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11

MASTERING HSC MATHEMATICS

YEAR 11 MATHEMATICS ADVANCED

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

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Errors:
This is a second print-run and many of the errors identified in the first print have been corrected. However, whilst the authors have taken utmost care with the editing and checking of this book, the authors cannot guarantee that all errors have been found and corrected. If you identify an error, please contact the authors directly at authors@masteringhscmathematics.com.au

Features of this book

This book is suitable for all students studying the HSC Mathematics Advanced and HSC Mathematics Extension 1 course. It has been designed in a thoroughly organised manner to help students master each syllabus topic in the new Stage 6 HSC Mathematics Advanced course. This book will teach, consolidate, test and challenge students. It is an essential resource for all students and teachers.

In flavour with the new course, this book has the following features:

- Technology-based questions.
- Interpretation questions.
- Modelling and application problems.
- Verification questions.

Within each chapter, there are subsections divided as follows.

Fundamentals

The carefully constructed *fundamentals* section appears before the main body of questions. The purpose of this section is to

- test all key formulae, definitions, concepts and theory.
- test essential mathematical terms and language through cloze-passages.
- ensure that the student has knowledge of the essential prerequisites.
- provide a summary of basic requirements for the topic.

Questions

This is the main body of questions with the following features.

- Step-by-step questions to assist the student with more difficult problems.
- Carefully graded exercises.
- "Show"-type questions, both guides the student, and offers good exam preparation.
- Proofs and explanations to strengthen understanding and develop problem-solving skills.
- Application questions to demonstrate future uses of learned theory.
- Technology-based questions to teach and reinforce concepts.

Challenge

These are more difficult questions that provide

- a challenge for students wishing to test their mastery of the topic.
- rigour and higher-order thinking skills.
- extension and more in-depth treatment of the unit of work.

Chapter Review

This section appears at the end of every chapter, and offers the following.

- Revision and consolidation of the previous exercises.
- Questions that require a combination of ideas from previous exercises.

Investigations

These tasks are potential assignments and research projects. Teachers may use and adapt these to cover the new NESA requirements on investigative assessment tasks. This section provides for the student

- application and modelling scenarios.
- research tasks involving data collection and analysis.
- scaffolding of learning tasks.
- open-ended style problems for discussions.
- opportunity to use appropriate technology effectively in a range of contexts.
- opportunity for students to demonstrate critical thinking.

Answers

- Quick answers to questions.
- "Show" and "prove" answers can be found in the full worked solutions.

Full worked Solutions

- Can be found online for free, or a full-colour hard copy purchased for convenience.
- Provide complete worked solutions to all questions, except investigative tasks to maintain the open-ended nature of the tasks.
- Includes several alternative solutions to problems, where possible.

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Table of Contents

Chapter 6:
Radians

Chapter 7:
Differentiation and Applications

Chapter 8:
Logarithmic and Exponential Functions

Chapter 9:
Probability

Chapter 10:
Probability Distributions

1

ALGEBRAIC TECHNIQUES

- Expanding and factorising
- Algebraic fractions
- Indices
- Surds

Exercise 1A

Expanding and factorising

Fundamentals

Fundamentals 1

Simplify the following.

(a) $(x+y)^2 =$ ________

(b) $(x-y)^2 =$ ________

(c) $(x+y)(x-y) =$ ________

(d) $(ax+y)^2 =$ ________

(e) $(ax+by)^2 =$ ________

(f) $\dfrac{x-y}{y-x} =$ ________

Question 1 Write down the expansion of the following.

(a) $(2x+1)^2$

(b) $(3x-4)^2$

(c) $(3xy-4)^2$

(d) $(3x-4y)^2$

(e) $\left(x+\dfrac{1}{x}\right)^2$

(f) $\left(x+\dfrac{1}{x}\right)\left(x-\dfrac{1}{x}\right)$

(g) $(2x+1)(3x-4)$

(h) $(2x-3y)(2x+3y)$

(i) $(2x+y)(3x-4y)$

Question 2 Factorise the following.

(a) x^2+6x+9

(b) $x^2+2x-15$

(c) $4x^2-8x-5$

(d) $48x^2-27y^2$

(e) $25-(x+y)^2$

(f) $2x^2-7x-15$

(g) $x^2-y^2+2x-2y$

(h) $(x-3)-x(3-x)$

(i) x^4-y^4

Question 3

(a) Expand $(x-1)(1+x+x^2)$.

(b) Deduce that $1+x+x^2 = \dfrac{x^3-1}{x-1}$, for $x \neq 1$.

Challenge Problems

Problem 1 Factorise the following.

(a) $16 - 81x^4$

(b) $a^2(a+2) - 4(a+2)$

(c) $6x^2 - 5x - 6$

(d) $6x^2 - 5xy - 6y^2$

Problem 2 Use the formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ to factorise the following.

(a) $x^3 - 1$

(b) $27x^3 - 8$

(c) $(2x+1)^3 - 8$

Problem 3 Expand the following.

(a) $(x-y)^3$

(b) $(x+y)^3$

Problem 4 [Geometric series]

Let $n \geq 2$ be a positive integer.

(a) Show that

$$x^n - 1 = (x-1)\left(1 + x + x^2 + \ldots + x^{n-1}\right).$$

(b) Deduce that for $x \neq 1$,

$$\frac{x^n - 1}{x - 1} = 1 + x + x^2 + \ldots + x^{n-1}.$$

(c) Hence, simplify $1 + 2 + 2^2 + \ldots + 2^{99}$.

Problem 5 [Factorisation of $z^6 + 1$ over $\mathbb{R}$]

Show that

$$z^6 + 1 = \left(z^2 + 1\right)\left(z^2 - z\sqrt{3} + 1\right)\left(z^2 + z\sqrt{3} + 1\right).$$

Problem 6 [Key step in proving AM/GM inequality]

Show that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)\left(a^2 + b^2 + c^2 - ab - bc - ac\right).$$

Exercise 1B

Algebraic fractions

Fundamentals

Fundamentals 1

Simplify the following expressions.

(a) $\frac{a}{x} + \frac{b}{x} = $ ________ (b) $\frac{a}{x} + \frac{b}{x^2} = $ ________

(c) $\frac{a}{x} + \frac{b}{y} = $ ________ (d) $\frac{a}{xy} + \frac{b}{y^2} = $ ________

Question 1 Simplify the following expressions.

(a) $\frac{2x}{5} + \frac{4x}{5}$ (b) $\frac{3x}{10} + \frac{11x}{15}$ (c) $\frac{3x-1}{3} - \frac{7-2x}{3}$

(d) $\frac{5x+3}{6} - \frac{1-4x}{8}$ (e) $\frac{5}{4x} + \frac{3}{4x} + \frac{1}{8x}$ (f) $\frac{5}{4x} - \frac{2}{x}$

(g) $\frac{x}{x-1} - \frac{x}{x+1}$ (h) $\frac{3x+5}{x-2} - \frac{x+7}{x-2}$ (i) $\frac{2}{x-2} + \frac{1}{x}$

(j) $\frac{3}{4x^2} + \frac{5}{6x}$ (k) $\frac{5}{x(x-2)} + \frac{2}{x(x+2)}$ (l) $\frac{3}{x(x-4)} - \frac{2}{x^2-3x+2}$

Question 2 Simplify the following expressions.

(a) $\frac{5x}{5y} \times \frac{3y}{20x^2}$ (b) $\frac{5x}{6} \div \frac{15xy}{9}$

(c) $\frac{x-2}{x+3} \times \frac{x^2-x-6}{x^2-6x+9}$ (d) $\frac{x^2-9}{x^2-2x-15} \div \frac{x^2-5x+6}{3x^2-12}$

Question 3 Simplify the following expressions.

(a) $\frac{4x^2+6xy}{8x^2}$ (b) $\left(\frac{1}{x} - \frac{1}{y}\right) \div (x-y)$ (c) $\left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{x} - \frac{1}{y}\right)$

Question 4 Complete the following.

(a) $$\frac{x - \frac{1}{x}}{x + \frac{1}{x}} = \frac{x - \frac{1}{x}}{x + \frac{1}{x}} \times \frac{x}{x}$$
$$= \underline{\qquad\qquad}$$

(b) $$\frac{\frac{1}{x} + \frac{1}{x+1}}{\frac{1}{x} - \frac{1}{x+1}} = \frac{\frac{1}{x} + \frac{1}{x+1}}{\frac{1}{x} - \frac{1}{x+1}} \times \frac{x(x+1)}{x(x+1)}$$
$$= \underline{\qquad\qquad}$$

Question 5 Simplify the following by multiplying the numerator and denominator by an appropriate term.

(a) $\dfrac{\frac{1}{x}}{1+\frac{1}{x}}$ (b) $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$ (c) $\dfrac{\frac{y}{x}+\frac{x}{y}}{\frac{y}{x}-\frac{x}{y}}$ (d) $\dfrac{\frac{2}{x}+\frac{1}{x+3}}{\frac{3}{x}-\frac{1}{x+3}}$

Question 6 Simplify the following.

(a) $\dfrac{1}{x-1}-\dfrac{x}{x-1}$ (b) $\dfrac{t^2-9}{3t-9}$.

(c) $\dfrac{3}{x^2-4}-\dfrac{2}{x^2-3x+2}$ (d) $\dfrac{2}{x^2-9}-\dfrac{1}{x-3}-\dfrac{5}{x+3}$

Question 7 Suppose a and b are numbers such that $a+b=5$ and $ab=3$. Find the following *without* explicitly finding the value of a and b.

(a) $\dfrac{1}{a}+\dfrac{1}{b}$ (b) $\dfrac{1}{a^2}+\dfrac{1}{b^2}$ (c) $\dfrac{1}{a^2+ab}+\dfrac{1}{b^2+ab}$

Question 8 Simplify

$$\frac{x}{x+1}+\frac{1}{(x+1)(x+2)}.$$

Challenge Problems

Problem 1 Show that the following expression will always have the same value, regardless of the value of n.

$$\frac{1}{n}+\frac{(n+1)(n-1)}{n}-(n+1)$$

Problem 2 Simplify the following.

(a) $\dfrac{a+\frac{1}{a}}{a-\frac{1}{a}}$ (b) $\dfrac{\frac{a}{b}+\frac{b}{a}}{\frac{a}{b}-\frac{b}{a}}$

Problem 3 Suppose $p=\dfrac{1}{a}$, $q=\dfrac{1}{1-p}$ and $r=\dfrac{q}{q-1}$. Show that $r=a$.

Problem 4 [Harmonic mean/Arithmetic mean inequality]

(a) Show that the average of $\dfrac{1}{a}$ and $\dfrac{1}{b}$, where a and b are positive, is $x=\dfrac{a+b}{2ab}$.

(b) Bob claims that the average of $\frac{1}{2}$ and $\frac{1}{10}$ is $\frac{1}{6}$ because 6 is the average of 2 and 10. Use the formula from (a) to calculate the actual average value, and show that Bob's claim is incorrect.

(c) Let y be the 'average', according to Bob's argument. Show that $y = \dfrac{2}{a+b}$.

(d) Show that

$$x - y = \frac{(a-b)^2}{2ab(a+b)},$$

and deduce that Bob's 'average' y will always be less than the actual average x.

Problem 5 [Thin lens equation]

Consider the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

(a) Show that $u = \dfrac{vf}{v-f}$.

(b) Briefly explain why $v \neq f$.

(c) Let $x_1 = u - f$ and $x_2 = v - f$. Show that $x_1 x_2 = f^2$.

Exercise 1C

Indices

Fundamentals

Fundamentals 1
Complete the following.

(a) $a^m \times a^n$ (b) $a^m \div a^n$ (c) $(a^m)^n$

(d) $a^{\frac{m}{n}}$ (e) a^0 (f) $a^{\frac{1}{n}}$

(g) $(ab)^n$ (h) a^{-1}

Fundamentals 2
Complete the following.

(a) a^{-n} (b) $(ab)^0$ (c) ab^0 (d) $\dfrac{1}{a^{-n}}$

(e) $a^{-\frac{1}{2}}$ (f) $\left(\dfrac{a}{b}\right)^n$ (g) $\left(\dfrac{a}{b}\right)^{-1}$ (h) $\left(\dfrac{a}{b}\right)^{-n}$

Fundamentals 3
If $a^m = a^n$, where $a, b > 1$, then $m =$ ___ .

Fundamentals 4
Factorise the following.

(a) $a^{n+1} - a^n$. (b) $a^{3n} - a^n$. (c) $(ab)^n - a^n$.

Question 1 Evaluate, without the use of a calculator.

(a) 5^0 (b) $49^{\frac{1}{2}}$ (c) $125^{\frac{1}{3}}$ (d) $8^{\frac{2}{3}}$

(e) $(9^3)^{\frac{1}{2}}$ (f) $\left(\dfrac{125}{64}\right)^{\frac{2}{3}}$ (g) $(-8)^{\frac{2}{3}}$

Question 2 Evaluate, without the use of a calculator.

(a) $9^{-\frac{1}{2}}$ (b) 5^{-2} (c) $\dfrac{1}{3^{-3}}$ (d) $\left(\dfrac{2}{5}\right)^{-2}$

(e) $\left(\dfrac{1}{25}\right)^{-\frac{3}{2}}$ (f) $4^{-\frac{1}{2}}$ (g) $125^{-\frac{2}{3}}$ (h) $8^{-3} \times 2^8$

Question 3 Simplify the following.

(a) $2x^0 + x^0$ (b) $3^{2x} \times 9$ (c) $3^{2x} \times 3^{x+1}$

(d) $(6x^3y^4)^2$ (e) $(2x^{-3})^2$ (f) $\dfrac{16x^4}{4x^{-3}}$

Question 4 Simplify the following.

(a) $2^n \times 4^n \times 8^n \times 16^n$ (b) $\dfrac{49^{-n} \times 7}{7^{3n}}$ (c) $\dfrac{3^n \times 9^{n+1}}{27^{n-2}}$

(d) $\dfrac{9^n - 1}{3^n - 1}$ (e) $\dfrac{5^{2n} - 5^n}{5^n - 1}$

Question 5 Solve the following for x.

(a) $2^x = 16$ (b) $4^{x+1} = 16$ (c) $4^{x+1} = 1$

(d) $4^{x+1} = \dfrac{1}{8}$ (e) $27^{x+1} = 3 \times 9^{x-1}$ (f) $x^{-2} = 49$

Question 6 Factorise and hence simplify the following.

(a) $3^8 - 3^7$ (b) $2^n + 2^{n+1}$ (c) $3^{2n} - 3^n$ (d) $6^n - 3^n$ (e) $3^{3n} - 3^n$

Challenge Problems

Problem 1 Solve the following for x.

(a) $16^{-x} = \dfrac{1}{32}$ (b) $8^x = 4^{x+1}$ (c) $16^{2-x} = \dfrac{1}{8^x}$

(d) $8^x = 2^{x+3} + 2^{x+3}$ (e) $32^x = 2^{x+5} - 2^{x+4}$ (f) $4^x - 2^{x+1} - 8 = 0$

Problem 2 Simplify the following.

(a) $\dfrac{12^x}{8^x} \times \dfrac{6^x}{3^x}$ (b) $\sqrt{2^{4x} \times 4^{2x} \times 8^{4x}}$

Problem 3 Find the value of x if 2^x is half of 2^{200}.

Problem 4 Factorise $5^{n+1} + 5^n$, and hence simplify $\dfrac{5^{101} + 5^{100}}{6}$.

Problem 5 Factorise the following.

(a) $5^{2x} - 2^{2y}$ (b) $2^{4x} - 2^{6x}$

Problem 6 Simplify $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6$ and express your answer in index form.

Problem 7 If $2^a + 3^b = 17$ and $2^{a+1} - 3^{b+1} = -11$, find the values of a and b.

Exercise 1D

Surds

Fundamentals

Fundamentals 1

Complete the following surd laws.

(a) $\sqrt{ab} = $ ________

(b) $\sqrt{\frac{a}{b}} = $ ________

Fundamentals 2

Express the following in index form

(a) $\sqrt{x}$

(b) $x\sqrt{x}$

(c) $x^2\sqrt{x}$

(d) $\frac{1}{\sqrt{x}}$

(e) $\sqrt[n]{x}$

(f) $\sqrt[n]{x^m}$

Fundamentals 3

Expand the following.

(a) $\left(\sqrt{x} + \sqrt{y}\right)^2$

(b) $\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)$

(c) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

(d) $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

Fundamentals 4

(a) Similarly how you can only add and subtract ‘like’ terms, you can only add and subtract ‘like’ s______.

(b) When simplifying a surd, always look for p________-square factors.

Fundamentals 5

(a) If $a + b\sqrt{c} = x + y\sqrt{c}$, then $a = $ ___ and $y = $ ___.

(b) If $a\sqrt{b} = \sqrt{x}$, then $x = $ ______.

Fundamentals 6

Rationalise the denominator of the following, provided that a and b are positive integers.

(a) $\frac{1}{\sqrt{a}}$

(b) $\frac{1}{b\sqrt{a}}$

(c) $\frac{1}{a + \sqrt{b}}$

(d) $\frac{1}{\sqrt{a} + \sqrt{b}}$

(e) $\frac{1}{a + b\sqrt{c}}$

(f) $\frac{1}{\sqrt{a} + b\sqrt{c}}$

Question 1 Simplify the following.

(a) $\sqrt{45}+\sqrt{80}-\sqrt{72}$

(b) $3\sqrt{24}+6\sqrt{150}-5\sqrt{54}$

(c) $\dfrac{4+\sqrt{12}}{2}$

(d) $\dfrac{2\sqrt{6}\times 3\sqrt{7}}{12\sqrt{21}}$

Question 2 Simplify the following.

(a) $\sqrt[3]{125}-\sqrt[3]{8}$

(b) $\sqrt[3]{24}$

(c) $\sqrt[3]{2000}$

(d) $\sqrt[5]{64}$

Question 3 Simplify the following.

(a) $\sqrt{a^2b}$

(b) $\sqrt{a^4b^2}$

(c) $\sqrt[3]{a^6b^3}$

(d) $\sqrt{a^5b^3}$

Question 4 Simplify the following by first rationalising the denominator.

(a) $\dfrac{1}{2\sqrt{5}}$

(b) $\dfrac{5\sqrt{2}}{3\sqrt{10}}$

(c) $\dfrac{5\sqrt{3}}{3\sqrt{5}}$

(d) $\dfrac{\sqrt{2}}{5-\sqrt{2}}$

(e) $\dfrac{5+\sqrt{2}}{5-\sqrt{2}}$

(f) $\dfrac{1}{\sqrt{3}+1}+\dfrac{\sqrt{3}-1}{4}$

(g) $\dfrac{42}{\sqrt{63}}-\dfrac{6}{\sqrt{7}+2}$

Question 5 Rationalise the denominator.

(a) $\dfrac{a\sqrt{b}}{c\sqrt{d}}$

(b) $\dfrac{1}{a+\sqrt{b}}$

(c) $\dfrac{1}{\sqrt{a}}-\dfrac{1}{\sqrt{b}}$

(d) $\dfrac{1}{a\sqrt{b}-c}$

(e) $\dfrac{1}{a\sqrt{b}+c\sqrt{d}}$

Question 6 Simplify the following.

(a) $(2+\sqrt{5})(2-\sqrt{5})$

(b) $(5+\sqrt{3})^2$

(c) $(-5+2\sqrt{3})^2$

(d) $(\sqrt{2}+\sqrt{3})^2$

(e) $\left(1+\dfrac{1}{\sqrt{2}}\right)^2$

(f) $\left(\sqrt{3}-\dfrac{1}{\sqrt{3}}\right)^2$

Question 7 Find the values of a and b in each of the following.

(a) $3+4\sqrt{5}=a+2\sqrt{b}$

(b) $\left(2-3\sqrt{5}\right)^2=a+b\sqrt{5}$

(c) $(2\sqrt{3}-3\sqrt{6})^2=a+b\sqrt{2}$

(d) $(1-3\sqrt{2})^2-(2+3\sqrt{2})^2=a+b\sqrt{2}$

Question 8 Find x in each of the following.

(a) $\sqrt{x}=\sqrt{18}+\sqrt{8}$

(b) $x\sqrt{2}=\sqrt{18}+3\sqrt{8}-\sqrt{32}$

Question 9

(a) Let $x=\sqrt{3}+\sqrt{2}$. By first simplifying $x+\dfrac{1}{x}$, find the value of $x^2+\dfrac{1}{x^2}$.

Hint: You do not need to find the value of x^2 to do this question.

(b) Use a similar technique to calculate $x^2-\dfrac{1}{x^2}$, if $x=-2+\sqrt{5}$.

Question 10 Show that $x=-3+2\sqrt{2}$ satisfies the equation $x^2+6x+1=0$

Challenge Problems

Problem 1 Show that $\dfrac{4}{2+\sqrt{5}} - \dfrac{1}{9-4\sqrt{5}}$ is a rational number.

Problem 2 [Telescoping sum]

(a) Rationalise the denominator of

$$\frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

(b) Hence, simplify

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

Problem 3 Suppose x and y are rational numbers. Show that

$$xy\left(\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}}\right)^2$$

is also a rational number.

Chapter 1 Review

Algebraic Techniques

Review

Question 1 [Review of word-problems]

Solve the following by constructing a suitable linear equation.

(a) Zach is 4 years older than Joel. The sum of their ages is 16. Find their ages.

(b) When a number is decreased by a quarter of the number, it equals 12. Find the number.

(c) A book and a card cost \$28. If the book costs 6 times as much as the card, find the price of each.

(d) Natalie is 5 times as old as her son Jeremy. In 8 years' time, she will be 3 times as old as her son. Find Natalie's age now.

(e) In a multiple choice exam with 30 questions, a correct answer earns 2 marks, an incorrect answer deducts 3 marks and an unanswered question scores 0 marks. If Glen left out 6 questions and his score was 28, how many questions did he answer correctly?

(f) A rectangular pool is 50% longer than it is wide. It is surrounded by a path 2 m wide. If the area of the path is 116 m^2, find the area of the pool.

Question 2 Change the subject of the formula to y in the following

(a) $x = \dfrac{y}{y+3}$

(b) $\dfrac{1}{x} = \dfrac{1}{z} + \dfrac{1}{y}$

(c) $ay^2 - by^2 = 4$

(d) $x = \sqrt{y^2 - 4ab}$

Question 3

(a) The volume of a square pyramid with base length x and height h is $V = \dfrac{1}{3}x^2h$.

(i) Express x in terms of V and h.

(ii) Find x if the volume is 15 cm^3 and height is 5 cm.

(b) The volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.

(i) Express r in terms of V and h.

(ii) Find the radius if the volume is 100 cm^3 and height is 4π cm.

(c) The volume of a cylinder $V = \dfrac{4}{3}\pi r^3$, where r is the radius.

(i) Express r in terms of V.

(ii) Find r if $V = \dfrac{9}{2}\pi$ cm^3.

Question 4 Solve the following problems.

(a) The breadth of a rectangle is 6 cm shorter than the length. The perimeter of the rectangle is 60 cm. Find the length of the rectangle.

(b) Robin is 23 years older than Emma. In 6 years' time, Robin will be twice as old as Emma. How old are they now?

Question 5 Factorise the following.

(a) $x^2 - 9x - 10$ (b) $x^2 + 2x - 63$ (c) $5x(x-2) - 4(x-2)$

(d) $5x^2 - 2x - 3$ (e) $20 - 3x - 2x^2$ (f) $5(x-5) - x(5-x)$

(g) $25 - (x-1)^2$ (h) $x^2 - y^2 + 3x - 3y$ (i) $(x+y)^2 - 4$

(j) $80x^4 - 5y^4$ (k) $x^3 - 3x^2 - 4x + 12$ (l) $2x^2 - xy - 3y^2$

Question 6 Simplify the following.

(a) $\sqrt{2} + \sqrt{8}$ (b) $2\sqrt{2} + \sqrt{18}$ (c) $\sqrt{20} + \sqrt{45}$

(d) $\sqrt{90} - \sqrt{40}$ (e) $(2 - 3\sqrt{2})^2$ (f) $(\sqrt{6} - \sqrt{3})(\sqrt{6} + \sqrt{3})$

Question 7 Simplify the following.

(a) $3x^0 + x^0$ (b) $a^{\frac{3}{8}} \times a^{\frac{5}{8}} \times a^{-2}$ (c) $\left(3x^2\right)^2 \div 6x^4$ (d) $\dfrac{9^{m-1} \times 2^{m+3}}{6^{m+2}}$

Question 8 Evaluate the following.

(a) $8^{\frac{2}{3}}$ (b) $25^{-\frac{3}{2}}$ (c) $\left(\dfrac{1}{16}\right)^{-\frac{3}{4}}$

Question 9 Express $5^5 + 5^5 + 5^5 + 5^5 + 5^5$ in the form x^y.

Question 10 Solve the following for x.

(a) $2^{x+3} = 16$ (b) $8^x = \dfrac{1}{4}$ (c) $4^x = \sqrt{2}$

(d) $3^{2x-1} = 1$ (e) $3^x + 3^{x+1} - 36 = 0$ (f) $4^x - 5\left(2^x\right) + 4 = 0$

Question 11 Simplify the following.

(a) $\dfrac{1}{\sqrt{2} - 1}$ (b) $\dfrac{2 + \sqrt{5}}{2 - \sqrt{5}}$ (c) $\dfrac{4 + \sqrt{2}}{1 + \sqrt{2}}$ (d) $\dfrac{3}{\sqrt{6} - 2}$

Question 12 Find a and b.

(a) $\dfrac{2 - \sqrt{3}}{2 + \sqrt{3}} = a - 4\sqrt{b}$ (b) $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} = a - \sqrt{b}$

Question 13 Solve the following for x.

(a) $8(x-4)-3(4-x)=2x$

(b) $\frac{2}{3}x-1=\frac{x}{2}-\frac{1}{6}$

(c) $\frac{3x-1}{2}-\frac{2x+1}{3}=-5$

(d) $\frac{1-2x}{x-2}=\frac{1-4x}{2x+3}$

Question 14 Simplify the following expressions.

(a) $\frac{2x-3}{5}-\frac{3x+1}{10}$

(b) $\frac{1}{x-y}-\frac{2}{y-x}$

(c) $\frac{2}{x^2-4}+\frac{1}{x-2}$

(d) $\frac{x}{x-y}-\frac{y}{x^2-y^2}$

(e) $\frac{3}{x+1}-\frac{3}{x-1}$

(f) $\frac{2}{x^2-4}-\frac{3}{x^2+5x-14}$

(g) $x-y-\frac{x^2}{x+y}$

(h) $\frac{4}{x^2-4}-\frac{1}{x-2}+\frac{5}{x+2}$

(i) $\frac{1}{x+3}-\frac{x}{(x+3)^2}$

(j) $\frac{1}{(x+1)^2}-\frac{1}{x+1}$

(k) $\left(x^{-1}+y^{-1}\right)^{-1}$

(l) $(x+y)^{-1}\left(x^{-1}+y^{-1}\right)$

Question 15 Expand the following.

(a) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

(b) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$

Question 16 Let $a=2+\sqrt{3}$.

(a) Find the exact value of $a+\frac{1}{a}$.

(b) Hence, without calculating a^2, find the exact value of $a^2+\frac{1}{a^2}$.

Question 17 [Medley of algebraic techniques]

Simplify the following, if possible.

(a) $y\times y$

(b) $y+y^2$

(c) $y\times y^2$

(d) $(2y)^2$

(e) $2(x^2)^3$

(f) $(2x^2)^3$

(g) $a^m\times a^n$

(h) $(3a)^0$

(i) $\frac{x+y}{x}$

(j) $\frac{x+y}{xy}$

(k) $\frac{x+xy}{x}$

(l) $\frac{1}{2}(4x+6y)$

(m) $\frac{4x+6y}{2x}$

(n) $\frac{4(x+y)}{4x}$

(o) $\frac{6xy}{3+6y}$

(p) $\frac{4x+x}{4}$

(q) $\frac{\frac{1}{x}+\frac{1}{y}}{xy}$

(r) $\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$

(s) $\frac{2}{x+y}$

(t) $\frac{x-y}{y-x}$

Investigation Task

Continued Fractions

A *continued fraction* is a series of fractions inside fractions. This process is sometimes called *nesting* fractions.

For example, the following fraction is a continued fraction nested 3 times.

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}$$

The above can be short-hand denoted as

$$[1;2,2,2,2]$$

Question 1

(a) Write down the equivalent continued fraction for $[1;2,2]$ and $[1;2,2,2]$, and calculate their values correct to three decimal places where necessary.

(b) Complete the following table that shows the difference between the continued fractions, with various levels of nesting, against the value of $\sqrt{2}$.

Continued fraction	Value	$\sqrt{2}$	Difference
$[1;2]$		1.41421	
$[1;2,2]$		1.41421	
$[1;2,2,2]$		1.41421	
$[1;2,2,2,2]$		1.41421	
$[1;2,2,2,2,2]$		1.41421	

What do you notice about the difference?

(c) Hypothesise the value of the infinite continued fraction $[1;2,2,2,\ldots]$

Question 2 [Solving for the convergent value]

(a) Suppose that $[1; 2, 2, 2, \ldots]$ converges to X. In other words, let

$$X = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

Explain briefly why $X = 1 + \dfrac{1}{1+X}$.

Hint: Try to substitute X back into itself.

(b) Hence, solve for X and confirm your hypothesis from Question 1 (c).

(c) Bob guesses that $[1; 2, 2, 2, \ldots]$ converges (approaches) to $\sqrt{2}$ since there are lots of 2's. He then claims that $[1; 3, 3, 3, \ldots]$ must converge to $\sqrt{3}$ since there are lots of 3's. Determine if Bob's hypothesis is correct or incorrect. If incorrect, state the correct value of $[1; 3, 3, 3, \ldots]$

Question 3 [Another way of deriving the continued fraction]

(a) Show that

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{1 + \sqrt{2}}.$$

(b) Substitute this expression for $\sqrt{2}$ back into itself to show that

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}.$$

(c) Repeat this process a few more times. What familiar-looking continued fraction do you get?

(d) Investigate a similar process to obtain a continued fraction expression for $\sqrt{3}$.

(e) Prove that

$$\sqrt{n} = a + \frac{n - a^2}{a + \sqrt{n}}$$

and explain how this can be used to generate continued fractions.

2

FUNCTIONS AND RELATIONS

- Function definitions and notation
- Domain and range
- Even and odd functions
- Composite functions

Exercise 2A

Function definitions and notation

Fundamentals

Fundamentals 1

(a) A r________ between two sets is a collection of ordered pairs containing an element from each set.

(b) If the relation associates an element x to only one element ___, then the relation is called a f________.

(c) If an element x is associated with two or more elements y, then it is just called a r________.

(d) The 'vertical line test' determines whether a graph represents a f________ or a r________.

(e) If the vertical line intersects the curve at ______ or more locations, then the graph represents a r________.

(f) If the vertical line intersects the curve at ______ and only ______ location, then the graph represents a f________.

(g) The equation ________ is read as "f of a is equal to b".

(h) Piece-wise defined functions are functions that have different e________ in different d________.

Question 1 Determine whether the following represent functions or relations.

(a) $\{(2,2),\ (3,3),\ (4,6),\ (4,9)\}$

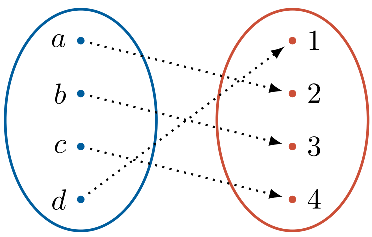
(b) $\{(-4,1),\ (3,1),\ (5,2),\ (7,-2)\}$

(c) $(x,3)$ for all $x \in \mathbb{R}$.

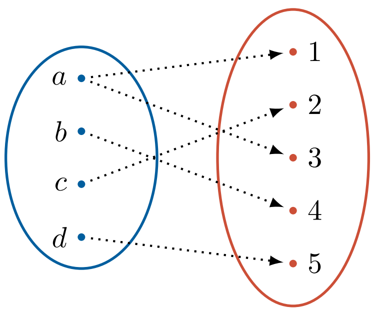
(d) $(3,y)$ for all $y \in \mathbb{R}$.

Question 2 Categorise the following as either one-to-one, one-to-many, many-to-one or many-to-many.

(a)



(b)



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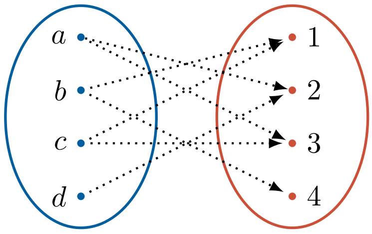
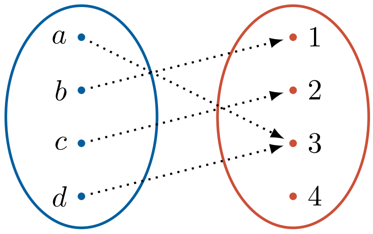
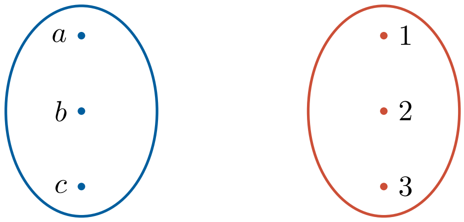


(d)

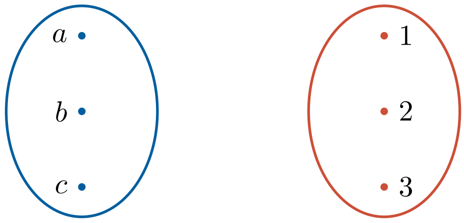


Question 3 Draw arrows in each of the following diagrams to indicate a relation being

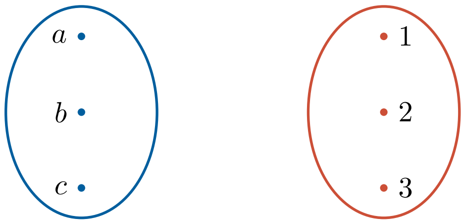
(a) one-to-one.



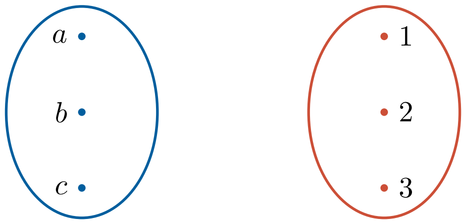
(b) one-to-many.



(c) many-to-one.

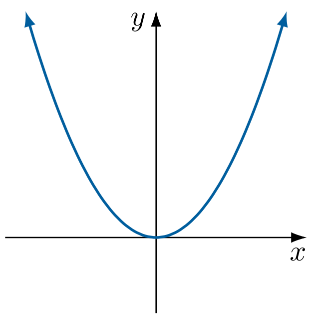


(d) many-to-many.

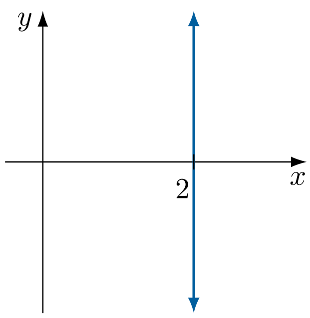


Question 4 Determine whether the following relations are also functions. Categorise each graph as either one-to-one, one-to-many, many-to-one or many-to-many.

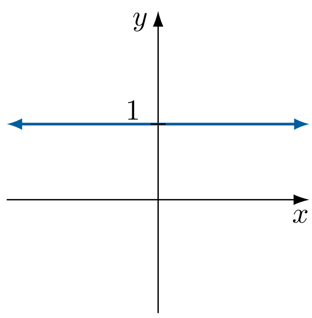
(a)



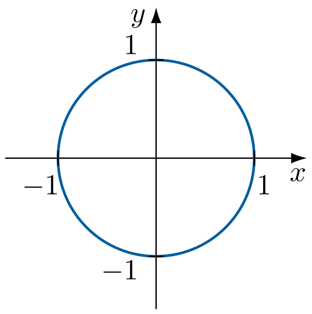
(b)



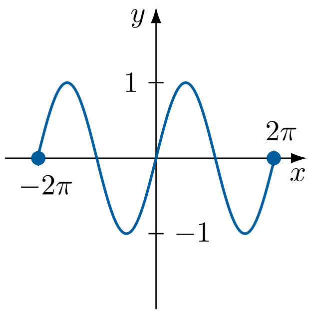
(c)



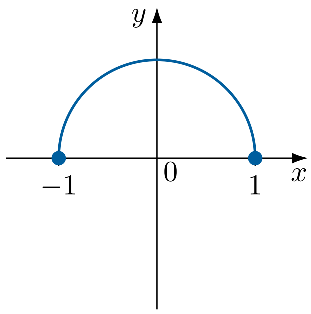
(d)



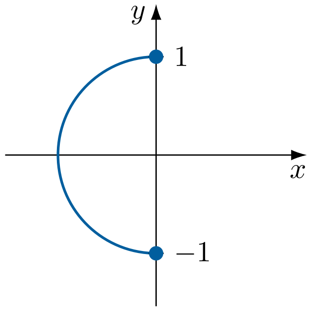
(e)



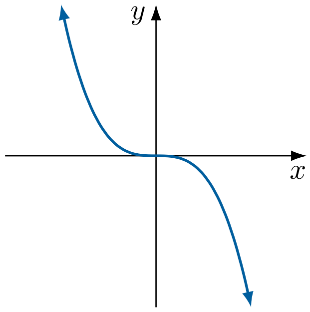
(f)



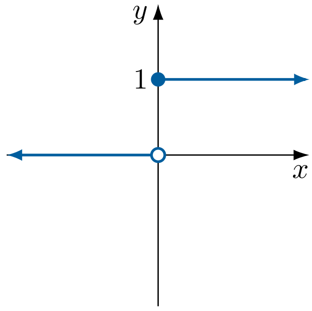
(g)



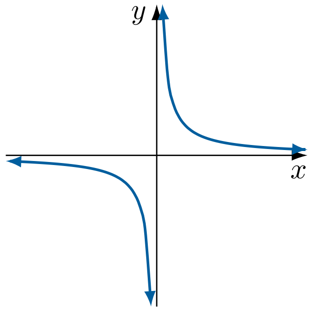
(h)



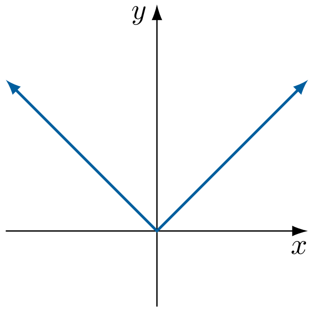
(i)



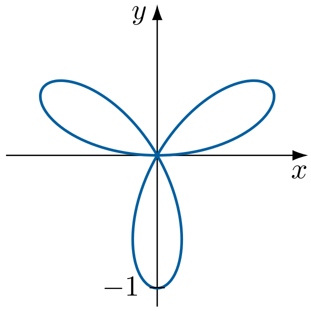
(j)



(k)



(l)



Question 5 For each of the following functions below, find $f(2)$

(a) $f(x) = 3x + 1$ (b) $f(x) = 1 - x^2$

(c) $f(x) = \sqrt{5x - 1}$ (d) $f(x) = \dfrac{x+1}{x-1}$

Question 6 Let $f(x) = 2x + 1$. Find the following.

(a) $f(-3)$ (b) $f(-a)$ (c) $f(a^2)$ (d) $f(2a)$ (e) $f(\sqrt{a})$

(f) $f(1+a)$ (g) $f(3a-2)$ (h) $f(a+b)$ (i) $f(ab)$ (j) $f\left(\dfrac{1}{a}\right)$

Question 7 Let $f(x) = 3x^2 - 2$. Find the following.

(a) $f(3t)$ (b) $f(-2t)$ (c) $f(\sqrt{t})$ (d) $f(t-1)$

Question 8 Let $f(x) = \dfrac{1}{x^2}$. Find the following.

(a) $f\left(\dfrac{1}{x}\right)$ (b) $f\left(\dfrac{1}{\sqrt{x}}\right)$ (c) $f(xy)$ (d) $f(x+y)$

Question 9 Consider the piece-wise defined function

$$f(x) = \begin{cases} x^2, & \text{for } x \le 1 \\ 2 - x, & \text{for } x > 1 \end{cases}$$

Find the following.

(a) $f(-2)$ (b) $f(1)$ (c) $f(3)$ (d) $f(-3)$

Question 10 Consider the piece-wise defined function

$$f(x) = \begin{cases} 2-x, & \text{for} \quad x < -2 \\ 4, & \text{for} \quad -2 \le x \le 2 \\ x+2, & \text{for} \quad x > 2 \end{cases}$$

Find the following.

(a) $f(0)$ (b) $f(5)$ (c) $f(-3)$ (d) $f(3) - f(-1)$

Question 11 Let $f(x) = x^2$. Determine whether the following statements are true or false.

(a) $f(4x) = 16f(x)$

(b) $f(xy) = f(x) \times f(y)$

(c) $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$

(d) $f(x^2) = (f(x))^2$

(e) $f(x+y) = f(x) + f(y)$

(f) $f(\sqrt{x+y}) = f(\sqrt{x}) + f(\sqrt{y})$

(g) $f(x+y) + f(x-y) = 2[f(x) + f(y)]$

(h) $f\left(x + \frac{1}{x}\right) = 2 + f(x) + f\left(\frac{1}{x}\right)$

Question 12 Find $f(k) - f(k-1)$ in simplest form if

(a) $f(x) = x^2$

(b) $f(x) = 3x^2 + 2x$

(c) $f(x) = \frac{1}{x}$

(d) $f(x) = 2^x$

Question 13 Let $f(x) = 4^x$. Show the following results.

(a) $f(2x) = 16^x$ (b) $4f(x) = f(x+1)$ (c) $f(x+1) - f(x) = 3 \times f(x)$

Question 14 Let $f(x) = \frac{2}{x}$. Simplify $f(x) \div f\left(\frac{1}{x}\right)$.

Question 15 [Recursions]

For the following functions, find $f(3)$.

(a) $f(n) = f(n-1) + 3$, where $f(1) = 3$

(b) $f(n) = n \times f(n-1)$, where $f(1) = 1$

Question 16 Let $f(x) = 2^x$. Show that $f(x) \times f(-x) = 1$

Question 17 Let $f(x) = x - x^3$. Show that $f(-x) = -f(x)$.

Question 18 Let $f(x) = \frac{x}{x+1}$. Show that $f(x) + f\left(\frac{1}{x}\right) = 1$.

Question 19 Let $f(x) = \frac{x}{x^2+1}$. Show that $f\left(\frac{1}{x}\right) = f(x)$.

Question 20 Let $f(x) = px^2 - x$. Find the value of p if $f(-3) = 2f(2)$.

Challenge Problems

Problem 1 Find the original function $f(x)$ if

(a) $f(x+3) = x+7$

(b) $f(x-2) = x^2-4$

(c) $f(x+1) = x^2-3x+4$

(d) $f(2x) = \dfrac{x}{3x-1}$

Problem 2 Let

$$f(x) = \frac{x}{x^2-1}.$$

Show that

$$f(x) + f\left(\frac{1}{x}\right) = 0,$$

where $x \neq 0$.

Problem 3 Define the functions

$$f(x) = 2^x + 2^{-x}$$
$$g(x) = 2^x - 2^{-x}$$

Show that

$$(f(x))^2 - (g(x))^2 = 4$$

Problem 4 [Self-inverse functions]

Let a and b be any non-zero real numbers, and define the function

$$f(x) = \frac{a-x}{1+bx}.$$

Show that

$$f\left(\frac{a-x}{1+bx}\right) = x.$$

Exercise 2B

Domain and range

Fundamentals

Fundamentals 1

(a) The interval ________ can be re-written as $x \in [a, b]$.

(b) The interval $a < x < b$ can be re-written as ______

(c) The interval ________ can be re-written as $x \in [a, b)$.

(d) The interval $a < x \leq b$ can be re-written as ______.

Fundamentals 2

(a) The domain of a function is the set of __-values that a function can accept.

(b) The r____ of a function is the set of y-coordinates that a function can output.

Fundamentals 3

(a) The domain of $f(x) = \dfrac{1}{x}$ is all real x except $x =$ __.

(b) The domain of $f(x) = \sqrt{x}$ is x____.

(c) The domain of $f(x) = \dfrac{1}{\sqrt{x}}$ is x____, noting that x cannot be z____.

Fundamentals 4

(a) The range of $y = x^2$ is y ____ since a perfect square cannot be n______.

(b) The range of $y = \sqrt{x}$ is y ____ since square rooting a number cannot result in a n______ number.

(c) In general, the r______ of a function can be found by examining the behaviour of the outputs in response to various inputs.

Question 1 Convert each of the following to interval notation.

(a) $1 \leq x \leq 3$

(b) $-2 < x \leq 5$

(c) $-7 \leq x < -3$

(d) $x \geq 4$

(e) $x > -2$

(f) $x \leq 3$

Question 2 Re-write the following intervals using inequalities.

(a) $x \in (2, 6)$ (b) $x \in [-5, 10]$ (c) $x \in (1, 4]$

(d) $x \in [-2, 3)$ (e) $x \in (-\infty, 3]$ (f) $x \in (-2, \infty)$

Question 3 Convert each of the following to interval notation.

(a)


(b)

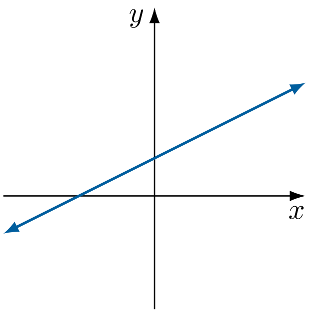

(c)


(d)


(e)


(f)


Question 4 State the domain and range of the following graphs.

(a)


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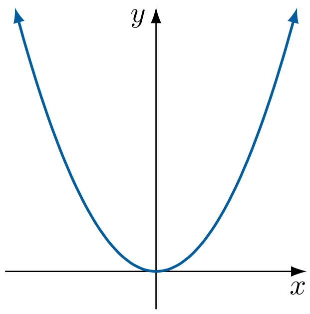

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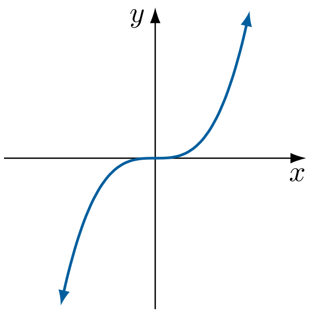

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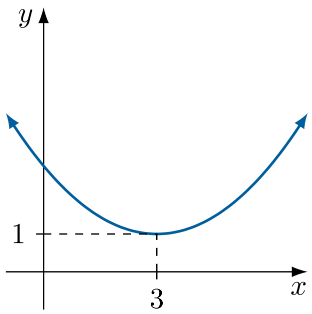

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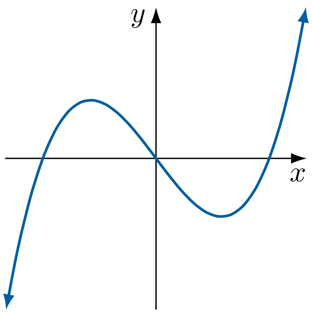

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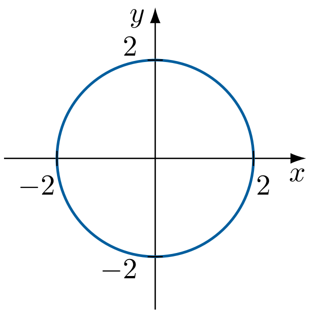

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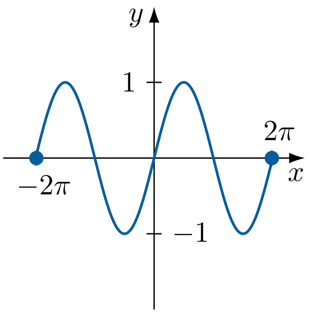

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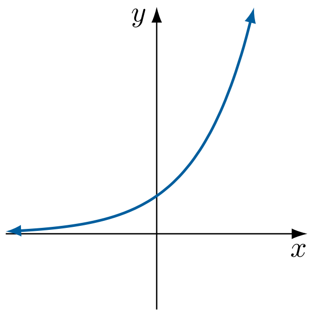

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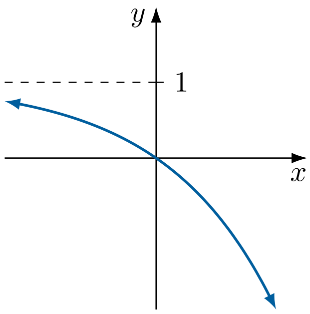

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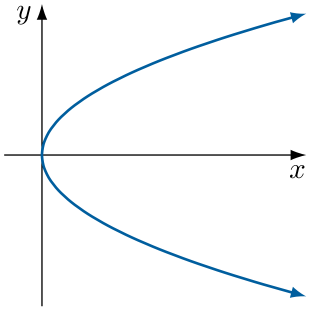

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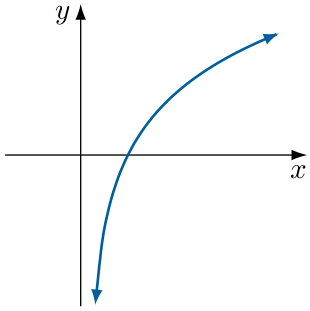

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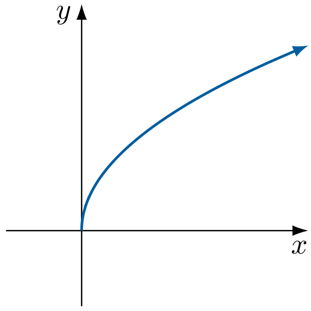
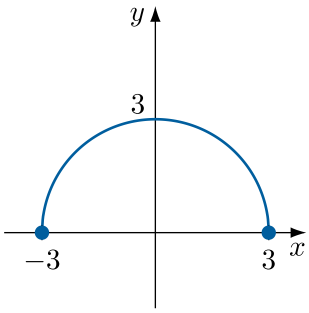


(m)



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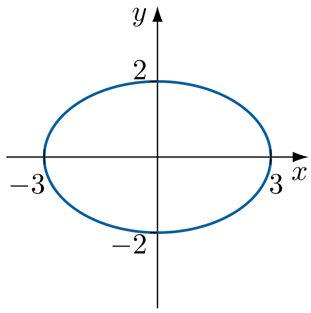

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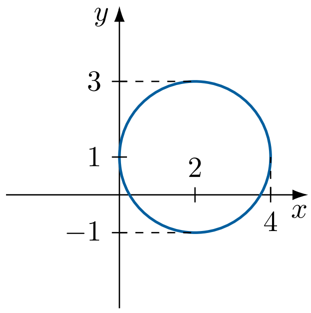
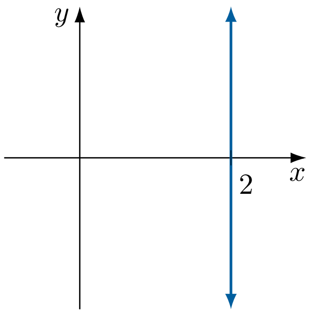


Question 5 State the domain and range of the following graphs.

(a)



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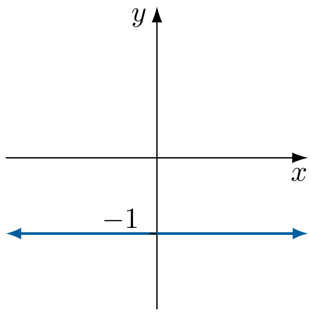

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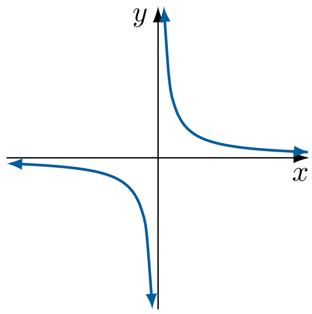

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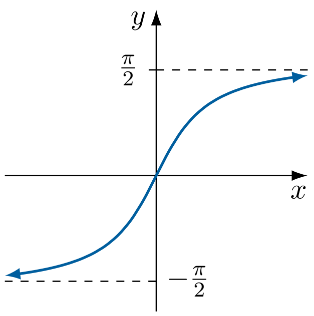

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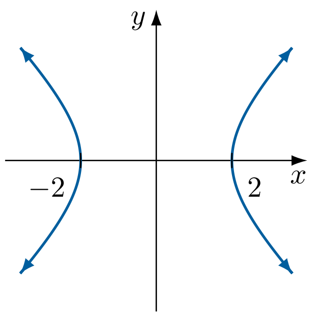

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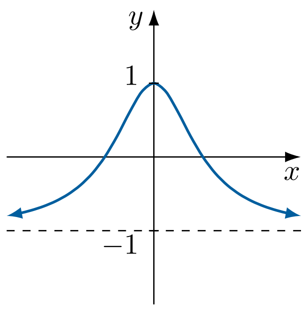


Question 6

(a) We can find the domain of functions with x's inside a square root by letting the inside be p______.

(b) Hence, find the domain of the following functions.

(i) $f(x) = \sqrt{x}$

(ii) $f(x) = \sqrt{x-2}$

(iii) $f(x) = \sqrt{5-x}$

(iv) $f(x) = \sqrt{7-2x}$

Question 7

(a) We can find the domain of functions with x's in the denominator by letting the denominator equal z___ and then excluding these values.

(b) Hence, find the domain of the following functions.

(i) $f(x) = \dfrac{1}{x}$

(ii) $f(x) = \dfrac{1}{x-1}$

(iii) $f(x) = \dfrac{1}{2x-3}$

(iv) $f(x) = \dfrac{x}{x+2}$

(v) $f(x) = \dfrac{2}{x-7} + 3$

(vi) $f(x) = \dfrac{1}{(x-3)^2}$

Question 8 Use a combination of techniques from Question 6 and Question 7 to find the domain of the following functions.

(a) $f(x) = \dfrac{1}{\sqrt{x}}$

(b) $f(x) = \dfrac{x+1}{\sqrt{x+4}}$

(c) $f(x) = \dfrac{2}{\sqrt{3x-9}}$

(d) $f(x) = \dfrac{3}{\sqrt{4-2x}}$

Question 9

(a) Find the solutions to the equation $(x-1)(x-2) = 0$.

(b) Hence, find the domain of $f(x) = \dfrac{1}{(x-1)(x-2)}$.

(c) Use a similar technique, find the domain of the following functions.

(i) $f(x) = \dfrac{1}{(x+1)(x-3)}$

(ii) $f(x) = \dfrac{1}{(2x-1)(3x-1)}$

(iii) $f(x) = \dfrac{x}{x^2-4}$

(iv) $f(x) = \dfrac{x^2-1}{x^2-4x+3}$

Question 10 Use graphing software to obtain the graph of the following, and hence state the domain and range.

(a) $f(x) = 2^x + 2^{-x}$

(b) $f(x) = \dfrac{1}{1+x^2}$

(c) $f(x) = x + \dfrac{1}{x}$

(d) $f(x) = \dfrac{1}{x^2-1}$

Question 11

(a) State the coordinates of the vertex of the equation $f(x) = (x-2)^2 + 1$.

(b) Sketch a graph of the parabola, labelling the vertex and the y-intercept.

(c) Hence, state the range of $f(x) = (x-2)^2 + 1$.

(d) Similarly, find the range of the following functions.

(i) $f(x) = (x+1)^2 - 4$

(ii) $f(x) = (x-2)^2 + 3$

(iii) $f(x) = x^2 - 4x$

(iv) $f(x) = x^2 + 6x + 11$

(e) In all of the above, you had to complete the square first and obtain the form $f(x) = (x-h)^2 + k$. What did you notice about the value of k and the range of the function?

Question 12 Recall that the range of $f(x) = x^2$ is $y \geq 0$ because x^2 cannot be negative. Use this fact to determine the range of the following.

(a) $f(x) = 4x^2$

(b) $f(x) = x^2 + 3$

(c) $f(x) = 3x^2 - 4$

(d) $f(x) = -x^2$

(e) $f(x) = -x^2 + 6$

(f) $f(x) = 5 - x^2$

(g) $f(x) = 3 - 4x^2$

(h) $f(x) = (x-3)^2 + 2$

(i) $f(x) = 10 - (x+2)^2$

Question 13 Recall that the range of $f(x) = \sqrt{x}$ is $y \geq 0$ because $\sqrt{x}$ cannot be negative. Use this fact to determine the range of the following.

(a) $f(x) = 2\sqrt{x}$

(b) $f(x) = \sqrt{x} + 3$

(c) $f(x) = \sqrt{x} - 5$

(d) $f(x) = -\sqrt{x}$

(e) $f(x) = -\sqrt{x} + 4$

(f) $f(x) = 3 - \sqrt{x}$

Question 14

(a) Determine the range of $y = 9 - x^2$ and hence find the maximum value of y.

(b) Hence, find the range of $f(x) = \sqrt{9 - x^2}$.

(c) Use a similar technique to find the range of $g(x) = \sqrt{9 + x^2}$.

Challenge Problems

Problem 1 Find the domain of

$$f(x) = \frac{\sqrt{2-x} + \sqrt{2+x}}{4-x^2}$$

Problem 2

(a) Use graphing software to sketch the graph of $f(x) = \dfrac{x}{x^2-1}$.

(b) Hence, state the range of $f(x)$.

Problem 3 Explain why the domain of $f(x) = \dfrac{x}{x}$ is $x \in \mathbb{R}$, $x \neq 0$

Problem 4 Find the domain of the following functions.

(a) $f(x) = \dfrac{x+1}{x+1}$

(b) $f(x) = \dfrac{x+1}{x^2-1}$

(c) $f(x) = \dfrac{x-2}{x^2-5x+6}$

(d) $f(x) = \dfrac{x^2}{\sqrt{x^2+1}}$

Problem 5 Find the domain and range of the function

$$f(x) = \sqrt{x-1} + \sqrt{1-x}$$

Hint: A graph is not needed for this question.

Problem 6 Find the domain and range of the following functions by thinking about the possible inputs and the resultant outputs.

(a) $y = \dfrac{1}{\sqrt{x}}$

(b) $y = \dfrac{1}{x^2}$

(c) $y = \sqrt{x^2-4}$

(d) $y = \dfrac{1}{\sqrt{x^2+4}}$

Exercise 2C

Even and odd functions

Fundamentals

Fundamentals 1

(a) If a function is even, then $f(-x) =$ ______.

(b) If a function is odd, then $f(-x) =$ ______.

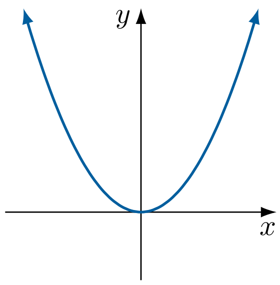
Fundamentals 2

(a) Even functions are symmetric across the ______-axis.

(b) Odd functions have rotational symmetry about the o________.

Question 1 Classify the following graphs as either even, odd or neither.

(a)



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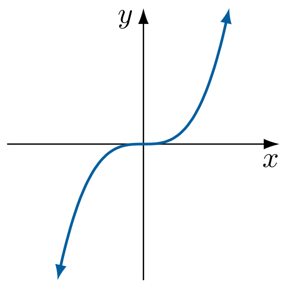

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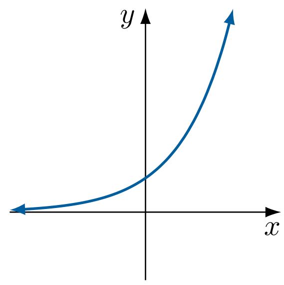

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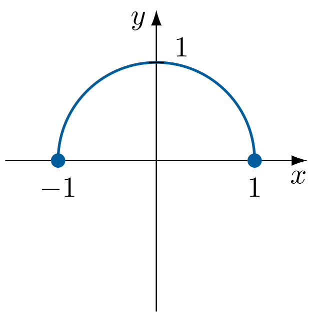

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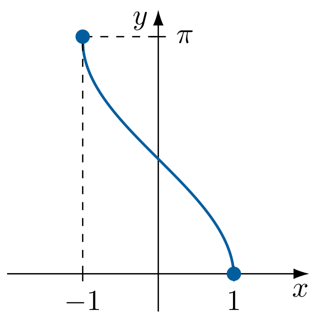

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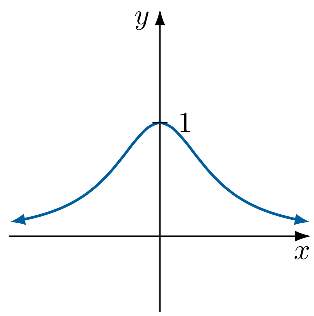


Question 2 For each of the following functions, find $f(-x)$ and hence determine whether they are even, odd or neither.

(a) $f(x) = 2x$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^2 + x$

(d) $f(x) = x^3$

(e) $f(x) = x - x^3$

(f) $f(x) = \dfrac{1}{x}$

(g) $f(x) = 2^x$

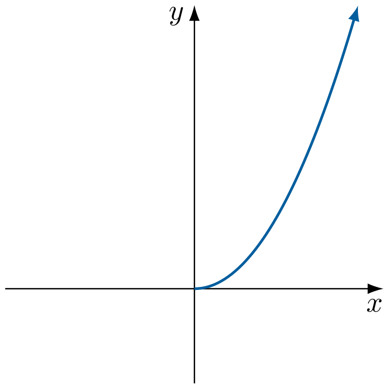
(h) $f(x) = x^4 - x^2 + 1$

(i) $f(x) = \sqrt{1 - x^2}$

(j) $f(x) = \dfrac{x}{x^2+1}$ (k) $f(x) = x + \dfrac{1}{x}$ (l) $f(x) = \dfrac{x-1}{x+1}$

Question 3 Complete the following diagrams to make the graphs represent even functions.

(a)



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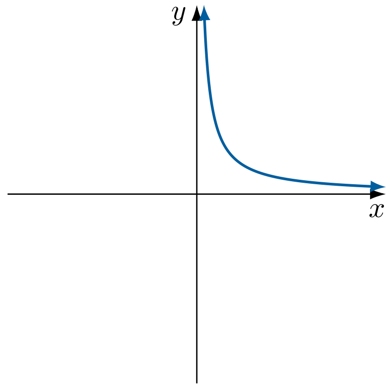
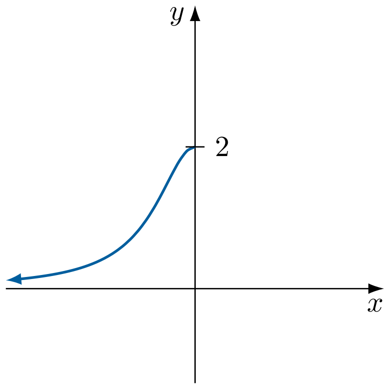
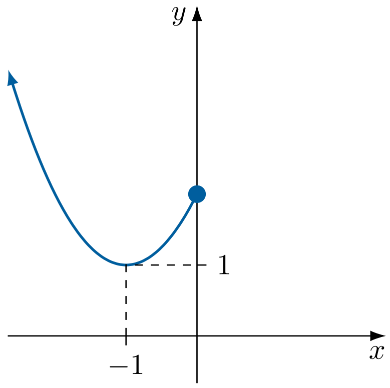


(c)

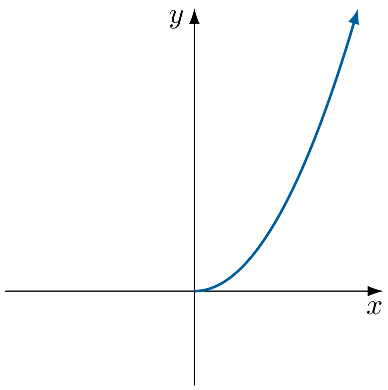


(d)



Question 4 Complete the following diagrams to make the graphs represent odd functions.

(a)



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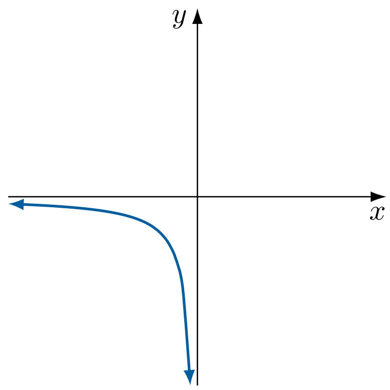

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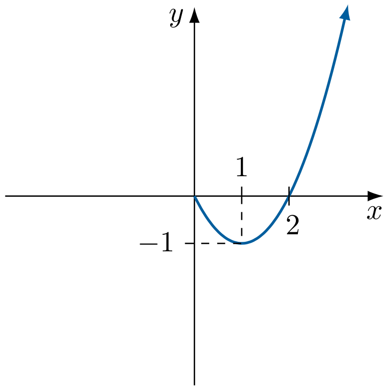
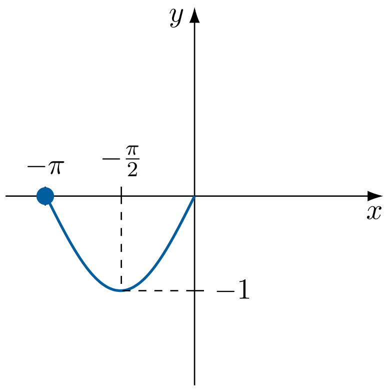


(d)



Challenge Problems

Problem 1 Show that $f(x) = \dfrac{2^{2x}+1}{2^x}$ is an even function.

Problem 2 Let $f(x)$ and $g(x)$ both be even functions. Define the function

$$h(x) = f(x) + g(x).$$

(a) Show that $h(-x) = f(x) + g(x)$.

(b) Hence, what kind of function is $h(x)$?

(c) Complete the following statement, that you just proved.

The sum of two even functions is ________.

Problem 3 By defining a function

$$h(x) = f(x) \times g(x)$$

or

$$h(x) = f(x) + g(x)$$

and then simplifying $h(-x)$, prove the following statements.

(a) The sum of two odd functions is odd.

(b) The product of two even functions is even.

(c) The product of two odd functions is even.

Problem 4 Determine whether the following statements are true or false.

(a) The difference of two even functions is neither.

(b) The difference of two odd functions is odd.

(c) The quotient of an even and odd function is even.

Exercise 2D

Composite functions

Fundamentals

Fundamentals 1

In a composite function, the output of the 'inside' function becomes the i________ of the 'o________' function.

Fundamentals 2

(a) When calculating $f(g(x))$, the function ______ is applied first to x, and then the function ______ is applied afterwards to the new expression.

(b) When calculating $g(f(x))$, the function ______ is applied first to x, and then the function ______ is applied afterwards to the new expression.

Question 1 Let $f(x) = 3x + 4$ and $g(x) = 3 - 2x$.

(a) Find $f(g(2))$.

(b) Find $g(f(2))$.

(c) Show that $f(g(x)) = -6x + 13$.

(d) Show that $g(f(x)) = -6x - 5$.

Question 2 Let $f(x) = \sqrt{x}$ and $g(x) = x + 4$.

(a) Find $f(g(x))$

(b) Find $g(f(x))$

(c) Find $g(g(x))$

(d) What combination of compositions will produce $\sqrt{x+8}$?

Question 3 For the following pairs of functions $f(x)$ and $g(x)$, find and simplify $f(g(x))$ and $g(f(x))$.

(a) $f(x) = x - 2, \quad g(x) = 2x + 5$

(b) $f(x) = 2x, \quad g(x) = 3x$

(c) $f(x) = x^2, \quad g(x) = x^3$

(d) $f(x) = \frac{1}{x}, \quad g(x) = \sqrt{x}$

Question 4 Let $f(x) = \frac{x}{x+1}$. Find $f(f(x))$.

Question 5 A function $f(x)$ is said to be an *self-inverse* if $f\left(f(x)\right) = x$. Determine which of the following are self-inverse functions.

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \dfrac{x}{x-1}$

(c) $f(x) = \dfrac{x-1}{x}$

(d) $f(x) = \dfrac{x+1}{x}$

Question 6 Let $f(x) = \dfrac{2}{x-9}$ and $g(x) = x^2$.

(a) Find $f\left(g(x)\right)$.

(b) Find $g\left(f(x)\right)$.

(c) Find domains of $f\left(g(x)\right)$ and $g\left(f(x)\right)$. Are they the same?

Question 7 [Commutativity under composition]

Two functions $f(x)$ and $g(x)$ are said to *commute under composition* if $f\left(g(x)\right) = g\left(f(x)\right)$. Determine which of the following pairs of functions commute under composition.

(a) $f(x) = x - 2, \quad g(x) = x + 5$

(b) $f(x) = ax, \quad g(x) = bx$

(c) $f(x) = 3x + 1, \quad g(x) = 2x - 1$

(d) $f(x) = x^m, \quad g(x) = x^n$

Question 8 Let $f(x) = x - 2$ and $g(x) = \sqrt{x}$.

(a) Find $f(g(x))$.

(b) Hence, state the domain of $f(g(x))$.

(c) Find the domain of $g(f(x))$.

Question 9 Let $f(x) = 2x - 3$, $g(x) = \sqrt{x}$ and $h(x) = x^2 + 1$.

Simplify the following, and hence state the domain.

(a) $f(g(x))$

(b) $g(f(x))$

(c) $h(f(x))$

(d) $g(h(x))$

Question 10 Let $f(x) = \dfrac{1}{x}$. For simplicity, denote $f\left(f(x)\right) = f_2(x)$ and so on.

(a) Simplify $f_2(x)$.

(b) Simplify $f_3(x)$.

(c) Hypothesise a formula for $f_n(x)$, where n is a positive integer.
Hint: Express your answer as a piecewise-defined function.

Challenge Problems

Problem 1 Let $g(x)$ be an even function and $f(x)$ be any function. Define the composition

$$h(x) = f\left(g(x)\right).$$

(a) Show that $h(-x) = f\left(g(x)\right)$.

(b) What can we say about the function $h(x)$?

(c) Hence, what can we say about the composition of two functions if the 'inside' function is even?

Problem 2 By defining

$$h(x) = f\left(g(x)\right)$$

and using a similar technique, prove the following statements.

(a) The composition of two even functions is even.

(b) The composition of two odd functions is odd.

(c) The composition of an even and odd function is even.
Hint: You will have to consider two different cases here.

Problem 3 Let $f(x) = \sqrt{x}$ and $g(x) = -x^2 - 1$.

(a) Simplify $g(f(x))$.

(b) Simplify $f(g(x))$.

(c) One of the above does not work. Find out which one, and explain why it does not work.

Problem 4 Let a and b be any non-zero real numbers.
Show that the function

$$f(x) = \frac{a - x}{1 + bx}$$

satisfies the equation $f(f(x)) = x$.

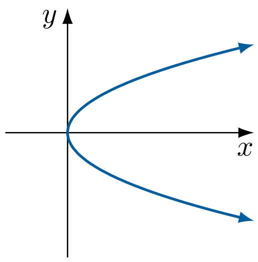
Chapter 2 Review

Functions and Relations

Review

Question 1 Categorise the following as either one-to-one, one-to-many, many-to-one, or many-to-many.

(a)



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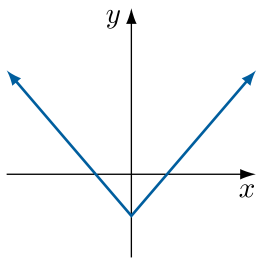

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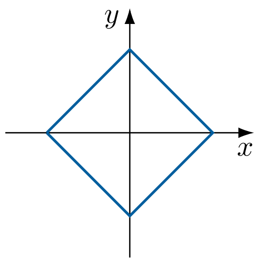

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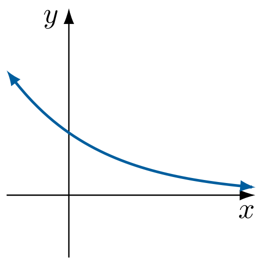
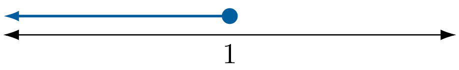


Question 2 Which of the graphs in the above questions represent functions?

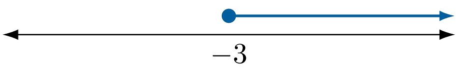
Question 3 Convert each of the following to interval notation.

(a)



(b)

(c)



Question 4 For each of the following, find the value of a if

(a) $f(x) = 3ax + 1$; $f(-2) = 7$

(b) $f(x) = ax^2 - 3$; $f(-2) = 2f(1)$

Question 5 Re-write the following intervals using inequalities.

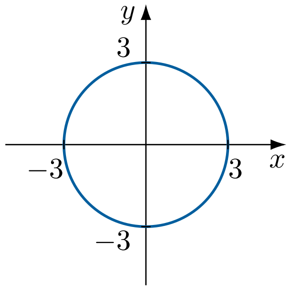
(a) $x \in [-2, \infty)$

(b) $x \in (-1, 4]$

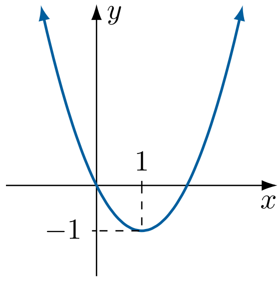
(c) $x \in (-\infty, 6]$

Question 6 For each of the following graphs, state if it is a function, write down the domain and write down the range.

(a)



(b)



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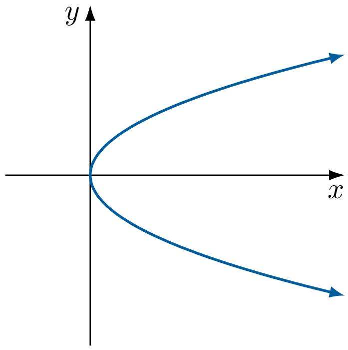

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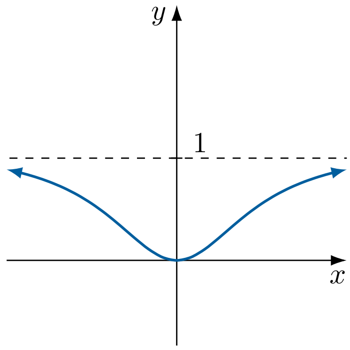

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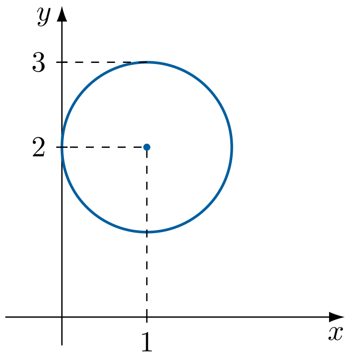
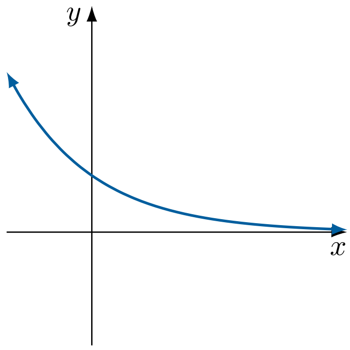


(f)



Question 7 Let $f(x) = x^2 + 4$ and $g(x) = x - x^3$. Define the function $h(x) = f(x)g(x)$.

(a) Show that $h(-x) = -h(x)$.

(b) Note that $f(x)$ and $g(x)$ are even and odd functions respectively. What do you hypothesise is true about the product of an even and odd function?

(c) Prove your guess from part (b).

Question 8 Let $f(x) = 3^x$. Find $f(x+1) - f(x)$ and express your answer in terms of $f(x)$.

Question 9 [Factorials]

Let $f(n) = 1 \times 2 \times 3 \times \ldots \times n$. Simplify $\dfrac{f(n)}{f(n-1)}$.

Question 10 Define the following piece-wise defined function.

$$f(x)=\begin{cases} x+3 & \text{for} \quad x<-1 \\ x^2 & \text{for} \quad -1\le x\le 1 \\ \sqrt{x} & \text{for} \quad x>1 \end{cases}$$

Evaluate the value of

$$3f(4)+f(-2)-f\left(-\frac{1}{2}\right)$$

Question 11 Define the following piece-wise defined function.

$$f(x)=\begin{cases} c(x-2)^2 & \text{for} \quad x<0 \\ ax+4 & \text{for} \quad 0\le x\le 2 \\ bx-4 & \text{for} \quad x>2 \end{cases}$$

Find the values of a, b and c if $f(3)=5$, $f(1)=3$ and $f(-2)=8$.

Question 12 Let $f(x)=\dfrac{3^x+3^{-x}}{2}$. Show that $f(2x)=2\left[f(x)\right]^2-1$.

Question 13 Let $f(x)=\dfrac{x^2}{x-1}$. Show that $f\left(\dfrac{x}{x-1}\right)=f(x)$.

Question 14 Let $f(x)=x+\dfrac{1}{x}$. Find and simplify

(a) $f\left(\dfrac{1}{x}\right)$ (b) $f\left(x+\dfrac{1}{x}\right)$ (c) $f\left(2+\sqrt{3}\right)$

Question 15 State the domain of the following.

(a) $f(x)=\sqrt{x+3}$ (b) $f(x)=\dfrac{1}{\sqrt{x+3}}$ (c) $f(x)=\dfrac{1}{x^2+9}$

(d) $f(x)=\dfrac{1}{x^2-9}$ (e) $f(x)=\dfrac{1}{x^2-5x+6}$ (f) $f(x)=\sqrt{6-3x}$

(g) $f(x)=\dfrac{x^3}{x^2-1}$ (h) $f(x)=\dfrac{1}{\sqrt{6-x}}$ (i) $f(x)=\dfrac{x}{x^3+x}$

Question 16 Categorise the following functions as either even, odd, or neither, and provide reasoning for each answer.

(a) $f(x)=x^3-2x$ (b) $f(x)=x^4+x^3$ (c) $f(x)=\dfrac{2x^2-1}{3-x^2}$

(d) $f(x)=\dfrac{2^x-2^{-x}}{x}$ (e) $f(x)=\dfrac{x^3}{x+1}$ (f) $f(x)=\dfrac{x}{x^3-x}$

(g) $f(x)=\dfrac{x}{x^2-5}$ (h) $f(x)=x^3-x+1$ (i) $f(x)=\dfrac{3^{2x}+1}{3^x}$

Question 17

(a) Can a quadratic function be an odd function? Explain your answer.

(b) Can a quadratic function be an even function? Explain your answer.

(c) If a quadratic function is an even function, then what is the coefficient of x?

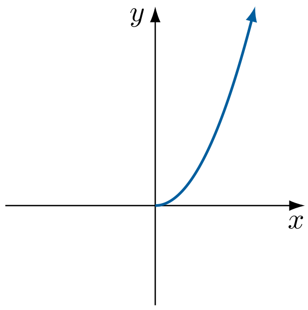
(d) Hence, or otherwise, find the value of p so that $f(x) = (x-1)^2 + px$ is an even function.

Question 18 Find the domain of $f(x) = \dfrac{\sqrt{5-x}}{x+1}$.

Question 19 If $f(x) = ax^2 + bx + c$, then simplify $f(x) - f(-x)$.

Question 20 Complete the graphs below so that they represent even and odd functions. Draw your graphs on separate axes.

(a)



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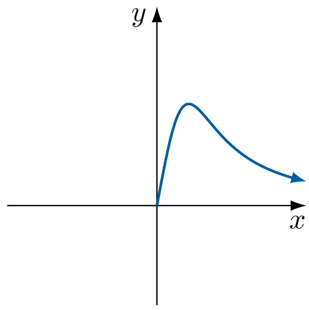
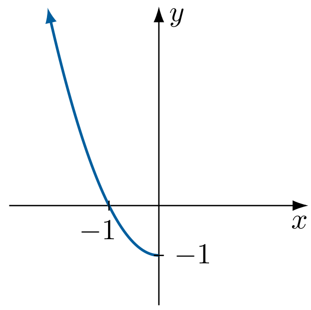


(c)



Question 21 Define the functions $f(x) = x^2 - 1$, $g(x) = 3x - 2$ and $h(x) = \dfrac{1}{x}$. Find the following.

(a) $g(-3a)$

(b) $f(3-a)$

(c) $f(g(1))$

(d) $g(f(0))$

(e) $f(h(2))$

(f) $f(g(x))$

(g) $g(g(x))$

(h) $f(f(x))$

(i) $h(f(x))$

Question 22 Find $f(x)$ if

(a) $f(x+2) = x + 5$

(b) $f(x-1) = x^2 - 1$

(c) $f(2x) = 4x^2 - 4x + 1$

Investigation Task

Group Theory

Define the following set G of functions over the domain $x \in \mathbb{R}$, $x \neq 0, 1$.

$$a(x) = x, \quad b(x) = 1 - x, \quad c(x) = \frac{x-1}{x}, \quad d(x) = \frac{1}{x}, \quad e(x) = \frac{1}{1-x}, \quad f(x) = \frac{x}{x-1}$$

Question 1

(a) Find $c\,(f(x))$ and show that it becomes another element of G. Which function does it become?

(b) Calculate the composition the other way around. In other words, calculate $f\,(c(x))$. Which element of G does this become? Is it the same one?

Question 2 A set of functions like the above is said to form a *group* under composition, if the composition of any two functions always yields another function in the same set. Show that G forms a group.
Hint: There should be 36 ways to do the compositions in total, but you will find that many of them are immediately obvious. Don't forget about functions being composed with themselves.

Investigation Task

Odd and Even Functions

So far, we have covered functions that are themselves odd or even functions. We also covered functions that were neither. However, it turns out that actually many 'neither' functions have a close relationship with odd and even functions. This investigation task aims to show what this relationship is.

Let $f(x)$ be any function with domain being all real x. Furthermore, define the following functions.

$$E(x) = \frac{f(x) + f(-x)}{2}$$
$$O(x) = \frac{f(x) - f(-x)}{2}$$

Question 1

(a) Show that $E(x)$ is an even function and that $O(x)$ is an odd function.

(b) Show that $f(x) = E(x) + O(x)$.

(c) What is the significance of this result?

Question 2 Let $f(x) = 2^x$.

(a) Show that $f(x)$ is neither even nor odd.

(b) Express $f(x)$ as the sum of an even and odd function.

(c) Use graphing software to sketch $E(x)$ and $O(x)$ and verify graphically that they are even and odd respectively.

(d) Ask a friend to provide you with some new functions $f(x)$ that are neither nor odd. Repeat part (b) for these new functions.

Question 3 What happens if $f(x)$ itself is already even or odd? Experiment with a few even and odd functions and comment on your findings. In hindsight, were these findings obvious?

Question 4 Earlier in this investigation task, $f(x)$ was defined to be a function with domain being all real x. Investigate to find out why this is so important and give a few examples demonstrating what can go wrong if the domain is *not* all real x.

3

LINEAR AND QUADRATIC FUNCTIONS

- Linear functions
- Features of a parabola
- Solving quadratic equations
- Discriminant
- Simultaneous equations and applications

Exercise 3A

Linear functions

Fundamentals

Fundamentals 1

(a) The gradient of a straight line is a measure of how s________ or shallow a line is. It also determines the d__________ of the straight line.

(b) A line with a larger gradient will be s__________ than a line with a smaller gradient.

(c) If the gradient is p__________, then the straight line will go up from left-to-right.

(d) If the gradient is negative, then the straight line will go d_____ from left-to-right.

(e) If two lines with gradients m_1 and m_2 have the same gradient, then _____ = _____.

(f) If two lines with gradients m_1 and m_2 are perpendicular, then ________.

Fundamentals 2

(a) The formula for the gradient is

$$m = \frac{\text{change in } y}{\text{change in } x} = \underline{\hspace{2cm}}$$

(b) The gradient-intercept form for a straight line is $y =$ __________.

(c) The point-gradient form for a straight line is $y - y_1 =$ __________.

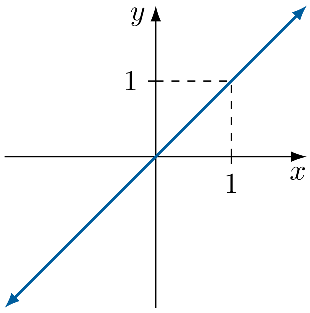
Fundamentals 3

(a) The equation of a line parallel to the x-axis is ________.

(b) The equation of a line parallel to the y-axis is ________.

Question 1 Find the gradient of the following linear functions.

(a)



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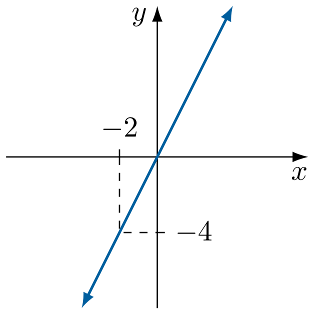

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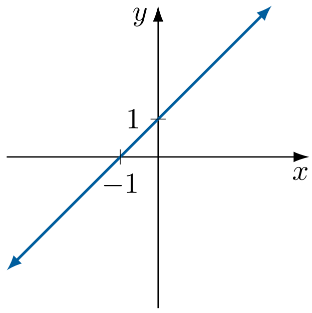

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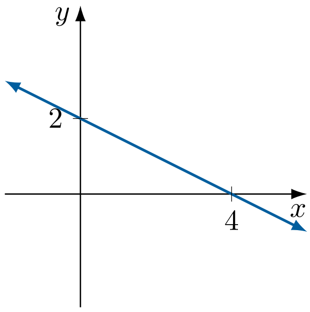

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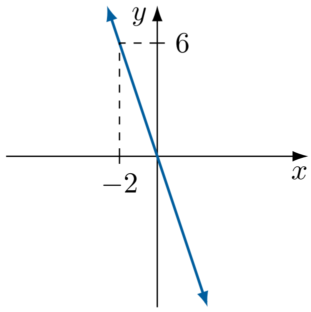

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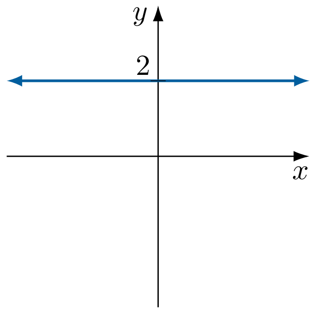

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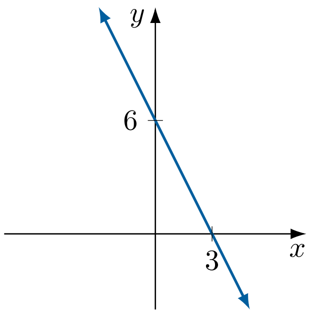

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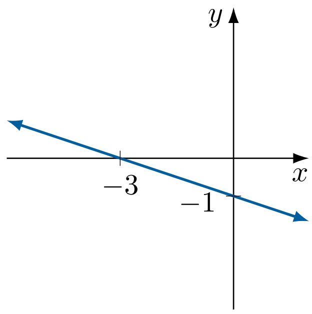
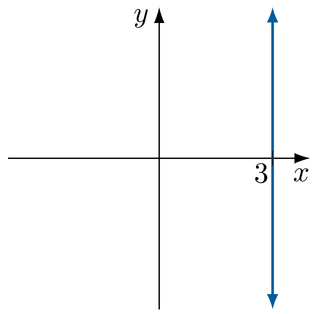


(i)



Question 2 Consider the linear function $\ell: \ y = 2x + 4$.

(a) State the gradient of ℓ.

(b) State the y-intercept of ℓ.

(c) Find the x-intercept of ℓ.

(d) Plot the x and y-intercepts on a set of axes, and hence sketch the graph of ℓ.

Question 3 Use a similar technique to Question 2 to sketch the following linear functions.

(a) $y = 2x + 1$

(b) $y = 0.5x - 1$

(c) $y = 2 - 4x$

(d) $y = -2x - 3$

(e) $3x + 2y - 1 = 0$

(f) $5x - 4y + 3 = 0$

Question 4 For parts (b) to (f) inclusive, express your answers in the form $y = mx + c$.
Find the equation of the straight line with

(a) gradient m that passes through (x_1, y_1).

(b) gradient 2 that passes through $(1, 4)$.

(c) gradient -3 that passes through $(2, -5)$.

(d) gradient $-\frac{3}{2}$ that passes through $(-4, 3)$.

(e) gradient $\frac{2}{3}$ that has an x-intercept of -2.

(f) gradient $\frac{5}{2}$ that has a y-intercept of 1.

Question 5 Draw a graph of the straight line that satisfies the following features. Label any intersections with the coordinate axes where possible.

(a) Gradient 3 with x-intercept at $x = 6$.

(b) Gradient -2 with y-intercept at $y = 2$.

(c) Gradient 2 and passing through the point $(1, 4)$.

(d) Gradient $\frac{1}{2}$ and passing through the point $(-4, 1)$

Question 6 Find the gradient of the straight line that passes through the following pairs of points.

(a) $(1, 2)$, $(0, 4)$

(b) $(-4, 1)$, $(-2, 3)$

(c) $(-7, -3)$, $(1, -4)$

(d) $(2, -6)$, $(3, -4)$

(e) $(5, 2)$, $(-3, 2)$

(f) $(-2, 4)$, $(-2, -3)$

Question 7 Find the equation of the straight line that passes through each of the two points in the previous question.

Question 8 Find the equation of the straight line that passes through $(-2, 4)$ and is parallel to the line $y = 4x - 7$.

Question 9

(a) What is the gradient of the line perpendicular to $y = 3x - 4$?

(b) Hence, find the equation of the line that passes through $(4, 6)$ and is perpendicular to $y = 3x - 4$.

Question 10 A line with gradient 4 passes through $(1, 5)$ and $(-2, k)$. Find the value of k.

Question 11

(a) State the equation of the line that is parallel to the x-axis and passes through $(-2, 6)$.

(b) State the equation of the line that is parallel to the y-axis and passes through $(3, 5)$.

Question 12 Show that the points $A(-2, 10)$, $B(1, 1)$ and $C(3, -5)$ lie on the same line using two different methods.
Hint: One method requires finding the equation of a straight line, but the other method does not.

Question 13 Determine whether the following points lie on the given lines.

(a) $y = 3x + 4$, $(-2, 1)$

(b) $y = -4x + 2$, $(3, -8)$

(c) $2x - 7y - 5 = 0$, $(6, 1)$

(d) $\dfrac{x}{3} - \dfrac{y}{5} = 1$, $(-3, -10)$

Question 14 Show that $A(3, 4)$, $B(5, 10)$ and $C(-6, 7)$ form a right-angled triangle.
Hint: Calculate the gradient of the intervals joining the points.

Question 15

(a) Find the value of a if $2x + 5y - 6 = 0$ passes through $(a, -2)$.

(b) Find the value of a if $ax - 8y + 3 = 0$ passes through $(-7, 3)$.

Question 16 Match every line to another parallel line from the list below.

(I): $y = 4x - 9$ **(II)**: $2x + 3y - 4 = 0$

(III): $\dfrac{x}{4} - \dfrac{y}{16} = 1$ **(IV)**: $y = -\dfrac{2}{3}x + 1$

Question 17 Define the points $A(-5, -2)$, $B(-1, -5)$, $C(5, 3)$ and $D(1, 6)$.

(a) Plot the points on a set of axes.

(b) Show that AB is parallel to CD.

(c) Show that BC is parallel to AD.

(d) Based on the above information alone, what kind of quadrilateral can $ABCD$ be?

(e) Show that $AB \perp BC$. Hence, what type of quadrilateral must $ABCD$ be?

Question 18 Determine whether the points $A(-6, -3)$, $B(3, -8)$, $C(8, -1)$ and $D(-2, 5)$ form a rectangle or not.

Question 19 The quadrilateral $PQRS$ has vertices $P(1, 0)$, $Q(3, 1)$, $R(4, 3)$ and $S(2, 2)$.

(a) Show that $PQRS$ is a parallelogram.

(b) Show that PR is perpendicular to QS.

(c) Hence, what type of quadrilateral does $PQRS$ form?

Question 20 The line $ax + by + 1 = 0$ is perpendicular to $4x + 3y = 7$ and passes through $(9, 7)$. Find the values of a and b.

Challenge Problems

Problem 1 Find the value(s) of a if $ax + y = 6$ passes through $(a, 5a)$.

Problem 2 Consider the points $A(-7, -11)$, $B(-4, -6)$, $C(-1, 1)$ and $D(2, 7)$. Only three of these points lie on the same line. Determine which three points lie on the same line.

Problem 3 [Locus]

Consider the points $A(1, 3)$ and $B(3, 9)$, and let $P(x, y)$ be a point such that $\angle APB = 90°$.

(a) Show that $x^2 - 4x + y^2 - 12y + 30 = 0$.

(b) By completing the square, show that this is the equation of a circle centred at $(2, 6)$ with radius $\sqrt{10}$.

(c) Explain the relationship between P and the circle.

Problem 4 Show that the line passing through (a, b) and (b, a) is $y = a + b - x$.

Problem 5 Define the linear function ℓ with equation

$$\frac{x}{a} + \frac{y}{b} = 1.$$

(a) Find the x and y-intercepts of ℓ and hence explain why this is called the *intercept form* of a straight line.

(b) Use the intercept form to find the equation of the line with x and y-intercepts at -2 and 3 respectively.

Exercise 3B

Features of a parabola

Fundamentals

Fundamentals 1

(a) The shape of a quadratic function is a p________.

(b) All p________ have a v______, which is referred to as the maximum or minimum point.

(c) A parabola going 'up' is said to be c________ up, whereas a parabola going 'down' is said to be c________ down.

(d) A parabola may or may not intersect the ___-axis. If it does, then the x-coordinate of those points are called the r______ of the quadratic function.

(e) Depending on the position of the vertex, a parabola may have ___, ___ or ___ roots.

Fundamentals 2

Consider the quadratic in the form $y = ax^2 + bx + c$.

(a) If a parabola is concave up, then a___. If a parabola is concave down, then a___.

(b) State the y-intercept.

(c) State the x-coordinate of the vertex.

(d) The vertical line that goes through the vertex is called the axis of s________.

(e) How do you find the y-coordinate of vertex, if you have the axis of s________?

Fundamentals 3

Consider the quadratic in the form $y = a(x - \alpha)(x - \beta)$.

(a) What is the significance of the values of α and β? How do they appear graphically?

(b) What is the x-coordinate of the vertex, in terms of α and β?

(c) To sketch parabolas in this form, first find and mark the r______, then find the x-coordinate of the v______. Find the y-coordinate of the v______ and the ___-intercept, and then sketch the parabola.

Fundamentals 4

Consider the quadratic in the form $y = a(x - h)^2 + k$.

(a) What is the significance of the values of h and k?

(b) To sketch parabolas in this form, first find and mark the v______, then find and mark the y-intercept. If asked, also find and add the ___-intercepts, then sketch.

Question 1 Sketch the graphs of $y = x^2$, $y = (x-2)^2$ and $y = (x+3)^2$. Hence, make a general statement as to how the graph of $y = (x-h)^2$ is related to the graph of $y = x^2$.

Question 2 For each of the following quadratics

(i) find the x-intercept.

(ii) the coordinates of vertex.

(iii) find the y-intercept.

(iv) find the axis of symmetry.

(v) sketch the parabola.

(a) $y = (x-2)^2$ (b) $y = -(x+2)^2$ (c) $y = 2(x-3)^2$ (d) $y = -5(x+1)^2$

Question 3

(a) Sketch the graph of the following.

(i) $y = (x-2)^2$ (ii) $y = (x-2)^2 + 3$ (iii) $y = (x-2)^2 - 2$

(b) Hence, describe how the graph of to $y = (x-h)^2 + k$ is related to $y = (x-h)^2$.

(c) Write down the coordinates of the vertex of $y = (x-h)^2 + k$

(d) Does the axis of symmetry or vertex change if the form of the above graph is changed to $y = a(x-h)^2 + k$ instead?

Question 4 By finding the vertex, axis of symmetry and y-intercept, graph the following quadratics.

(a) $y = 2(x+1)^2 + 3$ (b) $y = \frac{1}{2}(x-3)^2 - 1$ (c) $y = -3(x+2)^2 + 1$

Question 5 For each of the following quadratics

(i) find the x-intercepts.

(ii) find the y-intercept.

(iii) find the axis of symmetry.

(iv) find the coordinates of the vertex.

(v) state whether the y-coordinate of the parabola has a maximum or a minimum.

(vi) state the maximum or minimum y-value of the parabola.

(vii) sketch the parabola.

(a) $y = (x-2)(x-6)$ (b) $y = (x+2)(x-6)$ (c) $y = (x+2)(x+6)$

(d) $y = -(x+2)(x-6)$ (e) $y = 3(x+1)(x-7)$ (f) $y = -2(x-3)(x-5)$

Question 6 For each of the following quadratics

(i) factorise the equation.

(ii) find the x and y-intercepts.

(iii) find the axis of symmetry.

(iv) find the coordinates of the vertex.

(v) state whether the y-coordinate of the parabola has a maximum or a minimum.

(vi) state the maximum or minimum y-coordinate of the parabola.

(vii) sketch the parabola.

(a) $y = x^2 - 16$ (b) $y = x^2 + 5x$ (c) $y = 2x^2 - 5x$

(d) $y = x^2 + 4x - 12$ (e) $y = 12 - 4x - x^2$ (f) $y = 8x - x^2$

Question 7 Sketch the parabola $y = x(10 - x)$ and using your graph, find the maximum value of y.

Question 8 For each of the following quadratics

(i) State the axis of symmetry

(ii) Find the coordinates of the vertex

(iii) State whether the vertex is a maximum or a minimum.

(iv) State the coordinates of the y intercept.

(v) Find where the curve cuts the x-axis.

(vi) Hence, sketch the parabola.

(a) $y = -x^2 + 3x - 2$ (b) $y = 3x^2 + 5x - 2$

Question 9 Match each graph drawn with the equation of the parabola. Note not all equations have a matching graph.

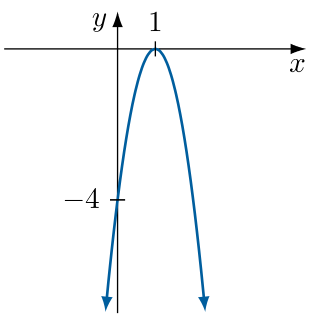
(a) $y = (x + 2)(x - 3)$ (b) $y = -2(x + 2)(x - 4)$ (c) $y = 2(x - 3)(x + 4)$

(d) $y = -(x + 1)^2 + 4$ (e) $y = \frac{1}{2}(x + 3)^2 + 1$ (f) $y = 4(x - 1)^2$

(g) $y = -(x - 2)^2 + 5$ (h) $y = -4(x - 1)^2$ (i) $y = \frac{1}{2}(x + 3)^2 + 1$

(j) $y = -(x - 1)(x - 5)$ (k) $y = (x + 2)(x - 6)$ (l) $y = 6x - x^2$

(A)



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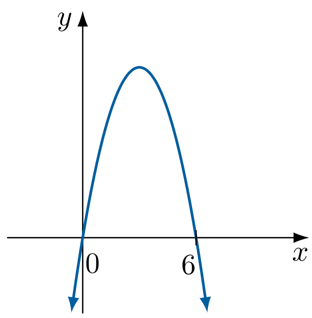

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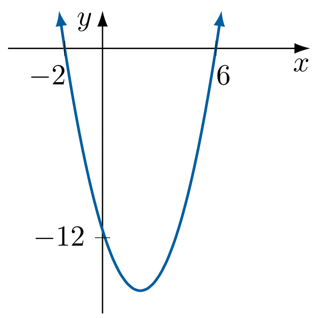

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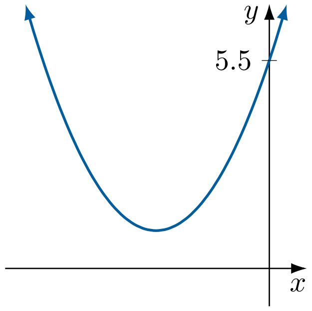

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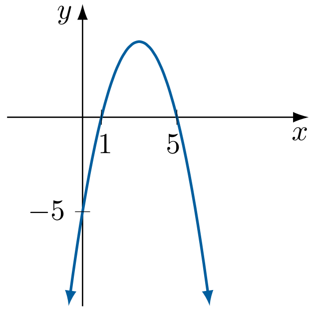

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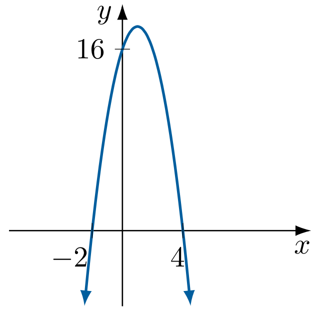

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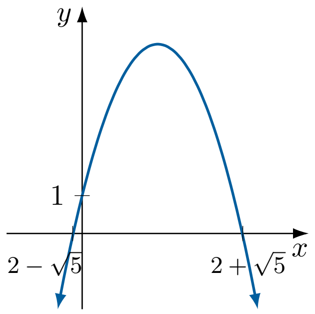

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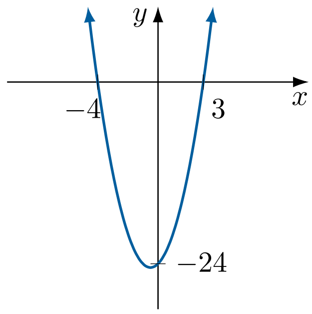

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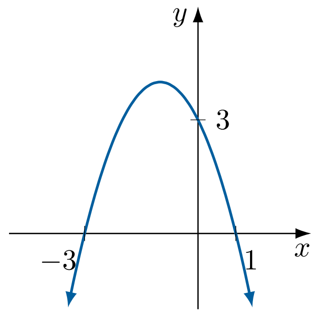


Question 10 Use the given information to sketch and find the equation of the quadratic with the following features.

(a) Roots are at $x = -1$ and $x = 6$, and the y-intercept is 12.

(b) Roots are at $x = -3$ and $x = 3$, and the y-intercept is $-\dfrac{9}{2}$

Question 11 A parabola has roots at $x = 1$ and $x = -3$.

(a) The intercept-form of the parabola is

$$y = a(x - \alpha)(x - \beta).$$

State the value of two of the unknowns.

(b) The parabola also passes through the point $(0, -6)$. Find the value of the last pronumeral.

(c) Hence, state the equation of the parabola.

Question 12 A parabola has vertex $(-2, 1)$.

(a) The vertex-form of a parabola is

$$y = a(x - h)^2 + k.$$

State the value of two of the pronumerals.

(b) The parabola also passes through the point $(1, 2)$. Find the value of the last pronumeral and hence, find the equation of the parabola.
Hence, the equation is $y = \dfrac{1}{9}(x + 2)^2 + 1$.

Question 13 There is a unique parabola whose axis of symmetry is parallel to the y axis, and it passes through the following three points. By first identifying a suitable general form and then finding the pronumerals, find the equation of the parabola.

(a) $(0,0)$, $(-1,9)$, $(8,0)$

(b) $(0,7)$, $(1,4)$, $(3,4)$

(c) $(0,-4)$, $(1,1)$, $(-2,-2)$

Question 14

(a) Express the parabola $y = x^2 - 6x + 10$ in vertex-form.

(b) Hence, explain why the parabola is always above the x-axis, without drawing a sketch.

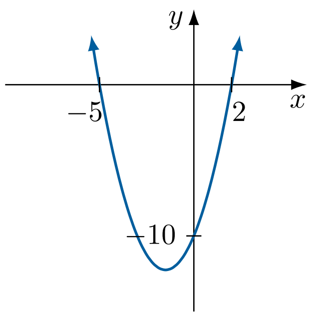
(c) Use a similar technique to show that the following parabolas are either always above or below the x-axis.

(i) $y = x^2 - 4x + 6$

(ii) $y = -x^2 + 2x - 4$

Question 15 Find the equations of the quadratic functions sketched below.

(a)



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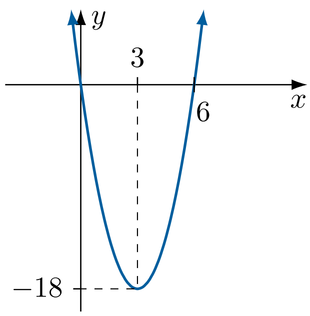

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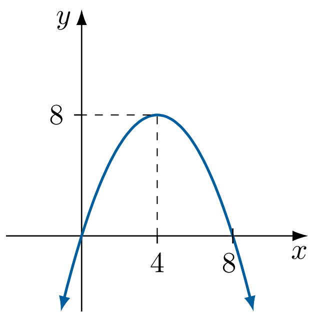
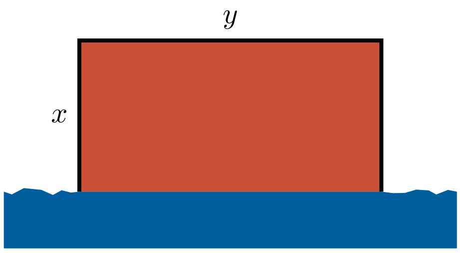


Challenge Problems

Problem 1 [Optimisation problem]

A farmer has 400 metres of fencing to construct a rectangular field. One of the sides of the field is a river, so the farmer only needs to fence three sides of the rectangle.

(a) Find the perimeter of the field that requires fencing if the field has length y metres and width x metres as shown in the diagram below.



(b) Show that the area A of the field is given by $A = 400x - 2x^2$

(c) By completing the square of the above expression, or otherwise, find the maximum area of the field that he can construct.

(d) What restrictions are there on your variables?

Problem 2 [Optimisation problem]

A burger truck normally sells 60 burgers a day for \$10 per burger. The owner is advised that they should increase the price of the burger. However, for every 50 cents extra that the burger costs, the number of daily burger sales will drop by 2. The owner decides to increase the price in 50-cent increments. Let x be the number of 50-cent price increases.

(a) Show that the daily revenue is $R(x) = (30 - x)(20 + x)$.

(b) Hence, find the price that the owner should charge per burger in order to maximise the daily revenue, and state what the maximum daily revenue is.

Exercise 3C

Solving quadratic equations

Fundamentals

Fundamentals 1

(a) There are three main ways of solving quadratic equations. The first is by reading off the solutions from the f________ equation. The second is by c________ the square. The third is by using the q________ formula.

(b) A quadratic may have ___, ___ or ___ solutions.

Fundamentals 2

(a) If a quadratic is expressed in factorised form $y = a(x-\alpha)(x-\beta)$, then the solutions of the quadratic are $x =$ ___ and $x =$ ___.

(b) Sometimes the factorised quadratic equation is a perfect square $y = a(x-\alpha)^2$. In this case, the solution is just $x =$ ___.

Fundamentals 3

(a) The solution to $x^2 - k = 0$ is $x =$ _____.

(b) The solution to $(x-h)^2 - k = 0$ is $x =$ __________.

(c) A general quadratic $y =$ ________ may be expressed in the form $y = a(x-h)^2 - k$ by completing the square. Afterwards, it can be solved like the question above.

Fundamentals 4

(a) If a quadratic is too difficult to factorise, then the q________ formula may be used to find the solutions.

(b) For the equation $ax^2 + bx + c = 0$, the solution is $x =$ ____________.

(c) If the inside of the square root sign is negative, then the quadratic has no r_____ s________.

(d) If the inside of the square root sign is positive, then the quadratic has ___ d_____ real solutions.

(e) If the inside of the square root sign is zero, then the quadratic has ___ d_____ real solution.

Question 1 Complete the following.

(a) $x^2 + 8x + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$ (b) $x^2 - 10x + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$

(c) $x^2 + 5x + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$ (d) $x^2 - 3x + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$

(e) $x^2 + 10xy + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$ (f) $x^2 - 12xy + \_\_\_\_\_\_ = (\_\_\_\_\_\_\_\_)^2$

Question 2 Solve the following quadratic equations by first factorising.

(a) $x^2 - 2x = 0$ (b) $x^2 + 5x = 0$ (c) $x^2 - 4x + 4 = 0$

(d) $x^2 - 5x + 6 = 0$ (e) $x^2 - x - 2 = 0$ (f) $x^2 + 4x - 12 = 0$

(g) $x^2 - 10x + 25 = 0$ (h) $x^2 - 6x + 5 = 0$ (i) $x^2 - 5x - 24 = 0$

(j) $x^2 + 2x - 15 = 0$ (k) $x^2 + 24x + 144 = 0$ (l) $x^2 - 10x + 24 = 0$

Question 3 Solve the following quadratic equations by first factorising.

(a) $2x^2 + 5x + 2 = 0$ (b) $2x^2 - 5x + 2 = 0$ (c) $3x^2 + 2x - 8 = 0$

(d) $4x^2 - 4x - 3 = 0$ (e) $6x^2 - x - 1 = 0$ (f) $6x^2 + 11x - 2 = 0$

Question 4 Find the values of a and b such that $x^2 - 8x + 7 = (x + a)^2 + b$, and hence solve the equation $x^2 - 8x + 7 = 0$ by instead solving $(x + a)^2 + b = 0$.

Question 5 Solve the following quadratic equations.

(a) $(x - 5)^2 = 3$ (b) $(x + 2)^2 - 3 = 0$

(c) $(x - 1)^2 - \dfrac{1}{9} = 0$ (d) $(x + 2)^2 - 6\frac{1}{4} = 0$

Question 6 Solve the following quadratic equations by completing the square.

(a) $x^2 - 2x - 1 = 0$ (b) $x^2 - 4x + 1 = 0$

(c) $x^2 - 10x + 13 = 0$ (d) $4x^2 - 16x + 11 = 0$

Question 7 Solve by using the quadratic formula.

(a) $x^2 - x - 1 = 0$ (b) $4x^2 - 8x + 1 = 0$

Question 8 Solve the following for x using the an appropriate method for the question.

(a) $x^2 = 4$ (b) $x^2 = 4x$ (c) $x(x - 6) = 0$

(d) $(x - 2)^2 = 25$ (e) $(2x + 5)(x - 3) = 0$ (f) $x^2 - 3x - 4 = 0$

(g) $x^2 - x - 12 = 0$ (h) $2x^2 + 11x + 15 = 0$ (i) $3x^2 + 8x - 16 = 0$

(j) $x + \dfrac{1}{x} = \dfrac{5}{2}$ (k) $x^3 = x$ (l) $2x^2 + 4x - 7 = 0$

(m) $\dfrac{4}{x} = \dfrac{x}{4}$ (n) $2x - 3 = \dfrac{9}{2x - 3}$ (o) $\dfrac{6}{x - 2} - \dfrac{10}{3x + 2} = 1$

Question 9 Solve the following.

(a) If one side of a square is increased by 5 and another is decreased by 5, a rectangle is formed whose area is 56 cm^2. Find the side length of the original square.

(b) The number of diagonals of an n-sided polygon is given by the formula

$$D = \frac{n}{2}(n-3).$$

A polygon has 65 diagonals. How many sides does it have?

(c) Let x and y be positive numbers that have a sum of 20 and product is 84. Find x and y.

(d) Let $C(x) = 2x^2 - 12x + 25$ represent the model for the cost of manufacturing products in a day, where x is the number of machines operating and $C(x)$ is in ten thousand dollars.

(i) Find the cost of production if one machine is operating.

(ii) If the cost of production is \$90000, how many machines are operating?

(iii) Find how many machines should be operated in order to minimise the cost of production, and hence find the minimum cost of production.

Challenge Problems

Problem 1 Find a, b and c if

(a) $x^2 \equiv a(x-1)^2 + b(x-1) + c$

(b) $2x^2 - 7x - 5 \equiv a(x+2)^2 + b(x+2) + c$

(c) $4x^2 + x - 5 \equiv ax(x+1) + b(x+1) + c$

Problem 2 Solve the following equations

(a) $2^{2x} - 2^x - 12 = 0$
Hint: Let $y = 2^x$

(b) $3^{2x} - 8 \times 3^x - 9 = 0$
Hint: Let $y = 3^x$

(c) $3^{2x} - 12 \times 3^x + 27 = 0$

(d) $3 \times 9^x - 3^{x+1} - 18 = 0$

(e) $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$

Exercise 3D

Discriminant

Fundamentals

Fundamentals 1

Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are rational numbers.

(a) The discriminant is the i_____ of the square root sign in the q_______ formula.

(b) The formula for the discriminant is $\Delta =$ __________

(c) If Δ _____, then there are two distinct real solutions.

(d) If Δ _____, then there is one distinct real solution.

(e) If $\Delta < 0$, then there are __ real solutions.

Fundamentals 2

Consider the quadratic equation $ax^2 + bx + c = 0$.

(a) What can we say about a and Δ for a quadratic that is always above the x-axis?

(b) What can we say about a and Δ for a quadratic that is always below the x-axis?

(c) Sketch the 6 different types of parabolas depending on Δ and if $a > 0$ or $a < 0$.

Question 1 Find the discriminant of each of the following quadratic functions, and hence determine whether the graph touches, crosses or does not intersect the x-axis.

(a) $y = x^2 - 8x + 16$

(b) $y = -4x^2 + 2x - 1$

(c) $y = x^2 - 13x + 12$

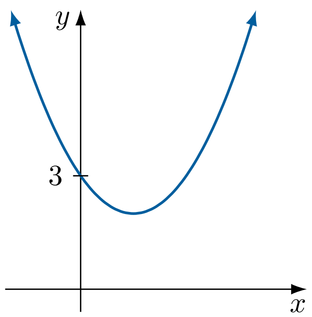
(d) $y = 2x^2 - 3x + 3$

(e) $y = 8 - 3x - 2x^2$

(f) $y = 4x^2 - 12x + 9$

Question 2 For each of the quadratic functions below, state the sign of the discriminant.

(a)



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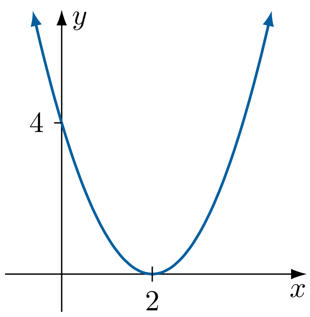

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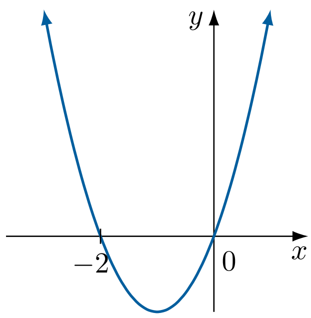

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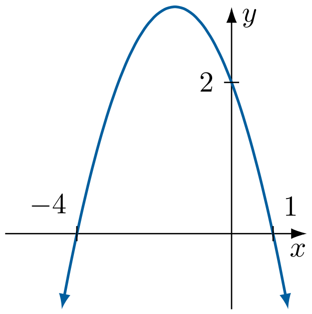

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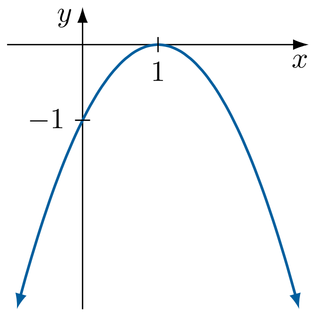

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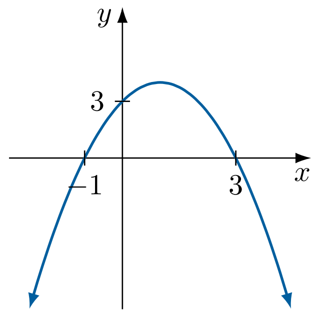


Question 3 Consider the quadratic equation with integer coefficients $ax^2 + bx + c = 0$.

(a) Explain why if $\Delta = 0$, then the roots are rational numbers.

(b) What can we say about the roots if Δ is a perfect square?

(c) What can we say about the roots if $\Delta > 0$, but is not a perfect square?

Question 4 By examining the discriminant, state the number of distinct solutions and if the roots are rational or irrational. Assume that the coefficients of the quadratic are integers.

(a) $9x^2 - 4 = 0$

(b) $x^2 + 3x + 2 = 0$

(c) $-2x^2 + 5x + 2 = 0$

(d) $-4x^2 + 20x - 25 = 0$

(e) $3x^2 + 4x + 2 = 0$

(f) $25x^2 - 20x + 4 = 0$

Question 5 Show that the following quadratic equations have exactly one real root.

(a) $25x^2 - 10x + 1 = 0$

(b) $9x^2 - 12x + 4 = 0$

Question 6 Find the values of k for which the following quadratic equations have exactly one real root.

(a) $kx^2 - 6x + k = 0$

(b) $x^2 - (k + 4)x + 9 = 0$

(c) $kx^2 - 3kx + 3 = 0$

(d) $2kx^2 + 2(k - 2)x - 3 = 0$

Question 7 Find the value(s) of k for which the following equations have

(i) one real root.

(ii) two real roots.

(iii) no real roots.

(a) $kx^2 - (2k - 1)x + k = 0$

(b) $4(k + 1)x^2 + 4kx + (k - 2) = 0$

Question 8 Show that the following equations have no real roots.

(a) $x^2 - 3x + 5 = 0$

(b) $-5x^2 + 2x - 4 = 0$

Question 9 Show that for any non-zero real value of k, the following quadratic equations can *never* have real roots.

(a) $x^2 - 2kx + 3k^2 = 0$

(b) $x^2 + 2x + 1 + k^2 = 0$

(c) $kx - 5k^2 = x^2$

(d) $x^2 + 2kx + 2k^2 + k + 1 = 0$

Question 10 Let k be a rational number. Show that the equation $kx^2 + (k+1)x + 1 = 0$ will always have real and rational roots. State the specific value of k so that the equation has exactly one real root.

Question 11 Let k be any real number.

(a) Complete the square of $k^2 + 2k + 9$ and show that $k^2 + 2k + 9$ is always positive.

(b) Hence, show that $kx^2 + (k+3)x + 1 = 0$ will always have two distinct real solutions.

(c) Use a similar technique from above to show that the following quadratic equations will always have two real roots.

(i) $x^2 + (4k+1)x + 2k = 0$

(ii) $2x^2 + (2k+1)x - 1 = 0$

Question 12 Let k be any real number.

(a) Find the discriminant of $x^2 - kx = 0$.

(b) Explain why the equation $x^2 - kx = 0$ will always have at *least* one real root.

Question 13 Let k be any real number.

(a) Find the discriminant of $x^2 + kx + k^2$.

(b) State the value of k for which $x^2 + kx + k^2 = 0$ will have one real root.

(c) For values of k other than this value, how many real roots does this quadratic have?

Question 14 Find the value(s) of m for which the quadratic equation

(a) $2x^2 - 8x + m = 0$ has real roots.

(b) $x^2 - (m-5)x + 4 = 0$ has equal roots.

(c) $8x^2 - 3x + 4m + 9 = 0$ has $x = 0$ being one of the roots.

Challenge Problems

Problem 1 The line $y = mx - 4$ is a tangent to the parabola $y = x^2 - 2x + 5$. Find the value of m.

Problem 2 Show that the discriminant of $x^2 - (p+1)x + p - 1 = 0$ is

$$\Delta = (p-1)^2 + 4$$

and hence discuss the roots of $x^2 - (p+1)x + p - 1 = 0$ for real values of p.

Problem 3 If $f(x) = px^2 + (p+1)x + \frac{p}{4}$, find p so that $f(x) > 0$ for all real x.

Problem 4 Show that for any real values of m and n, the graph of $y = x^2 + 2mx + m^2 + n^2$ is always touching or above the x-axis.

Problem 5 Show that $(p-3)x^2 + 6x - (p+3) = 0$ will have rational roots for all integer values of p.

Problem 6 Find the condition for the pronumerals p, q, and r if the equation $px^2+2qx+r = 0$ has one distinct real root.

Problem 7 Prove that the roots of the following equations are always real

(a) $px^2 + q = (p+2q)x$

(b) $p(1+x) = q(1-x-x^2)$

Problem 8 Show that $ax^2 + bx + c = 0$ cannot have real roots if $\frac{b}{a} = \frac{c}{b}$.

Problem 9 Find the values of p for which $y = px - 3p - 5$ is a tangent to $x^2 = 8y$. Hence find the equations of the tangents.

Exercise 3E

Simultaneous equations and applications

Fundamentals

Fundamentals 1

Briefly describe three different ways of solving a set of linear simultaneous equations.

Fundamentals 2

(a) Suppose two linear functions ℓ_1 and ℓ_2 were solved simultaneously, and it resulted in the solution $x = \alpha$. What is the geometric significance of $x = \alpha$?

(b) Suppose a linear function ℓ and a quadratic function $\mathcal{Q}$ were solved simultaneously, and it resulted in two solutions $x = \alpha$ and $x = \beta$. Describe the geometric significance of $x = \alpha$ and $x = \beta$

(c) Suppose a linear function ℓ and a quadratic function $\mathcal{Q}$ were solved simultaneously. Describe three possible scenarios that can result from this.

Fundamentals 3

(a) The income from a company or business is called the r________.

(b) For a simple business model, r________ can be calculated by multiplying the s________ price with the amount sold.

(c) The n___ profit is the r________ minus the costs.

(d) When a company's costs are e________ to the revenue, then we say that the company has reached it's b________-e______ point.

Fundamentals 4

There are two types of costs. They are the f________ costs, which are independent on the sales volume, and the v________ costs, which are dependent on the sales volume.

Question 1 Solve the following set of linear equations simultaneously. Use the method that will be more convenient.

(a) $x + y = 2$, $3x - y = 2$

(b) $4x - y = 30$, $3x + y = 19$

(c) $4x - 3y = 18$, $2x - y = 10$

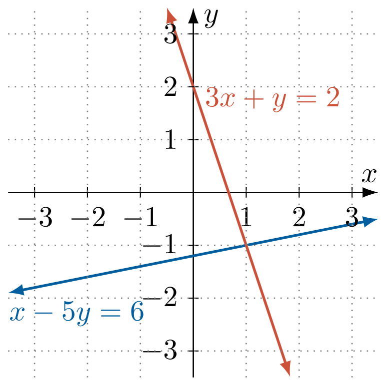
(d) $3x - 2y = 15$, $5x - 4y = 27$

(e) $4x + 3y = 21$, $3x + 2y = 15$

(f) $5x + 2y = 34$, $4x - 5y = 14$

Question 2 For each of the following diagrams, find the coordinates of the point of intersection.

(a)



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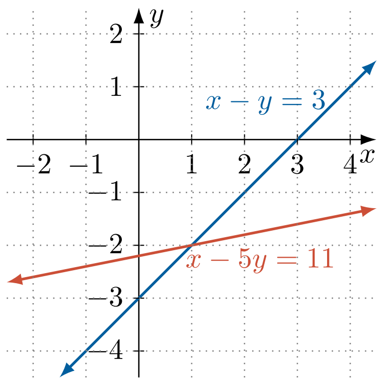

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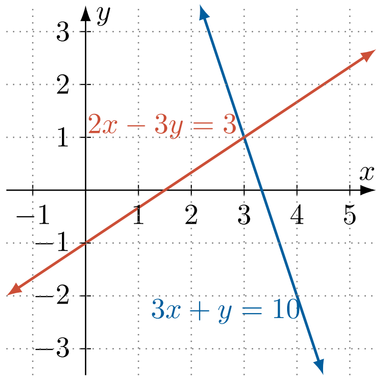

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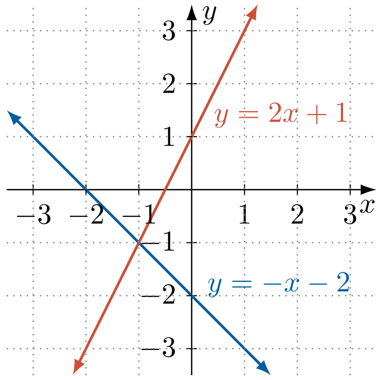


Question 3 Solve the following word-problems by forming a set of simultaneous equations representing the information, then solving it.

(a) The sum of two numbers if 20 and their difference is 4. Find the two numbers.

(b) Bob is three times as old as his daughter Mary. The sum of their ages is 56. How old are Bob and Mary?

(c) Bob has \$760 consisting of \$20 and \$50 dollar notes. In total, he has twenty notes. How many of each kind of note does Bob have?

(d) It costs \$160 per day to hire a labourer and a \$240 per day to hire a manager. In a single day, \$1760 was paid to all ten workers. How many were managers and how many were labourers?

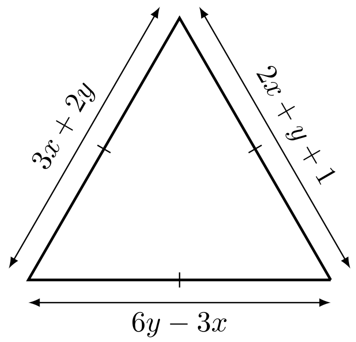
(e) Adam took his wife and three children to the circus and paid \$103, while Barry only went with his 4 children and paid \$94. Find the price of both the adult and child tickets.

(f) Mary, Bob and Jane have currently unknown weights. Mary and Bob have a combined weight of 121 kg. Jane and Bob have a combined weight of 130 kg and finally Mary and Jane have a combined weight of 99 kg.

  (i) Find the combined weight of Mary, Bob and Jane.

  (ii) Find all of their weights.

Question 4 The diagram below shows an equilateral triangle with unknown side lengths. Find x and y.



Question 5 Use graphing software to sketch the following, and hence find the solution graphically.

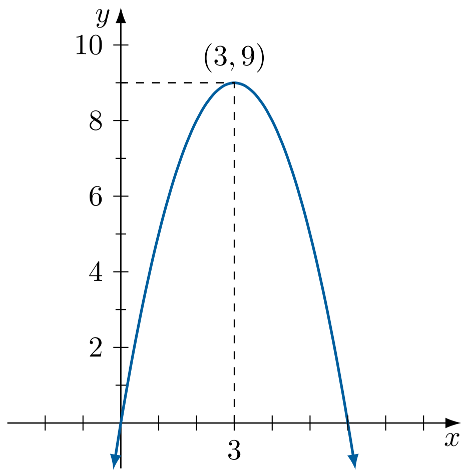
(a) $y = x^2$, $y = x$

(b) $y = x^2 - 4$, $y = x + 8$

(c) $y = x^2$, $y = 2x - 1$

(d) $y = x^2 + 2x + 2$, $y = -3x - 2$

Question 6 The diagram below shows the graph of $f(x) = 6x - x^2$. Copy this diagram into your book.



(a) On your diagram, draw the line $y = 8$. At what x-coordinates does the graph of $y = f(x)$ intersect the line $y = 8$?

(b) How many solutions does the equation $f(x) = 8$ have?

(c) Using your diagram, how many solutions would you expect the equation $f(x) = 6$ to have?

(d) Using a similar technique, how many solutions would you expect $f(x) = 10$ to have?

(e) Consider the more general equation $f(x) = k$. For what values of k would $f(x) = k$ have

(i) one solution?

(ii) two solutions?

(iii) no solutions?

(f) Make a general statement about the relationship between the equation $f(x) = k$ and the graphs of $y = f(x)$ and $y = k$.

Question 7 Bob solves a linear function simultaneously with a quadratic function to find any points of intersection, if they exist.

(a) Describe what happens geometrically if Bob finds that there are

(i) two solutions. (ii) one solution. (iii) no solutions.

(b) Suppose Bob is given a line that is tangential to a parabola. If he solves them simultaneously, he will acquire a new quadratic function. If he attempts to solve it, how many roots should he expect this new quadratic function to have?

(c) Is there a quicker way to check how many solutions (and therefore intersections) there will be *without* solving for them explicitly?
Hint: What topic was covered in the previous exercise?

Question 8 Determine how many times the line intersects the parabola.

(a) $y = 2 - 2x$
$y = x^2 - 4x + 3$

(b) $y = 3x + 4$
$y = 2x^2 - 7x + 1$

(c) $y = 2x - 3$
$y = 5x^2 - x + 3$

(d) $y = 3x - 4$
$y = x^2 - 3x + 5$

Question 9 Find the value(s) of m such that

(a) $y = mx$ is a tangent to the parabola $y = (x + 1)^2$

(b) $y = 2x + m$ is a tangent to the parabola $y = 2x^2 + 6x - 5$

(c) $y = x + m$ is a tangent to the circle $x^2 + y^2 = 4$

(d) $y = mx + 2$ is a tangent to the hyperbola $xy = 4$

Question 10 A company sells a product for \$50 per unit. The cost to produce x items is given by the cost-function

$$C(x) = 200 + 10x.$$

The revenue-function

$$R(x) = 50x$$

models the gross revenue from selling x units.

(a) What is the cost and revenue after selling 2 units? Has the company made profit yet?

(b) How about after selling 10 units?

(c) How many units does the company need to sell in order to have a profit of zero dollars i.e., break even?

Question 11 Bob the Baker wishes to bake and sell cakes. Bob buys the baking equipment for \$600. The cost of the ingredients per cake is \$2. He then sells each cake for \$10. Let $C(x)$ be the cost-function, that shows how much it costs to bake x cakes. Let $R(x)$ be the revenue-function, that shows his gross revenue after selling x cakes.

(a) Show that $C(x) = 600 + 2x$ and $R(x) = 10x$

(b) Hence, determine how many cakes that Bob must sell in order to break even.

(c) Sketch the graph of $y = C(x)$ and $y = R(x)$ on the same set of axes, labelling all important features.

(d) Suppose that Bob decided to give away the first 10 cakes for free as a way to promote his business. How many more cakes would he need to sell in order to break even?

Question 12 A company sells electronic devices for \$130 per unit. The company spent \$60,000 for the machines that produce the device, and each device requires \$10 worth of raw materials to produce.

(a) Write down the cost and revenue functions $C(x)$ and $R(x)$ respectively.

(b) Hence, find the break-even quantity and sketch a graph showing this.

Challenge Problems

Problem 1 Let the speed of a boat be x and let the speed of a river, of length d, streaming downwards be y. Assume that d, x and y are all constant. When the boat travels with the river, the relative speed of the boat is $x + y$. It takes 3 hours to travel downstream from point P to point Q, and 5 hours to travel upstream from Q to P.

(a) Write down an expression, in terms of x and y, for the relative speed of the boat when it travels upstream.

(b) How long would this boat take to go from P to Q in still water?

Problem 2 In a two-digit number, the units digit is three times the tens digit. When 36 is added to the number, the digits swap positions. What is the original two-digit number?

Problem 3 If 5 is added to both the numerator and denominator of a fraction, the resultant number simplifies to $\frac{2}{3}$. If 1 is subtracted from both the numerator and denominator, the resultant number simplifies to $\frac{1}{3}$. What is the original fraction?

Chapter 3 Review

Linear and Quadratic Functions

Review

Question 1 Find the equation of the parabola in the following

(a) A parabola has x intercepts at $x = -2$ and $x = 5$ and passes through $\left(0, \frac{5}{2}\right)$.

(b) The coordinates of vertex is $(2, -3)$ and the parabola passes through the point $(3, -5)$.

Question 2 $x = -2$ is the axis of symmetry of the parabola $y = 2x^2 - kx + 1$, find the value of k.

Question 3 For the function $y = 2x^2 - 4x - 6$

(a) Find the equation of the axis of symmetry and hence coordinates of the vertex.

(b) Find the vertex of this function by completing the square.

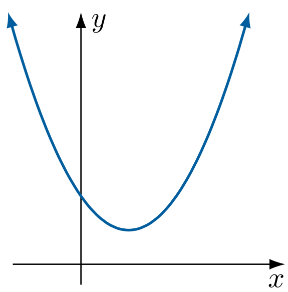
(c) Factorise the function and sketch this curve labelling all important features.

Question 4 Consider the function $f(x) = x^2 - 4x + 7$ for the domain $x \geq 2$. State the range of this function.

Question 5 A mathematics teacher wants to make up a quadratic equation $ax^2 + bx + c = 0$, so that a, b and c are integers, and the correct solutions are $x = -\frac{1}{2}$ and $x = 5$. Form the quadratic equation and find the values of a, b and c. Is there more than one equation that will work?

Question 6 The considering the curve $y = ax^2 + bx + c$ below, comment on the sign of

(a) a (b) b (c) c



Question 7 Show that $2x^2 - x + 4 > 0$ for all x.

Question 8 Prove that the graph of $y = 5x^2 + 4mx + m^2$ is always above the x-axis for all non-zero m.

Question 9 Show that the roots of the equation $(3m-5)x^2 - 3m^2x + 5m^2$ are rational if m is rational.

Question 10 What can be said about the graph of $y = ax^2 + bx + c$ if

(a) $a > 0$ and $c < 0$

(b) $a < 0$ and $c > 0$

Question 11 Prove that for all values of m the line $x = \frac{y}{m} + m$ is a tangent to the parabola $x^2 = 4y$.

Question 12 Find the value(s) of p so that the line $y = px - 4$ will touch the parabola $y = x^2 - 4x$.

Question 13 Show that the roots of $(5-x)(2-x) = p^2$ are always real and distinct.

Question 14 Solve the following

(a) $x^6 - 7x^3 - 8 = 0$

(b) $2^{2x} - 3 \times 2^x - 4 = 0$

Question 15 Gidon has just bought 80 metres of fencing to create a border around a new rectangular garden that is still being designed.

(a) If one of the sides of the rectangle is 9 metres in length, what would the other dimension be? What would the enclosed area be? Write this data in the first row of the table and hence complete the table.

Width	Length	Area
9		
16		
22		
35		
x		

(b) Find the restrictions on the value of x.

(c) Graph the rectangle's area versus x.

(d) Find the point on the graph that corresponds to the largest rectangular area that Gidon can enclose using the 80 metres of fencing.

Question 16 Solve the following simultaneous equations.

(a) $x + y = 10$
$x - y = -2$

(b) $5x + 2y = 43$
$x - y = 3$

(c) $5x - 2y = 37$
$3x - 2y = 23$

(d) $4x + 3y = 26$
$3x - 2y = 11$

Question 17 Peter hits a golf ball from the 1st tee down the fairway so that it follows the parabolic path $y = 0.5x - 0.002x^2$. This relates the height y of the ball above the ground to the ball's progress x down the fairway. Distances are measured in metres.

(a) How far from the 1st tee does the ball hit the ground?

(b) At what distance x does the ball reach the highest point of its arc? What is the maximum height attained by the ball?

(c) Peter now hits the golf ball from a plateau of height 10 metres from level ground. What is the maximum height attained by the ball from the ground now?

Question 18 The table gives some values for a function of the form $y = ax^2 + bx + c$. Find the values of a, b and c.

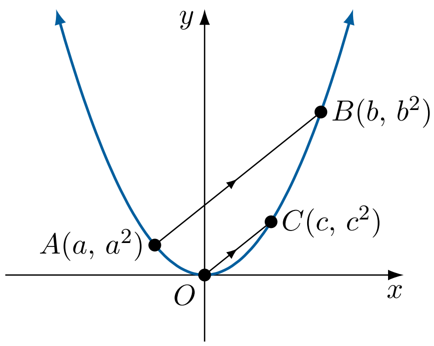
x	0	1	2	3
y	0	2	6	12

Question 19 Find the coordinates of the points of intersection of the parabola $y = (x-4)^2$ and the line $y = 9$.

Question 20 Express $2x^2 - 5x - 12$ in the form $a(x-h)^2 + c$. Hence sketch the quadratic function $y = 2x^2 - 5x - 12$.

Question 21 A farmer has 300 metres of fencing to construct a rectangular field. One of the sides of the field already has an existing fence in place and all he needs to do is use the fencing on the other three sides. Find the maximum area of the field that he can construct. What restrictions are there on your variables?
Hint: Form a function where the area A is a function of the side length x.

Question 22 Let $A(a, a^2)$, $B(b, b^2)$ and $C(c, c^2)$ be three points on the parabola $y = x^2$ such that AB is parallel to OC, as shown in the diagram below. Prove that $a + b = c$.



Investigation Task

Viète's Formulas and Irrational Roots

Consider a general quadratic function $y = ax^2 + bx + c$, where $a,\ b\,,c \in \mathbb{Z}$, with two roots $x = \alpha$ and $x = \beta$. Normally, to find values such as $\alpha + \beta$ or $\alpha\beta$, you would need to find the roots and then manually add or multiply them. However, Viète's formulas allows us to find such values *without* even knowing the roots themselves.

Question 1

(a) Since the roots are $x = \alpha$ and $x = \beta$, then the quadratic can be expressed in the form

$$y = a(x - \underline{\quad})(x - \underline{\quad}).$$

(b) Show that expanding this yields

$$y = a(x^2 - (\alpha + \beta)x + \alpha\beta).$$

(c) Deduce that

$$\alpha + \beta = -\frac{b}{a}$$
$$\alpha\beta = \frac{c}{a}$$

Hint: Remember that we still have the original equation $y = ax^2 + bx + c$.

(d) Hence, find the sum and product of the roots of $y = x^2 - 2x - 1$ *without* explicitly finding the roots themselves.

(e) Verify your result by finding the roots, using whichever suitable method, and then manually calculating their sum and product.

(f) Will this formula always work, regardless of whether the roots are actually real or not (the inside of the square root is negative)? Investigate the different cases.

Question 2 Consider a monic quadratic function $y = x^2 + bx + c$.

(a) Find $\alpha + \beta$ and $\alpha\beta$, in terms of b and c respectively.

(b) Deduce that the quadratic equation with roots $x = \alpha$ and $x = \beta$ is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

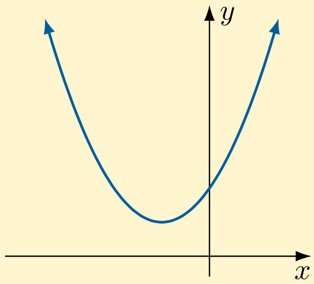
(c) Hence, find the equation of the quadratic with roots $x = 1 \pm \sqrt{2}$.

(d) Find the equation of the quadratic with roots $x = p \pm \sqrt{q}$, where q is not a perfect square.

Investigation Task

Relationship between graph and coefficients

Consider a random parabola with some equation $y = ax^2 + bx + c$. If we move the parabola around, then the values of a, b and c may change. For example, the parabola below has $a > 0$, $b > 0$ and $c > 0$



This investigation task allows the student to explore the effect of the coefficients on the graphs, and vice-versa.

Question 1 Explain how you would find the sign of a, b and c without knowing explicitly what the equation of the parabola is. What did you find is a good order to find the coefficients? For example, is it best to find c first, and then b, and then a? Explain why you think the order you have chosen is the best one.

Question 2 Draw a number of parabolas with different positions and concavities. For each graph, state the sign of a, b, and c. Can any of the coefficients be zero? Explain your answer.

Question 3 Ask a friend to fill in the following table with either 'Zero', 'Positive' or 'Negative'. Once it has been filled, draw a parabola that satisfies those values of a, b and c. Repeat this but for varying signs and values. An example has been given below.

a	b	c
Positive	Positive	Negative

4

FURTHER FUNCTIONS AND RELATIONS

- Polynomial functions
- Absolute value functions
- Inverse proportion and the hyperbola
- Circles and semi-circles
- Reflections

Exercise 4A

Polynomial functions

Fundamentals

Fundamentals 1

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. State the following features of $P(x)$.

(a) Degree.
(b) Leading coefficient.
(c) Leading term.
(d) Constant term.
(e) Value of a_n, if monic.
(f) Max # of terms.

Fundamentals 2

(a) The roots of a polynomial are called the z________ of the polynomial.

(b) Geometrically, the roots or zeroes of a polynomial are the ___-intercepts of the graph of the polynomial.

(c) A quadratic function will have at most _____ roots, so it will cut the ___-axis in at most ________ distinct places.

(d) A cubic function will have at most _____ roots, so it will cut the ___-axis in at most ________ distinct places.

(e) In general, a polynomial of degree n will have at most ________ distinct roots.

Fundamentals 3

(a) A single root occurs when the polynomial has a factor of $(x - \alpha)$. Geometrically, the graph cuts straight through the ___-axis at $x =$ _____

(b) A double root occurs when the polynomial has a factor of $(x - \alpha)^2$. Geometrically, the graph 'bounces' off ___-axis at $x =$ ________

(c) A triple root occurs when the polynomial has a factor of __________. Geometrically, the graph has a 'wiggle' off ___-axis at $x =$ _____

Fundamentals 4

(a) The sum, difference and product of two polynomials will always be a p________.

(b) When two polynomials are divided, the result is/is not (circle one) guaranteed to be a polynomial.

(c) Two quadratic polynomials $P(x) = ax^2 + bx + c$ and $Q(x) = Ax^2 + Bx + C$ are equal if and only if $a =$ ___, $b =$ ___ and $c =$ ___.

(d) Two polynomials $P(x)$ and $Q(x)$ are equal if and only if their corresponding coefficients are e________.

Question 1 Find the value of each pronumeral.

(a) $4x^2 - 6x + 2 \equiv 2ax^2 + 3bx + c$

(b) $2x^2 - 4x - 5 \equiv 2(x-1)^2 + b$

(c) $x^2 - 4x + 8 \equiv a(x-2)^2 + b$

(d) $x^2 - 4x + 8 \equiv a(x-1)^2 + b(x-1) + c$

(e) $(x+1)^3 \equiv ax^3 + bx^2 + cx + d$

(f) $x^3 + 3x^2 + 2x - 7 \equiv (x-2)(x^2 + 5x + a) + b$

(g) $2x + 1 \equiv a(x-1) + b(x^2-1) + c(x+1)$

(h) $x^2 \equiv (ax+b)(x-1) + c(x^2+1)$

Question 2 For each of the following polynomials, state the roots and categorise the root as either a single root, double root, or triple root.

(a) $P(x) = (x-1)(x-2)(x-3)$

(b) $P(x) = (x+1)(x-3)(x+5)$

(c) $P(x) = (x-5)^2$

(d) $P(x) = (x+4)^3$

(e) $P(x) = (x-2)^2(x+4)$

(f) $P(x) = (x+7)^3(x+3)^2$

(g) $P(x) = (x+7)^3(x+3)^2(x-1)$

(h) $P(x) = (x^2-1)^2$

Question 3 Write down, in factorised form, the monic polynomial that has

(a) a single root at $x = 1$, -2 and 3.

(b) a single root at $x = \pm 1$ and a double root at $x = 4$

(c) a double root at $x = 2$ and a triple root at $x = -2$

Question 4 Let $P(x) = 3x - 2$ and $Q(x) = x^2 - 2x + 1$. Find the following expressions.

(a) $P(x) + Q(x)$

(b) $P(x) - Q(x)$

(c) $Q(x) - P(x)$

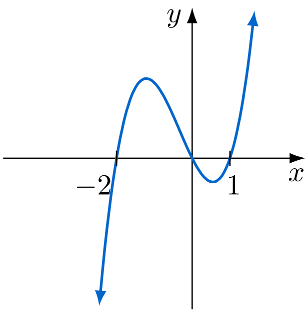
(d) $2Q(x) - 3P(x)$

(e) $P(x)Q(x)$

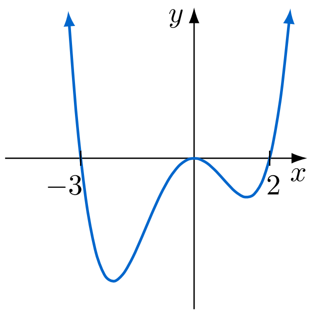
(f) $P(x)Q(x) + 2Q(x)$

Question 5 The diagrams below have leading coefficients either 1 or -1. Write down the equation of each polynomial.

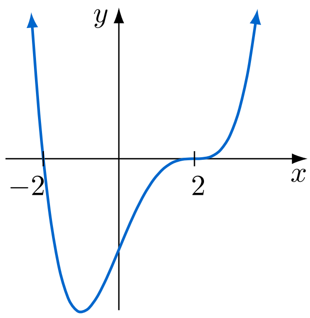
(a)



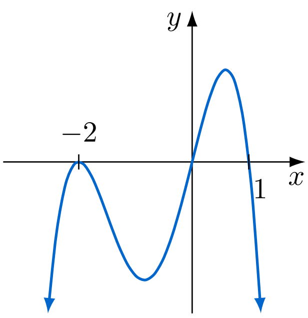
(b)



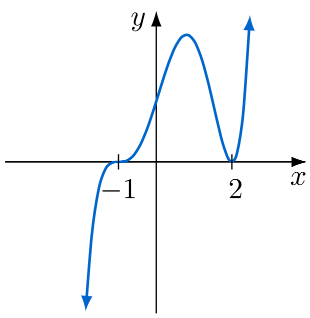
(c)



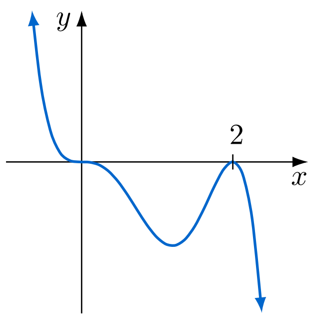
(d)



(e)



(f)



Question 6 Sketch the following graphs and clearly indicate all intercepts with the coordinate axes.

(a) $f(x) = x^3$

(b) $f(x) = x^3 + 8$

(c) $f(x) = -2x^3$

(d) $f(x) = (x-1)^3$

(e) $f(x) = -(x+1)^3$

(f) $f(x) = -(x+1)^3 + 2$

(g) $f(x) = (x-3)\left(x^2-1\right)$

(h) $f(x) = x^2(x-3)$

(i) $f(x) = x^2(3-x)$

Question 7 Sketch the following graphs and clearly indicate all intercepts.

(a) $f(x) = (x-1)^2(x+3)^2$

(b) $f(x) = x(x-2)^3$

(c) $f(x) = x(x-2)^3(x+2)$

(d) $f(x) = -x(x+1)^3(x-2)^2$

Challenge Problems

Problem 1 Sketch the graph of $P(x) = x^4 - 18x^2 + 81$.

Problem 2 Let $P(x)$ and $Q(x)$ both be polynomials of degree n. Determine whether the following statements are true or false.

(a) $P(x) + Q(x)$ must have degree n.

(b) $P(x) - Q(x)$ must have degree n.

(c) $P(x) \times Q(x)$ must have degree $2n$.

Problem 3 [Even and odd polynomials]

(a) Consider the general quartic polynomial $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. Show that if $P(x)$ is an even function, then $b = d = 0$.

(b) Consider the general quintic polynomial $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Show that if $P(x)$ is an odd function, then $b = d = f = 0$.

Problem 4 Use the result from Problem 3 to explain why

(a) all odd polynomials pass through the origin.

(b) in an even polynomial, if $x = \alpha$ is a root, then $x = -\alpha$ is also a root.

Problem 5 Write down the equation of a monic polynomial that

(a) is an even quartic with two roots being 1 and 3.

(b) is an odd cubic with a single root at $x = -2$.

Exercise 4B

Absolute value functions

Fundamentals

Fundamentals 1

(a) The absolute value $|x|$ of a real number x is the magnitude or s______ of x without regard to its sign, or the distance on the number line from the o______.

(b) Complete the following.

$$|x| = \begin{cases} \_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_ \end{cases}$$

(c) $|x - a|$ is the distance of x from ______ on the number line.

(d) Complete the following.

$$|f(x)| = \begin{cases} \_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \text{if } \_\_\_\_\_\_ \end{cases}$$

(e) $\sqrt{x^2} =$ ________

(f) $|x|^2 =$ ________

(g) If $|x| = k$, where $k > 0$, then $x =$ ________

Fundamentals 2

(a) The output of an absolute value is always p________.

(b) Absolute value graphs can often have s________ points where the points change direction. Sometimes, these are called c________.

Fundamentals 3

(a) $|-x| =$ ________ for all real x.

(b) $|x - y| =$ ________ for all real x and y.

(c) $|xy| =$ ________ for all real x and y.

(d) $\left|\frac{x}{y}\right| =$ ________ for all real x and y, where $y \neq$ ________.

Fundamentals 4

(a) Give a definition of $|x| = a$ as a distance on the number line.

(b) Give a definition of $|x - b| = a$ as a distance on the number line.

Question 1 Evaluate the following.

(a) $|5|$ (b) $|-3|$ (c) $|7-15|$

Question 2 Verify that the following properties of absolute values are true for the values $x = 3$ and $y = -5$.

(a) $|-x| = |x|$ (b) $|xy| = |x|\,|y|$ (c) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (d) $|x-y| = |y-x|$

Question 3 [Solving equations using the distance definition]

Complete the following statements.

(a) $|x| = 2$ means that the distance of x from the origin is ___. Hence x is equal to ___ or ___.

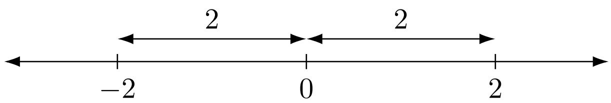
(b) $|x-2| = 5$ means that the distance of x from ___ is ___. Hence x is equal to ___ or ___.

(c) $|x+1| = 4$ means that the distance of x from ___ is ___. Hence x is equal to ___ or ___.

(d) To solve $|2x+1| = 5$, we can first divide both sides by ___ to obtain the equation ________. This means that the distance of x from ___ is ___. Hence x is equal to ___ or ___.

(e) To solve $|1-2x| = 5$, remember that $|1-2x| = |$________$|$. This can now be solved similarly to the above part.

Question 4 The diagram below shows the visual representation of part (a) above.



Draw similar diagrams for all the other parts.

Question 5 [Solving equations using the distance definition]

Use the techniques from Question 3 to solve the following equations.

(a) $|x| = 6$ (b) $|-x| = 3$ (c) $|2x| = 4$ (d) $|x+4| = 2$

(e) $|x-5| = 7$ (f) $|2x+1| = 3$ (g) $|3-2x| = 5$ (h) $|3x+5| = 16$

Question 6 [Solving equations algebraically]

Complete the following statements.

(a) $|x| = 2$ means that $x = \pm$______.

(b) $|x-2| = 3$ means that $x - 2 = \pm$______.

(c) $|2x+1| = 5$ means that $2x + 1 = \pm$______.

(d) To solve $|1-2x| = 5$, remember that $|1-2x| = |$________$|$. This can now be solved similarly to the above part.

Question 7 Use the techniques from Question 6 to solve the following equations.

(a) $|x| = 4$ (b) $|3x| = 6$ (c) $|x+2| = 5$

(d) $|x-9| = 3$ (e) $|5-x| = 3$ (f) $|2x+1| = 13$

(g) $|3-2x| = 11$ (h) $\left|\dfrac{x+2}{5}\right| = 1$ (i) $\left|\dfrac{2x+5}{3}\right| = 5$

Question 8 Complete the following method to solve the equation $|x-5| = 2x-6$.

$$\begin{aligned} |x-5| &= 2x-6 \\ x-5 &= \pm\rule{3em}{0.4pt} \\ &= \rule{2em}{0.4pt} \text{ or } \rule{2em}{0.4pt} \end{aligned}$$

Solving this, we get $x = 1$ or $x =$ _____.
However, we must test to see if the solutions are valid. We can do this either by substituting or by checking the domain.

[Substituting]
Substituting $x = 1$, we get $|1-5| = 2-6$, which becomes $|-4| = -4$, which is invalid. Hence $x = 1$ is not a solution.
Substituting $x =$ _____, we get _____ $=$ _____, which is valid/invalid (circle one).

[Checking domain]
Recall that the output of an absolute value cannot be negative.
This means that if $|x-5| = 2x-6$, then $2x-6 \geq$ _____. This means that any possible solution must satisfy $x \geq$ _____.
Only $x =$ _____ satisfies this, so the solution is $x =$ _____.

Question 9 Use a similar technique to Question 8 to solve the following equations.

(a) $|2x+1| = x$ (b) $|2x-3| = x-5$ (c) $|1-3x| = 2x+4$ (d) $|2-5x| = 6-3x$

Question 10 Determine whether the following functions are even, odd or neither.

(a) $f(x) = x + |x|$ (b) $f(x) = x \times |x|$

(c) $f(x) = \dfrac{x}{|x|-x}$ (d) $f(x) = |x-1| + |x+1|$

Question 11 Do you think the following statements are true or false for all values of x? If you think a statement can false, provide a specific value of x showing this.

(a) $|x-3| = |3-x|$ (b) $|x-1| = 1-x$ (c) $|5x| = 5|x|$

(d) $|x|^2 = x^2$ (e) $|-3x| = 3x$ (f) $\sqrt{x^2} = x$

Question 12 Sketch the graph of the following.

(a) $y = |x|$ (b) $y = |x-2|$ (c) $y = |x+1|$ (d) $y = |3-x|$

Question 13 For each of the following, state the coordinates of the vertex and the gradients of the branches, and hence sketch the graph.

(a) $y = |2x|$ (b) $y = |2x - 4|$ (c) $y = |3x + 6|$ (d) $y = \left|\frac{x}{2} + 2\right|$

Question 14 Recall the following definition of the absolute value.

$$|X| = \begin{cases} X, & \text{for} \quad X \geq 0 \\ -X, & \text{for} \quad X < 0 \end{cases}$$

Use the definition to find the equation of the branches, and their respective domains, for each of the following.

(a) $y = |x - 4|$ (b) $y = |x + 3|$ (c) $y = |2x - 3|$

Question 15 Consider the equation

$$f(x) = \frac{|x|}{x}.$$

(a) What is the x-coordinate of the location where the branches change equation?

(b) Complete the following.

$$\frac{|x|}{x} = \begin{cases} \_\_\_\_\_, & \text{for} \quad x \geq 0 \\ \_\_\_\_\_, & \text{for} \quad x < 0 \end{cases}$$

(c) Complete the following table of values.

$\boldsymbol{x}$	-2	-1	0	1	2
$\boldsymbol{f(x)}$					

(d) Hence, sketch the graph of $f(x) = \frac{|x|}{x}$.

Question 16 Use a similar technique to Question 15 to sketch the following and state the equation of $f(x)$ for each domain.

(a) $f(x) = x + |x|$ (b) $f(x) = x - |x|$ (c) $f(x) = x \times |x|$

Question 17 [Solving equations geometrically]

(a) On the same set of axes, sketch $y = |x - 2|$ and $y = \frac{x}{3}$.

(b) Find the equations of the branches of $y = |x - 2|$, and label them on your diagram.

(c) Explain how the solutions of the equation $|x - 2| = \frac{x}{3}$ are represented on your diagram.

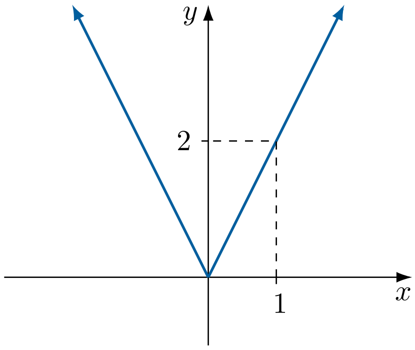
(d) Hence, find the solutions of the equation $|x - 2| = \frac{x}{3}$.

Question 18 Use a similar technique to Question 17 to solve the following geometrically.

(a) $|2x+1| = 3$ (b) $|x+1| = x$ (c) $|2x+1| = 3x$

Question 19 The diagrams below show the graph of various absolute value functions in the form $f(x) = |ax+b|$. Find the equation of each one.

(a)



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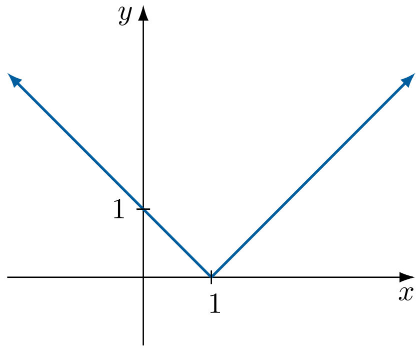
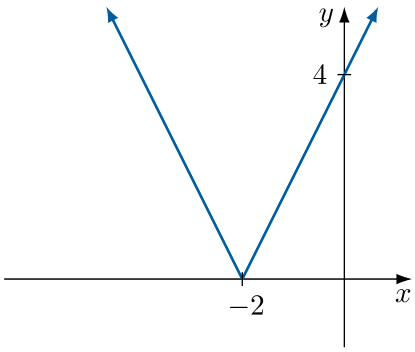
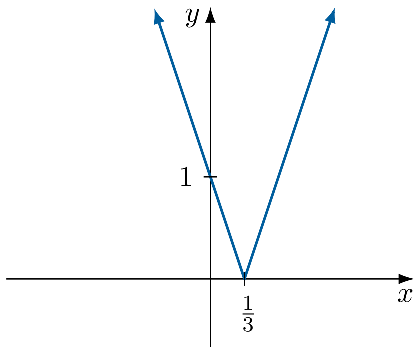


(c)



(d)



Question 20 A function is defined by the rule:

$$f(x) = \begin{cases} 9 - x^2, & \text{if} \quad x \geq 0 \\ |x+9|, & \text{if} \quad x < 0 \end{cases}$$

(a) Draw a neat sketch of this function.

(b) Find all possible values of a if $f(a) = 9$.

(c) Find all possible values of b if $f(b) = 1$.

Challenge Problems

Problem 1 By expressing as a piecewise-defined function, sketch the function $f(x) = x^2 + |x|$.

Problem 2 Simplify $f(x) = \dfrac{x^2 - 1}{|x - 1|}$ and hence sketch the graph of $y = f(x)$.

Problem 3 Explain why the equation $|x| = -1 - x^2$ has no solution, without explicitly solving for them.

Problem 4 Solve the equation $|3x - 2| = |x + 2|$ geometrically.

Problem 5

(a) Explain why when solving, we need to test the solutions of the equation $|2x - 1| = x$, but not the solutions of the equation $|2x - 1| = 1$.

(b) Do we need to test the solutions of $|2x - 1| = |x|$? Justify your answer.

Problem 6 Consider the following statement.

If $|x| = k$, then $x = \pm k$.

Is the statement always true? If not, then provide a counter-example.

Problem 7 Sketch the relation $|y| = |x|$.

Problem 8 Sketch the relation $|y| = x$.

Exercise 4C

Inverse proportion and the hyperbola

Fundamentals

Fundamentals 1

(a) If y is inversely proportional to x, then $y = \frac{k}{\rule{1cm}{0.4pt}}$, where $k \in \mathbb{R}$, $k \neq 0$.

(b) This means that as x gets l_____ in the positive direction, y becomes smaller.

(c) Conversely, it also means that as ___ gets larger in the positive direction, ___ gets small.

(d) The value k is called the c_____ of proportionality.

Fundamentals 2

For these questions, assume that $k \in \mathbb{R}$, $k \neq 0$.

(a) The graph of $y = \frac{k}{x}$ has ___ asymptotes.

(b) The asymptotes are $x =$ ___ and $y =$ ___.

(c) The domain of $y = \frac{k}{x}$ is __________.

(d) The range of $y = \frac{k}{x}$ is __________.

(e) If $k > 0$, then the graph of $y = \frac{k}{x}$ will lie in quadrants ___ and ___.

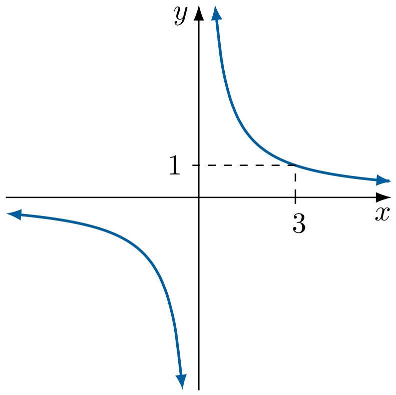
(f) If $k < 0$, then the graph of $y = \frac{k}{x}$ will lie in quadrants ___ and ___.

Question 1 Sketch the following hyperbolas. Make sure to label a unique point on your graph.

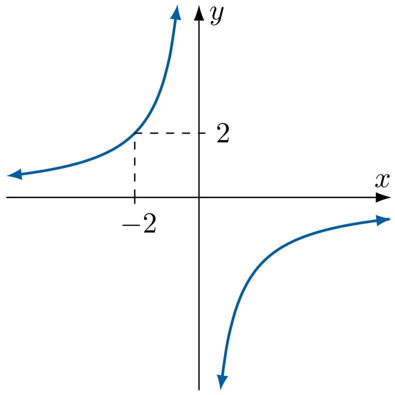
(a) $y = \frac{1}{x}$ (b) $y = \frac{4}{x}$ (c) $y = -\frac{2}{x}$ (d) $y = -\frac{1}{2x}$

Question 2 Write down the equations of the graphs shown and identify the asymptotes

(a)



(b)



Question 3 Suppose x and y are inversely proportional to each other.

(a) Write down the equation that models the relationship between x and y.

(b) When $x = 2$, $y = 3$. Find the constant of proportionality.

(c) Hence, find the value of y when $x = 12$.

(d) Sketch the graph of y against x, showing all the data above.

Question 4 Recall the formula

$$\text{Distance } (D) = \text{Speed } (S) \times \text{Time } (T)$$

(a) To which variable is time inversely proportional?

(b) To which variable is speed inversely proportional?

(c) Suppose distance is constant, so the relationship between speed and time is

$$T = \frac{k}{S}.$$

It takes 3 hours to travel to a destination at 100 km/hr. Find the value of k.

(d) How long will the same journey take at 120 km/hr instead?

(e) The journey takes 4 hours, what was the speed?

Question 5 [Boyle's Law]

Boyle's Law is a gas law that states that the pressure P in a container is inversely proportional to the volume V of the container. At a given time, the volume of the canister is 27.5 cm^3 and the pressure is 2.5 kg/cm^2

(a) Find the volume if the pressure is 3.75 kg/cm^2.

(b) Find the pressure if the volume is reduced to 20 cm^3.

Question 6 Suppose y is inversely proportional to x, so $y = \frac{k}{x}$. When x is some value x_1, y is some value y_1.

(a) When x is doubled, y is some new value y_2. Show that $y_2 = \frac{y_1}{2}$.

(b) Hence, state the effect on y if x is doubled.

(c) Use a similar technique, or otherwise, to state the effect on y if x is

(i) halved. (ii) tripled.

Question 7 It takes 15 people to do a job in 20 days.

(a) Explain briefly why this is a suitable scenario for inverse variation.

(b) Assume that the relationship is inverse variation. Form an equation that models the above scenario.

(c) Hence, how many days will it take 10 people to do the same job?

(d) If it took 75 days to do the job, how many people were working on it?

Question 8 Use graphing software to sketch $y = \frac{k}{x}$, and create a slider for $k \in (0, 20]$.

(a) Describe the effect of increasing k on the shape of the graph.

(b) Describe the effect of decreasing k on the shape of the graph.

(c) Draw $y = x$ on the same diagram. By experimenting with the slider, what value of k makes the line intersect the hyperbola at $(4, 4)$?

(d) Verify this result algebraically.

Challenge Problems

Problem 1 An army bunker currently has 60 soldiers and enough food to last them 20 days. They are joined by 40 more soldiers. Assuming inverse variation, how much less time will their food supply last?

Problem 2 Let (x_1, y_1) and (x_2, y_2) be points satisfying the same inverse variation equation

(a) Show that $x_1 y_1 = k$, where $k \in \mathbb{R}$, $k \neq 0$.

(b) Deduce that

$$\frac{x_1}{x_2} = \frac{y_2}{y_1}.$$

(c) Use this formula answer Problem 1 without needing to create a model and finding the constant of proportionality.

Exercise 4D

Circles and semi-circles

Fundamentals

Fundamentals 1

(a) The equation of the circle centred at the origin with radius r is ______________.

(b) The equation of the circle centred at (h, k) with radius r is ______________

Fundamentals 2

(a) The curves $y = \pm\sqrt{r^2 - x^2}$ represent s______-circles centred at ______ with radius ___.

(b) The semi-circle defined by $y = \sqrt{r^2 - x^2}$ is the u________ half of the circle. This is because the y coordinate of any point is a square root, which must be p______.

(c) The semi-circle defined by $y = -\sqrt{r^2 - x^2}$ is the l________ half of the circle. This is because the y coordinate of any point is a n______ square root, which must be n______.

Question 1 Write down the equation of the circles with the following features.

(a) Centre $(0, 0)$ and radius 2.

(b) Centre $(3, -2)$ and radius 2.

(c) Centre $(-2, 1)$ and radius $\sqrt{3}$.

(d) Centre $(-1, 1)$ and radius $\frac{1}{4}$.

(e) Centre (h, k) and radius r.

(f) Centre $(-p, -q)$ and radius $2r$.

Question 2 Draw a sketch of the circles in Question 1 from parts (a) to (d) inclusive.

Question 3 State the domain and range of all of the circles in Question 1 from part (a) to (d) inclusive.

Question 4 By completing the square, find the centre and radius of the following circles, and hence state the domain and range.

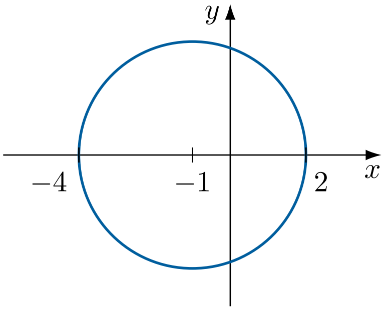
(a) $x^2 + y^2 + 10x = 0$

(b) $x^2 + y^2 - 8x + 6y - 11 = 0$

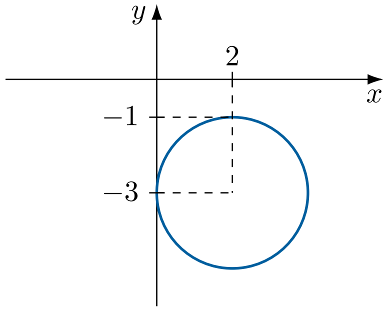
(c) $x^2 + y^2 + 6x - y = \frac{27}{4}$

Question 5 Find the centre, radius and equation of the following circles.

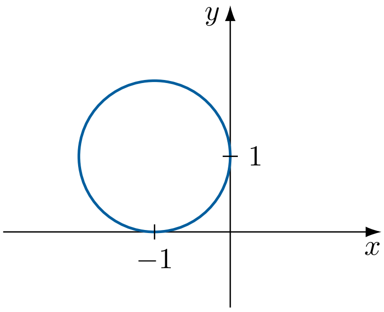
(a)



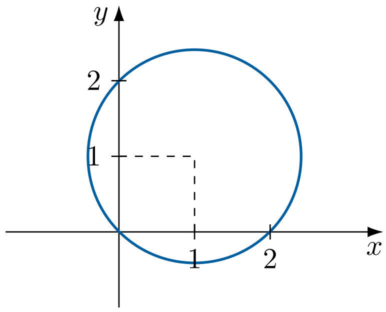
(b)



(c)



(d)



Question 6 Draw the graphs of the following semi-circles.

(a) $y = \sqrt{4 - x^2}$

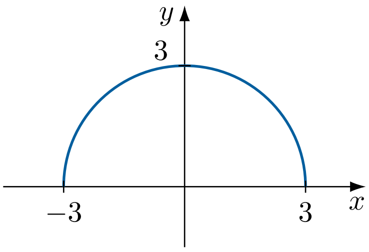
(b) $y = -\sqrt{9 - x^2}$

(c) $y = \sqrt{\frac{1}{4} - x^2}$

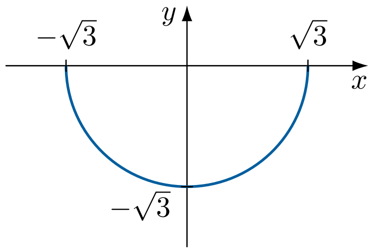
(d) $y = \sqrt{\frac{16}{9} - x^2}$

Question 7 Write down the equation of the following semi-circles.

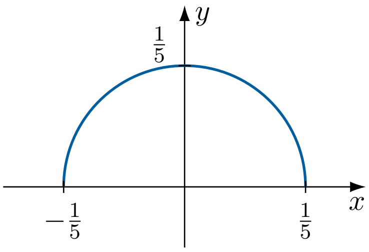
(a)



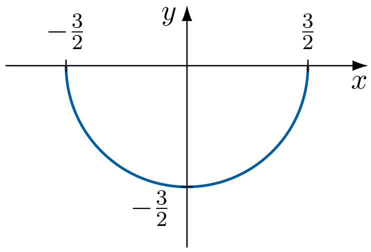
(b)



(c)



(d)



Question 8

(a) Plot the points $A(-4, 5)$ and $B(6, -3)$ on the same set of axes.

(b) Find the equation of the circle with AB being a diameter.
Hint: Find the centre of the circle first.

Challenge Problems

Problem 1 Draw the graph of the semi-circle $y = \frac{1}{3}\sqrt{1 - 9x^2}$.

Problem 2 A naval ship P is equipped with a scanner that is able to detect objects inside the circle $x^2 + y^2 - 80x - 40y - 500 = 0$, where the radius is measured in kilometres.

(a) Find the location of the ship P and the radius of the scanner.

(b) Another ship Q is located at $(60, -20)$ and it can detect objects in a region within a radius of 40 kilometres.

(i) Will ship P be able to detect ship Q?

(ii) Will ship Q be able to detect ship P?

Problem 3 [Auxiliary circle of a rectangular hyperbola]

The equation $(x + y)^2 + (x - y)^2 = a^2$ is an alternative way to represent a circle.

(a) Find the centre and radius of this circle, in terms of a.

(b) What kind of curve is $(x + y)^2 - (x - y)^2 = a^2$?
Hint: Expand and simplify.

(c) Use graphing software to sketch $(x + y)^2 + (x - y)^2 = a^2$ and $(x + y)^2 - (x - y)^2 = a^2$ on the same set of axes, and create a slider for $a \in [0, 10]$.

(d) Your diagram should show the circle and hyperbola touching each other in the first and third quadrants. By experimenting with various integer values of a, hypothesise the coordinates of the intersection points in terms of a.

(e) Verify your hypothesis algebraically by solving the two curves simultaneously.

Exercise 4E

Reflections

Fundamentals

Fundamentals 1

Consider the graph of some function $y = f(x)$.

(a) Write down the equation of the result when $f(x)$ is reflected across the y-axis.

(b) For the equation of the reflection across the y-axis, replace all instances of x with ___ .

(c) All positive x-coordinates now become n________ .

(d) All negative x-coordinates now become p________ .

(e) Although the x-coordinates change, the ___-coordinates stay the same.

Fundamentals 2

Consider the graph of some function $y = f(x)$.

(a) Write down the equation of the result when $f(x)$ is reflected across the x-axis.

(b) For the equation of the reflection across the x-axis, replace all instances of y with _____ .

(c) Since we usually define functions to be in the form $y = f(x)$, this is the same as multiplying the entire function by ___ .

(d) All positive y-coordinates now become n________ .

(e) All negative y-coordinates now become p________ .

(f) Although the y-coordinates change, the ___-coordinates stay the same.

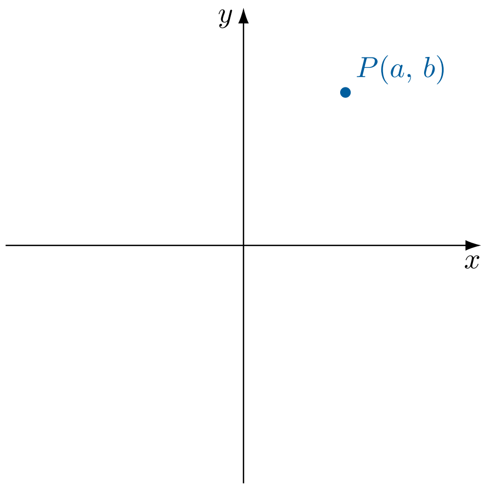
Fundamentals 3

Consider the graph of $y = f(x)$.

(a) Write down the equation of the result when $f(x)$ is reflected across the x-axis, and then across the y-axes.

(b) Write down the equation of the result when $f(x)$ is reflected across the y-axis, and then across the x-axes.

Question 1 The point $P(a, b)$ is shown in the diagram below.



(a) The point P is reflected across the x-axis. Label the result Q, and write down the coordinates.

(b) The point P is reflected across the y-axis. Label the result R, and write down the coordinates.

(c) The point R is reflected across the x-axis. Label the result S, and write down the coordinates.

(d) The point Q is reflected across the y-axis. What do you notice?

Question 2 Define the function $f(x) = x^2 - 2x$.

(a) Sketch the graph of $y = f(x)$, labelling the vertex and x-intercepts.

(b) Simplify the equation $y = -f(x)$ and sketch the graph on the same set of axes using a different coloured pen, labelling the new vertex and x-intercepts.

(c) What is the relationship between the graph of $y = f(x)$ and $y = -f(x)$?

Question 3 Define the function $f(x) = 4x - x^2$.

(a) Sketch the graph of $y = f(x)$, labelling the vertex and x-intercepts.

(b) Simplify the equation $y = f(-x)$ and sketch the graph on the same set of axes using a different coloured pen, labelling the new vertex and x-intercepts.

(c) What is the relationship between the graph of $y = f(x)$ and $y = f(-x)$?

Question 4 Define the function $f(x) = x^2 + 2x$.

(a) Sketch the graph of $y = f(x)$, labelling the vertex and x-intercepts.

(b) Write down the equation of the reflection across the y-axis, and call this function $g(x)$.

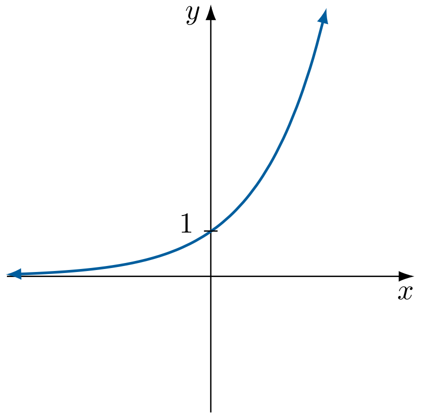
(c) Write down the equation of the reflection of $g(x)$ across the x-axis. Express your answer in the form $y = \dots$

(d) Simplify $y = -f(-x)$. What do you notice about this answer and part (c)?

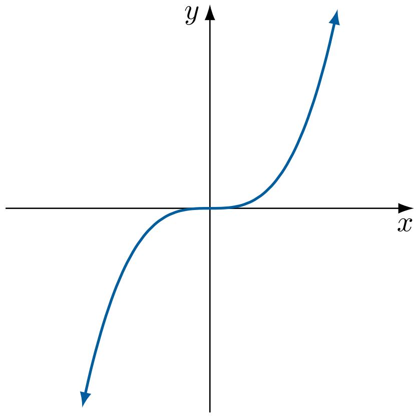
(e) What is the relationship between the graph of $y = f(x)$ and $y = -f(-x)$?

Question 5 The diagrams below show the graphs of $y = f(x)$. For each of the graphs below, sketch on the same set of axes and label all possible features: (i) $y = -f(x)$. (ii) $y = f(-x)$. (iii) $y = -f(-x)$.

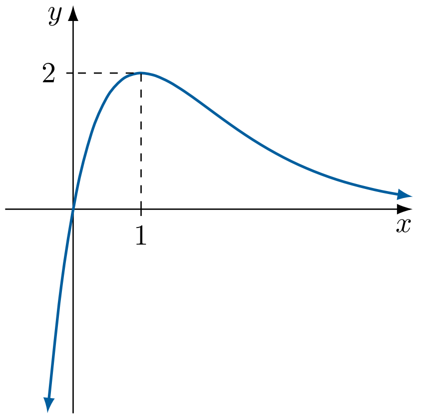
(a)



(b)



(c)



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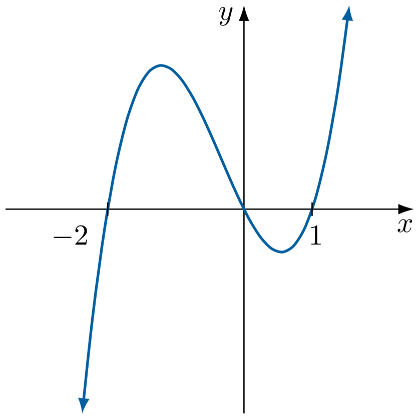
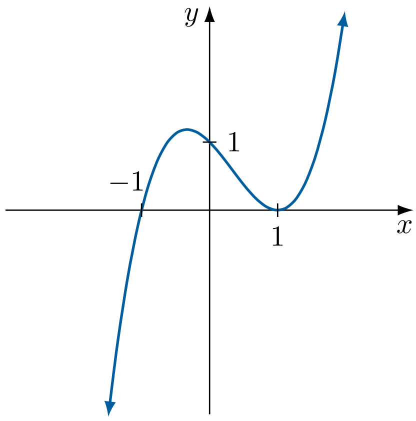


Question 6 For each of the following functions, find the equation of and sketch the: (i) reflection across the x-axis. (ii) reflection across the y-axis. (iii) reflection across both axes.

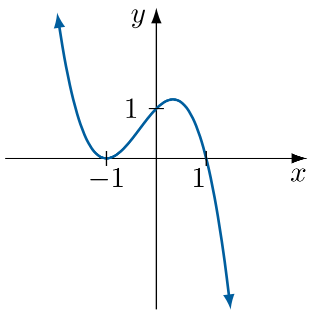
(a) $f(x) = x^2 - 4x$ (b) $g(x) = 4 - x^2$ (c) $h(x) = x^3$

Question 7 The diagram below is a sketch of $y = f(x)$.

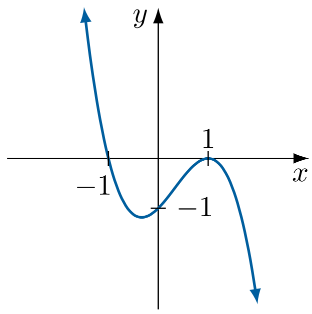


The diagrams below show sketches of either $y = -f(x)$, $y = f(-x)$ or $y = -f(-x)$. Identify which transformation was applied to the graph of $y = f(x)$ to obtain the following.

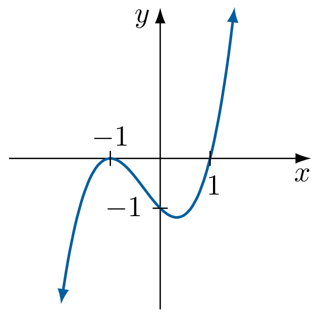
(a)



(b)



(c)



Challenge Problems

Problem 1 Consider the function $y = |x - 2|$.

(a) Write down the equation of the reflection across the x-axis.

(b) Now, write down the reflection of *that* curve across the y-axis.

(c) Repeat parts (a) and (b), but the other way around. Verify that you obtain the same result as part (b).

Problem 2 Which of the following equations is the reflection of $y = (x - 1)(x + 1)^2$ across both the x and y-axes?

(a) $y = (x^2 - 1)(x + 1)$ (b) $y = (x^2 - 1)(x - 1)$ (c) $y = (x^2 - 1)(1 - x)$

Problem 3 Let $f(x)$ be an odd function. Prove that reflecting the graph of $f(x)$ across both the x and y-axes will result in the exact same graph.

Chapter 4 Review

Further Functions and Relations

Review

Question 1 Write down the equation of the monic polynomial that has the following features.

(a) Single roots at $x = 0$ and $x = \pm 3$.

(b) Single roots at $x = \pm 2$ and a triple root at $x = 0$.

(c) Double roots at $x = 0$ and $x = 2$.

Question 2 Sketch the following polynomials

(a) $f(x) = (x-1)(x-2)(x+2)$

(b) $f(x) = (x+2)^2(5-x)$

(c) $f(x) = (1-x)^2(x+2)$

(d) $f(x) = (x-5)^2(x-1)(x+3)$

Question 3 Factorise the following, and hence sketch the graph.

(a) $y = x^3 - x^2 - 12x$

(b) $y = x^3 + 6x^2 + 9x$

(c) $y = 3x^3 + 10x^2 - 8x$

Question 4 Write a monic cubic equation with roots being

(a) $1, 3, -2$

(b) $-3, 4, 4$

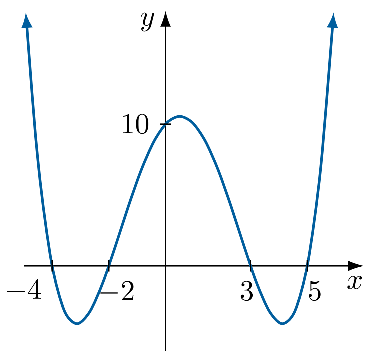
Question 5 Find the value of k if

(a) $x = -3$ is a zero of $P(x) = x^3 + 9x^2 + kx + 27$

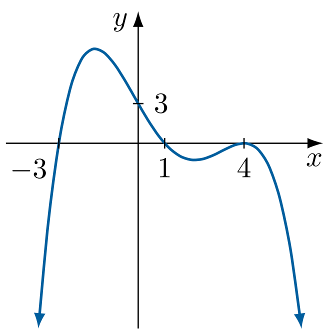
(b) $x = 2$ is a zero of $P(x) = kx^3 + 2x^2 - 41x + 10$

Question 6 Find the equation of the quartic polynomial satisfying the following shapes below.

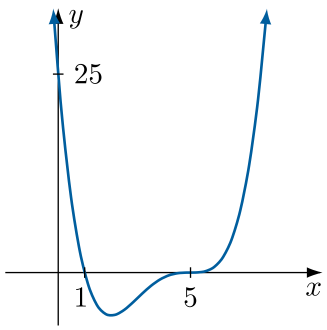
(a)



(b)



(c)



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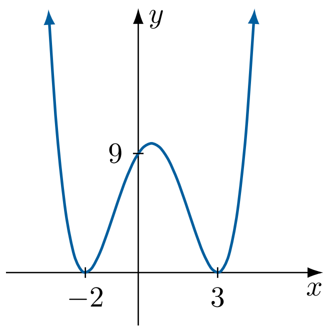


Question 7 Solve the following absolute value equations for x.

(a) $|x+5|=3$ (b) $|7-2x|=5$ (c) $|5-3x|=x+1$

(d) $|3x-4|=2x-5$ (e) $|3x-1|=3-x$ (f) $|3x-1|=|3-x|$

(g) $|x^2-5|=5x+9$ (h) $\dfrac{1}{|x-1|}=5$

Question 8

(a) Explain why the equation $|x+3|=-5$ has no solutions.

(b) Justify your answer graphically.

Question 9 Sketch the following.

(a) $y=|x-1|$ (b) $y=|2x+3|$ (c) $y=-|2-x|$ (d) $y=-|6-3x|$

Question 10 Sketch the following, and label the coordinates of any point on the curve.

(a) $y=\dfrac{4}{x}$ (b) $y=-\dfrac{3}{x}$

Question 11 Suppose that y is inversely proportional to x.

(a) Write down a general formula that shows the relationship between x and y.

(b) Suppose that when $x=3$, $y=2$. Find the constant of proportionality.

(c) Hence, find the value of y when $x=6$.

Question 12 Suppose that y is inversely proportional to x such that when $x=\dfrac{1}{2}$, $y=4$

(a) Find the equation that relates x and y.

(b) If x is doubled, then what is the effect on y?

(c) If x is divided by 3, then what is the effect on y?

(d) If y is halved, then what is the effect on x?

(e) Are the results from the above three parts true in general for any quantities that are inversely proportional?

Question 13 A company can complete a job in 3 hours with 12 workers.

(a) How many workers will it need to complete the same job in 2 hours?

(b) If the company allocates 15 workers, how long will it take to complete the same job?

Question 14 Find the radius and centre of the following circles, and hence draw a sketch.

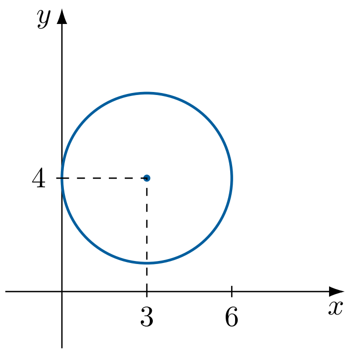
(a) $x^2 + y^2 - 10x = 0$

(b) $x^2 + y^2 - 4x - 6y - 3 = 0$

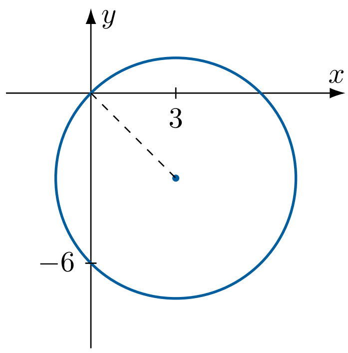
Question 15 State the domain and range, using interval notation, of the circles in the previous question.

Question 16 Write down the equations of the following circles, and state any intercepts with the coordinate axes.

(a)



(b)



Question 17 Write down the equations of the four circles with radius $2\sqrt{2}$ that pass through the origin.

Question 18 Write down the equation of the result when the following curves are reflected across the x-axis, y-axis, and both axes.

(a) $f(x) = x^3$

(b) $f(x) = (x - 2)^2$

(c) $f(x) = \dfrac{1}{x}$

(d) $f(x) = x^2 - x - 2$

(e) $f(x) = 2^x$

(f) $f(x) = x^3 - x$

Question 19 For each of the transformations in the previous question, sketch the original and sketch the reflection on the same set of axes, but using a different colour pen.

Investigation Task

Multiple Roots of Polynomials

This investigation task will allow students to further explore the nature of multiple roots, and how they affect the shape of the graph of the polynomial.

Question 1 Investigate what the term *multiple root* means in the context of polynomials. Categorise the different types and draw sketches demonstrating how they are different. Your answer should include discussion of roots that have powers beyond 3.

Question 2 Use graphing software to investigate the nature of the roots of the following products of polynomials, for $n = 1, 2, 3, 4$. What kinds of roots from your classification in Question 1 form for various values of n? Does it form a maximum or a minimum, or neither? Does it affect whether the polynomial is 'negative' or 'positive'?

(a) $f(x) = x^n$

(b) $f(x) = x^n(x - 4)$

(c) $f(x) = x^n(4 - x)$

(d) $f(x) = (x - 4)^n$

(e) $f(x) = x(4 - x)^n$

(f) $f(x) = x(4 - x^n)$

Question 3 Use graphing software to investigate the number of roots and the 'type' of roots of the following products of polynomials, for $n = 1, 2, 3, 4$ and $m = 2, 3, 4$.

(a) $f(x) = x^n(x - 4)^m$

(b) $f(x) = x^n(x - 4)^m(x + 4)^m$

Investigation Task

Effect of Parameters on the Roots of Polynomials

This investigation task aims to provide a geometric understanding of solutions of polynomials, and how various parameters can affect the number and nature of the roots.

Question 1 Consider the quadratic polynomial $y = 6x - x^2$ and a horizontal line $y = k$.

(a) Sketch the parabola and label the coordinates of the vertex.

(b) On your diagram, draw the line $y = k$ for $k = 6$, 9 and 12.

(c) For what values of k will the line intersect the parabola twice? How about once or even no times at all?

(d) Solve the quadratic polynomial simultaneously with the horizontal line and show that this results in the quadratic equation $x^2 - 6x + k = 0$.

(e) What is the geometric significance of the solutions of this equation?

(f) Use the discriminant to verify your answer from part (c).

Question 2 Consider the quadratic polynomial $P(x) = x^2 - m$.

(a) Sketch $y = m$, for various values of m, and sketch $y = x^2$. Comment on which values of m yield differing numbers of intersection points.

(b) What is the relationship between the intersection points of $y = x^2$ and $y = m$, with respect to roots of the polynomial $P(x) = x^2 - m$?

(c) Hence, use your diagram to state explicitly for what value(s) of m the polynomial has two real roots, one real root, and no real roots.

Question 3 Consider the cubic polynomial $P(x) = x^3 - 9x^2 - m$.

(a) Sketch $y = x^3 - 9x^2$ and $y = m$ using graphing software, and create a slider for m.

(b) What is the relationship between the graphs of $y = x^3 - 9x^2$ and $y = m$, with respect to roots of the polynomial $P(x) = x^3 - 9x^2 - m$?

(c) Describe the effect of m on the number of intersection points, and state any special values of m where the number of intersection points changes. What is the relationship between these values of m and the roots of $P(x)$?

(d) Verify your answer by sketching $y = P(x)$, with a slider for m.

Investigation Task

Adding and Subtracting Absolute Value Graphs

We are now quite familiar with sketching the graphs of equations such as $y = |3x-2|$ or $y = x \times |x|$. In all cases covered thus far, there has only been one set of absolute values to worry about. However, when there is more than one, the scenario can become a lot more complicated. This investigation task will allow the student to explore these scenarios more thoroughly.

Question 1 Use graphing software to sketch various functions in the form

$$f(x) = |ax+b| \pm |cx+d|,$$

for varying values of a, b, c and d.

(a) How many different general 'shapes' can you observe forming?

(b) List all the possible general shapes that you find, and state the particular values of a, b, c and d that resulted in each one.

(c) What is the significance of $x = -\dfrac{b}{a}$ and $x = -\dfrac{d}{c}$?

Question 2 Consider the equation

$$f(x) = |x+1| + |x-2|.$$

(a) From your findings in Question 1, what kind of shape should the graph have?

(b) What are the 'critical' x-coordinates where the shape may change?

(c) These critical points split the x-axis into three regions. Write down what the three regions are as inequalities.

(d) Complete and simplify the following

$$|X| = \begin{cases} X, & \text{for } \quad x > \underline{\quad} \\ \underline{\quad}, & \text{for } \quad X < 0 \end{cases}$$

(e) Hence, complete and simplify the following for $f(x)$ as defined above.

$$f(x) = \begin{cases} (\underline{\qquad}) + (\underline{\qquad}) & \text{for } \quad x > \underline{\quad} \\ (\underline{\qquad}) + (\underline{\qquad}), & \text{for } \quad \underline{\quad} \leq x \leq \underline{\quad} \\ (\underline{\qquad}) + (\underline{\qquad}), & \text{for } \quad x < \underline{\quad} \end{cases}$$

(f) Hence, sketch the graph of $y = f(x)$ and label the branches.

Question 3 Explain why if if $a \neq b$, then there will be no horizontal sections whereas if $a = b$ then there will be.

5

TRIGONOMETRY

- Right-angled triangles
- Exact values
- Quadrants and related angles
- Trigonometric identities
- Solving trigonometric equations with degrees
- Sine rule and the area formula
- Cosine rule
- Three-dimensional trigonometry

Exercise 5A

Right-angled triangles

Fundamentals

Fundamentals 1

(a) State the formula of Pythagoras' theorem.

(b) Define the angle of elevation.

(c) Define the angle of depression.

(d) Draw a clear diagram that illustrates both of the above on a diagram.

Fundamentals 2

Complete the following trigonometric ratios.

(a) $\sin\theta = \dfrac{\text{opposite}}{}$ (b) $\cos\theta = \dfrac{}{}$ (c) $\tan\theta = \dfrac{}{}$

Fundamentals 3

Complete the following reciprocal ratios.

(a) $\operatorname{cosec}\theta = \dfrac{1}{}$ where $\sin\theta \neq$ ___.

(b) $\sec\theta = \dfrac{1}{}$ where __________.

(c) $\cot\theta = \dfrac{1}{}$ where __________.

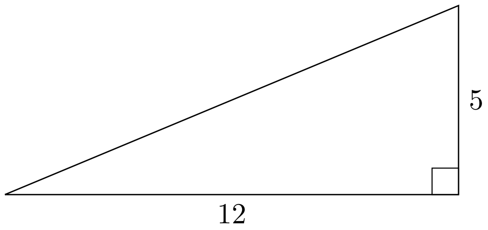
Fundamentals 4

(a) Describe the rules that define compass bearings, in terms of how it is measured.

(b) Describe the rules that define true bearing, in terms of how it is measured.

Question 1 Find the missing lengths in the diagrams below.

(a)



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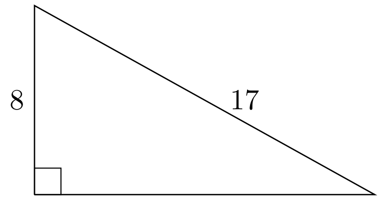


Question 2 Complete the following Pythagorean triads.

(a) $\{3, 4, \ldots\}$ (b) $\{7, 24, \ldots\}$ (c) $\{9, 40, \ldots\}$

Question 3 wUse your calculator to find the following, correct to two decimal places.

(a) $\sin 40°$ (b) $\cos 75°$ (c) $\tan 22°30'$

(d) $\sec 50°23'$ (e) $\operatorname{cosec} 37°52'$ (f) $\cot 17°47'$

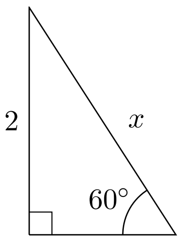
Question 4 Use your calculator to find the acute angle θ, correct to the nearest minute.

(a) $\sin\theta = 0.1$ (b) $\cos\theta = 0.73$ (c) $\tan\theta = \dfrac{1}{3}$

(d) $\sec\theta = 4$ (e) $\operatorname{cosec}\theta = \dfrac{7}{5}$ (f) $\cot\theta = 0.27$

Question 5 For each of the following, use a suitable trigonometric ratio to find the value of x, correct to one decimal place.

(a)



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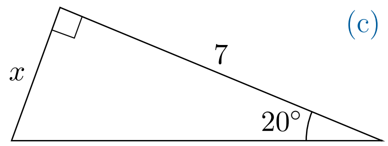
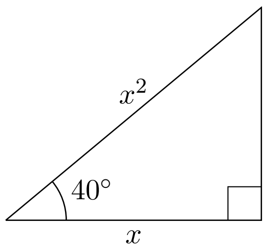
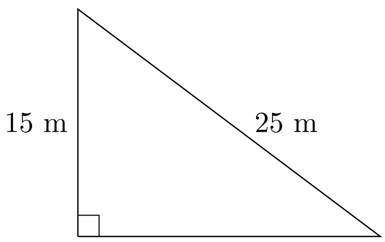


(c)



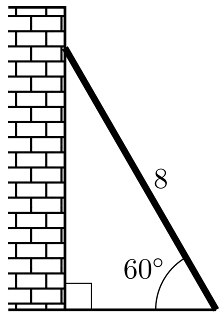
Question 6 A 25 m pole is propped against a vertical wall and it reaches 15 m up the wall.



(a) How far is the foot of the pole from the wall?

(b) If the top of the pole slips 8 m down the wall, how much did the foot slip away from the wall?

Question 7 A ladder 8 m long standing on level ground leans against a vertical wall and makes an angle of 60° with the ground. Calculate the following.

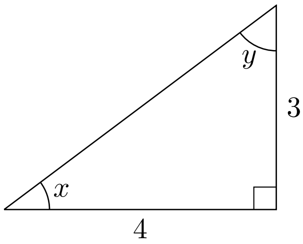


(a) How high up the wall the ladder reaches.

(b) The distance of the foot of the ladder from the wall.

Question 8

(a) Use Pythagoras' theorem to find the missing side in the following triangle.



(b) Hence, write down the value of

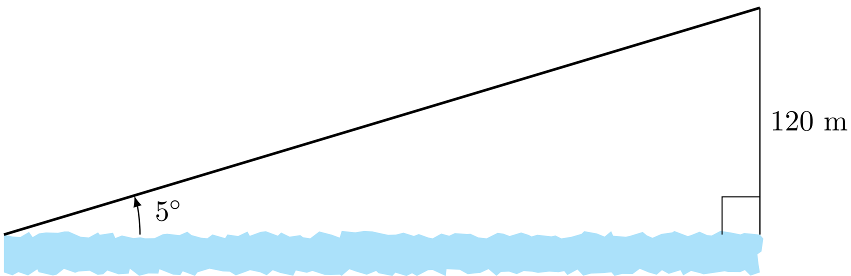
(i) $\sin y$ (ii) $\cos y$ (iii) $\tan x$

(iv) $\cot x$ (v) $\operatorname{cosec} y$ (vi) $\sec x$

Question 9 Draw a diagram to illustrate

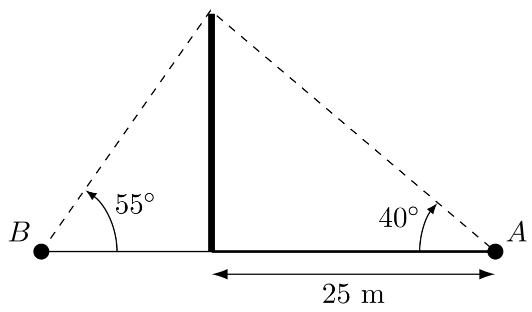
(a) an angle of depression of 30°. (b) an angle of elevation of 40°.

Question 10 A boat out at sea observes a house on top of a 120m cliff at an angle of elevation of 5°.



How far out to sea is the boat?

Question 11 From two points A and B on opposite sides of a telegraph pole, the angles of elevation of the top of the pole are found to be 40° and 55° respectively, as shown in the diagram below.



If A is 25 m from the base of the pole, find

(a) the height of the pole to the nearest metre

(b) how far B is from the base of the pole.

Question 12 The elevation of the top of a tower T 120 m high taken from two points P and Q on opposite sides of it were measured as 60° and 45° respectively.

Find the distance from P to Q correct to 1 decimal point.

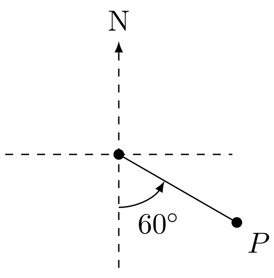
Question 13 From a point 6 m above the ground, the angle of depression of the bottom of a wall is 18° and the angle of elevation of the top of the wall is 30°. Find the following.

(a) The horizontal distance from the point of observation to the wall.

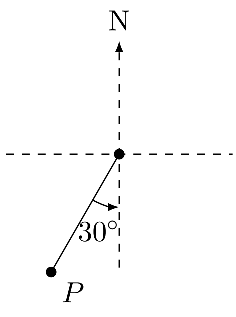
(b) The height of the wall.

Question 14 Express the bearings shown in each diagram as a compass bearing and a true bearing.

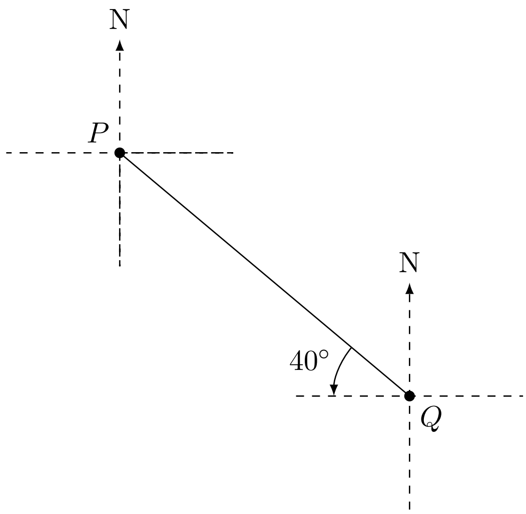
(a)



(b)



Question 15 Express the bearings shown in each diagram as a compass bearing and a true bearing.



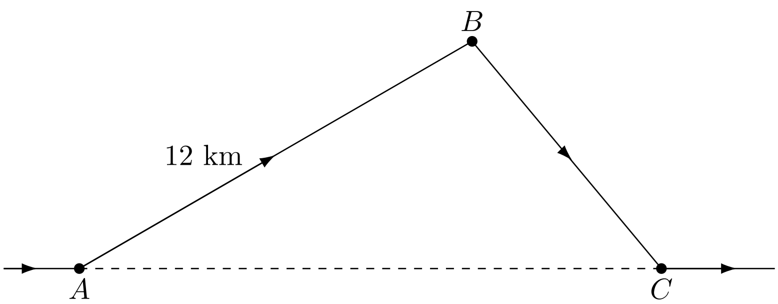
(a) From P to Q

(b) From Q to P.

Question 16 Town B is 800 km due south of town A, while town C is due west of A. The bearing of C from B is 300°. Find the distance BC.

Question 17 A bushwalker has walked 12 km North from base B and then 5 km in an Easterly direction to a lookout at L. What is the bearing of L from B (to the nearest degree)?

Question 18 A mining surveyor is travelling due East and to avoid a flooded area, a detour is made by first driving from A to B, a distance of 12 kilometres, on a bearing of 060°. From B the surveyor travels south-east until re-joining the original line of travel at C.



(a) Calculate the how far north the surveyor had to travel.

(b) By considering two different triangles, find the distance AC (to the nearest tenth of a kilometre).

Question 19 From a point O, the point P has a bearing of 135° and is 12 km away. The point Q is 16 km away from O on a bearing of 225°. Calculate the distance PQ.

Challenge Problems

Problem 1 A ship leaves point M and travels 300 km to point P, on a bearing of $N65°E$. It then turns to travel 400 km at a bearing of $S25°E$ to the point Q.

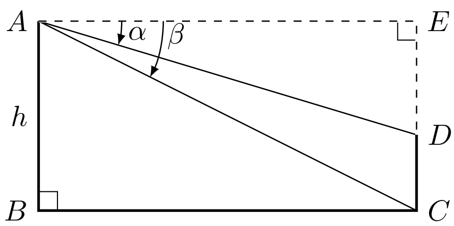
(a) What is the size of $\angle MPQ$?

(b) How far is the ship now from its original position?

(c) Find $\angle PQM$.

(d) Hence, find the true bearing of Q from M and the true bearing of M from Q.

Problem 2 From an observation tower AB, an observer measured an angle of depression of α and β to the top and bottom, respectively, of another tower CD. Let E be the point directly above tower CD such that CE is the same height h as AB.



(a) Show that $DE = BC\tan\alpha$.

(b) By first finding an expression for BC, show that $DE = \dfrac{h\tan\alpha}{\tan\beta}$.

(c) Hence, show that tower CD has height

$$h\left(\frac{\tan\beta - \tan\alpha}{\tan\beta}\right).$$

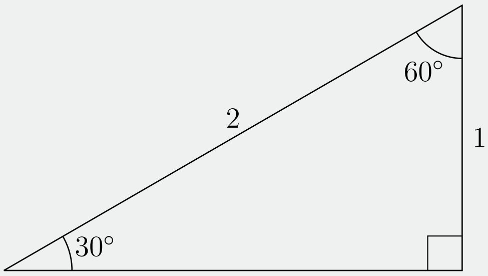
Exercise 5B

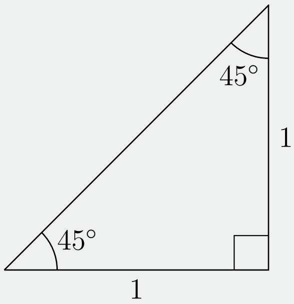
Exact values

Fundamentals

Fundamentals 1

(a) Find the missing sides in the triangles below.





(b) Hence, fill the following table

	30°	**60°**	**45°**
$\sin\theta$			
$\cos\theta$			
$\tan\theta$			

Question 1 Find the exact values of the following reciprocal trigonometric ratios.

	30°	**60°**	**45°**
$\operatorname{cosec}\theta$			
$\sec\theta$			
$\cot\theta$			

Question 2 Simplify the following and express your answer as an exact value in simplest form.

(a) $\sin 45° + \cos 45°$

(b) $\sin^2 45°$

(c) $\cos 45° \sec 45°$

(d) $\sec^2 60° - \tan^2 60°$

(e) $2\cos^2 30° - 1$

(f) $\sin 60° \cos 30° + \cos 60° \sin 30°$

Question 3 Show that

(a) $\dfrac{1+\tan 30^\circ}{1-\tan 30^\circ} = 2+\sqrt{3}$

(b) $\sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

(c) $\dfrac{\tan 60^\circ - \tan 45^\circ}{1+\tan 60^\circ \tan 45^\circ} = 2-\sqrt{3}$

(d) $\dfrac{1}{\sec 30^\circ - \tan 30^\circ} = \sqrt{3}$

Question 4 Suppose that θ is an acute angle such that $\sin\theta = \dfrac{2}{3}$.

(a) Draw a right-angled triangle with an angle θ such that $\sin\theta = \dfrac{2}{3}$.

(b) Use Pythagoras' Theorem to find the missing side

(c) Hence, find the exact value of the following.

(i) $\cos\theta$ (ii) $\tan\theta$

Question 5 Suppose θ is an acute angle. Use a similar technique to Question 4 to find

(a) $\tan\theta$ given that $\sin\theta = \dfrac{3}{5}$

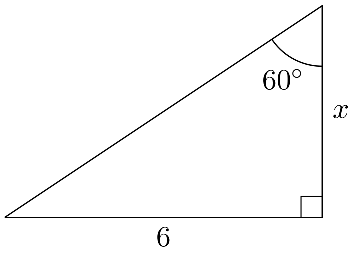
(b) $\sin\theta$ given that $\cot\theta = \dfrac{2}{3}$

(c) $\cos\theta$ given that $\sin\theta = \dfrac{1}{\sqrt{3}}$

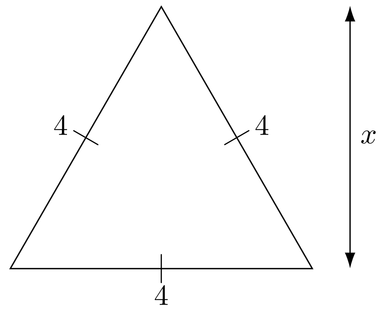
(d) $\operatorname{cosec}\theta$ given that $\tan\theta = \dfrac{5}{12}$

Question 6 Find the exact value of x in each of the following diagrams.

(a)



(b)



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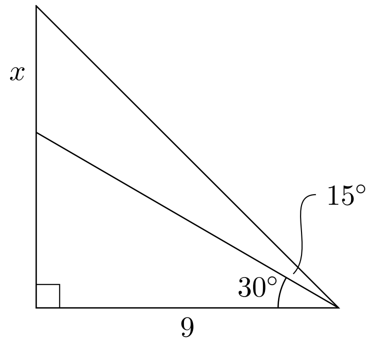
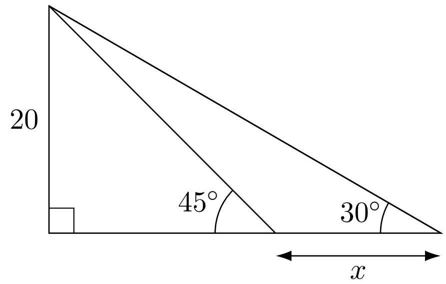
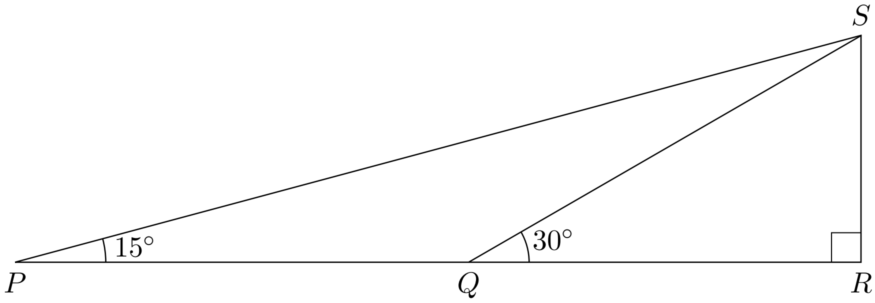


(d)



Question 7 Consider the diagram below.



(a) Show that $PQ = QS$.

(b) Given that the length of SR is 1 unit, write down the lengths of QR, QS and PQ.

(c) Deduce that $\tan(15°) = 2 - \sqrt{3}$.

Challenge Problems

Problem 1 [Deriving the exact value of $\tan 15°$ and $\tan 22.5°$]

For this question, you may assume the identity $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$.

(a) Let $t = \tan 15°$. Show that $\dfrac{2t}{1-t^2} = \dfrac{1}{\sqrt{3}}$.

(b) Show that $t^2 + 2\sqrt{3}t - 1 = 0$.

(c) Hence, show that $\tan 15° = 2 - \sqrt{3}$.

(d) Use a similar technique to show that $\tan 22.5° = \sqrt{2} - 1$.

Problem 2 [Deriving the exact value of $\sin 15°$]

For this question, you may assume the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

(a) Let $s = \sin 15°$. Show that $s = \dfrac{\sqrt{2-\sqrt{3}}}{2}$.

(b) Expand and simplify $\dfrac{(\sqrt{3}-1)^2}{2}$.

(c) Hence, show that $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

Exercise 5C

Quadrants and related angles

Fundamentals

Fundamentals 1

(a) θ is positive for angles measured in an a____________ direction.

(b) θ is negative for angles measured in a c________ direction.

Fundamentals 2

(a) $\sin\theta$ is positive only in quadrants _____ and _____.

(b) $\cos\theta$ is positive only in quadrants _____ and _____.

(c) $\tan\theta$ is positive only in quadrants _____ and _____.

Fundamentals 3

(a) The angle $180° - \theta$ is said to be s________ to θ.

(b) When we apply a trigonometric ratio to either $180° \pm \theta$ or $360° \pm \theta$, we will always get the m________ of the same ratio applied to just θ, except with a plus or minus depending on what q________ the angle is in.

Fundamentals 4

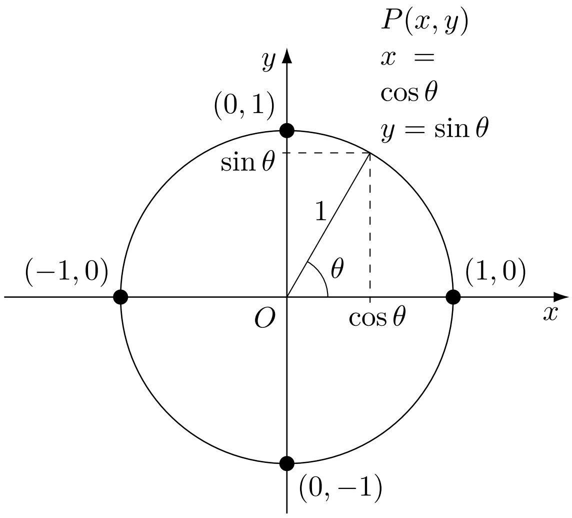
(a) The angle $90° - \theta$ is said to be c________ to θ.

(b) When we apply a trigonometric ratio to $90° - \theta$, we will always get the magnitude of the c__________ ratio applied to just θ.

(c) For example, the sine of $90° - \theta$ is equal to the complementary sine of θ, otherwise known as c_____ of θ.

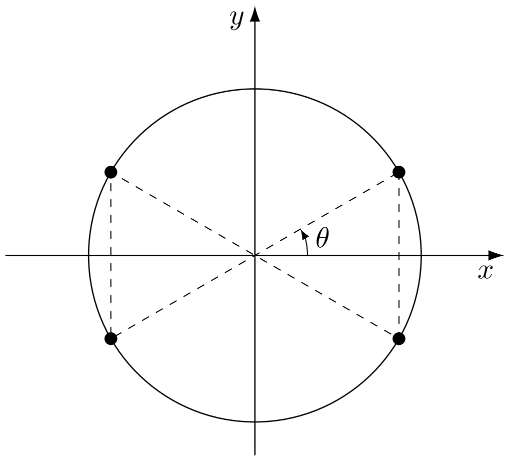
(d) Similarly, the tan of $90° - \theta$ is equal to the complementary tan of θ, otherwise known as c________ of θ.

Question 1 Use the diagram below to find the exact value of the following.



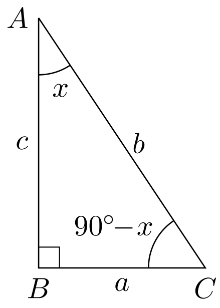
(a) $\sin(0°)$	(b) $\cos(0°)$	(c) $\sin(90°)$	(d) $\cos(90°)$
(e) $\sin(-90°)$	(f) $\cos(-90°)$	(g) $\sin(180°)$	(h) $\cos(180°)$
(i) $\sin(-180°)$	(j) $\cos(-180°)$	(k) $\sin(270°)$	(l) $\cos(270°)$
(m) $\sin(-270°)$	(n) $\cos(-270°)$	(o) $\sin(360°)$	(p) $\cos(360°)$

Question 2 Use the diagram below to find the exact value of the following.



(a) $\sin(180-\theta)$	(b) $\cos(180-\theta)$	(c) $\tan(180-\theta)$	(d) $\sin(180+\theta)$
(e) $\cos(180+\theta)$	(f) $\tan(180+\theta)$	(g) $\sin(360-\theta)$	(h) $\cos(360-\theta)$
(i) $\tan(360-\theta)$	(j) $\sin(-\theta)$	(k) $\cos(-\theta)$	(l) $\tan(-\theta)$

Question 3 Use the diagram below to simplify the following.



(a) $\sin(90 - x)$ (b) $\cos(90 - x)$ (c) $\tan(90 - x)$

Question 4 Let β be an acute angle. Find the value of β if

(a) $\tan\beta = \cot 64°$ (b) $\sin\beta = \cos 60°$ (c) $\cos\beta = \sin 45°$ (d) $\sin\beta = \cos 10°$

Question 5 Simplify

(a) $\tan x \cot x$ (b) $\sin x \operatorname{cosec} x$ (c) $\cos x \sec x$ (d) $\cot x \sin x$

Question 6 Simplify without using a calculator

(a) $\tan 20° + \cot 70°$ (b) $\tan 20° \tan 70°$

(c) $\tan 35° - \dfrac{1}{\tan 55°}$ (d) $\dfrac{\sin 20° + \cos 70°}{\sin 20°}$

(e) $\dfrac{\cos 42° + 2\sin 48°}{5\sin 48°}$ (f) $\dfrac{2\tan 56° + 3\cot 34°}{7\tan 56°}$

Question 7 Write the following as exact values

(a) $\cos 60°$ (b) $\tan 45°$ (c) $\tan 225°$ (d) $\cos 225°$

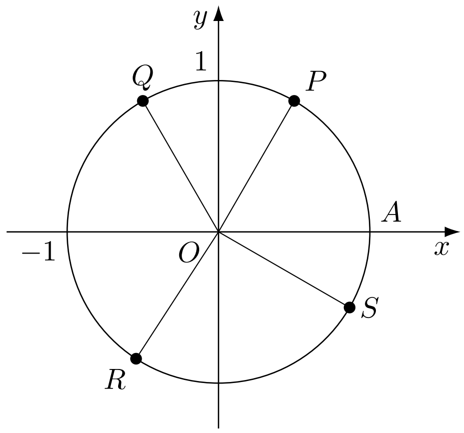
(e) $\sin 150°$ (f) $\sin 120°$ (g) $\cos 240°$ (h) $\cos 330°$

(i) $\sin 300°$ (j) $\tan 135°$ (k) $\tan 210°$ (l) $\cos 210°$

Question 8 Simplify the following by referring to the unit circle

(a) $\sin(x + 360°)$ (b) $\sin(x + 720°)$ (c) $\cos(x - 360°)$ (d) $\cos(x + 360°)$

Question 9 The points P, Q R, S are on the unit circle $x^2 + y^2 = 1$. $A(1, 0)$ and O is the origin. Find the exact coordinates of these points if



(a) $\angle AOP = 60°$ (b) $\angle AOQ = 120°$ (c) $\angle AOR = 240°$ (d) $\angle AOS = 30°$

Question 10 Consider the unit circle from Question 9.

(a) Let the point T be in the first quadrant with a y-coordinate of $\frac{1}{3}$. Find the exact value of

(i) the x coordinate of point T.
(ii) $\cos(180° - \angle TOA)$
(iii) $\tan(180° - \angle TOA)$
(iv) $\tan(180° + \angle TOA)$

(b) Let the point W be in the second quadrant with a y-coordinate of $\frac{1}{\sqrt{2}}$. Find the exact value of

(i) the x-coordinate of W.
(ii) $\angle WOA$
(iii) $\sin(\angle WOA)$
(iv) $\cos(\angle WOA)$

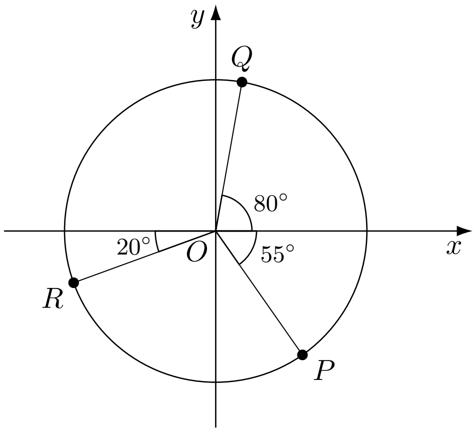
Question 11 Write the following as exact values

(a) $\sin(-60°)$ (b) $\tan(-60°)$ (c) $\tan(-120°)$
(d) $\cos(-45°)$ (e) $\cos(-120°)$ (f) $\sin(-585°)$

Question 12 Write the following as exact values

(a) $\sin 480°$ (b) $\tan 405°$ (c) $\sin 750°$
(d) $\tan 420°$ (e) $\cos 420°$ (f) $\sin 570°$

Question 13 The diagram below shows three points P, Q and R on the unit circle. Find the coordinates of P, Q and R. Leave your answer in exact form.



Question 14 Show that the following relationships are satisfied by the given values

(a) $\sin 2x = 2\sin x \cos x$ when $x = 120°$

(b) $\cos 2x = \cos^2 x - \sin^2 x$ when $x = 150°$

(c) $\sin(A + B) = \sin A \cos B + \cos A \sin B$ when $A = 210°$ and $B = 330°$

Question 15 Simplify

(a) $\dfrac{\sin(180° + \beta)}{\cos(360° - \beta)}$

(b) $\dfrac{\cos(180° - \beta)}{\sin(90° - \beta)}$

(c) $\sin B \cos(90° - B) + \cos B \sin(90° - B)$

(d) $\sin(180° + \beta)\cos(90° - \beta)$

(e) $\dfrac{\tan(180° + \beta)}{\sec(180° - \beta)}$

(f) $\dfrac{\cos(360° + \beta)}{\tan(90° - \beta)}$

Question 16 Show that

$$\frac{\cos(90° - \theta)\sin(180° - \theta)}{\cos(180° + \theta)\cos(-\theta)} = -\tan^2\theta.$$

Question 17 Determine whether the following statements are true or false.

(a) $\sin(180° + \beta) = -\sin(\beta)$

(b) $\cos(90° - \beta) = \sin\beta$

(c) $\tan\beta = \cot(90° - \beta)$

(d) $\cos(180° - \beta) = -\sin\beta$

(e) $\cot(180° - \beta)\tan(360° - \beta) < 0$

(f) $\dfrac{\sec(-\beta)}{\cos(360° - \beta)} > 0$

Challenge Problems

Problem 1

(a) If $\sin\alpha = \frac{3}{5}$ and $90° < \alpha < 180°$ find the value of $\cos\alpha$ and $\tan\alpha$.

(b) If $\cos\alpha = -\frac{4}{5}$ and $\sin\alpha < 0$ find the exact value of $\tan\alpha$.

(c) If $\operatorname{cosec}\alpha = -\frac{13}{5}$ and $\cos\alpha > 0$ find $\cot\alpha$.

Problem 2 Show that the relationship $\sin(A+B) = \sin A\cos B + \cos A\sin B$ is satisfied when $A = 300°$ and $B = 210°$.

Exercise 5D

Trigonometric identities

Fundamentals

Fundamentals 1

Complete the following Pythagorean identities.

(a) $\sin^2\theta + \cos^2\theta = \_\_\_$ (b) $1 + \_\_\_\_\_ = \sec^2\theta$ (c) $1 + \cot^2\theta = \_\_\_\_\_$

Fundamentals 2

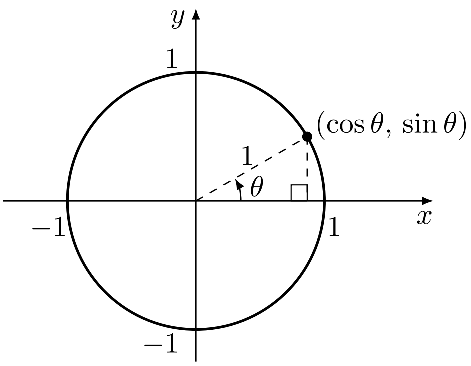
Complete the following results from the Pythagorean identities.

(a) $\sin^2\theta = \_\_\_\_\_\_\_\_\_\_$ (b) $\cos^2\theta = \_\_\_\_\_\_\_\_\_\_$

(c) $\tan^2\theta = \_\_\_\_\_\_\_\_\_\_$ (d) $\sec^2\theta = \_\_\_\_\_\_\_\_\_\_$

(e) $\text{cosec}^2\theta = \_\_\_\_\_\_\_\_\_\_$ (f) $\cot^2\theta = \_\_\_\_\_\_\_\_\_\_$

Question 1



(a) Using the unit circle above and Pythagoras theorem, complete the following identity

$$\sin^2\theta + \cos^2\theta = \_\_\_\_\_$$

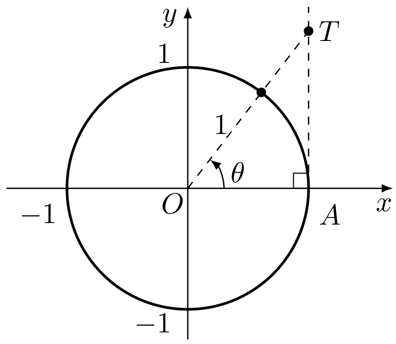
(b) Divide both sides of the above identity by $\sin^2\theta$ and complete and simplify the following.

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \_\_\_\_\_ \quad \text{so} \quad 1 + \_\_\_\_\_ = \_\_\_\_\_$$

(c) Divide both sides of the above identity in (a) by $\cos^2\theta$ and complete and simplify the identity

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \_\_\_\_\_ \quad \text{so} \quad \_\_\_\_\_ + 1 = \_\_\_\_\_$$

Question 2



Note $AT = \tan\theta$ and $OT = \sec\theta$. Hence, it follows that $1 + \tan^2\theta =$ _____

Question 3 Simplify the following.

(a) $\sin^2\theta \operatorname{cosec}^2\theta$ (b) $\cot^2\theta - \operatorname{cosec}^2\theta$ (c) $\tan^2\theta - \dfrac{1}{\cos^2\theta}$

(d) $\sin^2\theta - \sin^2\theta\cos^2\theta$ (e) $\sin^4\theta - \cos^4\theta$ (f) $(\sin\theta + \cos\theta)^2$

(g) $1 - \tan^2\theta + \sec^2\theta$ (h) $\tan^2\theta\cos^2\theta + \cot^2\theta\sin^2\theta$

Question 4 Prove the following identities

(a) $\tan\theta + \cot\theta = \sec\theta\operatorname{cosec}\theta$

(b) $\tan\theta(1 - \cot^2\theta) + \cot\theta(1 - \tan^2\theta) = 0$

(c) $2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

(d) $\tan\theta\sin\theta + \cos\theta = \sec\theta$

(e) $\dfrac{1}{\sin\theta\cos\theta} - \tan\theta = \cot\theta$

(f) $\dfrac{1}{1+\sin^2\theta} + \dfrac{1}{1+\operatorname{cosec}^2\theta} = 1$

(g) $\dfrac{1+\cos\theta}{\sin^2\theta} = \dfrac{1}{1-\cos\theta}$

(h) $\dfrac{1-\cos\theta}{\sin^2\theta} = \dfrac{1}{1+\cos\theta}$

(i) $\sec^4\theta - \tan^4\theta = 2\sec^2\theta - 1$

(j) $\dfrac{\operatorname{cosec}\theta + \sec\theta}{1+\tan\theta} = \operatorname{cosec}\theta$

(k) $\dfrac{\cos^2\theta}{1+\sin\theta} + \dfrac{\cos^2\theta}{1-\sin\theta} = 2$

(l) $\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta$

(m) $\dfrac{\cot^2\theta - 1}{\cot^2\theta + 1} = 1 - 2\sin^2\theta$

(n) $(\sec\theta + \tan\theta)^2 = \dfrac{1+\sin\theta}{1-\sin\theta}$

(o) $\tan^2\theta\cos\theta = \sec\theta - \cos\theta$

(p) $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

Challenge Problems

Problem 1 [Parametrisation of a circle]

If $x = a\cos\theta$ and $y = a\sin\theta$, show that $x^2 + y^2 = a^2$.

Problem 2 Eliminate θ from each of the following by using one of the trig identities.

(a) $x = \operatorname{cosec}\theta$ and $y = 2\cot\theta$

(b) $x = 3\sec\theta$ and $y = 3\tan\theta$

Problem 3 Show that $x = \sec\theta + \tan\theta$ and $y = 2\sec\theta$ satisfy $x^2 - xy + 1 = 0$.

Exercise 5E

Solving trigonometric equations with degrees

Fundamentals

Fundamentals 1

To solve a basic trigonometric equation in the domain $\theta \in [0°, 360°]$...

(a) First re-a________ to make the trigonometric function the subject, and acquire a form like $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$.

(b) Depending on whether k is p________ or n________, and also the type of trigonometric function you have, identify which q________ the solutions lie in.

(c) If k is negative, then temporarily ignore this fact and use your calculator to find an a______ value of θ. If k is positive, then proceed as usual.

(d) Find angles in other quadrants, according to the quadrants you need, by adding or subtracting θ from either ______° or ______°.

(e) If the domain is larger than the usual $\theta \in [0°, 360°]$, then keep adding or subtracting multiples of ________ to these two related angles, until you fill up the required domain. If the function is a tan function, then add or subtract multiples of ________ instead.

(f) If you have a sine or c________ function, and $k > 1$ or $k <$ ______, then there will be no solution. This is because the r______ of both functions is within the interval [________].

Fundamentals 2

Suppose you are solving a trigonometric equation in the domain $x \in [0°, 360°]$.

(a) Sometimes, the equation provided is in the form $\sin(n\theta) = k$ instead of $\sin\theta = k$. In these cases, you need to first m________ the domain accordingly.

(b) If the equation is $\sin(2\theta) = k$, then the domain is $2\theta \in$ ___________.

(c) If the equation is $\sin\left(\frac{\theta}{2}\right) = k$, then the domain is ________________.

(d) Solve for $n\theta$ in this modified domain, and then make ___ the subject again.

Fundamentals 3

(a) Sometimes, a trigonometric i________ can be useful for solving trigonometric equations, especially if there is a s______ involved.

(b) Useful identities include $\sin^2\theta =$ ___________ and $\tan^2\theta =$ ________, though there are many useful variations of these.

Question 1 Consider the trigonometric equation $\sin\theta = \frac{1}{2}$, which has two solutions in the interval $\theta \in [0°, 360°]$.

(a) Write down the acute value of θ that satisfies the equation.

(b) Note that $\sin\theta$ is positive. Using **A**ll **S**tations **T**o **C**entral or a similar mnemonic, in what quadrant should the other solution lie?

(c) Find the related angle, in the correct quadrant, to the acute angle that you found in (i). Hence write down the two solutions to $\sin\theta = \frac{1}{2}$.

Question 2 Find all values of $\theta \in [0°, 360°]$ satisfying the following equations. For each of these, the related acute angle is either 30°, 45° or 60°.

(a) $\cos\theta = \frac{1}{2}$ (b) $\sin\theta = -\frac{1}{\sqrt{2}}$ (c) $\tan\theta = \sqrt{3}$

(d) $\sin\theta = -\frac{\sqrt{3}}{2}$ (e) $\tan\theta = -1$ (f) $\cos\theta = -\frac{\sqrt{3}}{2}$

(g) $\sec\theta = \sqrt{2}$ (h) $\operatorname{cosec}\theta = -2$ (i) $\cot\theta = -\frac{1}{\sqrt{3}}$

Question 3 Solve the following for $\theta \in [0°, 360°]$.

(a) $\sin\theta = \frac{1}{3}$ (b) $\cos\theta = -0.7$ (c) $\cot\theta = 2.74$

Question 4 Solve the following for $\theta \in [0°, 360°]$.

(a) $5\cos\theta - 2 = 0$ (b) $-3\operatorname{cosec}\theta = 4$ (c) $2\sin\theta = 5\cos\theta$

Question 5 Solve the following equations in the provided domain.

(a) $\sin\theta = \frac{\sqrt{3}}{2}$; $\theta \in [-180°, 180°]$ (b) $\cos\theta = -\frac{1}{\sqrt{2}}$; $\theta \in [-180°, 180°]$

(c) $\tan\theta = -\sqrt{3}$; $\theta \in [-360°, 360°]$ (d) $\sec\theta = 2$; $\theta \in [-270°, 270°]$

Question 6 Solve the following for $\theta \in [0°, 360°]$.

(a) $\sin^2\theta = 1$ (b) $\cos^2\theta = \frac{3}{4}$

(c) $\tan^2\theta = 3$ (d) $2\cos^2\theta - 1 = 0$

(e) $3\tan^2\theta - 1 = 0$

Question 7 Solve the following for $\theta \in [0°, 360°]$.

(a) $(2\sin\theta - 1)(\sin\theta + 1) = 0$ (b) $2\sin^2\theta - 3\sin\theta + 1 = 0$

(c) $2\cos^2\theta - \cos\theta - 3 = 0$ (d) $\sec^2\theta + \sec\theta - 2 = 0$

Question 8 Consider the equation

$$\sin 2\theta = \frac{1}{2},$$

in the domain $\theta \in [0°, 360°]$. We will use a substitution to help us solve this equation. Complete the following.

(a) If $u = 2\theta$, then

$$u \in [\_\_, \_\_].$$

(b) The equation to solve is now

$$\sin \_\_ = \frac{1}{2}.$$

(c) Solving this in the domain from (a), we get

$$u = 30°, 150°, \_\_\_\_, \_\_\_\_.$$

(d) Hence, the solutions in terms of θ are

$$2\theta = \_\_\_\_, \_\_\_\_, \_\_\_\_, \_\_\_\_$$

$$\theta = \_\_\_\_, \_\_\_\_, \_\_\_\_, \_\_\_\_$$

Question 9 Use a similar technique to Question 8 to solve the following equations for θ in the domain $\theta \in [0°, 360°]$.

(a) $\tan 2\theta = 1$

(b) $\cos 2\theta = -\frac{1}{2}$

(c) $\sin \frac{\theta}{2} = -\frac{1}{2}$

(d) $\cos 3\theta = \frac{1}{\sqrt{2}}$

Question 10 Use a similar technique to Question 8 to solve the following equations for θ in the domain $\theta \in [0°, 360°]$.

(a) $\cos(\theta + 60°) = \frac{\sqrt{3}}{2}$

(b) $\tan(\theta - 45°) = \frac{1}{\sqrt{3}}$

(c) $\sin(\theta + 150°) = -\frac{1}{2}$

(d) $\tan(\theta + 210°) = -\sqrt{3}$

Challenge Problems

Problem 1 Solve the following for $\theta \in [0^\circ, 360^\circ]$

(a) $\sin(2\theta - 60^\circ) = 1$

(b) $\cos(3\theta + 45^\circ) = -\dfrac{1}{\sqrt{2}}$

Problem 2 Solve the following for $\theta \in [0, 360^\circ]$.

(a) $\cos^2\theta = \sin\theta - 1$

(b) $\sec^2\theta - \tan\theta - 1 = 0$

(c) $\cos^2\theta - \sin^2\theta + 3\cos\theta + 2 = 0$

(d) $\tan^2\theta - \sec\theta - 1 = 0$

Problem 3 Given the fact that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$$

solve the equation

$$8\cos^3\theta - 6\cos\theta + 1 = 0$$

in the domain $\theta \in [0, 180^\circ]$.

Problem 4 What is wrong with the following solution to solving the equation $\sin\theta = \cos\theta$ in the domain $\theta \in [0, 360^\circ]$?

$$\begin{aligned}
\sin\theta &= \cos\theta \\
\sin^2\theta &= \cos^2\theta \\
\sin^2\theta - \cos^2\theta &= 0 \\
\sin^2\theta - (1 - \sin^2\theta) &= 0 \\
2\sin^2\theta - 1 &= 0 \\
\sin\theta &= \pm\frac{1}{\sqrt{2}} \\
\therefore \theta &= 45^\circ, 135^\circ, 225^\circ, 315^\circ
\end{aligned}$$

Exercise 5F

Sine rule and the area formula

Fundamentals

Fundamentals 1

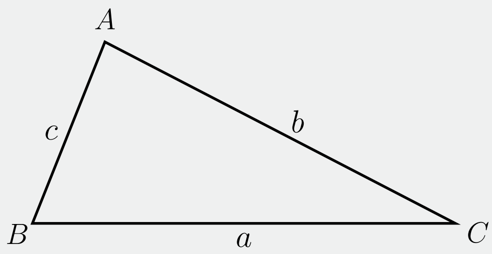
(a) The sine rule can be used to calculate side l________ and a________ in non-right-angled triangles.

(b) The sine rule can be used when we know _____ angles and a side length.

(c) It can also be used when we know _____ side lengths, and a non-included angle.

Fundamentals 2

Consider the following triangle with vertices A, B, C, and side lengths a, b and c.



(a) The sine rule states that $\dfrac{a}{\sin A} = \underline{\quad\quad} = \underline{\quad\quad}$.

(b) An equivalent form of the sine rule is $\dfrac{\sin A}{a} = \underline{\quad\quad} = \underline{\quad\quad}$.

(c) The sine area formula of a triangle is Area = __________.

Fundamentals 3

(a) Explain what the ambiguous case of the sine rule is.

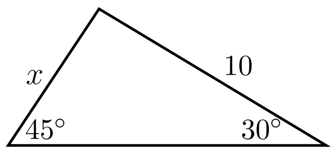
(b) When can it occur?

(c) When using the sine rule to solve for an a________, you cannot simply accept the first solution θ from your calculator. You must also check if the angle __________ also works.

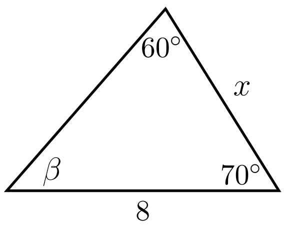
(d) To check if the other angle __________ also works, add it to the other known angle in the triangle, and see if the sum of the angles exceeds ________. If it does, then accept/reject (circle one) that second answer.

Question 1 Find the exact value of the unknowns in each of the diagrams below.

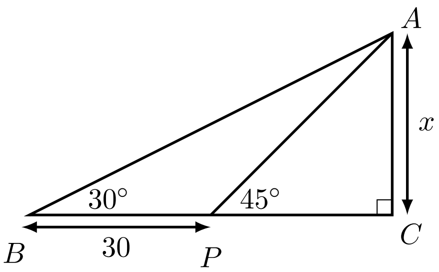
(a)



(b)



Question 2

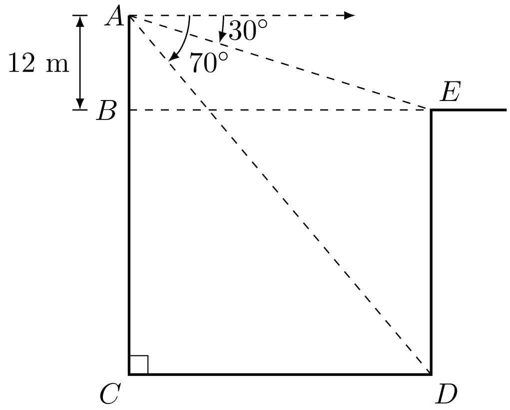


Talia finds that the angle of elevation of the top of a pole, A from B is 30°, but when she walks 30 metres towards it to point P, the angle of elevation is 45°.

(a) Show that $AP = \dfrac{30\sin 30°}{\sin 15°}$.

(b) Hence, calculate the height of the pole correct to the nearest metre

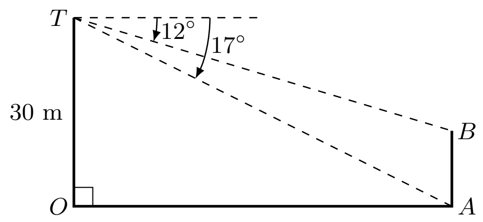
Question 3 A 12 m high antenna is on top of a building BC. Opposite the road is another building DE with the same height. From the top of the antenna, the angles of depression to the base and top of building DE respectively are 70° and 30°.



(a) Find the length of AE.

(b) Find the height of building BC, correct to the nearest metre.

Question 4 An observation tower, OT, is 30 m high. From the top of the tower T, the angle of depression to the base A of a small tree is 17°, while the angle of depression to the top of the tree, B, is 12°.



(a) State the value of $\angle OAT$, and hence find the length of AT, correct to 2 decimal places.

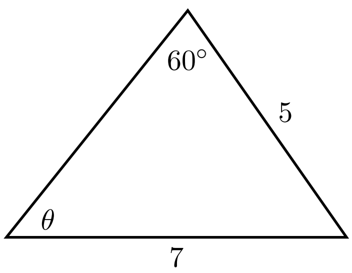
(b) Find the size of each of the angles in $\triangle ABT$.

(c) Hence, find the height of the tree.

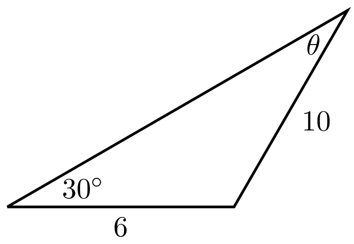
Question 5 [Ambiguous case of the sine rule]

Find θ in the following questions, and be careful to check if there is more than one solution. For the cases that *do* have more than one solution, draw both possible triangles.

(a)



(b)



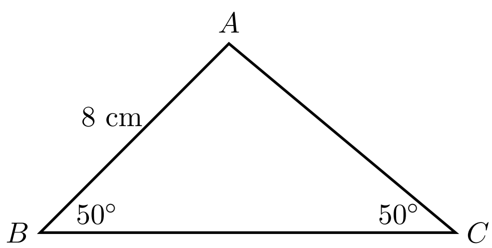
Question 6 Determine the possible dimensions for triangle ABC given $AB = 5.3$ cm, $\angle BAC = 33°$ and $BC = 3.4$ cm.

Question 7

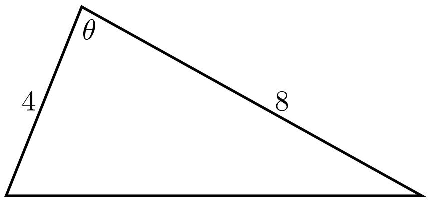
(a) An equilateral triangle has each of its sides 10 cm. find the area.

(b) If the area of the equilateral triangle is $9\sqrt{3}$ cm^2, find the length of each side.

Question 8 Find the area of $\triangle ABC$.

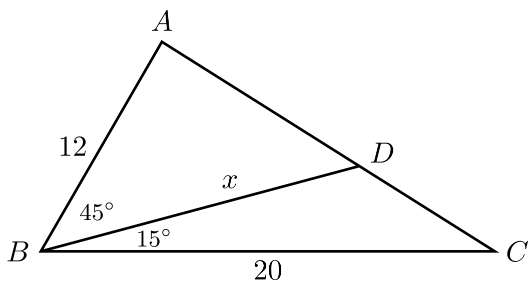


Question 9 A triangle with two sides with lengths 4 cm and 8 cm, and an included angle θ, has an area of 14 cm^2. Find the value(s) of θ.



Challenge Problems

Problem 1 Consider the diagram below.

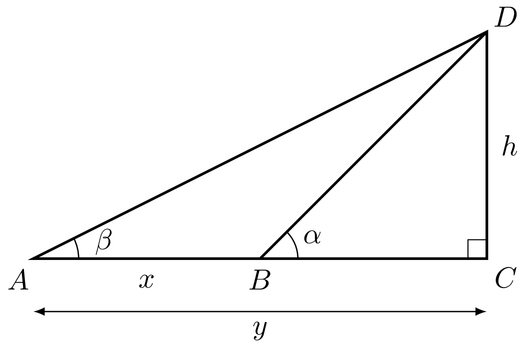


(a) Find the area of $\triangle ABC$.

(b) Hence, find the value of x.

Problem 2 [Preliminary step to prove the sine compound-angle formula]

In the diagram below $AC = y$, $AB = x$



(a) Write an expression for the length of AD in terms of h.

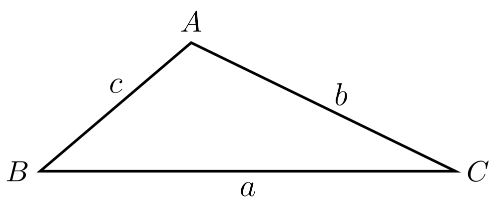
(b) Express $\angle BDA$ and $\angle ABD$ in terms of the unknown angles in the triangles.

(c) Show that

$$h = \frac{x \sin \beta \sin \alpha}{\sin(\alpha - \beta)}.$$

Problem 3 [Demonstrating the necessary logic for proof-type problems]

The diagram below shows a general triangle ABC with side-lengths a, b and c.



(a) Complete the formula Area $\triangle ABC = \frac{1}{2}ab \sin$ ___.

(b) What variable appears in our equation, but not in the required equation in part (d) below?

(c) Write down an equation involving that variable.
Hint: Look at the required expression.We can see $\sin A$ and $\sin B$. What formula might be useful here?

(d) Hence, show that

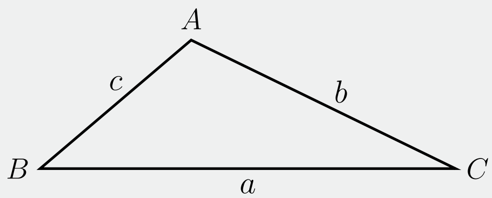
$$\text{Area } \triangle ABC = \frac{a^2 \sin B \sin C}{2 \sin A}.$$

Exercise 5G

Cosine rule

Fundamentals

Fundamentals 1



In a non-right-angled triangle, if we are given

(a) the lengths of two sides and the i________ angle, use the formula

$$c^2 = \_\_\_\_\_\_\_\_$$

(b) the lengths of t_____ sides, use the formula

$$\cos A = \_\_\_\_\_\_\_\_$$

Fundamentals 2

(a) The largest angle in a triangle is opposite the l_______ side.

(b) The smallest angle in a triangle is opposite the s_______ side.

Question 1 The adjacent sides of a parallelogram have lengths 5 cm and 4 cm. The angle included between them is $60°$. Calculate the lengths of the diagonals.

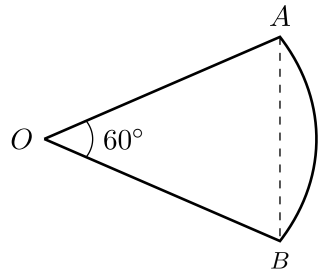
Question 2 In a triangle of sides 5 cm, 3 cm and 7 cm

(a) Show that the largest angle is $120°$.
Hint: The largest angle is opposite which side?

(b) Find the exact area of the triangle.

Question 3 A soccer goal is 8 m wide. A man shoots for goal when he is 15 m from one goal post and 20 m from the other. Within what angle must a shot be made in order to score a goal?

Question 4 In the diagram, OAB is a sector of the circle with centre O and radius 8 cm, where $\angle AOB = 60°$.



Find the length of the chord AB.

Question 5 Two geologists on a large flat area drive 20 km from point A on a bearing of 150°T to point B. They then drive 45 km on a bearing of 020°T to point C.

(a) Show that $\angle ABC = 50°$.

(b) Use the cosine rule to find the distance of point C from point A, rounded to the nearest kilometre.

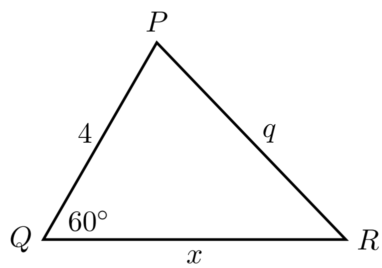
Question 6 Car A is travelling along a road at 45 km/h on a bearing of 048°. A second car, B, has just passed car A at an intersection and is travelling on a bearing of 320°. Car B is travelling at 50 km/h.

(a) Find the distance travelled by each car 3 hours after they pass each other.

(b) Find how far apart are they are after 3 hours

(c) what is the bearing of A from B?

Question 7 The diagram below shows $\triangle PQR$ such that $PQ = 4$, $QR = x$, $PR = q$, and $\angle PQR = 60°$.



(a) Use the cosine rule to show that $q^2 = x^2 - 4x + 16$.

(b) Use graphing software, or otherwise, to show that the minimum value of $y = x^2 - 4x + 16$ is 12, and that it occurs when $x = 2$.

(c) Hence, find the minimum value of q.

(d) Using $x = 2$ and the minimum value of q, investigate what type of triangle $\triangle PQR$ is in this scenario.

(e) Find $\angle QPR$ for this minimum value of q.

Question 8 From a lighthouse L, a ship S bears $051°30'$T and is at a distance of 8 km. From L a boat B bears $291°30'$T and is at a distance of 6 km

(a) Draw a clear diagram marking on it the information supplied.

(b) Find the distance of ship S from boat B. Give your answer as a surd.

(c) Find the bearing of the ship S from boat B. Give your answer to the nearest degree.

Question 9 Nathan leaves island A in a boat and sails 152 km on a bearing of 072° to island B. Nathan then sails on a bearing of 196° for 240 km to island C.

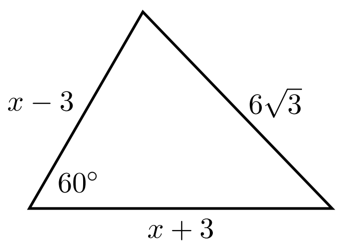
(a) Draw a diagram in your book displaying all the information above.

(b) Show that $\angle ABC = 56°$

(c) Show that the distance from island C to island A is approximately 200 km.

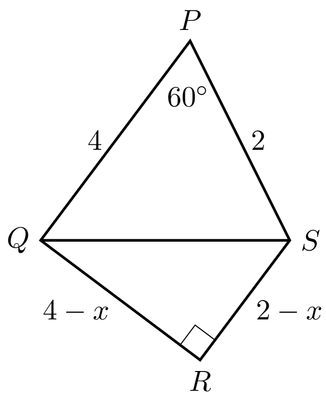
(d) Nathan wants to sail from island C directly to island A. On what bearing should he sail? Give your answer correct to the nearest degree.

Question 10 Find the value of x in the following diagram.



Challenge Problems

Problem 1 Consider the following triangle.



(a) Show that x satisfies the quadratic equation $x^2 - 6x + 4 = 0$.

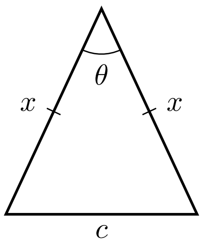
(b) Hence, find the exact value of x.
Hint: Only one solution works!

Problem 2 The cosine rule states that

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Suppose $C = 90°$. Substitute this in to the formula above and comment on the familiar-looking result. Was this a surprise?

Problem 3 The diagram below shows an isosceles triangle.



(a) Show that $c^2 = 2x^2\left(1 - \cos\theta\right)$.

(b) Suppose that $c = \dfrac{x}{2}$. Substitute this in the result from (a) and solve for θ, correct to the nearest degree.

(c) Suppose instead that $c = x$. Substitute this in the result from (a) and solve for θ. Explain briefly why this answer is obvious and could have been acquired by inspection.

Problem 4 [Triangle Inequality]

The cosine rule states that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

where $A \neq 0$.

(a) Show that $b^2 + c^2 - a^2 < 2bc$

(b) Hence, show that $b < a + c$.

(c) What does this say about the side lengths of triangles in general?

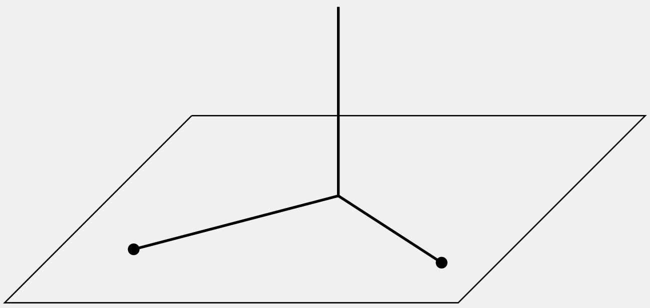
Exercise 5H

Three-dimensional trigonometry

Fundamentals

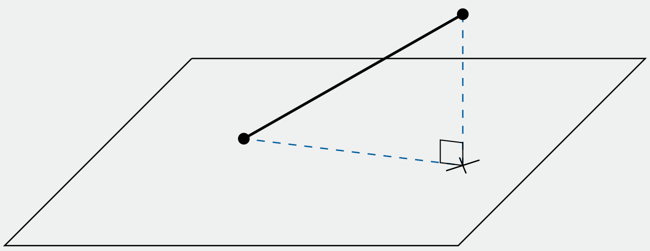
Fundamentals 1

If a line is perpendicular to horizontal plane, like a pole on the ground, then it will be p__________ to any lines on the horizontal plane that meet it. Label the right angles in the diagram below.

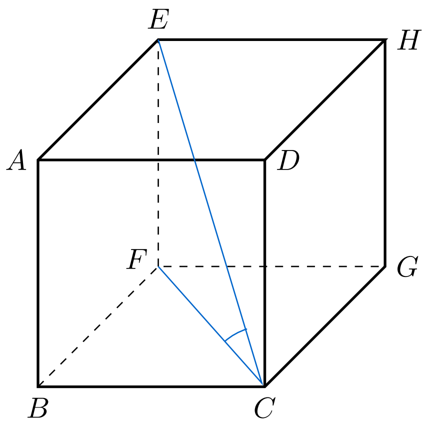


Fundamentals 2

The angle between a line and a plane is the angle between the line and the line directly below it, but on the plane. Label the angle between the line and the plane on the diagram below.



Question 1 The diagram below shows a cube with side-length a.

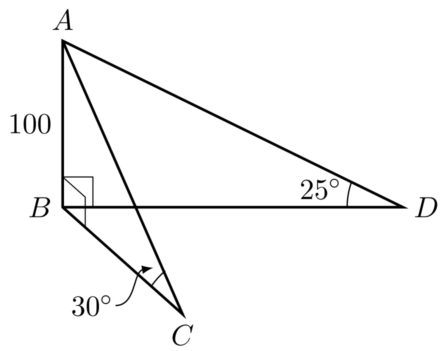


(a) Show that $CF = a\sqrt{2}$.

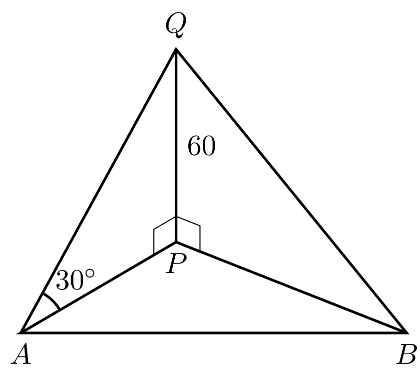
(b) Hence, show that $CE = a\sqrt{3}$.

(c) Find the exact value of $\angle ECF$.

Question 2 AB is a building of height 100 m. From points C and D on the same level as the foot of the building B, the angles of elevation of the top of A are 30° and 25° respectively. Find the distances BC and BD correct to 2 decimal places.



Question 3 A person A stands due south of a 60 m high tower PQ. The angle of elevation from A to the top of the tower is 30°, as shown in the diagram below.



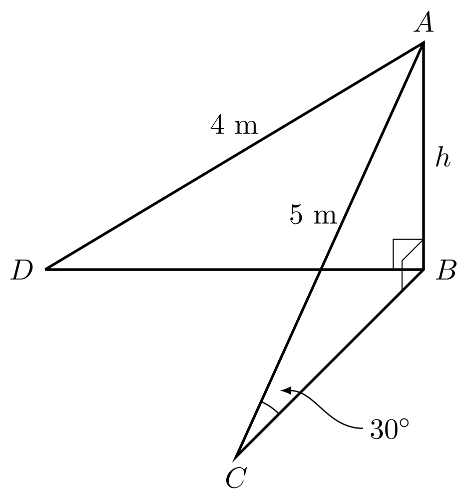
(a) Find the length of AP.

(b) The person now walks due east a distance of 150 m to a point B. Find the length of BP and the angle of elevation of Q from B.

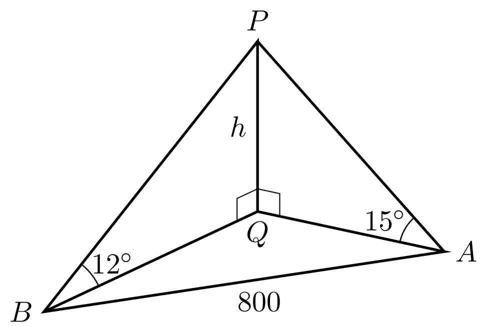
Question 4 AB is a vertical post. AC and AD are ropes of length 5 m and 4 m respectively. The rope AC is inclined at 30° to the horizontal. Find the

(a) height h of the post.

(b) angle at which the rope AD is inclined to the horizontal.



Question 5 The angle of elevation of a tower PQ of height h metres at a point A due east of it is $15°$. Point B is due south of the tower and the angle of elevation is $12°$. The points A and B are 800 metres apart and on the same level as the base Q of the tower.



(a) Consider the triangle APQ and show that $AQ = h\tan 75°$

(b) Find a similar expression for BQ

(c) Find $\angle AQB$

(d) Use triangle AQB to calculate h to the nearest metre.

Question 6 The figure is a right square pyramid with base edge of 10 cm and perpendicular height $EF = 8$ cm. G is the midpoint of DC. Calculate

(a) AC

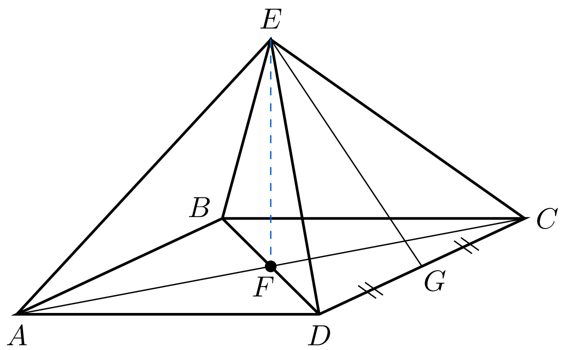
(b) EG

(c) $\angle EDG$

(d) $\angle EGF$

(e) $\angle EAF$

(f) $\angle AEC$

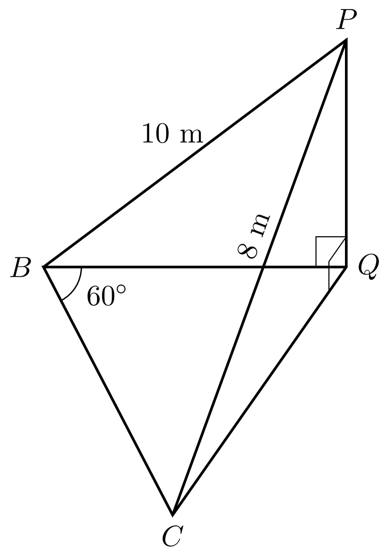


Question 7 A vertical post PQ with height h metres is held by two ropes PB due west and PC due south of it. It is known that $\angle QBC = 60°$, $PB = 10$ m and $PC = 8$ m.

(a) Show that $\dfrac{CQ}{BQ} = \sqrt{3}$.

(b) Hence, show that $\dfrac{64 - h^2}{100 - h^2} = 3$.

(c) Find the value of h.



Question 8 The base of a pole is situated 90 metres due north of a point A. The angle of elevation of the top of the pole from A is $30°$.

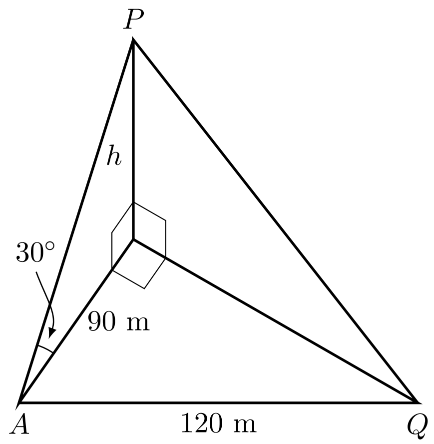
(a) Find the height of the pole in exact form.

(b) Q is a point 120 metres due east of A. Find:

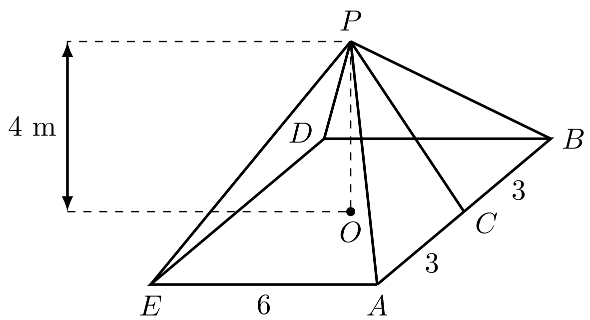
(i) The distance of Q from the base of the pole.

(ii) The angle of elevation of the top of the pole from Q.

(iii) The bearing of the pole from Q.



Question 9 The diagram below shows a square pyramid with base side length 6 units and height 4 units. Let α be the angle between the slant edge and the base of the pyramid. Let β be the angle between the oblique face and the base.

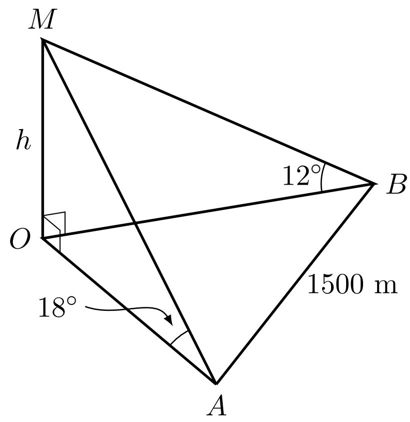


(a) Copy the diagram into your book, and label α and β.

(b) Show that $\tan\alpha = \dfrac{2\sqrt{2}}{3}$.

(c) Show that $\tan\beta = \dfrac{4}{3}$.

Question 10 A person walks 1500 metres due North along a road from point A to point B. The point A is due East of a mountain OM, where M is the top of the mountain. The point O is directly below point M, and is on the same horizontal plane as the road. The height of the mountain above point O is h metres. From point A, the angle of elevation to the top of the mountain is 18°. From point B, the angle of elevation to the top of the mountain is 12°.

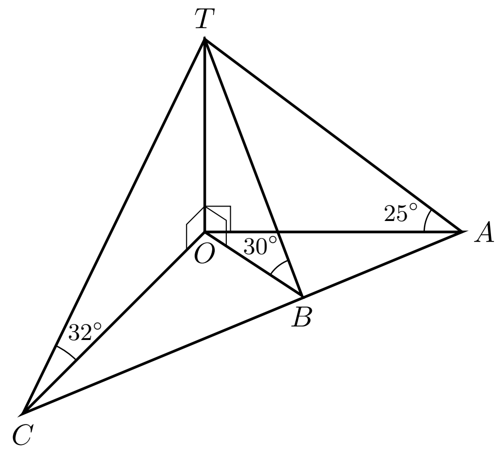


(a) Show that

$$h = \frac{1500}{\sqrt{\tan^2 78° - \tan^2 72°}}.$$

(b) Hence, find the height of the mountain, correct to two decimal places.

Question 11 Two people at A and C observe the top of a tower T with angles of elevation 25° and 32° respectively. A is due north from the base O of the tower and C is due east from O. A third person stands at B, on the line AC. The angle of elevation from B to T is 30°. Let the height of the tower be h.



(a) Show that A is $h\tan 65°$ from the base of the tower.

(b) Similarly, find the distances of B and C from the base of the tower, in terms of h.

(c) What of kind of triangle is AOC?

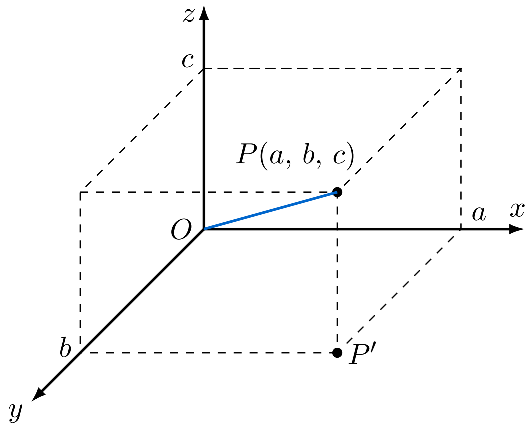
(d) In $\triangle AOC$ find $\angle OAC$.

(e) Using the sine rule, find $\angle ABO$.

(f) Hence, find the bearing of B from the tower.

Challenge Problems

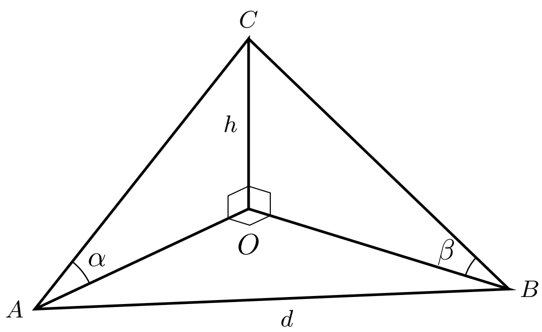
Problem 1 Consider the diagram below, which shows a point $P(a, b, c)$ in a 3-dimensional system.



Prove that the distance of P from the origin O is

$$OP = \sqrt{a^2 + b^2 + c^2}$$

Problem 2 Consider the diagram below.



(a) Show that $OA = h \cot \alpha$ and write down a similar result for OB.

(b) Hence, show that

$$h = \frac{d \tan \alpha \tan \beta}{\sqrt{\tan^2 \alpha + \tan^2 \beta}}$$

Chapter 5 Review

Trigonometry

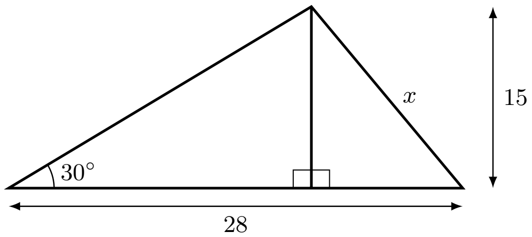
Review

Question 1 Complete the following Pythagorean triads.

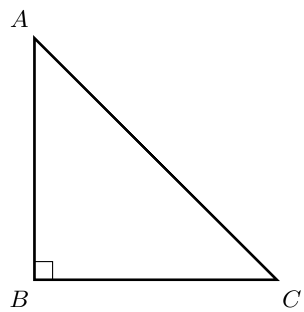
(a) $\{20, 21, \ldots\}$ (b) $\{12, 35, \ldots\}$ (c) $\{6, 8, \ldots\}$

(d) $\{33, 44, \ldots\}$ (e) $\{10, 24, \ldots\}$ (f) $\{50, 120, \ldots\}$

Question 2 Find the length x in the diagram below.



Question 3 Find $\angle BAC$ and $\angle BCA$ if $\sin A = \cos A$



Question 4 From the top of a 150 m high building, the angle of depression to the bottom of a second building is 25° while the angle of depression to the top of the building is 12°. Calculate the height of the second building.

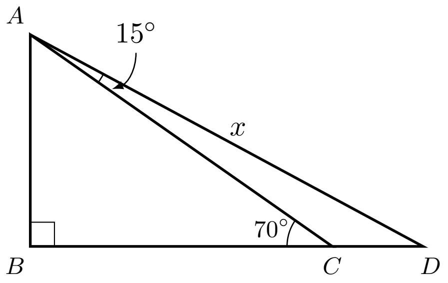
Question 5 The angle of elevation of a tower T from a point A is 45°. From a point B, 25 metres closer to the tower, the angle of elevation of the top of the tower is 78°. Find:

(a) the length of BT correct to 2 decimal places

(b) the height of the tower correct to 2 decimal places

Question 6 A ship travels from P to Q on a course of 060° for 40 km and from Q to R on a course of 045° for 25 km. Draw a clear diagram using this information and find how far east of P is R in km? Give your answer in exact form.

Question 7 Prove that $AB = x \sin 55°$



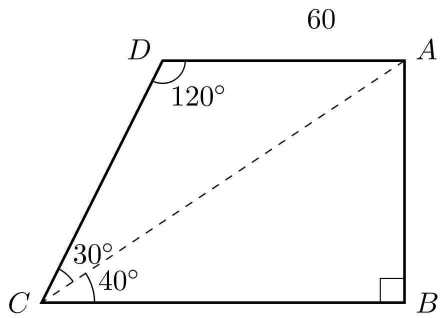
Question 8 By using the fact that $\cos(90° - \alpha) = \sin\alpha$ and $\sin(90° - \alpha) = \cos\alpha$, solve the following for $\theta \in [0°, 90°]$.

(a) $\cos\theta = \sin 30°$

(b) $\sin 2\theta = \cos 42°$
Hint: This one has 2 solutions.

Question 9 If $\tan A = \dfrac{4}{3}$ and A is acute find the value of $\dfrac{3\sin A - \cos A}{4\sin A - 5\cos A}$.

Question 10 In the figure below,



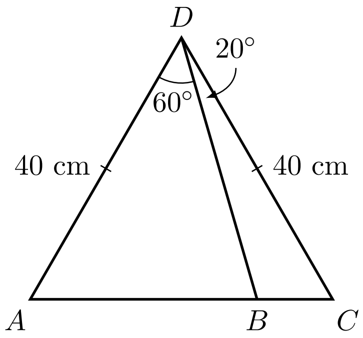
(a) Show that $AC = 60\sqrt{3}$

(b) Find the area of quadrilateral $ABCD$.

Question 11 In a triangle with sides 9 cm, 7 cm and 4 cm, find the $\cos\alpha$ where α is the smallest angle and hence show that the exact area is $6\sqrt{5}$.

Question 12 In $\triangle ABC$, $AB = 2x$, $AC = 3x$ and $\cos A = \dfrac{3}{5}$. Find the value of x if the area of $\triangle ABC = 60 \text{ cm}^2$.
Hint: You do not need to find the size of A explicitly to do this question.

Question 13 Using the information in the diagram below find

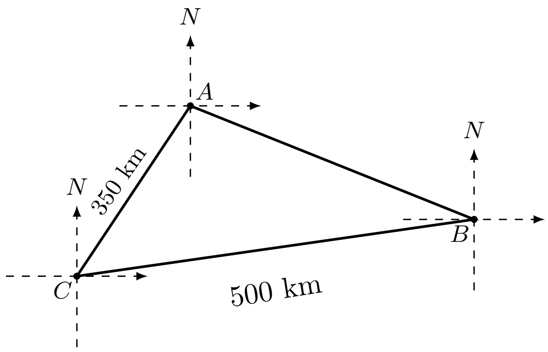


(a) the size of $\angle DAB$

(b) the size of $\angle DBA$

(c) the length of AB (to the nearest centimetre) using two different methods.

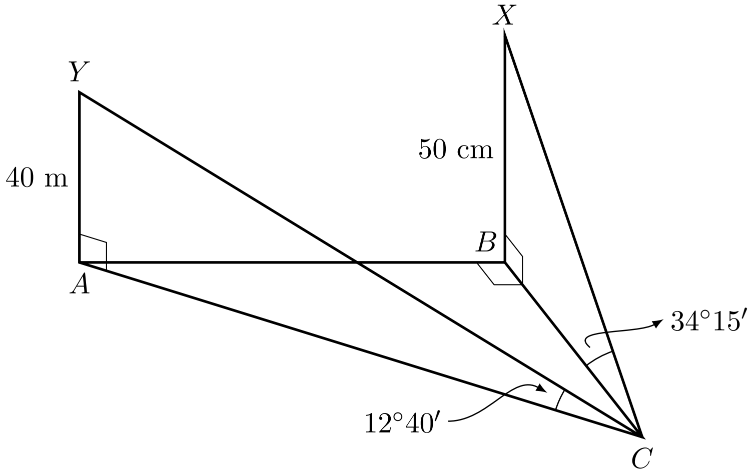
Question 14 The diagram shows the path of a plane flying from town A on a bearing of 240°T to town C, which is 350 kilometres away. At town C, the plane changes course, and travels 500 km on a bearing of 080°T to town B.



(a) Show that $\angle ACB = 20°$.

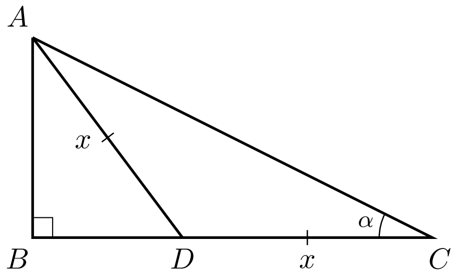
(b) Write down how you would find the distance of town B from town A, without evaluating your answer.

Question 15 The diagram below shows two buildings AY and BX that are 40 m and 50 m tall, respectively. The bases A and B are x metres apart. An observer due South of B observes the points X and Y at angles of elevation $34°15'$ and $12°40'$ respectively.



Find the distance between the two buildings.

Question 16 The triangle ABC is right angled at B. Suppose that $AD = DC = x$ and $\angle ACB = \alpha$.



(a) Explain why $\angle ADB = 2\alpha$

(b) Show that $AB = x \sin 2\alpha$

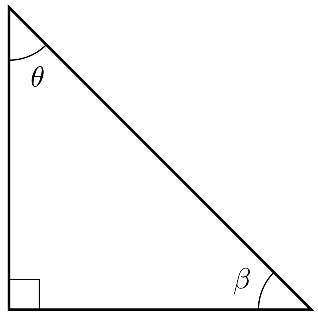
(c) Show that $BC = x + x \cos 2\alpha$

(d) Hence prove that

$$\tan \alpha = \frac{\sin 2\alpha}{1 + \cos 2\alpha}$$

Question 17 A school auditorium is 50 metres long and 35 metres wide. If the angle of elevation from the bottom left hand corner to the opposite corner in the ceiling is $4°30'$, then what is the height of the auditorium, correct to one decimal place?

Question 18 Let θ and β be acute angles. Let $x = \sin\theta$.



(a) Express $\cos\theta$ and $\tan\theta$ in terms of x.

(b) Suppose $\sin\beta = \dfrac{x}{2}$. Find $\tan\beta$ in terms of x.

(c) Suppose $\tan\theta = 3\tan\beta$. Show that

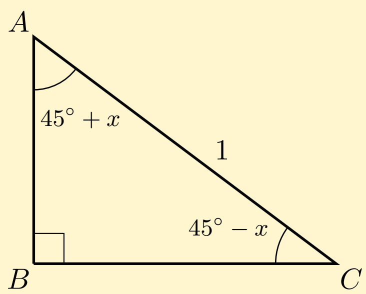
$$\frac{x}{\sqrt{1-x^2}} = \frac{3x}{\sqrt{4-x^2}},$$

and hence show that $x = \sqrt{\dfrac{5}{8}}$.

Investigation Task

Trigonometric Series

Consider the diagram of a right-angled triangle $\triangle ABC$ below.



Question 1

(a) Write down the lengths of sides AB and BC in terms of the angles given.

(b) Use Pythagoras' theorem to prove that

$$\sin^2(45° - x) + \sin^2(45° + x) = 1$$

(c) Use graphing software to draw $y = \sin^2 x$ for $x \in [0°, 360°]$.

(d) Using your graph as a guide, write three equivalent statements for

$$\begin{aligned}
\sin^2 1° &= ____ = ____ = ____ \\
\sin^2 2° &= ____ = ____ = ____ \\
\sin^2 3° &= ____ = ____ = ____ \\
&\vdots \\
\sin^2 89° &= ____ = ____ = ____
\end{aligned}$$

(e) Write down the values of $\sin^2 90°$, $\sin^2 180°$, $\sin^2 270°$, and $\sin^2 360°$.

(f) Explain why $\sin^2\theta + \sin^2(90° - \theta) = 1$.

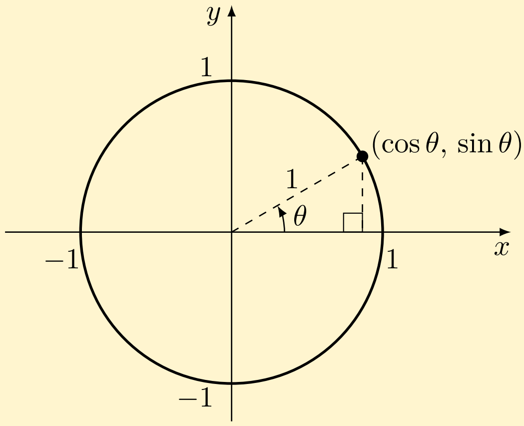
(g) Using all of the above, find the value of

$$\sin^2 1° + \sin^2 2° + \sin^2 3° + \ldots + \sin^2 359° + \sin^2 360°.$$

Investigation Task

Trigonometric Functions

The diagram below shows how the value of $\sin\theta$ and $\cos\theta$ can be visualised using the unit circle. The value of $\cos\theta$ is the x-coordinate of where θ is on the unit circle and similarly the value of $\sin\theta$ is the y-coordinate.



Question 1 Investigate and use the web to find where $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are on the unit circle. Use this to give clear geometric reasoning for the exact trigonometric ratios for the following values of θ. Also, by observing how the ratio is constructed, give a brief explanation as to why the trigonometric ratio is named as such. For example, why is $\tan\theta$ read as 'tangent of theta'? Where is the tangent?

(a) $\theta = 0°$ (b) $\theta = 90°$ (c) $\theta = 45°$

Question 2 The trigonometric functions that you know so far are the sine (sin), cosine (cos), tangent (tan), secant (sec), cosecant (cosec) and cotangent (cot) functions. However, there are around *ten* more lesser-used trigonometric functions. For example, one of these is the *versine* function.

(a) Research to find what some of (bonus points if all) these lesser used ones are, and show where they exist on the unit circle for some given value of θ.

(b) Express each of these new trigonometric functions in terms of the 'old' ones, and hence find their exact values for the commonly used exact-value angles such as $\theta = 30°, 45°, 60°$ etc.

Question 3 Write a paragraph discussion on the history of some of these functions. Your discussion should include:

- what they were used for.
- why they are not used so often nowadays.
- any modern applications still being used.

6

RADIANS

- Radian measure
- Arcs and sectors
- Graphing trigonometric functions
- Solving trigonometric equations with radians

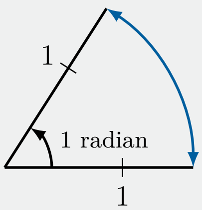
Exercise 6A

Radian measure

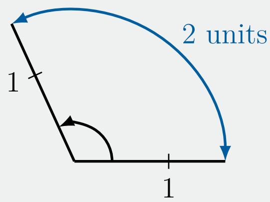
Fundamentals

Fundamentals 1

(a) One radian is the angle subtended at the centre of the u______ circle by an arc of length ___ unit.

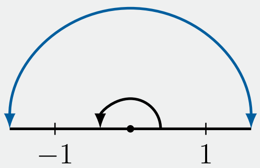


(b) If the arc length is 2 units in the unit circle, then the angle has a measure of ___ radians.



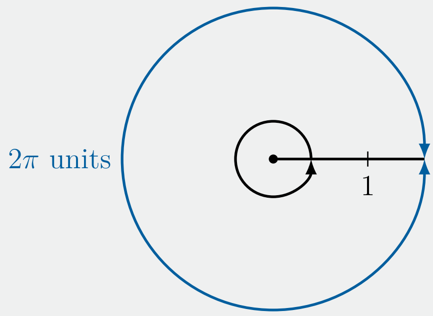
Fundamentals 2

(a) In a semi-circle of radius 1, the arc length is ___ units. This means that the angle represents ___ radians.



Since the angle is also 180°, we can conclude that 180° = ___ radians.

(b) It follows that 360° = ___ radians.



Fundamentals 3

(a) 1 radian = _____°

(b) 1° = _____ radians.

Question 1 Convert the following from radians to degrees.

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

(e) $\frac{5\pi}{3}$ (f) $\frac{7\pi}{6}$ (g) $\frac{4\pi}{3}$ (h) $\frac{13\pi}{6}$

(i) $\frac{3\pi}{2}$ (j) $\frac{5\pi}{6}$ (k) $\frac{11\pi}{6}$ (l) $\frac{7\pi}{3}$

Question 2 Convert the following from degrees to radians.

(a) 30° (b) 45° (c) 60° (d) 90°

(e) 300° (f) 180° (g) 210° (h) 135°

(i) 330° (j) 120° (k) 360° (l) 150°

(m) 270° (n) 315° (o) 240° (p) 225°

Question 3 Convert the following from radians to degrees, rounded to the nearest degree where necessary.

(a) $\frac{\pi}{5}$ (b) $\frac{\pi}{10}$ (c) $\frac{2\pi}{7}$

(d) $\frac{7\pi}{9}$ (e) $\frac{3}{2}$ (f) 0.27

Question 4 Convert the following from degrees to radians, rounding to one decimal place where necessary.

(a) 36° (b) 72° (c) 108°

(d) 100° (e) 200° (f) 305°

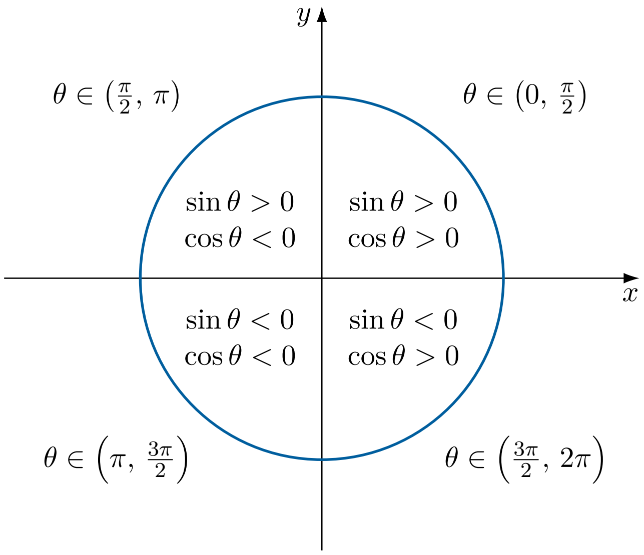
Question 5 Fill in the following table.

$\boldsymbol{\theta}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
sin $\boldsymbol{\theta}$								
cos $\boldsymbol{\theta}$								
tan $\boldsymbol{\theta}$								

Question 6 Fill in the following table.

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
sin $\boldsymbol{\theta}$									
cos $\boldsymbol{\theta}$									
tan $\boldsymbol{\theta}$									

Question 7 Use the diagram below to simplify the following.



(a) $\sin(\pi - \theta)$ (b) $\cos(\pi - \theta)$ (c) $\tan(\pi - \theta)$

(d) $\sin(\pi + \theta)$ (e) $\cos(\pi + \theta)$ (f) $\tan(\pi + \theta)$

(g) $\sin(2\pi - \theta)$ (h) $\cos(2\pi - \theta)$ (i) $\tan(2\pi - \theta)$

Question 8 Suppose $\sin\theta = 0.7$. Use diagram from Question 7 to help you find the value of

(a) $\sin(\pi - \theta)$ (b) $\sin(2\pi - \theta)$ (c) $\sin(\pi + \theta)$

(d) $\sin(-\theta)$ (e) $\sin(2\pi + \theta)$ (f) $\sin(4\pi - \theta)$

Challenge Problems

Problem 1 Suppose $f(x) = \sin(2x)$, $g(x) = x + 2$, and $h(x) = \cos(x)$. Find, correct to two decimal places, the value of

(a) $f(1) + h(1)$

(b) $f(g(1))$

(c) $h(g(0))$

(d) $g(f(1))$

Problem 2 Recall that if we have a unit circle, then one unit measured along the circumference subtends an angle of 1 radian, by definition. Determine whether the following statements are true or false.

(a) 1 unit along the circumference of a circle with radius 2 units subtends an angle of 2 radians.

(b) 2 units along the circumference of a circle with radius 2 units subtends an angle of 2 radians.

(c) 1 unit along the circumference of a circle with radius 2 units subtends an angle of 0.5 radians.

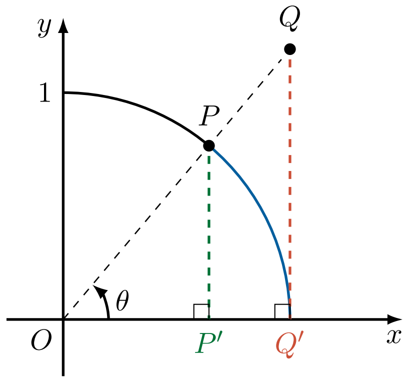
(d) 2 units along the circumference of a circle with radius 1 units subtends an angle of 2 radians.

(e) 0.5 units along the circumference of a circle with radius 1 units subtends an angle of 2 radians.

(f) 2 units along the circumference of a circle with radius 2 units subtends an angle of 1 radians.

Problem 3 [Small angle approximation]

The diagram below shows θ radians on the unit circle, where $\theta \in \left(0, \frac{\pi}{2}\right)$.



(a) Show that $PP' = \sin\theta$.

(b) Show that $QQ' = \tan\theta$.

(c) Briefly explain why $PQ' = \theta$.

(d) What can we say about the lengths PP', QQ' and PQ', in comparison to each other, if θ is a small angle?

(e) Hence find the approximate value of $\dfrac{\sin\theta}{\theta}$ and $\dfrac{\tan\theta}{\theta}$ if $\theta \approx 0$.

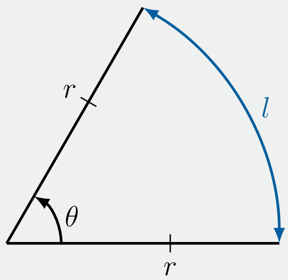
Exercise 6B

Arcs and sectors

Fundamentals

Fundamentals 1

A sector with radius r is subtended by an angle θ. Let the arc length be l, as shown in the diagram below.



Write down the formula for the

(a) arc length.

(b) area of a sector.

Fundamentals 2

(a) If the angle θ is in degrees we must convert it to r________ before we can substitute it into any of the usual formulae.

(b) $\theta^\circ =$ ________ radians.

Question 1 Complete the following table.

Arc length (cm)	Angle (radians)	Angle (degrees)	Radius (cm)
		45°	14
10	$\frac{\pi}{6}$		
8			20
	$\frac{3\pi}{4}$		12

Question 2 A circle has radius 5 cm. If the angle subtended at the centre is 2 radians, find the

(a) length of the arc

(b) area of the sector.

Question 3 A circle has radius 6 cm. if the angle subtended at the centre is 120°, find in exact form the

(a) length of the arc.

(b) area of the sector.

Question 4 A chord of a circle of radius 3 cm subtends an angle of 150° at the centre. Find the

(a) length of the arc.

(b) area of the sector.

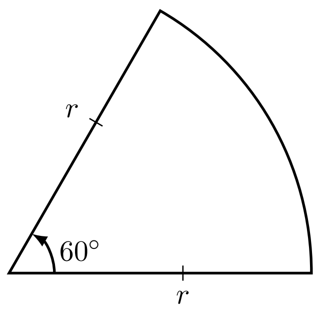
Question 5 Let θ be the the angle at the centre of a circle with radius 10 cm that subtends an arc of length 8 cm. Find the value of θ in

(a) radians.

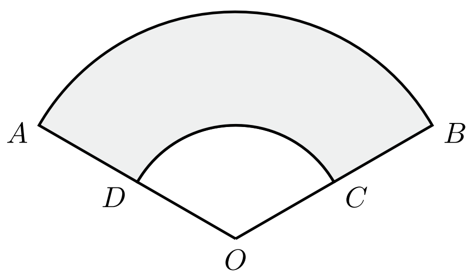
(b) degrees.

Question 6 A sector with radius 5 has area 10π cm^2. Find the arc length of the sector.

Question 7 The sector below has an area of 15π square units. What is the value of r?



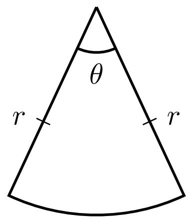
Question 8 A car windscreen wiper traces out the area $ABCD$ where AB and CD are arcs of circles with centre O and radii 30 cm and 15 cm respectively. Angle AOB measures 120°. Find the area of $ABCD$ in exact form.



Question 9 Let the sector subtended by angle θ have area 625π cm^2.

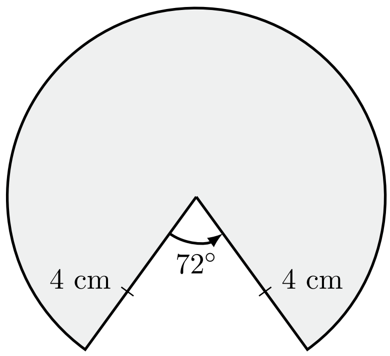
(a) Find the value of r if $\theta = \dfrac{\pi}{5}$.

(b) Find the value of θ if $r = 8$ cm.



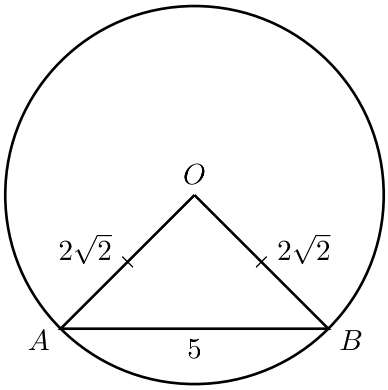
Question 10 A sector subtending 72° is cut out from a circle of radius 4 cm. Find the

(a) shaded area above.



(b) length of the remaining arc.

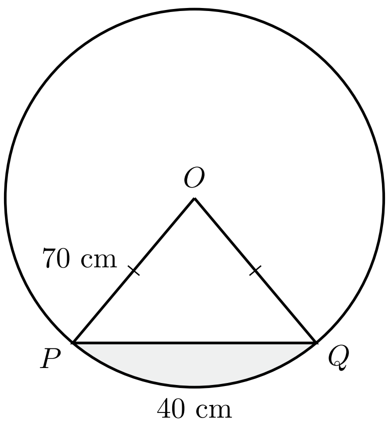
Question 11 A chord of length 5 cm is drawn in a circle with centre O and radius $2\sqrt{2}$ cm.



(a) Use the cosine rule to find $\angle AOB$ in radians, correct to three decimal places.

(b) Hence, find the arc length cut off by the chord, correct to two decimal places.

Question 12 The diagram below shows an arc PQ of length 40 cm, subtended by some angle in a circle with radius 70 cm. Find



(a) $\angle POQ$, in radians.

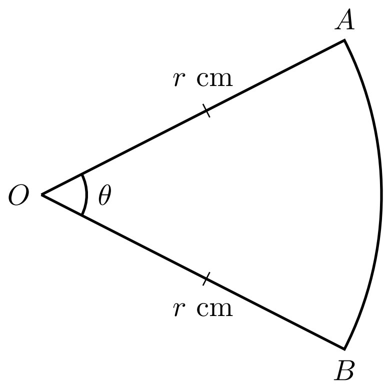
(b) the area of sector POQ.

(c) the area of $\triangle POQ$, to the nearest cm^2.

(d) the shaded area correct to the nearest cm^2.

Question 13

(a) The sector OAB below has an area of $\frac{\pi}{2}$ cm^2. The arc has length $\frac{\pi}{4}$ cm. By solving a pair of equations simultaneously, find the exact values of r and θ.



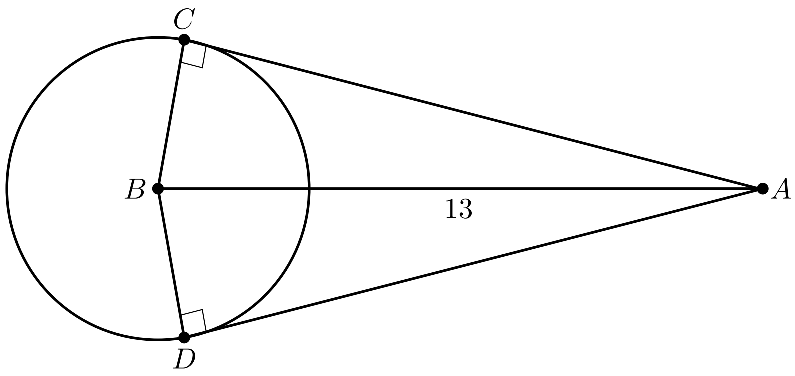
(b) The area of a sector is $\frac{3\pi}{10}$ cm^2 and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the exact value of r and θ.

Question 14 AC and AD are equal tangents to the circle radius 5 cm. The point A is 13 cm from the centre B. Note that BC is perpendicular AC and BD is perpendicular to AD.

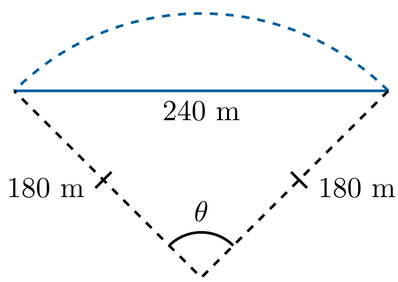
(a) Calculate $\angle ABC$ in radians, correct to three decimal places.

(b) Calculate the length of the minor arc CD, correct to two decimal places.

(c) Calculate the length of the major arc CD, correct to two decimal places.



Question 15 A straight road was constructed to cut a dangerous bend on a country road. It was found that the bend was part of an arc of radius 180 metres and the straight road was 240 metres long.



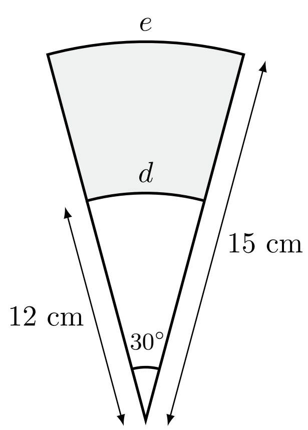
(a) Use the cosine rule to find the size of θ, in radians, correct to two decimal places.

(b) Find the distance by which the old road was shortened, correct to the nearest metre.

Question 16 Calculate the following and leave your answer in exact form.

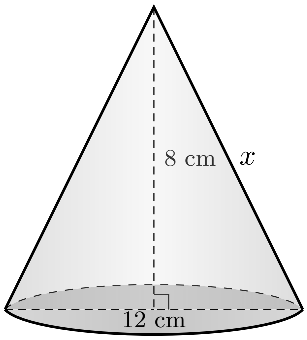
(a) The area of the region between the two circular arcs.

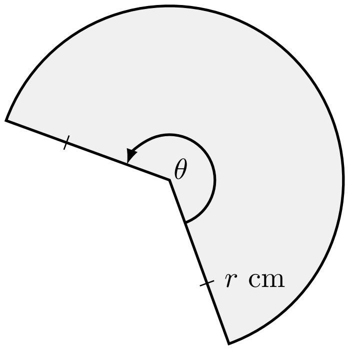
(b) The perimeter of this shaded region.



Challenge Problems

Problem 1 The cone below is made from a sector with radius r and angle θ by fastening the two straight-line edges together.





(a) Find the radius of the cone.

(b) Hence, find the value of r.

Hint: What does r become on the cone, when the sector is folded?

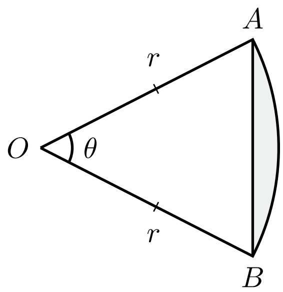
(c) Find the arc length of the sector.

Hint: What does the arc length become on the cone, when the sector is folded?

(d) Find the sector angle θ, in radians.

Problem 2 A bicycle wheel of radius 36 cm turns through an angle of $\frac{2\pi}{3}$ radians in 0.16 seconds. At what speed is the bicycle being ridden, in metres per minute? Give your answer correct to one decimal place.

Problem 3 [Area of a segment]
The shaded region below is called a *segment*, and it is the part of the circle that has been cut off by the chord AB.

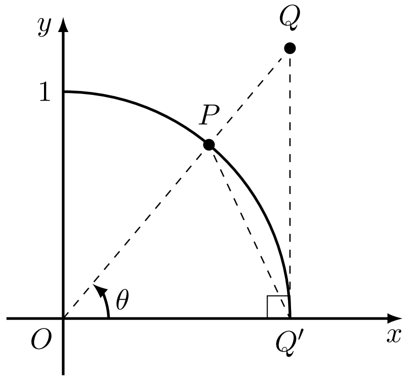


(a) Write down expressions for the area of $\triangle OAB$ and sector OAB.

(b) Deduce that the area of the segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

Problem 4 [Important inequality for developing calculus of trigonometric functions]

The diagram below shows θ radians on the unit circle, where $\theta \in \left(0, \frac{\pi}{2}\right)$.



(a) Show that the area of $\triangle OPQ' = \frac{1}{2}\sin\theta$.

(b) Show that the area of $\triangle OQQ' = \frac{1}{2}\tan\theta$.

(c) By finding the area of sector OPQ' and comparing it to the areas from part (a) and (b), show that for $\theta \in \left(0, \frac{\pi}{2}\right)$.

$$\sin\theta < \theta < \tan\theta.$$

(d) Use graphing software to sketch $y = \sin x$, $y = x$, and $y = \tan x$, and verify that this is the case.

Exercise 6C

Graphing trigonometric functions

Fundamentals

Fundamentals 1

(a) The functions $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$ are collectively referred to as the c________ functions.

(b) A function that repeats itself in regular intervals is called a p________ function.

(c) The interval where it repeats one complete c______ is called the p______ of the function.

(d) The functions $y = \sin x$ and $y = \cos x$ o________ up and down repeatedly from a mean position.

(e) As a result, these functions have a m________ and a m________ ___-value.

(f) The distance of either of these from the mean position is called the a________, and is denoted by a.

(g) The a________ of $y = \sin x$ and $y = \cos x$ is ___.

Fundamentals 2

(a) The functions $y = \operatorname{cosec} x$, $y = \sec x$ and ________ are sometimes referred to as the r________ trigonometric functions because of the way that they are defined.

(b) The graph $y = \tan x$ has asymptotes at $x = \frac{k\pi}{2}$, where k is an o______ number.

Fundamentals 3

(a) $\sin(x + \_\_) = \sin x$

(b) $\cos(x + \_\_) = \cos x$

(c) $\tan(x + \_\_) = \tan x$

(d) These three results show that $\sin x$, $\cos x$ and $\tan x$ have periods of ______, ______ and ______ respectively.

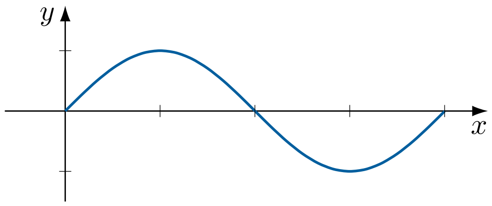
Question 1

(a) Fill in the exact values in the table below.

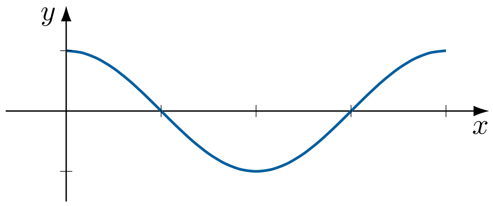
$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{\sin\theta}$					
$\boldsymbol{\cos\theta}$					

(b) The diagram below shows the sketches of $y = \sin x$ and $\cos x$ in the domain $x \in [0, 2\pi]$. Identify which one is $y = \sin x$ and which one is $y = \cos x$, then label all relevant information using your values in (a).

(i)



(ii)

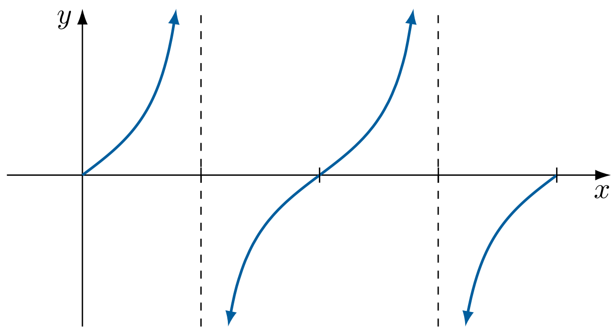


Question 2

(a) Fill in the exact values in the table below.

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{\tan\theta}$					

(b) The diagram below shows the sketch of $y = \tan x$ in the domain $x \in [0, 2\pi]$. Label all relevant information using your values in (a).



Question 3 Sketch the graphs of the following in the domain $x \in [0, \pi]$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

Question 4 Sketch the graphs of the following in the domain $x \in [-\pi, \pi]$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

Question 5 Sketch the graphs of the following in the domain $x \in [-2\pi, 2\pi]$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

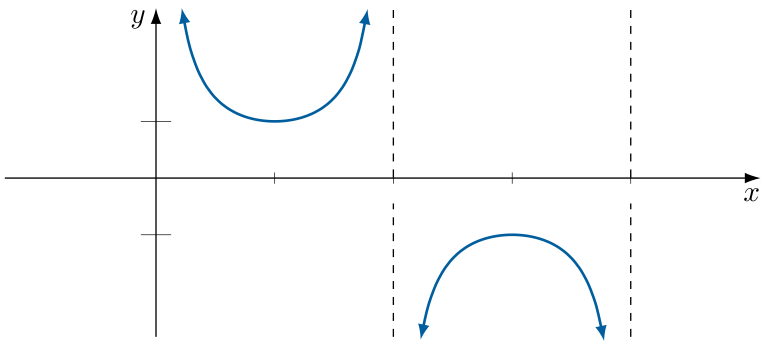
Question 6

(a) Fill in the exact values in the table below.

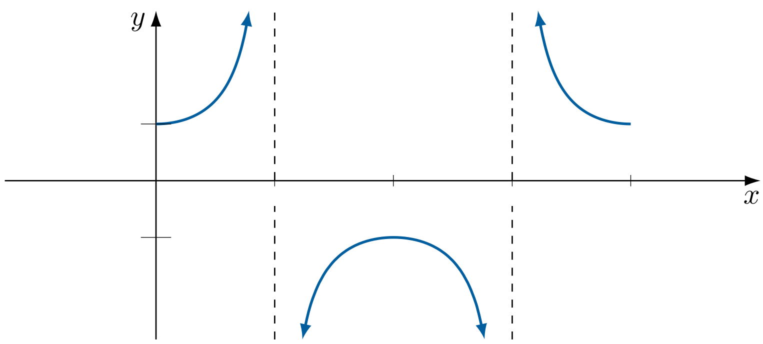
$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sec $\boldsymbol{\theta}$					
cosec $\boldsymbol{\theta}$					

(b) The diagram below shows the sketches of $y = \operatorname{cosec} x$ and $\sec x$ in the domain $x \in [0, 2\pi]$. Identify which one is $y = \operatorname{cosec} x$ and which one is $y = \sec x$, then label all relevant information using your values in (a).

(i)



(ii)

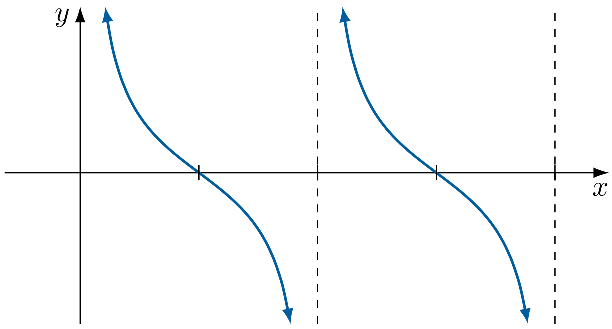


Question 7

(a) Fill in the exact values in the table below

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cot $\boldsymbol{\theta}$					

(b) The diagram below shows the sketch of $y = \cot x$ in the domain $x \in [0, 2\pi]$. Label all relevant information using your values in (a).



Question 8

(a) Write down the definition of $y = \sec x$.

(b) Solve $\cos x = 0$ in the domain $x \in [0, 2\pi]$.

(c) What is the geometric significance of these x values on the graph of $y = \sec x$.

(d) Use a similar technique to algebraically find the asymptotes of $y = \operatorname{cosec} x$ and $y = \cot x$ in the domain $x \in [0, 2\pi]$.

Question 9

(a) Is $f(x) = \sin x$ an odd or even function?

(b) Hence, complete the following identity $\sin(-x) =$ ________ .

(c) Use a similar technique to establish similar identities for the following.

(i) $f(x) = \cos x$ (ii) $f(x) = \tan x$ (iii) $f(x) = \operatorname{cosec} x$
(iv) $f(x) = \sec x$ (v) $f(x) = \cot x$

(d) Does reciprocating a function seem to affect its symmetry?

Question 10

State the range of the following in the domain $x \in \left[0, \frac{\pi}{2}\right]$.

(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$
(d) $y = \operatorname{cosec} x$ (e) $y = \sec x$ (f) $y = \cot x$

Question 11 State the range of the following in the domain $x \in [-\pi, \pi]$.

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \tan x$

(d) $y = \operatorname{cosec} x$

(e) $y = \sec x$

(f) $y = \cot x$

Challenge Problems

Problem 1

(a) Sketch $y = \sin x$ in the domain $x \in [-\pi, \pi]$.

(b) On the same set of axes, but in a different colour pen, sketch $y = \sin\left(x + \frac{\pi}{2}\right)$.

(c) What do you notice about your new curve?

(d) Complete the trigonometric identity $\sin\left(x + \frac{\pi}{2}\right) = \_\_\_\_\_$.

Problem 2 Sketch $y = \frac{\sin x}{\sin x}$ in the domain $x \in [-\pi, \pi]$.

Hint: This is not simply the graph of $y = 1$. There is more to it than that!

Problem 3 Mary sketches $y = \cot x$ in the domain $x \in [0, 2\pi]$, and Bob is asked to sketch $y = \frac{1}{\tan x}$ in the same domain. Bob argues the following.

$$\begin{aligned} y &= \frac{1}{\tan x} \\ &= \frac{1}{\frac{\sin x}{\cos x}} \\ &= \frac{\cos x}{\sin x} \\ &= \cot x \end{aligned}$$

Bob then claims that his sketch will therefore be identical to Mary's sketch. However, his teacher tells him that this is not quite correct. Explain why Bob's answer is incorrect.

Exercise 6D

Solving trigonometric equations with radians

Fundamentals

Fundamentals 1

(a) Suppose a question asks you to solve either $\sin\theta = k$ or $\cos\theta = k$ over a domain *other* than $x \in [0, 2\pi]$. First, find all the solutions in the domain $x \in [\quad]$, and then add or subtract multiples of ______ to generate equivalent solutions until all possible solutions in the given domain are found.

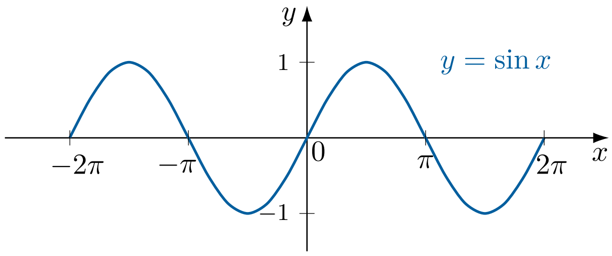
(b) If you are solving $\tan\theta = k$, then instead of adding ______ to each solution, you instead add ______ to each solution.

Fundamentals 2

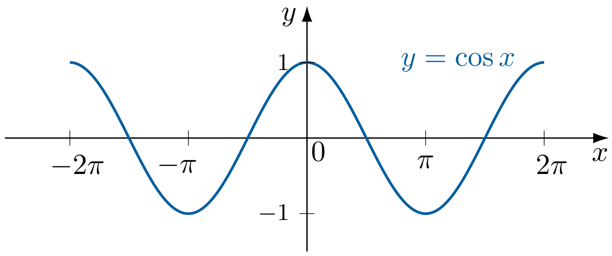
(a) $\sin\left(\frac{\pi}{2} - x\right) =$ ______ (b) $\cos\left(\frac{\pi}{2} - x\right) =$ ______ (c) $\tan\left(\frac{\pi}{2} - x\right) =$ ______

Question 1 The trigonometric graphs below are drawn to help you to solve the following equations for θ in the domain $-2\pi \leq \theta \leq 2\pi$

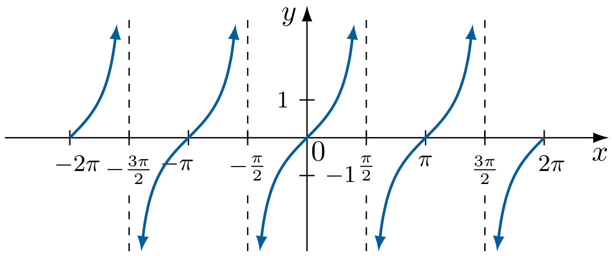
$y = \sin\theta$



$y = \cos\theta$



$y = \tan\theta$

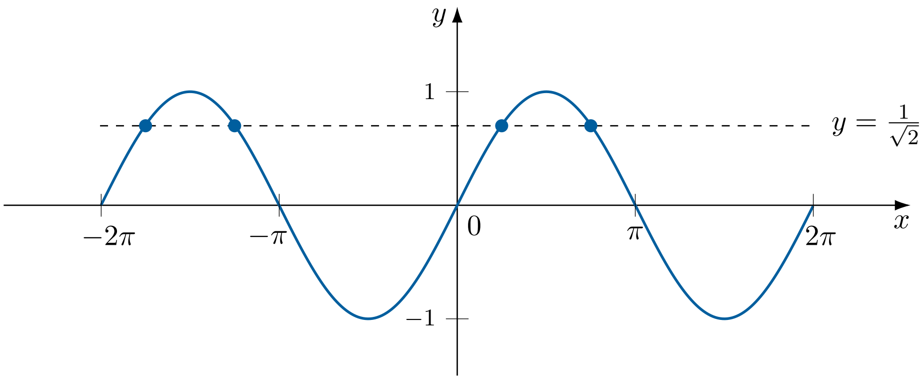


(a) $\sin\theta = 0$ (b) $\sin\theta = 1$ (c) $\cos\theta = -1$ (d) $\tan\theta = 0$

Question 2

(a) The graphs of $y = \sin x$ and $y = \dfrac{1}{\sqrt{2}}$ are drawn in for $x \in [-2\pi, 2\pi]$. Use the diagram to show that there will be 4 solutions to the equation $\sin x = \dfrac{1}{\sqrt{2}}$ in this domain.

(b) Find the exact values of x in this domain.



Question 3

Show that the solutions of $\sin\theta = \dfrac{1}{2}$ in the domain $\theta \in [0, 4\pi]$ are $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$.

Question 4

Solve for $x \in [0, 2\pi]$.

(a) $\sin x = -1$

(b) $\cos x = \dfrac{1}{\sqrt{2}}$

(c) $\cos x = 1$

(d) $\tan x = -\dfrac{1}{\sqrt{3}}$

(e) $\cos x = -\dfrac{1}{2}$

(f) $\tan x = -1$

Question 5

Solve for $x \in [-2\pi, 2\pi]$.

(a) $\sin x = \dfrac{\sqrt{3}}{2}$

(b) $\cos x = -\dfrac{1}{\sqrt{2}}$

(c) $\tan x = 1$

(d) $\tan x = -\dfrac{1}{\sqrt{3}}$

Question 6

Solve the following for $x \in [0, 2\pi]$. Round to three decimal places, where necessary.

(a) $\cos x = \sec x$

(b) $\sin^2 x = \dfrac{1}{2}$

(c) $2\sin x = \cos x$

(d) $\cos^2 x - \sin^2 x = 0$

(e) $\cos^2 x = \cos x$

(f) $(2\sin x + 1)(3\cos x - 1) = 0$

(g) $\tan x + 3\cot x - 4 = 0$

(h) $(\sqrt{3}\tan x + 1)(\tan x - \sqrt{3}) = 0$

(i) $(\sin x + 2)(2\sin x + 1) = 0$

(j) $2\sin^2 x + 3\sin x - 2 = 0$

(k) $6\sin x\cos^2 x - 2\sin x = 0$

(l) $2\cos^2 x + 5\cos x - 3 = 0$

Question 7

Solve the following for $\theta \in [0, 2\pi]$, correct to three decimal places.

(a) $\sin\theta = 0.316$

(b) $2\cos\theta = -0.689$

(c) $\tan\theta = \dfrac{3}{7}$

Question 8 Solve the following for θ in the domain $\theta \in [0, 2\pi]$. You may need to use some trigonometric identities that you have learned earlier.

(a) $\sec^2 x + \tan x = 1$ (b) $2\sin^2 x + \cos x - 2 = 0$ (c) $\sec^2 x - 2\tan x = 4$

Question 9

(a) Name the quadrant in which $\sin\theta > 0$ and $\tan\theta > 0$ and hence find the exact value of $\cos\theta$ if $\sin\theta = \dfrac{3}{5}$.

(b) Name the quadrant in which $\cos\theta < 0$ and $\tan\theta > 0$ and hence find the exact value of $\sin\theta$ if $\cos\theta = -\dfrac{4}{5}$.

(c) Name the quadrant in which $\sin\theta > 0$ and $\tan\theta < 0$ and hence find the exact value of $\cot\theta$ and $\cos\theta$ if $\sin\theta = 0.7$.

(d) Name the quadrant in which $\sin\theta < 0$ and $\cos\theta < 0$ and hence find the exact value of $\cot\theta$ if $\sin\theta = -\dfrac{1}{\sqrt{2}}$ and $\cos\theta = -\dfrac{1}{\sqrt{2}}$.

Question 10 Solve for a and x, without evaluating the left hand side.

(a) $a\tan x = 2\tan\dfrac{\pi}{6} + 4\cot\dfrac{\pi}{3}$ (b) $a\sin x = \sin\dfrac{\pi}{12} + 5\cos\dfrac{5\pi}{12}$

Challenge Problems

Problem 1

(a) Write down a possible value of α for $\sin x = \cos(\alpha - x)$.

(b) Hence, solve

$$\cos\left(\frac{\pi}{12} + x\right) = \sin(x)$$

in the domain $x \in \left[0, \dfrac{\pi}{2}\right]$.

Problem 2 Solve the following for $x \in [-\pi, \pi]$.

(a) $\cos^2 x = \sqrt{3}\sin x\cos x$ (b) $\cos^2 x - \sin^2 x = 3\sin x - 1$

Problem 3

(a) Show that $\dfrac{1}{\sin x\cos x} - \cot x = \tan x$.

(b) Hence, or otherwise, solve in the domain $x \in [0, 2\pi]$

$$\frac{\cos^2 x - 1}{\sin x\cos x} = \sqrt{3}$$

Chapter 6 Review

Radians

Review

Question 1 Convert the following to radians.

(a) $150°$ (b) $300°$ (c) $80°$ (d) $200°$

Question 2 Convert the following to degrees.

(a) $\frac{\pi}{10}$ (b) $\frac{5\pi}{9}$ (c) $\frac{7\pi}{6}$ (d) $\frac{2\pi}{5}$

Question 3 Convert the following to degrees, correct to 2 decimal places.

(a) 2 (b) 0.5 (c) 3.21

Question 4 Simplify the following.

(a) $\dfrac{\tan(2\pi - \theta)}{\cos\left(\frac{\pi}{2} - \theta\right)}$ (b) $\dfrac{\sin(\pi + \theta)}{\cos(\pi - \theta)}$

Question 5 Find the exact value of the following.

(a) $\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{11\pi}{6}\right)$ (b) $\operatorname{cosec}\left(\frac{\pi}{4}\right) + \sec\left(\frac{7\pi}{4}\right)$

(c) $\sec\left(\frac{2\pi}{3}\right) + \tan\left(\frac{\pi}{6}\right)$ (d) $\tan\left(\frac{3\pi}{4}\right) + \cos\left(\frac{5\pi}{6}\right)$

(e) $\cos(3\pi) + \cos\left(-\frac{\pi}{3}\right)$ (f) $\cot\left(\frac{5\pi}{3}\right) - \sec\left(\frac{7\pi}{6}\right)$

Question 6 Use your calculator in radian mode to evaluate the following, correct to four decimal places.

(a) $\sin(1)$ (b) $\tan(0.8)$ (c) $\cos(2)$

Question 7 Show that $\cos 2\theta = 1 - 2\sin^2\theta$ is true when

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = -\frac{5\pi}{6}$

Question 8 A sector is subtended by an angle of 0.7 radians, and an area of 20 cm^2. Find the following in exact form.

(a) The radius of the circle.

(b) The perimeter of the sector.

Question 9 Find the exact value of the radius r and angle θ of the sector that has an

(a) area of 100 cm^2 and arc length of 20 cm.

(b) area of π cm^2 and arc length of $\frac{\pi}{2}$ cm.

Question 10

(a) Suppose $\cos\theta = \frac{4}{5}$ and $\sin\theta < 0$. Find the exact value of $\tan\theta$ and $\sin\theta$.

(b) Suppose $\sin\theta = -\frac{3}{4}$ and $\cos\theta < 0$. Find the exact value of $\cos\theta$ and $\tan\theta$.

(c) Suppose $\tan\theta = \frac{4}{5}$ and $\operatorname{cosec}\theta < 0$. Find the exact value of $\sin\theta$ and $\sec\theta$.

Question 11 Solve the following for $\theta \in [0, 2\pi]$.

(a) $\sin\theta = -\frac{\sqrt{3}}{2}$

(b) $\tan\theta = -\frac{1}{\sqrt{3}}$

(c) $\tan^2\theta = 3$

(d) $4\sin^2\theta = 3$

Question 12 Solve the following for $\theta \in [-\pi, \pi]$.

(a) $\sin(2\theta) = \frac{1}{2}$

(b) $\cos(2\theta) = -\frac{1}{2}$

(c) $\sin\left(\frac{\theta}{2}\right) = -\frac{\sqrt{3}}{2}$

(d) $\tan\left(\theta - \frac{2\pi}{3}\right) = \sqrt{3}$

Question 13 Solve the following for $\theta \in [0, 2\pi]$.

(a) $2\sin^2\theta + \sin\theta = 0$

(b) $2\sin^2\theta - 3\sin\theta + 1 = 0$

(c) $2\sin^2\theta - 5\sin\theta + 3 = 0$

(d) $\sin^2\theta = 2\cos\theta + 2$

Question 14 What is the greatest value of the function $y = 3 - 2\sin x$?

Question 15 Write down the range of the following, using interval notation.

(a) $y = 3\cos x$

(b) $y = -4\sin x$

(c) $y = 5\tan x$

(d) $y = 2 - 3\cos x$

(e) $y = -2 + 6\sin x$

(f) $y = -4 - 4\cos x$

Question 16 Eliminate θ from the following by using a relevant trigonometric identity.

(a) $x = \cos\theta$ and $y = 2\sin\theta$

(b) $x = 3\sec\theta$ and $y = 4\tan\theta$

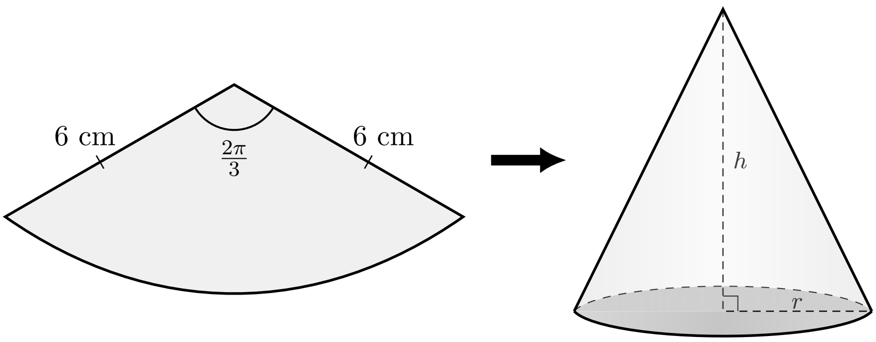
(c) $x = 2 - \cos\theta$ and $y = 1 + \sin\theta$

Question 17 Solve the equation

$$\frac{\cos^3\theta}{\sin\theta} + \sin\theta\cos\theta = 1$$

in the domain $\theta \in [0, 2\pi]$.

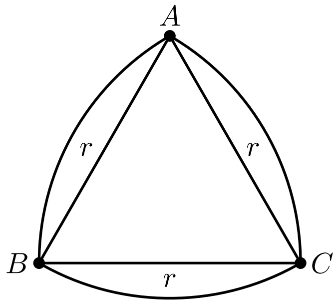
Question 18 The diagram below shows a sector subtended by an angle of $\frac{2\pi}{3}$ from a circle with radius 6 cm.



The straight edges of the sector are joined so that the sector forms a cone. Find the volume of the cone that it forms.

Question 19 [Reuleaux triangle]

The diagram below shows a *Reuleaux triangle*, which is formed by obtaining an equilateral triangle of side length r, and forming circular arcs along each side of the triangle.



Show that the area of the Reuleaux triangle is

$$\frac{1}{2}\left(\pi - \sqrt{3}\right)r^2$$

Question 20 Solve the equation

$$\sin\theta\cos\theta = \frac{1}{2}$$

in the domain $\theta \in [0, 2\pi]$.
Hint: Re-arrange to make the "1" the subject, then use a familiar trigonometric identity.

Investigation Task

Trigonometric Graphs

This investigation task will further develop the students' understanding of the effect of changing various parameters on the graph of the sine and cosine functions.

Question 1 Use graphing software to sketch

$$y = A\sin(nx + \alpha),$$

and create sliders for A, n and α.

(a) Set $\alpha = 0$ and $n = 1$. Adjust the slider for A and comment on your results.

(b) Set $\alpha = 0$ and $A = 1$. Adjust the slider for n and comment on your results.

(c) Set $A = 1$ and $n = 1$. Adjust the slider for α and comment on your results.

Question 2 [Verifying trigonometric identities graphically]

Use graphing software to sketch the graphs of the following, and comment on your result. What familiar-looking graph do you obtain?

(a) $y = \sin(-x)$

(b) $y = \cos(-x)$

(c) $y = \sin\left(\frac{\pi}{2} - x\right)$

(d) $y = \sin\left(\frac{\pi}{2} + x\right)$

(e) $y = \cos\left(\frac{\pi}{2} - x\right)$

(f) $y = \cos\left(\frac{\pi}{2} + x\right)$

(g) $y = \sin(\pi - x)$

(h) $y = \cos(\pi - x)$

Question 3 [Double-angle formulae]

Use graphing software to sketch the graph of $y = \sin^2 x$. Using methods from Mathematics Extension 1, it can be shown that this function can be expressed in the form

$$y = A\cos(nx) + c,$$

for the right values of A, n and c.

Experiment with the sliders until you find a suitable value of the parameters that results in the same graph as $y = \sin^2 x$.

Investigation Task

Angular Speed and Linear Speed

Consider a bicycle wheel with some radius r. We measure how 'fast' it spins in terms of radians. For example, a wheel spinning at one revolution per second spins at an *angular speed* of 2π radians per second, since one revolution is equivalent to 2π radians.

But of course, for a bicycle, we are not interested in how fast the wheels are spinning. Rather, we are interested in knowing how fast the cyclist is moving forwards. This is called the *linear speed*, usually measured in metres per second.

Question 1 Suppose a wheel with radius r metres spins with a constant angular speed ω radians per second.

(a) How many metres does the wheel roll after one second?

(b) Hence, what is the linear speed s of the wheel?

Question 2 [Application to cycling]

A fixed-gear bicycle has the pedals fixed to the front gear. This means that as long as the bicycle is moving, the pedals will always be spinning at a proportional rate. The rear-wheel gear, called the rear sprocket, is much smaller than the front gear, and is connected to it by a chain. It is also directly connected to the rear wheel, which means that the angular speed of the rear wheel will be the same as the angular speed of the rear sprocket.

Consider a bike with a 58/14 gear-ratio, which means that the front gear has 58 teeth on it whereas the rear sprocket has 14 teeth. Suppose that the radius of the rear sprocket is 3 cm.

(a) Suppose that the number of teeth on the gear is directly proportional to the radius of the gear itself. What is the radius of the front gear?

(b) An Olympic cyclist cycles 50 km after exactly one hour of cycling. Calculate the linear speed of the bicycle, in metres per second.

(c) This linear speed gets transferred from the rear sprocket to the front gear via the chain. Find the angular speed of the front gear.

(d) Hence, determine how many revolutions per minute the Olympian had to output, on average, for the whole hour.

Question 3 Research the linear speeds of a typical Olympic cyclist when doing the 'hour record'. Using the techniques from Question 2, find how many revolutions per minute they were pedalling at (called the *cadence*), on average. Now research the typical speed of an average cyclist (or even yourself!) and compare.

7

DIFFERENTIATION AND APPLICATIONS

- Limits
- Differentiation by first principles
- Basic differentiation rules
- Chain rule
- Product rule
- Quotient rule
- Average rate of change
- Instantaneous rate of change
- Gradient function
- Continuity

Exercise 7A

Limits

Fundamentals

Fundamentals 1

(a) The limit of a function as $x \to \alpha$ is the ___-coordinate that the graph approaches.

(b) The actual point $f(\alpha)$ does/does not (circle one) need to exist for the limit to exist.

(c) Sometimes, the limit of $f(x)$ as $x \to \alpha$ may be found by evaluating $f(\_\_)$.

(d) If substitution does not work, we can try s________ $f(x)$ before substituting.

Fundamentals 2

(a) We have the limit

$$\lim_{x\to\infty} \frac{1}{x} = \_\_$$

because as we divide 1 by a larger number, the result gets larger/smaller (circle one).

(b) Similarly, we also have

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = \_\_,$$

and

$$\lim_{x\to\infty} \frac{1}{x^n} = \_\_,$$

where $n > 0$.

Question 1 Find the following limits by substituting the given value.

(a) $\lim\limits_{x\to 6}(4 - x)$ (b) $\lim\limits_{x\to 3}(x^2 - x)$ (c) $\lim\limits_{x\to 4}\left(\frac{12}{x}\right)$ (d) $\lim\limits_{x\to -2}\left(\sqrt{x+2}\right)$

Question 2

(a) Try to find the following limit using substitution

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}.$$

Explain why this did not work.

(b) By first simplifying $\frac{x^2 - 1}{x - 1}$ before substituting in $x = 1$, evaluate the same limit.

Question 3 Find the following limits by first simplifying.

(a) $\displaystyle\lim_{x\to 0}\frac{x^2-9x}{x}$

(b) $\displaystyle\lim_{x\to 1}\frac{x-1}{x^2-1}$

(c) $\displaystyle\lim_{x\to 5}\frac{x^2-25}{x-5}$

(d) $\displaystyle\lim_{x\to 0}\frac{x^2+5x}{x^2-x}$

(e) $\displaystyle\lim_{h\to 0}\frac{2x^2h-9h^2}{h}$

(f) $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x^2-x-6}$

(g) $\displaystyle\lim_{x\to -3}\frac{x^2+2x-3}{x+3}$

(h) $\displaystyle\lim_{x\to 2}\frac{x^2-5x+6}{x-2}$

(i) $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2-x-2}$

(j) $\displaystyle\lim_{x\to -1}\frac{x^4-1}{x^2-1}$

Challenge Problems

Problem 1 Find $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Hint: Try rationalising the denominator first.

Problem 2 Find $\lim_{x \to 0} \dfrac{\sqrt{x^2+25}-5}{x^2}$.

Hint: Try rationalising the numerator first.

Problem 3 Find $\lim_{x \to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

Problem 4 Define the function $f(x) = x^2$.

(a) Find $f(1+h)$.

(b) Hence, find the value of

$$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$$

Problem 5 [Differentiation by first principles]

Define the function $f(x) = x^2$.

(a) Simplify $f(x+h) - f(x)$.

(b) Hence, show that

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = 2x$$

Exercise 7B

Differentiation by first principles

Fundamentals

Fundamentals 1

(a) The difference quotient is the gradient of a s________ on the graph of a function.

(b) This secant passes through a point $P(x, f(x))$ and another point Q, which is h units away horizontally. The coordinates of Q are ____________.

(c) The formula for the difference quotient is ________

(d) We may vary the value of ___, which will adjust the distance of Q from P.

(e) If we make the value of ___ smaller, then Q will move further/closer (circle one) to P.

Fundamentals 2

(a) A tangent to a curve at a point is a straight line that t________ the curve at that point.

(b) The tangent is a limiting case of a s________ PQ as the two points get closer.

(c) In other words, the gradient of the tangent is the limit of the gradient of the s________.

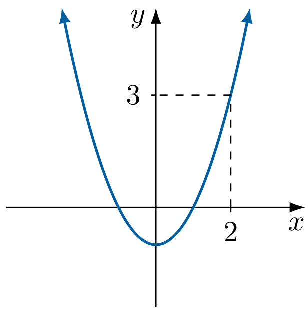
(d) This means that the gradient of the tangent is the limit of the d________ q________, as $h \to$ ___.

(e) The formula for the tangent at any point x is therefore $\lim\limits_{h \to 0}$ ________

(f) This process finds a function that outputs the g________ of the tangent at any point x. This special function is called the d________ function, denoted by $f'(x)$.

(g) The process of finding this function, as described above, is called d__________ by first principles.

Question 1 The diagram below shows the point $P(2, 3)$ on a function $f(x) = x^2 - 1$.



(a) Find the coordinates of the point Q that has x coordinate $2 + h$.

(b) Show that the interval PQ has gradient $h + 4$.

(c) Describe what happens to the point Q as we make $h \to 0$.

(d) As h gets smaller, what value does m approach?

(e) Describe the geometric significance of this value of m.

Question 2 Define $f(x) = x^2$.

(a) Find and expand $f(1 + h)$.

(b) Show that $\dfrac{f(1 + h) - f(1)}{h} = 2 + h$.

(c) Hence, find $\lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$.

(d) What is the geometric significance of the numerical value you obtained?

Question 3 Define the function $f(x) = x^2 + x$.

(a) Find the gradient of the secant that connects $P(1, f(1))$ to $Q(1 + h, f(1 + h))$.

(b) Find the limit of the gradient as $h \to 0$.

(c) Describe the significance of this result.

Question 4 Find the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ of the following functions.

(a) $f(x) = 2x + 1$

(b) $f(x) = 3 - 4x$

(c) $f(x) = x^2 - 1$

(d) $f(x) = x^2 + x - 5$

Question 5 By taking the limit as $h \to 0$ for each of the difference quotients in the previous question, state the derivative of each function above and hence find the gradient m of the tangent when $x = 1$.

Question 6 Find $f'(x)$ using first principles, and hence find the gradient of the tangent to each of the following when $x = 2$.

(a) $f(x) = x^2$

(b) $f(x) = 2 - x^2$

(c) $f(x) = x^2 - 2x$

(d) $f(x) = 3x^2 + x + 1$

Question 7

(a) Use graphing software to plot the point $(2, 1)$ on $y = \frac{1}{4}x^2$.

(b) Plot the point on the curve where $x = 2 + h$, where $h > 0$, on the same diagram. Create a slider that continuously varies the value of h from 0 to 1.

(c) Use your diagram to read off the gradient of the interval PQ as h approaches zero.

(d) Verify this result algebraically using differentiation by first principles.

Question 8

(a) Expand $(a + b)^3$.

(b) Hence, differentiate $y = x^3$ by first principles.

Challenge Problems

Problem 1 [Differentiating functions involving square roots by first principles]

Define the function $f(x) = \sqrt{x}$.

(a) By rationalising the denominator, show that the difference quotient is equal to

$$\frac{1}{\sqrt{x+h} + \sqrt{x}}$$

(b) Hence, write down what $f'(x)$ is.

(c) Use a similar technique to calculate $f'(x)$ if $f(x) = \frac{1}{\sqrt{x}}$.

Problem 2 [Differentiating $y = x^{-1}$ by first principles]

Define the function $f(x) = \frac{1}{x}$.

(a) Show that $f(x + h) - f(x) = -\frac{h}{x(x + h)}$.

(b) Hence, differentiate $y = \frac{1}{x}$ using first principles, and show that $f'(x) = -\frac{1}{x^2}$.

Exercise 7C

Basic differentiation rules

Fundamentals

Fundamentals 1

Complete the following.

(a) If $y = x^n$, where n is any rational number, then $y' =$ ________ .

(b) If $y = k \times x^n$, where n is any rational number and $k \in \mathbb{R}$, then $y' =$ ______ .

(c) If f and g are functions of x, then $(f \pm g)' =$ __________ .

(d) The notation $\dfrac{dy}{dx}$ means "differentiate ___ with respect to ___".

(e) If the derivative is $f'(x)$, then the gradient of the tangent at $x = \alpha$ is ________ .

Fundamentals 2

To find the equation of the tangent to the function $y = f(x)$ at $x = \alpha$:

(a) First, d________ the function to find ______ .

(b) Calculate $f'($___$)$ to obtain the g________ of the tangent.

(c) If the corresponding ___-coordinate is not provided, then calculate it by finding $f($___$)$.

(d) Substitute the values that you have found into the p_____-g________ formula, which is __________ .

(e) The normal is the line that is p________ to the tangent, but sharing the same point-of-contact.

(f) To find the gradient of the normal, first find the gradient of the t________ using the steps outlined earlier.

(g) Since the normal is p________ to the tangent, then the gradient of the normal must be the n________ r________ of the gradient of the tangent.

(h) Now, use the newly obtained gradient but the same p_____ to find the equation of the normal using the p_____-g________ formula.

Fundamentals 3

(a) If $f'(\alpha) = 0$, then that means the gradient of the tangent is ___. This means that the tangent is h________ at $x = \alpha$.

(b) In this scenario, the normal will be a v________ line, and have equation ________ .

Question 1

(a) Use the results from the previous exercise to write down the derivative of the following.

(i) $f(x) = x$ (ii) $f(x) = x^2$ (iii) $f(x) = x^3$

(b) Hence, hypothesise a formula for the derivative of $f(x) = x^n$, where n is a positive integer.

Question 2

The power rule of differentiation states that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is a rational number. By first expressing in the form x^n using the index laws, use the power rule to differentiate the following.

(a) x^6 (b) $\frac{1}{x}$ (c) $\frac{1}{x^2}$ (d) $\sqrt{x}$

(e) $\sqrt[3]{x}$ (f) $\sqrt{x^5}$ (g) $\frac{1}{\sqrt{x}}$ (h) 5

Question 3

Recall that if $y = k \times x^n$, then $y' = k \times nx^{n-1}$ and that $(f \pm g)' = f' \pm g'$. Use these rules, or a combination of them, to differentiate the following with respect to x.

(a) $f(x) = 3x^5$ (b) $f(x) = -4x^2$ (c) $f(x) = x^2 + 2x$

(d) $f(x) = x^3 - 4x^2$ (e) $f(x) = 5x - \frac{1}{2}x^2$ (f) $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2$

Question 4

By first expanding and/or simplifying, find $\frac{dy}{dx}$.

(a) $y = (x+1)^2$ (b) $y = x(x+1)$ (c) $y = \frac{x^2+1}{x}$

(d) $y = \frac{x^2 - 5x + 6}{x - 2}$ (e) $y = \left(x + \frac{1}{x}\right)^2$

Question 5

Differentiate the following with respect to their variables.

Express your answer as $\frac{d}{d\_\_\_}(\_\_\_\_) = \_\_\_\_\_$

For example,

$$\frac{d}{dx}(x^2 + x) = 2x + 1.$$

(a) $y^2 + 2y$ (b) $6z^3 - 5z^2$ (c) $t^3 - 2t^2$

(d) $p - p^3$ (e) $4u^2 - u$ (f) $w^3 + w^2$

Question 6 Find the gradient of the tangent on the following curves where $x = 1$.

(a) $y = x^2 - x$ (b) $y = \sqrt{x}$ (c) $y = (x-2)^2$

(d) $y = \dfrac{1}{x^2}$ (e) $y = \dfrac{x^2 - 1}{x^2}$

Question 7

(a) Sketch the graph of $y = x^2 - x$, labelling both x intercepts.

(b) Find the gradients of the tangents drawn from the x intercepts.

(c) What do you notice about the gradients from part (b)? Was this result expected?

Question 8 Let P be the point where $x = 4$ on the curve $y = \sqrt{x}$.

(a) Find the gradient of the tangent at P.

(b) Find the coordinates of P.

(c) Hence, find the equation of the tangent.

(d) State the gradient of the normal.

(e) Hence, find the equation of the normal.

Question 9

(a) Sketch the graph of $y = \sqrt{x}$.

(b) Find the equation of the normal to the curve $y = \sqrt{x}$ at $x = 1$.

(c) The normal from (b) intersects the x and y axes at A and B respectively. Draw this in your sketch in part (a) and find the coordinates of A and B.

(d) Find the area of $\triangle OAB$, where O is the origin.

Question 10 Find the equation of the tangent and normal to $y = 2x - x^2$ at $x = 2$.

Question 11 Find the x-coordinate of the point(s) on the curve $y = x^3 - 12x + 1$ where:

(a) the tangent is horizontal.

(b) the tangent has angle of inclination $45°$ from the positive x-axis.

(c) the normal has gradient $\dfrac{1}{9}$.

Question 12 Find the points on the curve $y = x^3 - 6x^2 + 10x - 1$ where the tangents are parallel to the line $y = x + 2$.

Question 13 Find the x-coordinate of the point on the parabola $y = ax^2 + bx + c$ where the tangent is horizontal. The result should look familiar. Explain your observation.

Question 14 Find the equation of the tangent and normal to $y = x + \dfrac{1}{x}$ at $x = 1$. You may notice some unusual results compared to other tangent/normal questions done earlier in this exercise. Use graphing software to sketch $y = x + \dfrac{1}{x}$, and explain why this question is different.

Challenge Problems

Problem 1 By first simplifying the expression, differentiate $f(x) = \dfrac{x^4 - 1}{x - 1}$.

Problem 2 Differentiate $f(x) = \dfrac{x - 1}{\sqrt{x} + 1}$.
Hint: Try rationalising the denominator.

Problem 3 The line $x + y + 2 = 0$ is a tangent to $y = x^3 - 4x$ at a point P. Find the coordinates of P.
Hint: Do not attempt to solve the two equations simultaneously.

Problem 4 Find the equation of the tangent and normal to $y = \sqrt{1 - x^2}$ at $(0, 1)$.
Hint: Try drawing a sketch of the curve. You will see that differentiation is not necessary here!

Problem 5

(a) State the gradient of a tangent that is horizontal.

(b) Hence, show that there are no points on $y = x - \dfrac{1}{x}$ where the tangent is horizontal.

(c) Verify your result using graphing software.

Problem 6

(a) Show that $P\left(cp, \dfrac{c}{p}\right)$, where c is a constant, lies on the hyperbola $xy = c^2$.

(b) Draw a sketch of the hyperbola and label any point P on the curve.

(c) Show that the tangent at P has equation $x + p^2 y = 2cp$. Draw the tangent on your sketch.

(d) The tangent intersects the x and y axes at A and B respectively. Show that the area of $\triangle OAB$ will always be the same, regardless of where the tangent was drawn.

Problem 7

(a) Expand and simplify the expression $(x - 1)(1 + x + x^2 + x^3 + x^4)$.

(b) Hence, differentiate $\dfrac{x^5 - 1}{x - 1}$ with respect to x.

Exercise 7D

Chain rule

Fundamentals

Fundamentals 1
The following are different variations of the statement of the chain rule.

(a) If $h(x) = f\,(g(x))$, then $h'(x) =$ ____________.

(b) If y is a function of u, and u is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times$ ________ .

Fundamentals 2
The following are useful formulas to remember.

(a) If $y = (ax+b)^n$, then $y' =$ ____________.

(b) If $y = (f(x))^n$, then $y' = n\,(f(x))^{n-1} \times$ __________.

Question 1

(a) Differentiate $(x^3+1)^2$ by first expanding and then differentiating term-by-term.

(b) Differentiate $(x^3+1)^2$ using the chain rule.

(c) Verify that the answers in (a) and (b) are the same.

Question 2

Use the chain rule to differentiate the following

(a) $y = (2x+1)^5$

(b) $y = (x^2+1)^5$

(c) $y = \dfrac{1}{2-x}$

(d) $y = \sqrt{2x-1}$

(e) $y = \dfrac{1}{1+x^2}$

(f) $y = \dfrac{1}{\sqrt{2x+1}}$

(g) $y = \sqrt{1-x^2}$

(h) $y = \left(x - \dfrac{1}{x}\right)^4$

(i) $y = \dfrac{1}{(3x-4)^2}$

Question 3

(a) Differentiate $f(x) = \sqrt{4-x^2}$.

(b) Find the gradient of the tangent where $x = 2$. What do you notice? Explain your observation.

Question 4

(a) Differentiate $y = \sqrt{2x+1}$.

(b) Show that $y \times y' = 1$.

(c) What does this imply about the behaviour of y' as y gets large?

(d) Verify this result by drawing a graph of $y = \sqrt{2x+1}$ either by hand or using graphing software.

Question 5 [A way to bypass the quotient rule]

(a) Find constants a and b such that $\dfrac{x}{x+1} = a + \dfrac{b}{x+1}$.

(b) Hence, differentiate $y = \dfrac{x}{x+1}$.

Question 6

(a) Differentiate $f(x) = (ax+b)^n$, where a and b are constants, and $n = 1, 2, 3, \ldots$

(b) Find $f\left(-\dfrac{b}{a}\right)$.

(c) Find $f'\left(-\dfrac{b}{a}\right)$ for $n = 2$ and $n = 3$. What does this mean geometrically? Was this answer expected?

Question 7 Consider the function $f(x) = \dfrac{\sqrt{x}+1}{x-1}$.

(a) Show that $\dfrac{\sqrt{x}+1}{x-1} = \dfrac{1}{\sqrt{x}-1}$

(b) Hence, differentiate $f(x)$.

Question 8 Consider the function $f(x) = \dfrac{x^2}{x-1}$.

(a) Show that $f(x) = x + 1 + \dfrac{1}{x-1}$.

(b) Hence, differentiate $f(x)$.

Challenge Problems

Problem 1 [A way to bypass the product rule]

Define the function $f(x) = x(x+1)^5$.

(a) By turning the x at the front into $(x+1-1)$, or otherwise, show that

$$f(x) = (x+1)^6 - (x+1)^5.$$

(b) Hence, differentiate $f(x) = x(x+1)^5$.

Problem 2

(a) Expand and simplify the expression $(x+1)(1-x+x^2-x^3+x^4)$.

(b) Differentiate $f(x) = \dfrac{x^5+1}{x+1}$ with respect to x.

(c) Hence, show that

$$\frac{d}{dx}\left(\frac{x^5}{x+1}\right) = 4x^3 - 3x^2 + 2x - 1 + \frac{1}{(x+1)^2}.$$

Exercise 7E

Product rule

Fundamentals

Fundamentals 1

The product rule of differentiation is used to differentiate the p________ of two functions of x when expanding is too impractical.

Fundamentals 2

The following are variations of the statement of the product rule.

(a) If u and v are functions of x, and $y = uv$, then $y' = u'v + \_\_\_\_\_$.

(b) If u and v are functions of x, then $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + \_\_\_\_\_$.

(c) If $h(x) = f(x)g(x)$, then $h'(x) = \_\_\_\_\_\_\_\_\_\_\_\_$.

Question 1 Differentiate the following using the product rule, and simplify your answer.

(a) $y = x(x+1)^5$

(b) $y = (3x-1)(5-4x^2)$

(c) $y = (3x^2+2x+9)(4-5x)$

(d) $y = x(3x-2)^4$

(e) $y = x^2(x-1)^3$

(f) $y = (x+1)^2(x+2)^3$

Question 2 Differentiate the following using the product rule.

(a) $y = (x^2+1)(3x-1)^2$

(b) $y = (x^2-1)^3(5x+2)^4$

(c) $y = (2x+1)^4(2x-1)^5$

(d) $y = x\sqrt{x+1}$

(e) $y = x^2\sqrt{1-x}$

(f) $y = x\sqrt{1+x^2}$

Question 3 Let n be a positive integer. Show that the function

$$f(x) = x\left(x^2+1\right)^n$$

will never be horizontal at any points on the curve.

Question 4 Define the function

$$f(x) = x^m(x-1)^n,$$

where m and n are positive integers. Show that the graph is horizontal at $x = 0, \dfrac{m}{m+n}, 1$.

Challenge Problems

Problem 1 [Proving the Quotient Rule]

Let u and v be functions in terms of x with derivatives u' and v' respectively. Define the quotient $f(x) = \frac{u}{v}$.

(a) By expressing the quotient as $f(x) = uv^{-1}$, show that

$$f'(x) = \frac{u'v - v'u}{v^2}.$$

(b) Use this formula to differentiate $f(x) = \frac{3x-1}{2x+1}$.

Problem 2 [Multiple Root Theorem]

(a) Define the polynomial

$$P(x) = (x-1)^3(1+x^2).$$

Show that $P(1) = P'(1) = 0$.

(b) Define the polynomial

$$P(x) = (x-\alpha)^n Q(x),$$

where $Q(x)$ is some polynomial and $n \geq 2$ is an integer. Show that

$$P'(x) = (x-\alpha)^{n-1}\left[nQ(x) + (x-\alpha)Q'(x)\right],$$

and hence state the value of $P'(\alpha)$.

Exercise 7F

Quotient rule

Fundamentals

Fundamentals 1

(a) The quotient rule is used to differentiate the q__________ of two functions of x.

(b) The quotient rule is/is not (circle one) always needed to differentiate the quotient of two functions of x.

Fundamentals 2

The following are variations of the statement of the quotient rule.

(a) If u and v are functions of x and $y = \dfrac{u}{v}$, then $y' = \dfrac{vu' - \quad}{\qquad}$.

(b) If u and v are functions of x, then $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - \quad}{\qquad}$.

(c) If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(x) = \dfrac{g(x)f'(x) - \quad}{(\qquad)^2}$

Question 1 By first simplifying, differentiate the following *without* using the quotient rule.

(a) $y = \dfrac{x}{\sqrt{x}}$ (b) $y = \dfrac{x^4 - x^2}{x}$ (c) $y = \dfrac{x^2 + 1}{x}$

(d) $y = \dfrac{x^2 - 1}{x + 1}$ (e) $y = \dfrac{x - 1}{x^2 - 1}$ (f) $y = \dfrac{x^2 - 5x + 6}{x - 3}$

Question 2 Differentiate the following using the quotient rule.

(a) $y = \dfrac{x - 1}{x + 1}$ (b) $y = \dfrac{3x + 1}{2x - 1}$ (c) $y = \dfrac{x^2}{3x + 5}$

(d) $y = \dfrac{1 - x^2}{1 + x^2}$ (e) $y = \dfrac{a - x}{a + x}$ (f) $y = \dfrac{ax + b}{bx + a}$

Question 3 Differentiate $y = \dfrac{x + 1}{x - 1}$ and show that the derivative is always negative.

Question 4 Consider the function

$$y = \frac{x + x^{-1}}{x - x^{-1}}.$$

(a) Show that $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$. (b) Hence, find $f'(x)$.

Question 5 Differentiate the following using the quotient rule.

(a) $y = \dfrac{\sqrt{x}}{x+1}$

(b) $y = \dfrac{\sqrt{x+1}}{x}$

(c) $y = \dfrac{x}{\sqrt{x+1}}$

Question 6 Define the function

$$f(x) = \frac{x-a}{x-b},$$

where $x \neq b$.

(a) Find $f'(x)$.

(b) Show that if $a > b$, then $f'(x) > 0$. What is the geometric significance of this?

(c) Show that if $a < b$, then $f'(x) < 0$. What is the geometric significance of this?

(d) Show that $f'(x) = 0$ when $a = b$. Was this result expected?

(e) Verify the results from (b) and (c) using graphing software, by graphing some sample functions where $a > b$ and $a < b$ respectively.

Challenge Problems

Problem 1 [Application to curve sketching]

Define the function $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$.

(a) Show that $f'(x) = \dfrac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}$.

(b) Explain why the gradients of the tangents are always positive.

(c) Examine the behaviour of $f'(x)$ as x gets smaller. What does this mean about the gradients of the tangents?

(d) Find the x and y intercepts of $y = f(x)$.

(e) State the domain of $f(x)$.

(f) Show that $\displaystyle\lim_{x\to\infty}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = 1$.
Hint: Divide top and bottom by $\sqrt{x}$ first.
What is the geometric significance of the limit being 1?

(g) Fill in the following table of values.

x	$\frac{1}{4}$	1	9	100
$f(x)$				

(h) Hence, draw a sketch of $y = f(x)$ and check that it agrees with all the data found in previous parts. Verify your answer using graphing software.

Exercise 7G

Average rate of change

Fundamentals

Fundamentals 1

Let $A(a, f(a))$ and $B(b, f(b))$ be two points on a continuous function $f(x)$.

(a) The average rate of change between A and B is the net change in ___-coordinate, divided by the length of the interval of x between the two points.

(b) This means that the average rate of change from A to B is the g________ of the chord AB.

(c) The formula for the average rate of change of a function $f(x)$ over the interval $x \in [a, b]$ is

$$\text{Average Rate} = \frac{\quad - \quad}{b - a}$$

Fundamentals 2

(a) The function may represent the quantity Q of something as a function of t______ t.

(b) In this case, the average rate of change is

$$\text{Average Rate} = \frac{\text{change in } \quad}{\text{change in time}}.$$

(c) Another way of expressing this is

$$\text{Average Rate} = \frac{Q_{\text{final}} - \quad}{\quad - t_{\text{initial}}}.$$

Fundamentals 3

Suppose t is time, in minutes. Write down the equivalent time intervals, in the form $t \in [a, b]$, that accurately depict the following descriptions.

(a) During the first minute.

(b) During the second minute.

(c) During the n^{th} minute.

(d) Over the first five minutes.

Question 1 A container is continuously filled with water, and the volume of water after t minutes is given by

$$V(t) = t^2 + t + 1,$$

where V is measured in litres.

(a) Find the volume of water in the container initially ($t = 0$) and when $t = 3$.

(b) Hence, find the average rate of change of V over the interval $t \in [0, 3]$.

(c) Find the average rate of change of V over the interval $t \in [1, 4]$.

Question 2 A bucket full of water has a leak, and the volume after t minutes is given by

$$V(t) = t^2 - 4t + 4,$$

where V is measured in litres.

(a) Find the average rate of change of V.

(i) during the first minute. (ii) over the first 2 minutes.

(b) Explain why the model is only valid for $t \in [0, 2]$.
Hint: What happens to a leaky bucket eventually?

Question 3 The value of a car, in dollars, is modelled by the equation

$$V(t) = 50000(1.15)^{-t},$$

where t is time in years. What is the average rate of change in the value of the car over the first five years?

Question 4 Find the average rate of change of y with respect to x, over the specified intervals.

(a) $y = 2x^2$, $[0, 1]$ (b) $y = x^3$, $[-1, 3]$ (c) $y = \dfrac{1}{x}$, $[-4, -2]$

(d) $y = \dfrac{1}{x^2}$, $[2, 3]$ (e) $y = \sqrt{x}$, $[0, 4]$ (f) $y = \dfrac{1}{\sqrt{x}}$, $[1, 2]$

Question 5 The *average rate of return* is how much an investment makes, on average, over a specified time period. Find the average rate of return of an investment of \$5000 that increases in value by 6% per annum for 4 years.

Challenge Problems

Problem 1 The volume of water in a container after t minutes is given by the equation

$$V(t) = t^2 + 4t + 20,$$

where V is measured in litres. It is known that the average rate of change in volume over the first k seconds is 10 litres per minute. Find the value of k.

Problem 2 [Linking to instantaneous rate of change]

The temperature (°C) of a metal rod after being heated for t minutes is given by

$$T(t) = t^2 + t.$$

(a) Find the average rate of change over $t \in [1, 1.1]$, $t \in [1, 1.01]$ and $t \in [1, 1.001]$. What value does the average rate of change appear to be approaching?

(b) In part (a), you were finding the rate of change over $t \in [1, 1+h]$ as $h \to 0$. What process does this resemble?
Hint: Think about earlier in this chapter!

(c) The *instantaneous rate of change* is the rate of change at an instant, or in other words at a specific value of t rather than across an interval of t. Use the results from previous parts to briefly explain the relationship between the average rate of change and the instantaneous rate of change.

Exercise 7H

Instantaneous rate of change

Fundamentals

Fundamentals 1

(a) The i__________ rate of change of a quantity Q is the rate at a specific value of ___.

(b) In contrast, the a________ rate of change of a quantity Q is the rate of change across an i________ of time.

(c) The instantaneous rate of change of $Q(t)$ at $t = t_1$ is denoted by $Q'($___$)$.

(d) This means that the instantaneous rate of change at $t = t_1$ is just the g________ of the t________ at that point.

(e) The instantaneous rate of change is the l________ of the average rate of change, as the interval of time gets smaller/larger (circle one).

Fundamentals 2

(a) Use the a________ rate of change if you are given a time interval $t \in [a, b]$, but use the i__________ rate of change if you are given an instant value of time $t = t_1$.

(b) The instantaneous rate of a quantity Q is the d__________ of Q with respect to t______.

(c) Hence, the rate of change of a quantity Q is denoted by $\dfrac{d}{d}$___.

Fundamentals 3

(a) A quantity is increasing at $t = t_1$ if $Q'(t_1)$ _____

(b) A quantity is decreasing at $t = t_1$ if $Q'(t_1)$ _____

(c) A quantity is not changing at $t = t_1$ if $Q'(t_1) =$ _____

Question 1 The volume of water in a container can be modelled by the equation

$$V = \frac{10 - t}{2 + t},$$

where t is time in minutes and V is measured in litres.

(a) How much water was in the container initially?

(b) Show that $V'(t) = -\dfrac{12}{(2 + t)^2}$.

(c) Explain why the container is always losing water.
Hint: Look at the sign of $V'(t)$

(d) Find the rate at which V is decreasing initially.

(e) Find the rate at which V is decreasing when $t = 2$.

Question 2 A ball is dropped from the top of a building, and the distance of the ball from the ground after t seconds is given by the equation

$$h = 100 - 5t^2,$$

where h is measured in metres.

(a) How tall is the building?

(b) How long does it take for the ball to hit the ground? Hence, state the time interval $t \in [a, b]$ where this model is valid.

(c) What is the rate of change $\dfrac{dh}{dt}$ of the height

(i) initially?

(ii) when it hits the ground?

(d) When is the rate at which h decreases the largest?

(e) Bob claims that when the ball is half-way to the ground, the rate at which h decreases is also half of the maximum. Is this correct?

Question 3 A circular disc expands as it is being heated. The area, in of the disc after t minutes is given by

$$A = 3t^2 + t,$$

where A is in cm^2.

(a) Find the rate at which the area increases after 4 minutes.

(b) Find the average rate of change during the 4^{th} minute.

Question 4 The rate of water flow into a tank is given by

$$\frac{dV}{dt} = 10(2 - t),$$

where $\dfrac{dV}{dt}$ is measured in litres per minute. The tank initially had 60 litres of water in it.

(a) Describe the difference in the water flow when $t = 1$ and $t = 4$.

(b) Show that the volume of water increases for a certain amount of time, and then decreases afterwards.

(c) Show that the formula $V = 20t - 5t^2 + 60$ satisfies the given information.

(d) Hence, find when the tank is empty.

Question 5 A block of ice of mass 8 kg melts according to the formula

$$M = 8 - \frac{t^2}{32},$$

where M is the mass remaining, in kilograms, after t minutes.

(a) Find the time taken for all the ice to melt.

(b) Find the rate at which the ice is melting after 4 minutes.

(c) Find the rate at which the ice is melting when 2 kg of ice has melted.

(d) Is the melting rate faster initially, or just before it is gone completely?

Question 6 Water is poured into a tank and the volume V, in litres, after t seconds is

$$V = 200 + 80t - 2t^2.$$

(a) Find the amount of water initially.

(b) If the container was full in 20 seconds, what is the capacity of the bottle?

(c) Find the average rate of change during the first 10 seconds, and compare it with the instantaneous rate of change after 10 seconds.

(d) When was the flow rate the greatest?

Challenge Problems

Problem 1 A pharmaceutical drug is administered and the the amount A of the drug in a person's bloodstream is given by the function

$$A = 8t^2 - t^3,$$

where t is time in hours and A is in milligrams.

(a) Explain intuitively why the model is only limited up to a certain value of t, and find it.
Hint: Sketch a graph.

(b) Find $\frac{dA}{dt}$ and provide a description of its physical meaning.

(c) When did the body stop absorbing more of the drug?
Hint: What is the value of $\frac{dA}{dt}$ when the body stops absorbing the drug?

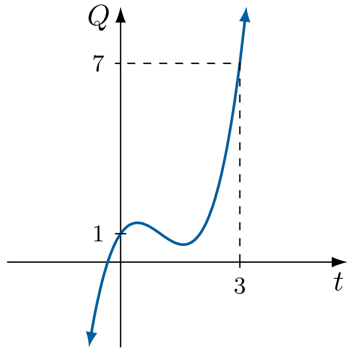
(d) During which interval of t was the drug amount increasing, and during which interval of t was the drug amount decreasing?

(e) What was the highest amount of the drug in the bloodstream?

Problem 2 [Mean value theorem]

The diagram below shows the graph of a quantity Q against time t, given by the equation

$$Q = t^3 - 3t^2 + 2t + 1$$



(a) Calculate the average rate of change over the interval $t \in [0, 3]$.

(b) Find the specific value(s) of $t_0 \in [0, 3]$ such that the instantaneous rate of change is equal to the average rate of change from (a).

Exercise 7I

Gradient function

Fundamentals

Fundamentals 1

(a) A function is said to be increasing in a domain, if as x increases in that domain, ___ increases in that domain.

(b) Hence, the g_____ must be p________ in that domain.

(c) A function is said to be decreasing in a domain, if as ___ decreases in that domain, ___ decreases in that domain.

(d) Hence, the g_____ must be n________ in that domain.

Fundamentals 2

(a) The gradient function of $y = f(x)$ is $y =$ ________.

(b) In the domain where a function is increasing, then the gradient function is p________.

(c) In the domain where a function is decreasing, then the gradient function is n________.

(d) Where the tangent is horizontal for $f(x)$, the gradient function is z_____.

Question 1 For the piecewise-defined functions given below,

(i) sketch the graph of $y = f(x)$.

(ii) write down the gradient along each branch of your graph.

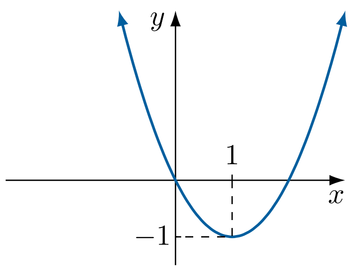
(iii) write down the equation of the derivative as a piecewise-defined function.

(iv) sketch the graph of $y = f'(x)$ on a separate set of axes.

(a) $f(x) = \begin{cases} -2x+3, & \text{if } x < -1 \\ 5, & \text{if } -1 \le x \le 2 \\ x+3, & \text{if } x > 2 \end{cases}$

(b) $f(x) = \begin{cases} 3, & \text{if } x < -3 \\ 3x-5, & \text{if } -3 \le x \le 1 \\ -2, & \text{if } x > 1 \end{cases}$

Question 2 Consider the following graph of $y = f(x)$.



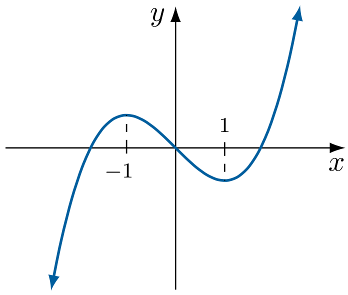
(a) State the domain where the function is increasing. What does this say about the y-coordinates of $y = f'(x)$ in this domain?

(b) State the domain where the function is decreasing. What does this say about the y-coordinates of $y = f'(x)$ in this domain?

(c) State where the function has a horizontal tangent. What does this say about the y-coordinates of $y = f'(x)$ at this point?

(d) Hence, sketch the graph of the derivative function $y = f'(x)$.

Question 3 The diagram below shows a sketch of $y = f(x)$.



(a) State where $f'(x) > 0$.

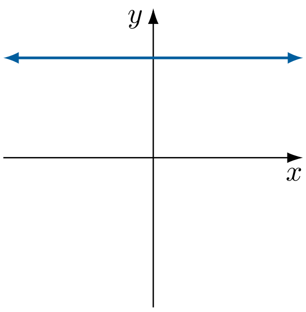
(b) State where $f'(x) < 0$.

(c) State where $f'(x) = 0$.

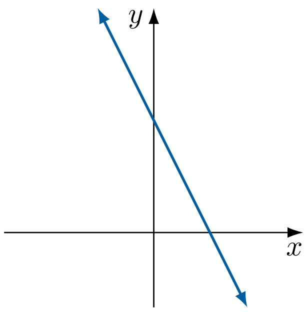
(d) Hence, sketch the graph of $y = f'(x)$.

Question 4 Graphs (a) to (i) below are sketches of $y = f(x)$.

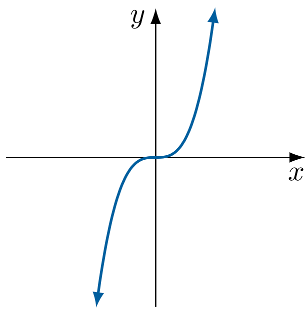
(a)



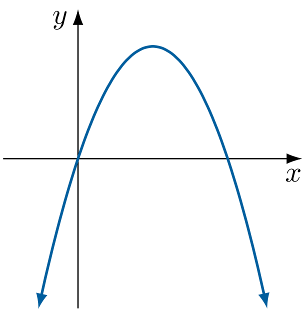
(b)



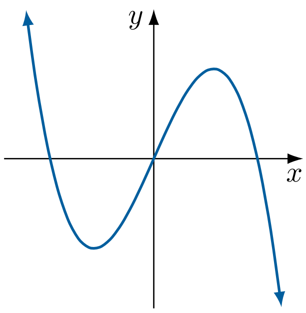
(c)



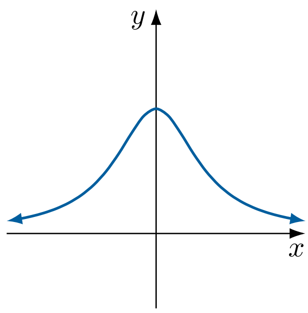
(d)



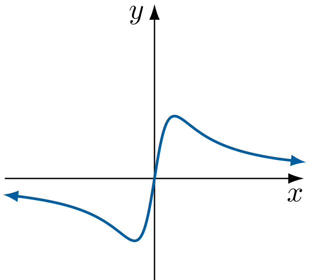
(e)



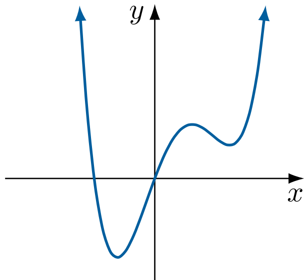
(f)



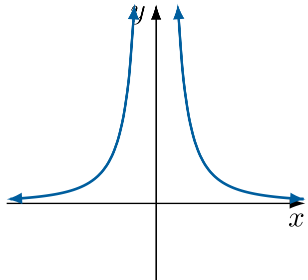
(g)



(h)

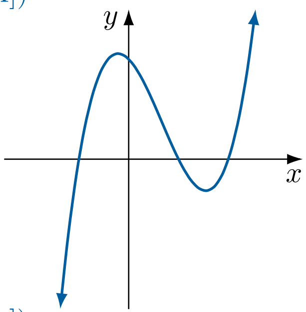


(i)

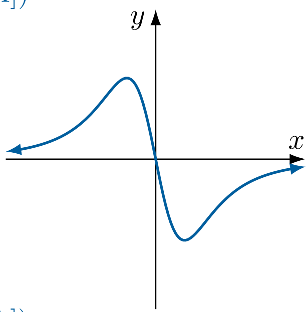


Graphs (A) to (I) below are sketches of $y = f'(x)$, but in random order. Match the graphs of $y = f(x)$ to their corresponding derivative graphs.

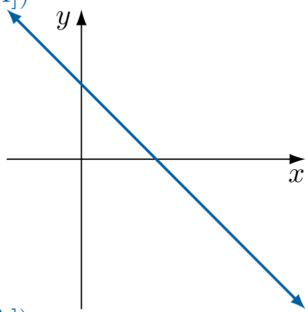
(tsk[A])



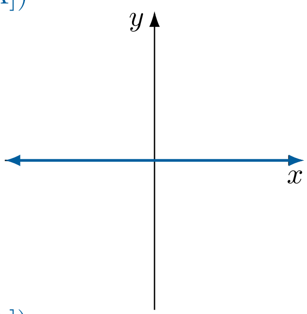
(tsk[A])



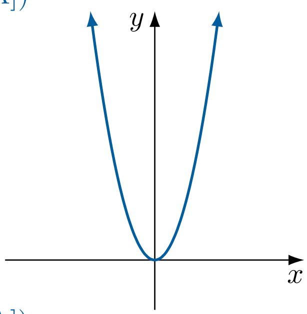
(tsk[A])



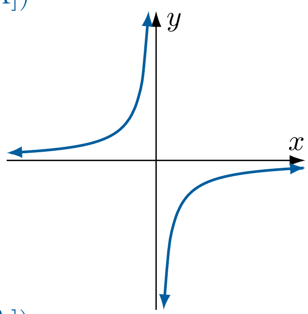
(tsk[A])



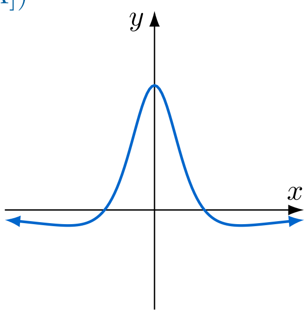
(tsk[A])



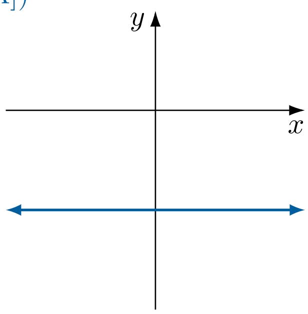
(tsk[A])



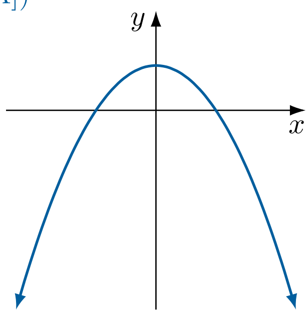
(tsk[A])



(tsk[A])

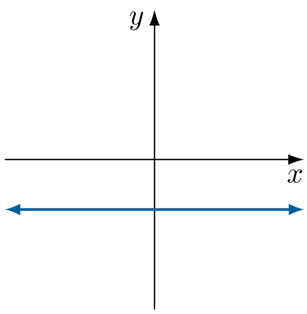


(tsk[A])

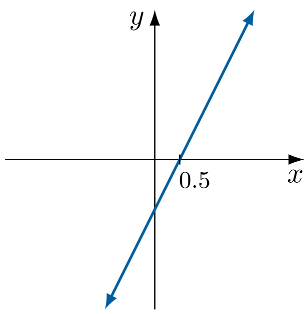


Question 5 Sketch the derivative function of the following graphs.

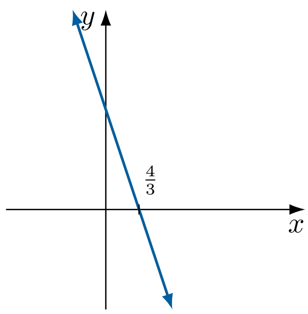
(a)



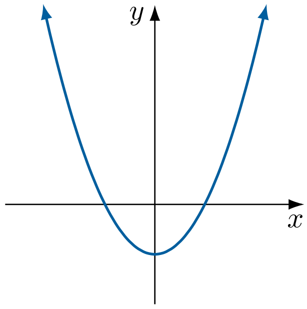
(b)



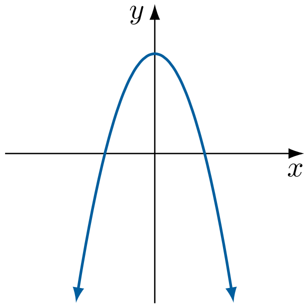
(c)



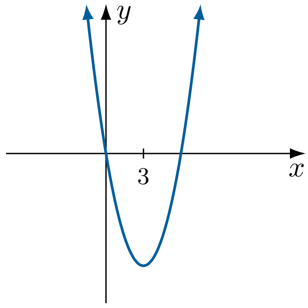
(d)



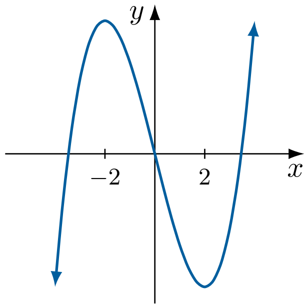
(e)



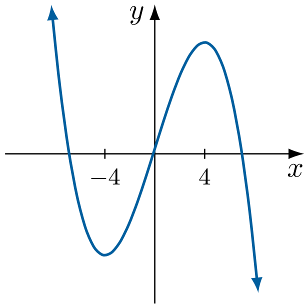
(f)



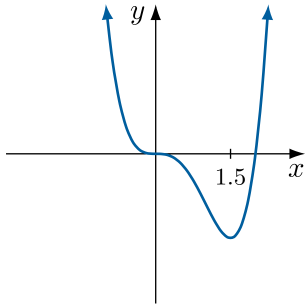
(g)



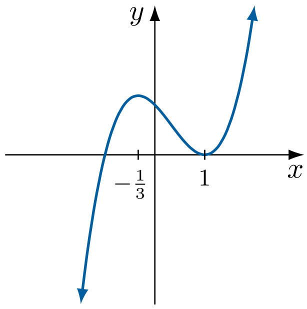
(h)



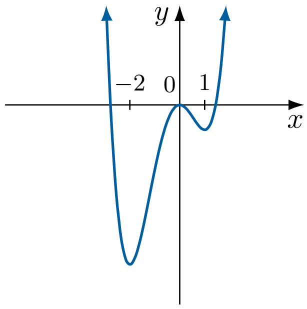
(i)



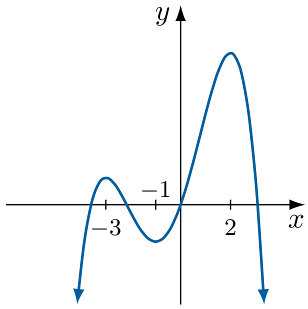
(j)



(k)



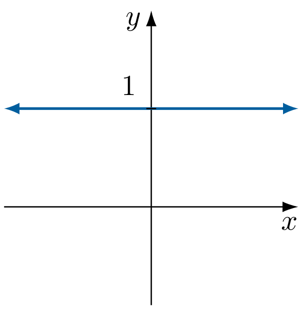
(l)



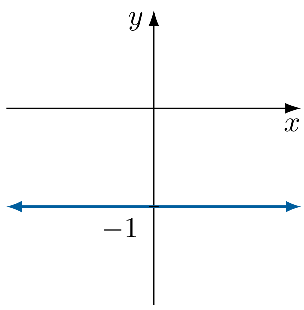
⚙ Challenge Problems

Problem 1 What is the shape of $y = f(x)$ if the graph of $y = f'(x)$ looks like the following?

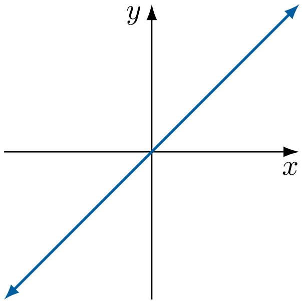
(a)



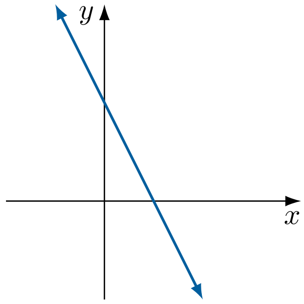
(b)



(c)

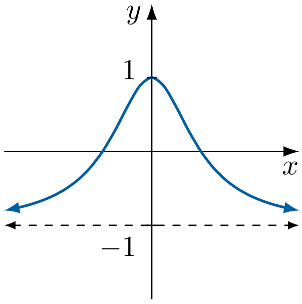


(d)

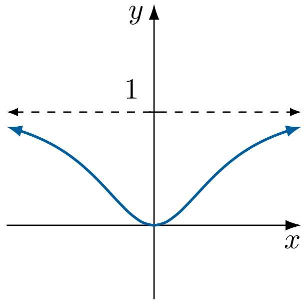


Problem 2 The diagrams below show sketches of $y = f(x)$. Sketch a graph of $y = f'(x)$.

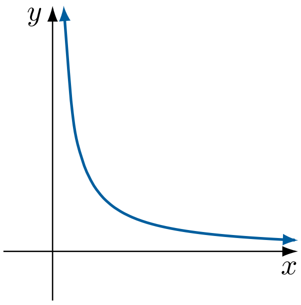
(a)



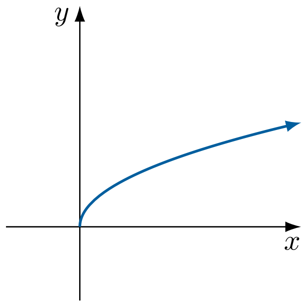
(b)



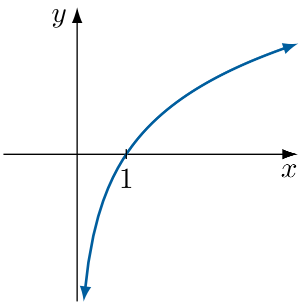
(c)



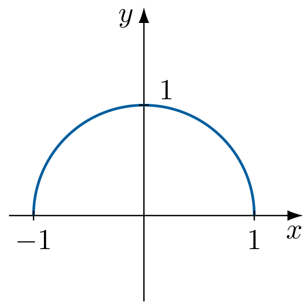
(d)



(e)



(f)



Exercise 7J

Continuity

Fundamentals

Fundamentals 1

(a) Informally, a c__________ f________ is a function that can be drawn 'without lifting the pen off the paper'.

(b) If in a given domain $x \in [a, b]$, the graph of a function cannot be drawn in such a way, then we say that the function is d__________ in the domain $x \in [a, b]$.

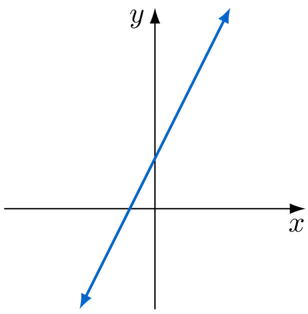
(c) Name and draw three types of discontinuities that can occur.

(d) To find a discontinuity in a function that has x's in the denominator, we can let the denominator equal z_____.

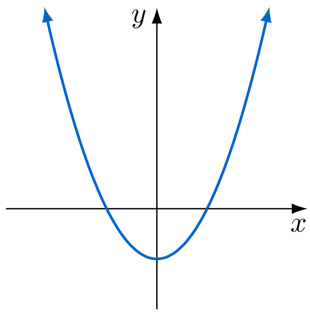
(e) Continuity of a function cannot be discussed unless a d________ has been specified. This is because a function may be c__________ in one part of the graph, but d__________ in another.

Question 1 Categorise the following as either continuous or discontinuous functions over the domain $x \in \mathbb{R}$. For the functions that are discontinuous in that domain, state the value(s) of x where the discontinuity has occurred.

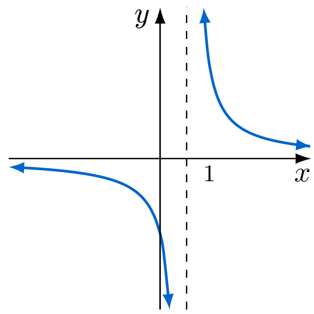
(a)



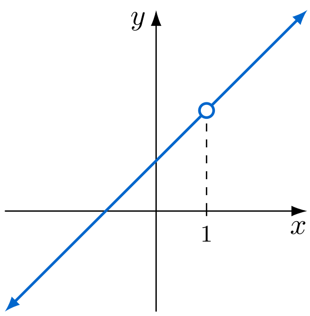
(b)



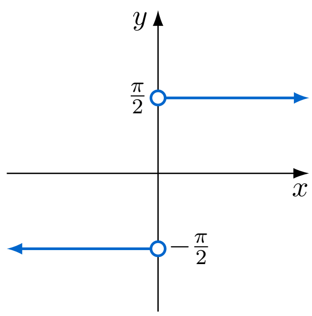
(c)



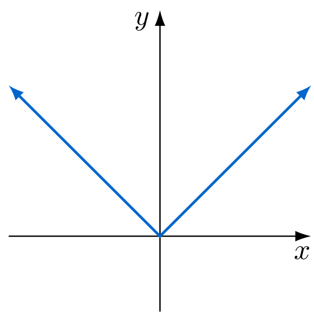
(d)



(e)



(f)



Question 2 Without sketching the following functions, categorise them as continuous or discontinuous functions over the domain $x \in \mathbb{R}$. For the functions that are discontinuous, state the value(s) of x where the discontinuity has occurred.

(a) $f(x) = x^2 - 1$

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = |x|$

(d) $f(x) = \dfrac{1}{1/x}$

(e) $f(x) = \dfrac{1}{x^2 - 1}$

(f) $f(x) = \dfrac{x^2 - 1}{x - 1}$

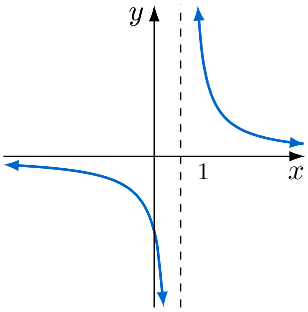
(g) $f(x) = \dfrac{1}{x^2 + 1}$

(h) $f(x) = \dfrac{x + 1}{x^2 + 5x + 4}$

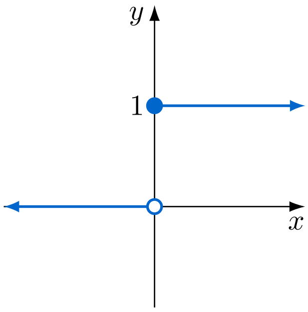
(i) $f(x) = \dfrac{x}{4x^2 + 4x + 1}$

Question 3 The diagrams below show functions that have discontinuities. Categorise them as either jump discontinuities, holes, or asymptotic discontinuities.

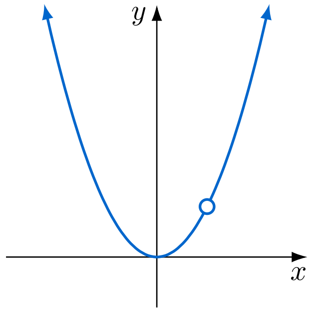
(a)



(b)



(c)



Question 4 Determine whether the following functions are continuous or discontinuous at the point where the curve changes equation.

(a) $f(x) = \begin{cases} 2x + 1, & \text{for} \quad x < 0 \\ -4x + 1, & \text{for} \quad x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{x}{x + 4}, & \text{for} \quad x < -1 \\ -\dfrac{1}{3}, & \text{for} \quad x \geq -1 \end{cases}$

(c) $f(x) = \begin{cases} x^2 + 1, & \text{for} \quad x \leq 0 \\ -1, & \text{for} \quad x > 0 \end{cases}$

(d) $f(x) = \begin{cases} 13 - x, & \text{for} \quad x < 3 \\ 10, & \text{for} \quad x = 3 \\ x^2 + 1, & \text{for} \quad x > 3 \end{cases}$

Question 5 For each of the following, find the value(s) of a so that the function is continuous

(a) $f(x) = \begin{cases} x^2 + ax, & \text{for} \quad x \leq 1 \\ 5x - 2, & \text{for} \quad x > 1 \end{cases}$

(b) $f(x) = \begin{cases} 3x + a, & \text{for} \quad x < 0 \\ x^2 - 2x + 1, & \text{for} \quad x \geq 0 \end{cases}$

(c) $f(x) = \begin{cases} 2x + 3, & \text{for} \quad x < 1 \\ -x^2 + a, & \text{for} \quad x \geq 1 \end{cases}$

(d) $f(x) = \begin{cases} ax + 2, & \text{for} \quad x \leq 2 \\ 2a^2x, & \text{for} \quad x > 2 \end{cases}$

Challenge Problems

Problem 1 Find the value of a and b so that $f(x)$ is continuous.

$$f(x) = \begin{cases} a, & \text{for} \quad x < 2 \\ (x-3)^2, & \text{for} \quad 2 \le x \le 6 \\ b - x, & \text{for} \quad x > 6 \end{cases}$$

Problem 2 What is wrong with the following 'solution' that concludes that $f(x) = \dfrac{x}{1+\sqrt{x}}$ is discontinuous at $x = 1$?

> To find discontinuities, let the denominator be zero.
>
> $$\begin{aligned} 1 + \sqrt{x} &= 0 \\ \sqrt{x} &= -1 \\ x &= (-1)^2 \quad \text{(square both sides)} \\ &= 1 \end{aligned}$$
>
> Hence, the function is discontinuous at $x = 1$.

Problem 3 Recall from trigonometry the circular functions $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \operatorname{cosec} x$, $y = \sec x$, and $y = \cot x$. Which of these functions have points of discontinuity over the domain $x \in [0, 2\pi]$?

Problem 4 [Why 'hole' discontinuities are said to be *removable*]

(a) Sketch the graph of $y = x - 2$.

(b) Sketch the graph of $y = \dfrac{x^2 - 4}{x + 2}$. What is the difference between this graph and the graph in part (a)?

(c) Hence, complete the following so that the function becomes continuous

$$f(x) = \begin{cases} \dfrac{x^2-4}{x+2}, & \text{for } x \_\_\_\_ \\ \_\_\_\_\_\_, & \text{for } x \_\_\_\_ \end{cases}$$

Chapter 7 Review

Differentiation and Applications

Review

Question 1 Differentiate the following with respect to x.

(a) $\dfrac{x^2}{x^4}$ (b) $x\sqrt{x}$ (c) $\dfrac{1}{\sqrt[3]{x^2}}$ (d) $\dfrac{\sqrt{x}}{x^2}$

Question 2 Differentiate the following using first principles.

(a) $f(x) = 4x^2$ (b) $f(x) = x - x^2$

(c) $f(x) = 2x^2 + 3x$ (d) $f(x) = x^2 - 4x + 2$

Question 3 Differentiate the following with respect to x.

(a) $y = x^n$ (b) $y = 4x^n$ (c) $y = -3$ (d) $y = 7x$

(e) $y = \dfrac{x}{5} + \dfrac{5}{x}$ (f) $y = \dfrac{3}{4x}$ (g) $y = \dfrac{3x^2 - 6x + 8}{x}$

(h) $y = x\sqrt[3]{x}$ (i) $y = \dfrac{3}{x^2}$ (j) $y = \dfrac{\sqrt[3]{x}}{x^2}$

Question 4 Differentiate the following with respect to x.

(a) $y = (3x - 2)^2$ (b) $y = (3x^2 - 5)^4$ (c) $y = \sqrt{x^2 + 1}$

(d) $y = \dfrac{1}{x^3 - x^2}$ (e) $y = \dfrac{1}{\sqrt{1 - x^2}}$ (f) $\left(x + \dfrac{1}{x}\right)^4$

Question 5 Differentiate the following with respect to x.

(a) $y = x^2(2 - 3x)^5$ (b) $y = (3x + 1)^4(2x - 1)^5$

(c) $y = x\sqrt{x + 1}$ (d) $y = x\left(x^2 + 1\right)^3$

Question 6 Differentiate the following with respect to x.

(a) $y = \dfrac{x + 1}{x - 1}$ (b) $y = \dfrac{x^2}{2x - 1}$ (c) $y = \dfrac{x^2 - 1}{x^2 + 1}$ (d) $y = \dfrac{x^2}{\sqrt{x + 1}}$

Question 7 Consider the curve $y = x^2 - 5x$.

(a) Find the point on the curve where the gradient is 1.

(b) Hence, find the equation of the tangent.

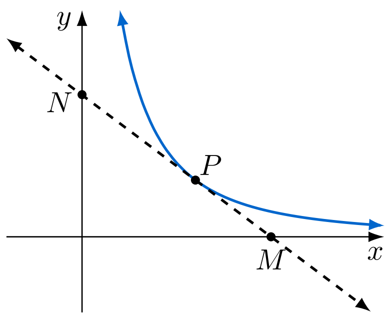
Question 8 Find the coordinates of the point on $y = x^2 - 2x - 3$, where the tangent is perpendicular to the line $x + 4y - 1 = 0$.

Question 9 Find the coordinates of the point on $y = 2x^3 - 2x^2$, where the tangent is perpendicular to the line $2x - y = 5$.

Question 10 Find the equation of the normal to the curve $y = 2x^2 - 3$ that is parallel to $x - y + 4 = 0$.

Question 11 The line $y = mx + c$ is a tangent to the curve $y = x^3 - x + 4$ at the point $(1, 4)$. Find the value of m and c.

Question 12 The diagram below shows the graph of $y = \dfrac{4}{x^2}$. The tangent drawn from the point P cuts the x and y-axes at M and N respectively.



(a) If the gradient of the tangent at P is -1, then show that P has coordinates $(2, 1)$.

(b) Find the equation of the tangent at P.

(c) Find the area of $\triangle OMN$.

Question 13 Consider the function $f(x) = x^2 - 4x$. Find the average rate of change over the intervals

(a) $x \in [2, 3]$. (b) $x \in [-2, 5]$. (c) $x \in [-4, -1]$.

Question 14 Find the average rate of change of $y = 2 \times 3^x$ over the interval $x \in [0, 2]$.

Question 15 A circular disc expands as it is heated. The area, in cm^2 after t minutes is given by $A = 8t^2 + 2t$. Find the rate of change of the area after 2 minutes, and the average rate of change during the second minute.

Question 16 The curve $y = ax^3 - x^2 - 3x + 5$ has a gradient of 5 when $x = -1$. Find the value of a.

Question 17 The graph of $y = ax^2 + bx + c$ has an x-intercept at $x = -2$. The tangent at $(1, -18)$ is horizontal. Find the values of a, b, and c.

Question 18 Let $f(x) = (2x - 3)^2$. Find the values of x for which

(a) $f(x) = 0$ (b) $f'(x) = 0$ (c) $f'(x) > 0$ (d) $f'(x) = 10$

Question 19 The population of trout in a trout farm is growing. Let the population after t months be $P = 10 \times 1.4^t$. Find

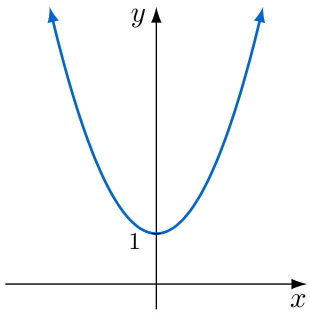
(a) the number of trout in the farm initially.

(b) Find the average rate of growth of the population during the first 5 months.

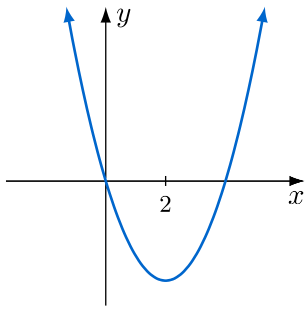
(c) Find the average rate of growth of the population during the fifth month.

Question 20 The diagrams below show sketches of $y = f(x)$. Copy the same diagram in your book and on the same set of axes, but with a different coloured pen, draw the graph of $y = f'(x)$.

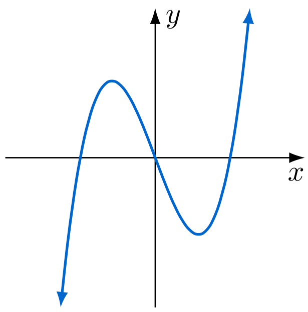
(a)



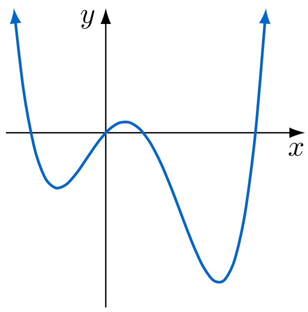
(b)



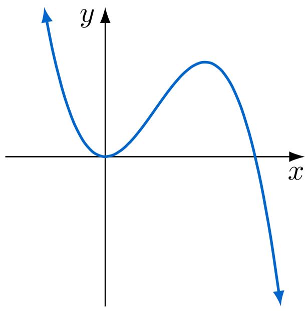
(c)



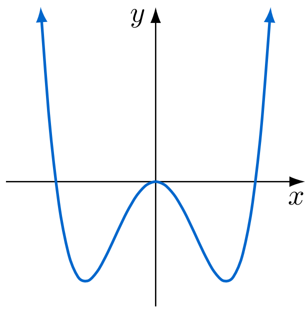
(d)



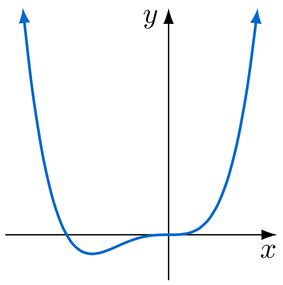
(e)



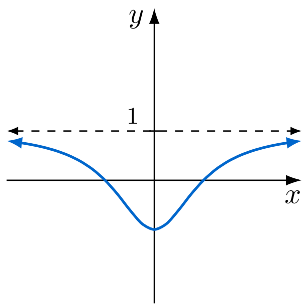
(f)



(g)



(h)



Investigation Task

Differentiation by First Principles

You are already familiar with the following formula for differentiation by first principles.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

However, this is not the only one. There are many other similar versions that do the same job and sometimes, they can be more convenient to differentiate specific functions.

Question 1 [Revising the mechanics of the standard version]

Explain, using a clear diagram, how the above formula finds the gradient of the tangent to the curve $y = f(x)$ at $P(x, f(x))$.

Question 2 [Alternate version]

Draw a clear diagram to illustrate how the following formulae give the gradient of the tangent to $y = f(x)$ at $x = c$.

(a) $f'(c) = \lim_{u \to c} \frac{f(u) - f(c)}{u - c}$

(b) $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$

(c) Find the gradient of the tangent at $x = c$ for the curve $y = x^2$ using both techniques above, as well as the standard formula. Verify that you get the same answer. Which technique do you think was most convenient to use?

(d) Repeat part (c) but now with $y = \frac{1}{x}$, and again comment on which technique you think was most convenient to use.

Question 3 [Experimenting]

Now that you have investigated more deeply the mechanics of differentiation by first principles, and alternative versions, make up your own formula and test it on $y = x^2$.

8

LOGARITHMIC AND EXPONENTIAL FUNCTIONS

- The logarithmic function
- Logarithmic laws
- Differentiating the exponential function
- Solving exponential and logarithmic equations
- Graphing exponential and logarithmic equations
- Applications of exponential and logarithmic functions

CHAPTER 8

Exercise 8A

The logarithmic function

Fundamentals

Fundamentals 1

Complete the following.

(a) The logarithm of a number is the p________, or i________, to which the b________ must be raised to give that number.

(b) The log form of $y = a^x$ is ______

(c) The log form of $y = a^x$ is only defined when $a >$ ___ and $a \neq$ ___

(d) The domain of $\log_a(x)$ is x ______

(e) The logarithmic function is the i________ of the exponential function.

(f) The graph of $y = a^{-x}$ is the graph of $y = a^x$ r________ across the y axis.

Fundamentals 2

Complete the following.

(a) $\log_a(a) =$ ______

(b) $\log_a(1) =$ ______

(c) $\log_a(a^x) =$ ______ for $x \in \mathbb{R}$

(d) $a^{\log_a(x)} =$ ______ for x ______

Fundamentals 3

(a) The d______ of $y = a^x$ is the r______ of $y = \log_a(x)$.

(b) The r______ of $y = a^x$ is the d______ of $y = \log_a(x)$.

Question 1 Convert the following logarithmic forms to index form.

(a) $\log_{10}(10) = 1$

(b) $\log_9(81) = 2$

(c) $\log_5(1) = 0$

(d) $\log_2(32) = 5$

(e) $\log_4(2) = \frac{1}{2}$

(f) $\log_5\left(\frac{1}{5}\right) = -1$

Question 2 Convert the following index forms to logarithmic form.

(a) $2^3 = 8$

(b) $4^2 = 16$

(c) $4^{1/2} = 2$

(d) $8^{1/3} = 2$

(e) $5^{-2} = \frac{1}{25}$

(f) $16^{-1/2} = 0.25$

Question 3 Solve the following by rewriting each equation in index form.

(a) $x = \log_{10}(1000)$ (b) $x = \log_{10}(0.01)$ (c) $x = \log_2(4)$

(d) $x = \log_4\left(\frac{1}{4}\right)$ (e) $x = \log_{\frac{1}{2}}(16)$ (f) $x = \log_{12}(12)$

(g) $x = \log_{12}(1)$ (h) $\log_x(25) = 2$ (i) $\log_x(64) = 3$

(j) $\log_x\left(\frac{1}{25}\right) = -2$ (k) $\log_x(3) = \frac{1}{2}$ (l) $\log_x(27) = \frac{3}{2}$

(m) $\log_3(x) = -2$ (n) $\log_2(x) = 3$

Question 4

(a) Complete the following table.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(x)$						

(b) Hence, draw a sketch of $y = \log_2(x)$.

Question 5 On the same set of axes, sketch the following sets of curves.

(a) $y = 2^x, \quad y = 3^x, \quad y = 4^x$

(b) $y = 2^{-x}, \quad y = 3^{-x}, \quad y = 4^{-x}$

(c) $y = 2^x, \quad y = \log_2 x$

Question 6 Use the identity $\log_a(a^x) = x$ to simplify the following expressions.

(a) $\log_2(2^4)$ (b) $\log_3(81)$

(c) $\log_5\left(\sqrt{5}\right)$ (d) $\log_2\left(\sqrt{32}\right)$

Question 7 Find the domain of the following logarithmic functions.

(a) $y = \log_2(x-2)$ (b) $y = \log_{10}(4-x)$

(c) $y = \log_8(2x+4)$ (d) $y = \log_{10}\left(\frac{1}{x}\right)$

(e) $y = \log_7\left(\sqrt{x}\right)$ (f) $y = \log_4(1+x^2)$

Question 8 Use the identity $a^{\log_a(x)} = x$ to simplify the following expressions.

(a) $2^{\log_2(5)}$ (b) $4^{\log_4(3)}$

(c) $5^{\log_5\left(\frac{1}{3}\right)}$ (d) $6^{\log_6\left(\sqrt{2}\right)}$

Challenge Problems

Problem 1

(a) Complete the following

$$\begin{aligned} 4^{3\log_4(5)} &= 4^{\log_4(5^{-})} \\ &= 4^{\log_4(\_\_\_)} \end{aligned}$$

(b) Hence find the exact value of $4^{3\log_4(5)}$.

(c) Use a similar technique to find the exact value of $25^{\log_5(4)}$.

Problem 2 Find the domain of $y = \log_{10}(x) + \log_{10}(x - 2)$.

Exercise 8B

Logarithmic laws

Fundamentals

Fundamentals 1

(a) $\log_a(m) + \log_a(n) = \_\_\_\_\_$

(b) $\log_a(m) - \log_a(n) = \_\_\_\_\_$

(c) $\log_a(m^n) = \_\_\_\_\_$

(d) $\log_a(x) = \dfrac{\log_b(\quad)}{\_\_\_\_\_}$

Fundamentals 2

(a) $\log_a(a) = \_\_\_\_\_$

(b) $\log_a(1) = \_\_\_\_\_$

(c) $\log_a\left(\frac{1}{a}\right) = \_\_\_\_\_$

(d) $\log_a \sqrt{a} = \_\_\_\_\_$

(e) $\log_a\left(\frac{1}{x}\right) = \_\_\_\_\_$

(f) $\log_a\left(\sqrt{x}\right) = \_\_\_\_\_$

Question 1 Evaluate the following, without the use of a calculator

(a) $\log_2 16$

(b) $\log_4(4)$

(c) $\log_5(625)$

(d) $\log_3(81)$

(e) $\log_4(0.25)$

(f) $\log_3\left(\frac{1}{27}\right)$

(g) $\log_9(3)$

(h) $\log_2\left(\frac{1}{16}\right)$

(i) $\log_{25}(0.2)$

Question 2 Evaluate the following, without the use of a calculator

(a) $\log_2(32) - \log_2(64)$

(b) $\log_5(100) - \log_5(4)$

(c) $\log_{10}\left(\sqrt{10}\right) \times \log_{10}\left(\sqrt[3]{10}\right)$

(d) $\log_3\left(\sqrt{3}\right) - \log_3\left(\frac{1}{9}\right) + \log_3(27)$

Question 3 Calculate the following to two decimal places.

(a) $\log_3 10$

(b) $\log_5 7$

(c) $2\log_2 6$

(d) $\log_{0.5} 3$

Question 4 Estimate the value of the following, to the nearest integer, by considering powers of 2, 3 or 10 where appropriate.

(a) $\log_{10}(97)$

(b) $\log_{10}(0.0098)$

(c) $\log_3(28)$

(d) $\log_2(33)$

(e) $\log_2(0.123)$

(f) $\log_3(240)$

(g) $\log_2(130)$

(h) $\log_3(83)$

(i) $\log_3(0.031)$

Question 5 Let $x = \log(a)$, $y = \log(b)$ and $z = \log(c)$.
Express the following in terms of x, y and z.

(a) $\log(ab)$ (b) $\log\left(\frac{c}{b}\right)$ (c) $\log\left(\sqrt{ab}\right)$

(d) $\log\left(\frac{1}{abc}\right)$ (e) $\log\left(ab^2c^3\right)$ (f) $\log\left(\frac{1}{\sqrt{b^2c}}\right)$

(g) $\log\left(\frac{\sqrt{ac}}{b}\right)$ (h) $\log\left(a^3\sqrt{bc}\right)$ (i) $\log\left(\frac{c^2}{a^2b^3}\right)$

Question 6 Let $a = \log_5(2)$ and $b = \log_5(3)$. Express the following in terms of a and b.
Hint: Notice that the numbers below are combinations of 2, 3 and 5.

(a) $\log_5(6)$ (b) $\log_5(1.5)$ (c) $\log_5(0.2)$

(d) $\log_5\left(\frac{8}{9}\right)$ (e) $\log_5\left(\sqrt{30}\right)$ (f) $\log_5(6.4)$

Question 7 Write the following as a single logarithm.

(a) $4\log_3(2) - \log_3(4) + \log_3(16)$ (b) $\frac{1}{2}\log_2(x) - \log_2(3)$

(c) $1 + \log_{10}(5)$ (d) $\log_{10}(800) - 3\log_{10}(2)$

(e) $\log_5(25x) - \log_3(9)$ (f) $-2\log_4(5) + \frac{1}{2}\log_4(20)$

Question 8 Determine whether the following are true or false.

(a) $\log_6\left(\frac{1}{6}\right) = -1$ (b) $\log_6(1) = 0$ (c) $\log_6(6) = 1$

(d) $\frac{1}{2}\log_3(10) = \log_3(5)$ (e) $\log(5) + \log(2) = 10$ (f) $\log(a) - \log(b) = \frac{\log(a)}{\log(b)}$

(g) $\log(a) - \log(b) = \log\left(\frac{a}{b}\right)$ (h) $\log_3(9) = 3$ (i) $\log_9(3) = \frac{1}{2}$

(j) $\log_5(20) = 1 + \log_5(4)$ (k) $\log_5\left(\frac{1}{25}\right) = -\frac{1}{2}$ (l) $\log_6(2) \times \log_6(3) = 1$

(m) $\log_6(2) + \log_6(3) = 1$ (n) $\frac{\log(x^4)}{\log(x)} = 4$ (o) $\frac{\log_c(a)}{\log_c(b)} = \log_b(a)$

Question 9 By changing the base to the specified value, simplify the following.

(a) $\log_8(32)$, [base 2] (b) $\log_{0.25}(32)$, [base 2]

(c) $\log_{\frac{1}{9}}(27)$, [base 3] (d) $\log_{25}(125)$, [base 5]

Question 10

(a) Show that $\log_a(b) = \dfrac{1}{\log_b(a)}$.

(b) Hence, find the value of $\log_{\sqrt{27}}(3)$.

Question 11 Show that $\log_a(b) \times \log_b(c) \times \log_c(a) = 1$.

Question 12 Show that $\log_{a^x}(b^x) = \log_a(b)$.

Challenge Problems

Problem 1

(a) Make y the subject of $\log_5(y) = 3 - 2\log_5(x)$.

(b) What is the domain of $\log_5(y) = 3 - 2\log_5(x)$?

Problem 2 Suppose $a^2 + b^2 = 7ab$.

(a) Simplify $\left(\dfrac{a+b}{3}\right)^2$.

(b) Hence, show that

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\big(\log(a) + \log(b)\big).$$

Problem 3 Simplify

$$\log_{10}\left(\frac{1}{2}\right) + \log_{10}\left(\frac{2}{3}\right) + \log_{10}\left(\frac{3}{4}\right) + \cdots + \log_{10}\left(\frac{99}{100}\right).$$

Problem 4 Let $p = \dfrac{\log(a)}{\log(x)}$. Show that $x^p = a$.

Problem 5 Show that $\log_{xy}(a) = \dfrac{\log_x(a)}{1 + \log_x(y)}$.

Problem 6 Show that $\log_{a^x}(b) = \dfrac{\log_a(b)}{x}$.

Problem 7

(a) Show that $c^{\log_c(a) \times \log_b(c)} = a^{\log_b(c)}$.

(b) Hence, or otherwise, show that $a^{\log_b(c)} = c^{\log_b(a)}$.

Exercise 8C

Differentiating the exponential function

Fundamentals

Fundamentals 1

(a) The function $\ln(x)$ is sometimes referred to as the n________ l________

(b) $y = e^x$ and $y = \ln(x)$ are i________ functions of each other.

(c) If $0 < x < 1$, then $\ln(x)$ is n________

Fundamentals 2

(a) $\dfrac{d}{dx}(e^x) = $ ________

(b) $\dfrac{d}{dx}(e^{ax+b}) = $ ________

(c) $\dfrac{d}{dx}(e^{f(x)}) = $ ________

(d) $\ln(x) = \log_{\_\_}(x)$

(e) $\ln(e^x) = $ ________ for x ________

(f) $e^{\ln(x)} = x$ for x ________

(g) $e^{a\ln(x)} = $ ______, for x ______.

(h) $e^{\ln(f(x))} = $ ______, for $f(x)$ ______.

Question 1 Differentiate the following.

(a) $y = e^{2x}$

(b) $y = e^{-x}$

(c) $y = e^{-6x}$

(d) $y = e^{\frac{x}{2}}$

Question 2 Differentiate the following.

(a) $y = e^{2x+5}$

(b) $y = e^{2x-1}$

(c) $y = e^{3-2x}$

(d) $y = e^{\frac{2x-1}{3}}$

(e) $y = e^{\frac{2-3x}{5}}$

Question 3 By first simplifying and expressing in the form e^{kx}, differentiate the following.

(a) $y = (e^x)^3$

(b) $y = \dfrac{1}{e^x}$

(c) $y = \sqrt{e^x}$

(d) $y = e^x\sqrt{e^x}$

(e) $y = \dfrac{1}{\sqrt{e^x}}$

(f) $y = \dfrac{e^x + 1}{e^x}$

Question 4

(a) Complete the following $2 = e^{\_\_\_}$.

(b) Hence, express 2^x in the form e^{ax} for some value of a.

(c) Use the expression in part (b) to show that $\dfrac{d}{dx}(2^x) = 2^x \times \ln(2)$.

Question 5 Differentiate the following using the technique above.

(a) 4^x

(b) 3^x

(c) 3^{2x}

(d) $4^{\frac{x}{2}}$

Question 6 Differentiate the following by using the fact that $e^{\ln x} = x$ for $x > 0$.

(a) $e^{\ln(x)}$

(b) $e^{-\ln(x)}$

(c) $e^{2\ln(x)}$

(d) $e^{0.5\ln(x)}$

Question 7

(a) Find the second derivative of e^{2x}.

(b) Hence, show that the n^{th} derivative of e^{2x} is $2^n e^{2x}$.

Question 8 Let P_0 and k be any constants. Show that $P = P_0 e^{kt}$ satisfies the differential equation $\frac{dP}{dt} = kP$.

Question 9 Show that $y = Ae^x + Be^{-x}$ satisfies the differential equation $y'' - y = 0$ for any real values of A and B.

Question 10 Show that $y = Ae^{-2x} + Be^{-3x}$ satisfies the differential equation $y'' + 5y' + 6y = 0$ for any real values of A and B.

Question 11 Find the value of k such that $y = e^{kx}$ is a solution to the differential equation $y'' + 5y' = 6y$.

Question 12 Define the functions $c(x) = \dfrac{e^x + e^{-x}}{2}$ and $s(x) = \dfrac{e^x - e^{-x}}{2}$. Show that the following are true.

(a) $c(x) + s(x) = e^x$

(b) $c(x) - s(x) = e^{-x}$

(c) $c'(x) = s(x)$

(d) $s'(x) = c(x)$

(e) $s(2x) = 2s(x)c(x)$

(f) $c(2x) = s^2(x) + c^2(x)$

(g) $c^2(x) - s^2(x) = 1$

(h) $c(2x) = 2s^2(x) + 1$

(i) $s(x+y) = s(x)c(y) + c(x)s(y)$

(j) $c(x+y) = c(x)c(y) + s(x)s(y)$

Challenge Problems

Problem 1 Let a be a positive constant.

(a) Simplify $e^{x \ln(a)}$.

(b) Hence, show that

$$\frac{d}{dx}(a^x) = a^x \times \ln(a).$$

Problem 2 Let α and β be distinct real roots of the quadratic equation $y = ax^2 + bx + c$.

Show that

$$y = Ae^{\alpha x} + Be^{\beta x},$$

where A and B are any real numbers, is a solution of the differential equation

$$ay'' + by' + cy = 0.$$

Exercise 8D

Solving exponential and logarithmic equations

Fundamentals

Fundamentals 1

(a) If $\log(X) = \log(Y)$, then $X =$ ________

(b) Suppose that $a > 1$. If $a^X = a^Y$, then $X =$ ________

(c) If $a^x = b$, then $x = \dfrac{\ln(b)}{\quad}$

Question 1 Solve the following equations by expressing both sides with a common base.

(a) $3^x = 9$ (b) $4^x = 2$ (c) $5^x = \dfrac{1}{25}$

(d) $4^x = \dfrac{1}{2}$ (e) $3^{4x-1} = 1$ (f) $25^x = 125$

(g) $25^x = \dfrac{1}{125}$ (h) $4 \times 32^x = \dfrac{1}{16}$ (i) $4 \times 6^x = \dfrac{1}{54}$

Question 2 Solve the following equations.

(a) $4^{x-2} = 2^{3x-4}$ (b) $4^{x+2} = 16^{x+1}$ (c) $25^{x+1} = 125^{2x-1}$

Question 3 Solve the following pairs of simultaneous equations.

(a) $\begin{aligned} 2^{x+y} &= 4 \\ 3^{x-y} &= 27 \end{aligned}$ (b) $\begin{aligned} 5^{2x+y} &= 1 \\ 2^{x-2y} &= 32 \end{aligned}$ (c) $\begin{aligned} 4^{2x-y} &= 64 \\ 2^{3x+y} &= 4 \end{aligned}$

Question 4 Solve the following equations.

(a) $49^x - 6 \times 7^x - 7 = 0$ (b) $9^x - 2 \times 3^x = 3$ (c) $4^x + 2^{x+1} - 8 = 0$

Question 5 Solve the following logarithmic equations

(a) $\log_3(2x+1) = 2$ (b) $\log_x(2) = \dfrac{1}{2}$ (c) $\log_x(81) = 4$

(d) $\log_9(x) = 1.5$ (e) $\log_x\left(\dfrac{1}{64}\right) = -3$ (f) $\log_4(x) = 0.25$

Question 6 Solve the following logarithmic equations. Remember to check that your answer is valid!

(a) $\log_2(x-1)+\log_2(x+1)=3$

(b) $\log_{10}(x+1)-\log_{10}(x-8)=1$

(c) $\log_2(x-1)=2\log_2(x+1)$

(d) $(\ln(x))^2-\ln(x^2)+1=0$

(e) $(\log_3(x))^2-\log_3(x^5)=-6$

(f) $\ln(2x+4)+\ln(x-1)=2\ln(x+1)$

Question 7 Solve $2^x=5$, correct to 2 decimal places, by taking the natural log of both sides.

Question 8 Use a similar technique to solve the following equations. Express your solutions to 2 decimal places.

(a) $2^x=10$

(b) $5^{2x}=8$

(c) $2^{-2x-1}=-4$

(d) $3^{3x-2}=15$

(e) $3e^{-5x}=2$

(f) $3e^{2x}-2e^x=0$

(g) $e^{x+\ln(4)}=3e^x+2$

(h) $e^{2x}+e^x=0$

Question 9 Solve the following equations and express your solutions to two decimal places where necessary.

(a) $4^x-6\times 2^x+5=0$

(b) $9^x-3^{x+1}+2=0$

(c) $25^x-5^{x+1}=6$

Question 10 A collectible is currently valued \$2000, and it appreciates at a rate of 5% per annum.

(a) Find the value of the collectible after two years.

(b) Find an equation that models the value of the collectible after t years.

(c) Find how long it takes for the collectible to double in value.

(d) Bob hypothesises that it will take the same amount of time for the collectible to double in value again to reach \$8000. Verify whether Bob's claim is correct or incorrect.

Question 11 An object is dropped from an aeroplane. Let s be the speed of the particle in metres per second, and t be time in seconds. The speed of the particle is given by the formula $s=10(1-e^{-t})$.

(a) Find the speed of the particle two seconds after release.

(b) How long does it take for the particle to reach a speed of 5 metres per second?

(c) Does this particle ever reach a speed of 10 metres per second? Justify your answer.

Question 12 When antibiotics are introduced into the human body, the concentration the antibiotic in the bloodstream can be modelled by the equation

$$C(t) = 9(e^{-0.2t} - e^{-0.4t}),$$

where t is time in hours and C is measured in μg mL^{-1}.

(a) What is the concentration of the antibiotic after 1 hour?

(b) Let $x = e^{-0.2t}$ and show that when $C = 2$, the resultant quadratic is $9x^2 - 9x + 2 = 0$.

(c) Solve for x, and hence show that it will take just over 2 hours to reach 2μg mL^{-1}.

(d) You should have acquired two solutions. Can you think of a biological reason as to why there may be two times when the concentration is 2μg mL^{-1}?

Challenge Problems

Problem 1 Solve the following equations for x.

(a) $3 \times 9^{x+1} = 27^{x-2}$ (b) $16^{2-x} = \sqrt{8^x}$ (c) $\left(\frac{8}{27}\right)^x = \left(\frac{9}{4}\right)^{2-x}$

Problem 2 An object falls from a hot-air balloon. The speed v of the object after t seconds can be modelled by the equation $v = b(1 - e^{-kt})$, where b and k are positive constants and v is measured in metres per second.

(a) The speed of the object is measured at $t = 1$ and $t = 2$, and found to be $v = 10$ and $v = 15$ respectively. Show that $2e^{-2k} - 3e^{-k} + 1 = 0$.

(b) Let $p = e^{-k}$ and solve the resultant quadratic equation for p.

(c) Hence, find the value of k and b.

(d) Use graphing software to sketch the graph of v against t.

(e) What is the limiting value of v as $t \to \infty$?

(f) What is the physical significance of this result?

Problem 3 Solve $\log_8(x) + \log_4(x) = 5$.

Exercise 8E

Graphing exponential and logarithmic equations

Fundamentals

Fundamentals 1

(a) Sketch the graph of $y = e^x$.

(b) Sketch the graph of $y = \ln(x)$.

Fundamentals 2

Consider $y = a^x$ for $a > 0$.

(a) The graph has a horizontal a________ at $y = 0$.

(b) The domain is x ______

(c) The range is y ______

(d) a^x ______ 0 for all real x.

(e) If $a > 1$, then the graph is i________

(f) If $a < 1$, then the graph is d________

(g) All graphs pass through the point ________

Fundamentals 3

Consider $y = \log_a(x)$ for $a > 1$.

(a) The expression $\log_a(x)$ is undefined when x ______

(b) The graph of $y = \log_a(x)$ has a v______ asymptote at $x =$ ______

(c) The graph is an i________ function for all x ______.

(d) The domain is x ______

(e) The range is y ______

(f) All graphs pass through the point ______

Fundamentals 4

Complete the following.

(a) $\lim\limits_{x\to\infty} (e^x) =$ ______

(b) $\lim\limits_{x\to-\infty} (e^x) =$ ______

(c) $\lim\limits_{x\to\infty} (e^{-x}) =$ ______

Question 1 Use graphing software to sketch the family of functions $y = e^x + c$ for $c = \pm1, \pm2$ and describe the effect of varying values of c on the shape of the curve.

Question 2 Use graphing software to sketch the family of functions $y = a^x$ for $a = 2, 3, 4, 5$ and describe the effect of varying values of a on the shape of the curve for the parts where $x > 0$.

Question 3 Use graphing software to sketch the family of functions $y = k \times e^x$ for $k = 1, 2, 3, 4, 5$ and describe the effect of varying values of k on the shape of the curve for the parts where $x > 0$.

Question 4 Use graphing software to sketch the family of functions $y = e^{x+c}$ for $c = 0, \pm 1, \pm 2$ and describe the effect of varying values of c on the shape of the curve.

Question 5 Compare the graph when $k = e^2$ in Question 3, and the graph when $c = 2$ in Question 4. What do you notice? Explain your observations algebraically.

Question 6 Sketch the graph of the following exponential functions. Label x and y-intercepts, wherever possible.

(a) $y = e^{2x}$ (b) $y = 2 \times e^x$ (c) $y = e^{x-1}$

(d) $y = e^{x+1}$ (e) $y = -e^x$ (f) $y = e^x + 1$

(g) $y = e^x - 1$ (h) $y = 1 - e^x$ (i) $y = -1 - e^x$

Question 7 Use your understanding of the graph of $y = e^x$ to find the range of the following.

(a) $y = e^{2x}$ (b) $y = 2e^x$ (c) $y = -e^{2x}$

(d) $y = 2e^x - 3$ (e) $y = e^{-x}$ (f) $y = -e^{-x}$

Question 8 Let $f(x) = 2 - e^{-x}$.

(a) Find the value of $f(0)$ and $f'(0)$.

(b) Show that $f'(x) > 0$ for all values of x. What is the geometric significance of this?

(c) Find the coordinates of the point on the graph where the gradient is $\frac{1}{2}$.

(d) Find $\lim_{x \to \infty} f(x)$.

(e) Find $\lim_{x \to \infty} f'(x)$.

(f) Hence, sketch the graph of $f(x)$.

Question 9 Sketch the following exponential functions, labelling any intercepts with the coordinate axes as well as any asymptotes. Verify your answers using graphing software.

(a) $y = e^{-x}$ (b) $y = -e^{-x}$ (c) $y = 4 + e^{-x}$

(d) $y = 3 - e^{-x}$ (e) $y = -2 + e^{-x}$ (f) $y = -5 - e^{-x}$

Question 10 Find the equation of the resultant graph when $y = e^x$ is transformed in the following ways.

(a) Translated left by 2 units.

(b) Translated upwards by 1 unit.

(c) Reflected across the x-axis.

(d) Reflected across the y-axis.

(e) Reflected across the x-axis, and then across the y-axis.

Question 11

(a) Sketch the graph of $y = \ln(x)$.

(b) Write down the equation of the resultant equation when the above graph is transformed in the following ways, and use graphing software to sketch your answer along with $y = \ln(x)$ to compare the curves.

(i) Shifted down by 1 unit.
(ii) Shifted up by 2 units.
(iii) Translated to the left by 1 unit.
(iv) Dilated vertically by a factor of 2.
(v) Dilated vertically by a factor of 0.5.
(vi) Reflected across the y axis.

Question 12 Sketch the graph of the following logarithmic functions. Label x and y intercepts, wherever possible.

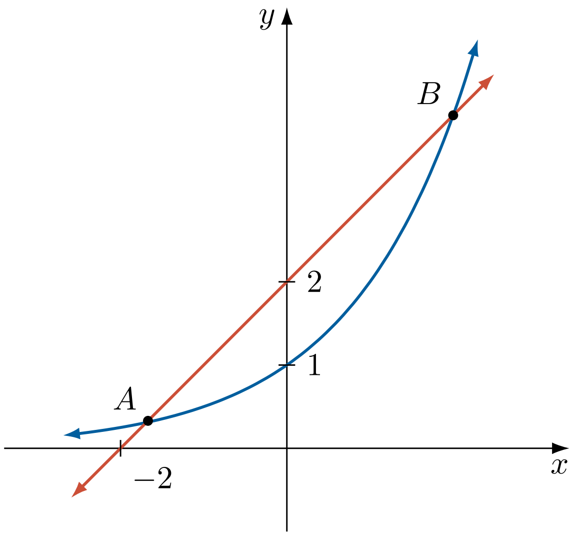
(a) $y = \ln(x + 2)$
(b) $y = 2\ln(x)$
(c) $y = 3\ln(x) + 2$
(d) $y = -\ln(x)$
(e) $y = 2 - \ln(x)$
(f) $y = -1 - \ln(x)$

Question 13 Find the following limits.

(a) $\lim_{x\to\infty} (e^x + 1)$
(b) $\lim_{x\to\infty} (5 - e^x)$
(c) $\lim_{x\to\infty} (7 - e^{-2x})$
(d) $\lim_{x\to-\infty} (5 + e^{3x})$
(e) $\lim_{x\to-\infty} (3 + e^{-0.1x})$
(f) $\lim_{x\to-\infty} \left(\frac{3}{e^x}\right)$
(g) $\lim_{x\to\infty} \left(\frac{1 - e^{-x}}{1 + e^{-x}}\right)$

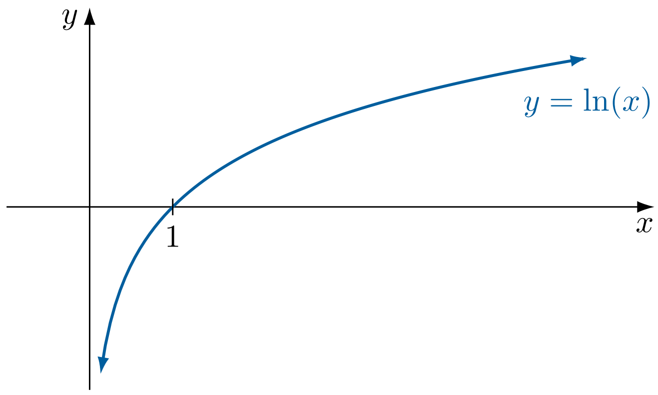
Question 14 Would the results from the previous question change if all the exponential terms were base 2 instead of e?

Question 15 The diagram below is a sketch of the graphs of $y = 2^x$ and $y = x + 2$ on the same set of axes.



(a) What is the relationship between the x coordinates of A and B, and the solutions of the equation $2^x = x + 2$.

(b) Hence, how many solutions does the equation $2^x = x + 2$ have?



Question 16 The diagram below is a sketch of the graph of $y = \ln(x)$. What function also needs to be drawn on the same set of axes above so that the intersection of the two curves is the solution to the following?

(a) $\ln(x) = x - 2$

(b) $\frac{1}{2}\ln(x) = 1 - x$

(c) $x\ln(x) = 1$

(d) $\frac{\ln(x)}{x} = x$

Question 17 Suppose a and b are numbers such that $a > b$.

(a) By drawing a graph of $y = e^x$, explain why $e^a > e^b$.

(b) Similarly, by drawing a graph of $y = \ln(x)$, explain why $\ln(a) > \ln(b)$.

Question 18

(a) Sketch the graph of $y = e^x - 3$.

(b) On the same set of axes, sketch $y = x$.

(c) Use your diagram to explain why $e^x - x - 3 = 0$ has exactly two real solutions.

Question 19

(a) Use graphing software to sketch the family of functions $y = \log_b(x)$ for $b = 2, 3, 4, 5$.

(b) Explain how changing the value of b affects the shape of the curve.

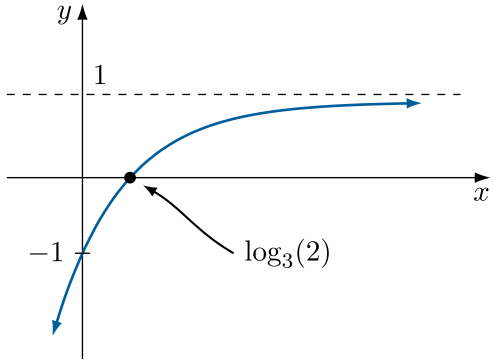
(c) How could this have been deduced algebraically *without* graphing various values of b?
Hint: Think about the change of base formula.

Challenge Problems

Problem 1 Solve $2^{2x} = 5^{x-1}$ and leave your answer in exact form.

Problem 2 Solve the equation $2^x + 4^x = 6$.

Problem 3 The diagram below shows the graph of a function $y = k \times a^{-x} + c$. Find the value of a, c and k.



(a) Find the value of c.

(b) By using the fact that the curve passes through $(0, -1)$, find the value of k.

(c) Suppose that the curve passes through $\left(1, \frac{1}{3}\right)$. Find the value of a and hence, find the equation of the curve.

(d) Verify that the x-intercept is $x = \log_3(2)$, as shown in the diagram above.

Problem 4 Without the use of a calculator, determine whether $\sqrt{2} \times \sqrt[3]{3}$ is larger or smaller than $\sqrt{3} \times \sqrt[3]{2}$.
Hint: Raise both sides by a common power.

Problem 5 Find $\lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$.

Problem 6 By first simplifying, sketch $y = \ln(x) + \ln\left(\frac{1}{x}\right)$.

Problem 7

(a) Use graphing software to sketch $y = xe^{-x}$.

(b) The 'peak' of the curve is called the *maximum stationary point*. State the value of y' at the maximum stationary point.

(c) Show that the maximum stationary point occurs at $(1, e^{-1})$.
Hint: What is the gradient of the tangent at the maximum stationary point?

(d) Hence, find the value(s) of k so that $xe^{-x} = k$ has exactly one real root.

Problem 8 Find the fallacy in the following 'proof' that $2 > 5$.

$$\frac{1}{2} > \frac{1}{5}$$

$$\log\left(\frac{1}{2}\right) > \log\left(\frac{1}{5}\right)$$

$$\frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{1}{5}\right)} > 1$$

$$\frac{\log(1) - \log(2)}{\log(1) - \log(5)} > 1$$

$$\frac{-\log(2)}{-\log(5)} > 1$$

$$\frac{\log(2)}{\log(5)} > 1$$

$$\log(2) > \log(5)$$

$$2 > 5$$

quod erat demonstrandum

Exercise 8F

Applications of exponential and logarithmic functions

Fundamentals

Fundamentals 1

Let A and B be any constants, and $k > 0$.

(a) $\lim_{t \to \infty} e^{-t} = $ ___

(b) $\lim_{t \to \infty} e^{-kt} = $ ___

(c) $\lim_{t \to \infty} A\left(1 - e^{-kt}\right) = $ ___

(d) $\lim_{t \to \infty} \left(A - Be^{-kt}\right)$ ___

Fundamentals 2

Let A, B and k be any real constants.

(a) When attempting questions involving time t, remember the restriction that t _____.

(b) The word "initially" or "beginning" refers to $t = $ ___.

(c) Suppose $P = Ae^{kt}$. When $t = 0$, $P = $ ___.

(d) Suppose $P = A + Be^{kt}$. When $t = 0$, $P = $ _____.

(e) Suppose $P = A - Be^{kt}$. When $t = 0$, $P = $ _____.

Fundamentals 3

(a) If __________, then $Q(t)$ is increasing for all $t \geq 0$.

(b) If __________, then $Q(t)$ is decreasing for all $t \geq 0$.

Fundamentals 4

Consider an object valued at $\$P$.

(a) If it appreciates at 10% per year, then after n years it is now valued at $P_n = $ __________.

(b) If it depreciates at 10% per year, then after n years it is now valued at $P_n = $ __________.

(c) If it appreciates at $r\%$ per year, then after n years it is now valued at $P_n = $ __________.

(d) If it depreciates at $r\%$ per year, then after n years it is now valued at $P_n = $ __________.

Question 1 The value of a brand new car is \$25,000. It depreciates at a rate of 10% per year.

(a) Find the value of the car after 1 year.

(b) Find the value of the car after 2 years.

(c) Make a formula that models the value of the car after n years.

(d) What percentage of the original price is the value of the car after 5 years?

(e) During which year will the value of the car drop below \$10,000?

Question 2 At the beginning of the year 2000, a learning centre had 180 students enrolled. The enrolment numbers increased by 20% each year.

(a) How many students were enrolled at the beginning of 2010?

(b) At the beginning of which year was the enrolment exceeding 1000?

Question 3 The population of an ant colony can be modelled using the equation

$$P = Ae^{kt},$$

where A and k are positive constants and t is time in weeks. Initially, the population is 1000. Two weeks later, the population has increased to become 1500.

(a) Find the value of A.

(b) Show that $k = \frac{1}{2}\ln\left(\frac{3}{2}\right)$.

(c) What is the population after four weeks?

(d) After how many weeks will the population exceed 1 million?

Question 4 The population of a diminishing bird species can be modelled using the equation

$$A = A_0e^{-kt},$$

where A_0 and k are positive constants and t is time in years. Initially, the population is 500. A year later, the population decreased down to 300.

(a) Find the value of A_0.

(b) Show that $k = \ln\left(\frac{5}{3}\right)$.

(c) What is the population after ten years?

(d) The bird species is considered extinct when the population is below 1. After how many years will the bird species become extinct?

Question 5 The population of an endangered species of bear can be modelled by

$$P = 200 - 10e^{0.05t},$$

where t is time in years from the 1^{st} January 2020.

(a) Show that the population is declining.

(b) Find how long it takes for the population to be half of what it was originally.

(c) Will the species of bear ever become extinct according to this model? If so, then when will it become extinct?

(d) Draw a graph of the model, labelling any important features.

Question 6 A bacterial population starts with 1000 bacteria and doubles every half hour.

(a) How many bacteria are there after 2 hours?

(b) Write down a general equation for the population of the bacteria after t hours.

(c) How long will it be until the population exceeds 1 million? Round to the nearest minute.

Question 7 A sheet of paper is roughly 0.05 mm thick. The sheet of paper is folded n times, which doubles the thickness with every fold.

(a) What is the thickness of the sheet of paper after two folds?

(b) If the thickness is 1.6 mm, how many folds have been made?

Question 8

(a) Show that $f(x) = \dfrac{1}{1+e^{-x}}$ satisfies $f(x) + f(-x) = 1$.

(b) Use graphing software to sketch $y = f(x)$ and hence determine the vertical translation of $f(x)$ that would result in an odd function.

Question 9

(a) Find the equation of the tangent to $y = e^{-x}$ when $x = 0$.

(b) Draw the graph of $y = e^{-x}$ and the tangent in (a).

(c) Find the area of the triangle enclosed by the tangent and the coordinate axes.

(d) Find the area of the triangle enclosed by the tangent, the normal and the x axis.

Question 10 The population of a rabbit species can be modelled by the equation $P = Ae^{kt}$, where A and k are positive constants and t is time in years. After two years, the population is 100. After five years, the population is 200. Find the value of A and k.

Question 11

(a) Express $3 \times 2^{3x+2}$ in the form $k \times 8^x$.

(b) Explain why the equation $y = k \times a^{bx+c}$ can be simplified down to $y = h \times A^x$ without any loss of information.

Question 12 Find the equation of the exponential function $y = k \times a^x$ where $a > 0$, that passes through the following points.

(a) $(1, 4)$ and $(4, 32)$

(b) $(1, 2)$ and $(4, 16)$

(c) $(0, 2)$ and $(2, 18)$

(d) $(1, 20)$ and $(-1, 0.8)$

Question 13 Find the domain of the following functions.

(a) $f(x) = \dfrac{1}{e^x - 1}$

(b) $f(x) = \dfrac{1}{e^x + 1}$

(c) $f(x) = \ln(x - 1)$

(d) $f(x) = \ln(2 - x)$

(e) $f(x) = \ln(3x - 6)$

(f) $f(x) = \ln(8 - 2x)$

Question 14

(a) Find the domain of $f(x) = \ln(x^2)$.

(b) Use graphing software to verify your answer.

Question 15 An unstable element undergoes radioactive decay, and the mass in kilograms can be modelled using the equation

$$M = M_0 e^{-kt},$$

where M_0 is the original mass and k is a positive constant.
Recall that the half-life of a radioactive substance is the amount of time that must elapse for the any mass to decay down to exactly half of what it was originally. Show that the half-life of the radioactive substance is independent of the original mass.

Question 16 The half-life of a radioactive substance is the amount of time that must elapse for the mass to decay down to exactly half of what it was originally. For example, the half-life of uranium-232 is approximately 68.9 years. This means that a 100 kg mass of uranium-232 will decay down to 50 kg after 68.9 years, and again down to 25 kg after another 68.9 years, and so on.
The equation for half-life is given by

$$M = M_0 \left(\frac{1}{2}\right)^{\frac{t}{H}},$$

where M_0 is the original mass, H is the half-life in years and t is time in years.

(a) What is the mass of a 100 kg sample of uranium-232 after 20 years?

(b) How long will it take for a 100 kg sample of uranium-232 to decay to less than 10 kg? Round to the nearest year.

Challenge Problems

Problem 1 Let $P(x_0, y_0)$ be the point on $y = e^x$ such that the tangent passes through the origin.

(a) Find the gradient of the tangent at P in terms of x_0.

(b) Find the equation of the tangent in terms of only x_0.

(c) Hence, find the exact coordinates of P.

Problem 2 Show that $f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$ is an odd function.

Problem 3 The population of many animal species often satisfies the *logistic equation*, which is given by

$$P(t) = \frac{a}{b + e^{-kt}},$$

where $P(t)$ represents the population, a, b and k are positive constants and t is time.

(a) Show that the population eventually approaches a fixed value, called the *carrying capacity* of the population, and find it.

(b) Let the carrying capacity be C, and let the initial population be P_0. Show that

$$\frac{1}{P_0} = \frac{1}{a} + \frac{1}{C}.$$

(c) By first finding $P'(t)$, show that populations satisfying the logistic equation are always increasing.

(d) The population of a bird species on an island satisfies the logistic equation, where $P(t)$ is measured in thousands and t is measured in years.

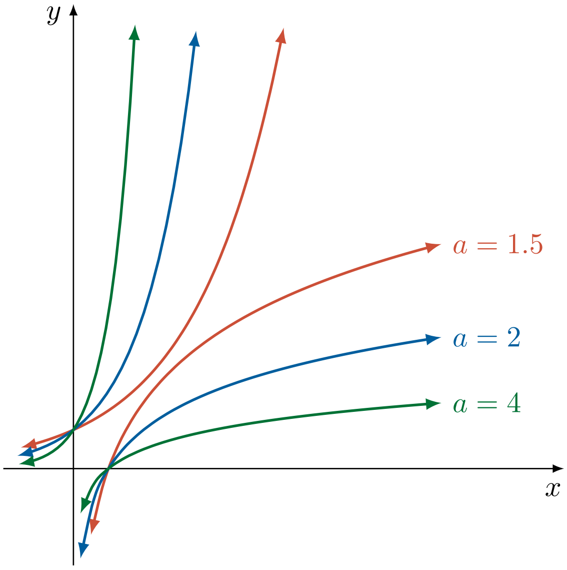
It is observed that initially, there were 800 birds on the island and that after many years the population plateaus at around 4000 without much change.

Find the value of a and b.

(e) Use graphing software to sketch the graph of $y = \dfrac{a}{b + e^{-kx}}$ for your values of a and b, and allow k to vary (remember that $k > 0$).

Does the value of k affect the either the limiting population or the carrying capacity of the bird species?

Problem 4 The diagram below shows the graph of $y = a^x$ and $y = \log_a(x)$ for varying values of a.



From the diagram, it can be observed $y = a^x$ and $y = \log_a(x)$ approach each other as a gets smaller. Eventually, if a is small enough, the two curves will be tangential. Let the point of contact P be at $x = \beta$.

(a) Explain briefly why P must lie on the line $y = x$ and hence explain why $a^\beta = \beta$.

(b) Show that $\dfrac{d}{dx}(a^x) = a^x \times \ln(a)$.

(c) Show that

$$a^\beta \ln(a) = 1.$$

(d) Hence, show that

$$\beta = \frac{1}{\ln(a)}.$$

(e) Explain why $\log_a(\beta) = \beta$.

(f) Deduce that $\beta = e$.

(g) Hence, show that $a = \sqrt[e]{e}$.

(h) Verify your result using graphing software.

Chapter 8 Review

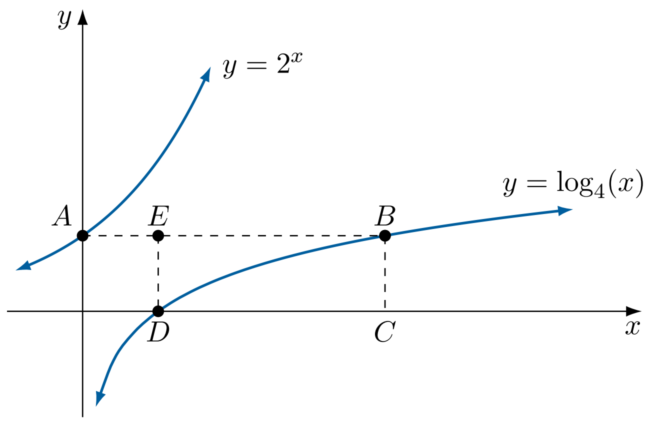
Logarithmic and Exponential Functions

Review

Question 1 Simplify the following.

(a) $\log_3(32) \times \log_2(9)$ (b) $\log_9(\log_2 8)$ (c) $\log_4(2)$

Question 2 Let A, B, C, D and E be defined as such in the diagram below.



(a) Find the co-ordinates of A, B, C, D and E.

(b) Hence, find the area of the rectangle $BCDE$.

Question 3 If $\log_x(p) = 1.6$, find the following.

(a) $\log_x(p^2)$ (b) $\log_x(\sqrt{p})$

(c) $\log_x\left(\frac{1}{p}\right)$ (d) $\log_x(p^x)$

Question 4 Solve the following equation for x.

$$\log_2\left(\frac{1}{x}\right) + \log_2\left(\frac{1}{x^2}\right) + \log_2\left(\frac{1}{x^3}\right) + \log_2\left(\frac{1}{x^4}\right) = -40$$

Question 5 Let $x = \log(a)$, $y = \log(b)$ and $z = \log(c)$. Express the following in terms of x, y and z.

(a) $\log\left(\frac{a}{b^2}\right)$ (b) $\log\left(\sqrt{b^2c}\right)$

Question 6 Consider the following equation.

$$\ln(y) = 2 - \ln(x^2) + \ln(x)$$

Express y as the simplest possible expression in terms of x.
Hint: $2 = \ln(\_\_\_\_\_)$

Question 7 Solve the equation $\ln(x) + \ln(x+2) = 1$, and write your answer(s) correct to two decimal places.

Question 8 Find the domain of the following.

(a) $y = \log_2(4 - 3x)$

(b) $y = \log_{10}(x^2 + 1)$

Question 9 Let $a = \log_6(3)$ and $b = \log_6(4)$. Find the following in terms of a and b.

(a) $\log_6(48)$

(b) $\log_6(8)$

(c) $\log_6(24)$

(d) $\log_6\left(\frac{2}{\sqrt{3}}\right)$

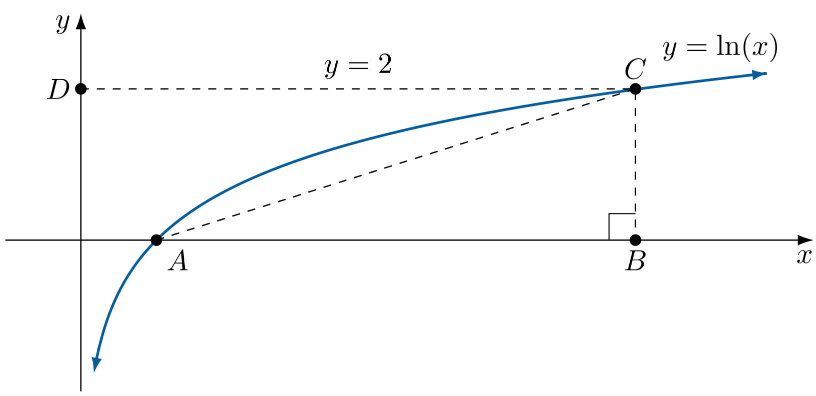
Question 10 Sketch the following graphs.

(a) $y = \ln(2x)$

(b) $y = \ln\left(\frac{1}{x}\right)$

(c) $y = 2\ln(x) - 1$

Question 11 Let A, B, C and D be defined as in the diagram below. Find the area of $\triangle ABC$.



Question 12

(a) Show that $3^{2x+2} - 3^{x+3} - 3^x + 3 = (9 \times 3^x - 1)(3^x - 3)$

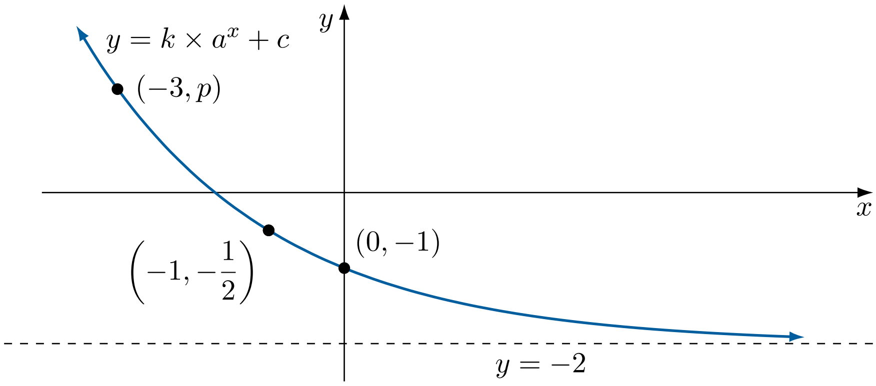
(b) Hence, solve the equation $3^{2x+2} - 3^{x+3} - 3^x + 3 = 0$.

Question 13 Solve the equation $(\ln x)^2 - 3\ln(x) - 4 = 0$.

Question 14 Solve the equation $e^x + e^{-x} = 2$.

Question 15 Show that $\log_{\frac{1}{b}}(a) = -\log_b(a)$.

Question 16 The point $(-3, p)$ lies on the graph of the exponential function $y = k \times a^x + c$, shown in the diagram below. The points $\left(-1, -\frac{1}{2}\right)$ and $(0, -1)$ also lie on the same graph.



(a) Find the values of a, c and k.

(b) Hence, find the value of p.

Question 17 Solve the following logarithmic equations.

(a) $2\log_3(x-3) = \log_3(49)$

(b) $\log_5(x+3) = 2\log_5(x-3)$

(c) $\log_3(2x-1) + \log_3(2x+1) = 1$

(d) $\ln(x) + \dfrac{12}{\ln(x)} = 7$

Question 18 Show that

$$\frac{1}{\dfrac{1}{\log_a(x)} + \dfrac{1}{\log_b(x)}} = \log_{ab}(x).$$

Investigation Task

Continuous Compounding

The compound interest formula states that

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where after n units of time,

A	is the amount that P grows to.
P	is the initial amount, called the principle.
r	is the annual interest rate
t	is the number of years

Question 1 Suppose a principle of $\$100,000$ is deposited into an account with an annual interest rate of 3% for five years. Find how much the principle grows if interest is compounded annually, quarterly, monthly, weekly, and per-second. Describe the general trend you notice as the compounding periods shorten. Does the principle appear have a limit if we continue to decrease the compounding periods?

Question 2

(a) Fill in the following table.

$\boldsymbol{x}$	10	100	1000	10000
$\left(\mathbf{1} + \frac{\mathbf{1}}{\boldsymbol{x}}\right)^{\boldsymbol{x}}$				

(b) What special number does this seem to approach?

Question 3

(a) Let $x = \dfrac{n}{r}$, where r is a fixed number. What happens to n as x gets large?

(b) Show that

$$A = \left[\left(1 + \tfrac{1}{x}\right)^x\right]^{rt},$$

where A is defined from the beginning of this investigation task.

(c) Explain how letting $x \to \infty$ is related to Question 1.

(d) Use Question 2 to deduce the formula for *continuous compounding* $A = Pe^{rt}$ and hence find an easier way of calculating the principle for the 'per-second' part of Question 1.

Question 4 Use a savings calculator online to find how much a deposit of $\$50,000$ will grow after varying amounts of time, assuming no further withdrawals or deposits from this account. Compare your findings to the theoretical values obtained using the formula from Question 1.

Investigation Task

Decibels and Intensity

The 'loudness' of a sound is called the *intensity* (I), and it is measured in watts per square metre (W/m^2). The smallest amount of sound that the human ear can detect is denoted by I_0, and it is approximately 10^{-12} W/m^2. For example, a close whisper is approximately $100I_0$.

However, physics and engineers often measure 'loudness' using a unit of measurement called the *decibel* (dB), which was named after the inventor of the telephone Alexander Graham Bell (1847-1922). The formula for decibels is given by

$$L = 10\log_{10}\left(\frac{I}{I_0}\right).$$

Question 1

(a) Draw a graph of decibels L against intensity I.

(b) Normal conversation has an intensity of around 10^{-6} W/m^2. Calculate the corresponding decibel value.

(c) A busy street is around 70 dB. Calculate the corresponding intensity.

(d) Plot the above on your graph.

(e) Explain why using only intensity to measure sound may be impractical.

Question 2

(a) Consider two sound sources, where one is 100 times more intense than the other. What is the difference in their decibel value?

(b) A car horn has a sound level of approximately 110 dB, whereas a space shuttle taking off nearby generates a sound level of approximately 150 dB. How many times louder is a space shuttle taking off compared to a car horn?

(c) A person will begin to suffer from ear damage when the decibel level exceeds 90 dB. The following are known decibel levels for certain events. Determine which events will cause ear damage.

Sound source	Intensity(W m^2)
Jet aircraft (15 metres away)	100
Rock concert	1
Ambulance siren 30 m away	0.01

Investigation Task

pH Scale

The pH of a solution is a measure of the acidity of the solution. The acidity of a solution is determined by the concentration of hydronium ions $[H_3O^+]$ in the solution.
The pH of a solution is given by the logarithmic formula

$$\text{pH} = \log_{10}\left(\frac{1}{C}\right),$$

where C be the concentration of $[H_3O^+]$ in a solution, measured in moles per litre (mol L^{-1}). If the pH is above 7, the solution is *basic*. If the pH is less than 7, the solution is *acidic*. If the pH is 7, the solution is *neutral*.

Question 1

(a) The concentration of in a solution is measured to be 2.85×10^{-6} mol L^{-1}. Is this solution acidic, basic or neutral?

(b) Find the concentration of $[H_3O^+]$ in a neutral solution.

(c) If a solution has a higher concentration of $[H_3O^+]$, does it increase or decrease the pH?

(d) Describe the behaviour of the pH as C gets large.

(e) Describe the behaviour of the pH as C approaches zero.

(f) Orange juice has a pH of around 3. On the other hand, milk has a pH of around 6. Bob claims that since the pH of milk is double the pH of orange juice, the $[H_3O^+]$ concentration is twice that of orange juice.

Is this true? If not, then correct his statement.

(g) Can the pH of a solution be negative?

Question 2 Draw a graph of pH against concentration C.

Question 3 Research some acids that are used in general everyday-use (whether we know it or not), and find what their pH is. Identify the *least* acidic from your list and use it as a 'base unit'. Then, rank your list of acids and write down next to each one how many times more the concentration of $[H_3O^+]$ is with respect to your weakest acid.

9

PROBABILITY

- Sample space and probability
- Set notation and Venn diagrams
- Mutually exclusive events
- Multi-stage experiments and independent events
- Conditional probability

Exercise 9A

Sample space and probability

Fundamentals

Fundamentals 1

(a) The process of performing an action like flipping a coin or rolling a die produces something called an o________.

(b) The collection of o________ satisfying a desired condition is called an e______.

(c) If a particular o________ satisfies the condition for an e______, then we say that it is a f________ o______.

(d) The set of all possible outcomes of an experiment is the s______ s______.

(e) An e________ is a subset of the s________ s________.

(f) An event can consist of several o________.

Fundamentals 2

Let A be some event in a sample space with probability $P(A)$.

(a) The probability $P(A)$, must satisfy ______ $\leq$ ________ $\leq$ ______.

(b) The sum of the probabilities of all outcomes in a sample space must be equal to ___.

(c) $P(A)$ cannot greater than ___ and it cannot be less than ___.

(d) The formula for $P(A)$ is $P(A) = \dfrac{\text{number of}\ \_\_\_\_\_\_\_\_}{\text{number of}\ \_\_\_\_\_\_\_\_}$.

(e) If $P(A) = 0$, then A c________ occur.

(f) If $P(A) = 1$, then A is c________ to occur.

(g) If two outcomes have the same probability, then we say that they are e_______ l_______.

(h) If A is an event, then the 'opposite' event i.e. A does *not* occur, is called the c______ event and it can be denoted using $\overline{A}$, A^c or A'.

(i) If $P(A) = p$, then $P(\overline{A}) =$ ______, which means that $P(A) + P(\bar{A}) =$ ___.

Fundamentals 3

When rolling a die three times, the probability of

(a) at *least* one '5' means P(one 5) or P(___ 5's) or P(___ 5's). Or, more easily done, it is the same as $1 - P$(___ 5's).

(b) at *most* one '5' means P(___ 5's) or P(___ 5).

Question 1 List the sample space for each of the following scenarios. Useful techniques may include listing the possibilities systematically, drawing a table or drawing a tree diagram.

(a) Two coins are tossed and you are counting how many heads.

(b) Three coins are tossed and you are counting how many heads.

(c) A coin is flipped and a dice is rolled.

(d) Two dice are rolled and their sum is recorded.

Question 2 A fair dice is rolled. What is the probability that the number facing upwards is

(a) greater than 2.

(b) divisible by 3.

Question 3 Two dice are tossed and the numbers facing upwards are recorded from each die.

	1	**2**	**3**	**4**	**5**	**6**
1						
2						
3						
4						
5						
6						

(a) Complete the table above. List all possible outcomes in the form $\{x, y\}$.

(b) Hence, find the probability of the following events.

(i) Only one of the dice shows a 2.

(ii) At least one of the dice shows a 2.

(iii) Both dice show at 2.

(iv) The sum is equal to 7.

(v) The sum exceeds 8.

(vi) Not rolling double 6.

(vii) Not rolling a double.

(viii) The product of the numbers is even.

Question 4 A card is drawn randomly from a standard deck of cards. Find the probability that the card drawn is

(a) a heart.

(b) a heart or a king.

(c) the queen of diamonds.

(d) not a diamond.

(e) a red jack.

(f) a queen or a red card.

Question 5 Consider the following statements. Determine whether the logic is valid or invalid, and give reasons for your answer.

(a) There are 26 letters in the alphabet. A random letter from a paragraph is chosen. Therefore, the probability of it being "w" is $\frac{1}{26}$.

(b) There are three teams A, B and C playing cricket. Therefore, the probability of team A winning is $\frac{1}{3}$.

(c) A bag contains red, blue and green marbles. A marble is drawn at random. The probability of drawing a red marble is $\frac{1}{3}$.

(d) The probability of getting a rainy day during a specific week is $\frac{1}{7}$. From Sunday to Friday inclusive, there was no rain. Therefore, it must rain on Saturday.

(e) When two coins are tossed, the possibilities of coins landing are either two heads, a head and a tail, or two tails. Therefore, the probability of getting a head and tail is $\frac{1}{3}$.

(f) A multiple choice test has 4 options. Bob guesses his answer. The probability that Bob answers a question correctly is $\frac{1}{4}$.

(g) There are more cat-owners in Town A than cat-owners in Town B. Therefore, you are more likely to meet a cat-owner in Town A than in Town B.

Question 6 At a fundraiser, there are three prizes on offer. The prizes are chocolates, movie tickets and a gift voucher. The number of tickets for the prizes, respectively, are in the ratio 21 : 12 : 7.

(a) What is the probability that the winner receives a movie ticket?

(b) What is the probability that Bob, who does not want the chocolates, recieves a gift that he wants?

Question 7 A bag of marbles contains five times more red marbles than blue marbles. A marble is drawn at random. What is the probability of it being blue?

Question 8 Suppose the probability of giving birth to a boy or girl is exactly 50%. Bob and Mary have three children. What is the probability that their children consist of:

(a) all boys? (b) one boy and two girls? (c) at least one girl?

Question 9 [Mark and recapture in biology]

Twenty tagged fish are released into a pond and allowed to mix with the rest of the fish population in the pond. A week later, fifty fish are captured, five of which were tagged. All fifty fish are then released back to the pond. Assuming that the fifty captured fish are a general representation of the entire population, estimate the total fish population.

Challenge Problems

Problem 1 A bag of marbles contains five more red marbles than blue marbles. The probability of drawing a blue marble is $\frac{1}{3}$. How many blue marbles are there in the bag?

Problem 2 [Symmetry in probability]

A bag contains an equal number of black and white marbles. Three marbles are drawn, with replacement. What is the probability of getting at most one black marble?
Hint: This question can be done without any actual calculations.

Exercise 9B

Set notation and Venn diagrams

Fundamentals

Fundamentals 1

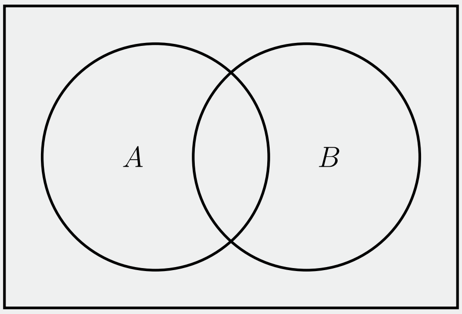
Let A and B be events in a sample space S.

(a) The set of combined outcomes from events A and B is called the u________ of A and B, and is denoted by A ___ B.

(b) The set of common outcomes that occur in both A and B is called the i__________ of A and B, and is denoted by A ___ B.

Fundamentals 2

The diagram below shows two intersecting sets A and B.



Shade the region corresponding to

(a) $A \cap B$

(b) $A \cup B$

Question 1 Define the following sets.

$$A = \{1, 2, 3, 4, 5\}$$
$$B = \{3, 4, 5, 6, 7\}$$

List the elements of

(a) $A \cap B$

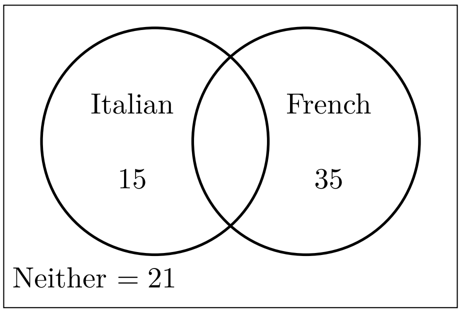
(b) $A \cup B$

Question 2 Define the following sets.

$$A = \{\text{even numbers less than } 72\}$$
$$B = \{\text{factors of } 60\}$$

List the elements of $A \cap B$.

Question 3 The Venn diagram below shows the number of students speaking French, Italian, both, or neither in a classroom of 80 students.



By first finding the missing information in the Venn diagram above, write down the probability that a randomly chosen student

(a) speaks neither Italian nor French.

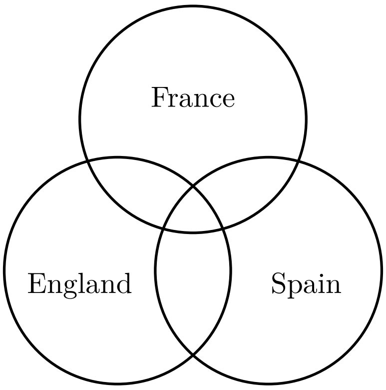
(b) speaks Italian or French only, but not both.

(c) speaks Italian and French.

(d) speaks French or Italian.

Question 4 Let I represent the event that somebody speaks Italian and let F represent the event that somebody speaks French. Note that both I and F contain the people who speak both. For the events in Question 3 (a) and (c), write down the equivalent statement using set notation. For example, the event of somebody speaking Italian or French is the union $I \cup F$.

Question 5 58 people took a trip to Europe to visit either France, England or Spain. Of this group, 26 visited France, 38 visited England, 29 visited Spain, 10 visited France and Spain, 12 visited only Spain, 13 visited only England and 8 visited all three countries.



A person from the group of 58 was chosen at random. What is the probability that the person visited

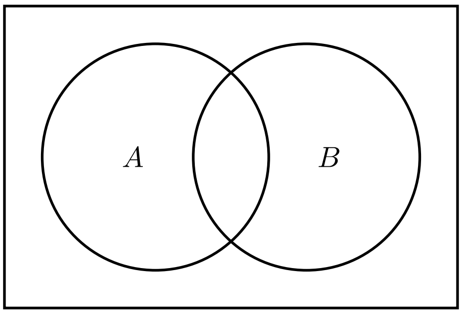
(a) France only?

(b) England and France?

(c) only one country?

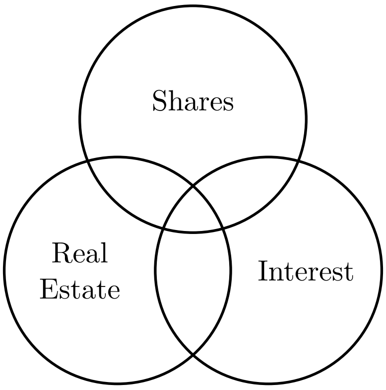
Challenge Problems

Problem 1 Consider the following statements. Determine whether they are true or false, by shading relevant areas on a two-way Venn diagram.



(a) $(A \cap B) = \bar{A} \cup \bar{B}$ (b) $(A \cup B) = \bar{A} \cup \bar{B}$ (c) $\bar{A} \cup \bar{B} = \overline{(A \cup B)}$

Problem 2 The Venn diagram below depicts the investment choices from a random sample of people. Every person invests either through shares, real estate or the interest from a bank account.



The following information was gathered. 320 did not want to buy real estate, 500 would not invest through bank account interest, 385 would not invest in shares, 240 invest in shares and real estate, 205 invest through shares and bank interest, 45 invest only through bank interest, and 110 invest using all three methods.

Find the probability that a randomly chosen person invests in shares.

Exercise 9C

Mutually exclusive events

Fundamentals

Fundamentals 1

(a) The notation for "A or B" is A ___ B.

(b) The notation for "A and B" is A ___ B.

Fundamentals 2

(a) Events A and B are said to be m________ e________ if they have no o________ in common.

(b) This means that A and B can/cannot (circle one) occur at the same time.

(c) For example, on a single coin toss, the outcome can be heads or tails, but not b______.

(d) On a Venn diagram, A and B will not i________.

Fundamentals 3
Let A and B be mutually exclusive events.

(a) Draw a Venn diagram demonstrating how A and B are mutually exclusive.

(b) $P(A \cup B) =$ ________ $+$ ________.

(c) $P(A \cap B) =$ ___.

Fundamentals 4
Let A and B be non-mutually exclusive events.

(a) Draw a Venn diagram demonstrating dependent events A and B.

(b) If A and B are not mutually exclusive, then we use the A________ Rule, which states that

$$P(A \cup B) = P(A) + P(B) - \text{\_\_\_\_\_\_\_\_}$$

Question 1

(a) Give an example of two events that are mutually exclusive.

(b) Give an example of two events that are independent.

Question 2 Draw a Venn diagram of two mutually exclusive events A and B.

Question 3 Are the following events mutually exclusive?

(a) Heads and tails.

(b) Kings and hearts.

(c) Spades and red card.

(d) Odd and even number.

(e) Multiple of 3 and multiple of 5.

(f) Prime number and even.

Question 4 Below are events that can occur when drawing from a standard deck of cards. Determine which combinations of outcomes, from the list below, are mutually exclusive.

(a) Drawing a club.

(b) Drawing a picture card.

(c) Drawing a king.

(d) Drawing a heart.

(e) Drawing a 5.

(f) Drawing a red card.

Question 5 State which of the following events are independent.

(a) Rolling a die twice and getting a six both times.

(b) Drawing two cards from a standard deck, with replacement.

(c) Drawing two cards from a standard deck, without replacement.

(d) Tossing a coin twice.

(e) Tossing a coin and then rolling a dice.

(f) Choosing two people in Australia over the age of 30.

Question 6 Let $P(A) = 0.4$ and $P(B) = 0.7$ be the probabilities of events A and B respectively.

(a) Explain why A and B cannot be mutually exclusive events.

(b) Find $P(A \cap B)$

(c) Find $P(A \cup B)$

Question 7 A card is drawn from a standard deck of 52 cards. What is the probability of drawing

(a) a diamond or the ace of spades?

(b) a diamond or an ace?

(c) a diamond or a red card?

(d) a diamond or a picture card?

Question 8 From a set of 12 cards numbered 1 to 12, a card is drawn at random. What is the probability that it is either a multiple of 3 or even?

Question 9 From a set of 20 cards numbered 1 to 20, a card is drawn at random. What is the probability that it is either a multiple of 5 or 4?

Challenge Problems

Problem 1 [Three-way Venn diagrams]

A class of 28 students is surveyed about what language they are able to speak, and the following results are found.

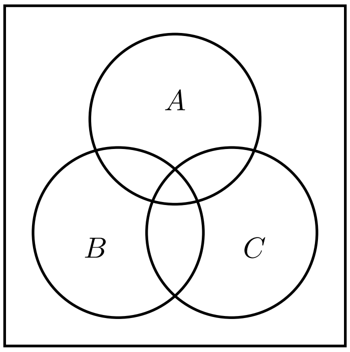
Five people could speak French, Italian and German. Two people speak none of those languages. 18 people speak French, 16 people speak Italian, 9 people speak French and German, and 14 people speak German. Two people speak Italian and German, but not French. Find the probability that a randomly selected person speaks French and Italian, but not German.

Problem 2 [Inclusion-Exclusion Principle]

Earlier, you used the formula for two-way Venn diagrams, also known as the *Addition Theorem*.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

This question will establish the equivalent formula but for three-way Venn diagrams.
Let A, B and C be finite sets.



Fill in the diagram above and use it to explain how the formula for three-way Venn diagrams is derived.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise 9D

Multi-stage experiments and independent events

Fundamentals

Fundamentals 1

(a) Events A and B are said to be i_______ if $P(A)$ will always be the same regardless of whether or not B occurs.

(b) For example, suppose we flip a coin and roll a die. $P(\text{heads})$ will always be ______ regardless of whether or not a six was rolled.

(c) On the other hand, picking two people randomly for a soccer team, one after the other, is an example of independent/dependent (circle one) events, because the second choice is/is not (circle one) directly affected by who was chosen first.

(d) If A and B are independent events, then $P(A \text{ and } B) = \_\_\_\_ \times \_\_\_\_$.

(e) This can also be written as $P(A \_\_ B) = \_\_\_\_ \times \_\_\_\_$.

(f) The converse is true/false (circle one). That is, if we verify numerically that

$$P(A \cap B) = \_\_\_\_ \times \_\_\_\_,$$

then we can conclude that A and B are independent events.

Fundamentals 2

(a) Experiments can have a number of stages. For example, flipping a coin twice is a two-stage experiment and rolling a die three times is a t______-stage experiment.

(b) In general, if $A_1, A_2, \ldots, A_n$ are events in a multi-stage experiment, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

(c) Another common type of multi-stage experiment is s_______, like drawing a marble from a bag.

(d) If s______ is done with replacement, then the second selection is i_______ of whatever was drawn from the first sampling.

(e) Most of the time, it is obvious if events are i_______. For example, flipping heads on a coin toss will not affect the o_______ of the next coin flip in any way.

Question 1 A bag contains two red jellybeans and two blue jellybeans.

(a) A jellybean is drawn randomly, the colour recorded, and then returned to the bag. Afterwards, a second jellybean is drawn. What is the probability that the second jellybean is red?

(b) A jellybean is drawn randomly, the colour recorded, and but it is then eaten. Afterwards, a second jellybean is drawn. What is the probability that the second jellybean is red?

(c) Out of parts (a) and (b), which one represented a series of *independent* events, and which one represented a series of *dependent* events?

Question 2 According to the Australian Red Cross Blood Service, the proportions of people with different blood groups is given as follows.

Blood Type	**O**	**A**	**B**	**AB**
Probability	0.49	0.38	0.1	0.03

Two people are selected at random from the population of Australia.

(a) What is the probability that both people have blood type B?

(b) Why are we allowed to multiply the probabilities?

Question 3 A class has 6 girls and 4 boys. Two students are selected at random. Find the probability that

(a) both students are boys?

(b) a boy and girl are selected?

Question 4 A coin is flipped twice. What is the probability of landing

(a) two heads?

(b) two tails?

(c) a head and tail, in that order?

(d) a head and tail?

Question 5 A coin is flipped and then a die is rolled. What is the probability that

(a) a tail is tossed and the number 5 is rolled?

(b) a head is tossed and an even number is rolled?

Question 6 A bag contains 5 red balls, 4 green balls and 2 blue balls. Kevin pulls out a red ball and keeps it. What is the probability that the next ball will also be red?

Question 7 20 blue tickets and 20 red tickets are sold in a raffle, in which there are three prizes. Three tickets are drawn one after the other, without replacement, for the first, second and third prizes respectively.

(a) What is the probability that all three prizes are won by blue tickets?

(b) What is the probability that at least one blue ticket wins a prize?

Question 8 A biased coin with outcomes either H or T has $P(H) = 0.55$. Find the probability of obtaining

(a) the sequence H, H, T in that order.

(b) exactly two heads.

(c) at least two heads.

(d) at least one head.

Question 9 A box contains twelve identical looking chocolates. However, four are milk chocolate and eight are dark chocolate. Nathan eats three chocolates chosen randomly from the box. Using a tree diagram, or otherwise, find the probability that

(a) the first chocolate eaten is a dark chocolate.

(b) Nathan eats three dark chocolates.

(c) Nathan eats exactly one dark chocolate.

Question 10 A box contains 7 red and 3 blue balls. Two balls are chosen randomly. What is the probability of both balls being the same colour if

(a) the balls were chosen with replacement.

(b) the balls were chosen without replacement.

Question 11 Daniel and Adam are playing a tennis tournament. They will play each other exactly twice and for their first game, each player has an equal chance of winning. However, if Adam wins the first game, the probability of him winning the second game is increased to 0.55. If he loses the first game, the probability of him winning the second game is reduced to 0.35.

(a) Find the probability that Adam wins exactly one game.

(b) Find the probability that Daniel wins at least one game.

(c) Find the probability that the same person wins both games.

Question 12 In a large university, the student population is 45% male and 55% female. Two students are selected at random to take part in a survey. Find the probability that

(a) both are female.

(b) both are of the same sex.

(c) one is male and one is female.

Question 13 Bob and Mary go for their driving test. Bob has a 80% chance of passing the test and Mary has a 90% chance of passing. What is the probability that

(a) only Bob passes.

(b) either Bob or Mary pass, but not both.

(c) at least one person passes.

Question 14 A fair die is rolled repeatedly until the number '5' is rolled.

(a) Find the probability of rolling the number '5' at least once after three rolls of the die.
Hint: You *could* find the probability of rolling '5' once, twice, and thrice. Alternatively, you could consider the c________ e______.

(b) Write down a formula for the probability of rolling the number '5' at least once after n rolls of the die.

(c) Hence, find the number of times we must roll the die so that the probability of scoring '5' at least once exceeds 99%.
Hint: You may need logs for this question.

Challenge Problems

Problem 1 A survey of students chewing gum was done in three schools. The results are as follows.

In School A, 50% of students chew gum.
In School B, 40% of students chew gum.
In School C, 30% of students chew gum.

A school is chosen at random and then two students are sampled at random from that school. Find the probability that

(a) both students chew gum.

(b) only one of those students chew gum.

Problem 2 It is known that 5% of light bulbs produced from a factory are faulty. What is the probability that in a random sample of 3 light bulbs

(a) no light bulbs are defective.

(b) exactly one light bulb is defective.

(c) at most two light bulbs are defective.

Problem 3 A bag contains 25 balls labelled 1 to 25. Two balls are drawn in succession, with replacement. Find the probability that the second number drawn is

(a) equal to the first.

(b) greater than the first.

Problem 4 If 3 random people are asked what day of the week they were born, find the probability that

(a) two or three people have the same day.

(b) only two people are the same.

Problem 5 Bob and Mary each take turns rolling a pair of dice. The first person to throw a double wins the game. Bob rolls first. Find the probability that Mary wins the game on her second turn.

Problem 6 A coin is flipped continuously, and the results recorded until the same side occurs twice in a row. Once this occurs, the sequence ends.

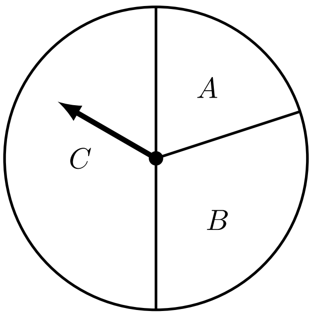
(a) Write down all possible sequence of H and T that finishes in exactly 5 coin tosses, and similarly exactly 6 coin tosses.

(b) What do you notice about the pattern in the sequence leading towards the end of the sequence?

(c) Calculate the probability that the sequence ends on the eighth toss of the coin.

Problem 7 Bob and Mary compete against each other in a game that involves taking turns spinning a counter that has three possible outcomes A, B and C.

$$P(A) = 0.2$$
$$P(B) = 0.3$$
$$P(C) = 0.5$$



If the counter lands on A, then Bob wins. If the counter lands on B, then Mary wins. If the counter lands on C, then nobody wins that round, and the other person spins the counter. This is done repeatedly until somebody wins. Bob spins the counter first.

(a) How many ways can Bob win within three turns of the game overall? Write down the sequences of A, B and C representing these scenarios.

(b) Hence, find the probability of Bob winning within three turns of the game.

(c) What is the probability that somebody has won within the first 5 turns of the game overall?

Exercise 9E

Conditional probability

Fundamentals

Fundamentals 1

(a) If we have two d_______ events A and B, and we know that B has already occurred, then the probability of event A will change.

(b) This is because the fact that B has already occurred r_______ the size of the sample space by eliminating all the outcomes that are not in ___.

(c) The new probability is called the c_______ p_______.

(d) The expression "Probability of A, given B has occurred" can be written as $P($_______$)$.

(e) The formula for conditional probability is

$$P(\underline{\qquad\qquad}) = \frac{\qquad}{\qquad}.$$

Fundamentals 2

Conditional probability questions can be done by reducing the s_______ space and e_____ space, or by using the formula.

Fundamentals 3

(a) If A and B are independent events, then $P(A \mid B) = P($___$)$.

(b) Conversely, if $P(B \mid A) = P($___$)$, then events A and B are independent.

(c) This means that if you wish to prove or disprove that A and B are i_______ events, then simply calculate $P($_______$)$ and $P($___$)$, and check if they are equal or not equal.

Question 1 In a particular class, 20 students study history, 15 study chemistry and 5 study both. A student is chosen randomly. What is the probability that they study chemistry, given that they study history?

Question 2 Amy buys 4 tickets in a raffle in which 50 tickets are sold. The tickets are drawn for first and second prizes. What is the probability that Amy winning

(a) first prize?

(b) second prize, given that she wins first prize?

(c) second prize, given that she does not win first prize?

Question 3 A school surveyed its students about wearing school uniform. The results are shown in the table below.

	Year 10	**Year 11**	**Year 12**	**Total**
For	50	55	25	130
Against	70	35	25	130
Uncertain	70	35	25	130
Total	190	125	75	390

What is the probability that a student chosen at random from this school

(a) is in favour of the uniform, given that they are in Year 11?

(b) is in Year 11, given that they are in favour of the uniform?

Question 4 Two dice are rolled simultaneously. Find the probability that

(a) both die show a 5, given that one of the die showed a 5.

(b) the sum is a 6, given that one of the die showed a 5.

(c) a die showed a 5, given that the sum is 6.

Question 5 A card is selected from a standard deck of 52 cards. Find the probability that the card is

(a) a queen given that the card is a picture card?

(b) a queen given that the card selected is red?

Question 6 A two-digit number is written down. Find the probability that

(a) it is palindromic, given that the unit digit is a 5.

(b) it is palindromic, given that the tens digit is a 5.

(c) the unit digit is a 5, given that the number is palindromic.

(d) it is divisible by 9, given that the tens digit is a 3.

(e) the tens digit is a 3, given that it is divisible by 9.

Question 7 All of Bob's friends either jog or go to the gym. He knows that 70% of his friends jog, and 35% jog and go to a gym. What percentage of Bob's friends go to the gym, but do not jog?

Question 8 Bob and Mary are discussing the days that they were born. They both know that at least one of them was born on a weekend. Find the probability that they were both born on the weekend.

Question 9 The manager of a team notices that the team has a probability of 0.6 of winning the game if it is raining, but 0.25 if it is dry. The probability that it will rain when they play is 0.3.

(a) Find the probability that they will not win.

(b) Given that the team has won a game, calculate the probability that it rained on the day of the match.

Question 10 The table below shows the results from a survey of 100 Year 12 students about their gender, and whether or not they intend on going overseas immediately after their HSC.

	Yes	**No**	**Total**
Female	30	30	60
Male	25	15	40
Total	55	45	100

(a) What is the probability that a randomly chosen student is female?

(b) What is the probability that a randomly chosen student will go overseas?

(c) Calculate the probability that a randomly chosen female student intends on going overseas by using the

(i) table.

(ii) conditional probability formula.

(d) What is the probability that a randomly chosen student is female, given that we know they intend on going overseas?

(e) Let Y be the event of choosing a student who says "Yes", and let F be the event of choosing a female student. Is $P(Y \mid F) = P(F \mid Y)$?

(f) What is the probability that a randomly chosen student is male, given that we know they intend on going overseas?

(g) What is the probability that a randomly chosen student intends on going overseas, given that they are male?

Question 11 Given that for two events A and B

(a) $P(A) = 0.6$, $P(B) = 0.35$, and $P(B \mid A) = 0.5$, find

(i) $P(A \cap B)$

(ii) $P(A \mid B)$

(b) $P(A) = 0.7$, $P(B) = 0.4$, and $P(A \cup B) = 0.8$, find

(i) $P(A \cap B)$

(ii) $P(A \mid B)$

(c) $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \mid B) = 0.8$, find

(i) $P(A \cap B)$

(ii) $P(B \mid A)$

Question 12 In a sports club A, 60% of the members are male and 40% are female. Of the male members, 20% prefer tennis, while 25% of the female members prefer tennis. Let M represent 'males', F represent 'females', T represent preferring tennis and $\bar{T}$ represent not preferring tennis.

(a) Fill in the following table.

	T	$\bar{T}$	**Total**
F			
M			
Total			

(b) What percentage of club members

(i) are female and prefer tennis?

(ii) are male and prefer tennis?

(iii) do not prefer tennis?

(iv) are male and do not prefer tennis?

(v) are female and do not prefer tennis?

(c) Find the probability that a person chosen at random from the club prefers tennis and is male.

(d) What is the probability that a randomly selected male from the club does not prefer tennis?

(e) In this club, does it appear that not preferring tennis and being a male are independent events?

Question 13 In a sports club B, the preferences for tennis with its members is given below.

	T	$\bar{T}$	**Total**
F	12%	8%	20%
M	48%	32%	80%
Total	60%	40%	100%

Show that in this club, not preferring tennis and being male are independent events.

Question 14 Michael found that in general, the probability that he watched TV on Sunday was 0.3. If he watched TV on Sunday, then the probability that he watched TV on Monday was 0.6. If he did not watch TV on Sunday, then the probability that he watched on Monday was 0.55. For a given week, find the probability that he watches TV

(a) on both Sunday and Monday.

(b) on Monday.

Question 15 Darius intends on opening a cafe, and the probability that he will make a profit in his first year is 0.8. Robin also has thoughts of opening a cafe but is cheeky and thinks of opening one next-door to Darius' cafe. The probability that Robin will go through with opening his own cafe is 0.3 and we know that if he does so, then Darius' probability of making profit in his first year reduces down to 0.6.

Let A denote the event that Darius makes a profit during his first year, and let B denote the event that Robin opens a cafe next-door. What is the probability that Darius will make a profit in his first year?

Question 16 [Proving the independence condition]

Suppose A and B are independent events. By considering the formula for $P(A \cap B)$, when A and B are independent events, prove that $P(A \mid B) = P(A)$.

Question 17 [Mutually exclusive events and conditional probability]

Suppose A and B are mutually exclusive events.

(a) Using your understanding of mutually exclusive events, what would you expect $P(A \mid B)$ to be equal to? Give a brief explanation of your answer.

(b) Prove your result from (a) using the formula for conditional probability.

Challenge Problems

Problem 1 In a certain population over a period of time, it was found that the probability $P(Y)$ of a new-born to reach the age of Y years is

$$\begin{aligned} P(50) &= 0.922 \\ P(60) &= 0.848 \\ P(70) &= 0.714 \\ P(80) &= 0.476 \end{aligned}$$

(a) What is the probability of a 50 year-old man reaching the age of 60?

(b) Joe the Farmer just turned 70 and has a 0.15 chance of dying within the next 10 years. What is the probability that Joe lives to see his 80^{th} birthday?

Problem 2 Bob and Mary have probabilities 0.3 and 0.2 respectively to score a goal in a given soccer match. Two goals are scored, one of which was from Mary. What is the probability that Bob scored the other one?

Problem 3 [Bayes' Theorem]

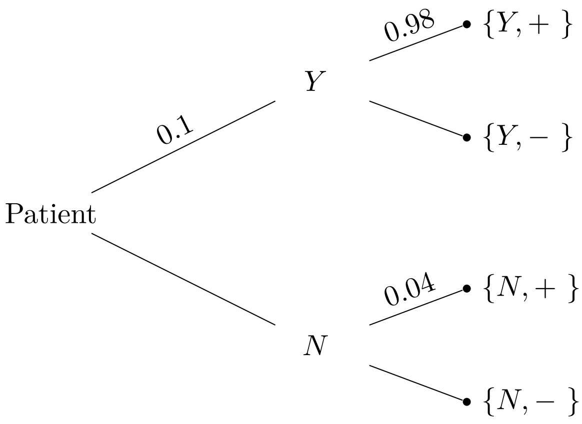
A hospital is trialling a new test that identifies if a patient has a certain disease. If the patient has that disease, the test is designed to return a 'positive' result, denoted by a "+" sign. If the patient does not have the disease, the test should return a 'negative' result. However, no test is perfect and false results may occasionally occur. The hospital wishes to see how accurate the test is.
The following facts are known.

98%	of patients who have the disease will test positive.
4%	of patients who do not have the disease will also test positive (false positive).
10%	of the general population has the disease.

Patients are randomly chosen and their test result is compared against definitive proof whether they actually have it or not.

(a) What is the probably that a positive result comes from patient who does *not* have the disease?

(b) Let Y and N represent the event where the patient does and does not have the disease, respectively. Let '+' and '−' represent the event where the patient tests positive and negative, respectively. Complete the information on this tree diagram.



(c) Find the probability that a patient has the disease and also tests positive.

(d) Show that the probability that a patient tests positive is 0.134.

(e) If a patient tests positive, what is the probability that they *actually* have the disease?

(f) Find the probability that a patient tests negative.

(g) What can you say about the sum of your answers in (c) and (f)? Was this answer expected?

Problem 4 [Independence is symmetric]

Suppose that A and B are independent events.

(a) Show that

$$\frac{P(B \mid A)}{P(A \mid B)} = \frac{P(B)}{P(A)}.$$

Hint: Use the formulas for $P(A \mid B)$ and $P(B \mid A)$.

(b) Deduce that $P(B \mid A) = P(B)$.
Hint: Remember that A and B are independent events.

(c) Complete the following statement that you just proved.

If A is independent of B, then ____________________.

(d) Bob claims that $P(A \mid B)$ can never be equal to $P(B \mid A)$. Mary claims that they *can* in fact be equal. By investigating the equation proved in (a), determine who is correct, and briefly justify your answer.

Problem 5 [Second Ace Problem]

Bob randomly picks two cards from the following four.

Ace	2	Ace	2
♠	♠	♡	♡

(a) What is the probability that Bob has two aces?

(b) Bob tells Mary "I have an ace". Explain why this increases the probability of Bob having two aces, and find this new probability.

(c) Bob now then tells Mary "I have an ace of spades". Do you think telling Mary what suit his ace is will change the likelihood of Bob having two aces? Justify your answer.

(d) Calculate the probability that Bob has two aces, given this additional information, and determine whether your answer from part (c) was correct.

Chapter 9 Review

Probability

Review

Question 1 In a group of students, there are 12 boys and 10 girls. Of these, there are 4 boys and 6 girls who have black hair. If one student is selected from the group at random, what is the probability that this person

(a) is a boy with black hair?

(b) has black hair?

(c) a girl who does not have black hair?

Question 2 If you toss a coin three times find the probability of getting

(a) a tail and two heads.

(b) at least one tail.

Question 3 A die is thrown 3 times. Find the probability that

(a) each toss is a 4.

(b) all three tosses are the same number.

(c) two 5's are thrown and then one 2.

(d) two 5's and a 2 are thrown.

(e) exactly one 5 is thrown.

(f) at most one 5 is thrown.

(g) exactly two 5's are thrown.

(h) exactly two of the numbers are same.

(i) there is at least one 4.

(j) all three numbers are different.

(k) the middle toss of the die is a 4.

(l) the numbers 6 or 1 are not thrown.

Question 4 Bob guesses three questions in his multiple choice exam, which has four options per question. Find the probability that Bob gets

(a) all three correct.

(b) at least one correct.

(c) exactly two questions correct.

Question 5 In a class of 30 students, 18 study Chinese, 15 study Japanese and 6 study both of these subjects. If a student is chosen at random from the class, find the probability that the student

(a) studies neither Chinese nor Japanese.

(b) studies Japanese, given that they also study Chinese.

Question 6 Julia and Sam are playing in a tennis tournament. They will play two rounds and each have an equal chance of winning the first round. If Julia wins the first round, her probability of winning the second round is increased to 0.7. If Julia loses the first round, her probability of winning the second round is reduced to 0.4.

Find the probability that Julia wins exactly one round.

Question 7 Two tetrahedral dice each having 4 faces numbered 1, 2, 3, 4 are thrown. When the die is thrown, there is exactly one side facing down and three sides facing upwards. The sum S of the upwards-facing sides from the two dice is recorded. Find the probability that S is

(a) divisible by 2 or 5.

(b) not divisible by 4.

(c) 15, given the sum from one of the die is 7.

Question 8 Three people are asked what day of the week they were born. Find the probability that

(a) all three are born on Tuesday.

(b) all three are born on the same day of each week.

(c) all three are born on different days of the week.

(d) exactly two people are born on the same day of the week.

Question 9 Sixty tickets are sold in a raffle with two prizes. Anthony buys 5 tickets. Find the probability that Anthony

(a) wins both prizes? (b) wins one prize only? (c) wins at least one prize?

Question 10 The probability that it will rain on Monday is 0.8. If it rains, the probability that an employee will be absent is 0.25. If it is fine, the probability that an employee will be at work is 0.9. Find the probability that the employee will be at work on Monday.

Question 11 Two dice are rolled and the sum of the upwards-facing sides is recorded. Find the probability that the sum is 8, given that one of the die is showing a six.

Question 12 There are 3 identical boxes X, Y and Z, each containing identically sized marbles.

Box X contains 5 blue and 3 green marbles
Box Y contains 2 blue and 4 green marbles
Box Z contains 3 blue and 1 green marbles

A box is chosen at random and from that box, a marble is selected at random. Find the probability that the marble is blue.

Question 13 Bob and Mary each roll a die. Find the probability that

(a) they throw the same number.

(b) Bob rolls a number greater than Mary's.

Question 14 Bob and Mary roll a die in turns, and the game is won by the first person to roll the number six. Mary rolls first. Find the probability that Mary wins

(a) on her second turn.

(b) on her third turn.

(c) in at most three of her turns.

Question 15 Suppose A and B are independent events such that $P(A) = 0.2$ and $P(B) = 0.6$. Find the value of

(a) $P(A \cap B)$ (b) $P(A \mid B)$ (c) $P(B \mid A)$

Question 16 A survey of 100 students was conducted, and they were asked about their gender, and if they have their drivers licence. The results are in the table below.

	Male	**Female**	**Total**
Licence	45	30	75
No licence	15	10	25
Total	60	40	100

(a) Find the probability that a randomly selected

(i) female has a licence.

(ii) male has a licence.

(iii) person who has a licence is a male.

(iv) person who has a licence is female.

(b) According to the data, is having a drivers licence dependent on gender?

Question 17 Two team leaders are to be chosen randomly from the candidates Adam, Bob, Carrie and David.

(a) List the sample space.

(b) What is the probability that the team leaders will be Adam and Carrie?

(c) What is the probability that David will be one of the team leaders?

(d) What is the probability that Carrie is a team leader, but Bob is not?

Question 18 The table below shows the results of a survey of 100 students about their gender and whether their phone was from brand A, or not.

	A	$\bar{A}$	Total
F	27	33	60
M	18	22	40
Total	45	55	100

(a) Find the probability that a student selected at random owns a phone from brand A.

(b) Find the probability that a female student selected at random owns a phone from brand A.

(c) Explain briefly how parts (a) and (b) imply that being female has nothing to do with owning a phone from brand A?

(d) Does this automatically mean that being male also has nothing to do with owning a phone from brand A? Verify your answer.

Question 19 A standard die is rolled twice. The sum S of the numbers that are upwards-facing is calculated.

(a) Find the probability that S is greater than 9.

(b) It is known that a 4 appears on the die at least once in the two throws. Find the probability that S is greater than 9.

Question 20 An insurance company has calculated that the probability of a 20 year-old woman having a car accident is 0.3 and a 20 year old male is 0.6 in a given year. A random 20 year-old male and female are chosen. What is the probability that

(a) they will both have an accident that year?

(b) only one of them will have an accident that year?

Investigation Task

Bayes' Theorem

Bayes' Theorem is named after Reverend Thomas Bayes (1701-1761), an English statistician and philosopher. He first provided the mathematical equation that allows new-found data to update the probabilities of various hypotheses, which were based on previous data.

$$P(A \mid B) = \frac{P(A) \times P(B \mid A)}{P(B)}$$

This formula has huge applications in the real-world including machine learning, medical diagnosis, and email spam-filtering.

Question 1 [Medical application]

Carol is worried that she may have a rare disease and decides to get tested. The sensitivity of the test is 99% and the specificity of the test is also 99%. Suppose that the prevalence of this rare disease is 0.01%. When Carol takes the test, she tested positive.

(a) Research the definition of sensitivity, specificity, and prevalence in this context.

(b) Use Bayes' Theorem to find the probability that Carol *actually* has the disease.

(c) In the medical scenario above, there is an underlying assumption that can affect the probability of Carol having the disease. Find some possible factors that may result in the calculations being invalid.

Question 2 [Demonstrating how it can be used to update a probability]

Bob draws a card at random from a standard deck of cards, and places it face-down in front of him. He bets that the card he drew was a King.

(a) What is the probability that the card drawn is a King?

(b) Mary, who is secretly helping Bob, sees that the card he placed down was a picture card. She tells Bob "You drew a picture card, but I did not see which one it was". Explain why intuitively speaking, we can expect that the probability of Bob having a King is now $\frac{1}{3}$.

(c) The following is the formal calculation of the same problem. Complete the following and verify your answer from (b).

$$P(\text{King} \mid \text{Picture}) = \frac{P(\text{King}) \times P(\text{Picture} \mid \text{King})}{P(\text{Picture})}$$

Hint: If you think very carefully about what $P(\text{Picture} \mid \text{King})$ means, it is obvious what it is equal to.

Question 3 Research a scenario on the web that require Bayes' Theorem to solve, and provide a solution for that scenario.

Investigation Task

Monty Hall Problem

Explore the Monty Hall problem and write a discussion about various aspects of the problem.

Your discussion should include and address each of the following (write a short paragraph for each one).

- A statement of the Monty Hall problem.
- Explicitly defining the role of the host.
- The most common misconception and why people make that mistake.
- What would happen if the host opened the door randomly instead.
- An explanation of the optimal choice to make, supported by probability calculations.
- How the Monty Hall problem is related to conditional probability.

10

PROBABILITY DISTRIBUTIONS

- **Random variables and probability distributions**
- **Expected value**
- **Variance and standard deviation**

Exercise 10A

Random variables and probability distributions

Fundamentals

Fundamentals 1

(a) Recall that the result of a random experiment, like flipping a coin, is called an o________.

(b) Sometimes, the o________ can be listed in an ordered manner, like the number of heads when flipping n coins or the number of times you roll a 'six' when rolling a die n times.

(c) If you construct a table listing each of these outcomes, and write down the associated probability underneath, then you have constructed a probability d__________ table. It is named as such because the table shows how probabilities are d__________ across various outcomes of the s________ s______.

(d) If the outcome can be listed, as described above, then we say that the probability distribution is d________.

(e) Furthermore, if the d________ probability distribution has all outcomes being equally likely, then we say that the the probability distribution is u________.

Fundamentals 2

(a) Let X be a variable that represents a set of possible outcomes in a sample space. For example, if X represents the number of possible heads when tossing three coins, then all possible values of X are $X =$ ___, ___, ___, ___.

(b) The variable X is called a r________ variable, since it represents the outcome of a r________ experiment.

(c) This gives us an easier way of referring to outcomes. For example, if X represents the number of heads when tossing three coins, then instead of saying P(two heads, one tail), we can instead say $P(X =$ ___$)$.

Fundamentals 3

(a) A random variable can be d________ or c__________.

(b) A d________ random variable is a random variable X that usually arises from scenarios involving counting, and is often listed as $X = 0, 1, 2, 3, \ldots$ This 'counting' list may be f________ or infinite.

(c) A c__________ random variable is a random variable X that usually arises from scenarios involving measurement. For this reason, the random variable X can be any real number, and therefore cannot be listed.

Question 1 Determine whether the following are best represented using *discrete* random variables D or *continuous* random variables C.

(a) The number of heads when tossing five coins simultaneously.

(b) The height of a randomly chosen person.

(c) The volume of water in a basin at a random point in time.

(d) The ATAR of a randomly chosen student.

(e) The number of times you have to roll a die before you finally roll a 'six'.

(f) The time that a randomly chosen Olympian swims 400 m.

(g) The number of times you have to flip a coin before you finally flip 'tails'.

(h) The product of the numbers facing up when rolling two dice.

Question 2 Two coins are tossed simultaneously. Let X represent the number of heads that land.

(a) List all the possible values of X.

(b) Find the associated probability for each value of X.

(c) Copy and complete the table directly below, and hence present your data more formally in the probability distribution table provided.

# of heads	__	__	__
Probability	__	__	__

x	__	__	__
$P(X = x)$	__	__	__

Question 3 For each of the following scenarios, define a suitable random variable X and construct a probability distribution table, but do NOT fill in the actual probabilities.

(a) The number of heads when tossing a coin two times.

(b) The number of heads when tossing a coin three times.

(c) The product of the numbers facing up when rolling two dice.

(d) The number of the side facing up when rolling a die.

(e) The sum of the numbers facing up when rolling two dice.

(f) A number is chosen randomly from 1 to 100 and the number of even non-zero digits is recorded.

Question 4 By thinking about which scenarios have equally likely outcomes, determine whether the probability distribution of the following will be uniform.

(a) The number of heads when tossing a coin.

(b) The number of heads when tossing two coins.

(c) The number of the side facing up when rolling a die.

(d) The sum of the numbers facing up when rolling two dice.

Question 5 The table below represents a discrete probability distribution. Calculate the following.

x	0	1	2	3	4
$P(X = x)$	0.1	0.4	0.3	0.15	0.05

(a) $P(X = 2)$ (b) $P(X = 0)$ (c) $P(X \geq 2)$

(d) $P(X > 2)$ (e) $P(1 \leq X \leq 4)$ (f) $P(X \text{ even})$

Question 6 The following are typical probability questions that are similar to those from Chapter 10. For each question below: (i) define a suitable random variable X. (ii) state the range of values that X can take. (iii) state the value(s) of X that is needed to answer the question.

(a) A coin is tossed three times. What is the probability that there were exactly two tails?

(b) A bag contains four red marbles and six blue marbles. Five marbles are drawn from the bag. What is the probability that exactly 3 of them are red?

(c) A die is rolled repeatedly until the number six is rolled. What is the probability that it took at least 4 failed attempts before finally rolling the number six?

(d) Two dice are rolled and the sum of the upwards-facing sides is recorded. What is the probability that the sum an odd number?

(e) Two dice are rolled and the product of the upwards-facing sides is recorded. What is the probability that the product is an even number?

Question 7 In a table tennis game, the first person to win 4 games in total wins the match. Let X represent the number of table tennis games played in a match.

(a) List all the possible values of X.

(b) What value(s) of X correspond to the game lasting

(i) exactly 4 games? (ii) exactly 6 games? (iii) at least 5 games?

Question 8 Determine whether the following represent valid probability distributions.

(a)

x	1	2	3	4
$P(X = x)$	0.1	0.5	0.2	0.3

(b)

x	0	1	2	3
$P(X = x)$	0.3	0.05	0.5	0.15

(c)

x	1	2	3	4
$P(X = x)$	0.2	1.2	-0.6	0.2

(d)

x	-2	0	2	4
$P(X = x)$	0.1	0.1	0.1	0.7

Question 9 Find the value of p such that the following tables represent probability distributions.

(a)

x	0	1	2
$P(X = x)$	p	$2p$	$3p$

(b)

x	0	1	2	3
$P(X = x)$	p	$\frac{p}{2}$	$\frac{p}{3}$	$\frac{p}{4}$

Question 10 Two marbles are drawn randomly, with replacement, from a bag containing 4 red marbles and 6 blue marbles. Let X represent the number of red marbles selected. Construct a probability distribution table for X.

Question 11 Construct a probability distribution table for X in Question 10, if replacement were not permitted.

Question 12 Construct a probability distribution table for an appropriate random variable X for each of the following scenarios.

(a) The number of heads when two coins are tossed simultaneously.

(b) The number of heads when three coins are tossed simultaneously.

(c) The number of red marbles drawn after two draws, without replacement, from a bag containing three red marbles, four blue marbles and five green marbles.

Challenge Problems

Problem 1 Construct a probability distribution table for the number of coin flips needed before tossing your first 'tails'.
Hint: There are infinitely many values of X here!

Problem 2 Show that the following can never represent a probability distribution table.

x	1	2	3	4
$P(X = x)$	$2p+1$	$1-p$	$3p$	p

Problem 3 Show that the following can never represent a probability distribution, no matter the value of p.

x	0	1	2
$P(X = x)$	p^2	$p+1$	$p+2$

Problem 4 Find the values of p and q such that the following probability distribution is uniform.

x	0	1	2
$P(X = x)$	$p+q+2$	$p-2q+5$	$2p-3q$

Exercise 10B

Expected value

Fundamentals

Fundamentals 1

(a) The expected value of a distribution is the p_____ weighted average.

(b) Informally, it can be thought of as the m_____ of the probability distribution.

(c) The expected value of a distribution with random variable X is/is not (circle one) always an actual value of X.

Fundamentals 2

(a) For a random variable X with values x_k and associated probabilities p_k for $k = 1, 2, 3, \ldots, n$, the formula for the expected value is $E(X) =$ ____________________.

(b) Alternatively if $P(X = x)$ is short-hand denoted by $p(x)$, then the expected value can be written as

$$E(X) = \sum_x x \_\_\_\_\_$$

(c) The expected value is often denoted using the Greek letter ___.

Fundamentals 3

(a) In a game situation where money is to be won or lost, the expected value is sometimes called the expected r_______ of the game.

(b) A game is said to be 'fair' if the expected return is z_____.

(c) A game is said to be 'favourable' if the expected return is p_______.

(d) A game is said to be 'unfavourable' if the expected return is n_______.

Question 1 By completing each of the following tables, calculate the expected value $E(X)$ for each probability distribution.

(a)

x_k	0	1	2	3
p_k	0.1	0.2	0.3	0.4
$x_k p_k$				

(b)

x_k	1	2	3	4	5
p_k	0.2	0.3	0.2	0.25	0.05
$x_k p_k$					

Question 2 Matthew tries his luck on a dating app. The table below shows the probability of Matthew getting matches on any given day.

Number of matches	0	1	2	3
Probability	0.9	0.05	0.03	0.02

How many matches does Matthew expect to receive per day?

Question 3 Jono is a life-saver at Bondi Beach and the probabilities of him having to do rescues on a given patrol day are given below.

Number of rescues	0	1	2	3
Probability	0.2	0.4	a	0.1

(a) Find the value of a.

(b) Hence, find the expected number of rescues that Jono will have if he goes on patrol.

Question 4 Anne is well-known around her suburb and the probability of her running into friends on a given shopping trip is given by the following.

Number of friends	1	2	3	4
Probability	a	$2a$	$3a$	$4a$

(a) Find the value of a.

(b) Hence, how many friends would Anne expect to bump into during her shopping trip?

Question 5 Suppose $\mu = 2.85$. Find the value of a and b in the probability table below.

x_k	1	2	3	4
p_k	0.1	0.2	a	b

Question 6 Nina plays a game where she rolls a die and she wins, in dollars, whatever number she rolls. For example, if she rolls a '4', then she will win \$4.

Number rolled	1	2	3	4	5	6
Winnings	1	2	3	4	5	6

(a) Calculate Nina's expected return from one round of the game.

(b) Suppose that it costs \$3 to enter the game. By comparing it with Nina's expected return, determine whether the game is favourable for her.

Question 7 In a game, two coins are flipped. If there are two heads, then you win \$10. If there is exactly one head, then you win \$5. However, if there are two tails, then you lose \$20.

(a) Complete the table below.

Number of heads	0	1	2
Probability			
\$ gained			

(b) Hence, calculate the expected return.

(c) Is this game fair?

Question 8 Albert plays a game where the probability of winning is 0.1 and the probability of losing is 0.9. If he wins, the prize is \$100. It costs \$15 to play the game.

(a) What is Albert's expected return?

(b) In the long run, would it be profitable for Albert to repeatedly play the game? Justify your answer.

(c) How much would the entry fee have to be, at most, for it to be worthwhile for Albert to enter the game?

Question 9 A person selects a marble from a bag containing 10 red marbles, 4 blue marbles and 1 white marble. The person wins \$1 for the red marble, \$5 for the blue marble and \$20 for the white marble. It costs \$4 to enter the game.

(a) Calculate the expected return from playing the game and explain why this shows that the game is *unfair*.

(b) How much should the prize for the following marbles be, so that the game is fair?

(i) Red marble. (ii) Blue marble. (iii) White marble.

(c) Suppose the prize of drawing the red, blue and white marbles respectively are k, $2k$ and $3k$. What is the value of k such that the entry cost of \$4 is considered fair?

Challenge Problems

Problem 1 [Linearity of $E(X)$]

Recall the formula for the expected value

$$\begin{aligned} E(X) &= \sum_k x_k p_k \\ &= x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_n p_n \end{aligned}$$

where x_k is the outcome of a random variable X with n outcomes, and p_k is the associated probability.
Define the formula

$$\begin{aligned} E(aX + b) &= \sum_k (ax_k + b) \times p_k \\ &= (ax_1 + b)\, p_1 + (ax_2 + b)\, p_2 + (ax_3 + b)\, p_3 + \ldots + (ax_n + b)\, p_n \end{aligned}$$

(a) By expanding and re-arranging, prove that $E(aX + b) = aE(X) + b$.
Hint: What is $p_1 + p_2 + p_3 + \ldots + p_n$ equal to?

(b) Use this formula to find a faster way of doing Question 8.

Problem 2 [Sum of expectation]

Let X and Y be two random variables. Prove that

$$E(X + Y) = E(X) + E(Y).$$

Problem 3 [Recursion-approach to expected value]

Suppose Bob plays a game where he flips a biased coin. If he flips 'heads', which has probability p, then he wins the game. If he flips 'tails', then he must flip again. This repeats until he finally flips 'heads'. Let X be the number of flips he makes until he wins, and let $E(X)$ be the expected number of flips to win.

(a) Suppose Bob attempts his first flip, then scores 'tails', and therefore must flip again. Explain briefly why his expected value at that point is exactly the same as when he first started.

(b) Use this to explain the relationship $E(X) = 1 + (1 - p)E(X)$.
Hint: When he makes his first flip, he either 'wins' or 'fails'

(c) Hence, show that $E(X) = \dfrac{1}{p}$.

(d) Suppose the coin is fair (unbiased). What is the value of p?

(e) Substitute this value of p in your answer from (c). What do you get? Was this answer *expected*?

Exercise 10C

Variance and standard deviation

Fundamentals

Fundamentals 1

(a) The variance of a random variable X is a measure of s_____ of the probability distribution.

(b) The variance of X is denoted by _____.

Fundamentals 2

(a) The variance is the e_______ value of the s_______ of the distance of x from μ.

(b) Hence, a formula for variance is $\text{Var}(X) = E($__________$)$.

(c) Alternatively, $\text{Var}(X) = E($_____$) - \mu^2$.

(d) The more practical formula to use, for calculation purposes, is $\text{Var}(X) =$ ____________.

Fundamentals 3

(a) The s_______ d_______ is also a measure of s_______.

(b) The more standard deviations that a particular value x is from μ, the further/closer (circle one) x is from the mean.

(c) The standard deviation is denoted by ___, and it is the square root of the v_______.

(d) Hence, we have the formula $\sigma = \sqrt{\quad}$ and hence _____ $= \text{Var}(X)$.

Question 1 Fill in the following table, and hence find $\text{Var}(X)$ and σ using $\text{Var}(X) = E\left(X^2\right) - \mu^2$.

$\boldsymbol{x}$	1	2	3	4	5
$\boldsymbol{p(x)}$	0.1	0.3	0.4	0.15	0.05
$\boldsymbol{x^2}$					
$\boldsymbol{x^2 p(x)}$					

Question 2 By first finding μ, fill in the following table, and hence calculate $\text{Var}(X)$ and σ using the (less practical) formula $\text{Var}(X) = E\left((X-\mu)^2\right)$. They should be the same as your values from Question 1.

$\boldsymbol{x}$	1	2	3	4	5
$\boldsymbol{p(x)}$	0.1	0.3	0.4	0.15	0.05
$\boldsymbol{x-\mu}$					
$\boldsymbol{(x-\mu)^2}$					
$\boldsymbol{(x-\mu)^2 p(x)}$					

Question 3 For the following probability distribution tables, calculate: (i) μ. (ii) variance. (iii) standard deviation.

(a)

$\boldsymbol{x}$	1	2	3	4
$\boldsymbol{P(X=x)}$	0.1	0.5	0.1	0.3

(b)

$\boldsymbol{x}$	1	2	3	4	5
$\boldsymbol{P(X=x)}$	0.2	0.2	0.2	0.2	0.2

Question 4 A die is numbered 1, 2, 2, 3, 3, 3. Let X be the number facing upwards when the die is rolled once. Find

(a) the probability distribution of X.

(b) the expected value.

(c) the variance.

(d) the standard deviation.

(e) $P(X \leq \mu)$.

(f) $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

Question 5 A coin is flipped and a standard die is rolled simultaneously. Let the random variable X be defined as follows.

- If the coin lands on heads, then 1 is added to the number that the die rolls.
- If the coin lands on tails, then 1 is subtracted from the number that the die rolls.

Find the following.

(a) The probability distribution of X.

(b) The expected value.

(c) The variance.

(d) The standard deviation.

(e) $P(\mu - \sigma \leq X \leq \mu + \sigma)$.

(f) $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$.

Question 6 [z-score to compare results]

The z-score of a value x is how many standard deviations x is from the mean μ. It could be positive or negative, depending on whether x is respectively above or below μ. Nahar is a student who studies Biology and Chemistry. The average for his Biology class was 76% with a standard deviation of 10%. The average for his Chemistry class was 89%, with a standard deviation of 5%. Nahar scored 85% and 95% in Biology and Chemistry respectively.

(a) Which subject did Nahar perform better in, relative to his peers?

(b) Write down a general formula for the z-score of a value x from the mean μ.

Challenge Problems

Problem 1 Recall the formula $\text{Var}(X) = E\left((X-\mu)^2\right)$.

(a) Use the results

$$E(X+Y) = E(X) + E(Y)$$
$$E(aX+b) = aE(X) + b$$

from the previous exercise to show that

$$\text{Var}(X) = E(X^2) - 2\mu E(X) + \mu^2.$$

(b) Hence, prove the familiar result $\text{Var}(X) = E(X^2) - \mu^2$.

(c) Deduce that $E(X^2) \geq \mu^2$.

Problem 2

(a) Use the fact that $\text{Var}(X) = E\left(X^2\right) - \left(E(X)\right)^2$ to prove that $\text{Var}(aX+b) = a^2\,\text{Var}(X)$.

(b) Deduce that $\text{Var}\left(\dfrac{X-\mu}{\sigma}\right) = 1$.

Chapter 10 Review

Probability Distributions

Review

Question 1 Consider the following probability distribution table.

x	0	1	2	3	4	5	6
$P(X = x)$	0.05	0.15	0.2	0.3	0.2	0.05	0.05

Calculate the following.

(a) $P(X \geq 4)$

(b) $P(X \leq 5)$

(c) $P(X \text{ odd})$

(d) μ

Question 2 Consider the following probability distribution table.

x	1	2	3	4	5
$P(X = x)$	0.1	0.3	0.4	0.1	0.1

(a) Find $P(X \geq 2)$ and $P(X \geq 4)$.

(b) Find the value of $P(X \geq 4 \mid X \geq 2)$.

(c) Find the value of $P(X \geq 2 \mid X \geq 4)$.

(d) Give an intuitive reason for your answer in (c).

Question 3 Consider the following table.

x	1	2	3	4	5
$P(X = x)$	a	$3a$	$5a$	$2a$	a

For what value of a does this represent a valid probability distribution?

Question 4 A bag contains six blue marbles and four red marbles. Three marbles are selected at random, with replacement, from the bag. Let X be the number of red marbles drawn. Find the value of $P(X = 2)$.

Question 5 A biased coin is flipped twice. The coin is biased so that the probability of flipping a tails is 0.3. Let X be the number of tails observed. Construct the probability distribution table for X.

Question 6 A bag contains three white balls and four black balls. Two balls are selected at random, without replacement, from the bag. Let X be the number of black balls drawn. Construct the probability distribution table for X and hence find the value of

(a) Construct the probability distribution table.

(b) Hence, find the value of

(i) μ (ii) $\text{Var}(x)$ (iii) σ

Question 7 A tutoring company offers up to two sessions of free one-to-one help for students who are enrolled in classes. Students pay \$200 per week for the classes.

$\boldsymbol{x}$	0	1	2
$\boldsymbol{P(X = x)}$	0.5	0.3	0.2

Every time a student uses the free service, it costs the company \$30 to pay for the tutor running the session, as well as any administrative costs to set up the session. How much money does the tutoring company expect to make per week from each student?

Question 8 Consider the following probability distribution table.

$\boldsymbol{x}$	$-a$	a
$\boldsymbol{P(X = x)}$	0.5	0.5

Without explicitly calculating anything, and by using purely your understanding of the mean, variance and standard deviation, find the values of the following. Justify your answer.

(a) μ (b) σ (c) $\text{Var}(x)$

Question 9 Jake pays x dollars to enter a game. In this game, he rolls a die. If he rolls anything from 1 to 5, inclusive, he wins that many dollars. He may continue rolling and winning money until eventually he rolls a six, in which case he gets knocked out.

(a) Calculate his expected value if $x = 6$ and $x = 18$.

(b) How much should the entry fee be for the game to be fair?

Question 10 The table below is a probability distribution table for a random variable X with $\mu = 21$ and $\sigma = 7$.

$\boldsymbol{x}$	10	20	30
$\boldsymbol{P(X = x)}$	a	b	0.3

Find the values of a and b.

Question 11 The table below is a probability distribution table for a random variable X with $\mu = 22$ and $\sigma = 6$.

$\boldsymbol{x}$	10	a	30
$\boldsymbol{P(X = x)}$	0.1	b	0.3

Find the values of a and b.

Investigation Task

Normal numbers

A number is said to be *normal* if the distribution of the digits is uniform.

Question 1 Research the difference between *sample mean* and *population mean*, and likewise for the variance and standard deviation. Are the formulas the same? If not, then why?

Question 2 Suppose you had a list of the first 100 decimal-place digits of a normal number, and then sorted them into a frequency table. From this frequency table, form a rough probability table.

(a) Discuss what you would expect to see if you were to calculate the mean and the variance of the probability distribution table that you constructed.

(b) Research some 'common' normal numbers and find the first 100 decimal digits. Use appropriate technology or otherwise to create a frequency distribution table and from that, construct a probability distribution table.

(c) Calculate the value of μ, $\mathrm{Var}(X)$ and σ. Comment on your findings and compare them to your answer from (a).

Question 3 What assumptions did you make by turning your frequency distribution table to a probability distribution table? Can this go wrong at any time? Research some examples that may cause problems, and find how these problems are amended. Some useful terms to consider include the *sample mean*, *sample variance* and *sample standard deviation*.

Investigation Task

Benford's Law

Benford's Law models the frequency of the leading (first) digit of a numerical data point. It has many real-life applications and this investigation task aims to show the student why it is so useful.

Question 1 Write a short paragraph describing the history of Benford's Law and how it is used in real-life. Your answer should include the formula used as well as the frequency distribution table of the digits.

Question 2 For the frequency distribution table found in Question 1, calculate μ and σ.

Question 3

(a) Research different ways to generate random numbers using everyday tools such as a calculator.

(b) Choose a method and use it to randomly generate 50 numbers.

(c) Record your results in a frequency distribution table and calculate the sample mean $\bar{x}$ and sample standard deviation s. Compare with your results from Question 2.

Question 4 Ask a group of friends (or a very dedicated friend) to each write down and each give you a list of twenty randomly chosen numbers. It is important that this list is created by themselves and not using technology (why?). Continue this process until you have a sample of *at least* 100 numbers. Construct the frequency distribution table and compare them against the table in Question 1. Comment on your findings. Your answer should include a calculation of the sample mean $\bar{x}$ and sample standard deviation s.

Question 5 Discuss any negatives of Benford's Law. Are there any scenarios that are not suitable for the use of Benford's Law? What are some constraints?

1. Algebraic Techniques

Exercise 1A

Expanding and factorising

F1

(a) $x^2 + 2xy + y^2$ (b) $x^2 - 2xy + y^2$

(c) $x^2 - y^2$ (d) $a^2x^2 + 2axy + y^2$

(e) $a^2x^2+2abxy+b^2y^2$ (f) -1

Q1

(a) $4x^2 + 4x + 1$ (b) $9x^2 - 24x + 16$

(c) $9x^2y^2 - 24xy + 16$ (d) $9x^2 - 24xy + 16y^2$

(e) $x^2 + 2 + \frac{1}{x^2}$ (f) $x^2 - \frac{1}{x^2}$

(g) $6x^2 - 5x - 4$ (h) $4x^2 - 9y^2$

(i) $6x^2 - 5xy - 4y^2$

Q2

(a) $(x + 3)^2$ (b) $(x + 5)(x - 3)$

(c) $(2x + 1)(2x - 5)$ (d) $3(4x+3y)(4x-3y)$

(e) $(5+x+y)(5-x-y)$ (f) $(2x + 3)(x - 5)$

(g) $(x + y + 2)(x - y)$ (h) $(x - 3)(1 + x)$

(i) $(x^2 + y^2)(x + y)(x - y)$

Q3

(a) $x^3 - 1$

(b) See full worked solutions.

P1

(a) $(4 + 9x^2)(2 + 3x)(2 - 3x)$

(b) $(a + 2)^2(a - 2)$

(c) $(3x + 2)(2x - 3)$

(d) $(3x + 2y)(2x - 3y)$

P2

(a) $(x - 1)(x^2 + x + 1)$

(b) $(3x - 2)(9x^2 + 6x + 4)$

(c) $(2x - 1)(4x^2 + 8x + 7)$

P3

(a) $x^3 - 3x^2y + 3xy^2 - y^3$

(b) $x^3 + 3x^2y + 3xy^2 + y^3$

P4

(a) See full worked solutions.

(b) See full worked solutions.

(c) $2^{100} - 1$

P5

See full worked solutions.

P6

See full worked solutions.

Exercise 1B

Algebraic fractions

F1

(a) $\frac{a+b}{x}$ (b) $\frac{ax+b}{x^2}$ (c) $\frac{ay+bx}{xy}$ (d) $\frac{ay+bx}{xy^2}$

Q1

(a) $\frac{6x}{5}$ (b) $\frac{31x}{30}$ (c) $\frac{5x-8}{3}$

(d) $\frac{32x+9}{24}$ (e) $\frac{17}{8x}$ (f) $-\frac{3}{4x}$

(g) $\frac{2x}{(x-1)(x+1)}$ (h) $\frac{2(x-1)}{x-2}$

(i) $\frac{3x-2}{x(x-2)}$ (j) $\frac{10x+9}{12x^2}$

(k) $\frac{7x+6}{x(x-2)(x+2)}$ (l) $\frac{x^2-x+6}{x(x-1)(x-2)(x-4)}$

Q2

(a) $\frac{3}{20x}$ (b) $\frac{1}{2y}$

(c) $\frac{(x-2)(x+2)}{(x-3)(x+3)}$ (d) $\frac{3(x+2)}{x-5}$

Q3

(a) $\frac{2x+3y}{4x}$ (b) $-\frac{1}{xy}$ (c) $\frac{y^2-x^2}{y^2x^2}$

Q4

(a) $\frac{x^2-1}{x^2+1}$ (b) $2x + 1$

Q5

(a) $\frac{1}{x+1}$ (b) $\frac{y+x}{y-x}$

(c) $\frac{y^2+x^2}{y^2-x^2}$ (d) $\frac{3(x+2)}{2x+9}$

Q6

(a) -1 (b) $\frac{t+3}{3}$

(c) $\frac{x-7}{(x+2)(x-2)(x-1)}$

(d) $\frac{2(7-3x)}{(x+3)(x-3)}$

Q7

(a) $\frac{5}{3}$ (b) $\frac{19}{9}$ (c) $\frac{1}{3}$

Q8

$\frac{x+1}{x+2}$

P1

-1

P2

(a) $\frac{a^2+1}{a^2-1}$ (b) $\frac{a^2+b^2}{a^2-b^2}$

P3

See full worked solutions.

P4

(a) See full worked solutions.
(b) See full worked solutions.
(c) See full worked solutions.
(d) See full worked solutions.

P5

(a) See full worked solutions.
(b) The denominator will be zero.
(c) See full worked solutions.

Exercise 1C

Indices

F1

(a) a^{m+n} (b) a^{m-n} (c) a^{mn} (d) $\sqrt[n]{a^m}$

(e) 1 (f) $\sqrt[n]{a}$ (g) $a^n b^n$ (h) $\frac{1}{a}$

F2

(a) $\frac{1}{a^n}$ (b) 1 (c) a (d) a^n

(e) $\frac{1}{\sqrt{a}}$ (f) $\frac{a^n}{b^n}$ (g) $\frac{b}{a}$ (h) $\frac{b^n}{a^n}$

F3

n

F4

(a) $a^n(a-1)$ (b) $a^n(a^{2n}-1)$ (c) $a^n(b^n-1)$

Q1

(a) 1 (b) 7 (c) 5 (d) 4

(e) 27 (f) $\frac{25}{16}$ (g) 4

Q2

(a) $\frac{1}{3}$ (b) $\frac{1}{25}$ (c) 27 (d) $\frac{25}{4}$

(e) 125 (f) $\frac{1}{2}$ (g) $\frac{1}{25}$ (h) $\frac{1}{2}$

Q3

(a) 3 (b) 3^{2x+2} (c) 3^{3x+1}

(d) $36x^6y^8$ (e) $\frac{4}{x^6}$ (f) $4x^7$

Q4

(a) 2^{10n} (b) 7^{1-5n} (c) 3^8

(d) 3^n+1 (e) 5^n

Q5

(a) 4 (b) 1 (c) -1

(d) $-\frac{5}{2}$ (e) -4 (f) $\pm\frac{1}{7}$

Q6

(a) 2×3^7 (b) 3×2^n

(c) $3^n(3^n-1)$ (d) $3^n(2^n-1)$

(e) $3^n(3^n+1)(3^n-1)$

P1

(a) $\frac{5}{4}$ (b) 2 (c) 8

(d) 2 (e) 1 (f) 2

P2

(a) 3^x (b) 2^{10x}

ANSWERS

P3
199

P4
5^{100}

P5
(a) $(5^x+2^y)(5^x-2^y)$ (b) $2^{4x}(1+2^x)(1-2^x)$

P6
6^7

P7
$a=3,\ b=2$

Exercise 1D
Surds

F1
(a) $\sqrt{a}\sqrt{b}$ (b) $\dfrac{\sqrt{a}}{\sqrt{b}}$

F2
(a) $x^{\frac{1}{2}}$ (b) $x^{\frac{3}{2}}$ (c) $x^{\frac{5}{2}}$
(d) $x^{-\frac{1}{2}}$ (e) $x^{\frac{1}{n}}$ (f) $x^{\frac{m}{n}}$

F3
(a) $x+2\sqrt{xy}+y$ (b) $x-y$
(c) $x+2+\dfrac{1}{x}$ (d) $x-\dfrac{1}{x}$

F4
(a) surds (b) perfect

F5
(a) x, b (b) a^2b

F6
(a) $\dfrac{\sqrt{a}}{a}$ (b) $\dfrac{\sqrt{a}}{ab}$ (c) $\dfrac{a-\sqrt{b}}{a^2-b}$
(d) $\dfrac{\sqrt{a}-\sqrt{b}}{a-b}$ (e) $\dfrac{a-b\sqrt{c}}{a^2-b^2c}$ (f) $\dfrac{\sqrt{a}-b\sqrt{c}}{a-b^2c}$

Q1
(a) $7\sqrt{5}-6\sqrt{2}$ (b) $21\sqrt{6}$
(c) $2+\sqrt{3}$ (d) $\dfrac{\sqrt{2}}{2}$

Q2
(a) 3 (b) $2\sqrt[3]{3}$
(c) $10\sqrt[3]{2}$ (d) $2\sqrt[5]{2}$

Q3
(a) $a\sqrt{b}$ (b) a^2b
(c) a^2b (d) $a^2b\sqrt{ab}$

Q4
(a) $\dfrac{\sqrt{5}}{10}$ (b) $\dfrac{\sqrt{5}}{3}$ (c) $\dfrac{\sqrt{15}}{3}$
(d) $\dfrac{2+5\sqrt{2}}{23}$ (e) $\dfrac{27+10\sqrt{2}}{23}$ (f) $\dfrac{3(\sqrt{3}-1)}{4}$
(g) 4

Q5
(a) $\dfrac{a\sqrt{bd}}{cd}$ (b) $\dfrac{a-\sqrt{b}}{a^2-b}$ (c) $\dfrac{\sqrt{a}}{a}-\dfrac{\sqrt{b}}{b}$
(d) $\dfrac{a\sqrt{b}+c}{a^2b-c^2}$ (e) $\dfrac{a\sqrt{b}-c\sqrt{d}}{a^2b-c^2d}$

Q6
(a) -1 (b) $28+10\sqrt{3}$ (c) $37-20\sqrt{3}$
(d) $5+2\sqrt{6}$ (e) $\dfrac{3}{2}+\sqrt{2}$ (f) $\dfrac{4}{3}$

Q7
(a) $a=3,\ b=20$ (b) $a=49,\ b=-12$
(c) $a=66,\ b=-36$ (d) $a=-3,\ b=-18$

Q8
(a) 50 (b) 5

Q9
(a) 10 (b) $-8\sqrt{5}$

Q10
See full worked solutions.

P1
See full worked solutions.

P2
(a) $\sqrt{n+1}-\sqrt{n}$ (b) 9

P3
See full worked solutions.

Chapter Review

R1

(a) Zach is 10, Joel is 6 (b) 16
(c) Book: \$24, Card: \$4 (d) 40
(e) 20 (f) 150 m^2

R2

(a) $y = \frac{3x}{1-x}$ (b) $\frac{zx}{z-x}$
(c) $y = \pm\frac{2}{\sqrt{a-b}}$ (d) $\pm\sqrt{x^2+4ab}$

R3

(a)
(i) $\sqrt{\frac{3V}{h}}$ (ii) $x = 3$ cm
(b)
(i) $\sqrt{\frac{V}{\pi h}}$ (ii) $\frac{5}{\pi}$ cm
(c)
(i) $\sqrt[3]{\frac{3V}{4\pi}}$ (ii) $\frac{3}{2}$ cm

R4

(a) 18 cm (b) Emma is 17 and Robin is 40

R5

(a) $(x-10)(x+1)$ (b) $(x+9)(x-7)$
(c) $(x-2)(5x-4)$ (d) $(5x+3)(x-1)$
(e) $(5-2x)(x+4)$ (f) $(x-5)(x+5)$
(g) $(6-x)(4+x)$ (h) $(x-y)(x+y+3)$
(i) $(x+y-2)(x+y+2)$
(j) $5(4x^2+y^2)(2x-y)(2x+y)$
(k) $(x-3)(x-2)(x+2)$
(l) $(2x-3y)(x+y)$

R6

(a) $3\sqrt{2}$ (b) $5\sqrt{2}$ (c) $5\sqrt{5}$
(d) $\sqrt{10}$ (e) $22-12\sqrt{2}$ (f) 3

R7

(a) 4 (b) $\frac{1}{a}$ (c) $\frac{3}{2}$ (d) $2\times3^{m-4}$

R8

(a) 4 (b) $\frac{1}{125}$ (c) 8

R9

5^6

R10

(a) 1 (b) $-\frac{2}{3}$ (c) $\frac{1}{4}$
(d) $\frac{1}{2}$ (e) 2 (f) 0, 2

R11

(a) $1+\sqrt{2}$ (b) $-9-2\sqrt{5}$
(c) $-2+3\sqrt{2}$ (d) $3+\frac{3\sqrt{6}}{2}$

R12

(a) $a=7,\ b=3$ (b) $a=4,\ b=15$

R13

(a) $\frac{44}{9}$ (b) 5 (c) -5 (d) $\frac{5}{13}$

R14

(a) $\frac{x-7}{10}$ (b) $\frac{3}{x-y}$
(c) $\frac{x+4}{(x+2)(x-2)}$ (d) $\frac{x^2+xy-y}{(x+y)(x-y)}$
(e) $-\frac{6}{(x+1)(x-1)}$ (f) $\frac{8-x}{(x-2)(x+2)(x+7)}$
(g) $-\frac{y^2}{x+y}$ (h) $\frac{4}{x+2}$
(i) $\frac{3}{(x+3)^2}$ (j) $-\frac{x}{(x+1)^2}$
(k) $\frac{xy}{x+y}$ (l) $\frac{1}{xy}$

R15

(a) $x+\frac{1}{x}-2$ (b) $x-\frac{1}{x}$

R16

(a) 4 (b) 14

ANSWERS

R17

(a) y^2 (b) $y+y^2$ (c) y^3
(d) $4y^2$ (e) $2x^6$ (f) $8x^6$
(g) a^{m+n} (h) 1 (i) $1+\frac{y}{x}$
(j) $\frac{1}{y}+\frac{1}{x}$ (k) $1+y$ (l) $2x+3y$
(m) $2+\frac{3y}{x}$ (n) $1+\frac{y}{x}$ (o) $\frac{2xy}{1+2y}$
(p) $\frac{5x}{4}$ (q) $\frac{x+y}{x^2y^2}$ (r) $\frac{y+x}{y-x}$
(s) $\frac{2}{x+y}$ (t) -1

2. Functions and Relations

Exercise 2A

Function definitions and notation

F1

(a) relation (b) y, function
(c) relation (d) function, relation
(e) 2, relation (f) one, one, function
(g) $f(a)=b$ (h) equations, domains

Q1

(a) relation (b) function
(c) function (d) relation

Q2

(a) one-to-one (b) one-to-many
(c) many-to-many (d) many-to-one

Q3

See full worked solutions.

Q4

(a) many-to-one function
(b) one-to-many relation
(c) many-to-one function
(d) many-to-many relation
(e) many-to-one function
(f) many-to-one function
(g) one-to-many relation
(h) one-to-one function
(i) one-to-one function
(j) one-to-one function
(k) many-to-one function
(l) many-to-many relation

Q5

(a) 7 (b) -3 (c) 3 (d) 3

Q6

(a) -5 (b) $-2a+1$ (c) $2a^2+1$
(d) $4a+1$ (e) $2\sqrt{a}+1$ (f) $2a+3$
(g) $6a-3$ (h) $2a+2b+1$ (i) $2ab+1$
(j) $\frac{a+2}{a}$

Q7

(a) $27t^2-2$ (b) $12t^2-2$
(c) $3t-2$ (d) $3t^2-6t+1$

Q8

(a) x^2 (b) x (c) $\frac{1}{x^2y^2}$ (d) $\frac{1}{(x+y)^2}$

Q9

(a) 4 (b) 1 (c) -1 (d) 9

Q10

(a) 4 (b) 7 (c) 5 (d) 1

Q11

(a) True (b) True (c) True (d) True
(e) False (f) True (g) True (h) True

Q12

(a) $2k-1$ (b) $6k-1$
(c) $-\frac{1}{k(k-1)}$ (d) 2^{k-1}

Q13

See full worked solutions.

Q14

$\frac{1}{x^2}$

Q15

(a) 9 (b) 6

Q16
See full worked solutions.

Q17
See full worked solutions.

Q18
See full worked solutions.

Q19
See full worked solutions.

Q20
-7

P1
(a) $f(x) = x + 4$ (b) $f(x) = x^2 + 4x$
(c) $f(x) = x^2 - 5x + 8$ (d) $f(x) = \dfrac{x}{3x - 2}$

P2
See full worked solutions.

P3
See full worked solutions.

P4
See full worked solutions.

Exercise 2B
Domain and range

F1
(a) $a \leq x \leq b$ (b) $x \in (a, b)$
(c) $a \leq x < b$ (d) $x \in (a, b]$

F2
(a) x (b) range

F3
(a) 0 (b) ≥ 0 (c) > 0, zero

F4
(a) ≥ 0, negative (b) ≥ 0, negative
(c) range

Q1
(a) $x \in [1, 3]$ (b) $x \in (-2, 5]$
(c) $x \in [-7, -3)$ (d) $x \in [4, \infty)$
(e) $x \in (-2, \infty)$ (f) $x \in (-\infty, 3]$

Q2
(a) $2 < x < 6$ (b) $-5 \leq x \leq 10$
(c) $1 < x \leq 4$ (d) $-2 \leq x < 3$
(e) $x \leq 3$ (f) $x > -2$

Q3
(a) $[-2, \infty)$ (b) $(-3, 1]$ (c) $[2, 4]$
(d) $[-4, -2)$ (e) $(-\infty, -3)$ (f) $(-2, 0)$

Q4
(a) D: $x \in \mathbb{R}$, R: $y \in \mathbb{R}$
(b) D: $x \in \mathbb{R}$, R: $y \geq 0$
(c) D: $x \in \mathbb{R}$, R: $y \in \mathbb{R}$
(d) D: $x \in \mathbb{R}$, R: $y \geq 1$
(e) D: $x \in \mathbb{R}$, R: $y \in \mathbb{R}$
(f) D: $-2 \leq x \leq 2$, R: $-2 \leq y \leq 2$
(g) D: $-2\pi \leq x \leq 2\pi$, R: $-1 \leq y \leq 1$
(h) D: $x \in \mathbb{R}$, R: $y > 0$
(i) D: $x \in \mathbb{R}$, R: $y < 1$
(j) D: $x \geq 0$, R: $y \in \mathbb{R}$
(k) D: $x > 0$, R: $y \in \mathbb{R}$
(l) D: $x \geq 0$, R: $y \geq 0$
(m) D: $-3 \leq x \leq 3$, R: $0 \leq y \leq 3$
(n) D: $-3 \leq x \leq 3$, R: $-2 \leq y \leq 2$
(o) D: $0 \leq x \leq 4$, R: $-1 \leq y \leq 3$

Q5
(a) D: $x = 2$, R: $y \in \mathbb{R}$
(b) D: $x \in \mathbb{R}$, R: $y = -1$
(c) D: $x \in \mathbb{R}$, $x \neq 0$, R: $y \in \mathbb{R}$, $y \neq 0$
(d) D: $x \in \mathbb{R}$, R: $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
(e) D: $x \leq -2$, $x \geq 2$, R: $y \in \mathbb{R}$
(f) D: $x \in \mathbb{R}$, R: $-1 < y \leq 1$

Q6
(a) positive
(b)
(i) D: $x \geq 0$ (ii) D: $x \geq 2$
(iii) D: $x \leq 5$ (iv) D: $x \leq \dfrac{7}{2}$

ANSWERS

Q7

(a) zero

(b)

(i) D: $x \in \mathbb{R}$, $x \neq 0$

(ii) D: $x \in \mathbb{R}$, $x \neq 1$

(iii) D: $x \in \mathbb{R}$, $x \neq \frac{3}{2}$

(iv) D: $x \in \mathbb{R}$, $x \neq -2$

(v) D: $x \in \mathbb{R}$, $x \neq 7$

(vi) D: $x \in \mathbb{R}$, $x \neq 3$

Q8

(a) D: $x > 0$ (b) D: $x > -4$

(c) D: $x > 3$ (d) D: $x < 2$

Q9

(a) $x = 1$, $x = 2$

(b) Domain: $x \in \mathbb{R}$, $x \neq 1$, $x \neq 2$

(c)

(i) D: $x \in \mathbb{R}$, $x \neq -1$, $x \neq 3$

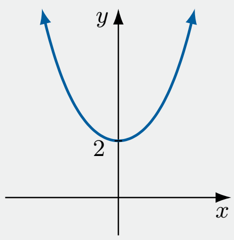
(ii) D: $x \in \mathbb{R}$, $x \neq \frac{1}{2}$, $x \neq \frac{1}{3}$

(iii) D: $x \in \mathbb{R}$, $x \neq -2$, $x \neq 2$

(iv) D: $x \in \mathbb{R}$, $x \neq 1$, $x \neq 3$

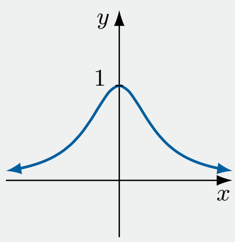
Q10

(a)



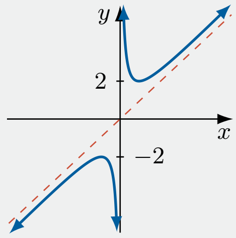
Domain: $x \in \mathbb{R}$, R: $y \geq 2$

(b)



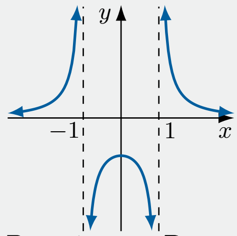
Domain: $x \in \mathbb{R}$, R: $0 < y \leq 1$

(c)



Domain: $x \in \mathbb{R}$, $x \neq 0$, R: $y \leq -2$, $y \geq 2$

(d)

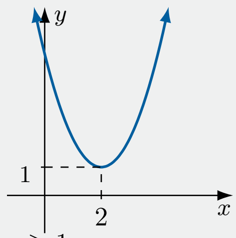


Domain: $x \in \mathbb{R}$, $x \neq -1$, $x \neq 1$, R: $y \leq -1$, $y > 0$

Q11

(a) $(2, 1)$

(b)



(c) $y \geq 1$

(d)

(i) $y \geq -4$ (ii) $y \geq 3$

(iii) $y \geq -4$ (iv) $y \geq 2$

(e) The range is $y \geq k$

Q12

(a) $y \geq 0$ (b) $y \geq 3$ (c) $y \geq -4$

(d) $y \leq 0$ (e) $y \leq 6$ (f) $y \leq 5$

(g) $y \leq 3$ (h) $y \geq 2$ (i) $y \leq 10$

Q13

(a) $y \geq 0$ (b) $y \geq 3$ (c) $y \geq -5$

(d) $y \leq 0$ (e) $y \leq 4$ (f) $y \leq 3$

Q14

(a) Range: $y \leq 9$, Maximum: $y = 9$

(b) $0 \leq y \leq 3$

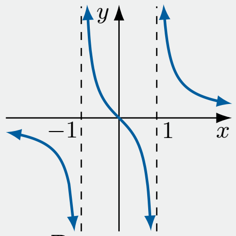
(c) $y \geq 3$

P1

$-2 < x < 2$

P2

(a)



(b) $y \in \mathbb{R}$

P3

The denominator cannot equal zero

P4

(a) $x \in \mathbb{R}, x \neq -1$

(b) $x \in \mathbb{R}, x \neq -1, x \neq 1$

(c) $x \in \mathbb{R}, x \neq 2, x \neq 3$

(d) $x \in \mathbb{R}$

P5

Domain: $x = 1$, Range: $y = 0$

P6

(a) D: $x > 0$, R: $y > 0$

(b) D: $x \in \mathbb{R}$, R: $y > 0$

(c) D: $x \geq 2, x \leq -2$, R: $y > 0$

(d) D: $x \in \mathbb{R}$, R: $0 < y \leq \frac{1}{2}$

Domain: $x = 1$, Range: $y = 0$

Exercise 2C

Even and odd functions

F1

(a) $f(x)$ (b) $-f(x)$

F2

(a) y (b) origin

Q1

(a) Even (b) Odd (c) Neither

(d) Even (e) Neither (f) Even

Q2

(a) $f(-x) = -2x$, Odd

(b) $f(-x) = x^2 + 1$, Even

(c) $f(-x) = x^2 - x$, Neither

(d) $f(-x) = -x^3$, Odd

(e) $f(-x) = -x + x^3$, Odd

(f) $f(-x) = -\frac{1}{x}$, Odd

(g) $f(-x) = 2^{-x}$, Neither

(h) $f(-x) = x^4 - x^2 + 1$, Even

(i) $f(-x) = \sqrt{1 - x^2}$, Even

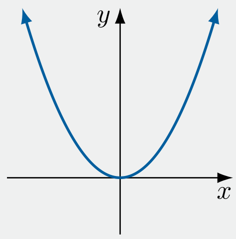
(j) $f(-x) = -\frac{x}{x^2 + 1}$, Odd

(k) $f(-x) = -x - \frac{1}{x}$, Odd

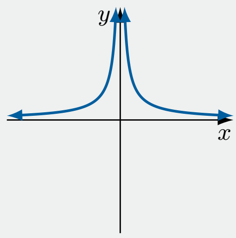
(l) $f(-x) = -\frac{1 + x}{1 - x}$, Neither

Q3

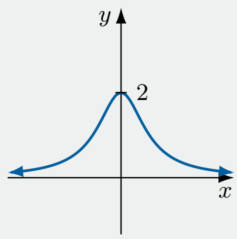
(a)



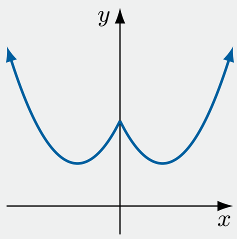
(b)



(c)

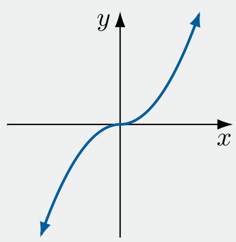


(d)

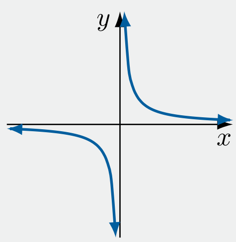


Q4

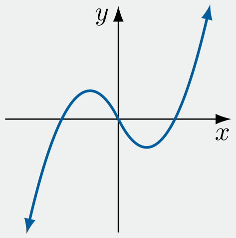
(a)



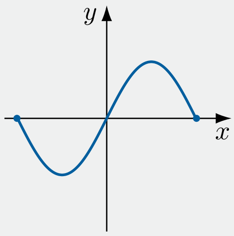
(b)



(c)



(d)



P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) Even

(c) Even

P3

See full worked solutions.

P4

(a) False (b) True (c) False

Exercise 2D

Composite functions

F1

input, outside

F2

(a) g, f (b) f, g

Q1

(a) 1 (b) -17

(c) See full worked solutions.

(d) See full worked solutions.

Q2

(a) $\sqrt{x+4}$ (b) $\sqrt{x}+4$

(c) $x+8$ (d) $f\left(g\left(g(x)\right)\right)$

Q3

(a) $f(g(x)) = 2x+3$, $g(f(x)) = 2x+1$

(b) $f(g(x)) = 6x$, $g(f(x)) = 6x$

(c) $f(g(x)) = x^6$, $g(f(x)) = x^6$

(d) $f(g(x)) = \dfrac{1}{\sqrt{x}}$, $g(f(x)) = \dfrac{1}{\sqrt{x}}$

Q4

$\dfrac{x}{2x+1}$

Q5

(a) and (b) are self-inverse

Q6

(a) $\dfrac{2}{x^2-9}$ (b) $\dfrac{4}{(x-9)^2}$

(c) No. $f(g(x))$ has domain $x \in \mathbb{R}$, $x \neq \pm 3$, but $g(f(x))$ has domain $x \in \mathbb{R}$, $x \neq 9$

Q7

(a), (b) and (d) commute under composition

Q8

(a) $\sqrt{x}-2$ (b) $x \geq 0$ (c) $x \geq 2$

Q9

(a) $2\sqrt{x}-3$, $x \geq 0$

(b) $\sqrt{2x-3}$, $x \geq \dfrac{3}{2}$

(c) $4x^2-12x+10$, $x \in \mathbb{R}$

(d) $\sqrt{x^2+1}$, $x \in \mathbb{R}$

Q10

(a) x

(b) $\dfrac{1}{x}$

(c) $f_n(x) = \begin{cases} x, & \text{for even } n \\ \dfrac{1}{x}, & \text{for odd } n \end{cases}$

P1

(a) See full worked solutions.

(b) $h(x)$ is an even function.

(c) It is always produces an even function.

P2

See full worked solutions.

P3

(a) $-x-1$

(b) $\sqrt{-x^2-1}$

(c) $f(g(x))$ does not work since $-x^2-1<0$

P4

See full worked solutions.

Chapter Review

R1

(a) one-to-many (b) many-to-one

(c) many-to-many (d) one-to-one

R2

(b) and (d)

R3

(a) $x \in (-\infty, 1]$ (b) $x \in (1, 5]$

(c) $x \in [-3, \infty)$

R4

(a) $a = -1$ (b) $a = -\frac{3}{2}$

R5

(a) $x \geq -2$ (b) $-1 < x \leq 4$

(c) $x \leq 6$

R6

(a) Not a function.
D: $x \in [-3, 3]$
R: $y \in [-3, 3]$

(b) Function.
D: $x \in \mathbb{R}$
R: $y \geq 1$

(c) Not a function.
D: $x \geq 0$
R: $y \in \mathbb{R}$

(d) Function.
D: $x \in \mathbb{R}$
R: $y \in (0, 1]$

(e) Not a function.
D: $x \in [0, 2]$
R: $y \in [1, 3]$

(f) Function.
D: $x \in \mathbb{R}$
R: $y > 1$

R7

(a) See full worked solutions.

(b) The product of an even and odd function is odd.

(c) See full worked solutions.

R8

$2f(x)$

R9

n

R10

$\frac{27}{4}$

R11

$a = -1,\ b = 3,\ c = \frac{1}{2}$

R12

See full worked solutions.

R13

See full worked solutions.

R14

(a) $x + \frac{1}{x}$ (b) $x + \frac{1}{x} + \frac{x}{x^2 + 1}$

(c) 4

R15

(a) $x \geq -3$ (b) $x > -3$

(c) $x \in \mathbb{R}$ (d) $x \in \mathbb{R},\ x \neq \pm 3$

(e) $x \in \mathbb{R},\ x \neq 2, 3$ (f) $x \leq 2$

(g) $x \in \mathbb{R},\ x \neq \pm 1$ (h) $x < 6$

(i) $x \in \mathbb{R},\ x \neq 0$

R16

(a) Odd (b) Neither (c) Even

(d) Even (e) Neither (f) Even

(g) Odd (h) Neither (i) Even

R17

(a) No (b) Yes

(c) 0 (d) $p = 2$

R18

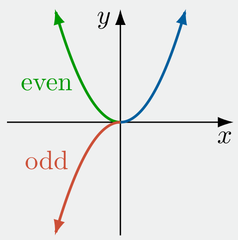
$x \leq 5,\ x \neq -1$

R19

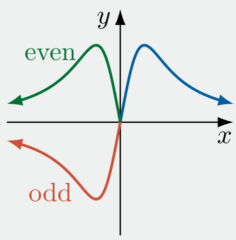
$2bx$

R20

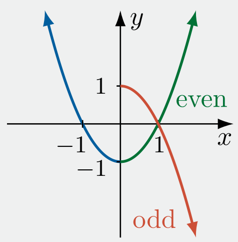
(a)



(b)



(c)



R21

(a) $-9a-2$ (b) a^2-6a+8

(c) 0 (d) -5

(e) $-\dfrac{3}{4}$ (f) $9x^2-12x+3$

(g) $9x-8$ (h) x^4-2x^2

(i) $\dfrac{1}{x^2-1}$

R22

(a) $f(x)=x+3$ (b) $f(x)=x^2+2x$

(c) $f(x)=x^2-2x+1$

3. Linear and Quadratic Functions

Exercise 3A

Linear functions

F1

(a) steep, direction (b) steeper

(c) positive (d) down

(e) m_1, m_2 (f) $m_1m_2=-1$

F2

(a) $\dfrac{y_2-y_1}{x_2-x_1}$ (b) $mx+c$ (c) $m(x-x_1)$

F3

(a) $y=k$ (b) $x=k$

Q1

(a) $m=1$ (b) $m=2$ (c) $m=1$

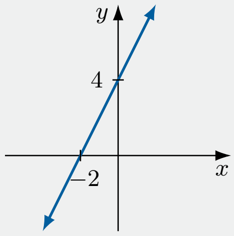
(d) $m=-\dfrac{1}{2}$ (e) $m=-3$ (f) $m=0$

(g) $m=-2$ (h) $m=-\dfrac{1}{3}$ (i) Undefined

Q2

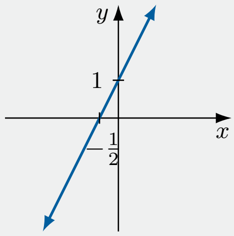
(a) 2 (b) 4 (c) -2

(d)

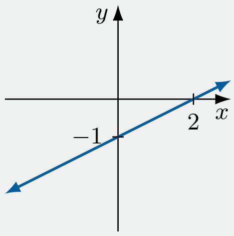


Q3

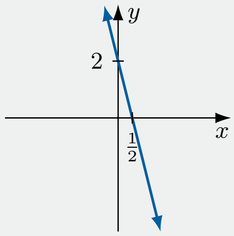
(a)



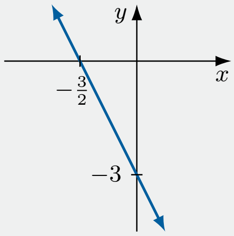
(b)



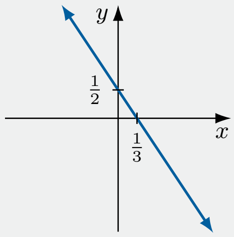
(c)



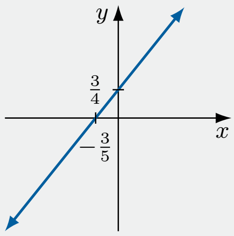
(d)



(e)



(f)



Q4

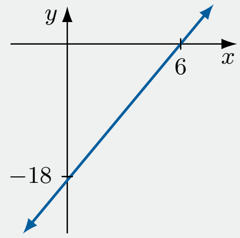
(a) $y-y_1=m(x-x_1)$ (b) $y=2x+2$

(c) $y=-3x+1$ (d) $y=-\dfrac{3}{2}x-3$

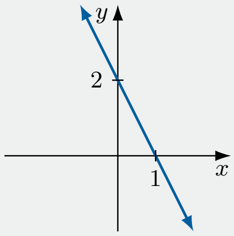
(e) $y=\dfrac{2}{3}x+\dfrac{4}{3}$ (f) $y=\dfrac{5}{2}x+1$

Q5

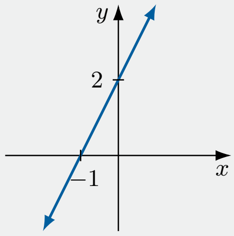
(a)



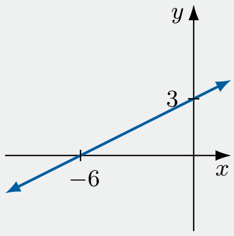
(b)



(c)



(d)



Q6

(a) -2 (b) 1 (c) $-\frac{1}{8}$ (d) 2

(e) 0 (f) Undefined.

Q7

(a) $y = -2x + 4$ (b) $y = x + 5$

(c) $y = -\frac{1}{8}x - \frac{31}{8}$ (d) $y = 2x - 10$

(e) $y = 2$ (f) $x = -2$

Q8

$y = 4x + 12$

Q9

(a) $-\frac{1}{3}$ (b) $x + 3y - 22 = 0$

Q10

-7

Q11

(a) $y = 6$ (b) $x = 3$

Q12

See full worked solutions.

Q13

(a) No. (b) No. (c) Yes. (d) Yes.

Q14

See full worked solutions.

Q15

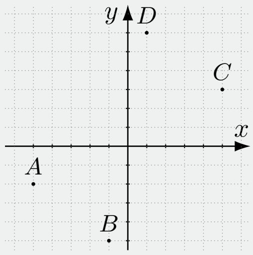
(a) $a = 8$ (b) $a = -3$

Q16

(I) and **(III)** are parallel, and
(II) and **(IV)** are parallel

Q17

(a)



(b) See full worked solutions.

(c) See full worked solutions.

(d) Parallelogram.

(e) Rectangle.

Q18

No.

Q19

(a) See full worked solutions.

(b) See full worked solutions.

(c) Rhombus.

Q20

$a = -3, b = 4$

P1

-6 or 1

P2

A, C, and D lie on the same line.

P3

(a) See full worked solutions.

(b) See full worked solutions.

(c) Any point P on this circle will result in $\angle APB = 90°$.

P4

See full worked solutions.

ANSWERS

P5

(a) x-intercept: a, y-intercept: b
The constants in the equation represent the x and y-intercepts of the line.

(b) $-\frac{x}{2}+\frac{y}{3}=1$

Exercise 3B

Features of a parabola

F1

(a) parabola (b) parabolas, vertex

(c) concave, concave (d) x, roots

(e) 0, 1, 2

F2

(a) $>0, <0$ (b) c

(c) $x=-\frac{b}{2a}$ (d) symmetry

(e) Substitute the x-coordinate $-\frac{b}{2a}$ back into the quadratic function.

F3

(a) They are the roots of the quadratic, and consequently the x-intercepts of the parabola.

(b) $x=\frac{\alpha+\beta}{2}$

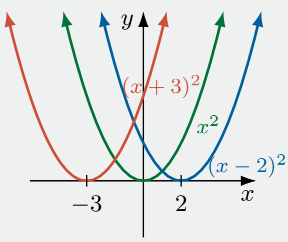
(c) roots, vertex, vertex, y

F4

(a) h and k are respectively the x and y-coordinates of the vertex

(b) vertex, x

Q1



$y=(x-h)^2$ is the horizontal translation of $y=x^2$.

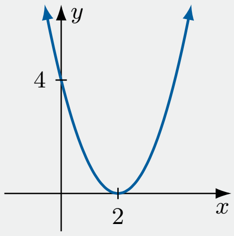
Q2

(a)

(i) 2 (ii) $(2,0)$

(iii) 4 (iv) $x=2$

(v)

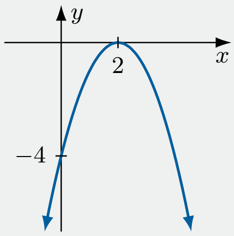


(b)

(i) -2 (ii) $(-2,0)$

(iii) -4 (iv) $x=-2$

(v)

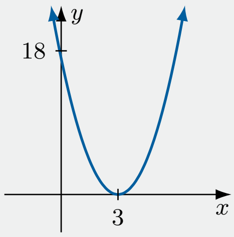


(c)

(i) 3 (ii) $(3,0)$

(iii) 18 (iv) $x=3$

(v)

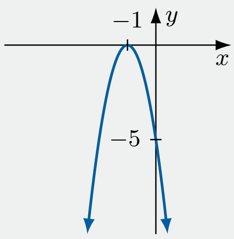


(d)

(i) -1 (ii) $(-1,0)$

(iii) -5 (iv) $x=-1$

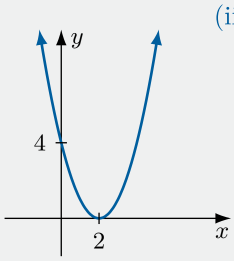
(v)



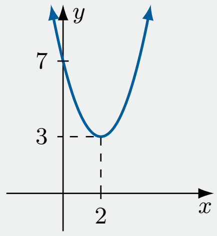
Q3

(a)

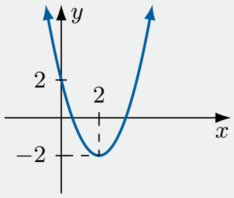
(i)



(ii)



(iii)



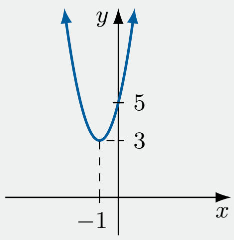
(b) $y = (x-h)^2 + k$ is a vertical translation of $y = (x-h)^2$. The translation is up if $k > 0$, and down if $k < 0$.

(c) (h, k)

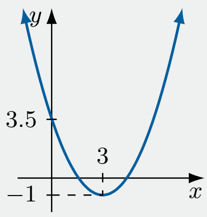
(d) No.

Q4

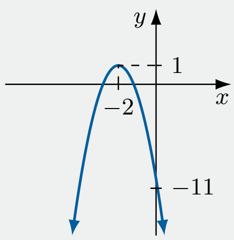
(a)



(b)



(c)



Q5

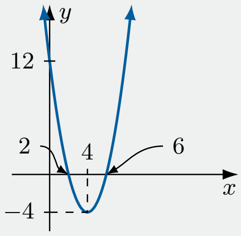
(a)

(i) 2, 6 (ii) 12

(iii) $x = 4$ (iv) $(4, -4)$

(v) Minimum. (vi) -4

(vii)



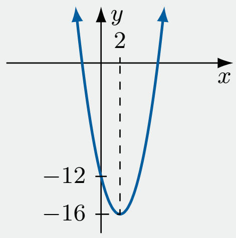
(b)

(i) -2, 6 (ii) -12

(iii) $x = 2$ (iv) $(2, -16)$

(v) Minimum. (vi) -16

(vii)



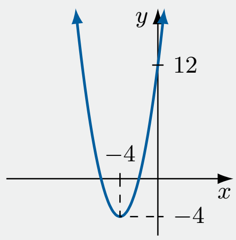
(c)

(i) -2, -6 (ii) 12

(iii) $x = -4$ (iv) $(-4, -4)$

(v) Minimum. (vi) -4

(vii)



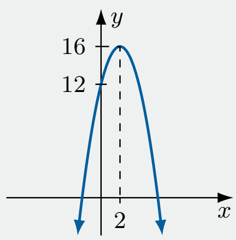
(d)

(i) -2, 6 (ii) 12

(iii) $x = 2$ (iv) $(2, 16)$

(v) Maximum. (vi) 16

(vii)



(e)

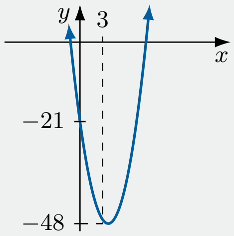
(i) -1, 7 (ii) -21

(iii) $x = 3$ (iv) $(3, -48)$

(v) Minimum. (vi) -48

ANSWERS

(vii)



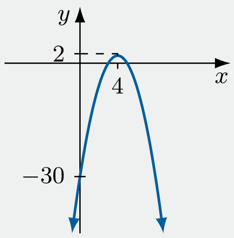
(f)

(i) 3, 5 (ii) −30

(iii) $x = 4$ (iv) $(4, 2)$

(v) Maximum. (vi) 2

(vii)



Q6

(a)

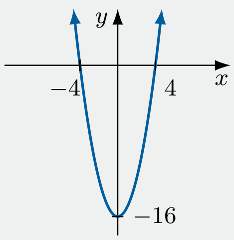
(i) $y = (x + 4)(x - 4)$

(ii) $(-4, 0)$, $(4, 0)$, $(0, -16)$

(iii) $x = 0$ (iv) $(0, -16)$

(v) Minimum. (vi) −16

(vii)



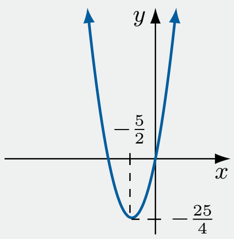
(b)

(i) $y = x(x + 5)$ (ii) $(0, 0)$, $(-5, 0)$

(iii) $x = -\dfrac{5}{2}$ (iv) $\left(-\dfrac{5}{2}, -\dfrac{25}{4}\right)$

(v) Minimum. (vi) $-\dfrac{25}{4}$

(vii)



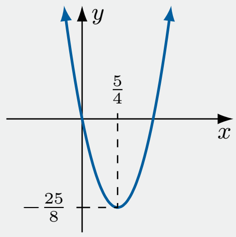
(c)

(i) $y = x(2x - 5)$ (ii) $(0, 0)$, $\left(\dfrac{5}{2}, 0\right)$

(iii) $x = \dfrac{5}{4}$ (iv) $\left(\dfrac{5}{4}, -\dfrac{25}{8}\right)$

(v) Minimum. (vi) $-\dfrac{25}{8}$

(vii)



(d)

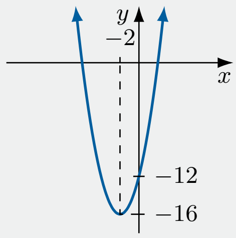
(i) $y = (x + 6)(x - 2)$

(ii) $(-6, 0)$, $(2, 0)$, $(0, -12)$

(iii) $x = -2$ (iv) $(-2, -16)$

(v) Minimum. (vi) −16

(vii)



(e)

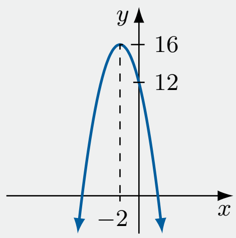
(i) $y = -(x + 6)(x - 2)$

(ii) $(-6, 0)$, $(2, 0)$, $(0, 12)$

(iii) $x = -2$ (iv) $(-2, 16)$

(v) Maximum. (vi) 16

(vii)



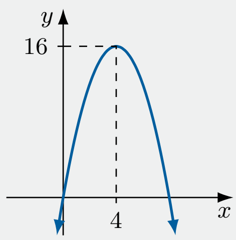
(f)

(i) $y = -x(x - 8)$ (ii) $(0, 0)$, $(8, 0)$

(iii) $x = 4$ (iv) $(4, 16)$

(v) Maximum. (vi) 16

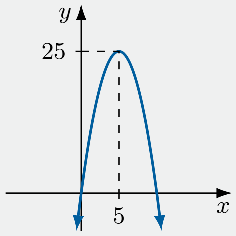
(vii)



ANSWERS

Q7

$y = 25$



Q8

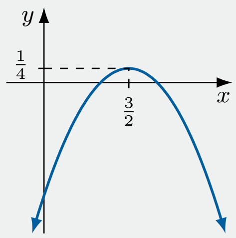
(a)

(i) $x = \dfrac{-3}{-2}$ (ii) $\left(\dfrac{3}{2}, \dfrac{1}{4}\right)$

(iii) Maximum. (iv) $(0, -2)$

(v) $(1, 0)$, $(2, 0)$

(vi)



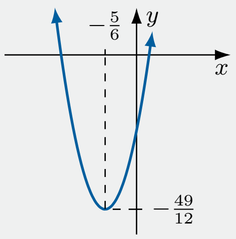
(b)

(i) $x = -\dfrac{5}{6}$ (ii) $\left(-\dfrac{5}{6}, -\dfrac{49}{12}\right)$

(iii) Minimum. (iv) $(0, -2)$

(v) $\left(\dfrac{1}{3}, 0\right)$, $(-2, 0)$

(vi)

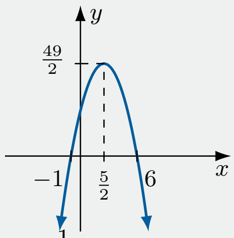


Q9

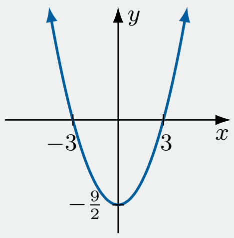
(a)	None	(b)	(F)	(c)	(H)
(d)	(I)	(e)	(D)	(f)	None
(g)	(G)	(h)	(A)	(i)	None
(j)	(E)	(k)	(C)	(l)	(B)

Q10

(a) $y = -2(x + 1)(x - 6)$



(b) $y = \dfrac{1}{2}(x - 3)(x + 3)$



Q11

(a) $\alpha = 1$, $\beta = -3$ (b) $a = 2$

(c) $y = 2(x-1)(x+3)$

Q12

(a) $h = -2$, $k = 1$ (b) $y = \dfrac{1}{9}(x + 2)^2 + 1$

Q13

(a) $y = x^2 - 8x$ (b) $y = x^2 - 4x + 7$

(c) $y = 2x^2 + 3x - 4$

Q14

(a) $y = (x - 3)^2 + 1$

(b) $(x - 3)^2 \geq 0$ implies $y = (x - 3)^2 + 1 > 0$

(c)

(i) $y = (x - 2)^2 + 2 > 0$

(ii) $y = -(x - 1)^2 - 3 < 0$

Q15

(a) $y = (x + 5)(x - 2)$ (b) $y = 2x(x - 6)$

(c) $y = -\dfrac{1}{2}x(x - 8)$

P1

(a) $P = 2x + y = 400$

(b) See full worked solutions.

(c) $20,000\ \text{m}^2$

(d) $A > 0$, $0 < x < 200$

P2

(a) See full worked solutions.

(b) He should charge \$12.50 per burger, and will achieve a maximum daily revenue of \$625.

Exercise 3C

Solving quadratic equations

F1

(a) factorised, completing, quadratic

(b) 0, 1, 2

F2

(a) α, β (b) α

F3

(a) $\pm k$ (b) $h \pm k$

(c) $ax^2 + bx + c$

F4

(a) quadratic

(b) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) real, solution

(d) 2, distinct

(e) 1, distinct

Q1

(a) $16, (x+4)^2$ (b) $25, (x-5)^2$

(c) $\frac{25}{4}, \left(x+\frac{5}{2}\right)^2$ (d) $\frac{9}{4}, \left(x-\frac{3}{2}\right)^2$

(e) $25y^2, (x+5y)^2$ (f) $36y^2, (x-6y)^2$

Q2

(a) 0, 2 (b) 0, −5 (c) 2

(d) 2, 3 (e) 2, −1 (f) −6, 2

(g) 5 (h) 1, 5 (i) 8, −3

(j) −5, 3 (k) −12 (l) 4, 6

Q3

(a) $-\frac{1}{2}, -2$ (b) $\frac{1}{2}, 2$ (c) $\frac{4}{3}, -2$

(d) $\frac{3}{2}, -\frac{1}{2}$ (e) $-\frac{1}{3}, \frac{1}{2}$ (f) $\frac{1}{6}, -2$

Q4

$a = -4,\ b = -9,\ x = 1, 7$

Q5

(a) $5 \pm \sqrt{3}$ (b) $-2 \pm \sqrt{3}$

(c) $\frac{2}{3}, \frac{4}{3}$ (d) $\frac{1}{2}, -\frac{9}{2}$

Q6

(a) $1 \pm \sqrt{2}$ (b) $2 \pm \sqrt{3}$

(c) $5 \pm 2\sqrt{3}$ (d) $2 \pm \frac{\sqrt{5}}{2}$

Q7

(a) $\frac{1 \pm \sqrt{5}}{2}$ (b) $1 \pm \frac{\sqrt{3}}{2}$

Q8

(a) ± 2 (b) 0, 4 (c) $x = 0$, $x = 6$

(d) −3, 7 (e) $-\frac{5}{2}, 3$ (f) −1, 4

(g) −3, 4 (h) $-\frac{5}{2}, -3$ (i) $\frac{4}{3}, -4$

(j) $\frac{1}{2}, 2$ (k) $0, \pm 1$ (l) $-1 \pm \frac{3\sqrt{2}}{2}$

(m) ± 4 (n) 0, 3 (o) −2, 6

Q9

(a) 9 cm (b) 13 (c) 6 and 14

(d)

(i) \$150,000

(ii) Either 2 or 4 machines.

(iii) He should operate 3 machines, and will achieve a minimum production cost of \$70,000.

P1

(a) $a = 1,\ b = 2,\ c = 1$

(b) $a = 2,\ b = -15,\ c = 17$

(c) $a = 4,\ b = -3,\ c = -2$

P2

(a) 2 (b) 2

(c) 1, 2 (d) 1

(e) −3, −2, 1, 2

Exercise 3D
Discriminant

F1

(a) inside, quadratic (b) $b^2 - 4ac$

(c) > 0 (d) $= 0$

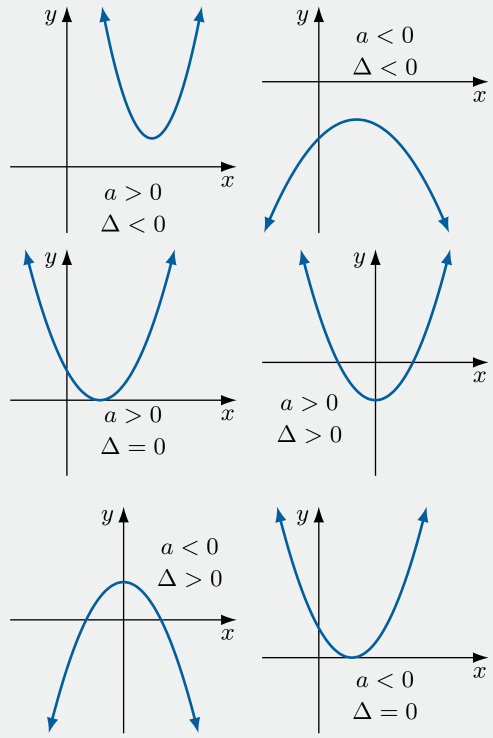
(e) no

F2

(a) $a > 0$, $\Delta < 0$

(b) $a < 0$, $\Delta < 0$

(c)



Q1

(a) $\Delta = 0$, touches

(b) $\Delta = -12$, does not intersect

(c) $\Delta = 121$, crosses

(d) $\Delta = -15$, does not intersect

(e) $\Delta = 73$, crosses

(f) $\Delta = 0$, touches

Q2

(a) $\Delta < 0$ (b) $\Delta = 0$ (c) $\Delta > 0$

(d) $\Delta > 0$ (e) $\Delta = 0$ (f) $\Delta > 0$

Q3

(a) The square root in the quadratic formula disappears, leaving us with $x = -\frac{b}{2a}$, a rational number.

(b) Distinct rational roots.

(c) Distinct irrational roots.

Q4

(a) 2 distinct rational roots.

(b) 2 distinct rational roots.

(c) 2 distinct irrational roots.

(d) 1 rational root.

(e) No real roots.

(f) 1 rational root.

Q5

See full worked solutions.

Q6

(a) $k = \pm 3$ (b) $k = -10, 2$

(c) $k = 0, \frac{4}{3}$ (d) No solution.

Q7

(a)

(i) $k = \frac{1}{4}$ (ii) $k < \frac{1}{4}$ (iii) $k > \frac{1}{4}$

(b)

(i) $k = -2$ (ii) $k > -2$ (iii) $k < -2$

Q8

See full worked solutions.

(a) See full worked solutions.

(b) See full worked solutions.

Q9

See full worked solutions.

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) See full worked solutions.

Q10

$k = 1$

Q11

(a) $k^2 + 2k + 9 = (k+1)^2 + 8 > 0$ since $(k+1)^2 \geq 0$

(b) See full worked solutions.

(c)

(i) See full worked solutions.

(ii) See full worked solutions.

Q12

(a) k^2

(b) $\Delta = k^2 \geq 0$

Q13

(a) $-3k^2$

(b) $k = 0$

(c) None.

Q14

(a) $m \leq 8$

(b) $m = 1, 9$

(c) $m = -\frac{9}{4}$

P1

4 or -8

P2

2 distinct roots since $\Delta > 0$.

P3

$p < -\frac{1}{2}$

P4

See full worked solutions.

P5

See full worked solutions.

P6

$q^2 = pr$

P7

See full worked solutions.

P8

See full worked solutions.

P9

$p = \frac{5}{2}$, $y = \frac{5}{2}x - \frac{25}{2}$
$p = -1$, $y = -x - 2$

Exercise 3E

Simultaneous equations and applications

F1

Algebraically using substitution, algebraically using elimination, and graphically by drawing and reading the coordinates of the intersection point.

F2

(a) It is the x-intercept of the intersection point of ℓ_1 and ℓ_2.

(b) It is the x-intercepts of the intersection point of ℓ and $\mathcal{Q}$.

(c) The quadratic function $\mathcal{Q}$ and the linear function ℓ can intersect in two distinct points ($\Delta > 0$), be tangential ($\Delta = 0$), or not intersect at all ($\Delta < 0$).

F3

(a) revenue

(b) revenue, selling

(c) net, revenue

(d) equal, break-even

F4

fixed, variable

Q1

(a) $x = 1, y = 1$

(b) $x = 7, y = -2$

(c) $x = 6, y = 2$

(d) $x = 3, y = -3$

(e) $x = 3, y = 3$

(f) $x = 6, y = 2$

Q2

(a) $(1, -1)$

(b) $(1, -2)$

(c) $(3, 1)$

(d) $(-1, -1)$

Q3

(a) 8 and 12.

(b) Bob is 42 and Mary is 14.

(c) 12 \$50 notes and 8 \$20 notes.

(d) 2 managers and 8 labourers.

(e) Adult: \$26, Child: \$17

(f)

(i) 175 kg

(ii) Mary: 45 kg, Bob: 76 kg, Jane: 54 kg

Q4

$x = \frac{2}{5}$, $y = \frac{3}{5}$

Q5

(a) $(0, 0), (1, 1)$

(b) $(-3, 5), (4, 12)$

(c) $(1, 1)$

(d) $(-1, 1), (-4, 10)$

Q6

(a) 2, 4 (b) 2 solutions.

(c) 2 solutions. (d) No solutions.

(e)

(i) $k = 9$ (ii) $k < 9$ (iii) $k > 9$

(f) The solutions of $f(x) = k$ are the x-coordinates of the intersection points of $y = f(x)$ and $y = k$.

Q7

(a)

(i) The line and parabola intersect at two distinct points.

(ii) The line is tangential to the parabola.

(iii) The line and parabola do not meet.

(b) One root.

(c) Determining the sign of the discriminant.

Q8

(a) Once. (b) Twice.

(c) Does not intersect. (d) Once.

Q9

(a) 0, 4 (b) -7 (c) $\pm 2\sqrt{2}$ (d) $-\frac{1}{4}$

Q10

(a) $C(2) = \$220,\ R(2) = \$100$
The company is making a loss.

(b) $C(10) = \$300,\ R(10) = \$500$
The company is making a profit.

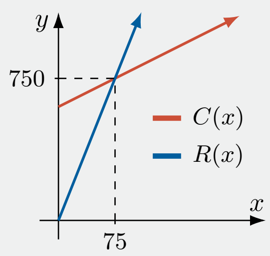
(c) 5

Q11

(a) See full worked solutions.

(b) 75

(c)

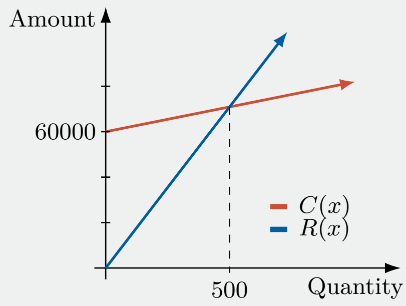


(d) 78

Q12

(a) $C(x) = 60000 + 10x,\ R(x) = 130x$

(b) 500 devices.



P1

(a) $x - y$

(b) 3 hours and 45 minutes.

P2

26

P3

$\frac{3}{7}$

Chapter Review

R1

(a) $y = -\frac{1}{4}(x+2)(x-5)$

(b) $y = -2(x-2)^2 - 3$

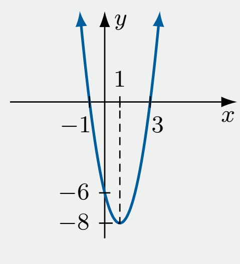
R2

-8

R3

(a) $x = 1,\ (1, -8)$ (b) $(1, -8)$

(c) $y = 2(x-3)(x+1)$



R4

$y \geq 3$

R5

$a = 2,\ b = -9,\ c = -5$
Any equation with integer multiples of these values will work.

R6

(a) $a > 0$ (b) $b < 0$ (c) $c > 0$

R7

See full worked solutions.

R8

See full worked solutions.

R9

See full worked solutions.

R10

(a) Concave up parabola with two distinct roots.

(b) Concave down parabola with two distinct roots.

R11

See full worked solutions.

R12

$0,\ -8$

R13

See full worked solutions.

R14

(a) $-1,\ 2$ (b) 2

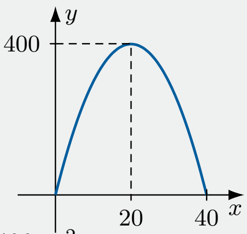
R15

(a)

Width	Length	Area
9	31	279
16	24	384
22	18	396
35	5	175
x	$40 - x$	$40x - x^2$

(b) $0 < x < 40$

(c)



(d) $400\ \text{m}^2$

R16

(a) $x = 4,\ y = 6$ (b) $x = 7,\ y = 4$

(c) $x = 7,\ y = -1$ (d) $x = 5,\ y = 2$

R17

(a) 250 m

(b) A maximum height of 31.25 m is attained when $x = 125$ m.

(c) 41.25 m

R18

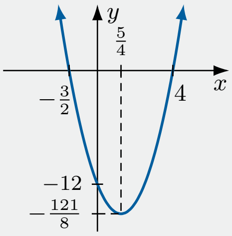
$a = 1,\ b = 1,\ c = 0$

R19

$(1, 9),\ (7, 9)$

R20

$$y = 2\left(x - \frac{5}{4}\right)^2 - \frac{121}{8}$$



R21

The maximum area is $11,250\ \text{m}^2$.
$A > 0,\ 0 < x < 150$

R22

See full worked solutions.

4. Further Functions and Relations

Exercise 4A

Polynomial functions

F1

(a) n (b) a_n (c) $a_n x^n$

(d) a_0 (e) 1 (f) $n + 1$

F2

(a) x, zeroes (b) x (c) $2,\ x,\ 2$

(d) $3,\ x,\ 3$ (e) n

F3

(a) x, α (b) x, α

(c) $(x-\alpha)^3$, x, α

F4

(a) polynomial (b) is not

(c) $a = A$, $b = B$, $c = C$ (d) equal

Q1

(a) $a = 2$, $b = -2$, $c = 2$

(b) $b = -7$

(c) $a = 1$, $b = 4$

(d) $a = 1$, $b = -2$, $c = 5$

(e) $a = 1$, $b = 3$, $c = 3$, $d = 1$

(f) $a = 12$, $b = 17$

(g) $a = \frac{1}{2}$, $b = 0$, $c = \frac{3}{2}$

(h) $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = \frac{1}{2}$

Q2

(a) Single roots at $x = 1, 2, 3$

(b) Single roots at $x = -5, -1, 3$

(c) Double root at $x = 5$

(d) Triple root at $x = -4$

(e) Single root at $x = -4$, Double root at $x = 2$

(f) Double root at $x = -3$, Triple root at $x = -7$

(g) Single root at $x = 1$, Double root at $x = -3$, Triple root at $x = -7$

(h) Double roots at $x = \pm 1$

Q3

(a) $P(x) = (x-1)(x+2)(x-3)$

(b) $P(x) = (x-1)(x+1)(x-4)^2$

(c) $P(x) = (x-2)^2(x+2)^3$

Q4

(a) $x^2 + x - 1$ (b) $-x^2 + 5x - 3$

(c) $x^2 - 5x + 3$ (d) $2x^2 - 13x + 8$

(e) $3x^3 - 8x^2 + 7x - 2$ (f) $3x^3 - 6x^2 + 3x$

Q5

(a) $P(x) = x(x+2)(x-1)$

(b) $P(x) = x^2(x+3)(x-2)$

(c) $P(x) = (x+2)(x-2)^3$

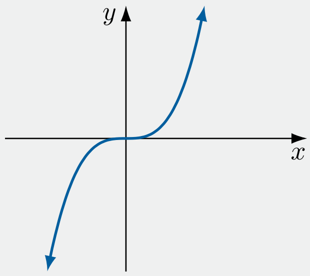
(d) $P(x) = -x(x-1)(x+2^2)$

(e) $P(x) = (x-2)^2(x+1)^3$

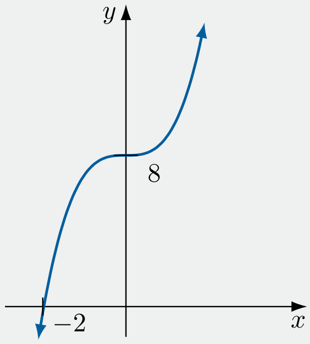
(f) $P(x) = -x^3(x-2)^2$

Q6

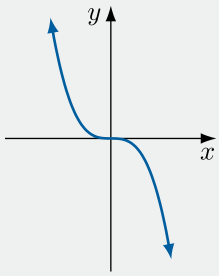
(a)



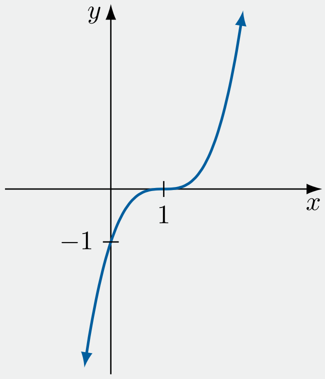
(b)



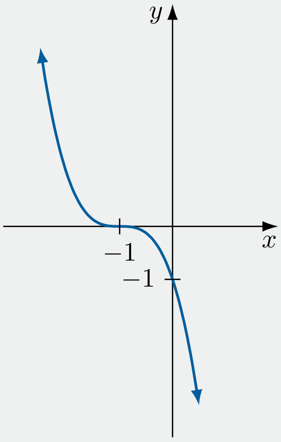
(c)



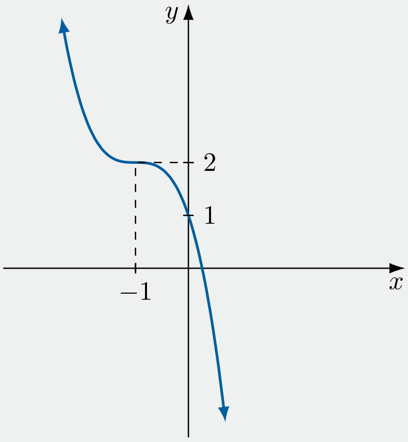
(d)



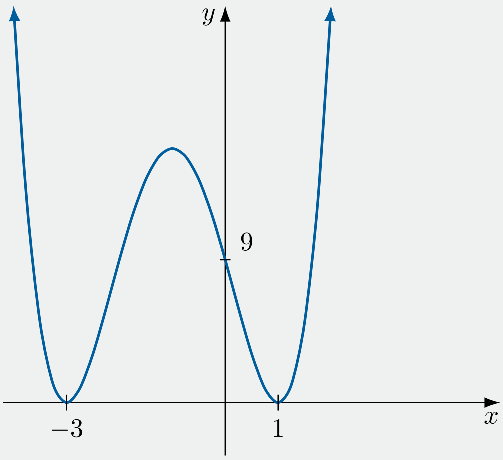
(e)



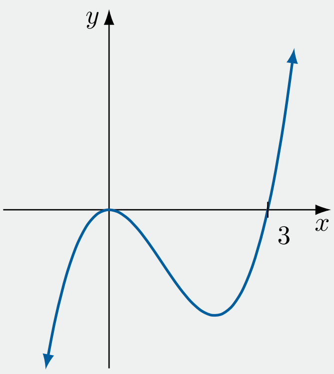
(f)



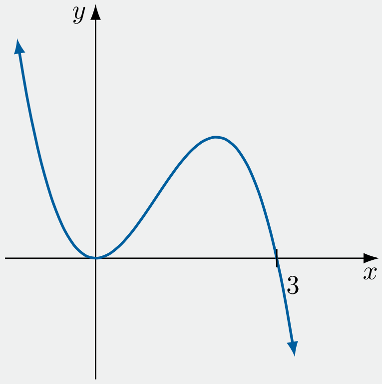
(g)



(h)

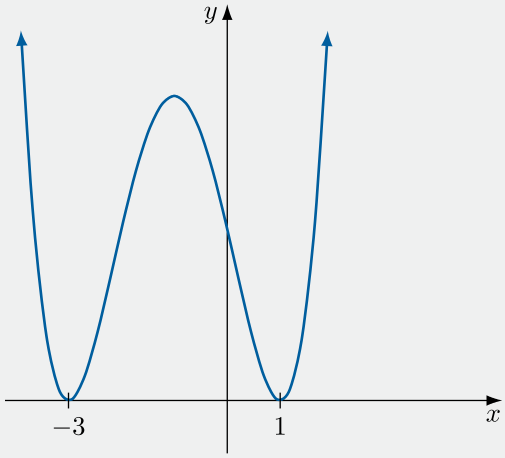


(i)

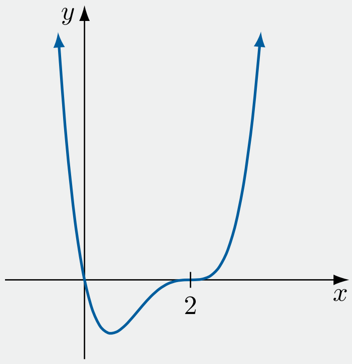


Q7

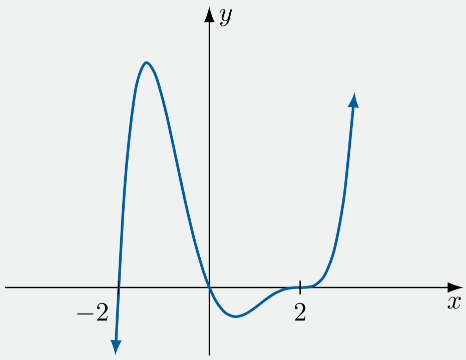
(a)



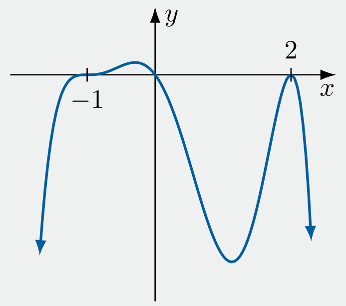
(b)



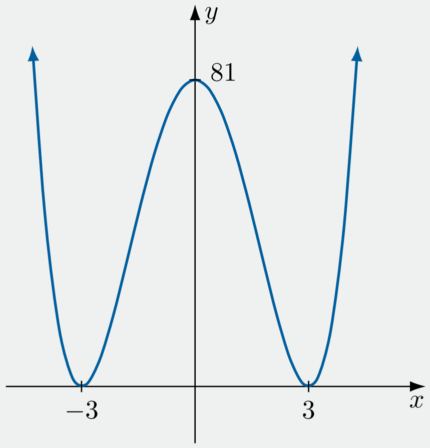
(c)



(d)



P1



P2

(a) False. (b) False. (c) True.

P3

(a) See full worked solutions.

(b) See full worked solutions.

P4

(a) See full worked solutions.

(b) See full worked solutions.

P5

(a) $P(x) = (x-1)(x+1)(x-3)(x+3)$

(b) $P(x) = x^3 - 4x$

Exercise 4B

Absolute value functions

F1

(a) size, origin

(b) $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

(c) a

(d) $|f(x)| = \begin{cases} f(x), & \text{if } x \geq 0 \\ -f(x), & \text{if } x < 0 \end{cases}$

(e) $|x|$

(f) x^2

(g) $\pm k$

F2

(a) positive (b) single, cusps

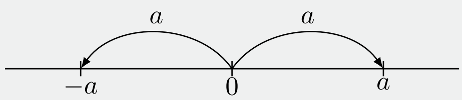
F3

(a) $|x|$ (b) $|y-x|$

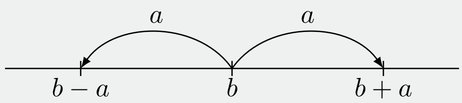
(c) $|x|\,|y|$ (d) $\dfrac{|x|}{|y|}$, 0

F4

(a) The distance from the origin is a



(b) The distance from b is a



Q1

(a) 5 (b) 3 (c) 8

Q2

(a) $|-3| = 3$, $|3| = 3$

(b) $|-15| = 15$, $|3|\,|-5| = 15$

(c) $\left|\dfrac{3}{-5}\right| = \dfrac{3}{5}$, $\dfrac{|3|}{|-5|} = \dfrac{3}{5}$

(d) $|3+5| = 8$, $|-5-3| = 8$

Q3

(a) 2, 2, −2

(b) 2, 5, 7, −3

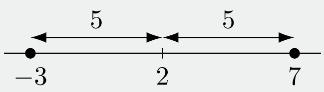
(c) −1, 4, 3, −5

(d) 2, $\left|x+\frac{1}{2}\right| = \frac{5}{2}$, $-\frac{1}{2}$, $\frac{5}{2}$, 2, −3

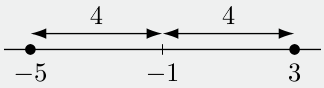
(e) $2x - 1$

Q4

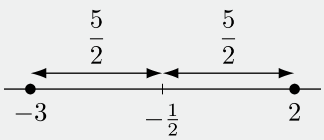
(b)



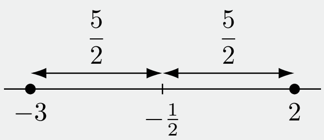
(c)



(d)



(e)



Q5

(a) $x = \pm 6$

(b) $x = \pm 3$

(c) $x = \pm 2$

(d) $x = -2, -6$

(e) $x = 12$ or $x = -2$

(f) $x = 1$, or $x = -2$

(g) $x = 4$ or $x = -1$

(h) $x = -7$ or $x = \frac{11}{3}$

Q6

(a) $x = \pm 2$

(b) $x - 2 = \pm 3$

(c) $2x + 1 = \pm 5$

(d) $|1 - 2x| = |2x - 1|$

Q7

(a) $x = \pm 4$

(b) $x = \pm 2$

(c) $x = 3$, $x = -7$

(d) $x = 12$, $x = 6$

(e) $x = 8$, $x = 2$

(f) $x = 6$, $x = -7$

(g) $x = 7$, $x = -4$

(h) $x = 3$, $x = -7$

(i) $x = 5$, $x = -10$

Q8

$(2x - 6)$, $2x - 6$ or $-2x + 6$

$\frac{11}{3}$

$\frac{11}{3}$, $\left|-\frac{4}{3}\right|$, $\frac{4}{3}$, is valid

0, $x \geq 3$

$x = \frac{11}{3}$, $\frac{11}{3}$

Q9

(a) No solution

(b) No solution

(c) $x = -\frac{3}{5}$, $x = 5$

(d) $x = 1$, $x = -2$

Q10

(a) neither

(b) odd

(c) neither

(d) even

Q11

(a) True

(b) False. Any $x > 1$ is a counter-example.

(c) True

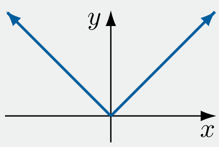
(d) True

(e) False. Any $x < 0$ is a counter-example.

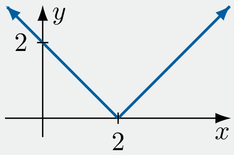
(f) False. Any $x < 0$ is a counter-example.

Q12

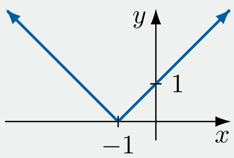
(a)



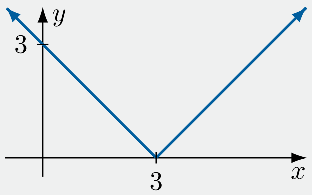
(b)



(c)

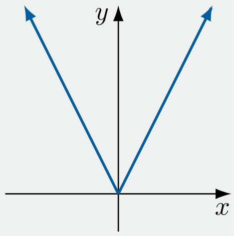


(d)

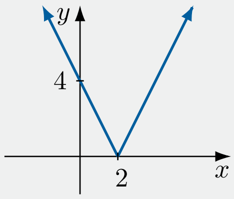


Q13

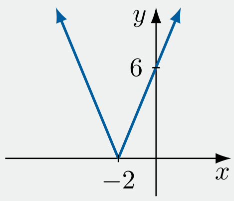
(a) $(0, 0)$, $m = \pm 2$



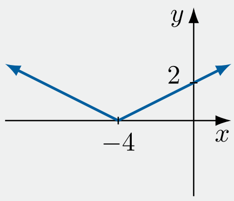
(b) $(2, 0)$, $m = \pm 2$



(c) $(-2, 0)$, $m = \pm 3$



(d) $(-4, 0)$, $m = \pm\frac{1}{2}$



Q14

(a) $y = \begin{cases} x - 4, & \text{if } x \geq 4 \\ -x + 4, & \text{if } x < 4 \end{cases}$

(b) $y = \begin{cases} x + 3, & \text{if } x \geq -3 \\ -x - 3, & \text{if } x < -3 \end{cases}$

(c) $y = \begin{cases} 2x - 3, & \text{if } x \geq \frac{3}{2} \\ -2x + 3, & \text{if } x < \frac{3}{2} \end{cases}$

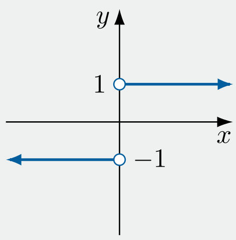
Q15

(a) $x = 0$

(b) $\frac{|x|}{x} = \begin{cases} 1, & \text{for } x \geq 0 \\ -1, & \text{for } x < 0 \end{cases}$

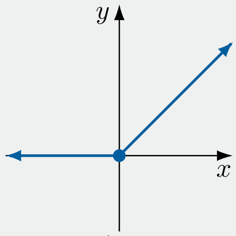
(c) -1, -1, undefined, 1, 1

(d)

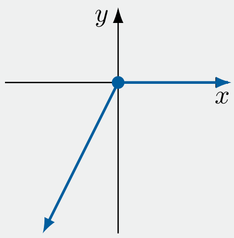


Q16

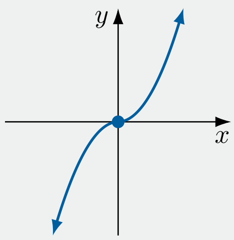
(a) $f(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$



(b) $f(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$

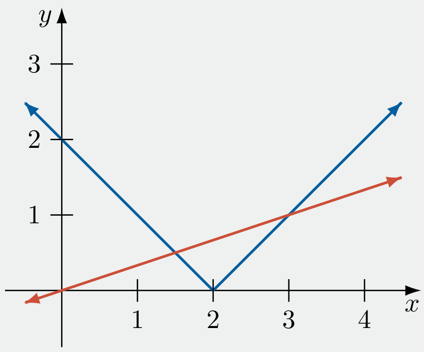


(c) $f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

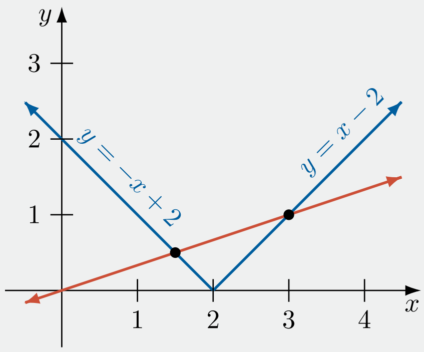


Q17

(a)



(b)



(c) They are the x-coordinates of the intersection points.

(d) $x = \frac{3}{2}, 3$

Q18

(a) $x = 1, -2$

(b) No solution.

(c) $x = 1$

Q19

(a) $f(x) = |2x|$

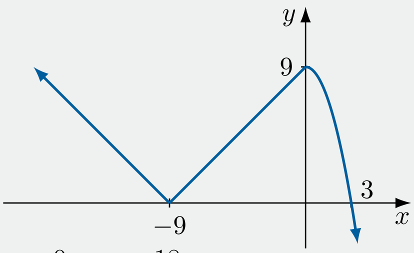
(b) $f(x) = |x - 1|$

(c) $f(x) = |2x + 4|$

(d) $f(x) = |3x - 1|$

Q20

(a)

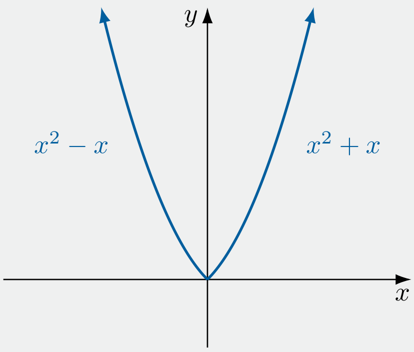


(b) $x = 0,\ x = -18$

(c) $x = -8,\ -10,\ \sqrt{8}$

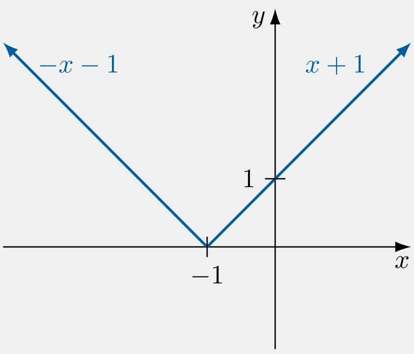
P1

$$f(x) = \begin{cases} x^2 + x, & \text{if} \quad x \geq 0 \\ x^2 - x, & \text{if} \quad x < 0 \end{cases}$$



P2

$$f(x) = \begin{cases} x + 1, & \text{if} \quad x \geq 1 \\ -x - 1, & \text{if} \quad x < 1 \end{cases}$$



P3

The left side must be positive, but the right side must be negative.

P4

$x = 0, 2$

P5

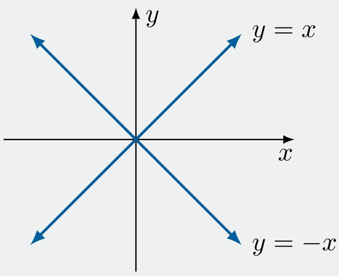
(a) x can be negative and that is invalid as $|2x - 1| \geq 0$ for all x

(b) No, as for all real x both LHS and RHS are positive.

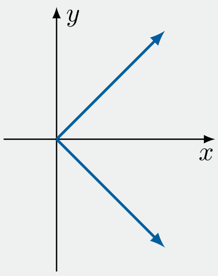
P6

No. All negative values of k do not work.

P7



P8



Exercise 4C

Inverse proportion and the hyperbola

F1

(a) $y = \frac{k}{x}$

(b) larger

(c) $y,\ x$

(d) constant

F2

(a) 2

(b) 0, 0

(c) $x \in \mathbb{R},\ x \neq 0$

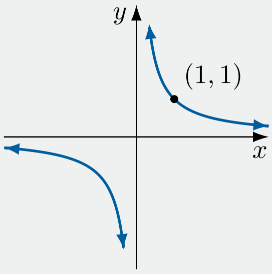
(d) $y \in \mathbb{R},\ y \neq 0$

(e) 1st, 3rd

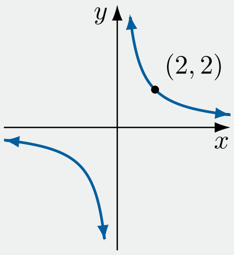
(f) 2nd, 4th

Q1

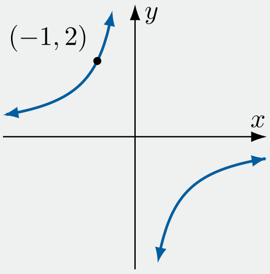
(a)



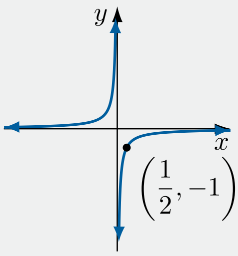
(b)



(c)



(d)



Q2

(a) $y = \frac{3}{x}$, $x = 0$ and $y = 0$.

(b) $y = -\frac{4}{x}$, $x = 0$ and $y = 0$.

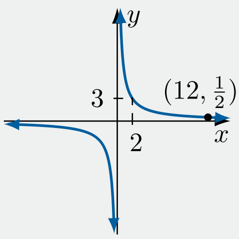
Q3

(a) $x = \frac{k}{y}$

(b) $k = 6$

(c) $y = \frac{1}{2}$

(d)



Q4

(a) Speed S

(b) Time T

(c) $k = 300$

(d) $2\frac{1}{2}$ hours

(e) 75 km/h

Q5

(a) $\frac{55}{3}$ cm^3

(b) $\frac{55}{16}$ km/cm^2

Q6

(a) See full worked solutions.

(b) y is halved.

(c)

(i) y doubles.

(ii) y divides by 3.

Q7

(a) The job to be completed has a fixed amount. The more people who do the job, the fewer days it (presumably) takes to complete. Conversely the fewer people working on the job, the longer it takes to complete. These are the characteristics of an inverse variation.

(b) $P = \frac{k}{D} = \frac{300}{D}$

(c) 30 days

(d) 4 people

Q8

(a) It ‘pulls’ the graph away from the origin.

(b) It ‘pushes’ the graph towards the origin.

(c) $k = 16$

(d) See full worked solutions.

P1

Food will last 12 days, so 8 days less.

P2

See full worked solutions.

Exercise 4D

Circles and semi-circles

F1

(a) $x^2 + y^2 = r^2$

(b) $(x - h)^2 + (y - k)^2 = r^2$

F2

(a) semi, $(0, 0)$, r

(b) upper, positive

(c) lower, negative, negative

Q1

(a) $x^2 + y^2 = 4$

(b) $(x - 3)^2 + (y + 2)^2 = 4$

(c) $(x + 2)^2 + (y - 1)^2 = 3$

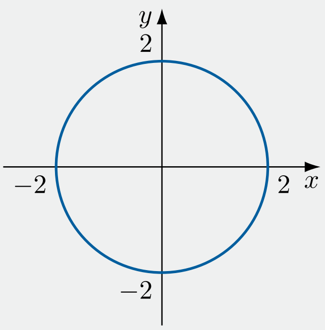
(d) $(x + 1)^2 + (y - 1)^2 = \frac{1}{16}$

(e) $(x - h)^2 + (y - k)^2 = r^2$

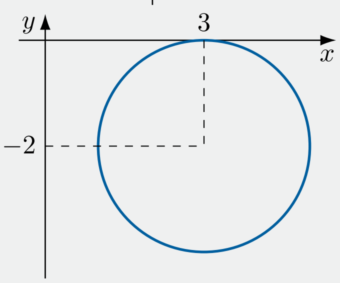
(f) $(x + p)^2 + (y + q)^2 = 4r^2$

Q2

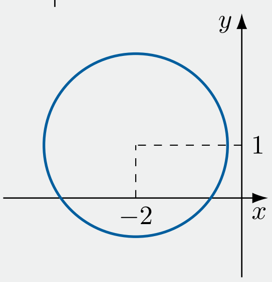
(a)



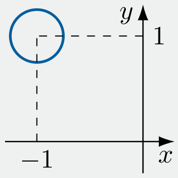
(b)



(c)



(d)



Q3

(a) $-2 \le x \le 2,\ -2 \le y \le 2$

(b) $1 \le x \le 5,\ -4 \le y \le 0$

(c) $-2 - \sqrt{3} \le x \le -2 + \sqrt{3}$,
$1 + \sqrt{3} \le y \le 1 - \sqrt{3}$

(d) $-\frac{5}{4} \le x \le -\frac{3}{4},\ \frac{3}{4} \le y \le \frac{5}{4}$

Q4

(a) $(-5, 0),\ r = 5,\ -10 \le x \le 0,\ -5 \le y \le 5$

(b) $(4, -3),\ r = 6,\ -2 \le x \le 10,\ -9 \le y \le 3$

(c) $\left(-3, \frac{1}{2}\right),\ r = 4,\ -7 \le x \le 1,\ -\frac{7}{2} \le y \le \frac{9}{2}$

Q5

(a) $(-1, 0),\ r = 3,\ (x + 1)^2 + y^2 = 9$

(a) $(-1, 0),\ r = 3,\ (x + 1)^2 + y^2 = 9$

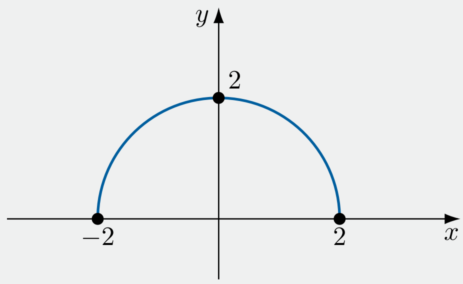
(b) $(2, -3),\ r = 2,\ (x - 2)^2 + (y + 3)^2 = 4$

(c) $(-1, 1),\ r = 1,\ (x + 1)^2 + (y - 1)^2 = 1$

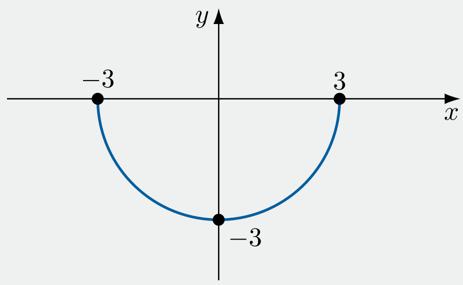
(d) $(1, 1),\ r = \sqrt{2},\ (x - 1)^2 + (y - 1)^2 = 2$

Q6

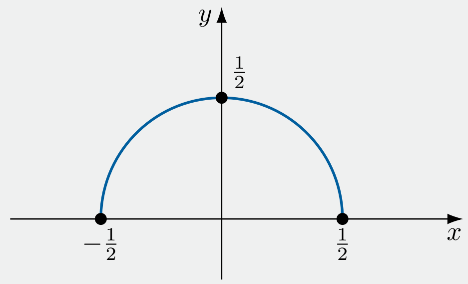
(a)



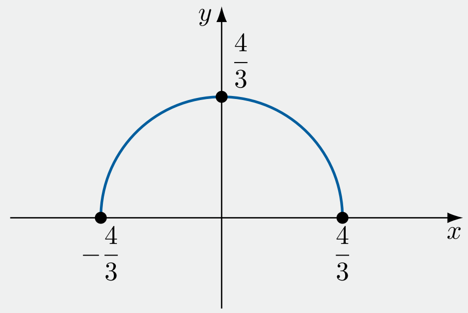
(b)



(c)



(d)



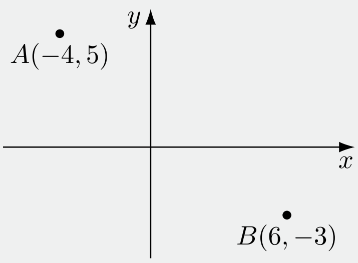
Q7

(a) $y = \sqrt{9 - x^2}$ (b) $y = -\sqrt{3 - x^2}$

(c) $y = \sqrt{\frac{1}{25} - x^2}$ (d) $y = -\sqrt{\frac{9}{4} - x^2}$

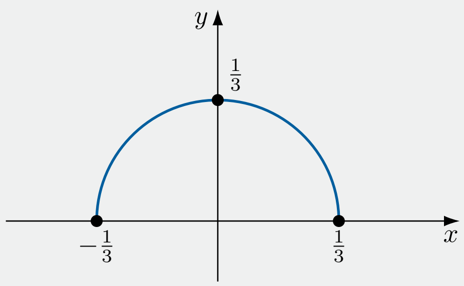
Q8

(a)



(b) $(x - 1)^2 + (y - 1)^2 = \sqrt{41}$

P1



P2

(a) $P(40, 20)$, $r = 50$

(b)

(i) Yes (ii) No

P3

(a) $(0, 0)$, $r = \dfrac{a}{\sqrt{2}}$

(b) It is a rectangular hyperbola with equation $xy = \dfrac{a^2}{4}$

(c) See full worked solutions.

(d) $\left(\frac{a}{2}, \frac{a}{2}\right)$ and $\left(-\frac{a}{2}, -\frac{a}{2}\right)$

(e) See full worked solutions.

Exercise 4E

Reflections

F1

(a) $y = f(-x)$ (b) $-x$

(c) negative (d) positive

(e) y

F2

(a) $y = -f(x)$ (b) $-y$

(c) -1 (d) negative

(e) positive (f) x

F3

(a) First $y = -f(x)$, then $y = -f(-x)$

(b) First $y = f(-x)$, then $y = -f(-x)$

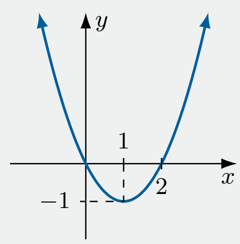
Q1

(a) $Q(a, -b)$ (b) $R(-a, b)$

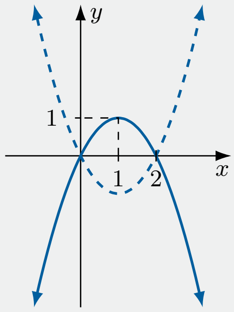
(c) $S(-a, -b)$ (d) It is $S(-a, -b)$

Q2

(a)



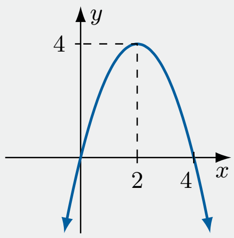
(b)



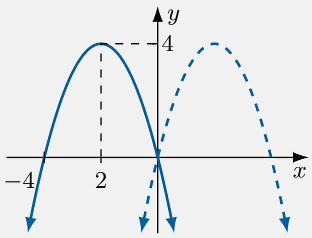
(c) Reflections across the x-axis

Q3

(a)



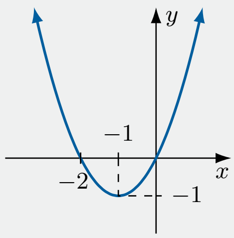
(b)



(c) Reflections across the y-axis

Q4

(a)



(b) $g(x) = x^2 - 2x$

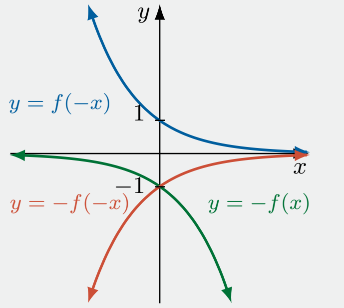
(c) $y = -g(x) = 2x - x^2$

(d) It is the same equation as the equation from (c).

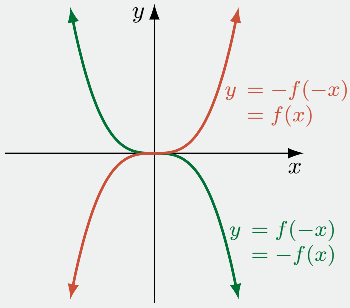
(e) $y = -f(-x)$ is the result of reflecting $y = f(x)$ across the y-axis and x-axis (order does not matter).

Q5

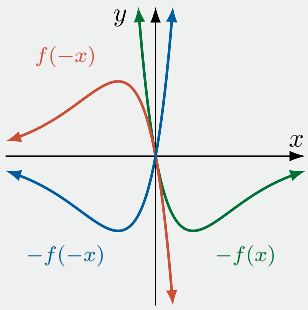
(a)



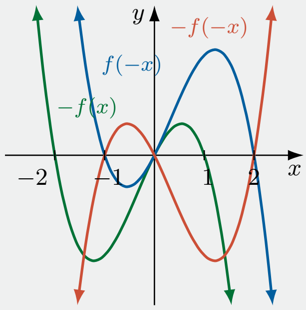
(b)



(c)



(d)



Q6

(a)

(i) $-f(x) = -x^2 + 4x$

(ii) $f(-x) = x^2 + 4x$

(iii) $-f(-x) = -x^2 - 4x$

(b)

(i) $-g(x) = x^2 - 4$

(ii) $g(-x) = 4 - x^2$

(iii) $-g(-x) = x^2 - 4$

(c)

(i) $-h(x) = -x^3$

(ii) $h(-x) = -x^3$

(iii) $-h(-x) = x^3$

Q7

(a) $y = f(-x)$ (b) $y = -f(x)$

(c) $y = -f(-x)$

P1

(a) $y = -|x-2|$

(b) $y = -|-x-2| = -|-(x+2)| = -|x+2|$

(c) See full worked solutions.

P2

(b)

P3

See full worked solutions.

Chapter Review

R1

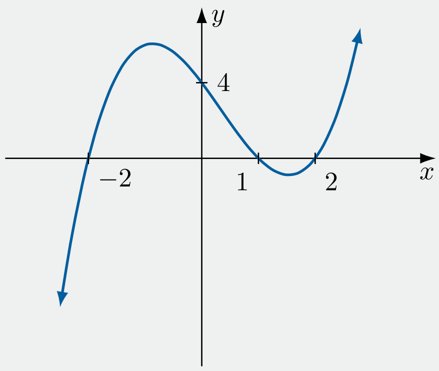
(a) $P(x) = x(x^2-9)$

(b) $P(x) = x^3(x-2)(x+2)$

(c) $P(x) = x^2(x-2)^2$

R2

(a)



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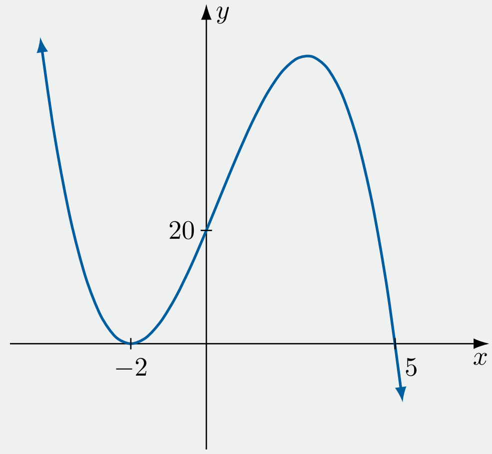
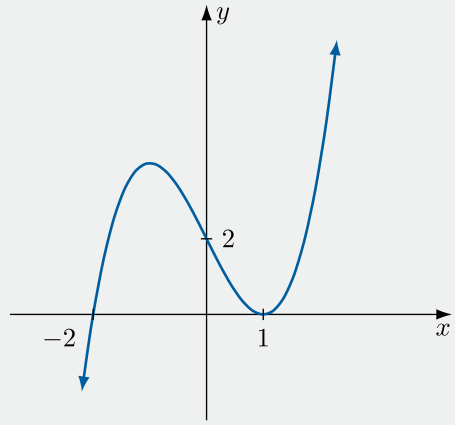
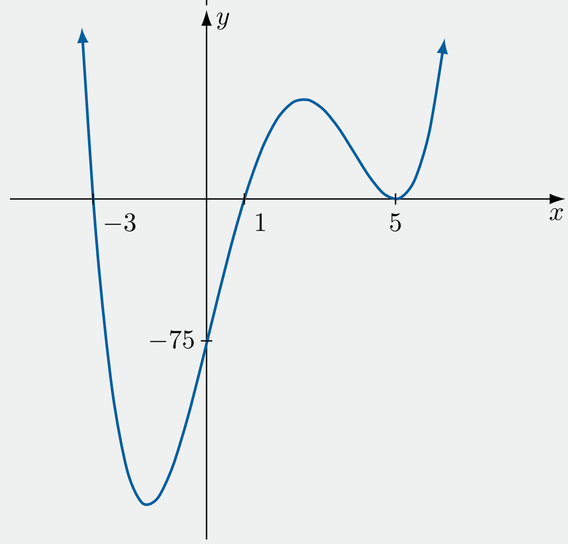


(c)



(d)



R3

(a) $y = x(x-4)(x+3)$

(b) $y = x(x+3)^2$

(c) $y = x(3x-2)(x+4)$

R4

(a) $y = (x-1)(x-3)(x+2)$

(b) $y = (x+3)(x-4)^2$

R5

(a) $k = 27$ (b) $k = 8$

R6

(a) $y = \frac{1}{12}(x+4)(x+2)(x-3)(x-5)$

(b) $y = \frac{1}{16}(1-x)(x+3)(x-4)^2$

(c) $y = \frac{1}{5}(x-1)(x-5)^3$

(d) $y = \frac{1}{4}(x+2)^2(x-3)^2$

R7

(a) $x = -2, -8$ (b) $x = 1, 6$

(c) $x = 1, 3$ (d) No solution

(e) $x = 1,\ x = -1$ (f) $x = 1,\ x = -1$

(g) $x = -1,\ x = 7$ (h) $x = 1\frac{1}{5},\ x = \frac{4}{5}$

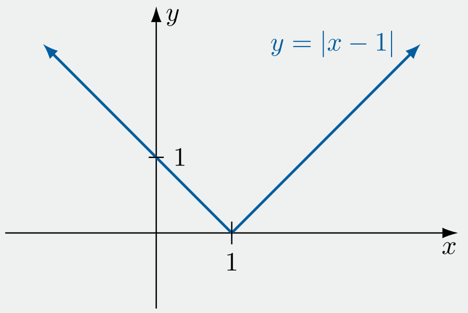
R8

(a) Absolute values cannot output negative numbers.

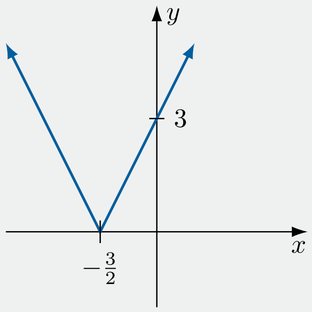
(b) See full worked solutions.

R9

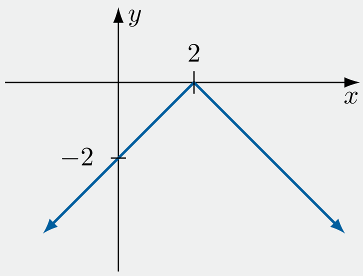
(a)



(b)



(c)



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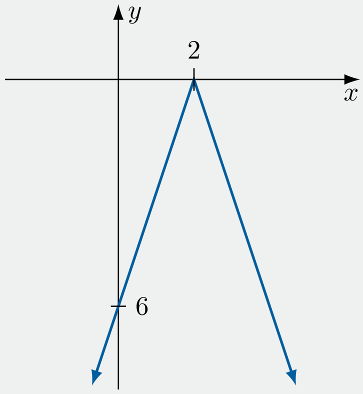
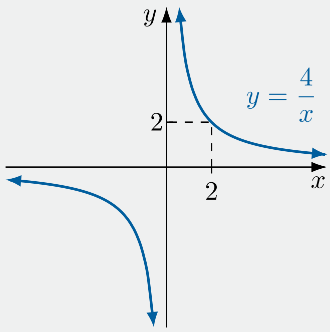
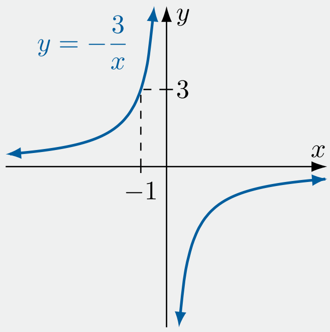


R10

(a)



(b)



R11

(a) $y = \frac{k}{x}$ (b) $k = 6$ (c) $y = 1$

R12

(a) $y = \frac{2}{x}$ (b) Halved (c) Tripled

(d) Doubled (e) Yes

R13

(a) 18 workers

(b) 2 hours and 24 minutes

R14

(a) $(5, 0)$, $r = 5$ (b) $(2, 3)$, $r = 4$

R15

(a) Domain: $x \in [0, 10]$, Range: $y \in [-5, 5]$

(b) Domain: $x \in [-2, 6]$, Range: $y \in [-1, 7]$

R16

(a) $(x-3)^2 + (y-4)^2 = 9$, $(0, 4)$

(b) $(x-3)^2 + (y+3)^2 = 18$, $(0, 0)$, $(6, 0)$, $(0, -6)$

R17

$(x \pm 2)^2 + (y \pm 2)^2 = 8$

R18

(a) y-axis: $f(-x) = -x^3$

x-axis: $-f(x) = -x^3$

Both axes: $-f(-x) = x^3$

(b) y-axis: $f(-x) = -(x-2)^2$

x-axis: $-f(x) = (x+2)^2$

Both axes: $-f(-x) = -(x+2)^2$

(c) y-axis: $f(-x) = -\dfrac{1}{x}$

x-axis: $-f(x) = -\dfrac{1}{x}$

Both axes: $-f(-x) = \dfrac{1}{x}$

(d) y-axis: $f(-x) = -x^2 + x + 2$

x-axis: $-f(x) = x^2 + x - 2$

Both axes: $-f(-x) = -x^2 - x + 2$

(e) y-axis: $f(-x) = -2^x$

x-axis: $-f(x) = 2^{-x}$

Both axes: $-f(-x) = -2^{-x}$

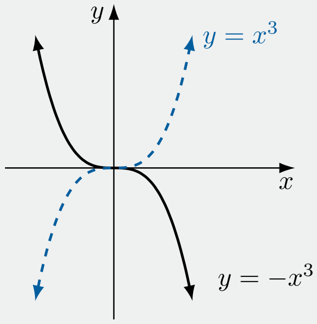
(f) y-axis: $f(-x) = -x^3 + x$

x-axis: $-f(x) = x^3 - x$

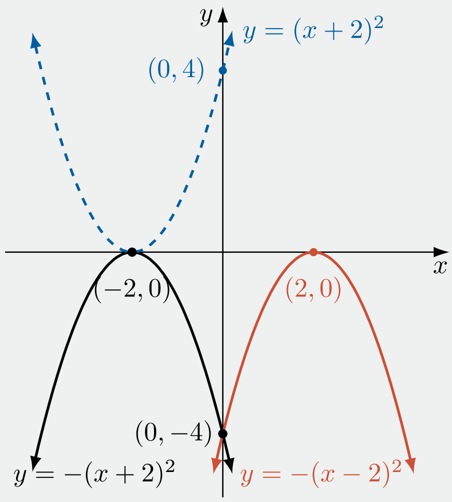
Both axes: $-f(-x) = -x^3 + x$

R19

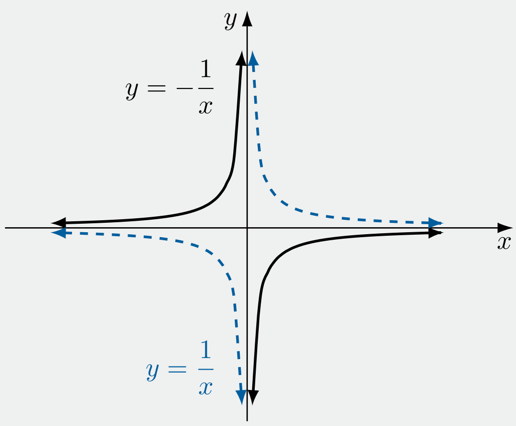
(a)



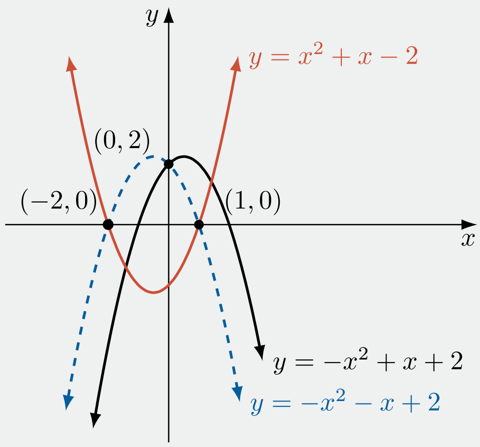
(b)



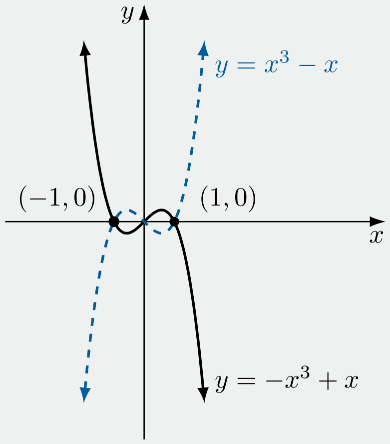
(c)



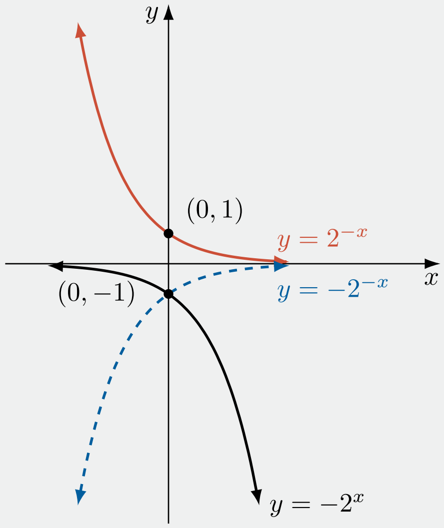
(d)



(e)



(f)



5. Trigonometry

Exercise 5A

Right-angled triangles

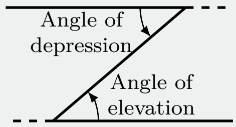
F1

(a) $a^2 + b^2 = c^2$

(b) The acute angle between the horizontal and the direct line-of-sight to a point above the horizontal.

(c) The acute angle between the horizontal and the direct line-of-sight to a point below the horizontal.

(d)



F2

(a) $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$

(b) $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

(c) $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

F3

(a) $\sin\theta$, 0 (b) $\cos\theta \neq 0$

(c) $\tan\theta$, $\tan\theta \neq 0$

F4

(a) Start with either north N or south S, then move either clockwise or anti-clockwise in an angle less than 90° towards either east or west respectively.

(b) True bearing is always measured in degrees from the North axis in a clockwise direction, and are written with three digits to specify the direction.

Q1

(a) 13 (b) 15

Q2

(a) 5 (b) 25 (c) 41

Q3

(a) 0.64 (b) 0.26 (c) 0.41

(d) 1.57 (e) 1.63 (f) 3.12

Q4

(a) 5°44′ (b) 43°7′ (c) 18°26′

(d) 75°31′ (e) 45°35′ (f) 74°53′

Q5

(a) 2.3 (b) 2.5 (c) 1.3

Q6

(a) 20 m (b) 4 m

Q7

(a) 6.93 m (b) 4 m

Q8

(a) 5

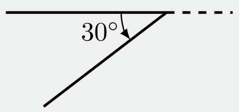
(b)

(i) $\frac{4}{5}$ (ii) $\frac{3}{5}$ (iii) $\frac{3}{4}$

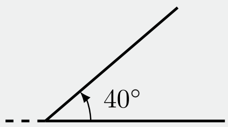
(iv) $\frac{4}{3}$ (v) $\frac{5}{4}$ (vi) $\frac{5}{4}$

Q9

(a)



(b)



Q10

1371.6 m

Q11

(a) 21 m (b) 15 m

Q12

189.3 m

Q13

(a) 18.47 m (b) 10.66 m

Q14

(a) $S60°E$, $120°T$ (b) $S30°W$, $210°T$

Q15

(a) $S50°E$, $130°T$ (b) $N50°W$, $310°T$

Q16

1600 m

Q17

$23°T$

Q18

(a) 6 km (b) 16.4 km

Q19

20 km

P1

(a) 90° (b) 500 km

(c) 37° (d) 118°, 298°

P2

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

Exercise 5B

Exact values

F1

(a) $\sqrt{3}$ and $\sqrt{2}$

(b)

	30°	60°	45°
sin θ	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tan θ	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

Q1

	30°	60°	45°
cosec θ	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$
sec θ	$\frac{2}{\sqrt{3}}$	2	$\sqrt{2}$
cot θ	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	1

Q2

(a) $\sqrt{2}$ (b) $\frac{1}{2}$ (c) 1

(d) 1 (e) $\frac{1}{2}$ (f) 1

Q3

(a) See full worked solutions.

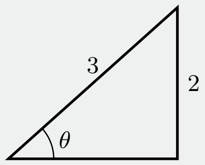
(b) See full worked solutions.

(c) See full worked solutions.

(d) See full worked solutions.

Q4

(a)



(b) $\sqrt{5}$

(c)

(i) $\dfrac{\sqrt{5}}{3}$ (ii) $\dfrac{2}{\sqrt{5}}$

Q5

(a) $\dfrac{3}{4}$ (b) $\dfrac{3}{\sqrt{13}}$ (c) $\dfrac{\sqrt{2}}{\sqrt{3}}$ (d) $\dfrac{13}{5}$

Q6

(a) $2\sqrt{3}$ (b) $2\sqrt{3}$

(c) $9 - 3\sqrt{3}$ (d) $20\sqrt{3} - 20$

Q7

(a) See full worked solutions.

(b) $QR = \sqrt{3}$, $QS = 2$ and $PQ = 2$

(c) See full worked solutions.

P1

See full worked solutions.

P2

(a) See full worked solutions.

(b) $2 - \sqrt{3}$

(c) See full worked solutions.

Exercise 5C

Quadrants and related angles

F1

(a) anti-clockwise (b) clockwise

F2

(a) 1 and 2 (b) 1 and 4 (c) 1 and 3

F3

(a) supplementary

(b) magnitude, quadrant

F4

(a) complementary (b) complementary

(c) cosine (d) cot

Q1

(a) 0 (b) 1 (c) 1 (d) 0

(e) -1 (f) 0 (g) 0 (h) -1

(i) 0 (j) -1 (k) -1 (l) 0

(m) 1 (n) 0 (o) 0 (p) 1

Q2

(a) $\sin\theta$ (b) $-\cos\theta$ (c) $-\tan\theta$

(d) $-\sin\theta$ (e) $-\cos\theta$ (f) $\tan\theta$

(g) $-\sin\theta$ (h) $\cos\theta$ (i) $-\tan\theta$

(j) $-\sin\theta$ (k) $\cos\theta$ (l) $-\tan\theta$

Q3

(a) $\cos x$ (b) $\sin x$ (c) $\cot x$

Q4

(a) 26° (b) 30° (c) 45° (d) 80°

Q5

(a) 1 (b) 1 (c) 1 (d) $\cos x$

Q6

(a) $2\tan 20°$ or $2\cot 70°$

(b) 1 (c) 0

(d) 2 (e) $\dfrac{3}{5}$ (f) $\dfrac{5}{7}$

Q7

(a) $\dfrac{1}{2}$ (b) 1 (c) 1 (d) $-\dfrac{1}{\sqrt{2}}$

(e) $\dfrac{1}{2}$ (f) $\dfrac{\sqrt{3}}{2}$ (g) $-\dfrac{1}{2}$ (h) $\dfrac{\sqrt{3}}{2}$

(i) $-\dfrac{\sqrt{3}}{2}$ (j) -1 (k) $\dfrac{1}{\sqrt{3}}$ (l) $-\dfrac{\sqrt{3}}{2}$

Q8

(a) $\sin x$ (b) $\sin x$

(c) $\cos x$ (d) $\cos x$

Q9

(a) $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (b) $Q\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

(c) $R\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ (d) $S\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Q10

(a)

(i) $\frac{2\sqrt{2}}{3}$ (ii) $-\frac{2\sqrt{2}}{3}$

(iii) $-\frac{\sqrt{2}}{4}$ (iv) $\frac{\sqrt{2}}{4}$

(b)

(i) $-\frac{1}{\sqrt{2}}$ (ii) $135°$

(iii) $\frac{1}{\sqrt{2}}$ (iv) $-\frac{1}{\sqrt{2}}$

Q11

(a) $-\frac{\sqrt{3}}{2}$ (b) $-\sqrt{3}$ (c) $\sqrt{3}$

(d) $\frac{1}{\sqrt{2}}$ (e) $-\frac{1}{2}$ (f) $\frac{1}{\sqrt{2}}$

Q12

(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\frac{1}{2}$

(d) $\sqrt{3}$ (e) $\frac{1}{2}$ (f) $-\frac{1}{2}$

Q13

$P(\cos 55°, -\sin 55°)$
$Q(\cos 80°, \sin 80°)$
$R(-\cos 20°, -\sin 20°)$

Q14

See full worked solutions.

Q15

(a) $-\tan\beta$ (b) -1 (c) 1

(d) $-\sin^2\beta$ (e) $-\sin\beta$ (f) $\sin\beta$

Q16

See full worked solutions.

Q17

(a) True (b) True (c) True

(d) False (e) False (f) True

P1

(a) $-\frac{4}{5}, -\frac{3}{4}$ (b) $\frac{3}{4}$ (c) $-\frac{12}{5}$

P2

See full worked solutions.

Exercise 5D
Trigonometric identities

F1

(a) 1 (b) $\tan^2\theta$ (c) $\operatorname{cosec}^2\theta$

F2

(a) $1-\cos^2\theta$ (b) $1-\sin^2\theta$

(c) $\sec^2\theta - 1$ (d) $1+\tan^2\theta$

(e) $1+\cot^2\theta$ (f) $\operatorname{cosec}^2\theta - 1$

Q1

(a) 1

(b) $\frac{1}{\sin^2\theta}$, $\cot^2\theta$, $\operatorname{cosec}^2\theta$

(c) $\frac{1}{\cos^2\theta}$, $\tan^2\theta$, $\sec^2\theta$

Q2

$\sec^2\theta$

Q3

(a) 1 (b) -1

(c) -1 (d) $\sin^4\theta$

(e) $\sin^2\theta - \cos^2\theta$ (f) $1+2\sin\theta\cos\theta$

(g) 2 (h) 1

Q4

See full worked solutions.

P1

See full worked solutions.

P2

(a) $4x^2 - y^2 = 4$ (b) $x^2 - y^2 = 9$

ANSWERS

P3

See full worked solutions.

Exercise 5E

Solving trigonometric equations with degrees

F1

(a) arrange

(b) positive, negative, quadrant

(c) acute

(d) 180, 360

(e) $360°$, $180°$

(f) cosine, -1, range, $[-1, 1]$

F2

(a) modify (b) $[0°, 720°]$

(c) $[0°, 180°]$ (d) θ

F3

(a) identity, square

(b) $1 - \cos^2 \theta$, $\sec^2 \theta - 1$

Q1

(a) $30°$

(b) 2^{nd} Quadrant

(c) $30°$, $150°$

Q2

(a) $60°$, $300°$ (b) $225°$, $315°$

(c) $60°$, $240°$ (d) $240°$, $300°$

(e) $135°$, $315°$ (f) $150°$, $210°$

(g) $45°$, $315°$ (h) $210°$, $330°$

(i) $120°$, $300°$

Q3

(a) $19°28'$, $160°32'$ (b) $134°26'$, $225°34'$

(c) $20°3'$, $200°3'$

Q4

(a) $66°25'$, $293°35'$ (b) $228°35'$, $311°25'$

(c) $68°12'$, $248°12'$

Q5

(a) $60°$, $120°$

(b) $-135°$, $135°$

(c) $-240°$, $-60°$, $120°$, $300°$

(d) $-60°$, $60°$

Q6

(a) $90°$, $270°$

(b) $30°$, $150°$, $210°$, $330°$

(c) $60°$, $120°$, $240°$, $300°$

(d) $45°$, $135°$, $225°$, $315°$

(e) $30°$, $150°$, $210°$, $330°$

Q7

(a) $30°$, $150°$, $270°$

(b) $30°$, $90°$, $150°$

(c) $180°$

(d) $0°$, $120°$, $240°$, $360°$

Q8

(a) $u \in [0°, 720°]$

(b) $\sin u = \dfrac{1}{2}$

(c) $u = 30°, 150°, 390°, 510°$

(d) $2\theta = 30°, 150°, 390°, 510°$
$\theta = 15°, 75°, 195°, 255°$

Q9

(a) $22.5°$, $112.5°$, $202.5°$, $292.5°$

(b) $60°$, $120°$, $240°$, $300°$

(c) No solution.

(d) $15°$, $105°$, $135°$, $225°$, $255°$, $345°$

Q10

(a) $270°$, $330°$ (b) $75°$, $255°$

(c) $60°$, $180°$ (d) $90°$, $270°$

P1

(a) $75°$, $255°$

(b) $30°$, $60°$, $150°$, $180°$, $270°$, $300°$

P2

(a) 90°

(b) 0°, 45°, 180°, 225°, 360°

(c) 120°, 180°, 240°

(d) 60°, 180°, 300°

P3

40°, 80°, 160°

P4

Squaring the equation introduces extra solutions.

Exercise 5F

Sine rule and the area formula

F1

(a) lengths, angles (b) two

(c) two

F2

(a) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(b) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

(c) $\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

F3

(a) It is when there are two possible triangles satisfying the same given dimensions.

(b) It can occur when the question involves using the sine rule to find an unknown angle.

(c) $180° - \theta$

(d) $180° - \theta$, 180°, reject

Q1

(a) $5\sqrt{2}$

(b) $\beta = 50°$, $x = 7.08$

Q2

(a) See full worked solutions.

(b) 41 m

Q3

(a) 24 m (b) 45 m

Q4

(a) $\angle OAT = 17°$, $AT = 102.61$ m

(b) $\angle ATB = 5°$, $\angle BAT = 73°$, $\angle ABT = 102°$

(c) 9.14 m

Q5

(a) $38°13'$, $141°47'$ (b) $17°27'$, $162°33'$

Q6

$AC = 6.2$ cm, $\angle BCA = 58°6'$, $\angle ABC = 88°54'$

$AC = 2.6$ cm, $\angle BCA = 121°54'$, $\angle ABC = 25°6'$

Q7

(a) $25\sqrt{3}$ cm^2 (b) 6 cm

Q8

31.51 cm^2

Q9

$\theta = 61°3', 118°57'$

P1

(a) $60\sqrt{3}$ (b) 15.2

P2

(a) $\dfrac{h}{\sin\beta}$

(b) $\angle BDA = \alpha - \beta$, $\angle ABD = 180° - \alpha$

(c) See full worked solutions.

P3

(a) $\text{Area} = \frac{1}{2}ab\sin C$ (b) b

(c) $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ (d) See full worked solutions.

Exercise 5G

Cosine rule

F1

(a) included, $c^2 = a^2 + b^2 - 2ab\cos C$

(b) three, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

F2

(a) longest (b) shortest

ANSWERS

Q1

4.58 cm, 7.81 cm

Q2

(a) See full worked solutions.

(b) $\dfrac{15\sqrt{3}}{4}$ cm^2

Q3

$20°46'$

Q4

8 cm

Q5

(a) See full worked solutions.

(b) 36 km

Q6

(a) Car A: 135 km, Car B: 150 km

(b) 198.27 km

(c) $97°7'$

Q7

(a) See full worked solutions.

(b) See full worked solutions.

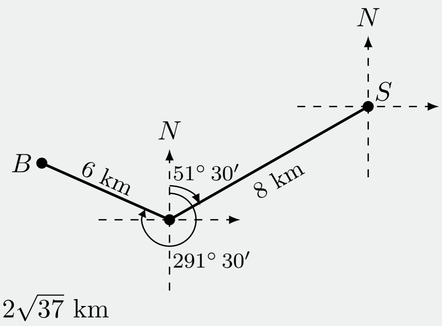
(c) $2\sqrt{3}$

(d) Right-angled triangle.

(e) 30°

Q8

(a)

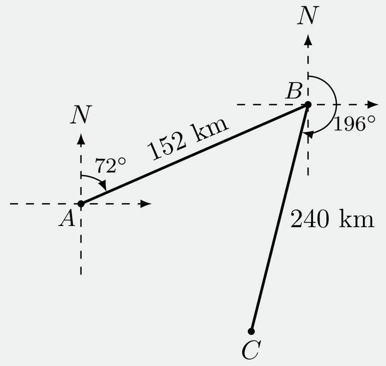


(b) $2\sqrt{37}$ km

(c) 077° T

Q9

(a)



(b) See full worked solutions.

(c) See full worked solutions.

(d) 337°

Q10

9

P1

(a) See full worked solutions.

(b) $3 - \sqrt{5}$

P2

You get Pythagoras' Theorem! This is not a surprise since $C = 90°$, so we have a right-angled triangle.

P3

(a) See full worked solutions.

(b) $\theta = 29°$

(c) $\theta = 60°$. If $c = x$ then the triangle is equilateral, so it is no surprise that $\theta = 60°$.

P4

(a) See full worked solutions.

(b) See full worked solutions.

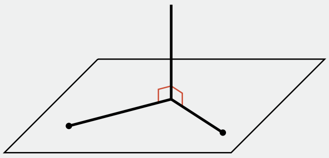
(c) The sum of two sides is strictly larger than the third side.

Exercise 5H

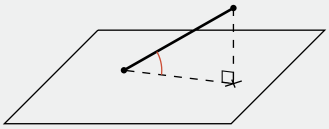
Three-dimensional trigonometry

F1

perpendicular



F2



Q1

(a) See full worked solutions.

(b) See full worked solutions.

(c) $35°16'$

Q2

$BC = 173.21$ m, $BD = 214.45$ m

Q3

(a) 103.92 m

(b) $BP = 182.48$ m, $\angle PBQ = 18°12'$

Q4

(a) 2.5 m (b) $38°41'$

Q5

(a) See full worked solutions.

(b) $h \tan 78°$

(c) $90°$

(d) 133 m

Q6

(a) $10\sqrt{2}$ cm (b) $\sqrt{89}$ cm (c) $62°5'$

(d) $58°$ (e) $48°32'$ (f) $82°57'$

Q7

(a) See full worked solutions.

(b) See full worked solutions.

(c) 10.86 m

Q8

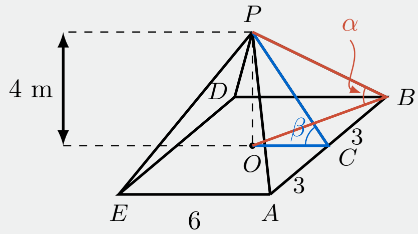
(a) $30\sqrt{3}$ m

(b)

(i) 150 m (ii) $19°6'$ (iii) $306°52'$

Q9

(a)



(b) See full worked solutions.

(c) See full worked solutions.

Q10

(a) See full worked solutions.

(b) 421.55 m

Q11

(a) See full worked solutions.

(b) $OB = h \tan 60°$, $OC = h \tan 58°$

(c) Right-Angled.

(d) $36°44'$

(e) $47°46'$ or $132°14'$

(f) $11°2'$

P1

See full worked solutions.

P2

(a) $h \cot \beta$

(b) See full worked solutions.

Chapter Review

R1

(a) 29 (b) 37 (c) 10

(d) 55 (e) 26 (f) 130

R2

15.14

R3

$\angle BAC = \angle BCA = 45°$

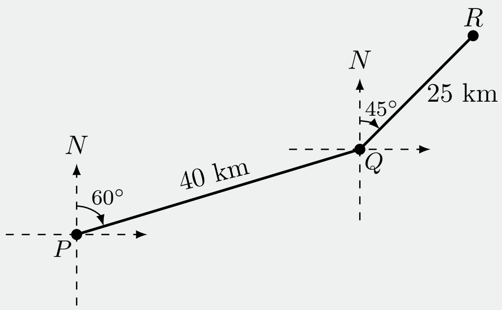
R4

81.63 m

R5

(a) 32.46 m (b) 31.75 m

R6



$\left(20\sqrt{3} + \frac{25}{2}\sqrt{2}\right)$ km

R7

See full worked solutions.

R8

(a) $60°$ (b) $24°$, $66°$

R9

9

R10

(a) See full worked solutions.

(b) 4217.83 cm^2

R11

$\frac{19}{21}$

R12

5 cm

R13

(a) $50°$ (b) $70°$ (c) 37 cm

R14

(a) See full worked solutions.

(b) $AB = \sqrt{350^2 + 500^2 - 2 \times 350 \times 500 \cos 20°}$

R15

162 m

R16

(a) The exterior angle of a triangle is equal to the sum of the interior opposite angles.

(b) See full worked solutions.

(c) See full worked solutions.

(d) See full worked solutions.

R17

4.8 m

R18

(a) $\cos\theta = \sqrt{1-x^2}$, $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

(b) $\frac{x}{\sqrt{4-x^2}}$

(c) See full worked solutions.

6. Radians

Exercise 6A

Radian measure

F1

(a) unit, 1 (b) 2

F2

(a) π, π, π (b) 2π

F3

(a) $\frac{180}{\pi}$ (b) $\frac{\pi}{180°}$

Q1

(a) $90°$ (b) $60°$ (c) $45°$ (d) $120°$

(e) $300°$ (f) $210°$ (g) $240°$ (h) $390°$

(i) $270°$ (j) $150°$ (k) $330°$ (l) $420°$

Q2

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(e) $\frac{5\pi}{3}$ (f) π (g) $\frac{7\pi}{6}$ (h) $\frac{3\pi}{4}$

(i) $\frac{11\pi}{6}$ (j) $\frac{2\pi}{3}$ (k) 2π (l) $\frac{5\pi}{6}$

(m) $\frac{3\pi}{2}$ (n) $\frac{7\pi}{4}$ (o) $\frac{4\pi}{3}$ (p) $\frac{5\pi}{4}$

Q3

(a) 36° (b) 18° (c) 51°

(d) 140° (e) 86° (f) 15°

Q4

(a) $\dfrac{\pi}{5}$ (b) $\dfrac{2\pi}{5}$ (c) $\dfrac{3\pi}{5}$

(d) $\dfrac{5\pi}{9}$ (e) $\dfrac{10\pi}{9}$ (f) $\dfrac{61\pi}{36}$

Q5

$\boldsymbol{\theta}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
sin $\boldsymbol{\theta}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
cos $\boldsymbol{\theta}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
tan $\boldsymbol{\theta}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

Q6

$\boldsymbol{\theta}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
sin $\boldsymbol{\theta}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0
cos $\boldsymbol{\theta}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
tan $\boldsymbol{\theta}$	0	1	Undef.	-1	0	1	Undef.	-1	0

Q7

(a) $\sin\theta$ (b) $-\cos\theta$ (c) $-\tan\theta$

(d) $-\sin\theta$ (e) $-\cos\theta$ (f) $\tan\theta$

(g) $-\sin\theta$ (h) $\cos\theta$ (i) $-\tan\theta$

Q8

(a) 0.7 (b) −0.7 (c) −0.7

(d) −0.7 (e) 0.7 (f) −0.7

P1

(a) 1.4496 (b) −0.2794

(c) −0.4161 (d) 2.9093

P2

(a) False. (b) False. (c) True.

(d) True. (e) False. (f) True.

P3

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) They are all approximately equal.

(e) Both are approximately 1.

Exercise 6B

Arcs and sectors

F1

(a) $l = r\theta$ (b) $A = \dfrac{1}{2}r^2\theta$

F2

(a) radians (b) $\dfrac{\theta\pi}{180}$

Q1

Arc length (cm)	Angle (radians)	Angle (degrees)	Radius (cm)
$\frac{7\pi}{2}$	$\frac{\pi}{4}$	45°	14
10	$\frac{\pi}{6}$	30°	$\frac{60}{\pi}$
8	$\frac{2}{5}$	22°55′	20
9π	$\frac{3\pi}{4}$	135°	12

Q2

(a) 10 cm (b) 25 cm^2

Q3

(a) 4π cm (b) $12\pi\ \text{cm}^2$

Q4

(a) $\dfrac{5\pi}{2}$ cm (b) $\dfrac{15\pi}{4}\ \text{cm}^2$

Q5

(a) $\dfrac{4}{5}$ (b) 45°50′

Q6

4π cm

Q7

$3\sqrt{10}$ units

Q8

225π cm^2

Q9

(a) $25\sqrt{10}$ cm (b) $\frac{625\pi}{32}\pi$ radians

Q10

(a) $\frac{64}{5}\pi$ cm^2 (b) $\frac{32}{5}\pi$ cm

Q11

(a) 2.168 (b) 6.13 cm

Q12

(a) $\frac{4}{7}$ (b) 1400 cm^2

(c) 1325 cm^2 (d) 75 cm^2

Q13

(a) $r = 4$ cm and $\theta = \frac{\pi}{16}$

(b) $r = 3$ cm and $\theta = \frac{\pi}{15}$

Q14

(a) 1.176 (b) 11.76 cm (c) 19.66 cm

Q15

(a) 1.46 (b) 23 m

Q16

(a) $\frac{27}{4}\pi$ cm^2 (b) $\frac{9}{2}\pi + 6$ cm

P1

(a) 6 cm (b) 10 cm (c) 12π cm (d) $\frac{6\pi}{5}$

P2

282.7 metres per minute

P3

(a) Area of triangle is $\frac{1}{2}r^2 \sin\theta$.

Area of sector is $\frac{1}{2}r^2\theta$

(b) See full worked solutions.

P4

(a) See full worked solutions.

(b) See full worked solutions.

(c) See full worked solutions.

(d) See full worked solutions.

Exercise 6C

Graphing trigonometric functions

F1

(a) circular (b) periodic

(c) cycle, period (d) oscillate

(e) maximum, minimum, y (f) amplitude

(g) amplitude, 1

F2

(a) $y = \cot x$, reciprocal (b) odd

F3

(a) 360° (b) 360°

(c) 180° (d) 360°, 360°, 180°

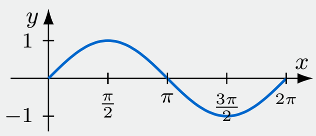
Q1

(a)

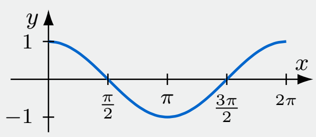
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin θ	0	1	0	−1	0
cos θ	1	0	−1	0	1

(b)

(i) $y = \sin x$



(ii) $y = \cos x$

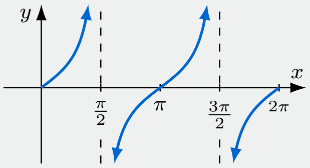


Q2

(a)

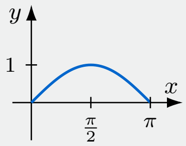
$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\tan\boldsymbol{\theta}$	0	Undef.	0	Undef.	0

(b)

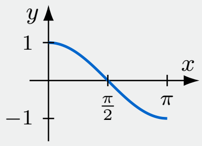


Q3

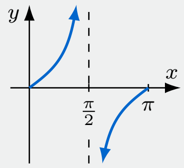
(a)



(b)

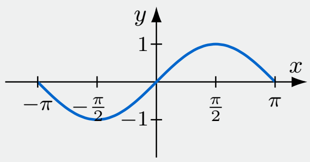


(c)

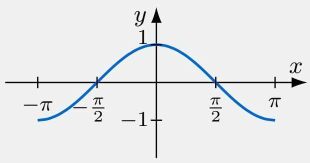


Q4

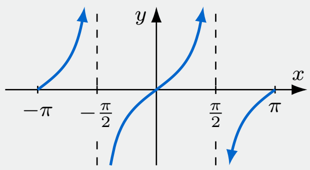
(a)



(b)

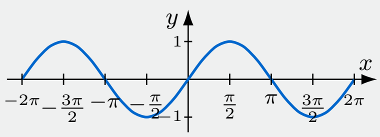


(c)

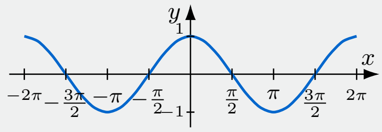


Q5

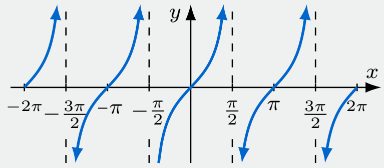
(a)



(b)



(c)



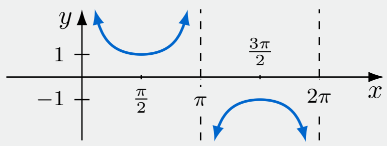
Q6

(a)

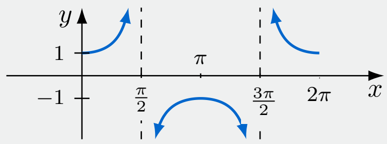
$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sec\boldsymbol{\theta}$	1	Undef.	-1	Undef.	1
$\operatorname{cosec}\boldsymbol{\theta}$	Undef.	1	Undef.	-1	Undef.

(b)

(i) $y = \operatorname{cosec} x$



(ii) $y = \sec x$

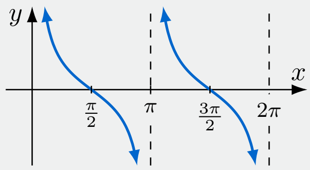


Q7

(a)

$\boldsymbol{\theta}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cot\boldsymbol{\theta}$	Undef.	0	Undef.	0	Undef.

(b)



Q8

(a) $\sec x = \dfrac{1}{\cos x}$

(b) $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

(c) They will be the locations of the vertical asymptotes.

(d) $y = \operatorname{cosec} x$ has asymptotes at $x = 0, \pi, 2\pi$, and $y = \cot x$ has asymptotes at $x = 0, \pi, 2\pi$.

Q9

(a) Odd. (b) $-\sin x$ (c)

- (i) even, $\cos(-x) = \cos(x)$
- (ii) odd, $\tan(-x) = -\tan(x)$
- (iii) odd, $\operatorname{cosec}(-x) = -\operatorname{cosec}(x)$
- (iv) even, $\sec(-x) = \sec(x)$
- (v) odd, $\cot(-x) = -\cot(x)$

(d) No, it does not.

Q10

(a) $y \in [0, 1]$ (b) $y \in [0, 1]$

(c) $y \in [0, \infty)$ (d) $y \in [1, \infty)$

(e) $y \in [1, \infty)$ (f) $y \in [0, \infty)$

Q11

(a) $y \in [-1, 1]$

(b) $y \in [-1, 1]$

(c) $y \in (-\infty, \infty)$

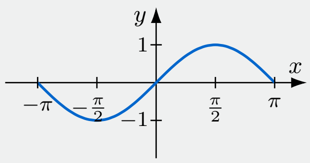
(d) $y \in \{(-\infty, -1] \cup [1, \infty)\}$

(e) $y \in \{(-\infty, -1] \cup [1, \infty)\}$

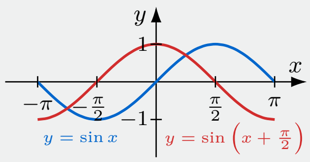
(f) $y \in (-\infty, \infty)$

P1

(a)



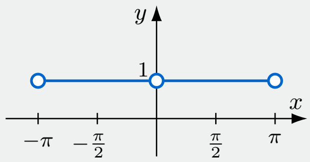
(b)



(c) It is the graph of $y = \cos x$.

(d) $\cos x$

P2



P3

$y = \cot x$ is defined at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ whereas $y = \frac{1}{\tan x}$ is not defined at those points.

Exercise 6D

Solving trigonometric equations with radians

F1

(a) $[0, 2\pi]$, 2π (b) 2π, π

F2

(a) $\cos x$ (b) $\sin x$ (c) $\cot x$

Q1

(a) $\theta = 0, \pm\pi, \pm 2\pi$ (b) $\theta = -\frac{3\pi}{2}, \frac{\pi}{2}$

(c) $\theta = \pm\pi$ (d) $\theta = 0, \pm\pi, \pm 2\pi$

Q2

(a) See full worked solutions.

(b) $x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

Q3

See full worked solutions.

Q4

(a) $x = \frac{3\pi}{2}$ (b) $x = \frac{\pi}{4}, \frac{7\pi}{4}$

(c) $x = 0, 2\pi$ (d) $x = \frac{5\pi}{6}, \frac{11\pi}{6}$

(e) $x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (f) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

Q5

(a) $x = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

(b) $x = \pm\frac{5\pi}{4}, \pm\frac{3\pi}{4}$

(c) $x = -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$

(d) $x = -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6}$

Q6

(a) $x = 0, \pi, 2\pi$

(b) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(c) $x = 0.464, 3.605$

(d) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

(e) $x = \frac{\pi}{2}, \frac{3\pi}{2}, 0, 2\pi$

(f) $x = 1.231, 5.052, \frac{7\pi}{6}, \frac{11\pi}{6}$

(g) $x = 1.249, 4.391, \frac{\pi}{4}, \frac{5\pi}{4}$

(h) $x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}$

(i) $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

(j) $x = \frac{\pi}{6}, \frac{5\pi}{6}$

(k) $x = 0, \pi, 2\pi, 0.955, 2.186, 4.097, 5.328$

(l) $x = \frac{\pi}{3}, \frac{5\pi}{3}$

Q7

(a) $\theta = 0.322, 2.820$ (b) $\theta = 1.923, 4.361$

(c) $\theta = 0.405, 3.547$

Q8

(a) $x = 0, \pi, 2\pi, \frac{3\pi}{4}, \frac{7\pi}{4}$

(b) $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}$

(c) $x = 1.249, 4.391, \frac{3\pi}{4}, \frac{7\pi}{4}$

Q9

(a) 1^{st} quadrant, $\cos\theta = \frac{4}{5}$

(b) 3^{rd} quadrant, $\sin\theta = -\frac{3}{5}$

(c) 2^{nd} quadrant, $\cot\theta = -\frac{\sqrt{51}}{7}$

(d) 3^{rd} quadrant, $\cot\theta = 1$

Q10

(a) $a = 6, x = \frac{\pi}{6}$ (b) $a = 6, x = \frac{\pi}{12}$

P1

(a) $\alpha = \frac{\pi}{2}$ (b) $x = \frac{5\pi}{24}$

P2

(a) $x = -\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$

(b) $x = \frac{\pi}{6}, \frac{5\pi}{6}$

P3

(a) See full worked solutions.

(b) $x = \frac{2\pi}{3}, \frac{5\pi}{3}$

Chapter Review

R1

(a) $\frac{5\pi}{6}$ (b) $\frac{5\pi}{3}$ (c) $\frac{4\pi}{9}$ (d) $\frac{10\pi}{9}$

R2

(a) $18°$ (b) $100°$ (c) $210°$ (d) $72°$

R3

(a) $114°35'$ (b) $28°39'$ (c) $183°55'$

R4

(a) $-\sec\theta$ (b) $\tan\theta$

R5

(a) 0 (b) $2\sqrt{2}$ (c) $-2 + \frac{1}{\sqrt{3}}$

(d) $-1 - \frac{\sqrt{3}}{2}$ (e) $-\frac{1}{2}$ (f) $\frac{1}{\sqrt{3}}$

R6

(a) 0.8415 (b) 1.0296 (c) -0.4161

R7

(a) See full worked solutions.

(b) See full worked solutions.

R8

(a) $\frac{20}{\sqrt{7}}$ cm (b) $\frac{54}{\sqrt{7}}$ cm

R9

(a) $r = 10, \theta = 2$ (b) $r = 4, \theta = \frac{\pi}{8}$

R10

(a) $\tan\theta = -\frac{3}{4}, \sin\theta = -\frac{3}{5}$

(b) $\cos\theta = -\frac{\sqrt{7}}{4}, \tan\theta = \frac{3}{\sqrt{7}}$

(c) $\sin\theta = -\frac{4}{\sqrt{41}}, \sec\theta = -\frac{\sqrt{41}}{5}$

R11

(a) $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$ (b) $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$

(c) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ (d) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

R12

(a) $\theta = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}$

(b) $\theta = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

(c) $\theta = -\frac{2\pi}{3}$

(d) $\theta = 0, \pm\pi$

R13

(a) $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$

(b) $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

(c) $\theta = \frac{\pi}{2}$ (d) $\theta = \pi$

R14

5

R15

(a) $y \in [-3, 3]$ (b) $y \in [-4, 4]$

(c) $y \in (-\infty, \infty)$ (d) $y \in [-1, 5]$

(e) $y \in [-8, 4]$ (f) $y \in [-8, 0]$

R16

(a) $4x^2 + y^2 = 4$ (b) $16x^2 - 9y^2 = 144$

(c) $(x-2)^2 + (y-1)^2 = 1$

R17

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

R18

$V = \frac{16\pi\sqrt{2}}{3}$ cm^3

R19

See full worked solutions.

R20

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

7. Differentiation and Applications

Exercise 7A

Limits

F1

(a) y (b) does not.

(c) α (d) simplifying

F2

(a) 0, smaller (b) $0, 0$

Q1

(a) -2 (b) 6 (c) 3 (d) 0

Q2

(a) The denominator is 0.

(b) 2

Q3

(a) -9 (b) $\frac{1}{2}$ (c) 10 (d) -5

(e) $2x^2$ (f) $\frac{6}{5}$ (g) -4 (h) -1

(i) $\frac{4}{3}$ (j) 2

P1

2

P2

$\frac{1}{10}$

P3

$-\frac{1}{4}$

P4

(a) $(1+h)^2$

(b) 2

P5

(a) $2xh+h^2$

(b) See full worked solutions.

Exercise 7B

Differentiation by first principles

F1

(a) slope

(b) $(x+h, f(x+h))$

(c) $\frac{f(x+h)-f(x)}{h}$

(d) h

(e) h, closer

F2

(a) touches

(b) secant

(c) secant

(d) difference, quotient, 0

(e) $\frac{f(x+h)-f(x)}{h}$

(f) gradient, derivative

(g) differentiation

Q1

(a) $(2+h, h^2+4h+3)$

(b) See full worked solutions.

(c) Q gets closer to P.

(d) 4

(e) It is the gradient of the tangent at $x=2$.

Q2

(a) h^2+2h+1

(b) See full worked solutions.

(c) 2

(d) It is the gradient of the tangent at $x=1$.

Q3

(a) $h+3$

(b) 3

(c) It is the gradient of the tangent at $x=1$.

Q4

(a) 2

(b) -4

(c) $2x+h$

(d) $2x+1+h$

Q5

(a) $f'(x)=2$, $m=2$

(b) $f'(x)=-4$, $m=-4$

(c) $f'(x)=2x$, $m=2$

(d) $f'(x)=2x+1$, $m=3$

Q6

(a) $f'(x)=2x$, $m=4$

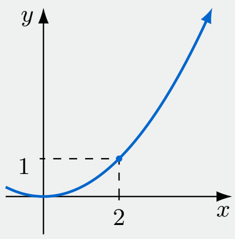
(b) $f'(x)=-2x$, $m=-4$

(c) $f'(x)=2x-2$, $m=2$

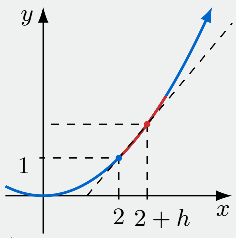
(d) $f'(x)=6x+1$, $m=13$

Q7

(a)



(b)



(c) 1

(d) See full worked solutions.

Q8

(a) $a^3 + 3a^2b + 3ab^2 + b^3$

(b) $3x^2$

P1

(a) See full worked solutions.

(b) $\frac{1}{2\sqrt{x}}$

(c) $-\frac{1}{2x\sqrt{x}}$

P2

See full worked solutions.

Exercise 7C

Basic differentiation rules

F1

(a) nx^{n-1} (b) knx^{n-1} (c) $f' \pm g'$

(d) y, x (e) $f'(\alpha)$

F2

(a) differentiate, $f'(x)$

(b) α, gradient

(c) y, α

(d) point, gradient, $y - y_1 = m(x - x_1)$

(e) perpendicular

(f) tangent

(g) perpendicular, negative, reciprocal

(h) point, point, gradient

F3

(a) 0, horizontal (b) vertical, $x = \alpha$

Q1

(a)

(i) 1 (ii) $2x$ (iii) $3x^2$

(b) nx^{n-1}

Q2

(a) $6x^5$ (b) $-\frac{1}{x^2}$ (c) $-\frac{2}{x^3}$

(d) $\frac{1}{2\sqrt{x}}$ (e) $\frac{1}{3\sqrt[3]{x^2}}$ (f) $\frac{5\sqrt{x^3}}{2}$

(g) $-\frac{1}{2\sqrt{x^3}}$ (h) 0

Q3

(a) $15x^4$ (b) $-8x$ (c) $2x + 2$

(d) $3x^2 - 8x$ (e) $5 - x$ (f) x^3+x^2+x

Q4

(a) $2x + 2$ (b) $2x + 1$ (c) $1 - \frac{1}{x^2}$

(d) 1 (e) $2x - \frac{2}{x^3}$

Q5

(a) $\frac{d}{dy}(y^2 + 2y) = 2y + 2$

(b) $\frac{d}{dz}(6z^3 - 5z^2) = 18z^2 - 10z$

(c) $\frac{d}{dt}(t^3 - 2t^2) = 3t^2 - 4t$

(d) $\frac{d}{dp}(p - p^3) = 1 - 3p^2$

(e) $\frac{d}{du}(4u^2 - u) = 8u - 1$

(f) $\frac{d}{dw}(w^3 + w^2) = 3w^2 + 2w$

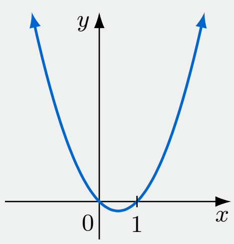
Q6

(a) 1 (b) $\frac{1}{2}$ (c) -2

(d) -2 (e) 2

Q7

(a)



(b) ± 1

(c) The gradients have opposite signs. This is expected because the parabola is symmetrical about its axis.

Q8

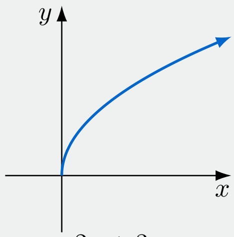
(a) $\frac{1}{4}$ (b) $(4, 2)$

(c) $y = \frac{1}{4}x + 1$ (d) -4

(e) $y = -4x + 18$

Q9

(a)



(b) $y = -2x + 3$

(c) $A\left(\frac{3}{2}, 0\right)$, $B(0, 3)$

(d) $\frac{9}{4}$ units ^{2}

Q10

Tangent: $y = -2x + 4$, Normal: $y = \frac{1}{2}x - 1$

Q11

(a) ± 2 (b) $\pm\sqrt{\frac{13}{3}}$ (c) ± 1

Q12

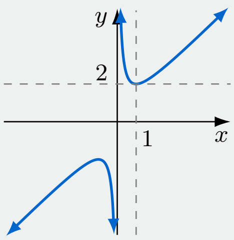
$(1, 4)$, $(3, 2)$

Q13

$x = -\frac{b}{2a}$, which is the x-coordinate of the vertex.

Q14

Tangent: $y = 2$, Normal: $x = 1$
At $(1, 2)$ the tangent is horizontal and the normal is vertical since $y' = 0$.



P1

$3x^2 + 2x + 1$

P2

$\frac{1}{2\sqrt{x}}$

P3

$P(1, -3)$

P4

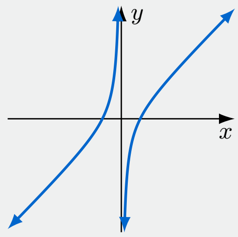
Tangent: $y = 1$, Normal: $x = 0$

P5

(a) 0

(b) See full worked solutions.

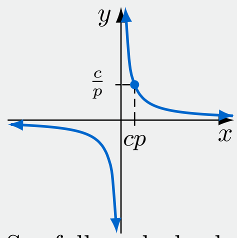
(c)



P6

(a) See full worked solutions.

(b)



(c) See full worked solutions.

(d) See full worked solutions.

P7

(a) $x^5 - 1$ (b) $4x^3 + 3x^2 + 2x + 1$

Exercise 7D
Chain rule

F1

(a) $f'(g(x))\, g'(x)$ (b) $\frac{du}{dx}$

F2

(a) $an(ax + b)^{n-1}$ (b) $f'(x)$

Q1

(a) $6x^5 + 6x^2$

(b) $6x^2(x^3 + 1)$

(c) See full worked solutions.

Q2

(a) $10(2x+1)^4$

(b) $10x(x^2+1)^4$

(c) $\dfrac{1}{(2-x)^2}$

(d) $\dfrac{1}{\sqrt{2x-1}}$

(e) $-\dfrac{2x}{(1+x^2)^2}$

(f) $-\dfrac{1}{\sqrt{(2x+1)^3}}$

(g) $-\dfrac{x}{\sqrt{1-x^2}}$

(h) $4\left(x-\dfrac{1}{x}\right)^3\left(1+\dfrac{1}{x^2}\right)$

(i) $-\dfrac{6}{(3x-4)^3}$

Q3

(a) $-\dfrac{x}{\sqrt{4-x^2}}$

(b) The gradient is undefined because the tangent is a vertical line.

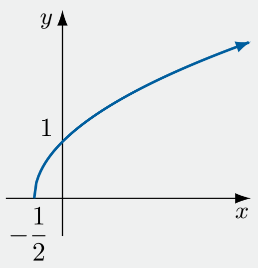
Q4

(a) $\dfrac{1}{\sqrt{2x+1}}$

(b) See full worked solutions.

(c) $y' \to 0$

(d)



Q5

(a) $a = 1$, $b = -1$

(b) $\dfrac{1}{(x+1)^2}$

Q6

(a) $an(ax+b)^{n-1}$

(b) $f\left(-\dfrac{b}{a}\right) = 0$

(c) $f'\left(-\dfrac{b}{a}\right) = 0$ for $n = 2$ and $n = 3$. This result shows that parabolas and cubic curves in the given form have horizontal tangents at their x-intercepts.

Q7

(a) See full worked solutions.

(b) $-\dfrac{1}{2\sqrt{x}(\sqrt{x}-1)^2}$

Q8

(a) See full worked solutions.

(b) $1-\dfrac{1}{(x-1)^2}$

P1

(a) See full worked solutions.

(b) $f'(x) = 6(x+1)^5 - 5(x+1)^4 = (6x+1)(x+1)^4$

P2

(a) x^5+1

(b) $4x^3-3x^2+2x-1$

(c) See full worked solutions.

Exercise 7E
Product rule

F1

product

F2

(a) uv'

(b) $v\dfrac{du}{dx}$

(c) $f(x)g'(x)+f'(x)g(x)$

Q1

(a) $(x+1)^4(6x+1)$

(b) $-36x^2+8x+15$

(c) $-45x^2+4x-37$

(d) $(3x-2)^3(15x-2)$

(e) $x(x-1)^2(5x-2)$

(f) $(x+2)^2(x+1)(5x+7)$

Q2

(a) $2(3x-1)(6x^2-x+3)$

(b) $2(5x+2)^3(x^2-1)^2(25x^2+6x-10)$

(c) $2(2x-1)^4(2x+1)^3(18x+1)$

(d) $\dfrac{3x+2}{2\sqrt{x+1}}$

(e) $\dfrac{x(4-5x)}{2\sqrt{1-x}}$

(f) $\dfrac{1+2x^2}{\sqrt{1+x^2}}$

Q3
See full worked solutions.

Q4
See full worked solutions.

P1
(a) See full worked solutions.

(b) $\dfrac{5}{(2x+1)^2}$

P2
(a) See full worked solutions.
(b) 0

Exercise 7F
Quotient rule

F1
(a) quotient
(b) is not

F2
(a) $y' = \dfrac{vu' - uv'}{v^2}$

(b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

(c) $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Q1
(a) $\dfrac{1}{2\sqrt{x}}$ (b) $3x^2 - 1$ (c) $1 - \dfrac{1}{x^2}$

(d) 1 (e) $-\dfrac{1}{(x+1)^2}$ (f) 1

Q2
(a) $\dfrac{2}{(x+1)^2}$ (b) $-\dfrac{5}{(2x-1)^2}$ (c) $\dfrac{x(3x+10)}{(3x+5)^2}$

(d) $-\dfrac{4x}{(1+x^2)^2}$ (e) $-\dfrac{2a}{(a+x)^2}$ (f) $\dfrac{a^2-b^2}{(bx+a)^2}$

Q3
See full worked solutions.

Q4
(a) See full worked solutions.

(b) $-\dfrac{4x}{(x^2-1)^2}$

Q5
(a) $\dfrac{1-x}{2\sqrt{x}(x+1)^2}$

(b) $-\dfrac{x+2}{2x^2\sqrt{x+1}}$

(c) $\dfrac{x+2}{2\sqrt{(x+1)^3}}$

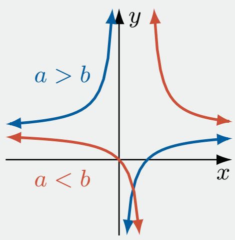
Q6
(a) $\dfrac{a-b}{(x-b)^2}$

(b) The gradients of the tangents are always positive.

(c) The gradients of the tangents are always negative.

(d) Yes, it is expected since $f(x) = 1$ when $a = b$.

(e)



P1
(a) See full worked solutions.

(b) $\sqrt{x} > 0$, since $x \neq 0$ for $f'(x)$, and $(\sqrt{x}+1)^2 > 0$.

(c) The gradients of the tangents increase as x gets smaller.

(d) $(0, -1)$, $(1, 0)$

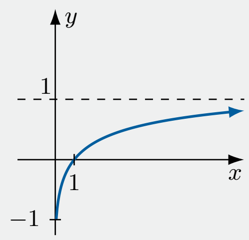
(e) $x \geq 0$

(f) It shows that $y = 1$ is a horizontal asymptote of $f(x)$.

(g)

x	$\frac{1}{4}$	1	9	100
$f(x)$	$-\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{9}{11}$

(h)



Exercise 7G
Average rate of change

F1

(a) y (b) gradient (c) $f(b) - f(a)$

F2

(a) time (b) quantity (c) $\dfrac{Q_{\text{final}} - Q_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$

F3

(a) [0, 1] (b) [1,2]
(c) $[n-1, n]$ (d) $[0, 5]$

Q1

(a) $V = 1$ L, $V = 13$ L
(b) 4 L/min
(c) 6 L/min

Q2

(a)
 (i) -3 L/min
 (ii) -2 L/min
(b) The bucket becomes empty when $t = 2$.

Q3

$-\$5028.23$/yr.

Q4

(a) 2 (b) 7 (c) $-\frac{1}{8}$
(d) $-\frac{5}{36}$ (e) $\frac{1}{2}$ (f) $\frac{1-\sqrt{2}}{\sqrt{2}}$

Q5

$328.10/yr

P1

6

P2

(a) 3.1°C/min, 3.01°C/min, 3.001°C/min
It is approaching a rate of 3°C/min.
(b) Differentiation by first principles.
(c) The instantaneous rate of change is the limit of the average rate of change.

Exercise 7H
Instantaneous rate of change

F1

(a) instantaneous, t (b) average, interval
(c) t_1 (d) gradient, tangent
(e) limit, smaller

F2

(a) average, instant (b) derivative, time
(c) $\dfrac{dQ}{dt}$

F3

(a) > 0 (b) < 0 (c) 0

Q1

(a) 5 L
(b) See full worked solutions.
(c) $V'(t) < 0$, since $(2+t)^2 > 0$, which tells us the volume is always decreasing.
(d) 3 L/min
(e) 0.75 L/min

Q2

(a) 100 m
(b) 4.47 seconds, $t \in [0, \sqrt{20}]$
(c)
 (i) 0 m/s
 (ii) $-10\sqrt{20}$ m/s
(d) $t = 4.47$ s, which is when the ball hits the ground.
(e) No. It is incorrect.

Q3

(a) 25 cm^2/min (b) 22 cm^2/min

Q4

(a) The rate is positive when $t = 1$ (water flowing in), but it is negative when $t = 4$ (water flowing out).
(b) See full worked solutions.
(c) See full worked solutions.
(d) $t = 6$ minutes

Q5

(a) 16 minutes

(b) -0.25 kg/min

(c) -0.5 kg/min

(d) It is faster just before it is gone.

Q6

(a) 200 L (b) 1000 L

(c) 60 L/s, 40 L/s (d) $t = 0$

P1

(a) A drops to 0 when $t = 8$ hours, and then becomes negative after that.

(b) $\dfrac{dA}{dt} = 16t - 3t^2$ is the rate of change of the amount of the drug in the person's bloodstream.

(c) $\dfrac{dA}{dt} = 0$ at $t = \dfrac{16}{3}$

(d) Increasing over $t \in \left(0, \dfrac{16}{3}\right)$, Decreasing over $t \in \left(\dfrac{16}{3}, 8\right)$

(e) 75.85 mg

P2

(a) 2 (b) 0, 2

Exercise 7I

Gradient function

F1

(a) y

(b) gradient, positive

(c) x, y

(d) gradient, negative

F2

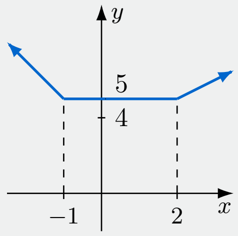
(a) $f'(x)$ (b) positive

(c) negative (d) zero

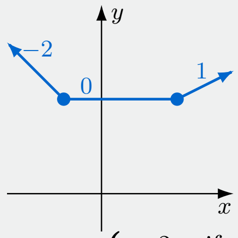
Q1

(a)

(i)

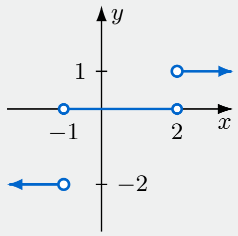


(ii)



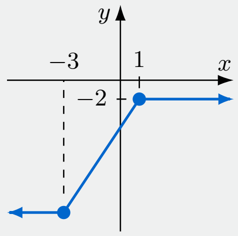
(iii) $f'(x) = \begin{cases} -2, & \text{if } x < -1 \\ 0, & \text{if } -1 \le x \le 2 \\ 1, & \text{if } x > 2 \end{cases}$

(iv)

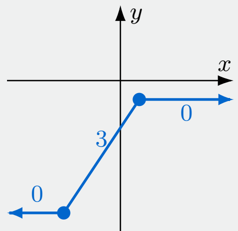


(b)

(i)

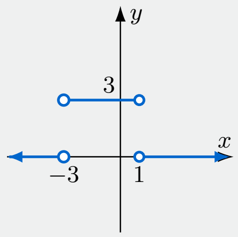


(ii)



(iii) $f'(x) = \begin{cases} 0, & \text{if } x < -3 \\ 3, & \text{if } -3 \le x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$

(iv)



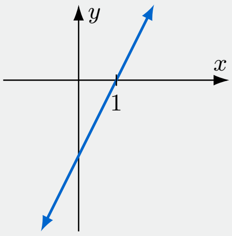
Q2

(a) $x > 1,\ y > 0$

(b) $x < 1,\ y < 0$

(c) $x = 1,\ y = 0$

(d)



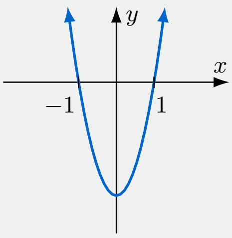
Q3

(a) $x < -1,\ x > 1$

(b) $-1 < x < 1$

(c) $x = -1,\ 1$

(d)

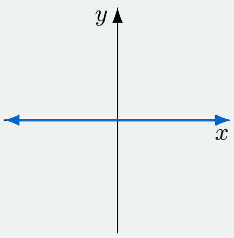


Q4

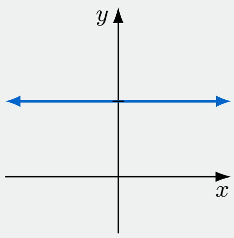
(a)	(D)	(b)	(H)	(c)	(E)	(d)	(C)
(e)	(I)	(f)	(B)	(g)	(G)	(h)	(A)
(i)	(F)						

Q5

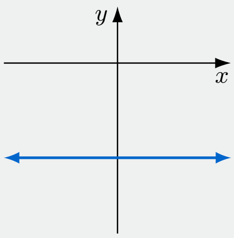
(a)



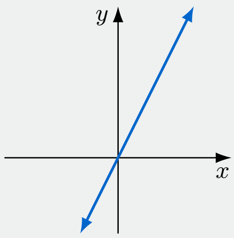
(b)



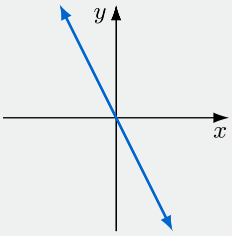
(c)



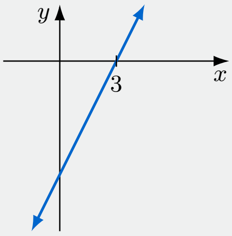
(d)



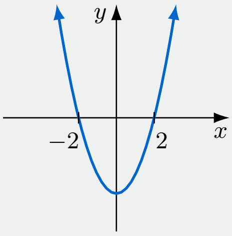
(e)



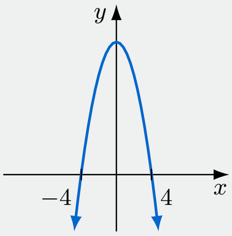
(f)



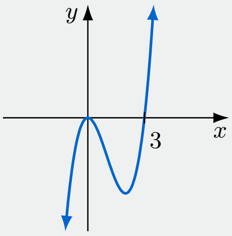
(g)



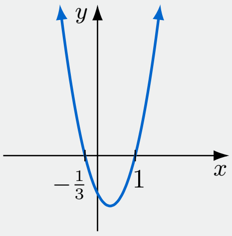
(h)



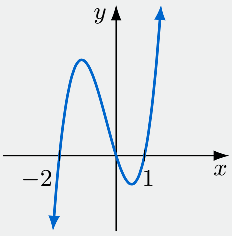
(i)



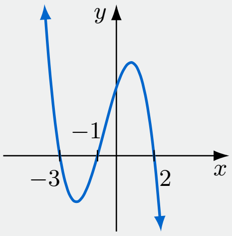
(j)



(k)



(l)



P1

(a) Straight line with gradient 1.

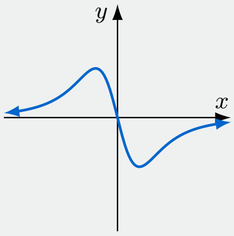
(b) Straight line with gradient −1.

(c) Concave up parabola, vertex at $x = 0$

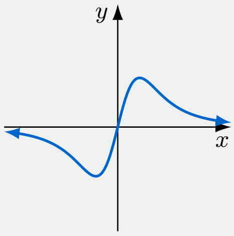
(d) Concave down parabola, vertex at some $x = c$ where the line cuts the x-axis.

P2

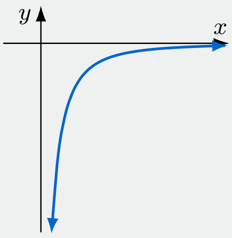
(a)



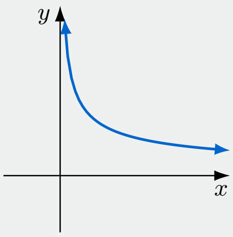
(b)



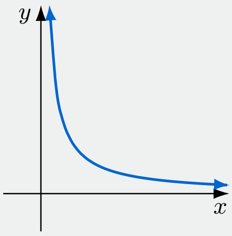
(c)



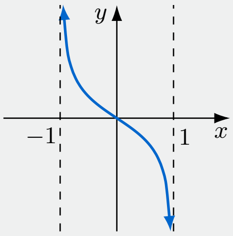
(d)



(e)



(f)



Exercise 7J

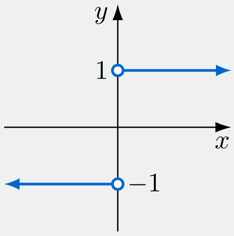
Continuity

F1

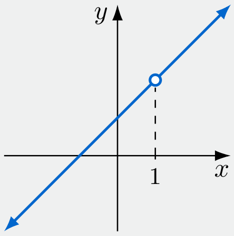
(a) continuous, function

(b) discontinuous

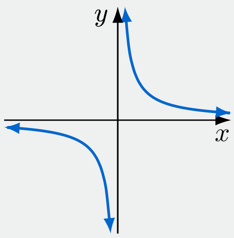
(c) **Jump discontinuities**



Holes



Asymptotic discontinuities



(d) zero

(e) domain, continuous, discontinuous

Q1

(a) Continuous.

(b) Continuous.

(c) Discontinuous at $x = 1$.

(d) Discontinuous at $x = 1$.

(e) Discontinuous at $x = 0$.

(f) Continuous.

Q2

(a) Continuous.

(b) Discontinuous at $x = 0$.

(c) Continuous.

(d) Discontinuous at $x = 0$.

(e) Disontinuous at $x = \pm 1$.

(f) Discontinuous at $x = 1$.

(g) Continuous.

(h) Discontinuous at $x = -4, -1$.

(i) Discontinuous at $x = -\frac{1}{2}$.

Q3

(a) Asymptotic discontinuity.

(b) Jump discontinuity.

(c) Hole.

Q4

(a) Continuous. (b) Continuous.

(c) Discontinuous. (d) Continuous.

Q5

(a) 2 (b) 1

(c) 6 (d) $-\frac{1}{2}, 1$

P1

$a = 1, b = 15$

P2

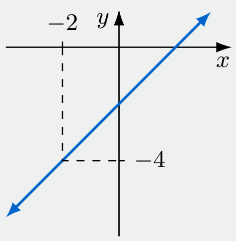
Squaring both sides introduces extra solutions.

P3

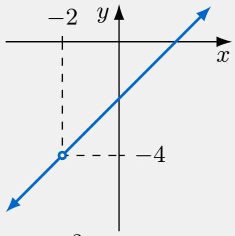
$y = \tan x$, $y = \operatorname{cosec} x$, $y = \sec x$ and $y = \cot x$.

P4

(a)



(b)



$y = \dfrac{x^2 - 4}{x + 2}$ has a hole at $x = -2$.

(c) $\begin{cases} \dfrac{x^2-4}{x+2}, & \text{for } x \in \mathbb{R}, x \neq -2 \\ -4, & \text{for } x = -2 \end{cases}$

Chapter Review

R1

(a) $-\dfrac{2}{x^3}$ (b) $\dfrac{3\sqrt{x}}{2}$

(c) $-\dfrac{2}{3\sqrt[3]{x^5}}$ (d) $-\dfrac{3}{2\sqrt{x^5}}$

R2

(a) $8x$ (b) $1 - 2x$

(c) $4x + 3$ (d) $2x - 4$

R3

(a) nx^{n-1} (b) $4nx^{n-1}$ (c) 0

(d) 7 (e) $\dfrac{1}{5} - \dfrac{5}{x^2}$ (f) $-\dfrac{3}{4x^2}$

(g) $3 - \dfrac{8}{x^2}$ (h) $\dfrac{4\sqrt[3]{x}}{3}$ (i) $-\dfrac{6}{x^3}$

(j) $-\dfrac{5}{3\sqrt[3]{x^8}}$

R4

(a) $6(3x - 2)$

(b) $24x(3x^2 - 5)^3$

(c) $\dfrac{x}{\sqrt{x^2 + 1}}$

(d) $-\dfrac{x(3x - 2)}{(x^3 - x^2)^2}$

(e) $\dfrac{x}{\sqrt{(1 - x^2)^3}}$

(f) $4\left(x + \dfrac{1}{x}\right)^3 \left(1 - \dfrac{1}{x^2}\right)$

R5

(a) $x(2-3x)^4(4-21x)$

(b) $2(3x+1)^3(2x-1)^4(27x-1)$

(c) $\dfrac{3x+2}{2\sqrt{x+1}}$

(d) $(x^2+1)^2(7x^2+1)$

R6

(a) $-\dfrac{2}{(x-1)^2}$ (b) $\dfrac{2x(x-1)}{(2x-1)^2}$

(c) $\dfrac{4x}{(x^2+1)^2}$ (d) $\dfrac{x(3x+4)}{2\sqrt{(x+1)^3}}$

R7

(a) $(3,-6)$ (b) $y=x-9$

R8

$(3,0)$

R9

$\left(\dfrac{1}{2},-\dfrac{1}{4}\right),\left(\dfrac{1}{6},-\dfrac{5}{108}\right)$

R10

$y=x-\dfrac{21}{8}$

R11

$m=2,\ c=2$

R12

(a) See full worked solutions.

(b) $y=-x+3$

(c) 4.5 units^2

R13

(a) 1 (b) -1 (c) -9

R14

8

R15

34 cm^2/min, 26 cm^2/min

R16

2

R17

$a=2,\ b=-4,\ c=-16$

R18

(a) $x=\dfrac{3}{2}$ (b) $x=\dfrac{3}{2}$

(c) $x>\dfrac{3}{2}$ (d) $x=\dfrac{11}{4}$

R19

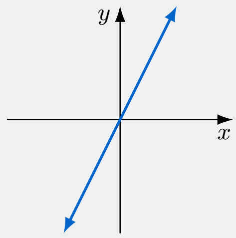
(a) 10

(b) 9 trout/month

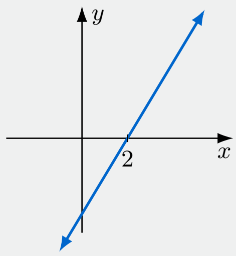
(c) 15 trout/month

R20

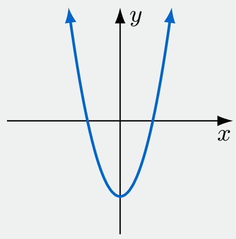
(a)



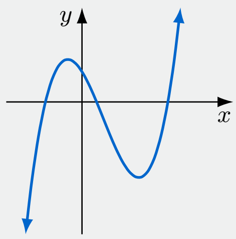
(b)



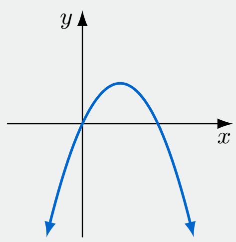
(c)



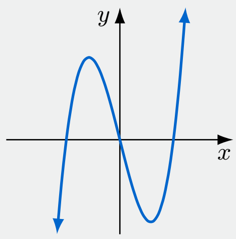
(d)



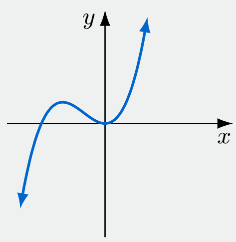
(e)



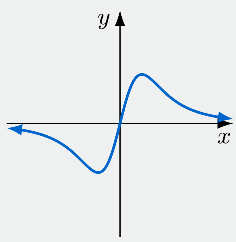
(f)



(g)



(h)



8. Logarithmic and Exponential Functions

Exercise 8A

The logarithmic function

F1

(a) power, index, base (b) $\log_a y = x$

(c) 0, 1 (d) > 0

(e) inverse (f) reflected

F2

(a) 1 (b) 0 (c) x (d) $x, > 0$

F3

(a) domain, range (b) range, domain

Q1

(a) $10^1 = 10$ (b) $9^2 = 81$ (c) $5^0 = 1$

(d) $2^5 = 32$ (e) $4^{\frac{1}{2}} = 2$ (f) $5^{-1} = \frac{1}{5}$

Q2

(a) $\log_2 8 = 3$ (b) $\log_4 16 = 2$

(c) $\log_4 2 = \frac{1}{2}$ (d) $\log_8 2 = \frac{1}{3}$

(e) $\log_5 \frac{1}{25} = -2$ (f) $\log_{16} 0.25 = -\frac{1}{2}$

Q3

(a) $x = 3$ (b) $x = -2$ (c) $x = 2$

(d) $x = -1$ (e) $x = -4$ (f) 1

(g) $x = 0$ (h) $x = 5$ (i) $x = 4$

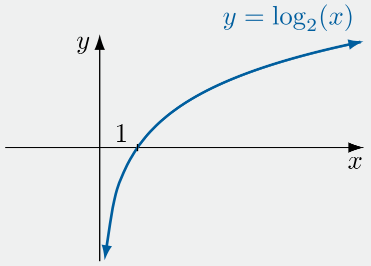
(j) $x = 5$ (k) $x = 9$ (l) $x = 9$

(m) $x = \frac{1}{9}$ (n) $x = 8$

Q4

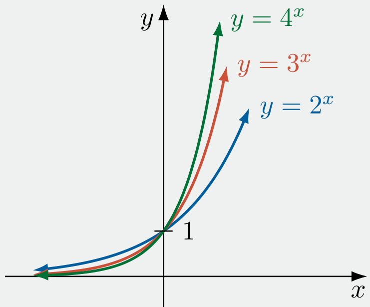
(a) $-2, -1, 0, 1, 2, 3$

(b)

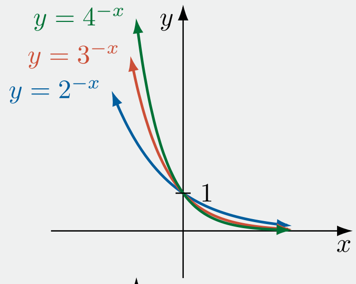


Q5

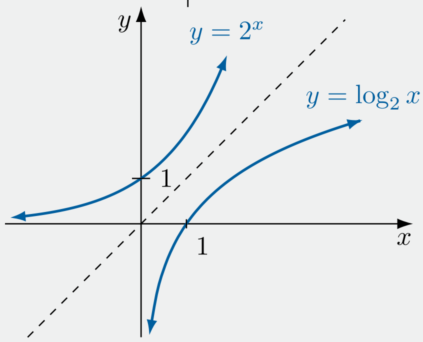
(a)



(b)



(c)



Q6

(a) 4 (b) 4 (c) $\frac{1}{2}$ (d) $\frac{5}{2}$

Q7

(a) $x > 2$ (b) $x < 4$ (c) $x > -2$

(d) $x > 0$ (e) $x > 0$ (f) $x \in \mathbb{R}$

Q8

(a) 5 (b) 3 (c) $\frac{1}{3}$ (d) $\sqrt{2}$

P1

(a) 3 (b) 125 (c) 16

P2

$x > 2$

Exercise 8B
Logarithmic laws

F1

(a) $\log_a mn$ (b) $\log_a \frac{m}{n}$

(c) $n \log_a m$ (d) x, $\log_b a$

F2

(a) 1 (b) 0

(c) -1 (d) $\frac{1}{2}$

(e) $-\log_a x$ (f) $\frac{1}{2}\log_a x$

Q1

(a) 4 (b) 1 (c) 4

(d) 4 (e) -1 (f) -3

(g) $\frac{1}{2}$ (h) -4 (i) $-\frac{1}{2}$

Q2

(a) -1 (b) 2 (c) $\frac{1}{6}$ (d) $5\frac{1}{2}$

Q3

(a) 2.10 (b) 1.21 (c) 5.17 (d) -1.58

Q4

(a) 2 (b) -2 (c) 3 (d) 5

(e) -3 (f) 5 (g) 7 (h) 4

(i) -3

Q5

(a) $x+y$ (b) $z-y$

(c) $\frac{1}{2}(x+y)$ (d) $-(x+y+z)$

(e) $x+2y+3z$ (f) $-\frac{1}{2}(2y+z)$

(g) $\frac{1}{2}(x+z)-y$ (h) $3x+\frac{1}{2}(y+z)$

(i) $2z-(2x+3y)$

Q6

(a) $a+b$ (b) $b-a$

(c) -1 (d) $3a-2b$

(e) $\frac{1}{2}(a+b+1)$ (f) $5a-1$

Q7

(a) $6\log_3(2)$ (b) $\log_2 \frac{\sqrt{x}}{3}$

(c) $\log_{10}(50)$ (d) 2

(e) $\log_5(x)$ (f) $\log_4 \frac{2}{5\sqrt{5}}$

Q8

(a) True (b) True (c) True (d) False

(e) False (f) False (g) True (h) False

(i) True (j) True (k) False (l) False

(m) True (n) True (o) True

Q9

(a) $\frac{5}{3}$ (b) $-\frac{5}{2}$ (c) $-\frac{3}{2}$ (d) $\frac{3}{2}$

Q10

(a) See full worked solutions.

(b) $\frac{2}{3}$

Q11

See full worked solutions.

Q12

See full worked solutions.

P1

(a) $y=\frac{125}{x^2}$ (b) $x>0$

P2

(a) ab

(b) See full worked solutions.

P3

-2

P4

See full worked solutions.

P5

See full worked solutions.

P6

See full worked solutions.

P7

(a) See full worked solutions.

(b) See full worked solutions.

Exercise 8C
Differentiating the exponential function

F1

(a) natural, log (b) inverse (c) negative

F2

(a) e^x (b) ae^{ax+b}

(c) $f'(x)e^{f(x)}$ (d) e

(e) x, all real x (f) $x > 0$

(g) x^a, $x > 0$ (h) $f(x)$, > 0

Q1

(a) $y' = 2e^{2x}$ (b) $y' = -e^{-x}$

(c) $y' = -6e^{-6x}$ (d) $y' = \frac{1}{2}e^{\frac{x}{2}}$

Q2

(a) $2e^{2x+5}$ (b) $2e^{2x-1}$

(c) $-2e^{3-2x}$ (d) $\frac{2}{3}e^{\frac{2x-1}{3}}$

(e) $-\frac{3}{5}e^{\frac{2-3x}{5}}$

Q3

(a) $y' = 3e^{3x}$ (b) $-e^{-x}$

(c) $y' = \frac{1}{2}e^{\frac{x}{2}}$ (d) $y' = \frac{3}{2}e^{\frac{3}{2}x}$

(e) $y' = -\frac{1}{2}e^{-\frac{1}{2}x}$ (f) $y' = -e^{-x}$

Q4

(a) $\ln 2$ (b) $2^x = e^{x\ln 2}$
$a = \ln 2$

(c) See full worked solutions.

Q5

(a) $4^x \ln 4$ (b) $3^x \ln 3$

(c) $3^{2x} \times 2\ln 3$ (d) $2^x \ln 2$

Q6

(a) 1 (b) $-\frac{1}{x^2}$ (c) $2x$

(d) $\frac{1}{2\sqrt{x}}$

Q7

(a) $4e^{2x}$

(b) See full worked solutions.

Q8

See full worked solutions.

Q9

See full worked solutions.

Q10

See full worked solutions.

Q11

$k = 1$, $k = -6$

Q12

See full worked solutions.

P1

(a) a^x

(b) See full worked solutions.

P2

See full worked solutions.

Exercise 8D
Solving exponential and logarithmic equations

F1

(a) Y (b) Y (c) $\ln a$

Q1

(a) $x = 2$ (b) $x = \frac{1}{2}$ (c) $x = -2$

(d) $x = -\frac{1}{2}$ (e) $x = \frac{1}{4}$ (f) $x = \frac{3}{2}$

(g) $x = -\frac{3}{2}$ (h) $x = -\frac{6}{5}$ (i) $x = -3$

Q2

(a) $x = 0$ (b) $x = 0$ (c) $x = \frac{5}{4}$

Q3

(a) $x = \frac{5}{2}$, $y = -\frac{1}{2}$ (b) $x = 1$, $y = -2$

(c) $x = 1$, $y = -1$

Q4

(a) $x = 1$ (b) $x = 1$ (c) $x = 1$

ANSWERS

Q5

(a) $x = 4$ (b) $x = 4$ (c) $x = 3$

(d) $x = 27$ (e) $x = 4$ (f) $x = \sqrt{2}$

Q6

(a) $x = 3$ (b) $x = 9$

(c) No solution. (d) $x = e$

(e) $x = 9,\ x = 27$ (f) $x = \sqrt{5}$

Q7

2.32

Q8

(a) 3.32 (b) 0.65

(c) No solution (d) 1.49

(e) 0.08 (f) −0.41

(g) 0.69 (h) No solution

Q9

(a) $x = 2.32,\ x = 0$ (b) $x = 0.63,\ x = 0$

(c) 1.11

Q10

(a) $V_2 = \$2205$ (b) $2000(1.05)^t$

(c) 15 years (d) correct

Q11

(a) 8.65 m/s

(b) $t = \ln 2 = 0.69$

(c) As t gets large, $v \to 10$, but never actually reaches it.

Q12

(a) $1.3357\mu\text{g mL}^{-1}$

(b) See full worked solutions.

(c) See full worked solutions.

(d) The first time occurs as the body takes on the antibiotic, and the second time occurs as the body metabolises and breaks it down.

P1

(a) $x = 9$ (b) $x = \dfrac{16}{11}$ (c) $x = -4$

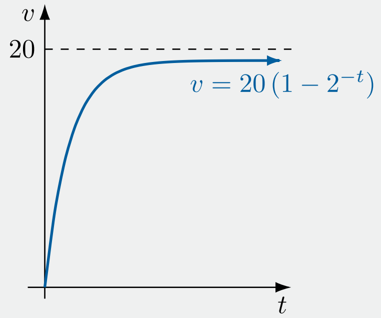
P2

(a) See full worked solutions.

(b) $p = \dfrac{1}{2}$ and $p = 1$

(c) $k = \ln 2,\ b = 20$

(d)



(e) $b = 20$

(f) This is called the *terminal velocity* of the object.

P3

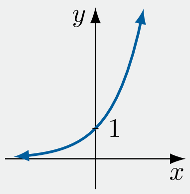
$x = 64$

Exercise 8E

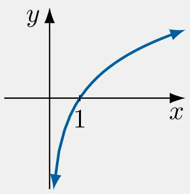
Graphing exponential and logarithmic equations

F1

(a)



(b)



F2

(a) asymptote (b) $\in \mathbb{R}$

(c) $(0, \infty)$ (d) $>$

(e) increasing (f) decreasing

(g) $(0, 1)$

F3

(a) ≤ 0 (b) vertical, $x = 0$

(c) increasing, $x > 0$ (d) $x > 0$

(e) $\in \mathbb{R}$ (f) $(1, 0)$

F4

(a) ∞ (b) 0 (c) 0

Q1

If $c > 0$, then the curve is translated upwards. If $c < 0$, then the curve is translated downwards.

ANSWERS

Q2

As a increases, the graph gets steeper.

Q3

If $k > 0$, then the curve gets steeper. If $k < 0$, then the curve gets less steep.

Q4

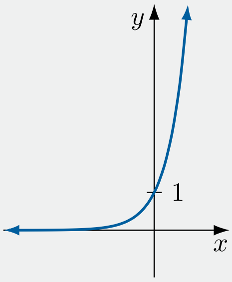
The graph $y = e^x$ is shifted to the left for $c > 0$ and to right if $c < 0$ and unshifted for $c = 0$.

Q5

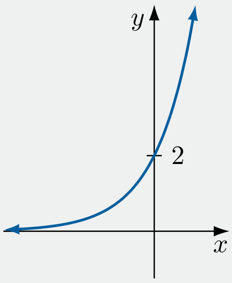
Same graph $e^2 e^x = e^{x+2}$

Q6

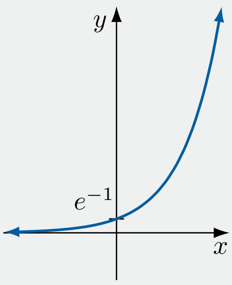
(a)



(b)



(c)



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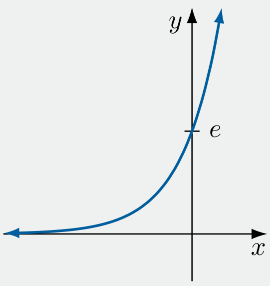
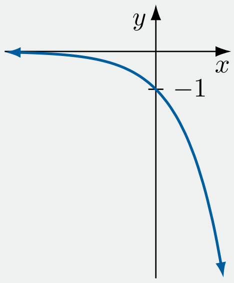
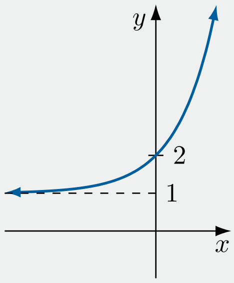


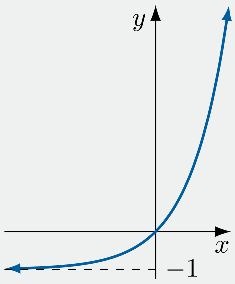
(e)



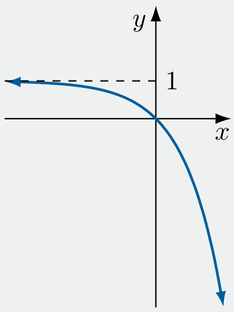
(f)



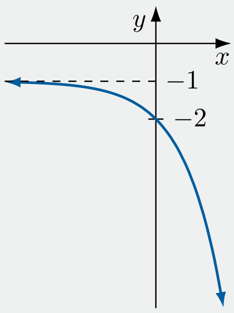
(g)



(h)



(i)



Q7

(a) $y > 0$

(b) $y > 0$

(c) $y < 0$

(d) $y > -3$

(e) $y > 0$ (f) $y < 0$

Q8

(a) $f(0) = 1,\ f'(x) = e^{-x},\ f'(0) = 1$

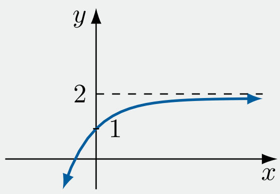
(b) $e^{-x} > 0$ for all x, $f(x)$ is an inceasing function $\left(\ln 2, 1\frac{1}{2}\right)$

(c) $\left(\ln(2), \frac{3}{2}\right)$

(d) 2

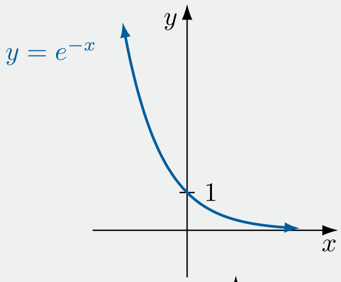
(e) 0

(f)

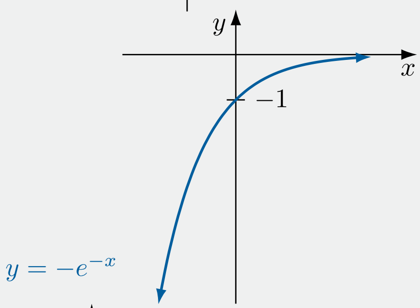


Q9

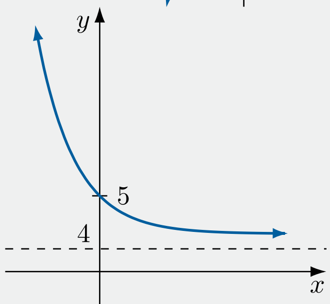
(a)



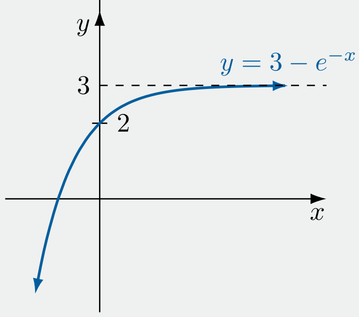
(b)



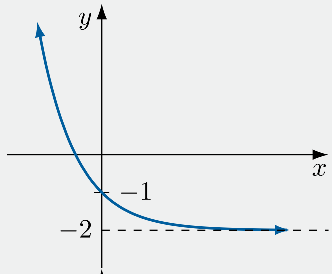
(c)



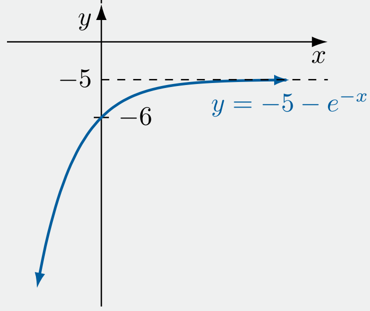
(d)



(e)



(f)



Q10

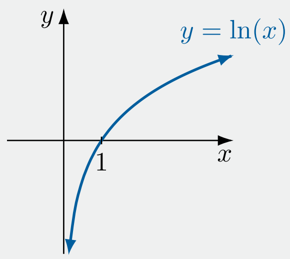
(a) $y = e^{x+2}$ (b) $y = e^{x} + 1$

(c) $y = -e^{x}$ (d) $y = e^{-x}$

(e) $y = -e^{-x}$

Q11

(a)



(a)

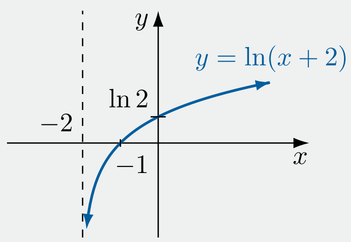
(i) $y = \ln x - 1$ (ii) $y = \ln x + 2$

(iii) $y = \ln(x + 1)$ (iv) $y = 2\ln x$

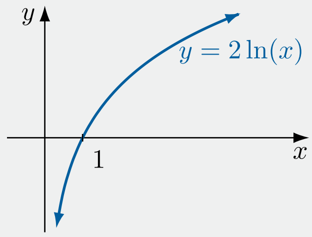
(v) $y = \frac{1}{2}\ln x$ (vi) $y = \ln(-x)$

Q12

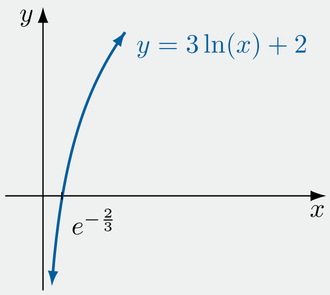
(a)



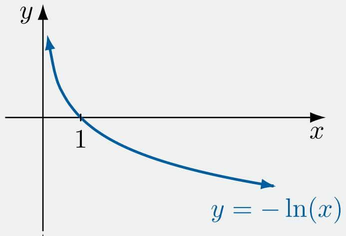
(b)



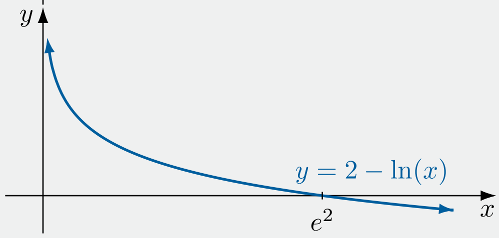
(c)



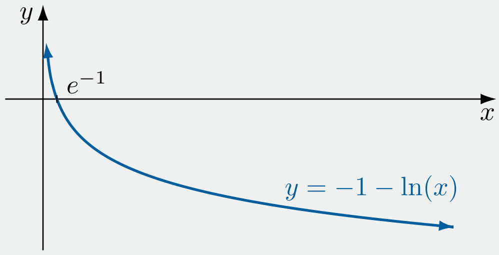
(d)



(e)



(f)



Q13

(a) ∞ (b) $-\infty$ (c) 7 (d) 5

(e) ∞ (f) ∞ (g) 1

Q14

No

Q15

(a) The x-coordinates of A and B *are* the solutions of $2^x = x + 2$.

(b) 2

Q16

(a) $y = x - 2$ (b) $y = 2 - 2x$

(c) $y = \dfrac{1}{x}$ (d) $y = x^2$

Q17

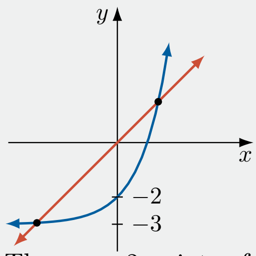
(a) $y = e^x$ is an increasing function, so if the input is larger, the output is also larger.

(b) $y = \ln(x)$ is also an increasing function, so if the input is larger, the output is also larger.

Q18

(a) See full worked solutions

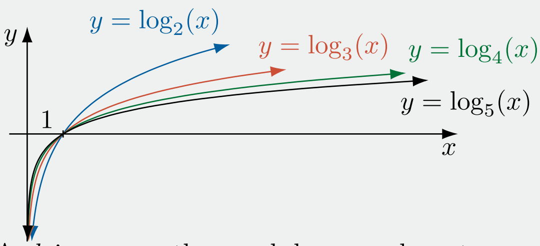
(b)



(c) There are 2 points of intersection

Q19

(a)



(b) As b increases, the graph becomes less steep.

(c) $y = \log_b x = \dfrac{\ln(x)}{\ln(b)}$. As b increases, y decreases.

P1

$$x = \frac{\ln 5}{\ln \frac{5}{4}}$$

P2

$x = 1$

P3

(a) 1

(b) -2

(c) $a = 3$, $y = -2 \times 3^{-x} + 1$

(d) See full worked solutions.

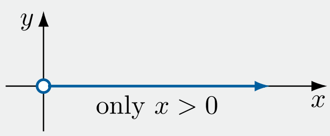
P4

$\sqrt{3}\sqrt[3]{2} > \sqrt{2}\sqrt[3]{3}$

P5

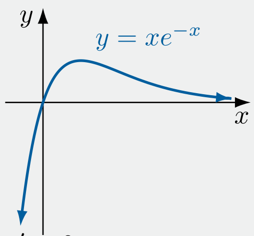
1

P6



P7

(a)



(b) $y' = 0$

(c) See full worked solutions.

(d) $k = \frac{1}{e}$, $k \leq 0$

P8

Dividing by $\log\left(\frac{1}{5}\right)$ should flip the inequality sign since $\log\left(\frac{1}{5}\right)$ is a negative number.

Exercise 8F

Applications of exponential and logarithmic functions

F1

(a) 0 (b) 0 (c) A (d) A

F2

(a) ≥ 0 (b) 0 (c) A

(d) $A + B$ (e) $A - B$

F3

(a) $Q'(t) > 0$ (b) $Q'(t) < 0$

F4

(a) $P(1.1)^n$ (b) $P(0.9)^n$

(c) $P\left(1 + \frac{r}{100}\right)^n$ (d) $P\left(1 - \frac{r}{100}\right)^n$

Q1

(a) \$22500 (b) \$20250

(c) $25000(0.9)^n$ (d) 59%

(e) 9th year

Q2

(a) 1114 (b) 2010

Q3

(a) 1000 (b) See full worked solutions. (c) 2250

(d) 34 weeks

Q4

(a) 500 (b) See full worked solutions.

(c) 3 (d) 13 years

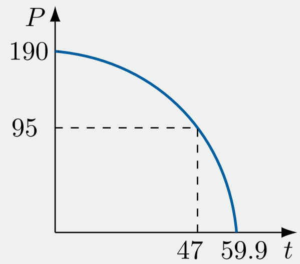
Q5

(a) Show $P' < 0$, $P' = -\frac{1}{2}e^{0.05t} < 0$ for all t

(b) 47 years

(c) Yes. After 60 years in 2080.

(d)



Q6

(a) 16000

(b) $P = 2^{2t} \times 1000$

(c) 4 hrs and 59 min.

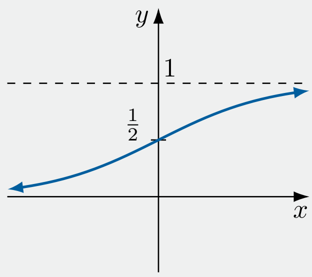
Q7

(a) 0.20 mm (b) 5 folds

Q8

(a) See full worked solutions.

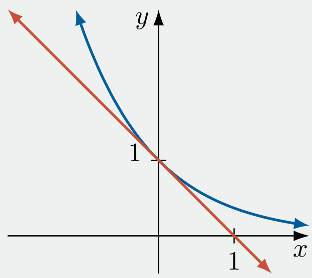
(b) Down by $\frac{1}{2}$,



Q9

(a) $y = -x + 1$

(b)



(c) $\frac{1}{2}$ u

(d) 1 u^2

Q10

$A = 63,\ k = \frac{\ln 2}{3}$

Q11

(a) 12×8^x

(b) $(a^b) = A,\ h = ka^c$

Q12

(a) $y = 2 \times 2^x$

(b) $y = 2^x$

(c) $y = 2 \times 3^x$

(d) $y = 4 \times 5^x$

Q13

(a) $x \in \mathbb{R},\ x \neq 0$

(b) $\mathbb{R}$

(c) $x > 1$

(d) $x < 2$

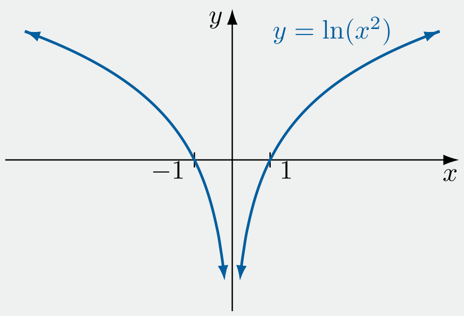
(e) $x > 2$

(f) $x < 4$

Q14

(a) $x \in \mathbb{R},\ x \neq 0$

(b)



Q15

$t = \frac{1}{k}\ln 2$ independent of M_0

Q16

(a) ≈ 82 kg

(b) 229 years

P1

(a) $m = e^{x_0}$

(b) $y = e^{x_0} + e^{x_0}(x - x_0)$

(c) $(1, e)$

P2

See full worked solutions.

P3

(a) $\frac{a}{b}$

(b) See full worked solutions.

(c) See full worked solutions.

(d) $a = 1000,\ b = \frac{1}{4}$

(e) No

P4

See full worked solutions.

Chapter Review

R1

(a) 10

(b) $\frac{1}{2}$

(c) $\frac{1}{2}$

R2

(a) $A(0,1),\ B(4,1),\ C(4,0),\ D(1,0),\ E(1,1)$

(b) 3 u^2

R3

(a) 3.2

(b) 0.8

(c) -1.6

(d) $1.6x$

R4

$x = 16$

R5

(a) $x - 2y$

(b) $\frac{1}{2}(2y + z)$

R6

$y = \frac{e^2}{x}$

R7

$x = 0.93$

R8

(a) $x < \frac{4}{3}$

(b) All real

R9

(a) $2b + a$

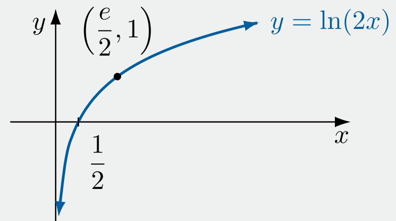
(b) $\frac{3b}{2}$

(c) $\frac{3b}{2} + a$ or $1 + b$

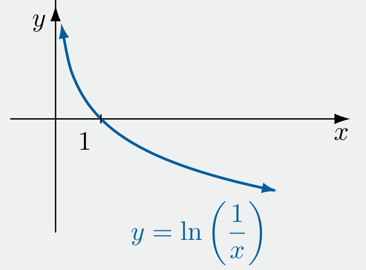
(d) $\frac{b}{2} - \frac{a}{2}$

R10

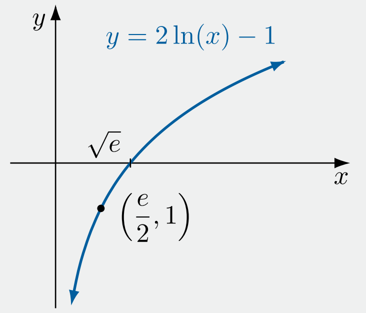
(a)



(b)



(c)



R11

$e^2 - 1$

R12

(b) $x = 1, -2$

R13

$x = e^4, \frac{1}{e}$

R14

$x = 0$

R15

R16

(a) $a = \frac{2}{3}, x = -2, k = 1$

(b) $\frac{11}{8}$

R17

(a) $x = 10$ (b) $x = 6$

(c) $x = 1$ (d) $x = e^3, x = e^4$

R18

See full worked solutions.

9. Probability

Exercise 9A

Sample space and probability

F1

(a) outcome

(b) outcomes, event

(c) outcome, event, favourable, outcome

(d) sample, space

(e) sample, space

(f) outcomes

F2

(a) $0 \le P(A) \le 1$

(b) 1

(c) 1, 0

(d) $\dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$

(e) cannot

(f) certain

(g) equally, likely

(h) complement

(i) $1 - p$, 1

F3

(a) two, three, no (b) no, one

Q1

(a) $\{HH, HT, TT\}$

(b) $\{HHH, HHT, HTT, TTT\}$

(c) $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

(d) $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Q2

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

Q3

(a)

	1	**2**	**3**	**4**	**5**	**6**
1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{1,6\}$
2	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{2,6\}$
3	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$	$\{3,4\}$	$\{3,5\}$	$\{3,6\}$
4	$\{4,1\}$	$\{4,2\}$	$\{4,3\}$	$\{4,4\}$	$\{4,5\}$	$\{4,6\}$
5	$\{5,1\}$	$\{5,2\}$	$\{5,3\}$	$\{5,4\}$	$\{5,5\}$	$\{5,6\}$
6	$\{6,1\}$	$\{6,2\}$	$\{6,3\}$	$\{6,4\}$	$\{6,5\}$	$\{6,6\}$

(b)

(i) $\frac{5}{18}$ (ii) $\frac{11}{36}$ (iii) $\frac{1}{36}$ (iv) $\frac{1}{6}$

(v) $\frac{5}{18}$ (vi) $\frac{35}{36}$ (vii) $\frac{5}{6}$ (viii) $\frac{3}{4}$

Q4

(a) $\frac{1}{4}$ (b) $\frac{4}{13}$ (c) $\frac{1}{52}$

(d) $\frac{3}{4}$ (e) $\frac{1}{26}$ (f) $\frac{7}{13}$

Q5

(a) Invalid (b) Invalid

(c) Invalid (d) Invalid

(e) Invalid (f) Invalid

(g) Invalid

Q6

(a) $\frac{12}{40}$ (b) $\frac{19}{40}$

Q7

$\frac{1}{6}$

Q8

(a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{7}{8}$

Q9

200

P1

5

P2

$\frac{1}{2}$

Exercise 9B

Set notation and Venn diagrams

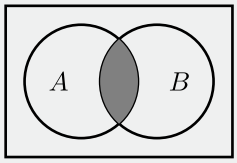
F1

(a) union, $A \cup B$

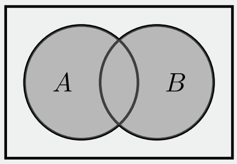
(b) intersection, $A \cap B$

F2

(a)



(b)



Q1

(a) $\{3, 4, 5\}$

(b) $\{1, 2, 3, 4, 5, 6, 7\}$

Q2

$\{2, 4, 6, 12\}$

Q3

(a) $\frac{21}{80}$ (b) $\frac{5}{8}$ (c) $\frac{9}{80}$ (d) $\frac{59}{80}$

Q4

(a) $\overline{(I \cup F)}$ (c) $I \cap F$

Q5

(a) $\frac{3}{29}$ (b) $\frac{9}{29}$ (c) $\frac{31}{58}$

P1

(a) False. (b) False. (c) True.

P2

$\frac{103}{180}$

Exercise 9C

Mutually exclusive events

F1

(a) $A \cup B$ (b) $A \cap B$

F2

(a) mutually, exclusive, outcomes

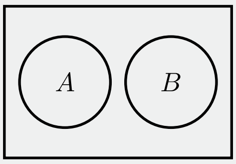
(b) cannot

(c) both

(d) intersect

F3

(a)

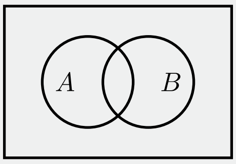


(b) $P(A)$, $P(B)$

(c) 0

F4

(a)



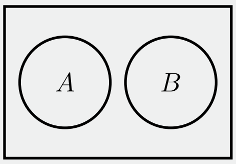
(b) Addition, Rule, $P(A \cap B)$

Q1

(a) Drawing a red card and drawing ace of spades from a standard deck of cards. Many more possible answers.

(b) Rolling a die and then flipping a coin. Many more possible answers.

Q2



Q3

(a) Yes (b) No (c) Yes

(d) Yes (e) No (f) No

Q4

(a) and (d); (a) and (f); (b) and (e); (c) and (e)

Q5

(a) Independent

(b) Independent

(c) Not independent

(d) Independent

(e) Independent

(f) Independent

Q6

(a) See full worked solutions.

(b) 0.1

(c) 1

Q7

(a) $\frac{7}{26}$ (b) $\frac{4}{13}$ (c) $\frac{1}{2}$ (d) $\frac{11}{26}$

Q8

$\frac{2}{3}$

Q9

$\frac{2}{5}$

P1

$\frac{3}{14}$

P2

See full worked solutions.

Exercise 9D

Multi-stage experiments and independent events

F1

(a) independent (b) 0.5

(c) dependent, is (d) $P(A)$, $P(B)$

(e) $\cap$, $P(A)$, $P(B)$ (f) $P(A)$, $P(B)$

F2

(a) three

(b) $P(A_1) \cap P(A_2) \cap P(A_3) \cap \ldots \cap P(A_n)$

(c) sampling

(d) sampling, independent

(e) independent, outcome

Q1

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) Part (a) is independent. Part (b) is dependent.

Q2

(a) $\frac{1}{100}$

(b) They are (virtually) independent events, since removing one person from Australia with blood type B has a negligible effect on the chance of the next person also having blood type B.

Q3

(a) $\frac{2}{15}$ (b) $\frac{8}{15}$

Q4

(a) $\frac{1}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Q5

(a) $\frac{1}{12}$ (b) $\frac{1}{4}$

Q6

$\frac{2}{11}$

Q7

(a) $\frac{9}{78}$ (b) $\frac{69}{78}$

Q8

(a) $\frac{1089}{8000}$ (b) $\frac{3267}{8000}$

(c) $\frac{2299}{4000}$ (d) $\frac{7271}{8000}$

Q9

(a) $\frac{2}{3}$ (b) $\frac{14}{55}$ (c) $\frac{12}{55}$

Q10

(a) $\frac{29}{50}$ (b) $\frac{8}{15}$

Q11

(a) $\frac{2}{5}$ (b) $\frac{29}{40}$ (c) $\frac{3}{5}$

Q12

(a) $\frac{121}{400}$ (b) $\frac{101}{200}$ (c) $\frac{99}{200}$

Q13

(a) $\frac{2}{25}$ (b) $\frac{13}{50}$ (c) $\frac{49}{50}$

Q14

(a) $\frac{91}{216}$ (b) $1-\left(\frac{5}{6}\right)^n$ (c) 26 rolls

P1

(a) $\frac{1}{6}$ (b) $\frac{7}{15}$

P2

(a) $\frac{6859}{8000}$ (b) $\frac{1083}{8000}$ (c) $\frac{7999}{8000}$

P3

(a) $\frac{1}{25}$ (b) $\frac{12}{25}$

P4

(a) $\frac{19}{49}$ (b) $\frac{29}{49}$

P5

$\frac{125}{1296}$

P6

(a) 5 tosses: $HTHTT$ or $THTHH$
6 tosses: $HTHTHH$ or $THTHTT$

(b) The last two are always the same, and it alternates leading up towards the last two.

(c) $\frac{1}{128}$

P7

(a) A or CCA (b) $\frac{1}{4}$ (c) $\frac{3124}{3125}$

Exercise 9E
Conditional probability

F1

(a) dependent

(b) reduces, B

(c) conditional probability

(d) $A \mid B$

(e) $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

F2

sample, event

F3

(a) A

(b) B

(c) independent, $A \mid B$, A

Q1

$\frac{1}{4}$

Q2

(a) $\frac{2}{25}$ (b) $\frac{3}{49}$ (c) $\frac{4}{49}$

Q3

(a) $\frac{11}{25}$ (b) $\frac{11}{26}$

Q4

(a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{2}{5}$

Q5

(a) $\frac{1}{3}$ (b) $\frac{1}{13}$

Q6

(a) $\frac{1}{9}$ (b) $\frac{1}{10}$ (c) $\frac{1}{9}$

(d) $\frac{1}{10}$ (e) $\frac{1}{10}$

Q7

30%

Q8

$\frac{1}{6}$

Q9

(a) 0.645 (b) 0.507

Q10

(a) $\frac{3}{5}$ (b) $\frac{11}{20}$

(c)

(i) $\frac{1}{2}$ (ii) $\frac{1}{2}$

(d) $\frac{6}{11}$

(e) No. (f) $\frac{5}{11}$

(g) $\frac{5}{8}$

Q11

(a)

(i) $\frac{3}{10}$ (ii) $\frac{6}{7}$

(b)

(i) $\frac{3}{10}$ (ii) $\frac{3}{4}$

(c)

(i) $\frac{8}{25}$ (ii) $\frac{8}{15}$

Q12

(a)

	T	$\bar{T}$	**Total**
F	10%	30%	40%
M	12%	48%	60%
Total	22%	78%	100%

(b)

(i) 10% (ii) 12%

(iii) 78% (iv) 48%

(v) 30%

(c) $\frac{3}{25}$

(d) $\frac{4}{5}$

(e) No, since $P(\bar{T} \mid M) \neq P(\bar{T})$

ANSWERS

Q13

See full worked solutions.

Q14

(a) $\frac{9}{50}$ (b) $\frac{113}{200}$

Q15

$\frac{37}{50}$

Q16

See full worked solutions.

Q17

(a) If B has already occurred, then A cannot occur since they are mutually exclusive. Hence, the probability must be zero.

(b) See full worked solutions.

P1

(a) 0.92 (b) 0.607

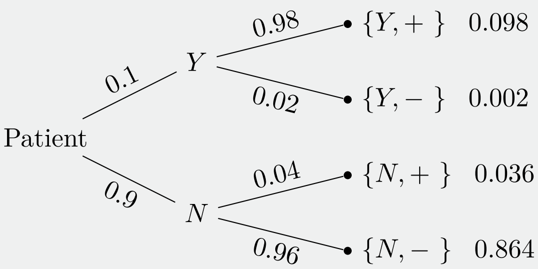
P2

$\frac{1}{3}$

P3

(a) $\frac{9}{250}$

(b)



(c) $\frac{49}{500}$

(d) See full worked solutions.

(e) $\frac{49}{67}$

(f) $\frac{433}{500}$

(g) The sum is 1.

P4

(a) See full worked solutions.

(b) See full worked solutions.

(c) B is independent of A.

(d) Mary is correct. When $P(A) = P(B)$, we have $P(A \mid B) = P(B \mid A)$.

P5

(a) $\frac{1}{6}$

(b) $\frac{1}{5}$

(c) Yes, as sample space has been reduced.

(d) $\frac{1}{3}$

Chapter Review

R1

(a) $\frac{2}{11}$ (b) $\frac{5}{11}$ (c) $\frac{2}{11}$

R2

(a) $\frac{3}{8}$ (b) $\frac{7}{8}$

R3

(a) $\frac{1}{216}$ (b) $\frac{1}{36}$ (c) $\frac{1}{216}$ (d) $\frac{1}{72}$

(e) $\frac{25}{72}$ (f) $\frac{25}{27}$ (g) $\frac{5}{72}$ (h) $\frac{5}{12}$

(i) $\frac{91}{216}$ (j) $\frac{5}{9}$ (k) $\frac{1}{6}$ (l) $\frac{8}{27}$

R4

(a) $\frac{1}{64}$ (b) $\frac{37}{64}$ (c) $\frac{9}{64}$

R5

(a) $\frac{1}{10}$ (b) $\frac{1}{3}$

R6

$\frac{7}{20}$

R7

(a) $\frac{3}{4}$ (b) $\frac{3}{4}$ (c) $\frac{1}{2}$

R8

(a) $\frac{1}{343}$ (b) $\frac{1}{49}$ (c) $\frac{30}{49}$ (d) $\frac{18}{49}$

R9

(a) $\frac{1}{177}$ (b) $\frac{55}{354}$ (c) $\frac{19}{118}$

R10

$\frac{39}{50}$

R11

$\frac{2}{11}$

R12

$\frac{41}{72}$

R13

(a) $\frac{1}{6}$ (b) $\frac{5}{12}$

R14

(a) $\frac{25}{216}$ (b) $\frac{625}{7776}$ (c) $\frac{2821}{7776}$

R15

(a) $\frac{3}{25}$ (b) $\frac{1}{5}$ (c) $\frac{3}{5}$

R16

(a)

(i) $\frac{3}{4}$ (ii) $\frac{3}{4}$

(iii) $\frac{3}{5}$ (iv) $\frac{2}{5}$

(b) No. They are independent events.

R17

(a) $\{AB, AC, AD, BC, BD, CD\}$

(b) $\frac{1}{6}$

(c) $\frac{1}{2}$

(d) $\frac{1}{3}$

R18

(a) $\frac{9}{20}$

(b) $\frac{9}{20}$

(c) $P(A \mid F) = P(A)$, hence independent.

(d) Yes, it does.

R19

(a) $\frac{1}{6}$ (b) $\frac{2}{11}$

R20

(a) $\frac{9}{50}$ (b) $\frac{27}{50}$

10. Probability Distributions

Exercise 10A

Random variables and probability distributions

F1

(a) outcome

(b) outcome

(c) distribution, distributed, sample, space

(d) discrete

(e) discrete, uniform

F2

(a) 0, 1, 2, 3 (b) random, random (c) 2

F3

(a) discrete, continuous

(b) discrete, finite

(c) continuous

Q1

(a) D (b) C (c) C (d) D

(e) D (f) C (g) D (h) D

ANSWERS

Q2

(a) 0, 1, 2

(b) $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$

(c)

# of heads	0	1	2
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Q3

(a) $X = \#$ heads, $X = 0, 1, 2$

(b) $X = \#$ heads, $X = 0, 1, 2, 3$

(c) $X =$ product, $X = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36$

(d) $X =$ number, $X = 1, 2, 3, 4, 5, 6$

(e) $X =$ sum, $X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12$

(f) $X =$ even non-zero digits, $X = 0, 1, 2$

Q4

(a) Uniform (b) Not uniform

(c) Uniform (d) Not uniform

Q5

(a) 0.3 (b) 0.1 (c) 0.5

(d) 0.2 (e) 0.9 (f) 0.45

Q6

(a) $X = \#$ failed, $X = 0, 1, 2, 3$, $X = 2$

(b) $X = \#$ red, $X = 0, 1, 2, 3, 4$, $X = 3$

(c) $X = \#$ failed attempts, $X = 0, 1, 2, \ldots$, $X = 4, 5 \ldots$

(d) $X =$ sum, $X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$, $X = 3, 5, 7, 9, 11$

(e) $X =$ product, $X = 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36$, $X = 2, 4, 6, 8, 10, 12, 16, 18, 20, 24, 30, 36$

Q7

(a) 4, 5, 6, 7

(b)

(i) 4 (ii) 6 (iii) 5, 6, 7

Q8

(a) Invalid (b) Valid

(c) Invalid (d) Valid

Q9

(a) $p = \frac{1}{6}$ (b) $p = \frac{12}{25}$

Q10

x	0	1	2
$P(X = x)$	0.36	0.48	0.16

Q11

x	0	1	2
$P(X = x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

Q12

(a)

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(b)

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(c)

x	0	1	2
$P(X = x)$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

P1

x	1	2	3	$\ldots$	n	$\ldots$
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\ldots$	$\frac{1}{2^n}$	$\ldots$

P2

See full worked solutions.

P3

See full worked solutions.

P4

$p = 6$, $q = 1$

Exercise 10B

Expected value

F1

(a) probability

(b) mean

(c) is not

F2

(a) $x_1p_1 + x_2p_2 + x_3p_3 + \ldots + x_np_n$

(b) $p(x)$

(c) μ

F3

(a) return (b) zero

(c) positive (d) negative

Q1

(a) $E(x) = 2$ (b) $E(x) = 2.65$

Q2

0.17

Q3

(a) 0.3 (b) 1.3

Q4

(a) 0.1 (b) 3

Q5

$a = 0.45,\ b = 0.25$

Q6

(a) 3.5 (b) Yes

Q7

(a)

Number of heads	0	1	2
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$ gained	−20	5	10

(b) 0

(c) Yes

Q8

(a) −$3.5

(b) No. His expected return is negative.

(c) $10

Q9

(a) The expected return is $-\frac{2}{3}$, which is negative.

(b)

(i) $\frac{157}{40}$ (ii) $\frac{197}{16}$ (iii) 40

(c) $k = \frac{60}{21}$

P1

(a) See full worked solutions.

(b) See full worked solutions.

P2

See full worked solutions.

P3

(a) Each attempt is an independent event. The fact that he 'failed' his first attempt has no bearing on any future attempt, so it is as if he started a whole new game.

(b) See full worked solutions.

(c) See full worked solutions.

(d) $\frac{1}{2}$

(e) $E(X) = 2$. Intuitively, this is expected. If you flip a typical coin, you would expect two attempts before you get your desired side.

Exercise 10C

Variance and standard deviation

F1

(a) spread (b) $\text{Var}(X)$

F2

(a) expected, square (b) $(X - \mu)^2$

(c) X^2 (d) $E\left(X^2\right) - \mu^2$

ANSWERS

F3

(a) standard, deviation, spread

(b) further

(c) σ, variance

(d) $\text{Var}(X)$, σ^2

Q1

x	1	2	3	4	5
$p(x)$	0.1	0.3	0.4	0.15	0.05
x^2	1	4	9	16	25
$x^2p(x)$	0.1	1.2	3.6	2.4	1.25

$\text{Var}(X) = 0.9875$, $\sigma = 0.9937$

Q2

x	1	2	3	4	5
$p(x)$	0.1	0.3	0.4	0.15	0.05
$X - \mu$	-1.75	-0.75	0.25	1.25	2.25
$(x-\mu)^2$	1.75^2	0.75^2	0.25^2	1.25^2	2.25^2
$(x-\mu)^2p(x)$	0.30625	0.16825	0.025	0.234325	0.253125

$\mu = 2.75$, $\text{Var}(X) = 0.9875$, $\sigma = 0.9937$

Q3

(a) $\mu = 2.6$, $\text{Var}(X) = 1.04$, $\sigma = 1.02$

(b) $\mu = 3$, $\text{Var}(X) = 2$, $\sigma = 1.41$

Q4

(a)

x	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

(b) $\frac{7}{3}$

(c) $\frac{5}{9}$

(d) $\frac{\sqrt{5}}{3}$

(e) $\frac{1}{2}$

(f) 1

Q5

(a)

x	0	1	2	3	4	5	6	7
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$

(b) 3.5

(c) $\frac{47}{12}$

(d) $\sqrt{\frac{47}{12}}$

(e) $\frac{2}{3}$

(f) 1

Q6

(a) Chemistry

(b) $z = \frac{x-\mu}{\sigma}$

P1

(a) See full worked solutions.

(b) See full worked solutions.

(c) $\text{Var}(X) \geq 0$

P2

(a) See full worked solutions.

(b) See full worked solutions.

Chapter Review

R1

(a) 0.3

(b) 0.95

(c) 0.5

(d) 2.8

R2

(a) 0.9, 0.2

(b) $\frac{2}{9}$

(c) 1

(d) If given $X \geq 4$, then of course $X \geq 2$ as well.

R3

$\frac{1}{12}$

R4

0.288

R5

x	0	1	2
$P(X = x)$	0.49	0.42	0.09

R6

(a)

x	0	1	2
$P(X=x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

(b) $\mu = \frac{8}{7}$, $\text{Var}(X) = \frac{20}{49}$, $\sigma = \frac{2\sqrt{5}}{7}$

R7

\$179 per week

R8

(a) 0 (b) a (c) a^2

R9

(a) $x = 6:$ \$1.50
$x = 18:$ $-$\$0.50

(b) \$15

R10

$a = 0.2$, $b = 0.5$

R11

$a = 20$, $b = 0.6$