NSW STAGE 6

NEW SENIOR

Mathematics

ADVANCED YEAR 11 | 4TH EDITION

J.B. FITZPATRICK
BOB AUS

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J.B. FITZPATRICK
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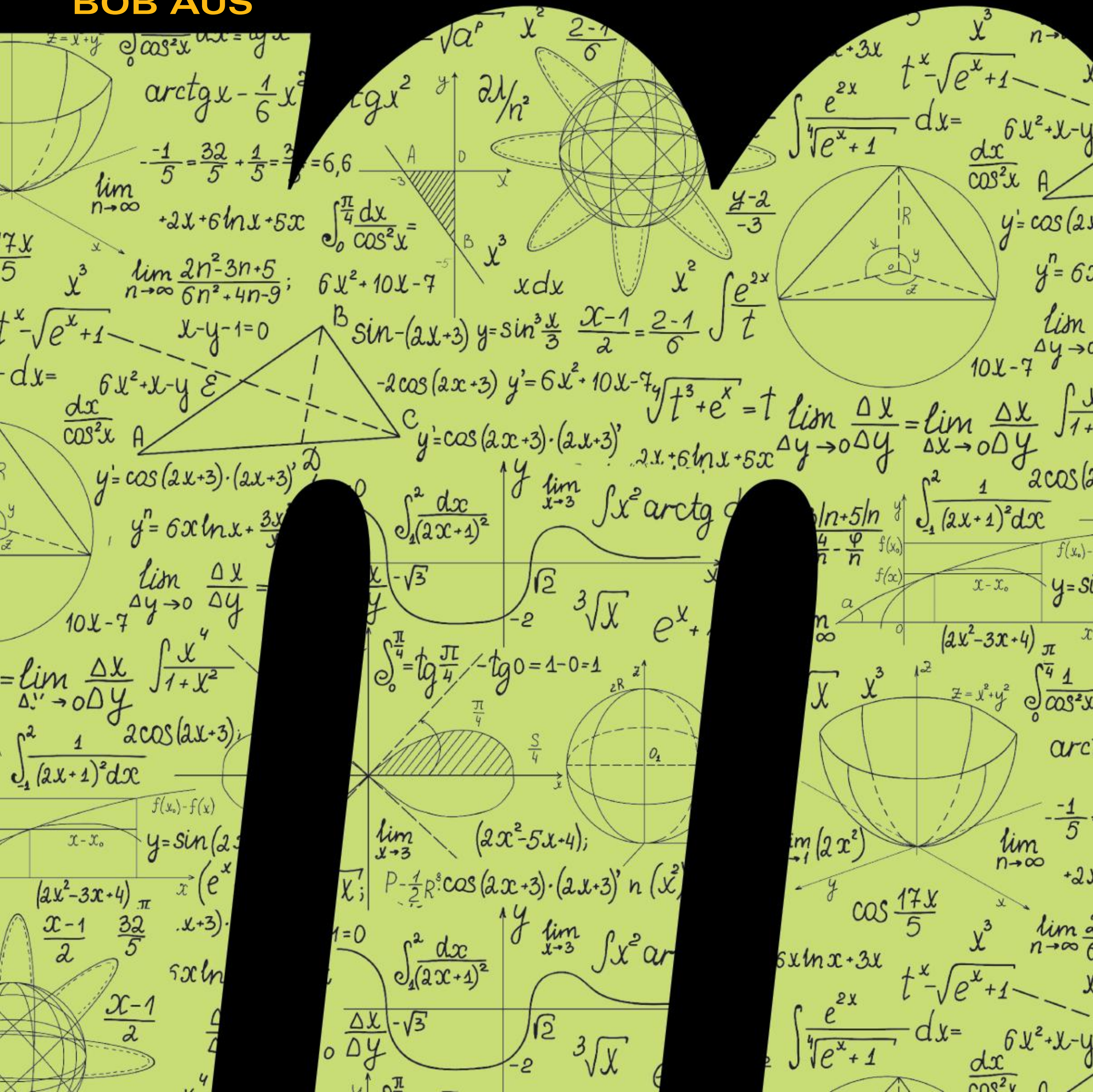
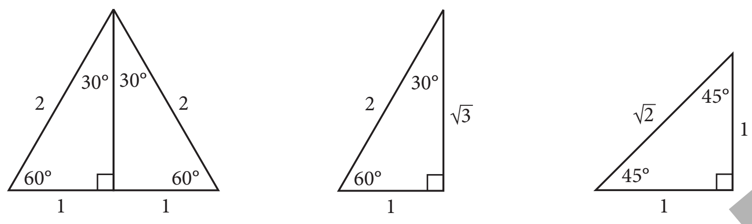


CHAPTER 5
Trigonometry and measures of angles

5.1 EXACT VALUES OF TRIGONOMETRIC RATIOS

The exact values of sin, cos and tan for 30°, 60°, and 45° can be found from the following diagrams.



You can use Pythagoras' theorem to calculate the side lengths of the right-angled triangles and relate the lengths to the trigonometric values, as summarised in the following table.

θ	30°	45°	60°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

You can either remember how to calculate these values from the triangles or remember the table of values. It is also easy to obtain the exact values for the reciprocal functions cosec, sec and cot by taking the reciprocal of each value above. These are given in the following table.

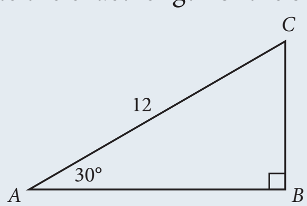
θ	30°	45°	60°
$\text{cosec}\,\theta$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$

You use the notation $\sin^n x$ for $(\sin x)^n$, but $\frac{1}{\sin x}$ cannot be represented by $\sin^{-1} x$.
$\frac{1}{\sin x} = (\sin x)^{-1} = \text{cosec}\,x$. The notation that you see on your calculator, $\sin^{-1} x$ means the angle whose sine is x.
In the Mathematics Extension course $\sin^{-1} x$ is used for the inverse function of $\sin x$.

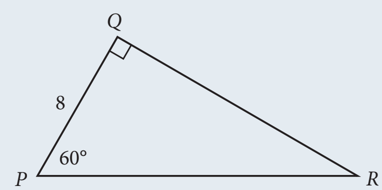
Example 1

Calculate the exact length of the sides in each triangle.

(a)



(b)



Solution

(a) $\sin 30^\circ = \dfrac{BC}{12}$

$BC = 12\sin 30^\circ$

$= 12 \times \dfrac{1}{2}$

$= 6$

$\cos 30^\circ = \dfrac{AB}{12}$

$AB = 12\cos 30^\circ$

$= 12 \times \dfrac{\sqrt{3}}{2}$

$= 6\sqrt{3}$

(b) $\tan 60^\circ = \dfrac{QR}{8}$

$QR = 8\tan 60^\circ$

$= 8\sqrt{3}$

$PR^2 = QR^2 + PQ^2$

$= (8\sqrt{3})^2 + 8^2$

$= 192 + 64 = 256$

$PR = \sqrt{256} = 16$

EXERCISE 5.1 EXACT VALUES OF TRIGONOMETRIC RATIOS

Give the exact answer to each of the following, expressing lengths in simplest surd form where necessary. (Do not use a calculator.)

1 In ΔABC, $B = 90°$, $A = 30°$, $AC = 20$ cm. Calculate the lengths of:

(a) BC (b) AB

2 In ΔABC, $C = 90°$, $A = 45°$, $BC = 10$ cm. The lengths of AC and AB are:

A $AC = 10\sqrt{2}$ cm, $AB = 10$ cm

B $AC = 10$ cm, $AB = 10\sqrt{2}$ cm

C $AC = 10$ cm, $AB = 5\sqrt{2}$ cm

D $AC = 10$ cm, $AB = 10$ cm

3 A vertical pole of height 15 m stands on level ground and a straight wire 30 m long joins the top of the pole to a point on the ground. Find:

(a) the distance of this point on the ground from the foot of the pole

(b) the angle the wire makes with the ground.

4 A ladder 10 m long, standing on level ground, leans against a vertical wall and makes an angle of 60° with the ground. Calculate:

(a) how high up the wall the ladder reaches

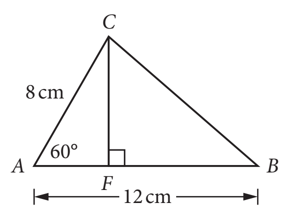
(b) the distance of the foot of the ladder from the wall.

5 In ΔABC, $AB = 12$ cm, $AC = 8$ cm, $A = 60°$. CF is drawn perpendicular to AB to meet AB at F. Calculate the length of:

(a) AF

(b) FC

(c) BC



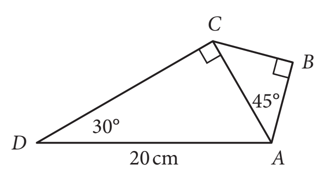
6 For the given diagram, calculate the length of:

(a) AC

(b) DC

(c) AB

(d) BC



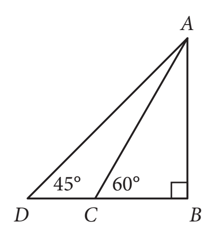
7 In the diagram, $AC = 12$ cm. Calculate the length of:

(a) AB

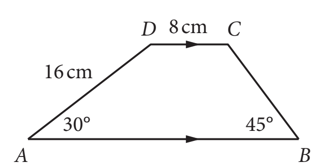
(b) BC

(c) DC

(d) AD



8 Calculate the perimeter of the trapezium $ABCD$ given that $AD = 16$ cm, $DC = 8$ cm, $A = 30°$, $B = 45°$.



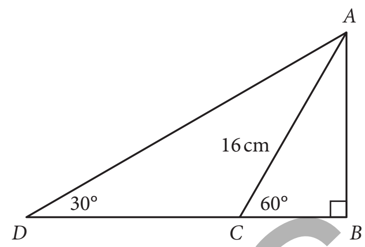
9 A stepladder stands on horizontal ground with its feet 2 m apart. If the angle formed by the legs is 60°, how high above the ground is the top of the ladder?

10 The magnitude of the angle formed by the diagonal of a rectangle and one of its longer sides is 30°. Find the dimensions of the rectangle if the length of the diagonal is 60 cm.

11 In the diagram, $AC = 16$ cm, $\angle ABD = 90°$, $\angle ACB = 60°$ and $\angle ADB = 30°$. Indicate whether each answer is correct or incorrect.

(a) $\angle DAC = 30°$ (b) $DC = 16$ cm

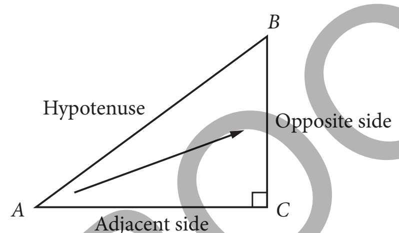
(c) $AB = 8$ cm (d) $CB = 8$ cm

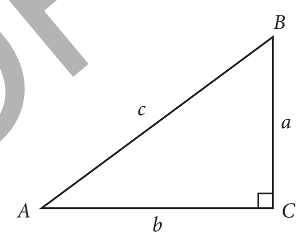


5.2 REVIEW OF RIGHT-ANGLED TRIANGLES

Trigonometric ratio definitions

From your study of the trigonometry of a right-angled triangle, you should already know the definitions of the trigonometric ratios for sine, cosine and tangent.





$$\sin A = \frac{\text{Opposite side}}{\text{Hypotenuse}} = \frac{a}{c} \qquad \cos A = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{b}{c} \qquad \tan A = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{a}{b}$$

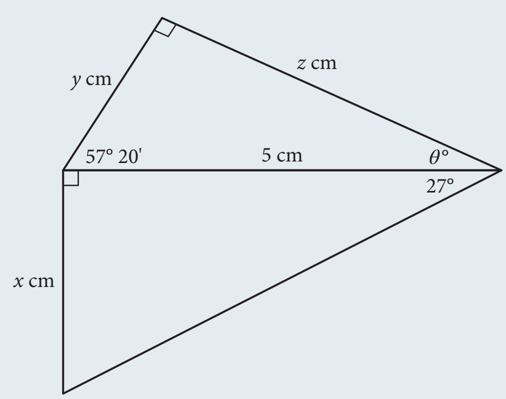
To remember these ratios you can use the mnemonic **SOHCAHTOA**, which stands for **S**in is **O**pposite over **H**ypotenuse, **C**os is **A**djacent over **H**ypotenuse, **T**an is **O**pposite over **A**djacent.

These trigonometric ratios describe the relationship between the angles and sides of a right-angled triangle, while Pythagoras' theorem $c^2 = a^2 + b^2$ describes the relationship between the side lengths without reference to the angles.

Example 2

Use the trigonometric ratios to find the value of each variable given on the diagram, giving the lengths correct to 2 decimal places.

Hence, calculate the area of the quadrilateral.



Solution

$\frac{x}{5} = \tan 27°$

$x = 5 \tan 27°$

$x = 2.55$ cm

$\frac{y}{5} = \cos 57°20'$

$y = 5 \cos 57°20'$

$y = 2.699 \approx 2.70$ cm

$\frac{z}{5} = \sin 57°20'$

$z = 5 \sin 57°20'$

$z = 4.21$ cm

$\theta = 180 - (90 + 57°20')$

$\theta = 32°40'$

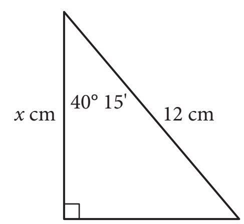
Area of quadrilateral $= \frac{1}{2}(5x + yz)$

$= \frac{1}{2}(5 \times 2.55 + 2.70 \times 4.21)$

$= 12.06 \text{ cm}^2$

EXERCISE 5.2 REVIEW OF RIGHT-ANGLED TRIANGLES

1 Which is the correct expression for the value of x in the following diagram?



A $x = 12 \sin 40° 15'$

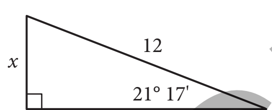
B $x = 12 \tan 40° 15'$

C $x = 12 \cos 40° 15'$

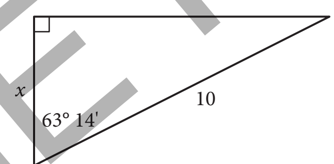
D $x = \frac{12}{\sin 49°15'}$

2 Find the value of x in each diagram. Where necessary give your answer correct to 2 decimal places.

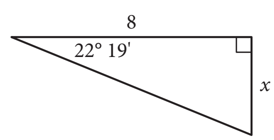
(a)



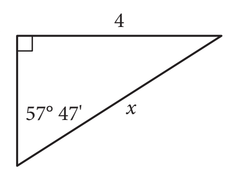
(b)



(c)

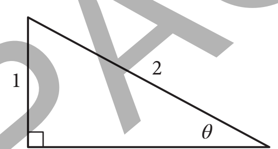


(d)

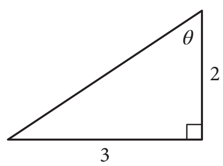


3 Find the value of θ in each diagram. Where necessary give your answer in degrees and minutes.

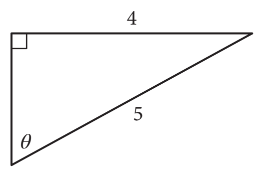
(a)



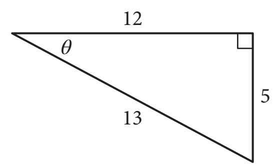
(b)



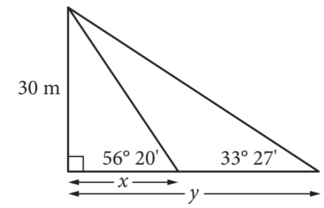
(c)



(d)



4 Use trigonometry to find the values of x and y, giving your answer correct to 1 decimal place.

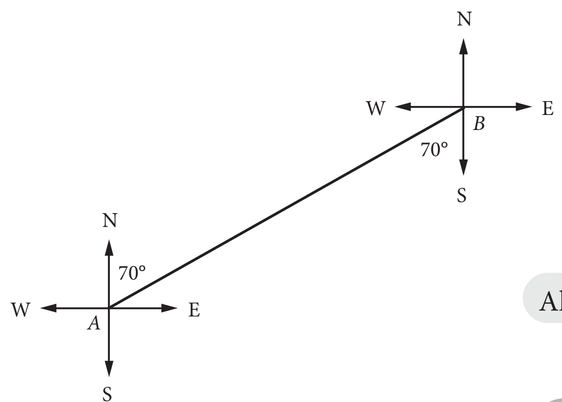


5.3 DIRECTION AND BEARING

Navigators and surveyors measure direction by reference to the points of the compass: north, south, east and west. Directions are indicated in terms of the number of degrees east or west of north or south, known as compass bearings, or are measured **clockwise from north** and written in standard three-figure notation, known as true bearings:

N 30° E or 030° means the direction is 30° east of north.

A **bearing** is a direction angle that indicates the direction of one point relative to another point. In this diagram, the bearing of *B* from *A* is N 70° E or 070°. (In bearings, the word 'from' indicates the starting point.) The bearing of *A* from *B* is S 70° W or 250°.



All bearings are in a horizontal plane.

MAKING CONNECTIONS

Bearings

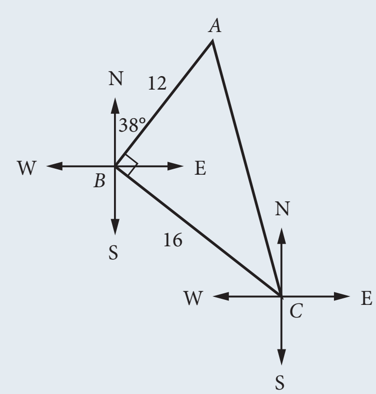
Move the compass point to view the relationship between compass bearings and true bearings.

Example 3

Two yachts sail in a straight line away from a buoy *B*. One sails 12 km in the direction 038° and the other sails 16 km in the direction 128°.

(a) How far apart are the yachts now?

(b) What is the bearing of the first yacht, as seen from the second yacht?



Solution

(a) $\angle ABC = 128° - 38° = 90°$

Use Pythagoras' theorem for ΔABC: $AC^2 = 12^2 + 16^2 = 400$

$AC = \sqrt{400} = 20$

The two yachts are 20 km apart.

(b) To find the bearing of *A* from *C* you first must calculate the size of $\angle ACN$.

In ΔABC: $\tan(\angle ACB) = \frac{12}{16} = 0.75$

$\angle ACB = 36° 52'$

From angle sum at *B*: $\angle EBC = 38°$

$BE \parallel CE$, alternate angles give: $\angle BCW = 38°$

$\angle ACN = 90° - (\angle ACB + \angle BCW)$

$= 90° - (36° 52' + 38°) = 15° 8'$

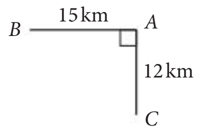
The bearing of A from C is N 15° 8′ W or 344° 52′.

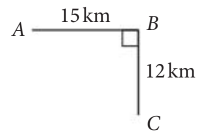
The bearing of C from A would be S 15° 8′ E or 164° 52′.

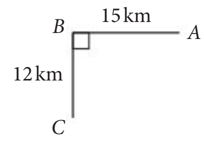
When working with bearings, draw a diagram to show all the given information. Use the diagram to calculate missing angles.

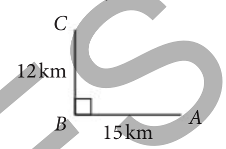
EXERCISE 5.3 DIRECTION AND BEARING

1 Two towns A and B are 15 km apart, with B due west of A. (For compass directions, 'due' means 'exactly'.) Town C is due south of B and 12 km away. Which diagram represents this information correctly?

A


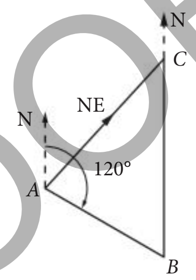
B


C


D


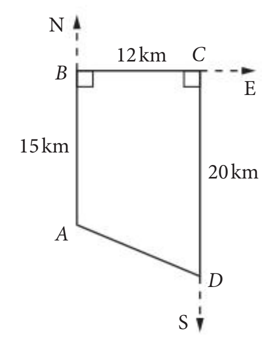
2 The bearing of B from A is 120°, the bearing of C from A is NE and the bearing of C from B is N. Find the bearing of:

(a) A from C
(b) A from B
(c) B from C.

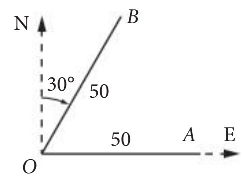


3 Two towns A and B are 15 km apart, with B due west of A. Town C is due south of B and 12 km away. Calculate the distance and bearing of A from C.

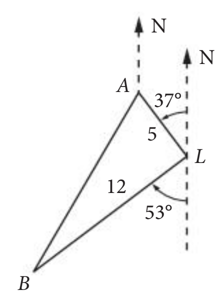
4 Ahmed cycles 15 km due north, then 12 km due east and finally 20 km due south. What are his distance and bearing from his original position?



5 On level ground, A is 50 m due east of O. The bearing of B from O is 030° and the distance of B from O is also 50 m. Find the distance and bearing of B from A.



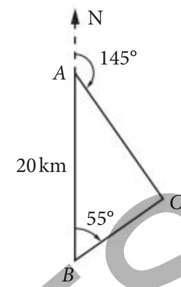
6 A is 5 km from a lighthouse, L, on a bearing N 37° W. B is 12 km from the same lighthouse on a bearing of S 53° W. Find the distance and bearing of B from A.



7 Karen and David set out from home at the same time. Karen cycles due north at 15 km h^{-1} and David cycles due east at 20 km h^{-1}. Find:

(a) how far apart they are after 1 hour
(b) after how many minutes they are 10 km apart
(c) the bearing of Karen from David at any time.

8 *A* and *B* are two lighthouses, *A* being 20 km due north of *B*. The bearing of a ship is 145° from *A* and 055° from *B*. Calculate the distance of each lighthouse from the ship.

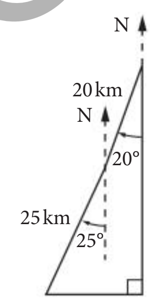


9 Two ports *A* and *B* are such that *B* is due west of *A*. *A* is due north of a ship *C*. The ship is on a course N 32° W and reaches *B* after travelling for 3 hours at 25 km h^{-1}. Calculate the distance between the two ports and the time it would take the ship to reach *A* from *C*.

10 A hiker walks 15 km from camp in the direction S 36° 52′ W and then walks 7 km due west. What are the distance and bearing of his new position from the camp?

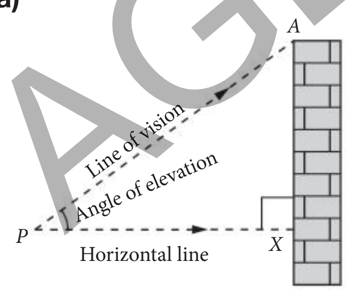
11 A ship sails for 20 km on a course S 20° W and then 25 km on a course S 25° W. Calculate:

(a) how far south the ship now is from its original position
(b) how far west the ship now is from its original position
(c) the bearing of the ship now from its original position.

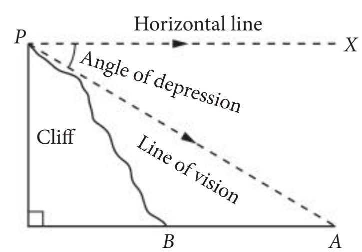


5.4 ANGLES OF ELEVATION AND DEPRESSION

(a)



(b)



In **(a)** above, if you look up at *A* from the point *P* then the **angle of elevation** of *A* from *P* is the angle between the horizontal line *PX* and the line of sight *PA*. $\angle APX$ is an angle of elevation.

The point *P* is the eye of the observer. *A* could be, for example, a point on top of a wall.

In **(b)** above, if you look down at *A* from the point *P* then the **angle of depression** of *A* from *P* is the angle between the horizontal line *PX* and the line of sight *PA*. $\angle APX$ is an angle of depression.

The point *P* is the eye of the observer. The observer could be, for example, at the top of a cliff looking down on a boat at *A* in the water below.

If we look up from *A* to *P*, then $\angle PAB$ is an angle of elevation. $\angle APX = \angle BAP$ because alternate angles between parallel lines are equal, so the angle of depression from *P* is the same as the angle of elevation from *A*.

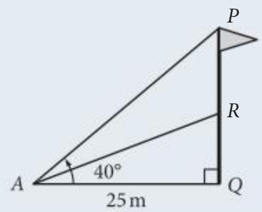
Angles of elevation and depression are in the same vertical plane.

Example 4

A point A is level with the foot of a vertical pole and 25 m away from it. The angle of elevation from point A to the top of the pole P is 40°. Calculate:

(a) the height of the pole, in metres to the nearest centimetre

(b) the angle of elevation from A of a point R, half-way up the pole, to the nearest minute.



Solution

In the diagram, PQ is the vertical pole and $\angle PAQ$ is the angle of elevation of P from A.

(a) In ΔPAQ: $\tan 40° = \dfrac{PQ}{25}$

$PQ = 25 \times \tan 40° = 20.98$

The height of the pole is 20.98 m.

(b) $PQ = 2RQ$: $RQ = 10.49$

In ΔRAQ: $\tan(\angle RAQ) = \dfrac{10.49}{25} = 0.4196$

$\angle RAQ = 22°\,46'$

The angle of elevation of R from A is $22°\,46'$.

Example 5

An observer in a lighthouse, 100 m above sea level, is watching a ship sailing towards the lighthouse. The angle of depression of the ship from the observer is 15°.

(a) How far is the ship from the lighthouse?

(b) Sometime later, the angle of depression is measured to be 25°. How far has the ship travelled in this time?

Solution

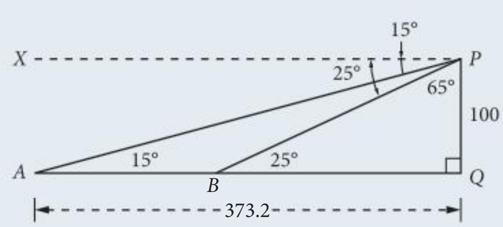
(a) In ΔPAQ: $\tan 15° = \dfrac{100}{AQ}$ **OR** $\tan 75° = \dfrac{AQ}{100}$

$AQ = \dfrac{100}{\tan 15°}$ $\qquad AQ = 100 \times \tan 75°$

$AQ = 373.2$ $\qquad AQ = 373.2$

The ship is 273.2 m from the lighthouse.

(b) In the diagram, the ship has moved from A to B. $\angle BPX$ is the angle of depression when the ship is at B.



In ΔBPQ: $\tan 25° = \dfrac{100}{BQ}$ **OR** $\tan 65° = \dfrac{BQ}{100}$

$BQ = \dfrac{100}{\tan 25°}$ $\qquad BQ = 100 \times \tan 65°$

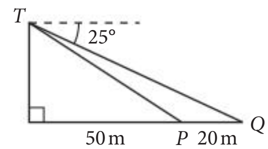
$BQ = 214.5$ $\qquad BQ = 214.5$

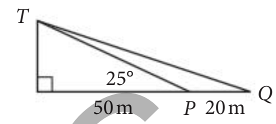
$$AB = AQ - BQ$$
$$= 373.2 - 214.5$$
$$= 158.7$$

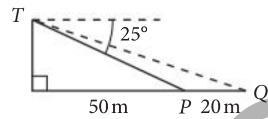
The ship has travelled 158.7 m.

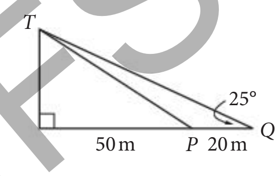
EXERCISE 5.4 ANGLES OF ELEVATION AND DEPRESSION

1 From the top of a cliff, T, a walker sees two ships P and Q at horizontal distances of 50 m and 70 m in a straight line. The angle of depression of P from T is 25°. Indicate whether each of the given diagrams is correct or incorrect.

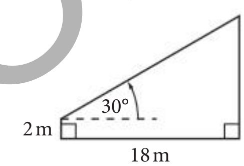
(a) 

(b) 

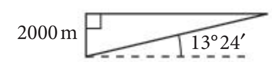
(c) 

(d) 

2 A person 2 m tall is standing on the ground and looking up at the top of a building. If the person is 18 m from the building and the angle of elevation of the top of the building is 30°, calculate the height of the building.



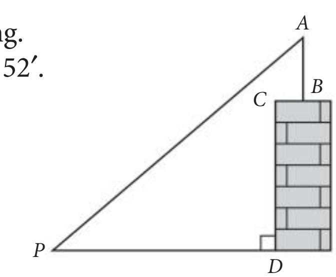
3 An aircraft flying in a horizontal straight line at an altitude of 2000 m passes directly over an observer on the ground. One minute later, the observer finds that the angle of elevation of the plane is 13° 24′. Calculate:



- **(a)** the distance flown by the aircraft in that time
- **(b)** the speed of the aircraft in km h^{-1}.

4 The diagram represents a vertical flagpole AB on top of a building. From a point P on the ground, the angle of elevation of A is 36° 52′. $PD = 55$ m, $CB = 5$ m, $AB = 12$ m. Calculate:

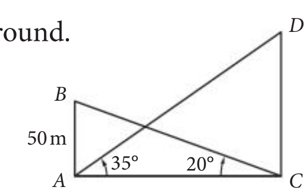
- **(a)** the height of A above the ground
- **(b)** the distance from A to P
- **(c)** the angle of elevation of A from C.



5 From an aircraft 1000 m above the ground, the angles of depression of the tops of two houses (the same height) in line with the aircraft are 40° and 60° respectively. How far apart are the two houses? (Ignore the height of the houses.)

6 AB and CD are two vertical buildings with their bases A and C on level ground. The height of AB is 50 m. The angle of elevation of B as seen from C is 20° and that of D as seen from A is 35°. Calculate:

- **(a)** the horizontal distance between the buildings
- **(b)** the height of CD
- **(c)** the angle of elevation of D as seen from B.

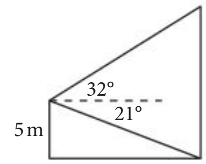


7 From the top of a cliff, T, an observer sees two ships P and Q in line with the observer and at horizontal distances of 50 m and 70 m. The angle of depression of P from T is 25°. Calculate:

- **(a)** the vertical height of the cliff
- **(b)** the angle of elevation of T from Q.

8 From a point 5 m above the ground, the angle of elevation of the top of a wall is 32° and the angle of depression of the bottom of the wall is 21°. Find:

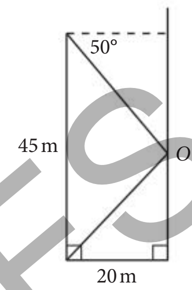
(a) the horizontal distance from the point of observation to the wall
(b) the height of the wall, correct to the nearest metre.



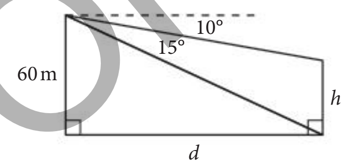
9 From a point A on the ground, the angle of elevation of the top of a tower is 38° and the angle of elevation of the top of a vertical flagpole on top of the tower is 41°. A is 80 m from the foot of the tower. The ground between A and the tower is horizontal. Calculate the length of the flagpole.

10 A city building is 45 m high. From the top of this building, the angle of depression of an object O on the wall of a building opposite is 50°. The width of the street is 20 m. Find:

(a) the height of O above street level
(b) the angle of elevation of O from the foot of the first building.



11 Two buildings of unequal height stand at a distance apart on horizontal ground. The taller building is 60 m high and from its top an observer finds that lines of sight to the bottom and the top of the shorter building are at angles of depression of 25° and 10° respectively. Calculate:



(a) the horizontal distance apart of the buildings **(b)** the height of the shorter building.

12 From the top of a lighthouse 75 m above sea level, the angles of depression of two buoys due north of the lighthouse are 60° and 30° respectively. Find, in simplest surd form:

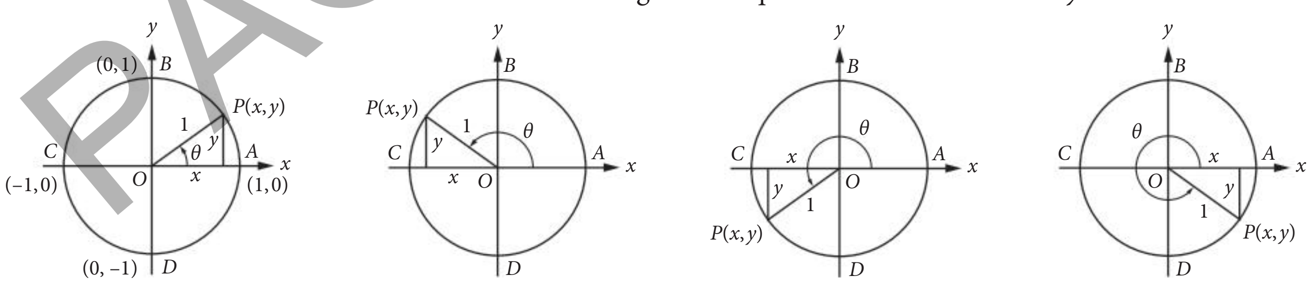
(a) the distance of each buoy from the lighthouse **(b)** the distance between the two buoys.

13 From a point P on horizontal ground, the angle of elevation of the top of a building 40 m high is 30°. From a point Q on the same horizontal level as P and in line with the foot of the building, the angle of elevation is 60°. Calculate the distance PQ in simplest surd form.

5.5 ANGLES OF ANY MAGNITUDE

The unit circle

Consider a circle of unit radius whose centre is at the origin. The equation of the circle is $x^2 + y^2 = r^2$ and $r = 1$.



Take any point P on the circumference of this unit circle whose coordinates are (x, y). Consider the point P as starting at A and rotating in an anticlockwise direction, taking various positions around the circumference as shown in the diagrams above. In each position $\angle AOP = \theta$.

You can define **cosine** (cos) and **sine** (sin) as:

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{1} = x = x\text{-coordinate of } P \quad \text{(the abscissa)}$$

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{1} = y = y\text{-coordinate of } P \quad \text{(the ordinate)}$$

The definitions of sine and cosine apply to angle θ of any magnitude. Also note:

- Because $-1 \le x \le 1$, $\cos\theta$ is from -1 to 1.
- Because $-1 \le y \le 1$, $\sin\theta$ is from -1 to 1.

Referring to the unit circle diagram, if the point P is:

(a) at A, $\theta = 0°$, the coordinates of A are $(1, 0)$ and hence $\cos 0° = 1$, $\sin 0° = 0$

(b) at B, $\theta = 90°$, the coordinates of B are $(0, 1)$ and hence $\cos 90° = 0$, $\sin 90° = 1$

(c) at C, $\theta = 180°$, the coordinates of C are $(-1, 0)$ and hence $\cos 180° = -1$, $\sin 180° = 0$

(d) at D, $\theta = 270°$, the coordinates of D are $(0, -1)$ and hence $\cos 270° = 0$, $\sin 270° = -1$

(e) all the way around to A again, $\theta = 360°$, the coordinates of A are $(1, 0)$ and hence $\cos 360° = 1$, $\sin 360° = 0$.

Note that $\cos 360° = \cos 0° = 1$, and $\sin 360° = \sin 0° = 0$.

There are four other trigonometric ratios: tangent (tan), cotangent (cot), secant (sec) and cosecant (cosec, or sometimes csc). These can all be defined in terms of cos and sin. The functions $\cot\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are the reciprocals of $\tan\theta$, $\cos\theta$ and $\sin\theta$ respectively.

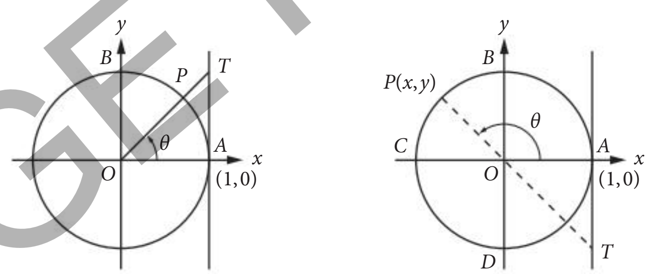
$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta},\ \cos\theta \ne 0 \qquad \cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta},\ \tan\theta \ne 0$$

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta},\ \cos\theta \ne 0 \qquad \operatorname{cosec}\theta = \frac{1}{y} = \frac{1}{\sin\theta},\ \sin\theta \ne 0$$

Because sin and cos appear as denominators, there are restrictions on the values of θ in these functions. The functions tan and sec are undefined when $\cos\theta = 0$, which is when $\theta = 90°, 270°, 450°, \ldots$ (i.e. $90° + n180°$, where n is any integer); cot and cosec are undefined when $\sin\theta = 0$, which is when $\theta = 0°, 180°, 360°, \ldots$ (i.e. $n180°$, where n is any integer).

The tangent ratio can also be considered as a ratio without reference to sin or cos.

At the point $A(1, 0)$ where the unit circle cuts the x-axis, draw a tangent line AT.



If $\angle AOP = \theta$, you can define $\tan\theta$ as the y-coordinate of T. (The tan function gets its name from this tangent line.)

If P is at B, $\theta = 90°$, OP (or OB) is parallel to AT and so $\tan 90°$ is undefined.

If instead of using the unit circle you used the circle with radius r, the results for the trigonometric ratios would have been:

$$\cos\theta = \frac{x}{r},\ r \ne 0 \qquad \sin\theta = \frac{y}{r},\ r \ne 0 \qquad \tan\theta = \frac{y}{x},\ x \ne 0$$

$$\sec\theta = \frac{r}{x},\ x \ne 0 \qquad \operatorname{cosec}\theta = \frac{r}{y},\ y \ne 0 \qquad \cot\theta = \frac{x}{y},\ y \ne 0$$

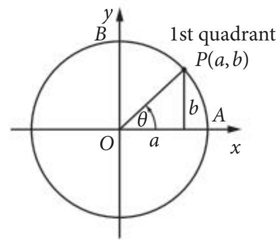
Symmetry properties of trigonometric ratios

The coordinate axes divide the circle into four quarters, called **quadrants**.

First quadrant $0° < \theta < 90°$

Consider the point $P(a, b)$, where P lies between A and B.

Both the x- and y-coordinates of P are positive numbers, so all the ratios are positive.



Second quadrant $90° < \theta < 180°$

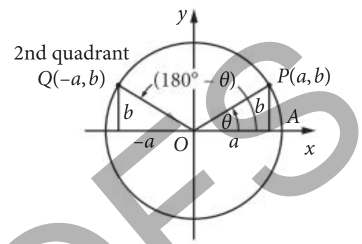
Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the second quadrant such that $\angle AOQ = 180° - \theta$.

By symmetry, the coordinates of Q are $(-a, b)$. Hence:

$$\cos(180° - \theta) = -a = -\cos\theta$$

$$\sin(180° - \theta) = b = \sin\theta$$

$$\tan(180° - \theta) = \frac{b}{-a} = -\tan\theta$$



Because the triangles are congruent, we can see that for every angle in the second quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting the second quadrant angle from 180°. (For example, $180° - 40° = 140°$, so the angle in the first quadrant corresponding to 140° is 40°.) Because the x-coordinate is negative in the second quadrant, the values of cos and tan are now negative; sin remains positive. For example:

$$\cos 140° = \cos(180° - 40°) = -\cos 40° \approx -0.7660$$

$$\sin 140° = \sin(180° - 40°) = \sin 40° \approx 0.6428$$

$$\tan 140° = \tan(180° - 40°) = -\tan 40° \approx -0.8391$$

Third quadrant $180° < \theta < 270°$

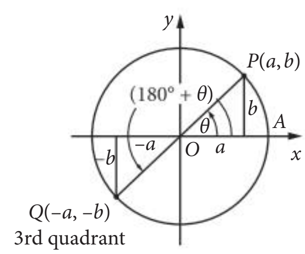
Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the third quadrant such that $\angle AOQ = 180° + \theta$.

By symmetry, the coordinates of Q are $(-a, -b)$. Hence:

$$\cos(180° + \theta) = -a = -\cos\theta$$

$$\sin(180° + \theta) = -b = -\sin\theta$$

$$\tan(180° + \theta) = \frac{-b}{-a} = \tan\theta$$



For every angle in the third quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting 180° from the third quadrant angle. (For example, $220° - 180° = 40°$ and $220° = 180° + 40°$, so the angle in the first quadrant corresponding to 220° is 40°.) Because in the third quadrant $x < 0$ and $y < 0$, only tan is positive, while sin and cos are both negative. For example:

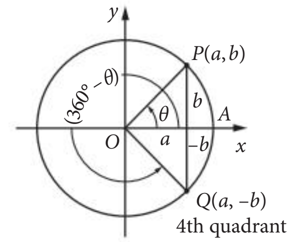
$$\cos 220° = \cos(180° + 40°) = -\cos 40° \approx -0.7660$$

$$\sin 220° = \sin(180° + 40°) = -\sin 40° \approx -0.6428$$

$$\tan 220° = \tan(180° + 40°) = \tan 40° \approx 0.8391$$

Fourth quadrant 270° < θ < 360°

Consider the point $P(a, b)$ in the first quadrant such that $\angle AOP = \theta$, and a point Q in the fourth quadrant such that $\angle AOQ = 360° - \theta$.



By symmetry, the coordinates of Q are $(a, -b)$. Hence:

$\cos(360° - \theta) = a = \cos\theta$

$\sin(360° - \theta) = -b = -\sin\theta$

$\tan(360° - \theta) = \dfrac{-b}{a} = -\tan\theta$

For every angle in the fourth quadrant there is a corresponding angle in the first quadrant whose sine, cosine and tangent ratios are numerically the same. You can find this angle by subtracting the fourth quadrant angle from 360°. (For example, 320° = 360° − 40°, so the angle in the first quadrant corresponding to 320° is 40°.) In the fourth quadrant only cos is positive. For example:

$\cos 320° = \cos(360° - 40°) = \cos 40° \approx 0.7660$

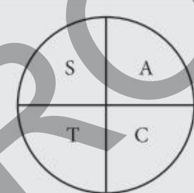
$\sin 320° = \sin(360° - 40°) = -\sin 40° \approx -0.6428$

$\tan 320° = \tan(360° - 40°) = -\tan 40° \approx -0.8391$

Sign of the trigonometric ratios

The sign of $\cos\theta$, $\sin\theta$ and $\tan\theta$ for the first four quadrants can be summarised as follows:

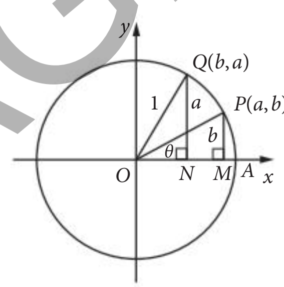
- First quadrant: all are positive (A)
- Second quadrant: $\sin\theta$ only is positive (S)
- Third quadrant: $\tan\theta$ only is positive (T)
- Fourth quadrant: $\cos\theta$ only is positive (C)

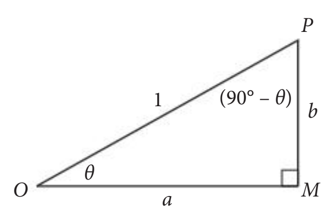


The trigonometric ratio positive signs in the different quadrants can be remembered with the mnemonic **ASTC**: **A**ll **S**tations **T**o **C**entral.

Complementary angles: θ and $(90° - \theta)$

Consider the point $P(a, b)$ on the unit circle such that $\angle AOP = \theta$, and a point Q such that $\angle AOQ = (90° - \theta)$.





From congruent triangles, the coordinates of Q are (b, a) because $ON = PM = b$ and $QN = OM = a$. Hence:

$\sin(90° - \theta) = a = \cos\theta$ $\quad$ $\cos(90° - \theta) = b = \sin\theta$

$\tan(90° - \theta) = \dfrac{a}{b} = \cot\theta$ $\quad$ $\cot(90° - \theta) = \dfrac{b}{a} = \tan\theta$

$\sec(90° - \theta) = \dfrac{1}{b} = \text{cosec}\,\theta$ $\quad$ $\text{cosec}(90° - \theta) = \dfrac{1}{a} = \sec\theta$

These relationships are said to be **complementary**. This is why the prefix ‘co-’ is in the words cosine, cosecant and cotangent.

The ratios sine and cosine, tangent and cotangent, secant and cosecant are complementary pairs. For example:

$\sin 50° = \cos(90° - 50°) = \cos 40°$ $\quad$ $\tan 75° = \cot(90° - 75°) = \cot 15°$

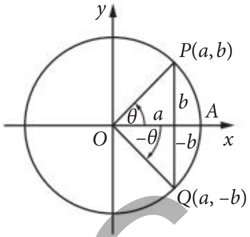
$\sec 80° = \operatorname{cosec}(90° - 10°) = \operatorname{cosec} 10°$ $\quad$ $\cos 60° = \sin(90° - 60°) = \sin 30°$

Use your calculator to verify these results.

Negative angles: $\theta < 0°$

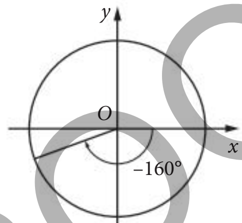
So far we have only considered $\theta > 0°$. If we start from the point A and rotate anticlockwise to P, then $\theta > 0°$. However, if we rotate clockwise to Q so that $\angle AOQ = \angle AOP$, then $\theta < 0°$.

Hence, by symmetry:



$\cos(-\theta) = a = \cos\theta$ $\qquad$ $\sin(-\theta) = -b = -\sin\theta$ $\qquad$ $\tan(-\theta) = \dfrac{-b}{a} = -\tan\theta$

For example:



$\cos(-40°) = \cos 40° \approx 0.7660$

$\tan(-25°) = -\tan 25° \approx -0.4663$

$\sin(-70°) = -\sin 70° \approx -0.9397$

$\cos(-160°) = \cos 160° = -\cos 20° \approx -0.9397$

$\tan(-245°) = -\tan 245° = -\tan 65° \approx -2.1445$

$\sin(-210°) = -\sin 210° = -(-\sin 30°) = \sin 30° = 0.5$

$\cot(-135°) = -\cot 135° = \cot 45° = \tan 45° = \dfrac{1}{\sqrt{2}} \approx 0.7071$

EXERCISE 5.5 ANGLES OF ANY MAGNITUDE

1 In which of the four quadrants is the following true?

(a) $\sin\theta > 0$ (b) $\tan\theta < 0$ (c) $\cos\theta < 0$ (d) $\sin\theta < 0$ and $\tan\theta < 0$
(e) $\sin\theta > 0$ and $\cos\theta < 0$ (f) $\cos\theta < 0$ and $\tan\theta > 0$ (g) $\cos\theta > 0$ and $\tan\theta > 0$

2 State the quadrant of each angle.

(a) 72° (b) 114° (c) 95° (d) 200° (e) 321°
(f) 183° (g) 83° (h) 216° (i) 300° (j) 155°

3 Express each of the following as a trigonometric ratio of angle A.

(a) $\sin(180° - A)$ (b) $\cos(90° - A)$ (c) $\tan(360° - A)$
(d) $\cos(180° + A)$ (e) $\sin(360° - A)$ (f) $\cot(90° - A)$

4 Use a calculator to evaluate $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$ for each of the following values of θ, writing each answer correct to 4 decimal places:

(a) (i) 125° (ii) 152° (iii) 117° (b) (i) 205° (ii) 217° (iii) 251°
(c) (i) 282° (ii) 301° (iii) 342° (d) (i) −25° (ii) −122° (iii) −215°

5 If $\sin\alpha = 0.2$, write the value of:

(a) $\sin(180° - \alpha)$ (b) $\sin(360° - \alpha)$ (c) $\sin(-\alpha)$
(d) $\cos(90° - \alpha)$ (e) $\sin(180° + \alpha)$ (f) $\operatorname{cosec}\alpha$

6 If $\tan\theta = t$, express in terms of t:

(a) $\cot\theta$ (b) $\cot(90° - \theta)$ (c) $\tan(180° - \theta)$
(d) $\tan(360° - \theta)$ (e) $\cot(180° - \theta)$ (f) $\tan(180° + \theta)$

7 If $\cos A = c$, express in terms of c:

(a) $\sec A$ (b) $\cos(-A)$ (c) $\cos(180° - A)$
(d) $\cos(360° - A)$ (e) $\sec(-A)$ (f) $\cos(180° + A)$

8 Use a calculator to evaluate, correct to 4 decimal places:

(a) $\tan 305°$ (b) $\sin 212°$ (c) $\cos(-140°)$
(d) $\sin(-160°)$ (e) $\cot 42°$ (f) $\cos 260°$

9 If θ is an angle in the second quadrant, state whether the following are positive or negative:

(a) $\cos(180° - \theta)$ (b) $\tan(180° - \theta)$ (c) $\sin(90° - \theta)$
(d) $\sin(360° - \theta)$ (e) $\cos(180° + \theta)$ (f) $\tan(90° - \theta)$

10 If $90° < \theta < 180°$, use a unit circle diagram to show that:

(a) $\cos(180° + \theta) = -\cos\theta$ (b) $\sin(360° - \theta) = -\sin\theta$

11 Which expression is equal to $\sin(360° + \theta)$?

A $\cos\theta$ B $-\sin\theta$ C $-\cos\theta$ D $\sin\theta$

12 Write 'correct' or 'incorrect' for each answer: $\cos\theta = \ldots$

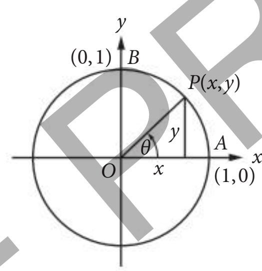
(a) $\cos(360° - \theta)$ (b) $\sin(180° - \theta)$ (c) $-\cos(180° + \theta)$ (d) $\sin(90° - \theta)$

5.6 MORE TRIGONOMETRIC EXACT VALUES

You have already found exact values of trigonometric ratios for the first quadrant angles of 30°, 45° and 60°. You can also find exact values for 0° and 90°.

Using the unit circle:

- $\theta = 0°$ when P is at A
- $\theta = 90°$ when P is at B



The table of exact values now becomes:

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

You should learn these results for sin, cos and tan. Once learnt, they can easily be used to find exact values for the reciprocal functions cosec, sec and cot.

Example 6

Find the exact value of each expression.

(a) $\sin 150°$ (b) $\cos 225°$ (c) $\tan 240°$ (d) $\sin 270°$ (e) $\cos(-300°)$ (f) $\cos 405°$

Solution

(a) $\sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}$

(b) $\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$

(c) $\tan 240° = \tan(180° + 60°) = \tan 60° = \sqrt{3}$

(d) $\sin 270° = \sin(180° + 90°) = -\sin 90° = -1$

(e) $\cos(-300°) = \cos 300° = \cos(360° - 60°)$
$= \cos 60° = \frac{1}{2}$

(f) $\cos 405° = \cos(360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

Example 7

Find the exact value of each expression.

(a) sec 150° (b) cosec 225° (c) cot 240° (d) cosec 270° (e) sec (–300°) (f) cot 405°

Solution

(a) $\sec 150° = \sec(180° - 30°)$
$= -\sec 30°$ (1st quadrant)
$= -\frac{2}{\sqrt{3}}$

(b) $\text{cosec } 225° = \text{cosec }(180° + 45°)$
$= -\text{cosec } 45°$ (3rd quadrant)
$= -\sqrt{2}$

(c) $\cot 240° = \cot(180° + 60°)$
$= \cot 60°$ (3rd quadrant)
$= \frac{1}{\sqrt{3}}$

(d) $\text{cosec } 270° = \frac{1}{y}$ (unit circle definition)
$= \frac{1}{-1}$ (In the unit circle an angle of 270° corresponds to a position of (0, –1).)
$= -1$

(e) $\sec(-300°) = \sec(-300° + 360°)$
$= \sec 60°$
$= 2$

(f) $\cot 405° = \cot(405° - 360°)$
$= \cot 45°$
$= 1$

Example 8

Find all values of θ, for $0° \le \theta \le 360°$, for which:

(a) $\cos\theta = \frac{1}{2}$ (b) $\sin\theta = -\frac{1}{\sqrt{2}}$ (c) $\tan\theta = 1$ (d) $\cos 2\theta = -\frac{\sqrt{3}}{2}$ (e) $\sin\theta = -1$

Solution

(a) $\cos\theta > 0$, so θ is in the 1st and 4th quadrants.
$\cos\theta = \frac{1}{2}$: $\theta = 60°, 360° - 60°$
$\theta = 60°, 300°$

(b) $\sin\theta < 0$, so θ is in the 3rd and 4th quadrants.
$\sin\theta = -\frac{1}{\sqrt{2}}$: $\theta = 180° + 45°, 360° - 45°$
$\theta = 225°, 315°$

(c) $\tan\theta > 0$, so θ is in the 1st and 3rd quadrants.
$\tan\theta = 1$: $\theta = 45°, 180° + 45°$
$\theta = 45°, 225°$

(d) $0° \le \theta \le 360°$, so $0° \le 2\theta \le 720°$. Because $\cos 2\theta < 0$, 2θ is in the 2nd and 3rd quadrants.

$\cos 2\theta = -\frac{\sqrt{3}}{2}$: $\quad 2\theta = 180° - 30°, 180° + 30°, 540° - 30°, 540° + 30°$

$2\theta = 150°, 210°, 510°, 570°$

$\theta = 75°, 105°, 255°, 285°$

(e) $\sin \theta < 0$, so θ is in the 3rd and 4th quadrants.

$\sin \theta = -1$: $\quad \theta = 180° + 90°, 360° - 90°$

$\theta = 270°$

Example 9

Find all values of θ, for $0° \le \theta \le 360°$, for which:

(a) $\sec \theta = 2$ (b) $\operatorname{cosec} \theta = -\sqrt{2}$ (c) $\cot \theta = 1$ (d) $\sec 2\theta = -\frac{2}{\sqrt{3}}$ (e) $\operatorname{cosec} \theta = -1$

Solution

(a) $\sec \theta > 0$, so θ is in the 1st and 4th quadrants.

$\sec 60° = 2$: $\quad \theta = 60°, 360° - 60°$

$\theta = 60°, 300°$

(b) $\operatorname{cosec} \theta < 0$, so θ is in the 3rd and 4th quadrants.

$\operatorname{cosec} 45° = \sqrt{2}$: $\quad \theta = 180° + 45°, 360° - 45°$

$\theta = 225°, 315°$

(c) $\cot \theta > 0$, so θ is in the 1st and 3rd quadrants.

$\cot 45° = 1$: $\quad \theta = 45°, 180° + 45°$

$\theta = 45°, 225°$

(d) $0° \le \theta \le 360°$ so $0° \le 2\theta \le 720°$. Since $\sec 2\theta < 0$, then 2θ is in the 2nd and 3rd quadrants.

$\sec 30° = \frac{2}{\sqrt{3}}$: $\quad 2\theta = 180° - 30°, 180° + 30°, 540° - 30°, 540° + 30°$

$2\theta = 150°, 210°, 510°, 570°$

$\theta = 75°, 105°, 255°, 285°$

(e) $\operatorname{cosec} \theta = -1$ so $\frac{1}{y} = -1$ (unit circle definition)

$y = -1$

The position is $(0, -1)$

$\theta = 270°$

EXERCISE 5.6 MORE TRIGONOMETRIC EXACT VALUES

For questions **1** to **4**, write the exact value without using a calculator.

1 (a) $\sin 90°$ (b) $\cos 120°$ (c) $\tan 150°$ (d) $\cos 180°$ (e) $\sin 120°$

2 (a) $\sin 180°$ (b) $\cos 210°$ (c) $\tan 225°$ (d) $\cos 240°$ (e) $\tan 180°$

3 (a) $\sin 270°$ (b) $\tan 300°$ (c) $\tan 315°$ (d) $\cos 330°$ (e) $\sin 300°$

4 (a) $\sin 360°$ (b) $\cos 390°$ (c) $\tan 405°$ (d) $\cos 450°$ (e) $\sin 420°$

5 The exact value of $\sin 210°$ is:

A $-\frac{1}{2}$ B $\frac{\sqrt{3}}{2}$ C $\frac{2}{\sqrt{3}}$ D 0

6 Write 'correct' or 'incorrect' for each answer: $-1 = \ldots$

(a) $\cos 180°$ (b) $\sin 45°$ (c) $\sin 270°$ (d) $\tan 495°$

For questions **7** to **18**, find all the correct values of θ for $0° < \theta < 360°$.

7 $\sin\theta = -\frac{\sqrt{3}}{2}$ **8** $\tan\theta = -1$ **9** $\cos\theta = -1$ **10** $\sin\theta = \cos\theta$

11 $\sin\theta = 0$ **12** $2\cos\theta + 1 = 0$ **13** $2\sin\theta = \sqrt{3}$ **14** $\sin\theta + \sqrt{3}\cos\theta = 0$

15 $\sin 2\theta = 0.5$ **16** $\cos\frac{\theta}{2} = \frac{1}{2}$ **17** $\tan 2\theta = 1$ **18** $\sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}$

19 Use a calculator to evaluate cosec θ, sec θ, and cot θ for each of the following values of θ, writing each answer correct to 4 decimal places:

(a) (i) 125° (ii) 152° (iii) 117° (b) (i) 205° (ii) 217° (iii) 251°
(c) (i) 282° (ii) 301° (iii) 342° (d) (i) −25° (ii) −122° (iii) −215°

20 If $\cos A = c$, express in terms of c:

(a) $\sec A$ (b) $\sec(180° - A)$ (c) $\sec(-A)$ (d) $\sec(180° + A)$

21 Use a calculator to evaluate, correct to 4 decimal places:

(a) cosec 305° (b) cot 212° (c) sec (−140°) (d) cot 42°

22 Which expression is equal to $\text{cosec}\,(360° + \theta)$?

A $\cos\theta$ B $\sin\theta$ C $\sec\theta$ D $\text{cosec}\,\theta$

23 Write 'correct' or 'incorrect' for each answer: $\sec\theta = \ldots$

(a) $\sec(360° - \theta)$ (b) $\text{cosec}\,(180° - \theta)$ (c) $\sec(180° + \theta)$ (d) $\text{cosec}\,(90° - \theta)$

24 Write the exact value without using a calculator.

(a) cosec 90° (b) sec 150° (c) cot 240° (d) cosec 180° (e) sec 300°
(f) cot 315° (g) sec (−30°) (h) cosec 510° (i) cot (−150°) (j) sec 120° + cosec 150°

25 The exact value of cosec 300° is:

A $-\frac{\sqrt{3}}{2}$ B $\frac{\sqrt{3}}{2}$ C $-\frac{2}{\sqrt{3}}$ D $\frac{2}{\sqrt{3}}$

26 Solve for $0° \le \theta \le 360°$.

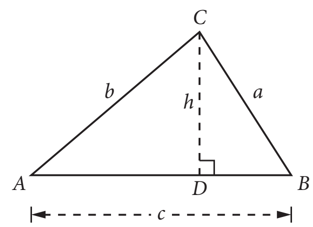
(a) $\text{cosec}\,\theta = -\frac{2}{\sqrt{3}}$ (b) $\cot\theta = \sqrt{3}$ (c) $\sec\theta = -1$ (d) $\text{cosec}\,\theta = -\sec\theta$

(e) $\text{cosec}\,\theta = 0$ (f) $\sec\theta = 2$ (g) $\cot 2\theta = \frac{1}{\sqrt{3}}$ (h) $\sec\frac{\theta}{2} = \frac{2}{\sqrt{3}}$

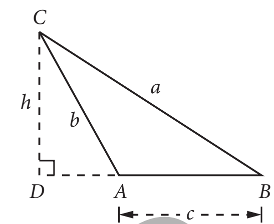
5.7 THE SINE RULE

Not all triangles are right-angled. You now consider triangles that have either three acute angles or one obtuse angle and two acute angles. To do this you must establish two rules: the **sine rule** and the **cosine rule**.

The sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



Acute-angled triangle



Obtuse-angled triangle

In triangle ABC, a, b and c are the lengths of the sides opposite the angles of magnitudes A, B and C respectively. A, B and C are also used to label the vertices of the triangle.

Proof

Let h be the length of the perpendicular from C to AB. The foot of this perpendicular lies on AB if ΔABC is acute-angled (diagram above left), or on BA produced if ΔABC is obtuse-angled (diagram above right).

In the right-angled triangles ΔACD and ΔBCD:

Acute-angled triangle:

$h = b\sin A = a\sin B$

So: $b\sin A = a\sin B$

Hence: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

Obtuse-angled triangle:

$h = b\sin(180° - A) = a\sin B$

But: $\sin(180° - A) = \sin A$

So: $b\sin A = a\sin B$

In a given triangle ABC, the ratio of a side to the sine of the opposite angle is a constant.

Similarly, by drawing a perpendicular from A to BC it can be seen that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Hence $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

The sine rule may also be written as $\dfrac{\sin A}{\sin B} = \dfrac{a}{b}$: the ratio of the sines = the ratio of the corresponding sides.

The sine rule can be used with a triangle when you are given:

(a) the size of two angles and the length of one side, **OR**

(b) the lengths of two sides and the size of an angle opposite one of these sides.

Important result

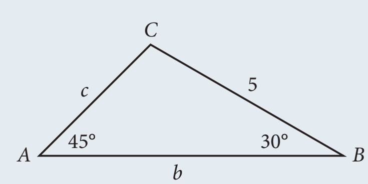
$\sin(180° - \theta) = \sin\theta$

Hence: $\sin 150° = \sin(180° - 30°) = \sin 30° = 0.5$

Solving an equation like $\sin\theta = 0.5$ gives two possible answers: $\theta = 30°$ or $(180 - 30)° = 150°$

Example 10

In ΔABC, given $A = 45°$, $B = 30°$ and $BC = 5$ cm, calculate the size of C, b and c.



Solution

Because $A + B + C = 180°$, it follows that $C = 180° - (45° + 30°) = 105°$.

Using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{5}{\sin 45^\circ} = \frac{b}{\sin 30^\circ}$$
$$b = \frac{5\sin 30^\circ}{\sin 45^\circ}$$
$$b = 5 \times \frac{1}{2} \times \frac{\sqrt{2}}{1}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 105^\circ} = \frac{5}{\sin 45^\circ}$$
$$c = \frac{5\sin 105^\circ}{\sin 45^\circ}$$
$$c = 6.8\,\text{cm}$$

The exact value of b is $\frac{5\sqrt{2}}{2}$ cm, which is 3.5 cm correct to one decimal place.

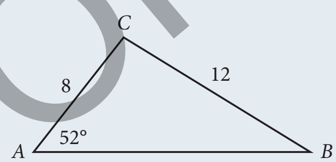
The ambiguous case

It is sometimes possible to find two different solutions that use the same values for the two sides of a triangle and an angle opposite one of these sides. This situation is investigated in the examples below.

Example 11

Solve the triangle ABC given $A = 52°$, $BC = 12$ cm and $AC = 8$ cm.

(In this context, to 'solve the triangle' is to find the sizes of all unknown angles and sides.)



Solution

Find B by writing the sine rule in the form with $\sin\theta$ in the numerator.

Sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 52^\circ}{12} = \frac{\sin B}{8}$$
$$\sin B = \frac{8\sin 52^\circ}{12}$$

If needed, write as a decimal: $\sin B = 0.5253$

Because $\sin(180° - \theta) = \sin\theta$ you must consider both the acute angle and its supplement as possible solutions.

$B = 31°\,41'$ or $180° - 31°\,41'$
$B = 31°\,41'$ or $148°\,19'$

Check whether $B = 148°\,19'$ is a possible answer. The angle sum of a triangle is 180°:

$A + B = 52° + 148°\,19' = 200°\,19' > 180°$

Thus the only value of B is $31°\,41'$.

Find C using A and B:

$C = 180° - (52° + 31°\,41')$
$C = 96°\,19'$

Find c using the sine rule:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\frac{c}{\sin 96^\circ 19'} = \frac{12}{\sin 52^\circ}$$
$$c = \frac{12\sin 96^\circ 19'}{\sin 52^\circ}$$
$$c = 15.14 \qquad \text{(correct to 2 decimal places)}$$

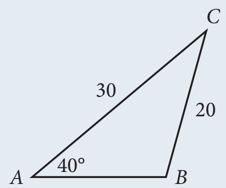
$\therefore B = 31°\,41'$, $C = 96°\,19'$, $c = 15.14$ cm

Example 12

In ΔABC, given $A = 40°$, $BC = 20$ cm and $AC = 30$ cm, find the sizes of the other angles.

Solution

Draw a diagram:



By the sine rule:
$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin 40°}{20} = \frac{\sin B}{30}$$
$$\sin B = \frac{30 \sin 40°}{20}$$
$$B = 74°\,37' \quad \text{or} \quad 105°\,23'$$

Test to see whether these values for B give a valid value for C:

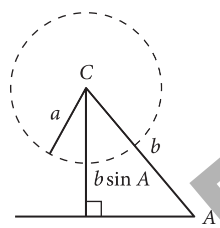
When $B = 74°\,37'$: $\quad C = 180° - (40° + 74°\,37') = 65°\,23'$

When $B = 105°\,23'$: $\quad C = 180° - (40° + 105°\,23') = 34°\,37'$

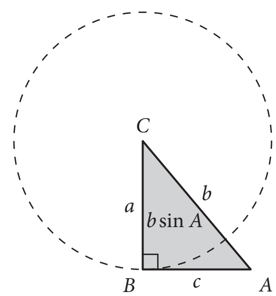
Both these values for C are valid, so there are two possible triangles for the given information. This is an example of the ambiguous case: there is not enough information for you to find a unique solution.

In general, for a triangle ABC if you are given two sides and a non-included angle (that is, values of a, b and A), then one of four situations can occur.

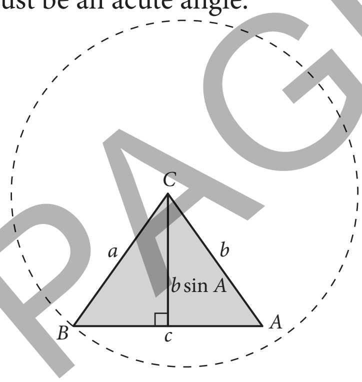
Case 1: If $a < b \sin A$ then no triangle can be constructed.



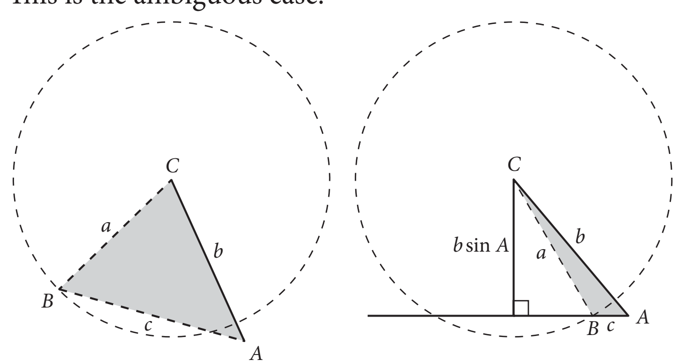
Case 2: If $a = b \sin A$ then one triangle exists and it is right-angled at B.



Case 3: If $a > b$, there is a unique solution as B must be an acute angle.



Case 4: If $b > a > b \sin A$, there are two distinct triangles. This is the ambiguous case.



The ambiguous case arises only when the given angle is opposite the shorter of the two given sides. Angle B may be either acute or obtuse. The two possible sizes for angle B are supplementary to each other.

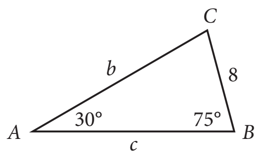
MAKING CONNECTIONS

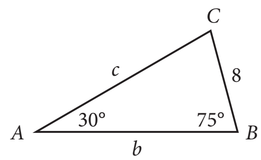
The ambiguous case of the sine rule

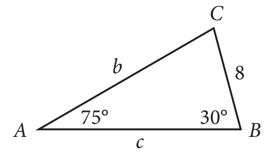
Use technology to explore the ambiguous case of the sine rule.

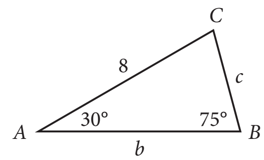
EXERCISE 5.7 THE SINE RULE

1 In a triangle ABC, $a = 8$, $A = 30°$ and $B = 75°$. The correct diagram is:

A


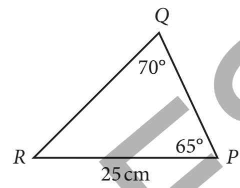
B


C


D


2 In ΔABC, $b = 10$, $B = 45°$ and $C = 120°$. Calculate the size of A, a and c.

3 In ΔPQR, $P = 65°$, $Q = 70°$ and $PR = 25$ cm. Calculate the length of PQ.

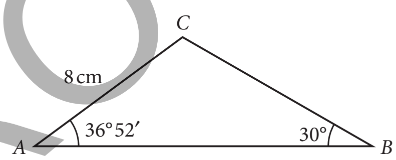


4 In ΔABC, $a = 5$, $A = 60°$ and $B = 45°$. Find the exact value of b.

5 In ΔABC, $\sin B = \frac{4}{5}$, $a = 6$ and $b = 9$. Find $\sin A$.

6 In ΔABC, $B = 2A$, $b = 5$ and $a = 3$. Show that $5 \sin A = 3 \sin 2A$.

7 Find the length of the longest side of a triangle ABC in which $A = 36° 52'$, $B = 30°$ and $AC = 8$ cm.



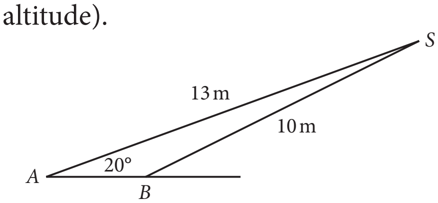
8 In a triangle ABC, $A = 42°$, $B = 28°$ and $BC = 6$ cm. Indicate whether each statement is correct or incorrect.

(a) $\frac{AB}{\sin 110^\circ} = \frac{6}{\sin 28^\circ}$ **(b)** $\frac{AB}{\sin 110^\circ} = \frac{6}{\sin 42^\circ}$ **(c)** $\frac{AB}{\sin 100^\circ} = \frac{6}{\sin 42^\circ}$ **(d)** $\frac{AB}{\sin 70^\circ} = \frac{6}{\sin 42^\circ}$

9 In a triangle ABC, $a = 16$, $b = 12$ and $\sin A = 0.4$. Calculate:

(a) $\sin B$ **(b)** the length of the perpendicular from C to AB (the altitude).

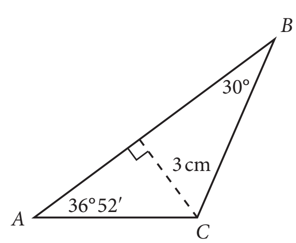
10 A wooden stake S is 13 m from a point A on a straight fence. SA makes an angle of 20° with the fence. If a goat is tethered to S by a 10 m rope, where on the fence is the closest point to A that the goat can reach?



11 In a triangle ABC, $AC = 30$ cm, $AB = 44$ cm and $B = 37°$. Find two possible values of angles A and C.

12 In a triangle PQR, $PQ = 20$ cm, $QR = 22$ cm and $R = 15°$. Find two possible values for $\angle RPQ$ and PR.

13 In a triangle ABC, $A = 36° 52'$, $B = 30°$ and the perpendicular distance from C to AB is 3 cm. Calculate the perimeter of ΔABC.



14 In ΔABC, the lengths of BC and AC are in the ratio $2 : 1$ and $\sin B = \frac{1}{4}$. Calculate:

(a) $\sin B$ **(b)** two possible sizes for A.

15 In ΔABC, $B = 25°$, $C = 55°$, $BC = 5$ m. Calculate:

(a) the length of AC
(b) the length of AX, where X is the foot of the perpendicular line from A to BC.

16 In ΔPQR, $PR = 3$ cm, $P = 40°$, $Q = 60°$. Calculate:

(a) the perimeter of ΔPQR
(b) the length of the perpendicular line from R to PQ.

17 In $\angle KLM$, $L = 55°$, $L = 35°$, $KL = 10$ cm.

(a) Use the sine rule to calculate the length of KM correct to 1 decimal place.
(b) What is the size of K? Comment on the significance of this result.

18 In ΔXYZ, $XY = 5$ m, $X = 68°$, $Y = 82°$. Calculate the length of XZ.

19 An aircraft flies 400 km from point A to point B on a course 040°. It then flies on a course 160° from B to C, 500 km from A. Calculate:

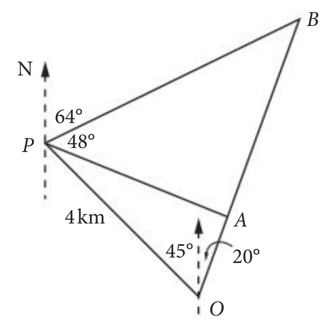
(a) the distance BC (b) the bearing of C from A.

20 The bearing of a ship from a lighthouse A is N 75° E. The ship's bearing from a second lighthouse B, 44 km south of A, is N 40° E. Find the distance of the ship from B.

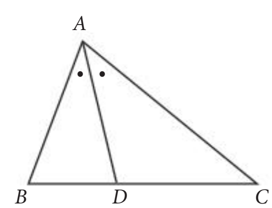
21 Three points A, B and C lie on a horizontal plane. B is 2000 m due south of A. C is on a bearing 145° from A and 052° from B. Calculate the distance of C from both A and B.

22 Two points A and B on the same bank of a river are 50 m apart. C is a point on the other bank. $A = 80°$, $B = 70°$. Calculate the width of the river, assuming the river is straight and its width is constant.

23 O, A and B are three points in a straight line (in that order). The bearings of A and B from O are 020°. From a point P, 4 km from O in a direction NW, the bearings of A and B are 112° and 064° respectively. Calculate the distance from A to B.

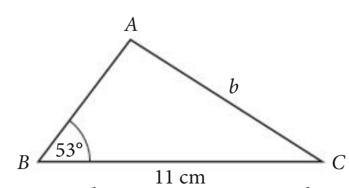


24 In ΔABC, AD bisects angle A. Use the sine rule to prove that $\dfrac{AB}{AC} = \dfrac{BD}{DC}$.



25 In ΔABC, $a = 11$ cm and $B = 53°$.

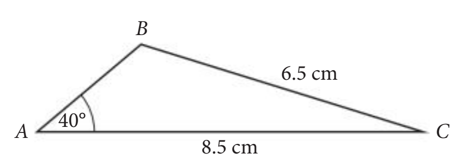
What can you say about the value of b if:
(a) two distinct triangles can be drawn to fit the information
(b) only one triangle can be drawn to fit the information
(c) no triangles can be drawn to fit the information.



26 In ΔABC, $a = 6$ cm, $b = 7$ cm, and either A or B is given. In which case is there certain to be a unique triangle?

27 In ΔABC, $C = 110°$, $b = 9$ and the value of c is given. If it is possible to find a unique value for a, what can you say about the value of c?

28 In ΔABC, $a = 6.5$ cm, $b = 8.5$ cm, $A = 40°$. Find the size of B.



29 In ΔABC, $b = 12$ cm, $c = 17$ cm, $B = 40°$. Find the size of C.

30 In ΔABC, $a = 5.4$ cm, $b = 6.7$ cm and $A = 53°40'$. Find the size of B.

31 In ΔABC, $a = 7.8$ cm, $c = 8.3$ cm and $C = 62°30'$. Calculate the size of the other sides and angles in this triangle.

5.8 THE COSINE RULE

For ΔABC, the cosine rule states $a^2 = b^2 + c^2 - 2bc\cos A$. In words:

- The square of one side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of those two sides and the cosine of the included angle.

To find a side, use:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

To find an angle, use:

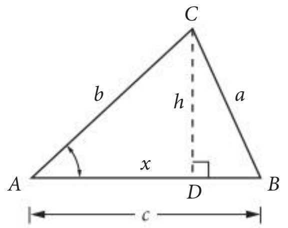
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Proof

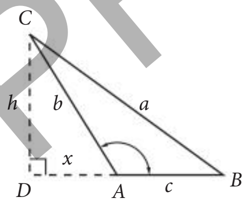
From C, draw the perpendicular to meet AB at D (for acute-angled triangles) or to meet BA produced at D (for obtuse-angled triangles).

Let $CD = h$ and $AD = x$-units.

Apply Pythagoras' theorem to ΔBCD:



Acute-angled triangle



Obtuse-angled triangle

Acute-angled triangle:

$$a^2 = h^2 + (c - x)^2$$
$$a^2 = h^2 + c^2 - 2cx + x^2$$

But: $h^2 + x^2 = b^2$

And: $x = b\cos A$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$

Obtuse-angled triangle:

$$a^2 = h^2 + (c + x)^2$$
$$a^2 = h^2 + c^2 + 2cx + x^2$$

But: $h^2 + x^2 = b^2$

And: $x = b\cos(180° - A)$

$x = -b\cos A$

$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$

The cosine rule can be used with a triangle when you are given:
(a) the lengths of three sides, **OR**
(b) the lengths of two sides and the size of the included angle.

If $A = 90°$ then $\cos A = \cos 90° = 0$. The size of A determines the proportions of the sides:

- If $A < 90°$ then $a^2 < b^2 + c^2$
- If $A = 90°$ then $a^2 = b^2 + c^2$ (Pythagoras' theorem)
- If $A > 90°$ then $a^2 > b^2 + c^2$

Important result

$$\cos(180° - \theta) = -\cos\theta$$

Hence: $\cos 120° = \cos(180° - 60°) = -\cos 60° = -0.5$

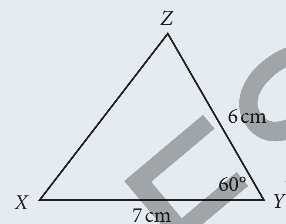
Solving an equation like $\cos\theta = -0.5$ gives $\theta = (180 - 60)° = 120°$.

Example 13

In triangle XYZ, $XY = 7$ cm, $YZ = 6$ cm and $Y = 60°$. Calculate the exact length of XZ.

Solution

Show the information on a diagram.



Two sides and the included angle are given.

Use the cosine rule:

$$y^2 = x^2 + z^2 - 2xz\cos Y$$
$$y^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos 60°$$
$$y^2 = 36 + 49 - 84 \times 0.5$$
$$y^2 = 43$$
$$y = \sqrt{43}$$

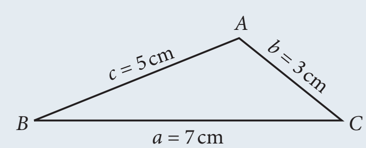
$\therefore\ XZ = \sqrt{43}$ cm

Example 14

Find the size of the largest angle of a triangle with side lengths 3 cm, 5 cm and 7 cm.

Solution

Show the information on a diagram.



The largest angle is opposite the longest side. Find the size of A.

Use the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos A = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5}$$
$$= \frac{9 + 25 - 49}{30}$$
$$= -\frac{15}{30} = -\frac{1}{2}$$

Because $\cos A < 0$, A is obtuse: $A = 180° - 60° = 120°$

The largest angle in the triangle is 120°.

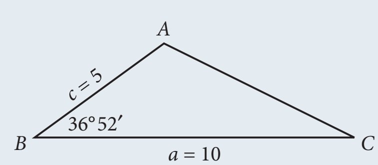
Example 15

In ΔABC, $a = 10$, $c = 5$ and $B = 36° 52'$. Calculate:

(a) The length b **(b)** the size of angle C

Solution

Show the information on a diagram.



(a) Two sides and the included angle are known.

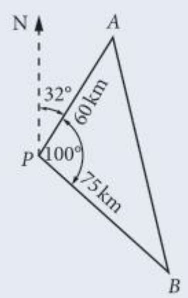
Use the cosine rule:
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = 10^2 + 5^2 - 2 \times 10 \times 5 \times \cos 36° 52'$$
$$b^2 = 125 - 100\cos 36° 52'$$
$$b^2 = 44.9966$$
$$b = 6.71$$

(b) Because you now know the length of the three sides and the size of one angle, you can use either the sine rule or the cosine rule to find the required angle. The sine rule is usually quicker and easier to use.

Use the sine rule:
$$\frac{\sin C}{c} = \frac{\sin B}{b}$$
$$\frac{\sin C}{5} = \frac{\sin 36° 52'}{6.71}$$
$$\sin C = \frac{5\sin 36° 52'}{6.71} = 0.4471$$
$$C = 26° 33'$$

Example 16

Two people set out from point P at the same time. One travels at 20 km h^{-1} along a straight road in the direction 032°. The other travels at 25 km h^{-1} along another straight road in the direction 132°. Find their distance apart after 3 hours.



Solution

After 3 hours, one person is at A, 60 km from P, and the other is at B, 75 km from P, as shown above.

$$\angle APB = 132° - 32° = 100°$$

Two sides and the included angle are known. Use the cosine rule:

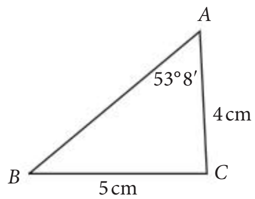
$$AB^2 = 60^2 + 75^2 - 2 \times 60 \times 75 \times \cos 100°$$
$$AB^2 = 10\,787.8$$
$$AB = 103.9$$

The two people are 103.9 km apart.

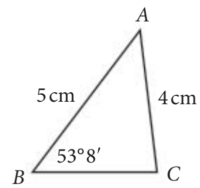
EXERCISE 5.8 THE COSINE RULE

1 In ΔABC, $b = 4$ cm, $c = 5$ cm, $A = 53° 8'$. The correct diagram for this information is:

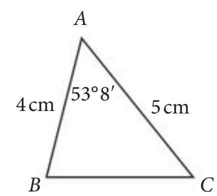
A



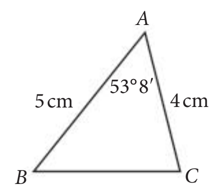
B



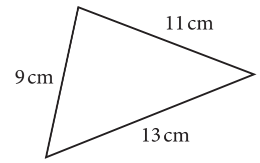
C



D

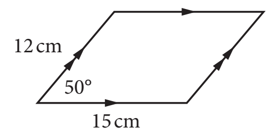


2 Calculate the cosine of the smallest angle of a triangle with side lengths 9 cm, 11 cm and 13 cm.



3 In ΔABC, $A = 36°\,52'$, $b = 7$ and $c = 10$. Calculate: **(a)** a **(b)** B.

4 Two adjacent sides of a parallelogram have lengths of 12 cm and 15 cm and the included angle is 50°. Calculate the lengths of the diagonals.



5 In ΔABC, $b = 4$ cm, $c = 5$ cm, $A = 53°\,8'$. Calculate the perimeter of the triangle, correct to the nearest centimetre.

6 Two sides of a triangle have lengths of 3.2 cm and 4.8 cm and the included angle is 65°. Calculate the length of the third side, correct to one decimal place. Indicate whether each statement is a correct or incorrect step in the working.

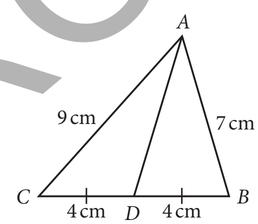
(a) $x = \dfrac{3.2\sin 65°}{4.8}$ **(b)** $x = \dfrac{4.8\sin 65°}{3.2}$ **(c)** $x^2 = 3.2^2 + 4.8^2 - 30.72\cos 65°$ **(d)** $x = 4.5$ cm

7 In ΔABC, $B = 126°\,52'$, $a = 12$ and $c = 15$. Find: **(a)** b **(b)** A.

8 The lengths of the sides of a triangle are in the ratio 5 : 6 : 9. Find the size of the largest angle.

9 In ΔABC, $BC = 8$ cm, $AC = 9$ cm and $AB = 7$ cm. If D is the midpoint of BC, calculate:

(a) the size of $\angle ABC$
(b) the length of AD
(c) the size of $\angle DAC$.



10 The two adjacent sides of a parallelogram have lengths of 8 cm and 10 cm. If the length of the longer diagonal is 14 cm, calculate:

(a) the size of the angles of the parallelogram **(b)** the length of the other diagonal.

11 In ΔPQR, $q = 12$, $r = 5$ and $\angle QPR = 108°$. Calculate: **(a)** p **(b)** $\angle PQR$.

12 In ΔABC, $BC = 11$ cm, $AC = 5$ cm and $AB = 8$ cm. P is a point on BC such that $BP = 4$ cm. Indicate whether each statement is correct or incorrect.

(a) $AP = 5$ cm **(b)** $\cos(\angle ABC) = \dfrac{10}{11}$ **(c)** $AP^2 = \dfrac{240}{11}$ **(d)** $\cos(\angle ACB) = \dfrac{41}{55}$

13 Two cars A and B depart from the same position. A travels along a straight road due east at 30 km h^{-1}. B departs 15 min after A and travels along another straight road in a north-east direction at 40 km h^{-1}. How far apart are they 15 min after B departs?

14 A, B and C are three towns such that B is 20 km from A in a direction 330° and C is 30 km from A in a direction $203°\,8'$. Find the distance from B to C.

15 P and Q are two towns 50 km apart with Q due east of P. A third town R, to the north of the line joining P and Q, is 70 km from P and 30 km from Q. Find the bearing of R from: **(a)** Q **(b)** P.

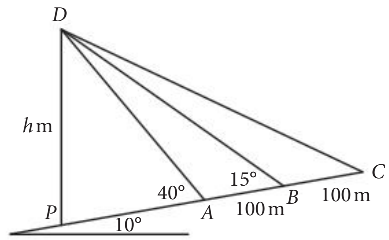
16 A lighthouse is 10 km north-west of a ship travelling due west at 16 km h^{-1}.

(a) How far is the ship from the lighthouse, 45 minutes later?
(b) What is the bearing of the lighthouse from the ship then?

17 P, A, B and C are four points (in that order) on a straight road that runs up a hill and makes a constant angle of 10° with the horizontal. A flagpole of height h m stands at P. From A and B, the top of the flagpole has elevations of 30° and 5° respectively above the horizontal.

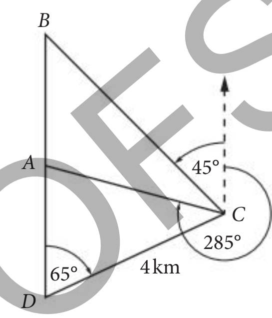
(a) If AB is 100 m, what is the height of the flagpole?

(b) If BC is also 100 m long, what is the elevation of the top of the flagpole from C?



18 From a point P, a person observes that the angle of elevation of the top of a cliff A is 40°. After walking 100 m towards A along a straight road inclined upwards at an angle of 15° to the horizontal, the angle of elevation of A is observed to be 50°. Find the vertical height of A above P.

19 The captain of a ship at D sailing on a bearing of 065° observes two lighthouses, A and B, in a line due north. After travelling 4 km to C, she notes that one lighthouse is on a bearing of 285° and the other 315°. Calculate the distance between the lighthouses.



20 Two cars leave a point A at the same time. One car averages 80 km h^{-1} along a straight road in the direction 025°. The other car averages 90 km h^{-1} along a straight road in the direction 135°. How far apart are they after 3 hours?

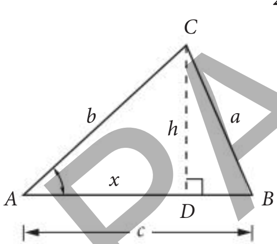
5.9 AREA OF A TRIANGLE

The formula for the area of a triangle is the same for both acute-angled and obtuse-angled triangles.

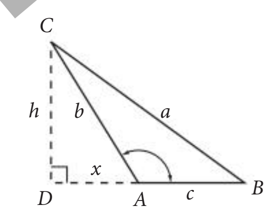
$$\text{Area of } \Delta ABC = \frac{\text{base} \times \text{altitude}}{2} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

In each triangle the altitude is of length h.

$$\therefore \text{ Area of } \Delta ABC = \frac{ch}{2}$$



Acute-angled triangle



Obtuse-angled triangle

Acute-angled triangle:

$$h = b\sin A$$

Obtuse-angled triangle:

$$h = b\sin(180° - A) = b\sin A$$

Hence: $\text{Area of } \Delta ABC = \text{Area of } \Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$

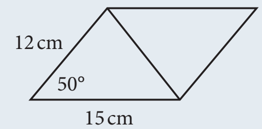
= half the product of two sides with the sine of the angle between them.

Example 17

The adjacent sides of a parallelogram are 12 cm and 15 cm and the angle between them is 50°. Calculate the area of the parallelogram correct to the nearest square centimetre.

Solution

Show the information on a diagram.



Draw the shorter diagonal.

The area of the parallelogram is twice the area of the triangle.

$$\text{Area of parallelogram} = 2 \times \frac{12 \times 15 \sin 50°}{2}$$
$$= 180 \sin 50° = 138\text{ cm}^2$$

EXERCISE 5.9 AREA OF A TRIANGLE

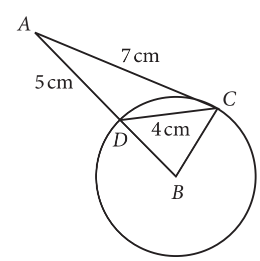
1 In ΔABC, $A = 36°\,52'$, $b = 7$ cm and $c = 10$ cm. Calculate:
(a) a (b) B (c) the area of ΔABC.

2 In ΔABC, $b = 4$ cm, $c = 5$ cm, $A = 53°\,8'$. Calculate the area of the triangle, correct to the nearest square centimetre.

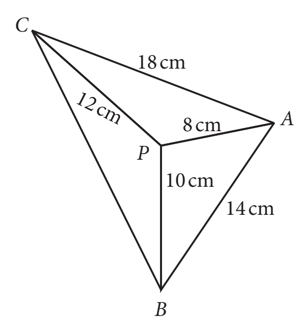
3 In ΔABC, $BC = 11$ cm, $AC = 5$ cm and $AB = 8$ cm. Calculate:
(a) the magnitude of A (b) the length of the perpendicular from A to BC
(c) the area of ΔABC.

4 The area of ΔABC if $a = 6$ cm, $b = 7$ cm and $c = 11$ cm is nearest to:
A 6 cm^2 B 19 cm^2 C 20 cm^2 D 22 cm^2

5 ABC is a triangle in which $AC = 7$ cm. A circle, centre B and radius BC, cuts AB internally at D. $AD = 5$ cm, $DC = 4$ cm. Calculate:
(a) the length of BC
(b) the area of ΔABC.



6 P, A, B and C are four points in a plane such that angles BPA and CPA are obtuse and on opposite sides of PA. $PA = 8$ cm, $BP = 10$ cm, $PC = 12$ cm, $AB = 14$ cm and $AC = 18$ cm. Calculate:
(a) the length of BC
(b) the area of ΔABC.



7 The equal sides AB and AC of an isosceles triangle ABC are each 5 cm and $BC = 4$ cm. D is a point on AC such that $DC = 1$ cm. Calculate:
(a) the size of A (b) the length of BD (c) the area of ΔABC.

8 $ABCD$ is a parallelogram in which $AB = 8$ cm, $BC = 5$ cm and $\angle DAB = 60°$. Calculate:
(a) the length of the diagonals AC and BD (b) the area of $ABCD$.

9 Two sides of a triangular field are 60 m and 50 m and the included angle is 140°. Indicate whether each statement is correct or incorrect.

(a) Third side = 103.4 m (b) Area = 964.2 m^2 (c) Third side = 38.8 m (d) Area = 1149 m^2

10 The sides of a triangular field have lengths 80 m, 90 m and 100 m. Calculate the area of the field.

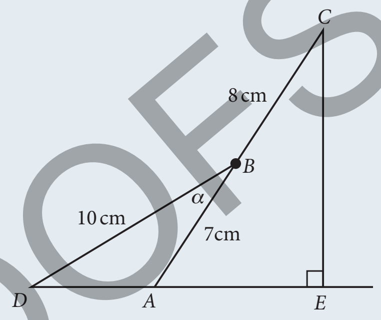
5.10 APPLIED TRIGONOMETRY

This section uses the skills from earlier in this chapter to solve harder two-dimensional problems and problems in three dimensions. In three-dimensional problems it is important that you break the 3D diagram into its 2D parts.

Example 18

In this diagram, $AC = 15$ cm, $AB = 7$ cm, $DB = 10$ cm and $\angle DBA = \alpha$.

Find the perpendicular distance CE of C from DA in terms of α and evaluate it for $\alpha = 25°$.



Solution

Use the cosine rule in ΔDBA: $DA^2 = 10^2 + 7^2 - 2 \times 10 \times 7 \cos \alpha$

$DA^2 = 149 - 140 \cos \alpha$

$\therefore \quad DA = \sqrt{149 - 140\cos\alpha}$ [1]

Use the sine rule in ΔDBA: $\dfrac{10}{\sin(\angle DAB)} = \dfrac{DA}{\sin\alpha}$

$\therefore \quad \sin(\angle DAB) = \dfrac{10\sin\alpha}{DA}$ [2]

In ΔCAE: $CE = 15\sin(\angle CAE)$ [3]

Now: $\angle DAB = 180° - \angle CAE$

Hence: $\sin(\angle DAB) = \sin(\angle CAE)$

Using [1], [2] and [3]: $CE = \dfrac{150\sin\alpha}{\sqrt{149 - 140\cos\alpha}}$

For $\alpha = 25°$: $CE = \dfrac{150\sin 25°}{\sqrt{149 - 140\cos 25°}}$

$= 13.48$

Perpendicular distance = 13.48 cm

EXERCISE 5.10 APPLIED TRIGONOMETRY

1 P, A, B, C are four points in a plane such that the angles BPA and CPA are obtuse and on opposite sides of PA. $PA = 8$ cm, $BP = 10$ cm, $PC = 12$ cm, $AB = 14$ cm and $AC = 18$ cm. Calculate the length of BC and the area of the triangle ABC.

2 P, A, B, C are four points (in order) on a straight road that runs up a hill at a constant angle of 10° to the horizontal. A flagpole with a height of h m stands at P. From A and B, the top of the flagpole has elevations of 30° and 5° respectively above the horizontal. If AB is 100 m long, what is the height of the flagpole? If BC is also 100 m long, what is the angle of elevation of the top of the flagpole from C?

3 From a point P, a person observes that the angle of elevation of the top of a cliff A is 40°. After walking 100 m towards A along a straight road that inclines upwards at an angle of 15° to the horizontal, the angle of elevation of A is observed to be 50°. Find the vertical height of A above P.

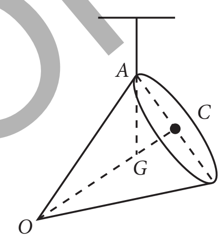
4 A ship sailing in a direction 065° observes two lighthouses in a line due north. After the ship has travelled 4 km, one of the lighthouses is on a bearing of 285° and the other 315°. The distance between the lighthouses is given by which two of the following expressions?

A $4\left(\frac{\sin 75°}{\sin 45°} - \frac{\sin 40°}{\sin 70°}\right)$ **B** $4\left(\frac{\sin 70°}{\sin 45°} - \frac{\sin 40°}{\sin 75°}\right)$ **C** $\frac{2\sin 65°}{\sin 75° \sin 45°}$ **D** $\frac{2\sin 75°}{\sin 65° \sin 45°}$

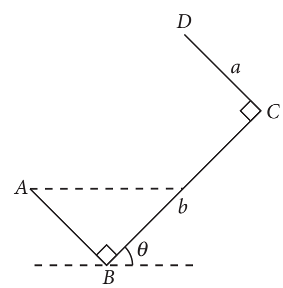
5 A flagpole 5 m high stands on top of a vertical tower. From a point on the ground, the angles of elevation of the top and bottom of the flagpole are 68° and 62° respectively. Show that the height of the tower (h metres) is given by $h = \frac{5\tan 62°}{\tan 68° - \tan 62°}$.

6 From a point P, an observer finds that the angle of elevation of the top of a vertical tower is $\alpha°$. After walking x metres horizontally towards the base of the tower, the observer finds the angle of elevation of the top of the tower is now $\beta°$. If the height of the tower is h metres, show that $x = h(\cot \alpha° - \cot \beta°)$.

7 A conical vessel with vertex at O and semi-vertical angle α is suspended from a point A on the rim of the base. G is a point on the axis OC of the cone such that $OG = \frac{2}{3}OC$. If the vessel rests with G vertically below A, show that the acute angle β that AO makes with the vertical is given by $\tan \beta = \frac{2\tan\alpha}{1 + 3\tan^2\alpha}$.



8 An open rectangular tank a units deep and b units wide holds water and is tilted so that the base BC makes an angle θ with the horizontal. When BC is returned to the horizontal, show that the depth of the water is $\frac{a^2\cot\theta}{2b}$ units.



Cross-section of tank

9 A and B are two towers, with B 4 km due east of A. A flagpole C has true bearings from A and B that are $\alpha°$ east of north and $\alpha°$ west of north respectively. A second flagpole D has true bearings from A and B that are $(\alpha + \beta)°$ east of north and $(\alpha - \beta)°$ west of north respectively. Draw a sketch to indicate the positions of A, B, C and D. Assuming that A, B, C and D are on level ground and that $\alpha = 25$, $\beta = 10$, find the distance between C and D.

10 In ΔABC, $AC = 7$ cm. A circle with centre B and radius BC cuts AB internally at D. $AD = 5$ cm, $DC = 4$ cm. The length of BC is:

A 10 cm **B** 5 cm **C** 7 cm **D** impossible to find

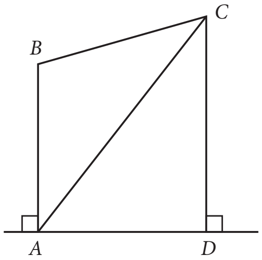
11 Two buildings of equal height are 40 m apart. At a point on the horizontal line joining the buildings' bases, the angles of elevation of the tops of the buildings are 47° and 28°. Show that the height h of the buildings is given by $h = \frac{40\tan 47° \tan 28°}{\tan 47° + \tan 28°}$.

12 A railway line runs south-west between two railway stations A and B. Two spires 3 km apart are both directly north-west of A. From B, the bearings of the spires are N 7.5° E and N 37.5° E. Show that $AB = 6\sin 52.5° \cos 7.5°$. If a train takes 4 minutes to travel from A to B, find its average speed.

13 From a point A, two points B and C are in line in a direction 049°. From a point D that is 100 m from A in a direction 139°, B is in a direction 352° and C is in a direction 022°. Calculate the distance between B and C.

14 An observer's eye is 2 m above the ground. A vertical pole fixed in the ground subtends an angle of 45° at the observer's eye. The angle of depression of the base of the flagpole from the eye is 15°. Show that the height of the flagpole is $2(1 + \tan 30° \tan 75°)$ metres.

15 In the diagram shown, $AB = 25$ cm, $BC = 27$ cm, $AC = 40$ cm. Calculate the length of AD.

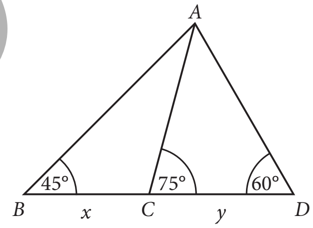


16 $ABCD$ is a cyclic quadrilateral in which $AB = 5$ cm, $BC = 6$ cm, $CD = 7$ cm, $AD = 8$ cm. Show that $\cos(\angle ADC) = \frac{13}{43}$.

17 O, A, B are three points in order in a straight line. The bearings of A and B from O are both 020° T. From a point P that is 4 km from O in a NW direction, the bearings of A and B are 112° T and 064° T. Calculate the distance from A to B.

18 From a point P an observer finds that the angle of elevation of the top of a vertical tower is $\alpha°$. After walking x m horizontally towards the foot of the tower, the observer finds that the angle of elevation is $\beta°$. If the height of the tower is h m, prove that $h = \dfrac{x\sin\alpha°\sin\beta°}{\sin(\beta - \alpha)°}$.

19 Using the sine rule, show that $\dfrac{x}{y} = \dfrac{\sqrt{3}}{2}$ and also that $\dfrac{\text{Area } \Delta ABC}{\text{Area } \Delta ADC} = \dfrac{\sqrt{3}}{2}$.



20 A side of a hill can be regarded as a plane making an angle of 20° with the horizontal.

(a) From a point A at the base of the hill, I walk 100 m up the hill along a straight line of steepest slope to the point B. Find the vertical height of B above A.

(b) A tower 40 m tall is constructed at B. Find the angle subtended by the tower at my eyes, which are 1.5 m above the ground, when I stand at A.

(c) A straight road is constructed on the hillside with a slope of 1 in 10, i.e. with a 1 m vertical rise for every 10 m horizontally forward. Find the angle this road must make with the line of steepest slope on the hillside.

CHAPTER REVIEW 5

1 Simplify:

(a) $\cos(90° - \theta)$ **(b)** $\sin(270° - \theta)$ **(c)** $\cos(90° + \theta)$
(d) $\tan(\theta - 180°)$ **(e)** $\tan(180° - \theta)$ **(f)** $\sin(\theta + 180°)$

2 Write the exact value.

(a) $\tan 315°$ **(b)** $\sin 225°$ **(c)** $\cos 180°$
(d) $\tan 360°$ **(e)** $\sin 60°$ **(f)** $\cos 210°$

3 Simplify:

(a) $\dfrac{\cos\theta}{\sin(90° - \theta)}$ **(b)** $\cos(90° + \theta) + \sin\theta$

4 If $\tan\theta = \frac{3}{5}$ and $180° < \theta < 270°$, write the exact value of: **(a)** $\sin\theta$ **(b)** $\cos\theta$

5 If $\sin\alpha = 0.6$ and $0° < \alpha < 90°$, write the exact value of:

(a) $\sin(180° - \alpha)$
(b) $\cos(90° - \alpha)$
(c) $\cos(180° + \alpha)$
(d) $\tan\alpha$
(e) $\tan(180° - \alpha)$
(f) $\sin(360° - \alpha)$

6 If $\tan\theta = t$, express in terms of t:

(a) $\tan(90° - \theta)$
(b) $\tan(180° + \theta)$
(c) $\cot(180° - \theta)$
(d) $\tan(360° - \theta)$
(e) $\tan(-\theta)$
(f) $\tan(90° + \theta)$

7 Calculate the cosine of the smallest angle of the triangle with side lengths 5 cm, 6 cm and 7 cm.

8 Find the size of the largest angle of the triangle with side lengths 5 cm, 6 cm and 8 cm. Hence, show that the triangle is obtuse-angled.

9 In ΔABC, $B = 53°$, $C = 48°$, $AC = 8$ cm. Calculate:

(a) the length of BC
(b) the area of ΔABC.

10 A ladder 8 m long rests against a wall and its foot makes an angle of 60° with the horizontal ground. The top of the ladder then slips down the wall until its foot makes an angle of 45° with the ground. Find, in simplest surd form, how far down the wall the ladder slips.

11 From a point A, level with the foot of a vertical pole and 30 m from it, the angle of elevation of the top of the pole is 40°. Calculate:

(a) the height of the pole
(b) the direct distance from A to the top of the pole
(c) the angle of elevation from A of a point half-way up the pole.

12 AB and CD are two vertical buildings with their bases at A and at C on horizontal ground. The height of AB is 30 m. The angle of elevation of B as seen from C is 25° and the angle of elevation of D as seen from A is 40°. Calculate:

(a) the horizontal distance between the buildings
(b) the height of CD
(c) the angle of depression of B as seen from D.

13 Two yachts sail in a straight line from a buoy B. One sails 10 km in the direction 040° and the other sails 20 km in the direction 160°.

(a) How far apart are the yachts?
(b) What is the bearing of the first yacht as seen from the second yacht?

14 (a) Find a simplified expression for r given that $r^2 = (100 - 50t)^2 + (80t)^2 - 4(100 - 50t) \times 80t \times \cos 60°$.

(b) Find the value of r to the nearest whole number when $t = \frac{30}{43}$.