVT = \angle SUT = 43°$ (angles in the same segment)

$\angle SOT = 86°$ (angle at the centre twice the angle at the circumference standing on the same arc)

Chapter 9

Recall 9 (p. 475)

1 (a) (i) 4 (ii) 5 (iii) 10
(b) (i) 10 (ii) 18 (iii) 0

2 (a) $x(x-7)$ (b) $2x(x+5)$ (c) $(x+2)(x-2)$

3 (a) $3x^2 + 12x$ (b) $x^2 + 7x + 10$ (c) $x^2 - 4x - 21$
(d) $x^2 + 16x + 64$ (e) $x^2 - 4x + 3$ (f) $x^2 - 16$

4 (a) $(x+5)(x-5)$ (b) $(3x-4)(3x+4)$
(c) $(x+\sqrt{5})(x-\sqrt{5})$ (d) $(x-4)^2$
(e) $(x+7)^2$ (f) $3(x-11)^2$

5 (a) $(x+5)(x+2)$ (b) $(x-3)(x-2)$ (c) $(x-14)(x+1)$
(d) $(x+8)(x-3)$ (e) $(x+9)(x-4)$ (f) $(x+1)(x+6)$
(g) $-(x+2)(x+1)$ (h) $-(x-5)(x-2)$ (i) $-(x+3)(x-4)$

6 (a) 27 (b) $\frac{1}{4}$ (c) $3\frac{1}{25}$

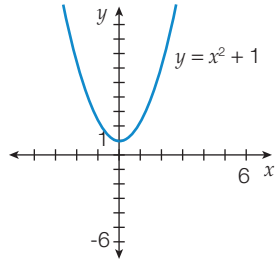
7 (a) $x = -3, 1$ (b) $x = -2, -4$
(c) $x = 1$ (d) $x = 0, 5$

8 (a) $x = 3$ (b) $x = \pm\frac{5}{2}$ (c) $x = -4$ (d) $x = \pm\frac{6}{11}$

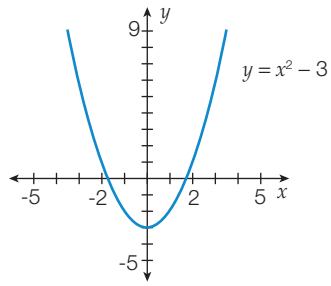
9 (a) $(x+2-\sqrt{2})(x+2+\sqrt{2})$
(b) $(x-3+\sqrt{3})(x-3-\sqrt{3})$
(c) $(x-\frac{3}{2}+\frac{\sqrt{29}}{2})(x-\frac{3}{2}-\frac{\sqrt{29}}{2})$

Exercise 9.1 (p. 483)

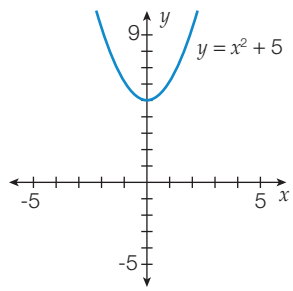
1 (a)



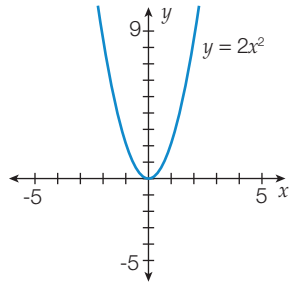
(b)



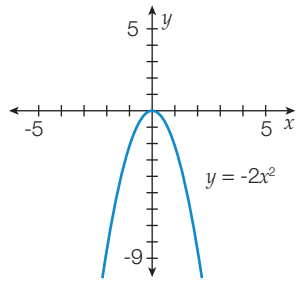
(c)



(d)



(e)



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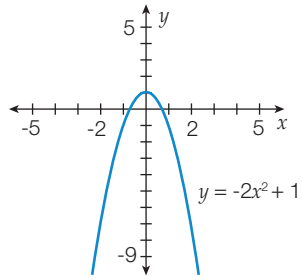
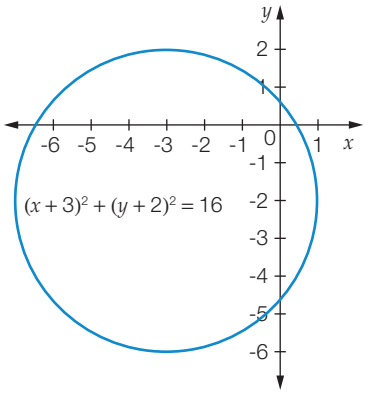


2 (a)



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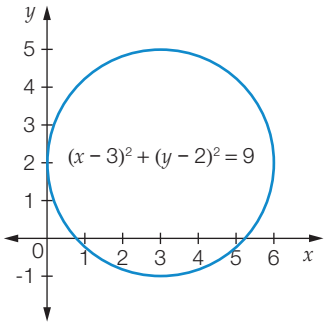
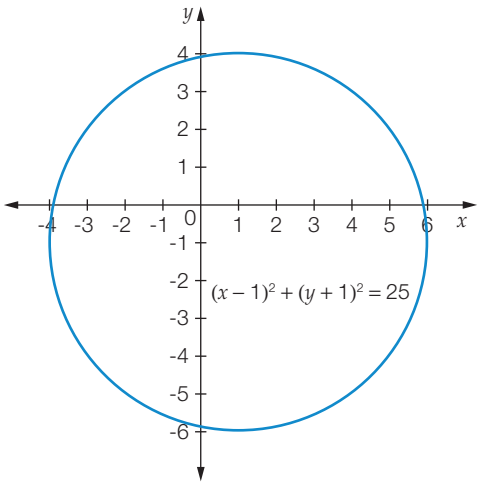


(c)



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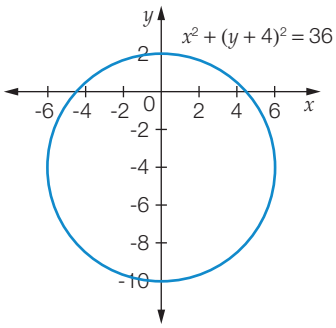

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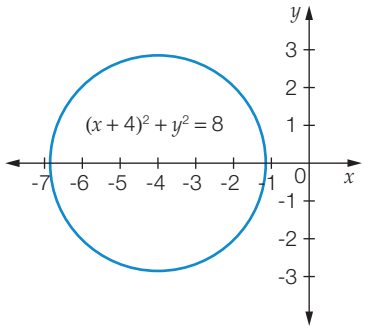

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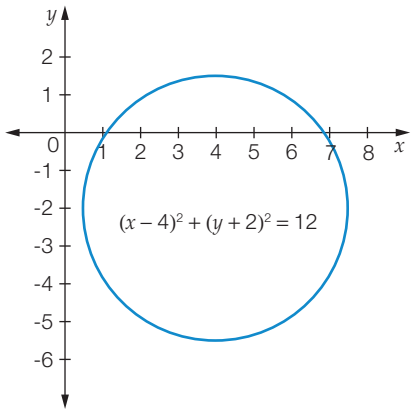
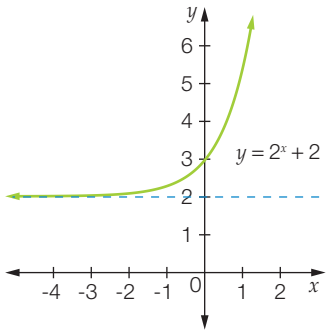
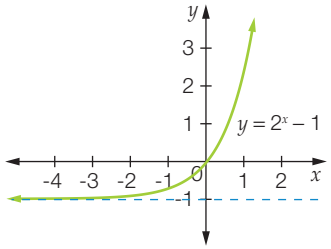


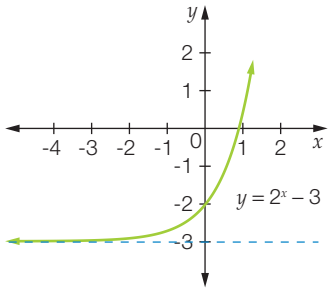
3 (a)



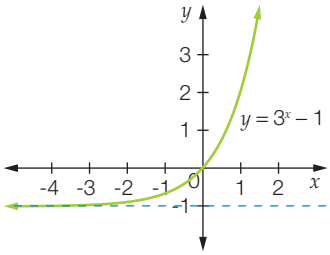
(b)



(c)



(d)



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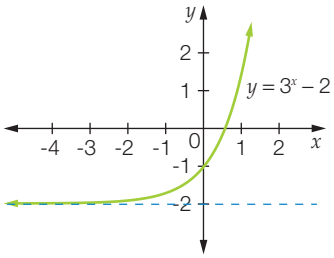
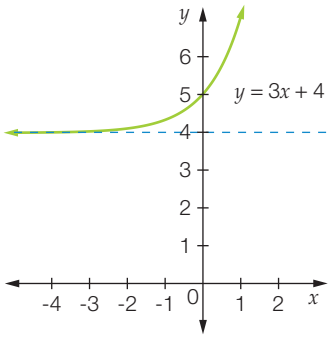
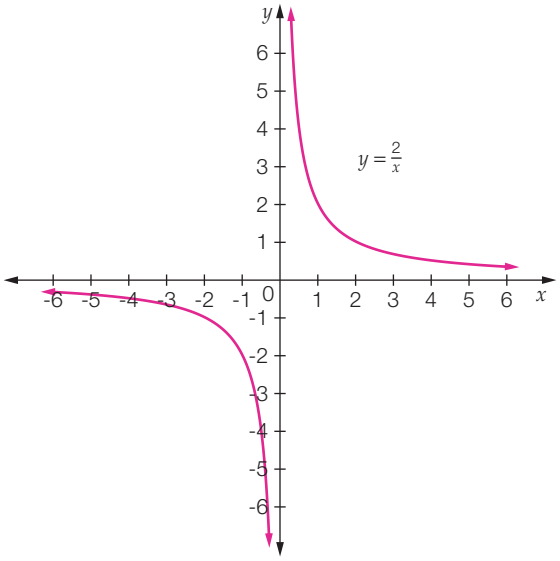


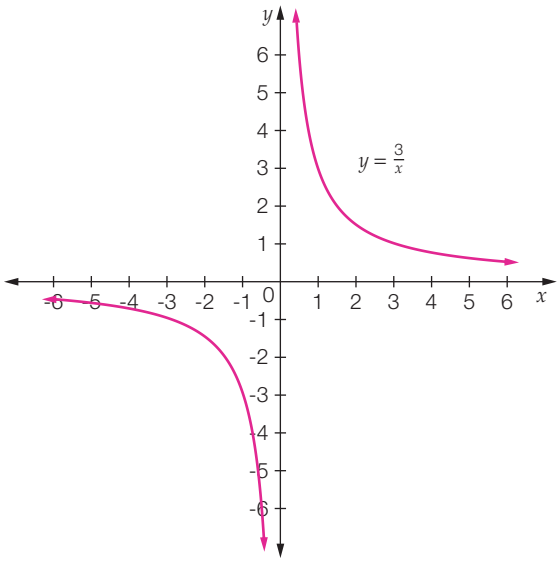
(f)



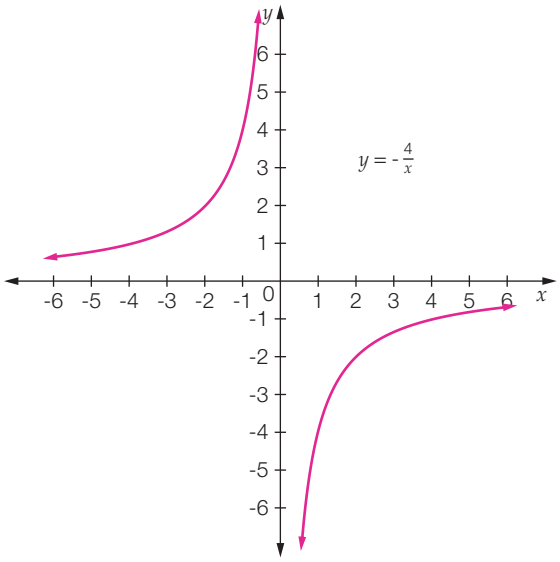
4 (a)



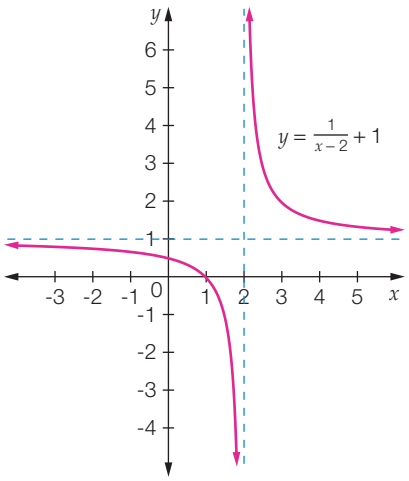
(b)



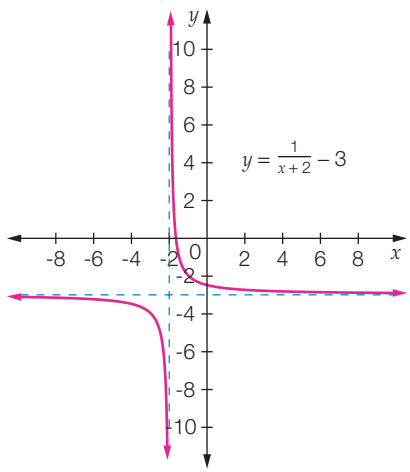
(c)



(d)



(e)



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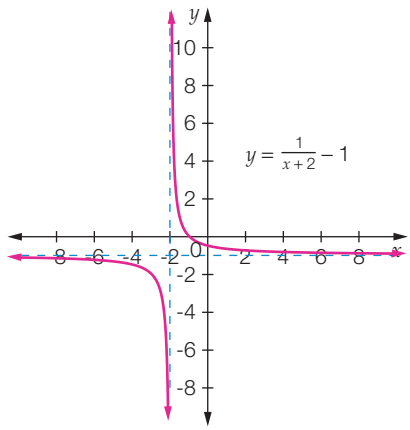
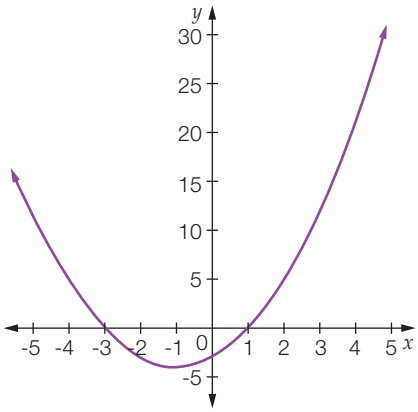


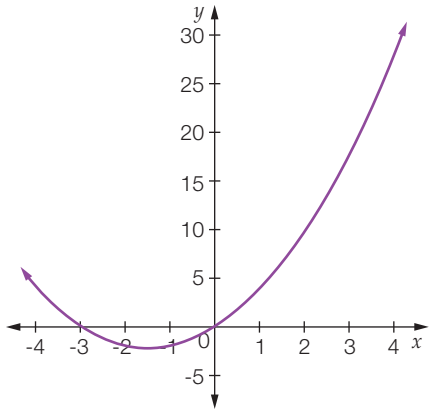
5 (a) (i)

x	-4	-3	-2	-1	0	1	2	3	4
y	5	0	-3	-4	-3	0	5	12	21



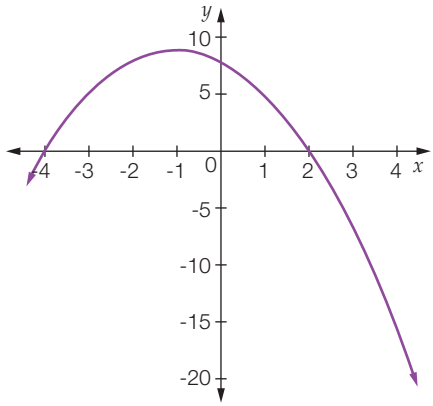
(ii)

x	-4	-3	-2	-1	0	1	2	3	4
y	4	0	-2	-2	0	4	10	18	28



(iii)

x	-4	-3	-2	-1	0	1	2	3	4
y	0	5	8	9	8	5	0	-7	-16



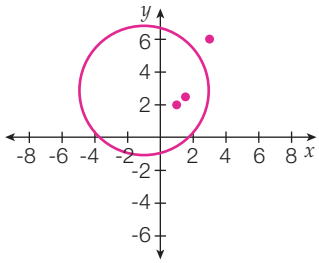
(b) (i) (-1, 4), minimum

(ii) (-1.5, -2.25), minimum

(iii) (-1, 9), maximum

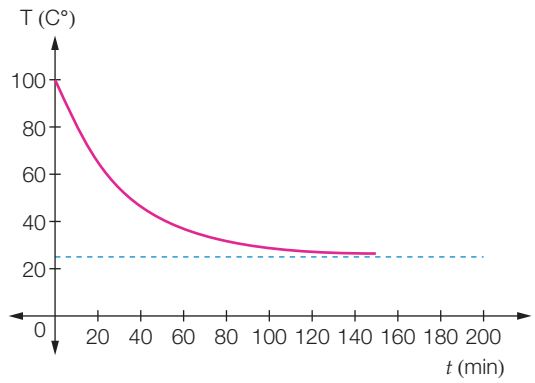
6 C

7 (a)



(b) (1, 2) and (1.5, 2.5) are inside and (3, 6) is outside.

8 **(a)**



(b) 100°C

(c) **(i)** 79°C **(ii)** 65°C **(iii)** 46°C **(iv)** 40°C

(d) The first 10 minutes. The coffee lost 21°C in the first 10 minutes, which is the most in any 10-minute interval. This is the steepest part of the graph.

(e) After 107 minutes

(f) After 3 hours, the temperature of the coffee will be approximately 25°C.

9 **(a)** $V = \pi r^2 h$

$= \pi r^2\left(\frac{3}{\pi}\right)$

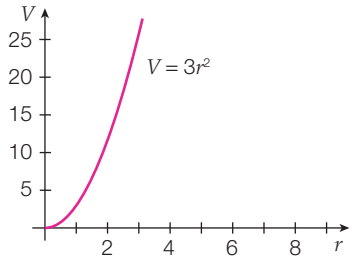
$= 3r^2$

(b) It is a quadratic because the highest power of r is 2.

(c)

r	0	1	2	3	4	5
r^2	0	1	4	9	16	25
V	0	3	12	27	48	75

(d)



(e) **(i)** 18.8 m^3 **(ii)** 43.3 m^3

(f) **(i)** 3.2 m **(ii)** 4.7 m

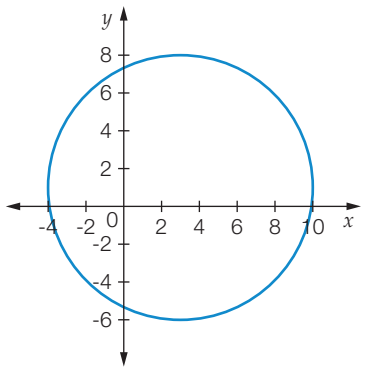
(g) 30 m^3

10 **(a)** **(i)** $(x-3)^2 + (y-1)^2 = 49$

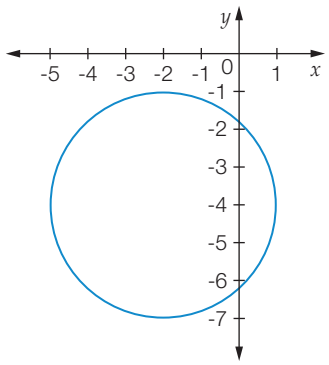
(ii) $(x+2)^2 + (y+4)^2 = 9$

(iii) $(x-2)^2 + (y+5)^2 = 4$

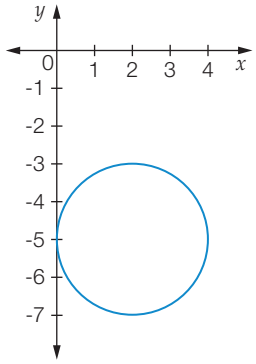
(b) **(i)**



(ii)



(iii)

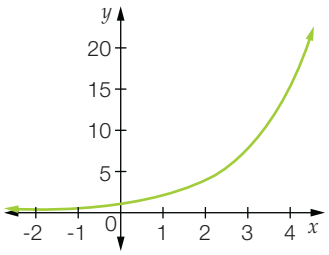


11 B **12** D

13 **(a)**

x	-2	-1	0	1	2	3	4
y	0.25	0.5	1	2	4	8	16

(b)



(c) All of these answers are approximate.

(i) 2.8 **(ii)** 1.7 **(iii)** 0.7

(iv) 0.4 **(v)** 1.4 **(vi)** 2.8

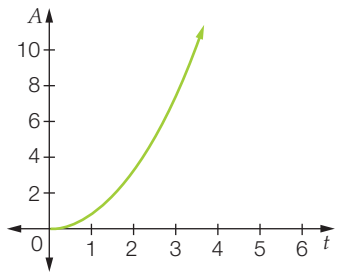
(d) Students are to check **(c)** using a calculator.

(e) All of these answers are approximate.

(i) 2.3 **(ii)** 3.6 **(iii)** 3.9

14 **(a)** $A = l(l - 0.4)$ m^2

(b)



(c) 2 m × 1.6 m

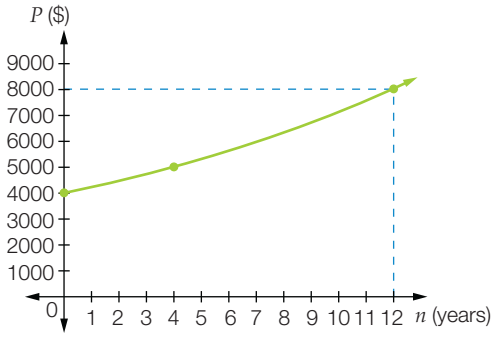
15 **(a)** 4240

(b)

n	0	1	2	3	4	5
P	4000	4240	4494	4764	5050	5353

(c) $P = 4000(1.06)^n$

(d)



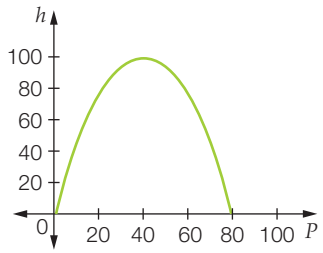
(e) 12 years

(f) $P = 4000(1.06)^{12}$

$= 8048.78$

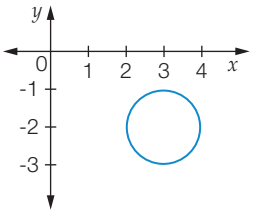
(g) Future time and population can only be positive numbers.

16 **(a)**



(b) 100 m **(c)** 80 m **(d)** 20 106.19 m^2

17 $(x - 3)^2 + (y + 2)^2 = 1$



The point (2.4, -2.7) is inside the circle, so a tree could not be planted there. If we substitute (2.4, -2.7) into the LHS of the equation we get 0.85. Any value less than 1 is inside the circle, exactly 1 is on the perimeter, and greater than 1 is outside the circle.

Open-ended – Sample answers

18 **(a)** Students to provide their own equation, but it must have at least $(x \pm h)^2$ and/or $(y \pm k)^2$ in it.

(b) For a maximum turning point the coefficient of x^2 must be negative.

(c) The equation will need to have -1 at the end.

(d) The equation will need to have $(x + 2)$ in it as well as + 6 at the end.

19 **(a)** Students to provide their own sketch.

(b) The graph should start at the origin, be symmetrical in shape with a maximum turning point. This means the turning point is midway between the two x-intercepts.

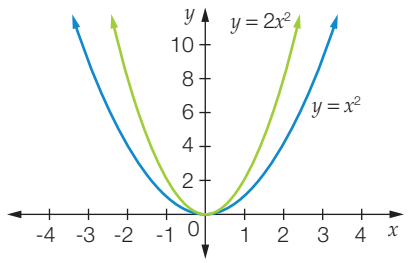
20 $(x + 3)^2 + (y + 1)^2 = 25$

$(x - 4)^2 + (y + 2)^2 = 25$

Exercise 9.2 (p. 493)

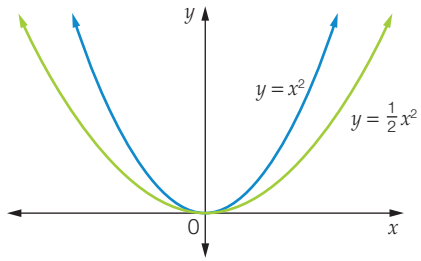
1 **(a)** **(i)** 2 **(ii)** narrower

(iii)

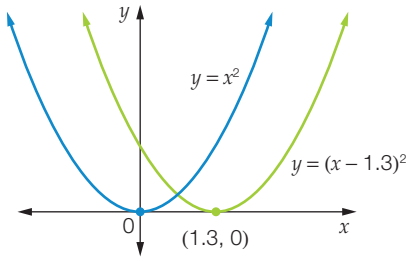


(b) **(i)** 0.5 or $\frac{1}{2}$ **(ii)** wider

(iii)

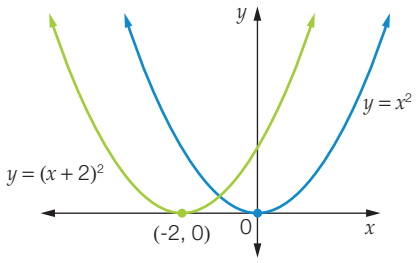


2 **(a)** **(i)**



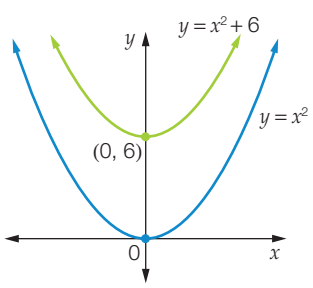
(ii) 1.3 units to the right

(b) **(i)**



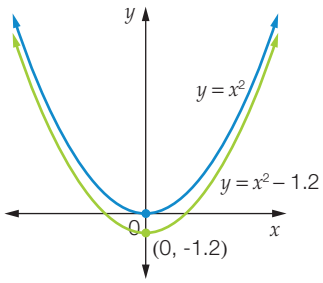
(ii) 2 units to the left

3 **(a)** **(i)**



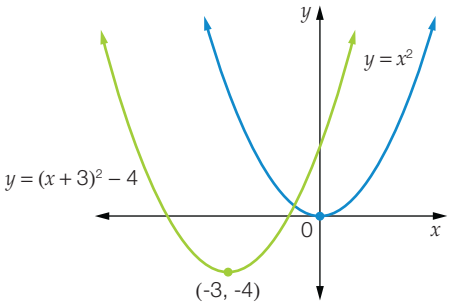
(ii) 6 units up

(b) **(i)**

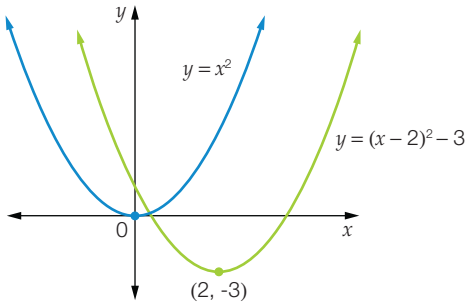


(ii) 1.2 units down

4 **(a)**



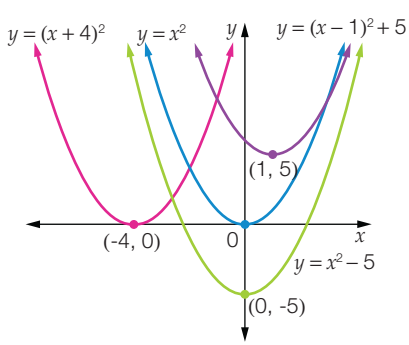
(b)



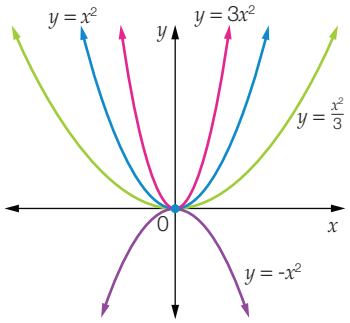
5 $y = (x - 2)^2 - 5$

6 B

7 **(a)–(d)**

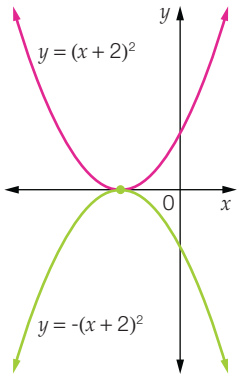


8 **(a)–(d)**

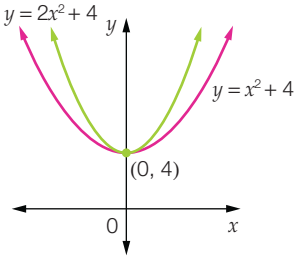


9 D

10 **(a)** and **(b)**

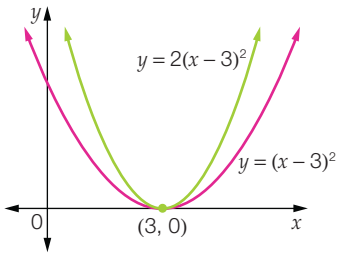


11 **(a)** and **(b)**



(c) $y = 2x^2 + 4$

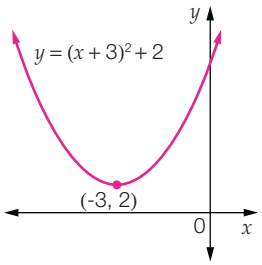
12 **(a)** and **(b)**



(c) $y = 2(x - 3)^2$

13 **(a)** $y = (x + 3)^2 + 2$

(b)



14 $y = (x + 2)^2 + 0.5$

15 (0, 0), (1, 5), (2, 8), (3, 9), (4, 8), (5, 5), (6, 0)

Open-ended – Sample answers

16 (a) The equation will be in the form $y = x^2 + k$.

(b) The equation will be in the form $y = (x - h)^2$.

17 When dealing with brackets you are looking for the value of x that will make the bracket equal zero. So, $(x + 6)$ requires x to equal -6 to make the bracket 0. This means the turning point will be in line with $x = -6$. Syd is correct in his thinking about the vertical shift.

Exercise 9.3 (p. 503)

1 (a) $x = 0$ or $x = 11$ (b) $x = 0$ or $x = -6$
(c) $x = 0$ or $x = -4$ (d) $x = 0$ or $x = 1$
(e) $x = 0$ or $x = 1$ (f) $x = 0$ or $x = 6$
(g) $x = 3$ or $x = -7$ (h) $x = 4$ or $x = -1$
(i) $x = 2$ or $x = 6$ (j) $x = 4$ or $x = 5$
(k) $x = 7$ or $x = -2$ (l) $x = 3$ or $x = -10$

2 (a) $x = 0$ (b) $x = 0$ (c) $x = 0$
(d) $x = -7$ (e) $x = -1$ (f) $x = 9$

3 (a) $x = 0$ or $x = 4$ (b) $x = 0$ or $x = -1$
(c) $x = 0$ or $x = 6$ (d) $x = 0$ or $x = 5$
(e) $x = 0$ or $x = \frac{1}{2}$ (f) $x = 0$ or $x = -3$

4 (a) $x = 5$ or $x = -5$ (b) $x = 9$ or $x = -9$
(c) $x = 2$ or $x = -2$ (d) $x = \frac{5}{4}$ or $x = -\frac{5}{4}$
(e) $x = \frac{7}{3}$ or $x = -\frac{7}{3}$ (f) $x = \frac{9}{10}$ or $x = -\frac{9}{10}$
(g) $x = 5$ or $x = -5$ (h) $x = 1$ or $x = -1$
(i) $x = 2$ or $x = -2$ (j) $x = 7$ or $x = -7$
(k) $x = 9$ or $x = -9$ (l) $x = 11$ or $x = -11$
(m) $x = \sqrt{13}$ or $x = -\sqrt{13}$
(n) $x = \sqrt{24}$ or $x = -\sqrt{24}$ (or $x = 2\sqrt{6}$ or $x = -2\sqrt{6}$)
(o) $x = \sqrt{32}$ or $x = -\sqrt{32}$ (or $x = 4\sqrt{2}$ or $x = -4\sqrt{2}$)

5 (a) $x = 7$ (b) $x = -4$ (c) $x = -8$
(d) $x = 2$ (e) $x = -1$ (f) $x = 3$
(g) $x = 5$ (h) $x = 1$ (i) $x = -2$
(j) $x = 3$ (k) $x = 5$ (l) $x = 4$

6 (a) $x = -1$ or $x = -2$ (b) $x = -3$ or $x = -6$
(c) $x = 4$ or $x = 7$ (d) $x = 1$ or $x = 9$
(e) $x = -5$ or $x = 6$ (f) $x = -4$ or $x = 2$
(g) $x = 4$ or $x = 8$ (h) $x = -5$ or $x = 3$
(i) $x = -2$ or $x = 8$ (j) $x = -6$ or $x = 2$
(k) $x = -1$ or $x = 2$ (l) $x = -1$ or $x = 3$

7 (a) $x = -4$ or $x = -3$ (b) $x = -6$ or $x = -2$
(c) $x = -7$ or $x = 9$ (d) $x = -8$ or $x = 3$
(e) $x = -3$ or $x = -2$ (f) $x = -3$ or $x = 5$
(g) $x = -10$ or $x = 5$ (h) $x = -3$ or $x = 9$
(i) $x = 6$ or $x = 11$ (j) $x = 1$ or $x = 5$
(k) $x = -8$ or $x = 2$ (l) $x = -7$ or $x = 3$

8 (a) D (b) A (c) D (d) A

9 (a) $x = -5$ or $x = 2$ (b) $x = -1$ or $x = 3$
(c) $x = -4$ or $x = 7$ (d) $x = -7$ or $x = -1$
(e) $x = -4$ or $x = 4$ (f) $x = -3$ or $x = 3$
(g) $x = -6$ or $x = -1$ (h) $x = -7$ or $x = 6$
(i) $x = 0$ or $x = 12$

10 (a) A (b) D (c) B (d) B (e) B (f) C

11 $t = 0$ seconds and $t = 50$ seconds

12 $x = -1$ and $x = 4$

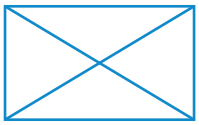
13 (a) $x = -12$ or $x = 12$ (b) $x = 0$ or $x = -4$
(c) $x = 6$

14 (a) 5°C (b) $t = 2$ and $t = 8$
(c) 10 a.m. and 4 p.m.

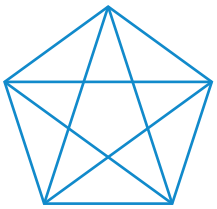
15 (a) $(x + 4)(x + 6)$ m^2 (b) length 8 m, width 6 m

16 60 m

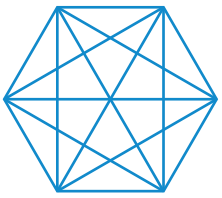
17 (a) $n - 3 = 0$ so $D = 0$
(b) quadrilateral: 2 diagonals



pentagon: 5 diagonals



hexagon: 9 diagonals



(c) (i) 20 (ii) 35 (iii) 54
(d) (i) 20 (ii) 26
(e) (i) 14 (ii) 42

Open-ended – Sample answers

18 (a) Students are to produce their own equations; they could be of the form $(x + a)(x + b) = 0$.

(b) Students are to produce their own equations; they could be of the form $(x + a)^2 = 0$.

19 Students are to produce their own equations; the easiest way to get the second equation is to make it the negative of the first.

20 Possible equations include: $x^2 + 2x - 2 = 0$; $6x^2 - 13x - 5 = 0$; $x^2 - x - 2 = 0$

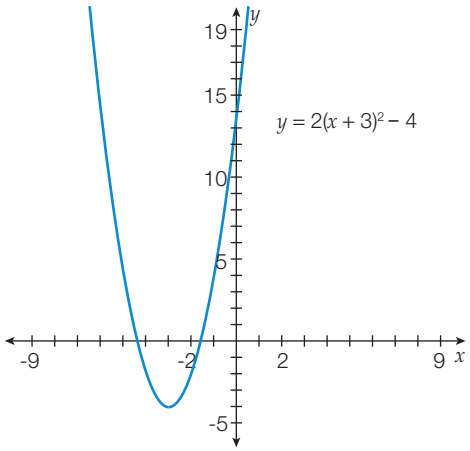
Half-time 9 (p. 507)

1 **(a)** $x = -2$ **(b)** $x = -1, 6$ **(c)** $x = 8, 1$

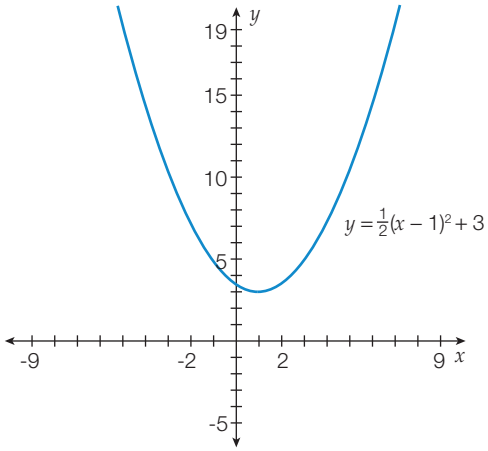
(d) $x = 5, -1$ **(e)** $x = -2, -1$ **(f)** $x = -5, 2$

2 radius 2 and centre (-1, 2)

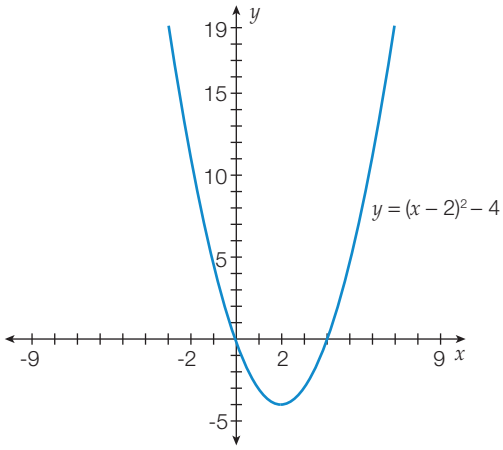
3 **(a)** dilated by a factor of 2, translated 3 units left and 4 units down



(b) dilated by a factor of a half, translated right 1 unit and up 3 units



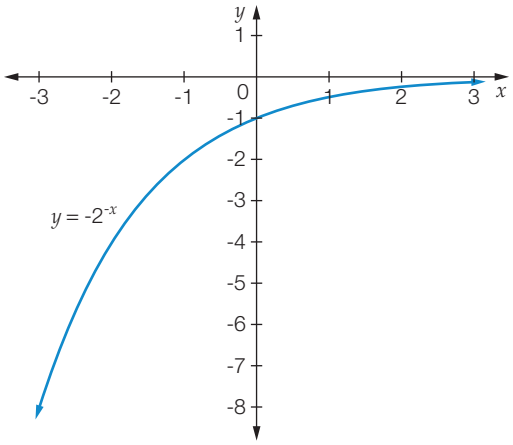
(c) translated 2 units right and 4 units down



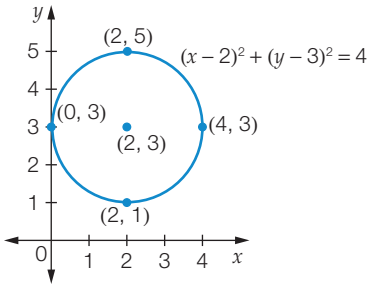
4 $y = (x - 3)^2 + 2$

5 **(a)** $x = 8, -8$ **(b)** $x = -1$ **(c)** $x = 4, 0$

6 **(a)**



(b)



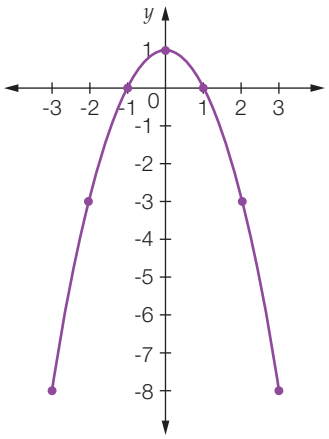
7 **(a)** $x = -5, -1$ **(b)** $x = 0, 9$ **(c)** $x = -5, 2$

(d) $x = -3, 2$ **(e)** $x = 1, 3$ **(f)** $x = 3$

8 **(a)**

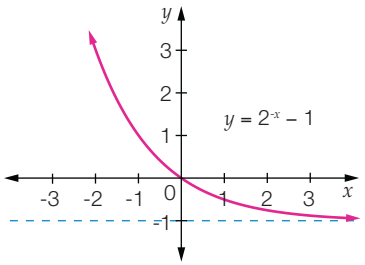
x	-3	-2	-1	0	1	2	3
y	-8	-3	0	1	0	-3	-8

(b)



(c) (0, 1) **(d)** $x = 0$

9 **(a)**



(b) $y = -1$ **(c)** (0, 0)

10 **(a)** rectangular hyperbola **(b)** $x = 3$ and $y = 4$

Exercise 9.4 (p. 510)

1 **(a)** **(i)** $(x+1-\sqrt{6})(x+1+\sqrt{6})$ **(ii)** $x=-1\pm\sqrt{6}$

(b) **(i)** $(x+2-\sqrt{11})(x+2+\sqrt{11})$ **(ii)** $x=-2\pm\sqrt{11}$

(c) **(i)** $(x+4-\sqrt{11})(x+4+\sqrt{11})$ **(ii)** $x=-4\pm\sqrt{11}$

(d) **(i)** $\left(x+\frac{3}{2}-\frac{\sqrt{13}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{13}}{2}\right)$ **(ii)** $x=\frac{-3\pm\sqrt{13}}{2}$

(e) **(i)** $\left(x+\frac{5}{2}-\frac{\sqrt{13}}{2}\right)\left(x+\frac{5}{2}+\frac{\sqrt{13}}{2}\right)$ **(ii)** $x=\frac{-5\pm\sqrt{13}}{2}$

(f) **(i)** $\left(x+\frac{3}{2}-\frac{\sqrt{29}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{29}}{2}\right)$ **(ii)** $x=\frac{-3\pm\sqrt{29}}{2}$

2 **(a)** $x=4\pm\sqrt{10}$ **(b)** $x=-3\pm\sqrt{5}$ **(c)** $x=\frac{3\pm\sqrt{5}}{2}$

(d) $x=\frac{5\pm\sqrt{33}}{2}$ **(e)** $x=-2\pm\sqrt{7}$

(f) $x=10\pm\sqrt{112}$ (or $x=10\pm4\sqrt{7}$)

(g) $x=2\pm\sqrt{10}$ **(h)** $x=-2\pm\sqrt{3}$ **(i)** $x=-2\pm\sqrt{7}$

3 **(a)** C **(b)** B **(c)** D

4 **(a)** $x=2\pm\sqrt{6}$ **(b)** $x=3\pm\sqrt{7}$

(c) $x=2\pm\sqrt{7}$ **(d)** $x=\frac{3\pm\sqrt{29}}{2}$

(e) $x=\frac{5\pm\sqrt{17}}{2}$ **(f)** $x=\frac{-7\pm\sqrt{69}}{2}$

5 **(a)** x^2+4x+7
$=x^2+4x+4+3$
$=(x+2)^2+3$

This cannot be factorised.

(b) You get $(x+2)^2+3$, which is not the difference of two squares.

(c) The graph does not cut the x-axis.

6 **(a)** width $=-2+2\sqrt{21}$ and length $=2+2\sqrt{21}$

(b) width $=7.2$ cm and length $=11.2$ cm

7 $y=-2+\sqrt{28}\approx3.3$ cm

8 **(a)** $x=\frac{1+\sqrt{801}}{2}$ m **(b)** $x=14.65$ m

9 **(a)** length $=l+4$, width $=l+2$

(b) $(l+4)(l+2)=215$

(c) $l=6\sqrt{6}-3$, $w=6\sqrt{6}-5$

(d) 15.7×13.7 cm

Open-ended – Sample answers

10 Students' own equations.

11 Students' own equations that do not result in the difference of two squares.

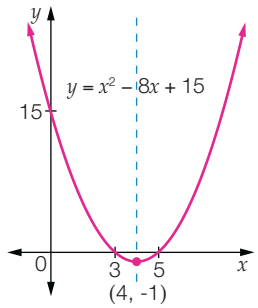
12 $x=-2\pm\sqrt{3}$, $x^2+4x+1=0$

Exercise 9.5 (p. 517)

1 **(a)** **(i)** parabola, min T.P. **(ii)** (0, 15)

(iii) (3, 0), (5, 0) **(iv)** (4, -1)

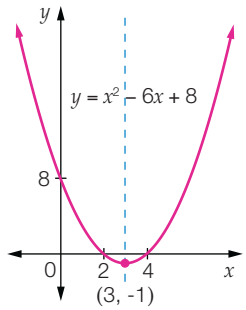
(v) $x=4$



(b) **(i)** parabola, min T.P. **(ii)** (0, 8)

(iii) (2, 0), (4, 0) **(iv)** (3, -1)

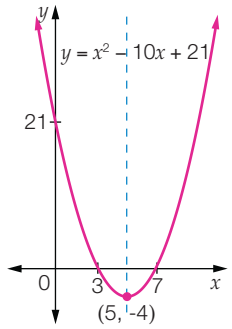
(v) $x=3$



(c) **(i)** parabola, min T.P. **(ii)** (0, 21)

(iii) (3,0), (7, 0) **(iv)** (5, -4)

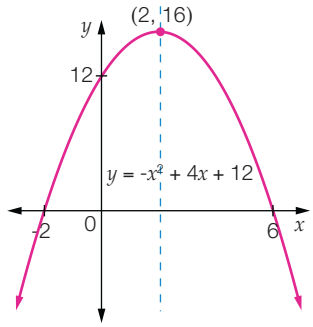
(v) $x=5$



(d) **(i)** parabola, max T.P. **(ii)** (0, 12)

(iii) (-2, 0), (6, 0) **(iv)** (2, 16)

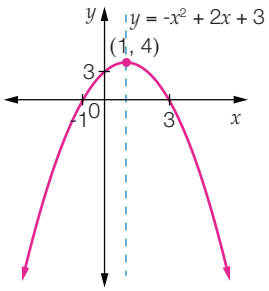
(v) $x=2$



(e) **(i)** parabola, max T.P. **(ii)** (0, 3)

(iii) (-1, 0), (3, 0) **(iv)** (1, 4)

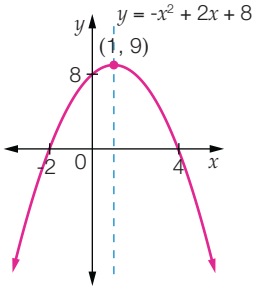
(v) $x = 1$



(f) **(i)** parabola, max T.P. **(ii)** (0, 8)

(iii) (-2, 0), (4, 0) **(iv)** (1, 9)

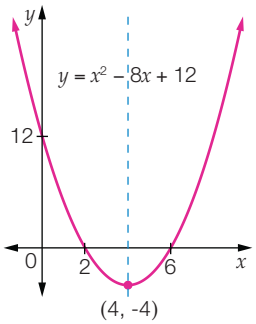
(v) $x = 1$



(g) **(i)** parabola, min T.P. **(ii)** (0, 12)

(iii) (2, 0), (6, 0) **(iv)** (4, -4)

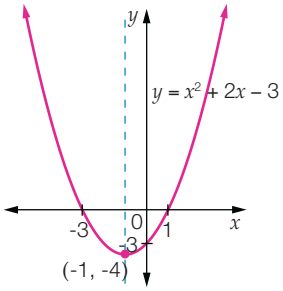
(v) $x = 4$



(h) **(i)** parabola, min T.P. **(ii)** (0, -3)

(iii) (-3, 0), (1, 0) **(iv)** (-1, -4)

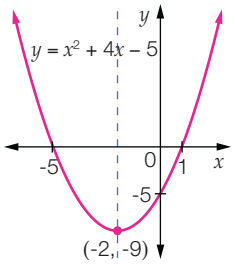
(v) $x = -1$



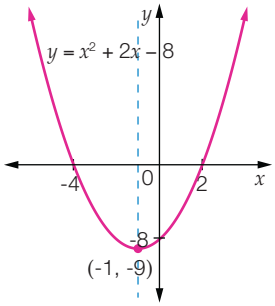
(i) **(i)** parabola, min T.P. **(ii)** (0, -5)

(iii) (-5, 0), (1, 0) **(iv)** (-2, -9)

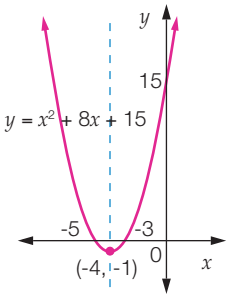
(v) $x = -2$



2 **(a)**



(b)



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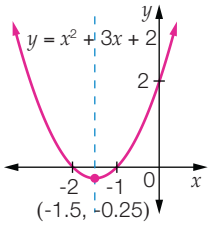

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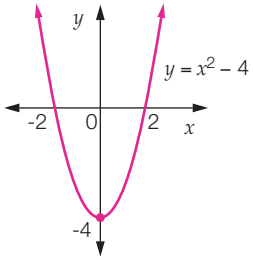

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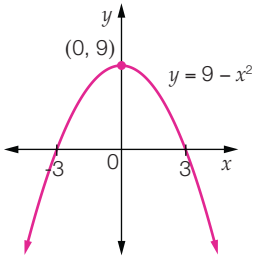

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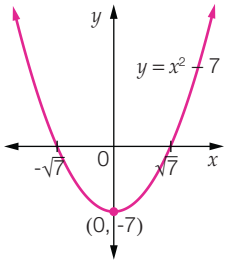

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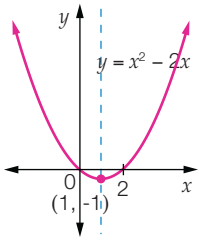

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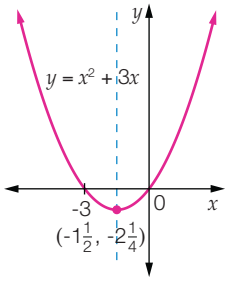

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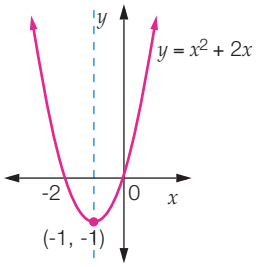

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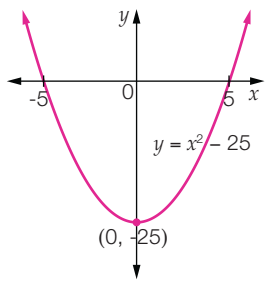

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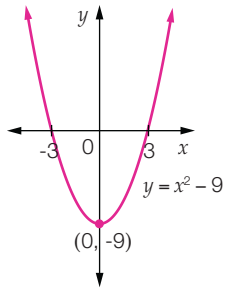

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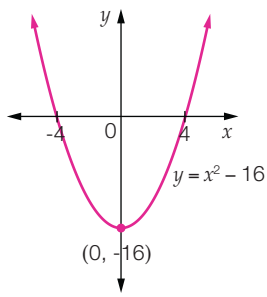

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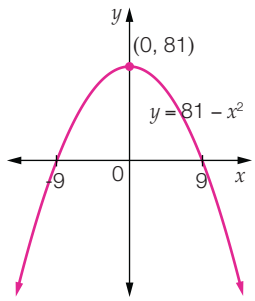

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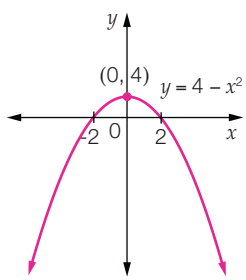

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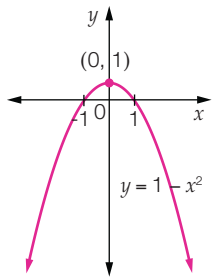

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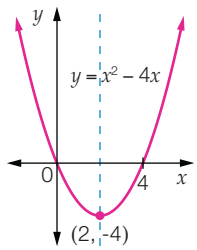

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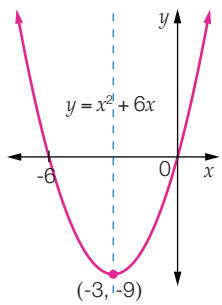

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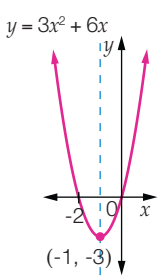

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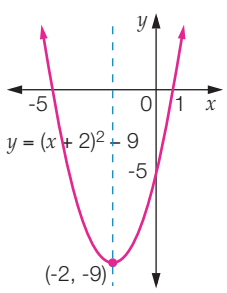

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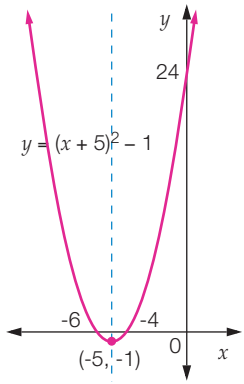

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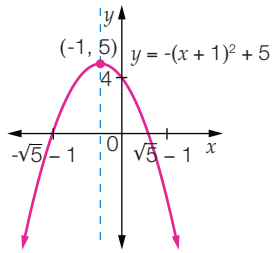

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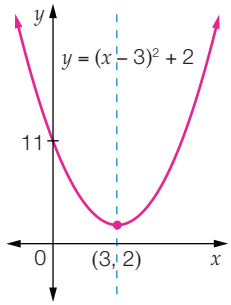

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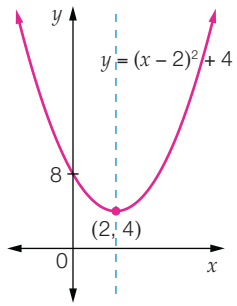

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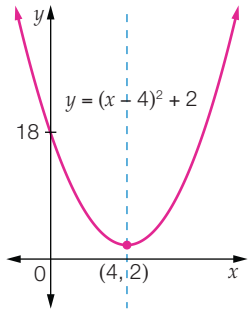
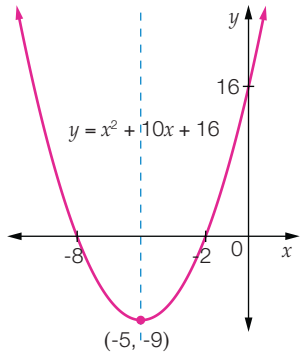


5 (a)



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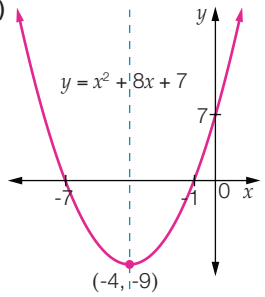

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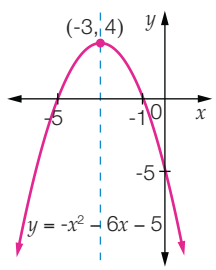

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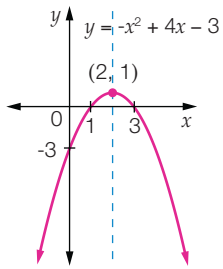

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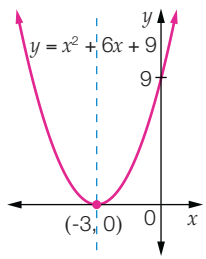

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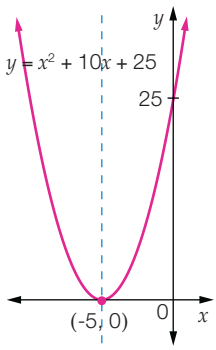

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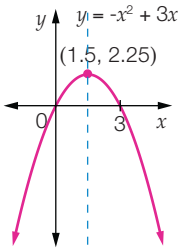

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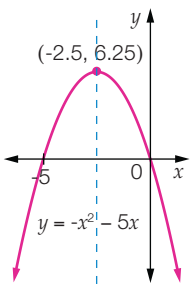
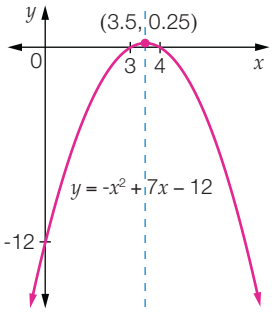


(i)

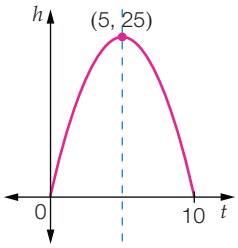


6 B

7 (a) D (b) B

8 (a) C (b) B

9

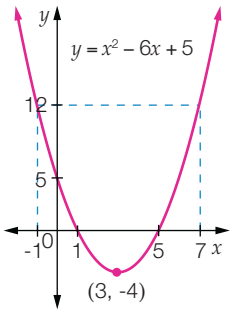


(a) 10 seconds (b) 25 m

10 (a) B (b) B

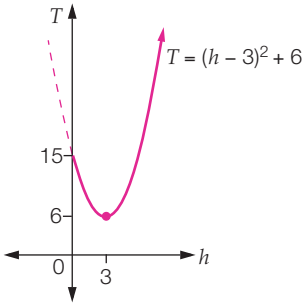
11 (a) D (b) (4, 2)

12 (a)



(b) $x = -1$ or $x = 7$

13 (a) $T = (h - 3)^2 + 6$



(b) 15°C (c) decreasing (d) increasing

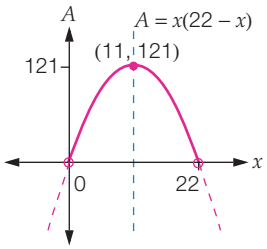
(e) 6°C (f) after 3 hours

(g) No, the temperature of the room is unlikely to increase, and certainly not at this rate. Using the equation, the temperature after 24 hours would be 447°C.

14 $y = (x + 3)(x - 5)$, $y = x^2 - 2x - 15$ or $y = (x - 1)^2 - 16$

15 (a) $A = x(22 - x)$

(b)



(c) $x > 0$ m (d) $x < 22$ m (e) 121 m^2

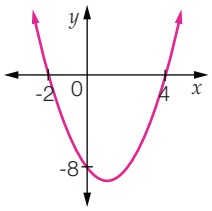
(f) square of length 11 m

Open-ended – Sample answers

16 If the equation is in factorised form then it will always have x-intercepts. If it is in turning point form, you can tell by inspection. If the TP is above the x-axis and it is a valley then no x-intercepts and if the TP is below the x-axis and it is a hill then no x-intercepts. If it is in standard form use completing the square to put it into turning point form and check it from there.

17 (a) The equation factorises to $y = (x - 4)(x + 2)$. Archie may have just found factors of 8 and ignored the sign. As it is negative one of the factors will be positive and the other negative. Archie needs to be more careful with his factorising.

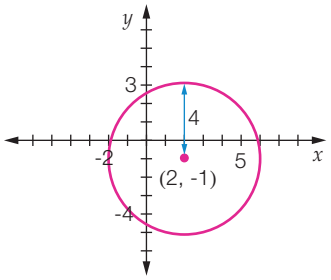
(b)



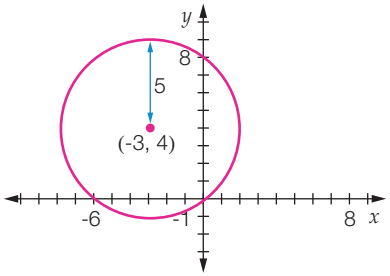
18 Students need to produce their own equations but using factorised form will make it easier. As an example, $(x - 4)(x - 6)$ has x-intercepts at 4 and 6. So the x-coordinate of the turning point is $x = 5$.

Exercise 9.6 (p. 527)

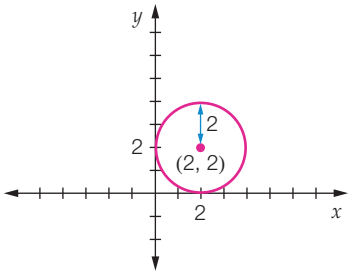
1 (a) $(x - 2)^2 + (y + 1)^2 = 16$



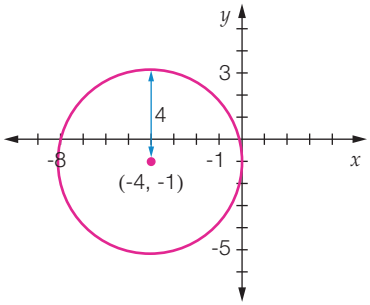
(b) $(x+3)^2+(y-4)^2=25$



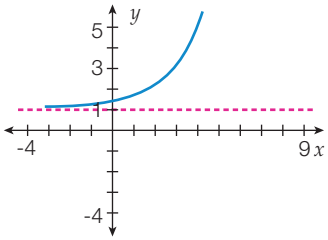
(c) $(x-2)^2+(y-2)^2=4$



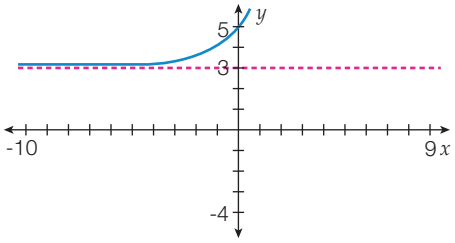
(d) $(x+4)^2+(y+1)^2=16$



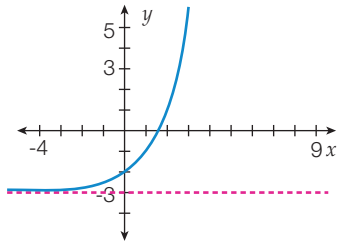
2 (a) $y=2^{x-2}+1$



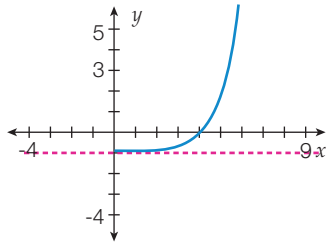
(b) $y=2^{x+1}+3$



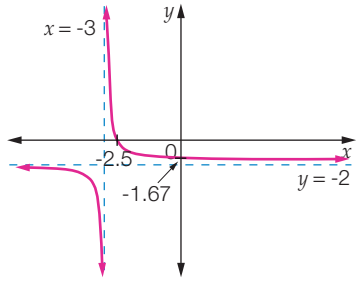
(c) $y=2^x-3$



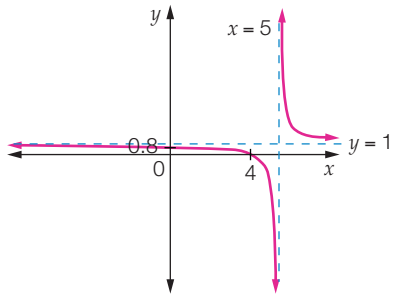
(d) $y=3^{x-4}-1$



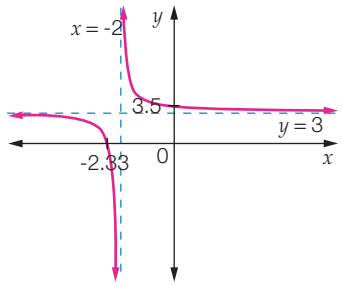
3 (a) $y=\dfrac{1}{x+3}-2$



(b) $y=\dfrac{1}{x-5}+1$



(c) $y=\dfrac{1}{x+2}+3$

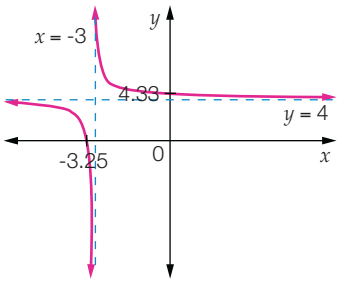


4 (a) $x^2+(y+1)^2=4$ **(b)** $(x+1)^2+(y+1)^2=9$

(c) $(x-2)^2+(y-3)^2=16$ **(d)** $(x+5)^2+(y+3)^2=4$

5 (a) $y = \frac{1}{x+3} + 4$

(b)



6 C

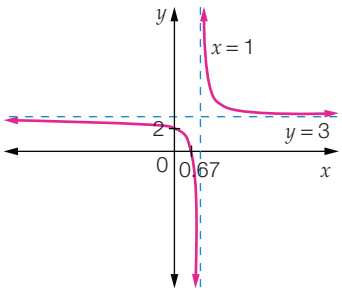
7 (a) $(x-1)^2 + (y+7)^2 = 25$

(b) $(x-3)^2 + (y+4)^2 = 64$

8 (a) $\frac{1}{x-1} + 3 = \frac{1}{x-1} + \frac{3}{x-1} = \frac{1+3(x-1)}{x-1} = \frac{1+3x-3}{x-1}$

$= \frac{3x-2}{x-1} = \text{RHS}$

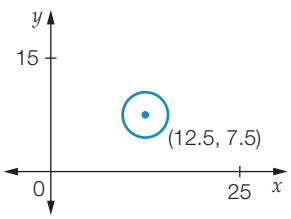
(b)



9 (a) (12.5, 7.5)

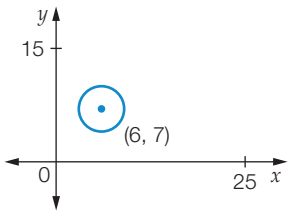
(b) $(x-12.5)^2 + (y-7.5)^2 = 9$

(c)



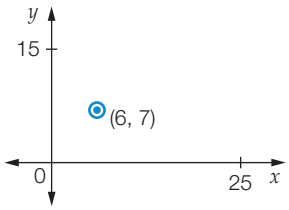
(d) (i) $(x-6)^2 + (y-7)^2 = 9$

(ii)



(e) (i) $(x-6)^2 + (y-7)^2 = 4$

(ii)



Open-ended – Sample answers

10 (a) Something in the form $y = \frac{1}{x-h} + k$.

(b) $x = h$ and $y = k$

(c) Students to sketch their own graph.

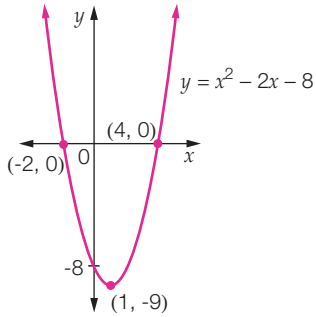
11 As an example $(x-2)^2 + (y-1)^2 = 1$

Chapter review 9 (p. 529)

1 (a) exponential; $y = 3$; asymptotes $x = 2$ and $y = 3$

(b) circle; centre (-1, 3) and $r = 3$

2



(a) x-intercepts: (-2, 0) and (4, 0); y-intercept: (0, -8)

(b) (1, -9) minimum (c) $x = 1$ (d) $y = -5$

(e) $x = 5$ and -3 (f) 5 and -3 (g) -2 and 4

3 (a) A (b) A (c) B

4 (a) $x = 0$ or 6 (b) $x = \pm\frac{9}{2}$ (c) $x = -2$ or 7

(d) $x = -3$ (e) $x = 0$ and 8 (f) $x = 6$

(g) $x = 0$ and -4 (h) $x = -1$ and 7 (i) $x = \pm 4$

5 (a) $x = -7$ and -5 (b) $x = -3$ and 2 (c) $x = 1$ and -9

(d) $x = 1$ and 8 (e) $x = 1$ and -4 (f) $x = 3$ and 1

6 (a) $x = -1$ and 5 (b) $x = -1$ and 6 (c) $x = -4$ and -3

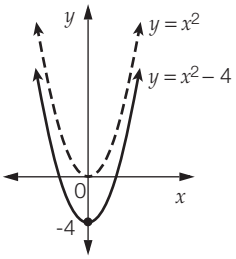
(d) $x = \pm 3$ (e) $x = -8$ and 4 (f) $x = -3$ and 4

7 (a) $x = 2 \pm \sqrt{2}$ (b) $x = -4 \pm \sqrt{7}$ (c) $x = -1 \pm \sqrt{5}$

8 (a) (i) Move 4 units down. (ii) (0, -4)

(iii) (0, -4), (-2, 0), (2, 0)

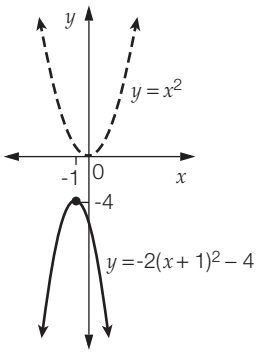
(iv)



(b) **(i)** Reflect in x-axis, dilate by a factor of 2 (make narrower) and translate 1 unit left and 4 units down.

(ii) (-1, -4) **(iii)** (0, -6)

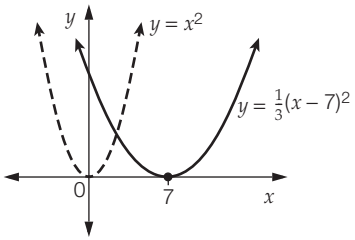
(iv)



(c) **(i)** Dilate by a factor of $\frac{1}{3}$ (make wider) and move 7 units right.

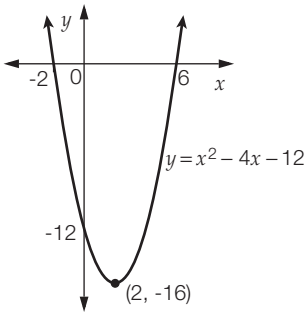
(ii) (7, 0) **(iii)** $(0, \frac{49}{3})$ (7, 0)

(iv)



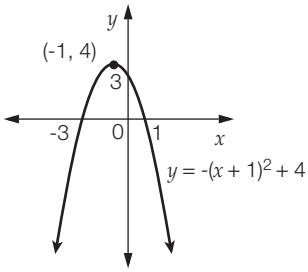
9 **(a)** **(i)** (0, -12) **(ii)** (-2, 0), (6, 0) **(iii)** (2, -16)

(iv)



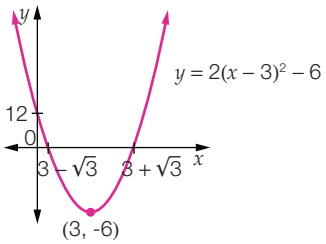
(b) **(i)** (0, 3) **(ii)** (-3, 0), (1, 0) **(iii)** (-1, 4)

(iv)



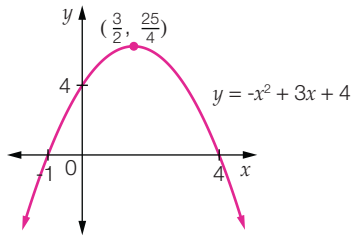
(c) **(i)** (0, 12) **(ii)** $(3 \pm \sqrt{3}, 0)$ **(iii)** (3, -6)

(iv)

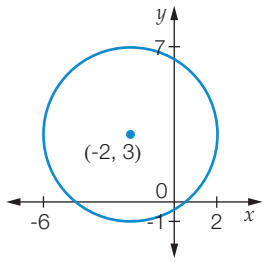


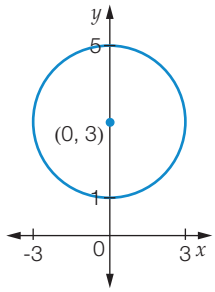
(d) **(i)** (0, 4) **(ii)** (-1, 0), (4, 0) **(iii)** $(\frac{3}{2}, \frac{25}{4})$

(iv)

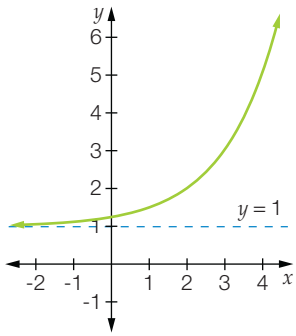


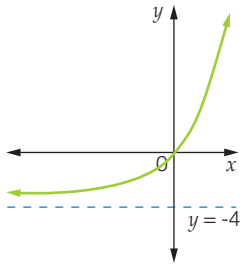
10 **(a)** $(x+2)^2 + (y-3)^2 = 16$ **(b)** $x^2 + (y-3)^2 = 4$



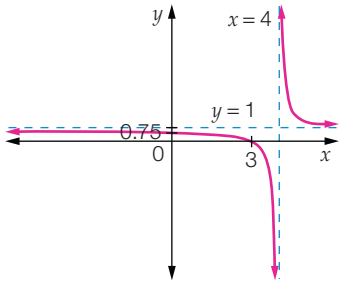


(c) $y = 2^{x-2} + 1$ **(d)** $y = 2^{x+2} - 4$

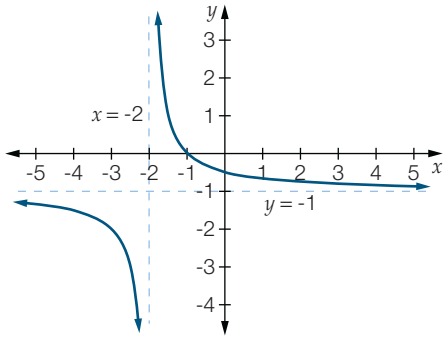




11 **(a)** $y = \dfrac{1}{x-4} + 1$



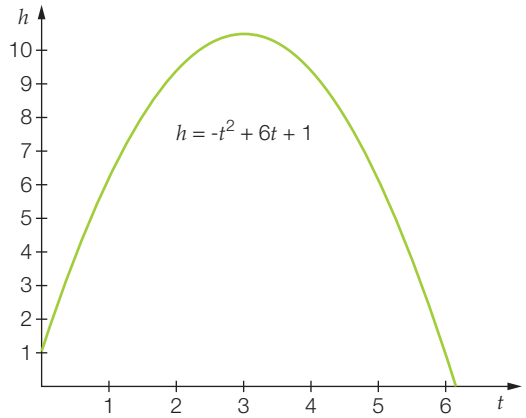
(b) $y = \dfrac{1}{x+2} + 1$



12 **(a)**

t	0	1	2	3	4	5	6
h	1	6	9	10	9	6	1

(b)

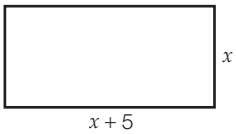


(c) 9 m **(d)** 1.27 seconds

(e) $2 < t < 4$ **(f)** 10 m

(g) 5.9 seconds

13 **(a)**



(b) $(x + 5)$ cm

(c) $x(x + 5)$ cm^2 **(d)** $x(x + 5) = 24$

(e) $x = -8$ or 3 **(f)** length 8 cm, width 3 cm

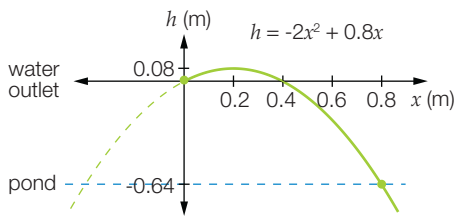
(g) $x = -8$ is not a feasible solution as a distance measurement must be positive.

14 **(a)** A horizontal distance of 0.4 m from the outlet.

(b) 0.64 m below the outlet

(c) 72 cm or 0.72 m

(d)



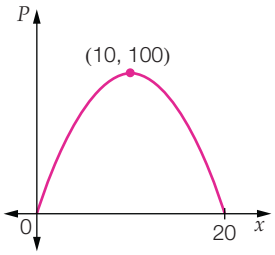
15 **(a)** 6 cm **(b)** 14 cm by 10 cm

(c) 36 cm^2 **(d)** 104 cm^2

(e) 64 cm^2

16 **(a)** $x + y = 20$ **(b)** $y = 20 - x$ **(c)** $x(20 - x) = P$

(d)



The maximum product is 100 and it is obtained by using $x = y = 10$.

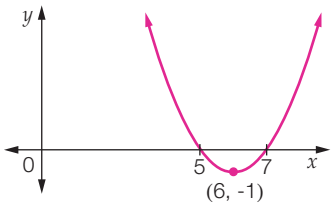
17 **(a)** 0.57 m

(b) The ball is 0.972 m high at the net, which is more than the height of the net (0.913 m).

(c) The ball lands 30 m from the fence, which is just outside the court.

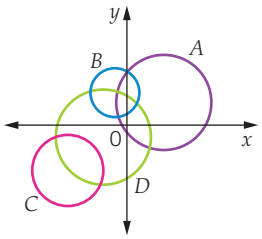
18 **(a)** (6, -1) **(b)** $y = (x - 6)^2 - 1$

(c)



19 **(a)** A and D

(b)



D intersects the other three graphs.

Mixed review C (p. 533)

1 \$20.23

2 **(a)** $(a - 9)(a + 4)$ **(b)** $(a - 6)(a - 2)$

(c) $(2a + 5)^2$

3 **(a)** 104 cm^2 **(b)** 37 m^2 **(c)** 103.5 cm^2

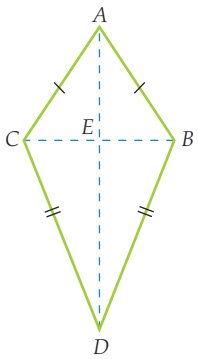
4 **(a)** -6 **(b)** $-\frac{15}{16}$ **(c)** $\frac{11}{6}$

5 **(a)** 42.7° **(b)** 23.9° **(c)** 50.5°

6 4.85 cm

7 **(a)** Range = 30, IQR ≈ 18 **(b)** Range = 30, IQR ≈ 13

8 The proof makes use of the following diagram, and its named vertices.

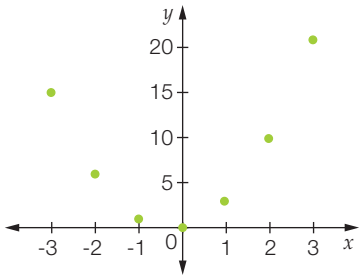


To show: $\Delta ACD \equiv \Delta ABD$
$AC = AB$ (given)
$CD = BD$ (given)
$AD = AD$ (shared)
So, $\Delta ACD \equiv \Delta ABD$
So, $\angle BAE = \angle CAE$
(matching angles in congruent triangles)
Now to show: $\Delta ABE \equiv \Delta ACE$
$\angle BAE = \angle CAE$ (from above)
$AC = AB$ (given)
$AE = AE$ (shared)
So, $\Delta ABE \equiv \Delta ACE$
So, $\angle AEB = \angle AEC$
(corresponding angles in congruent triangles)
But, $\angle AEB + \angle AEC = 180°$ (straight angle)
So, $\angle AEB = \angle AEC = 90°$
So, the diagonals of a kite intersect at right angles.

9 (a)

x	-3	-2	-1	0	1	2	3
y	15	6	1	0	3	10	21

(b)



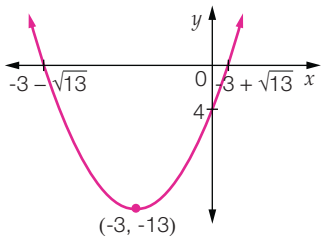
(c) (-0.25, -0.125)

(d) x-intercepts: -0.5 and 0; y-intercept: 0

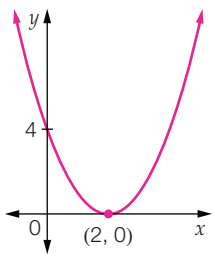
10 $x = -1$ and $y = -3$

11 (a) $x = -7$ and -6 **(b)** $x = \pm 10$ **(c)** $x = 7$

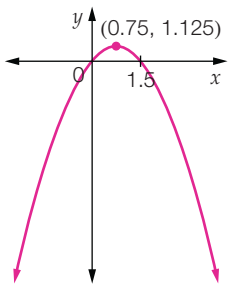
12 (a)



(b)



(c)



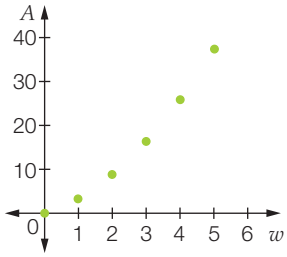
13 (a) $x = \pm 5$ **(b)** $x = 0$ and -0.5 **(c)** $x = 2$ and 3

14 (a) (2, 3) **(b)** (-4, -1) **(c)** (7, -6)

15 $(x - 4)^2 + (y - 2)^2 = 25$ **16** C **17** C

18 (a) $A = w(w + 2.5)$ cm^2

(b)



(c) $w = 5$ from the graph so the dimensions are 5 cm by 7.5 cm

19 (a) $x = 5$

(b) $y = (x - 4)(x - 6)$
$y = x^2 - 10x + 24$

Challenge C (p. 535)

1 C **2** B

3 ΔADC and ΔDBC are equal in area. They have the same height, the perpendicular distance from C to AB.

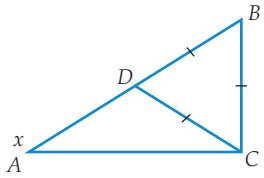
Hence their bases must be the same, so $AD = DB$.

ΔADC is isosceles.

ΔDBC is equilateral so $\angle BDC = 60°$

Hence, $\Delta ADC = 120°$ and $\angle DCA = \angle DAC = 30°$

$\therefore x = 150°$. (You can now say that ΔABC is right angled at C.)



4 B $AE = 25$, $AD = 24$, $ED = 7$, by Pythagoras' theorem

$AFD = 90°$ (angle in a semicircle)

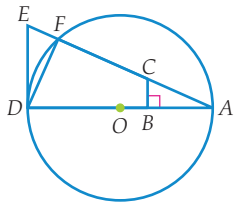
$AB = DF$, $BC = FE$ (matching sides of similar triangles)

$AC = ED = 7$, $CB = EF$

$FC = 25 - (EF + 7)$

$DB = 24 - AB$

Perimeter of $CBDF = DB + BC + CF + FD$
$= 24 - FD + EF + 25 - (EF + 7) + FD = 49 - 7 = 42$ units



5 45°

6 (a) $x^2 - 3x - 18 = -14$, $x^2 - 3x - 4 = 0$,
$(x - 4)(x + 1) = 0$. $x = -1, 4$

(b) $(x^2 - 3x)^2 - 3(x^2 - 3x) - 4 = 0$. $x^2 - 3x = -1$, $(x^2 - 3x) = 4$.

$x^2 - 3x + 1 = 0$, $x^2 - 3x - 4 = 0$. $x = \dfrac{3 \pm \sqrt{9 - 4}}{2}$,

$(x - 4)(x + 1) = 0$. $x = \dfrac{3 + \sqrt{5}}{2}, \dfrac{3 - \sqrt{5}}{2}, -1, 4$.

7 (a) $(x - 7)(x + 1) = 0$. $x = -1, 7$. Sum = 6.

(b) $(x - 2)(x + 2) = 0$. $x = -2, 2$. Product = -4.

(c) $x(x^2 - 6x + 5) = 0$. $x(x - 1)(x - 5) = 0$. $x = 0, 1, 5$. Average = 2

8 (a) $7 = (c - 2)(c - 8) + 7$
$(c - 2)(c - 8) = 0$
$c = 2$ or $c = 8$

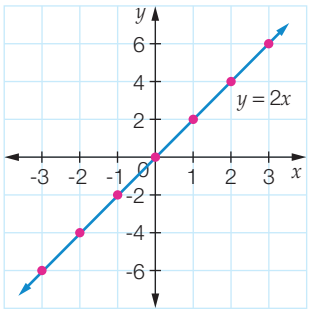
(b) vertex (5, -2)

9 (a) $y = -3, 2$ **(b)** $(-\sqrt{5}, 2)$, (0, -3), $(\sqrt{5}, 2)$

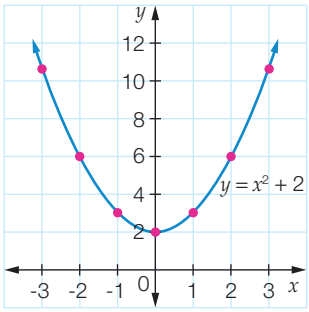
Chapter 10

Recall 10 (p. 537)

1 (a)



(b)

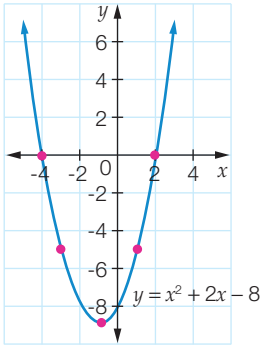


2 (a) neither (b) linear (c) quadratic

3 (a)

x	-5	-4	-3	-2	-1	0	1	2	3
y	7	0	-5	-8	-9	-8	-5	0	7

(b)



(c) (i) -8 (ii) -4, 2 (iii) (-1, -9)

(d) minimum (e) $x = -1$

4 (a) $(x-6)(x+1)$ (b) $(2x+9)(2x-9)$ (c) $3(x-3)(x+4)$

(d) $(x-7)^2$ (e) $-(x-10)(x+2)$

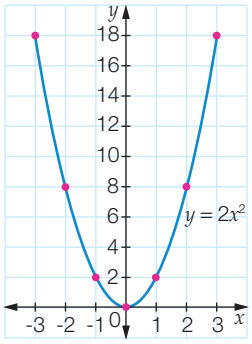
(f) $(x-3-2\sqrt{2})(x-3+2\sqrt{2})$

5 (a) $x = -2$ or 1 (b) $x = \pm 9$ (c) $x = 0$ or 2.5

Exercise 10.1 (p. 541)

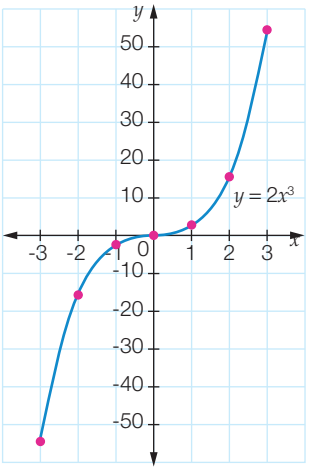
1 (a)

x	-3	-2	-1	0	1	2	3
y	18	8	2	0	2	8	18



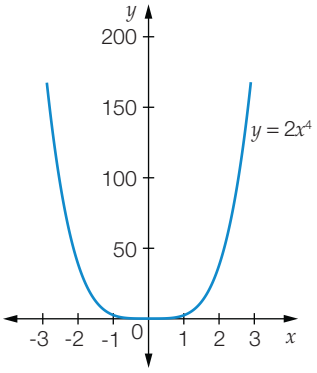
(b)

x	-3	-2	-1	0	1	2	3
y	-54	-16	-2	0	2	16	54



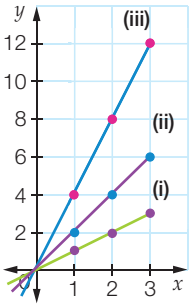
(c)

x	-3	-2	-1	0	1	2	3
y	162	32	2	0	2	32	162



2 (a) U (b) Z (c) W (d) Y (e) X (f) V

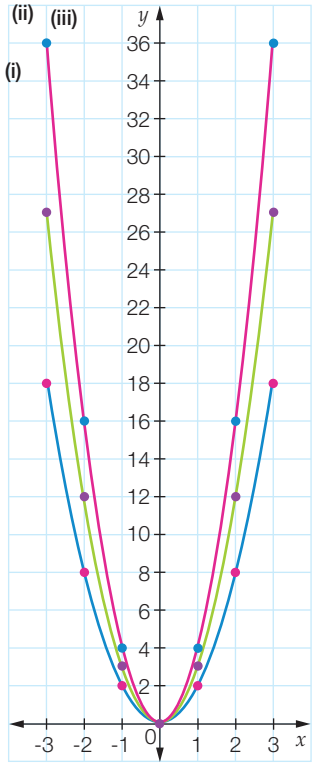
3 (a)



(b) The graph becomes steeper as the coefficient gets larger.

(c) No, graphs with a negative gradient become steeper as the coefficient of x decreases.

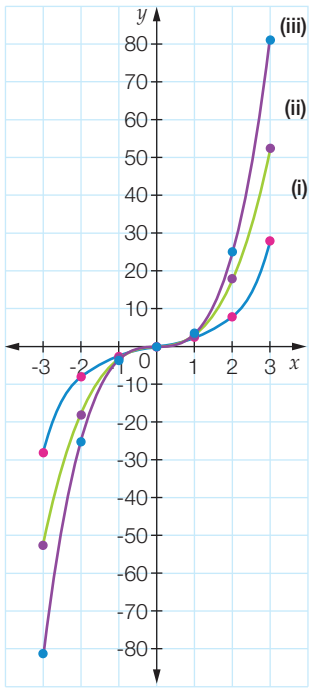
4 (a)



(b) The parabola becomes narrower about the y-axis as the coefficient increases.

(c) No, parabolas with a negative gradient become narrower as the coefficient of x decreases.

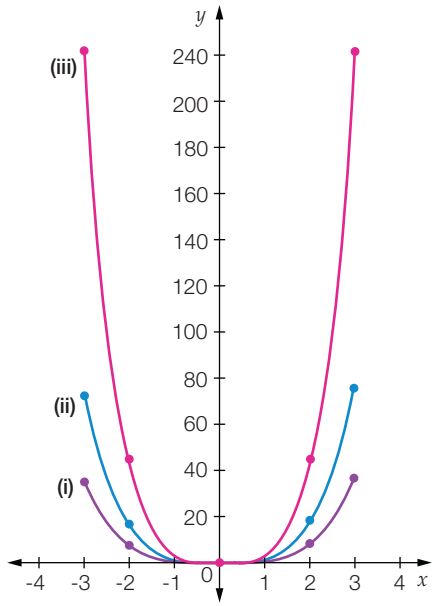
5 (a)



(b) The cubic gets steeper (gets closer to the y-axis) as the coefficient increases.

(c) No, cubics with a negative gradient become steeper as the coefficient of x decreases.

6 (a)



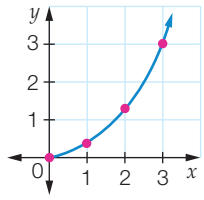
(b) The quartic becomes narrower about the y-axis as the coefficient increases.

(c) No, quartics with a negative gradient become narrower as the coefficient of x decreases.

7 B

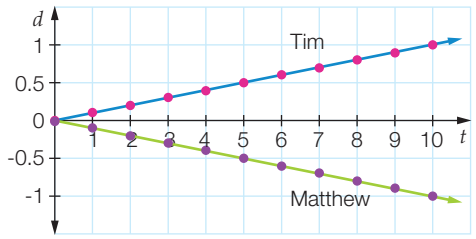
8 D

9 (a)



(b) $y = 0.75x$

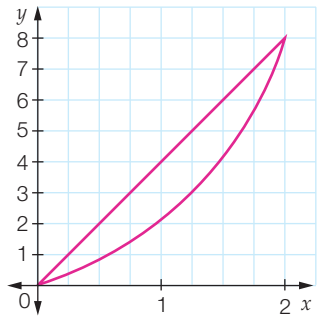
10 (a)



(b) 2 km

(c) $d = 0.12t$

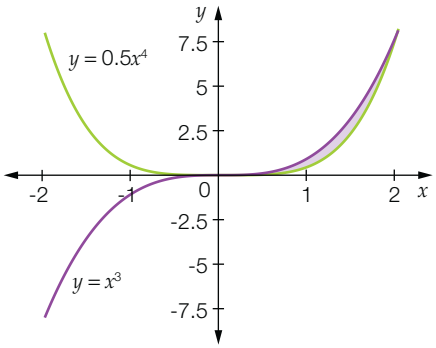
11 (a)



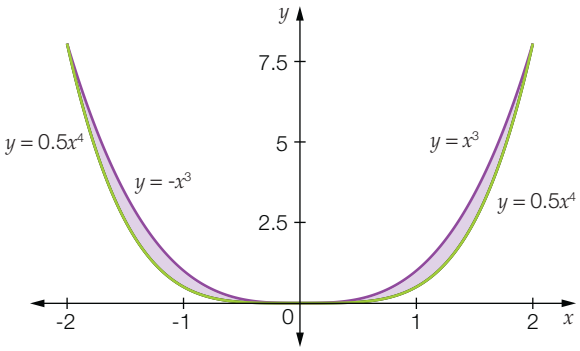
(b) $2\frac{2}{3}$ m^2

(c) $y = 4x$, $y = -2x^2$, $x = -2$ to 0

12 **(a), (b)**



(c)



(d) $y = -x^3$ for $-2 \le x \le 0$

Open-ended – Sample answers

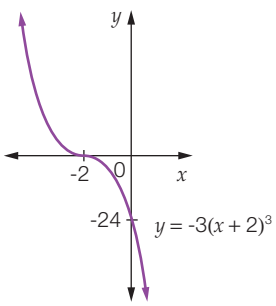
13 **(a)** $c = 10, 11$

(b) Gradient is undefined, the line is vertical.

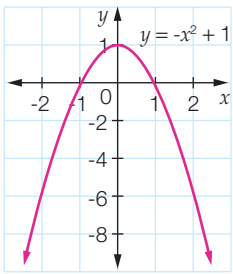
14 The graphs will intersect at $(-1, a)$ and $(1, a)$. This is true for both positive and negative values of a.

Exercise 10.2 (p. 549)

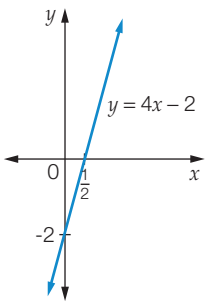
1



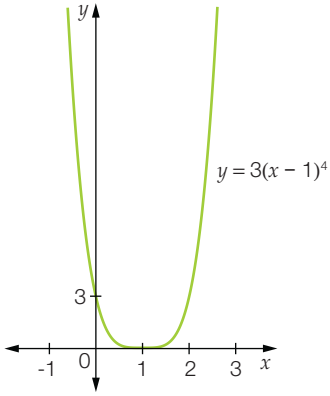
2



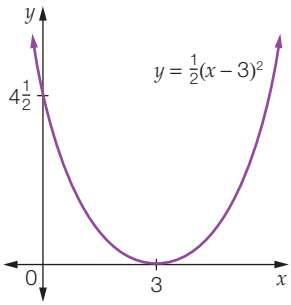
3



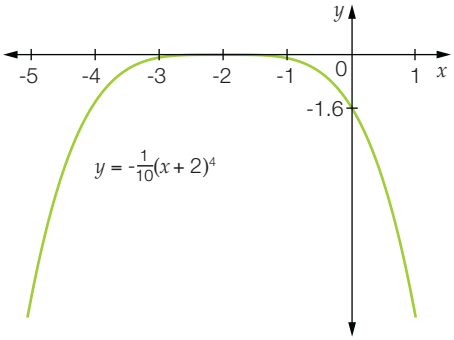
4



5



6



7 D **8** C **9** C

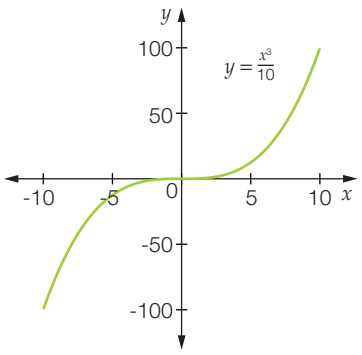
10 **(a)** $h = 1, k = 2$

(b) Negative, the turning point is a maximum.

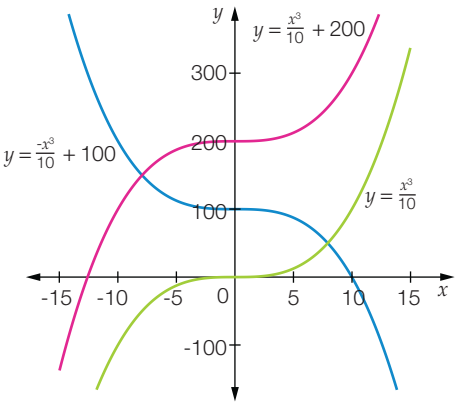
(c) $a = -3$

11 **(a)** $y = \dfrac{x^3}{10}$

(b)



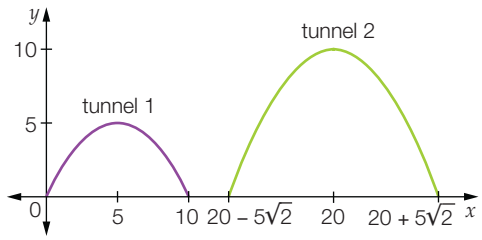
(c)



(d) The snake has moved up vertically 100 cm, the shape of the graph is exactly the same, but it is a mirror image of its initial shape. It is reflected in the line $y = 50$.

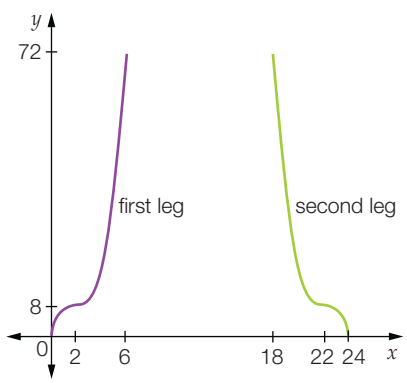
12 **(a)** $y = 10 - 0.2(x - 20)^2$

(b)



(c) 15.8 m

13 **(a), (c)**



(b) $y = -(x - 22)^3 + 8$ **(d)** 12 cm

Open-ended – Sample answers

14 $y = -3(x + 2)^3 + 4$, $y = -5(x + 1)^3 - 8$

Exercise 10.3 (p. 559)

1 **(a)** degree 3, leading coefficient 2, constant term 2

(b) degree 4, leading coefficient -4, constant term 5

(c) degree 1, leading coefficient 1, constant term 0

(d) degree 2, leading coefficient -1, constant term -6

2 **(a)** 62 **(b)** -142 **(c)** -28 **(d)** 8

3 **(a)** $5x^3 + 11x^2 - 13x - 3$ **(b)** $10x^4 - 3x^3 - 6x^2 + 29x + 6$

4 $2x^5 - 3x^4 - x^3 + 7x^2 + 15x - 20$

5 **(a)** $148\frac{15}{25}$ or $148\frac{3}{5}$ **(b)** $338\frac{5}{12}$

(c) $348\frac{8}{17}$ **(d)** $106\frac{4}{19}$

(e) $510\frac{7}{29}$ **(f)** $88\frac{122}{123}$

6 **(a)** quotient is $x + 3$, remainder is 1

(b) quotient is $x + 5$, remainder is 9

7 **(a)** quotient is $x^2 - 2x + 10$, remainder is -15

(b) quotient is $x^3 + 3x^2 + 6x + 18$, remainder is 55

8 $12x^3 - 55x^2 + 30x - 20$

9 **(a)** $x + 4$, 1 **(b)** $2x + 1$, 5

(c) $2x - 5$, 6 **(d)** $3x^2 + 6x + 14$, 24

10 A

11 $4x^2 + x - 3 = (4x - 3)(x + 1)$ so remainder is zero as it is a factor.

12 D **13** $x^3 - x^2 - 5x - 25 - \frac{120}{x - 5}$

14 **(a)** $R(x) = 16x^3 - 6x^2 + 4$; 108

(b) $R(x) = 3x^3 - 16x^2 - 17x + 22$; -152

(c) $R(x) = 4x^3 - 26x^2 - 28x + 36$; 34

15 **(a)** $x^4 - 3x - 6$ **(b)** 4

(c) $A(2) = 4$, $B(2) = 1$, $\frac{A(2)}{B(2)} = 4$, which is the same as the answer for **(b)**.

16 **(a)** $4x^3 + x + 8$

(b) **(i)** $4x^2 + 1 + \frac{8}{x}$ **(ii)** if $x = 0$, $\frac{8}{x}$ is undefined.

17 **(a)** $4x^2 - 10x + 6$

(b) $8x^3 - 30x^2 - 27x$

(c) **(i)** $8x - 22 - \frac{49}{x - 1}$

(ii) $x = 8$ cm, $x - 1 = 7$ cm $\Rightarrow$ 35 small rectangles

Open-ended – Sample answers

18 $2x^5 - x^4 - 3x^3 - 1$

19 $3x^3 - 14x^2 + 13x + 6$

20 **(a)** $1, x + \frac{1}{x + 1}, x^2 + 1, x^3 + x + \frac{1}{x + 1}, x^4 + x^2 + 1$

(b) If $P_n = x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$ then for n odd, answers are $1, x^2 + 1, x^4 + x^2 + 1$, so $\frac{P_n(x)}{x + 1} = Q_{n-1}$, where $Q_{n-1}(x) = x^{n-1} + x^{n-3} + \ldots$ which are even powers of x when n odd.

(c) If $P_n = x^n + x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1$ then for n even, answers are $x + \frac{1}{x + 1}, x^3 + x + \frac{1}{x + 1}$, so $\frac{P_n(x)}{x + 1} = Q_{n-1} + \frac{1}{x + 1}$, where $Q_{n-1}(x) = x^{n-1} + x^{n-3} + \ldots$ which are odd powers of x when n is even.

Exercise 10.4 (p. 566)

1 **(a)** $(2x + 1)(x + 2)$ **(b)** $(2y + 7)(y + 1)$

(c) $(5b + 3)(b + 1)$ **(d)** $(3k - 2)(k - 3)$

(e) $(11a - 12)(a + 1)$ **(f)** $(5x + 4)(x - 3)$

(g) $(2m - 3)(m + 3)$ **(h)** $(7x - 9)(x - 2)$

(i) $(11c - 3)(c - 2)$ **(j)** $(5y + 4)(y + 3)$

(k) $(3p - 4)(p + 4)$ **(l)** $(2x - 5)(x + 4)$

(m) $(x + 1)(3x + 5)$ **(n)** $(y - 2)(7y - 4)$

(o) $(3t + 4)(t - 1)$ **(p)** $(p + 2)(7p + 8)$

(q) $(q - 1)(2q + 5)$ **(r)** $(x + 2)(5x - 6)$

(s) $(r - 2)(7r - 4)$ **(t)** $(c - 3)(3c - 7)$

(u) $2(2t - 1)(3t - 1)$

2 (a) $(2x+3)(2x+1)$ (b) $(5y+4)(2y+1)$

(c) $(9c-1)(c-2)$ (d) $(3w+4)(2w+3)$

(e) $(5y-6)(3y+5)$ (f) $(8x+7)(x-2)$

(g) $(2a+9)(2a-3)$ (h) $(6k-5)(4k+3)$

(i) $(10x-7)(2x-3)$ (j) $(6b+1)(2b+5)$

(k) $(3g+2)(2g+1)$ (l) $(4a+1)(2a+5)$

(m) $(4x-3)(x-2)$ (n) $(10m+7)(m+1)$

(o) $(4p-5)(3p+2)$ (p) $(3d-2)(3d+4)$

(q) $(6x+11)(x-2)$ (r) $(9n-7)(2n+5)$

(s) $(8y-3)(3y-1)$ (t) $(5x+2)(3x+8)$

(u) $(4x+3)(3x-5)$

3 (a) $2\left(x-\frac{3}{2}-\frac{\sqrt{11}}{2}\right)\left(x-\frac{3}{2}+\frac{\sqrt{11}}{2}\right)$

(b) $3\left(x-2-\sqrt{\frac{19}{3}}\right)\left(x-2+\sqrt{\frac{19}{3}}\right)$

(c) $2\left(x-2-\sqrt{\frac{11}{2}}\right)\left(x-2+\sqrt{\frac{11}{2}}\right)$

(d) $3\left(x-\frac{3}{2}-\sqrt{\frac{11}{12}}\right)\left(x-\frac{3}{2}+\sqrt{\frac{11}{12}}\right)$

(e) $5\left(x-1-\sqrt{\frac{2}{5}}\right)\left(x-1+\sqrt{\frac{2}{5}}\right)$

(f) $2\left(x-\frac{7}{4}-\frac{\sqrt{33}}{4}\right)\left(x-\frac{7}{4}+\frac{\sqrt{33}}{4}\right)$

(g) not possible

(h) not possible

(i) not possible

4 (a) $(3x+1)(2x-3)$ (b) 3.5 m (c) 4 m

5 (a) $12t^2-12t+10-(4t^2+6t+1)$

(b) (i) $8t^2-18t+9$ (ii) $(4t-3)(2t-3)$

(c) The difference in the number of birds must be zero. $0=(2t-3)(4t-3)$. For this to be true either $2t-3=0$ ($t=1.5$) or $4t-3=0$ ($t=0.75$). Hence, when there are 75 or 150 trees there are equal numbers of birds.

6 (a) $(6x+5)(2x-3)$

(b) They each had \$10 initially.

(c) \$17

Open-ended – Sample answers

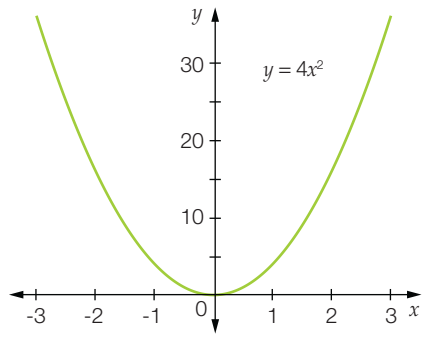
7 (a) $5x^2+10x=5x(x+2)$

(b) $60x^2+80x-15=5(6x-1)(2x+3)$

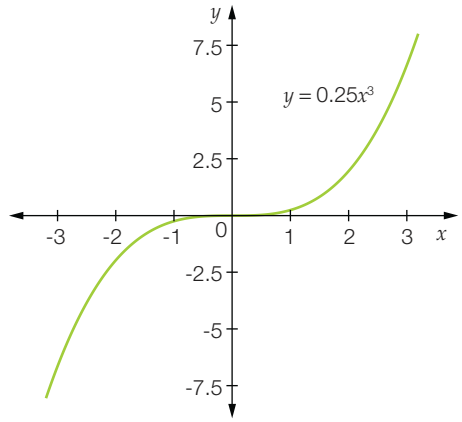
(c) $2x^2-4x+20$. This will not factorise completely as once you have factored out the 2 and completed the square, the last term is positive—not the difference of two squares.

Half-time 10 (p. 568)

1 (a)



(b)

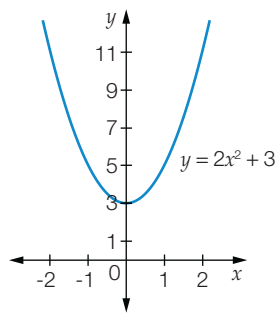


2 $x^2+x-7, 2$

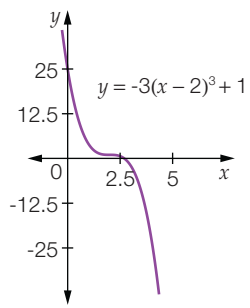
3 (a) (i) 8 (ii) 17 (iii) -42

(b) $9x^2-5$ (c) $9x^3-18x^2+11x-4$ (d) $2x+1$

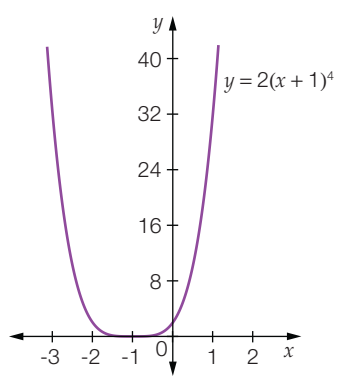
4 (a)



(b)



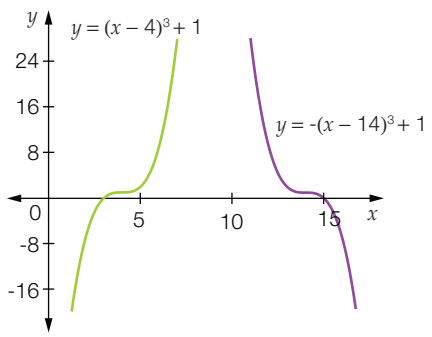
(c)



5 (a) 1 (b) $-\frac{47}{81}$

6 $2(x+2+\sqrt{\frac{5}{2}})(x+2-\sqrt{\frac{5}{2}})$

7 (a)



(b) $x = 9$

8 $\dfrac{x^3 + x - 5}{x - 1} = x^2 + x + 2 - \dfrac{3}{x - 1}$

9 (a) $(3x - 7)(2x + 5)$ (b) $(2x - 3)(5x - 1)$

(c) $(3x + 4)(4x + 1)$

10 (a) $(x - 1)(x^2 - 5x - 14)$ (b) $(x - 2)(x + 2)(x - 1)^2$

Exercise 10.5 (p. 573)

1 (a) $x = -\dfrac{2}{3}$ or 2 (b) $x = -\dfrac{2}{3}$ or -2

(c) $x = 1$ or 1.2 (d) $x = -2$ or 1.25

2 (a) $x = 1$ or 2.4 (b) $x = -1\dfrac{2}{3}$ or -1

(c) $x = -0.5$ or $\dfrac{2}{3}$ (d) $x = -1.5$ or 0.75

3 (a) $x = 0.5$ or 2 (b) $x = \dfrac{-5 \pm \sqrt{22}}{3}$

(c) $x = \dfrac{1 \pm \sqrt{31}}{5}$ (d) $x = \dfrac{-9 \pm \sqrt{129}}{12}$

4 (a) (i) $x = \dfrac{5 \pm \sqrt{73}}{6}$ (ii) $x = -0.59$ or 2.26

(b) (i) $x = \dfrac{-5 \pm 2\sqrt{5}}{5}$ (ii) $x = -1.89$ or -0.11

(c) (i) $x = \dfrac{11 \pm \sqrt{73}}{4}$ (ii) $x = 0.61$ or 4.89

(d) (i) $x = \dfrac{-5 \pm \sqrt{37}}{2}$ (ii) $x = -5.54$ or 0.54

5 (a) (i) $a = 1, b = 5, c = 2$ (ii) $x = \dfrac{-5 \pm \sqrt{17}}{2}$

(b) (i) $a = 2, b = -3, c = 1$ (ii) $x = \dfrac{1}{2}, 1$

(c) (i) $a = -3, b = -1, c = 3$ (ii) $x = \dfrac{-1 \pm \sqrt{37}}{6}$

(d) (i) $a = 5, b = 4, c = 0$ (ii) $x = -0.8, 0$

6 C

7 (a) $x = \dfrac{2}{3}$ or 1 (b) $x = \dfrac{-5 \pm \sqrt{15}}{5}$

(c) $x = -1.5$ or 1.5 (d) $x = \dfrac{-7 \pm \sqrt{85}}{2}$

(e) $x = -\dfrac{5}{6}$ or 2 (f) $x = -0.5$

(g) $x = \pm\sqrt{\dfrac{6}{7}}$ (h) $x = \dfrac{-3 \pm \sqrt{41}}{8}$

(i) $x = \dfrac{5}{6}$ (j) $x = \dfrac{11 \pm \sqrt{37}}{6}$

8 B 9 C

10 $4t^2 - 31t + 60 = 0$, $t = 3\dfrac{3}{4}$ or 4. After $3\dfrac{3}{4}$ min and 4 min.

11 (a) after 7 hours

(b) $2h^2 - 11h + 5 = 0$, After 30 mins and after 5 hours.

12 (a) 192 m (b) 6.2 seconds

13 (a) $x = \dfrac{3 \pm \sqrt{-7}}{4}$, no real solution

(b) $x = \dfrac{-5 \pm \sqrt{-15}}{10}$, no real solution

(c) $x = \dfrac{6 \pm \sqrt{-108}}{8}$, no real solution

(d) $x = \dfrac{-7 \pm \sqrt{-23}}{4}$, no real solution

14 (a) $3t^2 - 17t - 70 = 0$, $t = \dfrac{17 + \sqrt{1129}}{6} = 8.4$ min

(b) Try to solve $-3t^2 + 17t + 70 = 100$, $3t^2 - 17t + 30 = 0$,

$t = \dfrac{17 \pm \sqrt{-71}}{6}$ (no real solutions). Driver does not reach 100 km/h speed limit.

15 (a) Let kennel be x m wide, hence $2x$ m long.
Total painted area $= 4x^2 + 6x - 0.5$

(b) $4x^2 + 6x - 0.5 = 15.7$, $8x^2 + 12x - 1 = 31.4$,

$20x^2 + 30x - 81 = 0$, $x = \dfrac{-30 + \sqrt{7380}}{40} \approx 1.4$.

width 1.4 m, length 2.8 m, height 1 m

16 (a) Let x = side length (cm) of squares cut out from the corners.

(b) base length $= 30 - 2x$

(c) base width $= 25 - 2x$

(d) Tray 1: length of square = 3.7 cm

Tray 2: length of square = 4.3 cm

Tray 3: length of square = 5 cm

(e) Tray 1: 3.7 cm × 17.7 cm × 22.7 cm

Tray 2: 4.3 cm × 16.4 cm × 21.4 cm

Tray 3: 5 cm × 15 cm × 20 cm

17 $\frac{x}{2}(x+7) = 60$, $x^2 + 7x - 120 = 0$, $x = 8$.

hypotenuse = $\sqrt{8^2 + 15^2} = 17$ cm.
perimeter = 40 cm

18 $3n^2 + n - 1610 = 0$, $(3n + 70)(n - 23) = 0$, $n = 23$ terms

19 (a) $x \neq 0$; $2x^2 - x - 21 = 0$. $x = -3$ or 3.5

(b) $x \neq 0$; $x^2 - 9x + 14 = 0$, $x = 2$ or 7

(c) $x \neq 5$; $x^2 - 6x + 9 = 0$, $x = 3$

(d) $x \neq \pm 1$; $x^2 - 4x - 3 = 0$, $x = 2 \pm \sqrt{7}$

(e) $x \neq -2, 0$; $x^2 - 6x - 6 = 0$, $x = 3 \pm \sqrt{15}$

(f) $x \neq 0$; $6x^2 - x - 3 = 0$, $x = \frac{1 \pm \sqrt{73}}{12}$

Open-ended – Sample answers

20 $2x^2 + x - 15 = 0$ and multiples of this equation

21 $a = 0$; $4x^2 - 5 = 0$
$a = 1$; $x^2 + x - 1 = 0$ etc.

22 (a) $x^2 + x + 1 = 0$

(b) $4x^2 + 20x + 25 = 0$, $x = -2.5$

(c) $12x^2 + 7x - 10 = 0$, $x = \frac{2}{3}$ or -1.25

Exercise 10.6 (p. 582)

1 $x^2 + 6x + 7 + \frac{17}{x-2}$, $P(2) = 17$. The remainder = $P(2)$.

2 (a) $P(2) = 1$ (b) $P(-3) = 104$

3 $(x-1)(x-2)(x-3)$

4 $(x-1)(x+1)(x+2)(x-3)$

5 (a) 44 (b) -23 (c) $-\frac{21}{16}$

(d) $-\frac{31}{32}$ (e) 550 (f) -140

6 (a) $(x-1)(x-2)(x+4)$ (b) $(x+1)(x-3)^2$

(c) $(x+1)(x-2)(x-5)$ (d) $(x-1)(x-2)(x+6)$

(e) $(x+2)^3$ (f) $(x+1)^2(x-5)$

7 (a) -86 (b) -5 (c) -12 (d) -31

8 (a) $(x-1)(x+1)(x+2)(x+4)$ (b) $x(x-1)(x-2)(x-4)$

(c) $(x-1)^2(x-2)(x+3)$ (d) $(x-1)^3(x+5)$

(e) $(x-1)(x+1)(x-2)^2$ (f) $(x-1)(x+2)(x-3)(x+4)$

9 B 10 D 11 -6

12 (a) $(x-1)(x-2)(x-3) = 0$, $x = 1, 2$ or 3

(b) $(x+1)(x-2)(x+2) = 0$, $x = -2, -1$ or 2

(c) $(x+2)(x+3)(x+4) = 0$, $x = -2, -3$ or -4

(d) $(x-1)(x+1)(x-2)(x+3) = 0$, $x = -1, -3, 1$ or 2

(e) $(x-1)(x+1)(x-2)(x+2) = 0$, $x = \pm 1, \pm 2$

(f) $x(x-1)(x-\sqrt{6})(x+\sqrt{6}) = 0$, $x = \pm\sqrt{6}$, 0 or 1

13 (a) $(x-1)^2(x-2)^2$ (b) 4 cm (c) 1 s, 2 s

(d) The top is moving away from the fixed point.

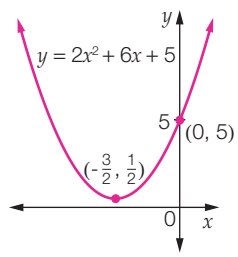
Open-ended – Sample answers

14 $x^3 - 3x^2 - x + 1$, $x^3 - 2x^2 - 3x - 2$

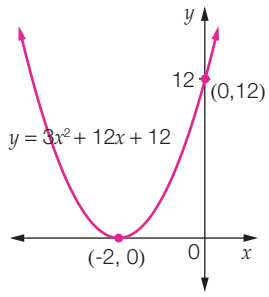
15 $x^4 - 4x^3 + x^2 + 6x = 0$ and multiples of the LHS.

Exercise 10.7 (p. 596)

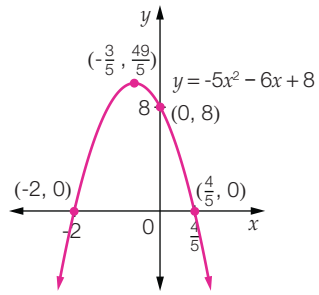
1 (a)



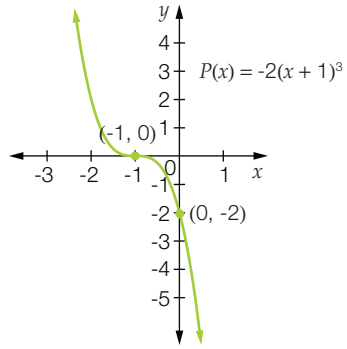
(b)



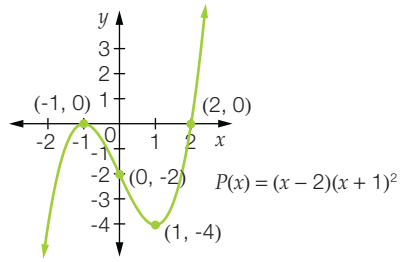
(c)



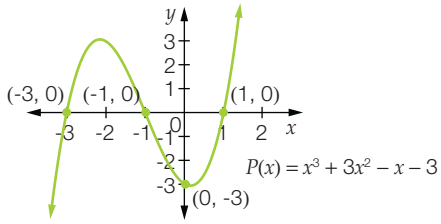
2 (a) (-1, 0)



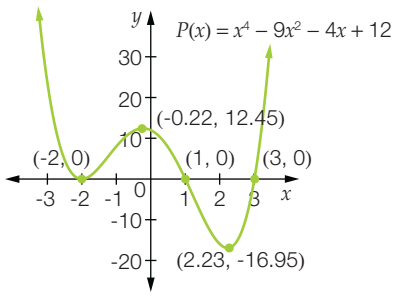
(b) (-1, 0) and (2,0)



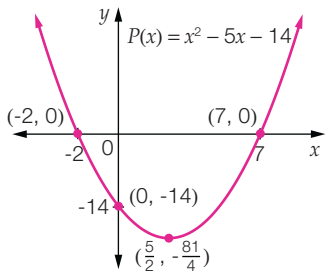
(c) (-3, 0), (-1, 0), (1, 0)



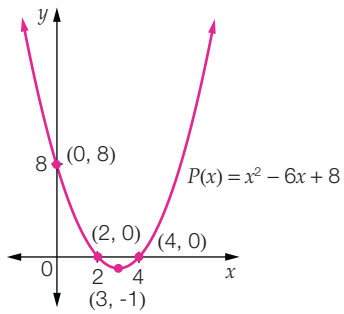
3



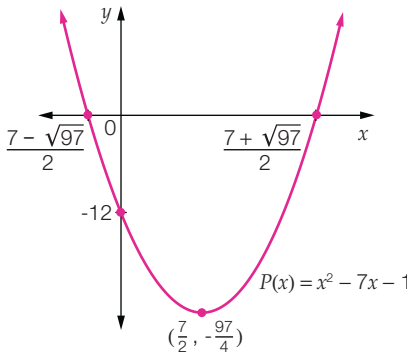
4 (a)



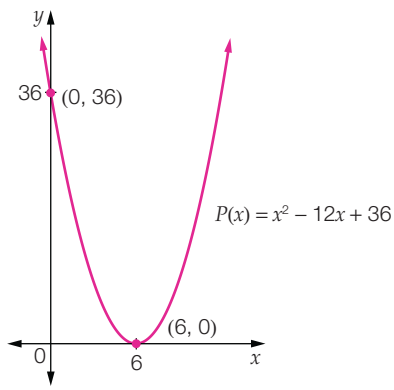
(b)



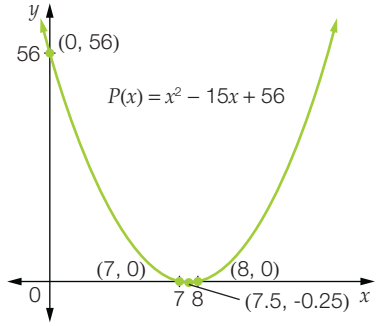
(c)



(d)

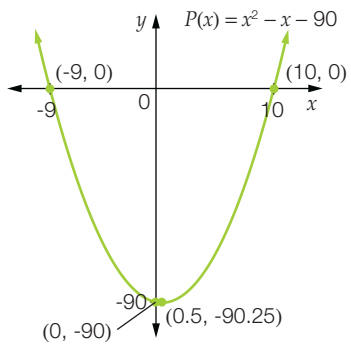


(e)

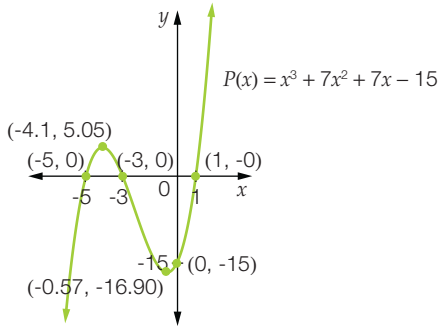


Note: y-intercept at (0, 56), turning point at (7.5, -0.25).

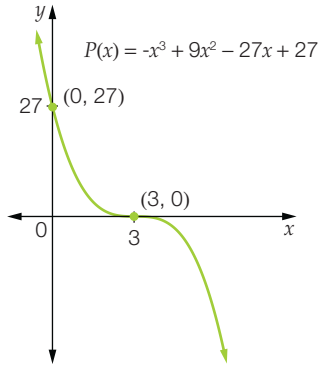
(f)



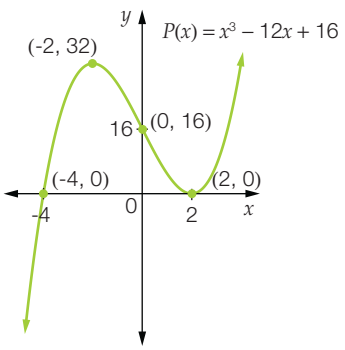
5 (a)



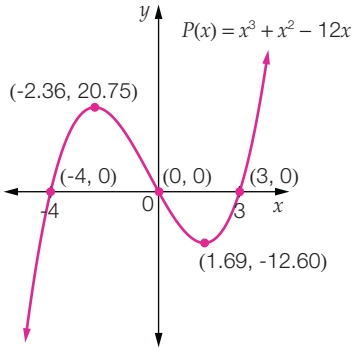
(b)



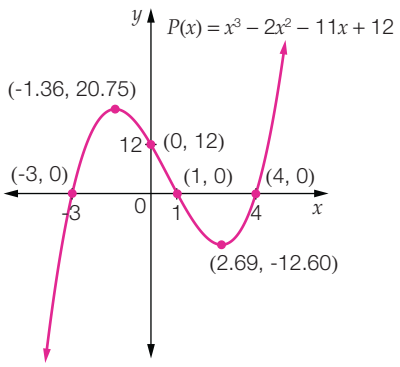
(c)


$P(x) = x^3 - 12x + 16$
(-2, 32)
16
(0, 16)
(-4, 0)
(2, 0)
-4
0
2
x
y

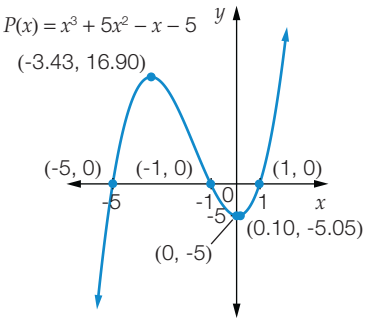
(d)


$P(x) = x^3 + x^2 - 12x$
(-2.36, 20.75)
(-4, 0)
(0, 0)
(3, 0)
-4
0
3
x
y
(1.69, -12.60)

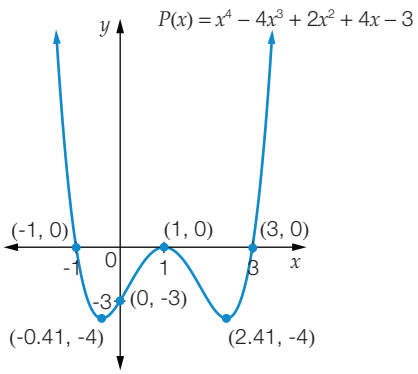
(e)


$P(x) = x^3 - 2x^2 - 11x + 12$
(-1.36, 20.75)
12
(0, 12)
(-3, 0)
(1, 0)
(4, 0)
-3
0
1
4
x
y
(2.69, -12.60)

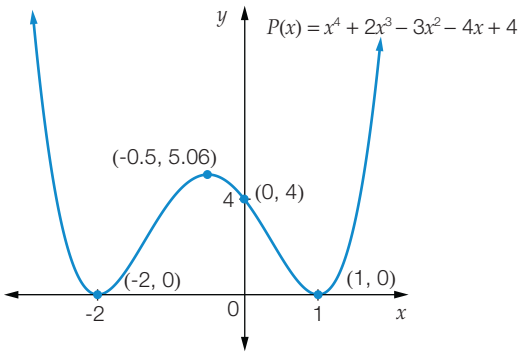
(f)


$P(x) = x^3 + 5x^2 - x - 5$
(-3.43, 16.90)
(-5, 0)
(-1, 0)
(1, 0)
-5
-1
0
1
x
y
-5
(0.10, -5.05)
(0, -5)

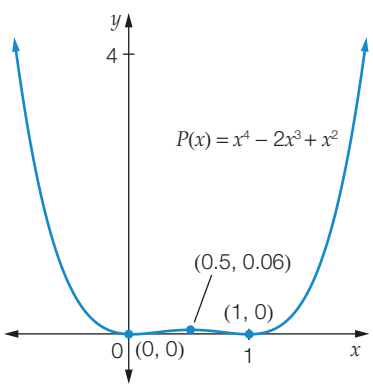
6 (a)


$P(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$
y
(-1, 0)
(1, 0)
(3, 0)
-1
0
1
3
x
-3
(0, -3)
(-0.41, -4)
(2.41, -4)

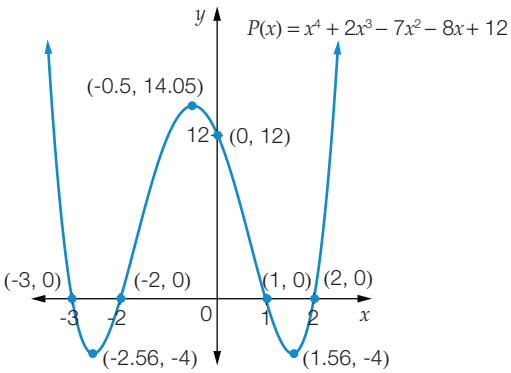
(b)


$P(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$
y
(-0.5, 5.06)
4
(0, 4)
(-2, 0)
(1, 0)
-2
0
1
x

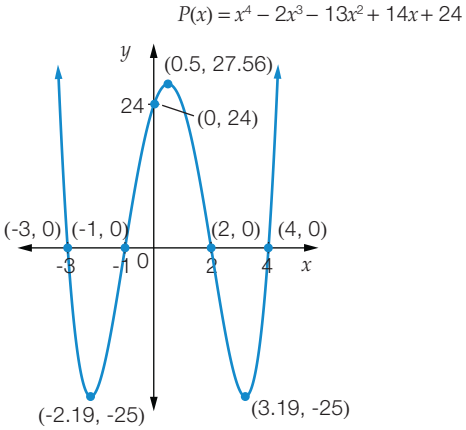
(c)


y
4
$P(x) = x^4 - 2x^3 + x^2$
(0.5, 0.06)
(1, 0)
0
(0, 0)
1
x

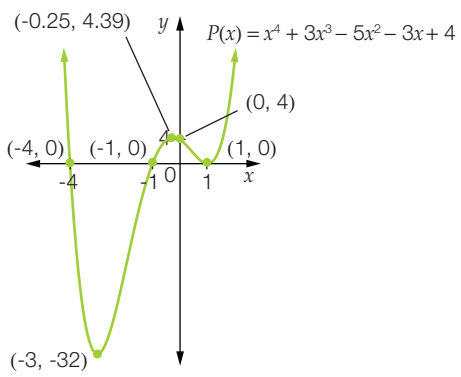
(d)


$P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$
y
(-0.5, 14.05)
12
(0, 12)
(-3, 0)
(-2, 0)
(1, 0)
(2, 0)
-3
-2
0
1
2
x
(-2.56, -4)
(1.56, -4)

(e)

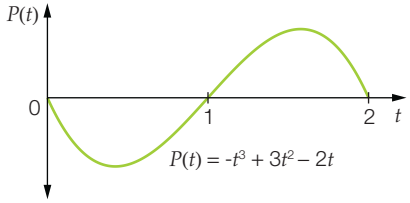

$P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$
y
(0.5, 27.56)
24
(0, 24)
(-3, 0)
(-1, 0)
(2, 0)
(4, 0)
-3
-1
0
2
4
x
(-2.19, -25)
(3.19, -25)

(f)



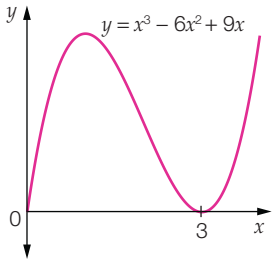
7 D **8** D **9** $x = 5, x = -7$ **10** $x = 1, x = -2$

11 **(a)** $P(t) = -t(t-1)(t-2)$



(b) 1 sec, 0.39 m **(c)** 1 sec, 0.39 m

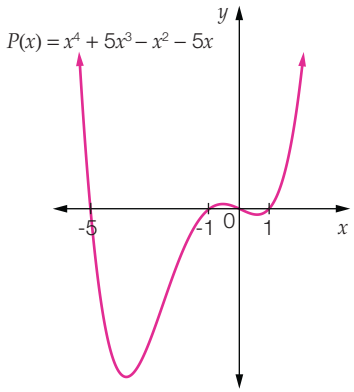
12 **(a)** $y = x(x-3)^2$



(b) (1, 4), (3, 0) **(c)** $y = -x(x-3)$

(d) Yes, as (2.5, 1.2) is not on the new parabola.

13 **(a)** $P(x) = x(x-1)(x+1)(x+5)$

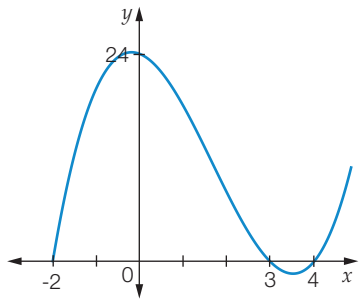


(b) negative pulse: $-5 < x < -1$ and $0 < x < 1$.
positive pulse: $-1 < x < 0$.

(c) **(i)** width 400 m, depth 61 m **(ii)** No

14 **(a)** $(x+2)(x-3)(x-4)$

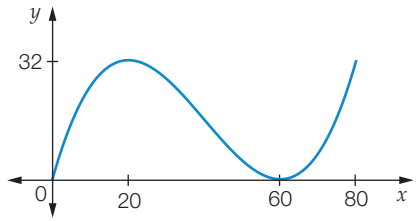
(b)



(c) 600 m

15 **(a)** $y = 0.001x(x-60)^2$

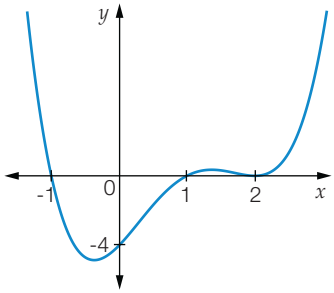
(b)



(c) 0.36 mins

Open-ended – Sample answers

16 $P(x) = (x+1)(x-2)(x-2)(x-1)$



17 **(a)** $P_1(x) = (x+1)^2(x-3)^2$, $P_2(x) = (x+1)(x-3)^3$

(b) $P_1(x) = (x+1)^2(x-3)(x-4)$, $P_2(x) = (x+1)(x-3)^2(x-5)$

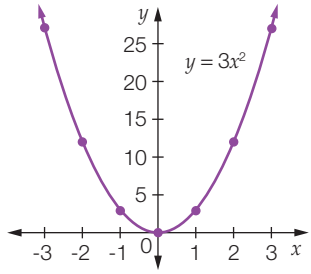
(c) $P_1(x) = x(x+1)(x-3)(x-4)$,
$P_2(x) = (x+1)(x-3)(x-5)(x+6)$

(d) $P_1(x) = 5(x+1)^3(x-3)$, $P_2(x) = 0.5(x+1)(x-3)^3$

Chapter review 10 (p. 599)

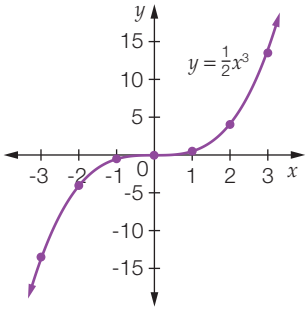
1 **(a)**

x	-3	-2	-1	0	1	2	3
y	27	12	3	0	3	12	27



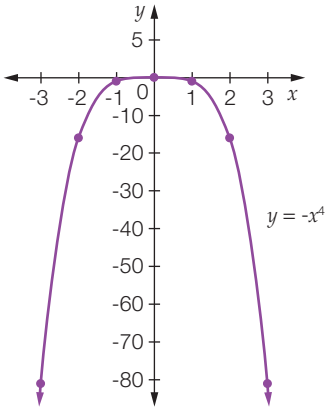
(b)

x	-3	-2	-1	0	1	2	3
y	-13.5	-4	-0.5	0	0.5	4	13.5

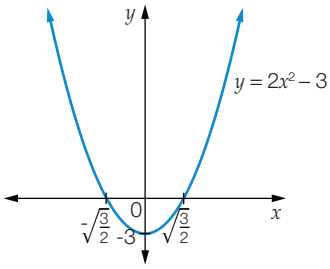


(c)

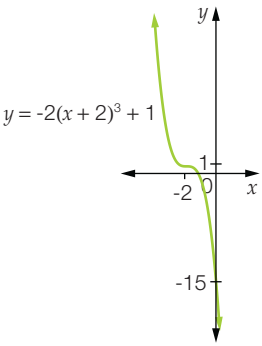
x	-3	-2	-1	0	1	2	3
y	-81	-16	-1	0	-1	-16	-81



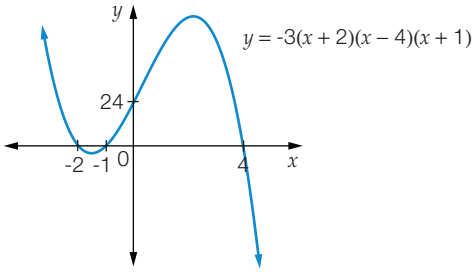
2 (a)



(b)



(c)



3 (a) (i) 13 (ii) 16 (iii) -42

(b) (i) $9x^2 + 5x$ (ii) $12x^3 - 13x^2 + 11x - 6$

(iii) $3x^2 - 5x + 5$

4 $x^2 + x + 6, 1$

5 (a) $(3x + 4)(4x - 5)$

(b) $(2x + 1)(6x - 5)$

(c) $(2x + 7)(3x - 2)$

(d) $3\left(x + \frac{5}{6} - \frac{\sqrt{13}}{6}\right)\left(x + \frac{5}{6} + \frac{\sqrt{13}}{6}\right)$

(e) $4\left(x - \frac{7}{8} - \frac{\sqrt{129}}{8}\right)\left(x + \frac{7}{8} + \frac{\sqrt{129}}{8}\right)$

(f) $2\left(x + \frac{3}{2} - \frac{\sqrt{3}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{3}}{2}\right)$

6 (a) $x = \pm 3$

(b) $(2x + 1)(x - 5) = 0$, $x = -0.5$ or 5

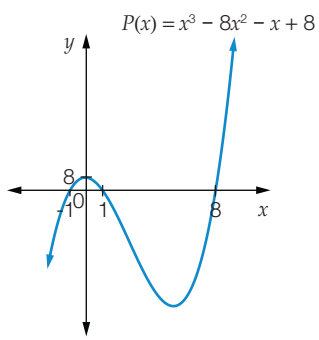
(c) $x = \frac{5 \pm \sqrt{3}}{2}$ (d) $x = \frac{-5 \pm \sqrt{17}}{4}$

(e) $x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$ (f) $x = \frac{-7 \pm \sqrt{205}}{26}$

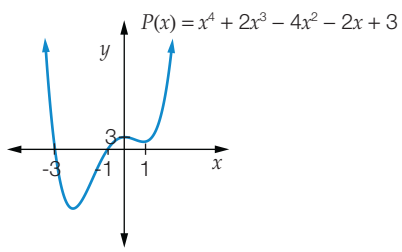
7 (a) -2 (b) $-\frac{5}{16}$

8 (a) $(x + 1)(x - 2)(x + 7)$ (b) $(x + 1)(x - 1)(x + 2)(x - 2)$

9 (a)

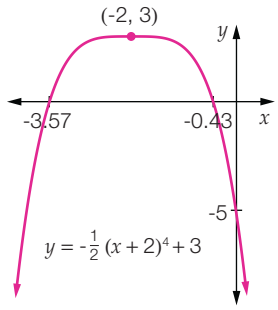


(b)



10 (a) $y = -\frac{1}{2}(x + 2)^4 + 3$

(b)



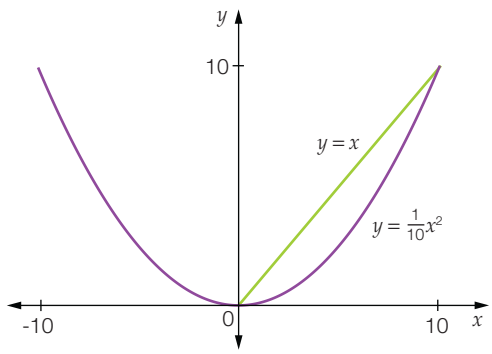
11 remainder = -7

12 $x \neq 0$; $40x^2 - 6x - 1 = 0$, $x = -0.1, 0.25$

13 $m^2 - m - 12 = 0$, $(m-4)(m+3) = 0$, $m = 4$ or -3, $x^2 = 4$, $x = \pm 2$

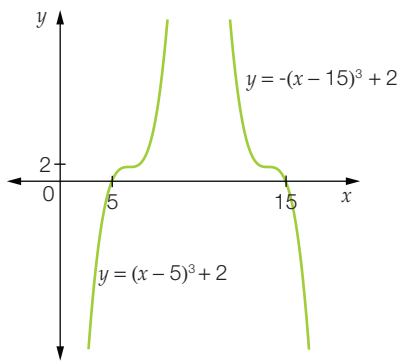
14 $\frac{n}{2}(10 + 4(n-1)) = 860$, $2n^2 + 3n - 860 = 0$, $(2n+43)(n-20) = 0$.
$n = 20$

15 (a)



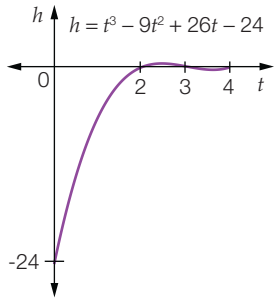
(b) 10 cm

16 (a)



(b) $x = 10$

17 (a) $h = (t-2)(t-3)(t-4)$



(b) The capsule is released at a depth of 24 m. It breaks the surface after 2 seconds and submerges 1 second later. It stays just below the surface for another second before breaking the surface again.

Chapter 11

Recall 11 (p. 602)

1 (a) $\frac{1}{5}$ (b) $\frac{4}{13}$

2 (a) $\frac{7}{8}$ (b) $\frac{11}{15}$ (c) $\frac{2}{5}$ (d) $\frac{3}{10}$ (e) 0.4 (f) 0.01

3 (a) boy, girl; boy, boy; girl, boy; girl, girl

(b) head, tail

4 (a) (i) 25 (ii) 25 (iii) 6 (iv) 9 (v) 15

(b) (i) 1, 2, 3, 4, 5, 6 (ii) 3

Exercise 11.1 (p. 611)

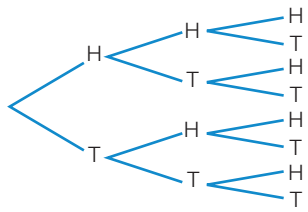
1 (a) $\frac{2}{7}$ (b) $\frac{5}{7}$

(c) We assume that the probability of the car being hard to start is the same every day. In essence, we are assuming that it is not dependent on the weather.

2 (a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{3}{8}$ (d) $\frac{1}{4}$ (e) $\frac{7}{8}$

3 (a) $\frac{1}{12}$ (b) $\frac{1}{2}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$ (e) $\frac{1}{6}$ (f) $\frac{1}{2}$

4 (a)

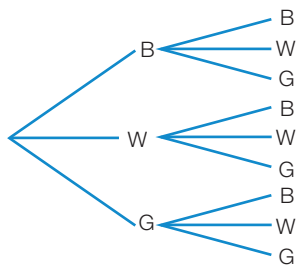


(b) (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)

(c) (i) $\frac{1}{8}$ (ii) $\frac{1}{8}$ (iii) $\frac{1}{8}$ (iv) $\frac{1}{4}$ (v) $\frac{1}{2}$

5 (a) $\frac{7}{12}$ (b) $\frac{5}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{24}$ (e) $\frac{1}{12}$ (f) $\frac{11}{12}$

6 (a)



(b) Each sock has an equal probability of being selected from each drawer, so the outcomes for the new event are also equally likely.

(c) (i) $\frac{1}{9}$ (ii) $\frac{2}{9}$ (iii) $\frac{1}{3}$ (iv) $\frac{2}{3}$

7 (a) $\frac{1}{12}$ (b) $\frac{11}{12}$ (c) $\frac{1}{6}$ (d) $\frac{5}{6}$

8 (a) B (b) D (c) C

9 (a) C (b) B (c) B

10 $\frac{4}{9}$

11 (a) A (b) C

12 $\frac{7}{9}$

13 (a) $\frac{1}{45}$ (b) $\frac{1}{5}$ (c $\frac{22}{45}$ (d) $\frac{23}{45}$ (e) 0 (f) $\frac{11}{45}$

14 **(a)** Possible totals

		Die 2					
		1	2	3	4	5	6
Die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(b) 36

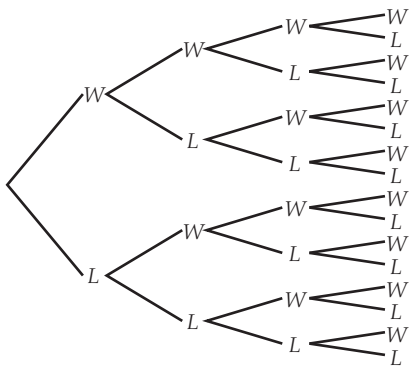
(c) Each die individually has an equal chance of producing 1, 2, 3, 4, 5, 6 so together the dice have an equal chance of producing each of the 36 possible outcomes.

(d) $\frac{1}{9}$ **(e)** $\frac{5}{12}$ **(f)** $\frac{1}{6}$

15 No, as some friends would be closer than others, so more likely to be contacted by Skye.

16 **(a)** $\frac{1}{4}$ **(b)** $\frac{1}{6}$ **(c)** $\frac{1}{4}$ **(d)** $\frac{1}{3}$ **(e)** $\frac{1}{2}$ **(f)** $\frac{2}{3}$

17 **(a)**



(b) (W, W, W, W), (W, W, W, L), (W, W, L, W), (W, W, L, L), (W, L, W, W), (W, L, W, L), (W, L, L, W), (W, L, L, L), (L, W, W, W), (L, W, W, L), (L, W, L, W), (L, W, L, L), (L, L, W, W), (L, L, W, L), (L, L, L, W), (L, L, L, L); 16 possible outcomes

(c) **(i)** $\frac{1}{16}$ **(ii)** $\frac{3}{8}$ **(iii)** $\frac{5}{16}$

(iv) $\frac{1}{2}$ **(v)** $\frac{1}{2}$ **(vi)** $\frac{1}{4}$

18 No, the existing cars in the court have no influence over what the new residents drive.

19 **(a)** Sample space

		Die 2					
		2	4	6	8	10	12
Die 1	1	(1, 2)	(1, 4)	(1, 6)	(1, 8)	(1, 10)	(1, 12)
	1	(1, 2)	(1, 4)	(1, 6)	(1, 8)	(1, 10)	(1, 12)
	2	(2, 2)	(2, 4)	(2, 6)	(2, 8)	(2, 10)	(2, 12)
	2	(2, 2)	(2, 4)	(2, 6)	(2, 8)	(2, 10)	(2, 12)
	3	(3, 2)	(3, 4)	(3, 6)	(3, 8)	(3, 10)	(3, 12)
	3	(3, 2)	(3, 4)	(3, 6)	(3, 8)	(3, 10)	(3, 12)

(b) Possible totals

		Die 2					
		2	4	6	8	10	12
Die 1	1	3	5	7	9	11	13
	1	3	5	7	9	11	13
	2	4	6	8	10	12	14
	2	4	6	8	10	12	14
	3	5	7	9	11	13	15
	3	5	7	9	11	13	15

(c) **(i)** $\frac{1}{3}$ **(ii)** $\frac{2}{9}$ **(iii)** $\frac{7}{18}$ **(iv)** $\frac{7}{18}$

(d) You cannot just look at one diagonal to find the totals; you have to check all over the lattice. However, it is not much more difficult in this case due to the nature of the numbers involved.

(e) Possible products

		Die 2					
		2	4	6	8	10	12
Die 1	1	2	4	6	8	10	12
	1	2	4	6	8	10	12
	2	4	8	12	16	20	24
	2	4	8	12	16	20	24
	3	6	12	18	24	30	36
	3	6	12	18	24	30	36

(f)

Product	2	4	6	8	10	12	16	18	20	24	30	36
Probability	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{6}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{18}$

20 **(a)** Even though the past long-term trend is $\frac{19}{20}$, short-term variation is possible. It is also possible that Aiden has switched jobs.

(b) The probability does not change. The balls do not have memories so they don't know if they have been drawn before.

Open-ended – Sample answers

21 **(a)** A is the event rolling a 1 or 2

(b) B is the event rolling a multiple of 3

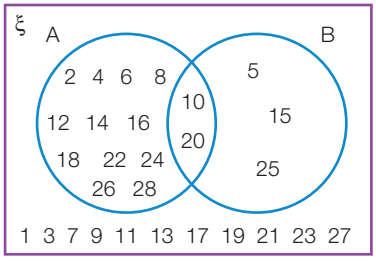
(c) C is the event rolling a number greater then 7

22 4W/1L, 8W/2L, 12W/3L

23 The correct answer is $\frac{7}{8}$. Sharanpreet has calculated the probability of exactly 1 boy, not at least 1 boy.

Exercise 11.2 (p. 620)

1 (a)



(b) (i) $\frac{1}{2}$ (ii) $\frac{1}{14}$ (iii) $\frac{23}{28}$

(c)

	B	not *A*
A	$\frac{1}{14}$	$\frac{3}{7}$
not *A*	$\frac{3}{28}$	$\frac{11}{28}$

2 (a) $\frac{3}{16}$ (b) $\frac{7}{16}$ (c) $\frac{1}{16}$ (d) $\frac{5}{16}$ (e) $\frac{1}{2}$ (f) $\frac{1}{4}$

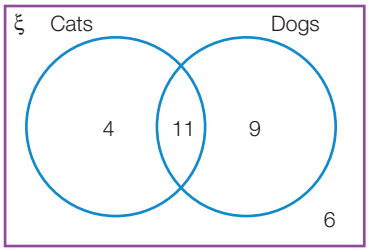
3 (a)

	Piano	not Piano	
Singing	4	2	**6**
not Singing	4	10	**14**
	8	**12**	**20**

(b) 4 take both lessons

(c) (i) $\frac{1}{10}$ (ii) $\frac{1}{5}$

4 (a)



(b) (i) $\frac{2}{15}$ (ii) $\frac{3}{10}$ (iii) $\frac{11}{30}$

5 (a)

	B	not *B*
A	$\frac{1}{8}$	$\frac{5}{16}$
not *A*	$\frac{1}{16}$	$\frac{1}{2}$

(b)

	C	not *C*
D	0.2	0.5
not *D*	0.2	0.1

6 (a) 1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 20

(b) 1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20

(c) 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 19, 20

(d) 3, 7, 12, 17, 20

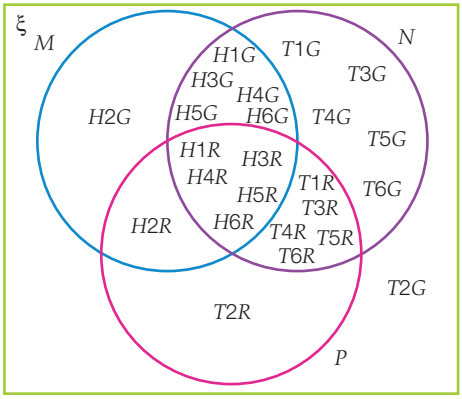
(e) 2, 7, 9, 12

(f) 5, 7, 12, 14

(g) 7, 12

(h) 6, 11, 16, 19

7 (a)



(b) not M = {T1R, T1G, T2R, T2G, T3R, T3G, T4R, T4G, T5R, T5G, T6R, T6G}

not N = {H2R, H2G, T2R, T2G}

not P = {H1G, H2G, H3G, H4G, H5G, H6G, T1G, T2G, T3G, T4G, T5G, T6G}

(c) No, because there are 4 red balls and only 3 green balls so there is a greater chance of H1R occurring than H1G.

8 (a)

	TV	not TV	
internet	236	1200	**1436**
not internet	554	10	**564**
	790	**1210**	**2000**

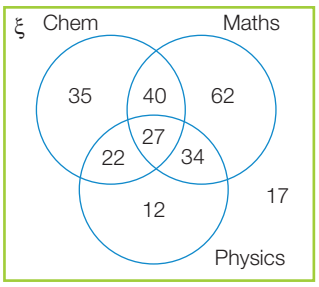
(b) $\frac{59}{500}$

9 (a)

	MySpace	not MySpace	
Facebook	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
not Facebook	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
	$\frac{3}{8}$	$\frac{5}{8}$	**1**

(b) (i) $\frac{1}{8}$ (ii) $\frac{5}{8}$

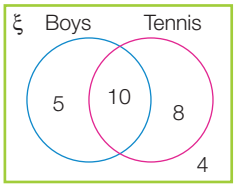
10 (a)



(b) 62 (c) 12 (d) 35 (e) 249

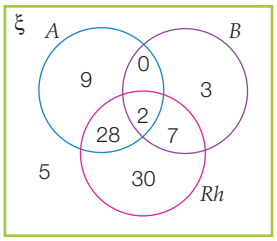
(f) $\frac{163}{249}$, it may be similar but it is unlikely to be exactly the same.

11 (a)



(b) (i) $\frac{8}{27}$ (ii) $\frac{5}{27}$ (iii) $\frac{2}{3}$ (iv) $\frac{1}{3}$

12 (a)



(b) 84 (c) $\frac{1}{2}$ (d) $\frac{5}{12}$

(e) A Positive: $\frac{1}{3} = 33.3\%$; A Negative: $\frac{3}{28} = 10.7\%$

B Positive: $\frac{1}{12} = 8.3\%$; B Negative: $\frac{1}{28} = 3.6\%$

AB positive: $\frac{1}{42} = 2.4\%$; AB Negative: 0%

O Positive: $\frac{5}{14} = 35.7\%$; O Negative: $\frac{5}{84} = 6\%$

(f) The A group is larger than expected and the O group is smaller than expected.

Open-ended – Sample answers

13 Mert has not taken into account students who may get their news from either the TV or the internet, but neither has taken into account there may be students who get their news from neither.

14 The existing figures add up to over 200 people because we are told both 36 and 44 people have never received a speeding fine.

Exercise 11.3 (p. 625)

1 (a) $\frac{1}{4}$ (b) $\frac{3}{8}$

2 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

3 (a) yes (b) no (c) yes

4 (a) D (b) A (c) D

5 (a) $\frac{4}{5}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$ (d) $\frac{3}{5}$ (e) $\frac{17}{20}$

6 (a) $\frac{1}{10}$ (b) $\frac{3}{10}$ (c) $\frac{2}{5}$ (d) $\frac{11}{20}$ (e) $\frac{11}{20}$

(f) $\frac{9}{10}$ (g) $\frac{1}{4}$ (h) $\frac{3}{20}$

7 (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{13}$ (d) $\frac{1}{52}$ (e) $\frac{27}{52}$

(f) $\frac{7}{26}$ (g) $\frac{1}{26}$ (h) $\frac{2}{13}$ (i) $\frac{9}{52}$ (j) $\frac{3}{26}$

(k) $\frac{11}{13}$ (l) $\frac{1}{4}$

8 (a) Six pairs: *AB*, *AC*, *AD*, *BC*, *BD*, *CD*

(b) *AC* and *BC* are mutually exclusive.

(c) $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{3}$, $\Pr(C) = \frac{1}{3}$, $\Pr(D) = \frac{1}{3}$

(d) $\Pr(A \text{ or } B) = \frac{1}{2}$, $\Pr(A \text{ or } C) = \frac{5}{6}$, $\Pr(A \text{ or } D) = \frac{2}{3}$,

$\Pr(B \text{ or } C) = \frac{2}{3}$, $\Pr(B \text{ or } D) = \frac{1}{2}$, $\Pr(C \text{ or } D) = \frac{1}{2}$

(e) $\Pr(A \text{ and } B) = \frac{1}{3}$, $\Pr(A \text{ and } C) = 0$, $\Pr(A \text{ and } D) = \frac{1}{6}$,

$\Pr(B \text{ and } C) = 0$, $\Pr(B \text{ and } D) = \frac{1}{6}$, $\Pr(C \text{ and } D) = \frac{1}{6}$

(f) $\Pr(A) + \Pr(B) = \frac{5}{6}$

$\Pr(A) + \Pr(C) = \frac{5}{6}$

$\Pr(A) + \Pr(D) = \frac{5}{6}$

$\Pr(B) + \Pr(C) = \frac{2}{3}$

$\Pr(B) + \Pr(D) = \frac{2}{3}$

$\Pr(C) + \Pr(D) = \frac{2}{3}$

(g) In general, $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B)$. However, if *A* and *B* are mutually exclusive $\Pr(A \text{ and } B) = 0$, and so $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$.

9 (a) The vehicle types listed are not mutually exclusive, so there could be arguments as to where a particular vehicle should be placed; or a particular vehicle may be registered in more than one category, thereby making later calculations difficult.

(b) The categories have to be clearly mutually exclusive but also cover every type of vehicle that may go past. Including the category 'other' would cover the second requirement.

Open-ended – Sample answers

10 'multiple of 5' and 'multiple of 7'; 'multiple of 4' and 'multiple of 9'; 'even number' and 'odd number'; 'composite number' and 'prime number' are some of the pairs that could be listed

Half-time 11 (p. 628)

1 (a)

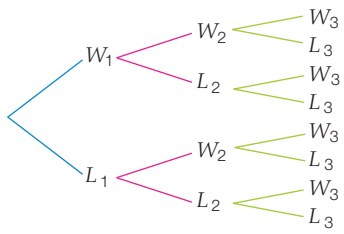
	iPhone	not iPhone	
iPad	60	15	75
not iPad	150	175	325
	210	190	400

(b) (i) $\frac{3}{8}$ (ii) $\frac{3}{80}$

2 (a) $\frac{2}{10} = \frac{1}{5}$ **(b)** $\frac{1}{2}$ **(c)** $\frac{6}{10} = \frac{3}{5}$ **(d)** $\frac{3}{10}$

3 0.09

4 (a) If W = Suzanne wins and L = Suzanne loses:



$W_1W_2W_3$, $W_1W_2L_3$, $W_1L_2W_3$, $W_1L_2L_3$, $L_1W_2W_3$, $L_1W_2L_3$, $L_1L_2W_3$, $L_1L_2L_3$

(b) 8

(c) (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$

(iv) $\frac{1}{2}$ (v) $\frac{1}{2}$ (vi) $\frac{1}{4}$

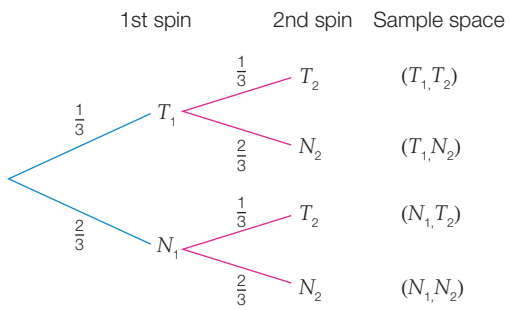
5 (a) A and B; A and C; B and C

(b) $\Pr(A) = \frac{1}{2}$; $\Pr(B) = \frac{1}{2}$; $\Pr(C) = \frac{1}{8}$

(c) A and C are mutually exclusive

Exercise 11.4 (p. 636)

1 (a)



Probability of each outcome

$Pr(T_1, T_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

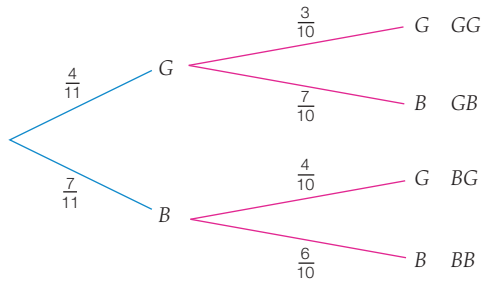
$Pr(T_1, N_2) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

$Pr(N_1, T_2) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

$Pr(N_1, N_2) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

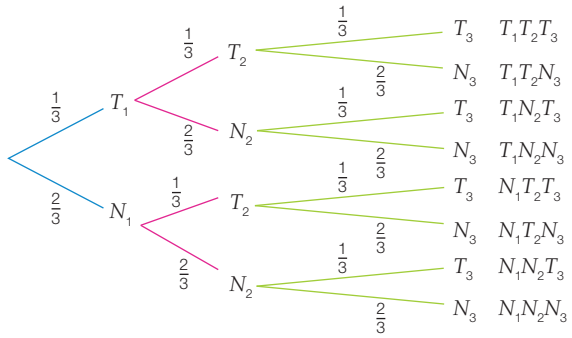
(b) (i) $\frac{1}{9}$ (ii) $\frac{4}{9}$ (iii) $\frac{4}{9}$ (iv) $\frac{5}{9}$

2



(a) $\frac{42}{110} = \frac{21}{55}$ **(b)** $\frac{98}{110} = \frac{49}{55}$

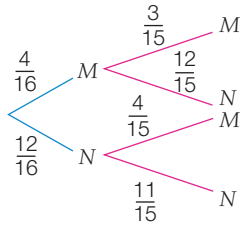
3 (a)



(b) $\frac{1}{27}$ **(c)** $\frac{8}{27}$

4 (a) B **(b)** D **(c)** B **(d)** C **(e)** B

5 (a)

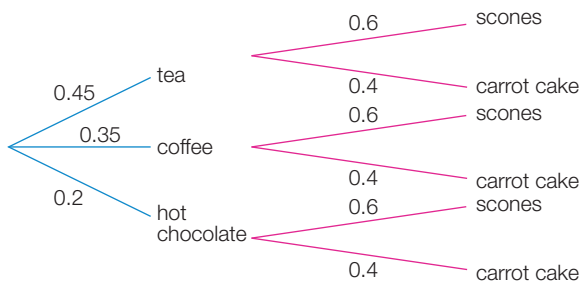


(b) MM, MN, NM, NN

(c) $\frac{1}{20}$ **(d)** $\frac{2}{5}$ **(e)** $\frac{11}{20}$

6 (a) 0.2 **(b)** 0.4

(c)



(d) (i) 0.27 (ii) 0.08 (iii) 0.35 (iv) 0.79

7 (a) (ii) $\Pr(G_1G_2) = 0.64$, $\Pr(G_1D_2) = 0.16$, $\Pr(D_1G_2) = 0.16$, $\Pr(D_1D_2) = 0.04$

(iii) 1

(b) (ii) $\Pr(F_1F_2) = \frac{1}{36}$, $\Pr(F_1N_2) = \frac{5}{36}$, $\Pr(N_1F_2) = \frac{5}{36}$,

$\Pr(N_1N_2) = \frac{25}{36}$

(iii) 1

(c) (ii) $\Pr(PA) = \frac{1}{60}$, $\Pr(P \text{ not } A) = \frac{1}{12}$, $\Pr(\text{not } PA) = \frac{3}{20}$,

$\Pr(\text{not } P \text{ not } A) = \frac{3}{4}$

(iii) 1

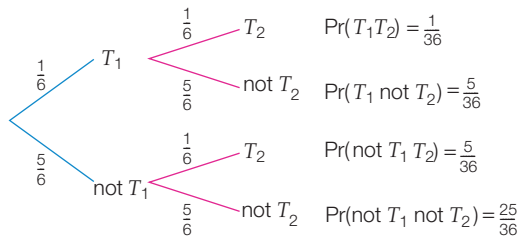
(d) (ii) $\Pr(B_1B_2B_3) = 0.064$, $\Pr(B_1B_2C_3) = 0.096$,
$\Pr(B_1C_2B_3) = 0.096$, $\Pr(B_1C_2C_3) = 0.144$,
$\Pr(C_1B_2B_3) = 0.096$, $\Pr(C_1B_2C_3) = 0.144$,
$\Pr(C_1C_2B_3) = 0.144$, $\Pr(C_1C_2C_3) = 0.216$

(iii) 1

8 (a) 0.008 (b) 0.096 (c) 0.488

9 (a) not T = a 3 is not rolled with a die

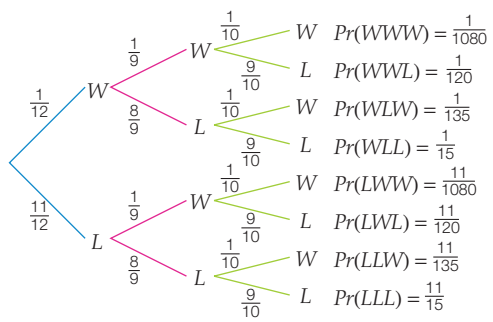
(b)



(c) (i) $\frac{1}{36}$ (ii) $\frac{25}{36}$ (iii) $\frac{5}{18}$ (iv) $\frac{11}{36}$ (v) $\frac{35}{36}$

10 (a) $\frac{1}{169}$ (b) $\frac{1}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{2}$

11 (a)



(b) (i) $\frac{1}{1080}$ (ii) $\frac{259}{1080}$ (iii) $\frac{288}{1080} = \frac{4}{15}$

12

Total	0	1	2
Probability	0.327 68	0.4096	0.2048

Total	3	4	5
Probability	0.0512	0.0064	0.000 32

13 (a)

Number of rolls	1	2	3
Probability	$0.1\dot{6}$	$0.13\dot{8}$	$0.115\,\dot{7}4\dot{0}$

Number of rolls	4	5	6
Probability	0.096 45	0.080 376	0.066 98

(b) approximately 0.665

(c) This means that it could still take more than six rolls to get the first 6. In fact, it could (theoretically) take an infinite number of rolls.

14 (a) $\frac{7}{29}$ (b) $\frac{3}{11}$

(c) $R3 = \frac{5}{16}$, $R4 = \frac{4}{11}$, $R5 = \frac{3}{7}$, $R6 = \frac{1}{2}$, $R7 = \frac{1}{2}$

(d) The probabilities add to more than one. This is because we assumed that elimination had not taken place before calculating the probability for the next round. This means the events did not all stem from the same tree but a new path was started after each round.

Open-ended – Sample answers

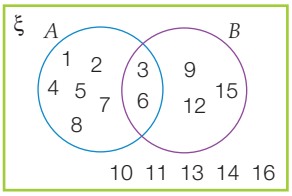
15 The Pr (leading team winning) would be greater than 0.5, an estimate would be 0.9. Leading 15 to 0 means this team only requires another 10 points whereas the trailing team still requires 25 points. Also, the result of the first 15 points challenges our assumption of the teams being equal in ability.

16 Students will produce their own diagrams and explanations.

17 The two events are not mutually exclusive, in fact, playing one of the sports increases the possibility of playing the other, as the person has a demonstrated interest in sport. Armapreet has treated them as mutually exclusive.

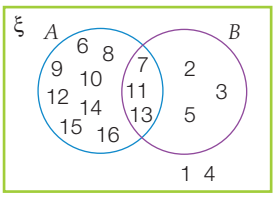
Exercise 11.5 (p. 644)

1 (a) (i)



(ii) $\frac{2}{5}$ (iii) $\frac{1}{4}$

(b) (i)



(ii) $\frac{1}{2}$ (iii) $\frac{3}{11}$

2 0.5746

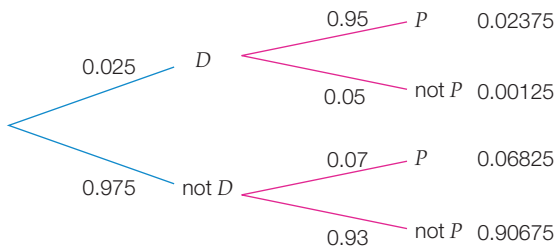
3 (a) (i) $\frac{2}{3}$ (ii) $\frac{3}{7}$

(b) (i) $0.1\dot{5}$ (ii) $0.2414 = \frac{7}{29}$

4 (a) C (b) D

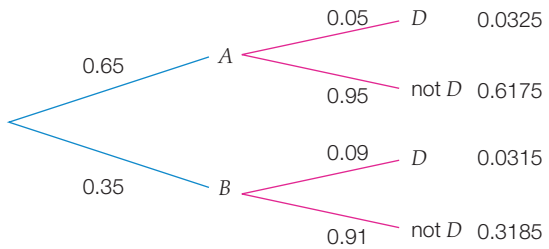
5 (a) $\frac{1}{13}$ (b) $\frac{1}{2}$ (c) $\frac{1}{13}$ (d) $\frac{1}{2}$

6 (a) D = 'is infected with the disease'
P = 'tests positive for the disease'



(b) 0.2582 (c) 0.9986

7 (a) D = 'shoe is defective'



(b) 0.5078 (c) 0.3403

8 (a) 0.9375 (b) 0.7895 (c) 0.8

9 (a) 0.784 (b) 0.91 (c) 0.7

(d) 0.83692 (e) 0.973

10 $0.\dot{5}$

Open-ended – Sample answers

11 (a) Not enough information has been provided to answer the question.

(b) Given $\Pr(A) = 0.6$, $\Pr(B) = 0.4$ and $\Pr(A \text{ and } B) = 0.1$, find $\Pr(A \text{ given } B)$. $\Pr(A \text{ given } B) = 0.25$

Exercise 11.6 (p. 651)

1 $\frac{1}{10}$

2 (a) $\frac{1}{78}$ (b) $\frac{9}{39} = \frac{3}{13}$

3 (a) $\frac{10}{27}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{16}{27}$

4 (a) $\frac{3}{40}$ (b) $\frac{3}{40}$ (c) $\frac{27}{40}$

5 (a) I (b) D (c) I (d) D (e) D (f) D

6 (a) $\frac{1}{35}$ (b) $\frac{2}{35}$ (c) $\frac{4}{15}$ (d) $\frac{2}{35}$

(e) $\frac{16}{105}$ (f) $\frac{16}{105}$ (g) $\frac{37}{105}$ (h) $\frac{68}{105}$

7 (a) A (b) D (c) A

8 (a) $\frac{3}{7}$ (b) $\frac{4}{35}$ (c) $\frac{2}{7}$

9 (a) $\frac{5}{51}$ (b) $\frac{28}{153}$ (c) $\frac{2}{51}$ (d) $\frac{49}{153}$

(e) $\frac{104}{153}$ (f) $\frac{8}{17}$ (g) $\frac{29}{51}$ (h) $\frac{91}{153}$

10 (a) $\frac{1}{8}$ (b) $\frac{1}{64}$ (c) $\frac{3}{64}$ (d) $\frac{1}{4}$

11 (a) 0.35 (b) 0.5

12 (a) 0.3 (b) 0.49 (c) 0.027 (d) 0.0081

13 (a) $\frac{4}{91}$ (b) $\frac{24}{91}$

14 (a) (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

(b) $\Pr(A) = \frac{1}{6}$; $\Pr(B) = \frac{1}{6}$; $\Pr(C) = \frac{5}{36}$; $\Pr(D) = \frac{1}{12}$

(c) 6 pairs: AB, AC, AD, BC, BD, CD

(d) See if $\Pr(A) \times \Pr(B) = \Pr(A \text{ and } B)$. If it is then the events are independent.

(e) Only AB are independent.

15 (a) 0.5

(b) Use $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B) - \Pr(A \text{ and } B) = 0.65$

16 If independent $\Pr(F) \times \Pr(M) = \Pr(F \text{ and } M)$, where F stands for 'female' and M stands for 'mathematics'. $\Pr(F) = \frac{475}{900}$, $\Pr(M) = \frac{870}{900}$, $\Pr(F \text{ and } M) = \frac{470}{900}$, $\Pr(F) \times \Pr(M) = \frac{551}{1080}$.
So, $\Pr(F) \times \Pr(M) \neq \Pr(F \text{ and } M)$.

17 (a) 0 (b) 0.1575

Open-ended – Sample answers

18 Students could define events such as 'rolling a 3 on the first roll', 'rolling a 5 on the second roll', 'rolling at least one multiple of 3', 'rolling a total of 6 or more'. Some questions then might be: 'Find the probability of each event', 'Are any of the pairs of events independent?' and so on.

19 (a) Drawing cards from a pack without replacement. Many pairs of events could be formed from this. Rolling a pair of dice and looking at a pair of events where one is to do with a particular number on the first die and the second is to do with the total of the two dice.

(b) Pairs of simple events based on separate flips of a coin, such as head on each and similar pairs of events based on rolling two dice.

20 The punter had knowledge that the bookmaker didn't have when the bet was made. By donating funds to the club the probability of success was affected so the original bet was no longer the same.

Chapter review 11 (p. 657)

1 (a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$ (e) $\frac{3}{5}$ (f) $\frac{4}{5}$

2 (a) $\frac{3}{15} = \frac{1}{5}$ (b) $\frac{8}{15}$ (c) $\frac{10}{15} = \frac{2}{3}$

3 $\frac{4}{6} = \frac{2}{3}$

4 (a) Possible outcomes

		Spinner 2			
		1	2	3	4
Spinner 1	1	(1, 1)	(2, 1)	(3, 1)	(4, 1)
	2	(1, 2)	(2, 2)	(3, 2)	(4, 2)
	3	(1, 3)	(2, 3)	(3, 3)	(4, 3)
	4	(1, 4)	(2, 4)	(3, 4)	(4, 4)

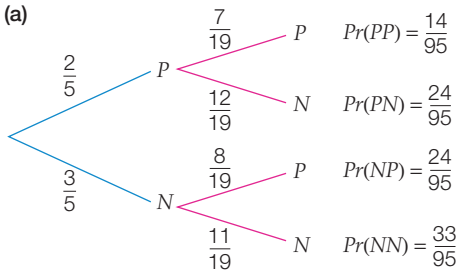
(b) (i) $\frac{1}{16}$ (ii) $\frac{3}{8}$ (iii) $\frac{9}{16}$ (iv) $\frac{1}{4}$

5 $\frac{1}{4}$

6 (a) 18 (b) 8 (c) 13 (d) 4 (e) 10 (f) 3

7 (a) D (b) B

8 (a)



(b) $\frac{14}{95}$ (c) $\frac{33}{95}$ (d) $\frac{62}{95}$

9 (a) (i) 0.4286 (ii) 0.3333

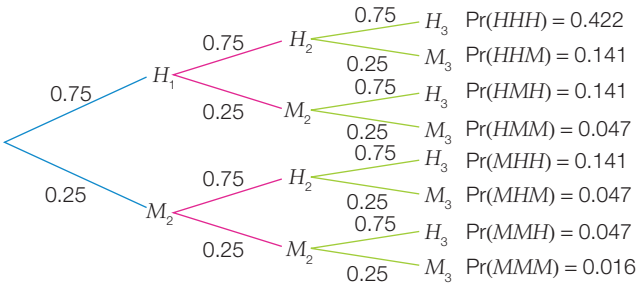
(b) (i) 0.4706 (ii) 0.2759

10 $\frac{6}{99} = \frac{2}{33}$

11 (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{3}{7}$ (d) $\frac{6}{7}$ (e) $\frac{2}{7}$ (f) $\frac{3}{7}$

12 (a) 0.25

(b) If H = hit and M = miss:



(c) (i) 0.422 (ii) 0.141 (iii) 0.984

13 (a)

	lemonade	not lemonade	
cola	12	18	30
not cola	28	2	30
	40	20	60

(b) (i) $\frac{2}{3}$ (ii) $\frac{7}{15}$ (iii) $\frac{1}{2}$ (iv) $\frac{3}{10}$

(v) $\frac{1}{5}$ (vi) $\frac{1}{3}$ (vii) $\frac{29}{30}$ (viii) $\frac{1}{30}$

14 (a) $\frac{7}{16}$ (b) $\frac{9}{16}$ (c) $\frac{13}{16}$ (d) $\frac{3}{8}$

15 (a) 0.35 (b) 0.275

(c) 0.015 (d) 0.012

16 (a) (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{3}{8}$ (iv) $\frac{1}{8}$

(b) 1. Yes, this is an expected result as these outcomes cover the sample space.

(c) Pr(0 tails) = Pr(3 heads); Pr(1 tail) = Pr(2 heads) etc.

(d) (i) $\frac{1}{8}$ (ii) $\frac{1}{8}$ (iii) $\frac{1}{8}$

(e) Any outcome with exactly 1 tail has exactly the same probability as any other outcome with exactly 1 tail. The order does not matter.

17 (a) $\frac{16}{81}$ (b) $\frac{344}{1215}$ (c) $\frac{296}{1215}$

18 (a) 0.015 (b) 0.16 (c) 0.032 (d) 0.12

(e) Sydney and Canberra are relatively close and if it rains in one city the probability of rain in both cities will be higher. The two events are not independent.

Chapter 12

Recall 12 (p. 662)

1 (a) $\pm\sqrt{3}$ (b) $\sqrt[3]{5}$ (c) $\pm\sqrt{7}$ (d) $\sqrt[3]{6}$ (e) $\pm\sqrt[4]{10}$

2 (a) 2 (b) 6 (c) 3 (d) 1 (e) 2

(f) 4 (g) -3 (h) 1 (i) 2 (j) -1

3 (a) 3.32 (b) -2.65 (c) 2.47 (d) 1156 (e) 19 683

4 (a) $3x + 12$ (b) $-8y + 12$

5 (a) m^5 (b) $12k^3p^9$ (c) x^7

(d) 3 (e) $m^{12}n^4$ (f) x^6y^{-4} $(= \frac{x^6}{y^4})$

6 (a) $x^2 - y^2$ (b) $c^2 - 81$ (c) $4p^2 - 9$

Exercise 12.1 (p. 665)

1 (a) irrational (b) rational (c) rational

(d) irrational (e) rational (f) irrational

(g) irrational (h) rational (i) irrational

(j) irrational (k) irrational (l) rational

2 (a) 8 (b) -5 (c) -6 (d) 9 (e) 42 (f) -28

(g) 13 (h) 7 (i) 69 (j) 30 (k) 16 (l) 73

3 (a) 3.146 (b) 0.592 (c) -0.960

(d) 5.916 (e) -1.732 (f) 2.571

(g) 2.154 (h) 2.449 (i) -4.160

4 (a) C (b) D (c) D (d) D

5 (a) C (b) A

6 (a) B (b) D (c) B

7 (a) 2.236 (b) 1.732 (c) 8.061
(d) 7.071 (e) 1.596 (f) -5.482
(g) -2.410 (h) -6.557 (i) 3.464
(j) 13.416 (k) -9.950 (l) 67.082

8 (a) 5 m (b) 6 cm (c) $\sqrt{46}$ mm (d) $\sqrt{17}$ m

9 (a) $\sqrt{60}$ m (b) 31 m
(c) The precision of measuring devices is important as far as accuracy is concerned. In this situation, a measurement to the nearest metre is therefore the most appropriate.

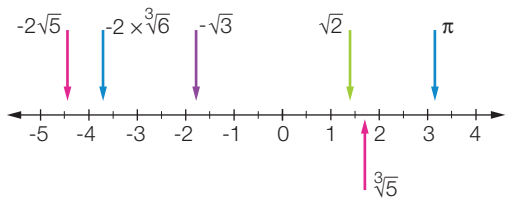
10 (a) 2 m (b) 3 m (c) $\sqrt[3]{35}$ cm (d) $\sqrt[3]{15}$ cm

11 (a) $\sqrt[3]{7}$ cm (b) 19 mm

Open-ended – Sample answers

12 (a) $\sqrt{2}, \pi, \sqrt[3]{5}$ (b) $-\sqrt{3}, -2\sqrt{5}, -2\sqrt[3]{6}$

(c)



13 Lie the metre ruler on a flat surface and rule a 1 m line with the pencil. Place an end of the ruler on the line and rule another metre line at right angles to the first. Run the string from one end of a metre line to the other (as if forming a triangle) this length is $\sqrt{2}$ metres.

Exercise 12.2 (p. 670)

1 (a) $\sqrt{6}$ (b) $\sqrt{55}$ (c) $\sqrt{70}$ (d) 7 (e) 4
(f) 8 (g) $6\sqrt{2}$ (h) $5\sqrt{7}$ (i) $3\sqrt{2}$ (j) $10\sqrt{6}$
(k) $4\sqrt{33}$ (l) $40\sqrt{2}$ (m) 35 (n) 24 (o) 40
(p) 42 (q) $15\sqrt{6}$ (r) $144\sqrt{2}$

2 (a) 3 (b) 4 (c) 5 (d) 2
(e) 6 (f) 3 (g) 1 (h) 2
(i) $\frac{5}{3}$ (j) $\frac{10}{7}$ (k) $\frac{2}{9}$ (l) $\frac{6}{11}$
(m) $\frac{1}{4}$ (n) $\frac{1}{3}$ (o) 4 (p) 5
(q) 1 (r) $\frac{1}{3}$ (s) 30 (t) 10
(u) $6\sqrt{6}$ (v) $\frac{\sqrt{5}}{5}$ (w) $\frac{2\sqrt{7}}{5}$ (x) $\frac{2\sqrt{3}}{3}$

3 (a) $3\sqrt{3}$ (b) $6\sqrt{3}$ (c) $9\sqrt{2}$ (d) $4\sqrt{6}$
(e) $3\sqrt{11}$ (f) $10\sqrt{5}$ (g) $9\sqrt{6}$ (h) $9\sqrt{5}$
(i) $10\sqrt{2}$ (j) $-8\sqrt{6}$ (k) $-15\sqrt{7}$ (l) $36\sqrt{3}$

4 (a) $\sqrt{12}$ (b) $\sqrt{63}$ (c) $\sqrt{72}$ (d) $\sqrt{27}$
(e) $-\sqrt{600}$ (f) $-\sqrt{40}$ (g) $-\sqrt{192}$ (h) $\sqrt{175}$
(i) $\sqrt{128}$ (j) $\sqrt{245}$ (k) $-\sqrt{108}$ (l) $-\sqrt{275}$

5 (a) C (b) B (c) C (d) D (e) B

6 (a) $45\sqrt{2}$ (b) $16\sqrt{7}$ (c) $-28\sqrt{15}$ (d) $-32\sqrt{5}$
(e) $-15\sqrt{5}$ (f) 120 (g) 48 (h) $36\sqrt{6}$
(i) $168\sqrt{21}$ (j) $100\sqrt{15}$ (k) $216\sqrt{14}$ (l) $54\sqrt{3}$

7 (a) $2\sqrt{2}$ (b) $3\sqrt{3}$ (c) $9\sqrt{5}$ (d) $-12\sqrt{6}$
(e) $\frac{3\sqrt{3}}{2}$ (f) $\frac{4\sqrt{2}}{5}$ (g) $6\sqrt{2}$ (h) $3\sqrt{7}$
(i) $\frac{5\sqrt{5}}{6}$ (j) $-\frac{2\sqrt{11}}{3}$ (k) $-\frac{3\sqrt{3}}{5}$ (l) $\frac{7\sqrt{3}}{10}$

8 (a) 75 m^2 (b) 16 m^2 (c) 12 km^2 (d) $9\sqrt{6}$ cm^2

9 One solution is $b = 9$, $c = 2$

10 (a) $2\sqrt{10}$ mm (b) $6\sqrt{5}\pi$ m

11 (a) $\sqrt{0.2825}$ m or $\sqrt{2825}$ cm (b) 6 windows

Open-ended – Sample answers

12 $\sqrt{24}, \frac{\sqrt{96}}{2}, \frac{\sqrt{216}}{3}$

13 (a) $\sqrt{5} \times \sqrt{5}$ (b) $\frac{8\sqrt{2}}{\sqrt{8}}$

Exercise 12.3 (p. 674)

1 (a) $10\sqrt{2}$ (b) $3\sqrt{5}$ (c) $5\sqrt{3}$
(d) $8\sqrt{7}$ (e) $-4\sqrt{3}$ (f) $-23\sqrt{11}$
(g) $11\sqrt{2}$ (h) 0 (i) $-3\sqrt{5}$
(j) $5\sqrt{6} - 5\sqrt{3}$ (k) $7\sqrt{2} - 5\sqrt{5}$ (l) $-\sqrt{3} - 4\sqrt{11}$

2 (a) $4\sqrt{2}$ (b) $\sqrt{7}$ (c) $3\sqrt{5}$
(d) $36\sqrt{2}$ (e) $\sqrt{7}$ (f) $-4\sqrt{10}$
(g) $9\sqrt{10}$ (h) $27\sqrt{2}$ (i) $8\sqrt{6}$
(j) $49\sqrt{3}$ (k) $-30\sqrt{5}$ (l) $19\sqrt{2}$
(m) $2\sqrt{7}$ (n) 0

3 (a) B (b) B (c) D

4 (a) C (b) D

5 (a) $8\sqrt{2}$ (b) $20\sqrt{3}$
(c) $11\sqrt{2}$ (d) $8\sqrt{10} + 12\sqrt{7}$

6 (a) $x = 4\sqrt{5}$ (b) $y = \sqrt{6}$

7 5π cm^2

8 $\frac{3\pi}{4}$ cm^2

9 (a) $6\sqrt{5} + 4\sqrt{3}$ m (b) $6\sqrt{15}$ m^2 (c) no

Open-ended – Sample answers

10 $7\sqrt{5} - 4\sqrt{5} = 3\sqrt{5}$; $-10\sqrt{5} + 13\sqrt{5} = 3\sqrt{5}$; $8\sqrt{5} - 5\sqrt{5} = 3\sqrt{5}$

11 length = $2\sqrt{6}$ cm, width = 2 cm, perimeter = $4\sqrt{6} + 4$ cm;

length = $2\sqrt{3}$ cm, width = $2\sqrt{2}$ cm,

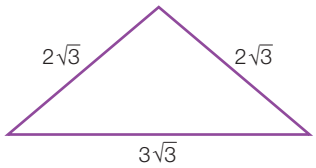
perimeter = $4\sqrt{3} + 4\sqrt{2}$ cm

12 **(a)** **(i)** Ruby has added the numbers underneath the square root.

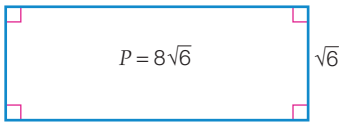
(ii) When subtracting one surd from another, Ruby has forgotten to include the surd part of the solution.

(b) **(i)** $7\sqrt{3}$ **(ii)** $3\sqrt{6}$

(c) **(i)**



(ii)



Exercise 12.4 (p. 678)

1 **(a)** $3\sqrt{7} + 6$ **(b)** $-4\sqrt{11} - 4\sqrt{3}$

(c) $18\sqrt{5} - 9\sqrt{2}$ **(d)** $3\sqrt{35} - 5\sqrt{14}$

(e) $8\sqrt{10} + 24\sqrt{6}$ **(f)** $3\sqrt{10} - 21\sqrt{15} + 9\sqrt{35}$

(g) $3 + 2\sqrt{3}$ **(h)** $15\sqrt{14} - 70$

(i) $24\sqrt{15} - 6\sqrt{5} - 15$ **(j)** $6\sqrt{7} - 12$

(k) $12 - 12\sqrt{2} + 20\sqrt{3}$ **(l)** $40 - 4\sqrt{5} - 20\sqrt{3}$

(m) $\sqrt{10} + 3\sqrt{5} + 4\sqrt{2} + 12$ **(n)** $3 + \sqrt{21} - \sqrt{15} - \sqrt{35}$

(o) $5 + 2\sqrt{6}$ **(p)** $88 - 16\sqrt{10}$

(q) 25 **(r)** -53

2 **(a)** B **(b)** D **(c)** C **(d)** C

3 **(a)** $\pi(19 - 6\sqrt{10})$ m^2 **(b)** $(2\sqrt{35} + 12\sqrt{7})$ cm^2

4 $\pi(21 + 8\sqrt{5})$ cm^2

5 **(a)** 21 km^2 **(b)** $(14 + 3\sqrt{2} + \sqrt{10})$ km

(c) 21 405 m

6 **(a)** **(i)** $(18\sqrt{2} - 12)$ m **(ii)** $(12\sqrt{2} - 8)$ m

(iii) $(15\sqrt{2} - 10)$ m **(iv)** $(21\sqrt{2} - 14)$ m

(b) **(i)** 13.46 m **(ii)** 8.97 m

(iii) 11.21 m **(iv)** 15.70 m

7 **(a)** 7π cm^2 **(b)** $(5 + 2\sqrt{7})$ cm by $(3 + 2\sqrt{7})$ cm

(c) $(43 + 16\sqrt{7})$ cm^2 **(d)** $(43 + 16\sqrt{7} - 7\pi)$ cm^2

(e) 63.34 cm^2

8 **(a)** 47.7 cm **(b)** 22 752.9 cm^2

(c) $(12\,110 + 1440\sqrt{55})$ cm^2 **(d)** 22 789.3 cm^2

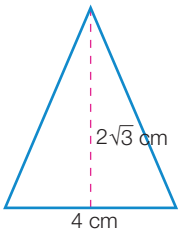
(e) part **(d)**

9 **(a)** $\sqrt{18}$ or $3\sqrt{2}$ cm **(b)** $150\sqrt{2}$ cm **(c)** 212.1 cm

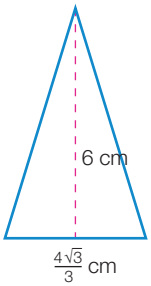
(d) 4.2 cm **(e)** 210 cm **(f)** part (c)

Open-ended – Sample answers

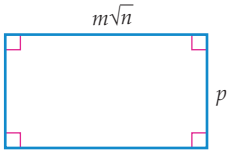
10 base 4 cm, height $2\sqrt{3}$ cm



base $\frac{4\sqrt{3}}{3}$ cm, height 6 cm



11 **(a)**



$P = 2(m\sqrt{n} + p)$

If $m = 10$, $n = 3$ and $p = 4$

Then $P = 2(10\sqrt{3} + 4) = 42.641$ cm

(b) $A = m\sqrt{n} \times p = mp\sqrt{n}$

(c) If $m = 2$, $n = 2$ and $p = 3$ then $A = 8.485$ m^2

Exercise 12.5 (p. 684)

1 **(a)** $\frac{\sqrt{7}}{7}$ **(b)** $\frac{4\sqrt{3}}{3}$ **(c)** $-2\sqrt{6}$ **(d)** $\frac{\sqrt{6}}{2}$

(e) $\sqrt{55}$ **(f)** $\frac{\sqrt{30}}{2}$ **(g)** $-\frac{3\sqrt{7}}{28}$ **(h)** $\frac{2\sqrt{15}}{7}$

2 **(a)** $\frac{3 + \sqrt{7}}{2}$ **(b)** $8 - 4\sqrt{3}$

(c) $\frac{\sqrt{15} + \sqrt{6}}{3}$ **(d)** $\frac{4\sqrt{7} - 10}{3}$

(e) $\frac{24 + 15\sqrt{3}}{3} = 8 + 5\sqrt{3}$ **(f)** $\frac{22 + 10\sqrt{5}}{4} = \frac{11 + 5\sqrt{5}}{2}$

(g) $\frac{6\sqrt{3} - \sqrt{42} - 3\sqrt{2} + \sqrt{7}}{11}$ **(h)** $\frac{5\sqrt{6} - 8\sqrt{3}}{6}$

3 **(a)** D **(b)** A **(c)** B

4 **(a)** B **(b)** D

5 (a) $\frac{2\sqrt{3}+\sqrt{5}+1}{4}$ (b) $\frac{\sqrt{7}+2\sqrt{6}+5}{6}$

(c) $\frac{5\sqrt{6}-5\sqrt{3}-12-3\sqrt{11}}{15}$ (d) $\frac{3\sqrt{5}-3\sqrt{3}-2\sqrt{7}-4}{6}$

6 (a) $\frac{5\sqrt{3}}{2}$ (b) 4 shelves (c) $(5\sqrt{6}-8\sqrt{2})$ m

7 (a) $15\,000\sqrt{6}$ cm^2 (b) $50\sqrt{6}$ tiles

(c) As $122 < 50\sqrt{6} < 123$, the minimum number of tiles is 123, but maybe buy 130 tiles to allow for wastage.

(d) $15\,000\sqrt{6}+6000\sqrt{15}+600\sqrt{35}+1500\sqrt{14}$ cm^2

8 $\frac{112\sqrt{3}-294}{249}$ or $\frac{14(8\sqrt{3}-21)}{249}$

Open-ended – Sample answers

9 Many answers are possible, just multiply the surd by a form of 1. As examples: $\frac{4\sqrt{6}}{3}, \frac{8}{\sqrt{6}}, \frac{4\sqrt{10}}{\sqrt{15}}$

10 (a) $\frac{\sqrt{2}}{3}, \frac{2}{3}$ (b) $\frac{2\sqrt{7}}{7}, \frac{4\sqrt{7}}{7}$

(c) $\frac{\sqrt{15}}{5}, \frac{3\sqrt{15}}{5}$ (d) $\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{2}$

11 $2-\sqrt{3}$ and $4-2\sqrt{3}$

12 When two surds that are the same are multiplied, the result is a whole number, e.g. $\sqrt{6}\times\sqrt{6}=6$. Multiplying by the conjugate will remove the square roots from the denominator, resulting in a rationalised denominator. It is also like the difference of two squares.

Half-time 12 (p 688)

1 (a) 6 (b) 88 (c) $\frac{9}{4}$

2 $5\sqrt{6}+3\sqrt{2}$ m

3 (a) $\frac{\sqrt{30}}{6}$ (b) $\frac{5\sqrt{3}}{6}$ (c) $\frac{12+3\sqrt{11}}{5}$

4 $3\sqrt{5}$ m

5 (a) rational (b) rational (c) irrational (d) rational

6 $(18+12\sqrt{15})$ m^2

7 (a) $6\sqrt{2}$ (b) 360 (c) $\frac{3}{5}$

8 (a) -15 (b) 17 (c) 34

Exercise 12.6 (p. 690)

1 (a) $12x^5$ (b) $\frac{a^5}{b^3}$ (c) $\frac{p}{q^3}$

(d) x^2 (e) $a^{20}b^9$ (f) $6m^{10}n$

(g) $\frac{1}{q^7}$ (h) $\frac{1}{c^9d^5}$ (i) 2

(j) $\frac{a^2}{b^2}$ (k) $\frac{m^5}{n^5}$ (l) 1

(m) q (n) $\frac{3k^3}{10}$ (o) $\frac{2a^6b^4}{3}$

2 (a) k^7p^6 (b) x^5y^5 (c) $\frac{6m^6n^5}{5}$

(d) $\frac{2q^{11}}{p^4}$ (e) $\frac{7y^{10}}{x^4}$ (f) $\frac{1}{m^6n^2}$

(g) $\frac{1}{k^7p^4}$ (h) $\frac{2a^5}{3b}$ (i) $\frac{5fg^4}{2}$

(j) x^2y^4 (k) $\frac{2m^3}{3}$ (l) $\frac{q^2}{5}$

3 (a) B (b) A (c) C

4 (a) x^9y^4 (b) $\frac{a^{12}}{b^{11}}$ (c) $\frac{4}{k^{18}p^2}$

(d) $9m^{20}n^2$ (e) $\frac{x^{27}}{y^{11}}$ (f) $\frac{m^6}{n^{20}}$

(g) $\frac{1}{q^{22}}$ (h) f^6g^{11} (i) $\frac{a}{b^7}$

(j) $\frac{m^{14}}{n^{11}}$ (k) $\frac{4x^{13}}{y^{18}}$ (l) $\frac{12d^{15}}{c^3}$

5 (a) x^2 (b) $\frac{a^3}{b^5}$ (c) pq (d) d^4f^2

(e) $\frac{1}{y^2}$ (f) $\frac{1}{a^3}$ (g) $\frac{n}{m^2}$ (h) $\frac{q^2}{p^7}$

(i) $\frac{b^9}{a^{16}}$ (j) $\frac{x^{13}}{y^7}$ (k) $\frac{12q^4}{p^{17}}$ (l) $\frac{9y^7}{8x^{14}}$

6 (a) $5^3, 5^2, 5^1, 5^0$ (b) $5^{-1}, 5^{-2}, 5^{-3}, 5^{-4}, 5^{-5}$

(c) $125, 25, 5, 1, \frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}$

7 (a) $\frac{m^9}{n^2}$ (b) $\frac{b^3}{a}$ (c) $\frac{x^5}{y^4}$ (d) $\frac{y^{19}}{x^6}$ (e) $\frac{9}{a^2b^7}$

(f) $\frac{8}{d^5f^7}$ (g) $\frac{10b}{a^9}$ (h) $\frac{n^3}{m^8}$ (i) $\frac{1}{x^2}$

8 (a) 1 divided by the original number (b) $192a$

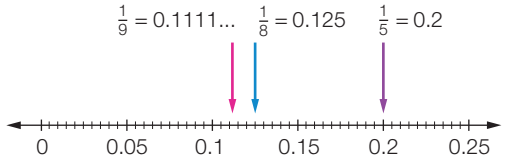
Open-ended – Sample answers

9 (a) $(6m^2n^3)^2\times n^2$; $9m^3n^4\times 4mn^4$

(b) $\frac{12m^7n^3}{15m^3n^9}$; $24m^4\div 30n^6$

10 $a=2, b=3$; $a=3, b=2$; $a=5, b=1$

These give the fractions: $\frac{1}{8}, \frac{1}{9}$ and $\frac{1}{5}$



11 (a) Murray just said $-2 \times -4 = 8$ and possibly put it as the power in the denominator because he thought the negatives needed to be accounted for. He is wrong in at least two ways.

(b) Raphael knows that indices are added when numbers are multiplied. $\frac{1}{a^6}$ may be Raphael's answer, which would be correct.

(c) When numbers with indices and the same base are multiplied, keep the base the same and add the indices.

$a^{-2} \times a^{-4} = a^{-6} = \frac{1}{a^6}$.

Exercise 12.7 (p 696)

1 (a) $x^{2/3}$ (b) $a^{5/6}$ (c) $k^{8/15}$ (d) $m^{1/4}$

(e) $p^{1/6}$ (f) $y^{1/4}$ (g) $5p^2$ (h) $7x^3$

(i) $2a^{1/3}b$ (j) $x^2y^{1/2}$ (k) $k^{1/2}n^4$ (l) $\frac{p^2}{q^5}$

2 (a) 14 (b) 23 (c) 19 (d) 8 (e) 6 (f) 8

(g) 5 (h) 4 (i) 7.94 (j) 14.42 (k) 2.15 (l) 2.29

3 4^{-1}, $6^{-\frac{1}{5}}$, $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$

4 (a) D (b) B (c) B (d) C

5 (a) a (b) d^3 (c) $\frac{x}{y}$ (d) $\frac{1}{m^2n^4}$

(e) $\frac{1}{q^2}$ (f) $\frac{1}{b^4}$ (g) $\frac{m^{1/3}}{2n^{1/6}}$ (h) $\frac{y^{7/6}}{5x^{1/3}}$

6 (a) B (b) A

7 (a) 0.5 (b) 0.437 (c) 0.164 (d) 0.196

(e) 0.369 (f) 0.343 (g) 0.398 (h) 0.359

8 (a) $x = \sqrt[3]{3}$ (b) $x = 4$ (c) $x = \frac{1}{2}$

9 (a) x^3 (b) a^3 (c) $2m^2n$

10 (a) $x = \pm 2$ (b) $x = 3$ (c) $x = -2$ (d) $x = 2$

11 (a) 7 (b) $x = 7, y = 5$ (c) 256

(d) $d = 256, e = \frac{3}{4}$

Open-ended – Sample answers

12 $1476.106^{\frac{4}{5}}$, 7^3, 1.793^{10}

13 $\left(n^{\frac{1}{4}}m^{\frac{1}{5}}\right)^3$, $(\sqrt[4]{n} \times \sqrt[5]{m})^3$; $(-\sqrt[4]{n^3})(-\sqrt[5]{m^3})$

Exercise 12.8 (p. 702)

1 (a) $\log_2 32 = 5$ (b) $\log_2 256 = 8$

(c) $\log_2 4096 = 12$ (d) $\log_3 27 = 3$

(e) $\log_3 243 = 5$ (f) $\log_3 6561 = 8$

(g) $\log_{10} 100 = 2$ (h) $\log_{10}(1\,000\,000) = 6$

(i) $\log_{10}(100\,000\,000) = 8$ (j) $\log_2\left(\frac{1}{4}\right) = -2$

(k) $\log_2\left(\frac{1}{8}\right) = -3$ (l) $\log_2\left(\frac{1}{16}\right) = -4$

2 (a) $10^2 = 100$ (b) $10^3 = 1000$

(c) $10^5 = 100\,000$ (d) $2^4 = 16$

(e) $2^7 = 128$ (f) $2^9 = 512$

(g) $10^{-3} = \frac{1}{1000}$ (h) $10^{-4} = \frac{1}{10\,000}$

(i) $10^{-6} = \frac{1}{1\,000\,000}$ (j) $2^{-3} = \frac{1}{8}$

(k) $2^{-5} = \frac{1}{32}$ (l) $2^{-7} = \frac{1}{128}$

3 (a) C (b) D

4 (a) 4 (b) 8 (c) 9 (d) -2 (e) -3

(f) -6 (g) -3 (h) -5 (i) 4

5 (a) 16 (b) 64 (c) 128 (d) -4 (e) -5

(f) -8 (g) 5 (h) 7 (i) 5

6 (a) (i) 10 (ii) 5 (iii) 7

(b) $\log_a(a) = 1$

7 (a) (i) $x = 1$ (ii) $x = 1$ (iii) $x = 1$

(b) $\log_a 1 = 0$

8 We cannot find the square root of a negative number.

9 (a) Students to verify that both equations are true.

(b) 6

(c) $2^{12} = 4096$ is the index equivalent of the first statement and $4^y = 4096$ is the second. But, this could be written as $(2^2)^y = 4096$ or $2^{2y} = 4096$, so $2y = 12$ and $y = 6$.

Open-ended – Sample answers

10 $\log_9(531\,441) = 6$ and $\log_{27}(531\,441) = 4$

11 Effectively Lee is saying $\log_6(-36) = -2$, which would mean that $6^{-2} = -36$, which is clearly incorrect. Lee should have divided both sides by -1 to get $(6)^2 = 36$, which has the equivalent form $\log_6(36) = 2$.

Exercise 12.9 (p. 708)

1 (a) $\log_2 32$ (b) $\log_2 64$ (c) $\log_2 256$

(d) $\log_{10} 30$ (e) $\log_{10} 21$ (f) $\log_{10} 36$

2 (a) $\log_2 2$ (b) $\log_2 5$ (c) $\log_2 5$

(d) $\log_{10} 14$ (e) $\log_{10} 5$ (f) $\log_{10} 6$

3 (a) $4\log_2 5$ (b) $3\log_2 6$ (c) $4\log_2 7$

(d) $3\log_{10} 7$ (e) $4\log_{10} 8$ (f) $5\log_{10} 11$

4 (a) 5 (b) 7 (c) 8 (d) 2 (e) 1 (f) 1

5 (a) A (b) C

6 (a) 3 (b) 11 (c) 1 (d) 2 (e) 8 (f) -9

7 (a) 2 (b) $\frac{5}{6}$ (c) 3 (d) 2 (e) $\frac{3}{2}$ (f) $\frac{4}{3}$

8 (a) $\frac{1}{4}$ (b) 2 (c) $\frac{2}{3}$

9 (a) $y = 4x^3$ (b) $y = 11x^5$

10 $5\log_2(y) = \log_2(x-1) + 2$

$$\log_2\left(\frac{y^5}{x-1}\right) = 2$$

$$2^2 = \frac{y^5}{x-1}$$

$$4(x-1) = y^5$$

Open-ended – Sample answers

11 No, the correct answer is 2. The mistake Danielle made was when she took out the $\frac{1}{2}$. This was really $\frac{1}{\frac{1}{2}}$ which is the same as 2.

Chapter review 12 (p. 710)

1 (a) 4.123 (b) 2.924 (c) 34.641 (d) 2.280

2 (a) $12\sqrt{30}$ (b) 70 (c) $\frac{8}{5}$ or $1\frac{3}{5}$ (d) $9\sqrt{2}$

(e) $2\sqrt{10}$ (f) $10\sqrt{5}$ (g) $30\sqrt{5}$ (h) $5\sqrt{3}$

3 (a) $\sqrt{72}$ (b) $-\sqrt{63}$

4 (a) $\sqrt{2}$ (b) $4\sqrt{7} - \sqrt{3}$

(c) $29\sqrt{3}$ (d) $46\sqrt{2} - \sqrt{6}$

5 (a) $4\sqrt{15} - 10 + \sqrt{10}$ (b) $12 - 18\sqrt{14} - 6\sqrt{2}$

(c) $5\sqrt{5} - 8$ (d) $95 + 24\sqrt{14}$

6 (a) $\frac{5\sqrt{2}}{2}$ (b) $2\sqrt{2}$ (c) $\frac{7\sqrt{5}}{10}$

7 (a) $\frac{21 + 7\sqrt{5}}{4}$ (b) $\frac{2\sqrt{6} - \sqrt{14} - 4\sqrt{3} + 2\sqrt{7}}{5}$

8 (a) $9a^2b^{10}$ (b) c^2d^{14} (c) u^2t

(d) $2xy^{10}$ (e) $m^{12}n^9$ (f) $a^{12}b$

9 (a) $\frac{6}{a^6}$ (b) $\frac{27x^4}{16y}$ (c) $\frac{n^6}{2}$

10 (a) $4mn^{1/3}$ (b) $p^3q^{1/2}$ (c) $a^{1/2}$

(d) $\frac{b}{a^9}$ (e) $\frac{2n^{4/3}}{m^{1/3}}$ (f) $\frac{729n^7}{m^{19}}$

11 (a) $k^{5/2}$ (b) $x^{3/4}$ (c) $m^{-2/3}$

12 (a) C (b) D

13 (a) $\log_4(16\,384) = 7$ (b) $\log_2(1024) = 10$

(c) $\log_{10}\frac{1}{1000} = -3$

14 (a) $2^{14} = 16\,384$ (b) $3^5 = 243$ (c) $10^{-5} = \frac{1}{100\,000}$

15 (a) 9 (b) -2 (c) $\log_3(50)$ (d) $\log_5(10)$

16 (a) $2\sqrt{58}$ cm (b) 15.23 cm

17 (a) length 25.3 cm, height 20.2 cm

(b) 1749.3 cm^2

(c) $(180\sqrt{6} + 108\sqrt{7} + 160\sqrt{15} + 48\sqrt{70})$ cm^2

(d) 1747.9 cm^2 (e) (b) was larger by 1.4 cm^2.

18 (a) 5 (b) 7 (c) -6 (d) $\frac{1}{16}$ (e) 8 (f) -4

19 (a) 3 (b) 3 (c) 2 (d) $\frac{3}{4}$

20 (a) $5\sqrt{2}$ cm (b) $\sqrt{\frac{160}{\sqrt{3}}}$ cm (c) $10\pi\sqrt{3}$ m

21 $-\frac{17 + 136\sqrt{5}}{319}$

22 (a) $y = 7x^3$ (b) $y = \frac{5}{x^2}$

Mixed review D (p. 713)

1 \$20.23

2 area = 36.2 cm^2, perimeter = 24.7 cm

3 (a) $2\sqrt{6}$ (b) 18 (c) 6

(d) $40\sqrt{2}$ (e) $12\sqrt{15}$ (f) $126\sqrt{2}$

4 (a) $4\sqrt{5}$ (b) $12\sqrt{15}$ (c) $35\sqrt{21}$

5 (a) $a = -6$ (b) $-\frac{15}{16}$ (c) $a = \frac{11}{6}$

6 (a) $\frac{4\sqrt{15}}{5}$ (b) $\frac{\sqrt{3+5}}{11}$ (c) $\frac{\sqrt{10} - \sqrt{6}}{2}$

7 (a) $(k-1)(k-3)$ (b) $(3t-5)(t+2)$

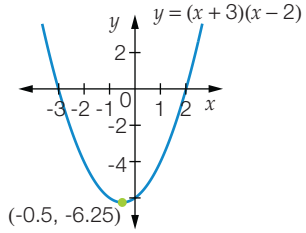
8 (a) $\frac{2a^6b^4}{3}$ (b) $a^{10}b^7$ (c) $\frac{1}{n^3k^{17}}$

9 (a) $a = 82.45°$ (b) 69.12° (c) 31.54°

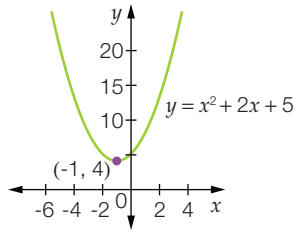
10 (a) $\frac{7}{3}$ (b) -2

11 (a) 33.36 (b) 125.24

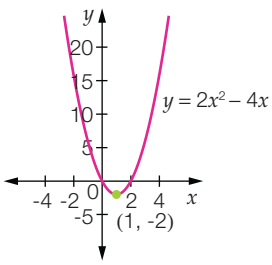
12 (a)



(b)



(c)



13 (a) $(8\sqrt{5} + 2\sqrt{3})$ cm (b) $15 + \sqrt{15})\text{cm}^2$

14 (a) A rectangle is a quadrilateral in which all angles are right angles.

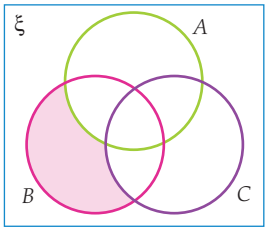
(b) A rhombus is a quadrilateral with all sides equal in length.

(c) A kite is a quadrilateral with two pairs of adjacent sides equal in length.

15 (a) $\frac{1}{12}$ (b) $\frac{5}{36}$ (c) $\frac{1}{6}$

16 (a) $3^{-5} < 5^{-3}$ (b) $9^{\frac{1}{2}} > 9^{-1}$ (c) $8^{-\frac{1}{3}} = \frac{2^{-3}}{2^{-2}}$

17

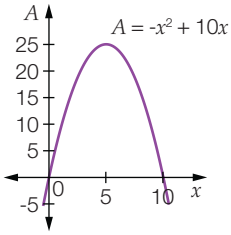


18 For an equation to be a quadratic equation, the highest powers of the variable must be a 2, and all powers of the variable must be positive integers.

19 (a) $10 - x$

(b) $A = x(10 - x) = -x^2 + 10x$

(c)



(d) 5 m by 5 m

(e) 25 m^2

20 $8\sqrt{2}$ cm

21 (a) Check students' diagrams carefully. Equal side lengths should be approximately 11.2 cm in length.

(b) Draw a line from the top of the triangle to the midpoint of the base to create two congruent right-angled triangles. You can now use Pythagoras' theorem to calculate the equal side lengths of the isosceles triangle:

$l^2 = 5^2 + 10^2$
$l^2 = 125$
$l = \sqrt{125} = 5\sqrt{5}$ cm

Challenge D (p. 715)

1 D.
There are 36 possible combinations from two rolls, 9 of which score a 3 or less: (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)

2 (a) $\frac{1}{6}$; the second spin is an independent event from the first.

(b) (i)

Prize	Frequency	Relative frequency
\$0	8	0.16
\$5	7	0.14
\$10	9	0.18
\$20	10	0.20
\$30	8	0.16
\$60	8	0.16
Total	**50**	**1**

(ii) \$1045; $5 \times 7 + 10 \times 9 + 20 \times 10 + 30 \times 8 + 60 \times 8 = 1045$

(c) (i) You would make a profit of \$40:

$5 \times 8 + 10 \times 8 + 20 \times 8 + 30 \times 8 + 60 \times 8 = 1000$
$48 \times 20 = 960$
$1000 - 960 = 40$

(ii) You would make a \$45 profit.

3 B

Side length of base $= 12\sqrt{10}$, so each triangular face a height of $\frac{1680}{12\sqrt{10}} = 14\sqrt{10}$. Using Pythagoras' theorem, the height of the pyramid, h, is given by $h^2 = (14\sqrt{10})^2 - (6\sqrt{10})^2$, so the height $= \sqrt{1600} = 40$.

4 A

Area of four semicircles $= 2\pi$, area of inner square $= 4$, radius of inner circle $= \sqrt{2}$, area of inner circle $= 2\pi$.

Area of shaded region $= 4 + 2\pi - 2\pi = 4$.

5 (a) $\frac{1}{x} = \frac{1}{3} + \frac{1}{4}$, $x = \frac{12}{7}$

(b) $x + 3 = 5$, $x = 2$

(c) $2^x = 2 \times 2^{48} + 2 \times 2^{48} = 2 \times 2^{49} = 2^{50}$, $x = 50$

(d) $2^{2x-4} = 2^3$, $x = 3.5$

(e) $2^{4x} = 2^{x+4}(2 - 1)$
$2^{4x} = 2^{x+4}$
$4x = x + 4$
$x = \frac{4}{3}$

Glossary and index

Addition Law for Mutually Exclusive Events $\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$; can only be used when A and B are mutually exclusive (p. 624)

adjacent side the side in a right-angled triangle next to the given angle that is not the hypotenuse (p. 214)

adjusted value the value of an item after depreciation (p. 34)

algebraic fractions expressions written in fraction form that contain at least one variable (p. 84)

amplitude half the distance between the minimum and maximum y-values of the graph of a trigonometric function (p. 275)

angle at the centre an angle that has its vertex at the centre of a circle (p. 445)

angle at the circumference an angle formed by the intersection of two chords at the circumference of a circle (p. 445)

angle in a semicircle an angle formed by joining the ends of the diameter of a circle to the circumference to form a right angle (p. 445)

angle of depression the angle between the horizontal and the line of sight when you look *down* (p. 233)

angle of elevation the angle between the horizontal and the line of sight when you look *up* (p. 233)

angles in the same segment angles subtended by the same arc of a circle at the circumference are equal (p. 445)

appreciation the increase in the cost or value of something over a period of time (p. 8)

arc a part of the circumference of a circle (p. 444)

asymptote a line representing a value on a graph that a curve approaches, but never cuts or crosses (pp. 277, 479)

axis of symmetry an imaginary line along which a mirror could be placed and the reflection would be indistinguishable from the original (p. 476)

binomial product the product of two expressions that each have two terms; e.g. $(x + 1)(x + 2)$ (p. 54)

bivariate data a data set that contains values for two variables (p. 351)

bisect when a point is placed on a line interval such that the distance from one end to the point is the same as the distance from the other end to the point (p. 422)

box plot a visual representation of the five-number summary (p. 327)

capacity the volume of gas or liquid that a container can hold, measured in litres (L) (p. 121)

chord a line segment connecting any two points on a circle (p. 444)

coincident lines two or more lines that lie on top of each other (p. 198)

compass bearing a direction given in terms of turning a number of degrees east or west of north or south (p. 240)

complementary event the event that is the opposite of the event under consideration (p. 604)

completing the square a method for factorising quadratic expressions that combines the use of the Perfect Square rule and the Difference of Two Squares rule (p. 78)

composite shape a shape or solid made up of more than one basic shape or solid (p. 106)

compound interest interest that is added, on a regular basis, to an investment or a loan, thus increasing the amount of the investment or loan (p. 6)

concentration the amount of a substance dissolved in a solution per unit quantity of the solution (p. 144)

conditional statement extra information is given in a probability situation that changes the universal set such as 'the probability of A given that B also happens' (p. 641)

cone a solid with a circular base that tapers through the height to a single vertex (p. 126)

congruent two figures that are exactly the same shape and size (p. 400)

conjugate surds two surd expressions such as $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{a} - \sqrt{b})$ that differ only in that one has an addition sign between the two terms and the other has a subtraction sign (p. 683)

constants fixed values in an equation (p. 44)

cosine one of the three basic trigonometric ratios; the ratio between the adjacent side and the hypotenuse in a right-angled triangle (p. 214)

cosine rule in any triangle, $a^2 = b^2 + c^2 - 2bc\cos A$, where a, b and c are the side lengths and A is the angle opposite side a. Can also be stated as $c^2 = a^2 + b^2 - 2ab\cos C$ (p. 287)

coterminal angles have a vertex at the origin of a Cartesian plane, start from the positive direction of the x-axis and share a common terminal side (p. 281)

cubic an expression or equation where the highest power of a variable is three (p. 538)

cumulative depreciation the total loss in value of an item over a number of years (p. 34)

cumulative frequency curve a graph that shows the number of results less than a given value in grouped data (p. 319)

cylinder a solid with a circular base and a uniform cross-section (p. 113)

definition the minimum amount of information needed to identify a particular figure (p. 412)

degree in a polynomial expression, the highest power of the variable (p. 551)

density the mass per unit volume of a substance (p. 142)

dependent events in probability, when the outcome of one event affects the outcome of another event (p. 649)

dependent variable the variable within a bivariate pair that varies because of changes in the other (independent) variable (pp. 157, 361)

depreciation the decrease over time of the value of an item (p. 34)

depreciation value the loss in dollar value of an item over the course of one year (p. 34)

Difference of Two Squares rule useful for factorising and expanding binomial expressions as $a^2 - b^2 = (a + b)(a + b)$ (p. 55)

dilation a transformation that makes the graph of a quadratic or quartic equation wider or narrower. For linear or cubic equations, varying the dilation factor makes the graph steeper or flatter (p. 487)

dilation factor the factor by which a graph has been made narrower or wider. For graphs with equations of the form $y = ax^n$, a is the dilation factor (p. 487)

Distributive Law a rule used to expand brackets: $a(b + c) = ab + ac$ (p. 677)

dividend a number being divided by another number e.g. in 300 ÷ 15, 300 is the dividend (p. 554)

divisor the number being divided into another number, e.g. in 300 ÷ 15, 15 is the divisor (p. 554)

effective interest rate the interest rate on a loan that takes into account the number of compounding periods per year. It is expressed as the percentage of the principal that is earned in the first year. Because the effect of compounding is taken into account, it may differ from the nominal interest rate (p. 27)

entire surd a surd expression where the entire number is under the root sign, e.g. $\sqrt{50}$ rather than $5\sqrt{2}$ (p. 670)

equidistant an equal distance from a point (p. 457)

equivalent angle an angle in the first quadrant that has the same sin, cos or tan value as an angle in another quadrant (p. 268)

event an outcome you are interested in when conducting a probability experiment (p. 603)

expected life the period of time for which an item (such as machinery or a computer) is expected to operate effectively, expressed as units of time or units of work output (p. 37)

exponential a relationship between variables where one of the variables is an exponent, such as $N = 2^t$ (p. 479)

exponential decay a relationship in which the value of one variable *decreases* by a fixed percentage for each unit change in the other (usually time) variable (p. 43)

exponential growth a relationship in which the value of one variable *increases* by a fixed percentage for each unit change in the other (usually time) variable (p. 42)

extrapolation making predictions from a graph outside the range of known values (p. 388)

factor theorem if, after dividing a polynomial by an expression, the remainder is zero, then the expression is a factor of the polynomial (p. 580)

factorise by grouping in pairs factorise an expression with four terms by grouping terms 'two and two' or 'three and one' (p. 62)

factorising writing an expression so that it is the product of two or more factors (p. 61)

five-number summary the five key values used to draw a box plot: minimum, lower quartile, median, upper quartile and maximum (p. 327)

flat rate a rate of interest used to calcualte the simple interest on an amount borrowed or invested (p. 8)

gradient a measure of the steepness or slope of a line joining two points, calculated as:

$$\frac{\text{change in vertical position}}{\text{change in horizontal position}}$$ (p. 168)

highest common factor (HCF) the highest factor that is common to all parts of an expression (p. 61)

hypotenuse the longest side of a right-angled triangle, opposite the right angle (p. 214)

included angle the angle between a pair of sides (p. 400)

independent events in probability, two or more events where the outcome of one event has no effect or influence on the others (p. 648)

independent variable the variable within a bivariate pair that causes change in the other (pp. 157, 361)

inflation rate the rate of increase in prices within an economy, usually expressed as a percentage per year (p. 8)

initial cost the original purchase price of an item, also called prime cost (p. 34)

interest money paid by the lender for the use of money borrowed, or by the bank for the used of money invested (p. 4)

interpolation making predictions from a graph within the range of known values (p. 388)

interquartile range the difference between the upper quartile and the lower quartile of a data set; the range in which the middle 50% of data values lie (p. 325)

irrational numbers numbers that cannot be written in fractional form; infinite non-recurring decimals (p. 663)

leading coefficient the coefficient of the leading term in a polynomial (p. 551)

leading term the first term of a polynomial expression (p. 551)

line of best fit a line drawn to approximate the relationship between two variables that have been plotted as a scatter plot, usually resulting from the collection of experimental or observational data (p. 386)

linear equation an equation that has 1 as the highest power of any variable (p. 157)

linear graph a graph that is a straight line (p. 157)

linear inequality a statement that involves an algebraic expression and one of the four inequality symbols, >, ≥, <, ≤; e.g. $x \geq 7$ is read as 'x is greater than or equal to 7'. $3 < x < 8$ is read as 'x is greater than 3 and less than 8' (p. 192)

linear relationship a relationship between two variables that, when graphed, results in a straight line (p. 157)

logarithm another word for index number, or exponent. Logarithms are used to write expressions involving indices in another way, i.e. $2^3 = 8$ is written as $\log_2(8) = 3$ (p. 701)

long-run proportion the fraction of trials that we would expect to perform in a particular way if a very large number of trials were held (p. 603)

major arc an arc that covers more than half the circumference of a circle (p. 444)

major segment a segment that covers more than half the area of a circle (p. 444)

minor arc an arc that covers less than half the circumference of a circle (p. 444)

minor segment a segment that covers less than half the area of a circle (p. 444)

minute one-sixtieth of a degree (p. 216)

monic quadratic a quadratic expression where the coefficient of the squared value is 1 (p. 65)

mutually exclusive impossible for two or more events to occur at the same time (p. 624)

net a series of connected shapes drawn on paper that can be folded together to form a solid (p. 112)

nominal interest rate an interest rate stated as 'per annum' by a bank that does not take into account the effect of compounding throughout the year. The actual amount of interst earned may be slightly higher (*see* effective interest rate) (p. 27)

non-monic quadratic a quadratic expression where the coefficient of the squared variable is not 1 (p. 561)

Null Factor Law if the product of two factors is zero, than at least one of the factors is zero. If $ab = 0$, then $a = 0$ or $b = 0$ (p. 498)

ogive another name for a cumulative frequency curve (p. 319)

opposite side the side in a right-angled triangle opposite the given angle (p. 214)

outlier a value that lies outside a certain range; a value more than 1.5 × IQR below the lower quartile, or more than 1.5 × IQR above the upper quartile, of a data set (p. 330)

parabola the symmetrical, curved shape of the graph of a quadratic equation (p. 476)

parallel box plot two or more box plots drawn on the same scale; useful for comparing data sets (p. 343)

parallel dot plot two or more dot plots drawn for comparative purposes; it is common to include a scale for each dot plot to serve as a dividing line between the plots (p. 343)

per annum 'for each year' in the Latin language (p. 4)

percentage cumulative frequency curve a cumulative frequency curve where the vertical axis (cumulative frequency) has been converted to percentages (p. 324)

percentile the value below which the stated percentage of results lie (p. 324)

Perfect Square rules useful in factorising and expanding binomial expressions as $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ (p. 55)

period the horizontal distance covered by one full cycle of the graph of a trigonometric function (p. 274)

perpendicular two lines that intersect or meet at right angles. The product of their gradients is -1: $m_1 \times m_2 = -1$. (p. 430)

perpendicular bisector a line that intersects another line at a right angle and cuts the other line in half (p. 430)

point of inflection a point on a curve where the shape of the curve changes direction. The curve can change from concave to convex, or vice versa. The points of inflection in cubics and quartics have a zero gradient (p. 539)

polynomial an expression involving a sum of terms raised to different powers (p. 551)

primary data data that you collect yourself; for example, recording the colour of cars passing the school (p. 351)

prime cost *see* initial cost (p. 34)

principal the original amount of money that was borrowed or loaned (p. 4)

prism a solid that has a regular shape, uniform cross-section and faces that are all polygons (p. 112)

probability the study of chance, or likelihood (p. 603)

probability tree diagram a tree diagram showing the probabilities on each of the branches (p. 631)

property a fact about a particular figure that can be proved from its definition (p. 412)

prove to demonstrate that a theorem is true in all cases (p. 408)

pyramid a solid made up of a polygonal base and triangular sides that all meet at a vertex (p. 126)

quadrant one quarter of a circle, formed when two perpendicular diameters are drawn in a circle (p. 267)

quadratic expression an expression in which the highest power of a pronumeral is 2 (p. 65)

quadratic trinomial an expression consisting of three terms in which the highest power of a pronumeral is 2 (p. 65)

quartic an expression or equation where the highest power of the variable is 4 (p. 538)

quantile a percentile expressed in decimal form (p. 324)

quotient the result of dividing one quantity by another (p. 554)

range the difference between the largest result and the smallest result in a data set; one of the measures of spread (p. 325)

rational numbers numbers that can be expressed in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$ (p. 663)

rationalising the denominator the process of changing a fractional expression that has a surd component in the denominator into an equivalent form that has a rational denominator (p. 682)

real numbers all rational and irrational numbers together (p. 663)

rectangular hyperbola a graph of an inverse relationship in which one variable decreases as the other increases (p. 481)

reducing balance depreciation when items depreciate by smaller and smaller amounts each year. (p. 35)

reflection a type of transformation where a quadratic graph is inverted or turned upside down in the x-axis (p. 487)

regression line a least squares line of best fit (p. 386)

remainder any amount remaining after a division process has been completed (p. 554)

remainder theorem if the polynomial $P(x)$ is divided by $(x - a)$, then the remainder is $P(a)$. If $P(x)$ is divided by $(ax - b)$, then the remainder is $P\left(\frac{b}{a}\right)$ (p. 580)

residual value the value of an item (such as a piece of machinery) at the end of its useful life (p. 37)

salvage value *see* residual value (p. 37)

sample space the full list of possible outcomes in a probability experiment (p. 603)

scatter plot a graph of bivariate data; each data pair is represented by a point on the graph (p. 352)

scrap value *see* residual value (p. 37)

second one-sixtieth of a minute (p. 216)

secondary data data that you do not directly collect yourself but access from another source; e.g. the internet (p. 351)

sector of a circle the area formed between an arc and two radii drawn from its end points (p. 444)

segment the area formed between a chord and the circumference of a circle (p. 444)

similar figures that are the same shape but not the same size (p. 401)

simple interest interest calculated on the original principal of the investment or loan. It does not take into account increases in the principal over time as interest is added (p. 4)

simultaneous equations two, or more, equations that are true for the same pair of x- and y-values (p. 198)

sine one of the three basic trigonometric ratios; the ratio between the opposite side and the hypotenuse in a right-angled triangle (p. 214)

sine rule in any triangle, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where A, B and C are the three angles and a, b, and c are the sides opposite the angles, respectively (p. 287)

sketch a drawing or graph that is not absolutely accurate, but shows all the key features required for the purpose (p. 174)

sphere a ball-shaped solid (p. 128)

standard deviation one of the measures of spread, related to the mean (p. 379)

straight-line depreciation decrease in value by a fixed amount each year (p. 34)

subtend to form an angle at a point (p. 444)

subtended supported at the circumference or centre of a circle by an arc; describes an angle (p. 444)

surds irrational numbers that can only be expressed exactly using the radical or root sign (p. 663)

surface area the sum of the areas of all the external surfaces of a solid (p. 112)

tangent one of the three basic trigonometric ratios; the ratio between the opposite side and the adjacent side in a right-angled triangle (p. 214)

tapered solids solids in which the cross-section gets smaller as you move from the base to the top (p. 126)

theorem a statement about general results or relationship —once proven, a theorem is known as a property (p. 421)

total possible depreciation the prime cost of an item, less any resale value or scrap value (p. 37)

transformation a change of shape or position for an object or graph (p. 487)

translation a transformation that involves sliding the shape, or graph, across the page (p. 487)

tree diagram a branching diagram that assists in identifying the sample space of each outcome in a multiple event (p. 608)

trigonometric ratio a ratio of the lengths of two sides in a right-angled triangle (p. 214)

true bearing a direction given in terms of the number of degrees turned in a clockwise direction from north (p. 240)

turning point the point on a parabola where the graph changes direction, also known as the vertex. The turning point is either a minimum or maximum value (p. 476)

turning point form the form of a quadratic equation that enables the turning point to be read off directly; $y = a(x - b)^2 + c$ has turning point (b, c) (p. 512)

two-way table in probability, a way of presenting information so that outcomes for one event are listed across the top and outcomes for a second event are listed down the side. The individual cells of the table represent the intersection of two outcomes (p. 608)

uniform cross-section a cross-section of a solid that is consistent from the base to the top (p. 112)

unique equations give non-parallel and non-coincident graphs, have only one point of intersection and one unique solution (p. 199)

unit circle a circle with a radius of one unit, used to define the trigonometric functions (p. 267)

unit cost depreciation the amount of depreciation, in dollars, per unit of an item's expected life (p. 37)

universal set the full list of outcomes under consideration (p. 617)

variables unknown values in a term, expression or equation (p. 44)

variance a statistical measure of spread; it is the mean of the squares of the deviations from the mean value (p. 379)

vertex another name for the turning point of a parabola (p. 476)

written-down value *see* adjusted value (p. 34)